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RELAXATION METHODS

Relaxation Methods

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PREFACE

It has been my aim in writing this book to explain as clearly as possible how to relax; its contents were entirely chosen with this end in view. The development of relaxation methods has advanced to a position where I believe such an attempt is justified and, according to the numerous enquiries which I have received in recent years, where it is also needed. The first application of the method was made in 1935 by Sir Richard Southwell, F.R.S., then Professor of Engineering Science in the University of Oxford; ever since then he has himself been energetically and almost continuously devoted to its growth as a powerful computational weapon for the solution of practical engineering and physical problems. It was my good fortune to join Sir Richard's research group in 1941 and to have remained in close contact with him ever since. I shall always remain grateful to him for the opportunities he has provided and for the encouragement and help he has given me during these years.

There are two points in particular which I wish to impress on any student of relaxation methods before he begins. First no one can hope to achieve success in their use without having spent some time at practice: just as the would-be pianist must practice his scales, so must the embryo relaxationist practice his art by working through a sufficiency of examples which may sometimes seem too easy, sometimes almost trivial, and perhaps at times extremely tedious. If he does this he will be repaid, and avoid later trouble for himself. The examples at the end of each chapter have been designed to give the student that practice; if they seem to be uninspiring drill problems, well, so they are. No relaxer can learn too early to take the rough with the smooth, the rough in this case being the patient and persevering application of simple arithmetical processes, and the smooth being the sense of final satisfaction and achievement when the answer is finally reached. I make no apologies whatsoever for providing my readers with some dirty work to do.

The second point that I want to make here is that concerning the inherent flexibility of the method when properly used. The student must realize that he is in no way the slave of the method but rather that he is its master, that his work will reflect his own personality. He must not allow himself to become a human computing machine. The relaxation

method is far from being a method carried out according to any set rules—instead it could more accurately be described as a method of breaking such rules as are necessary formally to explain it. The good relaxer is the man who knows, by experience, which rule to break, when to break it, and by how much to break it. It was, I believe, Prof. W. G. Bickley who provided relaxation with its motto, "Fortune favours the brave."

As I have already said, this book is intended to explain how to relax; it is not intended to be a reference book on the subject. I have therefore made no attempt to discuss the more advanced and difficult applications of the method; in particular no mention is made of the solution of non-linear equations. Nor have I thought it incumbent upon me ever to discuss the physical facts and assumptions or the mathematical theories which lead to the equations which I am taking as examples for solution. Most of these examples are taken from the fields of elasticity, hydrodynamics, or electricity and magnetism, and the accepted theories of these subjects which I have taken for granted are readily accessible in well-known treatises.* In no place, in fact, have I quoted any reference except where one is essential to the text itself. But I do include, as an appendix, a bibliography of papers and books which relate to relaxation. I would wish it to be comprehensive, and while I hope I have included in it every article of which I am aware, there must be some of which I am ignorant. I can only apologize in advance for such omissions and add that I shall be grateful to any reader who can draw my attention to them.

Lastly, in relation to the subject matter of this book, I must make some statement concerning the mathematical approach to relaxation. Its fundamentals extend into the theory of finite differences and overlap other computational methods. Another volume could be written dealing with such topics as convergence, finite-difference and truncation errors, parallel procedures used in relaxation and other computational methods, and so on. One day I may attempt to write such a philosophy of relaxation, for it has great intrinsic interest and importance. I have deliberately kept off these subjects as much as possible in the present book, for to deal with them inadequately would be useless, and to treat them adequately would here cloud my main issue, which is to show how relaxational solutions are obtained.

I owe a great deal to Mrs. E. G. Horoch and Miss M. J. Burns for the

*Love, A. E. H., "The Mathematical Theory of Elasticity," 3d ed., Cambridge University Press, 1920. Southwell, R. V., "The Theory of Elasticity," 2d ed., Oxford University Press, 1941. Lamb, H., "Hydrodynamics," 6th ed., Cambridge University Press, 1932. Jeans, J. H., "The Mathematical Theory of Electricity and Magnetism," 4th ed., Cambridge University Press, 1923.

typing of the original script, and to Miss M. J. Evenett, who worked out some of the illustrative examples and checked others. I am most grateful for their assistance. In particular my thanks are due to my friend and colleague, Dr. R. T. Severn, who has read through the entire manuscript and has also been of great assistance in checking proofs; he has made numerous suggestions, most of which I have been pleased to adopt, and has been a constant help throughout.

D. N. DE G. ALLEN

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CHAPTER 1

LINEAR ALGEBRAIC EQUATIONS

1. Introduction

The basic computational process which underlies all applications of the relaxation method can most easily be described in relation to the solution of a set of linear algebraic simultaneous equations; even the simplest of such equations will serve to illustrate both the fundamental idea of the method and also some of the modifications and devices which have been gradually added to the original concept. It cannot be too strongly emphasized that the method is founded on a single idea which forms an essential characteristic of all its applications, but that it owes its considerable power as a computational weapon to the many variations which are employed in its practice. These variations were evolved of necessity as the method was increasingly applied to problems more and more complex. The basic process provides one tool, *i.e.*, one arithmetical operation, which theoretically would be sufficient equipment to enable its user to arrive at the solution of any of the equations (algebraic or differential) which have yet been shown to yield to the relaxation technique; but, if only this single basic operation were available, the method would be virtually useless because the time and labor involved in its practice would be prohibitive. The variations and developments from the basic operation provide a set of tools whose judicious and appropriate use enables a large variety of equations to be solved with only a reasonable expenditure of time and effort on the part of the computer. The method is one which is intended to be of direct practical application for engineers and physicists, and it is therefore inherently required not merely to be able to produce results theoretically, but also to be able, practically, to produce them quickly and economically. This requirement will be stressed repeatedly as the method is described and extended; it will necessitate a somewhat unorthodox attitude on the part of the student toward his work, and he must learn to overcome a natural reluctance, in much of his arithmetic, to forsake temporary accuracy for ultimate speed.

The basic concept of relaxation and some of the more important and widely used modifications can be explained by employing them to solve a simple pair of algebraic equations:

$$\begin{aligned} -3x + y + 117 &= 0, \\ x - 2y + 51 &= 0. \end{aligned} \tag{1}$$

The solution to the problem contained in equations (1) is easily verified to be $x = 57$, $y = 54$; in this illustration, as in many others that will follow, the equations to be solved have been chosen to be as simple as possible. They have been selected not for any intrinsic interest in themselves or their solutions but only as a means of describing the various items of relaxational technique; it is hoped thereby initially to concentrate attention on the techniques involved by removing the attendant obscurity of lengthy arithmetic. Many of the preliminary examples, therefore, will be capable of solution by other well-known and more orthodox methods; these examples are used here only to serve the purpose of introducing, one at a time, the processes of the relaxation method. These processes will then immediately be seen to be capable of extended application to more difficult and complicated problems, for whose solution the older methods can, in practice, scarcely be contemplated. Thus, for example, the pair of equations (1) would normally be solved by elimination of one unknown, say y, in order to find x, followed by substitution back for x to obtain y.

If instead of two we had four equations in four unknowns, the method of elimination would still usually be preferable to the relaxation method. But when the number of equations becomes larger, say 10 or even 50, then the labor and effort required to use relaxation are frequently quite small compared with those needed in other methods.

2. Residuals

A set of algebraic equations to be solved should first be written with all their terms on one side and with only zero on the other side, as in (1). New quantities are then immediately introduced, called *residuals** (always denoted by the letter F), one of which stands identically for the nonzero side of each equation of the set:

$$\begin{aligned} F_x &\equiv -3x + y + 117, \\ F_y &\equiv x - 2y + 51. \end{aligned} \tag{2}$$

Now, for any values which might be assigned to the original unknowns x and y, corresponding values can be calculated for F_x and F_y directly from the defining relations (2). In using the relaxation method, a computer ceases to look upon the problem as the finding of those values of x and y which "satisfy" the equations (1) but regards it instead as the determination of a pair of values of x and y which happen to make the

* The origins of this and other nomenclature will be explained in Chapter 3.

residuals, as given by the relations (2), both zero. Any pair of values of x and y yield some values for F_x and F_y; attention is concentrated from the outset on these residual quantities with the aim of eventually making them become zero. When this is done, x and y will automatically, but almost incidentally, have been adjusted to acquire their wanted values. This predominance of the residuals in the mind of the computer, coupled with the virtually incidental role of the actual unknowns, is of the utmost importance in the development of the method.

3. Basic Unit Operations

The process of solution is started by the selection of an initial pair of values for x and y. Any choice or guess will serve as well as another, but usually (and unless there is some definite reason otherwise) we begin with all the unknowns taken to be zero. In this case, if x and y are put equal to zero, it follows immediately from the relations (2) that the corresponding initial values of the residuals are $F_x = 117$, $F_y = 51$. These residuals have to be changed to become zero, and the basic method of relaxation employs just one type of operation to effect this change in a number of computational steps. Each step of the calculation involves a single application of an operation of the basic kind and merely records in tabular form the operation used, the extent to which it is used, and the consequent effect on the residuals. One operation of this type corresponds to each of the unknowns in the problem and is called a *basic unit operation;* each such operation consists in the application of a unit positive increment to the value of the corresponding unknown, so that in our illustrative example the two basic unit operations available are* $\Delta x = 1$ and $\Delta y = 1$.

By reference to the relations (2) it is easy to see what changes in the residuals are produced by the use of each of these operations separately; whatever values x and y may have, whenever the operation $\Delta x = 1$ is applied, its effect is to change F_x by -3 and F_y by 1. Similarly the operation $\Delta y = 1$ results in a change in F_x of 1 and in F_y of -2. These unit operations can be used to any extent; thus if the second of them is applied n-fold, that is, $\Delta y = n$, then F_x is changed by n and F_y by $-2n$.

4. The Operations Table

N basic unit operations of this sort can clearly be written down when the problem involves the solution of N equations in N unknowns. It is convenient to exhibit them in an *operations table.* For the pair of equations (1) this table is recorded thus:

* The symbol Δ will always be used to denote an incremental addition.

	ΔF_x	ΔF_y
$\Delta x = 1$	-3	1
$\Delta y = 1$	1	-2

(3)

It will be noticed in this example, and in others instanced later in this chapter, that the numerical elements in the operations table are symmetrical about the leading diagonal. This is not a necessary feature of equations soluble by the relaxation method, but it is nevertheless convenient. Its assumption in the examples used does not really imply any lack of generality, because any set of equations which do not exhibit this property can always be transformed into an equivalent set which do, by the well-known algebraic process of normalization.* In such cases, however, it is usually better to avoid the labor involved in normalization, and instead to employ directly an unsymmetrical operations table.

It is clear that, in general, each operation listed in such an operations table will, when applied, effect some change to the values of all the residuals; the choice of which of the available basic unit operations should be used, at any stage in the process, is determined by the selection of one particular residual whose value it is most desired to change at that stage. If F_x is that residual, then the appropriate operation to use is $\Delta x = 1$; if F_y is that residual, then the appropriate operation is, correspondingly, $\Delta y = 1$, etc. The extent to which the operation is used depends on the extent to which it is desired to change the value of the corresponding residual and also, of course, on the magnitude of the appropriate coefficient in the operations table. Thus, according to the table (3), if it is desired to change the value of F_x by -30, then the operation should be $\Delta x = 10$ $(= -30 \div -3)$; and if it is desired to change the value of F_y by 56, then the operation should be $\Delta y = -28$.

5. The Basic Rule of Relaxation

The process of relaxation consists, then, in the application of the basic unit operations, repeatedly and to a correct extent, in order gradually, and eventually completely, to change the residuals from their initial values (whatever they may be) to zero. The fulfillment of this process is known as the *relaxation, or liquidation, of residuals*, and it is achieved in a number of steps, each of which is determined by the *basic rule of relaxation*. This rule lays down which operation should be used for each step and to what extent it should be so used. The rule states simply that

* See Appendix I, page 237.

the aim of each step is to change the value of the currently largest residual to zero. It is clear, therefore, that we have no means to decide of what any one step should consist until after the previous step has been completed; for until then it would not be possible to say which residual would become the largest, let alone what its magnitude would be. The succession in which, step by step, the basic operations are used is discovered only as the calculation proceeds, and in general there is no reason why it should display any systematic order or rotation.

6. The Relaxation Table

The computation is conveniently set out in tabular form in a *relaxation table,* each line of which corresponds to a single step of the process; its columns consist of one for each residual, showing the successive values taken by that residual, and one showing the nature of the successive operations used. The first line in the table records the values assigned initially to the unknown quantities together with the corresponding values of the residuals, calculated directly from their defining expressions. For the illustrative example of Sections 1 to 4 the relaxation table at starting would be (Section 3)

	F_x	F_y
$x = y = 0$	117	51

The aim of the first step in the relaxation of the residuals is, by the basic rule of relaxation (Section 5), to change the value (117) of the currently largest residual (F_x) to zero, *i.e.*, by -117. From the operations table (Section 4) the required operation is seen to be

$$\Delta x = -117 \div -3 = 39.$$

The application of this operation changes the value of F_x to zero but at the same time changes the value of F_y by 39×1 to become 90. This step is recorded as the second line in the relaxation table:

	F_x	F_y
$x = y = 0$	117	51
$\Delta x = 39$	0	90

The second and third steps, by similar reasoning, are found to require the operations $\Delta y = 45$ and $\Delta x = 15$, respectively, and after they have been effected the relaxation table reads

	F_x	F_y
$x = y = 0$	117	51
$\Delta x = 39$	0	90
$\Delta y = 45$	45	0
$\Delta x = 15$	0	15

The next step would seem to demand the operation $\Delta y = 7\frac{1}{2}$, but since, in any case, further increments are going to be added to y in later steps, there is little to be gained at this stage in recording the fractional part of this increment as calculated. Instead we take $\Delta y = 7$, thereby simplifying the arithmetic by using only whole numbers, and effectively adopt a loose interpretation of the word "zero" in the basic rule of relaxation, meaning by it "as near to zero as whole-number increments will allow." Throughout this book similar extended interpretations will be assumed whenever the term "zero" is used in reference to the value of a residual. After the operation $\Delta y = 7$ has been applied, the residuals F_x and F_y take the values 7 and 1, respectively. The liquidation of the residuals then continues in steps similar to those already described; the values of the residuals gradually converge together to zero, and the completed relaxation table is

	F_x	F_y
$x = y = 0$	117	51
$\Delta x = 39$	0	90
$\Delta y = 45$	45	0
$\Delta x = 15$	0	15
$\Delta y = 7$	7	1
$\Delta x = 2$	1	3
$\Delta y = 1$	2	1
$\Delta x = 1$	−1	2
$\Delta y = 1$	0	0
$x = 57, y = 54$	0	0

(4)

The final line in this table both summarizes and checks the calculation. In the first column is recorded the solution to the problem (cf. Section 1) obtained by summing for each unknown the value initially assigned to it and all increments subsequently added to it (and recorded in the first column); *e.g.*, $y = 0$ (initially) $+ 45 + 7 + 1 + 1 = 54$ (finally). The values of the residuals in the last line of the table are simply a check of those in the previous line; they are derived by substitution, of the solution found, back into the expressions (2) which define the residual quantities.

It was remarked in Section 5 that, in general, there need be no systematic order in which the basic operations are used in a relaxation table. It should be pointed out that in the table (4) of the above example a

rotation (*i.e.*, alternation) of the operations used in successive steps is seen only because of the simplicity inherent in there being only two unknowns. Since in successive lines of the table F_x and F_y alternately are made zero, it follows reciprocally that they must in turn become the "largest" residual and therefore that the basic operations must also be applied alternately. It is also a special property of this first example that all (*i.e.*, both) residuals are able eventually to be made exactly zero; this results from the fact that the given pair of equations *happen* to have a whole-number solution. In the next example (Section 8) this circumstance will not hold.

7. Arithmetical Mistakes in Computation

One of the advantages of the method is that it is not critically dependent on absolute arithmetical accuracy in the computation. Usually, in a numerical method, if a solution is thought to have been reached but a substitution back is found not to satisfy the original equations, then the computer must check all his work until the error is found, and then repeat over again all the calculations that had been performed after the occurrence of the error. The time taken in the discovery of the error, quite apart from its correction, may well be many times as large as that taken by the original computation (it is a well-established human failing to have difficulty in finding one's own arithmetical mistakes). In using the relaxation method, a computer is in no way concerned to discover his mistake but only to counteract its effect; this can usually be done in a fraction of the time taken by the original computation. As an illustration, suppose that a mistake had been made in the first step of the relaxation table (4), with the result that it was then continued as follows:

	F_x	F_y
$x = y = 0$	117	51
$\Delta x = 39$	0	80
$\Delta y = 40$	40	0
$\Delta x = 13$	1	13
$\Delta y = 6$	7	1
$\Delta x = 2$	1	3
$\Delta y = 1$	2	1
$\Delta x = 1$	−1	2
$\Delta y = 1$	0	0
$x = 55, y = 48$	0	10

That an error has been made is shown by the fact that the last line in the table does not check the previous line; F_y, which had been thought to be zero, is, instead, 10. It is immaterial how or when the mistake occurred; the important fact is that the residuals have not yet all been

made zero as we want them to be. The liquidation process is not complete and must therefore, for a few steps more, be carried on from the position shown in what, it had been hoped, was the final line of the relaxation table. The continuation of the table is

	F_x	F_y
$x = 55, y = 48$	0	10
$\Delta y = 5$	5	0
$\Delta x = 2$	−1	2
$\Delta y = 1$	0	0
$x = 57, y = 54$	0	0

When the liquidation is completed and the corrected solution obtained, a check must again be imposed. No relaxation can ever be properly or safely regarded as finished unless a final check has been made and does verify that all the residuals are zero.

It is usually found in practice that arithmetical mistakes which do take place in the relaxation process are such that the values of the residuals, when the check is imposed, are found to be small compared with what they were originally. In the above illustration the residuals were initially 117 and 51; at the first check they were found to be 0 and 10. The work done up to that point had liquidated about 94 per cent of the original residuals, and it was not wasted labor. At the position shown in that first check we were 94 per cent of the way toward the solution, and by carrying on from that position we had then only to achieve the remaining 6 per cent. It is most improbable that casual arithmetical mistakes should lead to values of residuals, when a check is taken, of more than about 10 to 20 per cent of their initial values. The effects of these mistakes can then be eradicated by a comparatively small amount of additional relaxation. If a check were to yield values of residuals comparable in magnitude with what they had been at the beginning, then it would probably mean that the computer had been making some error in principle in his use of the method. This may occasionally happen with a beginner, and, of course, then the errors should be traced so that misunderstandings of principles may be discovered and removed.

It might seem that undue emphasis is being given here to the occurrence of arithmetical errors in relaxation. It is true that it has no bearing on the theoretical and formal development of the subject, but the method is intended essentially to be practical; its aim is not only to produce results but to reach them in as short a time as possible. It is, indeed, no exaggeration to say that, if some mistakes in arithmetic are not made occasionally, then the student is spending too much time on the work. For while,

by taking care, he could no doubt arrive at a solution to a problem in, say, fifteen minutes, making no mistakes and finding a single check sufficient, it would nevertheless be considerably faster in the end to perform the first relaxation as rapidly (as distinct from carefully) as possible. This might well take only five minutes, and then, a check having revealed an error or errors, to complete some additional relaxation in compensation for those errors would probably take only one or two minutes more. Time saved by adopting this strongly recommended practical approach can easily amount to more than 50 per cent.

8. A Worked Example

A second example will now be taken, to solve, correct to two decimal places, the pair of equations

$$\begin{aligned} -7x + 3y + 96 &= 0, \\ 3x - 10y + 141 &= 0. \end{aligned} \tag{5}$$

In this case residuals are defined by

$$\begin{aligned} F_x &\equiv -7x + 3y + 96, \\ F_y &\equiv 3x - 10y + 141, \end{aligned}$$

and the operations table is

	ΔF_x	ΔF_y
$\Delta x = 1$	−7	3
$\Delta y = 1$	3	−10

The complete relaxation table, details of which will be explained later, is

	F_x	F_y	
$x = y = 0$	96	141	
$\Delta y = 14$	138	1	
$\Delta x = 20$	−2	61	
$\Delta y = 6$	16	1	
$\Delta x = 2$	2	7	
$\Delta y = 1$	5	−3	
$\Delta x = 1$	−2	0	(a)
$x = 23, y = 21$	−2.0	0.0	(b)
$\Delta x = -0.3$	0.1	−0.9	
$\Delta y = -0.1$	−0.2	0.1	(c)
$x = 22.7, y = 20.9$	−0.20	0.10	(d)
$\Delta x = -0.03$	0.01	0.01	(e)
$x = 22.67, y = 20.90$	0.01	0.01	(f)

(6)

The principal difference between the equations (5) of this example and the equations (1) of the previous example is that in this case the solution for the unknowns x and y consists of a pair of infinite decimals; consequently it can only be given correct to a certain number of places, which has been chosen to be, sufficiently, two. In these circumstances, which generally hold, the relaxation method should be applied first to obtain the solution correct to the nearest whole number; then, and only then, continued to find the solution correct to the first decimal place; then, in turn, to find it correct to the second decimal place, etc. In the table (6) at line (a), the residuals have been made as close to zero as possible using only whole-number increments; line (b) summarizes and checks the calculation up to this point, yielding, incidentally, the solution correct to the nearest whole number: $x = 23$, $y = 21$. At line (b) the first decimal place is introduced, and at line (c) the residuals have been made zero, using only first-decimal-place increments. Line (d) checks line (c) and introduces the second decimal place; it also shows the solution correct to the first decimal place: $x = 22.7$, $y = 20.9$. By line (e) the residuals have been made as near to zero as possible, using second-decimal-place increments. Finally line (f) checks line (e) and gives the required answer, $x = 22.67$, $y = 20.90$.

It is clear that as many further decimal places as might be required could be found in rapid succession. The number of steps required to advance the solution by each extra decimal place can never be more than a few because, for example, in advancing to the third decimal place from line (f), the total increment to be added to either x or y could not, in magnitude, exceed 0.005. It is desirable to impose checks (especially when there are more than two unknowns to be found) immediately before introducing each successive decimal place, as at lines (b), (d), and (f).

In the solution of equations where the initial residuals are very large compared with the coefficients in the operations table, it may be helpful in the first section of the relaxation table not to attempt to work to the nearest whole number in the unknowns but only to find the solution correct to the nearest ten or even to the nearest hundred; and then one by one to introduce successive significant figures, as many eventually as are called for in the answer.

9. Practicality of the Relaxation Method

The fundamental principle of the relaxation method has now been described, and its application to the solution of simultaneous algebraic equations illustrated in two simple examples. Theoretically there is nothing which could now prevent its extension to more difficult systems

of many simultaneous equations in many unknowns. If, however, there were nothing more in the method than has yet been suggested, it would remain of little more than academic interest, because in use it would yield solutions only after an impractical expenditure of time and labor. Thus, the relaxation table (4) is seen to require eight steps to achieve the solution of the pair of equations (1), than which a simpler pair could scarcely be discussed. The number of steps which would be needed to solve, say, 40 equations in 40 unknowns would certainly be prohibitive. Consequently, we must next consider several modifications of the basic method, all of which have been devised with the aim of speeding up the method in its convergence to the solution of a problem. These modifications or devices are essential weapons in making relaxation a useful method for the engineer and the physicist; with their use at disposal when necessary, the solution of a large number of simultaneous equations becomes not only possible but also practical. But while these various devices constitute the art of the method and are indispensable for its success and growth, nevertheless the roots of the method, its fundamentals and objectives, are the ideas and techniques described in Sections 2 to 7. Before he proceeds to acquire any of the art, it is essential that the student thoroughly appreciate the basic ideas. This he can do only by actually working through a few examples for himself, making each step in his calculations according to the basic rule of relaxation. By so doing, he will come to realize that the leading roles in a relaxational computation are played by the residuals and not by the unknowns, an unorthodox attitude which must become instinctive before much further progress can be made. He will also realize the necessity for developing modifications of the basic idea in order to render it practically useful.

PROBLEMS

1. Solve the equations

$$\begin{aligned} -10x + y + 728 &= 0, \\ x - 10y + 244 &= 0. \end{aligned}$$

(This example is rapidly convergent; three steps in the relaxation table should suffice.)

2. Solve the equations

$$\begin{aligned} -5x + y + 2z + 242 &= 0, \\ x - 3y - z + 32 &= 0, \\ 2x - y - 3z - 5 &= 0. \end{aligned}$$

(This example is reasonably quickly convergent; eight steps should suffice. Notice particularly the absence of any systematic order of use of the three basic operations.)

3. Solve the equations

$$\begin{aligned} -5x + y &= 107, \\ x - 3y &= 64. \end{aligned}$$

(This example is quickly convergent; four steps should be needed using whole-number increments, then three steps using first-decimal-place increments.)

4. Solve, correct to two decimal places, the equations

$$\begin{aligned} -10x + y + 2z + 3w - 42 &= 0, \\ x - 10y + 3z + 2w + 101 &= 0, \\ 2x + 3y - 10z + w + 78 &= 0, \\ 3x + 2y + z - 10w - 19 &= 0. \end{aligned}$$

(This example is more slowly convergent; to make certain of accuracy in the second decimal place, it is advisable to carry the computation eventually to three decimal places.)

CHAPTER 2

LINEAR ALGEBRAIC EQUATIONS (CONTINUED)

10. Introduction

In this chapter a number of modifications of the fundamental method of relaxation will be described and illustrated by their use in solving sets of linear algebraic equations. These modifications are computational devices which have all been developed with the aim of speeding up the convergence of the calculations, and they differ from the basic rule of relaxation (Section 5) in one most important property: Relaxation according to the basic rule is automatic, and each successive step in the calculation is quite definite and leaves nothing to the computer's judgment or experience; on the other hand, all the devices which are now to be described are flexible and are used entirely at the computer's discretion. No rules can be laid down to decide when any particular modification should be employed nor, indeed, precisely how it should be employed; these decisions depend on the personal inclination of the computer, and except in the simplest examples it would be unlikely that the calculations of two independent solvers would follow the same series of steps. Some of the devices may have more appeal to one computer than to another, and therefore would tend to be used more frequently by one than by another. It follows of course that, while the basic rule of relaxation can be quickly and easily learned and used by anyone, the various devices can be properly appreciated only after some experience of their use has been acquired, and it is therefore essential for the student to practice on simpler problems before tackling more difficult ones. He will then soon find that he is able to use the modifications naturally and instinctively.

In practice, of course, each of the devices may be helpful from time to time in the solution of any given problem. In learning what these devices are, however, it is best to take them singly, and they will therefore be dealt with quite separately in this chapter.

11. Overrelaxation

The first modification of the basic rule can be sufficiently illustrated by considering again the solution of the pair of equations used as a first example in Chapter 1:

$$\begin{aligned} F_x &\equiv -3x + y + 117 = 0, \\ F_y &\equiv x - 2y + 51 = 0. \end{aligned} \tag{1}$$

For convenience the operations table corresponding to this pair of equations* is given again here:

	ΔF_x	ΔF_y
$\Delta x = 1$	-3	1
$\Delta y = 1$	1	-2

(2)

It is evident at once from this table that each operation, when used to reduce one residual, automatically increases the other residual; thus the operation $\Delta y = 1$, used to decrease F_y by 2, at the same time increases F_x by 1. This circumstance would, of course, be an advantage if the initial residuals happened to be of opposite signs, so that, in order to bring them both to zero, one had to be decreased while the other needed to be increased (algebraically). But the property is a hindrance to convergence when, as in this example, the initial residuals are both of the same sign, here positive, F_x being 117 and F_y being 51. Even without consideration of numerical details, it is clear that the use of the basic rule will in the first step of the calculation reduce the value of F_x to zero but increase the value of F_y; and then in the second step F_y will be brought to zero but F_x increased to become again positive, and so on. Much work would be saved if we could anticipate the effect of the second step before making the first. With the knowledge that in the second step F_x is automatically going to be increased as F_y is reduced, it would obviously be helpful if F_x could be negative, and not merely zero, at the beginning of that second step. To achieve this, we have, of course, to modify the basic rule as originally used in the first step; it is better there not just to reduce F_x to zero but instead deliberately to go further and make F_x become negative. This device is known as *overrelaxation*. It has still to be decided by how much it is appropriate to overrelax, *i.e.*, to what extent F_x should be made negative; and in general it is impossible to say. It depends on the equations being solved, and a good decision is largely a matter of experience. As a rough rule it can be suggested that in the solution of a set of algebraic equations the overrelaxing operation to be used should be of a magnitude about half as much again as the operation which would be given by the basic rule of relaxation. In this example, according to that basic rule, the first operation would be $\Delta x = 39$; deliberately overrelaxing, we should instead choose as a first operation $\Delta x = 60$, a convenient increment about half as large again as $\Delta x = 39$. With this first operation and then carrying on

* The method of defining residuals F has been explained in Section 2. Henceforth the convention will be adopted, as in (1), of writing an equation and defining the corresponding residual in one line.

according to the basic rule, we find the solution in the complete relaxation table:

	F_x	F_y		
$x = y = 0$	117	51		
$\Delta x = 60$	−63	111	(a)	(3)
$\Delta y = 55$	−8	1	(b)	
$\Delta x = -3$	1	−2		
$\Delta y = -1$	0	0		
$x = 57, y = 54$	0	0		

The real effect of the overrelaxation in the first step at line (a) in this table is seen after the second step at line (b), where, F_y having been brought to zero, F_x is only -8; for comparison, in table (4) of Chapter 1, where only the basic rule was used, the value of F_x after two steps was 45, and the largest residual had been brought down to single figures only after four steps. In table (3) four steps complete the solution, as compared with eight in the solution in Chapter 1, so that the percentage saving in work is appreciable.

Situations in which overrelaxation is desirable frequently arise in all applications of relaxation; it is useful whenever the available operations are such (as they most often are) that, in reducing one residual, the other residuals are automatically increased, and when at the same time the individual residuals are predominantly of one sign.

12. Block Relaxation

A second, and even more important, device involves the creation of some new operations in the operations table (2). Although most useful in later applications of the method, this device can also be explained in relation to the solution of the same pair of equations (1) used to illustrate overrelaxation. This modification is known as *block relaxation*, and it consists in the use of other operations than the basic unit operations of Chapter 1. These new operations are called *unit block operations*, and they entail the simultaneous application of increments to more than one of the unknowns at the same time. For the example which we are using as an illustration there is, of course, only one block operation possible because there are only two unknowns, x and y. This unit block operation is $\Delta x = \Delta y = 1$; the changes to the residuals produced by applying it are easily calculated, either directly from the expressions defining the residuals or by adding together the changes produced by the basic unit operations which make up the block operation. The result is shown as a new line in the operations table:

	ΔF_x	ΔF_y
$\Delta x = 1$	−3	1
$\Delta y = 1$	1	−2
$\Delta x = \Delta y = 1$	−2	−1

In more complicated examples with more unknowns there are, of course, more block operations available; thus with three unknowns, x, y, and z, four such operations could be written down, three comprising increments added to the unknowns taken two at a time, and one containing increments added to all three unknowns together.

While block operations may be used at any stage in the relaxation process as may seem helpful to the computer, there is one particular use which deserves special mention: it is to use the block operation involving increments to all the unknowns, in the first step of the calculation, in order to reduce the total of all the residuals to zero in that one step. It has already been noticed, in the description of overrelaxation, that it is usually helpful to have the residuals not predominantly of one sign; this state is, of course, achieved if their total has been made approximately zero, and a block operation, applied as a first step, usually enables this to be done in a convenient manner.

In the example provided by the equations (1) the total residual is initially 168 (= 117 + 51), and the unit block operation effects a change in that total of −3; thus as a first step a block operation $\Delta x = \Delta y = 56$ (= −168 ÷ −3) must automatically reduce the total residual to zero. Then, with the individual residuals of opposite signs, the subsequent convergence may be expected to be rapid. The complete relaxation table is found to be

	F_x	F_y		
$x = y = 0$	117	51		
$\Delta x = \Delta y = 56$	5	−5	(a)	(4)
$\Delta y = -2$	3	−1		
$\Delta x = 1$	0	0		
$x = 57, y = 54$	0	0		

The effect of using the block operation is seen at the end of the first step in line (a) of table (4), where the total residual is zero and the individual residuals are also comparatively small. The number of steps required in the complete solution is now only three, comparing with four when over-

relaxation was used (Section 11) and with eight when only the basic rule was used (Section 6) to reach the solution.

13. Group Relaxation

The next device to be illustrated can be used with advantage in a kind of situation which can arise only in the solution of a rather more complicated set of equations than the pair (1). A set for which the convergence is slower is required, and a sufficient example is provided by the three simultaneous equations

$$\begin{aligned} F_x &\equiv -10x + 6y + 3z + 100 = 0, \\ F_y &\equiv 6x - 10y + 5z + 100 = 0, \\ F_z &\equiv 3x + 5y - 10z + 100 = 0. \end{aligned} \tag{5}$$

The operations table corresponding to these equations may be written down at once:

	ΔF_x	ΔF_y	ΔF_z
$\Delta x = 1$	−10	6	3
$\Delta y = 1$	6	−10	5
$\Delta z = 1$	3	5	−10

No block operations are included in this table because, although normally both block relaxation and overrelaxation would certainly be used in the solution of the equations (5), we are now concerned with another device; and, as was mentioned in Section 10, it is better that these various modifications should be introduced quite separately and so not be confused with one another.

Suppose that an attempt is made to solve the equations (5) by use only of the basic rule of relaxation; the relaxation table would begin in the following way:

	F_x	F_y	F_z	
$x = y = z = 0$	100	100	100	
$\Delta x = 10$	0	160	130	
$\Delta y = 16$	96	0	210	
$\Delta z = 21$	159	105	0	(6)
$\Delta x = 16$	−1	201	48	(a)
$\Delta y = 20$	119	1	148	
$\Delta z = 15$	164	76	−2	
$\Delta x = 16$	4	172	46	(b)

It soon becomes apparent that the process is very slowly convergent and that it would not be practicable to carry on until the solution is reached with only the use of the basic rule of relaxation. In cases like this, a halt must sooner or later be called to the calculations and a search made for some new device which will enable the liquidation of the residuals to proceed much more rapidly. Sometimes the answer may be found in that part of the relaxation table which has already been recorded, and in the above table a comparison of the values of the residuals at lines (a) and (b) reveals that the three steps between those two lines have had the combined effect of decreasing F_y by 29 while leaving F_x and F_z approximately unaltered. And, moreover, at line (b) the largest residual by far is F_y, the values of F_x and F_z, though not zero, being small enough in comparison to be ignored at this stage. Clearly, if F_y could be brought to zero without much change being produced in F_x and F_z, then a large step forward toward the solution would have been made; and, as we have seen, the three steps already recorded between lines (a) and (b), taken together, provide an operation which will effect precisely what is required. This new operation which has been discovered may be recorded as a new line in the operations table:

	ΔF_x	ΔF_y	ΔF_z
$\Delta x = 16, \Delta y = 20, \Delta z = 15$	5	−29	−2

(7)

This new operation is like a generalized block operation, differing from the latter in that the increments added to the various unknowns by its use are not all equal to one another; it is called a *group operation*, and the use of this device is known as *group relaxation*.

Like any of the unit operations, a group operation may be used to any extent, and to continue the relaxation process of table (6), the group operation (7) should next be applied to such an extent as will cause F_y to be reduced to zero while F_x and F_z are left comparatively unaltered. The necessary extent is clearly six times (*i.e.*, approximately $-172 \div -29$), and the continuation of the relaxation table reads

	F_x	F_y	F_z	
	4	172	46	(b)
$\Delta x = 96, \Delta y = 120, \Delta z = 90$	34	−2	34	(c)

(8)

The values of the residuals are now as recorded at line (c); while the solution has not, of course, yet been reached, the considerable improvement

that has been effected in one step by a use of group relaxation is obvious.

14. Multiplying Factors

One last device remains to be explained, and it can be well illustrated in the completion of the solution started in Section 13. It has already been discovered that the example there is slowly convergent, which means that some difficulty is experienced in the attempt to make all the residuals become zero. However, even in the most difficult example it is usually fairly easy at any stage in the calculation to make the residual values approximately have some prescribed *ratios* to one another, provided that their absolute values are allowed to adjust themselves. Thus, for example, if the solution in Section 13 is carried on from line (c) of table (8), there is no difficulty in making the residuals all become equal to one another; in fact two more steps suffice:

	F_x	F_y	F_z		
	34	−2	34	(c)	(9)
$\Delta z = 2$	40	8	14	(d)	
$\Delta x = 2$	20	20	20	(e)	
$x = 140,\ y = 156,\ z = 128$	20	20	20	(f)	

The two steps shown in lines (d) and (e) are aimed at reducing the values of F_z and F_x, respectively, not to zero but by sufficient amounts to become approximately equal to one another and to the increasing value of F_y. It happens in this example that it is possible to make the residuals exactly equal to one another; in general this will not be possible, any more than in general it is possible to make all the residuals exactly zero. We should say (cf. Section 6) that the values of the residuals had been made "equal" to one another when they had been made as nearly equal as whole-number increments allow.

In general, the ratios into which the values of the residuals should be put are the same as the ratios in which they were initially. In the example under consideration, the residuals were all initially 100, so in this case those ratios are ones of equality. It would have been no more difficult to have made the residuals in table (9) take values in the ratios 1:2:3 had their initial values been in these ratios, and no special significance should be attached to the fact that in this example the ratios happen to be ones of equality.

Once the values of the residuals have been brought into the appropriate

ratios, the complete solution can then be achieved in one further step. Before making this last step, it is desirable that the work so far should be checked; in table (9) this check is shown in line (f), where in the left-hand column all the increments applied to the unknowns have been added up and the totals recorded, and where the values, each 20, of the residuals have been checked by substitution of those totals into the residual formulas (5). In general at this stage, the residual values, being in the same ratio as they were initially, must each be the same fraction, say $1/n$, of their original values. Consequently each residual must have been reduced by a fraction $(n - 1)/n$ of its original value, and the unknowns must have acquired that same fraction, $(n - 1)/n$, of their final wanted values. The solution can therefore be immediately completed by multiplying the totals of the unknowns at the end of the last step in the relaxation table by the *multiplying factor* $n/(n - 1)$.

In our illustrative example the residuals at line (f) are each one-fifth of what they had been initially, so that the multiplying factor is $5/4$. Multiplication of the totals for x, y, and z shown in line (f) by this factor yields the final solution to the equations (5): $x = 175$, $y = 195$, $z = 160$.

15. A Worked Example

A typical example will now be taken and worked through to illustrate the methods of this chapter: to solve, correct to one decimal place, the set of equations:

$$\begin{aligned} F_x &\equiv -7x + y + 5z + 171 = 0, \\ F_y &\equiv x - 6y + 4z + 49 = 0, \\ F_z &\equiv 5x + 4y - 8z + 90 = 0. \end{aligned} \tag{10}$$

The operations table, including one unit block operation, is

	ΔF_x	ΔF_y	ΔF_z
$\Delta x = 1$	−7	1	5
$\Delta y = 1$	1	−6	4
$\Delta z = 1$	5	4	−8
$\Delta x = \Delta y = \Delta z = 1$	−1	−1	1

The complete relaxation table, details of which will be explained afterward, is

	F_x	F_y	F_z	
$x = y = z = 0$	171	49	90	
$\Delta x = \Delta y = \Delta z = 310$	−139	−261	400	(a)
$\Delta z = 50$	111	−61	0	(b)
$\Delta x = 16$	−1	−45	80	(c)
$\Delta z = 15$	74	15	−40	(d)
$\Delta x = 8$	18	23	0	(e)
$\Delta y = 3$	21	5	12	(f)
$x = 334, y = 313, z = 375$	21	5	12	(g)
$x = 380.8, y = 356.8, z = 427.5$	−0.3	−1.0	1.2	(h)
$\Delta z = 0.2$	0.7	−0.2	−0.4	(j)
$\Delta x = 0.1$	0.0	−0.1	0.1	(k)
$x = 380.9, y = 356.8, z = 427.7$	0.00	−0.10	0.10	(l)
$\Delta z = 0.02$	0.10	−0.02	−0.06	(m)
$\Delta x = 0.01$	0.03	−0.01	−0.01	(n)

In line (a), as a first step, a block operation involving increments to all the unknowns is applied, in order to reduce the total residual to zero; then for two steps, lines (b) and (c), relaxation proceeds according to the basic rule. In line (d) overrelaxation is used, because at line (c) the tendency for F_z to keep increasing positively is evident. At line (d) it is clear that the convergence is not rapid and that the residuals are again becoming predominantly positive; it is therefore decided to make use of a multiplying factor, and in lines (e) and (f) the residuals are made to become approximately in the same ratios as they were initially. Line (g) gives the total values of the unknowns at this stage of the calculation and checks the values of the residuals. A multiplying factor of 1.14 is then used, with results shown in line (h), where the opportunity is taken to introduce the first decimal place and where, of course, the residuals have had to be recalculated from the formulas (10). Two lines of the relaxation process follow which affect only the first decimal place; use is made of a little overrelaxation in line (j). Line (k) completes the solution, and the calculation is summarized and checked in line (l). In order to make quite certain that first-decimal-place accuracy has been attained, the relaxation process is carried further, in lines (m) and (n), with increments in the second decimal place. These are soon seen not to accumulate sufficiently to affect the first decimal place, and so the solution given in line (l) is confirmed.

16. Divergent Equations

It has been assumed thus far that the fundamental method of relaxation is always convergent when applied to solve a set of linear algebraic equa-

tions, and the modifications of that fundamental method have been regarded only as devices necessary to speed up this convergence and so make practical a computational method which in theory would always ultimately converge. Although they are not very likely to occur in practice, it is, however, possible to devise sets of equations for which the fundamental method would not converge but would instead diverge. Even in such cases the solution can still be found by the relaxation method, provided that use is made of the multiplying-factor device. In examples of this kind the multiplying factor will be found to have a negative value, but the principle underlying its use is exactly the same as in the more usual circumstances when it is positive. A somewhat extreme example illustrating this phenomenon is provided by the solution of the set of equations

$$\begin{aligned} F_x &\equiv -10x + 9y + 7z + 57 = 0, \\ F_y &\equiv 9x - 10y + 8z + 143 = 0, \\ F_z &\equiv 7x + 8y - 10z + 365 = 0. \end{aligned}$$

The operations table corresponding to these equations is

	ΔF_x	ΔF_y	ΔF_z
$\Delta x = 1$	−10	9	7
$\Delta y = 1$	9	−10	8
$\Delta z = 1$	7	8	−10

The relaxation table, details of which are explained subsequently, is

	F_x	F_y	F_z	
$x = y = z = 0$	57	143	365	
$\Delta z = 37$	316	439	−5	
$\Delta y = 44$	712	−1	347	
$\Delta x = 71$	2	638	844	(a)
$\Delta y = 30$	272	338	1,084	
$\Delta x = 7$	202	401	1,133	
$\Delta y = -4$	166	441	1,101	
$\Delta x = -1$	176	432	1,094	(b)
$x = 77, y = 70, z = 37$	176	432	1,094	(c)
$x = -38.3, y = -34.8, z = -18.4$	−2.0	−0.9	2.5	(d)
$\Delta z = 0.2$	−0.6	0.7	0.5	
$\Delta x = -0.1$	0.4	−0.2	−0.2	(e)
$x = -38.4, y = -34.8, z = -18.2$	0.4	−0.2	−0.2	(f)

In this table the first few steps are calculated according to the basic rule, but by line (a) it has become quite apparent that as a whole the residuals are gradually increasing and that the process is diverging. At this stage, then, the attempt to bring the residuals all to zero is temporarily discarded, and, instead, the relaxation continues with the aim only of making the residuals take values which are approximately in the same ratios as initially; this is achieved at line (b). Line (c) summarizes and checks the work at this stage, and in line (d) a multiplying factor of -0.497 is used, and at the same time the first decimal place is introduced. The residuals are immediately seen to become very small, and two further steps complete the solution, correct to the first decimal place. It might, of course, have happened that the process after line (d) again began to diverge, in which case a second multiplying factor would have been necessary; but the values at line (d) are so near to the ultimate solution that little difficulty is experienced in reaching line (e) by simpler means. The solution is shown in line (f), where also the final residuals are checked.

PROBLEMS

1. Solve, correct to the nearest whole number, the equations

$$\begin{aligned} -10x + 3y + 3z + 4w + 100 &= 0, \\ x - 10y + 5z + 3w + 100 &= 0, \\ 3x + 4y - 10z + w + 100 &= 0, \\ 4x + 3y + 2z - 10w + 100 &= 0. \end{aligned}$$

(Notice that the coefficients in these equations are *not* symmetrical about the leading diagonal; care must be taken to make sure that the operations table is written down correctly. This example is not rapidly convergent, and block relaxation will be found very useful.)

2. Solve, correct to two decimal places, the equations

$$\begin{aligned} -20x + y + z + 6w + 2u + v + 17 &= 0, \\ x - 20y + 2z + w + 4u + 3v + 25 &= 0, \\ x + 2y - 20z + 2w + u + 2v + 8 &= 0, \\ 6x + y + 2z - 20w + 3u + v + 60 &= 0, \\ 2x + 4y + z + 3w - 20u + 5v + 13 &= 0, \\ x + 3y + 2z + w + 5u - 20v + 1 &= 0. \end{aligned}$$

(This example is not difficult and should demonstrate how suitable the relaxation method is for solving sets of large numbers of equations.)

3. Solve, correct to the nearest whole number, the equations

$$\begin{aligned} -10x + 6y + 5z + 100 &= 0, \\ 6x - 10y + 7z + 100 &= 0, \\ 5x + 7y - 10z - 200 &= 0. \end{aligned}$$

(This example is more difficult; the equations are divergent in the sense of Section 16, and it may be necessary to use a multiplying factor as well as block relaxation in order to find the solution.)

CHAPTER 3

FRAMEWORK PROBLEMS

17. Introduction

In the previous two chapters we have described how a set of linear algebraic equations may be solved by the relaxation method; it is desirable now that we should go back, from a historical point of view, to learn how the method originated. In doing so, we shall discover a mechanical analogy to the arithmetical processes already described; the reasons for the nomenclature used will be explained; and, perhaps most important of all, a first practical application of the method will be introduced. This application is to the solution of framework problems; both pin- and stiff-jointed frames can be dealt with in both two and three dimensions. The only real difference between these various types of frames is in the number of degrees of freedom possessed by the joints of the frame: thus a three-dimensional stiff-jointed frame has six degrees of freedom at each joint, whereas a two-dimensional pin-jointed frame has only two. Each degree of freedom of each joint of a frame corresponds to one equation of a simultaneous set, and a three-dimensional stiff-jointed frame is more complicated to solve than a two-dimensional pin-jointed frame only because of the correspondingly greater number of unknowns to be found. However, the method of solution of a two-dimensional frame by relaxation can quite directly be extended to deal with a three-dimensional frame, just as the method of solution of a set of two or three simultaneous equations (Chapters 1 and 2) was clearly able to be applied to a set of 10 or 20 or more equations. The application of the method to solve framework problems in this chapter will therefore mostly be described in two-dimensional terms; nothing is lost thereby, and there is, for a beginner, a considerable gain in simplicity.

18. Two-dimensional Pin-jointed Frames

In all framework problems the solution will be regarded as complete when the displacements of the joints have been found in the equilibrium configuration produced by a specified loading system. These displacements are all measured from zero values corresponding to the positions of the joints when the framework is unstressed. Once the displacements

have been determined, it is, of course, a standard calculation in the theory of elasticity to evaluate the elongations, strains, and stresses, in the individual members or rods which make up the framework; in a solution by the relaxation method the problem is always worked in terms of these displacements as the unknowns which are to be found. In a pin-jointed frame in two dimensions, each joint requires two components of displacement to define its position in any deformation; these are denoted in two orthogonal directions (usually horizontally and vertically) by u and v, respectively, with suffixes to indicate displacements of particular joints.

In order to point out the physical analogy to the numerical processes of relaxation, it is necessary first to consider the set of equations which the displacements (*i.e.*, the u's and v's) of the joints must satisfy in a deformation of the framework. In Figure 1, A represents a typical joint of a two-dimensional frame and AB a typical one of those members of the frame which are pinned together at A; B, of course, is another of the joints. The length of the rod AB is denoted by L_{AB}, its Young's modulus by E_{AB}, its cross-sectional area by S_{AB}, and its inclination to the horizontal by θ_{AB}. The displacements of the joint A are u_A and v_A, horizontally and vertically, respectively, and those of the joint B are u_B and v_B. Part of the loading system acting on the framework may be applied* at A, and the components of external force there are denoted by U_A and V_A.

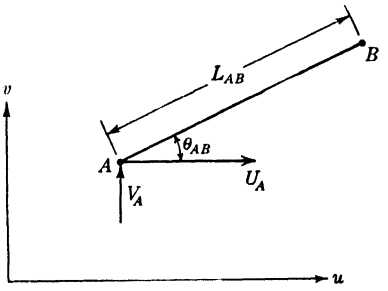


FIGURE 1

The total components of force acting on the joint A can now be found in terms of the displacements. The elongation of the rod AB is

$$(u_B - u_A) \cos \theta_{AB} + (v_B - v_A) \sin \theta_{AB};$$

therefore the tension in it is

$$\frac{E_{AB}S_{AB}}{L_{AB}} [(u_B - u_A) \cos \theta_{AB} + (v_B - v_A) \sin \theta_{AB}],$$

so that the two components of force (denoted by F_{Au} and F_{Av}) acting at A are

* For the present it is assumed that all loads are applied at joints of the framework. In cases where loading, either concentrated or distributed, acts at points along a rod, some preliminary work must be performed which, in effect, transfers the loading to the joints at the ends of the rod. This work does not involve any relaxation, and the example in Section 24 will sufficiently illustrate what must be done.

$$F_{Au} \equiv U_A + \sum_B \frac{E_{AB}S_{AB}}{L_{AB}} [(u_B - u_A) \cos \theta_{AB} + (v_B - v_A) \sin \theta_{AB}] \cos \theta_{AB},$$
$$F_{Av} \equiv V_A + \sum_B \frac{E_{AB}S_{AB}}{L_{AB}} [(u_B - u_A) \cos \theta_{AB} + (v_B - v_A) \sin \theta_{AB}] \sin \theta_{AB}, \quad (1)$$

where the Σ signs indicate summation for all the rods of the framework which are jointed together at A; a similar pair of expressions could be written down for each joint. In the equilibrium configuration all these forces must vanish, and, by putting each F equal to zero, we could obtain the set of simultaneous equations which must be satisfied by the wanted values of the displacements; clearly one such equation would correspond with each displacement. But it is not necessary actually to formulate these equations, because the expressions (1) correspond exactly with the definitions of residuals given in Section 2, and, as has already been seen, when the relaxation method is employed to solve a set of algebraic equations, no direct use is made of the actual equations except to define the residuals.

If the displacements had values assigned to them other than those that obtain in the equilibrium position of the framework, then some or all of the forces given by expressions of the type (1) would not vanish; they would instead be nonzero *unbalanced* forces acting on the joints; and in order that any set of assigned displacements be maintained, some external agencies, or constraints, would have to be provided to support those unbalanced forces.

19. Mechanical Interpretation of the Relaxation Process

To give a mechanical interpretation to the numerical processes of relaxation, we must imagine that the constraints mentioned in Section 18 are in fact provided at the joints of the frame. For a two-dimensional pin-jointed framework two constraints must be visualized at each joint: one which prevents any displacement in the u direction but presents no restriction to motion in the v direction, and another which prevents any displacement in the v direction but imposes no restriction to motion in the u direction. We must also suppose that these constraints can be removed or replaced at will.

We begin by placing the constraints in position before the framework is loaded, when all the displacements are therefore zero. Then the loads are applied, and the constraints prevent any deformation of the frame at all; the loads are immediately borne entirely by the constraints as

unbalanced or *residual forces*. This position in the mechanical analogy corresponds exactly with the first line in a relaxation table, where the unknowns (*i.e.*, displacements) are initially taken to be zero and where the corresponding initial residuals (*i.e.*, residual forces) are just the constant terms (*i.e.*, loads) in the equations being solved.

The first step in a relaxation table, according to the basic rule of relaxation, comprises the application of an increment, of appropriate magnitude, to the unknown which corresponds with the largest residual, in order to reduce that residual to zero. In the mechanical analogy the corresponding step is the removal of that constraint which bears the largest residual force, thereby allowing one joint to move* in one direction, under the action of the unbalanced force, until it comes of its own accord into equilibrium. In this motion the displacement associated with the removed constraint acquires such a value that the corresponding unbalanced force is reduced to zero; it becomes, of course, balanced by tensions in the members of the frame. In the process some of the other residual forces, which remain borne by the other (unremoved) constraints, may change their values, just as, in a relaxation table, an operation which reduces one residual to zero may alter the values of the other residuals. When equilibrium has been reached, the removed constraint is replaced to hold the joint fixed in its new position.

Now, in a relaxation table, after one residual has been reduced to zero in the first step, some other residual becomes the largest, and the process is repeated in order next to reduce it to zero, and so on. Similarly, in the framework analogy, after one constraint has been removed and one of the joints allowed to acquire some displacement, some other constraint must then bear the largest residual force, and that constraint in turn is temporarily removed. Each constraint is replaced in a new position after it has been removed and equilibrium reached. Later on, a constraint which has been removed once may again be required to bear some residual force, and eventually may have to be removed again, but gradually the residual forces borne by the constraints must decrease toward zero; the process must converge. This is evident because in each step some unbalanced force is allowed to do work, and therefore the potential energy of the system is continually decreasing toward a minimum value.

As in the solution of a set of equations, so in a framework problem, the residuals may never be able to be brought exactly to zero; in practice it would, indeed, be virtually meaningless to try to achieve this, because no load can really be specified precisely. A load might be given as, say, 20 tons, but there must in reality be some tolerance in its value; it could, for

* The motion is supposed to be *slow*, so that no dynamical considerations need apply.

instance, probably be more truthfully given to be 20 ± 0.1 tons. In describing a load in this way we recognize the existence of a *margin of uncertainty* associated with it; this is of no particular significance when an orthodox method of solution is employed, but it does have a useful meaning when the relaxation method is used. For one solution is as likely to be correct as another once the residual forces have been made sufficiently small to lie within the margin of uncertainty of the applied loads, and when that has been done, it is pointless to continue the relaxation any further.

The mechanical interpretation of the fundamental method of relaxation is now established, and the name itself—relaxation methods—is explained as a shortened form of the original title—the method of systematic relaxation of constraints.* The various devices explained in Chapter 2 can similarly be described, if it is desired, in terms of displacements, residual forces, etc. Thus block relaxation corresponds, not just to the movement of one joint of the frame until it is in equilibrium, but instead to the movement as a rigid body of some part of the framework, perhaps containing several joints, until that part is in equilibrium.

20. The Operations Table for a Framework

For the case of a two-dimensional pin-jointed frame the operations table can be written down immediately from the expressions (1) which define the residuals. Its form can, in general, be sufficiently illustrated by the following typical extract:

	ΔF_{Au}	ΔF_{Av}
$\Delta u_A = 1$	$-\sum_B \frac{E_{AB}S_{AB}}{L_{AB}} \cos^2 \theta_{AB}$	$-\sum_B \frac{E_{AB}S_{AB}}{L_{AB}} \cos \theta_{AB} \sin \theta_{AB}$
$\Delta v_A = 1$	$-\sum_B \frac{E_{AB}S_{AB}}{L_{AB}} \cos \theta_{AB} \sin \theta_{AB}$	$-\sum_B \frac{E_{AB}S_{AB}}{L_{AB}} \sin^2 \theta_{AB}$
$\Delta u_B = 1$	$\frac{E_{AB}S_{AB}}{L_{AB}} \cos^2 \theta_{AB}$	$\frac{E_{AB}S_{AB}}{L_{AB}} \cos \theta_{AB} \sin \theta_{AB}$
$\Delta v_B = 1$	$\frac{E_{AB}S_{AB}}{L_{AB}} \cos \theta_{AB} \sin \theta_{AB}$	$\frac{E_{AB}S_{AB}}{L_{AB}} \sin^2 \theta_{AB}$

(2)

The first two lines in this table show the changes effected in the residuals at a typical joint A of the frame by the application of unit displacements at that joint; the third and fourth lines similarly show the changes pro-

* This title was used in two papers by R. V. Southwell, published in 1935 (*Proc. Roy. Soc. A*, **151**, 56–95, and *Proc. Roy. Soc. A*, **153**, 41–76).

duced in the residuals at one joint A when unit displacements are applied at some other typical joint B. It is, of course, assumed that the joints A and B are directly connected together by a member of the frame; otherwise, of course, the expressions in the third and fourth lines of table (2) become meaningless and should be replaced by zeros. In many problems all the rods of the framework have the same Young's modulus E and cross-sectional area S; in such a case a preliminary transformation of the unknowns is helpful. For if we write $u' = ESu$ and $v' = ESv$, then the factors E and S can be omitted from the expressions (1) and the table (2), provided that each u is replaced by u' and each v by v'.

21. A Worked Example

A short worked example will sufficiently indicate the procedure. Figure 2 represents a plane framework of four rods, AB, AC, AD, and AF, of

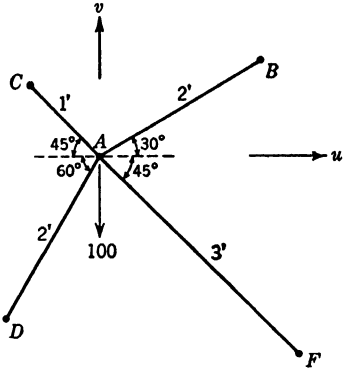


FIGURE 2

lengths respectively 2, 1, 2, and 3 ft. A is a movable pin joint, but B, C, D, and F are fixed pins; each rod has Young's modulus E and cross-sectional area S. A load of 100 lb acts as shown at A.

Substituting particular values in (1), we find expressions for the residual forces to be

$$F_{Au} \equiv -1.1667ESu_A + 0.2337ESv_A,$$
$$F_{Av} \equiv 0.2337ESu_A - 1.1667ESv_A - 100,$$

or, after u and v have been replaced by u' and v',

$$\begin{aligned} F_{Au} &\equiv -1.1667u_A' + 0.2337v_A', \\ F_{Av} &\equiv 0.2337u_A' - 1.1667v_A' - 100. \end{aligned} \tag{3}$$

It is not necessary to find residual forces at any of the joints B, C, D, or F, because, although these forces do exist, they do not have to be brought to

zero; whatever they may be, they are borne by the (rigid) supports to which the fixed pins at B, C, D, and F are attached. The operations table is at once seen to be

	ΔF_{Au}	ΔF_{Av}
$\Delta u_A' = 1$	−1.1667	0.2337
$\Delta v_A' = 1$	0.2337	−1.1667

and the relaxation table, giving u_A' and v_A' to one decimal place, is

	F_{Au}	F_{Av}	
$u_A' = v_A' = 0$	0	−100	
$\Delta v_A' = -100$	−23	17	
$\Delta u_A' = -20$	0	12	
$\Delta v_A' = 11$	3	−1	
$\Delta u_A' = 3$	−1	0	(a)
$u_A' = -17$, $v_A' = -89$	−0.9	−0.1	(b)
$\Delta u_A' = -0.9$	0.1	−0.3	
$\Delta v_A' = -0.3$	0.0	0.0	(c)
$u_A' = -17.9$, $v_A' = -89.3$	0.0	0.0	(d)

(4)

In the first section of this table, as far as line (a), the relaxation process is carried out with a use of only whole-number increments, and a little over-relaxation is employed. In this example, for the first time, the coefficients in the operations table contain decimal parts, but no attempt is made to keep the residual values accurate to more than the nearest whole number until line (a) is reached. As a consequence, small errors in the values of the residuals are acknowledged as being possible at line (a); these need not be a matter for concern because they are taken into account at line (b), where a check on the work up to line (a) is imposed, and where also the first decimal place is introduced. Of course, in the recalculation of the residuals at line (b), the expressions (3) must be used exactly. In the second section of table (4), between lines (b) and (c), increments are applied in the first decimal place; line (d) gives the solution and also checks the final values of the residuals.

In the equilibrium position the elongation of the rod AB is $-(u_A \cos 30° + v_A \sin 30°)$, and therefore the tension in AB is

$$-\frac{ES}{2}\left(u_A \cdot \frac{\sqrt{3}}{2} + v_A \cdot \frac{1}{2}\right) = -\frac{1}{2}\left(u_A' \cdot \frac{\sqrt{3}}{2} + v_A' \cdot \frac{1}{2}\right) = 30.1 \text{ lb.}$$

The tensions in the other rods, AC, AD, and AF, can similarly be evaluated to be 50.5, -43.1, and -16.8 lb, respectively.

22. Three-dimensional Pin-jointed Frameworks

For a three-dimensional pin-jointed framework an exactly similar method of solution is used as for a plane framework, the only difference being that now there are three components of displacement to find for each joint. We denote these by u, v, and w and likewise denote components of applied load by U, V, and W. The expressions for the components of residual force at a typical joint A become

$$F_{Au} \equiv U_A + \sum_B \frac{E_{AB}S_{AB}}{L_{AB}} [(u_B - u_A)l_{AB}^2 + (v_B - v_A)l_{AB}m_{AB} + (w_B - w_A)n_{AB}l_{AB}],$$

$$F_{Av} \equiv V_A + \sum_B \frac{E_{AB}S_{AB}}{L_{AB}} [(u_B - u_A)l_{AB}m_{AB} + (v_B - v_A)m_{AB}^2 + (w_B - w_A)m_{AB}n_{AB}],$$

$$F_{Aw} \equiv W_A + \sum_B \frac{E_{AB}S_{AB}}{L_{AB}} [(u_B - u_A)n_{AB}l_{AB} + (v_B - v_A)m_{AB}n_{AB} + (w_B - w_A)n_{AB}^2],$$

where l_{AB}, m_{AB}, and n_{AB} are the direction cosines of AB referred to the orthogonal directions of u, v, and w.

23. Stiff-jointed Frameworks

For a two-dimensional stiff-jointed frame each joint has three degrees of freedom, and three displacements are required to define its position in any deformation; these are two linear displacements u and v, horizontally and vertically, respectively, as in the case of a pin joint, and one angular displacement ϕ, measured positively in an anticlockwise sense. The same notation is used as in Section 18, with the additional definitions that Φ_A denotes the torque which may be applied at the joint A as part of the loading system; that I_{AB} is the second moment of the cross-sectional area of a typical member AB of the frame; and that $F_{A\phi}$ is the residual, or unbalanced, torque acting at the joint A at any stage in the calculation.

In this case a slightly different approach will be adopted from that used in Sections 18 and 20; there, for a pin-jointed framework, in order to emphasize the parallel between the numerical processes of relaxation and the mechanical analogy of a framework problem, the expressions (1) for the residual forces were found first, and the operations table (2) was

derived afterward directly from these expressions. Now, for a stiff-jointed framework, the operations table will be found first, and the expressions for the residual forces* left to be written down afterward. It will be sufficient, in general terms, to indicate the form of the operations table if we find the changes produced in the residual forces at a typical joint A by the application of unit displacements at that joint and also at some other joint B.

To be able to do this, we shall need to quote some standard results from the ordinary theory of the deflection of beams. These results are shown in Figure 3. In each of the diagrams (i) to (iii) of that figure, AB represents the member in some deformed position and A_0B indicates the position of the member in its original undeformed position. In all three cases the end B has undergone no displacement, either linear or angular; in case (i) the end A has been given a unit linear displacement only, in a direction parallel to A_0B; in case (ii) A has been given a unit linear displacement only, in a direction perpendicular to A_0B; and in case (iii) the end A has been given a unit angular displacement only. In all three cases the forces shown at the two ends of the deformed member are the reactions acting *on the joints* A and B when the member is so deformed. In this diagram the suffixes AB have been omitted from the quantities L, I, θ, etc.; it should not cause any confusion to leave them out always from here on.

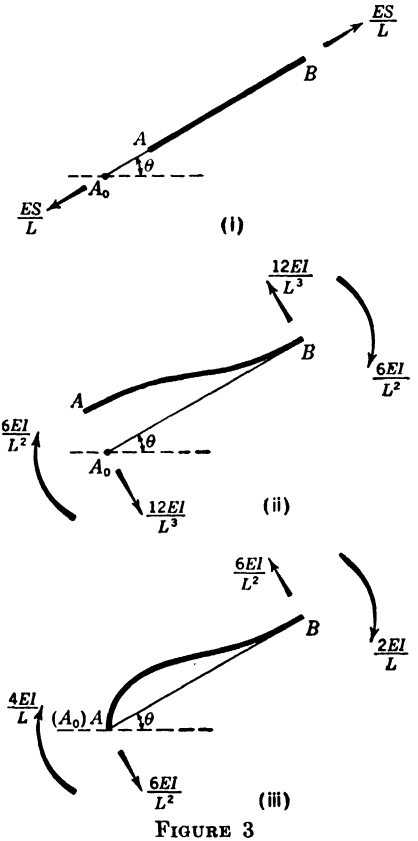


FIGURE 3

Now, the basic unit operations, $\Delta u_A = 1$, $\Delta v_A = 1$, and $\Delta \phi_A = 1$, represent displacements of the joint A, each of which can be made up of component displacements of the types (i), (ii), and (iii) of Figure 3. In fact, with an obvious convention, we may write

* The term *force* will, for the rest of this chapter, be used to include both forces and torques.

$$\begin{aligned} \Delta u_A &= 1 \equiv \text{(i)} \cdot \cos\theta - \text{(ii)} \cdot \sin\theta, \\ \Delta v_A &= 1 \equiv \text{(i)} \cdot \sin\theta + \text{(ii)} \cdot \cos\theta, \\ \Delta \phi_A &= 1 \equiv \text{(iii)}. \end{aligned} \tag{5}$$

So, with reference to Figure 3, it becomes possible immediately to construct the operations table; thus, for instance, if the operation $\Delta v_A = 1$ is applied, then the change produced in the residual F_{Av}, because of the tension and shear force created in the rod AB, is

$$-\frac{ES}{L}\sin^2\theta - \frac{12EI}{L^3}\cos^2\theta;$$

therefore the total change produced in F_{Av} is

$$-\sum\left(\frac{ES}{L}\sin^2\theta + \frac{12EI}{L^3}\cos^2\theta\right),$$

where the Σ sign indicates summation for all rods, such as AB, which are jointed together at A. In general, the complete operations table is sufficiently illustrated by the following extract:

	ΔF_{Au}	ΔF_{Av}	$\Delta F_{A\phi}$
$\Delta u_A = 1$	$-\sum\left(\frac{ES}{L}\cos^2\theta + \frac{12EI}{L^3}\sin^2\theta\right)$	$-\sum\left(\frac{ES}{L} - \frac{12EI}{L^3}\right)\cos\theta\sin\theta$	$\sum\frac{6EI}{L^2}\sin\theta$
$\Delta v_A = 1$	$-\sum\left(\frac{ES}{L} - \frac{12EI}{L^3}\right)\cos\theta\sin\theta$	$-\sum\left(\frac{ES}{L}\sin^2\theta + \frac{12EI}{L^3}\cos^2\theta\right)$	$-\sum\frac{6EI}{L^2}\cos\theta$
$\Delta\phi_A = 1$	$\sum\frac{6EI}{L^2}\sin\theta$	$-\sum\frac{6EI}{L^2}\cos\theta$	$-\sum\frac{4EI}{L}$
$\Delta u_B = 1$	$\frac{ES}{L}\cos^2\theta + \frac{12EI}{L^3}\sin^2\theta$	$\left(\frac{ES}{L} - \frac{12EI}{L^3}\right)\cos\theta\sin\theta$	$-\frac{6EI}{L^2}\sin\theta$
$\Delta v_B = 1$	$\left(\frac{ES}{L} - \frac{12EI}{L^3}\right)\cos\theta\sin\theta$	$\frac{ES}{L}\sin^2\theta + \frac{12EI}{L^3}\cos^2\theta$	$\frac{6EI}{L^2}\cos\theta$
$\Delta\phi_B = 1$	$\frac{6EI}{L^2}\sin\theta$	$-\frac{6EI}{L^2}\cos\theta$	$-\frac{2EI}{L}$

TABLE (6)

The last three lines in this table have been found by a direct use of Figure 3 and the relations (5) to determine the changes in the residual forces at B occasioned by unit displacements applied at A; then the changes in the residuals at A due to unit displacements applied at B were found by a simple interchange of the suffixes A and B. Care has to be taken in doing this, especially when some of the suffixes are eventually not recorded, for

whereas E_{BA} is the same as E_{AB} and a similar equivalence holds for the quantities L, S, and I, on the other hand $\theta_{BA} = \theta_{AB} + \pi$. It is desirable to adopt some uniform convention in recording the quantities which appear in the operations table, and here the convention is used that in each column of the table the *first* of the two suffixes, to be implied with every θ, describes the joint at which acts the residual force referred to in that column.

In any particular example the numerical quantities in the operations table can easily be evaluated by a direct substitution into the general expressions shown in table (6). When this has been done, numerical formulas for the residuals can be written down immediately and without difficulty (it would not be particularly helpful to give the formulas here in general terms). A worked numerical example will now explain the whole method of solution better than any attempt to continue further algebraically could do.

24. A Worked Example

Figure 4 represents a plane stiff-jointed framework with two movable joints A and B. The frame is built in to a rigid base at C and D and

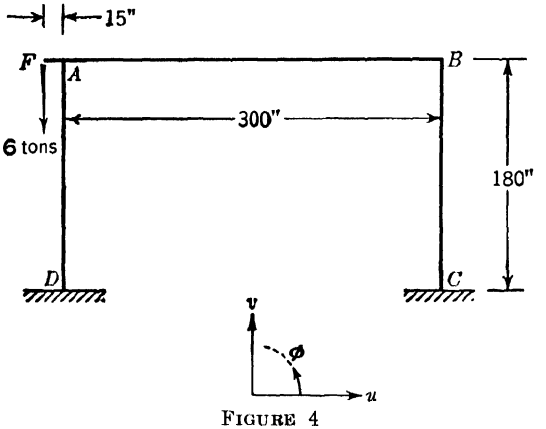


FIGURE 4

carries a concentrated load of 6 tons overhung at F. The dimensions shown are to the centers of the members, which are all rolled steel joists, 12 in. × 5 in. × 30 lb, for which EI has the value 2.69×10^6 ton-in.^{2} and ES the value 11.48×10^4 tons.

The complete solution is found in two parts, each with results which in the end may be superposed. The first part consists in the determination of the deformation produced in the various members by the preliminary transference (cf. footnote, page 25) of the applied loading system to act

at the joints of the frame. In this part of the solution the joints are allowed to undergo no displacement, and each member is treated separately as a beam clamped at both ends or, in the case of an overhang, as a simple cantilever. Thus the concentrated load of 6 tons acting at F may be transferred to the joint A, where it provides a contribution of -6 tons to the value of V_A together with a contribution of 90 ton-in. to the value of Φ_A; the deformation accompanying this transference of load is the ordinary deflection,* in AF, of a cantilever due to a concentrated load at the free end. Similarly the distributed load along AF, provided by its own weight, may be transferred to the joint A, where it yields a contribution to V_A of -37.5 lb and to Φ_A of 281.25 lb-in., accompanied by the ordinary deflection (in AF) of a cantilever due to a uniformly distributed load. In

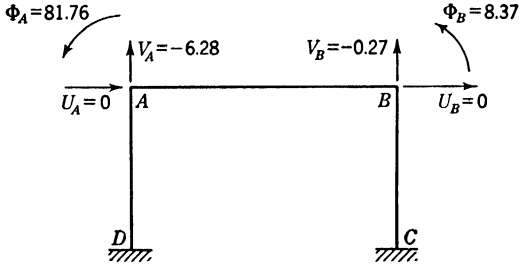


FIGURE 5

the same way the distributed load along AB, due to its own weight, may be transferred to the joints A and B, where it yields contributions to V_A and to V_B of -375 lb each, and to Φ_A and to Φ_B of $-18{,}750$ lb-in. and 18,750 lb-in., respectively, accompanied by the ordinary deflection (in AB) of a beam clamped at both ends and sustaining a uniformly distributed load. Last, the weight of the members AD and BC may be transferred to act at their ends, whereby additions are caused to both V_A and V_B of -225 lb.

When all the applied loads have thus been transferred to act at the joints of the framework, we are left with a second part of the problem to solve by relaxation. The initial unbalanced loads, or residual forces (in units of ton-inches), acting at the joints are shown in Figure 5, and these residuals must gradually be liquidated by the application of displacements at the joints; all that is needed now to be able to achieve this relaxation is the operations table. This table can be constructed at once

* The form of this deflection, together with the associated bending moment and shear force, is given in Appendix II, page 238, where a table is collected of those standard cases of deformation of beams which it may be necessary to use in the preliminary transference of loads in a framework problem.

by a substitution of particular values in the general form (6) for such a table; in this example it may easily be verified that the operations table is

	ΔF_{Au}	ΔF_{Av}	$10^{-2} \times \Delta F_{A\phi}$	ΔF_{Bu}	ΔF_{Bv}	$10^{-2} \times \Delta F_{B\phi}$
$\Delta u_A = 1$	−388.20	0	−4.98	382.67	0	0
$\Delta v_A = 1$	0	−638.98	−1.79	0	1.20	−1.79
$\Delta \phi_A = 10^{-2}$	−4.98	−1.79	−9.57	0	1.79	−1.79
$\Delta u_B = 1$	382.67	0	0	−388.20	0	−4.98
$\Delta v_B = 1$	0	1.20	1.79	0	−638.98	1.79
$\Delta \phi_B = 10^{-2}$	0	1.79	−1.79	−4.98	1.79	−9.57

TABLE (7)

In this table certain multiplying factors have been introduced for arithmetical convenience: the angular-displacement operations ($\Delta\phi_A$ and $\Delta\phi_B$) are given in units of 10^{-2} radian, and the residual torques ($F_{A\phi}$ and $F_{B\phi}$) are given as multiples of 10^{-2} ton-in.

25. The Moment-distribution Method

No new points arise in the actual relaxation of the residuals given in Figure 5 by use of the operations table (7); consequently the relaxation table and solution will not be given here but will be left as an example for the reader to complete. Instead, we shall consider a simplification of the problem of Section 24 which arises if we are concerned only to calculate the secondary (*i.e.*, flexural) stresses in the frame on the assumption that they can be dealt with entirely separately from the primary (*i.e.*, extensional) stresses. In doing this, we shall find that the relaxation method reduces, as a special case, to the moment-distribution method of Hardy Cross.* The exclusion of the primary stresses modifies the operations table (7) by causing the omission from it of all displacements which would result in an elongation of any member of the frame; the only permissible unit operations remaining are therefore the basic operations $\Delta\phi_A = 1$ and $\Delta\phi_B = 1$ and the block operation $\Delta u_A = \Delta u_B = 1$. We shall denote this block operation simply by $\Delta u = 1$, the increment Δu being taken to imply the increments Δu_A and Δu_B together. (They cannot be applied separately; otherwise a primary stress would be produced in AB.) The

* An account of the moment-distribution method was published by Hardy Cross in 1932 (*Trans. A.S.C.E.*, **96**, 1–10).

residuals which correspond to these three available operations are, respectively, $F_{A\phi}$, $F_{B\phi}$, and $F_u \equiv F_{Au} + F_{Bu}$, and their initial values are $F_{A\phi} = 81.8$, $F_{B\phi} = 8.4$, and $F_u = 0$. The operations table, derived from table (7), is

	$10^{-2} \times \Delta F_{A\phi}$	$10^{-2} \times \Delta F_{B\phi}$	ΔF_u
$\Delta\phi_A = 10^{-2}$	−9.57	−1.79	−4.98
$\Delta\phi_B = 10^{-2}$	−1.79	−9.57	−4.98
$\Delta u = 1$	−4.98	−4.98	−11.06

The liquidation of the residuals, performed with a use of this table, may be expected to be rapid, since a reduction of any one residual is seen to be automatically accompanied by a reduction of both the other residuals; this circumstance is, indeed, a usual feature of the moment-distribution method. In this example the solution is given in the relaxation table:

	$10^{-2} \times F_{A\phi}$	$10^{-2} \times F_{B\phi}$	F_u
$\phi_A = \phi_B = u = 0$	0.818	0.084	0
$\Delta\phi_A = 10 \times 10^{-4}$	−0.139	−0.095	−0.498
$\Delta u = -6 \times 10^{-2}$	0.160	0.204	0.166
$\Delta\phi_B = 2 \times 10^{-4}$	0.124	0.013	0.066
$\Delta\phi_A = 1.3 \times 10^{-4}$	0.000	−0.010	0.001
$\Delta\phi_B = -0.2 \times 10^{-4}$	0.004	0.009	0.011
$\Delta u = 0.1 \times 10^{-2}$	−0.001	0.004	0.000
$\phi_A = 11.3 \times 10^{-4}$ $\phi_B = 1.8 \times 10^{-4}$ $u = -5.9 \times 10^{-2}$	−0.002	0.003	0.000

26. Three-dimensional Stiff-jointed Frameworks

The method of solution of a two-dimensional stiff-jointed frame by relaxation, described in Section 23, can obviously be extended to solve three-dimensional framework problems. At each joint of such a framework there would be six components of displacement to find, three linear and three angular, and there would correspondingly be six components of residual force at each joint to be brought to zero in the process. Such problems are of considerable complexity to set up because it is necessary to take into account not only extensional and flexural actions in the members but also torsional actions. Consequently, and also because these problems contain nothing new from a purely relaxational point of view, they will not be discussed in any further detail in this book.

PROBLEMS

1. Verify the operations table (7) for the example of Section 24 and relax the residual forces shown in Figure 5, in order to find the displacements of the joints. (When the solution is obtained, it should be compared with the results of the simplified problem given in Section 25.)

2. Find the tensions in the five members of the plane pin-jointed framework shown in Figure 6. Each rod has the same uniform cross-sectional area and the

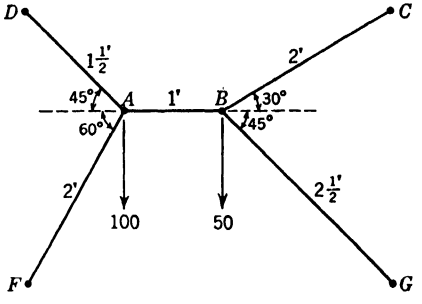


FIGURE 6

same Young's modulus. C, D, F, and G are fixed pins, and weights of 100 and 50 lb, respectively, are hung from A and B.

3. Figure 7 represents an electric network whose elements have resistances as shown. The junction E is earthed, and currents of 100 amp each are fed into

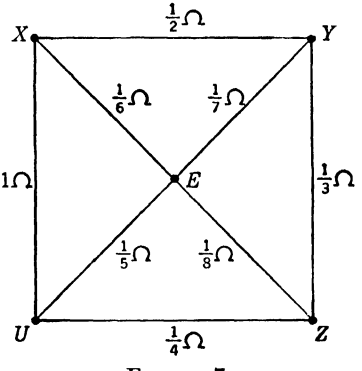


FIGURE 7

the network at X, Y, Z, and U. Find the current flowing in each element, correct to the nearest ampere.

(This problem is an example of another application of the relaxation method. It should be worked in terms of the junction potentials as unknowns, and the residuals should represent the unbalanced current flows into the various junctions.)

CHAPTER 4

ORDINARY DIFFERENTIAL EQUATIONS

27. Introduction

The most widespread, and perhaps the most important, application of the relaxation method has been to obtain particular solutions of partial differential equations in two dimensions, and the greater part of this book will be devoted to the solution of such equations (*e.g.*, Laplace's and Poisson's equations and the biharmonic equation). As a preliminary to the discussion of these partial differential equations, a simple type of ordinary differential equation in one dimension will be treated in this chapter. This equation, the beam-deflection equation, is, from the point of view of the relaxation method, the easiest of all differential equations to solve. At first sight this seems surprising because the equation is of the *second* order, but in general we shall find that equations of even order are much simpler to deal with relaxationally than equations of odd order. The reason for this circumstance is mostly to be found in the kind of boundary or end conditions that are required to be satisfied by any particular solution of a differential equation; indeed, it will be discovered later that the greatest difficulties encountered in the solution of differential equations are not contained in the equations themselves but instead in the boundary conditions which the solutions may have to satisfy.

28. The Beam-deflection Equation

In a beam of uniform flexural rigidity EI, the transverse deflection w is a function of position along the length of the beam and is related to the bending moment M by the equation

$$EI \frac{d^2w}{dx^2} = M(x),$$

where x measures distance along the length of the beam, usually from a convenient origin at the left-hand end. This equation is of the general form

$$\frac{d^2w}{dx^2} + W(x) = 0, \tag{1}$$

where W is a known function of x. By a "known function of x," we mean only to imply that a numerical value can be found for W corresponding to any given value for x. This value of W may be calculable by substitution in some analytical formula; it may be read off from a graph or interpolated from a set of tabular values; it could even be found as a direct experimental measurement; but it is to be emphasized that all that is necessary is that a numerical value for W should be determinable, by some means or other, for any chosen value of x in the range for which we wish to integrate equation (1).

To specify a particular solution of (1), two end conditions must be postulated. In general, these might be given either both at one end of the range of integration or one at each end; in this chapter we shall consider only the second of these possibilities. The first possibility—that both conditions be given at one end of the range—involves difficulties from a relaxational point of view which will not be resolved until Chapter 15. Also, to begin with, only the simplest type of end conditions will be assumed, *i.e.*, that the value of the wanted function w be given at both ends of the range of integration.

29. The Nature of a Relaxational Solution

It must first be explained what is meant by a *relaxational solution* to an equation such as (1). It is, of course, numerical, like all solutions found by a use of the relaxation method and, in fact, consists of values of the wanted function w at a number of equally spaced points of subdivision along the range of integration. In general, if that range extends from $x = a$ to $x = b$, then the end conditions will give the values of w at $x = a$ and at $x = b$, and the relaxation process will yield, as the required solution, values of w at the points $x = a + h$, $a + 2h$, . . . , $a + \overline{n - 1}h$, where h denotes the uniform interval between successive points of subdivision and where the range is divided into n such intervals, so that $b = a + nh$. The number of intervals, n, can be chosen as may be desired, and there are conflicting reasons why n should be both as small as possible and as large as possible; these will be discussed later in Section 37. For the present, in worked examples, n will be chosen to be fairly small, say 4 or 5, so that the method of solution can be sufficiently illustrated with a minimum of arithmetic.

30. Finite-difference Approximations

In any application of the relaxation method to solve a differential equation, certain finite-difference approximations to derivatives have to be used, and we shall at once proceed to establish two of these approxima-

tions which will be needed in this chapter; these are the approximations,* or replacements, for the derivatives dw/dx and d^2w/dx^2, holding at a typical point of subdivision of the range of solution. When working out a general result for any *typical* point of subdivision, we always denote that point by the number 0, and the adjacent points of subdivision are always denoted by the numbers 1 (to the right) and 3 (to the left),† as illustrated in Figure 8. Corresponding suffixes are used to indicate values of the functions w and W at these typical points of subdivision.

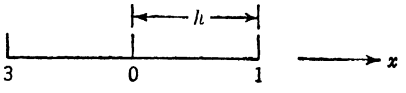


FIGURE 8

Now, in the neighborhood of any typical point 0, the function w can be supposed to be expanded in a Taylor's series:

$$w = w_0 + \left(\frac{dw}{dx}\right)_0 (x - x_0) + \frac{1}{2!}\left(\frac{d^2w}{dx^2}\right)_0 (x - x_0)^2 + \frac{1}{3!}\left(\frac{d^3w}{dx^3}\right)_0 (x - x_0)^3 + \frac{1}{4!}\left(\frac{d^4w}{dx^4}\right)_0 (x - x_0)^4 + \cdots.$$

In this relation, if x is put equal, in turn, to $x_0 + h$ and to $x_0 - h$, we find that

$$\begin{aligned} w_1 &= w_0 + h\left(\frac{dw}{dx}\right)_0 + \frac{h^2}{2}\left(\frac{d^2w}{dx^2}\right)_0 + \frac{h^3}{6}\left(\frac{d^3w}{dx^3}\right)_0 + \frac{h^4}{24}\left(\frac{d^4w}{dx^4}\right)_0 + \cdots, \\ w_3 &= w_0 - h\left(\frac{dw}{dx}\right)_0 + \frac{h^2}{2}\left(\frac{d^2w}{dx^2}\right)_0 - \frac{h^3}{6}\left(\frac{d^3w}{dx^3}\right)_0 + \frac{h^4}{24}\left(\frac{d^4w}{dx^4}\right)_0 + \cdots. \end{aligned} \tag{2}$$

Adding these two results together, we get

$$w_1 + w_3 = 2w_0 + h^2\left(\frac{d^2w}{dx^2}\right)_0 + O(h^4),$$

where all the terms containing fourth or higher powers of h are included together as $O(h^4)$. With neglect of this quantity, which becomes continually smaller as h is made smaller and smaller, we have the finite-difference approximation to $(d^2w/dx^2)_0$:

$$h^2\left(\frac{d^2w}{dx^2}\right)_0 = w_1 + w_3 - 2w_0. \tag{3}$$

* A list of these and other useful finite-difference approximations is collected for ease of reference in Appendix III, page 241. It may be mentioned here that finite-difference methods have long been used as a foundation for approximate numerical work.

† The reason for this apparently peculiar notation will become clear in Chapter 5, when the corresponding notation in two dimensions is defined.

There is an admitted error, of order h^4 at most, in the relation (3); the consequences of this error will be discussed later, in Section 37, but for the time being it may be ignored.

Similarly, by subtracting the results (2) from one another, and with neglect of terms of order h^3 at most, we obtain a finite-difference approximation to $(dw/dx)_0$:

$$h\left(\frac{dw}{dx}\right)_0 = \frac{w_1 - w_3}{2}. \tag{4}$$

31. Residuals

Now, according to the relation (3), satisfaction of the differential equation (1) at the typical point 0 requires that

$$w_1 + w_3 - 2w_0 + h^2W_0 = 0. \tag{5}$$

It must be emphasized that this equation is typical of relations which connect every set of three w values at every set of three successive points of subdivision. In all, then, there are $n - 1$ equations of this type, one when each point of subdivision is taken, in turn, as the typical point 0 (it should be noticed that we do not here regard the end points of the range as points of subdivision and do not contemplate the possibility of their being taken as the point 0). In effect, therefore, the differential equation has been replaced by a set of $n - 1$ simultaneous algebraic equations to be satisfied by $n - 1$ values of w at $n - 1$ points of subdivision. At each point of subdivision a residual is immediately defined, from (5), by the typical formula

$$F_0 \equiv w_1 + w_3 - 2w_0 + h^2W_0. \tag{6}$$

Because of the fact that every residual is defined, by (6), in a formula of the same form, a much more convenient method of setting out the work can be adopted than that involving the construction of an operations table and a relaxation table. It will soon be realized that the particularly simple way in which the work can, in fact, be set out is the real reason why the relaxation method is even more suited to solving differential equations than it is to solving sets of algebraic equations of the kind which were discussed in Chapters 1 to 3.

32. Relaxation Patterns

Instead of an operations table we construct what is known as a *relaxation pattern* directly from the formula (6), by means of which each residual is calculated. For it is at once obvious that the residual at any point of subdivision has its value changed by -2 if a unit increment is added to the value of w at that point, and has its value changed by 1 if a

unit increment is added to the value of w at either neighboring point of subdivision. Hence the complete effect of a unit increment applied to the value of w_0 at a typical point is to change the residual at that point by -2 and to change the residuals at the two neighboring points each by 1. This effect is illustrated in Figure 9 in the form of a relaxation pattern, where the changes to the residuals are recorded in circles in the same relative positions as would be occupied by any set of three typical points of subdivision actually marked out along the range of integration; the unit increment $(+1)$ is also recorded in Figure 9 in a position near to the typical point where it is applied. This relaxation pattern represents a basic unit operation available for use in the solution of equation (1). Being typical of the effect produced by the application of a unit increment to w at any point of subdivision, it is, in fact, a diagrammatic representation of all the basic unit operations available. This simple pattern replaces a whole operations table, and contains in itself as much information as was given in a lengthy table in some previous applications of the method.

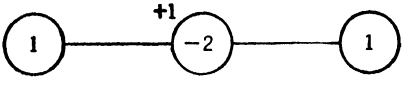


FIGURE 9

33. Relaxation Diagrams

The actual process of the liquidation of the residuals, together with the consequent determination of the solution, is carried out on a *relaxation diagram* instead of in a relaxation table. Figure 10 illustrates the beginning of such a diagram; it consists of a diagrammatic representation of a

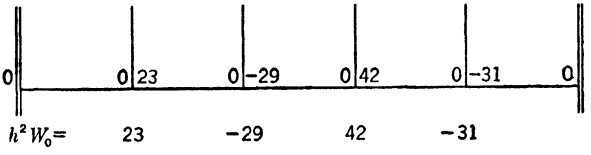


FIGURE 10

range of integration, divided, as an example, into five intervals by four points of subdivision. At the end points of the range are recorded the values of w which are given there as end conditions (in this example both conditions are supposed to be that $w = 0$). Below each point of subdivision is recorded the value of the quantity h^2W_0, which in this example is supposed to have already been found to be 23, -29, 42, and -31 at the first, second, etc., points of subdivision. (These values have been selected quite arbitrarily, purely for illustrative purposes; no matter, in practice, how W may be defined, it may be presumed that h^2W is immediately calculable at each point of subdivision.) Also in Figure 10 at each

point of subdivision are recorded initially *guessed* values of w (above and to the left) and corresponding initially *calculated* values of the residuals (above and to the right). In this example the values of w have been taken, as a beginning, all to be zero, so that the residuals, calculated from the formula (6), are each just equal to the appropriate value of h^2W_0. The position shown in Figure 10 corresponds exactly to the first line in a relaxation table.

The liquidation process is recorded directly on this same diagram, and in this first example the solution will be achieved by the fundamental method of relaxation with a use only of the basic rule of relaxation (Section 5). Increments added to the w values are written down in columns above the initially guessed values, and successively changed values of the residuals are similarly written down in columns above the initially calculated values. Thus, in Figure 10, applying the basic rule, we pick out the largest residual, which is 42 at the third point of subdivision, and apply at that point such an increment as will reduce that

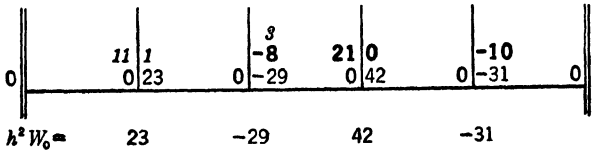


FIGURE 11

residual to zero. The required increment is seen, from the relaxation pattern of Figure 9, to be $\Delta w = 21$, which increment changes the residual at the point where it is applied by -42, and at the same time changes the residuals at the two adjacent points by 21. The complete operation is recorded in **heavy** type, in Figure 11; the residuals at the second, third, and fourth points of subdivision are seen to be changed in this first step of the liquidation process to become -8, 0, and -10, respectively. The second step similarly consists in the application of an increment $\Delta w = 11$ at the first point of subdivision in order to bring to zero the value of the residual there (which after the first step had become the largest, *i.e.*, 23). This step is also recorded in Figure 11, in *italic* type; the application of the increment $\Delta w = 11$ changes the residual at the first point by -22 to become zero and at the same time changes the residual at the adjacent (second) point by 11 to become 3. To the left of the first point of subdivision there is no residual for which to record any change. As has already been pointed out (Section 31), residuals cannot exist at the end points of the range, where the values of w do not have to be *found* but are instead *given* as end conditions. In effect, this means that the relaxation patterns for unit increments applied at the first and last points of sub-

division are, strictly, slightly modified from the pattern of Figure 9; they are, correctly, the patterns shown in Figures 12 (i) and (ii), respectively. In practice, of course, it is hardly necessary to think of the patterns in Figure 12 as being different from that of Figure 9.

The complete relaxation diagram for the example started in Figure 11 is shown in Figure 13, in the form in which it would be recorded in practice, with the exception that the sequence of largest residuals, at successive steps in the process, have been underlined so that the reader may more

FIGURE 12

easily follow through the progress of the calculations. When the liquidation of the residuals is completed, *i.e.*, when they have all been made no larger, in magnitude, than 1, the solution is obtained by adding up the columns of increments Δw for each point separately (including, of course, in general, the initial guessed values of w, which in this example happen to be zero); these totals are recorded in Figure 13 at the top of their respective columns, in **heavy** type. As a last step in the solution, the final values of the residuals, at the top of the right-hand columns, should be checked by a direct substitution of the totaled w values in the formula (6);

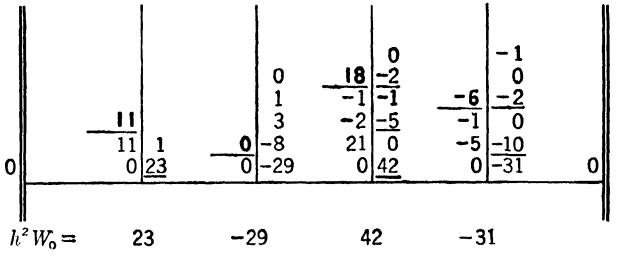


FIGURE 13

if any errors are then discovered, the relaxation must be continued until eventually the residuals are, correctly, brought to zero.

34. Block Patterns

In the relaxation method of solution of an ordinary differential equation by far the most useful of the devices described in Chapter 2 is block relaxation. By a block operation is meant, here, the simultaneous application of increments to w values at two or more adjacent points of subdivision. Unit *block patterns* are easily constructed, and Figure 14

(i) represents the simplest possible block pattern for the case when the increment $\Delta w = 1$ is applied at only two points of subdivision; Figure 14 (ii) shows the next largest block pattern, when the increment is applied at three points, and Figure 14 (iii) illustrates the general form of block pattern where an increment is applied at any number of points. Of course, once the form of the general block pattern (iii) has been realized—as an obvious generalization of the simpler patterns (i) and (ii)—it becomes apparent that this general pattern is the only one which really need be recorded, because it includes both (i) and (ii) as special cases, as well as, in fact, the *point pattern* of Figure 9 for the case of an increment applied at a single point only.

These patterns greatly facilitate the relaxation process; their use is, of course, known as *block relaxation*, in contrast to the use of the single-point

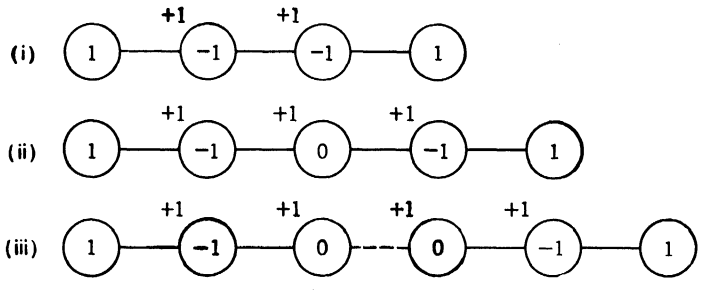


FIGURE 14

pattern which is usually referred to as *point relaxation*. In most problems the greater part of the liquidation of the residuals is achieved by block relaxation, with point relaxation reserved until the solution is nearly completed, in order to make small local adjustments. The other devices of Chapter 2 can be carried over also for use in the kind of relaxation described in this chapter, but no additional description of them is needed. It would rarely be necessary to employ either group relaxation or multiplying factors to solve an equation of the form (1), but their use is virtually imperative in order to solve some of the more difficult equations considered in later chapters.

One particularly important use of block relaxation should be emphasized, and it will now be illustrated to solve the problem contained in Figure 15. This time the range has been divided into nine intervals by eight points of subdivision, and the values of h^2W are again supposed to have been already calculated and to be as recorded. The values of w at the end points of the range are taken to be given as 0 and 100, and a reasonable first guess for the values of w at the eight points of subdivision is the approximately linear distribution shown; the corresponding initial

values of the residuals may be easily verified. Liquidation of these residuals with a use only of point relaxation, even with the help of some overrelaxation, would be a lengthy process, mostly because all the residuals are of the same sign (positive) and because the point pattern merely redistributes the residual values from one point to the next; it does not effect any change in their total except when an increment is applied at either the first or last point of subdivision, where, according to the patterns of Figure 12, one unit of residual is lost for each unit increment Δw applied. Relaxation can be complete, therefore, only when all the initial residual values have been gradually redistributed toward the two end points and eventually lost at these ends. The relaxation process must, as it were, sweep all the initial residuals away from the center of the range toward, and finally onto, the end points, where they are of no account, where, indeed, they can no longer exist or be recorded. This sweeping out

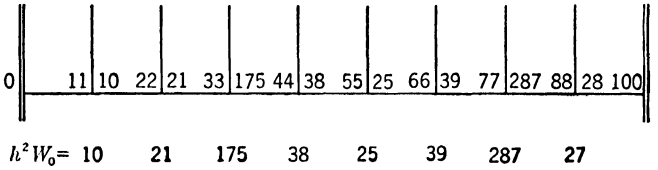


FIGURE 15

of the residuals may, in an over-all sense, be rapidly accomplished by what is known as the process of *blocking outward;* it consists in the application of a series of centrally situated and gradually more and more extensive block operations. These successive operations cover either two, four, six, eight, etc., points or one, three, five, seven, etc., points according as the total number of points of subdivision is even or odd. In the example of Figure 15 the number is even, so the first block operation to be applied extends over the central two points and should be of such a magnitude as to leave a zero total residual at these two points. Now the effect of any unit block operation (Figure 14) is to change the residuals at the two points at the ends of the block each by -1, while 1 is added to the residuals at the two adjacent points on either side of the block; thus each unit block operation has the effect of reducing the total residual at the points where it is applied by 2. In Figure 15 the total residual at the central two points is $(38 + 25 =)$ 63, so that a block operation covering these two points, of magnitude $\Delta w = 31$, must reduce that total at once to zero. When this has been done and the effects of the operation recorded, the total residual at the central four points may be verified to be 277, so that a block operation covering these four points, of magnitude $\Delta w = 138$, must in turn reduce that total also to zero. When this second step has

been achieved, similar block operations (of magnitude $\Delta w = 292$ and $\Delta w = 310$) applied to cover the central six and central eight points, respectively, reduce each total residual at these sets of points to zero. Each of these four block operations is shown recorded on the relaxation diagram in Figure 16. Their combined effect is to leave residuals, at the various points, which have zero totals over the whole range of eight points and also over the central six, four, and two points separately.

At this stage in the calculation the individual residuals are noticeably smaller than they were initially, and, moreover, they are, and must necessarily be, well distributed, some being positive and others negative, so that the solution can quickly be completed by point relaxation and small local applications of block relaxation. The solution will not be

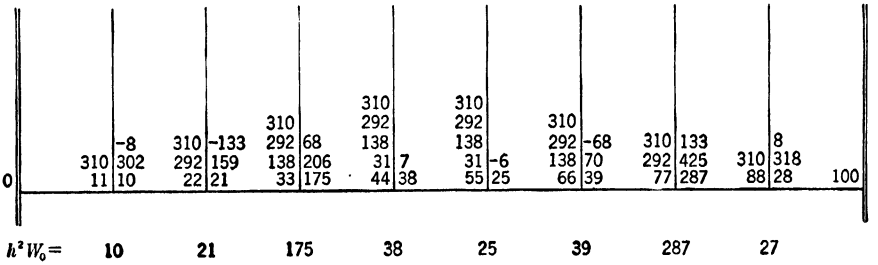


FIGURE 16

continued here but instead left as an example for the reader to finish; he will then soon realize that the greater part of the liquidation of the residuals has already been accomplished.

35. Criteria for Complete Liquidation of Residuals

It has been suggested (Section 33) that the liquidation of a set of residuals can be regarded as completed when, to the number of significant figures in use, each individual residual has been made as small as possible (*i.e.*, no larger in magnitude than 1 when the pattern of Figure 9 is being used). This criterion is necessary, but it is not quite sufficient, and now that the use of block patterns has been introduced, it can be supplemented by two additional criteria: first, that the total of all the residuals should be brought to zero, and, second, that those residuals which individually are not exactly zero should be well distributed along the range of integration. Thus, as an extreme example, the liquidation could not be regarded as complete if every residual were left with a value -1; nor would the liquidation be properly finished if, with, say, eight points of subdivision, the residuals at the first four points were each -1, and the residuals at the last four were each 1. In both cases an improvement could be effected by

a use of block relaxation, even though point relaxation would be no longer helpful.

36. Practical Aspects of Relaxation Technique

One or two practical aspects of the technique of relaxation can be appropriately mentioned now. First of all, it must be apparent that, in more complicated problems, the columns of figures which have to be written down in a relaxation diagram may become inconveniently lengthy as the work proceeds; this can be avoided if the work is done in pencil rather than ink, for, at any stage in the calculation, the only quantities that really matter are the current values of the residuals and the totals of the columns of w increments. So when any residual column becomes awkwardly long, it can be entirely erased, provided that the value at the top of the column is written down again in the space provided where the initial residual had been recorded; similarly a column of w increments can be erased provided that it is summed beforehand and the total recorded in the position previously occupied by the original guessed w value. By means of periodic erasures in this way it is possible to keep the number of figures actually on record at any one time comparatively small, so that it is easier to keep an eye on the ones that are important (*i.e.*, the current values of residuals) without any confusion arising from those that are no longer significant. It has been found that good-quality tracing paper is best for working on, for two reasons: first, it is much more durable than other types of paper in withstanding continual erasures, and, second, it becomes possible to draw out, on the reverse side of the paper, the framework of the relaxation diagram so that these lines are not also removed when figures are being erased.

For the initial calculation and subsequent checking of residual values, it is virtually essential that a calculating machine be available. It cannot be denied that the checking of these values is a tedious task, even though straightforward, and the value, to the success of the method, of relieving as much as possible of a computer's monotony is exceedingly important but sometimes overlooked. With a machine at hand it is not very much trouble to check the residuals fairly frequently in the course of the work, so that any errors which may occur in the actual arithmetic of liquidation can be taken into account without delay.

Last, it cannot be too strongly emphasized that attention should not be overmuch concentrated on the values of individual residuals until the very last stages in a relaxation calculation. For the most part the values of the residuals should be regarded as a whole, and the first aim should be to ensure that they become a well-balanced set; small local increments can

easily be made in the end to cause local redistributions and hence canceling out a set of residuals whose total value is approximately zero.

37. Finite-difference Inaccuracies

Having described the actual process of the method in its application to solve an equation of the type (1), we must now consider the consequences of the approximation which was used in Section 30, where, in the replacement of derivatives by finite-difference expressions, terms containing the higher powers of h were neglected. It is, of course, apparent that the consequent error introduced into the solution decreases as h is made smaller, and the only really satisfactory way of estimating what the error amounts to (apart from a certain facility acquired with experience) is to find the solution for a series of different values of h. An example will best illustrate how conclusions may be drawn from a comparison of solutions obtained with a use of differing values of the interval h. Suppose we want to solve the equation*

$$\frac{d^2w}{dx^2} + 4{,}096x(1 - x) = 0, \quad (7)$$

over the range $0 \leq x \leq 1$, with end conditions given to be that $w = 0$ at both $x = 0$ and at $x = 1$. In order to estimate the accuracy of the numerical results, the solution should be found successively for intervals $h = \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$, and this has been done on relaxation diagrams which are summarized in Figure 17. Since there is symmetry about the center point, only half the range need be shown, and in Figures 17 (i) to (iii), respectively, are given the final values of w and of the residuals at intervals $h = \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$. The convergence of these results to the exact solution as h is made smaller and smaller may be judged by a comparison of the values of w obtained, in these three cases, at the center of the range; these values are 128, 112, and 108, and, strictly, to make certain of absolute accuracy, the solution should be found for still smaller intervals, $h = \frac{1}{16}, \frac{1}{32}$, etc., until for two successive values of h there is no change in the value of w (to the number of significant figures quoted) at any point. Thus in this example the central value of w would be found to be 107 for both $h = \frac{1}{16}$ and $h = \frac{1}{32}$, so that we could then state, without

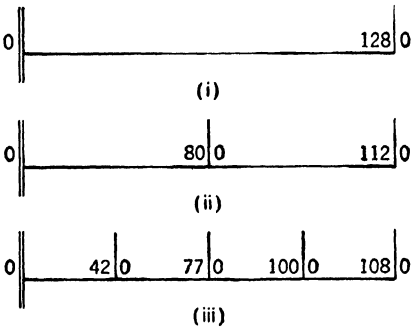


FIGURE 17

* This equation has been deliberately chosen as having an easily determined exact analytical solution for comparison with the relaxational results. The factor 4,096 is inserted in it only to avoid fractions or decimals in the working.

doubt, that for this problem an interval $h = 1/16$ is sufficiently small to yield w correctly to the nearest whole number. However, it is not usually necessary to go to these lengths because an estimate of the correct value of w can be made, within a stated percentage inaccuracy, with a use of the values found for larger intervals.

Thus suppose the exact central value of w is w_c; then the errors in the values obtained for $h = 1/2$, $1/4$, and $1/8$ are, respectively, $128 - w_c$, $112 - w_c$, and $108 - w_c$. It is reasonable to assume that the ratio of the errors obtained for two successive values of h is perpetuated as the number of intervals is repeatedly doubled. Thus we have

$$\frac{128 - w_c}{112 - w_c} = \frac{112 - w_c}{108 - w_c},$$

which yields $w_c = 107$, to the nearest whole number. It may be noticed that this value differs from the value 108, corresponding to $h = 1/8$, by less than 1 per cent. On a basis, therefore, of solutions found for three successive values of h ($= 1/2$, $1/4$, and $1/8$), we can estimate that the correct central value of w is 107, with a possible error of less than 1 per cent. If this were not regarded as a sufficiently good result, then it could be improved on a basis of solutions found for $h = 1/4$, $1/8$, and $1/16$.

In point of fact, the correct result, evaluated exactly by straightforward integration of equation (7), is $w_c = 106\frac{2}{3}$.

The method of extrapolation which we have used in the present example can be easily given as a formula for general use. If w', w'', and w''' are the values found for a quantity w for three successive values of h, then the extrapolated estimated value is given by

$$w = \frac{w'w''' - w''^2}{w' + w''' - 2w''}.$$

It will have been noticed that the successively smaller values of h used in the above process form a sequence in which each value is one-half the preceding one; it is important that this should be, so that the successive solutions are derived at as many common points of subdivision as possible. This circumstance is not only necessary for the comparison of values of w for successively smaller intervals, but also enables each solution itself to be found more rapidly than might otherwise be possible. Thus, with $h = 1/2$, the solution given in Figure 17 (i) can be found in less time than is taken to write it down, and then that solution enables a good initial guess to be made for the case when $h = 1/4$, so that the solution in (ii) can also be found very quickly without much relaxation being required. In turn, that solution provides very good initial values of w for the next case, when $h = 1/8$, and so on. Evaluation of a set of solutions with the

interval h successively halved may save much time in more complicated examples and, in particular, as we shall see later, in examples in two dimensions.

38. Slope End Conditions

It is worth while here, again mostly with a view to extension into two dimensions, to consider how a second type of end condition may be satisfied in a solution of an equation of the form (1). This is the condition that dw/dx have a given value at one end of the range, instead of w itself being given there. Suppose, in general, that it is required to satisfy the condition

$$\frac{dw}{dx} = k \tag{8}$$

at the right-hand end of the range of integration of the equation (1), where k is a given number. Now, w is no longer known initially at this end point but must instead be found there, just as it has to be found at the points of subdivision along the range. We need, therefore, to be able to define a residual at this end point. The formula (6), which defines residuals at ordinary points of subdivision, cannot be directly applied if we take the end point as the point 0, because then the point 1 would not exist and no w_1 would be available for substitution in (6). In a case like this, we say that the point 1 is a *fictitious point*, and any value, such as w_1 here, which might be associated with a fictitious point is called a *fictitious value*. Figure 18 indicates the position of such a fictitious point at the right-hand end of the range of integration. However, the governing differential equation must be satisfied at the end point 0 in Figure 18, so that a residual, as defined by the formula (6), must be made to vanish there in the determination of the solution, and therefore some means must exist to enable us to assign a numerical value to the fictitious w_1. This means is provided by the end condition (8), which, according to the finite-difference approximation (4), may be replaced by the relation

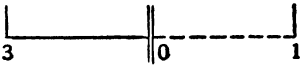


FIGURE 18

$$w_1 - w_3 = 2hk. \tag{9}$$

In effect, (9) enables a value to be attached to the fictitious w_1 by expressing it in terms of w_3, k, and h. In practice, we never actually evaluate the fictitious value but instead use (9) in order to eliminate w_1, by substitution, from the formula (6), and so derive

$$F_0 \equiv 2w_3 - 2w_0 + 2hk + h^2W_0, \tag{10}$$

as a special formula for the residual at the end point of the range; in this form the residual may be calculated directly from the real values of w actually recorded on the relaxation diagram. A very similar formula could be obtained, in the same way, if a slope condition were given at the left-hand end of the range. The modified formula (10) naturally gives rise to slightly modified relaxation patterns for increments applied at points in the neighborhood of the end of the range. These are shown in Figure 19, in which (i) is the pattern for a unit increment applied at the end point itself and (ii) is the pattern for an increment applied at the

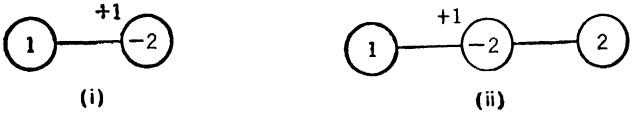


FIGURE 19

adjacent point of subdivision. No more difficulty will be found in the use of these patterns than has been experienced in the use of the other patterns illustrated in this chapter.

PROBLEMS

1. Find solutions to the equation $d^2w/dx^2 + W = 0$, satisfying end conditions that $w = 0$ at both ends of the range of integration, when it is given that the range is divided into five intervals and that the values of h^2W at the four points of subdivision are (1) 39, −66, 42, and −20, and (2) 39, 66, 42, and 20, from left to right, respectively.

2. Complete the solution of the example started in Section 34.

3. A nonuniform beam is 12 in. long and is simply supported at its ends. Young's modulus for the beam is 3×10^7 psi, and it carries a uniformly distributed load of 200 lb/in. along its whole length. The second moment (I in.^{4}) of the cross-sectional area of the beam is given at points distant x inches from one end in the table:

x	0, 12	1, 11	2, 10	3, 9	4, 8	5, 7	6
$100I$	1.200	1.245	1.281	1.308	1.327	1.338	1.342

Determine, to three significant figures, the maximum transverse deflection of the beam.

4. Solve the equation $d^2w/dx^2 + \sqrt{(1 + x^3)} = 0$ to the nearest whole number, over the range $0 \leq x \leq 8$, when $w = 1$ at $x = 0$ and $dw/dx = -10$ at $x = 8$. Use successive intervals $h = 4$, $h = 2$, and $h = 1$, and estimate the correct value of w at $x = 8$.

CHAPTER 5

LAPLACE'S AND POISSON'S EQUATIONS

39. Introduction

In two dimensions a direct extension of equation (1) of the previous chapter is provided by *Poisson's equation:*

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + W(x,y) = 0, \tag{1}$$

of which a special case is the important Laplace's equation:

$$\nabla^2 w = 0, \tag{2}$$

where the operator ∇^2 is used to stand for $\partial^2/\partial x^2 + \partial^2/\partial y^2$. Laplace's equation governs a large number of physical quantities and arises in almost every branch of applied mathematics. It is satisfied, for instance, by any two-dimensional distribution of electric or magnetic potential in regions, respectively, of constant resistivity and permeability; it governs both the stream function and the velocity potential in the two-dimensional irrotational flow of an incompressible fluid; and in the theory of heat it must be satisfied by any steady temperature distribution in a two-dimensional region of uniform conductivity. An example of Poisson's equation occurs in the solution of the "torsion problem" in elasticity (*i.e.*, in the determination of the shear stresses produced in a uniform cylinder subjected to a twisting action).

Since, from a relaxational point of view, (1) is no more difficult to solve than (2), the method will be described, in general terms, with reference to Poisson's equation, but to avoid unhelpful arithmetic, particular examples illustrating the method will mostly be of Laplace's type.

40. Relaxation Nets

In two dimensions, as in one, a relaxational solution to a differential equation consists of a finite number of values of the wanted quantity w at a number of points within the region of integration. Whereas in one dimension a range of integration was divided up by points of subdivision into a number of equal intervals, on the other hand in two dimensions an area of integration is subdivided by a uniform network, and the values of

w are calculated at the nodes of this network. It is geometrically possible to cover an area by three different shapes of uniform meshes, either square, equilateral-triangular, or equiangular-hexagonal, as illustrated in Figure 20. Such networks are known as *relaxation nets*. The hexagonal one is only a geometrical curiosity and is never utilized; almost always a square net is the one which is used, although on rare occasions there are special reasons why a triangular net should be employed. In all cases we refer

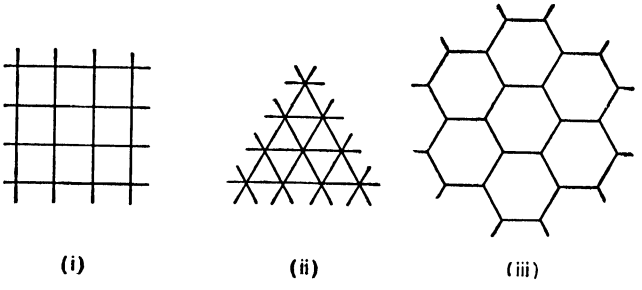


FIGURE 20

to the lines of the net joining up the nodes as *strings* and denote the length of each such string by h. To begin with, we shall be concerned with the solution of an equation, such as (1) or (2), only over a region which has a rectilinear boundary and for which the net can be so arranged that the boundary is entirely composed of strings of the net. This restriction will be removed in Section 47, where it will be explained how any shape of boundary may be dealt with.

41. Poisson's Equation

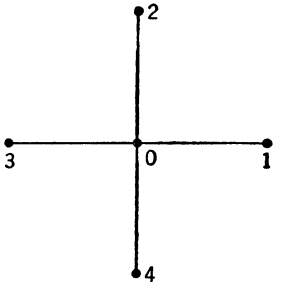


FIGURE 21

The first step in the solution of any differential equation is the replacement of that equation by a set of finite-difference approximations relating the values of the wanted quantity at the nodes of a relaxation net; for Poisson's equation this can be very easily done, since it involves only a use of the approximation (3) of Section 30 for a second derivative. Figure 21 illustrates the convention used in two dimensions to denote a typical node 0 of the net, and the four surrounding nodes 1, 2, 3, and 4;* as previously, suffixes will be used to indicate values of the variables at correspondingly numbered nodes. Now it has already been shown in Section 30 that

* The reason for the convention employed to denote a typical group of three points of subdivision in one dimension will now be apparent.

$$h^2\left(\frac{\partial^2 w}{\partial x^2}\right)_0 = w_1 + w_3 - 2w_0 \tag{3}$$

is the finite-difference approximation for the second derivative of w, at the point 0, with respect to x. In the same way it may be shown that, for the second derivative with respect to y, the approximation is

$$h^2\left(\frac{\partial^2 w}{\partial y^2}\right)_0 = w_2 + w_4 - 2w_0. \tag{4}$$

Adding (3) and (4) together, we at once obtain the finite-difference approximation for $\nabla^2 w$ at a typical node:

$$h^2(\nabla^2 w)_0 = w_1 + w_2 + w_3 + w_4 - 4w_0 = \Sigma w_1 - 4w_0,$$

where Σw_1 stands for the sum of the values of w at those four points, surrounding the point 0, which are typified by the point 1. Hence, the approximation to the equation (1), holding at a typical node 0, is seen to be

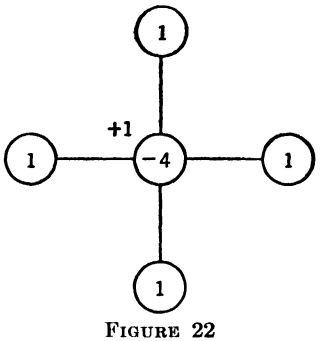


FIGURE 22

$$\Sigma w_1 - 4w_0 + h^2 W_0 = 0. \tag{5}$$

There is, again, an error, of $O(h^4)$, in this equation, and the effect of this error again diminishes with the mesh size h. Equation (5) must be satisfied by the wanted values of w at every group of five nodes. For any other values of w that, in general, might be assigned at the nodes of the net, (5) will not usually be true anywhere; instead a nonzero residual, given by the formula

$$F_0 \equiv \Sigma w_1 - 4w_0 + h^2 W_0 \tag{6}$$

will be calculable at each node, and the process of relaxation is aimed at eventually reducing all these residuals to zero by the application of increments Δw, so that automatically the w's at the same time acquire their wanted values. The basic unit operation by means of which this process may be accomplished is contained in the relaxation pattern of Figure 22, which shows diagrammatically the changes produced in the values of the residuals when a unit increment $\Delta w_0 = 1$ is added to the value of w at a typical node of the net.

42. Boundary Conditions

In order that a particular solution of Poisson's equation may be defined, certain conditions in terms of w must be specified to hold on the boundary of the area of integration. A single condition is usually speci-

fied at all points of the boundary, and the most frequently encountered types of conditions are either that the value of w itself be given on the boundary, or that the value of the normal gradient $\partial w/\partial \nu$ be given, where ν denotes the normal direction across the boundary at any point, directed outward from the area of integration. Sometimes mixed boundary conditions may be specified, by which is meant that w may be given at some parts of the boundary and $\partial w/\partial \nu$ may be given at other parts. A method of satisfying a normal-gradient boundary condition will be explained in Chapter 6; in this chapter we shall be concerned only with boundary conditions which specify the values of w.

43. An Example of Laplace's Equation

An example to illustrate how the work is set out is provided by the following problem: To determine the steady temperature distribution, in

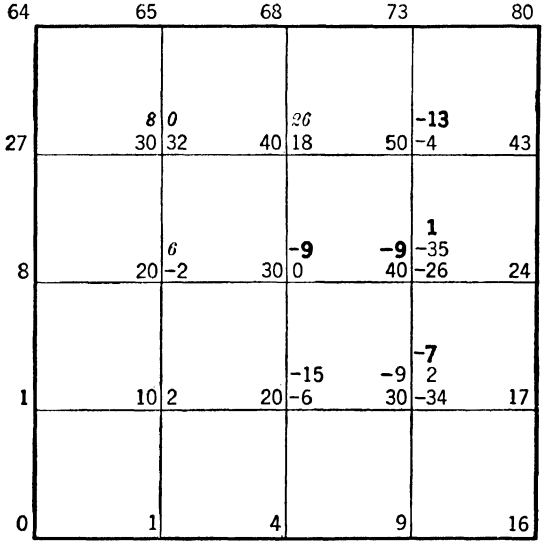


FIGURE 23

two dimensions, in a square of uniform conductivity when the temperature along the sides of the square is maintained at known values. Suppose that the square is bounded by the lines $x = 0$, $x = 4$, $y = 0$, and $y = 4$, that the boundary values of the temperature w are given to be $(x^2 + y^3)$°C, and that a square net of side $h = 1$ is used. Figure 23 shows the relaxation diagram and the first few steps in the calculation of the solution; at nodes on the boundary are recorded values of the temperature given by the formula $(x^2 + y^3)$°C. At the (nine) internal nodes, initially guessed values of w are recorded (above and to the left), and

corresponding initial residuals (above and to the right) calculated from the residual formula for Laplace's equation, *i.e.*, equation (6) with W_0 put equal to zero. With a use of point relaxation alone, the liquidation process picks out the largest of the initial residuals (−34, at the node nearest the bottom right-hand corner) and, as a first step, applies an increment to the w value there of sufficient magnitude to bring that residual (temporarily) to zero. The required magnitude of this increment is, according

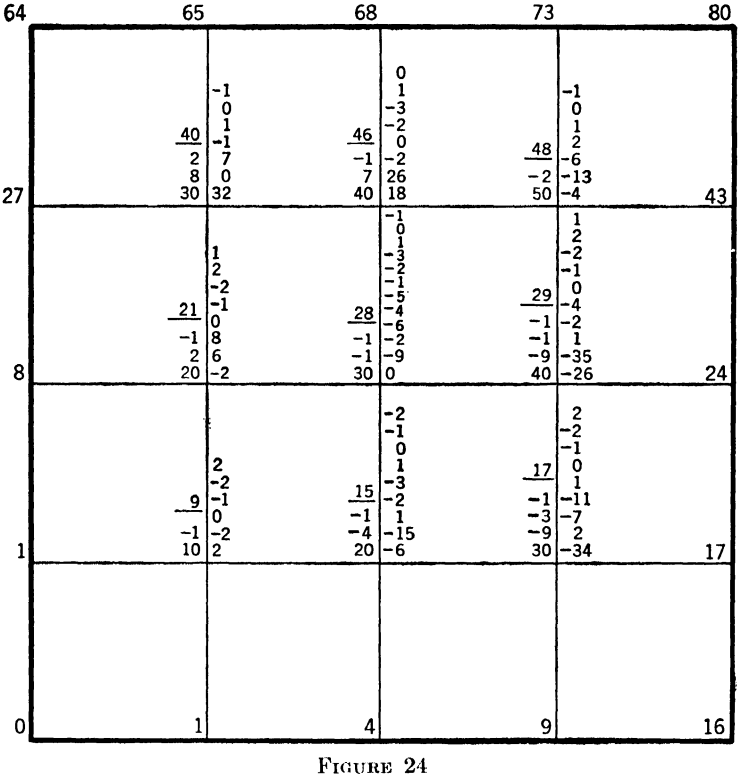


FIGURE 24

to the pattern of Figure 22, $\Delta w_0 = -9$; the effect of its application is to change the residual from −34 to become 2, while at the same time the residuals at the two neighboring nodes, one above and one to the left, are both changed, from values of −26 and −6, respectively, to become −35 and −15. The second step in the process, which is also shown in Figure 23 but in **heavy** type, is aimed at bringing to zero that residual which has become the largest after the completion of the first step, *i.e.*, the residual (−35) at the node nearest the middle of the right-hand side of the square. Again an increment $\Delta w_0 = -9$ is needed; the residual is thereby changed to become 1, and the residuals at three neighboring nodes are all changed

by -9. The third step continues the process in the same way; it is recorded in Figure 23 in *italic* type and consists of the increment $\Delta w_0 = 8$ applied in order to relax the residual (32) at the node nearest the top left-hand corner. By carrying on in this manner, we can eventually complete the liquidation of the residuals, and the whole calculation is shown in Figure 24, where at each node of the net are recorded two columns of figures: to the right the column consists of successive values of the residual at that node; to the left the column consists of the initial guess for w, at the bottom, and successive increments Δw applied during the computation. At the top of each of these columns of increments are recorded their respective totals; these are the w values which constitute our solution to the problem for a net of size $h = 1$. The final values of the residuals at the tops of the residual columns have been checked by a substitution of the w totals back into the residual formula (6). A glance at their values shows that, to the nearest whole number, the relaxation is properly completed: each residual is individually zero (*i.e.*, not greater in magnitude than 2); the total of all the residuals is only 1; and they are well distributed throughout the square.

The practical aspects of the technique of relaxation, mentioned in Section 36 in respect of one-dimensional problems, are equally, if not more, worthy of notice in two dimensions. In particular, the length of the columns which can be written down is limited by the size of the squares as drawn out in the relaxation diagram. A convenient size is $1\frac{1}{2}$ in. square, but this dimension might have to be made smaller if a fine net is being used; otherwise the size of the whole diagram could become inconveniently large.

44. Block Patterns

The use of block patterns is particularly suited to the solution of a Poisson-type equation, because their effects can be described in a very simple way. To begin with, we may build up the patterns corresponding to unit increments applied to the w values at small blocks of nodes; Figure 25 shows two such patterns: (i), when the block consists of three nodes in a line, and (ii), when it comprises four nodes in a square. Each may be verified by superposing, respectively, three and four point patterns of the type shown in Figure 22. In both the patterns of Figure 25 a dotted contour has been drawn in which surrounds the nodes at which the increment $\Delta w = 1$ is applied and which separates those nodes from the remaining nodes of the net. In both cases it may be seen that the effect, on the residuals, of the application of the block increment is *to transfer one unit of residual down each string of the net which crosses the bounding contour*, from the node at the end of each string inside the con-

tour to the node at the end outside. A little thought soon confirms that the effect of a unit increment added to the w values at the nodes of a block of any size or shape may be described in the same way. All, therefore, that it is necessary to do, in order to construct any unit block pattern, is to imagine a contour to be drawn in, surrounding the nodes which make up the block, and then to write changes, to the residuals, of $+1$ at the outer ends of all the strings which cross that contour and of -1 at their inner ends. Occasionally two or three strings may have ends at a common node, either inside or outside the contour; the total changes to the residuals at such points would then be -2, $+2$, -3, or $+3$, as the case may be.

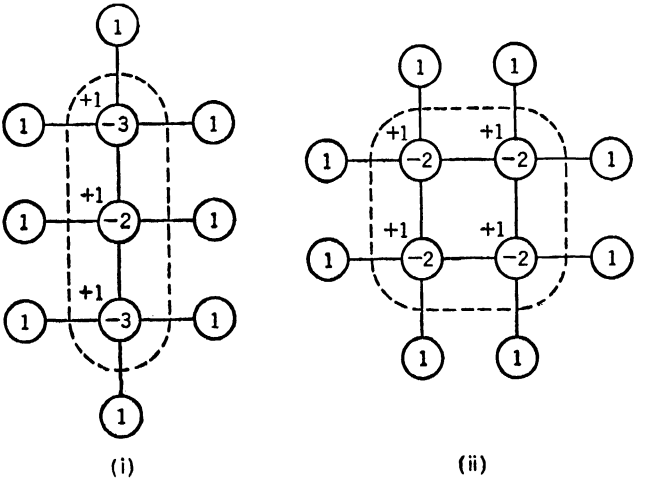


FIGURE 25

It should be noticed that even the simple point pattern of Figure 22 is included in this method of describing the effects of a unit operation. It should also be mentioned that there may be nodes in a block (particularly if it is a block of large extent) from which no strings radiate across the bounding contour; in such a case, of course, the residual at that node is unchanged by the application of the block operation, although it must not be forgotten that the increment is added to the w values at *all* the nodes which make up the block.

It is a matter of very elementary arithmetic to reduce the total residual at any block of nodes to zero by the use of a single block operation covering that group of nodes. All that is necessary is to add up the total of the residuals to begin with (say R), then count up the number of strings which radiate from the block to the rest of the nodes of the net (say N). Now, a unit block increment would reduce R by N, because one unit of residual would be transferred down each of the N strings. Hence a block

increment of magnitude R/N would reduce the total residual by R so that it would become zero. In this way the total residual over the whole of a relaxation diagram, or over any part of it, can always be made to become zero in a single step. An example of the repeated use of this particular technique will be given in Section 46.

45. Advance to a Finer Net

In many problems, in order to assess the accuracy of a relaxational solution, it is necessary to obtain solutions on at least two different sizes of net. Successively finer nets are always chosen, each to have strings one-half as long as the previous (coarser) net. The solution should always be worked first on the coarsest net, for it will contain the fewest nodes, and the liquidation process will be correspondingly easier; then a finer net is drawn by adding extra lines to the coarser net midway between the existing lines, as illustrated in Figure 26, where a part of a coarse net is shown in thick lines and where the extra lines which convert it into a finer net are shown as fine lines. In this advance to a finer net the nodes of the coarser net remain as nodes of the finer one; at these points the values of w found as the solution on the coarse net are available for use as initial values on the finer net, and, on the basis of these values, initial values of w at the new nodes which appear on the finer net can be readily interpolated. Thus, when a solution has been found on a coarser net, it may be confidently anticipated that the initial residuals on a finer net will be small, much smaller than if the work had been begun on the finer net with only a quite arbitrary guess of the solution to start from. Indeed, in some problems it may be that less work is involved in the end if the solution is first found on the coarsest possible net, followed by several advances to successively finer nets, until sufficient accuracy is achieved eventually on a very fine net. With a little experience the student may find he can do even better by means of a double advance at a time, but, like so much of the actual technique of relaxation, that is a matter for the personal inclination of the individual computer.

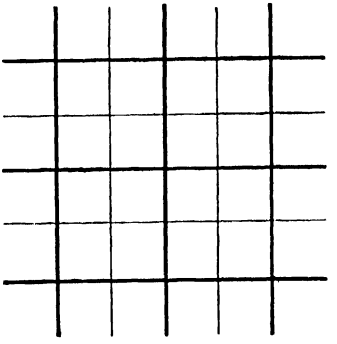


FIGURE 26

46. The Torsion Problem

A useful example, requiring the solution of a Poisson's equation, is provided by the well-known torsion problem. It can be shown that the

stresses produced in a uniform cylinder in torsion are expressible in terms of a single stress function ϕ. If we take orthogonal axes Ox,Oy in the plane of a cross section of the cylinder (Figure 27) and denote the direction of the generators of the cylinder by z, then the only nonzero components of stress at any point in the cylinder are the shear stresses τ_{zx} and τ_{yz}. These are independent of z but are given as functions of position in the cross section by the expressions

$$\tau_{zx} = \mu\tau \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\mu\tau \frac{\partial \phi}{\partial x}, \tag{7}$$

in terms of the stress function $\phi(x,y)$; μ is the modulus of rigidity of the material of the cylinder, and τ is the angular twist per unit of length along the cylinder. The stress function ϕ has to satisfy the Poisson-type equation

$$\nabla^2\phi + 2 = 0, \tag{8}$$

over the area of the cross section, together with a condition that $\phi = 0$ at all points of the boundary.

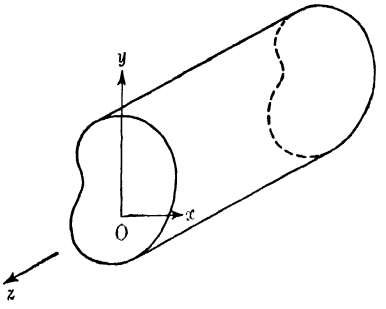


FIGURE 27

The dimensions of ϕ are those of the square of a length, and before any numerical work is attempted in the way of a solution of equation (8), it is desirable that the variables involved should be made nondimensional by the following substitutions:

$$\begin{aligned} x &= Lx', \\ y &= Ly', \\ \phi &= L^2\phi', \end{aligned} \tag{9}$$

where L represents some convenient dimension of the cross section (*e.g.*, the length of its side if the section is square). By effecting the transformation (9), we enable all the arithmetic to be carried out with pure numbers instead of dimensional quantities, and at the same time, in effect, then solve a given problem for any size, as distinct from shape, of cross section. The substitutions (9) do not alter the form of (8) in any way except to replace each variable involved by a corresponding primed quantity. We shall find it convenient, once having made the transformation, immediately to drop the primes in x', y', and ϕ', and so continue to use (8) as the governing equation, but with the understanding now that the variables are all nondimensional. When, with this interpretation, (8) has been solved for any particular shape of cross section, the stress

components may be determined from the formulas

$$\tau_{zx} = \mu\tau L \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\mu\tau L \frac{\partial \phi}{\partial x},$$

in replacement of (7).

With the use of a net of side h, which is also a nondimensional quantity and usually a fractional ratio of the actual mesh length to the representative dimension L, the finite-difference approximation to (8) leads to residuals being defined at the nodes of the net by the formula

$$F_0 \equiv \Sigma\phi_1 - 4\phi_0 + 2h^2. \tag{10}$$

As a worked example, we consider the case of a rectangular cross section with sides in the ratio 3:2. The smaller side of the rectangle is taken as the representative dimension L, and solutions will be given here for three successive sizes of net, respectively, $h = \frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$. To avoid working with fractional or decimal quantities, we introduce a multiplying factor of 3,200 throughout, thus calculating not ϕ directly but instead $3{,}200\phi$. For the three nets, and allowing for this factor, we find that the constant term in the expression (10) for the residuals takes the respective values, 1600, 400, and 100. The three solutions are shown in Figure 28; in (i), for the coarsest possible net, there are only two nodes at which ϕ must be found, and the solution is obtainable in a single step by the use of a block operation covering the two nodes. The value obtained for ϕ at these nodes, *i.e.*, 533, enables good guesses to be made to begin with on the next net, for which the completed solution is shown in (ii) of Figure 28, values being recorded of the final w values and the corresponding residuals. The value 533 may be seen to have changed on this finer net only to have become 577. In its turn the solution in (ii) allows a very good guess to be made for the ϕ values when a start is made to find the solution on the finest net used; the solution for this finest net is shown in (iii) of Figure 28, where the value 577 has changed again to become 590. In (iii) only one-quarter of the whole section is shown, for the reason that there is symmetry about both medians of the rectangle. A comparison of the values of ϕ for the solutions on the three successive nets suggests that the error in the solution on the finest of them is about 1 per cent.

This problem affords a good illustration of the use of block operations if it is worked from the beginning on the finest of the three nets, with no help being taken from the solutions on the two coarser nets. In that event we would begin with an initial guess that ϕ is zero at all nodes, and there would therefore be residuals of 100 at each of the 77 ($= 11 \times 7$) nodes of the net. The liquidation process is then best started by the application of a series of block operations, the first covering all the nodes

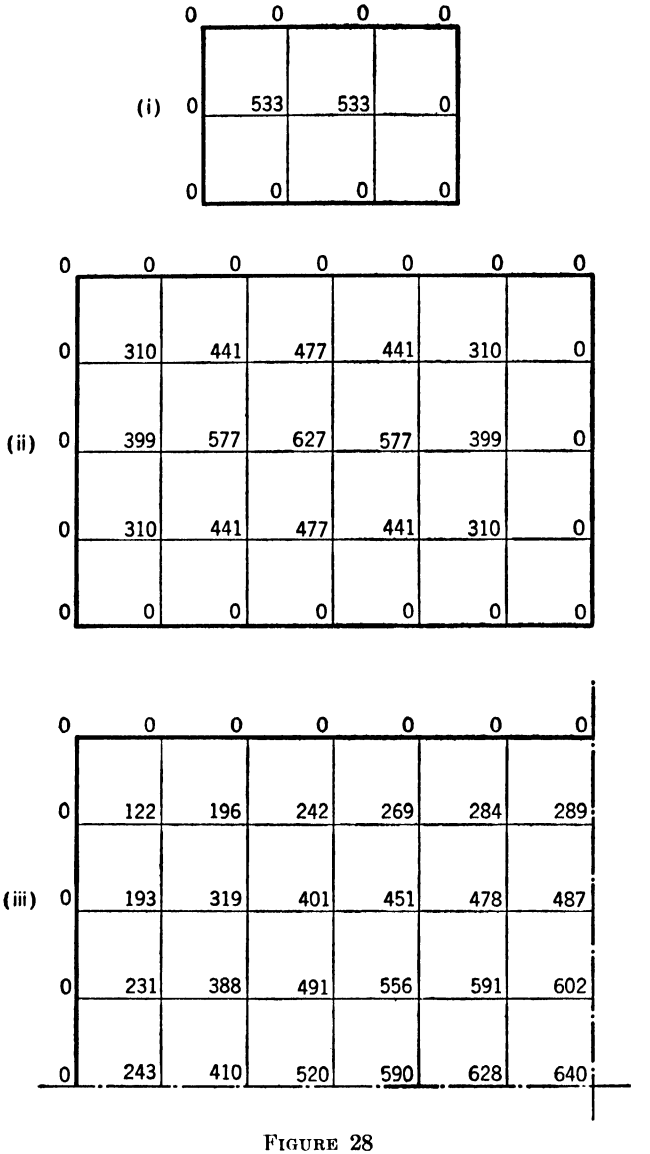


FIGURE 28

(*i.e.*, 11 × 7), the second covering a rectangle of nodes (9 × 5) placed centrally inside the first block, the third covering a still smaller central rectangle of nodes (7 × 3), etc. This process may be termed *blocking inward;* in the end it produces the same effect as the process of blocking outward (Section 34), and no significance attaches to the choice between these two processes. The magnitude of each of these block operations

should be such that no total residual is left at the points of the block after the operation has been applied. Thus, for the first block, the total initial residual is 7,700, and 36 strings radiate from that block, so that the required magnitude of the block increment is 214 (= 7,700 ÷ 36; cf. Section 44). Similarly for the second block the total residual is initially 4,500, and 28 strings radiate from it, so that a block increment of magnitude 161 should be applied. In the same way the magnitudes of the increments for the third and fourth blocks can be calculated to be 105 and 42, respectively. The state of the relaxation diagram after these four

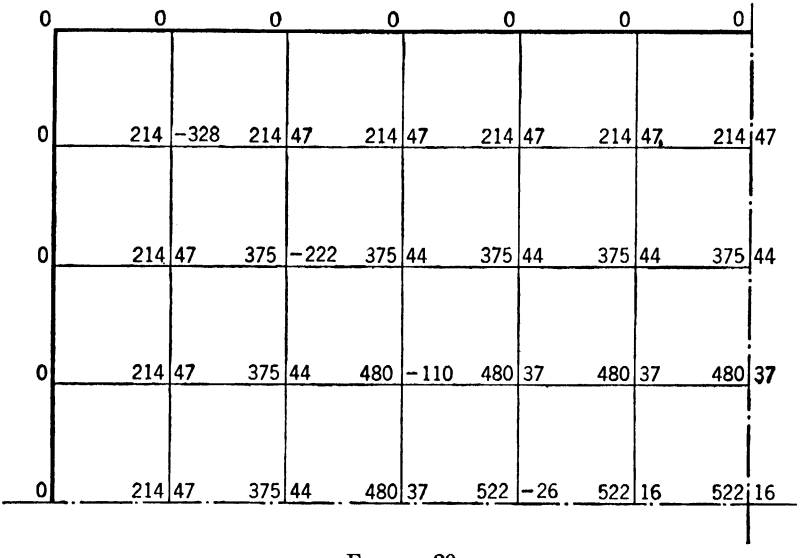


FIGURE 29

block operations have been effected is shown in Figure 29; again only one-quarter of the complete section is shown, and it can at once be seen that, although none of the residuals is individually zero, their distribution is reasonably good, and a large part of the liquidation has in fact been already achieved just by the use of the four rectangular-shaped block operations. The completion of the solution may be fulfilled by means of point relaxation and a use of smaller block operations.

47. Curved Boundaries

It is obviously of vital importance for the practical success of the method that it be capable of solving a problem for which the boundary is not rectilinear, as it has been in the examples so far treated, but for which the boundary may be of any given shape. It will be shown now how this may be done, at the expense of the introduction of modified

residual formulas and relaxation patterns at nodes near to such a (curved) boundary. Figure 30 illustrates the kind of situation which arises when a square net does not exactly fit into the region with a whole number of meshes; near the boundary, strings of the net are cut short by the boundary and do not exist (or extend) to their full normal length h. In Figure 30 we have taken the case where it is the string 01, radiating from a typical node 0 near the boundary, which is cut short by the boundary at the point B. Cases where others of the strings, 02, 03, or 04, are cut short can be dealt with in a way exactly similar to that which we now describe. The length $0B$ is denoted by ξh, where necessarily $0 < \xi < 1$; in Figure 30 the string $0B$ is drawn extended to where the node 1 would be if it did exist within the area of integration. Such a point is a fictitious point, and any values which may be postulated to hold at that point are fictitious values (cf. Section 38); the star of strings $0B$, 02, 03, and 04, radiating from the node 0, is called an *irregular star*. In a relaxational solution of, say, a Poisson-type equation over the region within the boundary, a value of the wanted quantity w must be found at the node 0, and consequently we must be able to define a corresponding residual F_0 at that point. In the ordinary way the residual at any node is defined by the formula (6), but here that formula is useless because it contains a fictitious w_1. For a point such as 0, then, a modified residual formula must be found which does not contain w_1; instead it will be found to include the value w_B of the wanted quantity at the point B on the boundary. This value will, of course, be known from the condition given to be satisfied on the boundary. As in Section 30, the finite-difference approximation for $(\partial^2 w/\partial x^2)_0$ is found by a consideration of the Taylor expansion of w in powers of $x - x_0$:

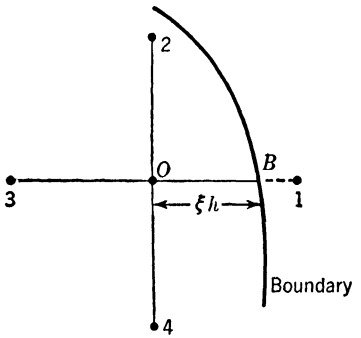


FIGURE 30

$$w = w_0 + \left(\frac{\partial w}{\partial x}\right)_0 (x - x_0) + \frac{1}{2!}\left(\frac{\partial^2 w}{\partial x^2}\right)_0 (x - x_0)^2 + \frac{1}{3!}\left(\frac{\partial^3 w}{\partial x^3}\right)_0 (x - x_0)^3 + \cdots . \quad (11)$$

In this expansion we put x, in turn, equal to $x_0 + \xi h$ and to $x_0 - h$; hence we find that

$$w_B = w_0 + \left(\frac{\partial w}{\partial x}\right)_0 \xi h + \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)_0 \xi^2 h^2 + O(h^3),$$

and

$$w_3 = w_0 - \left(\frac{\partial w}{\partial x}\right)_0 h + \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)_0 h^2 + O(h^3).$$

Between these last two relations $(\partial w/\partial x)_0$ may be eliminated so that, with neglect of terms of $O(h^3)$, we get

$$w_B + \xi w_3 = (1 + \xi)w_0 + \frac{\xi(1 + \xi)}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)_0 h^2,$$

or

$$h^2\left(\frac{\partial^2 w}{\partial x^2}\right)_0 = \frac{2w_B}{\xi(1 + \xi)} + \frac{2w_3}{1 + \xi} - \frac{2w_0}{\xi}, \tag{12}$$

as the finite-difference approximation for the second derivative of w with respect to x, in place of the usual approximation (3). When (12) is used instead of (3), the formula for the residual at the node 0 becomes

$$F_0 \equiv \frac{2w_B}{\xi(1 + \xi)} + \frac{2w_3}{1 + \xi} + w_2 + w_4 - \left(2 + \frac{2}{\xi}\right)w_0 + h^2 W_0, \tag{13}$$

which formula has to be used in place of (6) when the string 01 radiating from a node 0 is cut short by the boundary. It should be noticed that in the derivation of this formula it has been necessary to neglect terms of $O(h^3)$, so that, for a given size of net, h, there is a greater error introduced into a relaxational solution when the boundary is curved than there is when it is rectilinear. All this means, in practice, is that the nets which must eventually be used to obtain any prescribed accuracy are rather finer when there are any irregular stars in the relaxation diagram than when there are none. It may also be noticed that the formula (13) reduces, as of course it should, to the formula (6) when ξ attains the value 1 and the point B coincides with the point 1.

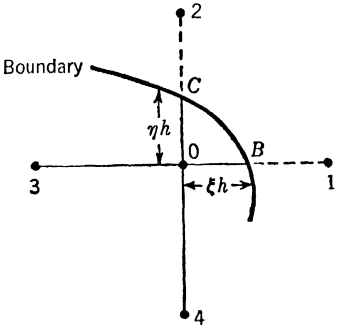


FIGURE 31

Similar formulas can be immediately written down for the cases when one of the other strings, 02, 03, or 04, is cut short by the boundary. Sometimes two strings may be cut short, as illustrated in Figure 31, where the strings 01 and 02 are ended at the points B and C, respectively. If the lengths of $0B$ and $0C$ are denoted by ξh and ηh, respectively, then a generalization of (13) for the residual F_0 is obvious:

$$F_0 \equiv \frac{2w_B}{\xi(1 + \xi)} + \frac{2w_C}{\eta(1 + \eta)} + \frac{2w_3}{1 + \xi} + \frac{2w_4}{1 + \eta} - \left(\frac{2}{\xi} + \frac{2}{\eta}\right)w_0 + h^2 W_0.$$

In no event should it happen that an irregular star exhibits three short strings; if it were to happen, it could only mean that an absurdly coarse net was being employed, and a finer net should then be introduced immediately and the coarse net completely discarded.

The relaxation patterns are also, of course, modified in the neighborhood of a curved boundary; in any particular case they may be easily worked out. Thus Figure 32 shows the patterns (i) for a unit increment Δw applied at the point 0 of Figure 30, and (ii) for a unit increment Δw

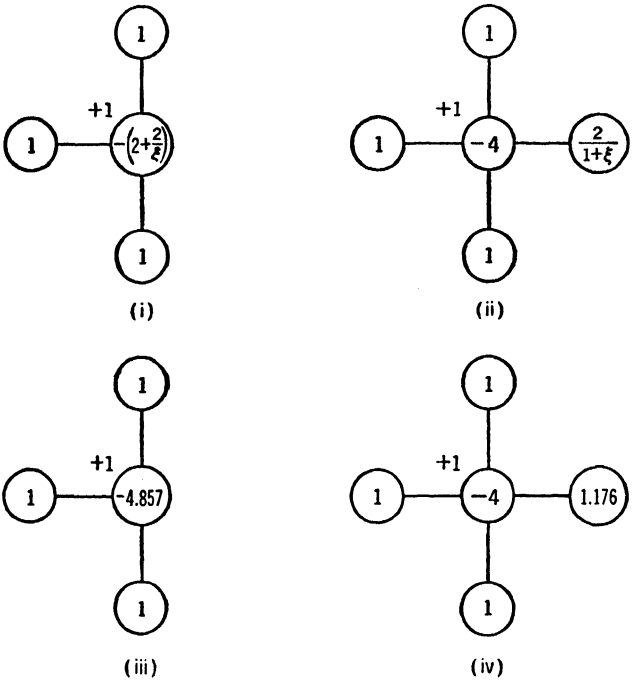


FIGURE 32

applied at the point 3 of Figure 30. These patterns repay a little study, and while there is nothing more to be said about their construction, the student should convince himself of their correctness before reading further.

It is desirable, however, to make an important point about their use. In general, the coefficients containing ξ will not be whole numbers; thus, for instance, if $\xi = 0.7$, then $2 + (2/\xi) = 4.857$, and $2/(1 + \xi) = 1.176$, and the patterns are as shown in Figure 32 (iii) and (iv). To use them as they stand would be most troublesome, and in any case we do not attempt to keep the residual values to more than the nearest whole number. It is far easier, and quicker in the long run, to ignore the decimal parts

which occur in such patterns, to use the pattern (iii) during the liquidation process as though the central coefficient were simply -5 instead of -4.857, and to use the pattern (iv) similarly as though the coefficient 1.176 were just 1. By doing this, of course, we simplify the arithmetic enormously at the expense of knowingly allowing small errors to creep in. These will slowly accumulate as the work proceeds, but they will be recovered each time the residuals are checked by recalculation from their various formulas. It has been found that in this way, by the use of *approximate patterns*, the time taken to find a solution is much reduced, even though the liquidation cannot hope to be achieved in only one attempt. Of course, the residuals must be initially calculated and always recalculated by substitution into their *exact* formulas.

48. Graded Nets

In some problems, a particularly fine net is needed in some small or local part of the complete area of integration. There may be various reasons for this: *e.g.*, we may be interested in the solution in only a certain part of the field; or we may require greater accuracy in one part of the area than in the remainder; or even to get the same accuracy throughout a region may demand a smaller value of h in one part than in another because the wanted function may be changing more rapidly in the former. Now it would, of course, always be possible to extend throughout the whole region a net as fine as that which it may be decided is necessary in any one part; but if this were done, it might well entail a very large total of nodes and consequently an unnecessary expenditure of time and effort in the computations.

For all these reasons together we see, therefore, that it is desirable to be able to use different sizes of net in different parts of the area of solution. To show how this may be done, it is sufficient to illustrate how the net size may be halved in one part of a region but unaltered in a neighboring part. A net which exhibits this feature is said to be a *graded net*, and it will be clear that a net which has been graded once could equally well be graded twice or three or more times, so that it is possible to have a net of side h in one part of the area, and a net of side $h/2$, or $h/4$, or $h/8$, etc., in another part of the area.

Figure 33 shows how a single grading can be effected between a net of side h in the left-hand part of the diagram and a net of side $h/2$ in the right-hand part. It will be seen at once that an intermediate net of side $h/\sqrt{2}$ is required to connect up the coarse and fine nets and that this intermediate net is orientated at 45° to these two nets. The various nodes in the diagram have been indicated as follows: a square □ indicates a node properly thought of as belonging to the coarse net; a full circle ● indicates

a node properly belonging to the fine net; and an open circle ○ indicates a node belonging to the intermediate net. When we say that a node "belongs" to a particular net, we mean primarily that the residual there is calculated from a formula based on the use of that net. Thus in Figure 33, at those nodes which belong to the coarse net, the residual is given by the expression

$$F_0 \equiv \Sigma w_1 - 4w_0 + h^2 W_0, \tag{14}$$

and, at those nodes which belong to the fine net, the residual is given by

$$F_0 \equiv \Sigma w_1 - 4w_0 + \left(\frac{h}{2}\right)^2 W_0. \tag{15}$$

That in all cases these formulas are applicable may be verified by inspection of Figure 33 to observe that for each node marked □ there do exist

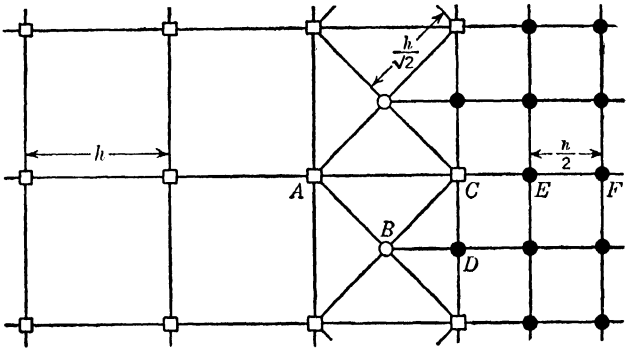


FIGURE 33

four surrounding nodes at a distance h, and that for each node marked ● there do exist four surrounding nodes at a distance $h/2$. Thus in both (14) and (15) the quantity Σw_1 is calculable, and these formulas for the residuals do have a meaning. For the nodes marked ○, belonging to the intermediate net, it may be observed in Figure 33 that each is surrounded by four nodes at a distance $h/\sqrt{2}$, but that these four surrounding nodes are in all cases in directions inclined at 45° to the strings of the coarse and fine nets. We must, therefore, make use of the property of the operator ∇^2, that it is invariant with respect to a rotation of the axes of coordinates, in order to be able to say that the residual at a node belonging to the intermediate net is given by the formula

$$F_0 \equiv \Sigma w_1 - 4w_0 + \left(\frac{h}{\sqrt{2}}\right)^2 W_0. \tag{16}$$

Thus, by one or other of the formulas (14), (15), and (16), residuals can be calculated at all the nodes of a graded net. They are liquidated in the

ordinary way, but a number of special relaxation patterns occur in the neighborhood of the intermediate net, specifically when unit increments are applied at nodes situated like A, B, C, D, E, and F in Figure 33. These patterns may be constructed by inspection, it being remembered that the unit pattern for any one selected node depends on the finding of all the residuals at other nodes which contain the w value at the selected

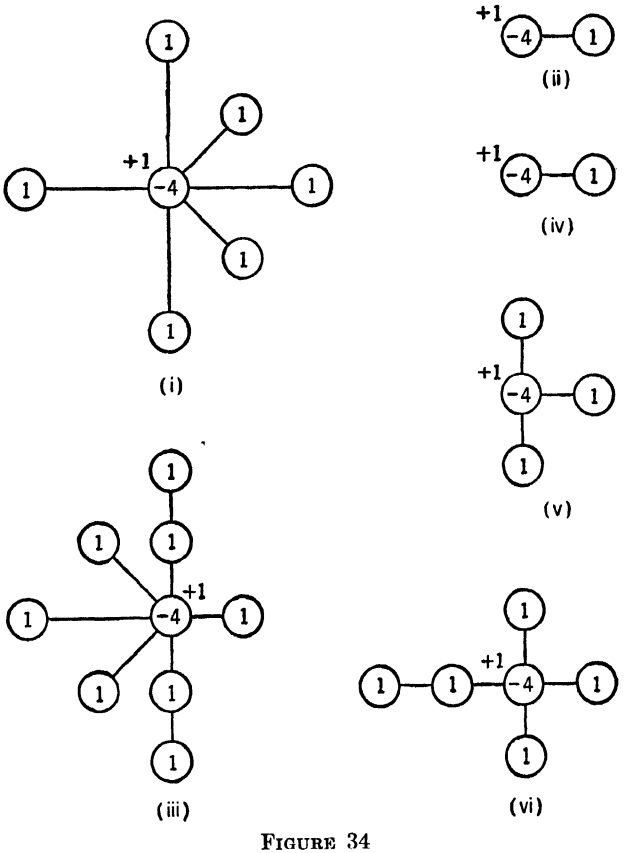


FIGURE 34

node in their residual expressions. Thus, for instance, the w value at B occurs in the residual formulas of B itself, of course, and also of D, but of no other point. In particular, the residual at C is not dependent on the w value at B (care must be taken that confusion does not arise because the converse is not true, *i.e.*, because the residual at B *is* dependent on the w value at C). The pattern for a unit increment $\Delta w_0 = 1$ applied at the node B is, therefore, that shown in (ii) of Figure 34; and other patterns (i) and (iii) to (vi) in that figure are similarly the patterns appropriate to the nodes A, C, D, E, and F, and they should all be verified by the reader.

49. Symmetry Conditions

It often happens that a problem, as set, is characterized by complete symmetry about some center line; an example has already been met in Section 46, where the solution of the torsion problem for a cylinder of rectangular section actually exhibits symmetry about two orthogonal center lines. It was seen then, in Figure 28 (iii), that the solution could sufficiently be presented on a diagram representing only one-quarter of the complete field. In a similar way the computation of the solution can also be accomplished on a diagram which represents only one-half (in the case of single symmetry) or one-quarter (in the case of double symmetry)

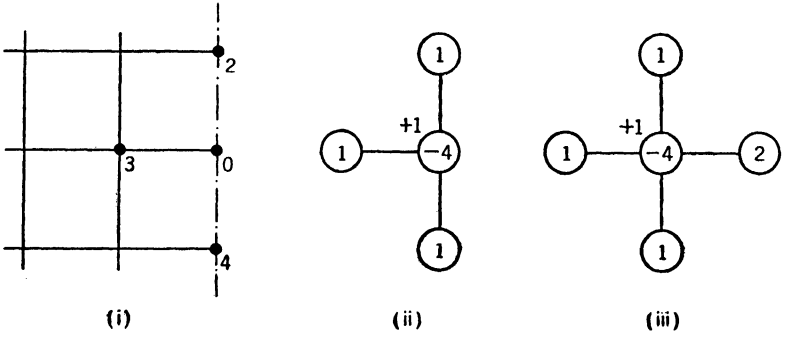


FIGURE 35

of the whole area. In such cases there is no reason why the center line(s) should not be taken as made up of strings of the net, and Figure 35 (i) illustrates a part of such a net in the neighborhood of a vertical center line. At a typical node 0 on the center line, a residual must be calculable (the center line is not a real boundary of the area of integration, but only an apparent boundary of the working diagram); the usual formula (6) cannot be immediately used to find F_0 because, on the diagram, there is no w_1 recorded. However, w_1 is not in this case a fictitious value; it does exist but is simply not recorded because, by symmetry, we know it to be identical with w_3. It is not, therefore, a question of having to eliminate a fictitious w_1, but rather of wanting to replace (6) by a more convenient formula, *i.e.*,

$$F_0 \equiv w_2 + 2w_3 + w_4 - 4w_0 + h^2W_0,$$

in which form the residual is expressed not merely in terms of real w values but in terms of real w values which are actually recorded on the relaxation diagram. Again there are corresponding modifications in the relaxation patterns in the neighborhood of a center line; these are easily seen to be as given in (ii) and (iii) of Figure 35, respectively, for the nodes 0 and 3 of (i) of that figure.

PROBLEMS

1. A cylindrical condenser consists of an outer conductor of square cross section of side 12 units and a centrally situated inner conductor of square cross section of side $4\sqrt{2}$ units, placed with its sides at 45° to those of the outer conductor. Calculate the electric potential distribution between the conductors at the nodes of a square net of side 1 unit when the potentials of the inner and outer conductors are maintained at 100 esu and 0, respectively.

(There is an obvious eightfold symmetry in this problem so that the number of nodes in the actual working relaxation diagram is not very large.)

2. Find the torsion stress function ϕ for the case of a cylinder of rectangular section with sides in the ratio 3:5, at the nodes of a square net which has 12 strings in the shorter side of the rectangle.

(Take the shorter side of the section to be the representative dimension L, and use a multiplying factor of 7,200. The solution may be found either directly on the net of side $h = 1/12$, or successively on nets of side $h = 1/3$, $1/6$, and $1/12$, according to individual preference.)

3. Figure 36 represents one-quarter of a uniform conducting plate; the outer edge ABC is kept at 0°, and the inner edge DEF is maintained at a temperature of

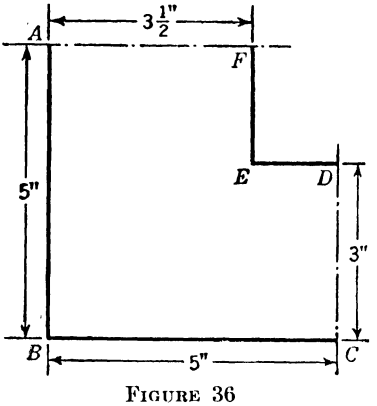


FIGURE 36

100°C. There is no heat loss from the surface of the plate. Find the steady temperature at the nodes of a square net, of side 1 in., of which AB and BC are both taken as mesh lines.

4. Find the solution to Laplace's equation over a semicircular region, given that the wanted quantity w has to satisfy boundary conditions that $w = 0$ on the diameter and $w = 100$ on the semicircumference. Use a square net of side one-quarter of the radius.

(The size of the net suggested is rather coarse to hope to find a very accurate solution, but the problem, as set, affords a sufficient exercise in the use of irregular stars.)

CHAPTER 6

NORMAL-GRADIENT BOUNDARY CONDITIONS

50. Introduction

One of the most difficult types of problem to tackle by the relaxation method is the determination of a solution of a Poisson's equation when a specified normal gradient of the wanted quantity is given as a boundary condition. Such a condition is easy enough to satisfy when the boundary is rectilinear and lies along mesh lines; we have, in effect, already met such a case in Section 49, since a condition of symmetry across a center line is equivalent to a condition that the normal gradient be specified to be zero at an actual boundary. But when, as is more usual, the boundary is curved, very considerable difficulties are encountered in any attempt to satisfy a normal-gradient boundary condition by a use of only arithmetical or algebraic ideas. A method has been proposed by Fox,* but it is not suitable for description here; instead a quite different method will be described which, although involving some tedious computations preliminary to the actual relaxation, does have the merit of being straightforward and easily workable. This method is based on a mechanical analogy of the relaxation method of solution of a Poisson-type equation, and before proceeding specifically to consider normal-gradient conditions, we must first discuss this analogy in detail.

51. The Soap-film Method

To begin with, mention must be made of the soap-film method of solution of an equation of Poisson's form; this method was evolved by Taylor and Griffith,† who made use of Prandtl's membrane analogue. It depends essentially on the fact that the differential equation

$$\nabla^2 w + W = 0 \tag{1}$$

governs the infinitesimal transverse deflection w of a uniformly tensioned film when it is subjected to a distributed transverse loading, of which, at any point, W is a measure of the intensity. To prove this, we consider the transverse equilibrium of an element of the film; in Figure 37 the xy

* Fox's account of his method may be found in *Quart. App. Maths.*, **2** (1944).

† The method is described by these authors in *Proc. Inst. Mech. Engrs.*, 755 (1917).

plane is the plane of the originally undeflected film, which we suppose to be stretched to a uniform tension T per unit length. $A_0B_0C_0D_0$ represents a typical elementary rectangle of the undeflected film of sides $A_0B_0 = \delta x$ and $B_0C_0 = \delta y$. When the film is deflected by the application of a distributed transverse loading, the typical element takes up a position of equilibrium $ABCD$, and it remains at rest in this position under the action of five forces shown in Figure 37: these are the four forces, each due to the tension T in the film, acting tangentially to the film and perpendicularly, respectively, to the four sides AB, BC, CD, and DA of the element (the magnitudes of these four forces are equal in pairs to $T\delta x$ and to $T\delta y$), together with a fifth force which is the resultant of the load distributed over the element (and of magnitude $P\delta x\delta y$, where P is the local intensity of loading). The pair of forces, $T\delta x$, combine together to yield a resultant $T(\partial^2 w/\partial y^2)\delta x\delta y$ in the transverse direction, and similarly the pair of forces, $T\delta y$, together provide a transverse force of magnitude $T(\partial^2 w/\partial x^2)\delta x\delta y$. Hence, at once, writing down the equation of transverse equilibrium of the element $ABCD$, we have

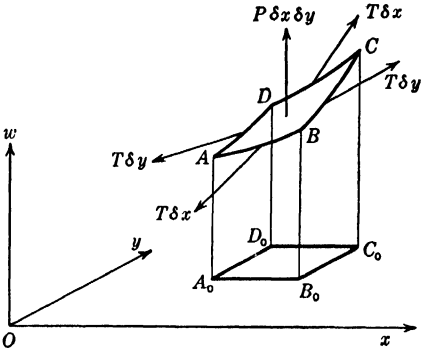


FIGURE 37

$$T\frac{\partial^2 w}{\partial x^2} + T\frac{\partial^2 w}{\partial y^2} + P = 0,$$

i.e.,

$$\nabla^2 w + \frac{P}{T} = 0. \tag{2}$$

This equation is immediately seen to be identical with equation (1) provided that we make

$$P = WT. \tag{3}$$

In other words, no matter how an equation of the form (1) may arise, it can always be interpreted as giving the transverse deflection w of a membrane, uniformly stretched to a tension T per unit length, when it is subjected to a distributed transverse loading of intensity WT. Such an interpretation of Poisson's equation provides, of course, an experimental method of determining particular solutions by means of the measurement of the actual deflections of a real membrane or soap film.

52. The String-net Analogy

Here, however, we are concerned with the soap-film interpretation of Poisson's equation only to demonstrate that the relaxation method of solution corresponds exactly with the replacement of the uniformly tensioned membrane by a network of uniformly tensioned strings, with the distributed loading on the membrane replaced by concentrated loads at the nodes of the net. Corresponding to Figure 37, which illustrates the equilibrium of a typical element of the membrane, Figure 38 now indicates the forces acting on a typical node of the string network. The xy plane is again the plane of the undeflected net, and in it the points 0, 1, 2, 3, and 4 represent a typical group of five nodes; when the net is deflected transversely, these nodes take up equilibrium positions 0′, 1′, 2′, 3′, and 4′ in Figure 38.

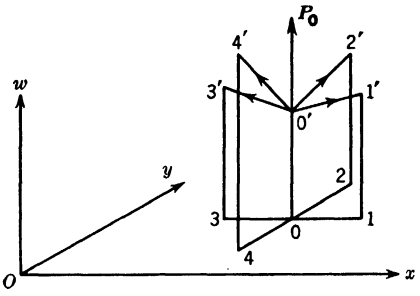


FIGURE 38

As usual, we denote the mesh length of the net by h, and let the tension in each string be, in the first instance, $\mathbf{T}$; suppose also that the transverse concentrated load acting at the node 0 is $\mathbf{P}_0$, and that the (small) deflections of the nodes are w_0, w_1, etc. Then the equation of transverse equilibrium of the typical node 0 may be immediately written down; it is

$$\sum\left(\mathbf{T}\cdot\frac{w_1 - w_0}{h}\right) + \mathbf{P}_0 = 0,$$

i.e.,

$$\Sigma w_1 - 4w_0 + \frac{h\mathbf{P}_0}{\mathbf{T}} = 0. \tag{4}$$

Now in the ordinary use of the relaxation method, as described in Chapter 5, we should replace the differential equation (2) by its finite-difference approximation

$$\Sigma w_1 - 4w_0 + \frac{h^2 P_0}{T} = 0, \tag{5}$$

and we can at once make equations (4) and (5) identical by putting

$$\begin{aligned} \mathbf{T} &= hT, \\ \mathbf{P}_0 &= h^2 P_0. \end{aligned} \tag{6}$$

The relations (6) are, in fact, exactly the relations which we could have anticipated to hold between the membrane and the network, for, on the

net, as is illustrated in plan view in Figure 39, a concentrated load at a typical node 0 must effectively replace, for the membrane, loading distributed over the square $STUV$, of area h^2, that is, $\mathbf{P}_0 = h^2 P_0$. Also the tension $\mathbf{T}$ in a string, 01 say, effectively replaces the tension in the membrane which acts across the line VS of length h, that is, $\mathbf{T} = hT$. The relaxational solution of a Poisson's equation corresponds exactly, therefore, with that obtained for a membrane-deflection problem by the approximate replacement of the membrane by a uniform string network, the distributed load on the membrane being concentrated at the nodes of the network. It may be recognized that this approximation of a network to a continuous membrane is mechanically correct, and that the approximation improves as the fineness of the network is increased. Clearly, in fact, the membrane is the limiting case of the network as the mesh length h tends to zero.

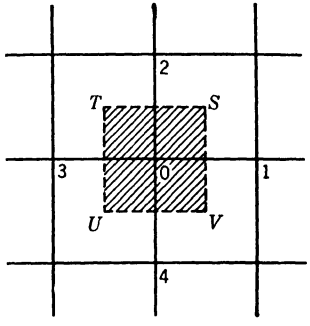


FIGURE 39

53. Interpretation of a Normal-gradient Boundary Condition

What has been done so far in this chapter offers an alternative approach to the formulation of the relaxation method of solution of a Poisson's equation. It is possible to interpret any such equation as governing the deflection of a membrane; then to *assert* that, physically, a membrane can be effectively and approximately replaced by a string network and that, when this is done, the relations (6) are the only feasible relations between the membrane and string tensions and between the intensity of loading on the membrane and the concentrated loads on the network. Last, residuals may be introduced which again, as in the case of a framework problem, are really residual forces; at any node of the network the residual would automatically be defined as the expression on the left-hand side of (4), which by (6) becomes the same as the expression on the left-hand side of (5) and which then by (3) becomes

$$F_0 \equiv \Sigma w_1 - 4w_0 + h^2 W_0, \tag{7}$$

which formula is, of course, identical with (6) of Chapter 5. In general, the methods of Chapter 5 are to be preferred to the alternative approach outlined in this chapter, because they are capable of extension to deal with other equations than Poisson's, whereas the membrane and string-network analogies are restricted by their very nature to dealing only with the "membrane equation," *i.e.*, Poisson's equation.

It would not have been necessary to make any mention at all of these

analogies if it were not that they happen to be particularly suitable to use in order to formulate a relaxation method of solution of Poisson's equation when a boundary condition of given normal gradients is to be satisfied. To describe how such a condition may be imposed, it is necessary first to interpret what it signifies in terms of the membrane analogy; then we shall be able to translate this meaning into terms of a string network and finally into the (numerical) terms of a relaxation process.

Figure 40 represents a small element QR, of length δs, of the boundary of a deflected membrane, Q_0R_0 being the position of this element when the membrane is not deflected. We denote by ν the direction, in the xy plane, normally across the boundary at any point, so that a specified value of the normal gradient $\partial w/\partial \nu$ implies a specified slope $\partial w/\partial \nu$ of the deflected membrane to the xy plane at a point of the boundary. Hence, acting normally inward across the element QR is a force $T\delta s$ at this inclination $\partial w/\partial \nu$; for transverse equilibrium it must be balanced by a distribution of transverse force, along the boundary, of intensity $T\,\partial w/\partial \nu$. In other words, a specified normal gradient of the membrane is achieved at a boundary if we impose a line intensity of transverse force, acting on the membrane at the boundary and of magnitude $T\,\partial w/\partial \nu$, and allow the membrane to take up its corresponding natural position of equilibrium.

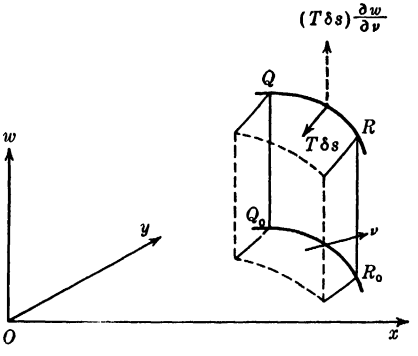


FIGURE 40

54. Concentration of a Line Intensity of Force on the Strings of a Network

The line intensity of transverse force which is imposed to act on the boundary of the membrane has now to be applied to act on the string network which we suppose, as the essence of the relaxation method, to replace that membrane. First of all, we can, using the first of (6), write that intensity as $(\mathrm{T}/h)(\partial w/\partial \nu)$ in terms of the string tension T which replaces the membrane tension T. Next, Figure 41 represents the plan view of a part of a region of integration of equation (1) with a part of the actual (real) boundary shown and also a square string net drawn in. In general, strings of the net are cut short by the boundary at the points E, G, H, etc., but these strings are all extended across the boundary in Figure 41 to the several fictitious nodes (α, β, γ, etc.). We imagine the string network actually to extend to these fictitious nodes and so, in effect, replace the real boundary by a substituted (polygonal) boundary,

formed by joining up the fictitious nodes and shown in Figure 41 as a broken line. This modification of the boundary can be easily seen, physically, to be unlikely to affect the solution to any appreciable extent if a reasonably fine net is used; for, in membrane terms, it implies the addition to the original membrane of only a very thin border of extra membrane, which border is completely unloaded. We still, of course, suppose that the line intensity $(\mathbf{T}/h)(\partial w/\partial \nu)$ of force remains applied along the real boundary, which is now thought of as being merely some curve (just inside the new substituted boundary) along which a line intensity of loading has to be applied.

The next step is to concentrate this distributed loading (along the real boundary) to act on the string network at the various points (E, G, H, etc.)

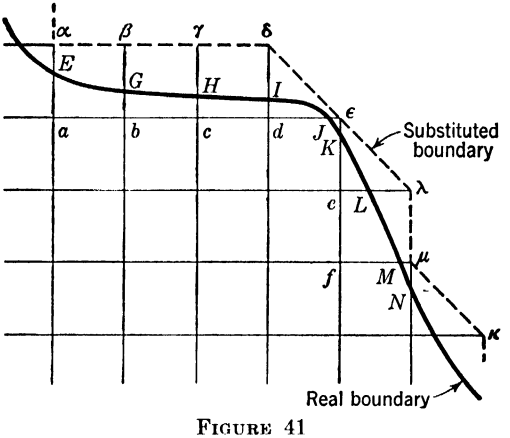


FIGURE 41

where the real boundary crosses the strings of the network. This concentration is achieved by a series of summations to find the total loads acting on the various separate portions (EG, GH, HI, etc.) into which the real boundary is divided by the net; then these total loads are transferred to act, in equal parts, at the ends of the corresponding portion of the real boundary. Thus, for instance, the total of the distributed load which acts along the portion HI is

$$\frac{\mathbf{T}}{h}\oint_I^H \frac{\partial w}{\partial \nu}\,ds, \tag{8}$$

where s denotes arc length measured along the real boundary in an anticlockwise direction. This total load (8) we now imagine to be concentrated into two equal parts, one each being applied to act on the string network at H and at I. Each portion of the real boundary must be dealt with in the same way, so that eventually membrane ideas may be forgotten and we are left, instead, with a string network, extending to the

broken-line boundary and subjected to a number of concentrated transverse loads. These transverse loads are of two kinds, those derived from integrations like (8) and acting at points like E, G, H, etc., and also those derived from the term W in equation (1), that is, $h\mathbf{T}W_0$ acting at each node of the net within the real boundary.

55. Preliminary Transference of Concentrated Loads

Before the actual relaxation process can be begun, one last preliminary calculation remains to be made; it involves the transference of the concentrated loads at points like E, G, and H to act at nodes of the network, *i.e.*, at the ends of strings. Basically this initial transference of the boundary loads is exactly similar to the preliminary calculations which are sometimes necessary in the solution of a framework problem (these were exemplified in Section 24, where, in a worked example, certain loads had initially to be transferred to act at the joints of a framework).

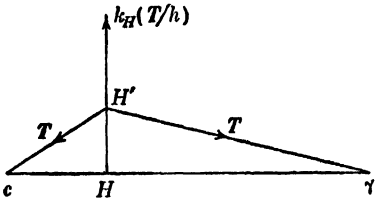


FIGURE 42

We suppose, in this initial transference of the loads, that the ends of all the strings of the net remain undeflected and that the redistribution of the loads is achieved by transverse displacements of only those strings, such as $a\alpha$, $b\beta$, and $c\gamma$ in Figure 41, which bear a load at some point along their length. Consider, for example, the string $c\gamma$; it is subjected to a concentrated load, at H, which has been calculated from expressions like (8) and which consequently is of the form $k_H(\mathbf{T}/h)$, where k_H is some determined numerical coefficient. This load is transferred to act in two parts at the ends, c and γ, of the string $c\gamma$ by a deformation of that string of the form illustrated in Figure 42. This deformation consists of a transverse displacement w_H of the point H to the position H', so that the load $k_H(\mathbf{T}/h)$ becomes balanced by the transverse components of the tension $\mathbf{T}$ in the two parts $H'c$ and $H'\gamma$ of the string; in this preliminary deformation the points c and γ, of course, being nodes of the network, are supposed to be restrained from undergoing any displacement. From the equilibrium of the point H' it follows easily that

$$w_H = \frac{k_H \cdot Hc \cdot H\gamma}{h^2}, \tag{9}$$

and that the load $k_H(\mathbf{T}/h)$ is transferred to the nodes c and γ in the inverse ratio of the distances of the point H from those nodes.

Preliminary deformations of the form (9) may be straightforwardly computed for each of the strings of the network which cross the real

boundary and which, therefore, to begin with, bear concentrated loads at some point along their length. These deformations have, in the end, to be superposed on the deflections of the network which are found in the second, or relaxational, part of the solution. This second part of the solution involves the determination of the deflection of the network due to the liquidation, by the ordinary relaxational processes, of the concentrated loads at the nodes of the network. These loads themselves are derived from two sources: first, at each node within the real boundary, there is a contribution $h\mathbf{T}W_0$, derived from the term W in equation (1); and second, at nodes like a, b, and c and α, β, and γ, there is a contribution derived from the preliminary transference to these nodes of the concentrated boundary loads. These latter contributions will be of the form $k_a(\mathbf{T}/h)$ and $k_\alpha(\mathbf{T}/h)$, where k_a and k_α are numerical coefficients which will have been calculated as described in this section.

56. Half-tension Strings

There remains now only to write down formulas for the calculation of residuals. In general, each residual is given directly from equation (4) by the formula*

$$F_0 \equiv \Sigma w_1 - 4w_0 + \frac{h\mathbf{P}_0}{\mathbf{T}}, \tag{10}$$

where $\mathbf{P}_0$ is the concentrated load acting at the typical node 0. When the node is an ordinary internal node of the net, so that $\mathbf{P}_0 = h\mathbf{T}W_0$ simply, (10) reduces, as we should expect, to (7). At a node just inside the real boundary, such as the point a, (10) becomes

$$F_0 \equiv \Sigma w_1 - 4w_0 + h^2W_0 + k_a, \tag{11}$$

where the number k_a has been explained in Section 55.

At nodes on the substituted boundary (like the points α, β, and γ), the correct formulas for the residuals need further consideration, for, in Figure 41, we have yet to decide whether or not we suppose strings of the net to exist joining up the pairs of nodes α and β, β and γ, γ and δ, and λ and μ. The answer to this question is that these strings should be supposed to exist but that they should be taken to be stretched to only one-half the normal string tension, *i.e.*, to a tension $\mathbf{T}/2$. The reason for this is that each of these strings, lying wholly along the substituted boundary, is effectively replacing a portion of the membrane on one side of the string only, and not, as is the case with all the other strings, replacing a portion

* It should be recorded that the residuals given by (10) are not strictly residual forces but instead, for arithmetical convenience, those forces multiplied by the factor $h/\mathbf{T}$.

of the membrane on both sides. These strings, therefore, which are part of the substituted boundary, are required to be only semieffective compared with the other strings of the network, and the simplest way of satisfying this requirement is to postulate that the tension in them is only T/2.

With this modification it becomes an easy matter to write down formulas for the residuals at the points α, β, γ, etc.; this must be done with reference to Figure 41, and the reader should check the following expressions:

$$
\begin{aligned}
F_\alpha &\equiv \frac{w_1}{2} + \frac{w_2}{2} + w_3 + w_4 - 3w_0 + k_\alpha, \\
F_\beta &\equiv \frac{w_1}{2} + \frac{w_3}{2} + w_4 - 2w_0 + k_\beta, \\
F_\gamma &\equiv \frac{w_1}{2} + \frac{w_3}{2} + w_4 - 2w_0 + k_\gamma, \\
F_\delta &\equiv \frac{w_3}{2} + w_4 - \frac{3w_0}{2} + k_\delta, \\
F_\epsilon &\equiv w_3 + w_4 - 2w_0 + k_\epsilon, \\
F_\lambda &\equiv w_3 + \frac{w_4}{2} - \frac{3w_0}{2} + k_\lambda, \\
F_\mu &\equiv \frac{w_2}{2} + w_3 + w_4 - \frac{5w_0}{2} + k_\mu.
\end{aligned}
\tag{12}
$$

These formulas are, of course, peculiar to the particular shape of the substituted boundary illustrated in Figure 41; they should, however, be sufficient to indicate how similar formulas are to be written down for any other shape of a substituted boundary.

The construction of corresponding special relaxation patterns need not be exemplified; it is sufficient merely to point out that, whereas an ordinary string transmits one unit of residual when an increment $\Delta w = 1$ is applied at one of its ends, a string which is stretched to only half the normal tension transmits only one-half unit of residual when a unit displacement is applied at one of its ends.

57. A Numerical Example

In order to illustrate the methods of this chapter, they will now be used to solve the following problem: To find a function w which satisfies Laplace's equation in the semicircular region bounded by one-half ($y > 0$) of the circle $x^2 + y^2 = 1$ and the diameter $y = 0$; with boundary conditions given that $w = 0$ on the diameter and that $\partial w/\partial \nu = y$ on the semicircle. There is, of course, symmetry about the center line $x = 0$, so that the solution need be worked on only one-half of the complete area;

Figure 43 (a) shows this half, with a square net drawn of mesh size $h = \frac{1}{4}$. The strings of this net are extended, in Figure 43 (a), to the various fictitious nodes which lie on the substituted boundary, but only that part of the substituted boundary which is actually made up of "half-tension" strings has been drawn (in broken lines).

The normal-gradient boundary condition which is required to be satisfied on the circular part of the real boundary has first to be interpreted (Section 53) as a line intensity of transverse force of magnitude $(\mathbf{T}/h)y$, or $(\mathbf{T}/h)\sin\theta$, where θ is the ordinary polar angle, measured from zero on

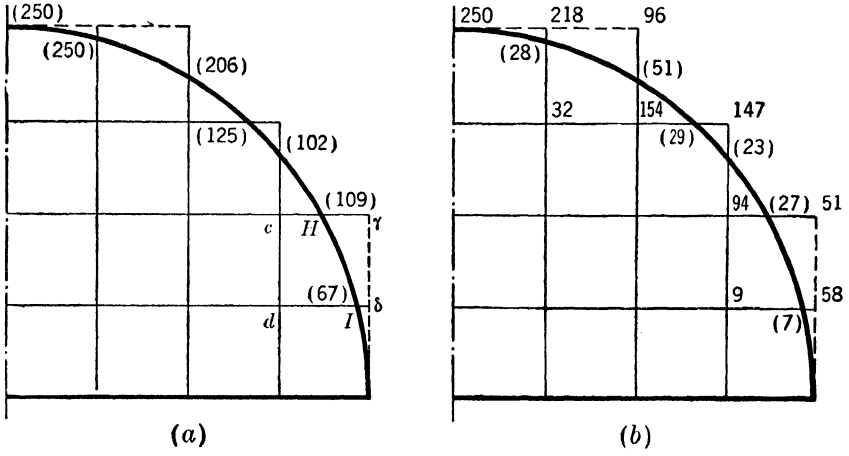


FIGURE 43

the x axis to $\pi/2$ on the y axis. This distributed load has next to be concentrated to act on the strings of the net where they cut the real boundary (Section 54); now the total load on any portion of the real boundary is

$$\frac{\mathbf{T}}{h}\oint_I^H \sin\theta\, ds = \frac{\mathbf{T}}{h}\oint_I^H \sin\theta\, d\theta = \frac{\mathbf{T}}{h}(\cos\theta_I - \cos\theta_H), \qquad (13)$$

where III [Figure 43 (a)] is a typical portion of the real boundary and where θ_I and θ_H are the values of θ at the points I and H, respectively. Quantities of the type of (13) can be immediately evaluated for all the portions of the real boundary; each of these quantities then represents a concentrated load which is supposed to be distributed to act on the strings at pairs of points, like I and H, in equal parts. At each point like H there is thus a contribution, to the load supposed concentrated there, from each of the two portions of the real boundary on either side of H. The values of these concentrated loads at the points I, II, etc., are shown in Figure 43 (a) (in brackets near each such point); the factor $\mathbf{T}/h$ has

been omitted, and also, for convenience, the values have all been multiplied by 10^3.

The next step (Section 55) is to transfer these concentrated loads from points like H to act at points like c and γ. This transference results in preliminary displacements of points like H which are calculated according to the formula (9); these displacements have been evaluated and are recorded in Figure 43 (b) (in brackets near the points H, etc., and again multiplied by 10^3). Also in Figure 43 (b) are recorded the transferred loads now acting at points like c and γ; these loads, again with the factor T/h omitted, are, according to (11), simply the initial residuals, at the points c and γ, etc., when, as an initial guess, the displacements of all the nodes are taken to be zero; they are therefore recorded in Figure 43 (b) in the usual positions occupied by initial residuals.

58. Relaxation of Initial Residuals

To complete the solution, there now remains only the liquidation of these initial residuals by the ordinary relaxation processes. Remembering the peculiar property of the broken-line half-tension strings, such as

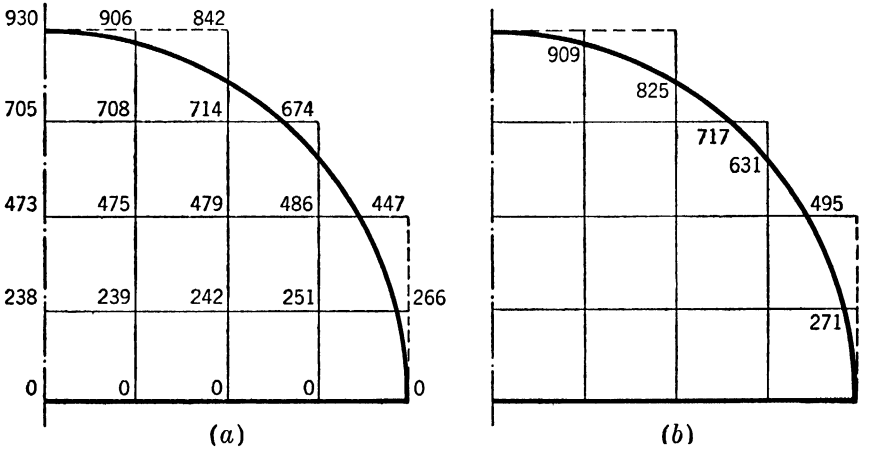


FIGURE 44

$\gamma\delta$, we can easily enough achieve this liquidation; the solution, in the form of displacements at all the nodes of the network, is given in Figure 44 (a), and the reader may verify that these displacements do in fact yield zero residuals. The solution values of w at points on the real boundary (like H) are given separately in Figure 44 (b). These are obtained in two additive parts: first, the contributions recorded in Figure 43 (b) due to the initial transference of concentrated loads to act at the nodes, and second, the contributions derived from Figure 44 (a) and evaluated from

the fact that, in that second (relaxational) part of the solution, strings like $c\gamma$ remain straight so that the displacement of H, say, can be linearly interpolated between the displacements of c and γ.

In both Figures 44 (a) and (b) the multiplying factor 10^3 persists. The values obtained may be compared with the exact solution, which is simply $w = y$; they are nowhere in error by more than 7 per cent, which, although by no means small, is no larger than is to be expected for a solution on such a coarse net of a problem involving a normal-gradient boundary condition.

59. Rectilinear Boundaries

The methods of this chapter are not necessary when a normal-gradient condition has to be satisfied on a boundary which is rectilinear and which can be entirely composed of mesh lines. In such a circumstance the problem is very considerably simplified, but we ought, for completeness, to show here how it may be tackled; only a very slight extension of Section 49 is required.

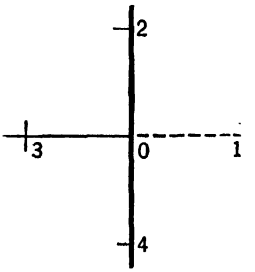


FIGURE 45

Suppose that a part of the rectilinear boundary lies along a vertical mesh line (Figure 45) and that 0 is a typical node of the net on this line so that the point 1 is a fictitious node. The usual residual formula,

$$F_0 \equiv \Sigma w_1 - 4w_0 + h^2 W_0, \tag{14}$$

cannot, therefore, be used directly because it contains the fictitious value w_1. However, if a normal-gradient condition, $\partial w/\partial \nu = N(x,y)$, is given to hold on the boundary, then its finite-difference approximation at 0 yields

$$w_1 - w_3 = 2hN_0.$$

Consequently the fictitious w_1 may be eliminated from (14) and so a residual formula, containing only real nodal values of w, obtained:

$$F_0 \equiv w_2 + 2w_3 + w_4 - 4w_0 + h^2 W_0 + 2hN_0. \tag{15}$$

PROBLEMS

1. Verify that the residual formula (15) can also be found as a special case of the more general formula (11).

2. Find the solution to the problem which is the same as that described in Section 57 except that the boundary condition on the semicircle is given to be $\partial w/\partial \nu = y^2$. Use successive nets of sizes $h = \frac{1}{4}$ and $h = \frac{1}{8}$.

CHAPTER 7

THE QUASI-PLANE-POTENTIAL EQUATION

60. Introduction

A generalized form of Poisson's equation has been termed the quasi-plane-potential equation; governing a dependent variable ϕ, it is

$$\frac{\partial}{\partial x}\left(\chi \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi \frac{\partial \phi}{\partial y}\right) + \Phi = 0, \tag{1}$$

where both Φ and χ are known functions of x and y. This equation reduces to be of Poisson's type, if χ is simply a constant, and to Laplace's equation, if also Φ is zero. A large number of second-order differential equations in two dimensions are, or may be written, in this form, and numerous examples will be quoted in this chapter. In many of these examples it will be found that the quantity Φ is actually zero, but since that circumstance involves no particular simplification from the point of view of the relaxation method, we shall describe the method of solution in general terms with reference to the full equation (1).

One of the simplest equations of the form (1) is Laplace's equation expressed in terms of cylindrical coordinates r and z, with axial symmetry. Although usually written as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{2}$$

it may be easily verified that it could equally well be written

$$\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r \frac{\partial \phi}{\partial z}\right) = 0, \tag{3}$$

which (with r and z replacing x and y) is the special case of (1) given by $\Phi = 0$ and $\chi = r$.

61. Adjoint Equations

A variety of physical quantities, all having the nature of a two-dimensional potential function, satisfy an equation of the form (1). Thus, with $\Phi = 0$, ϕ might be the magnetic potential in a two-dimensional region of variable permeability (χ); similarly ϕ could represent the elec-

tric potential in a two-dimensional sheet, for which χ would be the variable conductivity; or, in the field of hydrodynamics, ϕ might be the velocity potential for the irrotational flow of a compressible fluid in two dimensions, with χ being the variable density. In all these cases, Φ is zero, so that the existence of a solution to (1) implies that there is another function ψ, such that

$$\begin{aligned} \chi \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y}, \\ \chi \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}. \end{aligned} \tag{4}$$

Elimination of ϕ between these two relations shows at once that the function ψ also satisfies an equation of the same form as (1):

$$\frac{\partial}{\partial x}\left(\frac{1}{\chi}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\chi}\frac{\partial \psi}{\partial y}\right) = 0. \tag{5}$$

For the magnetic, electrical, and hydrodynamic interpretations of ϕ cited above, the function ψ is, respectively, the magnetic-flux function, the current-flow function, and the stream function. In general, for any equation of the form (1) with Φ equal to zero, it is always possible to construct a corresponding equation (5); the pair of equations are said to be *adjoint*, and the pair of functions ϕ and ψ are called adjoint functions.

In the case when Φ is not zero in (1) it is still possible, though by no means so easy, to find an adjoint equation. To do so, it is necessary to introduce a function Ω, which is any solution of the equation

$$\nabla^2 \Omega = \Phi;$$

then (1) may be written

$$\frac{\partial}{\partial x}\left(\chi \frac{\partial \phi}{\partial x} + \frac{\partial \Omega}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi \frac{\partial \phi}{\partial y} + \frac{\partial \Omega}{\partial y}\right) = 0, \tag{6}$$

on substitution for Φ in terms of Ω. Now the existence of a solution of (6) implies that there is another function ψ, such that

$$\begin{aligned} \chi \frac{\partial \phi}{\partial x} + \frac{\partial \Omega}{\partial x} &= \frac{\partial \psi}{\partial y}, \\ \chi \frac{\partial \phi}{\partial y} + \frac{\partial \Omega}{\partial y} &= -\frac{\partial \psi}{\partial x}. \end{aligned} \tag{7}$$

Elimination of ϕ between these relations gives the adjoint equation governing ψ to be

$$\frac{\partial}{\partial x}\left(\frac{1}{\chi}\frac{\partial \chi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\chi}\frac{\partial \chi}{\partial y}\right) + \frac{1}{\chi^2}\left(\frac{\partial \Omega}{\partial x}\frac{\partial \chi}{\partial y} - \frac{\partial \Omega}{\partial y}\frac{\partial \chi}{\partial x}\right) = 0. \tag{8}$$

The reason why it is desirable to show that, always, an equation (8) can be formulated which is adjoint to (1) is that in some problems it may be considerably easier to solve the adjoint than the original equation. Now there is obviously nothing to choose between (1) and (8) themselves—they are equations of precisely similar form—but there may well be reasons why we should prefer to work in terms of ϕ instead of ψ, or vice versa, when the boundary conditions come to be considered. For instance, if ν and s denote the normal and tangential directions at any point of a boundary, it follows from (4) that

$$\chi \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial s},$$
$$\chi \frac{\partial \phi}{\partial s} = -\frac{\partial \psi}{\partial \nu},$$

or, more generally, from (7), that

$$\chi \frac{\partial \phi}{\partial \nu} + \frac{\partial \Omega}{\partial \nu} = \frac{\partial \psi}{\partial s},$$
$$\chi \frac{\partial \phi}{\partial s} + \frac{\partial \Omega}{\partial s} = -\frac{\partial \psi}{\partial \nu}.$$

Hence, at once, we see that a boundary condition which, in terms of ϕ, specifies given *values*, on the other hand specifies given *normal gradients* when expressed in terms of ψ; conversely, a condition which specifies given normal gradients in terms of ϕ specifies given boundary values of ψ. (Strictly, if $\partial\phi/\partial\nu$ is given, then $\partial\psi/\partial s$ may be regarded as known, so that integration with respect to s round the boundary yields values there for ψ.) Now it has already been discovered, in the previous chapter, that it is much more laborious to impose a normal-gradient boundary condition than it is to impose given boundary values of the wanted function. Hence if a problem is governed by an equation of the form (1) when expressed in terms of ϕ, then we should, of course, solve that equation directly for ϕ if the boundary condition expressed given values for ϕ; but if the boundary condition specified normal gradients of ϕ, we would then solve the problem by working in terms of the adjoint function ψ, for which boundary values could be regarded as known. Sometimes, of course, the boundary condition might be of mixed type, *i.e.*, on one part of the boundary values of ϕ might be given, and on another part the normal gradient $\partial\phi/\partial\nu$ might be known; in such a case the problem would be best solved in terms of ϕ if the value of ϕ were given on the greater part of the boundary and would be best solved in terms of ψ if the value of ϕ were given only on a smaller part of the boundary. The above argument presupposes, of course, that the solution to the problem is effectively achieved

if *either* ϕ *or* ψ is determined. In many cases this is so, but if a particular one of the two functions is required, then no advantage accrues from a determination of the other.

62. The Residual Formula

We usually find a relaxational solution of an equation of the form (1) which consists of numerical values of the wanted function ϕ at the nodes of a square net (of mesh size h) drawn to cover the required area of integration. In order to write down a finite-difference replacement for (1) in terms of these nodal ϕ values, it is simplest, in general, first to rewrite (1) as

$$\nabla^2(\chi\phi) + \chi\nabla^2\phi - \phi\nabla^2\chi + 2\Phi = 0.$$

Then, with the usual convention for numbering a group of typical nodes, and using the known finite-difference approximation for the ∇^2 operator (Section 41), we get

$$\Sigma(\chi\phi)_1 - 4(\chi\phi)_0 + \chi_0(\Sigma\phi_1 - 4\phi_0) - \phi_0(\Sigma\chi_1 - 4\chi_0) + 2h^2\Phi_0 = 0,$$

i.e.,

$$\Sigma\phi_1(\chi_1 + \chi_0) - \phi_0(\Sigma\chi_1 + 4\chi_0) + 2h^2\Phi_0 = 0,$$

so that, after division through by $2\chi_0$, residuals may be defined by the formula

$$F_0 \equiv \sum\left(\frac{\chi_1 + \chi_0}{2\chi_0}\right)\phi_1 - \frac{\Sigma\chi_1 + 4\chi_0}{2\chi_0}\cdot\phi_0 + \frac{h^2\Phi_0}{\chi_0}. \tag{9}$$

This formula reduces, of course, when χ is just a constant, to the usual expression for a residual derived from Poisson's equation:

$$F_0 \equiv \Sigma\phi_1 - 4\phi_0 + \frac{h^2\Phi_0}{\chi}.$$

The use of the formula (9) to calculate residuals is exactly similar to the use of the simpler formula obtained from Poisson's equation; the only difference is that, in (9), the coefficients of the various nodal values of ϕ happen to be some other numbers than 1 and -4. We shall, however, not proceed any further with the discussion of quasi-plane-potential equations in general, but instead will consider in more detail a number of particular equations of this type.

63. Laplace's Equation in Cylindrical Coordinates

When a potential function ϕ is characterized by axial symmetry, its governing Laplace's equation, in terms of cylindrical coordinates, is equation (2). Although this equation can be rewritten as (3), in the standard form of a quasi-plane-potential equation, we should not usually

bother to do this but instead should treat (2) as it stands on its own merits. Similar remarks apply to many second-order partial differential equations: it may be possible to rewrite them in the form (1), but often they naturally arise in somewhat different shapes, and we usually prefer to deal with each such equation individually, although, of course, the general approach of Section 62 always remains as an alternative.

Thus, using the known approximations (Section 30) for first and second derivatives, we can immediately write down a finite-difference replacement for (2), holding at a typical node 0 of a square net* of side h:

$$\frac{\phi_2 + \phi_4 - 2\phi_0}{h^2} + \frac{1}{r_0} \cdot \frac{\phi_2 - \phi_4}{2h} + \frac{\phi_1 + \phi_3 - 2\phi_0}{h^2} = 0,$$

which, after multiplication through by h^2, enables residuals to be defined by the expression

$$F_0 \equiv \phi_1 + \phi_3 + \left(1 + \frac{h}{2r_0}\right)\phi_2 + \left(1 - \frac{h}{2r_0}\right)\phi_4 - 4\phi_0. \qquad (10)$$

In this result the coefficients, $1 \pm h/2r_0$, of the nodal values ϕ_2 and ϕ_4 are easily evaluated at any node of the net.† Usually it is arranged that the axis of symmetry, $r = 0$, is itself a nodal line of the network, so that $h/2r_0$ takes the values $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, etc., at nodes on successive lines distant h, $2h$, $3h$, etc., from the axis.

The pattern which should be used in the relaxation method of solution of equation (2) follows at once from the residual formula (10), although some care needs to be exercised in order to avoid an understandably common mistake. The correct pattern is shown in Figure 46, and it cannot be too strongly emphasized that the five elements which appear in this pattern are separately coefficients which occur in the formulas for residuals at five *different* nodes of the net. In order to explain this statement in complete detail,‡ the five elements of the pattern of Figure 46 have been lettered A, B, C, D, and E for reference. Now, this pattern shows the changes which must be made to the values of the residuals when an increment $+1$ is applied to the value of ϕ at the (typical) central point

* The strings 01 and 02 are here supposed to be drawn in the positive z and r directions, respectively.

† Nodes on the axis $r = 0$ are exceptional; there the term $(1/r)(\partial\phi/\partial r)$ in (2) is indeterminate and must be replaced by its limit as $r \to 0$, *i.e.*, by $\partial^2\phi/\partial r^2$. Since also $\phi_2 = \phi_4$ by virtue of symmetry, it follows that the residual formula at nodes on $r = 0$ reduces to

$$F_0 \equiv \phi_1 + 4\phi_2 + \phi_3 - 6\phi_0.$$

‡ It has been the author's repeated teaching experience that this particular point must be most carefully explained at some length; it is essential that the student should not only understand the explanation but also recognize the pitfall which has to be avoided.

C. So far as the residual at the point C itself is concerned, the increment is an addition of $+1$ to the value of ϕ_0; therefore the residual at C must be changed by an amount equal to the coefficient of ϕ_0 in the formula for the residual *there*, *i.e.*, by -4. Next consider the element in the pattern of Figure 46 at the point E; it represents the change made to the residual at E due to an increment $+1$ applied to ϕ at C. Now, in the residual formula at E, ϕ at C is denoted by ϕ_2, and its coefficient is $1 + (h/2r_4)$, where r_4 is the value of r at the point E. It must be emphasized that this coefficient is not a coefficient which appears in the formula for the residual at C; in particular it is *not* $1 - (h/2r_0)$, nor even $1 + (h/2r_0)$. Similarly, the element in the pattern of Figure 46 at the point A is the coefficient of ϕ_4 in the formula for the residual *at* A, and likewise the elements at the points B and D are the coefficients, respectively, of ϕ_1 and ϕ_3 in the residual formulas holding at those two points. To summarize, the five elements in the pattern of Figure 46 are coefficients of ϕ values taken one each from the five formulas for the residuals at the five different points concerned; they are *not* the same as the five coefficients which appear in any one residual formula. The reason why some confusion is apt to arise here is that no distinction between the two sets of five numbers (the ones which appear in the pattern and the ones which appear in a typical residual formula) needed to be made when, in Chapter 5, we were dealing with Poisson's equation, because then the two sets of numbers were, as it happened, identical (one being -4 and four being $+1$).

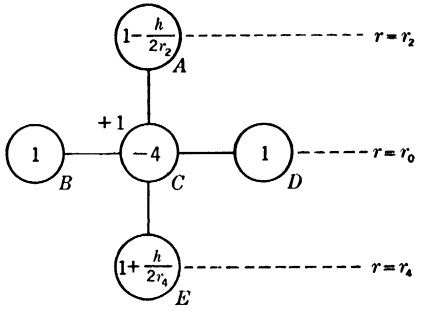


FIGURE 46

The principles which have been explained here regarding the formulation of the relaxation pattern appropriate to equation (2) apply in general for any quasi-plane-potential equation. Certain aspects of purely numerical technique also arise in the solution of these equations which could not be mentioned in respect of Poisson's equation because of its particular simplicity. These aspects will be illustrated in the worked solution of a typical example (Section 65) of a problem governed by a quasi-plane-potential equation which is similar in form but not identical to (2).

64. Torsional Stresses in an Incomplete Tore

A second example of a quasi-plane-potential equation arises in the determination of the torsional stresses in an incomplete tore. Figure 47

represents an incomplete solid of revolution generated by the partial revolution of an area A about the axis Oz; we are concerned to find the stress system set up in this solid by tractions on the plane ends A and A' applied so that the resultant force exerted over the end A' is of magnitude Z and directed along the axis Oz and so that the resultant force exerted over the end A is of magnitude Z and directed in the opposite direction along the axis of revolution. We may take cylindrical coordinates r, θ, and z, as shown in Figure 47, and it can then be proved* that the only nonzero components of stress at any point in the tore are the shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ (provided, of course, that the tractions applied over the plane ends are properly distributed). These shear stresses have values which are independent of θ and can be expressed in terms of a stress function ϕ by the relations

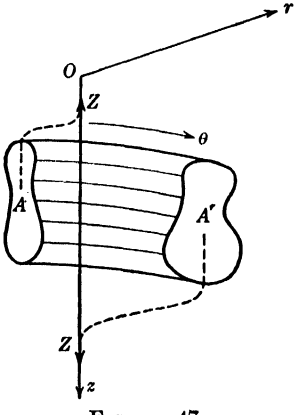


FIGURE 47

$$\tau_{r\theta} = \frac{k}{r^2}\frac{\partial \phi}{\partial z}, \qquad \tau_{\theta z} = -\frac{k}{r^2}\frac{\partial \phi}{\partial r}, \tag{11}$$

where ϕ itself is governed by the quasi-plane-potential equation

$$\frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 1 \tag{12}$$

and satisfies the condition $\phi = 0$ at all points on the boundary of the axial section of the tore. The constant k which appears in the expressions (11) depends on the value of the resultant force Z; for any particular cross section A of the tore, k can be found, after the function ϕ has been determined, from the relation

$$Z = \iint_A \tau_{\theta z}\, dA,$$

i.e.,

$$Z = -k \iint_A \frac{1}{r^2}\frac{\partial \phi}{\partial r}\, dA. \tag{13}$$

The governing equation (12) could be verified to be of the standard quasi-plane-potential form (1) with $\chi = \Phi = 1/r^3$, but it can also be taken directly as it stands, in the manner of Section 63, to yield a residual formula

* The results which are stated here were derived by Southwell, *Proc. Roy. Soc. A*, **180**, 367–396 (1942).

$$F_0 \equiv \phi_1 + \phi_3 + \left(1 - \frac{3h}{2r_0}\right)\phi_2 + \left(1 + \frac{3h}{2r_0}\right)\phi_4 - 4\phi_0 - h^2. \quad (14)$$

65. A Tore of Square Section

As a worked example we take the particular case of a tore with a square section. As a preliminary to any numerical work, the equations of Section 64 should all be expressed in terms of nondimensional variables (cf. Section 46). We make this transformation by denoting a representative dimension of the cross section by L (in the case of a square section we naturally choose L to be the side of the square); then, replacing ϕ, r, z, and k, respectively, by $L^2\phi$, Lr, Lz, and Lk, we can verify that the forms of (11), (12), and (14) are unchanged but that (13) becomes

$$Z = L^2 \iint_A \tau_{\theta z}\, dA = -kL^2 \iint_A \frac{1}{r^2}\frac{\partial \phi}{\partial r}\, dA, \quad (15)$$

where we now attach a nondimensional significance to the quantities ϕ, r, z, and A. In nondimensional terms the side of the square becomes 1, and we take the situation of the section relative to the axis of revolution to be given by the fact that its sides are the lines $z = \pm\frac{1}{2}$, $r = \frac{1}{2}$, and $r = \frac{3}{2}$. Figure 48 illustrates the working diagram; because of symmetry about the line $z = 0$, only one-half of the section need be drawn, and a rather coarse net of size $h = \frac{1}{4}$ is taken as sufficient for illustrative purposes. Known boundary values of ϕ ($= 0$) are recorded in Figure 48 at the nodes which lie on the sides of the section, and initially guessed values of ϕ (also zero) are recorded at all the internal nodes. For convenience all quantities are multiplied by a numerical factor of $-1{,}600$, so that initially all the residuals at the internal nodes are 100 ($= -1{,}600h^2$). To the right of the center line, and adjacent to each mesh line in the z direction, are recorded a pair of numbers: these are the values, appropriate to the corresponding values of r on these mesh lines, of $1 - (3h/2r_0)$ and $1 + (3h/2r_0)$, above and below, respectively. It is a great convenience to have these coefficients recorded in this way so that they may be easily

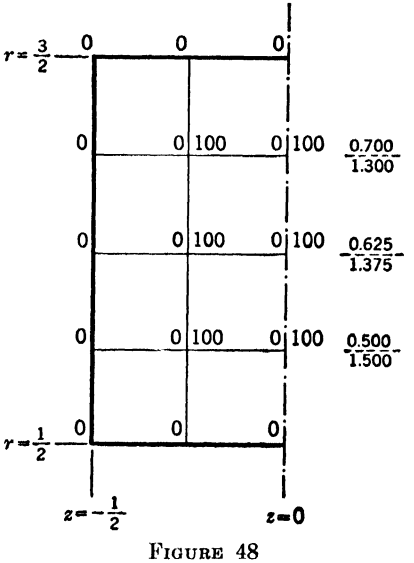


FIGURE 48

read off when they are needed in the calculation or checking of the residual values; each pair is required for use in the calculation of residuals at nodes on the same (horizontal) level.

The residuals are liquidated by means of the application of point or block increments to the nodal values of ϕ, strictly according to the pattern of Figure 49 (a). The numerical values of the elements in this pattern vary slightly for different values of r_0; Figure 49 (b) is the pattern, in numbers, for a typical value of r_0 [actually $r_0 = 1$, so that Figure 49 (b) is the pattern to be used when increments are added to ϕ at any of the nodes on the mesh line $r = 1$].

It is apparent at once that all the patterns contain numerical elements which have decimal parts and that it would consequently be a matter of

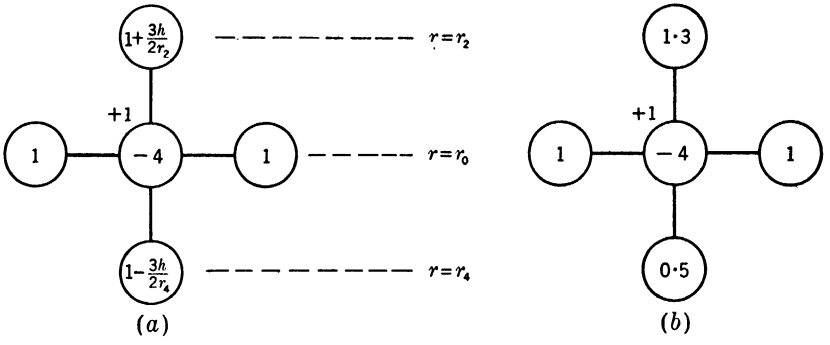


FIGURE 49

some labor and time to attempt, in the liquidation process, to keep an exact account of the residual values. It considerably simplifies the arithmetic of the relaxation and also saves time in the end if we deliberately discard the strictly correct patterns illustrated in Figure 49 and instead use a simpler approximate pattern. In doing this, we know, of course, that small errors are continuously being introduced during the relaxation; these errors accumulate to some extent, but they may be taken account of from time to time by a periodic recalculation of the residuals. It need hardly be added that, of course, the residuals must always be recalculated or checked by substitution into the exact formula

$$F_0 \equiv \phi_1 + \phi_3 + \left(1 - \frac{3h}{2r_0}\right)\phi_2 + \left(1 + \frac{3h}{2r_0}\right)\phi_4 - 4\phi_0 + 100. \quad (16)$$

The approximate pattern which might as well be used in the actual relaxation process is, in fact, the ordinary Poisson's equation pattern which is obtained from that of Figure 49 (a) by neglect of the quantities $3h/2r_0$; the approximate relaxation then becomes extremely simple and

rapid. The result of liquidating the initial residuals shown in Figure 48 according to this approximate pattern is given in Figure 50; in this figure are also recorded the residuals recalculated from the exact formula (16).

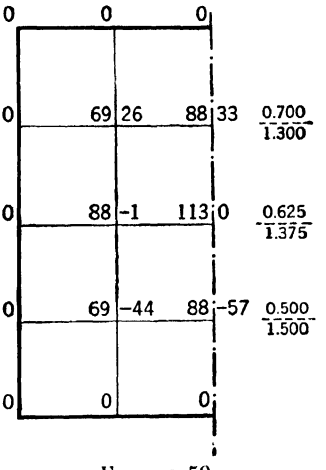


FIGURE 50

While they may be seen to have accumulated to values which individually are by no means small, nevertheless their total is only 62 compared with a total of 900 at the beginning, and as a whole the function ϕ already approximates quite closely to what it will become when the liquidation is eventually completed. The relaxation of the residuals shown in Figure 50 is continued with a use of the approximate pattern, and when they have been liquidated accordingly, the values of ϕ are as shown in Figure 51 (a); the corresponding residuals recorded in this diagram are the result of a second recalculation from (16). Carrying on in this way, we quickly arrive at the solution (for the net of size $h = \frac{1}{4}$), which is given in Figure 51 (b) with ϕ values correct to the nearest whole number.

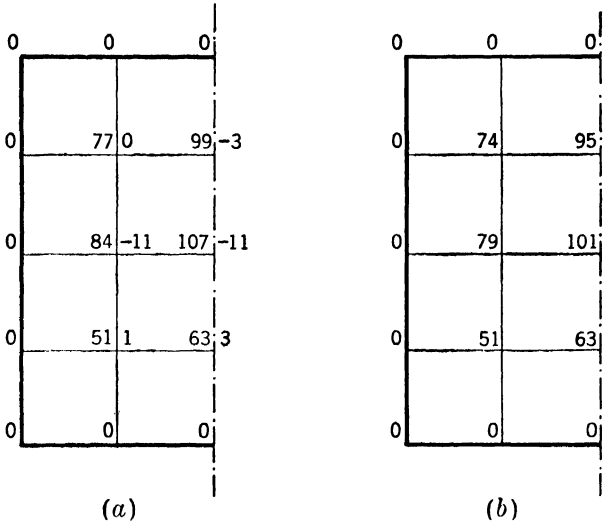


FIGURE 51

66. Calculation of Stresses

To complete the solution of the problem, we have still to compute values of the stress components $\tau_{r\theta}$ and $\tau_{\theta z}$. In practice, the solution for ϕ

would be found first for at least one finer net than the one used in Section 65, and the stresses would then be computed on a basis of the more accurate ϕ values so obtained. However, it is sufficient here, to illustrate how stresses are calculated, to continue to use the net of size $h = \frac{1}{4}$ only; with such a coarse net the results cannot be expected to be particularly accurate, but the method is the same in principle whatever size of net may be in use. It will similarly be sufficient to describe in detail the calculation of only one of the two stress components, that is, $\tau_{\theta z}$. This component is given in terms of ϕ by the second of (11), which relation yields as its finite-difference approximation

$$(\tau_{\theta z})_0 = -\frac{k}{r_0^2} \cdot \frac{\phi_2 - \phi_4}{2h},$$

or, with allowance for the multiplying factor, $-1{,}600$, which is included in the values of ϕ as given in Figure 51 (b),

$$(\tau_{\theta z})_0 = \frac{k}{100} \cdot \frac{2(\phi_2 - \phi_4)}{(4r_0)^2}. \tag{17}$$

By means of direct substitution in this expression, values of $\tau_{\theta z}$ can be found, as multiples of $k/100$, at all nodes of the net except those on the sides of the section $r = 3\frac{1}{2}$ and $r = \frac{1}{2}$; at nodes on those two sides (17) cannot be used immediately because, respectively, ϕ_2 and ϕ_4 are fictitious values which have not been determined and which therefore must be eliminated from (17), before it can be used to evaluate $\tau_{\theta z}$.

We consider first the side $r = 3\frac{1}{2}$. For a typical node 0 on this side, ϕ_2 is a fictitious value, and it is eliminated from (17) by use of the finite-difference approximation to the governing equation (12) holding at 0. It should be noticed that we have not yet had to use the fact that the governing equation is satisfied on any side of the section, for the reason that boundary values for ϕ were given and therefore we were not required to define residuals at nodes on the boundary. At a node on the line $r = 3\frac{1}{2}$ the finite-difference approximation to (12) reduces, because $\phi_1 = \phi_0 = \phi_3 = 0$, simply to

$$\frac{3\phi_2}{4} + \frac{5\phi_4}{4} + 100 = 0,$$

so that

$$\phi_2 = -\frac{5\phi_4 + 400}{3},$$

and therefore, after elimination of the fictitious ϕ_2, (17) is replaced by

$$(\tau_{\theta z})_0 = \frac{k}{100} \frac{-4\phi_4 - 200}{27}. \tag{18}$$

In a similar manner, at nodes on the line $r = \frac{1}{2}$, the fictitious ϕ_4 may be eliminated, and (17) is there replaced by

$$(\tau_{\theta z})_0 = \frac{k}{100}\,\frac{4\phi_2 + 200}{7}. \tag{19}$$

Thus, by use of (17) to (19), values of the shear-stress component $\tau_{\theta z}$ can be determined at all the nodes of the net; they are recorded, as multiples of $k/100$, in Figure 52. There now remains only to express k in terms of the resultant force Z; this may be done by a numerical evaluation of the integral in (15). An ordinary summation formula applied to the values recorded in Figure 52 yields

$$\iint_A \tau_{\theta z}\, dA = \frac{k}{100}\,\frac{110}{16};$$

therefore $k = 14.5Z/L^2$.

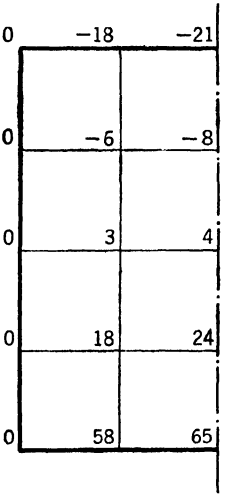


FIGURE 52

The evaluation of the other stress-component $\tau_{r\theta}$ can be performed in an exactly similar manner from the finite-difference approximation to the first of (11):

$$(\tau_{r\theta})_0 = \frac{k}{r_0^2}\cdot\frac{\phi_1 - \phi_3}{2h};$$

elimination from this formula of fictitious values of ϕ_3 is necessary for nodes on the side $z = -\frac{1}{2}$. The procedure is in no way different from that which was used to evaluate $\tau_{\theta z}$, and we shall not therefore describe it, nor are the values obtained for $\tau_{r\theta}$ recorded, since they would have little interest merely as results on such a coarse net.

67. Torsional Stresses in a Circular Shaft of Nonuniform Diameter

Another example of a quasi-plane-potential equation, very similar to (12), is presented for solution in the determination of the torsional stresses which are produced in a circular shaft of nonuniform diameter transmitting a torque of magnitude Z. With cylindrical coordinates r, θ, and z, and where the axis Oz is chosen as the axis of revolution of the shaft, the only nonzero stress components* are $\tau_{r\theta}$ and $\tau_{\theta z}$, and these shear stresses are again independent of θ and given by the expressions (11) in terms of a stress function ϕ. In this case ϕ no longer satisfies (12) but instead is governed by the equation

$$\frac{\partial^2\phi}{\partial r^2} - \frac{3}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{20}$$

* The footnote on page 92 applies again here.

and must also satisfy the conditions (i) that $\phi = 1$ at all points on the outer boundary of the axial section of the shaft, and (ii) that $\phi = 0$ on the axis of revolution or, if the shaft is hollow, at all points on the inner boundary. The constant k in the expressions (11) now depends on the value of the torque Z; the relation between k and Z can be found by taking moments about the axis Oz to find the torque transmitted across any circular section of the shaft. We see at once that

$$Z = \int r\,\tau_{z\theta}\,2\pi r\,dr = -2\pi k \int \frac{\partial \phi}{\partial r}\,dr = -2\pi k,$$

since, for every circular section, ϕ has the values 1 and 0, respectively, at the upper and lower limits of integration whether the shaft is solid or hollow.

68. A Uniform Shaft with a Collar

One new point of relaxational technique can be well illustrated here by consideration of a problem which is of the type most likely to arise in cases of stress determination in nonuniform circular shafts. It concerns

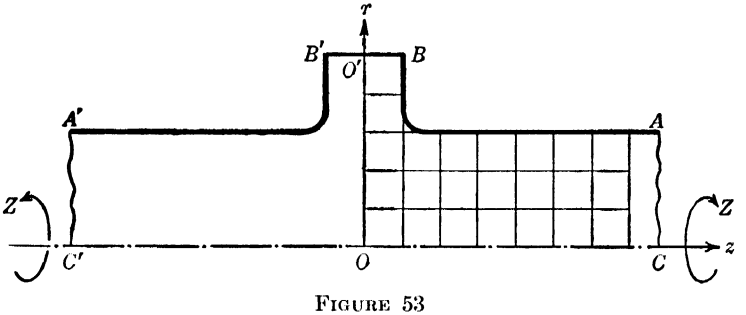


FIGURE 53

the difficulty, which we have not met previously but which often occurs in the practical solution of differential equations, that the area of integration may effectively extend to infinity in one or more directions. Figure 53 represents the axial section of a uniform shaft $A'A$, with a collar $B'B$, transmitting a torque Z. There is symmetry about the axis Or, and a coarse relaxation net has been drawn in to indicate what would be the working diagram for a relaxational solution. The computational problem is to find values of the stress function ϕ which satisfy the governing equation (20) over the area $OO'BACO$, the known boundary conditions being that $\phi = 1$ on $O'BA$, $\phi = 0$ on OC, and $\partial\phi/\partial z = 0$ on OO'. There remains to state a boundary condition on AC.

If the shaft is supposed to be long compared with the width $B'B$ of the collar, then, by Saint-Venant's principle, it is known that except in the

immediate neighborhood of the collar the stresses in the uniform part will approximate closely to the stresses in a completely uniform shaft. Hence although the shaft may effectively extend to infinity in the z direction, the ϕ values will settle down to uniformity in that direction when some section, such as AC, is reached; this section may not be many mesh lengths from the collar. We want to extend the field of computation as far as this section, but there is, of course, no point in extending it any further. We can accomplish this by means of a preliminary calculation of what the values of ϕ must be on the section AC, wherever it may be situated. For, where uniformity has been reached (to the number of significant figures in use), all z derivatives vanish, and therefore ϕ is governed by the one-dimensional equation

$$\frac{d^2\phi}{dr^2} - \frac{3}{r}\frac{d\phi}{dr} = 0. \tag{21}$$

This equation may be solved to yield nodal values of ϕ along a section like AC; these values should be taken as the initial guess for ϕ on all mesh lines parallel to Or in the uniform part of the shaft, and the network should be extended initially for, say, six mesh lengths to the right from B along this uniform part. Then there can be initial (nonzero) residuals only at nodes in the immediate neighborhood of the collar, and, to begin with, there are certainly no residuals to liquidate in the uniform part of the shaft. As the residuals are relaxed, by the usual methods, they gradually spread to the right along the uniform part of the shaft, although at the same time continually decreasing; should they eventually spread as far as the limit of the drawn network, it should then be extended an extra three mesh lengths; the same initial values of ϕ satisfying (21) should be assumed at the additional nodes, and the relaxation process continued. If, as they are further liquidated, the residuals again spread along Oz to the limit of the drawn network, more meshes can be introduced, a few at a time, as and when needed. By proceeding in this way we shall evidently complete the actual relaxation with the field extended along the uniform part of the shaft only just so far as necessary for the ϕ values to have settled down to uniformity. This technique is of frequent use in the solution of problems in which the area of integration extends in one direction to infinity, either actually or (as here) effectively.

A little must yet be said about the numerical solution of (21) from which the uniform values of ϕ are obtained. In order that this solution may provide no residuals, it must, of course, be the relaxational solution appropriate to the size of mesh in use. This may not be the *exact* solution, and it would be wrong, in general, to use the exact solution even if it were known. For instance, in the example of this section, the exact solu-

tion of (21) is easily verified to be $\phi \propto r^4$; but the uniform values of ϕ which should be used on a net in which there are four mesh lengths in the radius of the shaft are those obtained as the relaxational solution of the one-dimensional equation (21), with the range of integration divided into four intervals by three points of subdivision. These values, in fact, are $\phi = 0, -7, 28, 271, 1{,}000$ at nodes along CA, with a convenient multiplying factor, 10^3, introduced. The completion of the solution of the problem illustrated in Figure 53 may now be left to the reader as an exercise.

69. Continually Varying Patterns

Our next example of a quasi-plane-potential equation is one which is naturally expressed in the standard form (1); it is a typical case in which χ is not given algebraically as a function of x and y but can only be assessed numerically at the nodes of a network. In such cases it is a hindrance rather than a help to differentiate out the terms

$$\frac{\partial}{\partial x}\left(\chi \frac{\partial \phi}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\chi \frac{\partial \phi}{\partial y}\right)$$

in (1), and the equation is instead best treated in the manner of Section 62, with residuals consequently defined in terms of nodal values of ϕ and of χ by the formula (9).

We take as a worked example the determination of a two-dimensional steady temperature distribution. This means that Φ in (1) is to be taken to be zero, that ϕ represents the wanted temperature, and that χ is the nonuniform conductivity of the medium. We shall, in fact, not even suppose that χ is known numerically as a function of x and y, but instead treat the obviously more difficult case that χ derives its nonuniformity by virtue of its direct dependence on temperature. Thus χ, in general, has different values at each node of the net so that the residual formula (9) has different numerical coefficients, and different relaxation patterns are appropriate, at each node. Moreover, since the values of χ depend on the values of ϕ at corresponding nodes, the coefficients in the residual formulas, and hence the patterns, are continually varying.

The conductivity is taken to vary linearly* with the temperature according to a temperature coefficient β. Hence, in (9),

$$\chi_1 = \chi_0[1 + \beta(\phi_1 - \phi_0)], \tag{22}$$

* Any other law of variation could have been demonstrated, and linear variation was chosen partly for arithmetical simplicity and partly because it is a good approximation to what occurs in practice. The essential point here is not that there is a *linear* variation but that there is a variation of some sort.

so that the coefficient of ϕ_1 in the residual formula is $1 + (\beta/2)(\phi_1 - \phi_0)$. The coefficients of ϕ_2, ϕ_3, and ϕ_4 may be similarly written down, and the coefficient of ϕ_0 is minus the sum of the coefficients of ϕ_1, etc. Thus residuals may be calculated, at any stage in the relaxation process, according to formulas which are known but depend on the values of ϕ current at that stage. In the actual liquidation process the continually varying patterns need cause no difficulty because, as we have already seen in the solutions of other quasi-plane-potential equations, it is desirable anyway to use only approximate patterns. Whether or not they are changing as the solution proceeds, we should in any case use simply the approximate Poisson's equation pattern in the actual relaxation operations; but we must then check or recalculate the residual values from time to time according to the exact formulas which are appropriate at those times.

70. A Worked Example

Figure 54 shows a rectangular region on the boundaries of which the temperature is maintained at the values which are shown. At internal

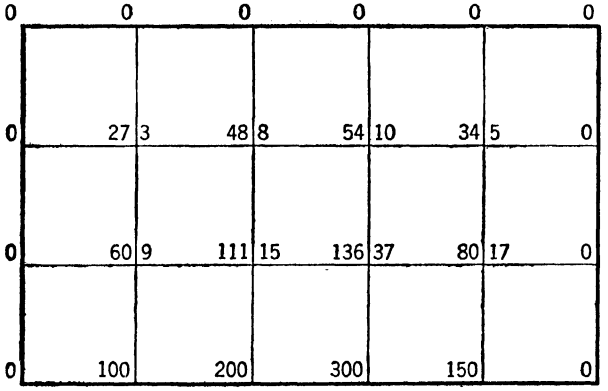


FIGURE 54

nodes are recorded the values of the temperature ϕ which are obtained by solution of Laplace's equation on the given network; these values correspond, of course, to the existence of a uniform conductivity over the rectangle, and they provide the obvious initial guess for the solution when a nonuniform conductivity is postulated. The residuals, which are also recorded in Figure 54, have been evaluated exactly, according to the method described in Section 69, when the value of β is given to be 0.002. These residuals are liquidated by use of the approximate pattern, and after only two recalculations the solution shown in Figure 55 is achieved.

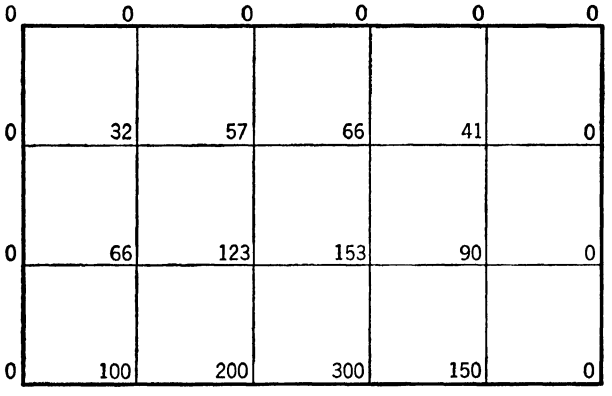


FIGURE 55

71. Two-dimensional Flow of a Compressible Fluid

An important application of the method described in Section 69 and illustrated in Section 70 is to determine the two-dimensional irrotational flow of a compressible fluid. Again, in (1), Φ is zero, and now ϕ represents the velocity potential of the flow; in most problems of compressible flow it would probably be desirable to work in terms of the stream function ψ, which is governed by the adjoint equation (5). In both (1) and (5), χ now represents the nonuniform density of the fluid; its dependence on ψ is of a somewhat complicated nature, being given by the relation

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \frac{2\chi^2}{\gamma - 1}\,(1 - \chi^{\gamma - 1}), \tag{23}$$

for a perfect gas, where γ is the ratio between the specific heats at constant pressure and at constant volume.

It is clear that the relation (23) must involve considerably greater computational difficulties than does the simple linear relation (22), but at the same time it should also be evident that in principle we may expect to be able to satisfy (23) by the method adopted in Section 70 in order to satisfy (22). The arithmetic involved in a use of (23) cannot be other than lengthy, so that a worked example would not be suitable here; instead the interested reader is referred to numerical solutions (of the flow of a compressible fluid through a two-dimensional nozzle) which have been found by Green and Southwell.*

72. Irregular Stars

The treatment of irregular stars, which occur at curved boundaries, in the solution of quasi-plane-potential equations is similar to that described

* These results were published in 1944 (*Phil. Trans. R.S. A*, **239**, 367–386).

in Section 47 in the case of Poisson's equation. There the fictitious values are expressed in terms of real values of the wanted function at nodes within the area of integration and at points on the boundary, and hence are then eliminated from the residual formulas. In the case of a quasi-plane-potential equation it is perhaps a rather simpler process to express each fictitious value in terms of the values at internal nodes but not then to eliminate these fictitious values from the residual formulas. Instead, the fictitious values are from time to time computed and recorded on the working diagram at the various fictitious nodes; in this way the normal residual formula may be used to calculate the residuals at all internal nodes.

Thus Figure 56 illustrates a typical irregular star, near a curved boundary which cuts short the string 01 at the point B; the point 1 is now fictitious, and we suppose that $0B = \xi h$, where h is the mesh length of the square network. Now we know that

$$h^2\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 = \phi_1 + \phi_3 - 2\phi_0 \quad (24)$$

as the standard finite-difference approximation for $\partial^2\phi/\partial x^2$ at the node 0 in terms of the fictitious value ϕ_1.

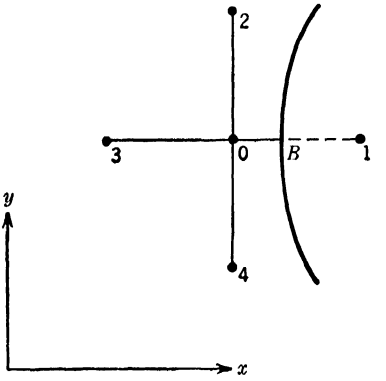


FIGURE 56

Also from (12) of Section 47, with replacement of w's by ϕ's, we have

$$h^2\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 = \frac{2\phi_B}{\xi(1+\xi)} + \frac{2\phi_3}{1+\xi} - \frac{2\phi_0}{\xi} \quad (25)$$

as the finite-difference approximation for $\partial^2\phi/\partial x^2$ at the node 0 in terms of the boundary value ϕ_B instead of ϕ_1. By equating the right-hand sides of (24) and (25), we can express the fictitious value ϕ_1 in terms of the real values ϕ_3, ϕ_0, and ϕ_B:

$$\phi_1 = \frac{2}{\xi(1+\xi)}\phi_B + \frac{1-\xi}{1+\xi}\phi_3 - \frac{2(1-\xi)}{\xi}\phi_0. \quad (26)$$

By use of this formula, fictitious values of ϕ may be calculated at any stage of the relaxation process in agreement with the real values of ϕ current at that stage; such fictitious values should be calculated as soon as initially guessed values of ϕ have been inserted at the internal nodes of the network. Then residuals may be calculated at all these internal nodes by substitution in the standard formula, *e.g.*, (9), (10), or (14), appropriate to the particular quasi-plane-potential equation which is

being solved. The fictitious ϕ values should, of course, be recalculated from time to time during the process of liquidation; one obviously convenient time to do so is always immediately before the residuals themselves are recalculated.

It may sometimes happen that two fictitious values of ϕ are attached at the same time to one fictitious point. Figure 57 illustrates how this could happen; there would always be two fictitious values at the point P, one calculated on a basis of real ϕ values on the line PQ and another on a basis of values on the line PR. There is *no* reason to suppose that these two fictitious values should be identical, although, on a fine enough net,

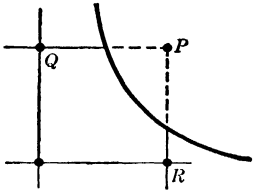


FIGURE 57

and when the liquidation process has been completed, they could usually be expected not to differ by any large amount. In the relaxation process, both fictitious values should be calculated, and both, indeed, should be used; the one based on real ϕ values on PQ would be used in the calculation of the residual at Q, and the one based on real values on PR would be used in the evaluation of the residual at R.

PROBLEMS

1. Determine values of the stress function ϕ for a tore of square section whose edges are the lines $r = L$, $r = 2L$, and $z = \pm L/2$. Use an eventual net of size $h = L/16$ and include a multiplying factor of $-256{,}000$.

(The student may check his results against the solution given by Allen, Fox, and Southwell.*)

2. Complete the solution of the problem described in Section 68.

3. Modify the solution to Problem 3, Chapter 5, to include a temperature coefficient of conductivity, $\beta = 0.001$.

* *Phil. Trans. R.S. C*, **1**, 99–135 (1943), and *A*, **239**, 501–537 (1945).

CHAPTER 8

THE BIHARMONIC EQUATION

73. Introduction

In previous chapters the application of the relaxation method to solve various second-order partial differential equations in two dimensions has been described; in this chapter the fourth-order biharmonic equation will be brought within the scope of the method. This equation is of considerable importance in the theory of flat elastic plates, both in cases of bending of a plate (*flexural* problems) and in cases of stretching of a plate in its own plane (*extensional* problems). The biharmonic equation also governs the slow motion (in two dimensions) of a viscous fluid; in this case the stream function ψ must satisfy the equation

$$\nabla^4\psi = 0, \tag{1}$$

where the biharmonic operator ∇^4 is given by

$$\nabla^4 \equiv (\nabla^2)^2 \equiv \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{2\partial^4}{\partial x^2\,\partial y^2}, \tag{2}$$

and x,y are rectangular cartesian coordinates in the plane of the flow.

74. The Bending of Flat Plates

The method of solution for a flexural, or bending, problem in a plate depends on the boundary conditions which hold at the edge of the plate. These may be given conditions of support (*i.e.*, of displacement), or given conditions of applied edge tractions (*i.e.*, of force), or given "mixed" conditions, *i.e.*, support conditions at parts of the boundary, traction conditions at other parts. In this chapter we shall be concerned, in flexural problems, only with cases of known support edge conditions.* The solutions in these cases are obtained in terms of w, the transverse deflection of the middle surface of the plate. With x,y as rectangular coordinates in the plane of this surface, w has to be found to satisfy the equation

$$\nabla^4 w = \frac{Z}{D}, \tag{3}$$

* Problems involving the other types of edge conditions will be discussed in Chapter 9.

where D is the flexural rigidity of the plate and Z is the intensity of distributed transverse loading, including, if such exist, transverse body forces. In flexure, we are not concerned with any forces acting in the plane of the plate; if such forces are additionally present, they are dealt with separately as an extensional problem and the resulting contributions to the stress system are superposed.

The edge conditions of support may be of two kinds, simple support and clamping. In the case of clamping these boundary conditions are expressed easily in terms of w as

$$\begin{aligned} w &= 0, \\ \frac{\partial w}{\partial \nu} &= 0, \end{aligned} \tag{4}$$

where ν denotes the direction of the outward-drawn normal at the edge. In using the relaxation method, we shall find it convenient to express the conditions (4) in the slightly different manner:

$$\begin{aligned} w &= 0, \\ \frac{\partial w}{\partial x} &= 0, \\ \frac{\partial w}{\partial y} &= 0. \end{aligned} \tag{5}$$

Although (5) contains three equations, these are not independent and mean no more than the two equations (4). It will be seen later (Section 83) why (5) is a more convenient form than (4) for the purpose of solving equation (3) by relaxation methods.

In the case of simple support, the boundary conditions become

$$\begin{aligned} w &= 0, \\ \frac{\partial^2 w}{\partial \nu^2} - \frac{1-\sigma}{\rho} \cdot \frac{\partial w}{\partial \nu} &= 0, \end{aligned} \tag{6}$$

where σ is Poisson's ratio for the material of the plate and ρ is the radius of curvature of the boundary. The second of equations (6) simplifies in the case of a straight edge: for then ρ is infinite, and the boundary conditions reduce to

$$\begin{aligned} w &= 0, \\ \frac{\partial^2 w}{\partial \nu^2} &= 0. \end{aligned} \tag{7}$$

When w has been determined to satisfy the governing equation (3) and appropriate boundary conditions such as (4) or (6), the stress actions in the plate can at once be derived. On the edges of a rectangular element of the plate there act line intensities of flexural couples G_x and G_y, of a

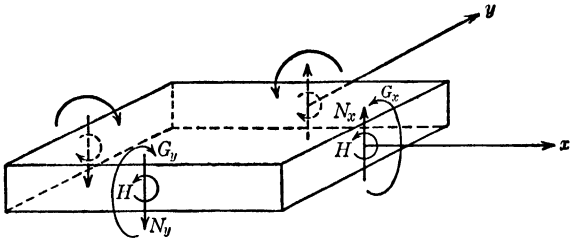


FIGURE 58

twisting couple H, and of shear forces N_x and N_y, as shown in Figure 58. These actions are given in terms of w by the expressions

$$\begin{aligned} G_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right), \\ G_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right), \\ H &= -(1-\sigma)D\frac{\partial^2 w}{\partial x\,\partial y}, \\ N_x &= D\frac{\partial}{\partial x}(\nabla^2 w), \\ N_y &= D\frac{\partial}{\partial y}(\nabla^2 w). \end{aligned} \tag{8}$$

75. The Stretching of Flat Plates

The method of solution of an extensional, or stretching, problem in a plate also depends on the boundary conditions, which again, as in the case of flexure (cf. Section 74), may be given conditions of displacement or of edge tractions, or may be mixed conditions. In this chapter we shall be concerned, in extensional problems, only with cases of known edge-traction conditions. The solution in these cases is obtained in terms of Airy's stress function χ. With x,y again as rectangular coordinates in the plane of the middle surface, and supposing first that no body forces or temperature stresses exist, we must find χ to satisfy the equation

$$\nabla^4\chi = 0. \tag{9}$$

The stresses in the plate, which act in planes parallel to the middle surface, are given in terms of χ by the expressions

$$\begin{aligned} \sigma_x &= \frac{\partial^2\chi}{\partial y^2}, \\ \sigma_y &= \frac{\partial^2\chi}{\partial x^2}, \\ \tau_{xy} &= -\frac{\partial^2\chi}{\partial x\,\partial y}, \end{aligned} \tag{10}$$

under conditions of either plane strain or generalized plane stress. These stresses act on the edges of a rectangular element of the plate as shown in Figure 59.

The edge conditions of known tractions yield values at all points of the boundary of the normal direct stress (σ_ν) and of the tangential shear stress ($\tau_{\nu s}$); ν denotes the direction of the *outward*-drawn normal at the edge and s the direction of the tangent in a counterclockwise circuit, as illustrated in Figure 60. These conditions can be interpreted in terms of χ because knowledge of

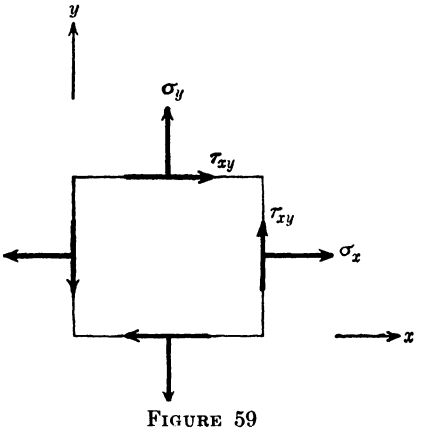


FIGURE 59

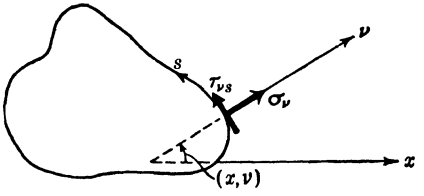


FIGURE 60

σ_ν and $\tau_{\nu s}$ means that, at the boundary, $\tau_{\nu x}$ and $\tau_{\nu y}$ also are known:

$$\begin{aligned} \tau_{\nu x} &= \sigma_\nu \cos(x,\nu) - \tau_{\nu s} \sin(x,\nu), \\ \tau_{\nu y} &= \sigma_\nu \sin(x,\nu) + \tau_{\nu s} \cos(x,\nu). \end{aligned}$$

But also

$$\begin{aligned} \tau_{\nu x} &= \sigma_x \cos(x,\nu) + \tau_{xy} \sin(x,\nu) \\ &= \frac{\partial^2 \chi}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 \chi}{\partial x\,\partial y}\frac{\partial x}{\partial s} \\ &= \frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial y}\right); \end{aligned}$$

and similarly

$$\begin{aligned} \tau_{\nu y} &= \sigma_y \sin(x,\nu) + \tau_{xy} \cos(x,\nu) \\ &= -\frac{\partial^2 \chi}{\partial x^2}\frac{\partial x}{\partial s} - \frac{\partial^2 \chi}{\partial x\,\partial y}\frac{\partial y}{\partial s} \\ &= -\frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial x}\right). \end{aligned}$$

Hence by line integration round the boundary we can obtain boundary values for $\partial\chi/\partial x$ and $\partial\chi/\partial y$:

$$\begin{aligned} \frac{\partial \chi}{\partial x} &= -\int \tau_{\nu y}\, ds, \\ \frac{\partial \chi}{\partial y} &= \int \tau_{\nu x}\, ds. \end{aligned} \tag{11}$$

When $\partial\chi/\partial x$ and $\partial\chi/\partial y$ have been evaluated, we can then determine the value, at all points of the boundary, of the expression

$$-\frac{\partial\chi}{\partial x}\sin(x,\nu)+\frac{\partial\chi}{\partial y}\cos(x,\nu).$$

Now

$$-\frac{\partial\chi}{\partial x}\sin(x,\nu)+\frac{\partial\chi}{\partial y}\cos(x,\nu)=\frac{\partial\chi}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial\chi}{\partial y}\frac{\partial y}{\partial s}=\frac{\partial\chi}{\partial s},$$

so that another line integration round the boundary yields boundary values of χ:

$$\chi=\int\left[\frac{\partial\chi}{\partial y}\cos(x,\nu)-\frac{\partial\chi}{\partial x}\sin(x,\nu)\right]ds. \tag{12}$$

The three relations contained in equations (11) and (12) comprise the boundary conditions which the stress function χ must satisfy. Although not independent of one another—equation (12) has, indeed, been derived directly from equations (11)—it is particularly convenient, for an application of the relaxation method, to have edge conditions stated in this way, exactly as the conditions of clamping support in flexure are more conveniently expressed in the form (5) than in the form (4).

76. Nondimensionalization of the Equation of Flexure

The determination of any solution for the transverse deflection w of a plate should always begin with a transformation of the governing equation (3) into nondimensional terms. If L denotes some representative linear dimension of the plate, and Z_0 is some convenient particular value of the transverse pressure Z, this nondimensionalization can be achieved by the substitutions

$$\begin{aligned} x &= Lx', \\ y &= Ly', \\ Z &= Z_0Z', \\ w &= \left(\frac{L^4Z_0}{D}\right)w', \end{aligned}$$

in which x', y', Z', and w' are all purely numerical quantities. After these substitutions have been made, we can conveniently omit the primes in x', etc., so that, on the understanding that now x, y, Z, and w are nondimensional, (3) becomes

$$\nabla^4 w = Z, \tag{13}$$

and the boundary conditions (4), (5), (6), or (7) remain unchanged. In (6), of course, ρ is now also understood to be nondimensional, and so also is ν in (4), (6), and (7).

77. The Biharmonic Residual Formula

The numerical values of w which comprise the solution to a biharmonic equation such as (13) are usually obtained at the nodes of a square network covering the region of integration, and we must first find the finite-difference approximation to (13) holding at a typical node 0 of such a network. The size of the mesh is again denoted by h, and Figure 61 illustrates a group of 13 typical nodes which we shall always denote by 0, 1, 2, 3, etc., as shown in that diagram. There are two methods available for the determination of the finite-difference approximation to $(\nabla^4 w)_0$ in terms of the nodal values w_0, w_1, w_2, etc. The first of these methods involves the expansion of w in a double Taylor's series of powers of $x - x_0$ and $y - y_0$ in the manner of Section 30; this method is lengthy but has the advantage that it yields the order of the error in the result. The alternative method is very simple but does not provide any indication of the error involved. It makes use of the known finite-difference approximation (Section 41) to the operator ∇^2 by writing

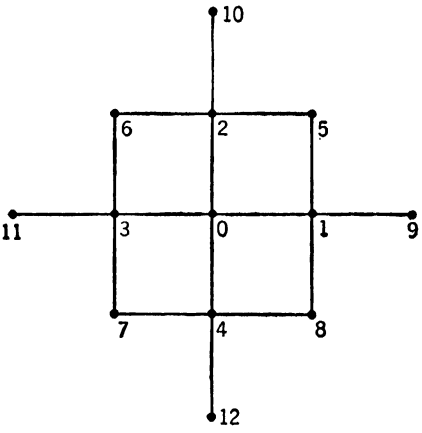


FIGURE 61

$$\nabla^2 w = \Theta,$$

so that

$$\nabla^4 w = \nabla^2 \Theta,$$

and therefore

$$h^4(\nabla^4 w)_0 = h^2(h^2\nabla^2\Theta)_0 \approx h^2(\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 - 4\Theta_0), \qquad (14)$$

where, as usual, suffixes denote values at the nodes correspondingly numbered in Figure 61. But

$$h^2\Theta_0 = h^2(\nabla^2 w)_0 \approx w_1 + w_2 + w_3 + w_4 - 4w_0,$$

and, in the same way,

$$h^2\Theta_1 = h^2(\nabla^2 w)_1 \approx w_9 + w_5 + w_0 + w_8 - 4w_1,$$

with similar expressions for $h^2\Theta_2$, etc., in terms of nodal values of w. Now, substituting for $h^2\Theta_0$, $h^2\Theta_1$, etc., in (14), we find at once that

$$h^4(\nabla^4 w)_0 = 20w_0 - 8\Sigma w_1 + 2\Sigma w_5 + \Sigma w_9, \qquad (15)$$

approximately, where Σw_1 is the sum of the values of w at the four points situated like 1 in Figure 61, and Σw_5 and Σw_9 have similar meanings. Hence the finite-difference approximation to (13), holding at a typical node 0, is

$$8\Sigma w_1 - 20w_0 - 2\Sigma w_5 - \Sigma w_9 + h^4 Z_0 = 0, \tag{16}$$

and a residual is defined at 0 by the formula

$$F_0 \equiv 8\Sigma w_1 - 20w_0 - 2\Sigma w_5 - \Sigma w_9 + h^4 Z_0, \tag{17}$$

in terms of any initially guessed values of w or of any subsequent w values in the course of the liquidation process. When the relaxation is completed, the residuals as defined by (17) are zero at every node; consequently the finite-difference approximation (16) is satisfied in respect of every set of 13 typical nodes, and therefore the relaxational solution to the governing equation (13) is determined for the particular size of net in use.

We shall not also deduce (17) by the first of the two methods suggested above, but it should be mentioned that, if that first method is employed, it demonstrates in addition that the error in (17) is of $O(h^6)$. It will be left as an exercise to the reader to verify this statement.

78. The Biharmonic Relaxation Pattern

Associated with the residual formula (17) is the biharmonic relaxation pattern; it shows the changes made to the residuals when, in the liquidation process, a unit increment $\Delta w_0 = 1$ is added to the value of w at a typical node of the network. The construction of this pattern, which is illustrated in Figure 62, needs no description, since the pattern is similar to, though more extensive than, that derived in Section 41 for Poisson's equation. It is necessary, however, to say more about the use of the biharmonic pattern, because, although in principle it is employed in the same way as the pattern for Poisson's equation, it is, in practice, much more difficult to use; this is mostly because biharmonic relaxation is deceptively slowly convergent.

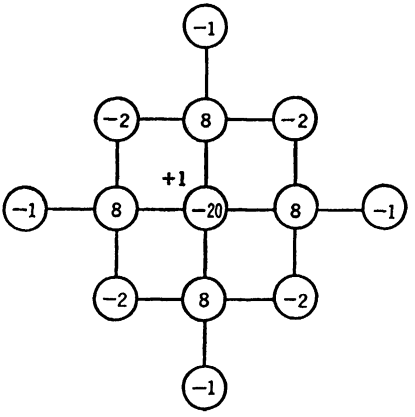


FIGURE 62

To realize why this is so, we must consider in detail the over-all effect

on the residuals of the application of a unit increment $\Delta w_0 = 1$. At first sight the biharmonic pattern appears to be more powerful than that for Poisson's equation, because it changes the residual, at the point where the increment is applied, by -20 instead of only -4. But the effectiveness of a pattern should not be measured by the magnitude of its central coefficient but rather by the extent to which it redistributes the residuals by transferring them from one node to another in directions radiating out from the central point. By this criterion the pattern for Poisson's equation is 100 per cent effective, because it does transfer 4 units of residual from the central point outward to the four surrounding nodes at a distance of one mesh length. On the other hand the biharmonic pattern is only 40 per cent effective, because, although it transfers 20 units of residual *outward* from the central point through a distance of one mesh length, at the same time it transfers *inward* 12 units of residual toward the central point. These 12 units may be counted in Figure 62; they are made up of the elements -1 and -2, which may be seen in that diagram forming an outer ring in the pattern. It is this feature of the biharmonic pattern which caused it to be described above as deceptive. While a unit increment appears to remove 20 units of residual outward from the central point, it, in fact, effectively so moves only 8 units. It has been said that in its operation the biharmonic pattern does its best to sabotage itself, and a striking demonstration will be given later (Section 81) of this unfortunate property. It can scarcely be emphasized too much here that considerable overrelaxation is almost always necessary in any biharmonic relaxation. It is impossible, of course, to say, in general, to what extent overrelaxation should be employed, but as a rough guide, it may be suggested that it should be, say, ten times the extent which might be used in the relaxation of a similar set of residuals with the pattern of Poisson's equation.

79. Rectilinear Boundaries

Two boundary conditions are appropriately associated with a biharmonic equation at all points of the boundary. We now discuss first how some typical conditions may be satisfied when the boundary is rectilinear and composed entirely of strings of the network; curved boundaries will be considered in Section 83.

Figure 63 shows a part of a straight boundary (parallel to the y direction) with a portion of the (square) network drawn in to the left of the boundary. We shall suppose in the first instance that the conditions (4) of clamping are given to hold on this boundary. The first of those conditions immediately yields the value for w (*i.e.*, zero) that we must take at all nodes on the boundary (like the point 1 in Figure 63); no residual,

therefore, is required to be defined at these nodes. But residuals do have to be calculated at all nodes inside the boundary, and at those which are only one mesh length within the boundary (like the point 0 in Figure 63), the residual formula (17) contains a fictitious value; thus for the node 0 in Figure 63, w_9 is a fictitious value. The second of the conditions (4) has to be used in order to eliminate such fictitious values from the residual formulas at points just inside the boundary. Thus in Figure 63 the second of (4) becomes $\partial w/\partial x = 0$, and applied to hold at the boundary point 1, it yields as its finite-difference approximation

$$w_9 = w_0.$$

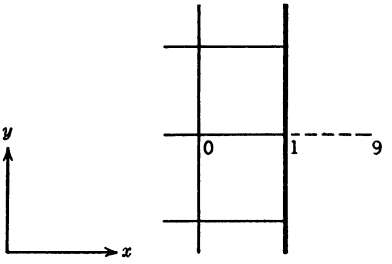


FIGURE 63

Thus w_9 can be eliminated from the residual formula at the point 0, and that formula becomes, therefore, in replacement of (17),

$$F_0 \equiv 8\Sigma w_1 - 21w_0 - 2\Sigma w_5 - w_{10} - w_{11} - w_{12} + h^4 Z_0. \qquad (18)$$

We may treat in a similar manner the conditions (7) of simple support. The first of (7) is the same condition ($w = 0$) as the first of (4), but the second of (7) becomes, for Figure 63, $\partial^2 w/\partial x^2 = 0$, and applied to hold at the point 1, it gives as its finite-difference approximation

$$w_9 + w_0 - 2w_1 = 0,$$

or, since w_1 vanishes,

$$w_9 = -w_0.$$

Again w_9 can be eliminated from the residual formula at 0, and that formula becomes now, in place of (17),

$$F_0 \equiv 8\Sigma w_1 - 19w_0 - 2\Sigma w_5 - w_{10} - w_{11} - w_{12} + h^4 Z_0. \qquad (19)$$

Boundary conditions of the form (4) or (7) can be applied in a similar manner for any part of a rectilinear boundary which lies along strings of the net; in the residual formulas at nodes near the boundary at least one of the values w_9, w_{10}, w_{11}, and w_{12} will be fictitious and hence have to be eliminated. The form of the resulting residual formula is illustrated by (18) in the case of boundary conditions of clamping, by (19) in the case of simple support.

80. A Worked Example

As a numerical example, we take the case of a uniform square plate which is clamped along one pair of opposite edges and simply supported

along the other pair, and suppose that the plate is subjected to a uniform intensity of transverse loading. We naturally take the side of the square to be the representative dimension L and the uniform intensity of loading to be Z_0; the nondimensional governing equation (13) thus becomes

$$\nabla^4 w = 1,$$

and the residual formula (17) similarly becomes

$$F_0 \equiv 8\Sigma w_1 - 20w_0 - 2\Sigma w_5 - \Sigma w_9 + 409{,}600h^4, \tag{20}$$

with the inclusion of a convenient multiplying factor, 409,600. We find the solution first on the coarsest possible net, for which $h = \frac{1}{2}$, so that the constant term in (20) takes the value 25,600. There is only one internal node on this net, so that the solution (Figure 64) is found immediately in one step; in that diagram the left- and right-hand edges are taken to be the pair which are clamped, and the top and bottom edges the pair which are simply supported. At the only internal node the initial value of the residual is 25,600 when, as a beginning, w is taken to be zero everywhere. For this node the values w_9, w_{10}, w_{11}, and w_{12} are all fictitious, and after they have each been eliminated in the manner of Section 79, the residual formula then becomes

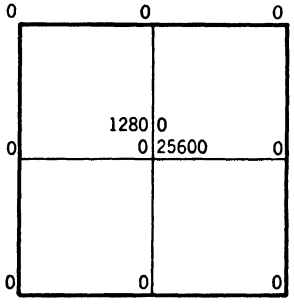


FIGURE 64

$$F_0 \equiv 8\Sigma w_1 - 20w_0 - 2\Sigma w_5 + 25{,}600.$$

All of w_1 to w_8 are given to have zero values (because $w = 0$ on the boundary), so that an increment $\Delta w_0 = 1{,}280$ applied at the central point reduces the residual there to zero, and the solution for the net $h = \frac{1}{2}$ is at once completed.

The solution is next found on the finer net for which $h = \frac{1}{4}$; for this net the constant term in (20) has the value 1,600, and by virtue of symmetry the calculation is sufficiently recorded (in Figure 65) on one-quarter of the whole area. First of all, initial values of w are chosen at the nodes of the new net with a use of the solution (such as it is) of Figure 64 as a guide. Corresponding residuals are then calculated according to the formula

$$F_0 \equiv 8\Sigma w_1 - 20w_0 - 2\Sigma w_5 - \Sigma w_9 + 1{,}600.$$

These initial w values and residuals are both recorded in Figure 65 at the bottom of the left-hand and right-hand columns, respectively, at each node. The complete liquidation process is also shown in Figure 65; at each node the left-hand column records the increments Δw applied during the relaxation, and the right-hand column records successive values of the

corresponding residual. These residuals are seen eventually (and on this comparatively coarse net, by a use of point relaxation only) to have been brought to zero; the totaled w values are also recorded, underlined at the tops of their respective columns. The central value of w is seen to have

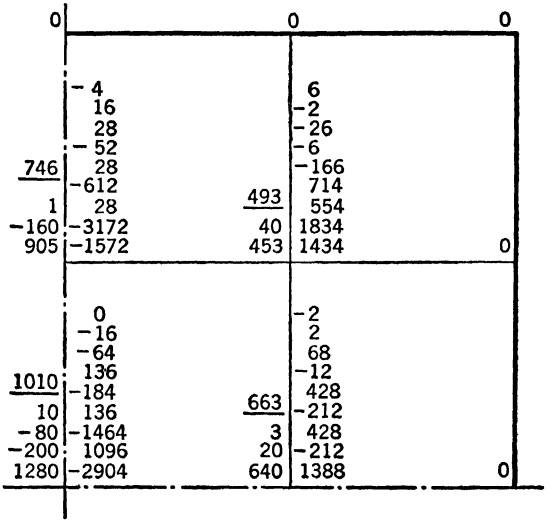


FIGURE 65

changed appreciably from Figure 64, and it can be still expected not to be very accurate, since, as yet, only very coarse nets have been introduced.

81. Biharmonic Block Patterns

However, before the above example could be continued on any finer nets, it would be necessary to employ some other devices than merely overrelaxation in order to accelerate the convergence of the process. Therefore, we now consider the possibility of introducing biharmonic block relaxation. It can be anticipated that biharmonic block patterns will be very complicated, and, in fact, in practice they are so complicated that only a few restricted shapes of block operations are worthy of consideration. These are mostly line blocks; by a unit line-block operation we mean the application of a unit increment $\Delta w = 1$ at a collinear set of nodes only. Such patterns are not difficult to construct, and they are illustrated in Figure 66: (i) is the pattern for a unit line block consisting of only two (adjacent) nodes; (ii) is the pattern for a block of three points; and (iii) indicates the form of the pattern for a block of four or more nodes. In (iii) the central (vertical) line of elements is included as many times as necessary to extend the block to any required (horizontal) length.

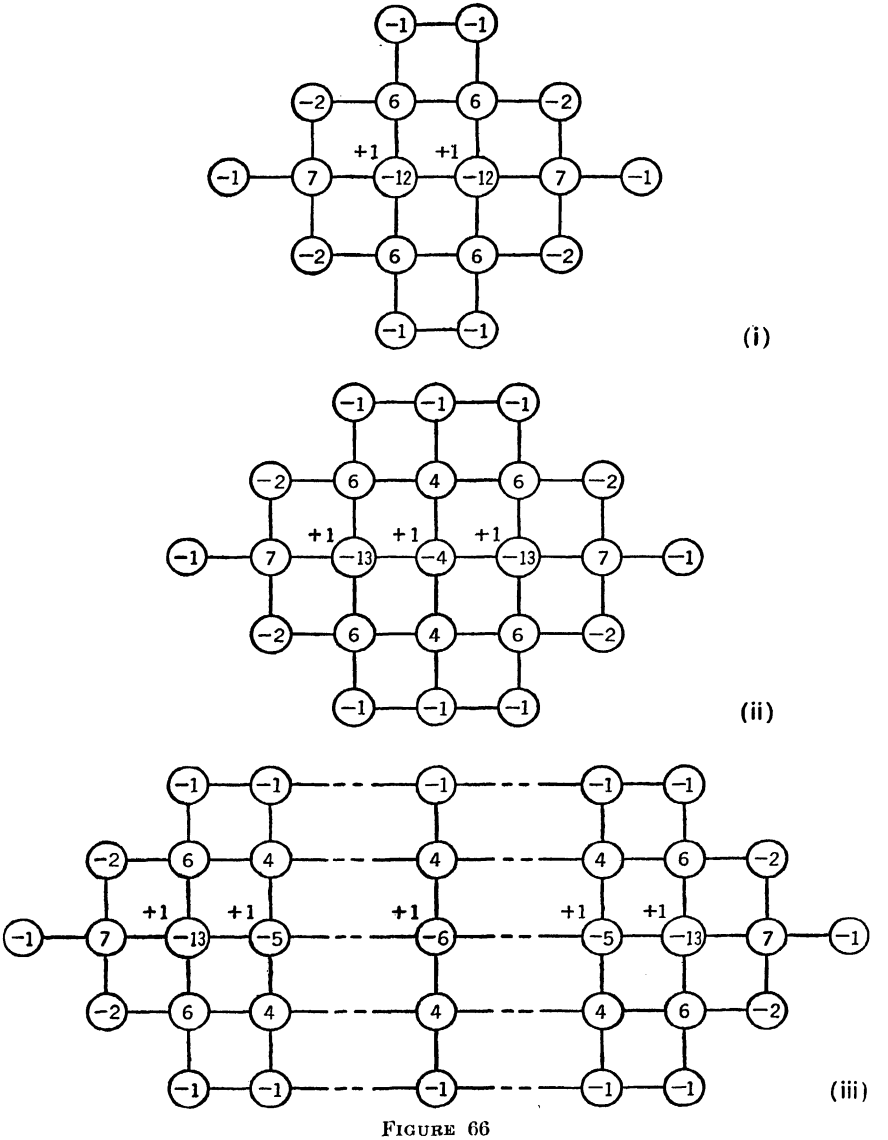


FIGURE 66

With a little practice in their use it will be found that a computer soon memorizes these line-block patterns; their employment is virtually essential to the success of biharmonic relaxation on any reasonably fine network. It is too much, however, to expect anyone to remember the patterns which correspond with other shapes of blocks, and their use is not therefore recommended. It is, nevertheless, very illuminating to consider the pattern associated with a square block of nine nodes; this is shown in Figure 67, and although the over-all effect of such a block operation is to transfer positive residual outward from the group of nine nodes, it should be observed that the effect on the residual at the node in the

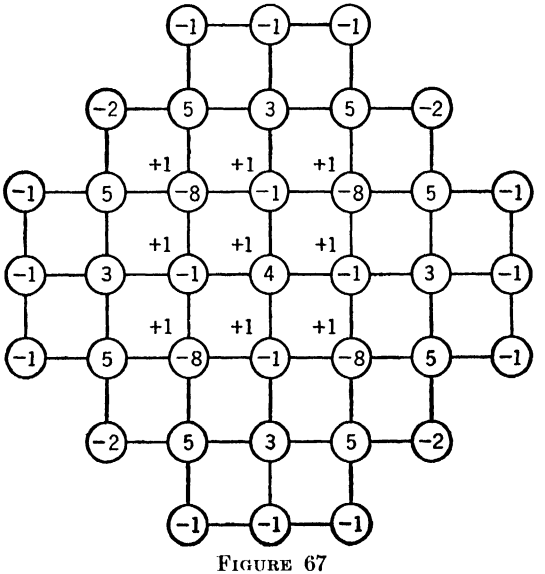


FIGURE 67

center of the square is an *increase* of four units. This diagram strikingly demonstrates the self-sabotaging property inherent in the biharmonic pattern; continually, as attempts are made to drive residuals outward from some area toward a boundary, they tend to reappear in the most inconvenient positions.

82. Multiplying Factors

The use of a multiplying factor as a device to speed up the convergence of a relaxation computation was introduced in Section 14 in relation to the solution of simultaneous algebraic equations; it is a particularly helpful device to employ in the solution of biharmonic differential equations. For instance, when we continue the example of Section 80 on the next finer net ($h = \frac{1}{8}$), the value of the constant term in the residual

formula (20) is 100; on this net there are 49 internal nodes, so that the total residual when the w values are all taken to be zero is 4,900. Now, suppose that at some stage in the relaxation the total of all the residuals is R; then at once we can reduce that total to zero by multiplying each w value by the factor $4{,}900/(4{,}900 - R)$. By virtue of the extremely slow convergence it is clearly a matter of the greatest importance that we can thus in one step bring the total of a set of biharmonic residuals to zero. Liquidation can then proceed by a use of line-block operations and point relaxation; at some later stage it may be helpful to introduce another multiplying factor if the total of the residuals happens to accumulate to any noticeable extent.

0	0	0	0	0
357	321	222	92	0
632	567	389	158	0
800	716	488	196	0
856	766	521	208	0

FIGURE 68

The solution to the problem of Section 80 for the finer net ($h = \frac{1}{8}$) was obtained in this way; it is given in Figure 68, and it provides a useful opportunity for the reader to practice biharmonic relaxation by finding this solution himself, starting with initial w values which may be guessed on a basis of the values given in Figure 65 as the solution on the previous net ($h = \frac{1}{4}$).

83. Curved Boundaries

The treatment of curved boundaries in the solution of a biharmonic problem does not involve any great difficulty. We shall describe the method in terms of the extensional problem for a plate, where the wanted quantity is an Airy's stress function χ satisfying the governing biharmonic equation (9) and the boundary conditions (11) and (12). These conditions number three in all, but only two of them are independent; we choose to satisfy (12) and the *first* of (11) at those points on the boundary where it is cut by a mesh line which is parallel to the x direction, and to satisfy (12) and the *second* of (11) at those points where the boundary is cut by a mesh line which is parallel to the y direction.

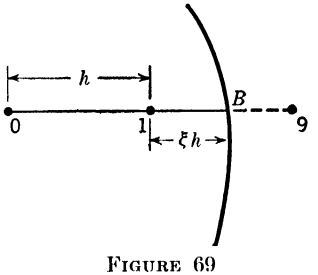


FIGURE 69

Figure 69 illustrates the first of these two possibilities, where a mesh line in the x direction cuts the boundary at B; the string is drawn extended to the fictitious point 9, and, within the boundary, mesh-points 0 and 1 are

denoted as shown. The two boundary conditions chosen to be satisfied at B are the given values there of χ and $\partial\chi/\partial x$; we denote these, respectively, by K and k and use those conditions in order to express the nodal values χ_1 and χ_9 in terms of the value χ_0. To do this, we suppose that, as usual, χ is expanded in a Taylor's series in the neighborhood of B along the line $01B9$:

$$\chi = K + k(x - x_B) + c(x - x_B)^2, \tag{21}$$

where c is an as yet undetermined coefficient and where powers of $x - x_B$ beyond the second are neglected. As previously, we denote the mesh length 01 by h and the short arm $1B$ by ξh.

Now if we put $x - x_B$ in (21) equal in turn to $(1 - \xi)h$, $-\xi h$, and $-(1 + \xi)h$, then we have at once the three relations

$$\begin{aligned}
\chi_9 &= K + k(1 - \xi)h + c(1 - \xi)^2h^2, \\
\chi_1 &= K - k\xi h + c\xi^2h^2, \\
\chi_0 &= K - k(1 + \xi)h + c(1 + \xi)^2h^2.
\end{aligned}$$

Next, eliminating c, first from the first and third of these equations, then from the second and third, we derive the results, with neglect of terms of $O(h^3)$:

$$\begin{aligned}
\chi_9 &= \frac{4\xi K}{(1 + \xi)^2} + \frac{2(1 - \xi)hk}{1 + \xi} + \frac{(1 - \xi)^2\chi_0}{(1 + \xi)^2}, \\
\chi_1 &= \frac{(1 + 2\xi)K}{(1 + \xi)^2} - \frac{\xi hk}{1 + \xi} + \frac{\xi^2\chi_0}{(1 + \xi)^2}.
\end{aligned} \tag{22}$$

Once values of χ have been attached to all other internal nodes, such as the node 0 in Figure 69, the pair of relations (22) enable consistent values of χ, automatically satisfying the boundary conditions, to be evaluated at those internal points, such as 1 in Figure 69, which lie within one mesh length of the boundary, and also at points, like 9, which are fictitious. Alternatively, we could say that pairs of equations like (22) enable values of χ to be calculated at each of the two ends (one real and one fictitious) of every string of the network which crosses the boundary, and when this has been done, residuals may be calculated from the ordinary formula

$$F_0 \equiv 8\Sigma\chi_1 - 20\chi_0 - 2\Sigma\chi_5 - \Sigma\chi_9,$$

at all other internal nodes. It should be carefully noticed that, in this treatment of curved boundaries in biharmonic problems, we distinguish between two different types of real nodes: those which are distant more than one mesh length inside the boundary and which are dealt with in the ordinary way, and those which are distant less than one mesh length from the boundary and which are dealt with in the same way as are the

fictitious nodes. To distinguish them from the other internal nodes, points like 1 in Figure 69 are referred to as *selvage points.*

The liquidation proceeds in the usual manner with the application of increments $\Delta\chi$ to the χ values at the properly internal nodes in order to reduce the corresponding residuals gradually to zero. From time to time during the process the χ values at the selvage and fictitious points should be recalculated according to the formulas (22) in order to bring those values into agreement with the current values of χ at the internal nodes.

It should be mentioned that it is quite possible that values of χ at fictitious and selvage points may be calculable from two different formulas and that these pairs of values may not be in complete agreement with one another, although on a reasonably fine net they should not show much difference as the solution is approached, particularly at the selvage points.

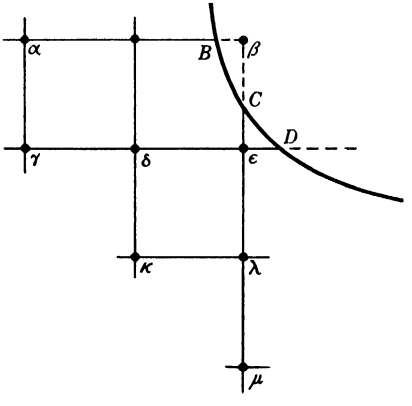


FIGURE 70

Figure 70 illustrates how it could happen that two such values of χ be calculable at some points. At β one fictitious χ value can be found in terms of the boundary conditions at B, and another fictitious value can be found in terms of the boundary conditions at C; similarly at ϵ one selvage χ value can be found by use of the boundary conditions at C and another by use of the conditions at D. The method of deciding which of a pair of χ values to use in the calculation of residuals should be made evident by the following illustrations: In Figure 70 the fictitious χ value at β calculated in terms of the conditions at B should be substituted in the residual formula at α, but the fictitious χ value at β calculated in terms of the conditions at C should be used in the residual formula at λ. The mean of the two fictitious χ values at β should be taken in the residual formula at δ. Similarly, the selvage χ value at ϵ obtained in terms of the conditions at C should be used in the calculation of the residuals at λ and at μ, but the selvage χ value at ϵ found in terms of the conditions at D should be substituted in the residual formulas at γ and at δ. The mean of the two selvage χ values at ϵ should be taken in the residual formula at κ.

84. Stresses in a Tension Specimen

As a numerical example we shall consider the formulation of the problem illustrated in Figure 71. The diagram represents one-quarter, $ABCDEA$ (*i.e.*, the sufficient working diagram by virtue of twofold

symmetry), of a uniform tension specimen. It is subjected to a uniformly distributed tensile action along the end AB, and a suitable square relaxation net has been drawn in the diagram. The notch CD is circular and of radius $L/2$ where the half breadth AB of the specimen is of length L and the half length EA is $2L$.

It will be assumed that the computational problem has been expressed in nondimensional form, L having been taken as the representative dimension of the specimen, and that the intensity of the applied tension along

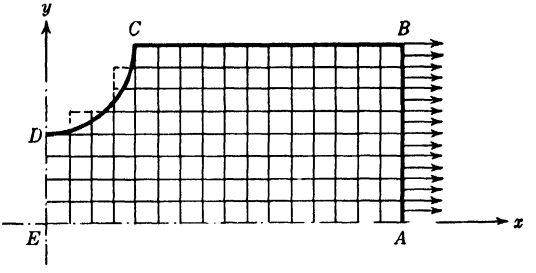


FIGURE 71

AB is 100 (also a nondimensional value). Then DE and EA are axes of symmetry, and the stress boundary conditions on $ABCD$ are that

$$\begin{aligned} \tau_{yy} &= 0 \text{ everywhere,} \\ \tau_{yx} &= 100 \text{ on } AB, \\ &= 0 \text{ on } BCD. \end{aligned}$$

At once, then, for this example, the boundary conditions (11) become

$$\begin{aligned} \frac{\partial \chi}{\partial x} &= 0 \text{ everywhere,} \\ \frac{\partial \chi}{\partial y} &= 100y \text{ on } AB, \\ &= 100 \text{ on } BCD, \end{aligned} \tag{23}$$

with a use of the conditions of symmetry that $\partial\chi/\partial x$ vanishes at D and $\partial\chi/\partial y$ vanishes at A in order to determine the constants of integration inherent in (11). Next it follows from (12) that

$$\begin{aligned} \chi &= 50y^2 \text{ on } AB, \\ &= 50 \text{ on } BC, \\ &= 100y - 50 \text{ on } CD, \end{aligned} \tag{24}$$

where the constant of integration in (12) is chosen to be such that χ vanishes at A. (It is clear from the form of the expressions (10) for the stress components that the addition of a constant everywhere to χ does not effectively change the solution.)

The method of satisfying the conditions (23) and (24) has been described in Section 79 (for the rectilinear part) and in Section 83 (for the curved part of the boundary). Formulas of the type of (22) must be determined for use in the calculation of χ values near the curved part of the boundary, CD. (It can be seen in Figure 71 that there are four fictitious points just outside CD and five selvage points just inside CD.) It is unnecessary to detail the arithmetic of the solution here, and the problem is left as an example for the reader to work.

85. Concentrated Loads

In both the flexural and extensional types of problem discussed in this chapter, the externally applied loading has so far been assumed to be *distributed* and therefore given by values of its intensity; we must now

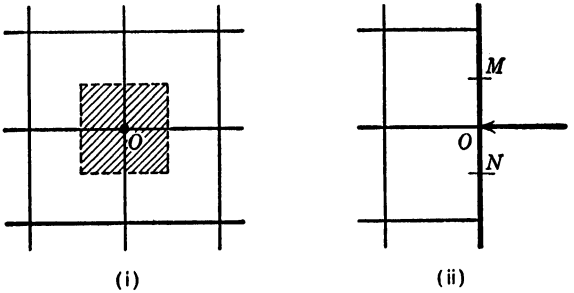


FIGURE 72

consider the possibility of the existence of applied *concentrated* loads and describe how they also may be included in the method. This can be done very simply by the approximate replacement of a concentrated load by a distributed intensity of loading over a small local region. Thus Figure 72 (i) illustrates a part of a square network covering a flat plate which might be subjected to a transverse concentrated load W_0 at the node 0. We can approximately represent this concentrated force by a uniformly distributed pressure over the shaded square of area h^2; *i.e.*, at the node 0 we may assume a value for Z in equation (3) equal to W_0/h^2. (At other nodes of the net, Z would, of course, be taken to be zero unless there were applied loads elsewhere in addition to W_0.)

Figure 72 (ii) similarly illustrates a part of the edge of a plate with a concentrated load P_0 acting at the point 0 in the plane of the plate. In this case the force P_0 would be approximately represented by a uniformly distributed intensity over the line MN; *i.e.*, at 0 we should take a value for σ_y equal to $-P_0/h$. It is clear that, in both these cases (i) and (ii), the assumed loading does, in fact, approximate to the postulated concentrated loading as the mesh length h is made smaller and smaller on

successively finer nets. In practice, indeed, provided that a reasonably fine net is used, the assumed loading may well give a better representation of the physical conditions than would the assumption that a load could be completely concentrated at a point, even if the latter assumption could be realized in computation.

86. Temperature Stresses

Allowances can be made, if desired, in an extensional problem for a plate, for the existence of temperature stresses. We let $T(x,y)$ denote the distribution of temperature excess above that datum temperature distribution (usually zero everywhere) which corresponds to the unstressed condition of the plate. Then in any stressed condition the three components of stress remain expressed in terms of the stress function χ according to the relations (10), but χ itself now satisfies

$$\nabla^4\chi + A\alpha E\,\nabla^2 T = 0, \tag{25}$$

in place of (9). In (25), E is the Young's modulus and α the temperature coefficient of expansion of the material of the plate; A is a constant which takes the value 1 in the case of plane stress and the value $1/(1-\sigma)$ in the case of plane strain, where σ is Poisson's ratio.

The additional term $A\alpha E\,\nabla^2 T$ in (25) makes it in no way more difficult to solve than equation (9). In fact, the form of (25) compares exactly with that of (3).

87. Body Forces

Body forces in the plane of a plate can also be included in solutions of extensional problems; we denote the components of body force (per unit mass) by X and Y in the axial x and y directions, respectively, and the density of the material of the plate by ρ. The stress components are no longer given by the expressions (10) but instead by

$$\sigma_x = \frac{\partial^2\chi}{\partial y^2} + \rho\Omega_x,$$

$$\sigma_y = \frac{\partial^2\chi}{\partial x^2} + \rho\Omega_y,$$

$$\tau_{xy} = -\frac{\partial^2\chi}{\partial x\,\partial y},$$

where Ω_x and Ω_y are body-force potential functions such that

$$X = -\frac{\partial\Omega_x}{\partial x},$$

$$Y = -\frac{\partial\Omega_y}{\partial y}.$$

In place of (9) the governing equation for χ becomes

$$\nabla^4\chi + \rho\nabla^2(\Omega_x + \Omega_y) - B\rho\left(\frac{\partial^2\Omega_x}{\partial x^2} + \frac{\partial^2\Omega_y}{\partial y^2}\right) = 0, \tag{26}$$

where B is a constant which takes the value $1 + \sigma$ in conditions of plane stress and the value $1/(1 - \sigma)$ in conditions of plane strain.

Again, as was the case when temperature stresses were included (Section 86), allowance for body forces in an extensional problem by means of the solution of (26) offers no more difficulty than does the solution of equation (3) or (9). It is, of course, equally possible to combine the results stated in this section and in Section 86, and so to make simultaneous allowance for both temperature stresses and body forces.

PROBLEMS

1. Continue the solution of the problem begun in Section 80 by introducing a finer net for which $h = \frac{1}{8}$.

2. Solve the similar problem for which the boundary conditions are those of clamping on all four edges of the square plate. Use an eventual net of size $h = \frac{1}{16}$ and include a multiplying factor of $16^4 \times 10$. (The solution to this problem, except for a different value of the multiplying factor, has been given by Fox and Southwell.*)

3. Solve the problem which was formulated in Section 84. (The solution to this problem, in the form of values of the stresses σ_x, σ_y, and τ_{xy}, but not of χ, has been quoted by Allen and Southwell.†)

* *Phil. Trans. R.S. C*, **1**, 15–56 (1941), and *A*, **239**, 419–460 (1945).

† *Phil. Trans. R.S. A*, **242**, 379–414 (1950).

CHAPTER 9

SIMULTANEOUS DIFFERENTIAL EQUATIONS

88. Introduction

In the previous chapter it was shown how to solve two types of problems for a flat plate: the flexural problem with given boundary conditions of displacement and the extensional problem with given edge conditions of traction. These two problems were seen to be mathematically identical, each requiring the solution of a biharmonic equation. The other two principal types of problems for plates are also virtually identical with each other from a mathematical point of view; these are the flexural problem with given edge conditions of traction and the extensional problem with given boundary conditions of displacement. Both are governed by a pair of simultaneous partial differential equations of the second order, and in this chapter we take the extensional problem for a plate with given boundary conditions of displacement as a first illustration of the application of relaxation methods to solve simultaneous differential equations. As a second illustration we discuss the simultaneous equations which are presented for solution in the determination of the stresses induced in axially symmetric solids of revolution; examples of direct application of the method to be described arise in the design of guns and shells, and a large amount of work has been undertaken in this connection.

89. The Extension of Flat Plates

When the boundary conditions in an extensional problem are given in terms of displacement, the governing equations are best taken in terms of the components of displacement, u and v, in the plane of the plate in the axial x and y directions, respectively. These equations are known to be

$$\begin{aligned} A\,\nabla^2 u + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\,\partial y} &= 0, \\ A\,\nabla^2 v + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x\,\partial y} &= 0, \end{aligned} \tag{1}$$

when, in the first instance, temperature stresses and body forces are not included. In the simultaneous second-order equations (1), A is a constant which takes the values $(1 - \sigma)/(1 + \sigma)$, in cases of plane stress,

and $1 - 2\sigma$, in cases of plane strain, where σ is Poisson's ratio. We can anticipate no difficulty in the satisfaction of the edge conditions, because when these express given displacements, then each of the wanted quantities u and v has specified values at all points of the boundary.

From the relaxational point of view the problem of the flexure of a flat plate when the edge conditions are of given applied tractions is identical with that expressed in equations (1); the solution is found in terms of two quantities U and V which satisfy a pair of equations of exactly the same form as (1). The derivation of these equations has been discussed by Southwell,* and a particular numerical example was given by Fox and Southwell† in 1941. We shall not discuss the flexural problem here but instead illustrate the solution of the *type* of equation involved by reference to an extensional example, in terms of the displacements u and v as the wanted quantities, and governed by equations (1).

90. Finite-difference Approximations

We work as usual on a square relaxation net, and finite-difference approximations to the equations (1), holding at a typical node 0, can be written down as soon as the approximations for the terms $\partial^2 u/\partial x\,\partial y$ and $\partial^2 v/\partial x\,\partial y$ have been established. These may easily be found in the following way, with the usual convention (Figure 61) for numbering a typical group of nodes:

$$\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_0 = \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right]_0 \approx \frac{(\partial u/\partial y)_1 - (\partial u/\partial y)_3}{2h}$$

$$\approx \frac{(u_5 - u_8) - (u_6 - u_7)}{4h^2};$$

so that, approximately,

$$4h^2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_0 = u_5 - u_6 + u_7 - u_8,$$

and, similarly,

$$4h^2\left(\frac{\partial^2 v}{\partial x\,\partial y}\right)_0 = v_5 - v_6 + v_7 - v_8.$$

Substituting these approximations and also the customary approximations (Section 41) for $\nabla^2 u$, $\partial^2 v/\partial x^2$, etc., we derive at once the finite-difference approximations to the equations (1):

* Southwell's discussion is particularly concerned with the mathematical analogue between the flexure and extension of plates. It is given in *Q.J.M.A.M.*, **3**, 257–270 (1950).

† *Phil. Trans. R.S. C*, **1**, 15–56 (1941), and *A*, **239**, 419–460 (1945).

$$\begin{aligned} A(\Sigma u_1 - 4u_0) + (u_1 + u_3 - 2u_0) + \tfrac{1}{4}(v_5 - v_6 + v_7 - v_8) &= 0, \\ A(\Sigma v_1 - 4v_0) + (v_2 + v_4 - 2v_0) + \tfrac{1}{4}(u_5 - u_6 + u_7 - u_8) &= 0. \end{aligned} \tag{2}$$

As a typical example, we consider the case of plane strain when σ has the value 0.3, so that A has the value 0.4 and the approximations (2) become

$$\begin{aligned} 1.4(u_1 + u_3) + 0.4(u_2 + u_4) - 3.6u_0 + 0.25(v_5 - v_6 + v_7 - v_8) &= 0, \\ 0.4(v_1 + v_3) + 1.4(v_2 + v_4) - 3.6v_0 + 0.25(u_5 - u_6 + u_7 - u_8) &= 0. \end{aligned} \tag{3}$$

91. Two-variable Technique

At each node of the relaxation net two quantities are required to be found; these are the values of the components of displacement, u and v. Correspondingly, two residuals are defined at each node; these are denoted by F_{u0} and F_{v0} at a typical node 0, and they are defined, respectively, by the left-hand sides of equations (3). Figure 73 illustrates how the four quantities are usually placed at each node of the network; above and to the left is recorded the initially guessed value for u with, above it, the successive increments Δu applied in the course of the liquidation process; above and to the right of each node is recorded a column of the successive values of the corresponding residual F_u. Similarly, below each node and extending downward are columns of increments Δv, to the left, and of successive values of the residual F_v, to the right. In this manner we are able to record on *one* working diagram a technique for the relaxational determination of the values of *two* wanted quantities. This technique is sometimes rather misleadingly referred to as a *two-diagram technique;* more properly it should be called a *two-variable technique*, for the essence of its usefulness is that all the arithmetic can be recorded on *one* diagram.

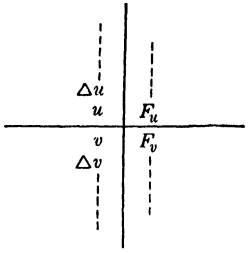


FIGURE 73

There are available, of course, two separate basic unit operations related to each node; these comprise the additions separately of unit increments to the values of u and v. Each gives rise to two relaxation patterns, one showing the changes made to the residuals F_u and the other the changes effected in the values of the residuals F_v. These four patterns may easily be deduced from equations (3), and they are illustrated in Figure 74, (i) and (ii) being the patterns for F_u and F_v, respectively, corresponding to an increment $\Delta u_0 = 1$, and (iii) and (iv) being the similar patterns for F_u and F_v, respectively, corresponding to an increment $\Delta v_0 = 1$.

In principle there is very little more which need be said about a two-variable problem. The aim of the relaxation process is eventually to

reduce the values of all the residuals to zero, and in order to achieve this end, increments Δu and Δv may be applied at each node for the purpose of liquidating corresponding residuals. The only difference from the technique required for the solution of, say, a Poisson's equation or a biharmonic equation is that unknowns and residuals are attached to the nodes of the net in pairs instead of singly.

However, for speed and efficiency in practice, it is desirable that we should adopt a slightly more systematic technique of liquidation than

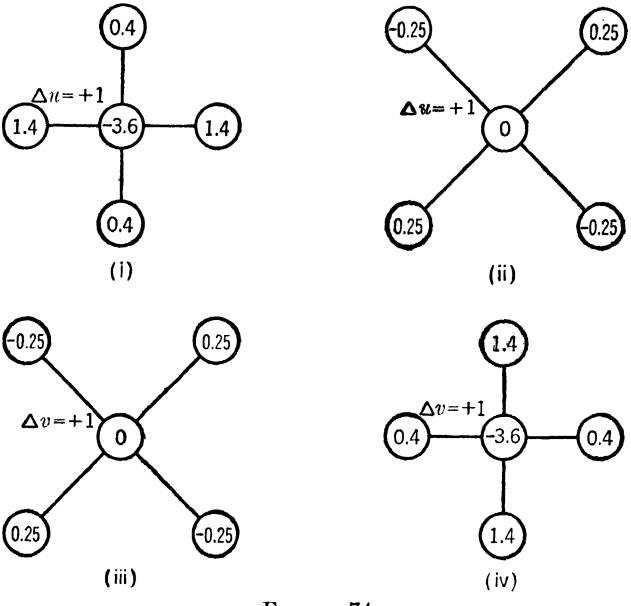


FIGURE 74

the previous paragraph would suggest. This is best achieved by working through an iteration of processes in the following sequence:

1. Initial values of both u and v having been chosen and recorded on the relaxation diagram, calculate the values of the residuals F_u *only*, according to the formula

$$F_{u0} \equiv 1.4(u_1 + u_3) + 0.4(u_2 + u_4) - 3.6u_0 + \tfrac{1}{4}(v_5 - v_6 + v_7 - v_8). \quad (4)$$

2. Begin to liquidate these residuals F_u by the application of increments Δu, according to the pattern of Figure 74 (i). There is no point *at this stage* in effecting their complete liquidation, and it usually suffices, roughly, to reduce them to about 10 per cent of their original values.

3. Now calculate the other set of residuals, F_v, according to the formula

$$F_{v0} \equiv 0.4(v_1 + v_3) + 1.4(v_2 + v_4) - 3.6v_0 + \tfrac{1}{4}(u_5 - u_6 + u_7 - u_8), \quad (5)$$

with a use, of course, of the values of u current at the end of stage 2 of this program.

4. Next liquidate these residuals F_v partially (*i.e.*, sufficiently, usually, to reduce them to about 10 per cent of their original values) by means of the application of increments Δv, according to the pattern of Figure 74 (iv).

5. Recalculate the residuals F_u by substitution in the formula (4), using the values of v current at the end of stage 4. This recalculation of the residuals F_u provides at the same time a check on the relaxation performed in stage 2 and also incorporates all at once the changes made to the residuals F_u during stage 4. (It was deliberately not suggested in stage 4 that any attempt should be made then to keep an account of the effect on the residuals F_u produced by the application of increments Δv.)

6. Continue the liquidation of the residuals F_u as described in stage 2.

7. Recalculate the residuals F_v according to the formula (5).

8. Continue the liquidation of the residuals F_v as described in stage 4.

The stages 5 to 8 are then repeated until eventually all the residuals are reduced to zero. Toward the end of the liquidation the strict iteration suggested here may be dispensed with, and small local increments added to either or both of u and v at the same time.

The ordered method of liquidation described above has been found by experience to be the speediest in the long run; it will now be illustrated by a worked example.

92. Compression of a Rectangular Plate

A typical example is illustrated in Figure 75; $ABCD$ represents one-half of a rectangular plate of breadth L and length $3L$; conditions of plane strain are assumed and σ given the value 0.3, so that the residuals are defined by the formulas (4) and (5). The edge conditions are that no movement is allowed on AB and BC and that on CD no movement is allowed in the x direction but a compression is applied in the y direction. Thus $u = 0$ on all of AB, BC, and CD, and $v = 0$ on AB and BC; the amount of the compressive displacement on CD is given by

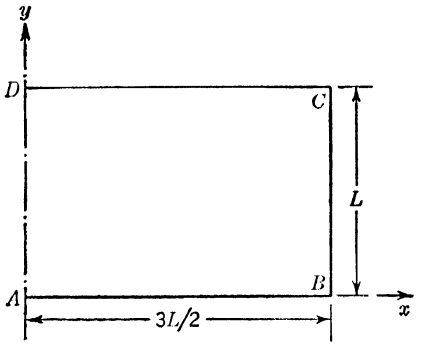


FIGURE 75

$$-v = 1{,}000\left(9 - \frac{4x^2}{L^2}\right), \tag{6}$$

0	0	0	0	0	0	0
−9000	−8750	−8000	−6750	−5000	−2750	0
0	0	0	0	0	0	0
−6750	150 −6562	148 −6000	150 −5062	148 −3750	150 −2062	148 0
0	0	0	0	0	0	0
−4500	100 −4375	101 −4000	100 −3375	101 −2500	100 −1375	101 0
0	0	0	0	0	0	0
−2250	50 −2187	48 −2000	50 −1687	48 −1250	50 −687	48 0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(i)

0	0	0	0	0	0	0
−9000	−8750	−8000	−6750	−5000	−2750	
0	0 0	125 0	250 0	377 0	509 0	658 0
−6625	−6437	−5875	−4939	−3632	−1967	0
0	0 0	125 0	250 0	375 0	499 0	616 0
−4357	−4232	−3857	−3234	−2367	−1271	0
0	0 0	125 0	249 0	372 0	491 0	592 0
−2161	−2098	−1911	−1600	−1168	−624	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(ii)

FIGURE 76

0	0	0	0	0	0	0
−9000	−8750	−8000	−6750	−5000	−2750	0
0	268	503	664	698	523	0
−6625	−170 −6437	−161 −5875	−122 −4939	−54 −3632	52 −1967	212 0
0	341	636	830	854	616	0
−4357	0 −4232	2 −3857	1 −3234	2 −2367	3 −1271	−4 0
0	266	497	653	678	496	0
−2161	172 −2098	158 −1911	123 −1600	55 −1168	−52 −624	−214 0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(iii)

0	0	0	0	0	0	0
−9000	−8750	−8000	−6750	−5000	−2750	0
0	273	511	674	707	527	0
−6687	−6495	−5919	−4956	−3610	−1904	0
0	355	662	863	884	626	0
−4356	−4231	−3857	−3233	−2366	−1272	0
0	271	506	663	687	501	0
−2098	−2040	−1867	−1581	−1189	−688	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

(iv)

FIGURE 76 (*Continued*)

in nondimensional terms and with the inclusion of a multiplying factor of 9×10^6 (so that the maximum compression, at D, is one-thousandth of the breadth of the plate). This applied compression varies parabolically from a maximum value at D to zero at C; it is symmetrical about D, so that the center line AD is a line of symmetry for the displacement v, and it is also a line on which the displacement u is known to vanish. We work the solution for u and v first on a net of size $h = L/4$, for which there are 18 mesh points inside the boundary on the working diagram. Figure 76 shows successive stages in the solution. Diagram (i) records the given boundary values of u and v, also their initially guessed values at the internal nodes (these are taken to be that $u = 0$ everywhere and that v varies linearly along each mesh line in the y direction), and the residuals F_v calculated according to (5). Diagram (ii) shows the values of v obtained by liquidation of these residuals F_v by the use of the pattern of Figure 74 (iv), and also the values *subsequently* calculated for the residuals F_u. Diagram (iii) gives the values of u which result from the liquidation of these residuals F_u, and also the subsequently calculated values of the residuals F_v. The solution almost converges after a second stage of both v relaxation and of u relaxation; these are followed by a little local point relaxation, and the completed solution is given in diagram (iv).

It is left to the reader as practice to find the solution on the next finer net ($h = \frac{1}{8}$); the results have been given by Allen and Southwell* correct to an extra significant figure beyond that shown in Figure 76.

93. Temperature Stresses and Body Forces

The governing equations (1) can be modified in order to allow for the existence of temperature stresses or body forces, or both, if that is desired. When $T(x,y)$ denotes the distribution of temperature excess above the datum temperature distribution which corresponds to the unstressed condition of the plate, when X and Y are the components of body force per unit mass in the axial x and y directions, and when α, G, and ρ are, respectively, the temperature coefficient of expansion, the modulus of rigidity, and the density of the material of the plate, the equations (1) are replaced by

$$\begin{aligned} A\,\nabla^2 u + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\,\partial y} + \frac{A\rho X}{G} - 2B\alpha\frac{\partial T}{\partial x} &= 0, \\ A\,\nabla^2 v + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x\,\partial y} + \frac{A\rho Y}{G} - 2B\alpha\frac{\partial T}{\partial y} &= 0, \end{aligned} \tag{7}$$

* *Proc. Roy. Soc. A*, **183**, 125–134 (1944).

where B is a constant which takes the value 1 under conditions of plane stress and the value $1 + \sigma$ under conditions of plane strain. The extra terms in equations (7) do not make them any more difficult to solve by relaxation methods than equations (1), for these terms contribute only calculable numbers, which have simply to be added in whenever residuals are evaluated.

94. Mixed Boundary Conditions

We have dealt so far with two types of extensional problems for flat plates: first, in Chapter 8, for cases when the boundary conditions are of given tractions, and results are then determined by the solution of a biharmonic equation for an Airy's stress function χ; second, in this chapter, for cases when the boundary conditions are of given displacements, when results are determined by the solution of a pair of simultaneous second-order equations for the component displacements u and v. A considerably more difficult problem is posed when the boundary conditions are mixed, *i.e.*, when they are conditions of given tractions at some parts of the boundary and conditions of given displacements at other parts. In such cases two alternative procedures are available: either the displacement boundary conditions can be expressed, by use of the ordinary stress-strain relations, in terms of stresses and hence in terms of χ, and then they can be applied in order to find χ by solution of the biharmonic equation; or, alternatively, the boundary conditions of given tractions can be expressed, again with a use of the stress-strain relations, in terms of displacements, and then they must be applied in order to solve the simultaneous equations. Neither prospect is at all attractive, and such problems are too difficult to warrant a detailed discussion in this book; there appears to be no method of escaping the difficulty which is inherent in such problems, no matter how one may choose to tackle them. We can, however, say in general that the lesser of two evils is faced if the problem is worked in terms of u and v and not in terms of χ. The interested reader is referred to a paper* by Fox in which two examples of problems involving mixed boundary conditions are solved.

Similar difficulties attend the solution of flexural problems for flat plates when the boundary conditions are mixed. The equations are mathematically of exactly the same form as those which arise in extensional problems, so that, presumably, they would have to be solved in a precisely similar manner in terms of the quantities U and V (cf. Section 89); but it is not known that any such solutions have ever, in fact, been attempted.

* *Proc. Roy. Soc. A*, **189**, 535–543 (1947).

95. Stresses in Axially Symmetric Solids of Revolution

Another set of problems which also involve the solution of a pair of second-order simultaneous equations are those which require the determination of the stresses set up in an axially symmetric solid of revolution when it is subjected to a given axially symmetric loading system. A typical example is illustrated in Figure 77; cylindrical coordinates r, θ, and z are naturally employed, with the z axis as the axis of revolution. All quantities are independent of θ, so from a mathematical point of view the problem is only two-dimensional and may be worked on any axial zr plane; Figure 77 represents such an axial section of a short cylindrical tube. The tube is generated by the revolution about Oz of the rectangle

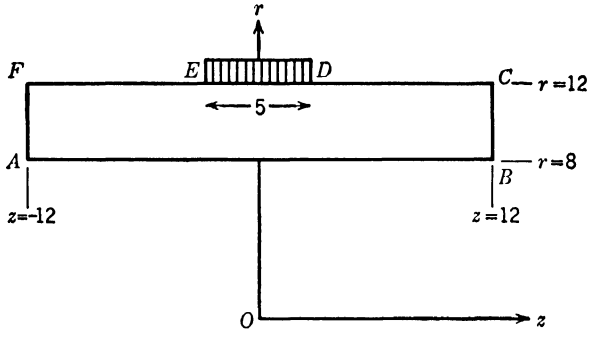


FIGURE 77

$ABCF$, and at any point of the tube there are four nonzero stress components to be found; these are the radial stress σ_r, the hoop stress σ_θ, the longitudinal stress σ_z, and the shear stress τ_{zr}. These stresses can all be expressed* in terms of two stress functions ϕ and ψ in the forms

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\left(\frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial r}\right) - \frac{1}{r^2}(\psi + \overline{1-\sigma}\phi), \\
\sigma_\theta &= \frac{\sigma}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}(\psi + \overline{1-\sigma}\phi), \\
\sigma_z &= -\frac{1}{r}\frac{\partial \psi}{\partial r}, \\
\tau_{zr} &= \frac{1}{r}\frac{\partial \psi}{\partial z},
\end{aligned} \tag{8}$$

where σ is Poisson's ratio for the material of the tube and where, of course, ϕ and ψ are functions of z and r only (but not of θ) which need to be deter-

* The formulation of the governing equations and the expressions for the stresses (in terms of stress functions) which are stated here were derived by Southwell, *Proc. Roy. Soc. A*, **180**, 367–396 (1942).

mined over the region of the axial section *ABCF*. The given loading is represented in Figure 77; it consists of a band of uniform pressure applied on that part of the outer surface of the tube which is generated by the line *DE*.

The simultaneous governing equations for the stress functions are known to be

$$\begin{aligned} \frac{\partial^2\phi}{\partial r^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} &= 0, \\ \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} &= \frac{\partial^2\phi}{\partial z^2}. \end{aligned} \tag{9}$$

On a square relaxation net, of size h, drawn to cover the area of integration *ABCF*, equations (9) have finite-difference approximations which lead to a pair of residuals being defined at each node by the formulas

$$\begin{aligned} F_{\phi 0} &\equiv \phi_1 + \phi_3 + \left(1 - \frac{h}{2r_0}\right)\phi_2 + \left(1 + \frac{h}{2r_0}\right)\phi_4 - 4\phi_0, \\ F_{\psi 0} &\equiv \psi_1 + \psi_3 + \left(1 - \frac{h}{2r_0}\right)\psi_2 + \left(1 + \frac{h}{2r_0}\right)\psi_4 - 4\psi_0 \\ &\qquad + 2\phi_0 - \phi_1 - \phi_3. \end{aligned} \tag{10}$$

These expressions are similar to those for the residuals F_{u0} and F_{v0} given in Section 91 and derived from the simultaneous equations (1); the method of solution here for ϕ and ψ is, indeed, very similar in principle to that described in Section 91 for u and v. The present problem is, however, considerably more difficult, and it receives detailed attention here because a number of additional points of relaxational technique must be mentioned before it can be regarded as tractable.

96. Particular Boundary Conditions

These points of technique are all concerned with the satisfaction of the appropriate boundary conditions. Once again we notice that the difficulties of solution are primarily associated with the boundary conditions and not with the governing differential equations. Before discussing the boundary conditions in general terms, we consider the particular conditions given for the problem of Figure 77. These can be immediately stated in terms of certain boundary stresses:

On *AB*, *CD*, and *EF*,

$$\begin{aligned} \sigma_r &= 0, \\ \tau_{zr} &= 0; \end{aligned} \tag{11}$$

On *BC* and *FA*,

$$\begin{aligned} \tau_{sr} &= 0, \\ \sigma_z &= 0; \end{aligned} \tag{12}$$

On *DE*,

$$\begin{aligned} \sigma_r &= -100, \\ \tau_{zr} &= 0, \end{aligned} \tag{13}$$

where the value of the applied pressure on the band *DE* is taken to be 100 units.

Now the second of each of the pairs of conditions (11), (12), and (13) enables us at once to integrate to find a value for ψ at all points of the boundary; for each of these conditions is, in fact, simply $\partial\psi/\partial s = 0$, where s is distance measured round the boundary. Hence at once we know that ψ must be constant all round the boundary, and it is sufficient to take the value of that constant to be zero [because, by reference to (8), it is clear that an addition to ψ of any constant magnitude A, everywhere, makes no alteration to the resulting values of the stresses, provided that it is accompanied by the addition to ϕ everywhere of a magnitude $-A/(1-\sigma)$].

Unfortunately, the other condition available on the various parts of the boundary, *i.e.*, the first of (11), (12), and (13), cannot be integrated to yield boundary values of the other stress function ϕ. If it could, then this problem would need no further discussion, for it would be in all respects similar to the problem explained in Section 91. As it stands, however, the first of each of (11), (12), and (13) are, in effect, normal-gradient conditions, and they have to be used to eliminate the fictitious values which occur in the expressions for the residuals F_ϕ at nodes on the boundary. (It may be noticed that we do not need to consider residuals F_ψ at these nodes; ψ having already been determined there, any associated residuals would be superfluous.)

We consider this elimination of fictitious values in respect of nodes on the various parts of the boundary separately, dealing firstly with the side *AB*. At any node on this side, $F_{\phi 0}$ is given by the first of (10), but that expression contains the fictitious value ϕ_4. The boundary condition which must be used to eliminate this fictitious value is the first of (11), *i.e.*, $\sigma_r = 0$; when this condition is expressed by means of the first of (8) in terms of the stress functions and then replaced by its finite-difference approximation, it becomes

$$\phi_2 + \psi_2 - (\phi_4 + \psi_4) - \frac{2h}{r_0}(1-\sigma)\phi_0 = 0, \tag{14}$$

when we remember that $\psi_0 = 0$.

Now if (14) is used in order to eliminate the fictitious ϕ_4 from the first of (10), it is clear that it does so only at the expense of introducing the equally fictitious ψ_4, which in turn has then to be eliminated. There are no unused boundary conditions left to accomplish this further elimina-

tion; what may be used, however, is the ψ-governing equation, which, as we have already pointed out, is not required for its usual purpose of defining a residual F_ψ, because ψ has already been determined to be zero on the boundary. Hence we may (and indeed must) use the finite-difference approximation to the second of (9) in order to eliminate ψ_4 from the residual formula for $F_{\phi 0}$ at nodes on AB, after (14) has been used to eliminate ϕ_4. The finite-difference approximation to the second of (9) is

$$\left(1 - \frac{h}{2r_0}\right)\psi_2 + \left(1 + \frac{h}{2r_0}\right)\psi_4 + 2\phi_0 - \phi_2 - \phi_4 = 0 \qquad (15)$$

(since $\psi_1 = \psi_0 = \psi_3 = 0$), and when both eliminations (of ϕ_4 and of ψ_4) have been accomplished, it is found that the formula for $F_{\phi 0}$ at a node on AB reduces simply to

$$F_{\phi 0} \equiv 2(\phi_2 + \psi_2) - \phi_0\left[2 + \frac{2(1-\sigma)h}{r_0}\left(1 + \frac{h}{2r_0}\right)\right]. \qquad (16)$$

The elimination of the fictitious ϕ_2 in the expression for $F_{\phi 0}$ at nodes on the side CF may be achieved in a similar way; the result is

$$F_{\phi 0} \equiv 2(\phi_4 + \psi_4) - \phi_0\left[2 - \frac{2(1-\sigma)h}{r_0}\left(1 - \frac{h}{2r_0}\right)\right] + 2hr_0\left(1 - \frac{h}{2r_0}\right)(\sigma_r)_0, \qquad (17)$$

which includes the formula for each of the parts of the boundary, CD, DE, and EF. At nodes on CD and EF the value of $(\sigma_r)_0$ in (17) is put equal to zero, and at nodes on DE it is put equal to -100. If either the point D or the point E is a node of the net, then at those nodes $(\sigma_r)_0$ is taken to be -50 (cf. the treatment of another type of theoretical singularity, *i.e.*, a concentrated load, in Section 85).

There remains to consider the elimination of fictitious values from the formulas for the residuals F_ϕ at nodes on the sides BC and FA; the formula for $F_{\phi 0}$ is the first of (10), which at nodes on BC contains a fictitious ϕ_1. In order to eliminate this fictitious value, we must expect to have to employ the as yet unused boundary condition, the first of (12). This condition, expressed by means of the fourth of (8) in terms of the stress functions and then replaced by a finite-difference approximation, yields simply

$$\psi_1 = \psi_3, \qquad (18)$$

which at first sight would not seem to be of much assistance to the elimination of ϕ_1. Again, as on the sides AB and CF, we have to make use of

the finite-difference approximation to the ψ-governing equation in order to accomplish the required elimination. On BC the finite-difference approximation to that equation, the second of (9), is

$$\psi_1 + \psi_3 + 2\phi_0 - \phi_1 - \phi_3 = 0, \tag{19}$$

when we remember that $\psi_2 = \psi_0 = \psi_4 = 0$. Now (19) can be used to eliminate ϕ_1 from the first of (10) at the expense of introducing the fictitious ψ_1; this fictitious value may then in turn be eliminated by use of (18). The result is that, at nodes on BC,

$$F_{\phi 0} \equiv \left(1 - \frac{h}{2r_0}\right)\phi_2 + \left(1 + \frac{h}{2r_0}\right)\phi_4 - 2\phi_0 + 2\psi_3. \tag{20}$$

Exactly similarly, of course, the formula at nodes on FA becomes

$$F_{\phi 0} \equiv \left(1 - \frac{h}{2r_0}\right)\phi_2 + \left(1 + \frac{h}{2r_0}\right)\phi_4 - 2\phi_0 + 2\psi_1. \tag{21}$$

We shall return to discuss the actual liquidation process for the problem of Figure 77 in Section 99; before that, it would be as well to consider in general terms the boundary conditions which have to be satisfied in the solution of any problem of stress determination in an axially symmetric solid of revolution.

97. General Boundary Conditions

For any given shape of boundary of the axial section of a solid of revolution, it is, in general, always possible to integrate one of the boundary conditions in order to determine boundary values for the stress function ψ, just as, in the preceding section, we were able in a particular example to find that ψ was, sufficiently, zero at all points of the boundary. That this integration can be achieved follows from the fact that boundary conditions of given tractions, in general, specify boundary values of the stresses $\tau_{\nu r}$ and $\tau_{\nu z}$, where ν is the direction of the outward-drawn normal at a point of the boundary; similarly, s will denote the tangential direction at a point on the boundary, taken, as illustrated in Figure 78, in an anticlockwise sense.

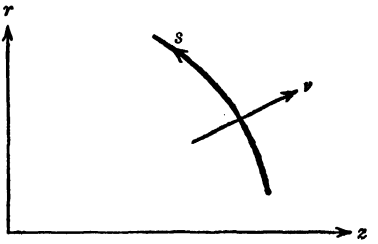


FIGURE 78

Now, given values of the normal and shear stresses across the boundary (which would usually be the way in which boundary tractions would be

expressed) would mean known values of σ_ν and of $\tau_{\nu s}$. But, at once,

$$\begin{aligned} \tau_{\nu z} &= \sigma_\nu \sin(\nu,r) - \tau_{\nu s} \cos(\nu,r), \\ \tau_{\nu r} &= \sigma_\nu \cos(\nu,r) + \tau_{\nu s} \sin(\nu,r), \end{aligned} \tag{22}$$

where (ν,r) is the angle between the ν direction and the r direction at a point of the boundary. Hence, by (22), boundary conditions of given tractions can always be interpreted in terms of specified values of the stresses $\tau_{\nu z}$ and $\tau_{\nu r}$.

We consider first the stress $\tau_{\nu z}$; in order to express it in terms of the stress functions ϕ and ψ, it will be necessary to make use of the geometrical relations that

$$\begin{aligned} \cos(\nu,r) &= -\frac{\partial z}{\partial s} = \frac{\partial r}{\partial \nu}, \\ \sin(\nu,r) &= \frac{\partial z}{\partial \nu} = \frac{\partial r}{\partial s}. \end{aligned} \tag{23}$$

Then, at once,

$$\tau_{\nu z} = \sigma_z \sin(\nu,r) + \tau_{zr} \cos(\nu,r) = -\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial s} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial s},$$

with a use of both the third and fourth of (8) and of (23). Thus we have

$$\tau_{\nu z} = -\frac{1}{r}\frac{\partial \psi}{\partial s},$$

i.e.,

$$\frac{\partial \psi}{\partial s} = -r\,\tau_{\nu z},$$

which relation can always be integrated round the boundary with respect to s to yield boundary values for ψ. Again, as in Section 96, there is an arbitrary constant of integration which may be chosen at our convenience.

As in the particular example of Section 96, so in general, it is the satisfaction of the second boundary condition which causes problems of this type to be of some considerable difficulty to solve. This second condition cannot be integrated; instead it is effectively a condition on ϕ in terms of boundary gradients. Its application has been illustrated in Section 96 in a particular case, and there is not much that can be usefully said about it in general, except to state it in terms of ϕ and ψ. This must be done by expressing the known stress $\tau_{\nu r}$ in terms of those stress functions; we have

$$\begin{aligned} \tau_{\nu r} &= \sigma_r \cos(\nu,r) + \tau_{zr} \sin(\nu,r) \\ &= \left[\frac{\partial}{\partial r}\left(\frac{\phi+\psi}{r}\right) + \sigma\frac{\phi}{r^2}\right]\cos(\nu,r) + \left[\frac{\partial}{\partial z}\left(\frac{\psi}{r}\right)\right]\sin(\nu,r), \end{aligned}$$

with a use of the first and fourth of (8). Now, by (23), it follows that

$$\tau_{\nu r} = \left[\cos(\nu,r)\frac{\partial}{\partial r} + \frac{\sigma}{r}\right]\frac{\phi}{r} + \frac{\partial r}{\partial \nu}\frac{\partial}{\partial r}\left(\frac{\psi}{r}\right) + \frac{\partial z}{\partial \nu}\frac{\partial}{\partial z}\left(\frac{\psi}{r}\right),$$

i.e.,

$$\tau_{\nu r} = \frac{\partial}{\partial \nu}\left(\frac{\psi}{r}\right) + \left[\cos(\nu,r)\frac{\partial}{\partial r} + \frac{\sigma}{r}\right]\frac{\phi}{r},$$

and this is the best form, in general, in which we can leave the second boundary condition. In most examples it simplifies somewhat when particular values are given to the angle (ν,r) and particular directions assigned to ν.

98. Conditions on the Axis of Symmetry

In the example of Section 95 the solid of revolution does not contain any part of the axis of symmetry, but, of course, there is no reason why in some problems we may not be concerned with solids which do extend to this axis. We must, therefore, consider how to deal with a node of the net when it lies on this axis; it requires special mention because r_0 then is zero, and the residual formulas (10) become meaningless. In the differential equations (9) the trouble is seen to lie in the terms $(1/r)(\partial\phi/\partial r)$ and $(1/r)(\partial\psi/\partial r)$, which must be indeterminate at $r = 0$. Their limits as $r \to 0$ are immediately seen to be $\partial^2\phi/\partial r^2$ and $\partial^2\psi/\partial r^2$, and therefore at $r = 0$ the governing differential equations (9) reduce simply to

$$\frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\phi}{\partial z^2} = 0,$$

so that both ϕ and ψ are just linear functions of z along the axis of symmetry.

99. Underrelaxation

We now return to discuss the actual process of liquidation of the residuals F_ϕ and F_ψ in a relaxational solution of the simultaneous equations (9), and we illustrate the technique by particular reference to the problem of Figure 77. In principle the two-variable technique, described in Section 91 in relation to solutions of the equations (1), is adopted, but with the following modifications:

1. At each stage of ψ relaxation the residuals F_ψ *must* be completely liquidated, even, with advantage, to an extra significant figure compared with that to which ϕ and F_ϕ are being taken.

2. At each stage of ϕ relaxation the residuals F_ϕ must be only very tentatively liquidated; it should not be attempted to reduce them, on an average, by more than 10 per cent before attention is turned back to the function ψ and the residuals F_ψ are again completely liquidated.

The reasons for these most important modifications may be most convincingly understood by consideration of the residuals F_ϕ at the nodes on the part of the boundary BC (Figure 77); a typical pattern corresponding to an increment $\Delta\phi = 1$ applied at one of these nodes is shown in Figure 79. It can be seen at once to make only a comparatively infinitesimal change to the total of the ϕ residuals on BC; Figure 79 is actually the pattern corresponding to an increment $\Delta\phi = 1$ applied at the node on BC at which $r = 10$ (when the mesh size is $h = 1$, so that there are 4 mesh lengths in the thickness of the tube and 12 in its half length). Even more striking than the point pattern of Figure 79 is the block pattern shown in Figure 80; it is the pattern corresponding to a unit block increment applied to ϕ at all the nodes on BC. In addition, it should also be noticed that the residuals F_ϕ on BC are completely unaffected by increments $\Delta\phi$ applied at any other nodes of the net; hence in all it is clear that to liquidate any initial residuals F_ϕ at nodes on BC by means of ϕ increments would require exceptionally large values of those increments, and if that were attempted the solution would soon tend to get out of hand. It is for this reason that in modification 2, above, we stated that at every stage ϕ relaxation must be very tentative, that in effect, when relaxing ϕ residuals by means of applying increments $\Delta\phi$, we should very deliberately and considerably underrelax. It has several times been necessary in this book to emphasize the virtues of overrelaxation; the present type of example is one of the few for which the converse technique of *underrelaxation* is essential.

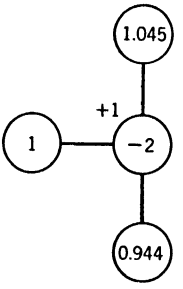


FIGURE 79

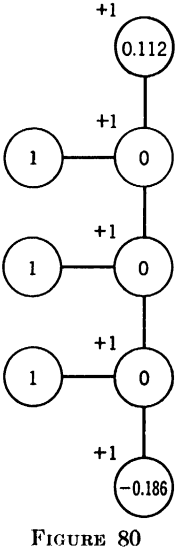


FIGURE 80

But if, of course, residuals F_ϕ do exist at nodes on BC, they must eventually be liquidated, and we have still to explain how in fact they are brought to zero. So far it has only been shown that extremely large ϕ increments would be necessary to remove those residuals, and it has been stated that, as a matter of experience, it is not helpful to apply such large ϕ increments. Whereas a unit block $\Delta\phi = 1$ applied at all the nodes on BC changes the total of the ϕ residuals at those nodes by only -0.074, on the other hand we can see, from (20), that a unit increment $\Delta\psi = 1$ applied at a node adjacent to BC changes one of the ϕ residuals on BC by $+2$. Consequently, the ϕ residuals on BC are comparatively critically dependent on the ψ values at internal nodes. It is for this

reason that in modification 1, above, we stated that it was desirable to try to keep the ψ residuals always fully liquidated, even to the extent of an extra significant figure over what we ultimately desire for the ϕ residuals. As for the liquidation of ϕ residuals at nodes on BC, it will be found in practice that, if very considerable underrelaxation is used in the stages of ϕ relaxation, then these residuals will gradually be automatically liquidated as a consequence of the stages of ψ relaxation; *i.e.*, as the process of the solution settles down, it will be found that the increments added to the ψ values at the internal nodes of the net automatically tend to reduce the residuals F_ϕ at the nodes on BC, and that these residuals do converge to zero predominantly by virtue of the effect of ψ increments at the adjacent (internal) nodes rather than by means of the effect of ϕ increments at the boundary nodes themselves.

100. Components of Stress

The solution of the problem of Figure 77 in terms of values of the stress functions ϕ and ψ will not be given here but instead will be left to the

0	0	0	0	0	0	0	0	0	0	-100	-100	-100	
-1	0	0	0	0	0	1	2	1	-3	-81	-85	-85	
1	1	0	0	-1	-1	-1	-2	-3	-12	-49	-58	-59	σ_r
2	1	0	-1	-2	-3	-4	-5	-6	-11	-22	-24	-27	
0	0	0	0	0	0	0	0	0	0	0	0	0	

(*a*)

25	15	7	-1	-9	-20	-33	-49	-69	-79	-193	-196	-201	
28	17	7	-3	-14	-26	-41	-59	-81	-108	-159	-178	-184	
32	19	7	-5	-19	-33	-50	-69	-92	-118	-148	-166	-171	σ_θ
36	21	7	-8	-24	-41	-60	-80	-102	-124	-144	-158	-162	
41	25	7	-11	-31	-51	-72	-94	-115	-133	-144	-152	-155	

(*b*)

FIGURE 81

reader to work as an exercise. We do, however, quote the solution, in Figure 81, in terms of the values of the four components of stress. These values have been calculated after the determination of the stress functions ϕ and ψ by a direct substitution into the finite-difference approximations to (8):

$$
\begin{aligned}
(\sigma_r)_0 &= \frac{1}{2hr_0}(\phi_2 + \psi_2 - \phi_4 - \psi_4) - \frac{1}{r_0^2}(\psi_0 + \overline{1-\sigma}\phi_0),\\
(\sigma_\theta)_0 &= \frac{\sigma}{2hr_0}(\phi_2 - \phi_4) + \frac{1}{r_0^2}(\psi_0 + \overline{1-\sigma}\phi_0),\\
(\sigma_z)_0 &= -\frac{1}{2hr_0}(\psi_2 - \psi_4),\\
(\tau_{zr})_0 &= \frac{1}{2hr_0}(\psi_1 - \psi_3).
\end{aligned}
\tag{24}
$$

At nodes on the boundary some of the ϕ and ψ values contained in (24) are fictitious. These have to be eliminated in a manner precisely similar to that employed in Section 96 to eliminate fictitious values from the

0	0	7	15	22	27	27	22	9	43	−191	−154	−159
0	1	4	7	9	10	9	4	−6	−23	−46	−60	−65
0	0	0	−1	−1	−1	−2	−2	−3	−2	9	10	10
0	−1	−5	−8	−11	−13	−11	−5	8	28	56	73	79
0	0	−8	−17	−25	−30	−29	−18	7	52	116	156	169

σ_z

(c)

0	0	0	0	0	0	0	0	0	0	0	0	0
0	3	4	5	3	0	−4	−10	−20	−38	−36	−14	0
0	4	6	6	4	0	−7	−17	−30	−43	−41	−21	0
0	3	5	5	3	0	−6	−14	−24	−33	−31	−17	0
0	0	0	0	0	0	0	0	0	0	0	0	0

τ_{zr}

(d)

FIGURE 81 (*Continued*)

residual formulas at nodes on the boundary; the consequent modified forms of the expressions (24) are not of sufficient general importance to quote here. The value taken for σ in the numerical work leading to Figure 81 was $\frac{1}{4}$.

PROBLEMS

1. Find the solution on a finer net ($h = \frac{1}{8}$) to the problem solved on a coarse net in Section 92.

2. Determine the values of the stress functions ϕ and ψ, on a net of size $h = 1$, for the problem described in Sections 95 and 96. Calculate ϕ and ψ to enough significant figures that the values of the stress components can be given to the nearest whole number, and verify the values given in Figure 81.

CHAPTER 10

TRIANGULAR NETS

101. Introduction

It was mentioned in Section 40 that three different shapes of net could be used to cover a two-dimensional region, these being nets of square mesh, of equilateral-triangular mesh, and of equiangular-hexagonal mesh, respectively. So far we have not considered or illustrated the use of any but a square network, and, indeed, that kind of net is in almost general usage. However, on rare occasions there are reasons why a triangular net could be employed to advantage. These occasions are restricted to solutions of Poisson's equation or of the biharmonic equation, and in this chapter the reasons why a triangular net may sometimes be preferable to a square net are explained, and the residual formulas and relaxation patterns appropriate to such a triangular net are derived.

102. Finite-difference Approximations

When an equilateral-triangular net is in use, we again denote the mesh length between pairs of nodes by h, and a group of 19 typical nodes are numbered as shown in Figure 82. In order to find the finite-difference approximations to the quantities $(\nabla^2 w)_0$ and $(\nabla^4 w)_0$, where w is the wanted quantity, we expand w in a double Taylor series in the neighborhood of the node 0:

$$\begin{aligned} w = w_0 &+ A(x - x_0)^2 + B(y - y_0)^2 \\ &+ C(x - x_0)^4 + D(y - y_0)^4 + E(x - x_0)^2(y - y_0)^2 \\ &+ \cdots ; \end{aligned} \tag{1}$$

in this expansion we have, for brevity, included only those terms which provide some contribution to the required results. (It could be verified that any term which includes an odd power either of $x - x_0$ or of $y - y_0$ makes no such contribution.)

Now on substituting in (1) values for the coordinates x and y corresponding to the various nodes 1, 2, 3, etc., we can easily derive the following relations:

$$\begin{aligned} \Sigma w_1 - 6w_0 &= 3h^2(A + B) + \tfrac{3}{4}h^4(3C + 3D + E), \\ \Sigma w_7 - 6w_0 &= 9h^2(A + B) + 27\tfrac{3}{4}h^4(3C + 3D + E), \\ \Sigma w_{13} - 6w_0 &= 12h^2(A + B) + 12h^4(3C + 3D + E), \end{aligned} \tag{2}$$

where terms of $O(h^6)$ have been neglected and where Σw_1 stands for the sum of the six w values typified by w_1 (that is, w_1 to w_6), Σw_7 for the sum of the six w values typified by w_7 (that is, w_7 to w_{12}), and Σw_{13} for the sum of the six w values typified by w_{13} (that is, w_{13} to w_{18}). In (2) the con-

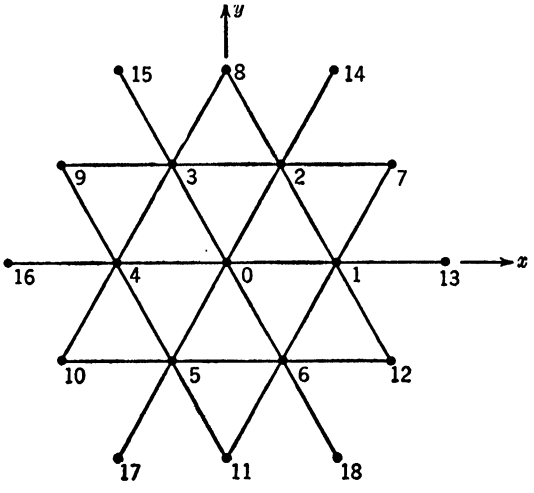


FIGURE 82

stants A, B, C, D, and E may now be replaced by their expressions as derivatives of w, that is, respectively,

$$\frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)_0, \quad \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2}\right)_0, \quad \frac{1}{24}\left(\frac{\partial^4 w}{\partial x^4}\right)_0, \quad \frac{1}{24}\left(\frac{\partial^4 w}{\partial y^4}\right)_0, \quad \frac{1}{4}\left(\frac{\partial^4 w}{\partial x^2\,\partial y^2}\right)_0;$$

the relations (2) then become

$$\begin{aligned} \Sigma w_1 - 6w_0 &= \frac{3h^2}{2}(\nabla^2 w)_0 + \frac{3h^4}{32}(\nabla^4 w)_0, \\ \Sigma w_7 - 6w_0 &= \frac{9h^2}{2}(\nabla^2 w)_0 + \frac{27h^2}{32}(\nabla^4 w)_0, \\ \Sigma w_{13} - 6w_0 &= 6h^2(\nabla^2 w)_0 + \frac{3h^4}{2}(\nabla^4 w)_0. \end{aligned} \tag{3}$$

103. Poisson's Equation

The first of (3) provides immediately the finite-difference approximation to Poisson's equation:

$$\nabla^2 w + W = 0, \tag{4}$$

holding at a typical node 0 of a triangular net; for it follows from (4) that

$$\nabla^4 w + \nabla^2 W = 0, \tag{5}$$

and when we substitute for $(\nabla^2 w)_0$ and $(\nabla^4 w)_0$ in the first of (3) by use of (4) and (5), we derive as the required finite-difference approximation

$$\Sigma w_1 - 6w_0 + \frac{3h^2}{2} W_0 + \frac{3h^4}{24} (\nabla^2 W)_0 = 0. \tag{6}$$

A residual F_0 can then be defined at once:

$$F_0 \equiv \Sigma w_1 - 6w_0 + \frac{3h^2}{2} W_0 + \frac{3h^4}{24} (\nabla^2 W)_0; \tag{7}$$

and the corresponding relaxation pattern is shown in Figure 83.

The advantage of a triangular net over a square net is apparent in the finite-difference approximation (6); it is that the error in that approximation is only of $O(h^6)$ compared with an unavoidable error of $O(h^4)$ in the corresponding approximation for a square net. This means that a solution of higher accuracy can be attained on a triangular net than on a square net for a given value of the mesh side h. This fact in itself does not necessarily imply that a triangular net would always be the more convenient to employ, and it is, in fact, more convenient in most problems to find the solution on a square net, with a use of a smaller value of h than would have been needed if a triangular net had been used.

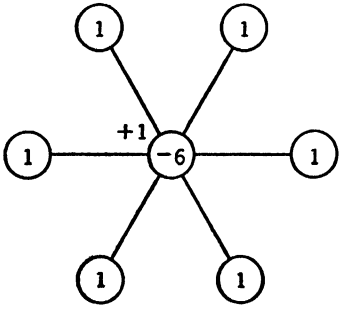


FIGURE 83

But if the shape of the boundary is such that a triangular net can be fitted into the region of integration, exactly, with a use of only whole meshes, whereas a square net could not be so arranged to avoid any irregular stars, then it would be more convenient as well as more accurate to employ a triangular net. These are usually the only circumstances under which a triangular net would be used, when the shape of the boundary and the extra accuracy in (6) combine to outweigh the inconvenience of recording the work on such a net.

It should be noticed that the reason why (6) is a more accurate approximation than that corresponding to a square net is simply that, in the first of (3), the term containing h^4, that is, $(3h^4/32)(\nabla^4 w)_0$, happens to contain w in such a way that it can, by (5), be expressed in terms of W instead of w; this is shown in (6), where that term has been rewritten as $(3h^4/32)(\nabla^2 W)_0$. This fortuitous circumstance has no parallel in the corresponding formula for a square net.

The use of the residual formula (7) and the relaxation pattern of Figure

83 in order to solve a Poisson's equation on a triangular net is exactly like the use of the very similar corresponding formula and pattern on a square net; no further description is needed here.

104. The Biharmonic Equation

In the solution of a biharmonic equation

$$\nabla^4 w = Z, \tag{8}$$

there is no gain in accuracy to be obtained by the use of a triangular net instead of a square net; in both cases the error involved in the finite-difference approximation to (8) is of $O(h^6)$. Consequently there is even less reason why a triangular net should ever be employed in the solution of any biharmonic equation. It is, however, just possible that the inconvenience of so doing may sometimes be overcompensated by reason of the shape of the boundary; if that shape were peculiarly suited to a triangular net, *e.g.*, if the boundary were itself an equilateral triangle, then it would be better to use a triangular net and so avoid any irregular stars. But this circumstance is likely to arise so infrequently that it would scarcely be worth while to record the residual formula and relaxation pattern for the biharmonic equation on a triangular net were it not for a point of some intrinsic interest which arises in the construction of that formula and pattern.

It is possible to eliminate $(\nabla^2 w)_0$ between the first of (3) and either the second or the third; these alternatives lead to the pair of relations

$$\begin{aligned} 3\Sigma w_1 - \Sigma w_7 - 12w_0 + \frac{9h^4}{16}(\nabla^4 w)_0 &= 0, \\ 4\Sigma w_1 - \Sigma w_{13} - 18w_0 + \frac{9h^4}{8}(\nabla^4 w)_0 &= 0, \end{aligned} \tag{9}$$

so that, by (8), a residual could be defined by either of the formulas

$$\begin{aligned} F_0 &\equiv 3\Sigma w_1 - \Sigma w_7 - 12w_0 + \frac{9h^4 Z_0}{16}, \\ F_0 &\equiv 4\Sigma w_1 - \Sigma w_{13} - 18w_0 + \frac{9h^4 Z_0}{8}, \end{aligned} \tag{10}$$

or, indeed, by any linear combination of these two. The patterns corresponding to each of (10) are shown in Figure 84.

The question naturally arises, which is the best formula to use for F_0? That different possibilities exist need cause no concern; each of (9) is an approximation and contains an error of $O(h^6)$, and different errors,

although of the same order, may be expected to correspond to different possible approximations. The best of the available formulas should, of course, be that which provides a pattern which is the easiest to use in the process of liquidation, and, in the event, only experience can determine finally which of the possible patterns that is.

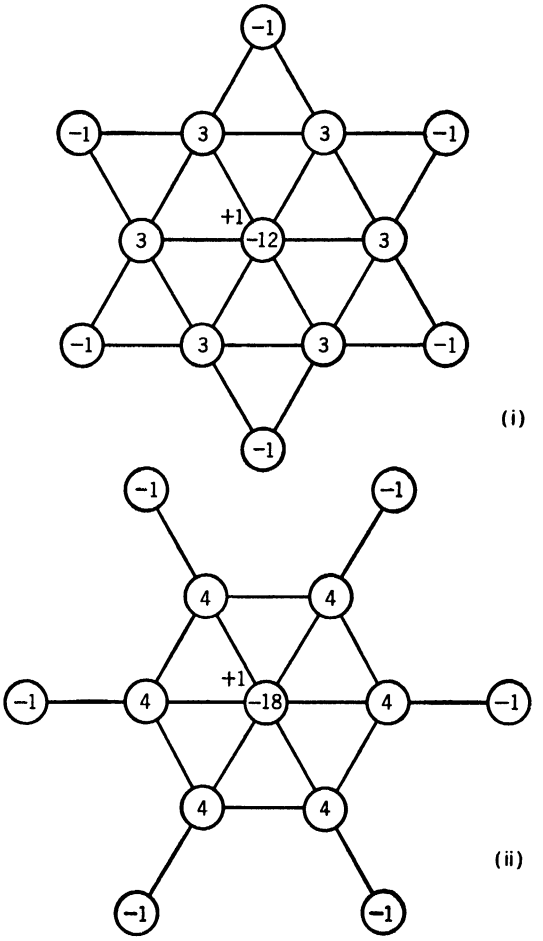


FIGURE 84

The only one of these patterns that has received any attention is the one derived from the residual formula given by adding the second of (10) to twice the first, *i.e.*,

$$F_0 \equiv 10\Sigma w_1 - 2\Sigma w_7 - \Sigma w_{13} - 42w_0 + \frac{9h^4 Z_0}{4}, \qquad (11)$$

for which formula the pattern* is given in Figure 85. This particular formula and pattern have no special virtue as compared with either of those in (10) and Figure 84; in fact, because the latter contain fewer elements (only 13 instead of 19), there can be little doubt that either of the

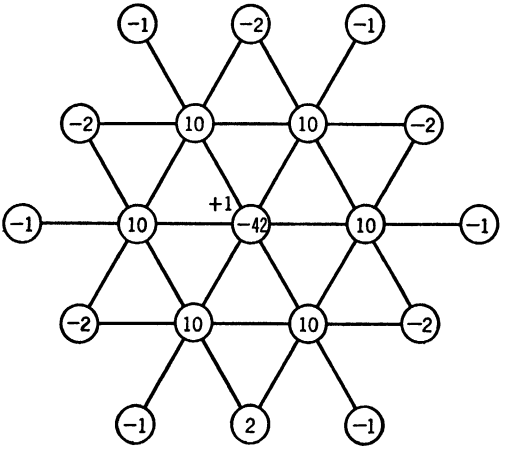


FIGURE 85

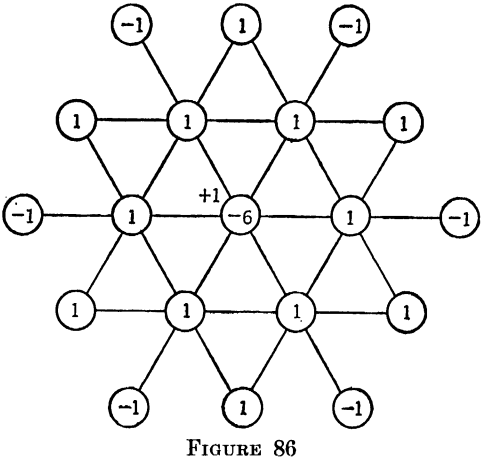


FIGURE 86

patterns of Figure 84 is more convenient to use than the pattern of Figure 85.

The best formula to use for F_0 is, however, likely to be that one which leads to a pattern whose central element bears the largest possible ratio to any of the other elements; this formula is derived by subtraction of the

* This pattern is given by Fox and Southwell, *Phil. Trans. R.S. C*, **1**, 15–56 (1941), and *A*, **239**, 419–60 (1945).

first of (10) from the second, which yields

$$F_0 \equiv \Sigma w_1 + \Sigma w_7 - \Sigma w_{13} - 6w_0 + \frac{9h^4 Z_0}{16},$$

for which the corresponding pattern is shown in Figure 86. In it the ratio of the central element to any other is as large as 6, compared with 4.2 in the pattern of Figure 85 and with 4.0 and 4.5 in the patterns of Figure 84.

105. Advance to a Finer Net

As when square nets are in use, so with triangular nets, solutions must usually be obtained on a succession of nets of different sizes. The sizes

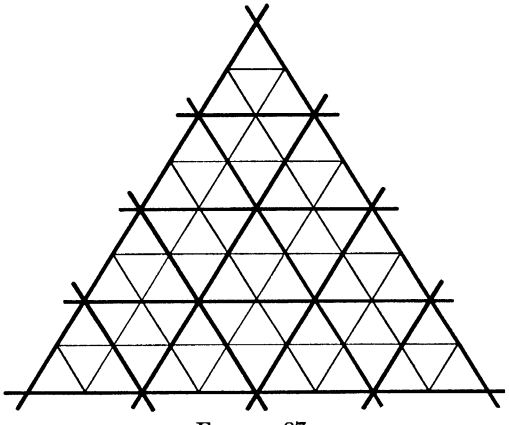


FIGURE 87

of the successively finer meshes are usually made, systematically, each to be one-half of the size of the mesh of the previous (coarser) net. Figure 87 illustrates how the advance from one net to the next finer net should be made. The coarser net is shown drawn in thick lines, and the extra lines which are drawn in to convert it into a finer net are shown as fine lines.

PROBLEM

1. Find the torsion stress function ϕ for the case of a cylinder of equilateral triangular section at the nodes of a triangular net which has 16 strings in the side of the section.

(Take the representative dimension L to be the side of the section, and use a multiplying factor of 2,560.)

CHAPTER 11

EIGENVALUE PROBLEMS (ALGEBRAIC EQUATIONS)

106. Introduction

In this chapter we revert to consider the solution of certain sets of algebraic equations; these are simultaneous linear equations which are similar to those discussed in Chapters 1 and 2, in that a number of unknowns x, y, z, etc., are to be found, but differ in having no constant terms and in having coefficients which are not completely known in the beginning. Thus as well as solving to find values for a set of unknowns, we must in addition, as part of the solution, find a quantity upon which depend the values of the coefficients in the set of equations. Because the equations have no constant terms, they are homogeneous as well as linear; therefore it is clear that strictly we cannot determine absolute values for the unknowns but really only their ratios $x:y:z:\ \ldots$. We express any set of such ratios which can satisfy the equations by the single symbol w and call it a *mode* of solution. The quantity on which the values of the coefficients in the equations depend is known as an eigenvalue, and we denote it by p^2. One of the most important ways in which equations of this type can arise for solution is in the determination of free vibrations of a dynamical system of n degrees of freedom. Such a system requires n coordinates to define its position in any displacement from its equilibrium configuration; we denote these coordinates by X, Y, Z, etc. The system is said to be vibrating in a normal mode when each coordinate varies sinusoidally with time with the same period and phase; thus a normal mode of vibration is given when the coordinates are expressed by a set of relations

$$\begin{aligned} X &= x \sin pt, \\ Y &= y \sin pt, \\ Z &= z \sin pt, \\ &\ldots\ldots\ldots \end{aligned}$$

The ratios $x:y:z:\ \ldots$ are said to define the mode of vibration, and the quantity p is a measure of its period $(2\pi/p)$. A system of n degrees of freedom, in general, possesses n distinct normal modes of vibration; we denote these by $w_1, w_2, w_3, \ldots, w_n$ with corresponding eigenvalues $p_1^2, p_2^2, p_3^2, \ldots, p_n^2$. It is also convenient to postulate that $p_1^2 \leq p_2^2$

$\leq p_3^2 \leq \cdots \leq p_n^2$, that is, that the n normal modes are arranged according to ascending eigenvalues.

107. Rayleigh's Principle

The form of the equations (governing the ratios $x:y:z:\ \ldots$) with which we are concerned in this chapter is

$$\begin{aligned}
(a_{11} - b_{11}p^2)x + (a_{12} - b_{12}p^2)y + (a_{13} - b_{13}p^2)z + \cdots &= 0,\\
(a_{12} - b_{12}p^2)x + (a_{22} - b_{22}p^2)y + (a_{23} - b_{23}p^2)z + \cdots &= 0,\\
(a_{13} - b_{13}p^2)x + (a_{23} - b_{23}p^2)y + (a_{33} - b_{33}p^2)z + \cdots &= 0,\\
\cdots\cdots\cdots

This expression of p^2 in terms of x, y, z, etc., is a Rayleigh quotient which has the property that it defines p^2 as a function of the variables x, y, z, etc., which is stationary when those variables have values corresponding to any normal mode. [It may be straightforwardly verified that the fraction on the right of (2) is an expression for p^2 which is such that the conditions that $\partial p^2/\partial x$, $\partial p^2/\partial y$, $\partial p^2/\partial z$, etc., should all vanish are precisely the equations (1); at a normal mode these equations are satisfied, and therefore p^2 as given by (2) has a stationary value.]

108. Solution by the Rayleigh Method

This property of the formula (2), that it expresses p^2 in a form which as a function of w has a stationary value at a normal mode, allows one relaxational method of solution of equations (1) to succeed. We describe this method as the "Rayleigh" method to distinguish it from a second relaxational method, the "intensification" method, which will be explained later in Section 112. In many examples the Rayleigh method suffers from a serious limitation which considerably reduces its usefulness; this shortcoming will be described in Section 110, where also it will be pointed out under what circumstances the method can be employed to advantage.

The manner in which the method converges to a solution depends on the fact that any approximation to a mode yields, on substitution in (2), a better approximation to the corresponding eigenvalue; as a solution is approached and a good approximation to a mode is reached, the stationary property of the formula (2) implies that it then yields an excellent approximation to p^2. The method begins with a guessed set of values to a mode, x, y, z, etc.; these are then substituted in (2) to determine a first approximation to the eigenvalue p^2. This value for p^2 is put back into the simultaneous equations (1); residuals are defined and calculated in the usual way, and by attempting to relax these residuals, we begin to *improve* the initial guessed values of x, y, z, etc. In general, at this stage, it would be found impossible to liquidate the residuals completely except by a degeneration into the null solution; this is because the first approximation to p^2 is most unlikely not to be in some error and, therefore, is a value for which, strictly, only the null solution for the mode exists. However, the liquidation of the residuals can be accomplished in part—no great effort should be made to take it very far—and in this process we improve the values initially guessed for x, y, z, etc. These improved values are then substituted in the formula (2), and a second (and better) approximation to the eigenvalue p^2 is thereby determined; this new value for p^2 can now be put back into the equations (1), the residuals then recalculated, and the mode (x,y,z, etc.) further improved by means of relaxation. This process is repeated until eventually all the residuals have been made

sufficiently small. In the actual stages of relaxation three main points should be borne in mind: First, increments should not be applied which are clearly tending to reduce *all* of x, y, z, etc., to zero; *i.e.*, the null solution must be deliberately avoided. Second, each individual stage of relaxation must not be overmuch prolonged; relaxation should cease as soon as there is any indication of difficulty in the reduction of the set of residuals (as a whole) toward zero. Third, it must not be forgotten that the solution for a mode really comprises only the ratios $x:y:z: \ldots$; at any time, a current set of values for x, y, z, etc., can themselves be multiplied through by any convenient constant. It is computationally convenient to multiply through in this way at the end of each stage of relaxation by such a constant as will bring the numerically greatest of x, y, z, etc., to, say, 100. This operation not only helps to prevent the solution drifting by chance toward the null solution; it also helps the computer to see more readily how his solution is converging, by making easier a comparison of the values of x, y, z, etc., at the end of successive stages of relaxation. A simple numerical example will now be solved to illustrate this method.

109. An Application of the Rayleigh Method

We take for this example the set of three equations

$$\begin{aligned} (2 - p^2)x + \quad y + \quad 2z &= 0, \\ x + (3 - p^2)y + \quad 3z &= 0, \\ 2x + \quad 3y + (4 - p^2)z &= 0. \end{aligned} \tag{3}$$

The Rayleigh formula (2) in this particular case becomes

$$p^2 = \frac{2x^2 + 3y^2 + 4z^2 + 2xy + 6yz + 4zx}{x^2 + y^2 + z^2}. \tag{4}$$

As the initial guess for a modal solution we choose

$$x = y = z = 100, \tag{5}$$

and substituting in (4), we find that the first approximation to p^2 is

$$p^2 = 7. \tag{6}$$

When this value for p^2 is taken in the equations (3), residuals become defined by the expressions

$$\begin{aligned} F_x &\equiv -5x + y + 2z, \\ F_y &\equiv x - 4y + 3z, \\ F_z &\equiv 2x + 3y - 3z. \end{aligned}$$

It is unnecessary to record the corresponding operations table. The first stage of relaxation is shown in the following relaxation table:

	F_x	F_y	F_z
$x = y = z = 100$	−200	0	200
$\Delta x = -40$	0	−40	120
$\Delta z = 40$	80	80	0
$\Delta x = 16$	0	96	32

After just three steps the liquidation process reaches a situation in which any attempt to reduce the residuals further can be verified by trial to have little success; perhaps, even, an optimum state was reached after only the first two steps. Thus the first stage of relaxation ends with the values of the unknowns given by

$$x = 76, \qquad y = 100, \qquad z = 140,$$

or, sufficiently, after multiplication through by 0.71 to bring the value of the largest unknown (z) to 100,

$$x = 54, \qquad y = 71, \qquad z = 100. \tag{7}$$

Substitution of these values into (4) leads to the second approximation for the eigenvalue:

$$p^2 = 7.4; \tag{8}$$

and with this new value for p^2 the residuals' expressions are modified to become

$$\begin{aligned} F_x &\equiv -5.4x + y + 2z, \\ F_y &\equiv x - 4.4y + 3z, \\ F_z &\equiv 2x + 3y - 3.4z. \end{aligned}$$

The second stage of relaxation follows in the table:

	F_x	F_y	F_z
$x = 54,\ y = 71,\ z = 100$	−21	43	−19
$\Delta y = 10$	−11	−2	11
$\Delta x = -2$	0	−4	7
$\Delta z = 2$	4	2	0

Again nothing is gained by any continuation of this table beyond the first three steps, when the values of the unknowns have become

$$x = 52, \qquad y = 81, \qquad z = 102,$$

or, after reduction of the greatest of these to become 100,

$$x = 51, \qquad y = 79, \qquad z = 100. \tag{9}$$

A third substitution in (4) yields

$$p^2 = 7.42. \tag{10}$$

The successive approximations (6), (8), and (10) to the eigenvalue show a very rapid convergence, and, in point of fact, the third approximation (10) could have been quoted with accuracy to another significant figure. However, the value given is sufficient for our purpose; it leads to new expressions for the residuals:

$$\begin{aligned} F_x &\equiv -5.42x + y + 2z, \\ F_y &\equiv x - 4.42y + 3z, \\ F_z &\equiv 2x + 3y - 3.42z; \end{aligned}$$

and thence to the third stage of relaxation:

	F_x	F_y	F_z
$x = 51,\ y = 79,\ z = 100$	3	2	−3
$\Delta x = 1$	−2	3	−1
$\Delta y = 1$	−1	−1	2

After two steps the liquidation is completed, in so far as the residuals can then be made smaller only by means of the introduction of increments affecting the first decimal place. Correct to the number of figures quoted, the modal solution has been found to be

$$x = 52, \qquad y = 80, \qquad z = 100. \tag{11}$$

The whole solution is contained in (10) and (11), and just as the convergence of p^2 could be observed from (6), (8), and (10), so the convergence of the mode (x,y,z) can be followed from stage to stage in the process in (5), (7), (9), and (11).

As has already been suggested in Section 108, the success of this method is entirely dependent on the fact that, by use of a Rayleigh quotient to express p^2 as a function of w, we are able throughout the calculation to keep the approximation to the eigenvalue one step ahead, as it were, in accuracy of the approximation to the mode.

110. Defects of the Rayleigh Method

It should have already been apparent that the Rayleigh method of calculation of a solution to equations of the type (1) suffers from the major

drawback that it gives no indication which of the n modal solutions is obtained. In the example of Section 109 there are, of course, three separate solutions, and if these are characterized by suffixes 1, 2, and 3, where $p_1^2 < p_2^2 < p_3^2$, then in fact the solution found in that section is actually the mode w_3, with the corresponding eigenvalue p_3^2 (= 7.42). Now in most practical problems interest is usually directed to the lowest, or gravest, mode, w_1, with the corresponding eigenvalue p_1^2; it is desirable that we should use a method of solution that not only indicates which mode results from the computation but that also can be arranged deliberately always to select the gravest one. The Rayleigh method, as we have seen, does not satisfy the first of these requirements, let alone the second. Another difficulty also occurs sometimes in the use of the Rayleigh method: this happens when two of the eigenvalues are nearly equal; the convergence to one of the two corresponding modes may then be impractically slow, if not indeed impossible to achieve.

These defects do not mean that the Rayleigh method is without its uses. In many examples, particularly those of a kind to be discussed in Chapter 12, the form of the lowest mode is so different from the other (higher) modes that a sufficiently good first guess can be made for w_1 to ensure that that mode is the one to be determined. In such cases the Rayleigh method may well have advantages in speed over the alternative method, which will be described in Section 112.

111. Expansion in Normal Modes

The alternative method is based on a well-known property of the normal modes: that *any* set of values for x, y, z, etc., which together comprise a quantity w, can be obtained by a linear combination of the sets of values which comprise the normal modes w_1, w_2, w_3, etc. Expressed algebraically, this means that, whatever values the components of w may have, we can expand w as a linear series in terms of the normal modes:

$$w = A_1 w_1 + A_2 w_2 + A_3 w_3 + \cdots + A_n w_n. \tag{12}$$

We shall make use of this result to say that any guessed w which we may choose is made up of linear contributions of the n normal modes: we shall not need to know the values of the coefficients A_1, A_2, etc.

112. The Intensification Method

The set of equations (1) can be concisely written as a single equation in the form

$$P(w) - p^2 Q(w) = 0, \tag{13}$$

where P is an operator which represents the coefficients a_{rs} and Q is an operator which represents the coefficients b_{rs} in (1). The satisfaction of (13) by the normal modes implies a set of relations

$$\begin{aligned} P(w_1) - p_1^2 Q(w_1) &= 0, \\ P(w_2) - p_2^2 Q(w_2) &= 0, \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{14}$$

Now consider the set of ordinary simultaneous equations represented by

$$P(w') - Q(w) = 0, \tag{15}$$

in which w is supposed to be given and w' to be found. Since, whatever w may be, it can be supposed to be expanded in the form (12), we have

$$\begin{aligned} P(w') &= Q(A_1 w_1 + A_2 w_2 + \cdots + A_n w_n) \\ &= A_1 Q(w_1) + A_2 Q(w_2) + \cdots + A_n Q(w_n), \end{aligned}$$

because the operator Q is linear and homogeneous. Hence by use of the set of relations (14) we get

$$\begin{aligned} P(w') &= \frac{A_1}{p_1^2} P(w_1) + \frac{A_2}{p_2^2} P(w_2) + \cdots + \frac{A_n}{p_n^2} P(w_n) \\ &= P\left(\frac{A_1 w_1}{p_1^2} + \frac{A_2 w_2}{p_2^2} + \cdots + \frac{A_n w_n}{p_n^2} \right), \end{aligned}$$

because the operator P is also linear and homogeneous. Thus we find that

$$w' = \frac{A_1}{p_1^2} w_1 + \frac{A_2}{p_2^2} w_2 + \cdots + \frac{A_n}{p_n^2} w_n, \tag{16}$$

a relation giving w' in its expansion in terms of the normal modes.

Now in any particular numerical example the solution of the set of equations represented by (15) is straightforward by the methods of Chapters 1 and 2, and by comparison of the expansions (12) and (16) we see that any initial guess for w leads by solution of (15) to a w' in which the *proportion* of the first mode w_1 relative to the other normal modes is increased compared with its proportion in w. This last statement follows at once from the fact that $p_1^2 < p_2^2 < \cdots < p_n^2$. It is a convenient description to say that, in deriving w' from w, we have *intensified* the gravest mode compared with the other normal modes. If this process is repeated a number of times, each time putting back the previous w' as a new w, then clearly we shall derive eventually a w' which is predominantly composed of the gravest mode only; in other words, as the process is repeated, w' converges to w_1. This is the *intensification method;* it ensures that we eventually find the gravest normal mode, and it is of particular

computational simplicity because it requires only a succession of solutions of sets of ordinary linear algebraic simultaneous equations. The method determines w_1 without any reference to the quantity p^2, and, of course, once the mode w_1 has been determined to a sufficient accuracy, the corresponding p_1^2 can be evaluated by a direct substitution of w_1 into the formula (2).

113. An Application of the Intensification Method

In any application of this method it may be remembered that the values of x, y, z, etc., in w are not defined absolutely; it is computationally convenient, therefore, each time a w' is determined, to multiply its components through by such a constant as brings the greatest of them to, say, 100.

We now illustrate the method by using it to find the gravest modal solution to the set of equations (3); one solution to these equations, not the gravest, has already been determined in Section 109 by means of the Rayleigh method. For these equations (3) the equations represented by (15) are

$$\begin{aligned} 2x' + y' + 2z' &= x, \\ x' + 3y' + 3z' &= y, \\ 2x' + 3y' + 4z' &= z, \end{aligned}$$

and residuals are defined by the expressions

$$\begin{aligned} F_{x'} &\equiv -2x' - y' - 2z' + x, \\ F_{y'} &\equiv -x' - 3y' - 3z' + y, \\ F_{z'} &\equiv -2x' - 3y' - 4z' + z. \end{aligned} \tag{17}$$

It is appropriate that the first guess for w should be the same as that taken in Section 109, that is, $x = y = z = 100$. Determination of a first w' is then shown in the relaxation table:*

	$F_{x'}$	$F_{y'}$	$F_{z'}$
$x' = y' = z' = 0$	100	100	100
$\Delta x' = 40$	20	60	20
$\Delta y' = 40$	−20	−60	−100
$\Delta z' = -40$	60	60	60
$x' = 100,\ y' = 100,\ z' = -100$	0	0	0

* It is not thought necessary to give any detailed explanation of the individual steps in the relaxation tables in this section.

In this table, after three steps, an obvious multiplying factor completes the relaxation in one further step; the first w' is therefore

$$x' = 100, \qquad y' = 100, \qquad z' = -100. \tag{18}$$

We now put these values back for x, y, and z in (17) and solve those equations again for x', y', and z'. The solution is found in the relaxation table:

	$F_{x'}$	$F_{y'}$	$F_{z'}$
$x' = y' = z' = 0$	100	100	−100
$\Delta z' = -100$	300	400	300
$\Delta x' = -100$	500	500	500
$\Delta x' = \Delta y' = 500$, $\Delta z' = -500$	0	0	0
$x' = 400$, $y' = 500$, $z' = -600$	0	0	0

After multiplication through to reduce the magnitude of its greatest component to 100, the second w' is found to be

$$x' = 67, \qquad y' = 83, \qquad z' = -100. \tag{19}$$

These values are put back for x, y, and z in (17), and those equations are solved again in the following table:

	$F_{x'}$	$F_{y'}$	$F_{z'}$
$x' = y' = z' = 0$	67	83	−100
$\Delta z' = -183$	433	632	632
$\Delta y' = 99$	334	335	335
$\Delta x' = \Delta y' = 335$, $\Delta z' = -335$	−1	0	0
$x' = 335$, $y' = 434$, $z' = -518$	−1	0	0

After appropriate reduction again, we find that the third w' is

$$x' = 65, \qquad y' = 84, \qquad z' = -100. \tag{20}$$

The convergence to the gravest mode can be seen by inspection of the successive approximations (18), (19), and (20) to be almost achieved in (20). In fact, to the number of figures quoted, one further stage in the process is required for complete convergence. The final values may be verified to be

$$x_1 = 64, \qquad y_1 = 84, \qquad z_1 = -100. \tag{21}$$

The corresponding eigenvalue, calculated by means of a substitution of this mode in the formula (4), is

$$p_1^2 = 0.194, \tag{22}$$

and the whole solution for the gravest mode is contained in (21) and (22) together.

114. Improvement of Basic Operations

A possible difficulty which may sometimes occur in the use of the intensification method is that the actual liquidation of the residuals in the individual stages of relaxation may be either very troublesome or, at best, very slow to achieve. An example in which this is the case is provided by the equations

$$\begin{aligned} (15 - 2p^2)x + 10y + 7z &= 0, \\ 10x + (7 - p^2)y + 4z &= 0, \\ 7x + 4y + (20 - 3p^2)z &= 0. \end{aligned} \tag{23}$$

These lead to the following expressions for the residuals $F_{x'}$, etc:

$$\begin{aligned} F_{x'} &\equiv -15x' - 10y' - 7z' + 2x, \\ F_{y'} &\equiv -10x' - 7y' - 4z' + y, \\ F_{z'} &\equiv -7x' - 4y' - 20z' + 3z, \end{aligned} \tag{24}$$

and a consequent operations table:

	$\Delta F_{x'}$	$\Delta F_{y'}$	$\Delta F_{z'}$
$\Delta x' = 1$	-15	-10	-7
$\Delta y' = 1$	-10	-7	-4
$\Delta z' = 1$	-7	-4	-20

(25)

This operations table is not easy to use, chiefly for the reason that the elements in the leading diagonal do not sufficiently predominate in numerical value over the remaining elements. This difficulty can be overcome by means of a simple transformation of the equations (23). In the latter we may replace p^2 by $(q^2 - K)$, where K is some convenient positive number, and then treat q^2 as the eigenvalue instead of p^2. Thus if we take K to be 8, equations (23) are replaced by

$$\begin{aligned} (31 - 2q^2)x + 10y + 7z &= 0, \\ 10x + (15 - q^2)y + 4z &= 0, \\ 7x + 4y + (44 - 3q^2)z &= 0. \end{aligned} \tag{26}$$

The expressions (24) for the residuals are similarly replaced by

$$\begin{aligned} F_{x'} &\equiv -31x' - 10y' - 7z' + 2x, \\ F_{y'} &\equiv -10x' - 15y' - 4z' + y, \\ F_{z'} &\equiv -7x' - 4y' - 44z' + 3y, \end{aligned}$$

and the operations table (25) is changed to become

	$\Delta F_{x'}$	$\Delta F_{y'}$	$\Delta F_{z'}$
$\Delta x' = 1$	-31	-10	-7
$\Delta y' = 1$	-10	-15	-4
$\Delta z' = 1$	-7	-4	-44

In this table the magnitudes of the elements in the leading diagonal are manifestly such that residuals can be liquidated rapidly and easily; the individual stages of relaxation in the intensification method of solution of (26) for the gravest normal mode, therefore, present no difficulties. We should specifically notice, of course, that the q^2 eigenvalues, like the p^2 eigenvalues, obey a set of inequalities, that is, $q_1^2 < q_2^2 < q_3^2$.

The question has now to be answered why in this illustration the value 8 was taken for K. Clearly, the greater the value chosen for K, the more the elements in the leading diagonal of the changed operations table predominate over the other elements, and therefore the easier becomes the actual liquidation of residuals required in the method. The answer to this point is that the greater K is chosen to be, the less effective is each individual cycle of intensification (for we have seen that in each cycle the gravest mode w_1 is intensified in comparison with any other normal mode w_r in the ratio q_r^2/q_1^2, and this ratio decreases nearer and nearer to 1 as K is made larger and larger); hence, in order that the method may converge in a minimum number of cycles of intensification, K should be chosen to be as small as possible.

Clearly the choice of a value for K must effect a compromise: It must be made as small as possible in order to keep the number of cycles of intensification required for convergence to a minimum, but it must be made just sufficiently large to make the relaxation in each of the cycles reasonably easy and rapid to achieve. In the numerical example in this section the value $K = 8$ is a fair compromise between the two conflicting demands. The determination of the gravest mode satisfying (26) and hence (23) follows computational steps similar to those illustrated in Section 113, and this solution is left as an example for the reader to obtain.

115. The Orthogonal Property of Normal Modes

Although, in many problems governed by a set of equations of the general type (1), the main interest of the solution lies in the gravest mode, it is sometimes the case that the other modes must be found as well. We

have seen how the intensification method enables the gravest mode to be found with certainty, and we must next describe how that method can be used also to determine the second, third, etc., normal modes. This can be achieved by use of a well-known property of normal modes: that they are orthogonal. The meaning of this statement as applied to two modal solutions w_r and w_s of the set of equations (1) is expressed in the relation*

$$b_{11}x_rx_s + b_{22}y_ry_s + b_{33}z_rz_s + \cdots + b_{12}(x_ry_s + x_sy_r) + b_{13}(z_rx_s + z_sx_r) + b_{23}(y_rz_s + y_sz_r) + \cdots = 0.$$

It is convenient to denote the left-hand side of this equation as the function $B(w_r,w_s)$ of the components x_r, y_r, z_r, etc., and x_s, y_s, z_s, etc., of the modes w_r and w_s; it may then be written simply as

$$B(w_r,w_s) = 0. \tag{27}$$

Now suppose that we have any set of values for x, y, z, etc., comprising together a quantity w; by (12) this quantity can be expanded in a series in terms of the n normal modes:

$$w \equiv A_1w_1 + A_2w_2 + \cdots + A_nw_n. \tag{28}$$

By use of relations of the type (27) we can express any coefficient A_r in this series in terms of w and the corresponding normal mode w_r; for, because B is a linear function of each of w_r and w_s and because w is a linear function of the set of modes w_1, w_2, w_3, etc., we have

$$B(w_r,w) = A_1B(w_r,w_1) + A_2B(w_r,w_2) + \cdots A_rB(w_r,w_r) + \cdots + A_nB(w_r,w_n).$$

With one exception the terms on the right of this equation vanish by virtue of (27), so that it reduces to

$$B(w_r,w) = A_rB(w_r,w_r),$$

i.e., to

$$A_r = \frac{B(w_r,w)}{B(w_r,w_r)}. \tag{29}$$

Thus, given any w and knowing a mode w_r, we can by substitution in (29) determine the coefficient of w_r in the expansion (28) of w. We can, therefore, by subtraction then remove from w the part A_rw_r contained in it. This immediately means that the intensification method can successively achieve the determination of the higher normal modes w_2, w_3, w_4, etc., satisfying a set of equations (1); for, having previously

* The proof of this relation is given in Appendix IV, page 243, where also are derived the orthogonal properties of modal solutions to other types of equations containing eigenvalues.

found the gravest mode w_1, we can then remove the contribution of w_1 in any initial guess for the second normal mode. Thus effectively it can be ensured that the initial guess for the second mode is some quantity of the form

$$w = A_2 w_2 + A_3 w_3 + \cdots + A_n w_n.$$

When now the intensification method is applied to a w of this form, in effect w_2 is the gravest component mode, to which, as the process is repeated, w' (cf. Section 112) will converge. In turn, when w_2 has been itself determined, both it and w_1 can be removed from any initial guess for the third mode, so that the resulting w after repeated intensification will then yield w_3, and so on.

116. A Worked Example

As an example to illustrate the method of the previous section, the second normal mode satisfying the equations (3) will be determined. The gravest mode w_1 in this case was found by the intensification method in Section 113 to be

$$x_1 = 64, \qquad y_1 = 84, \qquad z_1 = -100. \tag{30}$$

For the equations (3) the function $B(w_r,w_s)$ is $x_r x_s + y_r y_s + z_r z_s$, so that the formula (29) takes the form

$$A_1 = \frac{x_1 x + y_1 y + z_1 z}{x_1^2 + y_1^2 + z_1^2}, \tag{31}$$

when r is put equal to 1. Now suppose we choose as an initial guess for w_2 the values

$$x = 100, \qquad y = 100, \qquad z = 100, \tag{32}$$

making, since there is no reason not to, the same guess as we did initially for w_1. Substitution from (30) and (32) into (31) yields, at once, $A_1 = 0.227$, so that the contribution $A_1 w_1$ of the gravest mode (30), which has to be removed from (32), is

$$A_1 x_1 = 15, \qquad A_1 y_1 = 19, \qquad A_1 z_1 = -23.$$

After subtraction of these values from the corresponding components of (32), there remains

$$x = 85, \qquad y = 81, \qquad z = 123,$$

or, after, for convenience, multiplication through by such a constant as reduces the value of the largest component to 100,

$$x = 69, \qquad y = 66, \qquad z = 100. \tag{33}$$

This set of values may be properly termed the orthogonalized first guess for the second mode; after intensification in a manner similar to that employed in Section 113, we derive

$$x' = 100, \qquad y' = 5, \qquad z' = 74. \tag{34}$$

According to the theoretical treatment in Section 115, these values should comprise a w' which contains no contribution from the gravest mode w_1; however, in practice, because of rounding-off errors in the arithmetic, (34) may contain a small proportion of w_1. For this reason, and because it is very easily done, it is as well to orthogonalize (34) before we put its components back as values for x, y, and z in a second intensification. This orthogonalization of (34) yields as the second approximation to w_2

$$x = 100, \qquad y = 7, \qquad z = 70, \tag{35}$$

in the same way as, initially, orthogonalization of (32) yielded (33) as the first approximation.

Repetition of the process, *i.e.*, intensification followed by orthogonalization, produces as the third approximation

$$x = 100, \qquad y = -50, \qquad z = 21. \tag{36}$$

To the number of significant figures quoted, three further cycles of the process achieve convergence; the successive approximations are

$$\begin{aligned} x &= 100, \quad y = -66, \quad z = 8, \\ x &= 100, \quad y = -70, \quad z = 5, \\ x &= 100, \quad y = -71, \quad z = 4. \end{aligned} \tag{37}$$

The solution for the second normal mode is given completely by (37), together with the eigenvalue, calculated by substitution in (4) to be

$$p^2 = 1.39.$$

PROBLEMS

1. Find the gravest mode and corresponding eigenvalue satisfying the set of equations

$$\begin{aligned} (4 - p^2)x + 4y + z &= 0, \\ 4x + (5 - p^2)y + 3z &= 0, \\ x + 3y + (6 - p^2)z &= 0. \end{aligned}$$

2. Complete the solution in Section 114 to determine the gravest mode satisfying equations (23).

3. Find all the normal modes of solution of the set of equations

$$\begin{aligned} (5 - p^2)x + 3y + 2z &= 0, \\ 3x + (4 - p^2)y + 4z &= 0, \\ 2x + 4y + (5 - p^2)z &= 0. \end{aligned}$$

CHAPTER 12

EIGENVALUE PROBLEMS (DIFFERENTIAL EQUATIONS)

117. Introduction

Both the methods, described in Chapter 11, of solving eigenvalue problems governed by a set of simultaneous algebraic equations can be readily adapted to solve also a wide variety of eigenvalue problems governed by ordinary or partial differential equations. It would not be possible to explain here examples of all these types of problems, but among illustrations which will be used in this chapter to demonstrate the methods are the determination of the mode and frequency of transverse vibration of a stretched string, the calculation of the critical thrust and mode of distortion of a strut, and the solution for the mode and frequency of free vibration of a beam and of a uniformly tensioned membrane. In all these examples the system is continuous instead of being restricted, as in the examples of Chapter 11, to a finite number of degrees of freedom; as a consequence, there are in each case an infinite number of modes of solution. In practice, however, the gravest mode is of predominating interest, and we shall be concerned to demonstrate only how it may be found.

118. The Intensification Method

The theory behind the use of the intensification method of solution of a differential equation containing an eigenvalue is very similar to the theory underlying the application to solving sets of algebraic equations of the form (1) of Chapter 11; the differential equations to be considered are all of the same general type:

$$P(w) + \lambda Q(w) = 0, \tag{1}$$

in which w is the wanted function, a function of the independent variable x in one-dimensional problems and of the independent variables x and y in two-dimensional problems. P and Q are given differential operators in terms of the independent variable(s), and λ is the eigenvalue. We denote the normal-mode solutions of (1) by w_1, w_2, w_3, etc., with corresponding eigenvalues λ_1, λ_2, λ_3, etc., where it may be supposed that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. We use the known fact that any guessed func-

tion w can be expanded (cf. Section 111) in a series in terms of the normal modes:

$$w = A_1 w_1 + A_2 w_2 + A_3 w_3 + \cdots, \tag{2}$$

where the A's are constants which do not need to be found and where, of course, the series in (2) is now infinite. It follows from (2), in a manner exactly similar to that employed in Section 112, that the solution w' to the equation

$$P(w') + Q(w) = 0 \tag{3}$$

is given by the infinite series

$$w' = \frac{A_1 w_1}{\lambda_1} + \frac{A_2 w_2}{\lambda_2} + \frac{A_3 w_3}{\lambda_3} + \cdots, \tag{4}$$

and hence also that, when (3) is solved with w given by (2), the resulting solution (4) for w' contains a proportion of the gravest mode w_1 greater than that contained in w. This result follows immediately from the fact that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. The process of solution of (3), then, yields a w' in which the proportion of w_1 has been *intensified* in comparison with its proportion in w, and so if that process is continually repeated by using the values obtained for w' as values for successive substitution back for w in (3), a sequence of solutions for w' will be found which converge to w_1, the required gravest mode.

It should be mentioned that the determination of any one solution of a differential equation (1) is dependent on the existence of appropriate boundary or end conditions. Whatever these conditions may be given to be, the normal modes w_1, w_2, w_3, etc., of course, satisfy them; the first guess for w must also be chosen to obey these conditions, and they must be used as conditions to be satisfied by w' in the relaxational solution of (3).

When the gravest mode w_1 has been determined, the corresponding λ_1 is best evaluated by substitution of the modal values in a Rayleigh quotient. By virtue of the stationary property of such a quotient (cf. Section 107), this method of calculation will automatically yield the eigenvalue to a high degree of accuracy. The form of the Rayleigh quotient for particular differential equations of the type of (1) will be given in the discussions of particular examples.

119. Vibration of a Stretched String

The general equation governing the transverse motion of a stretched string is

$$\frac{\partial^2 W}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 W}{\partial t^2} \tag{5}$$

where ρ is the mass of the string per unit length and T is the tension; W is the transverse displacement of the string during its motion and is a function of distance x along the string and of time t. In any normal mode of vibration each element of the string moves with the same period and phase so that

$$W = w \sin pt, \tag{6}$$

where w is a function of x alone and $p/2\pi$ is the frequency of the vibration. On substitution of (6) into (5) we obtain as the governing equation for w

$$\frac{d^2w}{dx^2} + \frac{\rho p^2 w}{T} = 0.$$

This equation may be made nondimensional by replacement of x by Lx', where L is the length of the string; it then becomes

$$\frac{d^2w}{dx'^2} + \lambda w = 0, \tag{7}$$

where λ stands for $\rho p^2 L^2/T$ and is a purely numerical quantity. It is convenient to drop the prime from x' and to rewrite (7) as

$$\frac{d^2w}{dx^2} + \lambda w = 0, \tag{8}$$

with the understanding, now, that x is a nondimensional quantity which is a measure of distance along the length of the string from $x = 0$ at one end to $x = 1$ at the other.

Equation (8) is a particular example of an equation of the form (1) when $P \equiv d^2/dx^2$ and $Q \equiv 1$. To find its gravest normal mode of solution, the intensification method requires repeated solution of the corresponding particular example of (3), *i.e.*, of the equation

$$\frac{d^2w'}{dx^2} + w = 0. \tag{9}$$

This last equation can, of course, be very easily solved by the relaxation method when the range of integration $0 \leq x \leq 1$ is divided up into a number of intervals of length h by a set of equally spaced points of subdivision. The finite-difference approximation to (9), holding at a typical point of subdivision 0, is

$$w_1' + w_3' - 2w_0' + h^2 w_0 = 0; \tag{10}$$

in this approximation the factor h^2 can be dropped from the term $h^2 w_0$ because the absolute magnitude of w' is, in any event, indeterminate. From (10) we then at once derive the residual formula

$$F_0 \equiv w_1' + w_3' - 2w_0' + w_0, \tag{11}$$

and it clearly involves only very simple relaxation to determine the values of w' at the points of subdivision, given the values of w at those points.

120. String with Fixed Ends

For a numerical example, we take the case when the ends of the string are fixed, so that the end conditions to be satisfied are that

$$w = 0 \text{ at } x = 0, \quad w = 0 \text{ at } x = 1. \tag{12}$$

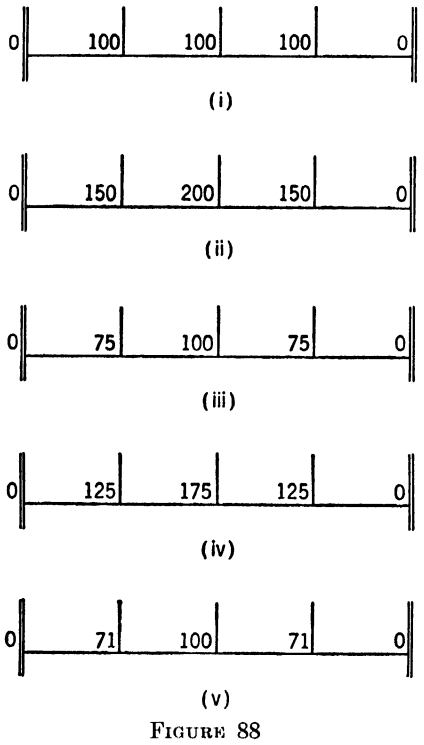


FIGURE 88

We find the solution first for a coarse subdivision of the range by three points into four intervals so that $h = \frac{1}{4}$, and as a first guess for the values of w we take 0 at the end points of the range (thereby satisfying the end conditions) and 100 at the three internal points. These values for w are shown in (i) of Figure 88. Substituting them in (11), we can then determine, by the ordinary methods of one-dimensional relaxation (cf. Chapter 4), the values for w' shown in Figure 88 (ii). These are then multiplied through by a constant in order that the greatest of them may be reduced to 100, and the resulting values are shown in Figure 88 (iii). These values may be described as constituting the second approximation in the solution for the gravest mode. The values in (iii) are put back in the formula (11) as w values; a second relaxational solution for w' is determined, as shown in (iv) of Figure 88, and reduction of the greatest value again to 100 yields the third approximation for w shown in Figure 88 (v). At this stage, correct to the number of figures quoted, the solution has converged, so that (v) is our answer for the lowest normal mode w_1. It has been obtained after only two cycles of the intensification method and is, in point of fact, identical with the exact answer (which is, of course, $w = \sin \pi x$).

121. Calculation of the Eigenvalue

When a mode has been obtained [as for instance in (v) of Figure 88], there remains, to complete the solution, the calculation of the corresponding eigenvalue; this is best done by a substitution of the modal values into a Rayleigh quotient for λ. For the case of equation (8) this quotient may be derived by multiplication through by w and integration over the range $0 \leq x \leq 1$ to yield

$$\lambda = -\frac{\int_0^1 w(d^2w/dx^2)\,dx}{\int_0^1 w^2\,dx}. \tag{13}$$

This formula has the property that it expresses λ in a form which has a stationary value as a function of w when w is a modal solution. [It can be verified by variational methods that the fraction on the right of (13) is an expression for λ in terms of w such that the condition that $\delta\lambda$ should vanish for all variations δw is precisely the equation (8). At a normal mode this equation is satisfied, and therefore λ as given by (13) has a stationary value.]

In order to apply (13) numerically, we must first replace it by a finite-difference approximation:

$$\lambda = \frac{-\Sigma w_0(w_1 + w_3 - 2w_0)}{h^2 \Sigma {w_0}^2}. \tag{14}$$

To obtain (14), we have replaced d^2w/dx^2 in the numerator of (13) by its usual approximation in terms of values of w at the points of subdivision, and we have also replaced the integrals in (13) by direct summations of the quantities involved at the various points of subdivision (it should be understood, of course, that the values of these quantities at the end points of the range of summation are included with a factor of only one-half).

When the w values given in Figure 88 (v) are substituted in the formula (14), the corresponding least eigenvalue is found to be

$$\lambda_1 = 9.37;$$

the exact answer ought, of course, to be π^2 ($=9.87$). The inaccuracy of the numerical method is no less than might have been expected in view of the small number of intervals into which the range was divided; if each interval is subdivided into two, and the solution determined when $h = \frac{1}{8}$, the eigenvalue will be found more accurately to be

$$\lambda_1 = 9.74.$$

122. The Rayleigh Method

The alternative to the intensification method of solution of eigenvalue problems is the Rayleigh method described in Section 108. It is often particularly suitable to use to find the gravest mode of solution of a differential equation for the reason that the form of that mode may be known beforehand in character if not in numerical detail. Thus the gravest normal mode of vibration of a uniformly tensioned membrane, held fixed round its edge, is known not to have any nodal lines (*i.e.*, the modal solution w_1 is everywhere of the same sign); in such a case we can always compel the Rayleigh method to determine the required mode by making a reasonable first guess for w and by not allowing any w value to change sign in the course of the actual relaxation. An example will be worked in Section 124.

The method finds a direct relaxational solution of a governing equation of type (1). Replacing that equation by its finite-difference approximation, we can define residual expressions in terms of w values and also containing λ. To begin with, the value taken for λ is that derived from the first guess for w by substitution into the Rayleigh quotient which corresponds with equation (1); with this value for λ the residuals may be calculated and their liquidation commenced. After they have been reduced as much as is easily possible, λ is recalculated by substitution of the modified w values again into the Rayleigh quotient, and this second approximation to the eigenvalue is used to yield a modified expression for the residuals; these are then recalculated and their liquidation carried further. This process is repeated until convergence is achieved. The first approximation for λ may not be particularly accurate, and when we are using it in the residual formula, we must not expect to be able to liquidate the residuals completely. That is why it was said above that in the first stage of liquidation the residuals should be reduced only "as much as is easily possible." As soon as it begins to be difficult to reduce the residuals, it becomes a waste of time and effort to try; much more is achieved at such a stage in the computation by a recalculation to determine a new and better approximation for λ. These same remarks hold true for each of the stages of liquidation in the process of solution, but it should be found in the ordinary way that, by virtue of the stationary property of the Rayleigh quotient for λ, the successive approximations to the eigenvalue converge rapidly. Consequently it is usually the fact that the residuals can be almost completely liquidated in the second or third cycle of the process.

A simple example will now explain the process better than any further attempt at description in general terms.

123. Vibration of a Stretched Membrane

For the example we choose this time the determination of the gravest normal mode of vibration of a uniformly tensioned uniform membrane; this problem is the two-dimensional counterpart of that of the vibration of a stretched string discussed in Section 119. The membrane is supposed to vibrate about a plane equilibrium position; when ρ is the mass per unit area and T is the tension per unit length in the membrane, the general equation of motion governing the transverse displacement W is

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 W}{\partial t^2}, \tag{15}$$

where W is, of course, a function of time t and of cartesian coordinates x and y in the equilibrium plane of the membrane. In any normal mode of vibration, W is of the form $w \sin pt$, where w is a function, of x and y only, which satisfies the differential equation

$$\nabla^2 w + \frac{\rho p^2}{T} w = 0. \tag{16}$$

This equation may be made nondimensional by replacement of x by Lx' and of y by Ly', where L is some representative dimension of the membrane. If at the same time we write λ to stand for $\rho p^2 L^2/T$, then equation (16) can be sufficiently written as

$$\nabla^2 w + \lambda w = 0, \tag{17}$$

with the understanding, now, that x and y are nondimensional measures of position in the membrane. Equation (17) is, of course, the particular example of equation (1) for which $P \equiv \nabla^2$ and $Q \equiv 1$.

The Rayleigh quotient for λ in the case of a membrane is analogous to its form (13) in the case of a string; it may similarly be verified to be

$$\lambda = -\frac{\int\int w \nabla^2 w \, dx \, dy}{\int\int w^2 \, dx \, dy}, \tag{18}$$

where both double integrals extend over the area of the membrane.

Using a square relaxation net of side h (also nondimensional), we replace (18) by the finite-difference approximation

$$\lambda h^2 = -\frac{\Sigma w_0(w_1 + w_2 + w_3 + w_4 - 4w_0)}{\Sigma w_0^2}, \tag{19}$$

where the Σ signs imply summation of the quantities in the numerator and denominator of (19) for all nodes of the net. The finite-difference approximation to (17), holding at a typical node of the net, can be

written down immediately; it leads to residuals being defined by the formula

$$F_0 \equiv \Sigma w_1 - (4 - \lambda h^2) w_0. \tag{20}$$

124. A Rectangular Membrane with Fixed Edges

As a numerical illustration of the application of the Rayleigh method, a rectangular membrane is taken with sides in the ratio 2:3; the shorter side is chosen as the representative dimension L, and boundary conditions are taken that the membrane be fixed at all points of its edge. This

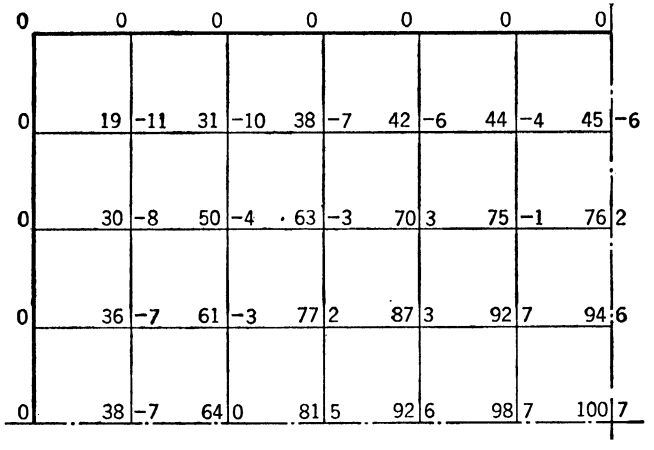


FIGURE 89

means that we are given that $w = 0$ at all nodes of the net which lie on the boundary of the rectangle. We work on a net of size $h = \frac{1}{8}$ and make use of the symmetry about both medians of the rectangle in order to be able to record the calculations on only one-quarter of the whole rectangle. As an initial guess for the values of w, we take those obtained for ψ in Figure 28 (iii) (Chapter 5) in the solution of the torsion problem for a cylinder of similar rectangular cross section. These values are reproduced in Figure 89, scaled down so that the greatest individual nodal w value is made to be 100. This distribution of w values conforms with our knowledge of the character of the wanted gravest mode: that it is one-signed throughout and likely to have a maximum value at the center of the rectangle. Otherwise there is no special significance in the values of Figure 28 (iii) relative to the present problem.

Using the w values recorded in Figure 89, we can substitute in the formula (19) to obtain a first approximation for λh^2, which may be verified to be 0.23. Hence the residual formula (20) should be taken to be

$$F_0 \equiv \Sigma w_1 - 3.77 w_0. \tag{21}$$

The next step is to apply this formula to the w values recorded in Figure 89; the corresponding residuals which result are also recorded in that diagram. We must next begin to liquidate these residuals. Strictly the pattern to be used in the relaxation ought to be that derived from (21) and shown in Figure 90; but since, in any event, we would not expect to be able to liquidate the residuals completely if this pattern were accurately used, there is, therefore, more reason than usual to employ an approximate pattern (cf. Section 65). Hence we liquidate the residuals shown in Figure 89 by use simply of the ordinary pattern corresponding to Poisson's equation, which is an approximation to the pattern of Figure 90 formed by taking the central element to be -4 instead of -3.77.

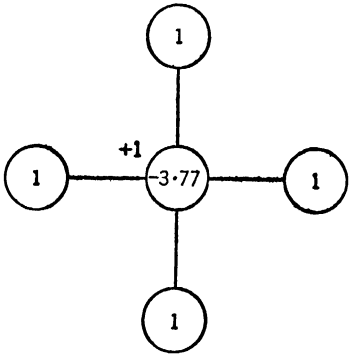


FIGURE 90

The resulting modified w values after this first stage of liquidation are shown in Figure 91, again having been scaled down so that the maximum

0		0		0		0		0		0		0	
0	11	0	21	0	29	0	35	-2	38	-1	39	0	
0	21	-3	39	-1	54	-4	63	2	69	1	71	2	
0	26	2	50	-1	69	0	82	1	90	1	93	0	
0	29	-4	54	0	75	-2	89	0	97	2	100	2	

FIGURE 91

(central) w value remains 100. From these values a second approximation for λh^2 is calculated by substitution in the Rayleigh formula (19); this second approximation is found to be 0.221. The modified residual formula is, therefore, correspondingly

$$F_0 \equiv \Sigma w_1 - 3.779 w_0,$$

and using this formula, we can recalculate the residual values; these are also recorded in Figure 91.

Now follows another stage of relaxation, and after a second cycle of the process we find the third approximation to λh^2 to be 0.221 and obtain w values and residuals which are recorded in Figure 92. At this stage in the computation the value of λh^2 has converged (to three significant figures), and increments Δw, nowhere greater in magnitude than 1, are needed to complete the liquidation of the residuals shown in Figure 92. Correct, therefore, to within ± 1, the relaxational solution for the gravest

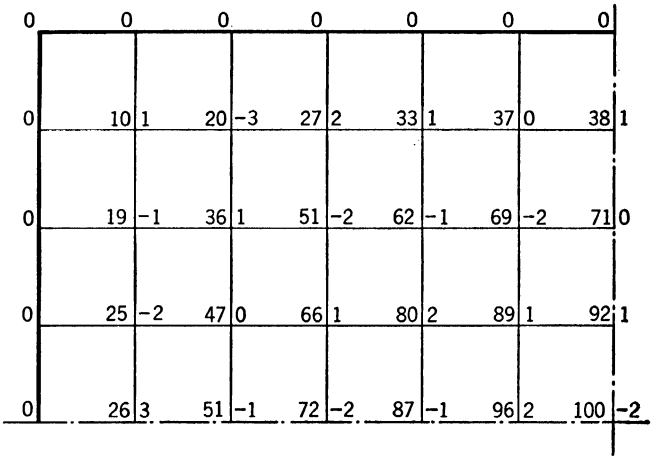


FIGURE 92

normal mode is given in Figure 92. Since $h = \frac{1}{8}$, the corresponding eigenvalue is

$$\lambda_1 = 14.14.$$

The exact answer is $13\pi^2/9$ ($= 14.26$), so that the value we have obtained, for a net of size $h = \frac{1}{8}$, is in error by rather less than 1 per cent.

125. Critical Thrust in a Strut with Clamped Ends

A third example of an eigenvalue problem which is governed by a differential equation is provided by the determination of critical thrusts for compression members. If a uniform strut of flexural rigidity EI is clamped at its ends and is subjected to a thrust P along its length, then any transverse deflection w which it could maintain must satisfy the equation

$$EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} = 0, \tag{22}$$

where w is a function of x, distance measured along the strut. We shall suppose that this equation is already expressed in nondimensional terms,

and that $x = 0$ at one end of the strut and $x = 1$ at the other end. When λ stands for P/EI, equation (22) may be rewritten as

$$\frac{d^4w}{dx^4} + \lambda \frac{d^2w}{dx^2} = 0, \tag{23}$$

which is again a particular example of an equation of the general type (1).

In order to find the gravest normal mode of solution of (23) and the corresponding lowest critical value for λ (and hence for P), we use the intensification method of repeated solution of the equation

$$\frac{d^4w'}{dx^4} + \frac{d^2w}{dx^2} = 0, \tag{24}$$

finding successive solutions w' to this equation and putting each, in turn, back into it as a new w. The finite-difference approximation to (24) is

$$4w_1' + 4w_3' - 6w_0' - w_9' - w_{11}' - h^2(w_1 + w_3 - 2w_0) = 0, \tag{25}$$

when the range $0 \leq x \leq 1$ is subdivided into equal intervals of length h. Because the absolute magnitude of the required modal solution is indeterminate, the factor h^2 in (25) can be dropped, and thus residuals may be defined at the points of subdivision by

$$F_0 \equiv 4w_1' + 4w_3' - 6w_0' - w_9' - w_{11}' + (2w_0 - w_1 - w_3). \tag{26}$$

The end conditions of clamping of the strut are expressed algebraically by the relations that

$$w' = \frac{dw'}{dx} = 0 \text{ at } x = 0,$$
$$w' = \frac{dw'}{dx} = 0 \text{ at } x = 1.$$

Thus the value of w' is given at the end points of the range of integration, and the second conditions, at each end, of zero slope of w' enable fictitious values to be eliminated from the residual formula (26) when it is applied at the two points of subdivision adjacent to the end points.

When the mode has been determined by the method of repeated intensification, the corresponding gravest eigenvalue can be calculated by substitution in the Rayleigh formula:

$$\lambda = -\frac{\int_0^1 w\,(d^4w/dx^4)\,dx}{\int_0^1 w\,(d^2w/dx^2)\,dx}. \tag{27}$$

(As for the similar formulas in the previous examples of this chapter, this may be verified by variational methods to express λ as a function of w

which has a stationary value at any normal mode.) The finite-difference replacement for (27) is

$$\lambda = \frac{\Sigma w_0(4w_1 + 4w_3 - 6w_0 - w_9 - w_{11})}{h^2 \Sigma w_0(w_1 + w_3 - 2w_0)}. \tag{28}$$

126. Intensification of the Buckling Deflection

The solution can be worked first with an interval $h = \frac{1}{8}$; with advantage taken of the available symmetry about the center of the range, this

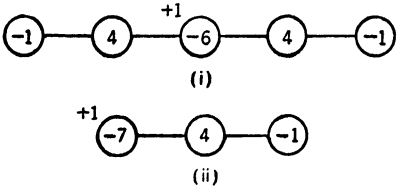


FIGURE 93

means that only four points of subdivision need be shown in the working diagram. The relaxation pattern corresponding to the residual formula (26) is shown in Figure 93 (i). The pattern (ii) in that diagram is the special pattern which applies at the point of subdivision nearest the left-hand end of the range; its construction makes use of the fact that, for that point of subdivision, w_{11} is fictitious but may be taken to be equal to w_0 by virtue of the end condition $dw/dx = 0$. As a first (and rather poor) guess for w we take its value to be 100 at all the points of subdivision inside the range; this guess is recorded in Figure 94 (i). When this w is substituted in (26) and the corresponding residuals are relaxed, values for w' are found which are recorded in Figure 94 (ii), after they have been scaled down to a maximum of 100; these values constitute the second approximation for w. Repetition of the process leads to successive approximations to the gravest mode of buckling deflection, and these are shown in Figures 94 (iii) to (vi). The values have converged, to the number of figures quoted,

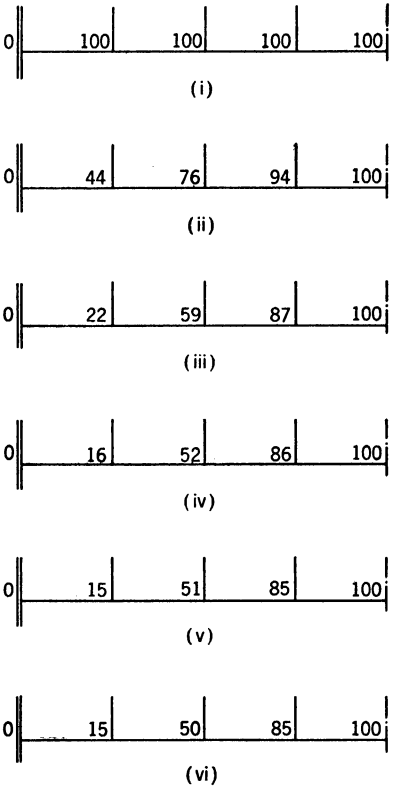


FIGURE 94

after five cycles of intensification (after only three cycles they are within ± 2 of their final values). The corresponding value of the gravest λ is found by substitution from Figure 94 (vi) into the Rayleigh formula (28); the result yielded is

$$\lambda = 37.5,$$

which is in error, according to the exact answer ($\lambda = 4\pi^2$), by about 5 per cent. A more accurate result could, of course, be obtained by a reduction of the size h of the interval between points of subdivision. However, this illustration has been taken far enough here to show how the intensification method should be applied, and it will be left to the reader to improve the results by use of a finer interval if he so desires.

127. Transverse Vibration of a Uniform Beam

The transverse motion of a uniform beam, of flexural rigidity EI and mass per unit length ρ, is governed by the equation

$$EI \frac{\partial^4 W}{\partial x^4} + \rho \frac{\partial^2 W}{\partial t^2} = 0, \tag{29}$$

where the transverse displacement W is a function of distance x along the beam and of time t. In a normal mode of free vibration, W is known to be of the form $w \sin pt$, where w is a function only of x and $p/2\pi$ is the frequency of the vibration. It follows from substitution in (29) that w must satisfy the equation

$$\frac{d^4 w}{dx^4} - \lambda w = 0, \tag{30}$$

where λ stands for $\rho p^2/EI$. The solution of a beam-vibration problem requires, then, the determination of a mode w satisfying (30) together with the corresponding eigenvalue λ. Various types of conditions may be imposed at the ends of the beam, *e.g.*, an end may be clamped ($w = dw/dx = 0$), or it may be pinned ($w = d^2w/dx^2 = 0$), or it may be free ($d^2w/dx^2 = d^3w/dx^3 = 0$).

In any particular example, either the intensification method or the Rayleigh method can be used to determine the gravest mode of vibration. If the Rayleigh method is employed, a direct solution of (30) is found by means of a replacement of that equation in finite-difference terms, and hence definition of residuals by the formula

$$F_0 \equiv 4w_1 + 4w_3 - 6w_0 - w_9 - w_{11} + \lambda h^4 w_0. \tag{31}$$

The relaxation process aims at eventually reducing the residuals at all points of subdivision to zero. In the formula (31) successively improved

approximations to λ are taken which are calculated, from time to time as the solution proceeds, from the Rayleigh formula (in nondimensional form):

$$\lambda = \frac{\int_0^1 w(d^4w/dx^4)\,dx}{\int_0^1 w^2\,dx},$$

or in finite-difference terms,

$$\lambda h^4 = -\frac{\Sigma w_0(4w_1 + 4w_3 - 6w_0 - w_9 - w_{11})}{\Sigma w_0{}^2}. \tag{32}$$

If, on the other hand, it is decided to use the intensification method, a succession of solutions for w' is found to the equation

$$\frac{d^4w'}{dx^4} - w = 0,$$

residuals being sufficiently defined by the formula

$$F_0 \equiv 4w_1' + 4w_3' - 6w_0' - w_9' - w_{11}' + w_0. \tag{33}$$

Each solution for w' is in its turn put back into (33) as a new w, and the repeated intensification causes the successive values of w' to converge to the gravest normal mode w_1. In this method the mode is found independently of the eigenvalue, which may subsequently be evaluated by substitution in (32).

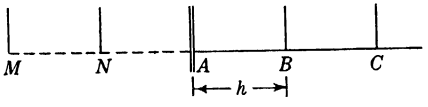


FIGURE 95

It is perhaps desirable that we should describe here, in detail, how the end conditions appropriate to a free end of a beam may be applied. Figure 95 represents the left-hand end of a beam where, say, these conditions may be given to hold; A is the end of the beam, and B and C are the first two points of subdivision along its length. Either of the residual formulas (31) or (33) applied at B will contain a fictitious value attached to the point N. This fictitious value may be eliminated by use of the first end condition, $(d^2w/dx^2)_A = 0$, which, in finite-difference terms, is

$$w_N + w_B - 2w_A = 0,$$

i.e.,

$$w_N = 2w_A - w_B. \tag{34}$$

In relation to the point of subdivision B, equation (34) gives

$$w_{11} = 2w_3 - w_0,$$

so that elimination of the fictitious w_{11} from, say, the residual formula (31) provides a modified formula for use at B:

$$F_0 \equiv 4w_1 + 2w_3 - w_9 - (5 - \lambda h^4)w_0.$$

Next we have to be able to calculate a residual at the end point A, for if the beam is free at this end, the value of w there is not known to begin with. This time either of the formulas (31) or (33), applied at A, includes two fictitious values, those at M and at N. Elimination of w_N can again be achieved by means of (34), and to eliminate w_M, we must use the second end condition $(d^3w/dx^3)_A = 0$. The finite-difference approximation to this condition is (Appendix II)

$$w_C - 2w_B + 2w_N - w_M = 0,$$

i.e., with use of (34),

$$w_M = w_C - 4w_B + 4w_A. \tag{35}$$

In relation to the end point A, equations (34) and (35) give

$$w_3 = 2w_0 - w_1,$$
$$w_{11} = w_9 - 4w_1 + 4w_0.$$

Substitution of these values in (31) yields the special residual formula for use at the point A:

$$F_0 \equiv 4w_1 - 2w_9 - (2 - \lambda h^4)w_0.$$

128. Whirling of Shafts

A problem mathematically identical to that of the vibration of a uniform beam is provided by the phenomenon of a whirling shaft. When a uniform circular shaft of flexural rigidity EI and mass per unit length ρ rotates with angular velocity ω, it normally does so with no transverse deflection of its central axis. But for certain critical, or whirling, speeds of rotation the central axis may deflect, in which case the shaft is said to whirl. The equation governing the deflection w as a function of distance x along the shaft is

$$EI \frac{d^4w}{dx^4} = \rho\omega^2 w,$$

which, if we write λ to stand for $\rho\omega^2/EI$, is exactly the same equation as (30). The whirling speed ω corresponds exactly with the circular frequency p in a normal vibration of a beam, and the whirling deflection of a shaft corresponds similarly with the modal deflection of a vibrating beam. Consequently, any solution for the gravest normal mode of a vibrating beam obtained by the methods outlined in Section 127 can also be inter-

preted as a solution for the whirling of a shaft. The end conditions of support of a beam are also directly comparable with the kinds of support conditions which may apply for a rotating shaft. Thus the conditions of clamping for a beam are the same as the conditions of support for a shaft running in a long bearing; the conditions of pinning for a beam are identical with the conditions of support for a shaft running in a short bearing; and the conditions at a free end of a beam are the same as the conditions at the free end of a shaft.

129. Electromagnetic Vibrations in a Wave Guide

A mathematical analogy also exists between the problem of the vibration of a membrane and that of electromagnetic oscillation in a uniform conducting tube. If an axis Oz is taken parallel to the generators of the tube (which may be of any cross-sectional shape) and axes Ox and Oy are taken in the plane of a cross section, then the electromagnetic oscillation is composed of two particular types of simple oscillation which may be superposed; these two particular oscillations are known respectively as E waves and H waves.* In an E wave, the axial component of magnetic force, H_z, vanishes, and the solution is expressible in terms of the axial component of electric force, E_z. The governing equation for E_z is

$$\nabla^2 E_z = \epsilon\mu \frac{\partial^2 E_z}{\partial t^2}, \tag{36}$$

where ϵ is the dielectric constant and μ the magnetic permeability of the medium. Equation (36) is of exactly the same type as the equation (15) governing the transverse motion of a stretched membrane. In a normal mode of oscillation of frequency $p/2\pi$, E_z is of the form $e_z \sin pt$, where e_z is a function of x and y only and satisfies the equation

$$\nabla^2 e_z + \lambda e_z = 0,$$

where λ stands for $\epsilon\mu p^2$. The boundary condition which e_z must satisfy on the surface of the *conducting* tube is, of course, that $e_z = 0$; this condition compares exactly with the condition for a membrane that it be fixed at its edge.

In an H wave, the axial component of electric force, E_z, vanishes, and H_z may be shown then to satisfy (36); in a normal mode of oscillation $H_z = h_z \sin pt$, so that

$$\nabla^2 h_z + \lambda h_z = 0.$$

The boundary condition for h_z is not so simple to apply as that for e_z; it is

* The theoretical results stated here are derived in a paper by Allen, Fox, Motz, and Southwell, *Phil. Trans. R.S. C*, **1**, 85–97 (1942), and *A*, **239**, 488–500 (1945). A number of solutions by the relaxational (Rayleigh) method are also given in this paper.

that $\partial h_z/\partial \nu = 0$, *i.e.*, a normal-gradient condition, ν being the direction of the normal at any point of the cross section of the tube.

130. More Difficult Eigenvalue Problems

In this chapter the important application of relaxation methods to solve eigenvalue problems has been exemplified by reference to a variety of physical problems all requiring the solution of an equation of the type (1). The numerical technique has been illustrated by means of the actual solutions of some comparatively simple cases—these cases are described as simple because each of them is capable of exact solution by an orthodox analytical approach. They have been chosen partly for the reason that their results can be easily verified, and partly because their aim was to illustrate our methods sufficiently but with a minimum of computational distractions. The value of the methods is, of course, that they can be immediately applied to solve more difficult problems, problems which are more difficult because they are not capable of exact analytical solution; from a relaxational point of view, the greater difficulty, if any, lies only in additional computation.

Thus in two-dimensional problems it is clear that our methods are able to solve eigenvalue problems, of the type discussed, for any given shape of boundary. In one dimension it is more likely that a problem could, if necessary, be solved analytically, but one example here will serve to indicate how more difficult problems could arise in one dimension. For orthodox analytical methods, great, if not insuperable, complications are introduced, but to a relaxational approach scarcely any additional labor is involved. The illustration concerns the vibration of a string, for which the nondimensional governing equation (8) is

$$\frac{d^2w}{dx^2} + \frac{\rho p^2 L^2}{T} w = 0. \tag{37}$$

In Section 119 we supposed the string to be of uniform line density, but if now ρ is given to be a variable function of x, we must make some slight modification to the method previously described. We should now let ρ_0 be the average mass per unit length and define λ to be $\rho_0 p^2 L^2/T$ and introduce a nondimensional density $\rho' = \rho/\rho_0$. Equation (37) may then be rewritten as

$$\frac{d^2w}{dx^2} + \rho\lambda w = 0, \tag{38}$$

when, as before, we drop the prime from ρ' after nondimensionalization. The intensification method now solves repeatedly the equation

$$\frac{d^2w'}{dx^2} + \rho w = 0,$$

instead of (9), and the Rayleigh quotient for the eigenvalue is now

$$\lambda = -\frac{\int_0^1 w(d^2w/dx^2)\,dx}{\int_0^1 \rho w^2\,dx},$$

instead of (13). Scarcely any additional labor is involved, because numerical values for ρ will be known at the points of subdivision along the range $0 \leq x \leq 1$, and these values have only to be inserted as additional factors when residuals are calculated from the formula

$$F_0 \equiv w_1' + w_3' - 2w_0' + \rho_0 w_0,$$

and when λ is calculated from the expression

$$\lambda = \frac{-\Sigma w_0(w_1 + w_3 - 2w_0)}{h^2 \Sigma \rho_0 w_0^2},$$

respectively, instead of (11) and (14).

Similar examples of more difficult one-dimensional problems could easily be instanced, such as the determination of the critical thrust for a nonuniform strut and the gravest mode of vibration of a nonuniform beam.

131. Higher Normal Modes

It was pointed out in Section 117 that, for the type of problem discussed in this chapter, the gravest of the normal modes is the one of greatest interest; and we have, in fact, in our examples, shown only how that gravest mode may be determined. If higher normal modes are required to be found also, this may be done by a method similar to that described in Section 115 for the solution of eigenvalue problems governed by a set of algebraic equations. With use of an appropriate orthogonal property the intensification method can be employed to yield successively the second, third, etc., normal modes; the forms of the orthogonal property for the various types of problems introduced in this chapter are given in Appendix IV.

PROBLEMS

1. Find the lowest critical thrust and mode of deflection for a strut, of length L and flexural rigidity EI, which is clamped at one end and pinned at the other end. Use successive intervals of subdivision $h = L/8$ and $h = L/16$, and employ the Rayleigh method.

2. Find the gravest mode of vibration and the corresponding frequency of vibration of a uniform beam, of length L, flexural rigidity EI, and mass per unit

length ρ, which is free at both ends. Use successive intervals of subdivision $h = L/8$ and $h = L/16$, and employ the intensification method.

(Do not *assume* for the initial guess that the mode is either symmetrical or antisymmetrical.)

3. Find the lowest whirling speed of a shaft, of length L, flexural rigidity EI, and mass per unit length ρ, which runs in short bearings at both ends. Use successive intervals of subdivision $h = L/8$, $h = L/16$, and $h = L/32$.

(Take advantage of the obvious fact that the required mode is symmetrical.)

4. A square membrane of side L is uniformly stretched to a tension T per unit length. The density of the membrane at any point is $\rho = \rho_0(1 + 4xy/L^2)$ per unit area, where x and y are cartesian coordinates parallel to the sides of the membrane with an origin at its center. Find the gravest mode of transverse vibration; use an ultimate net of side $h = L/16$, and find w values correct to the nearest whole number when the maximum w value is 1,000.

(Use may be made of the symmetry of the gravest mode about the diagonals of the membrane. The reader can compare his solution with that given by Allen, Fox, Motz, and Southwell in the reference cited in the footnote on page 182.)

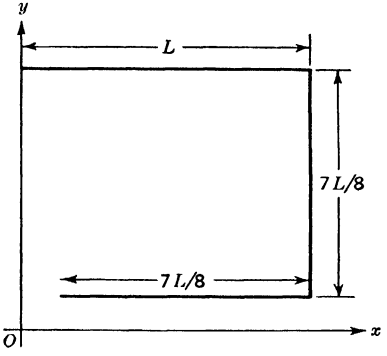


FIGURE 96

5. Figure 96 represents one-quarter of a section of a conducting tube. Find the gravest mode of oscillation for an E wave of the type for which Oy is a line of symmetry ($\partial h_z/\partial x = 0$) and Ox is a line of antisymmetry ($h_z = 0$). Use an ultimate net of size $h = L/16$, and make the maximum h_z value 1,000, working then to the nearest whole number.

(The solution to this problem may be found, for comparison, in the paper by Allen, Fox, Motz, and Southwell referred to in the footnote on page 182.)

CHAPTER 13

INTERNAL BOUNDARIES AND INTERFACES

132. Introduction

All the applications of the relaxation method to solving partial differential equations that have been described in previous chapters have had in common the fact that their domain of solution has been a singly connected region. When this is not the case (*i.e.*, when there may be one or more closed internal boundaries, or "holes," within the area of integration), the complete determination of the solution often requires the satisfaction of extra conditions. These conditions are usually ones of single-valuedness for some quantity which is derivable from the solution, and the relaxation method is peculiarly fitted to deal with such conditions and to incorporate them into its standard processes. The necessary extension of the method will be illustrated in this chapter by treatments of the torsion problem for a uniform hollow shaft, of the extensional problem for a flat plate containing holes, and of the stressing problem for an axially symmetric solid which contains cavities.

Another, not dissimilar, extension of the processes of relaxation enables solutions to be found when the area of integration contains internal interfaces subdividing it into two or more parts, where special conditions have to be satisfied across each interface; an example of this occurrence is provided by the phenomenon of refraction in plane-potential problems (*e.g.*, electric, magnetic, or thermal refraction).

133. Torsion of a Hollow Cylinder

The torsion problem for a solid cylinder was discussed in Section 46; we use the same notation now for the case of a hollow cylinder of any cross section. Figure 97 represents a typical cross section with an external boundary E and an internal boundary I; a square relaxation net is shown covering the entire region *including the hole within the internal boundary*. As for a cylinder of solid section, the solution for the torsion of a hollow cylinder is obtained in terms of a stress function ϕ, which again must satisfy the Poisson-type equation

$$\nabla^2\phi + 2 = 0, \tag{1}$$

at all points of the annulus between the boundaries E and I. The only nonzero components of stress, τ_{zx} and τ_{yz}, remain expressed in terms of ϕ by the relations (7) of Chapter 5, and the boundary conditions require that ϕ be constant on both the boundaries E and I. As for a solid cylinder, the constant value of ϕ on the boundary E may sufficiently be taken to be zero, but we cannot then also arbitrarily choose the constant value of ϕ on the boundary I. We suppose its value to be k and now investigate how k may be determined as part of the computational solution.

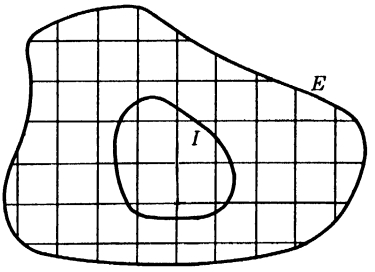


FIGURE 97

To do this requires that we introduce the components of displacement u, v, and w (in the axial directions Ox, Oy, and Oz, respectively) of a point in the twisted cylinder. The equations of stress equilibrium are automatically satisfied by the expressions for the stress components τ_{zx} and τ_{yz} in terms of ϕ; satisfaction of the strain-compatibility equations can be verified to imply that u and v are equal, respectively, to $-\tau yz$ and to τzx, and hence that

$$\begin{aligned} \frac{\partial w}{\partial x} &= \tau\left(\frac{\partial \phi}{\partial y} + y\right), \\ \frac{\partial w}{\partial y} &= -\tau\left(\frac{\partial \phi}{\partial x} + x\right). \end{aligned} \tag{2}$$

For a solid cylinder, the governing equation (1) for ϕ follows immediately by elimination of w from the equations (2). For a hollow cylinder, we proceed somewhat differently: Let C be any closed curve lying within E and without I (C may or may not encircle I), and denote arc length measured along the curve C by s. Then for a consistent single-valued w to exist we must have

$$\oint_C \frac{\partial w}{\partial s}\, ds = 0,$$

i.e.,

$$\oint_C \left(\frac{\partial w}{\partial x}\, dx + \frac{\partial w}{\partial y}\, dy\right) = 0.$$

Therefore, on substitution from (2),

$$\oint_C \left[\left(\frac{\partial \phi}{\partial y} + y\right) dx - \left(\frac{\partial \phi}{\partial x} + x\right) dy\right] = 0,$$

so that, by Green's theorem,

$$\iint_C (\nabla^2\phi + 2)\, dx\, dy = 0. \tag{3}$$

Now, in (3), since we may take C to be any curve not encircling I, it follows that at all points in the annulus between E and I the governing equation (1) must be satisfied. But also we may take C to be any curve encircling I, and, in particular, we may choose C to be the internal boundary I. In that case we find that

$$\iint_I (\nabla^2\phi + 2)\, dx\, dy = 0. \tag{4}$$

In the foregoing we have tacitly assumed that the function ϕ exists inside I as well as in the region between E and I; in fact ϕ does exist and is not even unique inside I, for all that we have been able to demand of ϕ within I is that (4) must hold, and that ϕ and its derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ should be continuous across I.

134. Hole Relaxation

The relaxational process of solution of the torsion problem for a hollow shaft, in fact, determines numerically inside I any one of the possible ϕ's. We define a residual F_0 at any node of the net, either inside or outside I, by the usual formula

$$F_0 \equiv \Sigma\phi_1 - 4\phi_0 + 2h^2; \tag{5}$$

then satisfaction of (1) in the region between E and I requires that the residual F_0 should be made to vanish at all nodes of the net which lie within this annular region; and satisfaction of (4) within I simply requires that the sum F_I of all the residuals F_0 at the nodes within the internal boundary should be made to vanish. Individually there need be no restriction on the values of residuals at nodes within I, provided only that their sum F_I is eventually brought to zero. Thus a comparatively straightforward relaxational problem is provided for the whole area inside E; the solution is subject only to the conditions stated above and to the condition that, on the curve I, ϕ should have some constant value k.

We must next describe how, in computation, this last condition can be achieved. Figure 98 represents a part of the internal boundary with a typical string of the net crossing it; the end of the string in the annular region is denoted by A, the end in the hole inside the internal boundary by H, and the point of intersection of the string and the internal boundary by I. Corresponding values of ϕ at these points are indicated by appro-

priate suffixes, and we take the length AI to be a fraction ξ of the mesh length h. At once, then, linear interpolation along AH yields

$$k = \phi_I = \xi\phi_H + (1 - \xi)\phi_A,$$

i.e.,

$$\phi_H = \frac{k - (1 - \xi)\phi_A}{\xi}. \tag{6}$$

To begin the computation, values of ϕ may be guessed at all nodes in the annulus; a value of k is also guessed, and then, according to (6), ϕ values at nodes typified by H may be immediately calculated. Values

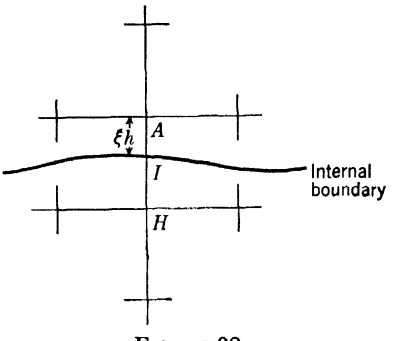


FIGURE 98

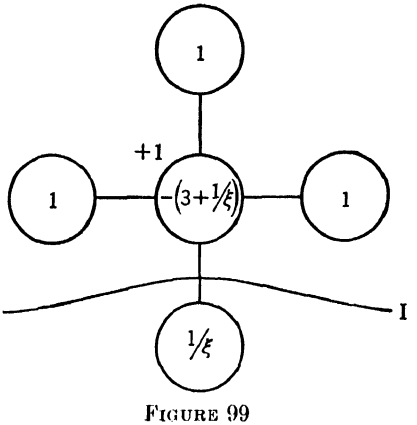


FIGURE 99

of ϕ at other nodes within the hole may be arbitrarily chosen. The next step is to evaluate residuals at all nodes by use of the formula (5) and hence to determine residuals F_0 at every node in the annulus and the total residual F_I at the nodes in the hole. The process of liquidation is then started; for nodes in the annulus the standard relaxation pattern (Figure 22 of Chapter 5) is used, with the exception that a special pattern must be constructed for use at nodes like A (Figure 98) which are "adjacent" to the hole. Also, in order to liquidate the *hole residual* F_I, we must investigate a special kind of unit increment $\Delta k = 1$.

We consider first the pattern appropriate to a node such as A. From (6) it is clear that a unit increment $\Delta\phi_A = 1$ should be automatically accompanied by an increment $\Delta\phi_H = -(1 - \xi)/\xi$. The resultant effect, on the residuals, of this complete operation can easily be verified to be that indicated in the pattern of Figure 99. In that pattern the element $1/\xi$ is the resultant change in the hole residual F_I. (Some obvious modifications would have to be made to deal with the cases when two or more strings radiate from an annular node A across the internal boundary, or when two or more strings cross this boundary and meet at the same

internal node H; it is not thought necessary to exemplify those possibilities here.)

Next the pattern corresponding to the operation $\Delta k = 1$ must be determined. From (6) it may be seen that this operation must be automatically accompanied by increments $\Delta\phi_H = 1/\xi$ at each internal node, typified by H (Figure 98), which is adjacent to the internal boundary. Although again slight modifications must be made to deal with cases where two or more strings radiate across the internal boundary from the same internal node H, we can here sufficiently give the pattern in general as that shown in Figure 100. In that figure the element $-\Sigma(1/\xi)$ is the change produced in the hole residual F_I consequent upon the increment $\Delta k = 1$, and the element $1/\xi$ represents the change in the residual at a typical annular node A.

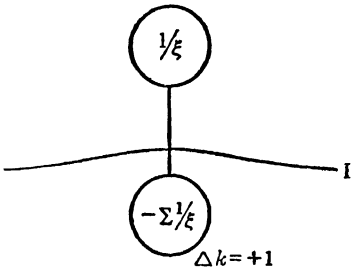


FIGURE 100

In the actual relaxation process the residuals F_0 at all the nodes in the annulus and the hole residual F_I are systematically liquidated by means of the application of increments to the ϕ values at individual nodes in the annulus and to k. During this liquidation it is not necessary to keep any account of the changing ϕ values at points like H, but, of course, before each check or recalculation of the residuals is performed, equation (6) must be used to bring each ϕ_H up to date.

In the initial stages of the computation before the solution is approaching convergence, and especially if a coarse net is being used, it is possible that differing ϕ values may be calculable at individual points H when they are at the ends of two or more strings which cross the hole boundary. In such an event the average value should be used pro tem. The anomaly must automatically dissipate itself as the solution converges and as finer and finer nets are employed.

The reader will realize that the process described above, which may be termed *hole relaxation*, is in its practice a very similar process to block relaxation, the hole being treated throughout effectively as a single block of nodes and an increment Δk being a kind of block increment. Because of this similarity no actual numerical illustration of hole relaxation is called for here.

135. Flat Plates Containing Holes

Another type of problem, which demands a technique similar to that described in the last section, is provided in the stretching of flat plates

which contain holes. Figure 97 (Section 133) can now be taken to represent a uniform flat plate with an external boundary E and a hole within an internal boundary I. The simpler case of a flat plate with no hole was discussed in Section 75; we use the same notation here, and the solution is again determined in terms of a biharmonic stress function χ. The stress components σ_x, σ_y, and τ_{xy} remain expressed in terms of χ by the relations (10) of Chapter 8. As in Section 75, edge conditions of known tractions on the external boundary yield values there of χ, $\partial\chi/\partial x$, and $\partial\chi/\partial y$. In the integrations round the boundary E to determine each of these quantities, the arbitrary constants of integration can be chosen at our convenience [because an addition to χ *everywhere* of any function of the form $\alpha + \beta x + \gamma y$, where α, β, and γ are constants, makes no difference to the stresses as calculated from (10) of Section 75]. But having chosen these constants, we cannot also arbitrarily choose the corresponding constants when the edge conditions of given tractions are integrated round the internal boundary I to find values there for χ, $\partial\chi/\partial x$, and $\partial\chi/\partial y$. All that can be done, at the start, is to determine values for these quantities on the boundary I which are relatively correct; the actual value of χ on this boundary will contain in addition a function $A + Bx + Cy$, and we have to investigate now how the constants A, B, and C may be determined as part of the computational solution. The problem is seen to involve a threefold extension of the technique described for solution of the torsion problem for a hollow cylinder. There (Section 133) we had to determine one constant, k, on the internal boundary, whereas now we have three constants, A, B, and C, to find. We may expect to be able to discover three conditions for this purpose, just as in Section 133 the condition (4) was derived and later used in order to discover the appropriate value for k.

136. Conditions for Single-valued Displacements

These three conditions are, in fact, those which ensure that the stress-strain relations may be integrated to yield single-valued displacements (in the plane of the plate). When these displacements in the axial Ox,Oy directions are taken to be u and v, respectively, the stress-strain relations, under conditions of plane stress, are

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{E}\left(\frac{\partial^2\chi}{\partial y^2} - \sigma\frac{\partial^2\chi}{\partial x^2}\right), \\ \frac{\partial v}{\partial y} &= \frac{1}{E}\left(\frac{\partial^2\chi}{\partial x^2} - \sigma\frac{\partial^2\chi}{\partial y^2}\right), \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= -\frac{2(1+\sigma)}{E}\frac{\partial^2\chi}{\partial x\,\partial y}, \end{aligned} \tag{7}$$

where E is Young's modulus and σ is Poisson's ratio for the material of the plate. In the ordinary way, for a plate without a hole, u and v may be eliminated from (7) to yield the governing equation

$$\nabla^4\chi = 0; \tag{8}$$

but when the plate contains a hole, it is necessary to proceed differently. First of all, differentiation of the first of (7) with respect to y, and elimination of v between the second and third of (7) yield

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) &= \frac{1}{E}\left(\frac{\partial^3\chi}{\partial y^3} - \sigma\frac{\partial^3\chi}{\partial x^2\,\partial y}\right), \\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) &= \frac{1}{E}\left(\frac{\partial^3\chi}{\partial x^3} + \overline{2+\sigma}\,\frac{\partial^3\chi}{\partial x\,\partial y^2}\right). \end{aligned} \tag{9}$$

If now C is any closed curve which lies between E and I and which may or may not encircle I, the condition that (9) should integrate to give a single-valued $\partial u/\partial y$ is

$$\oint_C \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)dy\right] = 0,$$

i.e.,

$$\oint_C \left[\left(\frac{\partial^3\chi}{\partial y^3} - \sigma\frac{\partial^3\chi}{\partial x^2\,\partial y}\right)dx - \left(\frac{\partial^3\chi}{\partial x^3} + \overline{2+\sigma}\,\frac{\partial^3\chi}{\partial x\,\partial y^2}\right)dy\right] = 0,$$

which, by Green's theorem, may be transformed into the condition

$$\iint_C \nabla^4\chi\,dx\,dy = 0. \tag{10}$$

This same condition (10) can similarly be shown to be necessary in order that $\partial v/\partial x$ also be single-valued.

Having established the condition which ensures that $\partial u/\partial y$ is single-valued, and being given $\partial u/\partial x$ in a single-valued form in the first of (7), we can now proceed to establish the condition that u itself be single-valued; it is

$$\oint_C \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) = 0;$$

i.e., with substitution from the first of (7) in the first term of this integral, and integration by parts of the second term,

$$\oint_C \left[\frac{1}{E}\left(\frac{\partial^2\chi}{\partial y^2} - \sigma\frac{\partial^2\chi}{\partial x^2}\right)dx - y\,\frac{\partial^2 u}{\partial y^2}\,dy\right] = 0.$$

We now substitute in the second term for $\partial^2 u/\partial y^2$ from the second of (9) to obtain

$$\oint_C \left[\left(\frac{\partial^2 \chi}{\partial y^2} - \sigma \frac{\partial^2 \chi}{\partial x^2} \right) dx + y \left(\frac{\partial^3 \chi}{\partial x^3} + \overline{2+\sigma} \frac{\partial^3 \chi}{\partial x \, \partial y^2} \right) dy \right] = 0,$$

and then Green's theorem gives

$$\iint_C \left[y \left(\frac{\partial^4 \chi}{\partial x^4} + \overline{2+\sigma} \frac{\partial^4 \chi}{\partial x^2 \, \partial y^2} \right) - \left(\frac{\partial^3 \chi}{\partial y^3} - \sigma \frac{\partial^3 \chi}{\partial x^2 \partial y} \right) \right] dx \, dy = 0,$$

or, after integration by parts with respect to y of the third-order terms,

$$\iint_C y \, \nabla^4 \chi \, dx \, dy = 0. \tag{11}$$

In a similar manner we can establish the condition that v be single-valued in the form

$$\iint_C x \, \nabla^4 \chi \, dx \, dy = 0. \tag{12}$$

Since C may first be taken as any closed curve between E and I which does not encircle I, any of (10), (11), or (12) proves that the governing equation (8) must hold throughout the annulus. Second, C may also be taken to encircle I and, in particular, may be chosen to be I itself; then (10), (11), and (12) become, in finite-difference terms,

$$\begin{aligned} \Sigma_I F_0 &= 0, \\ \Sigma_I x_0 F_0 &= 0, \\ \Sigma_I y_0 F_0 &= 0, \end{aligned} \tag{13}$$

where Σ_I denotes summation at all nodes of the relaxation net within I and F_0 is the residual, at any such node, defined in the ordinary way for the biharmonic equation as

$$F_0 \equiv 8\Sigma\chi_1 - 20\chi_0 - 2\Sigma\chi_5 - \Sigma\chi_9. \tag{14}$$

The eventual satisfaction of the three conditions (13) enables us to determine the constants A, B, and C in Section 135; precisely the same conditions (13) can be verified to hold under circumstances of plane strain in the plate.

137. Relaxational Technique for a Plate with a Hole

The relaxational technique of solution for χ in the case of a plate containing a hole will now be outlined. It is an extension of the technique described in Section 134.

First the edge conditions of known tractions on the external boundary E should be integrated in the manner of Section 75 to find boundary values on E of χ, $\partial\chi/\partial x$, and $\partial\chi/\partial y$; these values are applied throughout the computation in the manner described in Section 83. Similarly the edge conditions of known tractions on the internal boundary I can be integrated to find boundary values on I of χ, $\partial\chi/\partial x$, and $\partial\chi/\partial y$; these will be indeterminate to the extent of an additive constant C in $\partial\chi/\partial y$, of an additive constant B in $\partial\chi/\partial x$, and of an additive function $A + Bx + Cy$ in χ. To begin the numerical solution, some guess must be made for the values of the arbitrary constants A, B, and C; an initial guess is also required for the χ values at nodes of the relaxation net within the annulus between E and I.

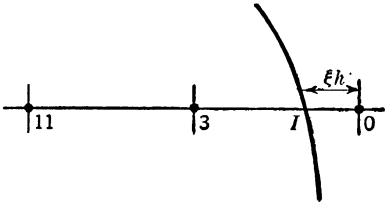


FIGURE 101

Next we can calculate compatible values of χ at nodes which lie not more than two mesh lengths within I (these nodes may be regarded as fictitious points with respect to the annular region). Figure 101 represents a part of the internal boundary which is cut by a typical mesh line (in the x direction) at the point I; 0 is the adjacent mesh point in the annulus, and 3 and 11 are nodes within the hole. We now express the values χ_3 and χ_{11} in terms of the already guessed value χ_0 and of the partially (in respect of A, B, and C) guessed values of χ and $\partial\chi/\partial x$ at the point I; we denote these, respectively, by K and k, and also suppose that the length $I0$ is a fraction ξ of the mesh size h. Then by expansion of χ in a Taylor's series of powers of $x - x_I$ along the line $3I0$, and with neglect of terms of $O(h^3)$, it is possible to show (cf. Section 83) that

$$\begin{aligned} \chi_3 &= -\frac{1-2\xi}{\xi^2}K - \frac{1-\xi}{\xi}kh + \left(\frac{1-\xi}{\xi}\right)^2\chi_0, \\ \chi_{11} &= -\frac{4(1-\xi)}{\xi^2}K - \frac{2(2-\xi)}{\xi}kh + \left(\frac{2-\xi}{\xi}\right)^2\chi_0. \end{aligned} \tag{15}$$

Formulas of the type (15) enable pairs of values like χ_3 and χ_{11} to be found at nodes just inside the hole on all mesh lines which cross the internal boundary; some nodes may lie on two such lines, and there two differing values of χ may be calculable. In such cases the average χ should be used, and as the solution converges, on a reasonably fine net, it may be anticipated that such pairs of values will not differ by any appreciable amount.

We have now reached a position where χ values have been guessed at

nodes in the annulus, and where consequent χ values have been calculated from formulas of the type (15) at a number of nodes within the internal boundary. At any other nodes inside this boundary (*i.e.*, at nodes distant more than two mesh lengths inside I), arbitrary values of χ can be assigned. The next computational step is the calculation of residuals according to the formula (14) at every node, both inside and outside of I. For the nodes within I we need to calculate individual residuals only in order to determine the values of the three quantities:

$$\begin{aligned} F_I &\equiv \Sigma F_0, \\ F_{Ix} &\equiv \Sigma x_0 F_0, \\ F_{Iy} &\equiv \Sigma y_0 F_0. \end{aligned} \tag{16}$$

The liquidation of the residuals can now be started. The ultimate aim is to reduce each individual F_0 within the annulus to zero, but within the hole, in order to satisfy the conditions (13), it is necessary to reduce only the quantities F_I, F_{Ix}, and F_{Iy}, defined in (16), to zero. To achieve this, ordinary biharmonic relaxation is employed in the annular region between E and I, and for liquidation of the three hole residuals F_I, F_{Ix}, and F_{Iy}, we have available the three unit operations $\Delta A = 1$, $\Delta B = 1$, and $\Delta C = 1$. It is unhelpful to attempt, in general terms, to express the effects, on the point residuals F_0 and on the hole residuals F_I, F_{Ix}, and F_{Iy}, of these three unit operations; in any particular example their separate effects can easily be discovered by means of individual trials. In principle their use is identical to the use of the unit operation $\Delta k = 1$, described in some detail in Section 134.

During the actual liquidation process there is no reason to attempt to record the modifications, to the χ values at nodes inside the hole, that must automatically accompany both the application of increments $\Delta\chi$ at nodes in the annulus and the application of increments ΔA, ΔB, and ΔC. From time to time in the course of the work the residual values should be checked, and each time, before this is done, the opportunity should be taken to bring the χ values inside the hole up to date according to relations of the type (15).

138. Cavities in an Axially Symmetric Solid

The stressing of an axially symmetric solid of revolution under an axially symmetric loading was discussed in Chapter 9; the possibility was not contemplated there that the solid might contain a cavity so that an axial section would exhibit an internal boundary I as well as an external boundary E. If, however, we now consider such an occurrence, the four nonzero components of stress may be again expressed in the form

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\left(\frac{\partial\phi}{\partial r} + \frac{\partial\psi}{\partial r}\right) - \frac{1}{r^2}(\psi + \overline{1-\sigma}\phi),\\
\sigma_\theta &= \frac{\sigma}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}(\psi + \overline{1-\sigma}\phi),\\
\sigma_z &= -\frac{1}{r}\frac{\partial\psi}{\partial r},\\
\tau_{zr} &= \frac{1}{r}\frac{\partial\psi}{\partial z},
\end{aligned} \tag{17}$$

in terms of the two stress functions ϕ and ψ. Then the equations of stress equilibrium require only that ϕ and ψ shall satisfy the equation

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\phi}{\partial r^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} = 0. \tag{18}$$

This equation is equivalent to one of the two equations (9) of Chapter 9 which were given in that chapter as the pair of simultaneous equations governing ϕ and ψ. To discover the second equation, we must write down the stress-strain relations in terms of the components of displacement u and v in the axial and radial directions, respectively; these are

$$\begin{aligned}
Ev &= \frac{1+\sigma}{r}(\psi + \overline{1-\sigma}\phi),\\
E\frac{\partial u}{\partial z} &= -\frac{1+\sigma}{r}\frac{\partial}{\partial r}(\psi + \sigma\phi),\\
E\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) &= \frac{2(1+\sigma)}{r}\frac{\partial\psi}{\partial z}.
\end{aligned} \tag{19}$$

The first of (19) ensures that v is single-valued if single-valued functions ψ and ϕ are obtained; and elimination of v between the first and third of (19) gives

$$E\frac{\partial u}{\partial r} = \frac{1+\sigma}{r}\frac{\partial}{\partial z}(\psi - \overline{1-\sigma}\phi). \tag{20}$$

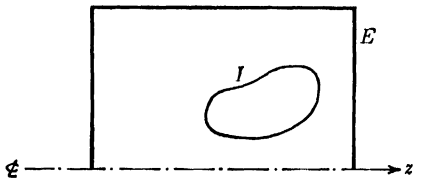


FIGURE 102

For a singly connected axial section, elimination of u between (20) and the second of (19) provides in effect the second of the equations (9) of Chapter 9. For the case when the axial section contains an internal boundary, as typified in Figure 102, we must ensure that u, in addition to v, is single-valued. If C is any closed curve in the region between E and I, we must have, therefore, that

$$\oint_C \frac{\partial u}{\partial s}\,ds = \oint_C \left(\frac{\partial u}{\partial r}\,dr + \frac{\partial u}{\partial z}\,dz\right) = 0,$$

which, after substitution from (20) and the second of (19) and application of Green's theorem, yields

$$\iint_C \left[\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} \right) - (1-\sigma)\left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right) \right] dr\, dz = 0, \quad (21)$$

or

$$\iint_C \left[\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \phi}{\partial z^2} \right) + \sigma \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right) \right] dr\, dz = 0. \quad (22)$$

If we first take C to be a curve which does not encircle the inner boundary I, then (21) combined with (18) shows that the equations governing ϕ and ψ may be taken in their simplest forms to be

$$\begin{aligned} \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} &= 0, \\ \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} &= \frac{\partial^2 \phi}{\partial z^2}. \end{aligned} \quad (23)$$

These are the equations (9) of Chapter 9 again, and at nodes of a relaxation net in the annular region between E and I they lead to a pair of residuals defined by the formulas

$$\begin{aligned} F_\phi &\equiv \phi_1 + \phi_3 + \left(1 - \frac{h}{2r_0}\right)\phi_2 + \left(1 + \frac{h}{2r_0}\right)\phi_4 - 4\phi_0, \\ F_\psi &\equiv \psi_1 + \psi_3 + \left(1 - \frac{h}{2r_0}\right)\psi_2 + \left(1 + \frac{h}{2r_0}\right)\psi_4 - 4\psi_0 \\ &\qquad + 2\phi_0 - \phi_1 - \phi_3. \end{aligned} \quad (24)$$

Boundary conditions of known tractions may be assumed to be given on both E and I; it was seen in Section 97 that one of the two conditions can be integrated round a boundary to determine values there of the stress function ψ. On the external boundary E the arbitrary constant in this integration may be chosen to suit our convenience, but when this has been done, it is not permissible also to fix arbitrarily the constant of integration, K, which is contained in the integrated expression for ψ on the internal boundary I. Instead we must be prepared to find K as part of the solution by first guessing a value for it and then adjusting that guess by means of increments ΔK. The unit operation $\Delta K = 1$ is available and must be used in order eventually to reduce to zero a hole residual. This hole residual is defined, from the finite-difference approximation to (22), to be

$$F_I \equiv \Sigma(F_\psi + \sigma F_\phi), \quad (25)$$

where F_ϕ and F_ψ are defined in (24) and the Σ sign implies summation at all nodes within the hole. In brief, the relaxational problem requires that each of F_ϕ and F_ψ be liquidated to zero at all nodes between E and I, by means of the relaxational technique of solution of a pair of simultaneous differential equations (Section 99); but within I it is required only that F_I be liquidated to zero. At nodes in the hole the values of ϕ and ψ may be arbitrarily assigned except in so far as they must be compatible with the boundary conditions given to hold on I.

139. Refraction at an Internal Interface

Another difficulty, similar to that presented by the existence of holes or cavities within the region of integration, has to be faced in potential problems which include an internal interface across which there is refraction. Suppose that we have to determine a potential function satisfying Laplace's equation and that the region of integration is made up of two parts, a and A, with a common boundary or interface I. It will be convenient to denote the potential function by ϕ at points within a and by Φ at points within A. The value of this function is required to be continuous across the interface, so that, on I, $\phi = \Phi$. At any point of I we take ν to represent the normal direction from a into A, and the sort of refraction condition which we contemplate here is that, on I,

$$\mu \frac{\partial \phi}{\partial \nu} = M \frac{\partial \Phi}{\partial \nu}, \tag{26}$$

where μ and M are constants associated, respectively, with the regions a and A (thus, for example, in a magnetic potential problem, μ and M would be the permeabilities of the different materials which occupy the two regions). Clearly some special formulas for residuals will have to be constructed to hold at nodes of any relaxation net in the neighborhood of the interface; there is considerable simplification of the problem in the special, but nevertheless important, case when the interface is a straight line, and we consider that possibility first before going on to consider, more generally, any curved interface.

140. Rectilinear Interfaces

Figure 103 represents part of a rectilinear interface with a square net chosen so that this interface lies along the mesh line 204; we denote the region to the left of this interface as a, the region to the right as A. Except at nodes on the interface, such as 0, residuals may be calculated from the usual formula for Laplace's equation, *i.e.*, at nodes within a,

$$F_0 \equiv \Sigma\phi_1 - 4\phi_0, \tag{27}$$

and at nodes within A,

$$F_0 \equiv \Sigma\Phi_1 - 4\Phi_0. \tag{28}$$

Neither of these formulas (27) and (28) is, however, of any immediate use for the evaluation of a residual at the point 0 (Figure 103) when it lies on the interface; for, in (27), ϕ_1 is a fictitious value of the function ϕ (which exists only in the region a), and, in (28), Φ_3 is similarly a fictitious value of the function Φ (existing only in the region A). In order to overcome this dilemma, we must make use of the refraction condition (26); its finite-difference approximation, holding at 0, implies

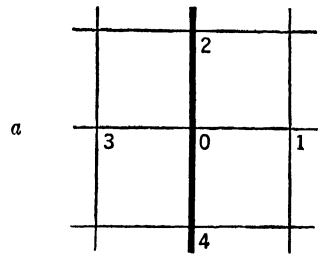


FIGURE 103

$$\mu(\phi_1 - \phi_3) = M(\Phi_1 - \Phi_3). \tag{29}$$

Also both ϕ and Φ must satisfy Laplace's equation at 0, so that, in finite-difference terms,

$$\begin{aligned} \phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0 &= 0, \\ \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_0 &= 0. \end{aligned} \tag{30}$$

If we eliminate the fictitious values ϕ_1 and Φ_3 from the three equations (29) and (30), we get

$$\mu(\phi_2 + 2\phi_3 + \phi_4 - 4\phi_0) + M(2\Phi_1 + \Phi_2 + \Phi_4 - 4\Phi_0) = 0. \tag{31}$$

Now, on I, $\phi = \Phi$, and no numerical distinction need be made between the two functions at any point of the interface. Algebraically we can denote the potential on the interface by ϕ, and then, after division through by $\mu + M$, (31) leads to a residual formula at a typical point 0 on the interface:

$$F_0 \equiv \frac{2M}{\mu + M}\Phi_1 + \phi_2 + \frac{2\mu}{\mu + M}\phi_3 + \phi_4 - 4\phi_0. \tag{32}$$

Now that (32) has been obtained, solutions of potential problems containing a rectilinear interface can be computed without much difficulty. To start the calculation, the values of the potential should be guessed at all nodes of the network covering the area of the required solution; residuals may then be calculated by use of the ordinary formulas (27) and (28) at all nodes except those which lie on the interface, where the special and slightly modified formula (32) should be employed. The liquidation of the residuals is accomplished in the usual manner with a use of the

customary pattern appropriate to Laplace's equation, with the exception only that modified unit patterns correspond to increments applied at nodes immediately *adjacent* to the interface; these modified patterns are easily constructed and are shown in Figure 104, (i) being the pattern to be

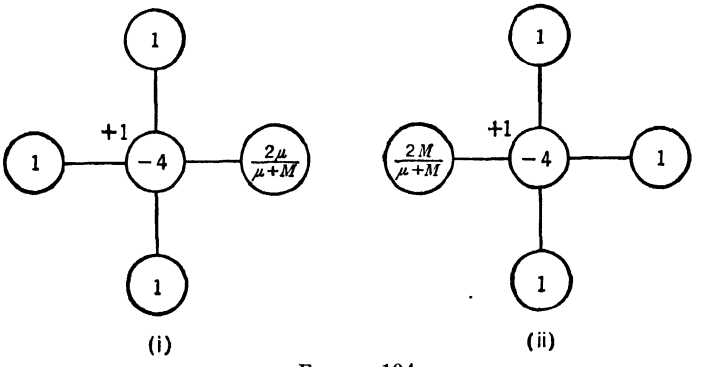


FIGURE 104

used when an increment $\Delta\phi = 1$ is applied at a point like 3 in Figure 103, and (ii) being the similar pattern corresponding to a point like 1.

141. Curved Interfaces

Satisfaction of the refraction condition (26) is much less easy when the interface is curved, as illustrated in Figure 105; again the usual formula (27) or (28) for Laplace's equation defines residuals at all nodes except those within one mesh length of the interface. It will be sufficient here to construct the special formula for use at a point such as 0 in Figure 105, which lies just inside the region A; an exactly similar type of residual formula will hold at a node which lies just inside the region a. In order to construct this special formula holding at the node 0, it is necessary to appeal to the string-net analogy described in Chapter 6. We must suppose that the tensions in the strings in the two regions a and A are different from one another, and we denote them by $\mathbf{t}$ and $\mathbf{T}$, respectively; we also interpret the potential ϕ (or Φ) as the transverse deflection of a uniform membrane. Now the normal gradient $\partial\phi/\partial\nu$ of the potential ϕ across the interface I implies (Section 54), in terms of the membrane, a line intensity of transverse force acting along I of magnitude $(\mathbf{t}/h)(\partial\phi/\partial\nu)$; similarly the gradient $\partial\Phi/\partial\nu$ of the potential Φ

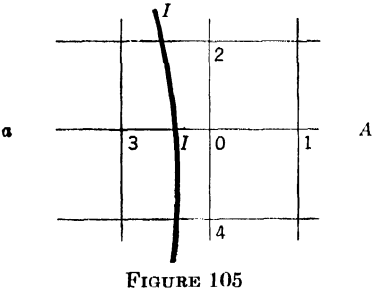


FIGURE 105

across I implies a line intensity of transverse force acting along I of magnitude $-(\mathbf{T}/h)(\partial\Phi/\partial\nu)$, where h is the size of the square mesh in Figure 105. Hence for transverse* equilibrium along I without the imposition of any external intensity of force, we must have

$$\frac{\mathbf{t}}{h}\frac{\partial\phi}{\partial\nu} = \frac{\mathbf{T}}{h}\frac{\partial\Phi}{\partial\nu},$$

and so, from (26),

$$\frac{\mathbf{t}}{\mu} = \frac{\mathbf{T}}{M}. \qquad (33)$$

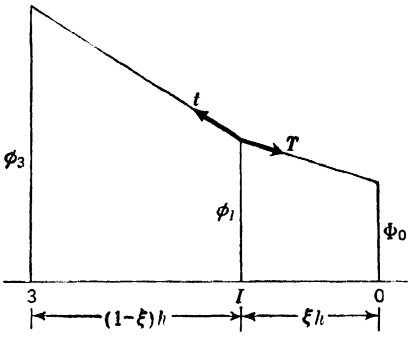


FIGURE 106

Having found the necessary relation (33) between the string tensions in the two regions a and A, we next consider the deflection of the string 03 of Figure 105. This string is shown, in elevation, in Figure 106, its transverse deflections being Φ_0 and ϕ_3 at its ends and ϕ_I at the point I where it crosses the interface; we suppose the distance $0I$ to be a fraction ξ of the full mesh length h. At once, then, for transverse equilibrium of the string at the point I, we get

$$\mathbf{t}\frac{\phi_3 - \phi_I}{(1-\xi)h} = \mathbf{T}\frac{\phi_I - \Phi_0}{\xi h}, \qquad (34)$$

each side of this equation being the transverse component of the tension acting in the string. It may easily be verified that, with a use of (33), each side of (34) is also equal to

$$\frac{\mathbf{T}}{h}\frac{\mu}{M(1-\xi) + \mu\xi}(\phi_3 - \Phi_0). \qquad (35)$$

When the string-net analogy is being employed to formulate expressions for the residuals at the nodes of the net, it was seen in Section 53 that a typical residual F_0 is, apart from an omitted factor $\mathbf{T}/h$, the sum of the transverse components of the tensions in the strings meeting at the typical node 0. For the node 0 in Figure 105 we have found the transverse component of tension in the string 03 to be given by (35); the transverse components of the tensions in the strings 01, 02, and 04 are simply $(\mathbf{T}/h)(\Phi_1 - \Phi_0)$, etc. Hence the residual formula for a node 0 situated,

* The membrane analogy is only concerned with *transverse* equilibrium; clearly when different tensions $\mathbf{t}$ and $\mathbf{T}$ act in the membrane in the separate regions a and A, equilibrium *in the plane of the membrane* can be maintained only by the supply of some external intensity of force along I in this plane.

as in Figure 105, near to an interface I is

$$F_0 \equiv \Phi_1 + \Phi_2 + \Phi_4 + \frac{\mu}{M(1-\xi)+\mu\xi}\phi_3 - \left(3 + \frac{\mu}{M(1-\xi)+\mu\xi}\right)\Phi_0. \quad (36)$$

Thus residuals are now calculable at all nodes like 0 just inside the region A; an exactly similar formula can be written down (by interchange of the constants M and μ) for the residual at a node just inside the region a. As in Section 140 special relaxation patterns are appropriate in the neighborhood of the interface; these can be constructed in the usual manner from the coefficients in residual formulas of the type (36).

PROBLEMS

1. Modify the solution to Problem 2 of Chapter 5 for the case when the rectangular section with sides in the ratio 3:5 contains a central and similarly orientated rectangular hole with sides in the ratio 1:1½.

2. Determine the temperature distribution inside a two-dimensional, square conducting region bounded by the lines $x,y = \pm L$ when the temperature is maintained at 1,000 on two adjacent sides and at 0 on the other two sides and when the conductivity of the medium inside the circle $x^2 + y^2 = L^2/4$ is three times the conductivity of the medium between this circle and the square boundary. Work to the nearest whole number on a net of size $h = L/16$.

CHAPTER 14

PROBLEMS INVOLVING UNKNOWN BOUNDARIES

142. Introduction

In Chapter 12 relaxational methods of solution of eigenvalue problems governed by differential equations were discussed. From one point of view it can be said that we were then concerned to find solutions to equations which were not completely known at the beginning because they contained a parameter (or eigenquantity) which had also to be found as part of the complete answer, but the range (in one dimension) or the area (in two dimensions) of integration of the equations concerned was defined in each problem. In this chapter we consider how to determine solutions of certain partial differential equations, in two dimensions, when those equations are completely known but when the boundaries of the areas over which the solutions are required may not be completely given at the beginning but have also to be found as part of the answer.

Three illustrative types of problem will be dealt with; two are concerned with fluid flow, and in both a free surface of the flow has to be determined in addition to the nature of the flow inside the region where it occurs. The first of these problems deals with the percolation of a fluid through a porous material; the second is concerned with the irrotational flow of an incompressible fluid. Both examples include free surfaces open to the atmosphere, and in both cases the governing differential equation is Laplace's, but along the free surfaces two boundary conditions can be stated instead of only one, as has hitherto been the case. We shall use one of these conditions in the usual manner in order to be able to solve the Laplace's equation throughout the area of integration; the second of the conditions will be employed in order, gradually and systematically as the solution proceeds, to fix the proper position of the originally unknown boundary.

The third example will illustrate a rather different possibility, that of an initially unknown interface, or internal boundary, within the region of integration. An example of this kind arises in the torsion problem for a uniform cylinder when the applied torque is increased sufficiently for a part of the material of the cylinder to become stressed to its elastic limit. When this happens, plastic conditions supersede elastic conditions, and

we have to determine, as part of any solution, over what parts of the cross section of the cylinder plastic conditions prevail and over what part elastic conditions remain operative; *i.e.*, an interface between what may be called the plastic and elastic parts of the cross section has to be found.

143. Percolation through a Porous Material

A typical problem involving the percolation of a fluid through porous material is illustrated in Figure 107, and it will be used here in order to

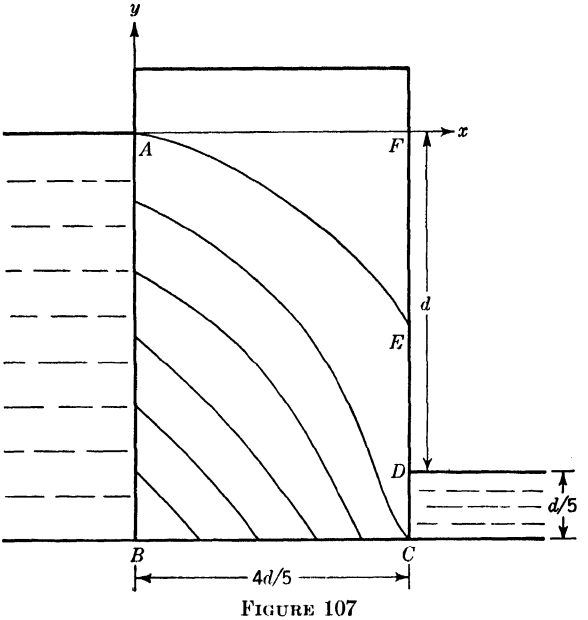


FIGURE 107

describe the technique required for solutions of this kind. $ABCF$ represents the porous retaining wall of a dam, standing on an impermeable base BC; a high water level (A) exists on one side of the wall, and there is a low water level (D) on the other side, the difference in height between the two levels being d. Water seeps slowly through the wall from left to right, and Figure 107 shows a few of the lines of constant pressure in the water in this seepage (these lines are obtained as part of the final solution; they are shown in Figure 107 so that the problem may be more easily understood by the reader). One of the lines, AE, is a free surface of the flow; as such, it forms part of the boundary of the relaxational problem, which is concerned with the determination of the flow within the region $ABCDEA$, and of that complete boundary the part AE is not initially known in position. In particular, the position of the point E is not known at the

beginning of the work, and it should be specifically mentioned that the water which seeps through to the surface of the wall at points between E and D then, in practice, runs down this surface to the low water level D.

144. Fluid Pressure in Percolating Flow

The work is most suitably carried out in terms of the fluid pressure. We denote the average excess pressure above atmospheric at any point in the dam by p (as a multiple of ρg, where ρ is the density of the fluid and g is the gravitational acceleration). We can immediately write down, in terms of this pressure p, some of the boundary conditions which must be satisfied by the flow. With coordinate axes of x and y chosen as shown in Figure 107, these conditions are that

$$\begin{aligned} p &= 0 \text{ on } AED, \\ p &= -y \text{ on } AB, \\ p &= -(y + d) \text{ on } DC, \end{aligned} \tag{1}$$

since, respectively, AED is open to the atmosphere, points on AB are at depths $-y$ below the high water level A, and points on DC are at depths $-(y + d)$ below the low water level D. There remains to be stated a boundary condition on BC and a second condition on AE; these will be apparent after we have first formulated the governing differential equation which p must satisfy throughout the region $ABCDEA$.

To discover this governing equation, we need to introduce components of percolation velocity of the fluid, which we denote as u and v, in the axial x and y directions; these velocity components must, of course, satisfy an equation of continuity of the flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

Next, the components of frictional force acting on the fluid are $-u/k$ and $-v/k$ per unit volume, measured as multiples of ρg, where k is the percolation coefficient, or coefficient of permeability, for the fluid through the porous material. The components of gravitational force per unit volume are simply 0 and -1, as multiples of ρg, so that, since the motion is supposed to be *slow*, we can write down equations of equilibrium:

$$\begin{aligned} \frac{\partial p}{\partial x} + \frac{u}{k} &= 0, \\ \frac{\partial p}{\partial y} + \frac{v}{k} + 1 &= 0. \end{aligned} \tag{3}$$

Substituting from (3) for u and v into (2), we find the governing equation for p to be Laplace's equation:

$$\nabla^2 p = 0. \tag{4}$$

The boundary condition on BC also follows immediately from the second of (3); it is that $v = 0$, and therefore, in terms of p,

$$\frac{\partial p}{\partial y} = -1 \text{ on } BC. \tag{5}$$

The second boundary condition on AE is derived, similarly, from the fact that the normal component of velocity across AE must clearly be zero; hence

$$u \cos(x,\nu) + v \sin(x,\nu) = 0, \tag{6}$$

where ν is the direction of the outward-drawn normal across AE at any point of that boundary and where (x,ν) is the angle between the x and ν directions at that point. Substituting in (6) for u and v from (3), we find

$$\frac{\partial p}{\partial x} \cos(x,\nu) + \frac{\partial p}{\partial y} \sin(x,\nu) = -\sin(x,\nu). \tag{7}$$

Now also along AE we know from (1) that $p = 0$, and therefore $\partial p/\partial s = 0$, so that

$$\frac{\partial p}{\partial x} \sin(x,\nu) - \frac{\partial p}{\partial y} \cos(x,\nu) = 0. \tag{8}$$

Combining (7) and (8), we can express the second boundary condition in either of two forms to be that

on AE,

$$\frac{\partial p}{\partial x} = -\sin(x,\nu)\cos(x,\nu),$$

or (9)

$$\frac{\partial p}{\partial y} = -\sin^2(x,\nu).$$

The problem is now completely set: we have to satisfy the governing Laplace's equation (4) over the region $ABCDEA$ subject to the boundary conditions stated in (1), (5), and (9). For any *assumed* position of the free surface AE the solution of (4), within the region thus defined and satisfying the conditions (1) and (5), can be achieved by the ordinary processes of relaxation described in Chapter 5. There remains for discussion here only the question of how the *position* of AE may be determined so that the extra boundary condition (9) may also be satisfied.

145. Determination of a Free Surface

The correct position of the free surface is, in practice, found only gradually (but systematically). We start with any guessed curve and solve (4) for p, subject to the boundary conditions (1) and (5); Figure 108 shows such a solution for a curve AE, which has been chosen, for illustra-

tive purposes, to be a deliberately bad guess. The size of mesh used is $h = d/5$, and the values of p, as shown, include a multiplying factor $500/d$. Also recorded in Figure 108 are the lengths of the short arms of the irregular stars in the vicinity of the free surface AE; these lengths are given as fractions of the mesh size h, and their values are shown (in brackets) just above the curve AE at the points where the short arms meet that curve. The values of angles recorded immediately above each of these lengths are the values of (x,ν) at the corresponding points on AE.

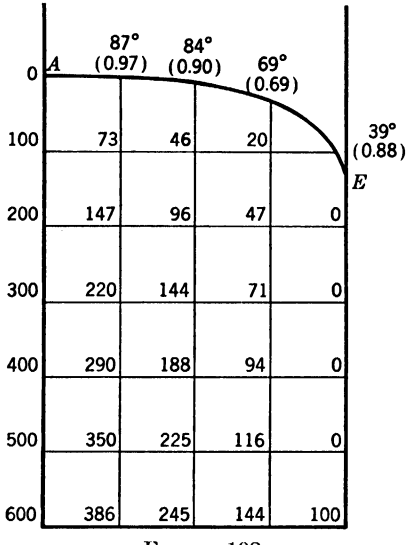


FIGURE 108

Having found the distribution of p values shown in Figure 108, we must next test whether or not the condition (9) is satisfied along AE. To do

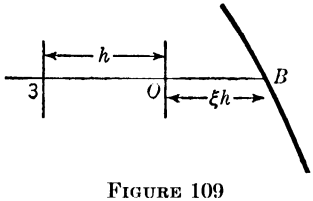


FIGURE 109

so requires the evaluation, from the p values, of $\partial p/\partial x$ at those points on AE where it is cut by horizontal mesh lines and of $\partial p/\partial y$ at those points where it is cut by vertical mesh lines. Figure 109 illustrates a typical (horizontal) mesh line cutting the boundary AE at the point B; 0 and 3 are the adjacent nodes of the net on this line, and ξ denotes, as usual, the fractional length of OB. Now it was shown in (12) of Chapter 5 that, with neglect of $O(h^3)$,

$$h^2\left(\frac{\partial^2 p}{\partial x^2}\right)_0 = \frac{2p_B}{\xi(1+\xi)} + \frac{2p_3}{1+\xi} - \frac{2p_0}{\xi},$$

and in the same manner it can also be shown that

$$h\left(\frac{\partial p}{\partial x}\right)_0 = \frac{p_B}{\xi(1+\xi)} - \frac{\xi p_3}{1+\xi} - \frac{(1-\xi)p_0}{\xi}.$$

It follows that

$$\begin{aligned} h\left(\frac{\partial p}{\partial x}\right)_B &= h\left(\frac{\partial p}{\partial x}\right)_0 + \xi h^2\left(\frac{\partial^2 p}{\partial x^2}\right)_0 \\ &= \frac{1+2\xi}{\xi(1+\xi)}p_B + \frac{\xi}{1+\xi}p_3 - \frac{1+\xi}{\xi}p_0, \end{aligned} \tag{10}$$

and an exactly similar formula can, of course, be written down for $h(\partial p/\partial y)$ at points on AE where it is cut by vertical mesh lines. Figure 110 shows the values, either of $\partial p/\partial x$ or of $\partial p/\partial y$, which have been calculated by substitution, into formulas of the form of (10), of p values taken from Figure 108; these values are shown just above AE at the points to which they refer. Corresponding values at each of the points are also shown (in brackets) of either $-\sin(x,\nu)\cos(x,\nu)$ or $-\sin^2(x,\nu)$, depending on whether the corresponding point of AE lies on a horizontal or a vertical mesh line.

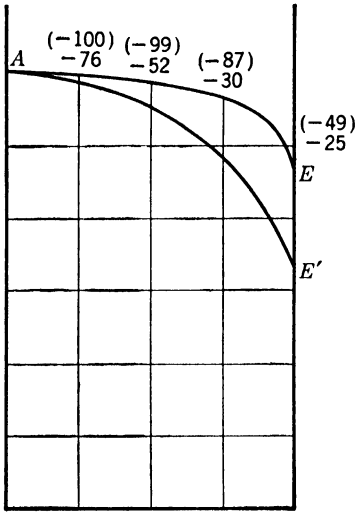


FIGURE 110

Now if the second boundary condition (9) were satisfied along AE, the recorded values in Figure 110 would be equal in pairs at each point. In fact, they are not even approximately equal, for the reason that the guessed curve AE was particularly badly chosen; in each pair the evaluated $\partial p/\partial x$, or $\partial p/\partial y$, is numerically smaller than the value of $-\sin(x,\nu)\cos(x,\nu)$, or $-\sin^2(x,\nu)$.

The next step is, therefore, one which is designed to correct the discrepancies between the paired values shown in Figure 110, by means of an attempt to increase numerically the gradients $\partial p/\partial x$ (or $\partial p/\partial y$). This must clearly be achieved by an alteration of the originally guessed position of the curve AE, and to increase the gradients of p at this free surface, we must manifestly move AE downward. The curve AE' in Figure 110 shows a possible position for the free surface which we might well take as a second trial.

The computation described above in relation to the first guessed curve AE must now be repeated for the second trial curve AE'; there will result a second set of pairs of values at points on AE' similar to the set shown in Figure 110 at points on AE. This new set is not given here because the process of calculating it is exactly the same as the process of finding the first set, which has been described in detail. On the basis now of two sets of pairs of values, one along the first guess AE and one along the second trial AE', for the free surface, a third trial position of that surface is drawn and the computation repeated again for this third trial curve.

The third trial should be approaching the real position of the free surface; for, although the first guess AE was poor and the second trial AE' may not have been very good (because it was known only that AE' should

be below AE, but not how much below), it can be expected that a comparison of the *discrepancies* in the pairs of values along AE *and* along AE' should enable the third trial free surface to be drawn as a good approximation to the true curve. Further trial curves are drawn until eventually one is found along which the computed values of $\partial p/\partial x$, or $\partial p/\partial y$, are equal to the values of $-\sin(x,\nu)\cos(x,\nu)$, or $-\sin^2(x,\nu)$. The problem will then have been solved.

Until he has actually carried through the necessary calculations to find a solution of the kind described here, it may appear to the reader that the suggested method is extremely lengthy, requiring, as it does, a number of solutions for the pressure distribution corresponding to different trial positions of the free surface. In fact, the process is quite rapid for the reason that the various pressure distributions do not show much difference from one another, so that, as each new trial free surface is drawn, only a very little relaxation is needed to modify the values of the pressure obtained in the previous trial and to find the values appropriate to the new curve. This fact becomes, of course, more and more relevant as the successive curves converge nearer and nearer to the final solution. To begin with, as was illustrated in Figure 110, we should normally expect the first, and probably also the second, position chosen for the free surface to be such that all the discrepancies, between the computed values of $\partial p/\partial x$ or $\partial p/\partial y$ and of $-\sin(x,\nu)\cos(x,\nu)$ or $-\sin^2(x,\nu)$, would be of the same sign. This means that we should normally expect at first to have to draw successive trials for which the free surface is moved either down all along its length or up at all points. But as the final solution is approached, we can expect that the discrepancies will become small, and for any one trial they may not all be of the same sign. If this is the case, then it follows that in the next trial we must draw a new curve which in part may be below the previous trial curve and in part may be above. In general, the curve should be lowered wherever along it we desire to increase the pressure gradient, and it should be raised wherever we wish to produce a decrease (in magnitude) of the pressure gradient.

The final solution to the problem described in Figure 107 was given on a net of size $h = d/20$ by Shaw and Southwell* in 1941. The results are not quoted here but are left for the reader to find as a suitable exercise on the determination of an initially unknown boundary.

146. Two-dimensional Irrotational Flow

From the relaxational point of view very similar problems are set in the discovery of irrotational flow patterns of an incompressible fluid, in two dimensions, when the flow is not completely bounded by fixed walls but

* *Proc. Roy. Soc. A*, **178**, 1–17.

may in part be open to the atmosphere along a free streamline. Such problems are suitably worked in terms of the stream function ψ rather than the velocity potential ϕ, for the reason that one boundary condition along any free streamline is then simply that

$$\psi = K, \tag{11}$$

where K is a known constant. Throughout the region of the flow, ψ must satisfy Laplace's equation in the ordinary way, and we need to deal here with the method of solution only in regard to the determination of the position of free streamlines. For any guessed or chosen position of a free streamline, the finding of ψ values satisfying Laplace's equation throughout the field, the condition (11) along the free streamline, and appropriate conditions along other parts of the boundary should by now need no further description.

147. Boundary Conditions on a Free Streamline

One of the two conditions which must hold on any free streamline has already been stated in (11); the other condition is derived from Bernoulli's theorem. Along a free surface the pressure is constant, and when gravitational effects are ignored, this means that the velocity also is constant, so that we derive the second condition on a free streamline to be that

$$\frac{\partial \psi}{\partial \nu} = k, \tag{12}$$

where ν denotes the normal direction drawn outward across the free streamline at any point of it and where k is a known constant. When gravitational effects are included and cartesian x,y axes are chosen such that the axis of y is vertically upward, the second condition on a free streamline becomes that

$$\frac{\partial \psi}{\partial \nu} = \sqrt{-2gy}, \tag{13}$$

where the origin of the coordinates is taken in the stagnation level of the flow. Whether the second condition is (12) or (13), it expresses the gradient $\partial\psi/\partial\nu$ in a form which is known numerically for any drawn position of the free streamline.

We use this second condition along a free streamline in a manner exactly similar to that in which we employed (9) as the second condition along a free surface in problems of fluid percolation through porous materials. We make, to begin with, a first guess for the position of a free streamline, and then, within the region of flow so defined, the governing

Laplace's equation is solved in conjunction with the condition (11) along the free streamline. Next the second condition (12) or (13) is used to test whether or not the guessed position of the free streamline is correct. At all points where this line is cut by mesh lines of the relaxation net, a pair of values are found and compared; one of these values is of k or $\sqrt{-2gy}$ [according as (12) or (13) applies] and the other is of $\partial\psi/\partial\nu$. The method of calculation of the values of $\partial\psi/\partial\nu$ makes use of the fact that, along a free streamline,

$$\frac{\partial\psi}{\partial s} = 0,$$

so that

$$\begin{aligned}\frac{\partial\psi}{\partial\nu} &= \frac{1}{\cos\,(x,\nu)} \cdot \frac{\partial\psi}{\partial x} \\ &= \frac{1}{\sin\,(x,\nu)} \cdot \frac{\partial\psi}{\partial y}.\end{aligned} \tag{14}$$

We use the first of (14) in order to calculate $\partial\psi/\partial\nu$ at points which lie on horizontal mesh lines and the second of (14) similarly at points on vertical mesh lines; $\partial\psi/\partial x$ and $\partial\psi/\partial y$ may be evaluated just as $\partial p/\partial x$ and $\partial p/\partial y$ were evaluated in Section 145.

148. Determination of a Free Streamline

We have seen that, once a first guess has been drawn for a free streamline, we can test its accuracy by means of the evaluation of pairs of values, of $\partial\psi/\partial\nu$ and of k or $\sqrt{-2gy}$, at points along the line. A survey of the discrepancies in the various pairs should indicate how a second trial curve for the free streamline can best be drawn. Normally at those parts along the first guess where the velocity $\partial\psi/\partial\nu$ turns out to be too small, we should move the free streamline in toward the region of flow, and, vice versa, where $\partial\psi/\partial\nu$ turns out to be too large, we should move the free streamline out away from the region of flow. Occasionally, in particular problems, it may be found that, if the above rule is adhered to, the discrepancies in the pairs of values along the free streamline are *increased* in the second trial compared with what they were for the first guess. It is neither possible nor necessary to say *precisely* what causes this nuisance, but it is certainly connected in some way with the local curvature of the free streamline. From a relaxational point of view all that need be said is that, if a movement of the free streamline in one direction increases the discrepancies, then a movement in the opposite direction will decrease them; and after two or three trial curves have been drawn and tested in any problem, sufficient information should have been accumulated to enable a very close approximation to the true curve to be found.

149. Axially Symmetric Irrotational Flow

The same type of problem can be equally well tackled for cases when the flow possesses axial symmetry instead of being two-dimensional as in Sections 146 to 148. The principle of the method of solution is identical, and the only differences of detail are that, with axial symmetry, (1) the equation governing the stream function ψ is

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0,$$

where, of course, z and r are cylindrical coordinates with the axis Oz along the axis of symmetry; and (2) the second boundary condition along a free surface is

$$\frac{\partial\psi}{\partial\nu} = kr,$$

when gravitational effects are ignored, and is

$$\frac{\partial\psi}{\partial\nu} = \sqrt{2gz}\, r,$$

when gravity is assumed to act in the axial z direction and where the origin O is taken to be in the stagnation level of the flow.

Many examples of the determination of free streamlines, both in two dimensions and with axial symmetry, have been solved by Southwell and Vaisey, and the interested reader is referred to their paper* published in 1946.

150. Plastic Torsion

The elastic (shear) stresses in a cylinder subjected to a purely twisting action were given in Section 46 to be

$$\tau_{zx} = \mu\tau L\frac{\partial\phi}{\partial y}, \qquad \tau_{yz} = -\mu\tau L\frac{\partial\phi}{\partial x}, \tag{15}$$

in nondimensional form, where μ is the modulus of rigidity of the material of the cylinder, τ is the angular twist per unit length along the cylinder, L is some representative dimension of its cross section, and ϕ is a stress function which satisfies Poisson's equation

$$\nabla^2\phi + 2 = 0, \tag{16}$$

over the area of the cross section, together with the edge condition $\phi = 0$, round the boundary of this area. It follows from (15) that the resultant shear stress at any point in the cross section acts in the direction which is

* *Phil. Trans. R.S. A*, **240**, 117–161.

tangential to the contour of constant ϕ through that point, and that its magnitude is given as a multiple of $\mu\tau L$ by the gradient of ϕ in the perpendicular direction. If q is the value of the shear stress at any point, then the gradient of ϕ there is $q/\mu\tau L$.

We suppose that the shear stress-strain relation for the material of the cylinder is of the kind shown in Figure 111. Provided that the shear stress at any point is less than some limiting "yield" value q_Y, it varies linearly with the shear strain γ, and the straining there is said to be elastic; but once the shear stress attains the yield value q_Y, the strain may increase indefinitely. The shear stress cannot exceed the limiting value q_Y, and so long as it maintains that value at any point, the straining there is said to be plastic. The existence of an upper limit to q can be immediately interpreted in terms of the stress function ϕ: that the gradient of ϕ can nowhere exceed $q_Y/\mu\tau L$.

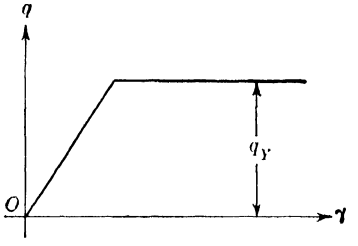


FIGURE 111

151. The Nádai Roof

For any given shape of cross section, (16) can be solved over the whole of the section to yield the distribution of ϕ appropriate to straining which is elastic everywhere; and if q_Y is large enough or τ is small enough, the gradient of ϕ in this solution will nowhere be as large as $q_Y/\mu\tau L$. But if the twisting is great enough, *i.e.*, if τ is large enough, then the gradient of ϕ may equal its limiting value $q_Y/\mu\tau L$ over some parts of the cross section. The condition that the gradient of ϕ may not exceed $q_Y/\mu\tau L$ is overriding; wherever in the section the gradient of ϕ has attained this limiting value and the straining has become plastic, the equation (16) need no longer be satisfied, as it would have to be if the straining were elastic. In torsion problems where the twisting is sufficient for the shear stress q to have attained the limiting yield value in some parts of the cross section but not in others, we are then presented with the difficulty of having to find as part of the solution an initially unknown interface between what may be called the elastic and plastic parts of the cross section. On one side of this interface, in the elastic part, ϕ remains governed by (16); on the other side, in the plastic part, it is governed by the overriding condition that its gradient is $q_Y/\mu\tau L$.

The determination of any solution is considerably aided by the fact that we are able to state what the value of ϕ *would* be at any point of the cross section *if* that point were in the plastic part, without any prejudice to the subsequent computation which actually decides the extents of the

separate plastic and elastic parts. That this can be done depends on the fact that, in the solution for which the straining is elastic everywhere, the gradient of ϕ takes its greatest value at some point on the boundary; it is at this point, as the twisting action increases, that plastic straining must commence; all plastic parts of the section, which develop as the twisting action increases further, must begin at and then spread from the boundary of the section. Consequently, wherever the straining is plastic, the ϕ values lie on the surface of constant maximum gradient, $q_Y/\mu\tau L$,

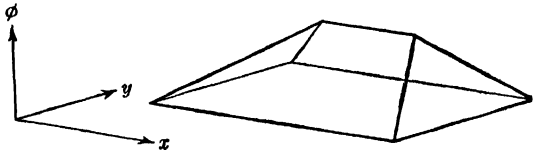


FIGURE 112

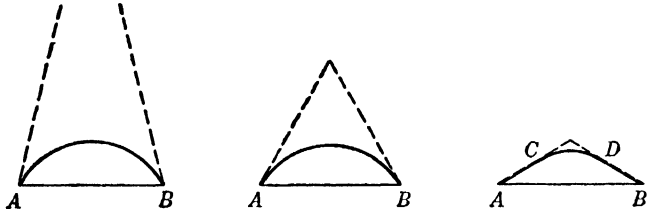


FIGURE 113

drawn from $\phi = 0$ on the boundary; this surface is known as a *Nádai roof*, and at no point of the cross section may the value of ϕ exceed the corresponding value on this roof. Figure 112 shows a typical one of these roofs for the case of a rectangular cross section.

In regions where the ϕ values have attained their corresponding roof values, the straining is plastic and equation (16) is not required to be satisfied; but in regions where the ϕ values are less than the corresponding roof values, the straining is elastic and (16) must hold. The nature of the possible solutions to a torsion problem is indicated in Figure 113, which represents a line AB in the plane of the cross section; A and B are points on the boundary of the cross section, and the heavy curve plots the variation of the stress function ϕ along AB, while the dotted lines represent the roof. In (a) the straining is elastic all along AB, and the curve nowhere touches the dotted lines. In (b), corresponding to an increased twisting action, the slope of the roof is reduced; the straining is still elastic at all points of AB, but the curve now touches the dotted lines at A and

B, at which points plastic straining is about to commence. In (c) the twisting action has been increased even more, and the slope of the roof is still further reduced; plastic straining extends from A to C and from B to D, where the curve is in contact with the dotted lines, but the straining remains elastic between C and D, where the curve lies below the roof and where (16) remains satisfied.

152. A Worked Example

For an illustrative numerical example, we take a case of plastic torsion for a uniform cylinder whose cross section is rectangular, with sides in the

0	0	0	0	0	0	0
0	250	250	250	250	250	250
0	250	500	500	500	500	500
0	250	500	750	750	750	750
0	250	500	750	1000	1000	1000

FIGURE 114

ratio 3:2. The solution for this section when the straining is everywhere elastic was discussed in Section 46, values of the stress function ϕ being given in Figure 28 on three successive sizes of net. We modify that solution here for a case when the straining is partly plastic, working on the finest of the three nets ($h = \frac{1}{8}$) and again including a multiplying factor of 3,200, so that the values recorded are those of $3{,}200\phi$ instead of simply ϕ. We suppose that the maximum gradient $q_Y/\mu\tau L$ which ϕ can attain has the value $10\frac{1}{16}$; when the multiplying factor 3,200 is included, this is a gradient of 250 per mesh length, and the roof values can be immediately written down; they are recorded in Figure 114.

The values given in Figure 28 (iii) can be seen to exceed the corresponding roof values shown in Figure 114 at three nodes (those near the center line and adjacent to the longer side of the section). As an initial set of values for this case of plastic torsion we take the values from Figure 28 (iii) except at those three nodes where they exceed the roof values; at

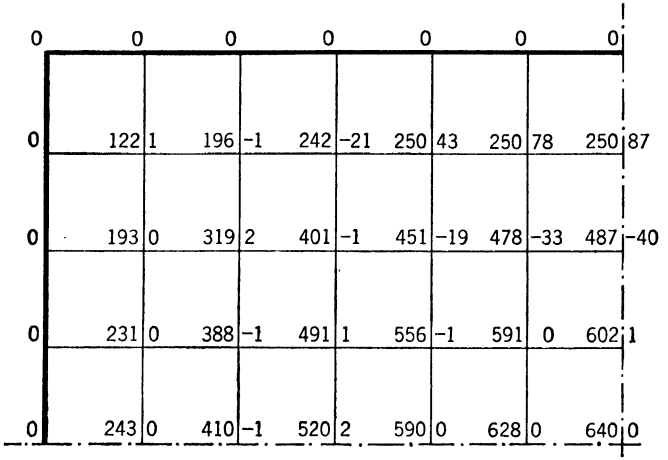


FIGURE 115

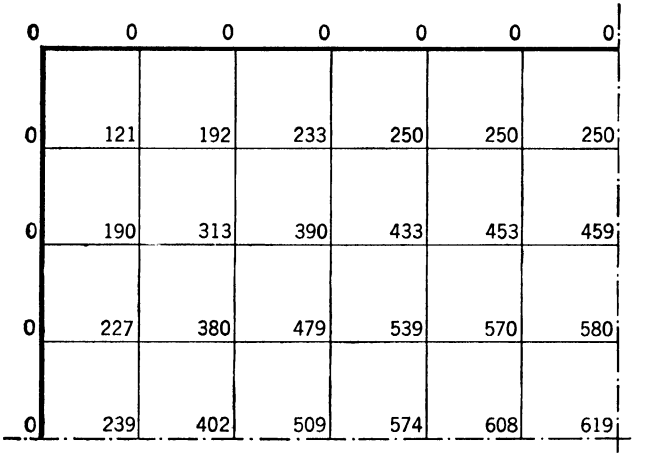


FIGURE 116

those nodes we naturally take the roof values themselves. Figure 115 shows these initial values, together with residuals calculated at all nodes according to the formula which may be derived from equation (16):

$$F_0 \equiv \Sigma\phi_1 - 4\phi_0 + 100. \tag{17}$$

The solution is found by the ordinary processes of relaxation; the residuals in Figure 115 must be liquidated as far as possible without any ϕ value being allowed to exceed the corresponding roof value. This means that residuals must be reduced to zero at all those nodes where ϕ has a value less than the limiting roof value, but that positive residuals

may be left, unliquidated, at those nodes where ϕ has attained the roof value. If, owing to increments applied at neighboring nodes, the residual at a node where ϕ has attained the roof value ever becomes negative, it should then be liquidated in the usual manner, for the liquidation of a negative residual requires a negative increment $\Delta\phi$, so that its application reduces the ϕ value to less than the limiting roof value.

Figure 116 shows the final solution for the stress function ϕ. The straining is elastic at all nodes where the ϕ values are less than the corresponding roof values and where necessarily the residuals are zero; the straining is plastic at those nodes where the ϕ values have attained their limiting roof values and where the residuals may, therefore, be nonzero but positive.

PROBLEMS

1. Complete the solution to the problem illustrated in Figure 107 and started in Section 145. Begin on a net of size $h = d/5$, but advance to successively finer nets ($h = d/10$ and $h^* = d/20$) as the solution converges.

2. Solve the problem of the plastic torsion of the cylinder in Section 152 for the cases when the maximum gradient $q_Y/\mu\tau L$ of ϕ is given to be (1) $9/16$, (2) $8/16$, and (3) $7/16$.

CHAPTER 15

NONELLIPTIC DIFFERENTIAL EQUATIONS

153. Introduction

We have shown in previous chapters how the relaxation method can be applied to solve various linear partial differential equations in two dimensions. Laplace's equation is an example which, though simple, is typical of all the second-order equations which have been discussed in that it is of the so-called "elliptic" form and in that all solutions which have been contemplated have been required to satisfy a single boundary condition at all points of a closed boundary. In many problems, however, either one or (usually) both of these properties do not hold, and in this chapter we show how the consequent difficulties may be overcome and such problems rendered capable of solution by relaxation methods. Typical examples are provided when solutions are required of the heat-conduction equation (which is of "parabolic" form) and of the wave equation (which is of "hyperbolic" form) and also, occasionally, of Laplace's equation itself, which, although of elliptic form, may not be associated with a single boundary condition at all points of a complete boundary.

In one dimension, inherently the same difficulties distinguish what are usually known as "jury" problems from "marching" problems (the difference between jury and marching problems was discussed in 1927 by Richardson,* who gave them those names). It will be convenient to introduce the basic ideas of this chapter in relation to such one-dimensional problems before we attempt to discuss their application to solve partial differential equations in two dimensions.

154. Availability of End Conditions

The difference between a jury and a marching problem in one dimension is entirely dependent upon the location of the end conditions which are required to be satisfied. Sufficiently generally we can suppose that we have to solve an ordinary differential equation of order n over the range $0 \leq x \leq L$, where x is the independent variable. In order that a particular solution of the equation may be defined, n end conditions must be postulated, and we may take m of these to be given conditions at one end

* *Phil. Trans. R.S. A*, **226**, 229–361.

of the range and p at the other end, where, of course, $m + p = n$. There is no loss of generality if we assume that $m \geq p$. A jury problem is one for which $m = p\ (= n/2)$, and all the problems in one dimension which we have hitherto shown how to solve have been of this kind (cf. Section 28). A marching problem is usually understood to be one for which $p = 0$ (and therefore $m = n$), but from a relaxational point of view we can more generally treat as a marching problem any one for which $m > p$, because the same type of difficulty has to be faced whenever $m \neq p$.

One of the simplest equations which can be used to illustrate the difference between jury and marching problems is the beam-deflection equation

$$\frac{d^2w}{dx^2} + W(x) = 0. \tag{1}$$

In order to define a particular solution of this equation, two end conditions must be stated; these may be given either one at each end of the range of integration (a jury problem) or both at one end of the range (a marching problem). The first of these possible cases was dealt with in Chapter 4 as our first example of the solution of a differential equation by relaxation methods; the second case presents difficulties which prevent a *direct* application of relaxation, for reasons which will become apparent in Sections 155 and 156.

155. Extent of Finite-difference Approximations

It is helpful here to recall some finite-difference approximations to low-order derivatives; in the customary notation we have (Appendix III)

$$\begin{aligned}
2h\left(\frac{dw}{dx}\right)_0 &= w_1 - w_3, \\
h^2\left(\frac{d^2w}{dx^2}\right)_0 &= w_1 + w_3 - 2w_0, \\
2h^3\left(\frac{d^3w}{dx^3}\right)_0 &= w_9 - 2w_1 + 2w_3 - w_{11}, \\
h^4\left(\frac{d^4w}{dx^4}\right)_0 &= -4w_1 - 4w_3 + 6w_0 + w_9 + w_{11}.
\end{aligned} \tag{2}$$

It may be seen in (2) that the approximations to the first and second derivatives are expressed in terms of values of w at points of subdivision which are distant from the central point 0 up to (but not exceeding) one interval h; similarly the approximations to the third and fourth derivatives are both expressed in terms of w values at points which are distant from 0 up to but not exceeding two intervals. We say that the *extent* of

the approximations to the first and second derivatives is one interval (on both sides of the point 0) and that the extent of the approximations to the third and fourth derivatives is, similarly, two intervals. More generally, the extent of the approximations to the derivatives of orders $2r - 1$ and $2r$ is r intervals.

The magnitude of the extent of the approximation to the highest-order derivative in an equation is important because it is equal to the number of fictitious values which the finite-difference approximation to that equation introduces at both ends of the range. To be able to use the relaxation method directly requires that we be able to eliminate these fictitious values from the various residual formulas in which they appear, and that means that we must have available at each end of the range as many end conditions as the magnitude of the extent of the approximation corresponding to the order of the equation.

It follows at once that it is not possible to solve any odd-order equation directly by the relaxation method; for, if the order is $2r - 1$, the extent of the approximation to the derivative of this order is r intervals, and therefore r end conditions would be wanted at both ends of the range. This is impossible because an equation of order $2r - 1$ cannot be made to satisfy (in all) $2r$ end conditions.

156. Proper Employment of End Conditions

Furthermore, even if the order of the governing differential equation is even, it can be solved by a direct application of relaxation only if the problem is of jury type. Suppose the order to be $2r$; then the extent of the approximation to the derivative of this order is r intervals, and therefore, of the $2r$ end conditions which are available, r are wanted at each end of the range.

Altogether we have now seen that a direct application of relaxation methods can be made to solve an ordinary differential equation only if the problem is of jury type (and the equation necessarily of even order); in the notation of Section 154 this is the case when $m = p$. We have now to consider how we can solve an equation for any case when $m > p$. The method to be described will be termed an *indirect* application of relaxation, because the required solution will be found, not as a direct solution of the given equation, but instead as a solution to some other equation which we have yet to show how to formulate.

When m ($> p$) conditions are postulated at one end of the range of integration, it is useless to attempt to associate them with the given equation of order n ($= m + p$), or indeed with any equation of order other than $2m$; for only when the order is $2m$ can the postulation of m conditions at one end of the range be reconciled with the problem having been

reformulated in jury form and therefore being soluble by relaxation. We say that m conditions at one end of a range are being properly employed only in conjunction with an equation of order $2m$; by themselves they would be insufficient if associated with an equation of order greater than $2m$ (even an equation of order $2m + 1$ would require $m + 1$ conditions at each end of the range); and we say that m conditions at one end of a range are being improperly employed in conjunction with an equation of order less than $2m$ (for then the number of conditions left to be available at the other end would be insufficient).

157. Transformation of a Marching into a Jury Problem

We see, therefore, that, when $m > p$, in order to solve an equation of order n $(= m + p)$, we must transform it into a new equation of order $2m$. The transformation must thus *increase* the order of the original given equation by $2m - n$, *i.e.*, by $m - p$. This can be done by a change of the dependent variable w by a substitution for it in terms of a new dependent variable v; clearly the function of v which is substituted for w must be a differential function of order $m - p$. We make no attempt to lay down any other rules according to which the substitution for w in terms of v should be chosen. In all cases there will be several possible substitutions which would suffice, and a compromise must be made, first, to choose the substitution itself as simply as possible, and second, to make the transformed equation as simple as possible and as suitable as we can for solution by relaxation.

The given end conditions are also, of course, transformed into terms of v, so that, after the substitutions for w have been effected, we find ourselves with a differential equation of order $2m$ to solve for v, together with m end conditions at one end of the range and p $(< m)$ conditions at the other end. To be able to find v uniquely requires the existence of $m - p$ extra conditions at this other end; these extra conditions can be arbitrarily chosen provided that they do not conflict with one another or with any of the original p conditions which have to be satisfied at that end of the range, and provided also that they cannot be expressed in terms of w or of w derivatives (so that they cannot be interpreted as implying any restriction on w or w derivatives). Normally the extra conditions consist of assigned values to v and the first $(m - p - 1)$ v derivatives.

The determination of the new variable v becomes a straightforward jury problem for solution by relaxation, and when v has been determined, it can be substituted back into the relation of transformation between w and v, and so w can be found immediately.

Some examples will now be given to illustrate the methods of this chapter.

158. A Worked Example

As a first example we take a simple equation of the type (1), for which the exact solution can be readily obtained, *i.e.*,

$$\frac{d^2w}{dx^2} = 6x, \tag{3}$$

to be solved over the range $0 \leq x \leq 4$ with end conditions given that

At $x = 0$,

$$w = 0 \qquad \text{and} \qquad \frac{dw}{dx} = 0. \tag{4}$$

This is a marching problem in the strict sense; the governing equation (3) is of order 2 ($= n$), and the end conditions are stated 2 ($= m$) at one end of the range and 0 ($= p$) at the other end. We need therefore to transform the equation into one of order 4 ($= 2m$) by a substitution for w in terms of a new variable v; for this example a substitution which is both simple and sufficient is

$$w = \frac{d^2v}{dx^2}. \tag{5}$$

In terms of v the governing equation (3) becomes

$$\frac{d^4v}{dx^4} = 6x, \tag{6}$$

and the end conditions (4) transform to be that

At $x = 0$,

$$\frac{d^2v}{dx^2} = 0 \qquad \text{and} \qquad \frac{d^3v}{dx^3} = 0. \tag{7}$$

Two extra end conditions have to be chosen at the other end of the range; we take them as simply as possible to be that

At $x = 4$,

$$v = 0 \qquad \text{and} \qquad \frac{dv}{dx} = 0. \tag{8}$$

These conditions (8) clearly do not conflict with one another or with any other conditions holding at $x = 4$, nor do they imply any restriction on w or w derivatives. It may be mentioned that any other valid choice of the conditions (8) merely adds to the solution for v a quantity of the form $A + Bx$, where A and B are some constants, and does not, therefore, make any difference after double differentiation to find w from equation (5).

The transformed problem is to satisfy equation (6) subject to the end conditions (7) and (8); it is a jury problem and presents no difficulty to a relaxational attack. The solution for v, at points of subdivision which divide the range into eight intervals each of length $h = \frac{1}{2}$, is given in Figure 117; the values shown in this figure include a multiplying factor of 16. The determination of the values of w follows immediately by substitution into the finite-difference approximation to (5):

$$w_0 = 4(v_1 + v_3 - 2v_0),$$

and the result is shown in Figure 118. This solution may be compared

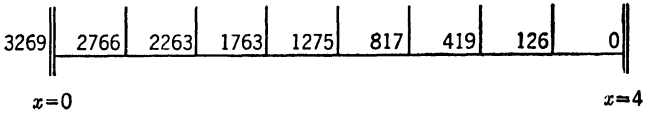


FIGURE 117

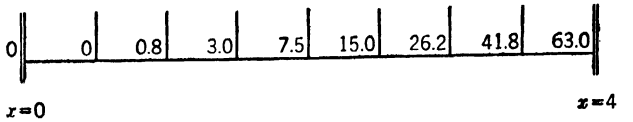


FIGURE 118

with the exact answer, which is $w = x^3$; the w values in Figure 118 are nowhere in error by as much as 2 per cent.

159. A First-order Equation

As a second example in one dimension, we consider the first-order equation

$$\frac{dw}{dx} = w + x, \tag{9}$$

to be solved over the range $0 \leq x \leq 1$, with the end condition to be satisfied that

At $x = 0$,

$$w = 1. \tag{10}$$

In order that it may be properly employed, this condition should be associated with a second-order equation, and we need, therefore, to make a suitable substitution for w in terms of a new dependent variable v. In this case an appropriate substitution is

$$w = \frac{dv}{dx} + v. \tag{11}$$

After the transformation (11) has been made, the governing equation (9) becomes

$$\frac{d^2v}{dx^2} - v - x = 0, \tag{12}$$

and the end condition (10) is that

At $x = 0$,

$$\frac{dv}{dx} + v = 1. \tag{13}$$

When this condition is used in conjunction with equation (12), it is properly employed; but to be able to solve (12) uniquely requires that we choose one extra end condition to hold at $x = 1$. This extra condition is most simply taken to be that

At $x = 1$,

$$v = 0. \tag{14}$$

If the interval between successive points of subdivision is of length h, the finite-difference approximation to (12) leads to the residual formula

$$F_0 \equiv v_1 + v_3 - (2 + h^2)v_0 - h^2x_0, \tag{15}$$

and the end conditions yield that

At $x = 0$,

$$v_1 - v_3 = 2h(1 - v_0),$$

At $x = 1$, (16)

$$v_0 = 0.$$

At $x = 0$, v_3 is a fictitious value, and it must be eliminated from the residual expression (15) by means of the first of (16). The result is that, at the special point $x = 0$, the residual formula is

$$F_0 \equiv 2v_1 - (2 - 2h + h^2)v_0 - 2h.$$

At the other end point of the range, $x = 1$, we do not need to define a residual because the second of (16) gives the value of v there.

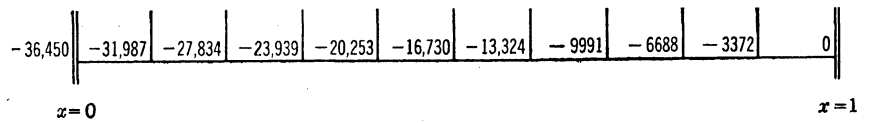


FIGURE 119

There is no difficulty in finding the relaxational solution for v, and it is given in Figure 119 at points of subdivision for which $h = 0.1$ and with

the inclusion of a multiplying factor of 10^4. The values of w can then be computed from the finite-difference approximation to the relation (11):

$$w_0 = 5(v_1 - v_3) + v_0.$$

Figure 120 shows these w values, which may be compared with the values

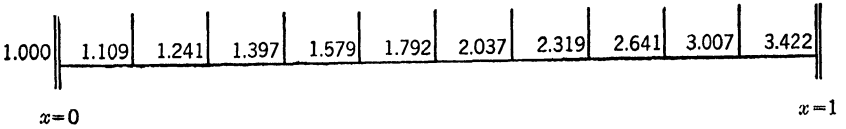


FIGURE 120

obtained from the exact solution, $w = 2e^x - x - 1$; they nowhere differ by as much as ½ per cent.

160. The Heat-conduction Equation

The ideas and methods of this chapter, which have been explained and illustrated in one dimension, can now be extended without difficulty to be applied to some partial differential equations in two dimensions. One most important example, for which the methods are needed, is provided by the heat-conduction equation

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}, \tag{17}$$

in which θ, a function of time t and distance x, measures the temperature in a linear flow of heat; κ is the diffusivity of the medium.

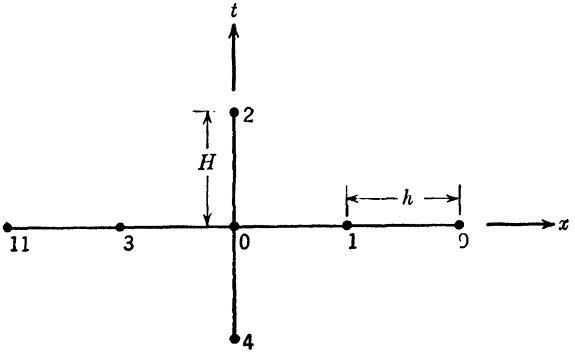


FIGURE 121

Two difficulties present themselves if an attempt is made to solve (17) directly by relaxation. A rectangular* net would be used to cover the region of integration, and Figure 121 shows a typical group of seven nodal

* A rectangular net is not really a new type of relaxation net; with appropriately chosen scales in the coordinate directions it could be converted into a square net.

points on such a net; the mesh lengths in the x and t directions are denoted, respectively, by h and H. The finite-difference approximation to (17) leads in the usual way to the definition of a residual at a typical node 0:

$$F_0 \equiv \theta_1 + \theta_3 - 2\theta_0 - \frac{h^2}{2\kappa H}(\theta_2 - \theta_4). \tag{18}$$

If we conveniently choose the mesh lengths so that $h^2 = 2\kappa H$, then the relaxation pattern which is derived from the formula (18) is that shown in Figure 122.

The first difficulty is apparent in this pattern; it enables the residuals to be easily moved from node to node along mesh lines in the x direction, but it may also be noticed that an increment $\Delta\theta$ at any node makes no change at all to the total of the residuals on the mesh line through that node in the x direction. Thus the pattern of Figure 122 does not allow residuals to be easily moved from one node to another in the t direction. This difficulty would not in itself be insurmountable, and we mention it here only because we shall have later an incidental opportunity of removing it.

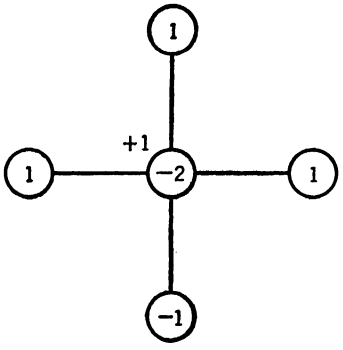


FIGURE 122

The second difficulty is much more fundamental: it is that the governing equation (17) is of an odd (*i.e.*, the first) order in the t derivative, so that whatever boundary conditions are given, they cannot be properly employed, and therefore a direct relaxational solution is impossible. Any problem which is governed by equation (17) must be of marching type in the t direction, and in order to be able to solve it by relaxation methods, we must transform (17) into a new equation of order 2 in the t derivative by means of a substitution for θ in terms of a new variable ϕ. We choose this substitution so far as possible to make the transformed equation both simple and convenient for the application of relaxation; the obvious substitution to use is

$$\theta = \frac{\partial \phi}{\partial t} + \kappa \frac{\partial^2 \phi}{\partial x^2}, \tag{19}$$

which transforms (17) into the new governing equation

$$\frac{\partial^2 \phi}{\partial t^2} - \kappa^2 \frac{\partial^4 \phi}{\partial x^4} = 0. \tag{20}$$

The finite-difference approximation to this equation, holding at a typical

node 0 of a rectangular net of sides h and H in the x and t directions, is

$$4\phi_1 + 4\phi_3 - 6\phi_0 - \phi_9 - \phi_{11} + \frac{h^4}{\kappa^2 H^2}(\phi_2 + \phi_4 - 2\phi_0) = 0. \quad (21)$$

The relaxation pattern which is derived from this approximation does not suffer from the disadvantage which we pointed out in connection with the pattern of Figure 122. Indeed, by a suitable choice of the mesh

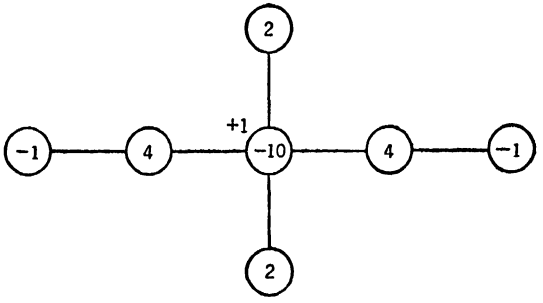


FIGURE 123

length H we can make the pattern which follows from (21) such that the effect of an increment $\Delta\phi$ at any node of the net is to move the residual away from that node *equally* in all four directions; to do this requires that we choose

$$H = \frac{h^2}{\kappa\sqrt{2}}, \quad (22)$$

and then (21) leads to residuals being defined by the formula

$$F_0 \equiv 4\phi_1 + 4\phi_3 + 2\phi_2 + 2\phi_4 - 10\phi_0 - \phi_9 - \phi_{11},$$

and the corresponding pattern is shown in Figure 123.

161. Choice of Extra Boundary Conditions

In any particular problem certain extra boundary conditions will have to be chosen in order that a solution for ϕ may be obtained. Before any attempt is made to lay down these extra conditions, it is usually helpful to transform the original conditions of the problem, known initially in terms of θ, into terms of the new variable ϕ. It has also been found very helpful in practice to display these transformed conditions diagrammatically; we can illustrate how this may be done by considering the following example: To determine the temperature in a medium which extends from $x = 0$ to $x = L$; which is initially at zero temperature throughout; where the end $x = 0$ is kept at zero temperature, but where, at time $t = 0$, the other end $x = L$ is raised to and subsequently maintained at a temperature of 100.

In terms of θ the boundary conditions implied in this statement of the problem are that

$$\theta = 0 \text{ on } x = 0,$$
$$\theta = 100 \text{ on } x = L,$$
$$\theta = 0 \text{ on } t = 0.$$

Expressed in terms of ϕ by means of the relation (19), these conditions become that

$$\frac{\partial \phi}{\partial t} + \kappa \frac{\partial^2 \phi}{\partial x^2} = 0 \text{ on } x = 0,$$
$$\frac{\partial \phi}{\partial t} + \kappa \frac{\partial^2 \phi}{\partial x^2} = 100 \text{ on } x = L,$$
$$\frac{\partial \phi}{\partial t} + \kappa \frac{\partial^2 \phi}{\partial x^2} = 0 \text{ on } t = 0.$$

These conditions are exhibited in Figure 124, each adjacent to that part of the boundary where it has to be satisfied. Figure 124 also indicates

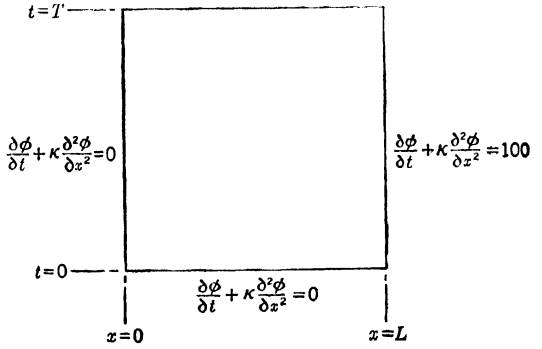


FIGURE 124

the area over which the solution for ϕ is to be found; T measures the extent in time for which the solution is required.

Now because the ϕ-governing equation (20) is of order 2 in the t derivatives and of order 4 in the x derivatives, the elimination of fictitious values round the boundary of the rectangular region of integration shown in Figure 124 makes necessary the existence of one boundary condition on each of the sides $t = 0$ and $t = T$ and two boundary conditions on each of the sides $x = 0$ and $x = L$. We have to choose one extra boundary condition, therefore, on each of the three sides $t = T$, $x = 0$, and $x = L$. These extra conditions are chosen as simply as possible to be that

$$\phi = 0 \text{ on } x = 0,$$
$$\phi = 0 \text{ on } x = L,$$
$$\frac{\partial \phi}{\partial t} = 0 \text{ on } t = T,$$

as illustrated in Figure 125. It may well be asked why the extra condition on $t = T$ has to be chosen to be $\partial\phi/\partial t = 0$ and is not, even more simply, $\phi = 0$. The answer to this is that, if ϕ were zero along $t = T$, then, in particular, $\partial^2\phi/\partial x^2$ would vanish at the point $x = L$, $t = T$; but also, since ϕ is zero along $x = L$, therefore $\partial\phi/\partial t$ vanishes at that point. Hence choosing ϕ to be zero along both $t = T$ and $x = L$ would make $(\partial\phi/\partial t) + \kappa(\partial^2\phi/\partial x^2)$ vanish at the point $x = L$, $t = T$, and this would conflict with the original condition that $(\partial\phi/\partial t) + \kappa(\partial^2\phi/\partial x^2)$ has the value 100 along $x = L$. This example provides a useful illustration of the care that is necessary in choosing the extra end conditions; it is worth repeating the statement made in Section 157 that the extra conditions can be arbitrarily chosen *provided that they do not conflict with one another or with any of the original conditions of the problem, and provided also that they do not imply any restriction on the original dependent variable or its derivatives.*

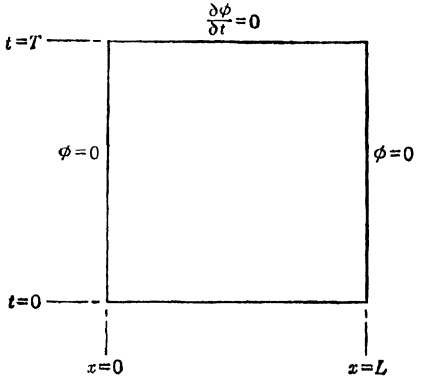


FIGURE 125

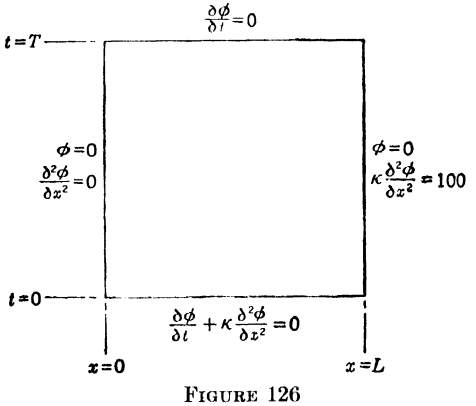


FIGURE 126

It should be noticed that some of the conditions shown in Figure 124 simplify because the conditions of Figure 125 are additionally imposed. The complete set of conditions are shown together in Figure 126, and it can be seen at once that they are of a kind whose satisfaction does not lead to any computational difficulty.

162. Nondimensional Variables

In discussing the example of Section 161 we made no mention of any transformation of the variables into nondimensional terms. As usual, however, the actual computation should be concerned with nondimensional quantities, and in the case of the heat-conduction equation, the transformation is perhaps best left until after the governing equation and boundary conditions have been expressed in finite-difference terms; they can then all be expressed nondimensionally in terms of a variable $\chi = \kappa\phi/h^2$. Thus in the example of Section 161 the only condition which is affected by any dimensional considerations is the original one on $x = L$ that $\kappa(\partial^2\phi/\partial x^2) = 100$; in its finite-difference approximation this condition is

$$\frac{\kappa}{h^2}(\phi_1 + \phi_3 - 2\phi_0) = 100,$$

but in terms of the nondimensional variable χ, it becomes simply

$$\chi_1 + \chi_3 - 2\chi_0 = 100.$$

The calculations are carried through strictly in terms of χ rather than ϕ; and, when χ has been found numerically by relaxation, θ can then be obtained immediately, because the finite-difference approximation to (19) shows that

$$\theta_0 = \frac{\kappa}{h^2}(\phi_1 + \phi_3 - 2\phi_0) + \frac{\phi_2 - \phi_4}{2II},$$

which, by virtue of (22) and the relation $\chi = \kappa\phi/h^2$, may be written as

$$\theta_0 = \chi_1 + \chi_3 - 2\chi_0 + \frac{\chi_2 - \chi_4}{\sqrt{2}}. \tag{23}$$

It would appear from (23) that values of the temperature can be found with no reference to the values of either L or κ. The solution is, of course, not independent of these quantities, but their values serve only to fix the time interval of the rectangular net at the nodes of which we find θ. This interval is given by (22); when n mesh lengths are used in the x direction, it follows from (22) that the interval in the time direction is $L^2/n^2\kappa\sqrt{2}$, and therefore that, if N intervals are used in this direction, the total time range covered by the solution is $T = NL^2/n^2\kappa\sqrt{2}$.

No arithmetical results are quoted here because the new ideas that have been described with reference to the heat-conduction equation are primarily concerned with the formulation of the computational problem rather than with its accomplishment.

PROBLEMS

1. Solve the equation

$$\frac{d^2w}{dx^2} = W(x),$$

over the range $0 \leq x \leq 8$, and subject to the end conditions that

At $x = 0$,

$$w = \frac{dw}{dx} = 0,$$

when W is given numerically as a function of x in the table:

x	0	1	2	3	4	5	6	7	8
W	10	15	22	31	34	30	17	7	0

2. Solve the equation

$$\frac{d^3w}{dx^3} - \frac{d^2w}{dx^2} = x^2,$$

over the range $0 \leq x \leq 10$, and subject to the end conditions that

At $x = 0$,

$$w = \frac{dw}{dx} = 0,$$

At $x = 10$,

$$w = 0.$$

(Use an interval of length $h = 1$.)

3. Solve the problem which has been formulated in Sections 161 and 162, taking the number of mesh lengths in the x direction to be $n = 6$ and the number of intervals in the t direction to be $N = 9$.

(The values eventually obtained for the temperature θ may be checked by comparison with the solution worked by Allen and Severn.*)

* Quoted in *Quart. J. Mech. and App. Math.*, **4**, 209–222 (1951).

CHAPTER 16

THREE-DIMENSIONAL RELAXATION

163. Introduction

So far as the solution of partial differential equations is concerned, we have in previous chapters been engaged in extending the scope of the relaxation method to deal with more and more advanced equations in two dimensions. In this last chapter we describe a different kind of extension of the method, to bring within its scope the solution of partial differential equations in three dimensions. Naturally the first equation to be tackled in three dimensions is Laplace's equation, and, in fact, in this chapter only that equation and the slightly more general Poisson's equation are discussed. Many of the statements and equations required in this extension of the relaxation method are counterparts of similar results which were proved in two dimensions in Chapter 5; it is not necessary to prove them again or to elaborate on their significance. Such results will, therefore, merely be stated in three-dimensional form, and we shall describe in detail only those aspects of the method which are peculiar to three dimensions.

164. Poisson's Equation

In three dimensions, Poisson's equation is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + W(x,y,z) = 0, \tag{1}$$

where x, y, and z are orthogonal cartesian coordinates. The relaxation method determines numerical values of the wanted quantity w, satisfying (1), at the joints of a uniform cubical lattice which extends throughout the volume of integration of the equation; W is a function of x, y, and z which is known in the sense that a numerical value can be assigned to it at all the joints of the lattice. The cubical lattice in three dimensions is the counterpart of the square net in two dimensions; this nomenclature is particularly convenient because it allows us to understand that three dimensions are under consideration whenever we use the term "lattice," and, similarly, use of the term "net" implies that we are working in only two dimensions.

Figure 127 represents a typical joint 0 of a lattice with the six (sur-

rounding) adjacent joints denoted by 1, 2, 3, 4, I, and II as shown. The mesh distance between neighboring joints is taken, as in two dimensions,

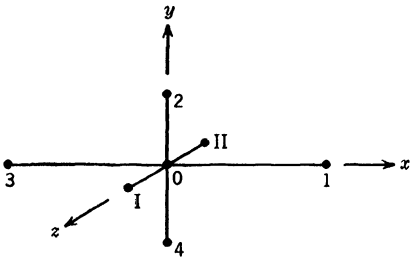


FIGURE 127

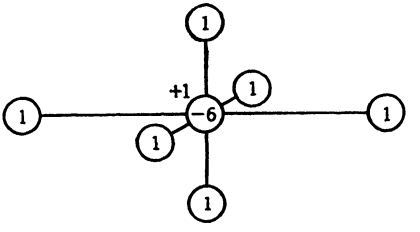


FIGURE 128

to be h, so that the finite-difference approximation to equation (1) leads to the definition of a typical residual at the joint 0:

$$F_0 \equiv w_1 + w_2 + w_3 + w_4 + w_{\mathrm{I}} + w_{\mathrm{II}} - 6w_0 + h^2 W_0. \qquad (2)$$

From this formula we can at once derive a corresponding three-dimensional relaxation pattern; it is given in Figure 128.

165. Isometric Projection of a Three-dimensional Lattice

The major difficulty, which for some time prevented most attempts at three-dimensional relaxation from being successful, was that involved in

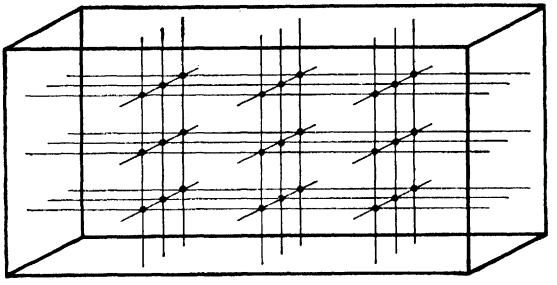


FIGURE 129

finding a satisfactory technique of recording on paper a calculation which in reality should take place on a three-dimensional lattice. It has often been suggested that the computation must be recorded on a number of superposed sheets, but that would be so unsatisfactory in practice as to make the method virtually useless. An alternative way of recording the relaxational computations is on an isometric projection of the lattice, and Figure 129 illustrates such a (typical) projected lattice drawn throughout the volume of a cube. The mesh length is one-quarter of the edge of the cube, and in Figure 129 27 internal joints can be counted. From each of these joints, six lattice lines radiate in the axial directions, and it can be

seen that the diagram allows at each joint two columns of values to be written down, one consisting of increments Δw and the other of successive values of the residual. It is necessary in drawing the lattice to allow only one set of plane sections to overlap one another (in Figure 129 it is the xy sections which overlap while the yz and zx sections do not); to achieve this and at the same time to compress the diagram into as small a space as possible necessitates considerable relative distortion of the mesh lengths in the three axial directions. It is only this "isometric" projection of the lattice which enables the work to be carried out on a single sheet of paper.

The three dimensions of the regions shown in Figure 129 are apparent in spite of the distortion already mentioned; this distortion is in no way troublesome because, in the actual computations made on it, the lattice should be thought of not in three dimensions but instead as a two-dimensional network with nodes at all points marked with dots; all other (unmarked) points where strings cross are not regarded as nodes of the net. This view of the actual working diagram greatly facilitates the liquidation; in particular, a block operation can be applied for any chosen group of nodes, and its effects on the residuals immediately recorded by the usual and simple two-dimensional technique of "counting strings."

166. Temperature Distribution in a Cube

A first numerical example concerns the steady temperature distribution inside a uniform cube when one face is maintained at a constant temperature (say 1,000) and the other five faces are kept at zero temperature. The equation governing the temperature w throughout the cube is Laplace's, so that residuals are defined by the formula (2) with W_0 put equal to zero, *i.e.*, by

$$F_0 \equiv w_1 + w_2 + w_3 + w_4 + w_{\mathrm{I}} + w_{\mathrm{II}} - 6w_0.$$

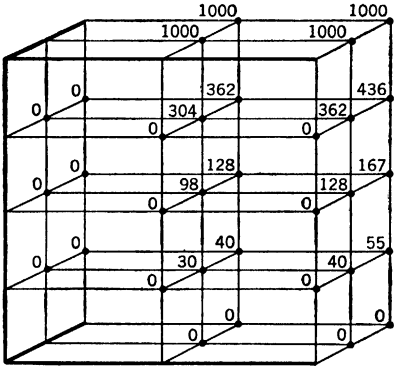


FIGURE 130

We may take Figure 129 to represent the cube; if the length of its side is L, then the comparatively coarse lattice in Figure 129 is the one for which $h = L/4$. The six faces of the cube may be taken to be the planes $x, y, z = 0$ or L, and we will suppose that the face $y = L$ is the one on which w is maintained at 1,000.

There is obvious symmetry in both the planes $x = L/2$ and $z = L/2$, so that only one-quarter of the whole cube need actually be used in the working diagram. The process of liquidation on a three-dimensional lattice is,

of course, at first rather more complicated than it is on a two-dimensional net, but with a little experience it soon becomes no more laborious, and the solution for the temperature distribution in the cube is easily obtained. It is shown in Figure 130, on the coarse lattice for which $h = L/4$. It is left as an exercise for the reader to find the solution on the finer lattice for which $h = L/8$.

167. Irregular Stars

In general, the bounding surface of the volume of integration may be curved, and then some of the strings of the lattice radiating from joints just inside the boundary would be cut short by it. Figure 131 illustrates such an irregular star, where the string 01 is cut short by the boundary at the point B, and the point 1 is now fictitious. A special formula for the residual at the joint 0 may be found which is similar to the corresponding formula for an irregular star in two dimensions. If the short arm $0B$ is a fraction ξ of the full mesh length h, then

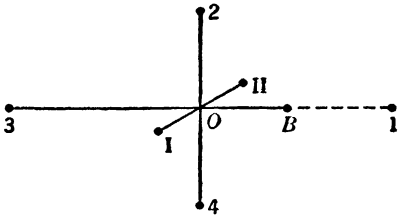


FIGURE 131

$$F_0 \equiv w_2 + w_4 + w_{\mathrm{I}} + w_{\mathrm{II}} + \frac{2}{1+\xi} w_3 + \frac{2}{\xi(1+\xi)} w_B - \left(4 + \frac{2}{\xi}\right) w_0.$$

168. Electric Potential Distribution in a Quadrant Electrometer

Our second example is given primarily to show the appearance of a three-dimensional lattice which includes a larger number of joints. The problem is to determine the electric potential distribution inside a quadrant electrometer whose dimensions are shown in Figure 132. The region of integration is bounded externally by a short, closed, circular cylinder divided axially into four quadrants, and internally by a plane vane, parallel to and midway between the ends of the cylindrical box; the shape of this vane is shown in plan view in Figure 132 (i), and its position along the axis of the box is shown in the elevations, Figures 132 (ii) and (iii).

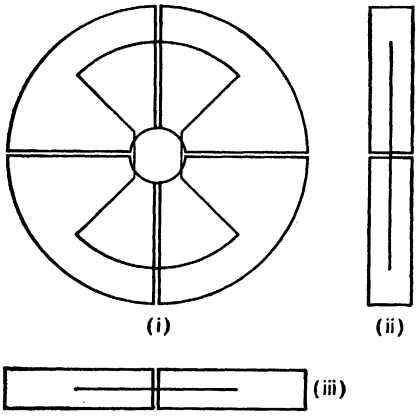


FIGURE 132

In the region between the vane and the box, the electric potential V satisfies Laplace's equation. The boundary conditions given are that the potential is maintained at prescribed values on the vane and on each pair of opposite quadrants; we take these values to be

On the vane,

$$V = 200,$$

On one pair of quadrants,

$$V = -4,$$

On the other pair of quadrants,

$$V = 4.$$

Because the thickness of the box is small compared with the diameter of its cross section, a smaller mesh length $(h/4)$ is best used in the direction parallel to its axis compared with the length (h) in the two perpendicular directions; this circumstance implies a modified residual formula

$$F_0 \equiv V_1 + V_2 + V_3 + V_4 + 16V_I + 16V_{II} - 36V_0,$$

and a correspondingly modified relaxation pattern. The solution is shown in Figure 133, which records values of V at the joints of a lattice in which there are four of the smaller mesh lengths in the half thickness of the box, and eight of the larger mesh lengths in the radius; conditions of symmetry make it sufficient to give results over only one-quarter of the whole volume.

PROBLEMS

1. Find the solution to the problem described in Section 166 on the finer lattice for which $h = L/8$.

(The values obtained may be checked by comparison with those given by Allen and Dennis.*)

2. A hemisphere is of radius 4 units being bounded by the curved surface $x^2 + y^2 + z^2 = 16$ $(z \geq 0)$ and the plane $z = 0$. Find values of a function w which satisfies Laplace's equation throughout the volume of the hemisphere and which is given to vanish on the curved surface and to be equal to $(16 - x^2 - y^2)$ $(16 - x - y)$ on the plane. Use a cubical lattice of mesh length $h = 1$.

* *Quart. J. Math. and App. Mech.*, **4**, 199–208 (1951).

APPENDIX I

THE NORMALIZATION OF ALGEBRAIC EQUATIONS

A set of linear algebraic equations is said to be in a normalized form if the matrix of the coefficients of the unknowns x, y, z, etc., in those equations is symmetrical about its leading diagonal. A simple process of normalization can be applied to any set of equations which is not in a normalized form, in order to replace them by an equivalent set which is normalized. Suppose the set of equations is given to be

$$f_r(x,y,z, \ldots) = 0 \qquad (r = 1,2, \ldots, N),$$

where f_r is a rational linear function of x, y, z, etc., and consider the function

$$U = \Sigma f_r^2.$$

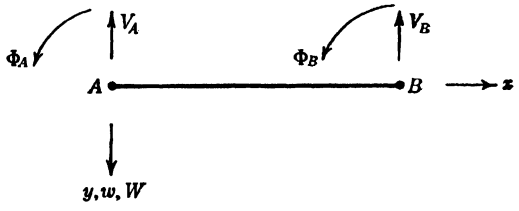
U is clearly a rational quadratic function of the variables x, y, z, etc., which attains a minimum value of zero when those variables have values which satisfy the set of equations $f_r = 0$. But the conditions for U, as a function of x, y, z, etc., to take a minimum value are

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = \cdots = 0,$$

and whatever quadratic expression U may be, the set of equations derived in this way must necessarily be in a normalized form (a fact which strictly depends on the equivalence of pairs of operations like $\partial^2/\partial x\,\partial y$ and $\partial^2/\partial y\,\partial x$). Hence the given set of equations are replaced by an equivalent normalized set.

APPENDIX II

STANDARD CASES OF BEAM DEFLECTION



When AB is a member (of a framework), of length L and flexural rigidity EI, which is subjected to applied transverse loads, the relaxation method of solution of the frame problem (Chapter 3) requires as a preliminary the transference of such loads to act at the joints of the frame. We list here the contributions to the actions at the ends A and B of a typical member by this transference, for several standard cases of support, when the load is either uniformly distributed of intensity w along the whole length of the beam or concentrated of magnitude W at the point $x = a$. x is a measure of position along the length of the beam from an origin at its left-hand end A, and the actions at A and B consist of transverse forces V_A and V_B and torques Φ_A and Φ_B, as illustrated in the above diagram. The deflection $y(x)$ of the beam, which accompanies this transference of load, is also given.

Case 1. *A Clamped, B Free; Distributed Load of Intensity w*

$$V_A = -wL, \qquad V_B = 0.$$

$$\Phi_A = -\frac{wL^2}{2}, \qquad \Phi_B = 0.$$

$$EIy = \frac{wx^4}{24} - \frac{wLx^3}{6} + \frac{wL^2x^2}{4}.$$

Case 2. *A Clamped, B Free; Concentrated Load W at* $x = a$

$$V_A = -W, \qquad V_B = 0.$$

$$\Phi_A = -Wa, \qquad \Phi_B = 0.$$

$$EIy = \frac{Wax^2}{2} - \frac{Wx^3}{6} \qquad (x \leq a),$$

$$= \frac{Wa^2x}{2} - \frac{Wa^3}{6} \qquad (x \geq a).$$

Case 3. *A Pinned, B Pinned; Distributed Load of Intensity w*

$$V_A = -\frac{wL}{2}, \qquad V_B = -\frac{wL}{2}.$$

$$\Phi_A = 0, \qquad \Phi_B = 0.$$

$$EIy = \frac{wx^4}{24} - \frac{wLx^3}{12} + \frac{wL^3x}{24}.$$

Case 4. *A Pinned, B Pinned; Concentrated Load W at x = a*

$$V_A = -\frac{W(L-a)}{L}, \qquad V_B = -\frac{Wa}{L}.$$

$$\Phi_A = 0, \qquad \Phi_B = 0.$$

$$EIy = \frac{Wa^2(L-a)(2L-a)x}{6L} - \frac{W(L-a)x^3}{6L} \qquad (x \leq a),$$

$$= \frac{Wax^3}{6L} - \frac{Wax^2}{2} + \frac{Wa(2L^2+a^2)x}{6L} - \frac{Wa^3}{6} \qquad (x \geq a).$$

Case 5. *A Clamped, B Clamped; Distributed Load of Intensity w*

$$V_A = -\frac{wL}{2}, \qquad V_B = -\frac{wL}{2}.$$

$$\Phi_A = -\frac{wL}{12}, \qquad \Phi_B = \frac{wL}{12}.$$

$$EIy = \frac{wLx^2}{24} - \frac{wLx^3}{12} + \frac{wx^4}{24}.$$

Case 6. *A Clamped, B Clamped; Concentrated Load W at x = a*

$$V_A = -\frac{W(L-a)^2(L+2a)}{L^3}, \qquad V_B = -\frac{Wa^2(3L-2a)}{L^3}.$$

$$\Phi_A = -\frac{Wa(L-a)^2}{L^2}, \qquad \Phi_B = \frac{Wa^2(L-a)}{L^2}.$$

$$EIy = \frac{Wa(L-a)^2x^2}{2L^2} - \frac{W(L-a)^2(L+2a)x^3}{6L^3} \qquad (x \leq a),$$

$$= \frac{Wa^2(3L-2a)x^3}{6L^3} - \frac{Wa^2(2L-a)x^2}{2L^2} + \frac{Wa^2x}{2} - \frac{Wa^3}{6} \qquad (x \geq a).$$

Case 7. *A Clamped, B Pinned; Distributed Load of Intensity w*

$$V_A = -\frac{5wL}{8}, \qquad V_B = -\frac{3wL}{8}.$$

$$\Phi_A = -\frac{wL^2}{8}, \qquad \Phi_B = 0.$$

$$EIy = \frac{wx^4}{24} - \frac{5wLx^3}{48} + \frac{wL^2x^2}{16}.$$

Case 8. *A Clamped, B Pinned; Concentrated Load W at* $x = a$

$$V_A = -\frac{W(L-a)(2L^2+2La-a^2)}{2L^3}, \qquad V_B = -\frac{Wa^2(3L-a)}{2L^3}.$$

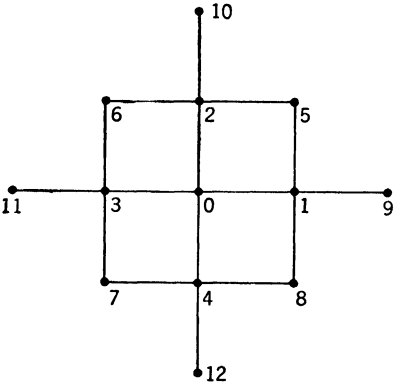
$$\Phi_A = -\frac{Wa(L-a)(2L-a)}{2L^2}, \qquad \Phi_B = 0.$$

$$EIy = \frac{Wa(L-a)(2L-a)x^2}{4L^2} - \frac{W(L-a)(2L^2+2La-a^2)x^3}{12L^3} \qquad (x \leq a),$$

$$= \frac{Wa^2(3L-a)x^3}{12L^3} - \frac{Wa^2(3L-a)x^2}{4L^2} + \frac{Wa^2x}{2} - \frac{Wa^3}{6} \qquad (x \geq a).$$

APPENDIX III

USEFUL FINITE-DIFFERENCE APPROXIMATIONS



In terms of the w values at the nodes of a square net (as indicated in the diagram), the following finite-difference approximations to low-order derivatives are frequently needed and have been collected together here for ease of reference:

First Derivatives

$$\left(\frac{\partial w}{\partial x}\right)_0 = \frac{w_1 - w_3}{2h},$$
$$\left(\frac{\partial w}{\partial y}\right)_0 = \frac{w_2 - w_4}{2h}.$$

Second Derivatives

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_0 = \frac{w_1 + w_3 - 2w_0}{h^2},$$
$$\left(\frac{\partial^2 w}{\partial y^2}\right)_0 = \frac{w_2 + w_4 - 2w_0}{h^2},$$
$$(\nabla^2 w)_0 = \frac{\Sigma w_1 - 4w_0}{h^2},$$
$$\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_0 = \frac{w_5 - w_6 + w_7 - w_8}{4h^2}.$$

Third Derivatives

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_0 = \frac{w_9 - 2w_1 + 2w_3 - w_{11}}{2h^3}$$

$$\left(\frac{\partial^3 w}{\partial y^3}\right)_0 = \frac{w_{10} - 2w_2 + 2w_4 - w_{12}}{2h^3}.$$

Fourth Derivatives

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_0 = \frac{6w_0 - 4w_1 - 4w_3 + w_9 + w_{11}}{h^4},$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_0 = \frac{6w_0 - 4w_2 - 4w_4 + w_{10} + w_{12}}{h^4},$$

$$(\nabla^4 w)_0 = \frac{20w_0 - 8\Sigma w_1 + 2\Sigma w_5 + \Sigma w_9}{h^4},$$

$$\left(\frac{\partial^4 w}{\partial x^2\, \partial y^2}\right)_0 = \frac{4w_0 - 2\Sigma w_1 + \Sigma w_5}{h^4}.$$

APPENDIX IV

ORTHOGONAL PROPERTIES OF NORMAL MODES

In Chapter 11 we quoted and made use of a certain "orthogonal property":

$$B(w_r,w_s) = 0,$$

of a pair of normal modes of solution, w_r and w_s, to the set of equations (1) of Section 107. The proof of this property follows from the fact that, since w_r is a mode satisfying that set of equations, we have

$$\begin{aligned}
&(a_{11} - b_{11}p_r^2)x_r + (a_{12} - b_{12}p_r^2)y_r + (a_{13} - b_{13}p_r^2)z_r + \cdots = 0,\\
&(a_{12} - b_{12}p_r^2)x_r + (a_{22} - b_{22}p_r^2)y_r + (a_{23} - b_{23}p_r^2)z_r + \cdots = 0,\\
&(a_{13} - b_{13}p_r^2)x_r + (a_{23} - b_{23}p_r^2)y_r + (a_{33} - b_{33}p_r^2)z_r + \cdots = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

where p_r^2 is the corresponding eigenvalue. Multiplying the above equations, respectively, by the components x_s, y_s, z_s, etc., of w_s and adding, we get

$$A(w_r,w_s) = p_r^2 B(w_r,w_s), \tag{1}$$

where $A(w_r,w_s)$ is defined like $B(w_r,w_s)$ and denotes the function

$$\begin{aligned}
a_{11}x_rx_s + a_{22}y_ry_s + a_{33}z_rz_s + \cdots + a_{12}(x_ry_s + x_sy_r)\\
+ a_{13}(z_rx_s + z_sx_r) + a_{23}(y_rz_s + y_sz_r) + \cdots.
\end{aligned}$$

Now both $A(w_r,w_s)$ and $B(w_r,w_s)$ are symmetrical functions of the two modes, so that from (1) it follows also that

$$A(w_r,w_s) = p_s^2 B(w_r,w_s), \tag{2}$$

and hence, from (1) and (2) together,

$$(p_r^2 - p_s^2)B(w_r,w_s) = 0.$$

Thus, since in general $p_r^2 \neq p_s^2$, we find the orthogonal property

$$B(w_r,w_s) = 0.$$

Various orthogonal properties can also be proved for the modal solutions to the differential equations discussed in Chapter 12. Thus the modes of vibration of a uniform stretched string satisfy equation (8) of

Section 119, so that, if w_r and w_s are two such modes,

$$\begin{aligned} \frac{d^2w_r}{dx^2} + \lambda_r w_r &= 0, \\ \frac{d^2w_s}{dx^2} + \lambda_s w_s &= 0, \end{aligned} \tag{3}$$

where λ_r and λ_s are the corresponding eigenvalues. Multiplying equations (3) respectively by w_s and w_r and subtracting, we get

$$(\lambda_r - \lambda_s) w_r w_s = w_r \frac{d^2w_s}{dx^2} - w_s \frac{d^2w_r}{dx^2}. \tag{4}$$

If (4) is now integrated over the length of the string ($0 \leq x \leq 1$), we find that

$$\begin{aligned} (\lambda_r - \lambda_s) \int_0^1 w_r w_s \, dx &= \int_0^1 w_r \frac{d^2w_s}{dx^2} \, dx - \int_0^1 w_s \frac{d^2w_r}{dx^2} \, dx \\ &= \left[w_r \frac{dw_s}{dx} \right]_0^1 - \int_0^1 \frac{dw_r}{dx} \frac{dw_s}{dx} \, dx \\ &\qquad - \left[w_s \frac{dw_r}{dx} \right]_0^1 + \int_0^1 \frac{dw_r}{dx} \frac{dw_s}{dx} \, dx \\ &= 0, \end{aligned}$$

because both w_r and w_s vanish at each end of the range of integration. Since $\lambda_r \neq \lambda_s$, we thus find the orthogonal property

$$\int_0^1 w_r w_s \, dx = 0.$$

For the case of a nonuniform string, the differential equation governing the modes is (38) of Section 130, and the same method then shows the orthogonal property to be

$$\int_0^1 \rho w_r w_s \, dx = 0.$$

In two dimensions, the equation governing the normal modes of vibration of a uniform stretched membrane is (17) of Section 123, and now we have, instead of (4),

$$(\lambda_r - \lambda_s) w_r w_s = w_r \nabla^2 w_s - w_s \nabla^2 w_r.$$

Integration over the area of the membrane yields

$$(\lambda_r - \lambda_s) \iint w_r w_s \, dx \, dy = \oint \left(w_r \frac{\partial w_s}{\partial \nu} - w_s \frac{\partial w_r}{\partial \nu} \right) ds, \tag{5}$$

by Green's theorem, where the contour integral is taken round the

boundary of the membrane in an anticlockwise direction and where ν denotes the normal direction (drawn outward) across this boundary at any point of it. Since both w_r and w_s vanish round the boundary, we have at once the orthogonal property

$$\int\int w_r w_s \, dx \, dy = 0.$$

For the buckling deflection of a uniform strut, the governing differential equation is (23) of Section 125, so that

$$\frac{d^4 w_r}{dx^4} + \lambda_r \frac{d^2 w_r}{dx^2} = 0,$$
$$\frac{d^4 w_s}{dx^4} + \lambda_s \frac{d^2 w_s}{dx^2} = 0.$$

Hence

$$\begin{aligned}\int_0^1 \left(\lambda_r w_s \frac{d^2 w_r}{dx^2} - \lambda_s w_r \frac{d^2 w_s}{dx^2}\right) dx &= \int_0^1 \left(w_r \frac{d^4 w_s}{dx^4} - w_s \frac{d^4 w_r}{dx^4}\right) dx \\ &= \left[w_r \frac{d^3 w_s}{dx^3} - w_s \frac{d^3 w_r}{dx^3}\right]_0^1 - \int_0^1 \left(\frac{dw_r}{dx}\frac{d^3 w_s}{dx^3} - \frac{dw_s}{dx}\frac{d^3 w_r}{dx^3}\right) dx \\ &= \left[w_r \frac{d^3 w_s}{dx^3} - w_s \frac{d^3 w_r}{dx^3}\right]_0^1 - \left[\frac{dw_r}{dx}\frac{d^2 w_s}{dx^2} - \frac{dw_s}{dx}\frac{d^2 w_r}{dx^2}\right]_0^1, \qquad (6)\end{aligned}$$

after two integrations by parts. Now, at a fixed end of a strut, w_r and w_s both satisfy $w = dw/dx = 0$, so that the right-hand side of (6) is seen to vanish. The left-hand side of (6) can also be integrated (once) by parts to yield

$$(\lambda_s - \lambda_r) \int_0^1 \frac{dw_r}{dx}\frac{dw_s}{dx} \, dx = 0,$$

so that we have the orthogonal property

$$\int_0^1 \frac{dw_r}{dx}\frac{dw_s}{dx} \, dx = 0.$$

Last, in Chapter 12, the mode of vibration of a uniform beam was given to be governed by equation (30) of Section 127, so that

$$(\lambda_r - \lambda_s) \int_0^1 w_r w_s \, dx = \int_0^1 \left(w_r \frac{d^4 w_s}{dx^4} - w_s \frac{d^4 w_r}{dx^4}\right) dx = 0,$$

as for the case of the strut, whether the ends of the beam are clamped ($w = dw/dx = 0$) or pinned ($w = d^2w/dx^2 = 0$) or free ($d^2w/dx^2 = d^3w/dx^3 = 0$). Hence in this case the orthogonal property is found to be

$$\int_0^1 w_r w_s \, dx = 0.$$

The remaining problems which were discussed in Chapter 12 are all governed by one or another of the differential equations with which we have already dealt in this Appendix. Thus the whirling deflections of a uniform rotating shaft satisfy the same equation as the modes of vibration of a uniform beam, and the modes of electromagnetic oscillation in a wave guide are governed by the same equation as the modes of vibration of a uniform stretched membrane. In all these problems, therefore, correspondingly similar orthogonal properties can be quoted. It may be noticed that, in the case of H waves in a wave guide, the proof of the orthogonal property depends on the fact that the right-hand side of (5) vanishes when both $\partial w_r/\partial \nu$ and $\partial w_s/\partial \nu$ (instead of w_r and w_s) are zero round the boundary.

BIBLIOGRAPHY

Allen, D. N. de G., La Méthode de libération des liaisons et les problèmes de charpentes, *Extrait du Colloque Méthodes de Calcul* (Marseille, 1947), 11–15.

Allen, D. N. de G., Compléments pour l'application de la méthode de libération, *Extrait du Colloque Méthodes de Calcul* (Marseille, 1947), 18–34.

Allen, D. N. de G., and S. C. R. Dennis, The application of relaxation methods to the solution of differential equations in three dimensions. I. Boundary value potential problems, *Quart. Journ. Mech. and App. Math.*, **4** (1951), 199–208.

Allen, D. N. de G., and S. C. R. Dennis, Graded nets in harmonic and biharmonic relaxation, *Quart. Journ. Mech. and App. Math.*, **4** (1951), 439–443.

Allen, D. N. de G., and S. C. R. Dennis, The application of relaxation methods to the solution of differential equations in three dimensions. II. Potential flow round aerofoils, *Quart. Journ. Mech. and App. Math.*, **6** (1953), 81–100.

Allen, D. N. de G., L. Fox, H. Motz, and R. V. Southwell, Relaxation methods applied to engineering problems. VII C. Free transverse vibration of membrane with an application (by analogy) to two-dimensional oscillations in an electro-magnetic system, *Phil. Trans. R.S. C*, **1** (1942), 85–97, and *A*, **239** (1945), 488–500.

Allen, D. N. de G., L. Fox, and R. V. Southwell, Relaxation methods applied to engineering problems. VII D. Stress distributions in elastic solids of revolution, *Phil. Trans. R.S. C*, **1** (1943), 99–135, and *A*, **239** (1945), 501–537.

Allen, D. N. de G., and R. T. Severn, The application of relaxation methods to the solution of non-elliptic partial differential equations. I. The heat-conduction equation, *Quart. Journ. Mech. and App. Math.*, **4** (1951), 209–222.

Allen, D. N. de G., and R. T. Severn, The application of relaxation methods to the solution of non-elliptic partial differential equations. II. The solidification of liquids, *Quart. Journ. Mech. and App. Math.*, **5** (1952), 447–454.

Allen, D. N. de G., and R. V. Southwell, Relaxation methods applied to engineering problems. X. The graphical representation of stress, *Proc. Roy. Soc. A*, **183** (1944), 125–134.

Allen, D. N. de G., and R. V. Southwell, Relaxation methods applied to engineering problems. XIV. Plastic straining in two-dimensional stress-systems, *Phil. Trans. R.S. A*, **242** (1950), 379–414.

Allen, D. N. de G., R. V. Southwell, and G. Vaisey, Relaxation methods applied to engineering problems. XI. Problems governed by the "quasi-plane-potential equation," *Proc. Roy. Soc. A*, **183** (1945), 258–283.

Atkinson, R. J., K. N. E. Bradfield, and R. V. Southwell, Relaxation methods applied to a spar of varying section, deflected by transverse loading combined with end thrust or tension, *Aero. Res. Cttee. R. and M.*, No. 1822 (1937).

Atkinson, R. J., and R. V. Southwell, On the problem of stiffened suspension-bridges, and its treatment by relaxation methods, *Journ. Inst. Civil Engrs.*, **5** (1939), 289–312.

Bickley, W. G., Finite difference formulae for the square lattice, *Quart. Journ. Mech. and App. Math.*, **1** (1948), 35–42.

Black, A. N., The method of systematic relaxation applied to survey problems, *Empire Surv. Rev.*, **4** (1938), 406–413.

Black, A. N., Approximate methods of solving normal equations, *Empire Surv. Rev.*, **7** (1944), 242–245.

Black, A. N., and R. V. Southwell, Relaxation methods applied to engineering problems. II. Basic theory, with applications to surveying and to electrical networks, and an extension to gyrostatic systems, *Proc. Roy. Soc. A*, **164** (1938), 447–467.

Bowie, O. L., A least square application to relaxation methods, *Journ. App. Phys.*, **18** (1947), 830–833.

Bradfield, K. N. E., D. G. Christopherson, and R. V. Southwell, Relaxation methods applied to engineering problems. IV. Problems relating to elastic stability and vibrations, *Proc. Roy. Soc. A*, **169** (1939), 289–317.

Bradfield, K. N. E., and R. V. Southwell, Relaxation methods applied to engineering problems. I. The deflexion of beams under transverse loading, *Proc. Roy. Soc. A*, **161** (1937), 155–181.

Christopherson, D. G., Relaxation methods applied to grid frameworks, *Aero. Res. Cttee. R. and M.*, No. 1824 (1937).

Christopherson, D. G., A theoretical investigation of plastic torsion in an I-beam, *Journ. App. Mech.*, **7** (1940), A1-4.

Christopherson, D. G., A new mathematical method for the solution of film lubrication problems, *Proc. Inst. Mech. Engrs.*, **146** (1941), 126–135.

Christopherson, D. G., L. Fox, J. R. Green, F. S. Shaw, and R. V. Southwell, Relaxation methods applied to engineering problems. VII B. The elastic stability of plane frameworks and of flat plating, *Phil. Trans. R.S. A*, **239** (1945), 461–487.

Christopherson, D. G., and R. V. Southwell, Relaxation methods applied to engineering problems. III. Problems involving two independent variables, *Proc. Roy. Soc. A*, **168** (1938), 317–350.

Cooper, J. L. B., The solution of natural frequency equations by relaxation methods, *Quart. App. Math.*, **6** (1948), 179–183.

Crandall, S. H., Iterative procedures related to relaxation methods for eigenvalue problems, *Proc. Roy. Soc. A*, **207** (1951), 416–423.

Crandall, S. H., On a relaxation method for eigenvalue problems, *Journ. Math. and Phys.*, **30** (1951), 140–145.

Dalton, G. C. J., and F. S. Shaw, Note on the calculation of vibration frequencies for an aero-engine installation, *Aero. Res. Cttee. R. and M.*, No. 1917 (1940).

Dalton, G. C. J., F. S. Shaw, and R. V. Southwell, Natural frequencies of vibration for a wing carrying wing-engines, *Aero. Res. Cttee. R. and M.*, No. 1918 (1940).

Eddy, R. P., and F. S. Shaw, Numerical solution of elastoplastic torsion of a shaft of rotational symmetry, *Journ. App. Mech.*, **16** (1949), 139–148.

Emmons, H. W., The numerical solution of heat-conduction problems, *Trans. A.S.M.E.*, **65** (1943), 607–612.

Emmons, H. W., The numerical solution of partial differential equations, *Quart. App. Math.*, **2** (1944), 173–195.

Fox, L., Solution by relaxation methods of plane potential problems with mixed boundary conditions, *Quart. App. Math.*, **2** (1944), 251–257.

Fox, L., Some improvements in the use of relaxation methods for the solution of ordinary and partial differential equations, *Proc. Roy. Soc. A*, **190** (1947), 31–59.

Fox, L., Mixed boundary conditions in the relaxational treatment of biharmonic problems (plane strain or stress), *Proc. Roy. Soc. A*, **189** (1947), 535–543.

Fox, L., A short account of relaxation methods, *Quart. Journ. Mech. and App. Math.*, **1** (1948), 253–280.

Fox, L., The solution by relaxation methods of ordinary differential equations, *Proc. Cambridge Phil. Soc.*, **45** (1949), 50–68.

Fox, L., The solution of elliptic differential equations when the boundary conditions involve a derivative, *Phil. Trans. R.S. A*, **242** (1950), 345–378.

Fox, L., and R. V. Southwell, Relaxation methods applied to engineering problems. VII A. Biharmonic analysis as applied to the flexure and extension of flat elastic plates, *Phil. Trans. R.S. C*, **1** (1941), 15–56, and *A*, **239** (1945), 419–460.

Fox, L., and R. V. Southwell, On the stresses in hooks, and their determination by relaxation methods, *Journ. Inst. Mech. Eng.*, **155** (1946), 1–19.

Gandy, R. W. G., and R. V. Southwell, Relaxation methods applied to engineering problems. V. Conformal transformation of a region in plane space, *Phil. Trans. R.S. A*, **238** (1940), 453–475.

Gilles, D. C., The use of interlacing nets for the application of relaxation methods to problems involving two dependent variables, *Proc. Roy. Soc. A*, **193** (1948), 407–433.

Green, J. R., and R. V. Southwell, Relaxation methods applied to engineering problems. VIII A. Problems relating to large transverse displacements of thin elastic plates, *Phil. Trans. R. S. C*, **1** (1944), 137–176, and *A*, **239**, 541–580.

Green, J. R., and R. V. Southwell, Relaxation methods applied to engineering problems. IX. High-speed flow of compressible fluid through a two-dimensional nozzle, *Phil. Trans. R.S. A*, **239** (1944), 367–386.

Jacobs, J. A., Relaxation methods applied to problems of plastic flow. I. Notched bar under tension, *Phil. Mag.*, **41** (1950), 349–361.

Jacobs, J. A., Relaxation methods applied to problems of plastic flow. II, *Phil. Mag.*, **41** (1950), 458–467.

Marshall, W. T., The application of relaxation methods to freely-supported flat slabs, *Engineering*, **170** (1950), 239–242.

Mitchell, A. R., Application of relaxation to the rotational field of flow behind a bow shock wave, *Quart. Journ. Mech. and App. Math.*, **4** (1951), 371–383.

Mitchell, A. R., and D. E. Rutherford, Application of relaxation methods to compressible flow past a double wedge, *Proc. Roy. Soc. Edinburgh A*, **63** (1951), 139–154.

Mitchell, A. R., and D. E. Rutherford, On the theory of relaxation, *Proc. Glasgow Math. Ass.*, **1** (1953), 101–110.

Motz, H., Calculation of the electromagnetic field, frequency and circuit parameters of high-frequency resonator cavities, *Journ. Inst. Elec. Engrs.*, **93** (1946), 335–343.

Motz, H., The treatment of singularities of partial differential equations by relaxation methods, *Quart. App. Math.*, **4** (1947), 371–377.

Motz, H., and W. D. Worthy, Calculation of the magnetic field in dynamo-electric machines by Southwell's relaxation method, *Journ. Inst. Elec. Engrs.*, **92** (1945), 522–528.

Pellew, A., and R. V. Southwell, Relaxation methods applied to engineering problems. VI. The natural frequencies of systems having restricted freedom, *Proc. Roy. Soc. A*, **175** (1940), 262–290.

Richards, J. C., Stress-determination for a three-dimensional rigid-jointed framework by the method of systematic relaxation of constraints, *Journ. Inst. Civil Engrs.*, **5** (1937), 196–216.

Shaw, F. S., The torsion of solid and hollow prisms in the elastic and plastic range by relaxation methods, *Australian Council Aeronaut.*, Report 11 (1944).

Shaw, F. S., The approximate numerical solution of the non-homogeneous linear

Fredholm integral equation by relaxation methods, *Quart. App. Math.*, **6** (1948), 69–76.

Shaw, F. S., and R. V. Southwell, Relaxation methods applied to engineering problems. VII. Problems relating to the percolation of fluids through porous materials, *Proc. Roy. Soc. A*, **178** (1941), 1–17.

Southwell, R. V., Stress-calculation in frameworks by the method of systematic relaxation of constraints. I and II, *Proc. Roy. Soc. A*, **151** (1935), 56–95.

Southwell, R. V., Stress-calculation in frameworks by the method of systematic relaxation of constraints. III, *Proc. Roy. Soc. A*, **153** (1935), 41–76.

Southwell, R. V., "Relaxation Methods in Engineering Science," Oxford University Press (1940).

Southwell, R. V., New pathways in aeronautical theory, *Journ. Aeronaut. Sci.*, **9** (1942), 77–89.

Southwell, R. V., On relaxation methods: A mathematics for engineering science, *Proc. Roy. Soc. A*, **184** (1945), 253–288.

Southwell, R. V., "Relaxation Methods in Theoretical Physics," Oxford University Press (1946).

Southwell, R. V., Relaxation methods: A mathematics for the engineer, *Trans. Inst. Chem. Engrs.*, **25** (1947), 1–25.

Southwell, R. V., Relaxation methods applied to engineering problems. XIII. The flexure and extension of perforated elastic plates, *Proc. Roy. Soc. A*, **193** (1948), 147–171.

Southwell, R. V., Relaxation methods: An engineering approach to computation, *Journ. Inst. Civil Engrs.*, (1948), 351–378.

Southwell, R. V., Relaxation methods as applied to structures, *The Structural Engineer*, **26** (1948), 463–506.

Southwell, R. V., On the computation of strain and displacement in a prism plastically strained by torsion, *Quart. Journ. Mech. and App. Math.*, **2** (1949), 385–397.

Southwell, R. V., Relaxation methods, *British Science News*, **3**, 113–117.

Southwell, R. V., and G. Vaisey, Relaxation methods applied to engineering problems. VIII. Plane-potential problems involving specified normal gradients, *Proc. Roy. Soc. A*, **182** (1943), 129–151.

Southwell, R. V., and G. Vaisey, Relaxation methods applied to engineering problems. XII. Fluid motions characterized by "free" stream-lines, *Phil. Trans. R.S. A*, **240** (1946), 117–161.

Synge, J. L., A geometrical interpretation of the relaxation method, *Quart. App. Math.*, **2** (1944), 87–89.

Temple, G., The general theory of relaxation methods applied to linear systems, *Proc. Roy. Soc. A*, **169** (1939), 476–500.

Tranter, C. J., The combined use of relaxation methods and Fourier transforms in the solution of some three-dimensional boundary value problems, *Quart. Journ. Mech. and App. Math.*, **1** (1948), 281–286.

Vaughan, D. C., Relaxation methods. A three-dimensional mechanical analogy, *Quart. Journ. Mech. and App. Math.*, **5** (1952), 462–465.

Vazsonyi, A., A numerical method in the theory of vibrating bodies, *Journ. App. Phys.*, **15** (1944), 598–606.

Whittrick, W. H., and W. Howard, Relaxation methods applied to two problems of two-dimensional stress distribution involving mixed boundary conditions, *Australian Journ. Sci. Research A*, **1** (1948), 135–160.

Wood, R. H., A special type of group displacement for use in the relaxation technique, *Quart. Journ. Mech. and App. Math.*, **4** (1951), 432–438.

Woods, L. C., The numerical solution of two-dimensional fluid motion in the neighbourhood of stagnation points and sharp corners, *Aero. Res. Cttee. R. and M.*, No. 2726 (1949).

Woods, L. C., Improvements to the accuracy of arithmetical solutions to certain two-dimensional field problems, *Quart. Journ. Mech. and App. Math.*, **3** (1950), 349–363.

Woods, L. C., A new relaxation treatment of flow with axial symmetry, *Quart. Journ. Mech. and App. Math.*, **4** (1951), 358–370.

Woods, L. C., The relaxation treatment of singular points in Poisson's equation, *Quart. Journ. Mech. and App. Math.*, **6** (1953), 163–185.

Woods, M. W., and E. J. Warlow-Davies, On the application to tubular frameworks of the method of "systematic relaxation of constraints," *Aero. Res. Cttee. R. and M.*, No. 1764 (1936).

Zienkiewicz, O. C., The stress-distribution in gravity dams, *Journ. Inst. Civil Engrs.*, **27** (1947), 244–271.

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