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LOGIC

PART II

DEMONSTRATIVE INFERENCE:
DEDUCTIVE AND INDUCTIVE

BY

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INTRODUCTION TO PART II

§ 1. Before introducing the topics to be examined in Part II, I propose to recapitulate the substance of Part I, and in so doing to bring into connection with one another certain problems which were there treated in different chapters. I hope thus to lay different emphasis upon some of the theories that have been maintained, and to remove any possible misunderstandings where the treatment was unavoidably condensed.

In my analysis of the proposition I have distinguished the natures of substantive and adjective in a form intended to accord in essentials with the doctrine of the large majority of logicians, and as far as my terminology is new its novelty consists in giving wider scope to each of these two fundamental terms. *Prima facie* it might be supposed that the connection of substantive with adjective in the construction of a proposition is tantamount to the metaphysical notions of substance and inherence. But my notion of substantive is intended to include, besides the metaphysical notion of substance —so far as this can be philosophically justified—the notion of occurrences or events to which some philosophers of the present day wish to restrict the realm of reality. Thus by a substantive *proper* I mean an existent; and the category of the existent is divided into the two subcategories: what continues to exist, or the continuant; and what ceases to exist, or the occurrent, every occurrent being referrible to a continuant. To exist is to be

in temporal or spatio-temporal relations to other existents; and these relations between existents are the fundamentally external relations. A substantive proper cannot characterise, but is necessarily characterised; on the other hand, entities belonging to any category whatever (substantive proper, adjective, proposition, etc.) may be characterised by adjectives or relations belonging to a special adjectival sub-category corresponding, in each case, to the category of the object which it characterises. Entities, other than substantives proper, of which appropriate adjectives can be predicated, function as quasi-substantives.

§ 2. The term adjective, in my application, covers a wider range than usual, for it is essential to my system that it should include relations. There are two distinct points of view from which the treatment of a relation as of the same logical nature as an adjective may be defended. In the first place the complete predicate in a relational proposition is, in my view, relatively to the subject of such proposition, equivalent to an adjective in the ordinary sense. For example, in the proposition, 'He is afraid of ghosts,' the relational component is expressed by the phrase 'afraid of'; but the complete predicate 'afraid of ghosts' (which includes this relation) has all the logical properties of an ordinary adjective, so that for logical purposes there is no fundamental distinction between such a relational predicate and an irrational predicate. In the second place, if the relational component in such a proposition is separated, I hold that it can be treated as an adjective predicated of the substantive-couple 'he' and 'ghosts'. In other words, a relation cannot be identified with a *class* of couples, i.e. be

conceived extensionally; but must be understood to *characterise* couples, i.e. be conceived intensionally. It seems to me to raise no controvertible problem thus to include relations under the wide genus adjectives. It is compatible, for example, with almost the whole of Mr Russell's treatment of the proposition in his *Principles of Mathematics*; and, without necessarily entering into the controvertible issues that emerge in such philosophical discussions, I hold that some preliminary account of relations is required even in elementary logic.

§ 3. My distinction between substantive and adjective is roughly equivalent to the more popular philosophical antithesis between particular and universal; the notions, however, do not exactly coincide. Thus I understand the philosophical term particular not to apply to quasi-substantives, but to be restricted to substantives proper, i.e. existents, or even more narrowly to occurrents. On the other hand, I find a fairly unanimous opinion in favour of calling an adjective predicated of a particular subject, a particular—the name universal being confined to the abstract conception of the adjective. Thus red or redness, abstracted from any specific judgment, is held to be universal; but the redness, manifested in a particular object of perception, to be itself particular. Furthermore, *qua* particular, the adjective is said to be an existent, apparently in the same sense as the object presented to perception is an existent. To me it is difficult to argue this matter because, while acknowledging that an adjective may be called a universal, I regard it not as a mere abstraction, but as a factor in the real; and hence, in holding that the objectively real is properly construed into an adjective

characterising a substantive, the antithesis between the particular and the universal (i.e. in my terminology between the substantive and the adjective) does not involve separation within the real, but solely a separation *for thought*, in the sense that the conception of the substantive apart from the adjective, as well as the conception of the adjective apart from the substantive, equally entail abstraction.

§ 4. Again, taking the whole proposition constituted by the connecting of substantive with adjective, I have maintained that in a virtually similar sense the proposition is to be conceived as abstract. But, whereas the characterising tie may be called constitutive in its function of connecting substantive with adjective to construct the proposition, I have spoken of the assertive tie as epistemic, in the sense that it connects the thinker with the proposition in constituting the unity which may be called an act of judgment or of assertion. When, however, this act of assertion becomes in its turn an object of thought, it is conceived under the category of the existent; for such an act has temporal relations to other existents, and is necessarily referrible to a thinker conceived as a continuant. Though, relatively to the primary proposition, the assertive tie must be conceived as epistemic; yet, relatively to the secondary proposition which predicates of the primary that it has been asserted by A, the assertive tie functions constitutively.

§ 5. In view of a certain logical condition presupposed throughout this Part of my work, I wish to remind the reader of that aspect of my analysis of the proposition, according to which I regard the subject as that which is *given* to be determinately characterised

by thought. Now I hold that for a subject to be characterised by some adjectival determinate, it must first have been presented as characterised by the corresponding adjectival determinable. The fact that what is given is characterised by an adjectival determinable is constitutive; but the fact that it is *presented* as thus characterised is epistemic. Thus, for a surface to be characterised as red or as square, it must first have been constructed in thought as being the kind of thing that has colour or shape; for an experience to be characterised as pleasant or unpleasant, it must first have been constructed in thought as the kind of thing that has hedonic tone. Actually what is given, is to be determined with respect to a conjunction of several specific aspects or determinables; and these determine the category to which 'the given' belongs. For example, on the dualistic view of reality, the physical has to be determined under spatio-temporal determinables, and the psychical under the determinable consciousness or experience. If the *same* being can be characterised as two-legged and as rational, he must be put into the category of the physico-psychical.

§ 6. The passage from topics treated in Part I to those in Part II, is equivalent to the step from implication to inference. The term inference, as introduced in Part I, did not require technical definition or analysis, as it was sufficiently well understood without explanation. It was, however, necessary in Chapter III to indicate in outline one technical difficulty connected with the paradox of implication; and there I first hinted, what will be comprehensively discussed in the first chapter of this Part, that implication is best conceived

as potential inference. While for elementary purposes implication and inference may be regarded as practically equivalent, it was pointed out in Chapter III that there is nevertheless one type of limiting condition upon which depends the possibility of using the relation of implication for the purposes of inference. Thus reference to the specific problem of the paradox of implication was unavoidable in Part I, inasmuch as a comprehensive account of symbolic and mechanical processes necessarily included reference to all possible limiting cases; but, apart from such a purely abstract treatment, no special logical importance was attached to the paradox. The limiting case referred to was that of the permissible employment of the compound proposition 'If p then q,' in the unusual circumstance where knowledge of the truth or the falsity of p or of q was already present when the compound proposition was asserted. This limiting case will not recur in the more important developments of inference that will be treated in the present part of my logic.

It might have conduced to greater clearness if, in Chapters III and IV, I had distinguished—when using the phrase *implicative proposition*—between the primary and secondary interpretations of this form of proposition. Thus, when the compound proposition 'If p then q' is rendered, as Mr Russell proposes, in the form 'Either not-p or q,' the compound is being treated as a *primary* proposition of the same type as its components p and q. When on the other hand we substitute for 'If p then q' the phrase 'p implies q,' or preferably 'p would imply q,' the proposition is no longer primary, inasmuch as it predicates about the proposition q the adjective 'implied by p' which renders the compound a secondary

proposition, in the sense explained in Chapter IV[1]. Now whichever of these two interpretations is adopted, the inference which is legitimate under certain limiting conditions is the same. Thus given the compound 'Either not-p or q' conjoined with the assertion of 'p,' we could infer 'q'; just as given 'p implies q' conjoined with the assertion of 'p,' we infer 'q.' It is for this reason that the two interpretations have become merged into one in the ordinary symbolic treatment of compound propositions; and in normal cases no distinction is made in regard to the possibility of using the primary or secondary interpretation for purposes of inference. The normal case, however, presupposes that p and q are entertained hypothetically; when this does not obtain, the danger of petitio principii enters. The problem in Part I was only a very special and technical case in which this fallacy has to be guarded against; in Part II, it will be dealt with in its more concrete and philosophically important applications.

§ 7. The mention of this fallacy immediately suggests Mill's treatment of the functions and value of the syllogism; but, before discussing his views, I propose to consider what his main purpose was in tackling the charge of petitio principii that had been brought against the whole of formal argument, including in particular the syllogism. In the first section of his chapter, Mill refers to two opposed classes of philosophers—the one of whom regarded syllogism as the universal type of all logical reasoning, the other of whom regarded syllogism

[1] The interpretation of the implicative form 'p implies q' as secondary is developed in Chapter III, § 9, where the modal adjectives *necessary*, *possible*, *impossible*, are introduced.

as useless on the ground that all such forms of inference involve *petitio principii*. He then proceeds: 'I believe both these opinions to be fundamentally erroneous,' and this would seem to imply that he proposed to relieve the syllogism from the charge. I believe, however, that all logicians who have referred to Mill's theory—a group which includes almost everyone who has written on the subject since his time—have assumed that the purport of the chapter was to *maintain* the charge of petitio principii, an interpretation which his opening reference to previous logicians would certainly not seem to bear. His subsequent discussion of the subject is, verbally at least, undoubtedly confusing, if not self-contradictory; but my personal attitude is that, whatever may have been Mill's general purpose, it is from his own exposition that I, in common with almost all his contemporaries, have been led to discover the principle according to which the syllogism can be relieved from the incubus to which it had been subject since the time of Aristotle. In my view, therefore, Mill's account of the philosophical character of the syllogism is incontrovertible; I would only ask readers to disregard from the outset any passage in his chapter in which he appears to be contending for the annihilation of the syllogism as expressive of any actual mode of inference.

Briefly his position may be thus epitomised. Taking a typical syllogism with the familiar major 'All men are mortal,' he substituted for 'Socrates' or 'Plato' the minor term 'the Duke of Wellington' who was then living. He then maintained that, going behind the syllogism, certain instantial evidence is required for establishing the major; and furthermore that the validity

of the conclusion that the Duke of Wellington would die depends ultimately on this instantial evidence. The interpolation of the universal major 'All men will die' has undoubted value, to which Mill on the whole did justice; but he pointed out that the formulation of this universal adds nothing to the positive or factual data upon which the conclusion depends. It follows from his exposition that a syllogism whose major is admittedly established by induction from instances can be relieved from the reproach of begging the question or circularity if, and only if, the minor term is not included in the ultimate evidential data. The Duke of Wellington being still living could not have formed part of the evidence upon which the universal major depended. It was therefore part of Mill's logical standpoint to maintain that there were principles of induction by which, from a limited number of instances, a universal going beyond these could be logically justified. This contention may be said to confer constitutive validity upon the inductive process. It is directly associated with the further consideration that an instance, not previously examined, may be adduced to serve as minor premiss for a syllogism, and that such an instance will always preclude circularity in the formal process. Now the charge of circularity or petitio principii is epistemic; and the whole of Mill's argument may therefore be summed up in the statement that the epistemic validity of syllogism and the constitutive validity of induction, both of which had been disputed by earlier logicians, stand or fall together.

In order to prevent misapprehension in regard to Mill's view of the syllogism, it must be pointed out that he virtually limited the topic of his chapter to cases in

which the major premiss would be admitted by all logicians to have been established by means of induction in the ordinary sense, i.e. by the simple enumeration of instances; although many of them would have contended that such instantial evidence was not by itself sufficient. Thus all those cases in which the major was otherwise established, such as those based on authority, intuition or demonstration, do not fall within the scope of Mill's solution. Unfortunately all the commentators of Mill have confused his view that universals cannot be intuitively but only empirically established, with his specific contention in Chapter IV. I admit that he himself is largely responsible for this confusion, and therefore, while supporting his view on the functions of the syllogism, I must deliberately express my opposition to his doctrine that universals can only ultimately be established empirically, and limit my defence to his analysis of those syllogisms in which it is acknowledged that the major is thus established. Even here his doctrine that all inference is from particulars to particulars is open to fundamental criticism; and, in my treatment of the principles of inductive inference which will be developed in Part III, I shall substitute an analysis which will take account of such objections as have been rightly urged against Mill's exposition.

[NOTE. There are two cases in which the technical terminology employed in Part II differs from that in Part I. (1) The phrase *primitive proposition*, in Part I, is to be understood psychologically; in Part II, logically as equivalent to *axiom*. (2) *Counter-implicative*, in Part I, applies to the form of a compound proposition; in Part II, to a principle of inference.]

CHAPTER I

INFERENCE IN GENERAL

§ 1. INFERENCE is a mental process which, as such, has to be contrasted with implication. The connection between the mental act of inference and the relation of implication is analogous to that between assertion and the proposition. Just as a proposition is what is potentially assertible, so the relation of implication between two propositions is an essential condition for the possibility of inferring one from the other; and, as it is impossible to define a proposition ultimately except in terms of the notion of asserting, so the relation of implication can only be defined in terms of inference. This consideration explains the importance which I attach to the recognition of the mental attitude involved in inference and assertion; after which the strictly logical question as to the distinction between valid and invalid inference can be discussed. To distinguish the formula of implication from that of inference, the former may be symbolised 'If p then q,' and the latter 'p therefore q,' where the symbol q stands for the conclusion and p for the premiss or conjunction of premisses.

The proposition or propositions from which an inference is made being called premisses, and the proposition inferred being called the conclusion, it is commonly supposed that the premisses are the propositions first presented in thought, and that the transition from these to the thought of the conclusion is the

last step in the process. But in fact the reverse is usually the case; that is to say, we first entertain in thought the proposition that is technically called the conclusion, and then proceed to seek for other propositions which would justify us in asserting it. The conclusion may, on the one hand, first present itself to us as potentially assertible, in which case the mental process of inference consists in transforming what was *potentially* assertible into a proposition *actually* asserted. On the other hand, we may have already satisfied ourselves that the conclusion can be validly asserted apart from the particular inferential process, in which case we may yet seek for other propositions which, functioning as premisses, would give an independent or additional justification for our original assertion. In every case, the process of inference involves three distinct assertions: first the assertion of 'p,' next the assertion of 'q,' and thirdly the assertion that 'p would imply q.' It must be noted that 'p would imply q,' which is the proper equivalent of 'if p then q,' is the more correct expression for the relation of implication, and not 'p implies q'—which rather expresses the completed inference. This shows that inference cannot be defined in terms of implication, but that implication must be defined in terms of inference, namely as equivalent to potential inference. Thus, in inferring, we are not merely passing from the assertion of the premiss to the assertion of the conclusion, but we are also implicitly asserting that the assertion of the premiss is *used* to justify the assertion of the conclusion.

§ 2. Some difficult problems, which are of special importance in psychology, arise in determining quite

precisely the range of those mental processes which may be called *inference*: in particular, how far assertion or inference is involved in the processes of association and of perception. These difficulties have been aggravated rather than removed by the quite false antithesis which some logicians have drawn between logical and psychological inference. Every inference is a mental process, and therefore a proper topic for psychological analysis; on the other hand, to infer is to think, and to think is virtually to adopt a logical attitude; for everyone who infers, who asserts, who thinks, *intends* to assert truly and to infer validly, and this is what constitutes assertion or inference into a logical process. It is the concern of the science of logic, as contrasted with psychology, to criticise such assertions and inferences from the point of view of their validity or invalidity.

Let us then consider certain mental processes—in particular processes of association—which have the semblance of inference. In the first place, there are many unmistakeable cases of association in which no inference whatever is even *apparently* involved. Any familiar illustration, either of contiguity or of similarity, will prove that association in itself does not entail inference. If a cloudy sky raises memory-images of a storm, or leads to the mental rehearsal of a poem, or suggests the appearance of a slate roof, in none of these revivals by association is there involved anything in the remotest degree resembling inference. The case of contiguity is that which is most commonly supposed to involve some sort of inference; but in this supposal there is a confusion between recollection and expectation. Our recollection of storms that we have experienced in

the past is obviously distinct from our expectation that a storm is coming on in the immediate future. It is to this latter process of expectation, and not to the former process of recollection, that the term inference is more or less properly applied; but even here we must make a careful psychological distinction. We may expect a storm when we notice the darkness of the sky, without at all having actually recalled past experiences of storms; in this case no inference is involved, since there has been only one assertion, namely, what would constitute the conclusion without any other assertion that would constitute a premiss. In order to speak properly of inference in such cases, the minimum required is the assertion that the sky is cloudy and that *therefore* there will be a storm. Here we have two explicit assertions, together with the inference involved in the word 'therefore.' It is of course a subtle question for introspection as to whether this threefold assertion really takes place. This difficulty does not at all affect our definition of inference; it would only affect the question whether in any given case inference had actually occurred. It has been suggested that, where there has been nothing that logic could recognise as an inference, there has yet been inference in a psychological sense; but this contention is absurd, since it is entirely upon psychological grounds that we have denied the existence of inference in such cases.

Let us consider further the logical aspects of a genuine inference, following upon such a process of association as we have illustrated. The scientist may hold that the appearance of the sky is not such as to warrant the expectation of an on-coming storm. He

may, therefore, criticise the inference as invalid. Thus, assuming the actuality of the inference from the psychological point of view, it may yet be criticised as invalid from the logical point of view. So far we have taken the simplest case, where the single premiss 'The sky is cloudy' is asserted. But, when an additional premiss such as 'In the past cloudy skies have been followed by storm' is asserted, then the inference is further rationalised, since the two premisses taken together constitute a more complete ground for the conclusion than the single premiss. This additional premiss is technically known as a *particular* proposition. If the thinker is pressed to find still stronger logical warrant for his conclusion, he may assert that in *all* his experiences cloudy skies have been followed by storm (a limited universal). The final stage of rationalisation is reached when the universal limited to all remembered cases is used as the ground for asserting the *unlimited* universal for all cases. But even now the critic may press for further justification. To pursue this topic would obviously require a complete treatment of induction, syllogism, etc., from the logical point of view. Enough has been said to show that, however inadequate may be the grounds offered in justification of a conclusion, this has no bearing upon the nature or upon the fact of inference as such, but only upon the criticism of it as valid or invalid.

As in association, so also in perception, a psychological problem presents itself. There appear to be at least three questions in dispute regarding the nature of perception, which have close connection with logical analysis: First, how much is contained in the percept

besides the immediate sense experience? Secondly, does perception involve assertion? Thirdly, does it involve inference? To illustrate the nature of the first problem, let us consider what is meant by the visual perception of a match-box. This is generally supposed to include the representation of its tactual qualities; in which case, the content of the percept includes qualities other than those sensationally experienced. On the other hand, supposing that an object touched in the dark is recognised as a match-box, through the special character of the tactual sensations, would the *representation* of such visual qualities as distinguish a match-box from other objects be included in the tactual perception of it as a match-box? The same problem arises when we recognise a rumbling noise as indicating a cart in the road: i.e. should we say, in this case, that the auditory percept of the cart includes visual or other distinguishing characteristics of the cart not sensationally experienced? In my view it is inconsistent to include in the content of the visual percept tactual qualities not sensationally experienced, unless we also include in the content of a tactual or auditory percept visual or similar qualities not sensationally experienced[1].

This leads up to our second question, namely whether in such perceptions there is an assertion (*a*) predicating of the experienced sensation certain specific qualities; or an assertion (*b*) of having experienced in the past similar sensations simultaneously with the perception of

[1] In speaking here of the *mental representation* of qualities not sensationally experienced, I am putting entirely aside the very important psychological question as to whether such mental representations are in the form of 'sense-imagery' or of 'ideas.'

a certain object. Employing our previous illustration, we may first question whether the assertion 'There is a cart in the road' following upon a particular auditory sensation, involves (*a*) the explicit characterisation of that sensation. Now if the specific character of the noise as a sensation merely *caused* a visual image which in its turn caused the assertion 'There is a cart in the road,' then in the absence of assertion (*a*) there is no explicit inference. In order to become inference, the character operating (through association) as *cause* would have to be predicated (in a connective judgment) as *ground*. On the other hand, any experience that could be described as hearing a noise of a certain more or less determinate character would involve, in my opinion, besides assimilation, a judgment or assertion (*a*) expressible in some such words as 'There is a rumbling noise.' The further assertion that there is a cart in the road is accounted for (through association) by previous experiences of hearing such a noise simultaneously with seeing a cart. Assuming that association operates by arousing memory-images of these previous experiences, it is only when by their vividness or obtrusiveness these memory-images give rise to a *memory-judgment*, that the assertion (*b*) occurs. We are now in a position to answer the third question as to the nature of perception; for, if either the assertion of (*a*) alone or of (*b*) with (*a*) occurs along with the assertion that there is a cart in the road, then inference is involved; otherwise it is not.

§ 3. Passing from the psychological to the strictly logical problem, we have to consider in further detail the conditions for the validity of an inference symbolised as '$p \therefore q$.' These conditions are twofold, and may be

conveniently distinguished in accordance with my terminology as constitutive and epistemic. They may be briefly formulated as follows:

Conditions for Validity of the Inference 'p .·. q'

Constitutive Conditions: (i) the proposition 'p' and (ii) the proposition 'p would imply q' must both be true.

Epistemic Conditions: (i) the asserting of 'p' and (ii) the asserting of 'p would imply q' must both be permissible without reference to the asserting of q.

It will be noted that the constitutive condition exhibits the dependence of inferential validity upon a certain relation between the *contents* of premiss and of conclusion; the epistemic condition, upon a certain relation between the *asserting* of the premiss and the *asserting* of the conclusion. Taking the constitutive condition first, we observe that the distinction between inference and implication is sometimes expressed by calling implication 'hypothetical inference'—the meaning of which is that, in the act of inference, the premiss must be categorically asserted; while, in the relation of implication, this premiss is put forward merely hypothetically. This was anticipated above by rendering the relation of implication in the subjunctive mood (p would imply q) and the relation of inference in the indicative mood (p implies q).

Further to bring out the connection between the epistemic and the constitutive conditions, it must be pointed out that an odd confusion attaches to the use of the word 'imply' in these problems. The almost universal application of the relation of implication in logic is as a relation between two propositions; but, in familiar language, the term 'imply' is used as a relation

between two assertions. Consider for instance (*a*) '*B*'s asserting that there will be a thunderstorm would imply his having noticed the closeness of the atmosphere,' and (*b*) 'the closeness of the atmosphere would imply that there will be a thunderstorm.' The first of these relates two mental acts of the general nature of assertion, and is an instance of 'the asserting of *q* would imply having asserted *p*'; the second is a relation between two propositions, and is an instance of 'the proposition *p* would imply the proposition *q*.' Comparing (*a*) with (*b*) we find that implicans and implicate have changed places. Indeed the sole reason why the asserting of the thunderstorm was supposed to imply having asserted the closeness of the atmosphere was that, in the speaker's judgment, the closeness of the atmosphere would imply that there will be a thunderstorm.

Recognising, then, this double and sometimes ambiguous use of the word 'imply,' we may restate the first of the two epistemic conditions and the second of the two constitutive conditions for the validity of the inference '*p* ∴ *q*' as follows:

Epistemic condition (i): the asserting of the proposition '*p*' should *not* have implied the asserting of the proposition '*q*.'

Constitutive condition (ii): the proposition '*p*' should imply the proposition '*q*.'

The former is merely a condensed equivalent of our original formulation, viz. that 'the asserting of the proposition '*p*' *must be permissible without reference to* the asserting of the proposition '*q*.'

Now the fact that there is this double use of the term 'imply' accounts for the paradox long felt as

regards the nature of inference: for it is urged that, in order that an inference may be formally valid, it is required that the conclusion should be *contained* in the premiss or premisses; while, on the other hand, if there is any genuine advance in thought, the conclusion must *not* be contained in the premiss. This word 'contained' is doubly ambiguous: for, in order to secure formal validity, the premisses regarded as propositions *must imply* the conclusion regarded as a proposition; but, in order that there shall be some real advance and not a mere *petitio principii*, it is required that the asserting of the premisses *should not have* implied the previous asserting of the conclusion. These two horns of the dilemma are exactly expressed in the constitutive and epistemic conditions above formulated.

§ 4. We shall now explain how the constitutive conditions for the validity of inference, which have been expressed in their most general form, are realised in familiar cases. The general constitutive condition 'p would imply q' is *formally* satisfied when some specific logical relation holds of p to q; and it is upon such a relation that the formal truth of the assertion that 'p would imply q' is based. There are two fundamental relations which will render the inference from p to q, not only valid, but *formally* valid; and these relations will be expressed in formulae exhibiting what will be called the *Applicative* and the *Implicative* Principles of Inference. The former may be said to formulate what is involved in the intelligent use of the word 'every'; the latter what is involved in the intelligent use of the word 'if.'

In formulating the Applicative principle, we take p

to stand for a proposition universal in form, and q for a singular proposition which predicates of some single case what is predicated universally in p. The Applicative principle will then be formulated as follows:

From a predication about 'every' we may formally infer the same predication about 'any given.'

In formulating the Implicative principle, we take p to stand for a compound proposition of the form 'x and "x implies y"' and q to stand for the simple proposition 'y.' The Implicative principle will then be formulated as follows:

From the compound proposition 'x and "x implies y"' we may formally infer 'y.'

§ 5. We find two different forms of proposition, one or other of which is used as a premiss in every formal inference; the distinction between which is fundamental, but has been a matter of much controversy among logicians. In familiar logic the two kinds of proposition to which I shall refer are known respectively as universal and hypothetical. As an example of the former, take 'Every proposition can be subjected to logical criticism'; from this universal proposition we may directly infer 'That "*matter exists*" can be subjected to logical criticism.' This inference illustrates what I have called the Applicative Principle, and its premiss will be called an Applicational universal. Take next the example 'If this can swim it breathes,' and 'it can swim'; from this conjunction of propositions we infer that 'it breathes'; here, the hypothetical premiss being in our terminology called implicative, the inference in question illustrates the use of the Implica-

tive Principle. It is the combination of these two principles that marks the advance made in passing from the most elementary forms of inference to the syllogism. For example: From 'Everything breathes if able to swim' we can infer 'This breathes if able to swim,' where the applicative principle only is employed. Conjoining the conclusion thus obtained with the further premiss 'This can swim,' we can infer 'this breathes,' where the implicative principle only is employed. In this analysis of the syllogism which involves the interpolation of an additional proposition, we have shown how the two principles of inference are successively employed. The ordinary formulation of the syllogism would read as follows: 'Everything that can swim breathes; this can swim; therefore this breathes.' In place of the usual expression of the major premiss, I have substituted 'Everything breathes if able to swim,' in order to show how the major premiss prepares the way for the inferential employment successively of the applicative and of the implicative principles.

§ 6. Now the two propositions 'Every proposition can be subjected to logical criticism' and 'everything that is able to swim breathes' must be carefully contrasted. Both of them are universal in form; but in the latter the subject term contains an explicit characterising adjective, viz. *able to swim.* The presence of a characterising adjective in the subject anticipates the occasion on which the question would arise whether this adjective is to be predicated of a given object. In the syllogism, completed as in the preceding section, the universal major premiss is combined with an affirmative minor premiss, where the adjective entertained cate-

gorically as *predicate* of the minor is the same as that which was entertained hypothetically as *subject* of the major. This double functioning of an adjective is the one fundamental characteristic of all syllogism; where it will be found that one (or, in the fourth figure, every) term occurs once in the subject of a proposition, where it is entertained hypothetically, and again in the predicate of another proposition where it is entertained categorically.

The essential distinction between the two contrasted universals (applicational and implicational) lies in the fact that an inference can be drawn from the former on the applicative principle alone, which dispenses with the minor premiss. We have to note the nature of the substantive that occurs in the *applicational* universal as distinguished from that which occurs in the *implicational* universal. The example already given contained 'proposition' as the subject term, and a few other examples are necessary to establish the distinction in question. 'Every individual is self-identical,' therefore 'the author of the *Republic* is self-identical'; 'Every conjunction of predications is commutative,' therefore 'the conjunction *lightning before* and *thunder after* is commutative'; 'Every adjective is a relatively determinate specification of a relatively indeterminate adjective,' therefore '*red* is a relatively determinate specification of a relatively indeterminate adjective.' These illustrations could be endlessly multiplied, in which we directly apply a universal proposition to a certain given instance. In such cases the implicative as well as the applicative principle would have been involved if it had been necessary or possible to interpolate, as an additional

datum, a categorical proposition requiring certification, to serve as minor premiss. Let us turn to our original illustration and examine what would have been involved if we had treated the inference as a syllogism; it would have read as follows: 'Every proposition can be subjected to logical criticism'; 'That *matter exists* is a proposition'; therefore 'That *matter exists* can be subjected to logical criticism.' In this form, the substantive word *proposition* occurs as subject in the universal premiss, and as predicate in the singular premiss. What I have to maintain is that this introduction of a minor premiss is superfluous and even misleading. It should be observed that, in all the illustrations given above of the purely applicative principle, the subject-term in the universal premiss denotes a general category. It follows from this that the proposed statement 'That *matter exists* is a proposition' is redundant as a premiss; for it is impossible for us to understand the meaning of the phrase 'matter exists' except so far as we understand it to denote a proposition. In the same way, it would be impossible to understand the word 'red' without understanding it to denote an adjective; and so in all other cases of the pure employment of the applicative principle. In all these cases, the minor premiss which might be constructed is not a genuine proposition—the truth of which could come up for consideration—because the understanding of the subject-term of the minor demands a reference of it to the general category there predicated of it. This proposed minor premiss, therefore, is a peculiar kind of proposition which is not exactly what Mill calls 'verbal,' but rather what Kant meant by 'analytic,' and which I propose to call '*struc-*

tural.' All structural statements contain as their predicate some wide logical category, and their fundamental characteristic is that it is impossible to realise the meaning of the subject-term without implicitly conceiving it under that category. The structural proposition can hardly be called verbal, because it does not depend upon any arbitrary assignment of meaning to a word;—this point being best illustrated by giving examples. For instance, taking as subject-term 'the author of the *Republic*,' then 'The author of the *Republic* wrote something,' would be verbal, while 'The author of the *Republic* is an individual,' would be structural. In reality the subject of a verbal proposition, and the subject of a structural proposition are not the same; the one has for its subject the *phrase* 'the author of the *Republic*,' and the other the *object* denoted by the phrase. This is the true and final principle for distinguishing a structural (as well as a genuinely real or synthetic statement) from a verbal statement.

§ 7. Since a category is expressed always by a general substantive name, the important question arises as to whether or how the name of a category such as 'existent' or 'proposition' is to be defined. Now the ordinary general substantive name is defined in terms of *determinate* adjectives which constitute its connotation; but, so far as a category can be defined, it must be in terms of adjectival *determinables*; e.g. an existent is what occupies some region of space or period of time: the determinates corresponding to which would be, occupying some specific region of space or period of time. Similarly, the category 'proposition' could be defined by the adjectival determinable 'that to which

some assertive attitude can be adopted,' under which the relative determinates would be *affirmed*, *denied*, *doubted*, etc. We may indicate the nature of a given category by assigning the determinables involved in its construction. Using capital letters for determinables and corresponding small letters for their determinates (distinguished amongst themselves by dashes), the major premiss of the syllogism would assume the following form: Every MP is p if m; where the determinables M and P serve to define the category so far as required for the syllogism in question. Here we substitute for the vague word 'thing' previously employed, the symbol MP to indicate the category of reference; namely, that comprising substantives of which some determinate character under the determinables M and P can be predicated. The statement that the given thing is MP is redundant where M and P are determinables to which the given thing belongs; for the thing could not be given either immediately or in an act of construction except so far as it was given under the category defined by these determinables. Hence any genuine act of characterisation of the thing so given would consist in giving to these mere determinables a comparatively determinate value. For example, it being assumed that the given thing is MP, we may characterise it in such determinate forms as 'm and p,' 'm or p,' 'p if m,' 'not both p and m,' where the predication of the relative determinates m and p would presuppose that the object had been constructed under MP. In defining the function of a proposition to be to characterise relatively determinately what is *given* to be characterised, we now see that what is '*given*' is not given in a merely abstract

sense, but—in being given—the determinables which have to be determined are already presupposed.

§ 8. We may now show more clearly why the force of the term 'every' is distinct from that of the term 'if'; and how, in the syllogism, the two corresponding principles of inference are both involved. The major premiss having been formulated in terms of the determinables M and P, the whole argument will assume the following form:

(a) Every MP is p if m,

from which we infer, by the applicative principle alone:

(b) The given MP is p if m.

Next we introduce the minor, viz.

(c) The given MP is m,

and finally infer, by the implicative principle alone:

(d) The given MP is p.

Now if we held that the inference from (a) to (b) required the implicative principle as well as the applicative, so that a minor premiss 'The given thing is MP' must be interpolated, the syllogism would assume the following more complicated form:

(a) Everything is p if m if MP (the reformulated major).

$\therefore$ (b) The given thing is p if m if MP (by the applicative principle alone).

Next we introduce as minor

(c) The given thing is MP.

$\therefore$ (d) The given thing is p if m (by the implicative principle alone);

finally, introducing the original minor, viz.

(e) The given thing is m.

$\therefore$ (f) The given thing is p (by the implicative principle alone).

Now this lengthened analysis of the syllogism, while involving the implicative principle twice, involves as well as the applicative principle the introduction of a new minor, viz. that the given thing is *MP*, which hints at the doubt whether what is given is given as *MP*. But if this were a reasonable matter of doubt requiring explicit affirmation, on the same principle we might doubt whether what is given is a 'thing,' in some more generic sense of the word 'thing.' If this doubt be admitted, the syllogism is resolved into *three* uses of the implicative principle, with *two* extra minor premisses. Such a resolution would in fact lead by an infinite regress to an infinite number of employments of the implicative principle. To avoid the infinite regress we must establish some principle for determining the point at which an additional minor is not required. The view then that I hold is not merely that what is given is a 'thing' in the widest sense of the term thing, but that what is given is always given as demanding to be characterised in certain definite respects—e.g. colour, size, weight; or cognition, feeling, conation—and that therefore such a proposition as 'The given thing is *MP*' is presupposed in its being given, i.e. in being given, it is given as requiring determination with respect to these definite determinables *M* and *P*. The above formulation, therefore, in which the syllogism is resolved into a process involving the applicative and the implicative principles each *only once*, is logically justified; for it brings out the distinction between the function of the term *every* as leading to the employment of the applicative principle alone, and the function of *if* as leading to the employment of the implicative principle

alone; and furthermore it distinguishes between the process in inference which requires the applicative principle alone from that which requires the implicative as well as the applicative principle.

The distinction between the cases in which the implicative principle can or cannot be dispensed with depends, so far, upon whether the subject-term of the universal stands for a logical category or not. But we may go further and say that, even if the subject of the universal is not a logical category, provided that it is definable by certain determinables, and that the subject of the conclusion is only apprehensible under those determinables, then again the use of the implicative principle may be dispensed with. For example: 'All material bodies attract; therefore, the earth attracts.' Here the term 'material body' is of the nature of a category in that it can only be defined under such determinables as 'continuing to exist' and 'occupying some region of space'; furthermore the earth is constructively given under these determinables: hence a proposed minor premiss to the effect that the earth is a material body is superfluous, and the above inference involves only the applicative principle. Again 'All volitional acts are causally determined; therefore, Socrates' drinking of hemlock was causally determined.' Here the subject of the conclusion is constructively given under the determinables involved in the definition of volitional act, which again justifies the use of the applicative principle alone. As a third example: 'Every denumerable aggregate is less than some other aggregate: therefore, an aggregate whose number is $\aleph_0$ is numerically less than some other aggregate.' Here the construction of the

notion of a class whose number is $\aleph_0$ involves its being denumerable, so that the given inference again requires only the immediate employment of the applicative principle.

§ 9. Incidentally the above analysis of the major premiss—Every MP is p if m—(or still more simply, 'Every M is m,' which may sometimes be true; or again, of the minor premiss—'The given MP is m' or 'The given MP is p')—accounts for the insistence by certain philosophers, notably Mr Bradley, that every proposition employs the relation of identity; i.e. that the adjective involved in the subject is the same as that involved in the predicate. This philosophical suggestion is, I hold, true, in the sense that the adjectival *determinable* in the subject is the same as that in the predicate; but the latter is a further determination of the former. Now, in this admission that the relation of identity of subject to predicate is involved in the general categorical proposition, I am not in any way withdrawing what was maintained as regards *identity* in my analysis of the proposition. For the identity which I denied was (as it has been expressed) identity in denotation with diversity of connotation, i.e. substantival identity with adjectival diversity. The identity I have accepted above is identity of an *adjectival* factor in the subject with an *adjectival* factor in the predicate. Moreover I should still deny that the proposition *asserts* this identity, and maintain that it simply *presupposes* it, in just the same way as a proposition presupposes the understanding of the meaning of the terms involved without asserting such meaning.

§ 10. We have discussed the case in which a *minor*

premiss may be dispensed with, namely that in which a certain mode of using the applicative principle is sufficient without the employment of the *implicative*. We will now turn to a complementary discussion of the case in which there is unnecessary employment of the *applicative* principle, entailed by the insertion of what may be called a redundant *major* premiss. It will be convenient to call the redundant minor premiss a sub-minor, and the redundant major premiss—to which we shall now turn—a super-major. In this connection I shall introduce the notion of a formal principle of inference, which will apply, not only to inferences that are *strictly* formal, but also to inferences of an inductive nature, for which the principle has not at present been finally formulated and must therefore be here expressed without qualifying detail. The discussion will deal with cases in which the relation of premiss or premisses to conclusion is such that the inference exhibits a formal principle.

We shall illustrate the point first by taking the principle of syllogism, and next, the ultimate (but as yet unformulated) principle of induction. As regards the syllogism, taking p and q to represent the premisses and r the conclusion, we may say that the syllogistic principle asserts that provided a certain relation holds between the three propositions p, q, and r, inference from the premisses p and q *alone* will formally justify the conclusion r. Now it might be supposed that this syllogistic principle constitutes in a sense an additional premiss which, when joined with p and q, will yield a more complete analysis of the syllogistic procedure. But on consideration it will be seen that there is a sort

of contradiction in taking this view: for the syllogistic principle asserts that the premisses p and q are *alone* sufficient for the formal validity of the inference, so that, if the principle is inserted as an additional premiss co-ordinate with p and q, the principle itself is virtually contradicted. In illustration we will formulate the syllogistic principle:

'What can be predicated of every member of a class, to which a given object is known to belong, can be predicated of that object.'

Now, taking a specific syllogism:

'Every labiate is square-stalked,
The dead-nettle is a labiate,
$\therefore$ The dead-nettle is square-stalked,'

if we inserted the above-formulated principle as a premiss, co-ordinate with the two given premisses, with a view to strengthening the validity of the conclusion, this would entail a contradiction; because the principle claims that the two premisses are *alone* sufficient to justify the conclusion 'The dead-nettle is square-stalked.'

Now the same holds, *mutatis mutandis*, of any proposed ultimate inductive principle. Here the premisses are counted—not as *two*—but as many, and summed up in the single proposition 'All examined instances characterised by a certain adjective are characterised by a certain other adjective'; and the conclusion asserted (with a higher or lower degree of probability) predicates of *all* what was predicated in the premiss of *all examined*. Now, in accordance with the inductive principle, the summary premiss is *sufficient* for asserting the unlimited universal (with a higher or lower degree of probability). To insert this principle, as an additional

premiss co-ordinate with the summary premiss, would, therefore, virtually involve a contradiction. In illustration, we will roughly formulate the inductive principle:

'What can be predicated of all examined members of a class can be predicated, with a higher or lower degree of probability, of all members of the class.'

Now, taking a specific inductive inference:

'All examined swans are white. ∴ With a higher or lower degree of probability, all swans are white,'

if we inserted the above-formulated inductive principle as a premiss, co-ordinate with the summary premiss 'All examined swans are white,' with a view to strengthening the validity of the conclusion, this would entail a contradiction; because the principle claims that this summary premiss is *alone* sufficient to justify the conclusion that 'With a higher or lower degree of probability, all swans are white.'

We may shortly express the distinction between a principle and a premiss by saying that we draw the conclusion *from* the premisses *in accordance with* (or through) the principle. In other words, we immediately see that the relation amongst the premisses and conclusion is a specific case of the relation expressed in the principle, and hence the function of the principle is to stand as a universal to the specific inference as an instance of that universal: where the latter may be said to be inferred from the former (if there is any genuine inference) in accordance with the Supreme Applicative principle. For example: from $x=y$ and $y=z$, we may infer $x=z$. This form of inference is expressed, in general terms, in the Principle: 'Things that are equal to the same thing are equal to one another.' Now, here,

the two premisses—$x=y$ and $y=z$—are *alone* sufficient for the conclusion $x=z$; the conclusion being drawn *from* the two premisses through or in accordance with the principle which states that the two premisses are *alone* sufficient to secure validity for the conclusion. The principle cannot therefore be added co-ordinately to the premisses without contradiction. Moreover the above-formulated principle (which expresses the transitive property of the relation of equality) cannot be subsumed under the syllogistic principle. In the same way the syllogistic or inductive principle may be called a redundant or super-major, because it introduces a misleading or dispensable employment of the applicative principle.

§ 11. There is a special purpose in taking the inductive and syllogistic principles in illustration of super-majors, for many logicians have maintained that any specific inductive inference does not rest on an independent principle, but upon the syllogistic principle itself; in other words, they have taken syllogism to exhibit the sole form of valid inference, to which any other inferential processes are subordinate. Now it is true that the inductive principle could be put at the head of any specific inductive inference, and thus be related to the specific conclusion as the major premiss of a syllogism is related to its conclusion; but the same could be said of the syllogistic principle: namely that it could be put at the head of any specific syllogistic inference to which it is related in the same way as the major premiss of a syllogism is related to its conclusion. But, if we are further to *justify* the specific inductive inference by introducing the inductive principle, then,

by parity of reasoning, we should have to introduce the syllogistic principle further to justify the specific syllogistic inference. But in the case of the syllogism this would lead to an infinite regress as the following illustration will show. Thus, taking again as a specific syllogism, that

from (p) 'All labiates are square-stalked'
and (q) 'The dead-nettle is a labiate'
we may infer (r) 'The dead-nettle is square-stalked,'

and, adding to this as super-major the syllogistic principle, namely (a), we have the following argument:

(a) For every case of M, of S and of P: the inference 'every M is P, and S is M, $\therefore$ S is P' is valid.

(b) The above specific syllogism is a case of (a).

(c) $\therefore$ The specific syllogism is valid.

But here, in inferring from (a) and (b) together to (c), we are employing the syllogistic principle, which must stand therefore as a super-major to the inference from (a) and (b) together to (c), and therefore as super-super-major to the specific inference from p and q to r. This would obviously lead to an infinite regress.

We may show that a similar infinite regress would be involved if we introduced, as super-major, the inductive principle, by the following illustration. Taking again as a specific inductive inference that from 'All examined swans are white' we may infer with a higher or lower degree of probability that 'All swans are white'; and adding to this as super-major the inductive principle, namely (a), we have the following argument:

(a) For every case of M and of P: from 'every examined M is P,' we may infer, with a higher or lower degree of probability, that 'every M is P';

(b) The above specific induction is a case of (a).

(c) $\therefore$ The specific induction is valid.

But, here we may argue in regard to this (a), (b), (c) as in the case of the previous (a), (b), (c). Thus, by introducing the inductive principle as a redundant major premiss, we shall be led as before, by an infinite regress, to a repeated employment of the *syllogistic* principle.

This whole discussion forces us to regard the inductive and syllogistic principles as independent of one another, the former not being capable of subordination to the latter; for we cannot in any way deduce the inductive principle from the syllogistic principle. Those who have regarded the syllogistic principle as ultimately supreme, have in fact arrived at this conclusion by noting that, as shown above, the inductive principle could be introduced as a major for any specific inductive inference, in which case the inference would assume the syllogistic form (a), (b), (c). But this in no way affects the supremacy of the inductive principle as independent of the syllogistic.

CHAPTER II

THE RELATIONS OF SUB-ORDINATION AND CO-ORDINATION AMONGST PROPOSITIONS OF DIFFERENT TYPES

§ 1. In the previous chapter we have shown that the syllogism which establishes material conclusions from material premisses involves the alternate use of the Applicative and Implicative principles. Now these two principles, which control the procedure of deduction in its widest application, are required not only for material inferences, but also for the process of establishing the formulae that constitute the body of logically certified theorems. All these formulae are derived from certain intuitively evident axioms which may be explicitly enumerated. It will be found that the procedure of deducing further formulae from these axioms requires only the use of the Applicative and Implicative principles; these, therefore, cover a wider range than that of mere syllogism. But a final question remains, as to how the formal axioms are themselves established in their universal form. By most formal logicians it is assumed that these axioms are presented immediately as self-evident in their absolutely universal form; but such a process of intuition as is thereby assumed is really the result of a certain development of the reasoning powers. Prior to such development, I hold that there is a species of *induction* involved in grasping axioms in their absolute generality and in conceiving of form as

constant in the infinite multiplicity of its possible applications. We therefore conclude that behind the axioms there are involved certain supreme principles which bear to the Applicative and Implicative principles the same relation as induction in general bears to deduction; and, even more precisely, that these two new principles may be regarded as inverse to the Applicative and Implicative principles respectively. This being so, it will be convenient to denominate them respectively, Counter-applicative and Counter-implicative. It should be pointed out that whereas the Applicative and Implicative principles hold for material as well as formal inferential procedure, the Counter-principles are used for the establishment of the primitive axioms themselves upon which the formal system is based. We will then proceed to formulate the Counter-principles, each in immediate connection with its corresponding direct principle.

§ 2. The Applicative principle is that which justifies the procedure of passing from the asserting of a predication about 'every' to the asserting of the same predication about 'any given.' Corresponding to this, the Counter-applicative principle may be formulated:

'When we are justified in passing from the asserting of a predication about some *one* given to the asserting of the same predication about some other, then we are also justified in asserting the same predication about *every*.'

Roughly the Applicative principle justifies inference from 'every' to 'any,' and the Counter-applicative justifies inference from 'any' to 'every'; but whereas the former principle can be applied universally, the latter holds only in certain narrowly limited cases; and,

in particular, for the establishment of the primitive formulae of Logic. These cases may be described as those in which we see the universal in the particular, and this kind of inference will be called 'intuitive induction,' because it is that species of generalisation in which we intuite the truth of a universal proposition in the very act of intuiting the truth of a single instance[1]. Since intuitive induction is of course not possible in every case of generalisation, we have implied in our formulation of the principle that the passing from 'any' to 'every' is justified only when the passing from 'any one' to 'any other' is justified. Now there *are* forms of inference in which we can pass immediately from any one given case to any other; if it were not so, the principle would be empty. For instance, we may illustrate the Applicative principle by taking the formula: 'For every value of p and of q, "p and q" would imply "p",' from which we should infer that 'thunder and lightning' would imply 'thunder.' If now we enquire how we are justified in asserting that for every value of p and of q, 'p and q' would imply 'p,' the answer will supply an illustration of the Counter-applicative principle. Thus, in asserting that '"thunder and lightning" would imply "thunder"' we see that we could proceed to assert that '"blue and hard" would imply "blue",' and in the same act, that '"p and q" would imply "p" for all values of p and of q.'

§ 3. The second inverse principle to be considered is the Counter-implicative. Before discussing this inverse principle, it will be necessary to examine closely the

[1] This is a special case of 'intuitive induction,' the more general uses of which will be examined in Chapter VIII.

Implicative principle itself, which may be provisionally formulated: 'Given that a certain proposition would formally imply a certain other proposition, we can validly proceed to infer the latter from the former.' Now we find that the one positive element in the notion of formal implication is its equivalence to potentially valid inference, and that there is no single relation properly called *the* relation of implication. We must therefore bring out the precise significance of the Implicative principle by the following reformulation: 'There are certain specifiable relations such that, when one or other of these subsists between two propositions, we may validly infer the one from the other.' From the enunciation of this principle we can pass immediately to the enunciation of its inverse—the Counter-implicative principle:

'When we have inferred, with a consciousness of validity, some proposition from some given premiss or premisses, then we are in a position to realise the specific form of relation that subsists between premiss and conclusion upon which the felt validity of the inference depends.'

Here, as in the case of the Counter-applicative principle, we must point out that there *are* cases in which we intuitively recognise the validity of inferring some concrete conclusion from a concrete premiss, before having recognised the special type of relation of premiss to conclusion which renders the specific inference valid; otherwise the Counter-implicative principle would be empty. In illustration, we will trace back some accepted relation of premiss to conclusion, upon which the validity of inferring the one from the other depends; and this

will entail reference to a preliminary procedure in accordance with the Counter-applicative principle; for every logical formula is implicitly universal. Thus we might infer, with a sense of validity from the information 'Some Mongols are Europeans' and from this datum *alone*, the conclusion 'Some Europeans are Mongols.' We proceed next in accordance with the Counter-applicative principle to the generalisation that the inference from 'Some M is P' to 'Some P is M' is always valid. Finally we are led, in accordance with the Counter-implicative principle, to the conclusion that it is the relation of 'converse particular affirmatives' that renders the inference from 'Some M is P' to 'Some P is M' valid.

§ 4. We have regarded the intuition underlying the Counter-applicative principle as an instance of 'seeing the universal in the particular'; and correspondingly the intuition underlying the Counter-implicative principle may be regarded as an instance of 'abstracting a common form in diverse matter.' But the *direct* types of intuition operate over a much wider field than the Counter-applicative and Counter-implicative principles: for, whereas the twin inverse principles operate only in the establishment of axioms, the *direct* types of intuition are involved wherever there is either universality or form. These direct types of intuition have been explicitly recognised by philosophers; but the still more purely intuitive nature of the procedure conducted in accordance with the twin inverse principles accounts for the fact that these principles have hitherto not been formulated by logicians. Moreover the point of view from which the inverse principles have been described

and analysed is purely *epistemic*; and the epistemic aspect of logical problems has generally been ignored or explicitly rejected by logicians. It follows also from their epistemic character that these principles, unlike the Applicative and Implicative principles of inference, cannot be formulated with the precision required for a purely mechanical or blind application.

§ 5. The operation of these four supreme principles is best exhibited by means of a scheme which comprises propositions of every type in their relations of super-, sub-, or co-ordination to one another. We propose, therefore, to devote the remainder of this chapter to the construction and elucidation of such a scheme.

I. *Superordinate Principles of Inference.*
  I*a*. The Counter-applicative and Counter-implicative.
  I*b*. The Applicative and Implicative.

II. *Formulae*: i.e. formally certified propositions expressible in terms of variables having general application.
  II*a*. Primitive formulae (or axioms) derived directly from III*a* in accordance with I*a*.
  II*b*. Formulae successively derived from II*a* by means of I*b*.

III. *Formally Certified Propositions expressed in terms having fixed application.*
  III*a*. Those from which II*a* are derived by use of the principles I*a*.
  III*b*. Those which are derived from II*b* by use of the Applicative principle I*b*.

IV. *Experientially Certified Propositions.*
  IV*a*. Data directly certified in experience.

IV*b*. Concrete conclusions inferred from IV*a* by means of implications of the type III, and therefore established in accordance with the Implicative principle, I*b*.

I. The highest type consists of those principles under one or other of which every inference is subordinated. These superordinate principles consist of I*a*: the Counter-applicative and Counter-implicative, to which *intuitional* inferences are subordinated; and of I*b*: the Applicative and Implicative, to which *demonstrative* inferences are subordinated. I*a* are those principles in accordance with which the primitive formulae (or axioms) of Logic are established. But the choice of logical formulae that are accounted primitive is (within limits) arbitrary, and since any comparatively self-evident logical formula, instead of being exhibited as derivative, could be regarded as established directly in accordance with these inverse principles, their scope must not be restricted to the establishment of the more or less arbitrarily selected axioms. It will be found later, when we discuss the types of proposition in level III, that the content or material upon which the inverse principles I*a* operate, is supplied by the propositions of type III*a*. On the other hand, the Applicative and Implicative principles I*b* stand in the relation of immediate superordination to the processes of inference by which from II*a* are derived II*b*, viz. the general formulae of deduction, induction, demonstration, probability, etc.

II. The characteristic common to all the propositions on the second level is that they are formally certified, and are expressible in terms of variable symbols. They are theoretically infinite in number, and may be divided

into two groups, 'primitive' and 'derivative'; but, as pointed out above, the line of demarcation between the two cannot be sharply drawn. Thus II*a* comprises a small number of primitive formulae which are directly established in accordance with the twin inverse principles I*a*: for example, the commutative and associative laws, the laws of identity and of negation, the modus ponendo tollens, etc., or such of these as have been selected as primitive. Next, II*b* comprises an indefinite number of formulae successively derived from the primitive formulae II*a*: for example, the dictum of the syllogism, and other more complicated logical formulae, as well as the rules of arithmetic and algebra. All the formulae of level II are implicitly universal in form; and most of those that are logical (as distinct from mathematical) assert relations of implication. Each formula in II*b* is derived from previously certified formulae, and ultimately from those in II*a*, the process of derivation being marked at each step by the relation 'therefore.' Now wherever a previously certified relation of implication is used for deriving a new formula (in which case its implicans must also have been previously certified in order that its implicate may be derivatively certified) the procedure is conducted in accordance with the implicative principle, to which therefore all such cases of inference are to be subordinated. Again, the process of successive derivation of the formulae of II*b* entails explicit recognition of the implicit universality of the formulae from which they are derived; and this allows us, by means of the Applicative principle, to replace the illustrative symbols occurring in an earlier formula by any other symbols, in order to derive a new formula.

III. The third level contains formally certified propositions expressed entirely in linguistic terms of fixed application; and, like its predecessors, is to be divided into two sections, the division being made on precisely the same grounds as that between II*a* and II*b*. Thus the propositions of III*a* constitute the intuited material for deriving II*a* in accordance with the inverse principles I*a*; and the propositions of III*b* are exhibited as derived from II*b* in accordance with the applicative principle I*b*[1]. It will be seen, however, that the relation of III*a* to III*b* differs from that of II*a* to II*b* in that the two parts of III are not inferentially connected, as are those of II. The propositions comprised in III*b* are obtained from II*b* by substituting words with fixed application for the variable symbols; these propositions, then, are specialised instances of the general formulae which constitute the second level, and are established from them in accordance with the applicative principle alone. Any logical text-book teems with examples of this procedure, where instances under such formulae as the modus tollendo tollens, or the syllogistic dictum are represented in words with fixed application, and then exhibited as derived (in accordance with the applicative principle) from the appropriate general formula. It is usual in these cases, however, to exhibit the conclusion as being *inferred* from the premisses, thus leading the reader to suppose that it is the conclusion which has been formally certified, whereas, properly speaking, what has been formally certified is the *relation of implication* of premisses to

[1] Hence the point of division between III*a* and III*b* cannot be precisely indicated.

conclusion. It will be found below that this distinction between implication and inference is the essential consideration in comparing III*b* with IV*b*.

IV. The fourth and lowest level consists of experientially certified propositions expressed in concrete terms; and again this level must be divided into two sections, viz. IV*a* the primitives and IV*b* the derivatives, these two sections standing in a relation to one another which in every respect agrees with the relation of II*a* to II*b*. Thus the propositions comprised in IV*b* are successively derived from experiential propositions that have been previously certified, and ultimately derived from the primitive experiential data which constitutes IV*a*. And again, as in the case of formally certified propositions, here, in that of experientially certified propositions, the point of division between the primitives and derivatives is not precisely fixed; the primitives of IV, like those of II, are supposed to be intuitively accepted, i.e. in this case perceptually guaranteed; but philosophers do not agree on the question of the kind and range of experiences that can be regarded as in this case immediate. Moreover, as regards experiential propositions admittedly derivative and not primitive, no logician or philosopher has as yet been able to show how they can be exhibited as derived ultimately from absolutely primitive data of experience. Hence, in expounding the logical nature of the propositions in this lowest level, attention must be chiefly directed to the mode in which any admittedly derived proposition is inferred from some previously certified proposition, without enquiring too closely as to the mode in which the previous certification had

been conducted, or whether this certification could properly be called perceptually immediate. The mode of deriving an experiential conclusion from experientially certified premisses may be explained quite briefly; the former is derived *directly* from the latter by means of some implication of type III, of which the implicans is composed of the previously certified premisses and the implicate is the conclusion required. Since in this process a relation of implication is transformed into the relation 'therefore,' it is obvious that the implicative principle alone is employed. But, to complete the exposition, we must trace the process of derivation one stage further back, namely to the general formulae of line II. Thus, while any conclusion in IV*b* is *directly* derived from premisses IV*a* by means of an implicative proposition of the type III, and so far employs the implicative principle alone; yet, since any proposition of type III is itself derived from some formula of type II in accordance with the applicative principle alone, it follows that both these principles are jointly involved in deriving experiential conclusions from experiential data. This mode of derivation is illustrated in any text-book example of a concrete syllogism, where from previous experiential certification of the premisses we infer the experiental certification of the conclusion. For the sake of variety we will choose, for illustrating the processes of deriving any conclusion IV*b*, the formula of pure induction, which, as was maintained in the preceding chapter, must be included amongst the formulae constituting the second level. Take for instance as premiss: 'Every examined case of an acquired characteristic is non-transmitted.'

This datum is regarded not, of course, as a mere summary of directly given experiences, but as the product of various constructive and inferential processes which may be supposed ultimately to be based on sense-data. Now by means of the concrete implication that 'every examined case of an acquired characteristic being non-transmitted would imply, with a certain degree of probability, that no acquired characteristic is transmissible,' conjoined with the certified fact that 'in all examined cases acquired characteristics are non-transmitted,' we infer the conclusion that 'with a certain degree of probability no acquired characteristic is transmissible.'

In this fourth line, we are representing propositions as proved, or as validly asserted on the basis of experiential knowledge, and this suggests an ambiguity in the use of the term 'ground' which is sometimes applied in philosophy to the experiential data which may be said to be *co-ordinate* with the experiential conclusions; the same term 'ground' being also applied to the logical formulae of induction or deduction which are *superordinate* to the experiential data and conclusions. This ambiguity in the use of the term is removed by thus recognising the distinction between superordinate and co-ordinate.

§ 6. In further elucidation of the scheme, we will show what exactly is involved in level II, where emphasis has been put upon the variable symbols. In logical text-books we find that an inference or implication is expressed in terms of variable symbols, such as *S*, *M*, and *P*, and this is always supplemented by a formula expressed entirely linguistically, but which is its mere equivalent. For example, it may be first

asserted that 'Every P is Q would imply that some Q is P'; and here the assertion of implication is understood as being implicitly universal, i.e. that it holds for all values of P and Q. This is usually supplemented by the so-called 'Rule for the Conversion of A,' viz. that 'Any universal affirmative proposition would imply the particular affirmative obtained by interchanging subject and predicate terms.' But this is merely an alternative formulation, and is not related to the former as a universal to its instance. We see therefore that the formulae of level II are not necessarily expressed in terms of variables, but may be expressed with precise equivalence in linguistic terms only. The possibility of this linguistic formulation depends upon the invention of a technical terminology which employs such terms as subject, predicate, conversion, universal, proposition, etc. The reason why what is called symbolic logic requires the employment of variable symbols is essentially because the logical formulae which it establishes are so complicated that a terminology could hardly be invented for dealing with them. There is therefore no difference of principle involved in the employment of variable symbols by symbolic logic and the employment of technical linguistic terms by ordinary logic. By the employment of the technical terminology of logic the variables entering into any formula are eliminated *en bloc*, leaving the formula with the same range of universality as before. In contrast with this, a proposition of level III, being obtained from level II by replacing *each* of the several variables by a particular word of fixed application, constitutes a single instance of the general formula. For instance, 'that every

trespasser will be prosecuted would imply that some prosecuted person is a trespasser[1],' is a specific assertion obtained by the applicative principle from the universal formula of conversion adduced above.

This last discussion of the distinction and connection between the use of variable symbols and that of linguistic terminology, points to certain respects in which the methods of symbolic logic differ, and others in which they agree with those of ordinary logic—a topic which will be treated at greater length in the following chapter.

[1] This illustration is chosen in order incidentally to suggest that the text-books are not always infallible, the form of implication in question being at least dubious.

CHAPTER III

SYMBOLISM AND FUNCTIONS

§ 1. THE value of symbolism, as is universally recognised, is due to the extreme precision which its employment affords to the process of logical demonstration. As a language it differs from all ordinary languages in three respects, viz. systematisation, brevity and exactness; and in these respects differs from all other languages in a way in which they do not differ from one another.

Now, when we examine the language of symbolism, we find that symbols are of two fundamentally distinct kinds, which I propose to call *illustrative* and *shorthand*. In such familiar logical forms as 'S is P,' 'Every M is P,' etc., S, M, P, exemplify illustrative symbols. Thus an illustrative symbol is represented by a single letter chosen from some alphabet. Shorthand symbols, on the other hand, are mere substitutes for words, and serve the obvious purpose of saving time in reading and space in writing. Some of them, in fact, are literal abbreviations, such as 'rel.' for 'relative,' 'prop.' for 'proposition,' 'indiv.' for 'individual.' Others again are arbitrarily shaped marks standing for simple words such as *not*, *and*, *or*, *if*, *is*, *identical with*. A third kind of shorthand symbol is one introduced in the course of a symbolic calculus, and defined in terms of combinations of other shorthand symbols, and ultimately in terms of the simple symbols introduced at the outset. So far,

a shorthand symbol has all the characteristics of a word or a word-complex—only differing from these in satisfying the essential symbolic requirements of systematisation, brevity and exactness. In one respect, however, these symbols differ from such word-complexes as 'that man,' 'the river,' 'Mr Smith,' 'this experience,' 'my present purpose,' in that these latter have a meaning or application—not universally fixed—but determined only by means of context; whereas the symbols of Logic have an unalterable meaning wholly independent of context, and resemble rather, such word-complexes as 'rational animal,' 'loud,' 'hard,' 'church,' differing from these however in being strictly unambiguous. Ordinary Logic generally dispenses with symbols of this kind—the most familiar exception being Dr Keynes's '*SaP*,' which is shorthand for 'Every *S* is *P*,' etc. On the other hand, Dr Keynes himself shows, in his Appendix C, how certain complicated problems, previously relegated to Symbolic Logic, can be solved without recourse to shorthand symbols, illustrative symbols only being introduced.

Now an important character of the shorthand symbol is that its constancy is logical or formal and not experiential or material. A formal constant is one whose meaning is to be understood by the logician as such; that is to say, logic pronounces it either as indefinable—because understood without requiring definition—or as definable in terms of logically understood constants alone. The following is a rough classification of formal constants expressed in ordinary language: (1) the articles or applicatives; *a*, *the*, *some*, etc. (2) the negative *not*; and the conjunctions *and*, *or*, *if*, etc. (3) the copula *is*;

and certain prepositions such as *of*, *to*, in some of their meanings. (4) certain relations such as *identical with*, *comprised in*. (5) such modal adjectives as *true*, *false*, *probable*, etc. Formal constants are to be contrasted with material in that the meanings of the latter are to be understood in terms of ideas or conceptions outside the sphere of logic. The division between formal and material constants, i.e. between what is and what is not required for the understanding of logical principles, can ultimately be rendered precise only after a complete logical system has been constructed. For instance, numerical adjectives such as *two* and *five* would have been pronounced as merely material at the stage at which the logical system had not been carried on into its mathematical developments. Ideas that are immediately recognised as material relatively to the essentials of logic are those of sense-qualities, or of the properties and characteristics of physical and mental entities. Temporal and spatial relations, being in one aspect subsumable under the conceptions of order, would, so far, be called formal or logical, but, inasmuch as these relations actually have a specific—over and above their generic—significance, they must be treated also as having an experiential or material source. The same holds of the determinates of a determinable, inasmuch as experience is required in order to present to the mind any single determinable and to distinguish one determinable from another, whilst the discussion of the formal relations of incompatibility, order, etc., between determinates under any determinable is purely logical.

Since shorthand symbols and the words or word-complexes of ordinary language function in the same

way, there is no essential difference between them if we take the symbols or words in isolation apart from consideration of the mental processes involved in their use. The psychological distinction is—not between words and symbols as such—but between the linguistic and the symbolic mode in which we think with their assistance. Thus, in linguistic thought, the words or symbols presented in imagination or vocalisation are the means or instruments by which we can attend to or think about the objects for which they stand. On the other hand, such a phrase as 'Waterloo was fought in 1815' might illustrate the symbolic use of language which consists—not in thinking about the objects for which the words stand—but in mentally rehearsing the language in which propositions previously accepted have been expressed. Now the previous acceptance of these propositions must have entailed genuine processes of thinking; but, when they are recalled, we need not repeat these mental processes. It is in this way that the symbolic is distinguished from the linguistic use of words or symbols. In the latter, we are thinking by the use of words; whereas, in the former, recall of the words serves merely as a substitute for a previous act of thought[1].

§ 2. These preliminary considerations bring us to the question: What actually happens in the mind of the symbolist, when he is either constructing or intelligently following the formulae of a symbolic calculus? In the first place, the axioms of the calculus can only be established by the use of what I have called the Counter-

[1] This subject will be found to be more fully treated in Dr Stout's *Analytic Psychology*.

applicative and Counter-implicative principles, and here genuine thought is required on the part of the symbolist. In the second place, the construction of any symbolic calculus involves the procedure of inference; and this is conducted always in accordance with the Applicative principle, and, in the case of the logical calculus, also in accordance with the Implicative principle. When proceeding in accordance with these principles, the symbolist is actually thinking; he is not merely recalling verbal formulae in which the results of previous acts of thought have been expressed. In the third place, even a perfectly constructed symbolic system would need to introduce some axioms, as also some propositions derived from axioms, that can only be expressed in non-symbolic terms. This necessary recourse to ordinary language in developing a deductive system shows that direct attention to meanings, presented linguistically, is entailed in the intelligent following of even a professedly symbolic exposition. Lastly, the extent to which thought can be dispensed with, when working a calculus, depends very largely and essentially upon the extent to which the system requires what may be called interpretation clauses such as 'when P stands for any proposition,' or 'where x is to be understood as a variable and a as a constant.' If the symbolic language is so constructed that a minimum of interpretation clauses is required, then there is a corresponding minimum in the extent to which actual thinking is involved. But, however few interpretation clauses are required, the intelligent use of symbolic formulae cannot be reduced to a merely mechanical process. This will be still more apparent from an examination of the nature of a symbolic system in which both

shorthand and illustrative symbols enter in combination with one another.

§ 3. For this purpose we will further consider the characteristics of illustrative symbols. These, being nothing but arbitrarily chosen letters of the alphabet, differ from words of ordinary language in that they cannot be interpreted as standing for this rather than for that specific object or idea; and hence, in the nature of the case, have a variable application. The writer or reader of a symbolic system must always bear in mind, however, that the variability in application of an illustrative symbol in any given case is not wholly unrestricted, but is limited within an understood range. Thus a single letter used illustratively must be understood to be restricted in one case (say) to any substantive; in another (say) to any adjective; and in another again (say) to any proposition,—these being the three most prominent categories to which illustrative symbols are applied. Symbolic devices may, indeed, be invented by which to distinguish one kind of symbol as applicable to a substantive, and another kind to some other specific category; but the range of application to be understood by letters taken in combination could not be indicated by any such device. When single letters are bound together into a complex by means of logical constants, then a further act of intelligence is required in interpreting such complex. For example, understanding in the first place the letters p, q, r, to stand for propositions, such constructs as 'p and q,' 'p or q,' 'p if q,' must be *further* interpreted as also constituting propositions. Thus, when a formula about any or all propositions has been established, we may proceed to

apply it to any complex such as '*p* and *q*' or '*p* if *q*' and so on in accordance with the Applicative principle, inasmuch as each of such complexes constitutes a proposition. Similarly, when such letters as *x*, *s*, *t* are understood to stand for substantives, and such letters as *p*, *q*, *r* to stand for adjectives, then a further act of intelligence is required to interpret such a complex as '*s* is *p*' as standing for a proposition. This presupposes that the logical analysis of the simple proposition into the form '*s* is *p*,' where *s* is understood to stand for a substantive and *p* for an adjective, has been discussed and established in a preliminary account in which words and *not* symbols were employed. Propositional significance having been attached to this form of construct, a distinct act of intelligence is required when, in uniting say '*s* is *p*' with '*t* is *q*' in some form of combination, the resulting construct is understood to stand for a proposition. As another example illustrating the need for intelligent activity in symbolic work, we may take the two propositional forms '*s e p*' and '*x i y*,' where '*e*' is shorthand for the copula '*is*' and '*i*' for *is identical with*. Not only must these forms be interpreted as standing for propositions, but the relation for which '*e*' stands must be understood to be different from that for which '*i*' stands. In consequence, when these two forms occur, reference must be made to one set of established formulae for the one case, and to a different set for the other. The necessity for using this modicum of intelligence is to be contrasted with the purely blind or mechanical process required of the reader or writer in making use of the formulae to which he refers; for, in this latter process, he need attach no significance to '*e*'

or to '*i*,' as each standing for its own specific relation. The examples adduced have been selected on the ground of their simplicity, but complex examples would have brought out more forcibly the importance of the distinction between the intelligent and the merely mechanical operations required in working a symbolic system.

§ 4. Now the variability that characterises illustrative symbols constitutes a special feature of symbolism, and its further discussion requires the introduction of the notion *function*. This term is used by logicians and mathematicians in a sense quite unconnected with the biological meaning of the term. The notion of a function is closely connected with the notion of a construct, but the former must be understood *relationally*, whereas this is not obviously the case with the term construct. Thus, we should speak of a certain construct as being a function *of* certain enumerated constituents. The notation for a function in general is $f(a, b, c, \ldots)$ where $a, b, c, \ldots$ stand for the constituents; and where the order in which these constituents are written is essential, so that $f(a, b, c, \ldots)$ is not necessarily equivalent to $f(a, c, b, \ldots)$. Thus any function of $a, b, c, \ldots$ is a construct involving $a, b, c, \ldots$. But, if this were all that could be said about a function, the term would have no special value, since it would be a mere synonym for 'construct involving.' The importance of the notion of function lies in the fact that we may speak of the *same* function in reference to *different* constituents, whereas the same construct would of course entail the same constituents. Thus, if C be a certain construct involving $a, b, c, \ldots$, and if D be another construct involving $p, q, r, \ldots$, then C is said to be the same

function of $a, b, c, \ldots$ as is D of $p, q, r, \ldots$, when the substitution of p for a, q for b, r for c, etc., would render D identical with C. Thus, in order to decide as regards two constructs, whether they express the same or a different function, we must specify the constituents of which the construct is regarded as a function; and, to avoid all possible ambiguity, *all* the constituents for which substitutions have to be made must be enumerated. To explain this necessity, it must be pointed out that a construct may involve, implicitly or explicitly, other constituents in addition to those *of* which it is to be regarded as a function. In order to indicate the sameness of function exhibited by different constructs, it is therefore essential to enumerate those constituents for which substitutions are contemplated. These constituents will be called *variants*[1], because it is these and these alone that have to be varied in order to obtain the *different* constructs that exhibit the *same* function. On the other hand, in exhibiting identity of function, terms entering into the construct that are not to be replaced by some other terms will be called constants or non-variants. Hence the distinction between a variant and a non-variant constituent of a construct has relevance only to functional identity. Since a function and its variants are to be understood relationally to one another, we may speak of the variants *for* a certain function just as we speak of a function *of* certain

[1] The word *variant* is here and throughout used in place of the mathematically technical word *argument*, partly in order to prevent confusion with the ordinary logical use of the latter word, and partly in order to bring out the distinction and connection between the notion of *variant* and that of *variable*.

variants. In a complicated symbolic system it is found to be convenient to use, in place of a singular or proper name, an illustrative symbol—which, *quâ* symbol, must be what is called *variable*. Variability is therefore the mark of an illustrative symbol as such, whereas the contrast between variant and non-variant holds—not of a symbol—but of that for which the symbol may stand; and, as has been said, this latter contrast has no significance apart from the notion of a function.

§ 5. In considering the constituents of a construct with a view of indicating which are to be variants and which non-variants for a function, we must first note the distinction between material and formal constituents. Now as regards the strictly formal constituents of a construct, logic never contemplates making substitutions for these; hence, in all applications of the notion of a function in reference to its variants, two cases only have to be considered; (1) the function for which *all* the material constituents are treated as variants, and (2) the function for which *some* of the material constituents are treated as constants and others as variants—in both cases the formal constituents being understood to be constants. When (1) all the material constituents are to be varied, then the function may be said to be formal; and the *form* of a construct is a brief synonym for the formal function which it exhibits. But, when (2) some of the material constituents are to be constant, then the function will be said to be non-formal. It follows that, when two constructs can be said to exhibit the same formal function, their reduction to identity is effected by taking all the formal constituents to be constant, and replacing all the material

constituents of the one by those of the other. But, when two constructs are said to exhibit the same non-formal function, their reduction to identity is effected by taking certain of the material, as well as all the formal, constituents to be constant, and replacing all the remaining material constituents of the one by those of the other. A formal function is a function of all the material constituents, since all these are to be varied; but a non-formal function is a function of only some of the material constituents, because only some of these are to be varied.

We may take the following as illustrations of formal functions: The construct 'a good boy' is the same function of the variants *good* and *boy* as is 'a difficult problem' of the variants *difficult* and *problem*; 'Socrates is wise' is the same function of *Socrates* and *wise*, as is 'London is populous' of *London* and *populous*; 'red or heavy' is the same function of *red* and *heavy* as is 'loud or pleasant' of *loud* and *pleasant*. We may compare these simple examples with similarly simple examples in arithmetic. The arithmetical construct 'three days plus seven days' is the same function of the two variants *three days* and *seven days* as is 'five feet plus four feet' of the two variants *five feet* and *four feet*; 'four days multiplied by three' is the same function of *four days* and *three* as is 'seven feet multiplied by two' of *seven feet* and *two*, etc. These illustrate formal functions because the only constituents which are constant are formal: namely 'a,' 'is,' 'or,' 'plus,' 'multiplied by,' respectively. Each of the above examples exhibits a *specific* formal function, and serves to explain the *general* notion of a formal function. We

may take similar examples to illustrate the general notion of a non-formal function. Thus taking *boy* as constant, 'a good boy' is the same function of *good* as is 'a tall boy' of *tall*; taking *good* as constant, 'a good boy' is the same function of *boy* as is 'a good action' of *action*; taking *pleasant* as constant, 'loud or pleasant' is the same function of *loud* as is 'bright or pleasant' of *bright*; taking *wise* as constant, 'Socrates is wise' is the same function of *Socrates* as is 'Plato is wise' of *Plato*; taking *Socrates* as constant, 'Socrates is wise' is the same function of *wise* as is 'Socrates is poor' of *poor*, etc., etc. And in general the specific function exhibited by a given construct varies according to the constituents of the construct that operate as variants[1].

[1] It will be observed that in the above illustrations of non-formal functions we have used adjectives and substantives indifferently as constants or as variables. Now in Mr Russell's first introduction of the notion of function, he appears to limit the application of the notion to the case where the substantive is variable and the adjective is constant. It is true that he extends the notion to include the cases in which the reverse holds; yet throughout he adopts an absolute distinction between the two constituents of a proposition which I have called substantive and adjective, inasmuch as he treats the substantive as the typical kind of entity which can stand by itself, the adjective never being allowed to stand by itself. Thus I am repeating his illustration in giving 'Socrates is wise' as the same function of *Socrates* as is 'Plato is wise' of *Plato*, since here the substantive terms *Socrates* and *Plato* are allowed to stand by themselves. But the parallel example, that 'Socrates is wise' is the same function of *wise* as is 'Socrates is poor' of *poor*, is not recognised by Mr Russell, because he does not allow such adjective-terms as 'wise' and 'poor' to stand by themselves. The consequences of this contrast, which I hold to be fundamentally fallacious, between the substantive and the adjective as constituents of a proposition, infect the whole of his logical

§ 6. A classification has been given, in an earlier section, of those formal constituents of a construct that are expressible *in words* or in shorthand symbols understood as equivalent to words. Such formal constituents may be called *explicit* in distinction from others which are more or less latent and not usually expressed in words. Reserving the name 'constituent,' for the material variants, and 'formal component' for those formal constants that are explicitly expressed, the implicit formal constants may be conveniently termed 'elements of form.' Of these, several different kinds are to be distinguished:

(1) *Ties.* These are more or less latent elements of form, inasmuch as it is a matter of accident whether they are expressed by some separate word or by some form of grammatical inflection.

(2) *Brackets.* A construct may be composed of sub-constructs, and these again of sub-sub-constructs

system. Without entering into elaborate detail, it would be impossible fully to justify my difference from Mr Russell on this matter; but what I take to be perhaps the root of the error is that he treats the general notion of function before giving examples of the simplest functional forms upon which the more complicated functions are built. It is true that he illustrates a function by such an elementary example as 'x is a man' where x stands indifferently for Socrates or Plato, etc., but he does not bring out the speciality of this form of proposition, which does in fact exhibit the specific function which is constructed by means of the copula 'is.' In mathematics the general notion of function is reached by building up constructs out of such elementary functions as those indicated by $+ \times -$ etc., but in Mr Russell's system it seems impossible to explain and reduce to systematic symbolisation the process by which any propositional function whatever is constructed.

I hope to treat more fully elsewhere this point of difference between Mr Russell's system and my own.

and so on, until we reach the ultimate constituents, namely those that are expressed, not as constructs, but as 'simples,' where by 'simple' is not meant incapable of analysis, but merely unanalysed. The operation of binding constituents into a unity to constitute a sub-construct I shall call *bracketing*. In speaking, the distribution of brackets is indicated by pauses or vocal inflections; and, in writing, by punctuation marks. But, as the employment of these signs is not governed by any systematic principle, they must be replaced in logical or mathematical symbolism by some conventional notation.

(3) *Connectedness.* Two sub-constructs will be called *unconnected* when one is a function of the simple terms *a*, *b*, *c* (say), and the other of the simple terms *d*, *e*—the terms of the one not recurring in the other. On the other hand, two sub-constructs will be called *connected* when one is a function of *a*, *b*, *c* (say), and the other of *a*, *e*—the term *a* recurring in the two. This distinction is of importance when we have to determine what constituents of a function can be taken as variants; for the several variants for a function must be independently variable, and in the case of any two complex constituents, if these are connected (in the sense explained), they cannot be made to vary independently the one of the other. Thus, in the above illustration of two sub-constructs that are respectively functions of *a*, *b*, *c* and of *a*, *e*, the variants for the function exhibited by the construct must be taken to be the 'simple' constituents *a*, *b*, *c*, *e*, and *not* the connected sub-constructs themselves. But, when a construct contains unconnected sub-constructs, as in the example of the

sub-constructs that are functions respectively of *a*, *b*, *c* and of *d*, *e*, then it may be regarded *either* as a function of the several ultimate terms involved in the different sub-constructs, namely *a*, *b*, *c*, *d*, *e*; *or* alternatively, as a function of the sub-constructs themselves.

(4) *Categories.* Every material, and therefore variable, constituent belongs to a specific logical category or sub-category which is not usually expressed in words. Thus the proposition 'Socrates is wise' is understood as it stands without being expanded into the form 'The substantive Socrates is characterised by the adjective wise.' Nevertheless the formal significance of the proposition for the thinker depends upon his conceiving of 'Socrates' as belonging to the category *substantive*, and of 'wise' as belonging to the category *adjective*. These must therefore be included amongst the latent elements of form. It further follows from the recognition of this formal element, latent in every material constituent, that the range of variation for any material constituent is determined by the logical category—substantive, adjective, relational adjective, as the case may be—to which it belongs. In other words, the material constituents which may replace one another, in order that the construct may exhibit the same function in its varied exemplifications, must all belong to the same logical category or sub-category.

§ 7. This account of the formal elements of a construct leads to an examination of different types of function. Amongst the functions of logic the conjunctional and the predicational are the most fundamental. A function is called *conjunctional* when the component that determines its form is the negative *not*

or some logical conjunction; and the variants for such a function are always, strictly speaking, propositions, as is also the construct itself. A function is called *predicational* when the component that determines its form is the characterising tie, which unites two variants related to one another as substantive to adjective. Thus there is only one elementary predicational function, namely the characterising function represented by the copula 'is'; whereas there are five elementary conjunctional functions represented respectively by the operators, 'not,' 'and,' 'if' and its converse, 'or,' 'not-both.' Just as a conjunctional function may exhibit any degree of complexity made up of these elementary conjunctional functions, so a predicational function may exhibit any degree of complexity made up of recurrences of the characterising function in sub-constructs and sub-sub-constructs, etc. An important distinction between these two types of function introduces the notion of functional homogeneity. A function is said to be *homogeneous* when all its variants belong to the same category as itself. Now, since a conjunctional function takes propositions as its variants and is itself a proposition, it illustrates a homogeneous function; but, since a predicational function constitutes a construct under the category proposition out of constituents under the respective categories, substantive and adjective, it illustrates a heterogeneous function. Under this head are also to be included secondary propositions which predicate adjectives of primary propositions, and propositions which predicate secondary adjectives of primary adjectives; for the subjects of these propositions are quasi-substantives, and the propositions themselves

are of a different order of category from their constituent terms.

§ 8. We will proceed to apply the notion of connectedness to these two types of function. A conjunctional function is a function of those propositional sub-constructs which are unconnected, but not of those which are connected with one another through identity of some of the terms involved. For such sub-constructs, though properly regarded as *constituents*, cannot be taken as *variants*, since they cannot be freely varied independently of one another. Thus the variants for a conjunctional function which is also connectional are not the connected sub-constructs themselves, but the ultimate propositions or 'simples' of which they are constituted; e.g. in the construct

$$\{(p \text{ and } q) \text{ or } (p \text{ and } r)\} \text{ and } (x \text{ or } y)$$

the constituents that may be taken as variants are p, q, r, $(x \text{ or } y)$; and in the construct

$$\{(p \text{ and } q) \text{ or } (p \text{ and } r)\} \text{ and } (q \text{ or } y)$$

the only constituents that can be taken as variants are p, q, r, y. In these symbolic illustrations, the ultimate constituents are unanalysed propositions; but the same distinction between connected and unconnected sub-constructs holds for a conjunctional function of propositions that are expressed analytically in terms of subject and predicate. For example, 'A is p or B is q' illustrates a conjunctional function of the two unconnected sub-constructs 'A is p,' 'B is q.' On the other hand, 'A is p or A is q' is not a function of the sub-constructs 'A is p,' 'A is q' because these are connected; but must be taken as a function of the three

ultimate constituents A, p, q. Again, 'A is p or B is p' is not a function of 'A is p,' 'B is p,' but of the ultimate constituents A, B, p. The connectedness in the former case is through identity of the substantive A; in the latter through identity of the adjective p. Similar examples of connectedness occur, in which 'if' or 'not-both' or 'and' enter in the place of 'or.'

Ordinary language adopts abbreviated expressions for propositions that are connected, through identity of *subject*, by constructing a compound *predicate*, e.g. 'A is p or q,' 'A is p and q'; as also for propositions that are connected, through identity of *predicate*, by constructing a compound *subject*, e.g. 'A or B is p,' 'A and B are p.' This is extended to any number of terms enumeratively assigned for which language supplies us with a special condensed mode of expression. Thus the alternative function is condensed into the form: 'Some one or other of the enumerated items is p'; and the conjunctive function into the form: 'Every one of the enumerated items is p.' Such forms are usually restricted to enumerations of substantival items: for example, 'Some one of the apostles was a traitor,' 'Every one of the apostles was a Jew.' But it is possible to extend the form to enumerations of propositional or of adjectival items; for example, 'Some one of the axioms of Euclid is unnecessary for the purpose of establishing the theorems of geometry'; or 'Every one of the qualities characterising A, B, C characterises D.'

§ 9. A special notation has been adopted by the symbolists for representing such condensed expressions. In this notation, an illustrative symbol such as x enters as an *apparent* variable (to use Peano's phraseology);

by which is meant that the proposition in which x occurs—though it appears to be, yet is not in reality—about x, inasmuch as its content is not changed when any other symbol, say y, is substituted for x. The typical mode of formulating propositions on this principle is: 'Every item, say x, is p,' or 'Some item, say x, is p,' where it is obvious that the force of the proposition would be unaltered if we substituted s, or y, or z, for x. If X is the name of the class that comprises all such items as x, then the above forms are equivalent to 'Every X is p,' and 'Some X is p' respectively. The ultimate constituents of such universal or particular propositions are the simple propositions of the form 'x is p' which are conjunctively combined for the universal, and alternatively combined for the particular. The phrases 'Every X,' 'Some X,' therefore, though obviously constituents of the *sentence*, do not denote genuine constituents of the proposition of which the sentence is the verbal expression. Since then the constituents of the general proposition are singular propositions of the form 'x is p,' such a class-name as X and such a variable name as x, which are in danger of being identified, must be carefully distinguished. To the former the distributives *some* or *every* can be prefixed, never to the latter. [See Part I, Chapter VII.]

When we use a symbolic variable or illustrative symbol x to construct the proposition 'x is p' say, x stands, not for a class-name, but for a special kind of singular name, only differing from the ordinary singular name in that it stands indifferently for *any* substantive name, such as 'Socrates' or 'Cromwell' or 'this table' or 'yonder chair.' To bring out more pre-

cisely the distinction between a symbolic variable and a class-name, we may suppose that in a certain context *s* stands indifferently for any *person* such as 'Socrates' or 'Cromwell'; or again, indifferently for any *article of furniture* such as 'this table' or 'yonder chair.' Now 'person' and 'article of furniture' are class-names, and in the instances adduced the symbolic variable *s* stands—not for the class-name—but in fact for *any* singular name (proper or descriptive) that denotes an individual comprised in the class 'person' for the one case, and the class 'article of furniture' for the other case. What holds of a substantive-name *s* holds also of an adjective-name *p* or of a class-name *c*. Thus, in the form '*s* is *p*,' where '*is*' represents the characterising tie, *p* stands for any one indifferently assignable adjective comprised, say, in the class *colour*, but not for the class itself to which the distributives 'every' or 'some' can be prefixed. Again, in the form '*s* is comprised in *c*,' *c* represents a singular class-name standing for any one indifferently assignable class; and the limits of variation for the variable *c* could be expressed in terms of a class of a higher order comprising it. Thus the symbol *c* is equivalent to a variable proper class-name, and, like the substantive-name *s* and the adjective-name *p*, is to be contrasted with the class in which it is comprised. The names *substantive*, *adjective*, *proposition*, etc., which denote logical categories, i.e. the ultimate comprising classes, are not *variable* proper names, but names bearing fixed or constant significance, having so far the character of shorthand symbols in that they stand for logical constants, not for material variables. Thus the employ-

ment of the illustrative symbol as an apparent variable —i.e. to stand indifferently for any one or another object—makes possible the use of the same symbol, recurring in a given context, to stand for the same object. It thus fulfils the same function in a complex symbolic formula as the proper name in ordinary narrative, where the use of the pronoun in complicated cases would be ambiguous. The construction of such formulae requires the use, in a symbolic system, of apparent variables in place of class-names.

§ 10. We have seen that certain phrases containing, implicitly or explicitly, the conjunctions *and* or *or*, though linguistically intelligible, do not really represent genuine constructs. This raises a wider and more fundamental problem in regard to the nature of logical conjunctions when used in constructing a compound out of enumerated items. Can conjunctions serve to construct compound substantives or compound adjectives in the same way as they operate in constructing compound propositions? Now I shall maintain that while the nature of an adjective is such that we may properly construct a compound adjective out of 'simple' adjectives just as we may construct a compound proposition out of 'simple' propositions, yet the nature of any term functioning as a substantive is such that it is impossible to construct a genuine compound substantive. Thus 'rational and animated' represents a genuine conjunctive adjective, since it is equivalent in meaning to the simple adjective 'human'; and 'one or other of the colours approximating to red' is a genuine alternative adjective, since it is equivalent in meaning to the simple adjective 'reddish.' And again, more generally, where no single adjectival

word represents such a conjunction of adjectives as 'square and heavy,' 'red or green,' these are still to be regarded as genuine adjectival constructs on the ground that they agree in all essentially logical respects with simple adjectives, from which in fact they cannot be distinguished by any universal criterion. It follows, therefore, that no contradiction will ensue from replacing a simple by a compound adjective in any general formula holding of all adjectives as such. At the same time it must be pointed out, as regards alternative adjectival constructs, that no single or *determinate* adjective can be *identified with* such an alternative or indeterminate adjective as 'red or green,' 'one or other of the colours approximating to red.' In this respect, as we shall see, an alternative adjectival construct precisely resembles a substantival construct. Turning then to substantival constructs, it is obvious in the first place that a conjunctive enumeration of substantives such as 'Peter and James' or 'Every one of the apostles' does not represent any single or determinate man. It might, however, be maintained that such phrases represent a couple of men or a class of men, and that a couple or a class comprising substantives is itself of the nature of a substantive. Such a view would, however, involve a confusion between the enumerative and the conjunctional *and*. A statement about 'Peter and James' or 'Every one of the apostles' is really not about the compound construct that appears to be denoted by its subject-term, but must be analysed into a conjunctive compound of singular propositions. Thus in the statement 'Peter and James were fishermen' the subject-term uses *and* enumeratively. The conjunctional *and* can be shown

to enter only when we analyse the statement into the form 'Peter was a fisherman and James was a fisherman.' The case of an *alternative* enumeration of substantives, such as 'Peter or James' or 'Some one of the apostles,' is less obvious than that of a conjunctive enumeration of substantives. To prove that the alternative enumeration does not represent a genuine substantive, it will be convenient to take a proposition in which the enumeration occurs in the predicate. Thus 'Nathaniel is one of the apostles' or 'Bartholomew is one of the apostles' would appear to be expressible in the form 'Nathaniel is-identical-with one or other of the apostles' or 'Bartholomew is-identical-with one or other of the apostles.' But, if this is allowed, the conjunction of these two propositions would imply that 'Nathaniel is-identical-with Bartholomew,' since things that are identical with the same thing are identical with one another. Now that 'Nathaniel is-identical-with Bartholomew' may or may not be the case; but it certainly would not follow from the fact that Nathaniel was one of the apostles and that Bartholomew was one of the apostles. In order correctly to formulate the proposition 'Nathaniel was one of the apostles' in terms of the relation of identity, it must be rendered: 'Nathaniel is-identical-with Peter or identical-with Bartholomew or identical-with Thaddeus, etc.' In this form, the alternants are not the proper or substantival names Peter, Bartholomew, Thaddeus, etc., but the adjectival terms 'identical with Peter,' 'identical with Bartholomew,' identical with Thaddeus,' etc. These latter being recognised as adjectives, the reconstructed proposition assumes the form 'A is p or q or r, etc.' where 'is'

represents the characterising tie, and p, q, r... stand for adjectives, so that (as alleged above) the new predicate expresses a genuine construct.

§ 11. A further and more general explanation may now be given of the principle according to which a proposition containing a fictitious construct must be reformulated. What holds of the relation of identity (as in the particular example concerning the apostles) holds of any relation whatever: that is to say, taking $\hat{r}$ to stand for any relation, the phrase '$\hat{r}$ to a or b or c...' does not express a genuine construct and must be replaced by the phrase '$\hat{r}$ to a or $\hat{r}$ to b or $\hat{r}$ to c...' which is an alternative of adjectives. For example, the proposition 'This action will injure either Germany or England' must be transformed into 'This action will either injure Germany or injure England.' The essential points in this transformation can best be indicated with the help of vertical lines for brackets. Thus:

$$x \mid \text{is } \hat{r} \text{ to } \mid a \text{ or } b \text{ or } c$$

is corrected into

$$x \mid \text{is } \mid \hat{r} \text{ to } a \text{ or } \hat{r} \text{ to } b \text{ or } \hat{r} \text{ to } c.$$

In the former the two principal constituents of the proposition are linked by the relational predication 'is $\hat{r}$ to,' in the latter by the characterising tie 'is.' In order that the predicate in the latter case should constitute a genuine construct, what is essential is, not that the subject term should stand for a substantive in any absolute sense, but only that it should function as a substantive relatively to the adjectival predicate; and it is the characterising tie which indicates this relative conception of substantive to adjective. Thus the term x

may be either a substantive proper, an adjective or a proposition, and the same holds of the terms a, b, c, with which x is connected by the relation $\hat{r}$.

Examples may be given of propositions based upon the forms 'x is $\hat{r}$ to a or b,' 'x is $\hat{r}$ to a and b,' in order further to illustrate the principles under discussion.

(1) In the example just given: 'This action will injure either Germany or England,' which must be rendered 'This action will either injure England or injure Germany,' the terms x, a, b, are all substantives proper. But taking

(2) 'p characterises either a or b or c,' which has to be transformed into 'p either characterises a or characterises b or characterises c,' the subject term is an adjective and may be called primary relatively to the predicate terms which function as secondary adjectives[1].

In (3): 'A has asserted p or q or r,' the subject term stands for a person (i.e. for a substantive proper), and the terms p, q, r in the predicate are propositions. Since here the terms alternatively combined are themselves propositions, the expression as it stands would be correct if its intention were to state that the compound proposition 'p or q or r' was asserted by A. But, if it were intended to state that one or other of the assertions p, q, r had been made by A, then (3) should be amended

[1] The predication *characterises*, like *injures* in the previous example, is expressed by a verb; but, as explained in Part I, Chapter XIII, section 5, any verb may be resolved into an adjective or relation preceded by the characterising tie. Thus, in order to show more explicitly that the principal constituents are united by the characterising tie, proposition (2) should be expanded into the form: 'p is characterised as either characterising a or characterising b or characterising c.' Similarly for other examples.

(as in the preceding examples) into the form: 'A has asserted p or has asserted q or has asserted r.'

(4) The proposition: 'g is characterised by all the adjectives that characterise a and b and c' exhibits a higher degree of complexity than those previously given since it introduces the two correlatives *characterising* and *characterised by*. It illustrates a type of proposition which plays an important part in the theory of induction; and is a specific case of the more general form: 'g is $\breve{r}$ to everything that is $\hat{r}$ to a and b and c.' As thus formulated it contains the fictitious conjunctive construct 'a and b and c,' where a, b, c function as substantives. To eliminate this fictitious construct, the statement must be reformulated thus: 'g is characterised by every adjective that characterises a and characterises b and characterises c.' But there still remains the fictitious construct prefaced by the distributive phrase 'every adjective.' The final correction must be made by introducing an apparent variable as was required in reformulating the elementary forms of proposition: 'Every M is p,' 'Some M is p.' Thus: 'Every adjective, say x, that characterises a and characterises b and characterises c also characterises g.'

§ 12. The above exposition of functions is fundamentally opposed to that given in the *Principia Mathematica*. The first point of difference to be emphasised concerns Mr Russell's view of the relation between what he calls a propositional function, and function in the sense in which it is universally understood in mathematics. The latter he terms a descriptive function, and maintains that it is derivable from the nature of the propositional function; whereas it appears to me that

the reverse is the case, and that his propositional function is nothing but a particular case of the mathematical function. The general nature of a descriptive function can be illustrated by taking a proposition say about 'The teacher of y.' This phrase illustrates what is meant by a descriptive function, the full meaning of which can be indicated only by showing how it may enter into a proposition such as (a): 'The teacher of y was a Scotchman.' Now we may agree with Mr Russell that this proposition could not be interpreted as true, unless y had one and only one teacher. On this interpretation the full force of the proposition is explicated as follows:

(a) There is a being, say b, of which the following statements may be made:

(1) that b was a Scotchman;
(2) that b taught y;
(3) that no being other than b taught y.

This analysis in which the describing relation is *teaching* is typical of all cases in which a descriptive function is used in a proposition. To illustrate a mathematical function of y, for *teacher-of* substitute *greater-by-3-than*; so that $y+3$ stands for '*the* quantity that is greater by 3 than y.' Again for the predication *is-a-Scotchman* substitute *is-divisible-by-4*. Thus, in place of the proposition 'The teacher of y was a Scotchman,' we have constructed the proposition (b): '$y+3$ is divisible by 4,' the full force of which is rendered as follows:

(b) There is a quantity, say b, of which the following statements may be made:

(1) that b is divisible by 4;
(2) that b is greater-by-3 than y;
(3) that no quantity other than b is greater-by-3-than y.

Thirdly, to illustrate a propositional function, for *divisible-by-4* substitute the predicate *dubious*; for the quantitative construct '$y + 3$' substitute the propositional construct 'y is p.' We have thus constructed the secondary proposition (*c*): 'That y is p is dubious,' of which the full force is rendered as follows:

(*c*) There is a proposition, say *b*, about y, of which the following statements may be made:

(1) that *b* is dubious;
(2) that *b* predicates-p-about y;
(3) that no proposition other than *b* predicates-p-about y.

Now in example (*a*) the ground for asserting uniqueness of the construct *the teacher of y* is merely empirical or factual; but in example (*b*) the necessary and sufficient condition for the uniqueness of the construct $y + 3$ is its mathematical form, as indicated by the symbol +; and in example (*c*) the uniqueness of the corresponding construct *y is p* similarly depends upon its *logical* form, as indicated by the logical constant *is*. Dismissing the empirical example which requires no further discussion, it must be pointed out as regards the quantitative function (*b*) and the propositional function (*c*) that these illustrate —not quantitative or propositional functions in general —but certain specific functions: in the former case that which is constructed by means of *plus*, and in the latter case, that which is constructed by means of *is*. The former may be called the *additive* and the latter the *characterising* function. Just as the quantitative construct $y + a$ would not yield a quantity unless y and a were themselves quantities of the same kind; so the propositional construct *y is p* would not yield a proposition unless the two constituents y and p were, in

their nature, relatively to one another as substantive to adjective. A specifically different form of quantitative construct would have been obtained if for $y+a$ we had substituted $y:a$. Similarly a specifically different propositional form of construct would have been obtained if for *s is p* we had substituted *x is identical with y*. In both cases the uniqueness of the construct is secured by the nature of the operator involved; viz., + which yields a sum, or : which yields a ratio for the two quantitative constructs; and *is* and *is identical with* for the two propositional constructs. If there is any difference between the uniqueness of the propositional construct when its constituents are given and that of the mathematical construct when its constituents are given, it is that the uniqueness in the former case is assumed on the ground of its intuitive evidence realised in the mental act of constructing the proposition, whereas in the latter the uniqueness may require and may be capable of formal demonstration.

§ 13. Before continuing the discussion of my differences from Mr Russell, I shall examine more precisely what he means by a descriptive function. A descriptive function (p. 245) is defined to be a phrase of the form: '*the* term x that has the relation $\hat{r}$ to the term y.' In this definition the sole emphasis is to be laid on the predesignation *the*. Now, just as we speak of *the* quantity '$s+p$,' so we speak of *the* proposition 's is p.' But these quantitative and propositional phrases differ from ordinary descriptive phrases such as 'the writer of *Waverley*' or 'the teacher of Xenophon'—which are of the general form: 'the thing x that is $\hat{r}$ to the thing y'—in that they do not explicitly contain any descriptive

relation $\hat{r}$ (writing or teaching). The arithmetical form '$s+p$' and the propositional form 's is p' having in common this negative characteristic, I shall proceed to maintain that they are, in all essential logical respects, identical in nature; and, if either of the two can be explicated into the form of a descriptive function, so can the other. We may attempt to express these forms explicitly as descriptive functions by introducing, as the describing relation, *constructed by*. Thus the propositional function may be rendered: 'the proposition x constructed by means of *is* out of the constituents s and p'; and the quantitative function may be rendered: 'the quantity x constructed by means of *plus* out of the constituents s and p.' This attempt reduces the statement of equivalence of the construct with the proposed descriptive phrase to a mere tautology; for 'the proposition x constructed by means of *is* out of the constituents s and p' is merely a lengthened expression for 'the proposition s is p'; and similarly 'the quantity x constructed by means of *plus* out of the constituents s and p' is merely a lengthened expression for 'the quantity $s+p$.' It thus turns out that the x thus introduced in the completely formulated descriptive phrase stands merely for the function itself, i.e. in the one case for 's is p' and in the other for '$s+p$.' Following Russell in his demand that a descriptive function must only be defined 'in use,' the statement that 's is p is dubious' or that '$s+p$ is divisible by 4' must be rendered 'the proposition x constructed etc. (as above) is dubious,' or 'the quantity x constructed etc. (as above) is divisible by 4.' In this way the original statements: 'the proposition s is p is dubious,' and 'the quantity

$s+p$ is divisible by 4,' which were supposed to require definition, are after all defined tautologically.

§ 14. Another way of attacking Russell's propositional function, which ,in fact presents only another aspect of the same criticism, is to ask: What are the variants for any given proposional function, and what function is it that a given propositional form exhibits? In his first introduction of the notion of propositional function, Mr Russell gives three quite different applications of the symbol for a function. According to his first definition, ϕx is called a propositional function when x is variable provided that when x is replaced by the constant a, ϕa represents a proposition. Now here the symbol for a function is first used along with a variable and then along with a constant; although Russell insists that ϕa is not a function but a proposition, and that ϕx is not a proposition but a function. It seems to me that he cannot attach the symbol for a function exclusively to a variable in this way without contradiction at every point; and it is for this reason that, in my account of functions, I have used the word *variant* to include both Russell's variable and his constant. There is yet a third application of the symbol for a function deliberately introduced in the very first paragraph of his exposition, by way of correcting his initial definition of propositional function. For his first account is that ϕx is to be called a propositional function, owing to the ambiguity—or as I should prefer to say indeterminateness—of the symbol x, and that it is not itself a proposition, and would only become a proposition when a is substituted for x. This is corrected, however, when he takes the example 'x is hurt' which he says illustrates, not a propositional

function, but an ambiguous (i.e. an indeterminate) value of a propositional function. Thus, as I have pointed out, he illustrates the use of the word *function* in his first paragraph in three different ways which are symbolised as follows: 'a is hurt,' 'x is hurt,' and '$\hat{x}$ is hurt.' The last application of the word function is that which he wishes to be finally adopted; but, in spite of this, he continually uses the word in both of the two other applications. It is still more surprising that, on page 6 of his Introduction, where he gives a preliminary account of the ideas and notations of logical symbolism, he uses the word function without any explanation of its meaning, and in deliberate defiance of his own later definition. Thus he speaks of the fundamental functions of propositions in these words: 'an aggregation of propositions considered as wholes, not necessarily unambiguously determined, into a single proposition more complex than its constituents, is a function with propositions as arguments.' This account appears clearly to suggest that unambiguously determined constituents are allowable as arguments for a function, which contradicts his explicit definition. He proceeds to enumerate the four fundamental functions of propositions which are of logical importance, viz. (1) the contradictory function, which I have called the negative function; (2) the logical sum or disjunctive function, which I have called the alternative function; (3) the logical product, which both he and I call the conjunctive function; (4) the implicative function, for which I have used the same term. These four functions I have called conjunctional functions, in contrast to the one fundamental predicational function. The recognition of this distinction, which does not appear

in Mr Russell's account, would have simplified and corrected his 'theory of types.' But, in thus introducing the specific conjunctional functions, he inevitably adopts the familiar meaning of the mathematical term 'function,' the essence of which lies, not in the indeterminateness of the constituent terms, but in the identity of form that is exhibited in the process of substituting indifferently any one term for any other.

That he is not only in disagreement with universal usage, but also logically mistaken, when he says that it is a *function* of which the essential characteristic is ambiguity—and thus that ϕx ambiguously denotes ϕa, ϕb, ϕc, where ϕa, ϕb, ϕc are the various values of ϕx—is shown by noting that the ambiguity attaching to ϕx is not due to the nature of ϕ as a function, but to the nature of the symbol x itself; that is to say, ϕx ambiguously denotes ϕa, ϕb, ϕc, etc., only because x ambiguously denotes a, b, c, etc. In short a propositional function has ambiguous denotation, if it contains a term having ambiguous denotation; whereas a propositional function has unambiguous denotation, if it contains no term having ambiguous denotation.

§ 15. Hitherto, in illustrating Russell's account, we have taken the propositional function to be a function of a single variable, viz., of the symbol for the subject of the proposition, the predicate standing for a constant. It is obvious, however, that no proposition can be regarded as a function of a single variant unless the proposition is represented by a simple letter; and we will therefore take the specific propositional form 'x is p' to illustrate a function of two variables. The variants of which this is a function would naturally be taken as the

symbols x and p themselves; but, since Russell refuses to allow a predicate or adjective to stand by itself, he takes as the two variables the subject term x together with the symbolic variable '$\hat{x}$ is p.' The symbolic expression '$\hat{x}$ is p' may be read 'x-blank is p'; by which is meant that instead of the full propositional form 'x is p,' we suppose that the subject-term x is omitted, leaving a blank. But, if we use a blank symbol for the subject-term, we ought in consistency to be allowed to use a similar blank symbol for the predicate term. This would give rise to nine combinations all of which are of the same propositional form: 'this is hurt,' 'x is hurt,' 'this is p,' '$\hat{x}$ is hurt,' 'this is $\hat{p}$,' '$\hat{x}$ is p,' 'x is $\hat{p}$,' '$\hat{x}$ is $\hat{p}$,' and finally 'x is p.' Of these nine phrases, Russell uses only 'this is hurt,' 'x is hurt' and '$\hat{x}$ is hurt'; of which the two latter illustrate the two admittedly different meanings or applications of the general notion ϕx, i.e. of the propositional function. Now, though his first reference is to a propositional function taking a single argument, nevertheless he allows that any proposition (as distinguished from a propositional function) when analysed contains at least two constituents. For example, the proposition 'this is hurt' as analysed contains the two constituents 'this' and '$\hat{x}$ is hurt.' In my view, there is no ground whatever for preferring this analysis either to that in which the constituents are 'hurt' and 'this is $\hat{p}$,' or to that in which the constituents are 'this is $\hat{p}$' and '$\hat{x}$ is hurt.' But, returning to his own analysis in which 'this' and '$\hat{x}$ is hurt' are assigned as the two constituents of 'this is hurt,' as also 'x' and '$\hat{x}$ is p' as the two constituents of 'x is p'; we must insist upon asking: What is the specific function for the

case of the proposition 'x is p' when its two arguments are taken to be 'x' and '$\hat{x}$ is p'? Mr Russell only tells us that 'x is p' = '$f(x, \hat{x}$ is $p)$' where the specific symbol f has nowhere been defined by him. It is as if he had said that the quantitative function '$x+p$' has for its two constituents, variants or arguments: (1) x and (2) $\hat{x}+p$. Now according to this analysis of the nature of a function, the process by which a function is constructed out of two variables is to substitute in one of these variables x for $\hat{x}$, so that taking a similar example to the above, the constituents of the quantitative construct '$x \div p$' would be x and $\hat{x} \div p$. Every mathematician would take as the two constituents of the construct $x \div p$ the two simple symbols x and p; as Russell himself does in his preliminary account of the alternative function x or p, of which the two constituents are the simple symbols x and p. In fact he can only take a function of a single variable as ambiguously denoting a proposition, by starting with what I have called a non-formal function, e.g. 'x is hurt' as a non-formal function of x; instead of starting with the essentially logical notion of a function, which is synonymous with the form of a construct such as 'x is p' where instead of one material or variable constituent there are two. In short the form of a proposition, if it has form at all and is not simply expressed by a simple symbol, must contain two independent constituents. When Mr Russell says that $\phi(x)$ is a propositional function, provided that $\phi(a)$ is a proposition, he provides us with no indication as to the *form* that $\phi(a)$ must assume in order that $\phi(a)$ *shall constitute* a proposition.

CHAPTER IV

THE CATEGORICAL SYLLOGISM

§ 1. As the relation between implication and inference has already been explained, we may treat the syllogism indifferently as a species either of implication or of inference: regarded as implication, the propositions concerned must be spoken of as implicants and implicate; regarded as inference, we speak of them as premisses and conclusion. The term syllogism is strictly confined to one only of the many forms of demonstrative inference; and in this strict usage must be defined as an argument containing two premisses and a conclusion, involving between them three terms, each of which occurs in two different propositions. That occurring as predicate in the conclusion is called the major term; that occurring as subject in the conclusion, the minor term; and that not occurring in the conclusion, the middle term. The distinction between the major and minor terms determines which of the premisses shall be called major and which minor: that which contains the predicate of the conclusion being called the major premiss; and that which contains the subject of the conclusion being called the minor premiss. Reference to the conclusion is thus required before the premisses can be distinguished as major or minor. The canonical order of the three propositions, viz. major premiss, minor premiss, conclusion, is purely artificial, and adopted only for general purposes of reference. The mood of a syllogism is defined by the forms (*A*, *E*, *I*,

or *O*) of the three propositions constituting the major premiss, minor premiss, and conclusion, in their canonical order. Furthermore syllogisms are distinguished according to figure: the first figure being that in which the middle term occurs as subject in the major premiss and predicate in the minor; the fourth figure being that in which the middle term occurs as predicate in the major and as subject in the minor: the second figure, that in which the middle term occurs as predicate in both premisses; and the third figure that in which the middle term occurs as subject in both premisses. Two syllogisms would be said to be of different form, although they might agree in mood, provided they differed in figure.

§ 2. There are two opposite tendencies in the choice of illustrations of the syllogism, both of which, in my view, should be avoided. The first is to select examples composed of propositions, each of which is universally accepted as true. But such illustrations hinder the learner from examining the validity of the inferential process from premisses to conclusion, since he is apt to assume validity because of his familiarity with the propositions as being generally accepted. The opposite course, which we find amusingly illustrated by Lewis Carroll, is to select propositions which are obviously false. But this leads the learner to regard the syllogism merely as a kind of game, and as having no real significance in actual thought procedure. It is preferable, therefore, to select propositions which are dubious, or which are affirmed by some persons and denied by others. Of such propositions important kinds are (1) those which deal with political, ethical, or similar

topics in *general*, e.g. 'Lying is sometimes right,' 'All countries that adopt free-trade are prosperous,' 'The suffrage should not be extended to uneducated persons; (2) those which exercise the faculty of judgment, in the Kantian sense, upon some *individual* case, e.g. 'This man is untrustworthy,' 'The Niche is finer than the Venus of Milo,' 'Esau is a more lovable person than Jacob.'

§ 3. Correlative to the syllogism we may here introduce the *antilogism*, in reference to which the above principle of selecting examples will be seen to have special significance. An antilogism may be defined as a formal disjunction of two, three, or more propositions, each of which is entertained hypothetically. When limited to three propositions constituting a disjunctive trio, the antilogism may be formulated in terms of illustrative symbols as follows: 'the three propositions P, Q, and R cannot be true together.' It is then seen that just as the disjunction of P and Q is equivalent to the implication 'If P is true, then Q is false,' so the disjunction of P, Q, and R is equivalent to each of the three implications:

(1) If P and Q are true, then R is false,
(2) If P and R are true, then Q is false,
(3) If R and Q are true, then P is false.

We may put forward the following example of an antilogism, no one of the propositions of which would be universally acknowledged either as true or as false, but which taken together are formally incompatible:

P. All tactful persons sometimes lie.
Q. Lord Grey is a tactful person.
R. Lord Grey never lies.

Something could be said in support of, as well as in opposition to, each of these three propositions; but it is obvious that they are together incompatible, and hence constitute an antilogism or disjunctive trio. This antilogism is equivalent to each of the three following syllogistic implications:

1st if All tactful persons sometimes lie
and Lord Grey is a tactful person,
then Lord Grey sometimes lies.

2nd if All tactful persons sometimes lie
and Lord Grey never lies,
then Lord Grey is not a tactful person.

3rd if Lord Grey never lies
and Lord Grey is a tactful person,
then Some tactful persons never lie.

§ 4. The propositions in each of these syllogisms are in the canonical order of major, minor, conclusion, and the syllogisms will be recognised as being in the first, second, and third figures respectively. In defining the figures of syllogism we may, in fact, separate the first three from the fourth in that the former contain one and only one term standing in one proposition as subject and in another as predicate, while in the fourth figure all three terms occupy this double position. Such a term may be called a class-term, on the ground that a class-term has a partly adjectival meaning, and as such serves appropriately as predicate; and partly a substantival meaning, and as such serves appropriately as subject. The first three figures, then, containing only one class-term, are distinguished from one another according as this term occupies one or another position. In the first figure it serves as the middle term; in the

second figure as the major term; and in the third figure as the minor term. Taking the above antilogism as illustrative, we may generalise by formulating the following antilogistic dictum for the first three figures:

It is impossible to conjoin together the three propositions:

Every member of a class has a certain property;
A certain object is included in that class;
This object has not that property.

From this single antilogistic dictum we construct the dicta for the first three figures of syllogism, thus:

Dictum for 1st Figure

if Every member of a class has a certain property
and A certain object is included in that class,
then This object must have that property.

Dictum for 2nd Figure

if Every member of a class has a certain property
and A certain object has not that property,
then This object must be excluded from the class.

Dictum for 3rd Figure

if A certain object has not a certain property
and This object is included in a certain class,
then Not every member of the class has that property.

These dicta bring out the normal function of each of the first three figures in thought-process. Thus we are reasoning in the first figure when, having established a certain characteristic as belonging to every member of a class, we bring forward an individual object known to belong to the class and proceed to assert that it will have the characteristic common to the class. We are reasoning in the second figure when, having similarly

established a certain characteristic as belonging to every member of a class, and having found that an individual object has not this characteristic, we proceed to assert that it does not belong to the class. We are reasoning in the third figure when we note that a certain object known to belong to a certain class has not a certain property, and proceed to assert that that property cannot be predicated universally of all members of the class; or otherwise, when, having noted that an object known to belong to a certain class has a certain character, we infer that at least one member of the class has this character. A peculiarity of the third figure is that it functions either destructively or constructively; as destructive, it disproves some universal proposition that may have been suggested; as constructive, it naturally suggests the replacement of the particular conclusion either by a universal whose subject is restricted by some further adjectival characteristic, or by an unrestricted universal to be obtained by induction *from* the particular conclusion.

§ 5. A second illustration of an antilogism developing into three syllogisms may be chosen with the purpose of showing how purely formal and elementary reasoning underlies even the most abstract arguments. Thus:

It is impossible to conjoin the three propositions:

P. All possible objects of thought are such as have been sensationally impressed upon us;

Q. Substance is a possible object of thought;

R. Substance has not been sensationally impressed upon us.

Since each of these propositions has been asserted by

some and denied by other philosophers, the three together constitute an antilogism having the same illustrative value as our previous example.

Taking, first, P and Q as asserted premisses and not-R as conclusion, we obtain the syllogistic inference:

P. All possible objects of thought have been sensationally impressed upon us;

Q. Substance is a possible object of thought;

$\therefore$ not-R. Substance has been sensationally impressed upon us.

With some explanations and modifications this syllogism represents roughly one aspect of the new realistic philosophy.

Taking, next, P and R as asserted premisses and not-Q as conclusion, we have:

P. All possible objects of thought have been sensationally impressed upon us;

R. Substance has not been sensationally impressed upon us;

$\therefore$ not-Q. Substance is not a possible object of thought.

This syllogism represents very fairly the position of Hume.

Taking, lastly, R and Q as asserted premisses and not-P as conclusion, we have:

R. Substance has not been sensationally impressed upon us;

Q. Substance is a possible object of thought;

$\therefore$ not-P. Not every possible object of thought has been sensationally impressed upon us.

This syllogism represents almost precisely the well-known position of Kant.

As in our previous example these three syllogisms are respectively in figures 1, 2, and 3; and, moreover, Kant's argument in figure 3 has both a destructive function in upsetting Hume's position; and a constructive function in suggesting the replacement of the particular conclusion by a limited universal which would assign the further characteristic required for discriminating those objects of thought which have not been obtained by experience from those which have been thus obtained.

§ 6. Since the dicta, as formulated above, apply only where two of the propositions are singular or instantial, they must be reformulated so as to apply also where *all* the propositions are general, i.e. universal or particular. Furthermore, they will be adapted so as to determine directly all the possible variations for each figure. As follows:

Dictum for Fig. 1

if *Every* one of a certain class *C* possesses (or lacks) a certain property *P*
and Certain objects *S* are *included* in that class *C*,
then These objects *S* must possess (or lack) that property *P*.

Dictum for Fig. 2

if *Every* one of a certain class *C* possesses (or lacks) a certain property *P*
and Certain objects *S* lack (or possess) that property *P*,
then These objects *S* must be *excluded* from the class *C*.

Dictum for Fig. 3

if Certain objects *S* possess (or lack) a certain property *P*

and These objects S are *included* in a certain class C
then *Not every* one of the class C lacks (or possesses) that property P.
i.e. *Some* of the class C possess (or lack) that property P.

In each of these dicta the word 'objects,' symbolised as S, represents the term that stands as subject in both its occurrences; the word 'property' P, the term that stands as predicate in both its occurrences; and the word 'class' C, that term which occurs once as subject and again as predicate. Hence, using the symbols S, C, P, the first three figures are thus schematised:

Fig. 1	Fig. 2	Fig. 3
C – P	C – P	S – P
S – C	S – P	S – C
∴ S – P	∴ S – C	∴ C – P

§ 7. In order systematically to establish the moods which are valid in accordance with the above dicta, it should be noted in each figure (1) that the proposition $S-P$ is unrestricted as regards both quality and quantity; (2) that the proposition $S-C$ is independently fixed in quality, but determined in quantity by the *quantity* of the unrestricted proposition; and (3) that the proposition $C-P$ is independently fixed in quantity, but determined in quality by the *quality* of the unrestricted proposition. Thus in Fig. 1, while the conclusion is unrestricted, the minor premiss is independently fixed in quality but determined in quantity by the quantity of the conclusion; and the major premiss is independently fixed in quantity but determined in quality by the quality of the conclusion. In Fig. 2, while the minor premiss is unrestricted, the conclusion

is independently fixed in quality but determined in quantity by the quantity of the minor premiss; and the major premiss is independently fixed in quantity, but determined in quality by the quality of the minor premiss. In Fig. 3, while the major premiss is unrestricted, the minor premiss is independently fixed in quality but determined in quantity by the quantity of the major premiss, and the conclusion is independently fixed in quantity but determined in quality by the quality of the major premiss.

Having in the above dicta italicised the phrase in each case which is directly restrictive, the proposition which is unrestricted, i.e. may be of the form *A* or *E* or *I* or *O*, is seen to be: in Fig. 1, the conclusion; in Fig. 2, the minor premiss; in Fig. 3, the major premiss. Hence each of these figures contains four fundamental moods derived respectively by giving to the unrestricted proposition the form *A*, *E*, *I* or *O*. Besides these four fundamental moods there are also supernumerary moods. These are obtained by substituting, in the conclusion, a particular for a universal; or, in the minor premiss, a universal for a particular; or, in the major again, a universal for a particular. These supernumerary moods will be said respectively to contain a weakened conclusion, a strengthened minor, or a strengthened major; and, in the scheme given in the next section, the propositions thus weakened or strengthened will be indicated by the raised letters ^{w} or ^{s} as the case may be.

§ 8. Adopting the method above explained, we may now formulate the special rules for determining the valid moods in each figure as follows:

Rules for Fig. 1.

The conclusion being unrestricted in regard both to quality and quantity,

(*a*) The major premiss must in quantity be universal, and in quality agree with the conclusion.

(*b*) The minor premiss must be in quality affirmative, and in quantity as wide as the conclusion.

Rules for Fig. 2.

The minor premiss being unrestricted in regard both to quality and quantity,

(*a*) The major premiss must be in quantity universal, and in quality opposed to the minor.

(*b*) The conclusion must be in quality negative, and in quantity as narrow as the minor.

Rules for Fig. 3.

The major premiss being unrestricted in regard both to quantity and quality,

(*a*) The conclusion must in quantity be particular, and in quality agree with the major,

(*b*) The minor premiss must in quality be affirmative, and in quantity overlap[1] the major.

Italicising in each case the unrestricted proposition, we may represent the valid moods for the first three figures in the following table:

Valid Moods for the "One-Class" Figures.

	Fundamentals				Supernumeraries	
					sw	sw
Fig. 1	AA*A*	EA*E*	AI*I*	EI*O*	AAI	EAO
					sw	sw
Fig. 2	E*A*E	A*E*E	E*I*O	A*O*O	EAO	AEO
					ss	ss
Fig. 3	*A*II	*E*IO	*I*AI	*O*AO	AAI	EAO

[1] The minor and major will necessarily overlap if one or the other is universal, not otherwise.

§ 9. Having established the valid moods of the first three figures from a single antilogism, we proceed to construct those of the fourth figure also from a single antilogism; thus:

Taking any three classes, it is impossible that
The first should be wholly included in the second
while The second is wholly excluded from the third
and The third is partly included in the first.

The validity of this antilogism is most naturally realised by representing classes as closed figures. Such a representation is in fact valid, although the relation of inclusion and exclusion of classes is not identical with the logical relations expressed in affirmative and negative propositions respectively; for, there is a true analogy between the relations between classes and the relations between closed figures; in that the relations between the relations of classes are identical with the corresponding relations between the relations of closed figures. Thus adopting as the scheme of the fourth figure:

$$C_1 - C_2 \qquad C_2 - C_3 \qquad C_3 - C_1$$

the above antilogism will be thus symbolised:

It is impossible to conjoin the following three propositions:

P. Every C_1 is C_2,
Q. No C_2 is C_3,
R. Some C_3 is C_1.

This yields the three fundamental syllogisms

(1) If P and Q, then not-R; i.e.
if Every C_1 is C_2
and No C_2 is C_3,
then No C_3 is C_1.

(2) If Q and R, then not-P; i.e.
if No C_2 is C_3
and Some C_3 is C_1,
then Not every C_1 is C_2.

(3) If R and P, then not-Q; i.e.
if Some C_3 is C_1
and Every C_1 is C_2,
then Some C_2 is C_3.

Since the propositions of these syllogisms are arranged in canonical order, the valid moods in the fourth figure can be at once written down: *AEE*, *EIO*, *IAI*. Moreover, since the conclusion of the first mood is universal, it may be weakened; since the minor of the second is particular, it may be strengthened; and since the major of the third is particular, it also may be strengthened. This yields:

Valid Moods of the Fourth Figure.

Fundamentals			Supernumeraries		
			w	s	s
AEE	EIO	IAI	AEO	EAO	AAI

Here each supernumerary can only be interpreted in one sense, viz., as containing respectively a weakened conclusion, a strengthened minor, and a strengthened major. In contrast to this, the supernumeraries of the first and second figures must be interpreted as containing *either* a weakened conclusion *or* a strengthened minor; and those of the third figure as containing *either* a strengthened major *or* a strengthened minor.

§ 10. An antiquated prejudice has long existed against the inclusion of the fourth figure in logical doctrine, and in support of this view the ground that has been most frequently urged is as follows:

Any argument worthy of logical recognition must be such as would occur in ordinary discourse. Now it will be found that no argument occurring in ordinary discourse is in the fourth figure. Hence, no argument in the fourth figure is worthy of logical recognition.

This argument, being in the fourth figure, refutes itself; and therefore needs to be no further discussed.

§ 11. Having formulated certain intuitively evident dicta, the observance of which secures the validity of the syllogisms established by their means, we will proceed to formulate equally intuitive rules the violation of which will render syllogisms invalid. These rules will be found to rest upon a single fundamental consideration, viz. if our data or premisses refer to *some* only of a class, no conclusion can be validly drawn which refers to *all* members of that class. This is technically expressed in the rule:

(1) 'No term which is undistributed in its premiss may be distributed in the conclusion.'

This rule *alone* is not sufficient *directly* to secure validity, but from it we can deduce other directly applicable rules which, taken in conjunction with the first, will be sufficient to establish directly the invalidity of any invalid form of syllogism. In the course of deducing these other rules we shall make use of certain logical intuitions that are obvious apart from their employment in this deductive process, of which the following may be mentioned:

(*a*) that if a term is distributed in any given proposition, it will be undistributed in the contradictory proposition; and conversely, if a term is undistributed in a given proposition, it will be distributed in the

contradictory proposition. That this is so is directly seen when it has been accepted on intuitive grounds that only universals distribute the subject term, and only negatives the predicate term; and that an *A* proposition is contradicted by an *O*, and an *I* proposition by an *E*.

(*b*) That any syllogism can be expressed as an antilogism and conversely. This principle follows from the intuitive apprehension of the relation between implication and disjunction.

(*c*) That it is formally possible for any three terms to coincide in extension. (This particular intuition is employed in the rejection of only one form of syllogism.)

We are now in a position to deduce from our original principle, i.e. from rule (1), by means of (*a*), (*b*), and (*c*), other rules, the direct application of which will exclude any invalid forms of syllogism.

(2) 'The middle term must be distributed in one or other of the premisses.'

To establish this, let us consider the antilogism which disjoins *P*, *Q* and *R*; this, by (*b*) is equivalent to the syllogism 'If *P* and *Q*, then not-*R*' and also to the syllogism 'If *P* and *R*, then not-*Q*.' Taking the first of these, if a term *X* is undistributed in the premiss *P*, it must be undistributed in the conclusion not-*R*, i.e. it must, by (*a*), be distributed in *R*. Applying this result to the second syllogism 'If *P* and *R*, then not-*Q*,' we have shown that if the middle term *X* is undistributed in the premiss *P*, it must be distributed in the premiss *R*. This then establishes rule (2).

(3) 'If both premisses are negative, no conclusion can be syllogistically inferred.'

For, taking any two universal negative premisses, these can be converted (if necessary) into 'No P is M' and 'No S is M'; which, by obversion, are respectively equivalent to 'All P is non-M' and 'All S is non-M,' in which the new middle term non-M is undistributed in both premisses. But this breaks rule (2). What holds of two universals will hold *a fortiori* if one or other of the two negative premisses is particular. Thus rule (3) is established.

(4) 'A negative premiss requires a negative conclusion.'

For, taking again the antilogism which disjoins P, Q and R, this is equivalent both to the syllogism 'If P and R, then not-Q,' and to the syllogism 'If P and Q, then not-R.' Taking the first of these two syllogisms, by rule (3), if the premiss P is negative, the premiss R must be affirmative. Applying this result to the second syllogism, we have, if the premiss P is negative, the conclusion not-R must be negative. This establishes rule (4).

(5) 'A negative conclusion requires a negative premiss.'

This is equivalent to the statement that two affirmative premisses cannot yield a negative conclusion. To establish this rule, we must take the several different figures of syllogism:

Fig. 1	Fig. 2	Fig. 3	Fig. 4
M – P	P – M	M – P	P – M
S – M	S – M	M – S	M – S
S – P	S – P	S – P	S – P

For the first or third figure, affirmative premisses with negative conclusion would entail false distribution

of the major term; which has been forbidden under our fundamental rule (1). Taking next the second figure, it would entail false distribution of the middle term, forbidden by rule (2). Finally taking the fourth figure, it would either entail some false distribution forbidden by rules (1) and (2); or else yield the mood *AAO* which would constitute a denial that three terms could coincide in extension, thus contravening (*c*). This establishes rule (5).

§ 12. The five rules thus established may be re-arranged and summed up into two rules of quality and two rules of distribution, viz.

A. *Rules of Quality.*

(a_1) For an affirmative conclusion both premisses must be affirmative.

(a_2) For a negative conclusion the two premisses must be opposed in quality.

B. *Rules of Distribution.*

(b_1) The middle term must be distributed in at least one of the premisses.

(b_2) No term undistributed in its premiss may be distributed in the conclusion.

These rules having been framed with the purpose of rejecting invalid syllogisms, we may first point out that, irrespective of validity, there are sixty-four abstractly possible combinations of major, minor and conclusion. The Rules of Quality enable us to reject *en bloc* all moods except those coming under the following three heads, viz. those which contain (i) an *affirmative conclusion* (requiring affirmative major and affirmative minor); (ii) a *negative major* (requiring affirmative

minor and negative conclusion); (iii) a *negative minor* (requiring affirmative major and negative conclusion). This leads to the following table, which exhibits the 24 possibly valid moods unrejected by the Rules of Quality.

	Maj. & Min. Univ.		Maj. Univ. Min. Part.		Maj. Part. Min. Univ.		Maj. & Min. Part.	
Concl. Aff.	AAI	AAA	AII	AIA	IAI	IAA	III	IIA
Maj. Neg.	EAO	EAE	EIO	EIE	OAO	OAE	OIO	OIE
Min. Neg.	AEO	AEE	AOO	AOE	IEO	IEE	IOO	IOE

§ 13. The Rules of Quality having thus been applied, it remains to reject such of the above 24 as violate the Rules of Distribution, (i) for the middle term, (ii) for the major term, (iii) for the minor term. This will require three special rules for each of the four figures:

Fig. 1	Fig. 2	Fig. 3	Fig. 4
M – P	P – M	M – P	P – M
S – M	S – M	M – S	M – S
S – P	S – P	S – P	S – P

The above scheme shows that it will be convenient to bracket Fig. 1 with Fig. 4 for the *middle term*, Fig. 1 with Fig. 3 for the *major term*, and Fig. 1 with Fig. 2 for the *minor term*; leading to the following:

RULES FOR CORRECT DISTRIBUTION.

1st of the Middle Term.

Fig. 1. If the minor is affirmative, the major must be universal.

Fig. 4. If the major is affirmative, the minor must be universal.

Fig. 2. One premiss must be negative; i.e. *conclusion must be negative.*

Fig. 3. One or the other of the premisses must be universal.

2nd of the Major Term.

Figs. 1 and 3. If the conclusion is negative, the major must be negative; i.e. (in either case) *the minor must be affirmative.*

Figs. 2 and 4. If the conclusion is negative, the major must be universal.

3rd of the Minor Term.

Figs. 1 and 2. If the minor is particular, the conclusion must be particular.

Figs. 3 and 4. If the minor is affirmative, the conclusion must be particular.

These rules have been grouped by reference to the *term* (middle, major or minor) which has to be correctly distributed. They will now be grouped by reference to the *figure* (1st, 2nd, 3rd or 4th) to which each applies. In this rearrangement we shall also simplify the formulations by replacing where possible a hypothetically formulated rule by one categorically formulated. As a basis of this reformulation we take the rules of quality for Figs. 1, 2 and 3, which have already been expressed categorically; viz. for Figs. 1 and 3: 'The minor premiss must be affirmative,' and for Fig. 2: 'The conclusion must be negative.' Conjoining the categorical rule (of quality) for Fig. 1 with its hypothetical rule, 'If the minor is affirmative the major must be universal,' we deduce for this figure the categorical rule (of quantity), 'The major must be universal.' Again, conjoining the

The Special Rules of Distribution for each Figure are:

Fig. 1	Fig. 2	Fig. 3	Fig. 4
a. Minor affirmative.	*a.* Conclusion negative.	*a.* Minor affirmative.	1. If major affirmative, then minor universal.
b. Major universal.	*b.* Major universal.	*b.* Conclusion particular.	2. If conclusion negative, then major universal.
c. If minor particular, then conclusion particular.	*c.* If minor particular, then conclusion particular.	*c.* One premiss must be universal.	3. If minor affirmative, then conclusion particular.

Applying these rules of distribution to the scheme of possibly valid moods unrejected by the rules of quality, we have:

		Unrejected by distribution
Fig. 1.	Rule *a* rejects last row (8 moods)	AAI AAA AII . . . . .
	Rule *b* rejects last 4 columns (12 moods)	EAO EAE EIO . . . . .
	Rule *c* rejects 4th and 8th columns (6 moods)	. . . . .
Fig. 2.	Rule *a* rejects 1st row (8 moods)	. . . . .
	Rule *b* rejects last 4 columns (12 moods)	EAO EAE EIO . . . . .
	Rule *c* rejects 4th and 8th columns (6 moods)	AEO AEE AOO . . . . .
Fig. 3.	Rule *a* rejects last row (8 moods)	AAI . AII . IAI . . .
	Rule *b* rejects even-numbered columns (12 moods)...	EAO . EIO . OAO . . .
	Rule *c* rejects last two columns (6 moods)	. . . . .
Fig. 4.	Rule 1 rejects from 1st and 3rd row the 3rd, 4th, 7th and 8th columns (8 moods)	AAI . . . IAI . . .
	Rule 2 rejects from 2nd and 3rd row the 5th, 6th, 7th and 8th columns (8 moods)	EAO . EIO . . . . .
	Rule 3 rejects from 1st and 2nd row the 2nd, 4th, 6th and 8th columns (8 moods)	AEO AEE . . . . .

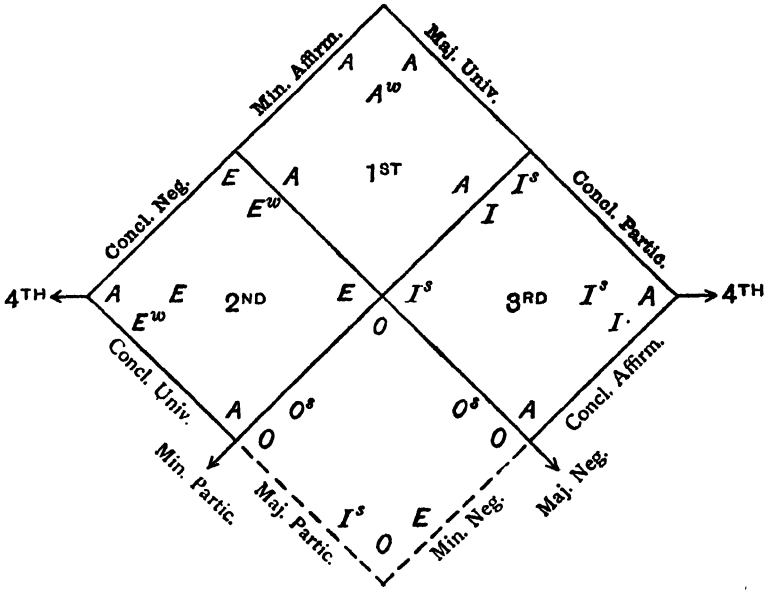
categorical rule (of quality) for Fig. 2 with its hypothetical rule 'If the conclusion is negative the major must be universal,' we deduce for this figure the categorical rule (of quantity), 'The major must be universal.' Lastly, conjoining the categorical rule (of quality) for Fig. 3 with its hypothetical rule, 'If the minor is affirmative the conclusion must be particular,' we deduce the categorical rule (of quantity) for this figure, 'The conclusion must be particular.' The remaining rules must be repeated without modification.

The *Special Rules of Distribution for each Figure* and the application of these rules of distribution to the scheme of possibly valid moods unrejected by the rules of quality are set out on the preceding page.

§ 14. We will now compare the results reached by the two methods—direct and indirect. The direct method determines, by means of certain intuited dicta, what moods are to be accepted as valid; the indirect method determines—on equally intuitive principles—what moods are to be rejected as invalid, and consequently what moods remain unrejected. We gather from this comparison that the 24 moods (6 for each figure) that are established as valid by the direct method are identical with the 24 that are not rejected as invalid by the indirect method. It follows that the two methods must be used as supplementary to one another. For, apart from the use of the indirect method we should not have proved that the moods established as valid were the *only* valid moods; and apart from the use of the direct method we should not have proved that the moods unrejected as invalid were themselves valid. In short, by the direct method we establish the conditions

that are *sufficient* to ensure validity, and by the indirect those that are *necessary* to ensure validity.

§ 15. The attached diagram, taking the place of the mnemonic verses, indicates which moods are valid, and which are common to different figures. The squares are so arranged that the rules for the first, second and third figures also show the compartments into which each



mood is to be placed, according as its major, minor or conclusion is universal or particular, affirmative or negative. The valid moods of the fourth figure occupy the central horizontal line.

§ 16. A very simple extension of the syllogism and of the corresponding antilogism is treated in ordinary logic under the name Sorites, which is a form of argument comprised of propositions forming a closed chain;

and may be defined as 'an argument containing any number of terms and an equal number of propositions, such that each term occurs twice and is linked in one proposition with one term, and in another with a different term.' E.g. an argument of this form, containing five terms, would be represented by the five propositions: *a*—*b*, *b*—*c*, *c*—*d*, *d*—*e*, *e*—*a*, where each term placed first may stand indifferently either for subject or for predicate. Now it will be found that the necessary and sufficient rules for inferences of this form are virtually the same as for the three-termed argument; viz.

A. *Rules of Quality.*

(a_1) For an affirmative conclusion, all the premisses must be affirmative.

(a_2) For a negative conclusion, all but one of the premisses must be affirmative.

B. *Rules of Distribution.*

(b_1) Any term recurring in the premisses must be distributed in (at least) one of its occurrences.

(b_2) Any term occurring in the conclusion must be undistributed, if it was undistributed in its premiss.

The rules for the corresponding antilogism reduce to two, viz.

A. *Rule of Quality:* All but one of the propositions must be affirmative.

B. *Rule of Distribution:* Every term must be distributed at least once in its two occurrences.

§ 17. There are certain irregular forms of syllogism or of sorites, which may be reduced to strict syllogistic form by the employment of certain logical principles, the nature of which we shall proceed to discuss. The

arguments to be considered are those which involve a larger number of terms than of propositions; and it is necessary, in order to test the validity or invalidity of such arguments, to substitute if possible, for one or more of the propositions, an equivalent proposition, which will diminish the terms to the number of propositions. This is done by means of obversion, conversion, and other logical modes. Until this substitution is made, the argument may be valid, and yet break one or more of the rules of syllogism. Thus two of the premisses may be negative, and the argument yet be valid, the apparent violation of the rule being due to the presence of more than the proper number of terms; for example,

No right action is inexpedient,
This is not a wrong action,
∴ This is expedient.

Here by merely obverting the two premisses we arrive at the standard syllogism of the first figure, namely:

Every right action is expedient,
This is a right action,
∴ This is expedient.

In all cases of substituting for a proposition some equivalent, we may require, besides simple conversion, the replacement of some term by one of its cognates. Thus in obversion, we replace P by not-P or conversely; P and not-P being the simplest case of cognates. Again any relative term may be replaced by its cognate correlative. Now in the previous illustration obversion alone was required, whereas if the major had been written 'No inexpedient action is right' conversion would have been required before obverting. To illustrate the

replacement of a relative by its corelative, we may take the old example from the Port Royal logic,

The Persians worship the sun,
The sun is a thing insensible,
∴ The Persians worship a thing insensible.

This argument contains five terms, viz., the Persians, worshippers of the sun, the sun, a thing insensible, and worshippers of a thing insensible. The process of reducing this argument to a strictly three-termed argument is effected by what is called 'relatively converting' the major premiss, and again 'relatively converting' the conclusion syllogistically arrived at from our new premisses. The transformed argument then assumes the form of a strict syllogism in the third figure:

The sun is worshipped by the Persians,
The sun is a thing insensible,
∴ A thing insensible is worshipped by the Persians,

where, by converting the conclusion, we reach that required,

The Persians worship a thing insensible.

§ 18. The question whether the syllogism is actually used in thought process is met by noting that, while in ordinary discourse it is rare to find three propositions constituting a syllogism explicitly propounded, arguments of a syllogistic nature are of frequent occurrence. These syllogisms are expressed as enthymemes, i.e with the omission of one at least of the requisite propositions. Now in an enthymeme there is one, and only one, proposition which could be introduced to render the corresponding syllogism valid. For this reason the enthymeme is liable to one or other of two forms of attack: first it may be attacked on the ground that the premiss

supplied by the hearer is true, and yet renders the argument invalid; or secondly, that the premiss supplied by the hearer is false, and is yet the only one which would render the argument valid. The former case would be said to involve a formal fallacy; and the latter a material fallacy. For example: Consider the enthymeme, 'This flower is a labiate, because it is square-stalked.' Here the premiss 'All labiates are square-stalked,' which is true, renders the argument formally invalid; on the other hand, the proposition which renders the argument formally valid, namely 'All square-stalked plants are labiates' is false. These fallacies arise, for the most part, in the case of disagreement between disputants with respect to the conclusion. An enthymeme is free from both kinds of fallacy when the premiss to be supplied is known or accepted by all parties, and at the same time, renders the argument formally valid. Thus a strictly valid argument is expressed in the form of an enthymeme when there is no question with regard to the truth of the omitted proposition which will render the argument formally valid.

This may be instructively illustrated by taking examples where either the major or the minor premiss in each of the first three figures of syllogism is omitted.

'Mr X is a profiteer, and therefore he ought to be super-taxed,'

this argument is acceptable on condition that the required major premiss 'All profiteers ought to be super-taxed' is admitted.

'All bodies attract, therefore the earth attracts,'

this requires the minor premiss 'The earth is a body.'

'Mr X ought not to be super-taxed, therefore he is not a profiteer,'

this requires the same major premiss as in the first example.

'Everyone present voted for Home Rule, therefore Mr Carson was not present.'

This requires as minor premiss 'Mr Carson would have voted against Home Rule.'

'Mr Carson was present, therefore someone there must have voted against Home Rule.'

This requires the major premiss 'Mr Carson would vote against Home Rule.'

'The earth is not self-luminous, therefore not all attracting bodies are self-luminous.'

This requires as minor premiss 'The earth is an attracting body.' These three pairs of arguments are respectively in the first, second and third figure of syllogism.

§ 19. Having restricted my technical treatment of the syllogism to a single chapter, it will be easily inferred that I attach considerable importance to this form of inference, while at the same time I hold it to be only one among many other equally important forms of demonstrative deduction. Syllogism is *practically* important because it represents the form in which persons unschooled in logical technique are continually arguing. It is *theoretically* important because it exhibits in their simplest guise the fundamental principles which underly all demonstration whether deductive or inductive. It is *educationally* important because the establishment of its valid moods and the systematisation and co-ordination of its rules afford an exercise of thought not inferior and in some respects superior to that afforded by elementary mathematics.

CHAPTER V

FUNCTIONAL EXTENSION OF THE SYLLOGISM

§ 1. THE categorical syllogism treated in the last chapter is correctly described as subsumptive. This term applies strictly to the first figure alone—which may be called the *direct* subsumptive figure, and since, either by antilogism or by conversion, the other figures can be reduced to the first, these may be called *indirectly* subsumptive figures. As explained in Chapter I, this form of inference employs in the most simple manner the Applicative followed by the Implicative principle. The ordinary subsumptive syllogism has a conclusion applying to the same range as the instantial minor, and its typical form is:

Everything is p if m;
This is m;
$\therefore$ This is p.

The first step in the extension of the ordinary syllogism to its functional form is to take a conjunction of disconnected syllogisms of the type:

'Everything is p if m; This is m; $\therefore$ This is p.'
'Everything is p' if m'; This is m'; $\therefore$ This is p'.'
'Everything is p'' if m''; This is m''; $\therefore$ This is p'',' etc.

We next take m, m', m'', etc., to be determinates under the determinable M, and p, p', p'', etc., to be determinates under the determinable P. If then we can collect these major premisses into a general formula holding for every value of M and P in accordance with

the mathematical equation $P=f(M)$, then we have an example of what may be called the functional extension of the syllogism, or (more shortly) of the functional syllogism, where the major or supreme premiss may be expressed in the simple form $P=f(M)$. Thus in the subsumptive syllogism the terms that occur in the minor and major premisses are merely repeated in the conclusion; but, in the functional syllogism which yields an indefinite number of different conclusions for the different minors, the terms which occur in these different minors and conclusions are specific values of the determinables presented in the supreme premiss. Now it will be seen that no other principles are used in the functional syllogism, except the Applicative and Implicative, which together are sufficient to extend deduction beyond the scope of merely subsumptive syllogism. As a concrete example, let us take the formula of gravitation, which may be elliptically expressed 'Acceleration P varies inversely as the square of the distance M,' and written in the form $P=\frac{1}{M^2}$. Then, by the Applicative Principle:

'If $M=7$, then $P=\frac{1}{49}$,'

and adding, as Minor Premiss:

'In this instance $M=7$,'

we infer, by the Implicative Principle:

'In this instance $P=\frac{1}{49}$.'

Similarly when the value of M is 11, the value of P will be $\frac{1}{121}$, and so on. The same form of inference holds for two or more independent variables: thus

Boyle's Law may be written '$T = 239\,PV$'; then, as before, we infer:

When the value of P is 5, and the value of V is 2, the value of T will be 2390.

When the value of P is 3, and the value of V is 7, the value of T will be 5019.

§ 2. A functional expression is of course familiar to the mathematician, but it will be important to examine the logical principles in accordance with which a universal functional formula operates in mathematical demonstration. In the first place we may observe that, as in ordinary syllogism, the supreme or major functional universal must always have been ultimately established by means of inductive generalisation, and in the last resort from intuitive or experiential data. Further, the functional universal may be said to be a universal of the second order, because it not only universalises over every *instance* of a given value m, but applies also to every *value* of M. In deducing from the major '$P = f(M)$' conjoined with the minor 'A certain given instance is m,' we reach the conclusion 'This given instance is p,' where p is found from the equation '$p = f(m)$.' Here it is to be observed: first, that this type of conclusion can be drawn, not only for the minor which predicates m, but also for minors which predicate any other value of M; and secondly, that the character predicated in each conclusion is not merely what is predicated in the functional major, but a determinate specification of this predicate.

In the functional syllogisms that we shall consider in this chapter, the functional major is to be understood to express a factual rule, or more particularly a Law of

Nature. The general conception of a Law of Nature has been discussed (in Chapter XIV of Part I) under the head of the Principles of Connectional Determination. There it is shown that a typical uniformity or Law of Nature may be expressed in the form that the variations of a certain phenomenal character depend upon an enumerable set of other phenomenal characters; of these the former is taken to be connectionally dependent upon the others, which are connectionally independent of one another. A specific universal, which expresses such a relation of dependence may also be called a Law of Covariation; for the nature of the dependence (say) of P upon ABC is such that all the possible variations of which P is capable are determined by the joint possible variations of A, B, C, which are themselves connectionally independent of one another.

§ 3. We have a special case of this relation of dependence or covariation when the determined character can be represented as a mathematical function of the determining characters; and it is this special case which gives rise to the functional syllogism. Now, in a functional major expressed (say) in the form $P=f(A, B, C)$, it may in general be assumed that the correlation of these variables is such that, not only can the value of P be calculated from any assigned values of A, B, C; but also, conversely, that the value of A can be calculated from the values of P, B, C; and that of B from the values of P, A, C; and that of C from the values of P, A, B; and similarly for a larger number of such connected variables. This process is expressed in mathematical terms as solving the equation $P=f(A, B, C)$, to find the value of A, which is thus calculated as a

certain function of P, B, C, and so on. A convenient symbolisation for these several equations will be as follows:

$$P=f_p(A, B, C):$$

from which we calculate

$$A=f_a(P, B, C);\ B=f_b(P, A, C);\ C=f_c(P, A, B).$$

It is here assumed not only that P (in mathematical phraseology) is a single-valued function of A, B, C, but also that in solving this equation to determine A or B or C respectively, these also are single-valued functions of the remaining variables. When this assumption holds, we may speak in a special sense of the *reversibility* of cause and effect; i.e. not only is the effect P uniquely determined by the conjunction of the cause-factors A, B, C; but also each of the cause-factors themselves, such as A, is uniquely determined by the effect-factor P in conjunction with the remaining cause-factors B and C. In the simplest cases reversibility follows immediately from the *form* of the function as seen in the example given of Boyle's Law. Here we have a correlation between temperature, pressure, and volume, in which a constant, say k, is involved, and which assumes indifferently the form:

$$\theta=\frac{pv}{k},\ \text{or}\ v=\frac{k\theta}{p},\ \text{or}\ p=\frac{k\theta}{v}.$$

In this simple case, the multiplier k indicates the special form of the function which in the *general* case was represented by the unassigned but constant symbol f. An equation which, in this way, solves uniquely for all the variables is known as linear. But even in the case of non-linear equations we must be able to deter-

mine, amongst the theoretically possible solutions for any one of the variables, that which is the sole factual value. In other words, a unique determination of all the variables, in terms of a given number of them, may be taken as expressing the actual concrete fact.

§ 4. In this connection it is important to note the *number* of variables entering into the functional formula. In Boyle's Law this number is three; i.e. there are three variables, any one of which is connectionally dependent upon the two remaining variables, so that the scope of dependence may be measured either as two or as three: for the functional formula contains three variables which are *notionally* independent of one another, namely p, v, θ; but of these two only are *connectionally* independent of one another. These two may be taken indifferently either as v and θ, or as p and θ, or as p and v, where, according to the alternative taken, p or v or θ is connectionally dependent upon the two others. In general, when there are r functional relations, connecting n notionally independent variables, then any $n-r$ of these can be taken as connectionally independent of one another, and each of the remaining r as connectionally dependent jointly upon the others.

Thus when $n=8$, and $r=3$, the three functional relations may be symbolised:

$p=f_p(a, b, c, d, e)$; $q=f_q(a, b, c, d, e)$; $r=f_r(a, b, c, d, e)$;

or adopting a shorter notation:

$$p=f_p.abcde\,;\ q=f_q.abcde\,;\ r=f_r.abcde.$$

Such a trio of equations are taken to be implicationally independent of one another; i.e. from neither one or

two of them could we infer the third. Otherwise the number three would reduce to two. Now the number of implicationally independent equations is necessarily the same as the number of connectionally dependent variables. Hence for the case under consideration we may express the three independent functional relations in either of four typical forms:

1.	$p = f_p \,.\, abcde;$	$q = f_q \,.\, abcde;$	$r = f_r \,.\, abcde.$
2.	$a = f_a \,.\, pbcde;$	$q = f_q \,.\, pbcde;$	$r = f_r \,.\, pbcde.$
3.	$p = f_p \,.\, abcqr;$	$d = f_d \,.\, abcqr;$	$e = f_e \,.\, abcqr.$
4.	$a = f_a \,.\, pqrde;$	$b = f_b \,.\, pqrde;$	$c = f_c \,.\, pqrde.$

The first trio expresses the three effect-factors separately in terms of the five cause-factors jointly; the second expresses one cause-factor and two effect-factors separately in terms of one effect-factor and four cause-factors jointly; the third expresses one effect-factor and two cause-factors separately in terms of three cause-factors and two effect-factors jointly; and the fourth expresses three cause-factors separately in terms of three effect-factors and two cause-factors jointly.

In illustration of this general principle we will consider the Law of Gravitation, which may be formulated

$$A = c\frac{m_1 m_2}{d^2},$$

where A is the force of attraction of any two masses m_1 and m_2 whose distance is d, c being constant for all variations of m_1 and m_2, as well as of d. In any application of the above formula we must first suppose m_1 and m_2 to be constant, so that the variation of A depends solely upon that of d. The algebraical equation here is, however, logically incomplete. In the first

place, as regards the effect A, we must add the statement that it is a force acting in the direction of the line joining m_1, m_2. In the second place, as regards the cause d, not only must the *distance* of the line joining m_1 and m_2 be taken as a cause-factor, but also its direction.

In comparing the Law of Gravitation with Boyle's Law, the constants k, c, m_1, m_2, represent unchangeable properties of the bodies concerned, while p, θ, v, d represent their changeable states or relations. It is necessary then to include amongst the independent cause-factors the *permanent* properties of bodies as well as their alterable states or relations.

§ 5. In our typical expression of a set of functional equations the number of variables taken to be connectionally independent was the *same* in all the several equations. But a very important type of connectional formulae is that in which equations enter involving *different* numbers of independent variables. Consider the following:

Let a body be allowed to fall *in vacuo*. Here the two independent cause-factors are the mass (m) of the body and the distance (d) from which it falls to the earth. The effect-factors to be considered are the time (t) of falling, and the impulse (p) of the body upon the earth. Since out of the four variables m, d, p, t, two of them, namely m and d, are (as cause-factors) connectionally independent, the standard form in which *both* of these would enter into the function is

$$p = f_p . md \text{ and } t = f_t . md.$$

But, where the body falls *in vacuo*, the time (t) is in-

dependent of the mass (m)[1]. Hence in this case the two formulae assume the form

$$p = f_p . md \text{ and } t = f_t . d.$$

In this case, since the solution of the equations gives uniquely determined roots, we have:

$$(1)\ p = f_p . md,\ (2)\ m = f_m . pd,\ (3)\ d = f_d . mp$$

and

$$(4)\ t = f_t . d,\ (5)\ d = f_d(t),$$

and, by substitution from (5) in (1), (2), (3) respectively,

$$(6)\ p = f_p . mt,\ (7)\ m = f_m . pt,\ (8)\ t = f_t . mp.$$

Now, since, of the four variables m, d, p, t, any two except t and d may be taken as connectionally independent, either one of the following pairs of connectional equations may be used, thus:

Taking m and d as independents:			$p = f . md$	with	$t = f . d$,
„	m and t „	„	$p = f . mt$	„	$d = f . t$,
„	p and d „	„	$m = f . pd$	„	$t = f . d$,
„	p and t „	„	$m = f . pt$	„	$d = f . t$,
„	m and p „	„	$d = f . mp$	„	$t = f . mp$.

Giving to the unassigned functions their actual form, we have here

$$(6)\ p = mgt \text{ and } (5)\ d = \tfrac{1}{2}gt^2,$$

where the constant g stands for the acceleration 32 ft. per second. Solving these equations so as to express the effect-factors p and t in terms of the cause-factors m and d, we have

$$(1)\ p = m\sqrt{2gd} \text{ and } (4)\ t = \sqrt{\left(\frac{2d}{g}\right)}.$$

§ 6. The example just given suggests a certain further characteristic of the connectional equations

[1] This illustrates the principle underlying the inductive method of agreement; where m is eliminated as a cause-factor relative to the effect t, since a variation in m does not entail a variation in t.

of applied mathematics. As will be seen in the above illustration, the connectional equations from which the deductive process derives other but equivalent equations are of a mixed nature as regards the variables that are taken as independents. Of the two equations $p = mgt$, and $d = \frac{1}{2}gt^2$, from which the other equations are derived, the former expresses the effect-factor p in terms of the cause-factor m and the other effect-factor t; while the second expresses the cause-factor d in terms of the effect-factor t. It is therefore necessary to solve this pair of equations by an appropriate process in order to derive the pair of equations which express the effect-factors in terms of the cause-factors; namely in the form

$$p = m\sqrt{2gd}, \quad t = \sqrt{\left(\frac{2d}{g}\right)}.$$

What holds in this particular example may be generalised. Instead of separating variables that are given from those which have to be deduced, we have a set of equations (corresponding in number to the *dependent* variables) which all the variables taken together have to satisfy. Thus, in the above example, the equations were not at first expressed by taking a pair of cause-factors as independent upon which the pair of effect-factors depended, but the first of the two equations was taken from the pair in which m and t were supposed to be independent, and the other from the pair in which m and t, or p and t were taken as independent. That the particular example of the falling body, originally taken to illustrate a different principle, should have lent itself to the principle now under consideration, is more or less accidental, and we will now put

forward an example which more naturally exhibits this new speciality of a set of determining equations.

Thus: consider the effect of mixing two substances at different temperatures in order to find the resultant temperature which will be reached when thermal equilibrium has been established. Here we must take as the causally determining factors, the two initial temperatures θ_1, θ_2, and the two thermal capacities of the substances k_1 and k_2. The factors to be determined are the heat H_1 entering into or passing from the one substance, and the heat H_2 passing from or entering into the other, together with the final temperature θ. Now the equations that must be here used express the conditions that are to be satisfied—the effects not being, in the first instance, expressible as functions of the cause-factors. These equations of condition are the following three:

$$H_1 = k_1(\theta - \theta_1);\ H_2 = k_2(\theta - \theta_2);\ H_1 + H_2 = 0;$$

from which we find

$$\theta = \frac{k_1\theta_1 + k_2\theta_2}{k_1 + k_2};$$

$$H_1 = \frac{k_1k_2(\theta_2 - \theta_1)}{k_1 + k_2};\ H_2 = \frac{k_1k_2(\theta_1 - \theta_2)}{k_1 + k_2}.$$

The solutions of these equations give the three values H_1, H_2, and θ (respectively) that were to be determined. Thus, in the final solution we have succeeded in expressing the factors to be determined in terms of the determining factors. But, in the equations expressing the conditions to be satisfied, the first two express an effect-factor as a function of two of the cause-factors and one of the effect-factors, and the third equation

expresses one effect-factor as a function of another effect-factor. The use of equations of this kind is necessitated by the inadequacy of our knowledge of the precise temporal process by which the causal conditions operate until the final issue is reached. Thus, in the actual process, heat will be passing to and fro from one to the other of the two substances, and this will entail a rise or fall of their temperatures in an incalculable way, which may be roughly expressed by suggesting that the quantum of heat entering the cooler body may be too great, so that the flow of heat will immediately be reversed; and this process might be conceived as involving even an infinite number of ingoings and outgoings of heat. What we know, however, is that at any stage of the process the heat that leaves one body must be equal to the heat that enters the other, whether this quantum is to be reversed in the next stage or not. It is this law which is expressed in our third equation, while the other two equations express a law or property, specific to the two substances, which correlates the effect upon the temperature with the quantum of heat which enters or leaves the body. What then we know, are these conditions of conservation of the total heat, and the several thermal capacities of the bodies, and from this knowledge the final effects can be calculated. It would appear, in fact, that the cases in which this logical principle is exhibited are those in which we know what is entailed in a final state of equilibrium, without having adequate knowledge for tracing in detail the perhaps oscillating processes which take place in the lapse of time before the final state of equilibrium is reached.

We have illustrated this in the simplest case, where only two substances are mixed, but the reader will easily be able to construct the corresponding equations for any given number of different substances. In all cases the final or resultant temperature is equivalent to the arithmetic mean of the initial temperatures, each 'weighted' by the corresponding thermal capacity. Thus

$$\theta = \frac{k_1\theta_1 + k_2\theta_2 + \ldots + k_r\theta_r}{k_1 + k_2 + \ldots + k_r}.$$

Now having given an illustration from physics, we will give a closely analogous illustration from economics. The formula of covariation which connects the *quantity* of a commodity that is exchanged with its *price* is such that the two opposed parties shall be satisfied at the rate of exchange finally agreed upon. Now the formula of covariation on the side of demand is assumed to be connectionally independent of that on the side of supply. That which represents the economic attitude of the consumers depends solely upon their relative desires for different commodities, their monetary resources, and—we may add—the prices at which they are able to buy commodities other than that under consideration. In the same way, the attitude of the producers is wholly independent of that of the consumers; and depends upon the contract-prices current for the employment of the several agents of production, and upon the efficiency of these agents when co-operating in producing the commodity. It will thus be seen that the several factors that determine the conditions of supply are independent of those that determine the conditions of demand. Here, as in the case of thermal equilibrium,

the equations of condition express, not the effect-factors as functions of the cause-factors, but the conditions taken together which satisfy the consumers and the producers regarded each as economically independent of the other.

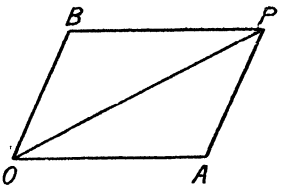
This economic illustration differs from the case of thermal equilibrium in the important respect that the two functions of demand and supply respectively replace the actually operating cause-factors, which are highly complex and do not explicitly enter into the equations to be solved.

§ 7. The above illustrations of the functional extension of the syllogism have shown how, by the use of a set of functional premisses standing as majors, we may take not only minors which enable us to infer an effect-factor from the knowledge of a given cause-factor, but also minors which enable us to infer a cause-factor from the knowledge of a given effect-factor. The supposition upon which this is based has been called the Principle of Reversibility. We shall now show that it is this principle which underlies the so-called method of Residues, and other similar deductive processes. The canon of this method is stated by Mill as follows:

'Subduct from any phenomenon such part as is known by previous inductions to be the effect of certain antecedents, and the residue of the phenomenon is the effect of the remaining antecedents.'

In using the term 'subduct' Mill intends no doubt to hint that, in the simplest cases, for 'subduct' we may substitute 'subtract.' Thus Jevons, in his *Elementary Lessons*, takes the case of 'ascertaining the exact weight of any commodity in a cart by weighing the cart and

load, and then subtracting the weight of the cart alone, which has been previously ascertained.' Here what corresponds to the effect (p) is the weight of the cart and load together, and its causes are (a) the weight of the commodity, and (b) the weight of the cart: so that the functional datum assumes its simplest form, viz. $p=a+b$, which by reversibility gives $a=p-b$. This is a case of solving an equation $p=f(a, b)$ to find a, and deducing $a=f(b, p)$, the equations being linear. The next simplest example of such reversibility is that of the composition of forces. Here the diagonal OP represents the effect, and the sides OA, OB, the cause-factors. Just as, given the two cause-factors OA and OB, we draw AP parallel and equal to OB to find the effect OP; so, given OA as one cause-factor and OP as effect, we may draw OB parallel and equal to AP to find the other cause-factor OB. Innumerable other examples may be given of reversibility for more or less complicated cases. But the classical example most frequently cited is the Adams-Leverier discovery of the planet Neptune from the observed movements of Uranus. Here we may represent the positions and masses of the Sun, of the Moon and of the unknown Neptune by the symbols a, b, c respectively; and the movement of Uranus by the symbol p. Thus p was theoretically known as a given function of a, b, c, say $p=f_p(a, b, c)$, where p stands elliptically for the effect, and a, b, c for the several cause-factors. The solution for c was then uniquely calculated in the form



$$c=f_c(p, a, b).$$

Now it will be observed that the so-called Method of Residues, which is based upon the assumption of reversibility is purely deductive, in that (1) it employs only the Applicative and Implicative principles of inference, and (2) the conclusion obtained applies solely to the specific instances for which the calculation is made. This consideration shows that there is no justification for putting Herschel's method of Residues under the head of methods of induction, along with such methods as those of Agreement and Difference; for, on the grounds above alleged, it is purely deductive. On this matter Mill sees half the truth; for, in comparing the Method of Residues with that of Difference, he remarks that the negative instance in the former is not the direct result of observation, but has been arrived at by deduction. And again, in his formulation of the Canon of Residues, he speaks of 'such part of the phenomenon as is known by previous induction,' where he fails to note that what is *known by previous induction* functions merely like the major premiss of a syllogism, and therefore does not in any way render the inference inductive. What holds for the method of Residues holds also of many less technical processes which, while purely deductive, have been obscurely conceived as inductive. For instance, the procedure in a judicial enquiry or by a police detective or of historical research in discovering the specific cause of a complicated set of circumstances constituting an observed effect, is purely deductive; for it employs as major premiss known laws of human or physical nature under which the known circumstances are to be subsumed in the minor; while the conclusion refers solely to the case *sub judice*.

§ 8. Something should be said in explanation of the fact that inferences of this kind are so frequently spoken of as inductive. It is not only because the major premiss must itself have been obtained by induction, but further because the minor premiss represents a fact obtained by observation, that logicians have made this mistake; for the notion of observation or experimentation as the method by which new knowledge is acquired is invariably associated with induction. But it should be pointed out that there is here a confusion between the matter and the form of an inference. Mere syllogism will obviously yield new material knowledge, provided that the minor premiss represents new material knowledge such as can only be obtained by observation. For example, from the observation that the importation of food has been taxed, we may infer the new material knowledge that the price of food will rise at a certain time in a certain economic market, if we have been otherwise assured of the major premiss appropriate to the circumstance. The *form* of such an inference is purely deductive, and the fact that historical research—and not a merely foreknown universal formula—has been required to establish the minor does not render the argument in any sense inductive; for the conclusion holds only of the period and region to which the causal occurrence which has been discovered applies, and does not involve any inductive generalisation from one period or region to others. A further explanation of this common error is to be found in the fact that the conclusion reached deductively for a given instance may often be *verified* by awaiting the occasion for observing the effect in that instance. Now this process of verification

merely assures us that we have adequately estimated the causes operating in the given instance; but it has been almost invariably confused with the process of verifying, or rather *confirming*, the major premiss itself regarded as a problematic hypothesis as yet unproven.

§ 9. We ought now to distinguish, in these functional extensions of the syllogism, the element which is purely subsumptive from that which is functional; for the two elements are practically always united in any concrete inference of the functional kind. It will be found that the factual formulae used in applied mathematics as major premisses for deduction necessarily involve two kinds of constituent, one of which is known as variable and the other as constant. The mathematical use of the term *constant* presents certain difficulties from the logical point of view. There are certain constants—e.g. the specific integers and the algebraical operators—which are absolutely constant in the sense that in all their occurrences they stand for the same thing and are entirely independent of context. But those so-called constants which are dependent upon context are only referentially constant, being actually variable in precisely the same sense as the symbols that mathematicians recognise as variable. To explain this we may select illustrations from an innumerable variety of formulae used in applied mathematics. Consider, for instance, the formula which expresses the elasticity of a solid body which can support tension. The rule upon which the extension of such a body depends is shortly expressed in the formula $T = kE$, where T stands for the variable tension and E for the variable extension; while k, which is said to be constant, stands for the elasticity of

the particular kind of solid for which the rule holds. Now this coefficient of elasticity, though constant for all possible variations of extension and tension of the body, yet varies from one kind of solid substance to another. We shall show, then, that such a coefficient, which mathematicians call constant, is used in the deductive process *subsumptively*, while that which is explicitly regarded as variable is used *functionally*. We may mark the real variability of a so-called constant by a subscript indicating the specific kind of substance of which the coefficient can be predicated. Thus k_s will stand for the coefficient of elasticity of the kind of substance named s; while $k_{s'}$ (say) will stand for that holding for the kind of substance called s'. To express the mathematical procedure in strictly explicit logical form:

Major Premiss. Every body, say b, which is k_s has the property expressed by the algebraical equation $T = k_s E$.

Minor Premiss. A certain body b is k_s.

Conclusion. The body b has the property expressed by the equation $T = k_s E$.

Now this is a merely subsumptive syllogism, in which the coefficient k_s and the body b recur unmodified in the conclusion as in the premisses. Thus, the coefficient which is called constant is used solely in a subsumptive form of syllogism; but, inasmuch as a similar formula applies to bodies of a different nature (such as s'), the coefficient k is not absolutely constant but varies according to the substance of the solid. In logical analysis, we must recognise the distinct ways in which the so-called constants and the so-called variables enter into the deductive process. This may be expressed

logically by defining the *order* in which the variations have to be made. For we have *first* to consider variations of the so-called variables, which determine the range of the conclusion as holding for every case over which the constant applies. Only *after* this range of variation has been taken into consideration may we proceed to vary the so-called constants, and for any new value carry out the same range of variations of the variables. In language borrowed from mathematical terminology, we may say that the variations of the explicit variables are to be made within the bracket, while the variations to be made of the so-called constants are to be made outside the bracket.

CHAPTER VI

FUNCTIONAL DEDUCTION

§ 1. UNDER this heading we shall discuss the principles underlying the deduction of formulae in the sciences of mathematics and logic. Although properly speaking pure mathematics is a development of logic, yet certain important points of distinction between the two sciences must be brought out. It has been very commonly assumed that the sole method of deductive procedure in pure mathematics, including Geometry, is syllogistic. Now although it will be found that no fundamental principle is employed in mathematical deduction other than the Applicative—which is essential for syllogism—yet the conclusions successively derived from previously established formulae are not such as could be inferred by means of any mere chain of syllogisms. To explain this, it is necessary to point out the peculiar nature of the relation between conclusion and premisses in mathematical processes. Ordinary syllogism, as has been explained, is of the comparatively simple type denominated subsumptive. If subsumptive inferences only were used in algebra or geometry, it would be impossible to demonstrate conclusions except for special cases subsumable under the primary intuited axioms or under some previously established formulae. Thus from such premisses as: 'Everything that is m is p' and 'Everything that is n is q' we could infer sub-

sumptively only that 'Everything that is m and n is p and q.' In other words, by means of subsumptive deduction, we can infer only that what holds universally of the members of a genus, m or n, holds universally of the members of their common species, viz. of the things that are characterised as being both m and n. For example: in geometry, having established a formula for all triangles and a formula for all right-angled figures, we could by merely subsumptive inference predicate of any species of triangles—say right-angled—only what could be predicated of all triangles; and similarly we could predicate of any species of right-angled figures—say three-sided—only what could be predicated of all right-angled figures. But actually in geometry we prove a property (viz. the Pythagorean) of all right-angled triangles which is not the same as any universal property either of three-sided or of right-angled figures. Similarly in algebra, we can deduce properties of all integers divisible by 2 and divisible by 3, which hold neither of all integers divisible by 2 nor of all integers divisible by 3. A predicate which holds for all members of a species, but not for all members of any genus to which by definition the species belongs, is technically known as a proprium or ἴδιον, either of which term may be translated *property*. It is one of the special objects of this chapter to analyse the process by which properties, in this technical sense, are deduced. It will be shown that, in the deductions peculiar to pure mathematics, the premisses and conclusions assume the form of functional equations; and that it is owing to this characteristic that properties in the technical sense can be deductively demonstrated. We therefore give the

name *functional deduction*, in antithesis to subsumptive or syllogistic deduction, to the specifically mathematical form of inference.

§ 2. Before entering upon the main discussion it will be well further to consider the nature of the Aristotelean ἴδιον. Many modern logicians have failed to grasp the important significance to be attached to this notion. Elementary textbooks, such as that of Jevons, define a property of a class as any character not included in the connotation, which can be predicated of *all*, as distinct from an accident which can be predicated only of *some*, members of the class. On the other hand, Mill attempts to define a proprium in closer connection with the scholastic development of Aristotle's doctrine, and distinguishes not merely between an invariable and a variable predicate of a class—which satisfies Jevons—but defines a proprium as a predicate not included in the connotation of the class (and therefore assertible in a proposition not merely verbal) but following necessarily from the connotation alone. But since a proposition which merely asserts connotation is verbal, this account of the proprium is incompatible with the theory—so clearly expounded in his chapters on Definition and on Verbal Propositions—that no conclusion can be drawn from merely verbal propositions that is not itself merely verbal. From this it follows that in order demonstratively to establish any invariable character that can be regarded as necessary, we require as premisses not only definitions but also real or genuine propositions, and, in mathematics, ultimately axioms. It is true that Mill distinguishes two ways in which the proprium may follow necessarily from the connotation:

'it may follow as a conclusion follows premisses, or it may follow as an effect follows a cause.' But this distinction is purely illusory and wholly irrelevant to the notion of necessity of demonstration; for, in both cases, the ground for Mill's account of a proprium as necessarily following from the connotation is that appropriate knowledge will enable us to infer demonstratively the proprium from the connotation. A legitimate distinction may be drawn according as the major premiss from which a proprium is inferred is of the nature of an axiom or of a causal law. Indeed Mill himself goes on to say that the necessity attributed to the proprium means that 'its not following would be inconsistent with'—i.e. *its following could be inferred from*—either an Axiom or a Law of Nature. Thus in both cases the notion of *following* is the same, and simply means *inferrible from*. The proprium, therefore, never follows from the connotation *alone*, but requires in addition one or other of the two species of real propositions, axiomatic or experiential, to serve as major premiss.

§ 3. The functional equations used in the deductions of pure mathematics in some respects differ from and in others agree with those used as major premisses in the process discussed under the head of the functional extension of the syllogism. The equation used in this latter process serves as a single major premiss for a number of specific conclusions found by replacing the variables by their specific values. Here the functional equation assumes the form $P=f(A, B, C)$ for all values of A, B, C. But the equations used in the process of functional deduction are of the form $f(A, B, C)=\phi(A, B, C)$ for all values of A, B, C, where all the variables are

independently variable, and the equation therefore contains no such symbol as P that can be exhibited as dependent upon the others. The distinction between these two types of equation is familiar to mathematicians; the former may be called a *limiting*, the latter a *non-limiting* equation. The limiting equation is generally used to determine one or other of the quantities P, A, B, or C, in terms of the remainder; so that here we associate the antithesis between dependent and independent with the antithesis between unknown and known; whereas, in the non-limiting equation, no one of the variables can be regarded as unknown and as such expressible in terms of the others regarded as known. The distinctions that have been put forward between these two types of functional process are tantamount to defining the functional syllogism as that which proves factual conclusions from factual premisses, and functional deduction as that which proves formal conclusions or formulae from formal premisses, i.e. from formulae previously established. It will further be observed, from the simple illustrations which follow, that whereas the functional syllogism requires only the one functional equation that serves as major premiss, the process of functional deduction will necessarily involve a conjunction of two or more functional equations, all of which are, as above explained, formal and not factual.

To illustrate the general formula used in functional deduction, viz.:

$$f(a, b, c, \ldots) = \phi(a, b, c, \ldots)$$

which is understood to hold for every value of the

variables A, B, C, ..., we may instance the following elementary examples:

$$(a+b)\times(a-b)=a^2-b^2$$

and

$$a\times b=b\times a,$$

both of which involve two variables; and again

$$(a+b)+c \quad =a+(b+c)$$

and

$$(a+b)\times c \quad =(a\times c)+(b\times c),$$

both of which involve three variables. The last three formulae are known respectively as the Commutative, the Associative and the Distributive Law.

§ 4. In the functional equations of mathematics it is important to realise the range of universality covered by any functional formula. This range depends upon the number of independent variables involved in the formula, the range being wider or narrower according as the number of independent variables is larger or smaller. For example, supposing that x, y, z have respectively 7, 5, 10 possible values; then the number of applications of the formula involving x alone is 7, that of a formula involving x and y alone is 35, and that of a formula involving x and y and z is 350. And in general, the number of applications of a formula is equal to the arithmetical product of the numbers of possible values for the variables involved. Now the number of possible values of any variable occurring in logical or mathematical formulae is infinite; hence, for the cases respectively of 1, 2, 3 ... variables, the corresponding ranges of application would be ∞, ∞^2, ∞^3..., constituting a series of continually higher orders of infinity; or rather, in accordance with Cantor's arithmetic, each of the ranges of application for 1, 2, 3 ... variables is a

proper part of that for its successor, although their cardinal numbers are the same.

Now it will be found that, in inferences of the nature of functional deduction, the derived formula may have a range of application—not narrower than but—equal to or even wider than that from which it is derived. Thus the word *deduction* as here applied does not answer to the usual definition of deduction (illustrated especially in the syllogism) as inference from the generic to the specific; although the only fundamental principle employed in the process is the Applicative, according to which we replace either a variable symbol by one of its determinates or one determinate variant by another. But here a distinction must be made according as the substituted symbol is simple or compound. If we merely replace any one of the simple symbols a, b, c by some other simple symbol we shall not obtain a really new formula, since the formula is to be interpreted as holding for all substitutable values, and hence it is a matter of indifference whether we express the formula in terms of the symbols a, b, c, (say) or of p, q, r. In order to deduce new formulae, it is necessary to *replace two or more simple symbols by connected compounds.*

For those unfamiliar with mathematical methods, it should be pointed out that, when any compound symbol is substituted for a simple, the compound must be enclosed in a bracket or be shown by some device to constitute a single symbolic unit. Though we may always replace in a general formula a simple by a compound symbol, the reverse does not by any means hold without exception. The cases in which such substitution is permissible have been partially explained in the

chapter on Symbolism and Functions. There it was shown that, if a formula involves such compound symbols or sub-constructs as $f(a, b)$, $f(c, d)$ etc., *and only such*, where none of the simple symbols used in the one bracketed sub-construct occur in any of the others, then these bracketed functions are called disconnected. It is in the case of disconnected functions that free substitutions of simple symbols for the compound are permissible. The reason for this is that, for the notion of a function of any given variants, it is essential that these shall be variable independently of one another. Now, when the different sub-constructs or bracketed functions are connected with one another through identity of some simple symbol, say a, it is clear that we cannot contemplate a variation of one of these compounds without its involving a variation of the other connected compounds. Hence we should be violating the fundamental principle of independent variability of the variants, if we freely substituted for such connected compounds simple symbols which would have to be understood as capable of independent variation. Hence, it is only when the various compounds involved in a function are unconnected, that for each of such compounds a simple symbol may be substituted.

§ 5. Returning to the problem under immediate consideration, a simple illustration from algebra will show how, by making appropriate substitutions in a given functional formula, we may demonstrate a new formula. Thus, having established the formula that for all values of x and y

$$\text{(i)}\quad (x+y)\times(x-y)=x^2-y^2$$

we may substitute for x and y, respectively, the connected

compounds $a+b$ and $a-b$; and so deduce (by means of the distributive law for multiplication etc.) that for all values of a and b,

$$\text{(ii)}\quad 4ab=(a+b)^2-(a-b)^2.$$

This is a *new* formula, different from the previous one, because the relation between a and b predicated in (ii) is different from the relation between x and y predicated in (i). Moreover the range of application for (ii) is no narrower than that for (i); for (i) applies for every diad or couple 'x to y,' and (ii) for every diad or couple 'a to b'; and therefore the ranges for (i) and (ii) are the same. Again, if we have established the Commutative, Associative, and Distributive formulae given above, the reader will see that, by means only of the Applicative principle, we can deduce from these three formulae what is in fact a new formula:

$$(a+b)(c+d)=ac+bc+ad+bd.$$

In this case, the formula deduced has a wider range of application than any of the formulae from which it is deduced. For the premisses for this deduction involve respectively 2, 3 and 3, independent variables, while the conclusion involves 4; showing, as explained in the previous paragraph, that the range of application of the conclusion is wider than that of even the widest premiss. To reach a conclusion inclusive of and wider than the premisses is in general considered the mark of an inductive inference; but we have shown by the above example that, where the premisses are functional formulae involving more than one independent variable, the mere employment of the Applicative principle enables us to reach a formula wider than any of the premisses. Now

it is in accordance with general usage to define deductive inference as that which employs no principles but the Applicative and the Implicative. In the purely deductive process of mathematics, in fact, it is only the Applicative principle that is required; and pure mathematics is regarded as specially typifying the power of mere deduction. It is true, however, that mathematicians have employed a method which involves also the Implicative principle, viz. what has always been known under the name of 'mathematical induction.' In these later days, this method has been regarded as more specifically characteristic of mathematics than any other. But the line of distinction between induction and deduction, in their extended potentialities for demonstrative inference, cannot be drawn on any logical principle that would be universally accepted. It is for this reason that I have attempted to treat in one large division of my Logic all varieties of demonstrative inference, on the ground that it is the demonstrative character of these inferences that brings them within one sphere, and that the distinction that might be drawn between deductive and inductive demonstration has no important logical significance comparable with that between demonstrative and problematic inference. Mathematics, as the above adduced inferences illustrate, provides a host of cases in which the Applicative principle alone is explicitly employed without any recourse to the Implicative principle. These inferences might be called purely Applicative[1] in contrast to the syllogism, which in our analysis has been shown to involve the Implicative as well as the Applicative principle. Again the construction of the logical

[1] Cf. Chapter I, p. 11 and onwards.

calculus involves the Implicative as well as the Applicative principle, and will be discussed later. Before proceeding to this topic, we must complete our account of mathematical demonstration by an analysis of *mathematical induction*, which also involves both principles.

§ 6. Mathematical induction assumes a unique place in logical theory. It resembles other forms of demonstrative induction, which will be discussed in a later chapter, where it will be shown that the universal mark of this type of induction is that the conclusion demonstratively inferred asserts for every case what has been asserted in one premiss for a single case. The possibility of such demonstration rests upon the logical character of *the other* premiss, which may be of different types, each type yielding a different form of demonstration. The distinctive characteristic of mathematical induction is that it is concerned with finite integers. These constitute a discrete series beginning with the integer 1, and proceeding step by step in the construction of successive integers. The generation of each successive integer from the preceding is indicated by the operation *plus* 1. Thus, using the illustrative symbol n to stand for any finite integer, the operation symbolised as $n+1$ will yield the next following integer. This construction defines the general conception of a finite integer which is fundamental for arithmetic. The method of mathematical induction introduces the notion of function. Thus $f(n)$ will be used to stand for any proposition[1]

[1] The functions previously adduced were *mathematical*, i.e. constructs yielding quantities, whereas the function here introduced is *propositional*, i.e. a construct yielding a proposition. And, in general, the *equating* of two mathematical functions yields a propositional function.

about the specific integer n, where variation of form will be represented by changing f into ϕ say, and variation of reference by changing n into m say. The argument in its general form will consist of the following assertions of two premisses and of the inferred conclusion:

Implicative Premiss: 'The proposition $f(n)$ would imply the proposition $f(n+1)$' for every finite integer n.

Categorical Premiss: '$f(1)$' holds.

Conclusion: Therefore '$f(n)$' holds for every finite integer n.

In this argument we observe that the conclusion states categorically what is stated hypothetically in the implicative premiss; and further that it predicates for every case what is predicated for a single case in the categorical premiss. Its demonstrative force may be shown by resolving the argument into a succession of steps. Thus, by the applicative principle, we may replace in the implicative premiss n by 1, and this yields the assertion '$f(1)$ would imply $f(2)$'; then, adding the categorical premiss '$f(1)$,' we infer, by the implicative principle, '$f(2)$.' Again, replacing n by 2, '$f(2)$ would imply $f(3)$,' and, adding to this the conclusion of the preceding inference, we may infer '$f(3)$.' If this process is indefinitely continued we are enabled, by use merely of the applicative and implicative principles, to infer successively $f(2), f(3), f(4)$, etc., for every finite integer. The whole argument therefore rests merely upon the same principles as are involved in ordinary deduction; and yet the inference is of the nature of induction, because the conclusion is a generalisation of the same

Boyle's Law may be written '$T = 239\,PV$'; then, as before, we infer:

When the value of P is 5, and the value of V is 2, the value of T will be 2390.

When the value of P is 3, and the value of V is 7, the value of T will be 5019.

§ 2. A functional expression is of course familiar to the mathematician, but it will be important to examine the logical principles in accordance with which a universal functional formula operates in mathematical demonstration. In the first place we may observe that, as in ordinary syllogism, the supreme or major functional universal must always have been ultimately established by means of inductive generalisation, and in the last resort from intuitive or experiential data. Further, the functional universal may be said to be a universal of the second order, because it not only universalises over every *instance* of a given value m, but applies also to every *value* of M. In deducing from the major '$P = f(M)$' conjoined with the minor 'A certain given instance is m,' we reach the conclusion 'This given instance is p,' where p is found from the equation '$p = f(m)$.' Here it is to be observed: first, that this type of conclusion can be drawn, not only for the minor which predicates m, but also for minors which predicate any other value of M; and secondly, that the character predicated in each conclusion is not merely what is predicated in the functional major, but a determinate specification of this predicate.

In the functional syllogisms that we shall consider in this chapter, the functional major is to be understood to express a factual rule, or more particularly a Law of

premisses (including the axioms) from which the conclusion is drawn. This summary account of the methods and results of deductive reasoning enables us to meet what has been called the paradox of inference in a more direct way than that explained in Chapter I. For the existence of the mathematical calculus, where the conclusions are absolutely unknown to those who start by admitting as self-evident the fundamental premisses, constitutes a direct refutation of the arbitrary dictum that for valid inference the conclusion must not contain more than what is already known in asserting the premisses.

The notion of a calculus is generally associated with elaborate symbolism, which renders possible the more complex deductive processes in logic and mathematics. As a question of history, there is no doubt that the introduction of such simple symbols as $+$, $-$, $\times$, created a revolution in mathematical science, and rendered it possible to make advances otherwise unattainable. Again it is an equally noteworthy historical fact that the best formal logicians, such as Leibniz and Lambert, were comparatively unsuccessful in their attempt to develop a logical calculus, which was first started by Boole on lines followed by all subsequent symbolists who advanced the science. Boole's method was simply to import the familiar symbols of elementary arithmetic into logic, making use of the fundamental formulae with which algebraists were already conversant. In this way he created the first great revolution in the study of formal logic, and one that is comparable in importance with that of the algebraical symbolists in the sixteenth century. I think, however, that Boole's procedure has

led to considerable confusion with regard to the relations between the logical and the algebraical calculus, inasmuch as he seems to have supposed—in common with many logicians of his time—that the advance achieved by introducing mathematical formulae into logic made logic into a department of mathematics. This attitude of Boole's obstructed, for a considerable period, the investigation of the foundations of mathematics, which demanded the reversal of the relationship between the two sciences. It is under the influence mainly of Peano and of the new mathematicians such as Cantor, that we now recognise mathematics to be a department of logic. The current phrase *mathematical logic* is ambiguous inasmuch as it may be understood to mean either the logic of mathematics or the mathematics of logic. Now, in my view, the logic or rather philosophy of mathematics is a study which ought to dispense entirely with symbolic language. It must, of course, explain the nature of symbols and of symbolic methods, and account for the extraordinary power of symbolism in deducing with absolute security previously unknown formulae. But the philosophical exposition of the deductive power of mathematics must be treated in language the understanding of which requires thought of a profounder nature than that required in merely following symbolic rules. As indicated in the chapter on Symbolism and Functions, the essential purpose of symbolism is to economise the exercise of thought; and thus symbolic methods are worse than useless in studying the philosophy of symbolism or of mathematics in particular. The phrase 'mathematics of logic,' on the other hand, merely indicates a certain line of development of logic, in which

deductive processes are reduced to strictly demonstrative form by means of a symbolism founded on explicitly logical axioms. The important advances in this direction have been systematised with extraordinary success in Whitehead and Russell's great work *Principia Mathematica*, where it is shown how pure mathematics can be actually developed from pure logic. The value of the work consists, therefore, in reducing mathematics to logic, and not at all in reducing logic to mathematics. I shall attempt hardly any criticism of their formal development of the science, and shall here confine myself to the principles which enter into its very elementary foundations.

§ 8. In contrasting the mathematical developments of logic with the ultimate foundations of the science, it will be convenient to use the terms prematheматical and mathematical logic, the latter of which introduces certain novel conceptions, strictly formal in character, in addition to those employed in the former. There are certain notions common to the prematheматical and mathematical departments of logic, and of these we have already discussed the nature of functions, illustrative and short-hand symbols, variables, brackets, etc., which before Peano and Russell had not received adequate recognition in logical teaching; they apply, however, over a wider field than mere mathematics, and must therefore be transferred without modification from the narrower science back to logic. The term 'formal' as applied to these conceptions means that they are to be understood by the logician as such, and they include, besides those primitive ideas which are to be understood without definition, also derivative ideas which are com-

pletely defined in terms of primitive ideas. For example, the notions of implication, alternation, disjunction and negation are formal, and of these we may take negation and alternation as understood without definition, while the others can be defined in terms of these two[1]. Again logical categories and sub-categories such as substantives proper, primary and secondary adjectives and propositions, come under the head of formal conceptions. There are also specific adjectives and relations, such as true, probable, characterised by, comprised in, identical with, which are formal; and, though some of them are ultimately indefinable, the understanding of all of them is essential to logical analysis. In contrast to these formal adjectives, such adjectives as *red*, *hard*, *popular*, *virtuous*, etc. are termed material, because their meaning is unessential to the explication of logical forms. In premathematical logic formulae are established for all adjectives as such, or for a limited set of adjectives comprised in such a sub-category as that of secondary adjective. The range over which these formulae hold must be said to be *material*, though it necessarily comprises adjectives which may happen to be formal, i.e. to have specifically logical significance. Passing from premathematical to mathematical logic, we find that *new* specific adjectives, having essentially logical significance and coming therefore under the head of formal conceptions, are introduced. We may specially mention *integers* and *ratios*. Integer is a logical sub-category

[1] This, at any rate, is the procedure of the *Principia Mathematica*; but, while undoubtedly permissible from the point of view of the logical *calculus*, it is open to serious philosophical criticism, which I have given elsewhere.

comprised in the general category *adjective*; and ratio is a logical sub-category comprised in the general category *relation*; but what constitutes the new feature in mathematical logic is that each specific integer and each specific ratio has itself essentially logical significance, while at the same time formulae hold for all integers and again for all ratios. Prematheматical logic on the other hand can only establish formulae holding for adjectives in general or for secondary adjectives in general. This distinction carries with it the further result that prematheматical logic can only use illustrative adjectival symbols as variables over a range of variation covering the whole category *adjective*, or the whole sub-category *secondary adjective*; while in mathematical logic there occur illustrative symbols for variables covering the range, in the one case *integer*, in the other *ratio*. Consider for example such an illustrative symbol as m in ordinary or prematheматical logic. The specific values that can be substituted for this variable are material; for the formal character of such of them as have specifically logical significance is irrelevant to the truth of the formulae. In mathematical logic, on the other hand, *all* the specific values which can be substituted for a symbol m standing for any integer, say, or a symbol t standing for any ratio, denote formal conceptions. Again it is obvious that, besides the formulae which hold for adjectives in general, there are innumerable additional formulae holding for integers or for ratios; and this accounts for the variety and complexity of mathematics as compared with prematheматical logic. But the essential distinction between the two sciences—or rather the two departments of logical science—lies in the point

already urged, namely that every specific adjective within a certain mathematical range has itself a logically determined value; whereas no logically determined value can be assigned to adjectives in general which enter into prematheматical logic. This distinction may be summed up in other words by taking the two antitheses material and formal, and constant and variable, which combined give the four cases formal variables, formal constants, material variables and material constants. Now prematheматical logic uses formal constants and material variables (and also in Mr Russell's work material constants), *but nowhere formal variables*. On the other hand mathematics uses formal constants, material variables, *and also formal variables*. It is therefore the use of formal variables that fundamentally distinguishes mathematics from prematheматical logic.

§ 9. To continue our account of the relation between the prematheматical and mathematical departments of logic, we must next define and illustrate the nature of those formal elements which are never expressed by variable symbols, and therefore come under the head of formal constants. To these, the name *connectives* will be given. The first division under this head includes what are known as operators in mathematics, such as *plus*, *minus*, *multiplied by*, *divided by*, as well as analogous logical operators such as *and*, *or*, *not*, *if*. Thus the operation '$m+n$,' where m, n stand for determinate numbers, yields a certain determinate number; and analogously the operation '*p and q*,' where p, q stand for determinate adjectives, yields a certain determinate adjective. This is most clearly seen when a proper name has been invented to stand for the compound

construct as well as for each of the constituents themselves: thus, the operation 'three-plus-five' yields the number 'eight'; the operation 'rational-and-animated' yields the adjective 'human.' The analogy goes one step further when, in place of the simple predication *yields*, we use the complex yields-what-is-yielded-by: thus, the operation '*m* plus *n*' yields-what-is-yielded-by the operation '*n* plus *m*'; the operation '*p* and *q*' yields-what-is-yielded-by the operation '*q* and *p*.' Now neither in logic nor in mathematics is it ever required to use illustrative or variable symbols to stand for formal operators like *plus* or *and*—the reason being that no formula which holds for one operator will hold if we substitute indiscriminately any other operator. Hence, if symbols are used for formal operators, these come under the head of short-hand symbols, and never under the head of illustrative or variable symbols. Thus the operators both of logic and of mathematics enter as formal constants, never as variables.

In the second division of connectives are to be included certain relational predications which must be systematically illustrated and classified according to their different properties. Of these, the five of most fundamental importance are the relational predications: identical with, implied by, characterised by, comprised in, included in, together with their cognates. These are formal, and to represent them I shall introduce the short-hand symbols: $\check{\iota}$, $\hat{\iota}$; $\check{\lambda}$, $\hat{\lambda}$; $\check{\chi}$, $\hat{\chi}$; $\check{\kappa}$, $\hat{\kappa}$; $\check{\nu}$, $\hat{\nu}$ respectively. These five formal connectives are absolutely distinct from one another, although they have been frequently confused by logicians; and this distinctiveness is sufficient to account for the fact that they are

never represented by variable symbols for which one could replace another. Thus: in the predication $x\check{\iota}y$, the symbols x and y must stand for entities belonging to one and the same assigned category; but, in the predication $x\check{\kappa}y$, x and y must stand respectively for an item or member and an enumeration or class; and, in the predication $x\check{\chi}y$, x and y must stand respectively for a substantive and an adjective. Again, while $\check{\iota}$ connects entities belonging to *any* the same category, $\check{\lambda}$ connects only propositions or adjectives or relations; and $\check{\nu}$ connects only classes or enumerations of the same order. And yet again: the relation *identity* is reflexive, symmetrical, and transitive; but the relations *characterised by* and *comprised in* are a-reflexive, a-symmetrical, a-transitive; while the relations *implying* and *included in* are reflexive and transitive but neither symmetrical nor a-symmetrical. The five connectives above enumerated may be said to be on the borderland between premathematical and mathematical logic. There are, however, many formal connectives which belong exclusively to mathematics, of which the most fundamental is *equals* universally represented by the shorthand symbol $=$. There is serious danger of confusing *equal-to* with *identical-with* because they agree in possessing the properties reflexive, symmetrical and transitive (to the consideration of which we shall have to return later). Other important connectives in logic and algebra are derivative from those above enumerated as fundamental. Classifying fundamentals and derivatives according to their properties we have the following table, where the initials F, S, T stand respectively for reflexive, symmetrical and transitive, and the suffix a

means 'for all cases,' *e* 'for no cases,' and *oi* 'for some but not all cases.'

Formal Relations	Properties
Identical, equal, co-implicant, coincident	F_a S_a T_a
Differing by unity, co-opponent, co-remainder ...	F_e S_a T_e
Greater, less, sub- and super-implicant, sub- and super-incident	F_e S_e T_a
Sub- and super-opponent, sub- and super-remainder ...	F_e S_e T_{oi}
Not greater, not less, implying, implied by, included in, including	F_a S_{oi} T_a
Other than, unequal, disjunct, alternant, co-exclusive, co-exhaustive	F_e S_a T_{oi}
Not disjunct, not alternant, not co-exclusive, not co-exhaustive...	F_a S_a T_{oi}
Not implying, not implied by, not included in, not including	F_e S_{oi} T_{oi}
Comprising, comprised in, characterising, characterised by	F_e S_e T_e.

§ 10. Not only formal constants but also material variables enter in the same way into mathematics as into premathematical logic. The particular values which any material variable may assume are unessential for pure logic and pure mathematics; and enter as significant factors only into applied logic or applied mathematics. For example, a variable representing *any* substantive or *any* adjective is replaced by a particular substantive or a particular adjective only when the general formulae established by logic are applied to concrete propositions. Similarly the purely formal notion of magnitude or of quantity, which enters into mathematics, is applied to several different species and sub-species such as mass, volume, intensities of different kinds, etc., the differentiae of which, not being expressible in terms of pure mathematical conceptions, must be determined materially. Thus, for instance, in the mathematical

formula $3q + 5q = 8q$, q enters as a material variable standing for any quantity; and 3, 5, 8, =, +, as also the category *quantity* itself, enter as formal constants. But in applying the material variable q to deduce the equation

3 feet + 5 feet = 8 feet, or 3 ohms + 5 ohms = 8 ohms,

the terms foot, ohm, as species of the genus quantity, have to be defined by means of conceptions outside the range of pure mathematics. In this way we see that variable symbols—material as regards their range of application—entering into premathematical and mathematical logic, assume their particular values when logical theorems are applied to experimental matter. Having shown then, as regards both formal constants and material variables, that general logic agrees in all respects with mathematics, the conclusion follows that the latter fundamentally differs from the former in the sole fact that it introduces formal variables.

§ 11. Before examining the characteristics of the specifically mathematical notion 'equals' upon which its symmetry and transitiveness depend, we will consider the wider problem of relations in general possessing these two properties. There is one mode of constructing such relations which has very wide application and is of great importance in logical theory, viz.

'x is $\hat{r}$ to *the* thing that is $\breve{r}$ to z.'

Here the word *the* indicates that $\hat{r}$ is a many-one relation. I shall call 'the thing' to which reference is made in the above formula the *intermediary* term, and the relation $\hat{r}$ the *generating* relation. Thus, given an intermediary term and a many-one generating relation, we can always construct by (what is called) *relative*

multiplication a derived relation which is symmetrical and transitive. Representing the intermediary by the symbol y, the relation of x to z may be otherwise expressed by the conjunctive proposition:

'x is $\hat{r}$ to y and z is $\hat{r}$ to y,'

where it is to be understood that there *is* some uniquely determined entity (say y) to which x and z stand in the relation $\hat{r}$; i.e. $\hat{r}$ is a many-one relation.

Now the theorem that any relation so constructed is symmetrical and transitive requires no discussion and is universally admitted; but the converse theorem —that any symmetrical and transitive relation can be exhibited by this mode of construction—cannot be assumed to be true without careful examination. To this converse theorem Mr Russell gives the name 'the principle of abstraction'; and professes to have proved its truth by a process involving highly complicated symbolism. It is quite easy, however, to explain the nature of his proof without recourse to such symbolism. Thus, let $\hat{t}$ be a symmetrical and transitive relation; then, in order to prove the theorem, we have to discover an intermediary entity and a generating relation in terms of which $\hat{t}$ may be constructed. The intermediary entity for the relational predication 'x is $\hat{t}$ to z' is, in Mr Russell's proof, 'the class of things comprising x together with everything such as z for which "x is $\hat{t}$ to z" holds.' The required generating relation $\hat{r}$ is the relation of *being comprised in*; hence the proposition 'x is $\hat{t}$ to z' is resolved into the form:

x is comprised in the class
(defined as comprising everything to which x is $\hat{t}$)
which comprises z.

Here the intermediary entity is a class uniquely defined in terms of x and $\hat{t}$, and therefore the relation in which x or any other item stands to the intermediary is a many-one relation. Now what Mr Russell has succeeded in proving in this way is proved with absolutely demonstrative validity; but my first comment is: has he proved what he undertook to prove? In one sense he has proved too much, and in another sense he has proved nothing whatever that is relevant. He has proved too much in the sense that he has discovered an intermediary entity which would, *mutatis mutandis*, apply to every possible symmetrical and transitive relation, such as contemporaneous, compatriot, co-implicant, co-incident, as well as equal. Thus he has proved that, for the resolution of the relation *equals*, we must take as intermediary 'the class of quantities equal to any given quantity'; for the relation *contemporaneous*, 'the class of events contemporaneous with any given event'; for the relation *compatriot*, 'the class of persons that are compatriots of any given person'; and so on. But what he set out to discover as the required intermediary was, in the case of equality, a *certain magnitude*; in the case of compatriot, *a certain country*; in the case of contemporaneous, *a certain date*; and so on. He has *not* proved that there is a certain magnitude that all equal quantities possess; nor a certain country to which all compatriots belong; nor a certain date to which all contemporaneous events are to be referred. Moreover, in taking as his intermediary a certain uniquely determined *class*, it seems obvious that Mr Russell's alleged proof is incomplete, unless we can assert that there *are* such entities as classes, and the

validity of this assertion is explicitly denied by him: or rather he holds that there is no necessity in the deductions of logic and mathematics to assume that there are classes, although without this assumption his proof of the principle of abstraction completely breaks down.

I do not, however, wish to press my criticism of Mr Russell further, but rather to expound what appears to me to be the true view on the nature of abstraction. The cases in which the principle comes into consideration may be distinguished according as the intermediary is of the nature of a substantive such as *country*, or of the nature of an adjective such as *magnitude*. In applying the attempted proof of the principle of abstraction to such a relation as *compatriot*, Mr Russell argues as if we knew this relation to be symmetrical and transitive independently of our knowledge that a person can belong-to ($\hat{r}$) only one country (y); whereas it is obvious that we have constructed the derivative relation *compatriot* by means of the prior notions *country* and *belonging-to*. Hence, no such case as *compatriot* can be used to prove the principle of abstraction, but only to illustrate the theorem of which the principle of abstraction is the converse. Where the intermediary is adjectival, e.g. colour, pitch, magnitude, the principle directly raises the issue of the connection and distinction between a determining adjective and the class that it determines. In the case of an adjectival intermediary, our general formula

'x is $\check{r}$ to the term (say y) that is $\hat{r}$ to z'

must be expressed in a special form in which the generating relation ($\check{r}$) is to stand for characterised-by ($\hat{\chi}$),

and the intermediary term (y) is to stand for *a specific determinate under a specific determinable*, thus:

'x is characterised-by the determinate adjective that characterises z.'

Here the uniqueness of the intermediary term is secured by the disjunctive principle of adjectival determination expressed (Part I, Chapter XIV) in the form: 'Nothing can be characterised by more than one determinate under any assigned determinable.' Now, since any one substantive may be characterised under many different determinables, the intermediary term must specify the determinable, or ground of comparison, upon which the symmetry and transitiveness of the derived relation depend. Thus,

'x is characterised by the colour that characterises z,' or 'x is characterised by the shape that characterises z,' or 'x is characterised by the size that characterises z.'

Any of these three propositions may be significantly asserted of the same subjects x and z, if these are bounded surfaces distinguished from one another by determinate localisation; and the relation of x to z thus constructed is transitive (as well as symmetrical) provided that the colour, shape or size is strictly determinate. With this proviso, we may say that x and z are equivalently coloured, equivalently shaped or equivalently sized, as the case may be. Such symmetrical and transitive relations between the *substantives* x and z must be distinguished from the symmetrical and transitive relation *identity* which holds between the *adjectives* described as the colour of x and the colour of z, the shape of x and the shape of z, or the size of x and the size of z.

Now *magnitude*—like any other adjectival determinable —must first be abstracted as a character in order that by its means we can construct the class of equally sized objects. Thus it is just as absurd to define the size of x in terms of 'the class of objects that are equal in size to x' as to define the colour of x in terms of 'the class of objects that are equivalent in colour to x.'

§ 12. To secure that the relations constructed by means of the above formula shall be symmetrical and transitive, it is necessary to specify, not only such differences as those between colour, shape, etc., but also differences within the general notion *magnitude*, constituting various kinds or species of magnitude. For just as colours and sounds are incomparable with one another, since they must be characterised under different determinables, so there are distinct determinables subsumable under the superdeterminable *magnitude*. Taking some of Mr Russell's suggestive examples, we note that the magnitude of pleasure predicable of an experience is incomparable with the magnitude of area predicable of a surface, and that these again are incomparable with the magnitude of duration predicable of an event. Hence pleasure-magnitude, area-magnitude, duration-magnitude, are three distinct determinables, predicable only of experiences, surfaces, and events respectively. In ordinary usage the word *magnitude* is omitted when reference is made to the determinables in question; but in specifying the 'area' of a surface, we are in point of fact specifying a kind of magnitude; so in specifying the 'duration' of an event we are specifying another kind of magnitude; and in specifying the 'pleasure' of an experience, we are specifying yet another kind of magnitude. The

analogy here drawn between area or duration on the one hand, and pleasure on the other will probably be disputed because pleasure is so often used in its concrete sense to mean 'pleasurable experience' as well as in its abstract sense to mean 'the pleasure of a (pleasurable) experience.' Now it happens that a pleasurable experience may be characterised under at least two different determinables of magnitude; viz. pleasure-magnitude and duration-magnitude, the latter of which applies in the same sense to any event whatever that may last through a period of time. Here it is important to note that pleasure-magnitude and duration-magnitude, etc. are not determinates under the one determinable *magnitude*, but different species included in the genus magnitude. They may therefore be conveniently termed sub-determinables of magnitude, each generating its own determinates, which are incomparable with the determinates generated by any other. Thus magnitude does not generate its sub-determinables in the way in which a determinable generates its determinates. An experience, a surface, an event are substantives belonging to different categories of which pleasure, area, or duration may be respectively predicated as adjectives; but a *specific* pleasure-magnitude, or area-magnitude, or duration-magnitude is related to its respective species of magnitude as a determinate to its determinable. We shall proceed in the next chapter to examine and classify the fundamental kinds of magnitude, to which reference is here made.

§ 13. It remains to point out one highly important characteristic which distinguishes pure or pre-mathematical logic from mathematics proper. In both branches, the two principles of inference termed *Applicative* and

Implicative are employed in the procedure of functional inference, and these alone. But the peculiarity of pre-mathematical deduction is that it lays down two *formulae of implication* (either as primitive or as derived) which are virtually equivalent respectively to the Applicative and Implicative Principles themselves. The formulae in question may be thus expressed:

(1) *Applicative formula*: Any predication that holds for every case x would formally imply that the same predication holds for a given case a.

(2) *Implicative formula*: For any case x, y, the compound '"x" and "x would imply y"' would formally imply 'y.'

We must, therefore, explain the distinction between *Principles of Inference*, on the one hand, and *Formulae of Implication*, on the other hand. In all formulae of implication, the implicans and implicate stand indifferently for propositions that are to be *materially* or *formally* certified. But, when a formula of implication is used as a premiss in the process of deduction, its implicans must first be formally certified in order that its implicate may be formally certified. This inference is made by a direct *application* of the *implicative* principle. And again, every formula of implication holds for *all* cases coming under an assigned form; hence the inferences from any formula of implication are made by a direct *application* of the *applicative* principle. The fact that every step by which we advance in the building up of the logical calculus requires both the Applicative and the Implicative principles of inference, *and these alone*, establishes their sovereignty over all deductive processes.

CHAPTER VII

THE DIFFERENT KINDS OF MAGNITUDE

§ 1. THE term *magnitude*, as is suggested by its etymology, denotes anything of which the relations greater or less can be predicated; and it is only if M and N (say) are magnitudes of the same kind that M can be said to be greater or less than N. I have taken *magnitude* to be an adjectival determinable, or rather a class of adjectival determinables including several distinct kinds. That of which a determinate magnitude of a specific kind may be predicated stands, relatively to its magnitude, as substantive to adjective; but it may be either an existent, i.e. substantive proper (in which case the magnitude predicated is a primary adjective) or itself an adjective (in which case the magnitude predicated is a secondary adjective). In order to keep clear the distinction between the adjectives of magnitude themselves and the substantives of which magnitude is predicable, a separate terminology ought strictly to be applied to the latter. A striking case where language supplies us with the logically required terminological distinction is that of 'longer' and 'shorter' predicated of *lines*—to the *lengths* of which the terms 'greater' and 'less' are applied. It would be convenient, for the purposes of a general exposition of magnitude, to restrict the application of the terms 'greater' and 'less' to magnitudes, and to adopt the corresponding terms 'larger' and 'smaller' for that of which the

magnitudes are predicated. For example: the class *compositae* is larger or smaller than the class *violaceae*, according as the number of compositae is greater or less than the number of violaceae[1]; the period 1815 to 1832 may be called larger than the period 1714 to 1720, inasmuch as the temporal magnitude of the former is greater than that of the latter. Now for every distinct kind of magnitude there is a corresponding distinct kind or category of entity of which it can be predicated; and hence, though it is strictly illogical, yet it is legitimate and usual to apply the same terms, such as extensive and intensive, to distinguish both between the different kinds of magnitude and between the corresponding different kinds of entities which bear to the magnitude the relation of substantive to adjective. From these preliminary remarks, we may pass to an examination of the nature of different kinds of magnitude, beginning with *number*, which is the most fundamental of all.

§ 2. Integral number is an adjective exclusively predicable of what we call classes, including enumerations; two classes being said to be numerically equal when the number predicable of the one is identical with that predicable of the other. I think it is legitimate to maintain that the two notions class and number are not independently definable, but each definable only in its relation, the one as the only appropriate substantive for

[1] This may mean either that the number of *existing plants comprised* in the genus is greater or less, or that the number of *infimae species included* in the genus is greater or less. It is obvious that these two modes of determining numerical comparison do not necessarily tally. It will be shown later that the same distinction holds as regards the number of *points* in a line and the number of linear *parts* (equal or unequal) into which it may be exhaustively and exclusively divided.

the other as its only appropriate adjective. The common habit of representing classes by closed figures may lead to the false supposition that the members of a class can as such be arranged in some kind of proximity to one another within an enclosed space. But when the items to be comprised in a class have relations metaphorically called near or far, they constitute not merely a class but a series or ordered class. Now in modern mathematics the appropriate number-adjective of a class conceived independently of any arrangement or order of its items, is known as a cardinal number; whereas of a series or ordered aggregate the appropriate numerical adjective is known as an ordinal number. When a class or enumeration comprises a finite number of items, then, in whatever order the items may be enumerated, we reach the same ordinal number, and this number agrees with the cardinal number; but for transfinite aggregates, which have been introduced into modern arithmetic, this agreement no longer holds; and consequently the fundamental distinction between ordinal and cardinal numbers is required. Readers are referred particularly to Mr Russell's *Principles of Mathematics* for the full development of this topic, which is outside the compass of my work.

§ 3. The psychological aspect of number is revealed by analysing the process of counting. In this process we establish numerical equality between a set of *things*, on the one hand, and a set of *number-names* temporarily attached to the things, on the other hand. Hence counting is a special, and, in some respects, a unique case of correlation between the things upon which names are imposed and the names that are imposed upon the

things. Ideally language requires that any given proper name should denominate one and only one thing, and conversely that any given thing should be denominated by one and only one proper name; or briefly, that there should be a one-one correlation between the names of things and the things named. If this relation held, it would follow that the class of names would be numerically equal to the class of things named. Actually, however, this ideal is not realised; for the same thing often has many names, and the same name is often attached to many things. It is worth pointing out that there may still be numerical equality in spite of there not being a one-one correlation between names and things named. For example: let R, Q, M, T, U be a set of names, and κ, σ, θ, χ, ϕ a set of things named. Then suppose that

> R names κ or σ; Q names κ or σ; M names σ or θ or χ; T names χ; and U names θ or ϕ or κ or χ;

so that

> κ is named R or Q or U; σ is named R or Q or M; θ is named M or U; χ is named M or T or U; and ϕ is named U.

Here the denominating correlation is not one-one but many-many, and yet the names and the things happen to be numerically equal. How then do we establish the fact that the number of items in the enumeration R, Q, M, T, U is the same as that in the enumeration κ, σ, θ, χ, ϕ? What we do, where there is no *factual* correlation, is to institute what I shall call a *factitious* correlation; by which I mean one which is not inherent or objective, but arbitrarily imposed by the counter. In the adduced instance—in order to establish the numerical

equality between the enumerations R, Q, M, T, U, and κ, σ, θ, χ, ϕ—we must mentally attach, either in thought or in figurative imagery, R say to θ, Q to ϕ, T to σ, U to κ, M to χ; where the items of the two sets have been indiscriminately permuted and attached. We can now analyse the mental act of counting as a special case of factitious correlation. The essential psychological requisite is that we should *learn to enumerate a set of arbitrary names in a fixed or invariable temporal order from the first onwards*; and these names are attached *temporarily* to the objects to be counted, in this respect differing from names in general which have *fixed* denotation. For example: let us arrange the names U, R, Q, M, T in the following order: M, Q, R, T, U; and *temporarily* attach these names as follows: M to χ, Q to ϕ, R to θ, T to σ, U to κ. Thus the set of names have to be attached in a *fixed* order, one by one, to the set of things taken in *any* order. What is logically required to avoid mistake is that the enumeration of the things should be both exhaustive and non-repetitive—a condition which children and savages often find difficult to fulfil. Now, inasmuch as the number-names M, Q, R, T, U are always attached in an invariable order, the *last* number named indicates unequivocally the number of the counted set of objects. In other words, the cardinal number of any enumerable set of objects is unambiguously indicated by the ordinal number of the correlated number-names. Historically the letters of the alphabet, having been memorised in a fixed order, served also as the written symbols for numbers; but their employment for this purpose could not be extended to *all* numbers, since an alphabet

necessarily consists of a limited number of letters. Moreover it is psychologically impossible to memorise an endless list of names. Hence it was necessary to invent some system which would render it possible to count any set of things, however large. The Roman, Greek and Hebrew alphabets were employed for this purpose with more or less success, but were finally superseded by the Arabic notation in which *place-value* was given to the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, and the symbol 0 was added. These ten symbols serve as *proper* names of numbers, all other numbers being expressed by names constructed out of these. Thus the compound word twenty-four or the compound symbol 24 is analysable as meaning 'two tens plus four,' and therefore to be understood in terms of the operations of multiplication and addition. Such compound symbols or words are not proper names of numbers like two, ten, or four, but may be called constructed names. The elementary learner of arithmetic must, in fact, reverse the logical order of thought, and understand the processes of multiplication and addition before he can intelligently learn to count beyond twenty or so, or understand what is known as the decimal system of notation.

§ 4. We now pass from the psychological analysis of counting to the consideration of its underlying logical principles. Counting is a special case of one-one correlation, the peculiar characteristics of which are (1) that a prescribed set of name-items have to be memorised in a definite serial order; and (2) that the correlations are factitious. As regards (1) the mental process of counting, which involves order, must be contrasted with

one-one correlations in general which are irrespective of order. As regards (2) factitious correlations must be contrasted with such correlations as husband and wife, denominating and denominated by (in an ideal language) etc., in which any given husband is correlated with a determinate wife, or any given proper name with a determinate thing. Now in my view, factitious correlations are essentially necessary in the *general* theory of numerical equality; though they never enter into the abstract deductions of arithmetic. On the necessity of factitious correlations, recognised authorities, above all Mr Russell, are opposed to me. Their definition of the numerical equality of two sets of things is, in effect, formulated as follows: 'There is a one-one relation of any member of the one set to some member of the other set.' But it seems to me essential to distinguish the statement that 'the items *can be* correlated one to one' from the statement '*there is* a one-one correlation'; the former points to a factitious, the latter to a factual correlation. There need be no relation at all depending on the nature of the items themselves comprised in the two sets, that would determine which item of the one set should be attached to any given item in the other. If relations are treated extensionally, i.e. as mere substantive-couples, then it is of course a matter of fact that two numerically equal classes contain couples of items, one of which is comprised in the one class and the other in the other; but I know of no sense in which the two members of the couple are *related* the one to the other, except that the one is temporarily attached in thought by some thinker to the other. Apart from this *factitious* coupling, there is *no* one-one relation,

subsisting between any given item of the first set and any determinate item of the second, that would not equally subsist between the given item and any *other* item arbitrarily selected from the second set. Thus the establishment of numerical equality between two finite classes requires *in general* factitious correlations. On the other hand, the only mode of establishing numerical equality between *infinite* classes is to discover *factual*, or more specifically *formal*, correlations. The formal correlations required in pure arithmetic, finite and transfinite, are what may be called functional; and, for the purposes of this elementary exposition of the logic of arithmetic, the notion of functional correlation must be introduced and explained.

§ 5. Using the symbol $\hat{f}$ for any function and $\check{f}$ for its converse, the relation n to $\hat{f}(n)$ will be one-one; provided that n determines uniquely the value of $\hat{f}(n)$ and $\hat{f}(m)$ determines uniquely the value of $\check{f}\{\hat{f}(m)\}$. For example: let $\hat{f}(n)$ stand for $n+7$, then $\check{f}(m)$ will stand for $m-7$; and the integers from 1 to n (inclusive) can be correlated one to one with the integers from 8 to $n+7$; each integer in the second series being given by adding 7 to the corresponding integer in the first, and each in the first series by subtracting 7 from the corresponding integer in the second. Similarly, if $\hat{f}(n)$ stands for $n \times 7$, then $\check{f}(m)$ will stand for $m \div 7$; and the integers from 1 to n can be correlated one to one with the multiples of 7 from 7 to $7n$. In general: if the relation of n to $\hat{f}(n)$ is one-one, then the series of values assumed by n is numerically equal to the series of values assumed by $\hat{f}(n)$.

An important application of this theorem is to the case where the integer n assumes all possible finite

values, obtained from unity by the successive addition of unity. In this case, the simplest illustration is afforded by taking $\hat{f}(n)$ to stand for $2n$. There is then established a one-one correlation of the successive integers 1, 2, 3, 4 ... with the successive integers 2, 4, 6, 8 .... In other words, the number of finite integers is the same as the number of finite *even* integers; although the former series comprises all the *odd* integers and these are not comprised in the latter. Thus, although the aggregate of even integers is a part proper of or *sub-included in* the aggregate of integers, yet the two aggregates are numerically equal. Now we may *define* an infinite number as the number of any aggregate that includes a part proper numerically equal to itself. Thus the instance above cited is the simplest of the many proofs that establish the theorem that *the number of finite integers is infinite.* If the integers are presented in ascending order of magnitude, the series so conceived has a first but no last term and also is discrete in the sense that each term has one and only one immediate successor. The cardinal number of any aggregate that can be so arranged in a series is called $\aleph_0$. This is the smallest of infinite cardinal numbers.

The reader must here be referred to the mathematical exponents of the theory of transfinite cardinals and ordinals for further instruction. The most comprehensive account of this theory will be found in Mr Bertrand Russell's work entitled *Principles of Mathematics.*

§ 6. As number and the magnitudes that are derived solely from number may be called abstract, so those which contain a material factor may be called concrete

magnitudes or quantities. Thus duration-magnitudes, stretch-magnitudes, magnitudes of qualitative difference are quantities, because the entities of which they are predicable are defined and differentiated in terms that are not purely logical. This use of the term *quantity* differs from that expressly enjoined by Mr Russell, who defines a quantity as 'an instance or specification of magnitude.' He then proceeds to identify the relation thus indicated in some cases with that of substantive to adjective, and in others with that of determinate to determinable; whereas, in the common language of mathematics, quantity stands to magnitude in the relation of species to genus, with which my use of the term quantity corresponds. With regard to quantities the three differentiae which I hold to be fundamental or primitive are extensive, distensive and intensive. The term distensive magnitude is new, and the reason for placing it intermediarily between extensive and intensive is that by some logicians it has been included under extensive and by others under intensive magnitude.

An extensive magnitude may be defined as one which can be predicated only of an entity that can appropriately be called a whole. The notion of whole is correlative to the notion of part; and, more precisely, a whole is to be conceived as having parts which can be specifically identified and distinguished independently of their relations of equality or inequality; e.g. a finite line is a whole of the simplest possible kind, under the figure of which all one-dimensional wholes may be metaphorically pictured. Thus a line CEG is represented as having the parts CE and EG, each of which is definitely identifiable for itself and distinguish-

able from the other. The several parts of a whole are of the same nature as the whole, and therefore the construction of a whole out of parts or the division of a whole into parts may always be called homogeneous[1]. The term extensive magnitude has, in fact, been popularly restricted to spatial and temporal wholes; but I shall follow Mr Russell in applying this term also to certain qualitative wholes, e.g. to a continuous aggregate of hues or of pitches. Thus we speak of a scale of hue and a scale of pitch in the sense of a class comprising all specific hues or pitches which are qualitatively intermediate between two terminal hues or pitches. Now the class comprising such determinate items constitutes what is now called a *stretch*; thus a qualitative stretch of hue or of pitch is formally analogous to the period comprising all determinate instants between one instant and another or to the geometrical line comprising all points intermediate between one point and another.

§ 7. It might appear, since the instants comprised in a period and the points comprised in a line are substantival, while the hues or pitches comprised in a qualitative stretch are adjectival, that there is some fundamental logical distinction between these two kinds of stretches. Thus: though either may be metaphorically represented by a line *CEG*, yet, if the points *C*, *E*, *G* stand *actually* for points or instants—these being substantival—the stretch represented is substantival; whereas if *C*, *E*, *G* represent three pitches—pitches

[1] The term *whole* is frequently applied to a construct constituted of *heterogeneous* elements, e.g. to a proposition; but for such a construct the term *unity* is preferable, unity being the genus of which whole is a species.

being characteristics of sound and therefore adjectival—the stretch itself is adjectival. It is open to question, however, whether points or instants are of a substantival nature; and this has been a matter of frequent philosophical dispute. If we regard time and space as existents, then the events which occur at a given date or occupy a given period, like the ink-spots which may be placed at different points or the ink-lines which may be drawn on paper, have as substantives a unique kind of relation to the substantives of a different category—date, period, point or line—to which they are attached. Such a relation, like that of characterisation, is unique;—in the sense that one of its terms necessarily belongs to a certain category and the other to a certain other category. The relations 'occupying' and 'occurring at' further resemble the characterising tie in being unmodifiable; thus, of any given date and any given event the only relevant assertion that can be made is that the event either did or did not occur at that date. In saying of an event that it occurs at a certain date or of a material body that it occupies a certain region, the predications may be, not ultimately analysable into definable relations to a definable period or region, but regarded rather as adjectivally unanalysable. 'Occurring at' and 'occupying' are therefore properly speaking *ties*. It is not, however, formally incorrect to regard them as relations, in the same way as we have allowed characterisation to be analysed as a relation involving the two correlatives characterising and characterised by. From this discussion it will be seen that I incline to the view that instants of time and points of space, as well as time

and space as wholes, are not substantival or existential but merely adjectival.

Now, under the head of the relativity of time and space two distinct philosophical problems are often confused. The view that position in space or time is definable not as absolute, but only as relative to other points or instants is to be distinguished from another view according to which temporal and spatial relations are relations, not between entities such as points and instants, but between what occupies the points or instants. The first of these two problems is appropriately described as the question of the absoluteness or relativity of time and space; the second as the question of the substantival or adjectival nature of time and space. In the *Principles of Mathematics* Mr Russell explicitly maintains the absolute view as regards both these problems; he deliberately asserts that position—a term conveniently used both for space and time—is absolute and not merely relationally definable in terms of other points or instants; and also that points and instants are existents. Now, in the foregoing analysis, I have taken the relative i.e. adjectival view on the second of these two problems, while not rejecting the absolute view on the first. The adjectival view of space and time, in which we deny such separable entities as instants and points, must not be confounded with the class-view: that identity of dating merely means being comprised in a certain assigned class of contemporaneous events. For, in holding that 'occupying a certain instant' is an unanalysable adjectival predicate, we maintain at the same time that, *quâ* predicate, it is an identifiable entity, in the same way as the adjective

'red' is an identifiable entity when predicated now of this patch and then again of some existentially other patch. This view is not inconsistent with my previous analysis; for I have repeatedly maintained, particularly in my analysis of the principle of abstraction, that adjectival identity cannot be resolved merely into membership of a certain definable class. My contention for the adjectival nature of space and time amounts to the statement that instants and points are substantival myths. It is not necessary, however, for the purposes of this exposition, to press the question of the substantiality of time and space, for any difference of view on this point does not affect the further development of the subject.

§ 8. Having shown the analogies between the three kinds of stretches—qualitative, temporal, and spatial or rather linear—we will now compare such extensive wholes with classes considered in extension, which may be called *extensional* wholes. It is not a mere accident of language that the term *extension* has two applications in philosophy, these generally occurring in such different contexts that they are not confused. But it is worth while drawing attention to the double use of the word; and, in so doing, to examine a topic, prominent in modern mathematics, concerning the formal agreements and differences between extensional and extensive wholes. An extensional whole, otherwise a class, is naturally associated with the notion of assignable items of which the class is composed; on the other hand, a linear whole—which illustrates an extensive whole of the simplest kind—is apprehended as a whole without thinking of the points it contains. In other words:

the items in an extensional whole are prior in thought to the whole, which appears to be the product of a constructive process; while the extensive whole is prior to any conception of points, which seem to be the result of a similar, but reversed, process of thought-construction. This psychological distinction Mr Russell seems to regard as philosophically negligible; and he devotes a large part of his exposition to a proof of the essential sameness in nature of extensional and extensive wholes. This question raises the same problem as that discussed by Hume and Kant—the former in his quarrel with the mathematicians, and the latter in his solution of the antinomies.

Let us then examine what common-sense would elicit from a consideration of these two kinds of wholes. With regard to extensional wholes, I have adopted the term 'comprise' to represent the relation of a class to any of its items or members, and 'include' to represent the relation of a genus to any of its species; and it is of the first importance to note that, for extensive wholes, an analogous distinction holds between the relation of a line to any of its *points* and the relation of a line to any of its *parts* which are themselves linear. For just as a class comprises items which have to one another the sole relationship of otherness, so a line comprises points which have to one another the sole relationship of otherness; and again, just as members of a species are members of the genus, so points in a linear part are points in the linear whole. Further, since a line or stretch contains parts in the same sense as a genus includes species, it follows that such purely logical or formal relations as overlapping, includent, excludent,

which apply to classes and not to items, apply also to the parts of a stretch but not to points. Now the parts into which a three-dimensional space can be divided are three-dimensional, and have, *quâ* three-dimensional, all the properties of the whole; similarly the parts of a two-dimensional space are two-dimensional; and the parts of a one-dimensional space one-dimensional. On the other hand, of the contiguous parts of a three-dimensional whole the common boundary is two-dimensional; of the contiguous parts of a two-dimensional whole the common boundary is one-dimensional; and of the contiguous parts of a one-dimensional whole the common boundary is zero-dimensional, i.e. a point. Restricting our discussion to the last case, we note a very substantial difference between extensive wholes and extensional wholes; for within a merely extensional whole there are no relations of contiguity, whereas every extensive whole is apprehended as containing parts which are either literally or metaphorically further from or nearer to one another. Hence the notion of a point as a boundary comprised in neither or in both of the two parts of a line has no analogy amongst members of a genus which belong either to one species or to another and cannot belong to both. It further follows that an extensive whole resembles a serial or ordered set of items rather than a mere unordered class or enumeration.

§ 9. Having considered the nature of an extensive whole, i.e. that of which extensive magnitude may be predicated, we will pass to the consideration of the kinds of entities of which distensive or intensive magnitude can be predicated. By distensive magnitude is

meant degree of difference, more particularly between distinguishable qualities ranged under the same determinable[1]. Thus the difference between *red* and *yellow* may be greater or less than that between *green* and *blue*; and similarly the difference between the pitches C and F may be greater or less than that between B and G. The notion of difference is apt to be associated with the arithmetical process of addition, for which the term 'addendum' or 'subtrahend' may be used in order to distinguish it from a distensive magnitude. Thus it is preferable to say that successive terms forming an arithmetical progression are obtained by a constant addendum, just as those forming a geometrical progression are obtained by a constant multiplier. This reference to arithmetical and geometrical progressions is needed because the measure of qualitative difference, in its logical and even its philosophical sense, is in some cases or on some grounds to be conceived as an addendum, and in other cases or on other grounds as a multiplier. For example, if a series of colours are presented as in the spectrum, in a continuous spatial order, we might conceive the magnitude of difference between any one hue and any other to be proportional to the length in the spectrum between the two hues. In this case, by taking any hue as origin, say O, such that A is between O and B, and representing the difference between A and B by the symbol AB, we should assume that its value was given by the equation $AB = OB - OA$. On the other hand, as regards the scale of pitch, the scientist would naturally connect the pitches with the physical process of aerial vibration, and measure each

[1] See Part I, p. 191.

pitch by the number of vibrations per second. On this assumption the 'difference' between C and G would be represented by the ratio of $\frac{3}{2}$, and that between G and B by $\frac{5}{4}$, and therefore the 'difference' between C and B would be $\frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$. These two examples of the two natural modes of estimating degrees of qualitative difference—viz. by an addendum or by a multiplier—are typical of all problems regarding distensive or even intensive magnitudes.

It will be important, however, to contrast either of these more physical modes of conceiving distensive magnitude with the mode that has become familiar to psychologists ever since Fechner's and Weber's experiments. According to Fechner it would appear that the magnitude of difference between the qualities or intensities of sensations should be determined by taking as unit-difference that which is just discernible in an act of perception directed to the sensations as experienced. It should be here noted that we are measuring psychical entities, and not, as in the previous discussion, their physical correlates. Fechner adopted the view that the proper sensational magnitude, either of qualitative or of intensive difference, was obtained by addition, in which equal units were those which were just perceptible. When he compared the resulting sensational magnitude with the magnitude of the stimulus as measured physically, he concluded that, while the sensations could be ranged in arithmetical progression, the corresponding stimuli would form a geometrical progression. By means of an elementary mathematical process it will be seen that this formula can be expressed by saying that the magnitude of the sensation

varies as the logarithm of the stimulus. But this technical development is not our concern here. I wish rather to draw attention to the extraordinary, and in my view baseless, assumption that the just discernible differences at the different points in a scale should be taken to indicate equal *addenda*. If he had assumed what appears to be more plausible that the just discriminable qualities were those which bore a common ratio to one another, the experimental results of Fechner or Weber would have been most naturally expressed in the formula that the sensational magnitude (σ) varies in proportion to the magnitude of the physical stimulus (s) measured from a certain constant (s_0): i.e.

$$\sigma = k(s - s_0),$$

where k is constant; instead of by the formula

$$\sigma = k \log (s - s_0).$$

So far from taking discriminability as equivalent to an addendum, it is more plausible to consider it as equivalent to a ratio[1]. For example, taking the visual magnitudes of four objects A, B, C, and D, if we can just discriminate between the magnitudes of A and B and also between those of C and D, then it is reasonable to infer that the ratio of B to A is equal to the ratio of D to C, rather than that the addendum by which B exceeds A is equal to the addendum by which D

[1] I am not here concerned with the accuracy of the experiments made by Fechner, nor with his right to make the very wide induction from the artificial nature and limited number of cases that he and his successors have examined. I am referring merely to a logical and not to a psychological question, namely, the justification for regarding our power of discrimination as equivalent to our power of perceiving additions of magnitude rather than ratios.

exceeds *C*. This principle for measuring psychical magnitude may be applied not only to cases of direct sense-perception, but also to those in which we are guided by general psychological considerations; for example, the difference of pleasure that we conceive to be produced by two different increases of income such as that from £100 to £200 and from £1000 to £1100 would not naturally be taken to be equal; the increase from £100 to £110 would rather be considered the equivalent of the increase from £1000 to £1100.

§ 10. We now pass explicitly to the third fundamental kind of magnitude, namely intensive, which has received considerable philosophical attention. Kant regarded intensity as so to speak equivalent to existence or reality, so that that which has greater intensity has for him greater reality. The point in which this view agrees with the modern theory is that intensity has a terminus in the value called zero; and it is in this respect that the distinction between distensive and intensive magnitudes is most clearly marked; the minimum or zero of distensive magnitude is *identity*, whereas the minimum or zero of intensive magnitude is *non-existence*. Another obvious distinction between the two kinds of magnitude is that distensive magnitude is a relation between determinates under some one given determinable, whereas intensive magnitude holds within each separate determinate, or even amongst different qualities under the same determinable. Thus, with regard to the comparative brightness of different hues, we may predicate equal to, greater than or less than, and so also with regard to the loudness of sounds of different pitch. It is impossible, however, to compare

two kinds of intensive magnitude such as the brightness of a light sensation with the loudness of a sound sensation; all we can say is that a colour of zero brightness would be non-existent, and a sound of zero loudness would be non-existent. The subtle point then arises whether the notion of zero-intensity of sound is distinguishable from the notion of zero-intensity of light. In popular language we might ask: Is there anything to distinguish absolute silence from absolute darkness? I think that apart from an organ of sensation having potentialities as a medium for receiving sensations we must say that zero-intensities are indistinguishable; it is only through the capacity of visual and auditory imagery, and indirectly through the possession of organs for conveying these two corresponding kinds of sensation, that distinctions between zeros can have for us any import.

§ 11. In conclusion I have to explain why distensive magnitudes have been confused on the one hand with extensive and on the other hand with intensive magnitudes. As regards the former, the confusion is due to identifying the distensive magnitude of difference, say between the pitches *C* and *G*, with the stretch including all the intermediary pitches. This stretch illustrates what we have called an extensive whole; and, in so far as it can be measured, its measure would be equivalent to that of the difference between *C* and *G*; i.e. its *measure* would be equivalent to that of a distensive magnitude, but the *natures* of the two are non-equivalent. As regards distensive and intensive magnitudes, these agree in so far as they both apply to qualities, and not obviously to things occupying a

quantum of space or time or forming a linear or temporal series; but it is necessary to distinguish them inasmuch as distensive magnitude requires the fundamental conception of different qualities which are yet comparable; while intensive magnitude requires—what has sometimes been paradoxically described as the conception of a thing as merely qualitative, and yet as susceptible of quantitative variation.

§ 12. Having distinguished different kinds of magnitude, we have now to consider how magnitudes of any given kind are to be compared; and we will begin by the simplest kind of magnitude, viz. that which can be predicated of a linear whole.

Mr Russell deliberately adopts the view that the ultimate parts of a line are points, of which the number may be assumed to be 2 exp ℵ, whatever be the magnitude of the line. In other words, any comparison of one line with another in regard to magnitude depends upon something other than the number of points which the lines contain. Hence the magnitude of an extensive whole, as illustrated by a line, cannot be estimated in terms of pure or abstract number. In this respect it is of a totally different nature from a class, the magnitude of which is entirely determined by the number of items it comprises, or by the number of exclusive sub-classes into which it may be divided. It follows then that magnitude, when applied to an extensive whole, has a different meaning from magnitude when applied to an extensional whole. For what I have called an extensive whole Mr Russell uses the term 'divisible whole,' because the notion of dividing is essential to our conception of the relation of part to

whole, particularly in temporal and spatial applications. But, in discussing the principle required for comparing the magnitude of one line with that of another, he uses the phrase '*magnitude* of divisibility.' This phrase appears to me unfortunate inasmuch as it conveys no meaning: entities may be distinguished according as they do or do not possess the quality of divisibility; and the term magnitude is of course required when we discuss whether one thing is greater or less than another. But I fail to see how we can regard one line as being greater than another on the ground that it possesses the quality of divisibility in a higher degree. It is quite certain that the number of parts into which a shorter line can be divided is exactly the same as the number of parts into which a longer line can be divided; as also are the number of points in the one and in the other. The term 'magnitude of divisibility' therefore appears to me merely to conceal what really is the problem involved in comparing things having extensive magnitude; namely the conception of equality of magnitude.

§ 13. What do we mean by the question, or how can we test, whether one given line or surface or bounded three-dimensional figure is greater or less than another? Or again whether one stretch of hue or of pitch is equal to or greater than another? In general, for two extensive wholes M and N of the same kind, if M includes but is not included in N it will be agreed that the magnitude of M is greater than that of N; or briefly, the relation of superincident to subincident, whole to part proper, entails the relation of greater to less. But, if the wholes M and N are *coexclusive*, then no such test of equality or inequality can be directly

applied; and in order to compare their magnitudes in this case, we must be able to find parts of M that can be equated to one another as also to parts of N. This would provide us with a unit magnitude, in reference to which the magnitudes of M and N could be numerically compared. If we further assume that the wholes satisfy the strict criterion of continuity as defined by Cantor, then the series of numbers rational and irrational will provide means for comparative measurement of all such magnitudes. On this assumption the only problem that remains is the provision of a test or definition of equality amongst unit parts. The possibility of such a test must be separately examined for the three cases of spatial, temporal and qualitative stretches. As regards spatial wholes of one, two or three dimensions, the classical method is that of superposition, the validity of which must be carefully considered. It is obviously absurd to think of the parts of space themselves as moving; and hence the so-called method of superposition can only have practical significance when we distinguish the material occupant of a place from the place which it occupies. When the material occupants of space are superposed one upon another, and the boundary of one is coincident with that of the other, they are said to be *conterminous*; and when the boundary of one is subincident to that of the other, they may be said to be partially conterminous. Thus the outer boundary of a liquid and the inner boundary of a closed receptacle which it fills are coincident; and, in this case, the volume occupied by the liquid is equal to the volume unoccupied by the receptacle. Again, if any two bodies have a common two-dimen-

sional boundary (which does not enclose a volume), then the boundary of the one has the same *areal* magnitude as that of the other; and, if two bodies have a common one-dimensional boundary, then the boundary of the one has the same *linear* magnitude as that of the other. If, moreover, several different bodies can in either of these three ways be made conterminous with some one given body, the volume, area or length of the corresponding boundary of the one is equal to that of the other. But this predication of equality assumes that the volume of the receptacle, or of the areal or linear boundaries of the superposed bodies remains unchanged; and the assumption that in the course of time a material body does not change its spatial magnitude is in general invalid; hence there is no literally logical justification for asserting equality or inequality in general, either with respect to the same body in different places, or with respect to the different places which the same body may occupy. Science in this case relies upon the constancy (under unchanged conditions) of the volume of *certain* bodies, and uses these as standards by which the changes of volume of *other* bodies are tested. In this process, we are continually acquiring more precise knowledge of causal conditions; but the final justification for comparisons of spatial magnitude is to be found in the coherency or consistency with which the systematisation of measurements and the construction of physical laws can be developed. The conclusion follows then that no directly logical test can be found, and we must be satisfied with the indirect principle according to which comprehensive universals are asserted on the mere ground that they do not lead to appreciable inconsistencies.

The problem of temporal magnitude, like that of spatial magnitude, is met first by the axiom that events that are conterminous at both ends have the same temporal magnitude, and secondly by the postulate that *under identical causal conditions* equal changes occupy equal lengths of time. We then employ some physical process, such as the movement of the hands of a watch, in which the mechanical conditions can be estimated with the closest approximation to exactitude, and adopt as standard time-units the times occupied by the changes thus effected. Conversely, where equal changes are effected during unequal times, we infer that the causal conditions are not identical. In all temporal changes, the means by which we can measure such changes as *equal*, itself depends upon the assumption that we can measure certain spatial, distensive or intensive magnitudes.

Turning now to qualitative magnitudes, we have to consider by what method stretches of hue and pitch can be quantitatively compared. If we agree that the stretch from A to E is equal to that from C to G in a scale of pitches, this cannot be tested by any such method as that of superposition, for there is no distinction here corresponding to that between the place which is occupied on the one hand, and that which is movable and can occupy indifferently one place or another on the other hand. If a qualitative stretch has magnitude, this involves the assumption that stretches of the same kind are comparable as greater or less. But how much greater, or by what ratio the two are to be compared must be determined, if at all, by some principle totally different from superposition. Mathematicians who have

written on this subject appear to agree in the view that two magnitudes may be comparable as greater or less, and yet not measurable in terms of number. But, if two stretches are mutually excludent, I can see no sense in which they can be compared as greater or less, unless we have a test of equality; and, when such test is forthcoming, a numerical measurement seems to me immediately to follow. Numerical measurement is not a merely arbitrary one-one correlation between numbers and magnitudes: for such correlation could only mean that for the greater magnitudes we apply higher numbers, and the precise numbers which we correlate would be absolutely arbitrary. Hence it appears to me that if a specific one-one numerical correlation has an objective ground, according to which it is to be preferred to any other, this must be because we have adopted some principle for determining a correct quantitative unit. For example, if we prefer the absolute measurement of temperature to the thermometric measurement as determined say by the changes of volume of mercury, this is because we believe that the differences of temperature indicated by the former scale *do* correspond to really equal differences of magnitude, whereas the other does not. Readers of Clerk Maxwell's *Heat* will learn that the *absolute* measurement of temperature depends upon measurements of heat and work, which are complex quantities, being partly extensive and partly intensive. In all such cases, where we cannot *directly* measure a cause or an effect, we measure it *indirectly* in terms of its effect or cause (as the case may be).

§ 14. The entities of which either extensive, distensive or intensive magnitude can be predicated alone may be termed simple or simplex, and from these kinds of entity we now pass to those which may be called compound or complex, on the ground that two or more magnitudes are combined in our conception of the quantity of the resultant complex. These latter may be illustrated by light sensations, which vary intensively according to their brightness, distensively as regards their hue, and in yet a third respect according to the proportion in which the chromatic and achromatic factors are combined to produce different degrees of saturation. Similarly sound sensations vary intensively according to their loudness, distensively as regards their pitch, and as regards timbre or klang-tint in accordance with the proportional intensities of their constituent tones, under-tones and over-tones. It is convenient to speak, then, of light and sound sensations as three-dimensional, in the sense that there are three distinct determinables under which any such sensation can be defined and quantitatively estimated. But the simplest case of a three-dimensional quantity is space. In space we may take three arbitrary directions; and, according to the ordinary view, the magnitudes (i.e. lengths) along these directions have the unique characteristic of being comparable. Any point in a space of three dimensions is therefore assignable by three ordinates drawn in determined directions from a given point as origin. In this way a surface in three dimensions, or a line in two dimensions, differs from what is called a *graph*, in that the magnitudes represented by the ordinates of a point in the graph are of different kinds and therefore incom-

parable. For example: a graph representing the co-variation of work done and hours expended uses two incomparable magnitudes.

The general topic which we have now to consider is that of a derived quantity that is constructed by some kind of combination of other quantities. In constructing a quantity comparable with each of those combined, the processes of addition and subtraction can alone be applied; and, conversely, addition and subtraction can only be applied to comparable quantities. Such addition and subtraction may be termed *concrete*, in antithesis to *abstract* in which pure numbers are concerned whose sum or difference is also a pure number. Now I shall maintain that processes analogous to multiplication and division may be employed in constructing a quantity of a different kind from any of those that are combined in its construction; and such multiplication or division may also be called concrete. Thus, considering first the three notions of length, area, and volume, I shall say that the multiplication of two differently directed lengths constitutes an area, and that of three differently directed lengths constitutes a volume. Here we are extending the operation called multiplication beyond its primary use. For, while it is universally agreed that we may multiply a pure number by a pure number, in constructing another pure number, or a quantity of any kind by a pure number, in constructing a quantity of the same kind, yet most mathematicians have refused to allow that by multiplying one quantity by another we may construct a third quantity different in kind from both the quantities multiplied. They maintain that what is multiplied is *the numerical measure* of the quantities

and not the quantities themselves. Similarly with regard to division: it is agreed that we may divide a pure number by a pure number in constructing another pure number, or a quantity of any kind by a pure number in constructing a quantity of the same kind; but we are prohibited from dividing one quantity by another in constructing a quantity different in kind from the quantities divided. In this case too, the so-called division is regarded as a division not of the quantities but of their numerical measures. My first objection to this view is that it offers no means of distinguishing between the multiplication or division of a quantity by a pure number, which yields a quantity of the same kind, from that very different kind of multiplication or division which yields a quantity different in kind from those multiplied or divided. My disagreement, however, with the almost unanimous opinion of mathematicians may perhaps be considered merely verbal; but the view that I maintain is, I think, based upon an important logical principle. Apart from any conception of numerical measurement which adopts numbers, integral, rational and irrational, it appears to me that we must conceive the process of multiplying say a foot by an inch (which involves no idea of number) as a construction by which, from two magnitudes of the same kind, a third magnitude of a different kind is derived. If no such magnitude were presented to perception or thought, it would follow that no meaning could be attached to such multiplication; but, inasmuch as an area is a genuine object of thought construction, I see no insurmountable objection to speaking of the process of multiplication as that by which area, for instance, is constructed out of two

directed lengths, or volume out of three. The mathematicians who reject this idea hold that the notion of *units* of different kinds is sufficient, without introducing the multiplication or division of units. It is agreed that the area of a rectangle whose sides are of unit length is a unit area, and the volume of a cube whose sides are of unit length is a unit volume. In this way the numerical measures of area and volume are obtained by multiplying the numerical measures of their sides; but in my view we must allow that the lengths themselves are multiplied, for otherwise we could not distinguish the different kinds of magnitudes constructed, since where only abstract numbers are concerned only abstract numbers are constructed, and there is nothing to indicate the difference between one quantity thus derived and another. Passing from concrete multiplication to concrete division we have what may be thought a more interesting and certainly a wider application of the same general principle. Those quantities which are derived by dividing one kind of quantity by another may be called rate-quantities, or in certain cases degree-quantities. A rate-quantity is expressed in familiar English by the Latin word *per*, of which it is easy to multiply examples: e.g. space traversed per second, wages earned per hour, pleasure experienced per minute, pressure per square foot, mass per cubic foot. The two constituent quantities in this kind of division may be themselves complex or of different kinds, extensive, distensive or intensive; but so far as the conception of concrete division is concerned, no logical distinctions are required in analysing the *general* notion of a rate-quantity. Each of the rate-quantities constructed by

this species of division is a quantity of a different kind from the quantities of which it is constituted; and, as in multiplication, it is always useful for arithmetical purposes to adopt as the *derived* unit-quantity that which is constructed out of fundamental unit-quantities. The general term 'rate' which I have introduced is in common use: thus we mean by the rate of wages the quantum of wages earned per unit of time, by the rate of speed the quantum of length traversed per unit of time, and by the rate at which pleasure is being experienced, the quantum of pleasure per unit of time—these being cases in which the rate is estimated in reference to time. Again the rate called hydrostatic pressure is the quantum of pressure per unit of area; the rate called density is the mass per unit of volume. The term degree, which is sometimes used instead of rate, is ambiguous inasmuch as it is often used as equivalent to intensity; but the terms rate and intensity or degree ought to be clearly distinguished, because the notion of intensity refers to a single determinate quality, whereas rate is always constituted out of two distinguishable quantities; moreover the notion of rate, which involves concrete division, is always correlated with concrete multiplication. For example velocity, i.e. rate of movement, which involves the division of space by time, involves the converse process of multiplying velocity by time in constructing space. But if we conceive of velocity only by its numerical measure, confusion results between an abstract number on the one hand and the very many different kinds of quantity on the other hand that may be measured by the same abstract number.

§ 15. The practical importance of recognising concrete multiplication and division is best indicated by explaining what is meant by the *algebraical dimensions* of a quantity. We have already spoken of dimensions in its geometrical sense; thus an area is of dimension *two* in regard to length, a volume of dimension *three* in regard to length. Symbolising the dimension length by $[L]$ that of area is symbolised by $[L^2]$ and that of volume by $[L^3]$. Similarly velocity, i.e. length per time, is dimensionally $\frac{[L]}{[T]}$ or $[L].[T^{-1}]$; acceleration, i.e. velocity per time is $[L].[T^{-2}]$; density, i.e. mass per volume is $\frac{[M]}{[V]}$, i.e. $[M].[L^{-3}]$; momentum, i.e. mass × velocity is $[M][L][T^{-2}]$; force, i.e. mass × acceleration is $[M].[L].[T^{-2}]$; hydrostatic pressure, i.e. force per area is $[M].[L^{-1}][T^{-2}]$, etc., etc. Now the one rule as regards dimensions is that the additions and subtractions that are involved in a quantitative equation must always operate upon homogeneous quantities; i.e. upon quantities all of which have the same dimensions—these dimensions being generally expressed in terms of the three fundamental incomparables *mass*, *length*, and *time*. Regarding multiplication and division, in accordance with my view, as real operations performed upon concrete quantities, the square bracket in the above symbols stands for a *concrete unit*. For example the velocity '320 feet per 60 seconds' means $\frac{320\text{ ft.}}{60\text{ sec.}} = \frac{16\text{ ft.}}{3\text{ sec.}} = \frac{16}{3}$ of *unit velocity*. Those mathematicians who hold that such an expression as ft.÷sec. is meaningless have to maintain that the mathematical equations which are

used to express physical facts are concerned only with the numerical measurement of concrete quantities, whereas I hold that they are concerned with the concrete quantities themselves.

§ 16. There is one very unique case in concrete division, viz. where the dividend and divisor are quantities of the same kind. In general the result of such division is to construct a pure ratio, i.e. a magnitude which, when entering as multiplier or divisor of a quantity of any kind yields a quantity of the same kind, like the processes of addition and subtraction of quantities. But when a length is divided by a length, or an area by an area, we often intend the result of such division to represent an *angle*. It is therefore necessary to distinguish those cases in which the division of a length by a length represents a mere ratio, from those in which it represents an angle. In the former case, the quotient being a pure number can be used as a multiplier or divisor for a quantity of any kind whatever; but in the latter case this is never possible; one angle can only be mathematically combined with another angle, and this only by the operation of addition or of subtraction. The further complication in respect of the measurement of an angle is that this measurement may be used in different algebraical applications alternatively either as an abstract ratio or as a concrete quantity, which is denoted by the term angle. But the special question which, in my view, requires a clear answer is how to distinguish the process of dividing length by length that yields a mere ratio, from what appears to be the same process and yet yields an angle. The answer seems to be that when we are merely *comparing* two lengths

which may be said to be *dissociated*, their comparison yields a mere ratio, while when connecting two *associated* lengths in the process of division, we are constructing an angle. Thus, when we define the magnitude of an angle by the ratio of the arc of a circle to its radius, the arc and the radius are associated in our conception of the mode in which the angle is constructed; but when we are merely comparing the length of one line with that of any other, no natural association between the two lines is involved. The same holds of the differential coefficient dy by dx, when used in geometry to represent the slope of a tangent of a curve, which is a concrete quantity in the same sense as the quotient foot by second representing velocity.

§ 17. To sum up: Of the different kinds of magnitude, the first division is between abstract and concrete, abstract magnitudes being represented by pure numbers, these falling into the three divisions of integral, rational and irrational. Amongst concrete quantities—namely those that involve conceptions obtained from special kinds of experience, and which are therefore not purely logical—we distinguish the fundamental or primitive from the complex or derivative; the former being subdivided into extensive, distensive and intensive magnitudes, out of which the various derived or complex quantities have been shown to be constructed by operations analogous to arithmetical multiplication and division. These complex magnitudes fall again into different kinds, the distinctions between which may be always indicated by expressing the quantity dimensionally, i.e. as involving a concrete product of different fundamental quantities, each entering with a positive or negative

index. Finally a fundamental distinction has been drawn between addition or subtraction on the one hand and multiplication or division on the other; inasmuch as the quantities added or subtracted must be of the same kind, i.e. represented as dimensionally equivalent; whereas the operations of multiplication and division yield a quantity different in nature from its factors, which, however, together determine its nature. Throughout the whole discussion of concrete magnitudes, the difficult problem of defining or testing equality has been examined for each fundamentally distinct kind of quantity. The treatment has been comparatively elementary, the reader being referred for more subtle distinctions and analyses to works which deal primarily with mathematics and its philosophy.

CHAPTER VIII

INTUITIVE INDUCTION

§ 1. INDUCTION in general may be contrasted with deduction in that for a universal conclusion deduction needs universal premisses, whereas in induction a universal conclusion is drawn from instances of which it is a generalisation. Here the emphasis is upon the word instances, because although the customary account of deduction is that the range of the conclusion is identical with that of the narrowest of the premisses, yet deduction must include cases in which the range of the conclusion is not identical with that of any one of the premisses, and may even be wider than the widest of them. Actually the antithesis between inductive and deductive inference is not so fundamental as that between demonstrative and problematic inference; for every form of induction, except the problematic, is based upon the same fundamental principles (and these alone), as syllogism and other forms of deduction; whereas it is impossible to establish a theory of problematic induction, without recourse to certain postulates that are not involved in either form of demonstration, whether deductive or inductive. Now the fundamental principles which underlie demonstrative forms both of induction and deduction are themselves based upon a kind of inference which may be called intuitive induction. This process is not limited to the establishment of the principles of demonstration, but applies also to certain material as well as formal generalisations.

We have so far referred to two types of induction, viz., intuitive and demonstrative; it will be convenient to distinguish in all four varieties, namely intuitive, summary, demonstrative and problematic. Of these the three former will be discussed in the present Part of this work, but problematic induction will be examined in detail in a separate Part, on the ground, specified above, of its dependence upon special postulates.

§ 2. Before treating the main topic of this chapter, we must discuss the necessarily preliminary process known as abstraction, the nature of which was a special subject of philosophical and psychological controversy amongst James Mill and his contemporaries. The discussions of that date started from the supposition that what was presented in our earliest acts of perception was a combination of impressions from different senses, such as those of sight and touch. From this presupposition, upon which both parties were agreed, the difficulty was raised as to how the percipient could single out an occurrent impression of one sense from the concurrent impressions of other senses. This presupposition, however, is fundamentally mistaken. For, in *fact*, our earliest acts of attention, which yield any product that could be called a percept, are directed to impressions of one sense at one time, and to impressions of another sense at another time. For example, the child when interested in the colour of a ball, is attending to his visual impressions apart from any motor or tactual sensations that he may be experiencing in handling the ball; that is to say, his attention is from the first exclusive, and it is only in further progress of attentive power that his attention becomes inclusive. The atten-

tion that includes visual with tactual impressions is a higher and later process than the attention which is directed either exclusively to the visual impressions or exclusively to the tactual impressions. The fact that we can and do attend to impressions of one order in disregard of concurrent impressions of other orders, explains how our primitive perceptual judgments, from the first, assume a logically universal form. For, in predicating a determinate colour, for instance, of any given impression, there is a recognition that the same determinate can be predicated of *all* impressions which agree with the given impression in respect of colour, however much they may disagree in other respects. Now, if this be granted, it has an important bearing upon another serious historical controversy—namely that between Mill and his opponents as to the foundations of geometry. Both parties to this dispute started with an obscure view, that there was an opposition between intuition and experience; whereas in truth intuition is a *form* of knowledge, in relation to which experience is the *matter*. The intuitionists seem to have held that the intuitive form of knowledge involved no reference to experience; whereas the empiricists forgot, when relying upon experience as the sole factor in knowledge, that knowing is a mode of activity, and therefore not of the same nature as sense-experience which is merely passive or recipient. The truth is that when we have asserted a predicate of a particular, we have apprehended the universal in the particular, in the sense that the adjective is universal and the object of which it is predicated is particular.

§ 3. There is another sense in which we may be

said directly to apprehend the universal in the particular, namely in regard to certain classes of propositions, where the terms universal and particular apply to the propositions themselves, and not to the distinction between the subject and the predicate within the proposition. It is at this stage that we pass, in our discussion, from abstraction to our main topic, viz., abstractive or intuitive induction. The term intuitive is taken to imply felt certainty on the part of the thinker; and it is characteristic of propositions established by means of intuitive induction that an accumulation of instances does not affect the rational certainty of such intuitive generalisations. The procedure by which these generalisations are established may be shown by psychological analysis to involve an intermediate step by which we pass from one instance to others of the same form and in this passage realise that what is true of the one instance will be true of all instances of that form.

§ 4. Two types of intuitive induction may be distinguished, experiential and formal, although these types are not precisely exclusive of one another.

The experiential type of intuitive induction may be illustrated from our immediate judgments upon sense-impressions and the relations amongst them. For example, in judging upon a single instance of the impressions red, orange and yellow, that the qualitative difference between red and yellow is greater than that between red and orange (where abstraction from shape and size is already presupposed) this single instantial judgment is implicitly universal; in that what holds of the relation amongst red, orange and yellow for this

single case, is seen to hold for all possible presentations of red, orange and yellow. Again in immediately judging that a single presented object, whose shape is perceived to be equilateral and triangular, is also equiangular (where abstraction from colour and size is presupposed) we are implicitly judging that *all* equilateral triangles are equiangular. Similarly when judging for a single instance that the sounds *A*, *C*, *F*, produced, say, from the human voice, are in an ascending scale of pitch, we are implicitly judging that *all* sounds—apart from differences of timbre or loudness such as those produced by the violin or piano—that can be recognised as of the same pitches *A*, *C*, *F*, are also in an ascending order of pitch. The universality of these experiential judgments extends over imagery as well as sense impressions: the fact that we can identify a specific image as corresponding to a specific impression is sufficient to enable us directly to transfer our judgments about the relations amongst impressions to those amongst the corresponding images. These elementary illustrations show that intuited universals about colours and pitches are of the same epistemological nature as those about geometrical figures, in that the judgment upon a single presented instance is sufficient for the establishment of a universal extending in range over imagery as well as impression.

§ 5. Passing now to other experiential judgments, which are not merely sensational, we may illustrate intuitive induction from introspective judgments. For instance, when I judge that it is the pleasure of this or that experience which causes me to desire it, I am implicitly universalising and maintaining that the

pleasure of any experience would cause me to desire it. And again, when I judge that the greater resultant desire for one possible alternative than for any other causes me to will that alternative, I am judging that this will hold for all my volitional experiences. An important sub-class of experiential judgments which are intuitively inductive consists of moral judgments. Thus, when anyone judges that a certain act characterised with a sufficient degree of precision is cowardly, or dishonest, or generous, he is implicitly judging that all acts of the same specific character would be characterisable by the corresponding moral attribute. That this is not a case of mere abstraction is clear when we consider that the characteristics used to define the nature of the action are other than ethical, and that the judgment is therefore synthetic. This intuitive aspect of moral judgments assumes importance as reconciling the two forms of ethical intuitionism to which Sidgwick refers as Perceptual and Dogmatic, the first of which stands for the particular, and the second for the universal, intuition. For, in my view, the Dogmatic form of intuition is not genuinely intuitive except so far as it is based on the Perceptual. Instead, therefore, of distinguishing moralists according to what they hold to be the nature of an ethical intuition, it is more important to distinguish them according as they base their doctrine upon genuinely intuitive judgments, e.g. Kant; or upon judgments accepted on authority as expressions of the voice of God, e.g. Butler.

§ 6. The gulf between experiential and formal intuition is bridged by considering certain intermediary forms of intuitive apprehension in which, according as

the range of universality increases, we depart further from the merely experiential and approach nearer to the merely formal type. A typical case is the merely experiential judgment that red and green cannot both be predicated of the same visual area by one person at one time. The judgment is first universalised when the experient sees that the same holds of all cases of the specific determinates red and green. But this judgment almost immediately passes into the wider universal that any two different determinates under the determinable colour are similarly incompatible. And when lastly the experient extends the range of his judgment to all determinables, he has reached a formal intuition, namely that any two different determinates under *any* determinable are incompatible.

To this formal type of intuition belong all intuitively apprehended mathematical, as well as purely logical, formulae. For instance, the algebraical formula known as the Distributive Law is intuitively reached in some such way as this: perceiving that

3 times 2 ft. + 3 times 5 ft. = 3 times (2 ft. + 5 ft.)

we immediately realise that

4 times 7 days + 4 times 9 days = 4 times (7 days + 9 days),

and in this step we are virtually apprehending the Distributive Law symbolically expressed thus:

$$n \text{ times } P + n \text{ times } Q = n \text{ times } (P + Q)$$

where n stands for any number, and P and Q for any two homogeneous quantities.

A logical example of a similar nature is the formula of the simple conversion of particular affirmative propositions. This is reached by perceiving, for instance,

that 'Some Mongols are Europeans' would imply that 'Some Europeans are Mongols,' and at the same time that

'Some beings incapable of speech have the same degree of intelligence as men' would imply that 'Some beings having the same degree of intelligence as men are incapable of speech.'

This leads to the virtual apprehension of the universally expressed implication:

'Some things that are p are q' would imply that 'Some things that are q are p'

where p and q stand for any adjective.

§ 7. This example of the establishment of logical formulae by means of intuitive induction has an educational importance in correcting a certain prevalent conception of the function of logic. What is called formal or deductive logic is usually taught by first presenting general principles in a more or less dogmatic form, with the result that the learner is apt to use these principles merely as rules to be applied mechanically in testing the validity of logical processes. Instead of leading him to conceive of these rules as externally imposed imperatives, an appeal should be made to him to justify all fundamental principles by the exercise of his own reasoning powers; and this exercise of power will involve the process of intuitive induction.

CHAPTER IX

SUMMARY (INCLUDING GEOMETRICAL) INDUCTION

§ 1. THE term summary induction is here chosen in preference to what, in the phraseology of the old logicians, was called 'perfect induction,' to denote a process which Mill regarded as not properly to be called induction; on the ground that the conclusion does not apply to any instances beyond those constituting the premiss. Mill's contention can certainly be justified inasmuch as the process involves precisely the same logical principles, and these alone, that govern ordinary deduction. In fact, the process of summary induction may be expressed in the form of a syllogism in the first figure. For example:

Major Premiss. 'Sense and Sensibility' and 'Pride and Prejudice' and 'Northanger Abbey' and 'Mansfield Park' and 'Emma' and 'Persuasion' deal with the English upper middle classes.

Minor Premiss. Every novel of Jane Austen is identical either with 'Sense and Sensibility' or with 'Pride and Prejudice' or with 'Northanger Abbey' or with 'Mansfield Park' or with 'Emma' or with 'Persuasion.'

Conclusion. ∴ Every novel of Jane Austen deals with the English upper middle classes.

Here the enumeration standing as subject in the major premiss is the same as the enumeration standing as predicate in the minor premiss. But, in the former, reference is made to every one of the collection, in the

latter to some one or other. This precisely corresponds to the characteristic of first figure syllogism; namely that the middle term is distributed as subject of the major and undistributed as predicate of the minor. In text-book illustrations of perfect induction the minor premiss is almost invariably omitted, because the illustrations chosen—such as the Apostles or the months of the year—are so familiar that the completeness of the enumeration is assumed to be known by every ordinary reader and therefore does not require to be expressed in a separate minor premiss. The same process is exhibited by an example in which each of the items enumerated is a universal instead of being a singular:

Every parabola and every ellipse and every hyperbola meet a straight line in less than 3 points.

Every conic section is either a parabola or an ellipse or a hyperbola.

∴ Every conic section meets a straight line in less than 3 points.

§ 2. Another case of perfect induction, which has specific bearing upon induction in general, may be expressed symbolically in the following syllogism:

s_1 and s_2 ... and s_n are p.

Every examined case of m is identical either with s_1 or with s_2 ... or with s_n.

∴ Every examined case of m is p.

A summary or perfect induction of this form is the necessary preparatory stage in gathering together the relevant instances for establishing an unlimited generalisation. For the *conclusion* thus obtained, constitutes the *premiss* from which we directly infer, with a higher or lower degree of probability, that 'Every case of m is p.'

Whewell pointed out the importance and difficulty of discovering 'the concept p under which the instances are colligated.' He, in agreement with other critics of Mill, accordingly held that the process of induction was completed in the discovery of this colligating concept, on the ground that this process alone required something like genius to perform, while it is the easiest thing in the world to pass from every examined instance to every instance. Mill, on the other hand, considered that this process only supplied the requisite premiss for a genuine inductive inference. To illustrate his view, Whewell had chosen Kepler's famous discovery of the formula for the orbit of the planets, and it was towards this illustration that Mill directed his criticism. Expressed in terms of the above used symbols.

Let m stand for 'positions of a certain moving planet,'

$s_1, s_2 \ldots s_n$,, ,, 'the several observed positions,'

and p ,, ,, 'being a point on a certain ellipse.'

The syllogism which expresses the process of perfect induction used by Kepler will then be as follows:

'Each of several observed positions is a point on a certain ellipse.

'Every examined position of a certain moving planet is identical either with one or with another of these several observed positions.'

$\therefore$ 'Every examined position of the moving planet is a point on that ellipse.'

This formula had not been discovered by any previous astronomer, and, on the grounds already assigned, Whewell maintained that the discovery constituted the completion of the induction. To this Mill demurred, because by induction he meant a process in which the

conclusion is an unlimited universal extending beyond examined instances; he, however, failed to observe that Kepler had actually gone beyond the examined instances and had described the *complete* orbit of the planet by inferring that what held of the examined positions would hold of all the interpolated positions. Kepler had thus unconsciously made a genuine induction in the sense required by Mill. Whewell was concerned with the art of discovery, and therefore held that the essential factor in induction was the discovery of the colligating concept; whereas Mill was concerned with the science of proof, and therefore held that the essential factor in any induction (that was not merely formal or demonstrative) was the inferential extension from examined to unexamined instances.

§ 3. Having illustrated the process of summary (or perfect) induction by familiar examples, in which the conclusion applies to a finite number of cases which are enumerable, we proceed to consider a more interesting type of summary induction in which the conclusion applies to an infinite number of cases which are non-enumerable. This type occurs in geometrical proofs of geometrical theorems, and has been more or less confused on the one hand with merely intuitive, and on the other hand with problematic induction. It differs, however, from the former in that its conclusion cannot be reached from an examination of one or of a few instances; and from the latter in that the conclusion does not extend beyond the range of the examined instances—these being apprehended in their infinite totality.

It is well known that there are two modes by which geometrical theorems may be proved, viz. 'analytical'

and 'geometrical.' Strict analytical proof has the same logical character as *algebraical* proof, and comes under the head of functional deduction. Such proofs do not require the aid of geometrical figures. But the *geometrical* method of proof depends essentially upon the use of such figures. It may further be pointed out that the analytical method has an indefinitely wider scope than the geometrical. For example, by employing mere analysis we can construct spaces of various different forms other than Euclidian; and certainly a geometrical method would be impossible except as applied to our space which is presumed to be Euclidian. The actual procedure in constructing any non-Euclidian space is to bring forward some four or five axioms which must be (*a*) independent of one another, and (*b*) mutually consistent. These axioms, however, are not put forward categorically, but purely hypothetically; it follows, therefore, that the theorems which, for convenience are said to be deduced from the axioms, should be more strictly said to be implied by the axioms. Such systems, therefore, are throughout implicative and not inferential. In other words, a supposed space, definable by any chosen set of axioms, would have such and such other characteristics which these axioms would formally imply. On the other hand, the geometrical method is a method of proof or inference, inasmuch as we accept its conclusions as true only because we have accepted its axioms as true.

§ 4. We must therefore examine the process by which the axioms of geometry are established. These, *quâ* axioms, are not reached by deduction; and, since they are universal in the specific sense that they apply

for an infinite number of possible instances, it would seem that some form of induction is required for their establishment; unless we adopt the view that they are obtained by a process which embraces all possible cases in a single act of direct intuition. This latter appears to be the view of Kant who held, as regards geometry, that our intuitions are from the first universal, and that they therefore function as premisses for deducing any, or any other, given case.

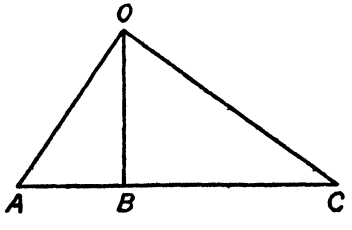
In order to examine this question let us take the familiar axiom conveniently expressed in the form: 'Two straight lines terminating at the same point cannot intersect at any other point.' This is the most important axiom which does not hold of non-Euclidian spaces in general. Independently, however, of the nature of any other kind of space, the axiom certainly represents the manner in which we actually intuite *our* space, whether falsely or truly. Now this axiom, in its universality, can be established only by means of imagery and not by mere perception; for the compass over which the axiom holds is beyond the range of actual perception. For in the first place it is only through imagery that we can represent a line starting from a certain point and extending indefinitely in a certain direction; and, in the second place, we cannot represent in perception the infinite number of different inclinations or angles that a revolving straight line may make with a given fixed straight line. We may, however, by a rapid act of ocular movement represent a line revolving through 360° from any one direction to which it returns. In this imaginative representation the entire range of variation, covering an infinite number of values, can be exhaus-

tively visualised because of the continuity that characterises the movement. It is only if such a process of imagery is possible that we can say that the axiom in its universality presents to us a self-evident truth. It is therefore this species of summary induction that is employed to establish geometrical axioms—differing, as explained above, on the one hand from mere intuitive induction, inasmuch as one or a few specific cases would not constitute an adequate premiss; and, on the other hand, from induction in Mill's specific sense, since the conclusion does not go beyond the premisses taken in their totality.

§ 5. I shall further maintain that if, in the course of a geometrical proof which may involve several successive steps, the perception or image of a figure is required for any single step, this is because we have to go through precisely the same process of summary induction, embracing an infinite number of specialised cases of which the figure under inspection is one—all of these being included in the subject of the universal conclusion to be proved at that step. Speaking generally, in any one demonstrative step, the major premiss is a universal previously established, and from this universal major it is required to establish a new universal conclusion. It is obvious that this can only be done by means of a universal minor; and it is in the establishment of the universality of the minor that consists the logical function of the figure. The arbitrarily chosen figure under inspection can only be used as a minor term to prove the conclusion about that single figure; and hence, to obtain the required universal conclusion, the minor must be universalised by the same logical process that

is used for establishing the explicit axioms. Now the Euclidian geometry might have been established by purely analytical methods; provided first, that a sufficient number of axioms had been explicitly formulated; and secondly that each of these axioms had been established for itself by the process of summary induction. Such an analytical system would dispense with the use of figures as objects either of perception or of imagery *in the course of the proof*, these being only required in the process of establishing the axioms themselves.

To show by specific illustration how the geometrical proof uses a figure, we will select a very frequently assumed, but not explicitly stated axiom, which, in Euclid's proofs is required to supplement the explicit axiom 'the whole is greater than its part,' or more precisely, 'the whole is equal to the sum of its parts.' Before this explicit axiom can be used, we must be satisfied that the two elements of the figure, one of which is to be greater than the other, do stand in the relation of whole to part. The axiom to which I refer is actually employed by Euclid and most geometricians in the propositions numbered 5, 6, 7, 16, 18, 20, 21, 24, and 26 in Euclid, Book I. It may be formulated as follows: The angle subtended at any point by a part of a line is part of the angle subtended by the whole line.' If the reader is not familiar with this new axiom, he must go through a process in which he imagines a line revolving in a plane through a point (*O*) from some initial direc-



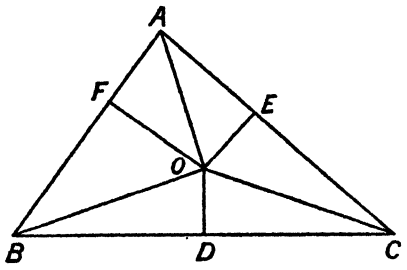
tion (OA) to a final direction (OC), so that it will intersect the whole line (AC) in a series of successive points. In this way, and in this way alone, can he accept the universality of the required conclusion that the angle AOC is greater than the angle AOB. In Euclid's theorems enumerated above it will be found that this axiom is required in every case to establish the conclusion that a certain angle is greater than another; and that this conclusion is a necessary step in the further progress of each proof.

Geometrical induction involves, in addition to the summary process above explained, two further processes which are of the nature of intuitive induction, as explained in the preceding chapter. Of these two, the first is concerned with absolute position, the second with absolute magnitude. Thus, having reached a universal by summary induction limited to figures occupying a certain position, it is by intuitive induction that we pass to figures of the same specific shape and magnitude occupying any other possible position; and again from a figure imaged as having a certain magnitude, to figures of the same specific shape but of any other possible magnitude. I have described these two processes as of the nature of intuitive induction, in which we universalise by abstracting from variable position and from variable magnitude; but they might otherwise be regarded as involving the conception of position and magnitude as being—not absolute—but relative to the percipient's own position and to his distance from the figure depicted in imagination.

§ 6. Having illustrated the proper use of the geometrical figure, we shall proceed to illustrate what may

be called its abuse; and give, by means of a figure, an alleged proof that every triangle is isosceles:

To prove that every Triangle is Isosceles.



Let the bisector of the vertical angle A meet the perpendicular bisector of the base BC, whose middle point is D, at the point O. Join BO, CO, and draw OE perpendicular to AC, and OF perpendicular to AB. Then,

(1) the triangles BOD, COD, are congruent; for $BD = CD$; OD is common; and
$$\angle BDO = \angle CDO;$$
$$\therefore BO = CO.$$

(2) the triangles AOE, AOF, are congruent; for AO is common; $\angle OAE = \angle OAF$; and
$$\angle OEA = \angle OFA;$$
$$\therefore AE = AF \text{ and } OE = OF.$$

(3) the triangles COE, BOF are congruent; for, by (1) $CO = BO$; and by (2) $OE = OF$; and hence $CO^2 - OE^2 = BO^2 - OF^2$;

i.e. (since CEO and BFO are rt $\angle$) $CE^2 = BF^2$.

Hence, by (2) $AE = AF$ and, by (3), $CE = BF$;
$\therefore$ by addition $AC = AB$. Q.E.D.

Here we see that the axiom: 'the whole is greater than its part' is used in its more precise form, 'the whole is equal to the sum of its parts.' Now before we can state as regards the straight line AFB, that

$$AB = AF + FB,$$

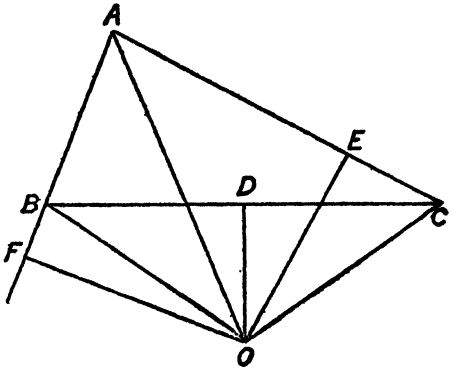
we must be sure that AF, FB are really parts of AB;

whereas if F was beyond AB, then AF would be the whole and AB, BF would be its parts.

The fallacy incurred in this proof arises from the mistaken intuition that the bisector of the vertical angle A meets the perpendicular bisector of the base BC at a point O *inside* the triangle. By drawing an incorrect figure and thus convincing ourselves of the false conclusion, we had unconsciously universalised from the figure before us that for every case the two bisectors would meet at a point within the triangle, this being indicated in the figure as drawn. In other words, we have swallowed the relation presented in the drawn figure as being universalisable, without having gone through the necessary summary induction.

We may proceed to draw the corrected figure. From this we reach, as before, the two conclusions

$$AE = AF \text{ and } CE = BF.$$



But now we see that

$$AC = AE + EC,$$

while

$$AB = AF - BF.$$

Euclidian demonstration professes to be based on pure reasoning, in such a manner that the figure may be drawn quite inaccurately, and yet the force of the proof be equally cogent. But it may happen, as in the case before us, that the figure is drawn with a degree of inaccuracy which affects the proof; because the particular demonstration, involving unconscious reference to the figure drawn, has been illegitimately universalised.

§ 7. My explanation of the logical function of the figure in geometrical demonstration differs fundamentally from that put forward by Mill, who maintains that it is by *parity of reasoning* that what is apprehended to be true for the one drawn figure, is apprehended to be true for any other figure (within the scope of the conclusion). But the passage from the demonstration for one case to that for any other case can only be said to exhibit 'parity of reasoning' when the two demonstrations have the same *form*. Taking for example the two demonstrations:

(1) every m is p; this S is m; therefore this S is p;

(2) every m is p; that S is m; therefore that S is p;

we may certainly pass by parity of reasoning from (1) to (2) inasmuch as both arguments are of the same form, the words 'this' and 'that' indicating difference in matter. In ascribing the same form to (1) and (2), what is meant is that the relation of implication between the premisses and conclusion of the one is the same as that between the premisses and conclusion of the other. But in order to use an implication for the purposes of inference, we should have to *assert* 'This S is m' for

case (1) and 'That S is m' for case (2); for although the relation of implication is the same in the two arguments, it does not follow from having asserted the minor of the one that we can, on this ground, assert the minor of the other. Now, in order to establish the required conclusion 'Every S is p' we must first establish the universalised minor 'Every S is m.' No reasoning process (in the accepted meaning of the term) would enable us to pass from the case 'This S is m' to 'That S is m' and to 'That other S is m' *ad infinitum*; and the only mode of establishing the required universal minor 'Every S is m' is through some process of induction, the nature of which we have been describing.

CHAPTER X

DEMONSTRATIVE INDUCTION

§ 1. HAVING so far examined intuitive and summary induction, we now pass to the third type of inductive inference distinguished at the outset, namely demonstrative induction. As its name suggests, this form of inference partakes both of the nature of demonstration and of induction. It includes several different forms, the characteristics common to them all being (1) that they are demonstrative, in the sense that the conclusion follows necessarily from the premisses; and (2) that they are inductive, in the sense that the conclusion is a generalisation of a certain premiss or set of premisses which, taken as a collective whole, may be spoken of as 'the instantial premiss.' The possibility of arriving demonstratively at a conclusion wider than the premisses, depends here upon the nature of the major premiss, which is not only universal but composite. In short demonstrative induction may be described as that form of inference in which one premiss is composite and the other instantial; the conclusion being a specification of the former and a generalisation of the latter.

§ 2. In explaining the nature of demonstrative induction as above described, the composite nature of the major premiss brings us back to those fundamental modes of inference specified in Part I, Chapter III on compound propositions. There P and Q are taken to stand for any propositions, and four composite relations

are distinguished in which P may stand to Q; (*a*) Implicative, leading to the *Ponendo Ponens*; (*b*) Counter-implicative, leading to the *Tollendo Tollens*; (*c*) Alternative, leading to the *Tollendo Ponens*; (*d*) Disjunctive, leading to the *Ponendo Tollens*:

(*a*) If P then Q, but P; therefore Q.
(*b*) If Q then P, but not P; therefore not Q.
(*c*) Either P or Q, but not P; therefore Q.
(*d*) Not both P and Q, but P; therefore not Q.

In these composite premisses, we shall take the implicates and alternants to stand for universal propositions, and the implicants and disjuncts to stand for particular propositions. This secures, for each case, a form of inference in which a particular or singular premiss yields a universal conclusion. Thus:

(*a*) If some S is p, then every T is q;
but this S is p,
$\therefore$ every T is q.

(*b*) If some T is q, then every S is p;
but this S is p',
$\therefore$ no T is q.

(*c*) Either every S is p, or every T is q;
but this S is p',
$\therefore$ every T is q.

(*d*) It cannot be that some S is p, and some T is q;
but this S is p,
$\therefore$ no T is q.

In the above formulae, it will be observed that the simple or categorical premiss is not the precise equivalent or contradictory, as the case may be, of the corresponding proposition that occurs in the composite premiss; for 'this S is p' is more determinate than

'some S is p,' being one of its superimplicants; and again 'this S is p' is not the mere contradictory of 'every S is p,' being one of its contraries or superopponents. The categorical premiss having been in this way strengthened, the conditions of valid inference are still satisfied. In short, we have taken as our instantial premiss a specific instance characterised determinately. The object of this is to illustrate the symbolic formulae by concrete examples which, when further developed, will exhibit the nature of demonstrative induction in its most important scientific forms. Consider the following illustrations of the symbolic formulae:

(*a*) If some one recorded miracle has been shown to have happened, then every natural phenomenon has a supernatural factor; but such or such recorded miracle has been shown to have happened; therefore every natural phenomenon has a supernatural factor.

(*b*) If some one female member of a Board had lowered the educational standard in her university, every woman would have submitted to exclusion from the Cambridge Senate; but Miss C. has not submitted to exclusion from the Cambridge Senate; therefore no female member of a Board has lowered the educational standard in her university.

(*c*) Either every Protectionist country is financially handicapped or every economist of the old school is mistaken; but America is commercially prosperous; therefore every economist of the old school was mistaken.

(*d*) It cannot be that some variations can be artificially produced in domesticated animals, while there are some species whose characters are unaffected by their environment; but some variations have been artificially produced in the pigeon; therefore there are no

species whose characters are unaffected by their environment.

These illustrations would be regarded by those logicians who divide *all* inferences into inductive and deductive, as being of the nature of deduction rather than of induction, because the universal conclusion is not a generalisation of the instantial premiss. In contrast to these we will therefore now select a set which will be recognised as of the nature of induction; inasmuch as here the universal conclusion in each case *is* a generalisation of the instantial premiss. These new examples are applications of the same symbolic formulae as the preceding set; they differ only in that the symbols S and T will now stand for the same class, whereas in the first set they stood for different classes.

(*a*) If some boy in the school sends up a good answer, then all the boys will have been well taught; the boy Smith has sent up a good answer; therefore all the boys have been well taught.

(*b*) If a single authoritative person had witnessed the alleged occurrence, then everyone would have believed it; but Mr S. is incredulous; therefore no authoritative person could have witnessed the occurrence.

(*c*) Either every act of volition is determined or every act of volition is free; but by introspection I am sure that a certain act of mine was undetermined; therefore every volition is free.

(*d*) It is impossible to suppose that any modern theologians are genuine scholars while others have remained orthodox; Dean I. is a genuine scholar; therefore no modern theologian could have remained orthodox.

§ 3. Returning to the symbolically expressed formulae, and substituting p or $\bar{p}$, as the case may be, for q, as well as S for T, the composite premisses will assume the following still more specialised form:

(a) If some S is p then every S is p.
(b) If some S is $\bar{p}$ then every S is $\bar{p}$.
(c) Either every S is p or every S is $\bar{p}$.
(d) Not both some S is p and some S is $\bar{p}$.

It will be seen that these four composite premisses are formally equivalent to one another, and that by adding the categorical premiss 'This S is p' we may conclude in each case that 'Every S is p.' Now we may transform the alternation of universals in (c) and the disjunction of particulars in (d) by substituting for p and $\bar{p}$ any set of predicates p, q, r, t, v ... for the alternative proposition (c), and the same set in pairs for the disjunctive proposition (d), thus:

(c) Either every S is p or every S is q or every S is r or ... etc.
(d) Not both 'some S is p and some S is q' and not both 'some S is p and some S is r' ... etc. etc.

In this transformation the two complexes (c) and (d) are no longer equivalents but rather complementaries to one another. If the categorical premiss 'This S is p' is now introduced we may infer by means of (d) that 'No S is q,' 'No S is r,' 'No S is t' etc., so that all but the first of the universal alternants in (c) is rejected, and again the universal conclusion 'Every S is p' is established. The need of combining the complementaries (c) and (d) in order to establish the required universal conclusion is apparent when we consider a con-

crete illustration. In the following example, where the predicates p, q, r... stand respectively for 'attacking the Coalitionists,' 'attacking the Liberals,' 'attacking the Labour Party'... it will be observed that the composites (*c*) and (*d*) retain the same logical force as in the above symbolisation, although somewhat differently worded:

(*c*) At least one of the political parties was attacked by every speaker at a certain sitting of the Congress, and (*d*) not more than one of the parties was attacked at that sitting.

Mr X. who spoke attacked the Coalitionist Party.

∴ Every speaker at the sitting attacked the Coalitionist Party.

§ 4. Now, if—instead of p, q, r...—we take determinates p, p', p''... under the same determinable P, then the disjunctive premiss (*d*) will not be explicitly required, because it is accepted *a priori* that nothing can be characterised by both of any two determinates under the same determinable. What remains then is the universal alternative proposition (*c*), established, we may assume, by problematic induction; namely: 'Either every S is p, or every S is p', or every S is p''... running through all the determinates under P,' and this may be summed up in the single phrase 'Every S is characterised by *some the same* determinate under the determinable P.' If to this composite premiss is added the instantial premiss 'This S is p,' the universal conclusion follows that 'Every S is p.' This trio of propositions represents the one immediate way of establishing a generalisation demonstratively from a single instance, and it will be termed

The Formula of Direct Universalisation

Composite Premiss: Every S is characterised by some the same determinate under the determinable P.

Instantial Premiss: This S is p.

Conclusion: $\therefore$ Every S is p.

§ 5. To take a typical illustration from science:

Every specimen of argon has some the same atomic weight.

This specimen of argon has atomic weight 39·9.

$\therefore$ Every specimen of argon has atomic weight 39·9.

In this, as in all such cases of scientific demonstration, the major premiss is established—not *directly*, by mere enumeration of instances—but rather by deductive application of a wider generalisation which has been ultimately so established. In the given example it is assumed that *all* the chemical properties of a substance, defined by certain 'test' properties, will be the same for all specimens; and this general formula is applied here to the specific substance *argon*, and to the specific property *atomic weight*. The assumption in this case is established by problematic induction, i.e. directly from an accumulation of instances. In practically all experimental work, a single instance is sufficient to establish a universal proposition: when instances are multiplied it is for the purpose of eliminating errors of measurement. It is owing to the fact that the general proposition, functioning as major or supreme premiss, has the special form of an alternation of universals that, by means of a minor premiss expressing the result of a single observation, we are enabled to establish a universal conclusion. This conclusion, in accordance with

our general account of demonstrative induction, is a specification of what is predicated indeterminately in the universal premiss, and a generalisation of the proposition recording the result of a single observed instance.

§ 6. The most important extension of demonstrative induction deals with such methods as those of agreement and difference that have been treated by Mill. We propose to give a formal account of methods similar to those explained by Mill, but so constructed as to render them strictly demonstrative. Many critics of Mill's methods have treated them disparagingly because of his failure to exhibit their formal cogency; while others have maintained that induction should not profess to exhibit the strictly formal character that is ascribed to syllogism and other deductive processes. I hold, on the contrary, that Mill's methods can and should be exhibited as strictly formal, by rendering explicit certain implicit premisses upon which the cogency of the argument from instances in any given case depends; and by indicating the precise conclusion which can be drawn from the instances in question. The implicit premiss is ultimately established by a process of problematic induction, which must be sharply distinguished from the demonstrative process exemplified by the methods. Mill's exposition differs from mine, then, in three preliminary respects. In the first place, he does not clearly distinguish the nature of direct or problematic induction from the nature of the process conducted in accordance with his 'methods of induction,' which he appears often to regard as demonstrative. This confusion is particularly noticeable when we contrast his different modes

of treating the methods of Agreement and of Difference: 'Agreement' he hardly distinguishes from the method of simple enumeration, which is admittedly problematic; whereas 'Difference' he attempts to exhibit as strictly demonstrative. In the second place, he professes to employ as the 'supreme major premiss' for his methods a very wide but at the same time undefined proposition called the 'Law of Causation.' In opposition to this prevalent view, I hold that it is impossible to present such methods as those of Agreement and Difference as strictly formal so long as we attempt to subsume them under so vague a proposition as the Law of Causation, and that each inference drawn in accordance with these methods requires *its own* specific major premiss. The formulation of such a major premiss is the necessary first step in rendering formally cogent any inference (drawn under methods similar to Mill's) from instances finite in number, presented either in passive observation or under experimental conditions. In the third place, whereas Mill retains or eliminates a determining factor according as it affects or does not affect a determined character, in my view the precise conclusion to be drawn is not correctly expressed in terms of the presence or absence of factors, but rather in terms of co-variation, thus: according as in two instances a single variation in any determining character does or does not yield a variation in the determined character, the same will hold for any and every further variation of that determining character.

§ 7. In order to obtain the requisite premisses for demonstrative induction, we must assume that by a preliminary inductive process based upon general ex-

perience, a number of variable circumstances have been eliminated as irrelevant to the formula to be proved. The exposition of this preliminary process by which irrelevant conditions are eliminated, must be postponed until we examine in detail the nature of problematic induction. The process itself must be regarded as pre-scientific; and science takes up the problem at the point where the character of a phenomenon is known to depend only upon a limited number of variable conditions. This knowledge is expressed in a proposition which constitutes the major premiss in the scientific process which we are about to examine as a species of demonstrative induction. The major in question is specifically different for different classes of phenomena, and is in this respect unlike the so-called Law of Causation which professes to be the same for every class of phenomena. If the symbols A, B, C, D, E are taken to illustrate the determining characters, and P the thereby determined character, then the instances collected in order to establish a given generalisation of the form $ABCDE \sim P$, must be characterised by the same set of determinables, and will be said to be of the same type or homogeneous with one another. The specific major premiss may then be expressed in the formula:

The variations of the phenomenal character P depend *only* upon variations in the characters A, B, C, D, E (say).

The conception of dependence, which the above formula introduces, requires more precise explanation. In the first place the formula must be understood to imply that the variations of A, B, C, D, E, upon which

variations of P depend, are independent of one another. For if, for example, a variation of A entailed a variation of B, then B being a *determined* character should be omitted from amongst the *determining* characters. It is only by observing this principle that we can apply the essential rule for all experimentation—that *one* only of the determining characters should be varied at a time. Again it is essential that A, B, C, D, E, should be simplex characters: for the nature of the dependence of P upon them is such that, if only one of these mutually independent determining characters varies, the character P will vary; whereas, if more than one of them varied, P might remain constant. This consideration shows that if any character such as A was not simplex, but resolvable into unknown factors X and Y which varied independently of one another, then a variation in A might involve such a variation in both X and Y that the character P would remain unchanged.

In the second place, the force of the term 'only' indicates that the dependence of P upon A, B, C, D, E is such that no variable circumstances other than these need be taken into consideration, all others having been previously eliminated in what we have called the prescientific or problematic stage of the induction. The conclusion that results from this prescientific induction is to be expressed by an alternation of universals in the form: 'Either every instance of *abcde* is p, or every instance of *abcde* is p', or every instance of *abcde* is p'', or ....' From this it follows that when a single instance is given of *abcde* that is p, this may be immediately universalised in the form 'Every *abcde* is p.

It should be pointed out that this immediate universalisation is not dependent upon any comparison of one instance with another, and is prior to the use of such methods as those of difference or agreement; being in fact exemplified above for the case of the atomic weight of argon.

The full significance of the notion of dependence is brought out by taking not only instances which agree in the determining characters and therefore in the determined character, but by taking also instances which differ in the determining, and consequently also in the determined characters. If a variation in any one of the characters A, B, C, D, E entails a variation in P, then, in accordance with the principle underlying Mill's method of Difference, that character cannot be eliminated; whereas, if no variation in P is entailed by a variation in some one of the characters A, B, C, D, E, then, in accordance with the principle underlying Mill's method of Agreement, that character *can* be eliminated.

§ 8. The forms of Demonstrative Induction to be now exhibited contain (1) the supreme premiss of dependence formulated above for a given set of determinables, and (2) a finite set of instantial premisses under the same determinables. These forms will be distinguished under four heads to be designated *figures* rather than *methods*; but will not correspond severally to Mill's methods, although primarily based upon his method of Difference, and with some important modifications upon his method of Agreement. The notion of 'figure' is substituted for that of 'method'; (*a*) because there is only one method employed in the four figures, namely that of varying one determining factor

at a time; and (*b*) because, as in the case of the figures of syllogism, the precise conclusion drawn from the instantial premisses will depend on the nature of the instances themselves, and the figure to be employed in any given case will not be foreknown until the instances have been examined and compared. I shall adopt the phrases 'Difference' and 'Agreement' for the first two figures but 'Composition' and 'Resolution' for the two remaining figures. All the four figures have the same demonstrative force, and the two last figures—though they have some resemblance to Mill's or rather Herschel's method of Residues, which, as shown in a previous chapter, is purely deductive—have precisely the same inductive nature as those of Difference and Agreement. In each figure, the first step in the demonstrative process is to universalise each single instance *taken separately* in accordance with the principle of Direct Universalisation enunciated above; and the second to draw the more specific conclusion that can be inferred from a *comparison* of instances.

We proceed to give an account of each of the four figures in turn.

§ 9. *Figure of Difference*

Given the supreme premiss: P depends only upon $ABCDE$: we shall suppose instantial premisses in which variations occur in the determining factor D, which is assumed to be simplex.

Then a single instance of *abcde* that is p is universalised into 'Every instance of *abcde* is p.'

Again a single instance of *abcd'e* that is p' is universalised into 'Every instance of *abcd'e* is p'.'

Comparing these two instances of *abce*, we note that a variation from d to d' entails a variation from p to p'.

From this we infer that the value of D is actually operative in determining the value of P. Hence any further variation of D—say from d to d''—will entail a further variation of P—say from p to p''. I.e. any value of D other than d or d' will yield a value of P other than p or p'.

Represented symbolically, the conclusion reached is that

'Every instance of $abcd''e$ will be p'','

where this universal is interpreted to signify that, within the range *abce*, any given difference in D will entail *some* difference in P, without however indicating what determinate value of P will be yielded by the given determinate value of D.

We may symbolise the form of inference which has just been explained in the following scheme:

Figure of Difference

Supreme Premiss: P depends only on A, B, C, D, E where D is simplex.

Instantial Premisses	Immediate Conclusions
1. A certain *abcde* is p.	∴ 1. Every *abcde* is p.
2. A certain *abcd'e* is p'.	∴ 2. Every *abcd'e* is p'.

Final Conclusion: ∴ Every $abcd''e$ is p''.

§ 10. *Figure of Agreement*

Given the supreme premiss: P depends only upon $ABCDE$: we shall suppose instantial premisses in which variations occur in the determining factor A, which is assumed to be simplex.

Then a single instance of *abcde* that is *p* is universalised into 'Every instance of *abcde* is *p*.'

Again a single instance of *a'bcde* that is *p* is universalised into 'Every instance of *a'bcde* is *p*.'

Comparing these two instances of *bcde*, we note that a variation from *a* to *a'* entails no variation in *P*.

From this we infer that the value of *A* is not actually operative in determining the value of *P*. Hence any further variation of *A*—say from *a* to *a''*—will entail no variation in *P*; i.e. any value of *A* will yield the same value *p* of *P*.

Represented symbolically, the conclusion reached is that:

'Every instance of *Abcde* will yield *p*,'

where this universal is interpreted to signify that within the range *bcde*, whatever value *A* may have, the value *p* will remain unaffected.

We may symbolise the form of inference which has just been explained in the following scheme:

Figure of Agreement

Supreme Premiss: *P* depends only on *A*, *B*, *C*, *D*, *E*, where *A* is simplex.

Instantial Premisses	Immediate Conclusions
1. A certain *abcde* is *p*.	∴ 1. Every *abcde* is *p*.
2. A certain *a'bcde* is *p*.	∴ 2. Every *a'bcde* is *p*.

Final Conclusion: ∴ Every *a''bcde* is *p*.

§ 11. *Figure of Composition*

Given the supreme premiss: *P* depends only upon *ABCDE*: we shall suppose instantial premisses in which variations occur in the determining factor *C*, which is assumed to be simplex.

Then a single instance of *abcde* that is p is universalised into 'Every instance of *abcde* is p.'

Again a single instance of *abc'de* that is p' is universalised into 'Every instance of *abc'de* is p'.'

Comparing these two instances of *abde* we could infer, as in the Figure of Difference, that a further variation of C would entail a variation in p. But we have to contemplate a third instance where c'' yields the same value p that was presented in the first instance. If the values *abe* are known to be the same as in this first instance, then a difference in the remaining factor d must have accounted for the recurrence of the same determined value p. Thus the first and third instances of *abe* determining p must have been due to the compounding of c with d in the first case, and to the compounding of c'' with d'' in the third case. Such a case arises when the factor D in the third instance has not been amenable to precise evaluation.

Represented symbolically the conclusion reached is that:

'Any instance of *abc''pe* will be d'','

where d'' is some unevaluated value of D other than d or d'.

Symbolically:

Figure of Composition

Supreme Premiss: P depends only on A, B, C, D, E, where C is simplex.

Instantial Premisses	Immediate Conclusions
1. A certain *abcde* is p.	∴ 1. Every *abcde* is p.
2. A certain *abc'de* is p'.	∴ 2. Every *abc'de* is p'.

Final Conclusion: ∴ Every *abc''pe* is d''.

§ 12. *Figure of Resolution*

Given the supreme premiss: P depends only upon $ABCDE$: we shall suppose instantial premisses in which variations occur in the determining factor E, which is not here assumed to be simplex.

Then the three single instances of

$$abcde \sim p, \quad abcde' \sim p', \quad abcde'' \sim p,$$

may be respectively universalised into

Every *abcde* is p, Every *abcde'* is p', Every *abcde''* is p.

Comparing the first and third of these instances, where under the range *abcd*, e and also e'' yield p, we conclude that E is complex, being resolvable say into the two independent factors X, Y; so that (say) $e=xy$, and $e''=x''y''$.

Represented symbolically, the conclusion reached is that

'Every *abcdxy* is p, and Every *abcdx''y''* is p,'

where xy and $x''y''$ represent the resolution of e and e'' to account for the same value p of P. Thus:

Figure of Resolution

Supreme Premiss: P depends only upon A, B, C, D, E.

Instantial Premisses	Immediate Conclusions
1. A certain *abcde* is p.	1. Every *abcde* is p.
2. A certain *abcde'* is p'.	2. Every *abcde'* is p'.
3. A certain *abcde''* is p.	3. Every *abcde''* is p.

Final Conclusion: E is resolvable into XY, where $e=xy$, and $e''=x''y''$.

§ 13. It will be seen that each of these figures of inductive implication is formally equivalent to a single

disjunction of four propositions. This fourfold disjunction may be called:

The Antilogism of Demonstrative Induction

Given three instances of the same type exhibiting three different values of a given determining character, then no case can arise in which:

(1) the *given* determining character is simplex;
(2) the values of *the other* determining characters agree throughout the three instances;
(3) the value of the determined character *differs* in two of the three instances;
(4) the value of the determined character *agrees* in two of the three instances.

Symbolically expressed, we cannot have

B simplex; and

an instance of $a\ b\ c\ d\ e$ that is p,
,, ,, ,, $a\ b'\ c\ d\ e$ that is p',
,, ,, ,, $a\ b''\ c\ d\ e$ that is p.

Expressing this fourfold disjunction in terms of its four equivalent implications, we can formulate the Figures of Demonstrative Induction thus:

(not-4) *Figure of Difference*: If 1 and 2 and 3; then not-4
(not-3) *Figure of Agreement*: If 1 and 2 and 4; then not-3
(not-2) *Figure of Composition*: If 1 and 3 and 4; then not-2
(not-1) *Figure of Resolution*: If 2 and 3 and 4; then not-1.

In symbols this becomes:

Figure of Difference.

If B is simplex, and we have

an instance of $a\ b\ c\ d\ e$ that is p,
and an instance of $a\ b'\ c\ d\ e$ that is p',
then every instance of $a\ b''\ c\ d\ e$ will be p''.

Figure of Agreement.

If B is simplex, and we have

an instance of $a\ b\ c\ d\ e$ that is p,
and an instance of $a\ b'\ c\ d\ e$ that is p,
then every instance of $a\ b''c\ d\ e$ will be p.

Figure of Composition.

If B is simplex, and we have

an instance of $a\ b\ c\ d\ e$ that is p,
and an instance of $a\ b'\ c\ d\ e$ that is p',
then any instance of $a\ b''c\ p\ e$ must be d''.

Figure of Resolution.

If we have an instance of $a\ b\ c\ d\ e$ that is p,
and an instance of $a\ b'\ c\ d\ e$ that is p',
and an instance of $a\ b''c\ d\ e$ that is p,
then B is complex.

§ 14. A simple illustration of the Figure of Difference is afforded by Guy-Lussac's law which connects variations in the pressure p, temperature t, and volume v, of a specific gas g. Suppose that in two instances without changing g and p, a change of temperature from t to t' is found to entail a change of volume from v to v'. From this it can be inferred under the Figure of Difference that, with the same gas at the same pressure, any further change of temperature, say from t to t'', would entail a further change of volume, say v to v''. This experiment does not prove that for any other gas or for any other pressure, a change of temperature would entail a change of volume; nor does it indicate what determinate value of the volume would be entailed by any supposed further change of temperature. It should be observed that the conditions required

for the Method of Difference—namely precise constancy in all but one of the determining factors—is much more easily realisable when dealing with the same body or substance and varying its alterable states than when we pass from one to another body or substance in one of which a character is present and in the other absent. Hence the conditions most favourable for the application of the Figure of Difference are those in which concomitant variations in the determining and determined factors are observed. For Mill, on the other hand, the so-called Method of Concomitant Variations was primarily distinguished from the Method of Difference in that the latter was concerned with presence and absence, and the former with variations in degree. He speaks of this method as the one necessarily required when we cannot wholly get rid of a phenomenon, and are obliged to be satisfied with noting the varying degrees with which it is manifested from instance to instance; as if this method were a sort of makeshift which had to be put up with when recourse to the Method of Difference was impossible. But it is precisely in those cases in which we can vary the degree of a phenomenon, and not in those that can be described as presence and absence, that we can be assured that the rigid conditions required by the Method of Difference are fulfilled. Mill in adopting this position neglected the consideration of the homogeneity in any collection of instances brought together for comparison under any method of induction whatever. In the conception of concomitant variations is included—not only quantitative variations or variations of degree but also qualitative variations under any

given determinable such as colour or sound. To illustrate Concomitant Variations, Mill chose the method employed in connecting the varying heights of the tides with the variations of the position of the sun and moon relatively to the earth; but he presented the matter as if the difference in the cogency of this method from that of Difference was due to the distinction between presence and absence in the latter and variations of degree in the former; whereas it is obvious that the real deficiency in this application of the Method of Concomitant Variations was due to the special nature of the case, which made it impossible to secure, in the different instances examined, exact agreement in regard to the circumstances not known to be irrelevant: e.g. the variations of height of the tides might have depended upon variations in the force or direction of the wind, or in the shape of the coast, etc. So far then from regarding the Method of Concomitant Variations as an inferior substitute for that of Difference, if by the former is meant variation in the alterable states or relations of some one body or substance, and by difference is meant comparison of two similar bodies in one of which some quality is present and in the other absent, we must regard the former method as superior to the latter. For example: if we attempt empirically to establish a causal connection between the prosperity or the reverse of a country and its adoption of free trade or protection, it would be impossible to find two different countries which agreed in all relevant respects with the exception of this difference in industrial policy; and hence a change in which one policy was replaced by the other within one and the same country

would afford incomparably more cogent evidence of causation than a comparison of the effects in two different countries which must necessarily differ in very many respects that could not be assumed to be irrelevant.

§ 15. To illustrate the Figure of Agreement we may take instances used to establish the law that the rate at which a body falls *in vacuo* to the earth is independent of its weight. In these instances we keep unchanged all the possibly relevant circumstances, such as distance from the earth, absence of air, substance and shape of the falling body, and vary only the weight. From two instances in which the weight alone differs, we find that the time occupied in falling through any given distance is unchanged. In this way we use the Figure of Agreement which might also be called the Figure of Indifference, since it picks out a determining condition which is naturally expected actually to modify the effect in question, and yet is shown by a comparison of instances to be *indifferent* as regards the determinate value of the effect. An illustration of this kind seems not to have occurred to Mill, because in his Method of Agreement every circumstance except one *differs* in the several instances; whereas, in my formulation of the corresponding figure, every circumstance except one *agrees* in the several instances. In other words, as regards the determining factors my Figures of Difference and of Agreement require the same condition, namely a single difference; whereas Mill contrasts the two by defining the Method of Difference as involving a single difference and the Method of Agreement as involving a single agreement. In fact Mill attempts

the elimination *en bloc* of all the varying circumstances which distinguish the different instances in which the same effect-value is observed, whereas what is required in order to give corresponding form to the two methods is that we should eliminate as indifferent or irrelevant only one circumstance at a time.

§ 16. Having illustrated the figures of Agreement and Difference, I will explain the strict procedure of using these figures in dealing with a *number* of cause factors and of effect factors conjoined in a set of examined instances. Taking as our original major premiss: $ABCDE \sim PQRT$, i.e. the conjunction of the cause factors A, B, C, D, E determines the conjunction of the effect factors P, Q, R, T: it is to be remembered that no cause-factors other than those enumerated are determinative of the enumerated effect-factors, as also that no effect-factors other than those enumerated are dependent upon the enumerated cause-factors. We then take in turn one cause-factor and another and find instances from which we may conclude, in regard to a given effect-factor either, in accordance with the Figure of Difference, that the factor that is varied *is* actually operative, or in accordance with the Figure of Agreement, that such factor is *not* actually operative: and this procedure is repeated for each of the effect-characters in turn. Each pair of instances compared in this way will lead to a universal conclusion under the Figure of Difference or of Agreement as the case may be. This *Complete* method (as it may be called) is by no means identical with Mill's Joint Method of Agreement and Difference, the use of which he advocates only to compensate for the failure to

secure variation in a *single* factor; in this Complete method, on the other hand, one cause-factor alone is varied in each pair of compared instances.

This process symbolically expressed serves as an exercise in the application of the principles underlying demonstrative induction. For example, take the following instances:

(1) $abcde \sim pqrt$; (2) $a'bcde \sim pq'r't$;
(3) $ab'cde \sim p'qr''t$; (4) $a'bc'de \sim p''q''r't$;
(5) $abcd'e \sim p'q'r't$.

From (1) and (2), we eliminate a and a' as irrelevant to p and t; and infer $Abcde \sim pt$. From (1) and (3), we eliminate b and b' as irrelevant to q and t; and infer $aBcde \sim qt$. From (2) and (4), we eliminate c and c' as irrelevant to r and t; and infer $a'bCde \sim r't$. On the other hand, from the comparison of (1) and (5) we infer that d and d' cannot be eliminated as ineffective as regards either p, p' or q, q' or r, r'. Hence, under the Figure of Difference, we infer $abcd''e \sim p''q''r''$. Since, however, in these two instances, the variation of D is inoperative on T, we also infer, under the Figure of Agreement, $abcDe \sim t$. We may now combine the conclusions $Abcde \sim pt$ and $abcd''e \sim p''q''r''$, and thus infer $Abcd''e \sim p''$. This conclusion expresses the fact that, under unchanged conditions bce, A is inoperative while D is operative upon P. It should be observed that the conclusion $abcd''e \sim p''q''r''$ is contrary to the inference drawn by Mill in his Method of Difference; for, according to his formulation of the Method, a difference in a single cause-factor entails a difference in a *single* effect-factor. Other inferences—such as

from (1) and (2) that $a''bcde \sim q''r''$—may be left to the ingenuity of the reader to discover.

§ 17. Before illustrating the two remaining figures, it is desirable to explain how the symbols employed in my notation are to be practically applied. When the characters of two or more cause-factors are represented by such symbols as a, b, c, ... two typical cases may arise; (1) where $a, b, c, \ldots$ represent determinates under different determinables A, B, $C\ldots$; (2) where two or more of them are determinates under the same determinable. In the latter case, supposing the symbol B to represent the same determinable character as A, the three factors a, b, c would more naturally be symbolised by a, a, c. Here the *recurrence* of the symbol a indicates that there are two factors conjoined which are *existentially* different the one from the other, although characterisable under the same adjectival determinable. In order to symbolise the two-fold manifestation of cause-factors characterised under the same determinable, we might use the subscripts 1, 2, to represent *existential* plurality; and thus, instead of writing ab, ab', $a'b$, $a'b'$, etc., we should write a_1a_2, a_1a_2', $a_1'a_2$, $a_1'a_2'$, etc. For example, the adjectival determinable *Force* may be represented by F, and, when two forces enter together as cause-factors in producing a certain effect, the possible variations in which they may be conjoined may be represented by $f_1f_2, f_1f_2', f_1'f_2, f_1'f_2'$, etc. Or again: taking the character of a chemical element to be indicated say by its atomic weight, we may use A to represent this adjectival determinable; and the possible variations in which two elements are conjoined in producing a compound may be represented by a_1a_2, a_1a_2', $a_1'a_2$, $a_1'a_2'$, etc.

Since, however, amongst symbolists, any difference of symbol such as *a* and *b*, is never understood to prohibit identity of meaning, while of course *an identity of symbol is always understood to prohibit a difference of meaning*, the notation that I have adopted in my schematisation of the Figures may still be retained without danger of confusion; and, in any case, it serves to represent the *general* principles of the Figures, although in specific cases the special notation indicated above may be preferred.

§ 18. As regards the two remaining figures of Composition and Resolution, we must point out their differences from the figures of Agreement and Difference, and explain what is meant by the composition of causes as contrasted with the combination of causes. Although these figures have been exhibited in a form according to which their demonstrative cogency is equivalent to that of Difference or of Agreement, they palpably differ from these latter in two respects. In the first place, the predicate of the universal conclusion drawn in the last two figures concerns one of the *determining* factors *D* or *E*, while that in the first two figures concerns the *determined* factor *P*. In the second place, the last two figures introduce the notion of 'composition' and its converse 'resolution'—these terms being used in a special and technical sense which requires explanation.

§ 19. The notion of composition has been long understood in mathematical physics, where the resultant of two directed forces regarded as components is represented by the diagonal of the parallelogram whose sides represent these components. The principle by

which the mechanical effect of two conjoined forces can thus be calculated, was contrasted by logicians and philosophers with the principle underlying *chemical* formulae, in which the properties of a compound substance could not be calculated in terms of those of the elements combined in the compound. This led to the view that there was a fundamental antithesis between mechanism and chemism, the former of which involved a 'composition,' the latter a 'combination' of cause-factors. Mill introduced and explained the phrase 'composition of causes' (i.e. of cause-factors) and contrasted this with the combination of cause-factors, specially characteristic of chemical phenomena, and also, in his opinion, of many psychological and sociological phenomena. Mill's explanation is not altogether satisfactory. I will therefore attempt my own explanation of the antithesis between composition and combination.

When two cause-factors represented, say, by the determinables B and C are such that there are certain pairs of values, say bc and $b'c'$, which jointly determine the *same* value p of an effect character P; then, referentially to P, the conjunction BC constitutes a *composition*. On the other hand, when there are no pairs of values under the determinables B and C, such as bc and $b'c'$, which jointly determine the *same* value of P; then, referentially to P, the conjunction BC constitutes a *combination*. What is important to note here is that the distinction between composition and combination is not absolute; for certain conjunctions of cause-factors may constitute a composition referentially to one assigned effect-character, and a combination re-

ferentially to another. For example, when chemical elements are conjoined in producing a compound substance, it is possible to take the weights of certain elements and different weights of other elements so as to produce a compound of the *same* weight; hence referentially to the effect *weight*, the conjunction of chemical elements comes under the principle of composition. But as regards the chemical *character* of the elements conjoined, it is impossible so to vary these as to produce a compound of the same chemical character in two different cases; for instance, the substance having the chemical properties of water can only be produced by the combination of hydrogen and oxygen.

This account of the distinction between composition and combination is to be regarded as an indication rather than as a definition. Expressed mathematically: the conjunction of the factors B and C constitute a composition, referentially to the effect P, when there is a certain function f such that p equals $f(b, c)$ for any and every value b and c of B and C. We might therefore replace the terms composition and combination respectively by the more suggestive terms *functional* and *non-functional* conjunction. The method of discovering and establishing such functional relations will be treated in the next chapter. But we cannot well illustrate the figures of Composition and Resolution without first modifying their formulation in view of the above explanation of the nature of composition.

§ 20. In the figure of Composition as symbolically formulated, we took two instances agreeing as regards the determining factors *abde*, and a third instance agreeing with both as regards *abe*, but in which the

factor D was unamenable to precise calculation. We then supposed that, while in the first two instances the differences c and c' in the determining factor C yielded a corresponding difference p and p' in the determined factor P; yet, in a third instance, c'' yielded the unexpected effect p equivalent to that yielded in the first. The unexpectedness of this result was thus accounted for either by our inability, in the third instance, to measure the factor D, or by our error in supposing that its value was still unchanged. Now, instead of illustrating our figure by supposing equivalence as regards P in the first and third instance—a somewhat artificial assumption—let us suppose rather that in the third instance the effect, say p_3, was other than that calculated by a foreknown formula in which the value of P would be given by $p''=f(a, b, c'', e)$. On the assumption that the correctness of this formula had been properly assured by means of the functional extension of the Figure of Difference or of Concomitant Variations, we should rightly infer that any instance of $abc''p_3e$ would entail d'' in place of d, so that the effect p_3, under the constant conditions abe, would be due to the *composition* $c''d''$, and not merely to c''.

In this modified form, the Figure of Composition can be illustrated by the irregular motions from p to p_3 of the planet Uranus, the positions a, b, e, of any other planets being effectively unaltered while that of the sun had changed from c to c''. The motion from d to d'' of an unknown planet, afterwards called Neptune, conjoined with that of the sun from c to c'' accounted for the unexpected movement of Uranus from p to p_3; in other words, a, b, e being constant, p_3 was the same

function of c'' and d'' as p was of c and d; so that P was a function, not of C alone, but of C and D compounded.

A similar illustration of the Figure of Resolution is found in the experiments by which the new chemical substance *argon* was discovered by Sir William Ramsay. Here the factor E would represent atmospheric nitrogen, and its greater weight—as compared with that of nitrogen prepared from chemical compounds—was accounted for by the *resolution* of the atmospheric nitrogen into the two components argon and pure nitrogen. It should be pointed out that the resolution here employed was not a chemical analysis, for argon does not combine with any other element (as far as is at present known) and therefore the resolution in question was a true instance of the converse of composition.

In regard to the illustration of *Composition* involving the discovery of Neptune, and that of *Resolution* involving the discovery of argon, the precise measurements finally made reduced the inference to a purely deductive form, which assumed the character of the method of Residues according to my interpretation of this method (see p. 118).

CHAPTER XI

THE FUNCTIONAL EXTENSION OF DEMONSTRATIVE INDUCTION

§ 1. In concluding the treatment of demonstrative inference I propose to recapitulate the results that have been so far reached, and to bring into focus the distinctions and connections between the several forms of inference, deductive, inductive and problematic. I have already examined the general notion of function, and shown how it is employed in mathematical and other processes of deductive inference; and it remains to exhibit this notion as it enters into inductive inference—this constituting the specifically new topic to be discussed in the present chapter.

Pure induction, by which is to be understood that which involves no assumption of universal laws, has been shown to be the sole direct and ultimate mode of generalising from instances examined and theoretically enumerable. This species of induction I have called problematic because, in my view, the universal propositions which it establishes must be regarded, not as absolutely certified, but as accepted only with a higher or lower degree of probability depending upon the collective character of the instances enumerated. The possibility of establishing such direct generalisations depends upon certain postulates, the discussion of which raises one of the most important and difficult problems of philosophical logic; and even then, the probability

to be attached to generalisations thus established has to be determined by reference to the formal principles of probability. But, so far as these generalisations enter into the account of demonstration, they function as major premisses. Demonstrative induction, then, so far resembles deduction in that it requires the conjunction of two types of premisses: (1) the major or supreme universal premiss, which expresses the relation of dependence between one specified set of variables and another; and (2) the minor or instantial premiss which sums up the results of single observations or experiments. The major premiss in this mixed form of demonstration is formulated, not as a uniformity pervading all nature, but as a specified universal holding only for the special class of phenomena to which the conclusion refers.

§ 2. A very general statement of the contrast between my exposition and Mill's is conveniently introduced at this point. I have deliberately separated the treatment of formal or demonstrative induction from that of problematic induction. In the latter, the accumulation of instances is all important; in the former, a precise major premiss, relating to a finite and enumerable set of determinables, is required in each step of the formal process. These major premisses are assumed to have been previously established, with a higher or lower degree of probability, on the principles of problematic induction. The essence of problematic as contrasted with formal induction is expressed in three statements: first; no wide generalisation, such as that which asserts the uniformity of nature, is involved; secondly; the instances compared are not

determinately analysed with respect to the variable characters upon which the proposed generalisation may depend; and hence, thirdly, an indefinite multiplication of instances is required in order to give any appreciable value to the probability of the conclusion. It is partly for this reason that Mill's account of the Method of Agreement differs so considerably from my extremely simple Figure of Agreement; for Mill is largely thinking, under the title Agreement, of a direct method of establishing empirical generalisations to which only an inferior degree of probability can be attached. The generalisations thus established by problematic induction function as major premisses in demonstrative processes in one of two ways: either as established with what may be called *experiential* as opposed to *rational* certitude; or as put forward hypothetically, and thus as exhibiting forms of *implication* rather than of inference—implication being defined, as in Chapter I, to be potential or hypothetical inference.

§ 3. The term hypothesis has been used by logicians in so very many senses that, in order to obviate logical confusion, it will be well to examine its various usages, showing how they have developed from one fundamental element. This element will be found to be definitely epistemic rather than constitutive, and for my own purposes I consequently prefer to use the phrase 'hypothetically entertained,' which has an epistemic significance quite independent of the form or content of the proposition so entertained. We may take in turn the various meanings of the substantive 'hypothesis' or the adjective 'hypothetical' that occur

in deductive or inductive logic, in order partly to connect and partly to contrast its epistemic with its other bearings. In traditional formal logic, propositions are called hypothetical which are in fact compounded out of two categorical propositions, say p and q. In this case, while the adjective *hypothetical* is traditionally used to denote a particular species of compound proposition, namely that of the form 'if p then q'; yet at the same time the term hypothesis clings firstly to the proposition p because in this form it is not actually asserted, and next to the proposition q because it is only assertible on condition that p has been asserted. Thus the adjective *hypothetical* is actually attached to three quite distinct propositions or forms of proposition: the compound 'if p then q'; the simple proposition p itself, which I call the implicans; and the simple proposition q which I call the implicate. Now in order to make a first approximation to justifying this confused terminology, we must consider its epistemic aspect, and we may say that normally both the implicans separately and the implicate separately are entertained hypothetically, while the compound proposition 'if p then q' is entertained assertorically. Hence, even where the term hypothetical is used in its most precise technical sense, it is applied to a form of proposition assumed to be entertained assertorically, the components alone of this assertoric compound being entertained hypothetically.

The recognition of this ambiguity in the use of the term hypothetical resolves the often disputed problem of the relation in general between induction and deduction. When we are concerned with the purely formal

relation of implication as subsisting between the premisses and conclusion of any argument of the general nature of a syllogism, then these premisses need only be entertained hypothetically; while, at the same time, the relation of implication itself is to be conceived, not only as assertorically advanced, but even as having the highest degree or kind of assertoric certitude. The conclusion of a syllogism thus deduced is usually spoken of as demonstrated, i.e. as having demonstrative certitude; although, taken by itself, any kind or degree of certainty attaching to it is wholly dependent upon the kind or degree of certitude with which the premisses are entertained. Taking full advantage, then, of Mill's account of the functions and value of the syllogism, we may say that the hypothetical conclusion has been hypothetically demonstrated, and can only be assertorically demonstrated when we have examined and tested the truth of the premisses. Only when the major premiss has been inductively established can the conclusion be entertained categorically, and even then with a degree of probability dependent upon that of the major premiss; and ultimately upon the mode of induction by which the major has been established.

§ 4. The problematic nature of the universal obtained by induction and functioning as major premiss in a deductive process has led to a confusion between the notions *problematic* and *hypothetical*, resulting in the use of the term 'hypothesis' for any proposition entertained with a degree of probability. Thus, when Jevons says that all induction is hypothetical, what he means is merely that an inductive conclusion has not certainty but probability. Thus any inductive generali-

sation is commonly called a hypothesis; and the term when applied to a scientific theory may have three alternative meanings: first, it may mean that the proposition is unproven; secondly, that the proposition has an appreciable degree of probability which renders it worth considering; thirdly, that the proposition has no appreciable probability at all, and may even be known to be false. Besides the epistemic significance revealed in all these three alternative meanings, the term hypothesis must also be understood to indicate the purpose which an unproven universal, definitely formulated, fulfils in calculating deductively the conclusions to which it would lead. In fact Jevons, in describing induction as hypothetical, uses the term in two quite different senses: first, in the formal sense, to indicate the provisional or tentative attitude towards a universal *before* we have confirmed it by a process involving deduction; and, secondly, to represent the final attitude towards a universal *after* it has been tested and confirmed with the highest attainable degree of probability. With the view indicated in the second application of the term *hypothesis*, I agree; but, as regards the first use of the term, it seems to me that we always adopt a tentative attitude towards a proposition entertained as a proposal, whether it is to be proved deductively or inductively; so that the term as applied to a proposition to be proved does not represent any characteristic peculiar to induction. Now the special topic with which this chapter is concerned involves both the contrasted ideas of hypothesis: namely, of a proposition having a certain degree of probability, and of one put forward to be tested by appropriate evi-

dence. Thus, while the functional formula in deduction is assumed to be true and therefore may serve as premiss for deducing an equally assured conclusion, the inductive aspect of such a functional formula presents the *inverse* problem; for we have now to examine by what kind of instances, and by what modes of comparison, the functional formula itself can be established. So far as this process of examination may be said to have a special characteristic by which it may be distinguished from problematic induction used for establishing the wide generalisations of science, its peculiarity is that a comparatively small number of instances will constitute the sufficient factual basis for the establishment of the formula, and that the actual procedure of mathematical physics, at least in the majority of cases, rightly attaches practical certitude to the formula thus inferred.

§ 5. In order to show how the functional formula is established, I must refer to my account of the figures of Demonstrative Induction. There the conclusion demonstratively drawn does not assign the specific value of the effect-character that is to be correlated with any given value of a cause-character. In popular language, the conclusions drawn would be termed qualitative not quantitative; that is to say, the figures establish causal *connection* without determination of a causal *law* or formula. In comparing the different figures, it is seen that the Figure of Difference, which stands first, is a direct expression of the principle of the dependence of change in the effect upon change in the cause; and that the Figure of Agreement or of Indifference is complementary to that of Difference in the

same sense as the universal or implicative 'if not-p then not-q' is the complementary of 'if p then q'; while the Figures of Composition and Resolution merely carry out the principle of Difference under certain more complicated circumstances. There is, therefore, one principle common to all the four figures, namely that underlying the Figure of Difference—the functional extension of which will be our principal concern.

The original formula of Difference may be restated in the following canon: When in two instances a difference in the cause-character D entails a difference in the effect-character P, all other cause-characters which might contribute to the determination of P being the same in the two instances, then we infer that any other difference in the cause-character will be correlated with some other difference in the effect-character, under the continued constancy of the remaining cause-characters. Now this canon, which applies to two instances only, may be obviously extended to any number of instances all of which conform to the figure of Difference: i.e. all other cause-factors remaining unchanged, we find a series of instances in which D alone varies, and in which the determinate values d, d', d'', d''', etc., say, are associated respectively with p, p', p'', p''', etc. Now, as in the simple case of two instances, these observations do not enable us to assign the specific value of P that is to be correlated with any given value of D: we can still only infer that any *further* change in D will be associated with *some* further change in P. The required extension of the figure of Difference consists, therefore, in the determination of P as a function of D which shall hold for all unexamined as well as examined

instances. A famous example of the determination of such a function is that formulated by Kepler who, after nineteen guesses, discovered a formula for the planetary movements about the sun which co-ordinated the spatio-temporal relations for the cases—necessarily finite in number—that he was able to examine and measure. The discovery of this formula involved nothing of the nature of inductive inference, but its application to all the planetary positions *intervening between those observed* constituted a genuine inductive inference, so easy to draw that neither Whewell nor Mill seems to have been aware that any such inference was implicitly made.

The canon for the Figure of Composition may be reformulated as follows: When in several instances variations in the single cause-character C have entailed variations in the effect-character P such that, in accordance with the functional extension of the Figure of Difference, P has been shown to be a certain function of C, then, if some similar instance of a further variation of C has entailed a variation of P *not satisfying this function*, we infer that, in this instance, besides C some other character, say D, has varied, and hence that P depends upon the composition of C with D. This simple use of the Figure of Composition does not, however, enable us to determine the value of D in the particular instance observed. In expanding this figure therefore we have to look for further instances in which both C and D can be evaluated; and thus construct a formula by which P is represented as a function both of C and of D. This method should be compared with that of Residues, which I have regarded

as purely deductive; for, in the method of Residues, the values of D are determined *deductively* from the known formula $p=f(c, d)$, whereas, in our extension of the Figure of Composition, the formula $p=f(c, d)$ is determined *inductively* from the observed values of D. The case of the irregularities in the movements of Uranus, instanced in the previous chapter, illustrates this type of functional extension.

§ 6. Now the formula which expresses an effect as a function of one or more cause-factors must at least satisfy the negative condition that it fits all the examined instances as regards the observed values of cause and effect. Many logicians, and certainly many experimenters in practical branches of science, are finally satisfied with this negative criterion. They assert, in effect, that provided the formula $p=f(d)$, where f has some specific form, agrees with the values of P and D as measured in the examined cases, then it has all the guarantee that experimentation requires for its universalisation. But the mathematician points out that, theoretically speaking, there are an infinity of different functions that would exactly fit any finite number of cases of covariation. Hence he demands in general a much more rigid defence for selecting one formula rather than another to represent the universal law.

In order to escape this threatening annihilation of inductive inference, we may indicate two fundamental principles upon which the highest attainable degree of certainty, which may be called practical or experiential certitude, depends. In the first place, reliance is placed upon the character of the formula itself, and in particular on its comparative simplicity; in the second

place, the higher credibility of a proposed formula depends upon its analogies with other sufficiently well-established formulae in similar classes of phenomena. Briefly, the criteria of simplicity and analogy, especially when conjoined, confer upon a formula of covariation that highest degree of probability which allows us to regard the induction, not as merely problematic, but as virtually demonstrative. For example, the experiments that have been conducted in regard to the covariations of temperature, pressure and volume of gases have always been treated by physicists as conferring absolute demonstrative certitude upon the formulae inferred, although they have been actually confirmed from a necessarily limited number of observations.

We may illustrate the notion of simplicity by taking the simplest of all possible functions, namely where p is proportional to d, or its inverse $\frac{1}{d}$. For example, if we have instances in which, weight being the determined factor, and some quantitatively measurable cause D varies so that where we double D we double P, and where we treble D we treble P, and so on for fractional as well as integral multipliers, we inductively infer that P, not merely varies *with* D, but in mathematical language, varies *as* D. There have been philosophers who, in effect, have imagined that, unless a causal formula can be expressed by a proportionate relation of cause to effect, it must be regarded as a mere empirical rule; and conversely, as soon as instances are found to fit some such simple formula, the generalisation may be regarded as absolutely certified. A slightly less simple kind of formula is exemplified by gravitation

where, for a given attracting mass, the acceleration of the attracted body varies inversely as the square of the distance, being in the direction towards the attracting body. The high probability of this formula is due, not only to its relative simplicity, but to its analogy with the independently known formula for the intensity of radiant light or heat. Moreover the formula in question could have been deduced from the assumption that radiation operates equally in all spatial directions, so that its magnitude upon any part of a spherical surface is inversely proportional to the area of that surface and therefore to the square of the distance. In the examples thus brought forward, indications are given of the kind of reasoning upon which the high probability attached to any formula that fits the examined instances is based.

§ 7. The criterion of simplicity is not often directly applicable; but, when in a relatively complex conjunction of circumstances that can be analysed, a formula is constructed that could have been deduced from a combination of wider and well-established formulae of comparative simplicity, then an empirical formula thus confirmed acquires problematic value corresponding to that of the laws from the combination of which it could have been deduced. Both Whewell and Mill have taken this kind of criterion as fundamental in their theories of induction; Whewell using the phrase 'consilience of inductions,' and Mill having in his earlier chapters put forward this deductive confirmation as the one principle dominating his whole theory. At first sight Mill's position is paradoxical, since he apparently attributes a higher probability-value to a law, merely

on the ground of its width, whereas it would appear that the narrower generalisation is the safer. I think, on this matter, we must recognise the value of the two opposed principles that have been put forward. On the one hand, mere simplicity has been elevated into a supreme criterion; but, so far from admitting that simplicity alone guarantees a formula, we must maintain that where a known complexity of circumstances is involved, a corresponding complexity must be expected to characterise their co-ordinating formula. Hence, when a class of phenomena that have not been definitively analysed resembles other classes for which a complex formula has been established, a corresponding complexity should be anticipated for the given class; whereas the formula for a class of phenomena analogous to others for which a simple formula holds may rightly be expected to be simple. The criterion of simplicity, when including its indirect as well as its direct form, is of value; but it is only when analogy is thus conjoined with simplicity that we may attach practical certitude to a formula which satisfies at least the negative criterion of fitting perhaps only a small number of well-examined cases.

§ 8. The theory of what I have called the functional extension of demonstrative induction constitutes a link between the Demonstrative and the Problematic forms of inference. For certain rules (of a strictly formal character) are required for deducing, amongst all the functions which fit the observed co-variations, the *most probable* function of the variable cause-factors by which an effect-factor may be calculated. The oldest and most usual method of determining this function is known as

the *method of least squares*. Its validity depends upon a certain assumption with regard to the form of the Law of Error, i.e. of the function exhibited by divergences from a mean or average, when the number of co-variational instances is indefinitely increased; and a different method must be employed for each corresponding different assumption. The reader must be referred to Mr J. M. Keynes's *Treatise on Probability*, Chapter XVII, for a very comprehensive and original discussion of this topic.

The inductive inference examined in the above is thus shown to be based upon purely formal and demonstrative principles of probability, whereas the discussion of problematic induction to be developed in Part III will introduce informal theorems of probability, based on postulates of a highly controversial nature. It is therefore legitimate, and even necessary, to include the functional extension of the figures of induction under the general title of demonstrative inference.

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