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**CAMBRIDGE MATHEMATICAL SERIES.**

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**KEY TO ELEMENTARY GEOMETRY.**

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A KEY  
TO  
ELEMENTARY GEOMETRY

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## KEY TO ELEMENTARY GEOMETRY.

### EXERCISES A.

14. Place a straight ruler with its edge against the two end points of the line. Trace a straight line between these points by means of the ruler. If the two lines coincide, the original line is straight.

15. Take any two points A and B on a piece of paper, and place the edge of the ruler against them. Trace a line from A to B along the edge of the ruler. Again place the edge of the ruler against the points A and B, the ruler being *on the other side* of the line AB. Again trace a line from A to B as before. If these two lines coincide, the ruler is straight.

16. Place the straight edge of a ruler on the surface. If in all positions of the ruler the surface touches the edge of the ruler at every point, the surface is flat.

18. Two planes intersect in a straight line.

### EXERCISES B.

1. 3 in. = 7.62 cms.  $\therefore$  1 in. = 25.4 mm.

2. 4 in. = 10.16 cms. nearly  $\therefore$  1 in. = 25.4 mm.

3. 6 cms. = 2.36 in.  $\therefore$  1 cm. = .39 in. = .4 in. to the nearest tenth.

4. 8 cms. = 3.15 in.  $\therefore$  1 cm. = .39 in. = .4 in. ...

5. 11.45 cms. =  $4\frac{1}{2}$  in. nearly.

**EXERCISES C.**

6. Cut them out and superimpose them.
8. 360, 120, 180, 240.
9. 30,  $7\frac{1}{2}$ ,  $22\frac{1}{2}$ .
10. 90, 150, 240, 15, 75,  $187\frac{1}{2}$ .
14. The angles are rt.  $\angle$ s.
17. If the perpendicular is drawn along the edge AB of the set-sq. ABC, turn the set-square over, so that C lies on the other side of AB. Trace another line along the edge AB. If the two lines coincide the perpendicular is a true one.
23. Any two sides together  $>$  the third.
30. In an isosceles  $\triangle$ .
31. They meet in a pt. This pt. is a pt. of trisection of each.

**EXERCISES D.**

2.  $90^\circ$ .
3.  $90^\circ$ .
4.  $90^\circ$ .
5.  $90^\circ$ .

**EXERCISES E.**

2. A regular hexagon.
3. A square.

**EXERCISES G.**

3. See Book I., Prop. 31, first method.

**EXERCISES H.**

3. Make a crease ABC. Fold again so that BC falls along BA. If DBE is the second crease, DBC is a rt.  $\angle$ . A fold along DC makes DBC a rt.  $\angle$ d.  $\triangle$ .
4. Make a crease ABC, and, as in Example 3 above, a second crease DBE at rt.  $\angle$ s to it. Fold the double thickness again along DC, and we have an isos.  $\triangle$ .
5. Fold so that BC falls along CA, and the crease is at rt.  $\angle$ s to ABC at C.
6. Fold the paper about AB. Mark (or prick through) the pt. D where C falls. CD when joined is the reqd. line.

9. Make two creases at rt.  $\angle$ s to any st. line AB as in Example 5. These two creases are parallel.

10. and 11. Use the methods of Examples 5 and 9.

### EXERCISES K.

11. By compasses set off an equal length on one of the lines of the paper.

12. 3.6 miles.

13. 3.6 miles.

14. 19.14 yards.

15. 8 ft.

16. Let ABCD be the larger sq., CEFG the smaller, G lying in CD, and BCE being a str. line. In CB take a pt. H such that CH = GD. Join FH and cut along the line FH. Also join AH and cut along the line AH.  $\triangle FEH$  fitted into  $\angle FGD$ , so that FE coincides with FG, and EH falls along GD, and  $\triangle AHB$  placed so that AB coincides with AD, and BH is in a str. line with GD, will form a square.

19. On squared paper take AB 4 half inches long, to represent the support ( $\frac{1}{2}$  in. represents 1 foot). Along the line thro. B take BC 3 half inches long.  $AC^2 = 3^2 + 4^2$  (Exercises D. 2), and  $AC = 5$  half in. Produce CA to D, making  $AD = AC$ . Read off the perp. distance of D from CB produced. This is equal to 8 half inches  $\therefore$  the end of the see-saw can rise 8 ft. from the ground. Also by measurement the  $\angle$  reqd. =  $\angle ACB = 53^\circ$

21. On squared paper take AC 24 units long, and AB at rt.  $\angle$ s to it 10 units long. BC represents the rope. With centre B and rad. BC describe a circle meeting BA produced at D. Read off BD and we find it to be 26 units  $\therefore$  the rope is 26 feet long.

22. Take AB 2.4 in long and BC at rt.  $\angle$ s to it .7 in. long. With centre A and rad. AC describe a circle cutting AB produced at D. Reading off AD, we find it to be 2.5.  $\therefore$  the length reqd. = 25 feet.

23. The angle is  $90^\circ$ ; length of crease  $3\frac{3}{4}$  inches. See Ex. P. 12.

### EXERCISES P.

4. 6 ft. 8ft.

9. See Ex. XXIII. 4.

12. If EOF is the crease, E lying in CD, O in AC, F in AB, EOC is a rt.  $\angle$ .  $AC^2 = AD^2 + DC^2 = 3^2 + 4^2 = 5^2 \therefore AC = 5$  ft.  $\triangle EOC$  is equiangular to  $\triangle ADC \therefore \frac{EO}{OC} = \frac{AD}{DC}$  i.e.  $\frac{EO}{\frac{5}{2}} = \frac{3}{4} \therefore EO = \frac{1.5}{8}$  ft.  $\therefore$  the crease  $= \frac{1.5}{4} = 3\frac{3}{4}$  ft.

### EXERCISES I.

1. The pencil mark has *some* width and is therefore not a true line.
2. The dot has *some* length and breadth and is therefore not a true point.
18. Two planes intersect in a str. line.

### EXERCISES II.

1. The ext.  $\angle$ s are respectively supplementary to the int. and equal  $\angle$ s (I. 1.) and are therefore equal.
2. Similar to the above.
3. If BE, BF are the bisectors,  $\angle EBF = \angle EBA + \angle FBA = \frac{1}{2}(\angle ABC + \angle ABD) =$  a rt.  $\angle$ .
4. If the crease DE meets AB at C, the adj.  $\angle$ s ACD, BCD coincide when we fold, and are therefore equal. They are also adj.  $\angle$ s and are therefore rt.  $\angle$ s.
5. The four  $\angle$ s at the meeting pt. are equal to four rt.  $\angle$ s  $\therefore$  two adj.  $\angle$ s = 2 rt.  $\angle$ s. The theorem thus follows by I. 2.
6.  $\angle AOP + \angle AOQ = \angle BOQ + \angle AOQ = 2$  rt.  $\angle$ s (I. 1.)  $\therefore$  POQ is a str. line (I. 2.).
7. Any two adj.  $\angle$ s = half all the  $\angle$ s at the meeting pt. = 2 rt.  $\angle$ s. The proposition thus follows by I. 2.
8. Let EO be the bisector of  $\angle AOD$ , and OF the bisector of the opp. vert.  $\angle BOC$ .  $\angle AOE + \angle AOF = \angle AOE + \angle AOC + \angle COF = \angle AOE + \angle AOC + \frac{1}{2}\angle COB = \angle AOE + \angle AOC + \frac{1}{2}\angle AOD = \angle AOD + \angle AOC = 2$  rt.  $\angle$ s  $\therefore$  EOF is a str. line (I. 2.).
9. Produce AB, AC, sides of  $\triangle ABC$  to D and E.  $\angle$ s ABC, DBC = 2 rt.  $\angle$ s (I. 1.)  $\angle$  ACB, ECB = 2 rt.  $\angle$ s (I. 1.)  $\therefore$  these four  $\angle$ s = 4 rt.  $\angle$ s.
10. They are in a str. line by I. 2.
11. They are in a str. line, as in Example 6, above.

**EXERCISES III.**

1.  $\triangle s$  ABC, AED are equal in all respects by I. 4.
2.  $\triangle s$  ABC, DEF are equal in all respects by I. 4.
3.  $\triangle s$  ACD, BCD are equal in all respects by I. 4.
4. As in Example 2 above  $AC=AE$  and  $AE=EC$ .
5.  $\angle AEC=\angle BED$  (I. 3.)  $\therefore \triangle s$  AEC, BED are equal in all respects by I. 4.
6. This follows at once from I. 4.
7. Join AC, XZ. From  $\triangle s$  ABC, XYZ,  $AC=XZ$  and  $\angle ACB=\angle XZY$  (I. 4.). From  $\triangle s$  ACD, XZW,  $AD=WX$  (I. 4.). Also by the same prop. the  $\angle s$  are equal respectively  $\therefore$  the quadr<sup>ls</sup> are equal in all respects.
8. Let E be the mid. pt. of CD, Join AE, BE. From  $\triangle s$  ACE, BDE,  $AE=BE$  (I. 4.).
9. This follows at once from  $\triangle s$  ABE, ACD (I. 4.).
10.  $\triangle ABF=\triangle DAE$  (I. 4.). Take  $\triangle AGE$  from each and fig.  $GEBF=\triangle AGD$ .
11. This follows from  $\triangle s$  ACD, BCD by I. 4.
12. This follows from I. 4.

**EXERCISES IV.**

1. In  $\triangle s$  ABF, ACF,  $AB=AC$ ; AF is common; and  $\angle BAF=\angle CAF$   $\therefore \angle ABF=\angle ACF$  (I. 4.)  $\therefore$  also their supplements  $\angle s$  DBC, ECB are equal (I. 1.).
2. In  $\triangle s$  BAO, CAO  $\angle ABO=\angle ACO$  (I. 4.)  $=\frac{1}{2}\angle ACB=\frac{1}{2}\angle ABC$ .
3. Let D, E, F be the mid. pts. of AB, AC, BC respectively. By I. 5  $\angle DBF=\angle ECF$ .  $\therefore$  from  $\triangle s$  DBF, ECF,  $DF=EF$  (I. 4.).
4. From  $\triangle s$  BAQ, CAP,  $BQ=CP$  (I. 4.).
5. From  $\triangle s$  BAQ, CAP,  $BQ=CP$  (I. 4.).
6. From  $\triangle s$  ACP, ABQ,  $\angle ACP=\angle ABQ$  (I. 4.). Also their parts  $\angle s$  ACB, ABC are equal (I. 5.)  $\therefore \angle BCP=\angle CBQ$ .
7. From  $\triangle s$  AFE, BDF,  $EF=DF$  (I. 4.). In the same way,  $DE=EF$   $\therefore \triangle DEF$  is equilateral.
8.  $\angle ABD=\angle ACE$  (I. 5.)  $\therefore$  from  $\triangle s$  ABD, ACE,  $AD=AE$  (I. 4.).

9. Produce AD to E. In  $\triangle$ s BAD, CAD,  $\angle$ BDA =  $\angle$ CDA (I. 4.).  $\therefore$  their supplements  $\angle$ s BDE, CDE are equal. For proof that CD bisects  $\angle$ ACB, see Exercise 2 above.

10. Each of the angles =  $66\frac{1}{2}^\circ$ . They are equal by I. 5.

### EXERCISES V.

1. Let the sides AB, AC of  $\triangle$ ABC be produced to D and E, so that  $\angle$ DBC =  $\angle$ ECB. Their supplements  $\angle$ s ABC, ACB are equal.  $\therefore$  AB = AC and the  $\triangle$ ABC is isosceles (I. 6.).

2. This follows at once from I. 6.

3. OB = OC  $\therefore$   $\angle$ OBC =  $\angle$ OCB (I. 5.)  $\therefore$   $2\angle$ OBC =  $2\angle$ OCB *i.e.*  $\angle$ ABC =  $\angle$ ACB  $\therefore$  AB = AC (I. 6.).

4. Produce DA, CB to meet in O. OD = OC (I. 6.).  $\angle$ DAB =  $\angle$ CBA  $\therefore$  their supplements  $\angle$ s OAB, OBA are equal (I. 1.).  $\therefore$  OA = OB (I. 6.). But OD = OC  $\therefore$  AD = BC.

5. In  $\triangle$ s DCB, ECB,  $\angle$ DBC =  $\angle$ ECB (I. 4.)  $\therefore$  BA = AC (I. 6.).

6.  $\angle$ DBC >  $\angle$ ABC. But  $\angle$ ABC =  $\angle$ ACB (I. 5.)  $\therefore$   $\angle$ DBC >  $\angle$ DCB.

7. From  $\triangle$ s AEB, CDE, EB = EC (I. 4.).

8. Let the sides AB, AC in  $\triangle$ ABC be unequal. If  $\angle$ ABC =  $\angle$ ACB, then AB = AC (I. 6.) which is contrary to the hypothesis.

9. Let the angles ABC, ACB of  $\triangle$ ABC be unequal. If AB = AC,  $\angle$ ABC =  $\angle$ ACB, which is contrary to the hypothesis.

10. CA should be equal to CB, by I. 6.

11. The exterior  $\angle$ s should be  $136^\circ$  and  $132^\circ$  by I. 1. The third  $\angle$  of the  $\triangle$  will be found to be  $88^\circ$ .

### EXERCISES VI.

1. This follows by I. 7 from  $\triangle$ s ACB, ADB.

2. From  $\triangle$ s ACB, ADB,  $\angle$ CAE =  $\angle$ DAE (I. 7.)  $\therefore$  the  $\triangle$ s CAE, DAE are equal in all respects by I. 4.

3. Let EO, FO bisect AD and BC at rt.  $\angle$ s at E and F. From  $\triangle$ s AEO, DEO, AO = DO (I. 4.). From  $\triangle$ s BFO, CFO, BO = CO (I. 4.).  $\therefore$   $\triangle$ s AOB, COD are equal in all respects (I. 7.)

4.  $\angle GBC = \angle GCB \therefore GB = GC$  (I. 6.)  $\therefore$  from  $\triangle s$  AGB, AGC  $\angle BAG = \angle CAG$  (I. 7.).

5. Take O the centre of the circle, and let AC meet BO at E. From  $\triangle s$  AOB, COB,  $\angle AOB = \angle CBO$  (I. 7.). Then from  $\triangle s$  ABE, CBE,  $AE = EC$ , and  $\angle AEB = \angle CEB$  (I. 4.). Also these  $\angle s$  are adj. and therefore rt.  $\angle s$ .

6. Let ADB be the new position of the  $\triangle ABC$ . Join CD, meeting AB at E. From  $\triangle s$  ACB, ADB,  $\angle ABC = \angle ABD$ , for one is a copy of the other. From  $\triangle s$  BEC, BED,  $CE = DE$  and  $\angle CEB = \angle DEB$  (I. 4.). Also these  $\angle s$  are adj. and therefore rt.  $\angle s$ .

7. Join EB. From  $\triangle s$  EDB, BCE,  $\angle EDB = \angle BCE$  (I. 7.).

8. From  $\triangle s$  ACB, ADB,  $\angle CBA = \angle DAB$ , and  $\angle CAB = \angle DBA$  (I. 7.)  $\therefore \angle DAC = \angle DBC$ . Also  $AO = BO$  (I. 6.)  $\therefore \triangle s$  AOC, BOD are equal in all respects (I. 4.).

9. Take a pt. P equidistant from A and D. Join OP. From  $\triangle s$  AOP, DOP  $\angle AOP = \angle DOP$  (I. 7.). But  $\angle AOB = \angle COD$  (hyp.)  $\therefore \angle POB = \angle POC \therefore$  from  $\triangle s$  POB, POC,  $PB = PC$  (I. 4.).

10. Let AC, DB meet at O.  $\angle ADB = \angle ABD$  (I. 5.)  $\therefore \angle CDB = \angle CBD \therefore CD = CB$  (I. 6.)  $\therefore$  from  $\angle s$  ADC, ABC,  $\angle DAO = \angle BAO$  (I. 7.)  $\therefore$  from  $\angle s$  AOD, AOB,  $DO = OB$  and  $\angle AOD = \angle AOB =$  a rt.  $\angle$  (I. 4.).

11. From  $\triangle s$  DAC, EBC,  $\angle DAC = \angle EBC$  (I. 7.)  $\therefore$  from  $\triangle s$  DAB, EBA,  $DB = EA$  (I. 4.).

12. From  $\triangle s$  ABC, CDA,  $\angle DAC = \angle BCA$  (I. 7.)  $\therefore \angle OAC = \angle OCA \therefore OA = OC$  (I. 6.). In the same way from  $\triangle s$  DAB, BCD,  $OB = OD$ .

13. From  $\triangle s$  AOD, COD,  $\angle AOD = \angle COD$  (I. 7.) from  $\triangle s$  AOB, COB,  $\angle AOB = \angle COB$  (I. 7.)  $\therefore \angle AOD + \angle AOB = \frac{1}{2}$  the 4  $\angle s$  round O  $= 2$  rt.  $\angle s \therefore DOB$  is a str. line (I. 2.).

14. From  $\triangle s$  AOE, COE,  $\angle AOE = \angle COE$  (I. 7.)  $= \frac{1}{2} \angle AOC = \frac{1}{2} \angle BOD$  (I. 3.). From  $\triangle s$  DOF, BOF,  $\angle DOF = \angle BOF = \frac{1}{2} \angle BOD \therefore \angle DOF = \angle COE$ . Add  $\angle EOD$  to each and  $\angle DOF + \angle DOE = \angle COE + \angle DOE = 2$  rt.  $\angle s$  (I. 1.)  $\therefore EOF$  is a str. line (I. 2.).

15.  $OA = OB \therefore \angle OAB = \angle OBA$  (I. 5.)  $\angle EOA = \angle DOB$  (I. 3.)  $\therefore$  from  $\triangle s$  EOA, DOB,  $\angle EAO = \angle DBO$  (I. 4.)  $\therefore \angle EAB = \angle DBA \therefore AC = BC$  (I. 6.).

16.  $BF = CF$  (I. 6.)  $\angle EFB = \angle DFC$  (I. 3.). But  $\angle AFE = \angle AFD$   
 $\therefore \angle AFB = \angle AFC \therefore$  from  $\triangle s$   $AFB, AFC, AB = AC$  (I. 4.).

17. From  $\triangle s$   $ADB, ADC, \angle BDA = \angle CDA$  (I. 4.) = a rt.  $\angle$ .

18. From  $\triangle s$   $AEO, BEO, \angle AOE = \angle BOE$  (I. 7.)  $\therefore \angle DOF = \angle COF$  (I. 3.)  $\therefore$  from  $\triangle s$   $DOF, COF, DF = CF$  (I. 4.).

19. From  $\triangle s$   $DOA, DOB, \angle DOA = \angle DOB$  (I. 7.) = a rt.  $\angle$ .  
 We have drawn a perp. to  $AB$  at its mid. pt.  $O$ .

### EXERCISES VII.

1. If possible let there be drawn from the point  $A$  to the str. line  $BC$  three equal str. lines  $AB, AC, AD$ .  $AD = AC \therefore \angle ADC = \angle ACD$  (I. 5.)  $AB = AC \therefore \angle ABC = \angle ACB$  (I. 5.)  $\therefore$  ext.  $\angle ABC =$  int. opp.  $\angle ADB$  which is impossible (I. 8.).

2. If a circle whose centre is  $A$  cut a str. line at three points,  $B, C, D$ , we should have three equal str. lines from a pt. to a str. line, which is impossible by the preceding.

3. If in  $\triangle ABC, \angle ABC = \angle ACB =$  a rt.  $\angle$ , produce  $BC$  to  $D$ .  $\angle ACD =$  a rt.  $\angle =$  int. opp.  $\angle ABC$  which is impossible (I. 8.).

4. This follows at once from I. 4.

5. As in the preceding  $\triangle AEB = \triangle CEF$ . Adding  $\triangle BEC$  to each,  $\triangle ABC = \triangle BCF$ .

6.  $\angle CBD =$  supplement of  $\angle CBA = 116^\circ \therefore \angle CBD$  is gr. than  $\angle CAB$  by  $77^\circ$ .

7. From  $\triangle s$   $BOD, AOD, \angle BDO = \angle ADO$  (I. 4.) = a rt.  $\angle$ .

8.  $\angle DOC = \angle AOB$  (I. 3.)  $\therefore$  from  $\triangle s$   $DOC, BOA, CD = AB =$  8 cms. (I. 4.).

9. With centre  $B$  and rad.  $BC$  describe a circle to meet  $BA$  produced at  $D$ . Read off the length of  $BD$ , estimating the second decimal place ( $BD = 5.83$  cms.).

### EXERCISES VIII.

1. Bisect  $BC$  at  $H$ . Join  $AH$  and produce it to  $K$ , making  $HK$  equal to  $AH$ . Join  $CK$ .  $\angle KHC = \angle AHB$  (I. 3.)  $\therefore$  from  $\triangle s$   $KHC, AHB, \angle KCH = \angle ABH$  (I. 4.)  $\therefore \angle HCG$  i.e.  $\angle ACD$  is gr. than  $\angle ABC$ .

2. Produce DA to E, BC to F. Join DB. Ext.  $\angle BAE > \text{int. opp. } \angle ABD$  (I. 8.) Also ext.  $\angle DCF > \text{int. opp. } \angle DBC$  (I. 8.)  $\therefore$  sum of the  $\angle$ s BAE, DCF is gr. than  $\angle ABC$ . In the same way the sum of the  $\angle$ s BAE, DCF is gr. than  $\angle ADC$ .

3. Let CO produced meet DB at E. Ext.  $\angle COD > \text{int. opp. } \angle DEO$  of  $\triangle DEO$  (I. 8.). Ext.  $\angle DEO > \text{int. opp. } \angle EBC$  of  $\triangle EBC$  (I. 8.)  $\therefore \angle COD > \angle DBC$ . Also in  $\triangle DAB$  2 rt.  $\angle$ s  $>$  the two  $\angle$ s DAB, DBA  $\therefore \angle COD + 2 \text{ rt. } \angle$ s  $> \angle DBC + \angle DAB + \angle DBA$  i.e.  $> \angle DAB + \angle CBA \therefore \angle DAB + \angle CBA$  differ from  $\angle COD$  by less than 2 rt.  $\angle$ s.

4. Take a quad<sup>l</sup> ABCD. Place a pencil on the line AB, pointing from A to B. Turn the pencil about the pt. A through the  $\angle BAD$ , the pencil then pointing from A to D. Next turn the pencil about D to the position CD, the pencil now pointing from C to D. Again turn the pencil about C into the position CB, the pencil now pointing from C to B. To complete a turn through 4 rt.  $\angle$ s we shall have to turn the pencil further about B through the angle CBA  $\therefore$  the three  $\angle$ s BAD, ADC, DCB are less than 4 rt.  $\angle$ s.

5. Join A to any pt. D in BC. Ext.  $\angle ADC > \text{int. opp. } \angle ABC$  (I. 8.). Ext.  $\angle BDA > \text{int. opp. } \angle ACB$  (I. 8.)  $\therefore$  the two  $\angle$ s BDA, CDA are together  $> \angle$ s ABC, ACB, i.e.  $\angle$ s ABC, ACB are together  $< 2 \text{ rt. } \angle$ s.

6. Two obtuse angles are together  $> 2 \text{ rt. } \angle$ s  $\therefore$  no  $\triangle$  can have two obtuse angles (I. 9.).

7. On sqd. paper take AB, a vertical line, 4 in. long to represent 48 ft. With centre A and rad. 5 in. (to represent 60 ft.) describe a circle cutting the horizontal line thro. B at C. Read off BC (= 3 in.) and we find the dist. reqd. is 36 ft.

8. ABD will be a rt. angle.

9. The  $\angle$ s will be rt.  $\angle$ s.

10. By folding as in Exercises C. 15, make a rt.  $\angle DEF$ . Fold so that EF falls along ED. The new crease will bisect  $\angle DEF$ , and  $\therefore$  make  $\angle$ s of  $45^\circ$  with ED and EF.

11. Repeat Exercise H. 3.

**EXERCISES IX.**

1. Let  $ABC$  be a  $\triangle$ , rt.  $\angle$ d. at  $B$ . Produce  $AB$  to  $D$ .  $\angle ABC = \angle CBD > \text{int. opp. } \angle BCA$  (I. 8.)  $\therefore AC > AB$ . Similarly  $AC > BC$ .

2. Let  $ABC$  be a  $\triangle$  having  $AB$  less than  $AC$   $\therefore \angle ACB$  is less than  $\angle ABC$ . If  $\angle ACB$  were obtuse or a rt.  $\angle$ ,  $\angle ACB + \angle ABC$  would be gr. than two rt.  $\angle$ s, which is impossible (I. 9.).

3. The greatest  $\angle$  is a rt.  $\angle$ .

4. If  $x$  and  $y$  are the other sides of the reqd.  $\triangle$ ,  $\frac{x}{44} = \frac{y}{54} = \frac{90}{60}$   $\therefore x = 66$  mms. and  $y = 81$  mms.

5.  $BC$  on paper = 5.05 cms. approx.  $\therefore$  the dist. reqd. = 253 yds. nearly.

6. Fold the paper so that  $BD$  falls on  $BC$ . If the crease coincides with  $BA$ ,  $BA$  is perp. to  $CD$ .

**EXERCISES X.**

1. Take any quadl  $ABCD$ . Join  $AC$ .  $AB + BC > AC$  (I. 12.)  $\therefore AB + BC + CD > AC + CD > AD$  (I. 12.).

2.  $AD + AC > DC$  (I. 12.)  $\therefore BD + BC + AD + AC > BD + BC + DC$  *i.e.* perimeter of  $\triangle ABC >$  perimeter of  $\triangle BDC$ .

3. Let  $PQ$ ,  $SR$  meet in  $O$ . Then  $OQ = OR$  (I. 6.). But  $\angle RSP$  is gr. than, equal to, or less than  $\angle SPQ$  according as  $OP$  is gr. than, equal to, or less than  $OS$  *i.e.* as  $PQ$  is gr. than, equal to, or less than  $RS$ .

4. Ext.  $\angle EDC > \text{int. opp. } \angle ABC$  (I. 8.)  $> \angle ACB$  for  $\angle ABC = \angle ACB$  (I. 5.)  $> \angle ECD$   $\therefore EC > ED$  (I. 11.)  $\therefore EC > EA$ .

5. See Exercises VII., Example 1.

6. Let  $ABCD$  be the quadl.  $AB + BC > AC$  (I. 12.) and  $AD + DC > AC$  (I. 12.)  $\therefore AB + BC + CD + DA > 2AC$ . Similarly  $AB + BC + CD + DA > 2BD$   $\therefore$  adding and dividing by 2,  $AB + BC + CD + DA > AC + BD$ .

7. Let the diagonals of the quadl.  $ABCD$  cut at the pt.  $O$ .  $AO + OB > AB$ ,  $BO + OC > BC$ ,  $CO + OD > CD$ ,  $DO + AO > AD$  (I. 12.)  $\therefore$  adding and dividing by 2,  $AC + DB > \frac{1}{2}(AB + BC + CD + AD)$ .

8. Produce AO to meet BC at D.  $\angle ADB > \text{int. opp. } \angle ACB$  (I. 8.)  $> \angle ABD \therefore AB > AD > AO$ . Also  $BO + OC > BC$  (I. 12.)  $> AB > AO$ .

9. Take any pt. O within the quadr. and not at their intersection.  $BO + OD > BD$  (I. 12.).  $AO + OC > AC$  (I. 12.)  $\therefore AO + BO + CO + DO > AC + BD$ .

10. Take O the common centre, A a pt. on the larger circle. Let OA meet the smaller circle at B. We have to prove that AB is shorter than any other line drawn from A to the smaller circle. Take any pt. C on the smaller circle. Join OC, CA.  $OC + CA > OA$  (I. 12.) *i.e.*  $OC + CA > OB + BA$   $\therefore CA > BA$ .

11. Let ABCDEF be any rectil. fig. O any point within it. Join OA, OB, etc.  $OA + OB > AB$ ,  $OB + OC > BC$ , and so on.  $\therefore$  adding,  $2(OA + OB + OC + OD + OE + OF) > \text{the perimeter}$ .  $\therefore OA + OB + \text{etc.} > \frac{1}{2} \text{ perimeter}$ .

12.  $\angle BAC$  is obtuse  $\therefore BD > BA$  and  $CE > CA$  (I. 11.)  $\therefore BD + CE > BA + CA > BE + CD + EA + DA$ , but  $EA + DA > ED$  (I. 12.)  $\therefore BD + CE > BE + CD + DE$ .

13. Produce AD to G, and make DG equal to DA.  $\angle ADC = \text{opp. vert. } \angle BDG$  (I. 3.)  $\therefore$  from  $\triangle$ s BDG, CDA,  $BG = AC$  (I. 4.) Also  $AB + BG > AG$  (I. 12.) *i.e.*  $AB + AC > 2AD$ . Similarly  $AC + CB > 2CF$  and  $CB + BA > 2BE \therefore$  adding  $AB + BC + CA > AD + BE + CF$ .

14.  $AD + DB > AB$  (I. 12.) *i.e.*  $2AD + BC > 2AB$ . Similarly,  $2BE + AC > 2BC$ ,  $2CF + AB > 2CA \therefore$  adding  $2(AD + BE + CF) > AB + BC + CA$ .

15. Thro. C draw ACD equal to the string. With centre A and rad. AB desc. a circle, and with centre C and rad. CD desc. a second circle meeting the first at B and E.  $CB = CD$  radii  $\therefore AC + CB = \text{the string}$   $\therefore AB$  is the reqd. position of the rod. The position AE gives a second solution. For a solution to be possible, the circles must meet.  $AC + CB$  must be  $> AB$ , *i.e.* the string must be longer than the rod.

### EXERCISES XI.

1.  $BA < BD + DA$  (I. 12.)  $CA < CD + DA$  (I. 12.)  $\therefore BA + CA < BD + CD + 2DA$  *i.e.* the diff. of  $BA + CA$  and  $BD + CD < 2DA$ .

2. Take any pt. O within the  $\triangle ABC$ . From  $\triangle s$  DAC, BAC,  $\angle DAC = \angle BAC$  (I. 7.)  $\therefore \angle DAO > \angle BAO$   $\therefore$  from  $\triangle s$  DAO, BAO, DO  $>$  BO (I. 14.).

3. Let D be any pt. on the bisector of the  $\angle BAC$ , and let DE, DF be drawn perp. to AB and AC respectively. From  $\triangle s$  AED, AFD,  $DE = DF$  (I. 16.).

4. Let BD, CE be drawn from the extremities of the base BC of the isos.  $\triangle ABC$  perp. to the opp. sides.  $\angle ABC = \angle ACB$  (I. 5.)  $\therefore$  from  $\triangle s$  EBC, DCB,  $CE = DB$  (I. 16.).

5. Let the diagonal AC of the quadl. ABCD bisect the  $\angle s$  at A and C, and let BD meet AC at O. From  $\triangle s$  ADC, ABC,  $CD = CB$  (I. 16.)  $\therefore$  from  $\triangle s$  DOC, BOC,  $DO = OB$  and  $\angle DOC = \angle BOC =$  a rt.  $\angle$  (I. 4.).

6. Let AD, BE meet at P, BE and CF at Q, CF and AD at R. The angles of  $\triangle ABC$  are all equal (I. 5.)  $\therefore \triangle s$  ADC, BEA, CFB are equal in all respects (I. 4.)  $\therefore \triangle s$  RDC, QFB, PEA are equal in all respects (I. 16.)  $\therefore \angle CRD = \angle APE = \angle BQF$   $\therefore \angle PRQ = \angle RPQ = \angle PQR$  (I. 3.)  $\therefore PQ = QR = RP$  (I. 6.).

7. From  $\triangle s$  BAD, ABC,  $\angle ABD = \angle BAC$  (I. 4.). Also  $\angle AEB = \angle BFA$  (hyp.)  $\therefore \triangle s$  AEB, BFA are equal in all respects (I. 16.).

## EXERCISES XII.

1.  $\angle FAD = \angle DAE$  (Hyp.) = alt  $\angle FDA$  (I. 20.)  $\therefore FA = FD$  (I. 6.). Then from  $\triangle s$  DFA, DEA,  $DE = DF$  and  $EA = FA$  (I. 16.)  $\therefore$  AEDF is equilateral.

2. Let PN be the perp. on the line thro. A, PM the perp. on that thro. B.  $\angle PAN =$  alt.  $\angle PBM$  (I. 20.)  $\therefore$  from  $\triangle s$  PAN, PBM,  $PA = PB$  (I. 16.).

3. Let FAE be  $\parallel$  to BC, DBF  $\parallel$  to CA, and ECD  $\parallel$  to AB.  $\angle ABC =$  alt.  $\angle BCD$  (I. 20.) = int. opp.  $\angle DEF$  (I. 20.) Similarly,  $\angle BAC = \angle FDE$  and  $\angle ACB = \angle DFE$ .

4. Let  $\triangle s$  ABC, DBC be on the same base BC, and between the same  $\parallel s$  AD and BC such that DC bisects AB at O.  $\angle OAD =$  alt.  $\angle OBC$  (I. 20.) and  $\angle DOA = \angle COB$  (I. 3.)  $\therefore DO = OC$  (I. 16.). Also  $\angle BOD = \angle COA$  (I. 3.)  $\therefore$  from  $\triangle s$  BOD, AOC,  $\angle BDO = \angle ACO$  (I. 4.)  $\therefore BD$  is  $\parallel$  to CA.

5.  $\frac{1}{2} \angle AED > \frac{1}{2} \angle BAC$  (I. 8.) *i.e.*  $\angle ACF > \text{alt. } \angle CAE \therefore CF$  is not  $\parallel$  to  $EA$ .

6.  $\angle CDE = \text{int. opp. } \angle CAB$  (I. 20.)  $= \angle ABC$  (I. 5.)  $= \text{ext. } \angle DEC$  (I. 20.)  $\therefore CD = CE$  (I. 6.)  $\therefore DA = EB \therefore$  from  $\triangle s DAB, EBA, \angle DBA = \angle EAB$  (I. 4.). Also  $\angle EDF = \text{alt. } \angle DBA$  (I. 20.)  $= \angle EAB = \text{alt. } \angle FED$  (I. 20.)  $\therefore DF = EF$  (I. 6.).

7. Let  $O$  be the centre of the circle. Join  $OC, OD$ .  $\angle OCA = \angle OAC$  (I. 5.)  $= \text{alt. } \angle OBD$  (I. 20.)  $= \angle ODB$  (I. 5.)  $\therefore$  from  $\triangle s AOC, DOB, AC = BD$  (I. 16.).

8. Let  $O$  be the centre of the circle, and  $EBF$  be drawn  $\parallel$  to  $CD$ .  $\angle EBC = \text{alt. } \angle BCO$  (I. 20.)  $= \angle CBO$  (I. 5.). Similarly  $\angle FBD = \angle OBD$ .

9. Let  $ABCD$  be a quadl. having  $BC \parallel$  to  $AB$  but not equal to it. If possible let  $AB$  be  $\parallel$  to  $CD$ . Then  $\angle BAC = \text{alt. } \angle ACD$  (I. 20.) and  $\angle BCA = \text{alt. } \angle CAD$  (I. 20.)  $\therefore$  from  $\triangle s ABC, CDA, BC = AD$  (I. 16.). Which is contrary to the hypothesis.

10. Let  $AF$  the perp. at  $A$  meet  $OB$  at  $E$  and let  $BG$  be the perp. at  $B$ . If possible let  $AF$  be  $\parallel$  to  $BG$ . Then  $\angle EBG = \text{ext. } \angle OEF$  (I. 20.) wh.  $> \text{int. opp. } \angle OAE$  (I. 8.) but  $\angle EBG = \angle OAE$  by hyp.  $\therefore AF$  and  $BG$  cannot be  $\parallel$  *i.e.* they must meet if produced.

11. From  $\triangle s DBC, ECB, DB = EC$  (I. 16.)  $\therefore AD = AE$ . If  $DE$  is not  $\parallel$  to  $BC$  let  $DF \parallel$  to  $BC$  meet  $AC$  at  $F$ .  $\angle ADF = \text{int. opp. } \angle ABC$  (I. 20.)  $= \angle ACB$  (I. 5.)  $= \text{ext. } \angle AFD$  (I. 20.)  $\therefore AF = AD = AE$  wh. is absurd  $\therefore DE$  must be  $\parallel$  to  $BC$ . Similarly,  $MN$  is  $\parallel$  to  $BC \therefore MN$  is  $\parallel$  to  $DE$ .

12. Draw  $AM$  perp. to  $BC$ , and  $EH \parallel$  to  $BC$  to meet  $AM$  at  $H$ . Draw  $DK \parallel$  to  $AH$  to meet  $EH$  in  $K$ . From  $\triangle s ADM, DEK, AM = DK$  (I. 16.). From  $\triangle s DKH, HMD, DK = MH$  (I. 20. and 16.)  $\therefore MH = AM \therefore H$  is a fixed point and  $EH$  is a fixed line.

### EXERCISES XIII.

1. Let  $n$  be the number of sides.  $8 \text{ rt. } \angle s + 4 \text{ rt. } \angle s = 2n \text{ rt. } \angle s$  (I. 22. Cor. 1.)  $\therefore n = 6$ .

2.  $\text{Ext. } \angle = 30^\circ \therefore 30n = 360$  (I. 22. Cor. 2.)  $\therefore n = 12$ .

3. Let  $x$  be the number of rt.  $\angle s$  in each angle. (1)  $8x + 4 = 16$  (I. 22. Cor. 1.)  $\therefore x = 1\frac{1}{2}$ . (2)  $10x + 4 = 20$  (I. 22. Cor. 1.)  $\therefore x = \frac{8}{5} = 1\frac{3}{5}$ . (3)  $7x + 4 = 14$  (I. 22. Cor. 1.)  $\therefore x = \frac{10}{7} = 1\frac{3}{7}$ .

4. Int.  $\angle s = 4$  rt.  $\angle s$  (I. 22. Cor. 2.)  $\therefore$  if  $n$  be the number of sides,  $4 + 4 = 2n$  (I. 22. Cor. 1.)  $\therefore n = 4$ .

5. Let the smallest  $\angle$  be equal to  $x$  rt.  $\angle s$ . Then  $(1 + 3 + 6 + 9 + 11)x + 4 = 10 \therefore x = \frac{6}{30} = \frac{1}{5} \therefore$  the angles are respectively  $\frac{1}{5}, \frac{3}{5}, \frac{6}{5}, \frac{9}{5}, \frac{11}{5}$  of a rt.  $\angle$ .

6. If  $n$  and  $2n$  be the numbers of sides of the polygons, and  $\alpha$  and  $\beta$  rt.  $\angle s$  in an angle of each,  $n\alpha + 4 = 2n$  (I. 22. Cor. 1.).  $2n\beta + 4 = 4n$  (I. 22. Cor. 1.)  $\therefore \alpha = \frac{2n-4}{n}$  and  $\beta = \frac{2n-2}{n}$   
 $\therefore \frac{\frac{n}{2n-2}}{\frac{n}{n}} = \frac{8}{9}$ , whence  $n = 10$ , and the polygons have 10 and 20 sides respectively.

7. Let each angle be equal to  $x$  rt.  $\angle s$ .  $5x + 4 = 10$  (I. 22. Cor. 1.)  $\therefore x = \frac{6}{5}$ .  $\therefore \angle ABE + \angle AEB = \frac{4}{5}$  of a rt.  $\angle$  (I. 22.) *i.e.*  $2\angle ABE = \frac{4}{5}$  of a rt.  $\angle$  (I. 5.).  $\angle ABC - \frac{1}{2}\angle ABE = (\frac{6}{5} - \frac{1}{5})$  of a rt.  $\angle =$  a rt.  $\angle$ .

8. Let the bisectors of the  $\angle s$  A and B of the quadl. ABCD meet at O.  $\angle BAO + \angle ABO = \frac{1}{2}(\angle DAB + \angle ABC) =$  a rt.  $\angle$  (I. 20)  $\therefore \angle AOB =$  a rt.  $\angle$  (I. 22.).

9.  $BD > AD \therefore \angle BAD > \angle DBA$  (I. 10.).  $CD > AD \therefore \angle CAD > \angle DCA \therefore$  adding  $\angle BAC > \angle ABC + \angle ACB \therefore \angle BAC$  is obtuse (I. 22.).

10. Not more than one  $\angle$  of a  $\triangle$  can be obtuse (I. 22.)  $\therefore$  since the ext.  $\angle s$  are supplements of the adj. int.  $\angle s$ , two, at least, of the ext.  $\angle s$  must be obtuse.

11. This follows at once from the fact that the three  $\angle s$  of a  $\triangle$  are equal to two rt.  $\angle s$  (I. 22.).

12. Draw a diagonal dividing the fig. into two  $\triangle s$ . The prop<sup>n</sup> follows at once from (I. 22.).

13. Two of the angles together are equal to a rt.  $\angle \therefore$  the third  $\angle$  is a rt.  $\angle$  (I. 22.).

14. The greatest  $\angle = \frac{1}{2}$  the sum of all three  $\angle s =$  a rt.  $\angle$  (I. 22.).

15. The greatest  $\angle$  of the  $\triangle >$  half all three  $\angle s$  and is  $\therefore$  gr. than a rt.  $\angle$  (I. 22.).

16. Let the isos.  $\triangle s$  ABC, DEF have their vertical  $\angle s$  at A and D equal.  $\angle ABC + \angle ACB = \text{supplement of } \angle A$  (I. 22.)  $\therefore \angle ACB = \frac{1}{2} \text{ supplement of } \angle A$  (I. 5.). Similarly,  $\angle DEF = \frac{1}{2} \text{ supplement of } \angle D = \frac{1}{2} \text{ supplement of } \angle A \therefore \angle DEF = \angle ACB \therefore \text{the } \triangle s \text{ are equiangular (I. 22.)}$ .

17. Let AD be drawn to the mid. pt. D of the base of the  $\triangle ABC$ . Produce AD to E making DE equal to DA. Join BE.  $\angle BDE = \angle CDA$  (I. 3.)  $\therefore$  from  $\triangle s$  BDE, CDA,  $BE = AC$  and  $\angle BED = \angle DAC$  (I. 4.)  $\therefore BE$  is  $\parallel$  to  $AC$  (I. 18.)  $\therefore \angle BAC + \angle ABE = 2 \text{ rt. } \angle s$  (I. 20.). Hence, if  $\angle BAC < a \text{ rt. } \angle$ ,  $\angle ABE > a \text{ rt. } \angle \therefore$  from  $\triangle s$  ABE, BAC,  $AE > BC$  (I. 14.) i.e.  $AD > \text{half the base } BC$ . Similarly, if  $\angle BAC > a \text{ rt. } \angle$ ,  $AD < \text{half the base } BC$ , and if  $\angle BAC = a \text{ rt. } \angle$ ,  $AD = \text{half the base } BC$  (I. 4.).

18. Let BAC be an acute  $\angle$ , and let BD, CD be perp. to AB and AC respectively. Produce CD to E. Join AD. The  $\angle s$  of the two  $\triangle s$  ABD, ACD = 4 rt.  $\angle s$  (I. 22.)  $\therefore \angle BAC + \angle BDC = 2 \text{ rt. } \angle s$ . But  $\angle EDB + \angle BDC = 2 \text{ rt. } \angle s$  (I. 1.)  $\therefore \angle EDB = \angle BAC$ .

19.  $\angle AMC + \angle LMN = 2 \text{ rt. } \angle s$  (I. 1.)  $\angle BLC + \angle MLN = 2 \text{ rt. } \angle s$  (I. 1.)  $\angle ANB + \angle MNL = 2 \text{ rt. } \angle s$  (I. 1.)  $\therefore$  since  $\angle s$  LMN, MLN, MNL = 2 rt.  $\angle s$  (I. 22.)  $\angle s$  AMC, BLC, ANB = 4 rt.  $\angle s$ .

20. Let M, N be the mid. pts. of AC and BC respectively. From  $\triangle s$  AMD, CMD,  $\angle DCM = \angle DAC$  (I. 4.). From  $\triangle s$  BME, CNE,  $\angle NCE = \angle EBC$  (I. 4.)  $\therefore \angle ACB = \angle DCE + \angle A + \angle B \therefore 2\angle ACB = \angle DCE + \angle A + \angle B + \angle ACB = \angle DCE + 2 \text{ rt. } \angle s$  (I. 22.)  $\therefore \angle DCE = \text{twice the excess, etc.}$

21. From  $\triangle s$  DAE, CAE,  $\angle DEA = \angle CEA = \frac{1}{2} \angle DEC = 30^\circ$  (I. 5. and 22.). Let AB, CD meet at O. From  $\triangle s$  DOA, COA,  $\angle DOA = \angle COA = a \text{ rt. } \angle \therefore \angle ADO = 30^\circ$  for  $\angle DAO = 60^\circ$ . But  $\angle EDO = 60^\circ \therefore \angle EDA = 30^\circ \therefore \angle DAE = 180^\circ - \angle ADE - \angle EDA = 120^\circ \therefore \angle DAE + \angle DAO = 180^\circ \therefore EAB$  is a str. line (I. 2.).

22. Let ED produced meet CB produced at O.  $\angle ODB = \text{supplement of } \angle s$  EDC and  $\angle OCB = \text{supplement of } \angle s$  CED and  $\angle DCB$  (I. 5.) = supplement of  $\angle s$  CED and  $\angle ACB$  (I. 5.) =  $\angle DOB$ . But  $\angle DCB = \angle ODB + \angle BOD$  (I. 22.) =  $2\angle DOB$ .  $\therefore \angle DOB = \angle OCB$  if BF bisects  $\angle ABC \therefore DE$  is  $\parallel$  to BF (I. 19.).

23. Let the quad<sup>l</sup> ABCD have its sides AB, DC produced to E and F. The four angles of the quad<sup>l</sup> = 4 rt.  $\angle s$  (Exercises

XIII. 12). But  $\angle EBC + \angle ABC + \angle BCF + \angle BCD = 4$  rt.  $\angle$ s (I. 1.)  
 $\therefore \angle EBC + \angle BCF = \angle BAD + \angle ADC$ .

24. Let  $\angle ADE = \angle DEA = a^\circ$ , so that  $\angle DAE = 6a^\circ$ ,  $8a = 180$   
 $\therefore a = 22\frac{1}{2}$ . But  $\angle DAC = \angle ACB - \angle ADC$  (I. 22.)  $= 45^\circ - 22\frac{1}{2}^\circ = 22\frac{1}{2}^\circ$   
 $= \angle CDA \therefore CD = CA$  (I. 6.)  $\therefore$  we must produce BC both ways, so  
that  $DC = BE = CA$  or  $AB$ .

25. Let the bisectors meet at E.  $\angle BEC = \angle ECD - \angle ECB$   
(I. 22.)  $= \frac{1}{2} [\angle ACD - \angle ABC] = \frac{1}{2} \angle BAC$  (I. 22.).

26. Let the quad<sup>l</sup> ABCD have the side AB held fast. Join  
AC. From  $\triangle$ s DAC, CBA,  $\angle DCA = \angle CAB$  (I. 7.)  $\therefore DC$  is  $\parallel$  to  $AB$   
in all positions, *i.e.* all positions of  $DC$  are parallel (I. 21.).

27.  $\angle A + \angle BDC = 2$  rt.  $\angle$ s.  $= \angle BDC + \frac{1}{2} \angle B + \frac{1}{2} \angle C$  (I. 22.)  $\therefore \angle A$   
 $= \frac{1}{2} \angle B + \frac{1}{2} \angle C = \frac{1}{2} [2 \text{ rt. } \angle \text{s} - \angle A]$  (I. 22.)  $\therefore \angle A = \frac{2}{3}$  of a rt.  $\angle = \angle$  of  
an equilateral  $\triangle$ .

28.  $360^\circ = \angle B + \angle C + \angle BDE + \angle DEC$  (Ex. XIII. 12.)  $= \angle B +$   
 $2\angle DEC$  (I. 5.)  $= \angle B + 360^\circ - 2\angle DEA$  (I. 1.)  $\therefore \angle B = 2\angle DEA = 2\angle A$   
(I. 5.).

29. Ext.  $\angle$ s at A, C, E + ext.  $\angle$ s at B, D, F = 4 rt.  $\angle$ s (I. 22,  
Cor. 2.). Also  $\angle$ s B, D, F + ext.  $\angle$ s at B, D, F = 6 rt.  $\angle$ s (I. 1.)  
 $\therefore \angle$ s B, D, F - ext.  $\angle$ s at A, C, E = 2 rt.  $\angle$ s.

30.  $\angle DEB = \text{alt. } \angle EBC$  (I. 20.)  $= \angle DBE \therefore DE = DB$  (I. 6.)  $= DA$   
 $\therefore \angle DEA = \angle DAE$ . Also  $\angle DEB = \angle DBE \therefore \angle AEB = \frac{1}{2}$  sum of  $\angle$ s  
of  $\triangle AEB = a$  rt.  $\angle$  (I. 22.).

31.  $\angle DEF = \angle BAE + \angle ABE$  (I. 22.)  $= \angle EBC + \angle ABE$  (Hyp.)  $=$   
 $\angle ABC$ . Similarly,  $\angle DFE = \angle ACB$  and  $\angle EDF = \angle BAC$ .

32. Let ABCD be a quad<sup>l</sup> such that  $\angle A + \angle B = 200^\circ$ . The  
four  $\angle$ s of the quad<sup>l</sup>  $= 360^\circ \therefore \angle D + \angle C = 160^\circ \therefore$  if the  
bisectors of these  $\angle$ s meet at O in  $\triangle DOC$ ,  $\angle DOC = 180^\circ - \frac{1}{2} \angle D$   
 $- \frac{1}{2} \angle C = 100^\circ$ .

33. From  $\triangle$ s ABD, ACD,  $BD = DC$  (I. 16.)  $\therefore$  from  $\triangle$ s BDF,  
CDF  $\angle FBD = \angle FCD$  (I. 4.)  $2\angle ACB = 2$  rt.  $\angle$ s  $- \angle A$  (I. 22.)  $= \frac{3}{2}$  rt.  $\angle$ s  
 $\therefore \angle ACB = \frac{3}{4}$  of a rt.  $\angle$ . Also  $\angle E = a$  rt.  $\angle \therefore \angle EBC = \frac{1}{4}$  rt.  $\angle$   
(I. 22.)  $\angle EFC = 2\angle FBC$  (I. 22.)  $= \frac{1}{2}$  rt.  $\angle$ . Also  $\angle FEC = a$  rt.  $\angle$   
 $\therefore \angle ECF = \frac{1}{2}$  rt.  $\angle$  (I. 22.)  $\therefore EC = EF$  (I. 6.).

34. Let AD bisect the  $\angle A$ , and AE be perp. to the base of  
 $\triangle ABC$ .  $90^\circ - B = \angle BAE = \angle DAE + \frac{1}{2} \angle BAC \therefore$  adding  $\frac{1}{2} \angle B + \frac{1}{2} \angle C$   
to both sides,  $90^\circ + \frac{1}{2} \angle C - \frac{1}{2} \angle B = \angle DAE + \frac{1}{2} \angle$ s of  $\triangle ABC = \angle DAE$   
 $+ 90^\circ$  (I. 22.)  $\therefore \angle DAE = \frac{1}{2} \angle C - \frac{1}{2} \angle B$ .

35. Let AN be perp. to BC.  $\angle ADN = \angle ABD + \angle BAD = \angle B + \angle C$  (I. 22.)  $\angle AEN = \angle ACE + \angle CAE = \angle C + \angle B$  (I. 22.)  $\therefore \angle ADN = \angle AEN \therefore$  from  $\triangle$ s ADN, AEN,  $DN = EN$  (I. 16.).

36. Let ADE be an equilateral  $\triangle$ , and let DE meet AC at O. From  $\triangle$ s ABD, ACD,  $\angle BAD = \angle CAD = 30^\circ$ . Also  $\angle ADO = 60^\circ \therefore \angle AOD = \text{a rt. } \angle$  (I. 22.)  $\angle EAD = 60^\circ$ . Also  $\angle BAD = 30^\circ \therefore \angle BAE$  is a rt.  $\angle$ . Also AD is perp. to BC by hypothesis.

### EXERCISES XIV.

1. Let O be the centre of the given circle, A any point on the locus. Let OA meet the given circle at B. Then  $OA = \text{sum of radii of the two circles}$ , and is therefore constant  $\therefore$  the locus of A is a circle whose centre is O and whose rad. is the sum of the radii of the two circles.

2. In this case,  $OA = \text{the diff. of the radii of the two circles}$ , and is constant, and the locus is a circle as in Example 1.

3. This locus will be found to be a str. line thro. the intersection of the given lines.

4. This will be found to be a str. line  $\parallel$  to the given line.

5. The pt. is at a constant distance from the centre of the given circle; therefore its locus is a concentric circle.

6. Take O the mid. pt. of AB and let OM, ON be drawn perp. to the lines through A and B respectively.  $\angle MAO = \text{alt. } \angle NBO$  (I. 20.)  $\therefore$  from  $\triangle$ s AMO, BNO,  $OM = ON$  (I. 16.). Thus we see that the locus of O is a str. line equidistant from the given lines.

7. Let AB, AC be the given lines, draw DE  $\parallel$  to AB and at a dist. from it equal to the given constant length. Let AC, DE meet in O. Let OP be that part of the bisector of  $\angle AOE$  which falls within the  $\angle CAB$ . Taking any pt. F in OP, and drawing FM perp. to AC, and EN perp. to DE and AB,  $FM = FE$  from  $\triangle$ s OMF, OEF (I. 16.)  $\therefore FM + FN = EN \therefore F$  is a pt. on the locus. Similarly if OQ bisect  $\angle DOA$  and meet BA produced in Q, QO is part of the locus. If CA is produced to R, similar parts of the locus are found within the  $\angle$ s QAR, RAP, and the complete locus is found to be the perimeter of a right-angled quad<sup>1</sup>.

**EXERCISES XV.**

1. With centre  $A$  and rad.  $AB$  describe a circle. With centre  $B$  and rad.  $BA$  describe another circle cutting the first in  $C$  and  $D$ .  $\triangle s$   $ABC$ ,  $ABD$  will both be equilateral  $\triangle s$  on  $AB$ .

2. With centres  $A$  and  $B$  and rad. equal to  $2AB$  describe circles cutting at  $C$  and  $D$ .  $\triangle s$   $ACB$ ,  $AEB$  both fulfil the reqd. conditions as in I. 25.

3. Let  $AB$  be the given base. With centres  $A$  and  $B$  and rad. equal to the given line, describe circles cutting at  $C$  and  $D$ .  $\triangle s$   $ACB$ ,  $ADB$  will both fulfil the reqd. conditions as in I. 25.

4. Let  $A$  and  $B$  be the given pts. With centres  $A$  and  $B$  and rad. equal to the given radius describe circles. Let  $C$  be one of the pts. at which they meet. Then since  $CA = CB =$  the given radius a circle described with centre  $C$ , and rad.  $CA$  is the reqd. circle. The circles will not cut unless the given rad. is gr. than  $\frac{1}{2}AB$ .

5. This can be done as in the previous exercise.

6. With any pt.  $O$  in a str. line  $AB$  as centre and any rad. describe a circle cutting  $AB$  at  $A$  and  $B$ . With centres  $A$  and  $B$  and any rad. gr. than  $OA$ , describe circles cutting in  $C$ . Join  $OC$ . From  $\triangle s$   $AOC$ ,  $BOC$ ,  $\angle AOC = \angle BOC =$  a rt.  $\angle$  (I. 7.).

7. Join the given pts.  $A$ ,  $B$ . Bisect  $AB$  at  $O$  (I. 27.) and draw  $OD$  at rt.  $\angle s$  to  $AB$  to meet the given line at  $D$ . From  $\triangle s$   $AOD$ ,  $BOD$ ,  $DA = DB$  (I. 4.)  $\therefore D$  is the reqd. pt. The problem is impossible unless  $OD$  meets the given line, i.e. when  $OD$  is  $\parallel$  to the given line, i.e. when  $AB$  is perp. to the given line. When  $AB$  is perp. to the given line and bisected by it, any pt. in the given line satisfies the reqd. condition.

8. Draw  $AD$  bisecting  $\angle BAC$  (I. 26.) and meeting  $BC$  at  $D$ . If  $DM$ ,  $DN$  be perp. to  $AB$ ,  $AC$ , then from  $\triangle s$   $AND$ ,  $AMD$ ,  $DM = DN$  (I. 16.)  $\therefore D$  is the reqd. pt.

9. With centre  $C$  and rad. 1 in., describe a circle. If  $OD$  bisects  $AB$  at rt.  $\angle s$ , any pt. in it is equidistant from  $A$  and  $B$  (I. 23.)  $\therefore$  the pts. where  $OD$  meets the circle are the reqd. points. Impossible when  $OD$  does not meet the circle.

**10.** Draw  $OD$  bisecting  $BC$  at rt.  $\angle$ s. Any pt. in  $OD$  is equidistant from  $B$  and  $C$  (I. 23.)  $\therefore$  the pt. where  $OD$  meets  $AB$  is the reqd. pt.

**11.** Produce  $BA$  to  $D$  making  $AD$  equal to  $AC$ . Bisect  $BD$  at  $O$ . With centre  $A$  and rad. equal to  $OB$  or  $OD$  describe a circle cutting  $BC$  at  $E$ .  $AE = OB = \frac{1}{2}BD = \frac{1}{2}(AB + AC)$   $\therefore D$  is the reqd. pt.

**12.** Bisect  $AD$ ,  $BC$  at rt. angles by str. lines meeting at  $E$ .  $E$  is the reqd. pt. (I. 23.).

### **EXERCISES XVI.**

**1.** Bisect the given angle, and also bisect the two angles thus formed.

**2.** Let  $A$  be the given pt.,  $BC$  the given line. Thro.  $A$  draw  $AD \parallel$  to  $BC$  (I. 31.). Make  $\angle DAB$  equal to the given angle.  $AB$  is the reqd. line, for  $\angle ABC = \text{alt. } \angle DAB$  (I. 16.) = the given angle.

**3.** Let  $AB$ ,  $AC$  be the given lines,  $O$  the given pt. Bisect  $\angle BAC$  by  $AD$ ; and draw  $ON$  perp. to  $AD$ , and produce it to meet  $AB$ ,  $AC$  in  $E$  and  $F$ . From  $\triangle$ s  $ANE$ ,  $ANF$   $\angle AFN = \angle AEN$  (I. 16.)  $\therefore EF$  is the reqd. line.

**4.** Thro.  $A$  the given pt. draw  $EAD \parallel$  to the given line and make  $\angle DAC$  equal to the given  $\angle$ . Bisect  $\angle EAC$  by  $AF$  meeting  $BC$  at  $F$ . If  $FE$  is  $\parallel$  to  $CA$ ,  $\angle EFA = \text{alt. } \angle FAC$  (I. 20.) =  $\angle EAF = \text{alt. } \angle AFC$ .  $\angle DAC = \text{alt. } \angle ACF$  (I. 20.) = ext.  $\angle BFE$  (I. 20.) =  $\angle BFA - \angle EFA = \angle BFA - \angle AFC \therefore AF$  is the reqd. line.

**5.** Let  $OA$ ,  $OB$ ,  $OC$  be the given lines,  $OB$  falling between  $OA$  and  $OC$ . In  $OB$  take any pt.  $D$ . Make  $BD$  equal to  $DO$ , and from  $B$  draw  $BA \parallel$  to  $CO$  to meet  $OA$  at  $A$ . Join  $AD$  and produce it to meet  $OC$  at  $C$ .  $\angle ADB = \angle CDO$  (I. 3.).  $\angle ABD = \text{alt. } \angle DOC$  (I. 20.)  $\therefore$  from  $\triangle$ s  $ADB$ ,  $CDO$ ,  $DA = DC$  (I. 16.).

**6.** Let  $BAC$  be the given  $\angle$ ,  $P$  and  $Q$  the given perpendiculars. At  $A$  draw  $AD$  perp. to  $AB$  and equal to  $P$ . At  $A$  draw  $AE$  perp. to  $AC$  and equal to  $Q$ . Draw  $DC \parallel$  to  $AB$  to meet  $AC$  at  $C$ , and draw  $EB \parallel$  to  $AC$  to meet  $AB$  at  $B$ .  $ABC$  is the reqd.  $\triangle$ . For if  $CM$  be perp. to  $AB$  and  $BN$  perp. to  $AC$ ,  $\angle DAC = \text{complement of } \angle CAM = \angle ACM \therefore$  from  $\triangle$ s,  $ADC$ ,  $CMA$   $\therefore CM = DA = P$  (I. 16.). Similarly,  $BN = Q$ , and  $\angle BAC$  is the given  $\angle$ .

7. Let  $\triangle ABC$  be a rt.  $\angle$ . On  $BC$  describe an equilateral  $\triangle BDC$ .  $\angle DBC = \frac{2}{3}$  of a rt.  $\angle$  (I. 22.)  $\therefore \angle ABD = \frac{1}{3}$  of a rt.  $\angle$   $\therefore$  bisecting  $\angle DBC$  by  $BE$ ,  $BD$  and  $BE$  divide  $\angle ABC$  into three equal parts.

8. Let  $A, B, C$  be the given pts. On  $AB$  describe an equilateral  $\triangle DBA$ . Thro.  $C$  draw  $ECF \parallel$  to  $AB$  to meet  $DA$  and  $DB$  in  $E$  and  $F$ .  $\angle DEF = \text{ext. } \angle DAB$  (I. 20.)  $= 60^\circ = \angle DBA$  (I. 5.)  $= \text{int. } \angle DFE$  (I. 20.) and  $\angle FDE$  also  $= 60^\circ \therefore \triangle DEF$  is equilateral (I. 6.).

9. If  $\alpha$  is an angle at the base,  $6\alpha = 180^\circ$  and  $\alpha = 30^\circ$   $\therefore$  if we bisect two angles of an equilateral  $\triangle$ , we obtain the  $\triangle$  reqd.

10. Draw  $BE \parallel$  to  $CA$ . Make  $\angle EBD$  equal to  $\angle ABC$  and let  $BD$  meet  $AC$  at  $D$ .  $\angle BDA = \text{alt. } \angle EBD$  (I. 20.)  $= \angle ABC$ .

11. Describe an equilateral  $\triangle ABC$  as in I. 25.  $\angle ABC = 60^\circ$  (I. 5. and 22.). Bisect  $\angle ABC$  by  $BD$  (I. 26.) and  $\angle DBC = 30^\circ$ . Bisect  $\angle DBC$  and we have  $\angle s$  of  $15^\circ$ , *i.e.* one-sixth of a rt  $\angle$ .

12. Let  $\angle BAE$  be the given  $\angle$ ,  $AB$  and  $P$  the given sides,  $\angle BAE$  being opp. to  $P$ . With centre  $B$  and rad.  $P$  describe a circle cutting  $AE$  in  $C$ .  $BAC$  is the  $\triangle$  reqd. Since the circle will generally cut  $AE$  in two pts. we generally obtain two solutions.

13. Let  $BAC$  be the given vertical  $\angle$ . Bisect it by  $AD$  (I. 26.) and make  $AD$  equal to the given perp. Draw  $EDF$  perp. to  $AD$  meeting  $AB, AC$  in  $E$  and  $F$ . From  $\triangle s$   $ADE, ADF$ ,  $AE = AF$  (I. 16.) and  $AD = \text{given perp.}$   $\therefore \triangle AEF$  is the  $\triangle$  reqd.

14. Let  $AB$  be the given perp<sup>r</sup>. Draw  $BC$  perp. to  $AB$  making  $BC$  equal to half the perimeter. Join  $AC$ , and make  $\angle CAD$  equal to  $\angle ACD$ ,  $AD$  meeting  $BC$  at  $D$ . In  $DB$  produced make  $BE$  equal to  $DB$ . Join  $AE$ .  $AD = DC$  (I. 6.)  $\therefore AD + DB = \text{half the given perimeter.}$  Also from  $\triangle s$   $ABE, ABD$ ,  $AE = AD$  (I. 4.)  $\therefore ADE$  is the  $\triangle$  reqd.

15. Let  $AB$  be the given side. Draw  $AC$  at rt.  $\angle s$  to  $AB$ . With centre  $B$ , and rad. equal to the hypotenuse, describe a circle cutting  $AC$  at  $D$ .  $DAB$  is the  $\triangle$  reqd.

16. From  $AC$  the perimeter cut off  $AB$  equal to the hypotenuse. At  $C$  make  $\angle BCD$  equal to half a rt.  $\angle$ ; and with centre  $B$  and rad.  $BA$  describe a circle meeting  $CD$  at  $D$ . Draw  $DE$  perp. to

**BC.**  $\angle DEC = \text{a rt. } \angle$ ,  $\angle DCE = \frac{1}{2} \text{ a rt. } \angle \therefore \angle EDC = \frac{1}{2} \text{ a rt. } \angle$  (I. 22.)  
 $\therefore ED = EC$  (I. 6.). Also  $BD = \text{the given hypotenuse}$   $\therefore DEB$  is the  $\triangle$  reqd.

**17.** Let  $AB$  be the given perp<sup>r</sup>. Draw  $BC$  perp. to  $BA$ , and on  $BC$  describe an equilateral  $\triangle DBC$ . From  $A$  draw  $AF \parallel$  to  $DB$  and  $AE \parallel DC$  to meet  $BC$  in  $F$  and  $E$ .  $\angle AFE = \text{ext. } \angle DBC = \frac{2}{3} \text{ of a rt. } \angle$ .  $\angle AEF = \text{int. opp. } \angle DCE = \frac{2}{3} \text{ of a rt. } \angle \therefore \angle FAE = \frac{2}{3} \text{ of a rt. } \angle$  (I. 22.)  $\therefore \triangle AFE$  is equilateral (I. 6.), and is the  $\triangle$  reqd.

**18.** Bisect  $\angle ABC$  by  $BE$  meeting  $AC$  at  $E$ . Draw  $ED \parallel$  to  $CB$  meeting  $AB$  at  $D$ .  $\angle DEB = \text{alt. } \angle EBC$  (I. 20.)  $= \angle DBE \therefore DB = DE$  (I. 6.)  $\therefore D$  is the pt. reqd.

### EXERCISES XVII.

**1.** Use the method of I. 25. The lengths of the sides in cms. are 8.9, 6.35, 10.2.

**2.** Use the method of I. 25. The lengths of the sides in inches are 1.57, 1.97, 2.36.

**3.** With a protractor make  $\angle BAC$  equal to  $35^\circ$ . Cut off  $AB$  equal to 5 cms. and draw  $BC$  perp. to  $AC$ .  $BAC$  is the  $\triangle$  reqd.

**4.** If we fold the quadl.  $ABCD$  about the diagonal  $BD$ , the pt.  $C$  will fall upon the pt.  $A \therefore AC$  is bisected by the crease, for its parts coincide. Similarly  $BD$  will be bisected if we fold about  $AC$ . Also if  $AC, BD$  meet at  $O$ , adj.  $\angle$ s  $AOD, COD$  coincide when we fold, and are therefore rt.  $\angle$ s. Hence, the diagonals of an equilateral quadl. (a rhombus) bisect one another at rt.  $\angle$ s.

**5.** Draw  $AB$  equal to 2 inches. At  $A$  and  $B$ , with a protractor, make  $\angle$ s  $CAB, CBA$  each equal to  $40^\circ$ .  $ABC$  is the  $\triangle$  reqd.

**6.** Each angle of the pentagon  $= \frac{6}{5}$  of a rt.  $\angle$  (p. 57)  $= 108^\circ$ . Draw  $AB = 3$  cms. and make, with a protractor,  $\angle$ s  $DAB, CBA$  each equal to  $108^\circ$ . Cut off  $AD = BC = 3$  cms. With centres  $D$  and  $C$  and radii 3 cms. describe arcs cutting at  $E$ .  $ABCED$  is the fig. reqd.

**7.** Draw  $AB$ , 4 cms. long. With centres  $A$  and  $B$  and radii  $AB$ , describe circles cutting at  $O$ . With centre  $O$  and same

rad. describe a circle  $ABCDEF$ . Let this circle cut the first two circles at  $F$ , etc. With centres  $F$  and  $C$  and same rad. describe circles cutting circle  $ABEF$  at  $D$  and  $E$ .  $ABCDEF$  will be a regular hexagon.

8. Draw  $AB$  2 in. long. At  $A$  and  $B$  with a protractor make  $\angle CAB = 30^\circ$  and  $\angle CBA = 50^\circ$ .  $ABC$  is the  $\triangle$  reqd. The sides are 4 cms. and 2.6 cms. long.

9. Construct an equilateral  $\triangle$  and bisect one angle. This will give two  $\triangle$ s each satisfying the given conditions.

10. Draw a str. line  $AB$ , and fold the paper so that the pt.  $A$  falls on  $B$ . Let  $COD$  be the crease, meeting  $AB$  at  $O$ . When we fold,  $\angle COA$  coincides with  $\angle COB$ , and they are adj. angles  $\therefore$  they are rt.  $\angle$ s  $\therefore$  the crease  $CD$  is perp. to  $AB$ .

11. Take any str. line  $AB$ . With centres  $A$  and  $B$  and rad. equal to  $AB$  describe circles meeting in  $C$ .  $ABC$  is an equilateral  $\triangle \therefore \angle CAB = 60^\circ$  (I. 22.)  $\therefore$  bisecting  $\angle CAB$  we obtain angles of  $30^\circ$ .

12. Draw any str. line  $AB$ , and with a protractor make  $\angle BAC$  equal to  $60^\circ$  and  $\angle ABC$  equal to a rt.  $\angle$ .  $\angle BCA = 30^\circ$  (I. 22.)  $\therefore ABC$  is the  $\triangle$  reqd. Produce  $AB$  to  $D$ , making  $BD = BA$ . Join  $CD$ . From  $\triangle$ s  $CBA$ ,  $CBD$ ,  $\angle CDB = \angle CAB = 60^\circ$ , and  $\angle DCB = \angle ACB = 30^\circ$  (I. 4.)  $\therefore \triangle ACD$  is equilateral (I. 6.)  $\therefore AC = AD = 2 AB$ .

13. At the mid. pt. of a line 1.4 in. long draw a perp<sup>r</sup>. 2.4 in. long. Joining the ends of the first line to the end of the perp<sup>r</sup>. we have the  $\triangle$  reqd. (I. 4.). The sides are 2.5 in. long.

14. Make  $\angle ABC = 33^\circ$  with a protractor. Cut off  $BA = 2.5$  in. and  $BC = 3.4$  in.  $ABC$  is the  $\triangle$  reqd.  $BC = 1.91$  in.

15. Draw  $AB$ ,  $BC$  at rt.  $\angle$ s to one another. Cut off  $BA = 3.7$  cms., and  $BC = 3.2$  cms.  $ABC$  is the  $\triangle$  reqd.  $AC = 4.9$  cms.

16. With centre  $C$  and rad. 1 in. describe a circle, meeting  $AB$  at  $E$  and  $F$ .  $CE$  and  $CF$  both give solutions. A circle generally meets a str. line at two points, and we therefore generally obtain two solutions. If the perp. from  $C$  upon  $AB$  is gr. than 1 in. a solution is impossible, for in that case the circle does not meet  $AB$ .

17. Use the method of I. 25.

18. Let BC represent the tower, A the pt. in the horizontal plane such that  $\angle CAB = 45^\circ$  and  $BA = 50$  ft.  $\angle CBA = 90^\circ \therefore \angle ACB = 45^\circ \therefore CB = BA = 50$  ft.

19.  $\angle$  reqd.  $= 97^\circ$ .

20. Make  $\triangle ABC$  as in I. 25. Bisect AC at D. (This can be done by making  $AD = 2\frac{1}{2}$  cms.)  $DB = 2.5$  cms.

21. Draw AB 3 in. long; at A with a protractor make  $\angle BAC = 40^\circ$ , and at B make  $\angle ABC = 60^\circ$ .  $CA = 2.64$  in. and  $CB = 1.96$  in.

22. Draw the  $\triangle$  as in Example 14 above. The third side  $= 1.61$  in.

23. 4 in.  $= 10.16$  cms.  $\therefore 1$  in.  $= 2.54$  cms. Any error made in measuring the 4 in. line is divided by 4, in finding 1 in. in cms.

24. Draw  $AB = 3$  in., AC perp. to AB, and (with a protractor)  $\angle ABC = 30^\circ$ . AC represents the tower.  $AC = 173.2$  ft.

25. Draw BA 3 cms. long to represent 30 feet, and produce it to D. With a protractor, make  $\angle DBC = 45^\circ$ , and  $\angle DAC = 60^\circ$ . Draw CD perp. to BA. CD represents the tower.  $CD = 7.1$  cms.  $\therefore$  the tower is 71 ft. high.

26. Make AB 3 in. Make  $\angle s$  BAC, ABC each  $40^\circ$ . Then  $\angle ACB = 100^\circ$ ; and by measurement  $AC = 1.96$  in. approx. which represents 196 feet.

27. Draw BC 1 in. long to represent the height of the mound. Draw BA perp. to BC, and make  $BA = 1$  in.  $\angle CAB = \angle ACB = 45^\circ$  (I. 5. and 22.). With a protractor, make  $\angle BAD$  equal to  $60^\circ$ , and let BC produced meet BD at D. CD is the flagstaff.  $AD = 2$  inches  $\therefore$  the dist. of A from the top of the flagstaff  $= 40$  feet.

28. With the protractor draw an  $\angle$  of  $45^\circ$ . Bisect it and we obtain angles of  $22\frac{1}{2}^\circ$ .

\*\*\* The additional Exercises, 29-39, will be found on pages 173-174.

### EXERCISES XVIII.

1. It is reqd. to draw a perp. to BA at the pt. A. With any centre C and rad. CA describe a circle cutting AB again at B. Join BC and produce it to meet the circle again at D. Join AD.  $\angle CAB = \angle CBA$  (I. 5.) and  $\angle CAD = \angle CDA$  (I. 5.)  $\therefore$

$\angle BAD = \angle CBA + \angle CBA \therefore \angle BAD = \frac{1}{2}$  sum of the  $\angle$ s of  $\triangle ABD =$  a rt.  $\angle \therefore AD$  is the reqd. line.

2. Let  $AB$  be the given pts.,  $CD$  the given line. Draw  $AF$  perp. to  $CD$  and produce it to  $G$ , making  $FG = AF$ . Join  $BG$  cutting  $CD$  at  $E$ . Join  $AE$ . From  $\triangle$ s  $AFE$ ,  $GFE$ ,  $\angle AEF = \angle GEF$  (I. 4.)  $= \angle BED$  (I. 3.)  $\therefore AE, EB$  make equal angles with  $CD$ .

3. (1) Let  $O$  be the given pt.,  $PQ$  the given line, and let  $OA$  be perp. to  $PQ$ . Draw any other line  $OB$  to meet  $PQ$  in  $B$ .  $\angle BAO = \angle QAO > \angle OBA$  (I. 8.)  $\therefore OB > OA$  (I. 11.). Similarly any other line from  $O$  to  $PQ > OA$ . (2) Draw  $OC$  further from  $OA$  than  $OB$  to meet  $PQ$  in  $C$ .  $\angle OBC > \angle OAB$  and is  $\therefore$  an obtuse  $\angle \therefore \angle OBC > \angle OBA > \angle OCB$  (I. 8.)  $\therefore OC > OB$  (I. 11.). (3) On the side of  $OA$  remote from  $OB$ , let  $\angle AOD = \angle AOB$ . From  $\triangle$ s  $OAB$ ,  $OAD$ ,  $OD = OB$  (I. 16.). If possible let  $OC = OB = OD$ ,  $OC$  and  $OB$  being on the same side of  $OA$ , and  $OC$  further from  $OA$  than  $OB$ .  $\angle OCB = \angle OBC$  (I. 5.) which  $> \angle OAB$  a rt.  $\angle$  (I. 8.)  $\therefore$  the two  $\angle$ s  $OCB, OBC$  of  $\triangle OBC$  are together gr. than two rt.  $\angle$ s, which is impossible (I. 9.).

4. Let  $AD$  be the line joining  $A$  to the mid. pt.  $D$  of the side  $BC$  of the  $\triangle ABC$ . Produce  $AD$  to  $E$ , making  $DE$  equal to  $AD$ .  $\angle CDE = \angle BDA$  (I. 3.)  $\therefore$  from  $\triangle$ s  $CDE, BDA$ ,  $CE = BA$  (I. 4.)  $\therefore BA + AC = EC + CA > AE$  (I. 12.)  $\therefore BA + AC > 2AD$ .

5. Let  $AB$  be the given base. Bisect it at  $C$  and draw  $CD$  at rt.  $\angle$ s to  $AB$  and equal to the given sum. Join  $AD$  and make  $\angle DAE$  equal to  $\angle CDA$ ,  $AE$  meeting  $CD$  at  $E$ . Join  $EB$ . From  $\triangle$ s  $ACE, BCE$ ,  $AE = BE$  (I. 4.). Also  $AE = DE$  (I. 6.)  $\therefore AE + EC = CD =$  given sum  $\therefore AEB$  is the  $\triangle$  reqd.

6. Let  $AB$  be the given base,  $\angle BAC$  the given  $\angle$ , and  $AC$  the given sum of the two sides. Join  $BC$ , and make  $\angle CBD$  equal to  $\angle ACB$ ,  $BD$  meeting  $AC$  at  $D$ .  $DB = DC$  (I. 6.)  $\therefore AD + DB = AC =$  given sum of sides  $\therefore ADB$  is the reqd.  $\triangle$ .

7. Let  $DE$  be the given perimeter. At  $D$  and  $E$  make  $\angle$ s  $EDA, DEA$  equal to half the given angles at the base. At  $A$ , where  $DA$  and  $EA$  meet, make  $\angle DAB$  equal to  $\angle ADB$ , and  $\angle EAC$  equal to  $\angle AEC$ ,  $AB$  and  $AC$  meeting  $DE$  at  $B$  and  $C$ .  $AB = DB$  and  $CA = CE$  (I. 6.)  $\therefore AB + BC + CA = DE =$  given perimeter. Also  $\angle ABC = 2\angle ADB =$  given base angle (I. 22.). Similarly  $\angle BCA =$  the other given base angle  $\therefore ABC$  is the  $\triangle$  reqd.

8. Let A, B be the given pts., CD the given line. Draw AE perp. to CD and produce AE to F, making EF equal to EA. Join BF cutting CD at G. Join AG. Take any other pt. H in CD, and join FH, BH. From  $\triangle$ s AEG, FEG,  $AG = FG$ . (I. 4.)  $\therefore FH + HB > FB$  (I. 12.)  $> AG + GB$ . Thus we see that AG + GB is a minimum.

9. Let D be the mid. pt. of the hypotenuse AB of the  $\triangle$  ABC. Join DC. If  $DC > DB$  or  $DA$ ,  $\angle DBC > \angle DCB$  and  $\angle DAC > \angle DCA$  (I. 10.)  $\therefore \angle ABC + \angle BAC > \angle BCA$   $\therefore \angle BCA < a$  rt.  $\angle$  (I. 22.), which is contrary to the hypothesis. Similarly, if  $DC < DB$  it may be proved from I. 22. that  $\angle BCA > a$  rt.  $\angle$ , which is contrary to the hypothesis  $\therefore$  DC must be equal to DB or DA.

### EXERCISES XIX.

1. With the fig. of II. 1 let AD, BC meet in O.  $\angle AOB = \angle DOC$  (I. 3.).  $\angle ABO = \text{alt.}$   $\angle OCD$  (I. 20.)  $\therefore$  from  $\triangle$ s AOB, COD,  $AO = OD$  and  $BO = OC$  (I. 16.).

2. Let ABCD be a quadl. having its opp. sides equal. From  $\triangle$ s ABD, CDB,  $\angle ABD = \angle CDB$  and  $\angle ADB = \angle CBD$  (I. 7.)  $\therefore AB$  is  $\parallel$  to  $CD$ , and  $AD$  to  $BC$  (I. 18.)  $\therefore$  ABCD is by def. a parm.

3. Let ABCD be the quadl. Its four  $\angle$ s are equal to four rt.  $\angle$ s (I. 22.)  $\therefore \angle DAB + \angle ADC = 2$  rt.  $\angle$ s  $\therefore AB$  is  $\parallel$  to  $CD$  (I. 19.). Similarly  $AD$  is  $\parallel$  to  $BC$   $\therefore$  the fig. is a parm.

4. Let AB be the given line. Draw AC, BD  $\parallel$  to one another by I. 31. Also AD and BC  $\parallel$  to one another. ADCB is a parm.  $\therefore$  its diagonals AB, CD bisect one another (II. 2., Cor. 3.).

5. Let ABCD be the parm. and let its diagonals cut at O.  $\angle$ s BAD, ADC = 2 rt.  $\angle$ s (I. 20.)  $\therefore \angle DAO + \angle ADO = a$  rt.  $\angle$   $\therefore \angle AOD = a$  rt.  $\angle$  (I. 22.)  $\therefore$  from  $\triangle$ s AOD, AOB,  $AD = AB$  (I. 16.)  $= CD = BC$   $\therefore$  ABCD is a rhombus.

6. Let ABCD be a rhombus. Draw its diagonals. From  $\triangle$ s ADB, CBD,  $\angle ABD = \angle BDC$  (I. 7.)  $\therefore AB$  is  $\parallel$  to  $CD$  (I. 18.). Similarly  $AD$  is  $\parallel$  to  $BC$ .

7. Let ABCD be the parm. From  $\triangle$ s DAB, CBA,  $\angle DAB = \angle CBA$  (I. 7.). But  $\angle DAB + \angle CBA = 2$  rt.  $\angle$ s (I. 20.)  $\therefore \angle DAB$  is a rt.  $\angle$   $\therefore$  ABCD is a rectangle.

8. Let the diagonals of ABCD bisect one another at O.  $\angle AOD = \angle BOC$  (I. 3.)  $\therefore$  from  $\triangle$ s AOD, COB,  $\angle ABD = \angle BDC$  (I. 4.)  $\therefore$  AB is  $\parallel$  to CD (I. 18.). Similarly AD is  $\parallel$  to BC.

9. Let ABCD be the quadr. formed by the rails. Draw AE perp. to BC and AF to CD.  $AE = AF$ . In  $\triangle$ s AFD, AEB,  $AE = AF$ ,  $\angle AFD = \text{a rt. } \angle = \angle AEB$ , and  $\angle ADF = \angle ABE$  (II. 2.)  $\therefore AD = AB$  (I. 16.). Also  $AD = BC$  and  $AB = CD$   $\therefore$  ABCD is a rhombus.

10. Let the diagonals of the rhombus ABCD intersect at O. From  $\triangle$ s DAC, BAC,  $\angle DAC = \angle BAC$  (I. 7.)  $\therefore$  from  $\triangle$ s DAO, BAO,  $DO = OB$ , and  $\angle AOD = \angle AOB = \text{a rt. } \angle$  (I. 4.). Similarly  $AO = CO$ .

11. From BC cut off BD equal to the given line, and thro. D draw DE  $\parallel$  to BA to meet CA at E. Draw EF  $\parallel$  to BC. By constr. FBDE is a parm.  $\therefore FE$  is  $\parallel$  and equal to BD, which is equal to the given line.

12. CD and EF are each  $\parallel$  and equal to AB  $\therefore$  they are themselves equal and  $\parallel$  (I. 21.)  $\therefore$  FECD is a parm. (II. 1.).

13. Let ABCD be the parm. such that  $AB = 2AD$ . Bisect AB at E. Join ED, EC. Also join E to the mid. pt. F of CD. AEFD is a parm. (II. 1.)  $\therefore EF = AD = DF = CF$  (II. 2.)  $\therefore \angle FDE = \angle FED$  and  $\angle FCE = \angle FEC$   $\therefore \angle DEC = \angle FDE + \angle FCE$   $\therefore$  DEC is a rt.  $\angle$  (I. 22.).

14. Let ACBD be the quadr. formed by joining the ends of the diameters AOB, COD.  $OD = OA = OC$   $\therefore \angle OAD = \angle ODA$ , and  $\angle OAC = \angle OCA$  (I. 5.)  $\therefore \angle DAC = \angle ADC + \angle ACD$   $\therefore \angle DAC = \text{a rt. } \angle$  (I. 22.). Also from  $\triangle$ s AOD, BOC,  $\angle DAO = \angle CBO$  and  $AD = BC$  (I. 3. 4.)  $\therefore AD$  is equal and  $\parallel$  to BC  $\therefore$  ADBC is a rectangle.

15. Join EG.  $\angle HAE = \angle FCG$  (II. 2.)  $\therefore$  from  $\triangle$ s HAE, FCG,  $HE = FG$  (I. 4.). Similarly  $HG = EF$   $\therefore$  from  $\triangle$ s EHG, GFE,  $\angle HEG = \angle EGF$   $\therefore HE$  is  $\parallel$  to  $FG$  (I. 18.), and it is also equal to it  $\therefore$  EFGH is a parm (II. 1.).

16. Let AB be  $\parallel$  to CD and AD equal, but not  $\parallel$ , to BC in the quadr. ABCD. Draw AE  $\parallel$  to BC. AEBC is a parm.  $\therefore AE = BC$  (II. 2.)  $= AD$   $\therefore \angle ADE = \angle AED$  (I. 5.)  $= \text{int. opp. } \angle BCD$  (I. 20.). Also  $\angle DAB + \angle ADC = 2 \text{ rt. } \angle$ s  $= \angle CBA + \angle BCD$  (I. 20.)  $\therefore \angle DAB = \angle CBA$ .

17. Let ABCD be the given parm., and let its diagonals cut at O. They bisect one another (II. 2. Cor. 3.). Let E in AB be the given vertex. Let EO produced meet CD in F, and draw

GOH perp. to EF to meet AD in G and BC in H.  $\angle AOE = \angle COF$  (I. 3.).  $\angle AEO = \angle OFC$  (I. 20.)  $\therefore$  from  $\triangle$ s AOE, COF,  $OE = OF$  (I. 16.). Similarly from  $\triangle$ s AOG, COH,  $OG = OH$ . Hence from  $\triangle$ s GOE, HOF,  $GE = HF$  (I. 4.). Similarly  $HF = HE = EG \therefore$  EHFG is a rhombus, and is described as reqd.

18. Let ABCD be the given parm., and let its diagonals meet at O. Let P be the given pt. thro. which a diagonal of the reqd. rhombus passes. Join OP, and let it when produced meet AB at E and CD at F. Drawing GOH perp. to EF to meet AD at G and BC at H, it may be proved, as in Example 17 above, that EHFG is the rhombus reqd.

19. From P a pt. in the base BC of the isos.  $\triangle ABC$  let PM, PN be drawn perp. to AB and AC respectively. Draw BK  $\parallel$  to AC, and let NP produced meet it at K.  $\angle BKP = \text{alt. } \angle KNC$  (I. 20.) = a rt.  $\angle \therefore$  KN is the perp. dist. between the  $\parallel$  lines BK, AC, and is therefore constant in length. Also  $\angle PBK = \text{alt. } \angle PCN$  (I. 20.) =  $\angle ABP$  (I. 5.)  $\therefore$  from  $\triangle$ s PMB, PKB,  $PM = PK$  (I. 16.)  $\therefore$   $PM + PN = PK + PN = \text{constant}$ .

20. Join AD, and produce it to E, making  $DE = DA$ . Draw EC  $\parallel$  to AB. Also join CD, and let it when produced meet AB at B.  $\angle EDC = \angle ADB$  (I. 3.).  $\angle ECD = \text{alt. } \angle ABD$  (I. 20.)  $\therefore$  from  $\triangle$ s ABD, ECD,  $BD = DC$  (I. 16.).

21. Let EF meet AD at O. From  $\triangle$ s AOF, AOE,  $AF = AE$  and  $OF = OE$  (I. 16.). Also  $\angle EDA = \text{alt. } \angle DAF$  (I. 20.) =  $\angle DAE \therefore$  from  $\triangle$ s AOE, DOE,  $AO = DO$  (I. 16.).

22. Let the diagonals of the parm. ABCD cut at O.  $AO = OC$  and  $BO = OD$  (II. 2. Cor. 3.).  $\angle AOD = \angle BOC \therefore \triangle AOD = \triangle BOC$  (I. 4.).  $\angle AOE = \angle COF$  (I. 3.).  $\angle EAO = \angle FCO$  (I. 20.)  $\therefore \triangle AOE = \triangle COF$  (I. 16.). Similarly  $\triangle DOF = \triangle EOB \therefore$  fig. ADFE = fig. EBCF.

23. Draw GCH  $\parallel$  to BDF to meet AB at G and EF at H.  $\angle ACG = \angle HCE$  (I. 3.).  $\angle AGC = \text{alt. } \angle EHC$  (I. 20.)  $\therefore$  from  $\triangle$ s ACG, ECH,  $CG = CH$  (I. 16.). Also GCDB, CHFD are parms.  $\therefore$   $CG = BD$  and  $CH = DF$  (II. 2.)  $\therefore$   $BD = DF$ .

24. Let the bisectors AE, DE of two angles of the quadrl. ABCD meet at E. AED is a rt.  $\angle$  by hyp.  $\therefore \angle DAE + \angle ADE =$  a rt.  $\angle$  (I. 22.)  $\therefore \angle BAD + \angle ADC = 2 \text{ rt. } \angle$ s  $\therefore$  AB is  $\parallel$  to CD (I. 19.). Similarly AD is  $\parallel$  to BC  $\therefore$  ABCD is a parm.

25. Let ABCD be a quadl. having AB equal to CD and obtuse  $\angle DAB$  equal to obtuse  $\angle BCD$ . Join BD. Then in  $\triangle$ s DAB, BCD,  $AB=CD$ , DB is common, and  $\angle DAB=\angle DCB \therefore \angle$ s ADB, DBC are either equal or supplementary (Prop. p. 44). But each of these  $\angle$ s is  $< a$  rt.  $\angle$  (I. 22.)  $\therefore$  they cannot be supplementary  $\therefore \angle ADB=\angle CBD \therefore \angle ABD=\angle CDB$  (I. 22).  $\therefore$  AB is equal and  $\parallel$  to CD (I. 20.)  $\therefore$  ABCD is a parm.

### EXERCISES XX.

1. Let D, E, F be the mid. pts. of the sides BC, CA, AB of  $\triangle ABC$ . Join CF, BE.  $AE=EC \therefore \triangle EBC=\triangle AEB=\frac{1}{2}\triangle ABC$  (II. 6.). Similarly  $\triangle FBC=\frac{1}{2}\triangle ABC \therefore \triangle EBC=\triangle FBC \therefore EF$  is  $\parallel$  to BC (II. 7.). Similarly DF is  $\parallel$  to AC, and DE to AB. Hence EFBD is a parm.  $\therefore EF=BD=\frac{1}{2}BC$ . Similarly  $DE=\frac{1}{2}AB$  and  $DF=\frac{1}{2}AC$ .

2. Let E, F, G, H be the mid. pts. of the sides AB, BC, CD, DA of the quadl. ABCD. EH is  $\parallel$  to BD and equal to  $\frac{1}{2}BD$  by the above example. Similarly FG is  $\parallel$  to BD and equal to  $\frac{1}{2}BD \therefore$  EH and FG are equal and  $\parallel$  (I. 21.)  $\therefore$  EFGH is a parm. (II. 1.). Also its diagonals bisect one another (II. 2. Cor. 3.), and this proves the third part of the exercise.

### EXERCISES XXI.

1. Let ABCD, ABEF be equal parms. on the same base AB and on the same side of it. If DCFE is not a str. line, produce DC to meet AF and BE in G and H. Parm. ABHG=parm. ABCD (II. 3.)=parm. ABEF Hyp., the part equal to the whole, which is impossible  $\therefore$  DCFE must be a str. line, *i.e.* the parms. ABCD, ABEF are between the same parallels.

2. Let ABCD be the given rhombus. Bisect BC at E, and AD at F, and join EF, thus dividing the rhombus into two parallelograms (II. 1.). Produce FE to H making FH equal to one-half the given perimeter. Bisect EH at K. With centre E and rad. EK describe a circle meeting AB at P. Draw FQ  $\parallel$  to EP to meet AB at Q. Produce PE, QF to meet CD at M and N.  $FQ+PQ+PE=FH$  and parm. QE=parm. AE. Also from  $\triangle$ s PEB, MCE,  $EM=EP$  (I. 3. 20. 16.)  $\therefore FN+NM+ME=FH \therefore$  parm. PQNM has the reqd. perimeter and is equal to the rhombus in area.

3. Bisect the sides  $AB$ ,  $CD$  of the parm.  $ABCD$  at  $E$  and  $G$ . Join  $EG$ .  $EG$  is  $\parallel$  to  $BC$  and  $AD$  (II. 1.)  $\therefore AG$  and  $EC$  are parms. Also we see by II. 4. that these parms are equal.

4. Let  $ABCD$ ,  $EFGH$  be two equal parms. between the same  $\parallel$ s  $BCFG$ ,  $ADEH$ . If  $FG$  is not equal to  $BC$ , make  $FK$  equal to  $BC$  and draw  $KM \parallel$  to  $EF$  or  $GH$  to meet  $AH$  at  $M$ . Parm.  $EFGH = \text{parm. } ABCD = \text{parm. } EFKM$  (II. 4.), the part equal to the whole, which is impossible  $\therefore$  the bases must be equal.

5. Produce  $HB$  to meet  $AC$  at  $M$  and  $KL$  at  $N$ . Also produce  $KA$  to meet  $EH$  at  $P$ , and  $LC$  to meet  $FG$  at  $Q$ . Parm.  $AN = \text{parm. } BP$  (II. 4.)  $= \text{parm. } ABDE$  (II. 3.). Similarly parm.  $CN = \text{parm. } BQ = \text{parm. } BCGF \therefore \text{Parm. } AKLC = \text{parm. } ABDE + \text{parm. } BFGC$ .

6. Let  $ABC$ ,  $DEF$  be equal  $\triangle$ s of equal altitudes,  $AB$ ,  $DE$  being their bases. Place the  $\triangle ABC$  so that  $A$  falls on  $D$  and  $AB$  along  $DE$ , the  $\triangle$ s being on the same side of  $DE$ . Let  $KDH$  be the new position of  $\triangle ABC$ . Since the altitudes are equal  $KF$  is  $\parallel$  to  $DH \therefore \triangle DEF = \triangle ABC = \triangle KDH = \triangle FDH$  (II. 5.), the part equal to the whole unless the point  $H$  falls on the pt.  $E \therefore$  the pt.  $H$  must fall on the pt.  $E \therefore AB = DE$ .

7.  $\angle ADO = \angle CBO$  (I. 20.),  $\angle AOD = \angle COB$  (I. 3.)  $\therefore \triangle AOD = \triangle COB$  (I. 16.). Also  $\triangle AOD = \triangle COD$  (II. 6.)  $\therefore \triangle COB = \triangle COD$ .

8. Let  $AC$ ,  $BD$  cut at  $O$ . Draw  $DM$ ,  $BM$  perp. to  $AC$ .  $DO = BO$  (II. 2. Cor. 3.).  $\angle DON = \angle BOM \therefore$  from  $\triangle$ s  $DON$ ,  $BOM$ ,  $DN = BM \therefore \triangle$ s  $BPQ$ ,  $DPQ$  are on the same base and have equal altitudes  $\therefore$  they are equal in area.

9. Let  $AB$  be the given base,  $CD \parallel$  to  $AB$ ,  $ACB$  an isos.  $\triangle$ . Produce  $AC$  to  $E$ , making  $CE$  equal to  $CA$ . Join  $DE$ ,  $DB$ .  $\angle ECD = \text{int. opp. } \angle CAB$  (I. 20.)  $= \angle CBA$  (I. 5.)  $= \text{alt. } \angle BCD$  (I. 20.)  $\therefore$  from  $\triangle$ s  $DCE$ ,  $DCB$ ,  $DE = DB \therefore AD + DB = AD + DE > AE$  (I. 12.), *i.e.*  $> AC + CB$ .

10. Let  $E$ ,  $F$ ,  $G$ ,  $H$  be the mid. pts. of the sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  of the parm.  $ABCD$ .  $EB = CG$  and is  $\parallel$  to it  $\therefore EBCG$  is a parm.  $\therefore \triangle EGF = \triangle EBG$  (II. 5.)  $= \frac{1}{2} \text{ parm. } EBCG$  (II. 2.). Similarly  $\triangle EHG = \frac{1}{2} \text{ parm. } EADG \therefore EFGH = \frac{1}{2} \text{ parm. } ABCD$ .

11.  $AD = DB \therefore \triangle ADC = \triangle DBC = \frac{1}{2} \triangle ABC$  (II. 6.). Also  $AE = EC \therefore \triangle BEC = \triangle BEA = \frac{1}{2} \triangle ABC$  (II. 6.)  $\therefore \triangle BEC = \triangle ADC$ . Take away the common  $\triangle EFC$ . Then  $\triangle BFC = \text{quadr. } ADFE$ .

12. Let AC, BD meet at O. Draw BM, DN perp. to AC.  $\triangle s$  ABC, ADC on the same base AC are equal  $\therefore$  their altitudes AN, BM are equal  $\therefore$  from  $\triangle s$  AON, BOM,  $DO = BO$  (I. 3. and 16.).

13. Let AC be greater than BC. From AC cut off AD equal to BC. Join PD.  $\triangle PAD = \triangle PCB$  (II. 6.)  $\therefore \triangle PDC$  is constant in area and on a fixed base DC  $\therefore$  its altitude is constant, *i.e.* P lies on one of two str. lines  $\parallel$  to AB and at a constant distance from it.

14.  $\triangle ABC$  is fixed and the area of ABCD is constant  $\therefore$  the area of  $\triangle ADC$  is constant, and the  $\triangle$  is on a fixed base AC  $\therefore$  D lies on a str. line  $\parallel$  to AC and at a constant distance from it.

15. Let the diagonals of the quadl. ABCD meet at O.  $\triangle AOD = \triangle COB$ . Add the  $\triangle COD$   $\therefore \triangle ADC = \triangle BDC$   $\therefore$  AB is  $\parallel$  to DC (II. 7.). Similarly AD is  $\parallel$  to BC  $\therefore$  ABCD is a parm.

16. AEDF is a parm.  $\therefore \triangle AED = \triangle AFD$  (II. 2.). Also  $\triangle ABD = \triangle ACD$  (II. 6.)  $\therefore \triangle EBD = \triangle FDC$ , and they are on equal bases  $\therefore$  EF is  $\parallel$  to BC (II. 8.).

17. Let ABC, DEF be two  $\triangle s$  on equal bases BC, EF, but such that the altitude AM of  $\triangle ABC =$  twice DN, the alt. of  $\triangle DEF$ . Bisect AM at H, and join BH, CH.  $\triangle AHB = \triangle BMH$  and  $\triangle AHC = \triangle CMH$  (II. 6.)  $\therefore \triangle ABC = 2\triangle BHC = 2\triangle DEF$  (II. 6.).

18. Join DE, and let AE meet BD at N.  $\triangle DEB = \triangle DCB$  (II. 5.)  $= \triangle ADB$  (II. 2.)  $\therefore$  their altitudes EN, AN are equal  $\therefore$  from  $\triangle s$  ENB, ANB,  $EB = AB$  (I. 4.)  $= CD$ .

19. Let F be the other pt. of trisection of AC and G the mid. pt. of BD. E, F, G are the mid. pts. of the sides of  $\triangle ABD$   $\therefore$  EF is  $\parallel$  to BD, EG to AD, and FG to AB (Exercises xx. 1.)  $\therefore$  EG is equal and  $\parallel$  to DC  $\therefore$  from  $\triangle s$  DOC, GOE,  $DO = OG$  (I. 16.)  $= \frac{1}{2}GD = \frac{1}{4}BD$ .

20. Take G the mid. pt. of BC. EF is  $\parallel$  to BC, EG to AB, FG to AC (Exercises xx. 1.)  $\therefore$  AFGE, BGEF, CGFE are parms.  $\therefore \triangle AEF = \triangle FEG = \triangle GEC = \triangle BFG$  (II. 2.)  $= \frac{1}{4}\triangle ABC$ .

21. Draw FG  $\parallel$  to CD or AB to meet AD at G.  $\triangle CFE + \triangle DFC = \triangle DEC = \frac{1}{2}$  parm. ABCD (II. 9.)  $= \frac{1}{2}$  parm. DGFC  $+ \frac{1}{2}$  parm. GABF  $= \triangle DCF + \triangle ABF$  (II. 9.)  $\therefore \triangle CFE = \triangle ABF$   $\therefore$  adding  $\triangle BFE$  to each  $\triangle CBE = \triangle AFE$ .

22. From  $\triangle s$  AEB, FEC,  $AB = CF$  (I. 3. 20. and 16.)  $\therefore$  DF  $= 2 \cdot AB$  (II. 2.)  $\therefore$  parm. GDFA  $= 2$  parm. ABCD (II. 4.).

23. Draw  $FOE \parallel$  to  $AB$  or  $CD$  to meet  $AD$  at  $E$  and  $BC$  at  $F$ .  
 $\triangle OAB + \triangle COD = \frac{1}{2}$  parm.  $AF + \frac{1}{2}$  parm.  $EC$  (II. 9.)  $= \frac{1}{2}$  parm.  
 $ABCD = \triangle ADC$  (II. 2.)  $= \triangle AOD + \triangle AOC + \triangle DOC \therefore \triangle OAB =$   
 $\triangle AOD + \triangle AOC$ , i.e.  $\triangle AOC =$  the diff. between  $\triangle s$   $OAB$  and  $AOD$ .

24. Let  $TS, RP$  meet at  $O$ . Draw  $SN$  and  $TM$  perp. to  $RP$ .  
 $\triangle s$   $PTR, PSR$  are on same base and equal in area  $\therefore$  their altitudes are equal, i.e.  $SN = TM \therefore$  from  $\triangle s$   $SON, TOM, SO = OT$  (I. 3. and 16.).

25. Having drawn the  $\triangle ABC$  as in (I. 25.) bisect  $BC$  at  $D$ .  
 $AD$  is perp. to  $BC$  (I. 7.).  $AD$  by measurement  $= 2.6$  in. Thro.  
 $A$  draw  $AE \parallel$  to  $BC$ , and thro.  $B$  draw  $BE \parallel$  to  $DA$  to meet  $AE$  at  
 $E$ . Parm.  $EBDA = 2\triangle BDA$  (II. 9.)  $= \triangle ABC$  (II. 6.).

26. Let  $CD$  be perp. to  $AB$  the base. The area of  $\triangle ABC = \frac{1}{2}$   
the product of  $AB$  and  $CD$ , and is therefore greatest when  $CD$   
is greatest. In the rt.  $\angle d.$   $\triangle CDB$ ,  $CB$  the hypotenuse is the  
greatest side, and  $CD$  increases as we increase the  $\angle CBD \therefore$  the  
 $\triangle$  is greatest when  $CD$  coincides with  $CB$ , i.e. when  $\angle ABC$  is a rt.  $\angle$ .

27. Let  $ABC$  be the  $\triangle$ ,  $O$  the pt. 1 ft. from  $BC$  and  $AC$ ,  $x$  ft.  
the reqd. dist. of  $O$  from  $AB$ .  $\triangle AOB + \triangle AOC + \triangle BOC = \triangle ABC$   
 $\therefore \frac{1}{2}x \times 5 + \frac{1}{2}4 \times 1 + \frac{1}{2}3 \times 1 = \frac{1}{2} \times 3 \times 4$ , whence  $x = 1$ .

28. Draw  $EPF \parallel$  to  $AB$  or  $CD$  to meet  $AD$  at  $E$  and  $BC$  at  $F$ .  
 $\triangle APB + \triangle DPC = \frac{1}{2}$  parm.  $AF + \frac{1}{2}$  parm.  $EC$  (II. 9.)  $= \frac{1}{2}$  parm.  
 $ABCD$ .

29.  $\triangle ABC = \triangle BCE$  and they have the same altitude  $\therefore$   
 $AC = CE$ . Also,  $CD$  is  $\parallel$  to  $BE \therefore \triangle BDC = \triangle EDC$  (II. 5.)  
 $= \triangle ACD$  for  $AC = CE$  (II. 6.)  $\therefore BD = DA$ , since  $\triangle s$   $BDC, ACD$   
have the same altitude.

30. Let  $OL, OM, ON$  be perp. to the sides  $BC, CA, AB$  of the  
equilateral  $\triangle ABC$ .  $\triangle ABC = \triangle BOC + \triangle AOC + \triangle AOB = \frac{1}{2}OL \cdot BC$   
 $+ \frac{1}{2}OM \cdot AC + \frac{1}{2}ON \cdot AB = \frac{1}{2}BC(OL + OM + ON) \therefore OL + OM +$   
 $ON$  is constant.

### EXERCISES XXII.

1.  $AB^2 = BD^2 + AD^2$  (II. 11.).  $AC^2 = CD^2 + AD^2$  (II. 11.)  $\therefore$   
 $AB^2 \approx AC^2 = BD^2 \approx CD^2$ .

2. Let  $AD$  be perp. to the base  $BC$  of  $\triangle ABC$ . From  $\triangle s$   $ABD,$   
 $ACD, BD = DC = \frac{1}{2}BC$  (I. 5. and 16.). Also  $AD^2 + BD^2 = AB^2 \therefore AD^2$   
 $+ \frac{AB^2}{4} = AB^2 \therefore AD^2 = \frac{3}{4}AB^2 \therefore AD = \frac{\sqrt{3}}{2} \cdot AB$ .

3. Let the diagonals of ABCD meet at rt.  $\angle$ s at O.  $AB^2 + CD^2 = (AO^2 + BO^2) + (DO^2 + CO^2)$  (II. 11.)  $= (AO^2 + OD^2) + (BO^2 + CO^2) = AD^2 + BC^2$  (II. 11.).

4.  $\angle DCA = a$  rt.  $\angle = \angle FCB \therefore$  adding  $\angle FCD$  to each,  $\angle DCB = \angle ACF \therefore$  from  $\triangle$ s DCB, ACF,  $AF = BD$  (I. 4.).

5. Draw AB 3 in. long, and BC, at rt.  $\angle$ s to it, 2 in. long.  $AC^2 = AB^2 + BC^2$  (II. 11.)  $= 9 + 4 = 13$  sq. in.  $\therefore$  the sq. on AC is the sq. reqd.

6. Draw DL perp. to BC, DM to BF, and DN to CG. From  $\triangle$ s DCN, DCL,  $DN = DL$  (I. 16.). From  $\triangle$ s DBM, DBL,  $DM = DL$  (I. 16.)  $\therefore DN = DM \therefore$  from the rt.  $\angle$ d.  $\triangle$ s DNA, DMA,  $\angle DAN = \angle DAM$  (I. 17.).

7.  $CE^2 + BD^2 = (EA^2 + AC^2) + (BA^2 + AD^2)$  (II. 11.)  $= (EA^2 + AD^2) + (AC^2 + BA^2) = DE^2 + BC^2$  (II. 11.).

8. Let AE and BK cut at O, and let BK cut AC at F. As in (II. 11.)  $\triangle$ s ECA, BCK are equal in all respects  $\therefore \angle EAC = \angle BKC$ . Also  $\angle OFA = \angle CFK$  (I. 3.)  $\therefore \angle FOA = \angle FCK$  (I. 22.)  $= a$  rt.  $\angle$ .

9. (1) If P, Q, R, S be the pts. of trisection of AB, BC, CD, DA nearest to A, B, C, D respectively. From  $\triangle$ s SAP, QBP,  $PS = PQ$ , and  $\angle APS = \angle BQP$  (I. 4.)  $=$  complement of  $\angle BPQ$  (I. 22.)  $\therefore \angle SPQ$  is a rt.  $\angle$  (I. 1.). In the same way  $\angle PQR = \angle QRS = \angle RSP = a$  rt.  $\angle \therefore PQRS$  is a parm. (I. 19.). It is also a rect. with two adj. sides equal  $\therefore$  it is a sq.

9. (2) Let P, S be the pts. of trisection of AB, AD nearest to A, and Q, R the pts. of trisection of CB, CD nearest to C.  $AS = AP \therefore \angle APS = \angle ASP = \frac{1}{2}$  a rt.  $\angle$  (I. 22.). Similarly  $\angle BPQ = \frac{1}{2}$  a rt.  $\angle \therefore \angle SPQ = a$  rt.  $\angle \therefore PQRS$  is a rectangle.

10. Draw DH perp. to FB produced  $\angle DBH =$  complement of  $\angle HBC = \angle CBA$ .  $\angle DHB = a$  rt.  $\angle = \angle BAC$ , and  $BD = BC \therefore \triangle DBH = \triangle ABC$  and  $BH = BA = BF$  (I. 16.)  $\therefore \triangle FBD = \triangle HBD$  (II. 6.)  $= \triangle ABC$ .

11. From  $\triangle$ s SAP, PBQ,  $PS = QR$  (I. 4.). Also  $\angle SPA = \angle PQB =$  complement of  $\angle BPQ \therefore \angle SPQ$  is a rt.  $\angle$ . Similarly  $\angle$ s at Q, R, S are rt.  $\angle$ s  $\therefore PQRS$  is a parm. (I. 19.) rt.  $\angle$ d. and having two adj. sides equal  $\therefore$  it is a sq. (Def.). Moreover  $SP^2 = SA^2 + AP^2$  (II. 11.)  $= 5 \cdot AB^2$ , for  $AP^2 = 4AB^2$ .

12. Join AO, BO, CO.  $AO^2 + BO^2 + CO^2 = (AF^2 + OF^2) + (BD^2 + OD^2) + (CE^2 + OE^2)$  (II. 11.)  $= AF^2 + BD^2 + CE^2 + OD^2 + OE^2 +$

OF<sup>2</sup>. Also  $AO^2 + BO^2 + CO^2 = (AE^2 + OE^2) + (BF^2 + OF^2) + (CD^2 + OD^2) = AE^2 + BF^2 + CD^2 + OD^2 + OE^2 + OF^2 \therefore AF^2 + BD^2 + CE^2 = AE^2 + BF^2 + CD^2$ .

13. Let DF be drawn perp. to AC produced and DH perp. to AB produced.  $\angle CBA =$  complement of  $\angle DBH$  (I. 1.)  $= \angle BDH \therefore$  from  $\triangle$ s DBH, BCA,  $HB = CA$  (I. 16.). DFAH is a rectangle  $\therefore DF = AH = AB + BH = AB + AC$ .

14. Let DF be drawn perp. to BE, and  $\therefore$  perp. to AD, and  $\parallel$  to AB.  $\angle ACD = \angle ADC = \frac{1}{2}$  a rt.  $\angle$  (I. 5. and 22.)  $\therefore \angle FDC =$  alt.  $\angle DCA = \frac{1}{2}$  a rt.  $\angle \therefore \angle EDF = \frac{1}{2}$  a rt.  $\angle \therefore \angle DEF = \frac{1}{2}$  a rt.  $\angle \therefore DE^2 = DF^2 + EF^2$  (II. 11.)  $= 2 \cdot DF^2$  (I. 6.)  $= 2 \cdot AB^2 = 8 \cdot AC^2 = 4 \cdot CD^2 \therefore DE = 2CD$ .

15. If the diagonals AC, BD of the quadl. meeting at O are not at rt.  $\angle$ s, let AN, CM be drawn perp. to BD.  $AB^2 + CD^2 = (AN^2 + BN^2) + (DM^2 + CM^2)$  (II. 11.). Also  $AD^2 + BC^2 = (AN^2 + DN^2) + (CM^2 + BM^2)$  (II. 11.)  $\therefore BN^2 + DM^2 = DN^2 + BM^2 \therefore BN^2 - BM^2 = DN^2 - DM^2 \dots (1)$ . But if  $BN > BM$ ,  $DN < DM \therefore (1)$  is impossible unless the pts. N, M coincide, in which case AC is perp. to BD.

16.  $10^2 - 8^2 = (10 - 8)(10 + 8) = 6^2 \therefore 10^2 = 8^2 + 6^2 \therefore$  the lengths form a rt.  $\triangle$ . II. 12.

17.  $39^2 - 36^2 = (39 - 36)(39 + 36) = 3 \times 75 = 15^2 \therefore 39^2 = 36^2 + 15^2 \therefore$  the lines form a rt.  $\triangle$ . II. 12.).

18.  $100^2 - 96^2 = (100 - 96)(100 + 96) = 4 \times 196 = 4^2 \times 49 = 28^2 \therefore 100^2 = 96^2 + 28^2 \therefore$  the lines form a rt.  $\triangle$ . II. 12.).

19. Let ABC be a  $\triangle$  such that  $AC^2 > AB^2 + BC^2$ . Let BD be drawn perp. to AB and equal to BC.  $AC^2 > AB^2 + BC^2 > AB^2 + BD^2 > AD^2$  (II. 11.)  $\therefore AC > AD \therefore$  from  $\triangle$ s ABC, ABD  $\angle ABC > \angle ABD$  (I. 15.) i.e.  $\angle ABC >$  a rt.  $\angle$ .

20. With the same construction as in the preceding example,  $AC < AD \therefore \angle ABC < \angle ABD$  (I. 15.)  $<$  a rt.  $\angle$ .

21. Let BE, CF, BK be diagonals of the sqs. on the sides AB, AC, BC of  $\triangle ABC$ , rt.  $\triangle$  at C.  $CF^2 = 2 \cdot AC^2$ ,  $BK^2 = 2BC^2$  and  $BE^2 = 2BA^2$  (II. 11.)  $\therefore CF^2 + BK^2 = 2 \cdot AC^2 + 2BC^2 = 2AB^2$  (II. 11.)  $= BE^2 \therefore$  the  $\triangle$  is rt.  $\triangle$ . II. 12.).

## EXERCISES XXIII.

1. Let  $ABC$  be the given  $\triangle$ ,  $D$  the given pt. in  $AB$ . Bisect  $CA$  at  $E$ , and draw  $BF \parallel$  to  $DE$  to meet  $AC$  at  $F$ . Join  $DF$ .  $\triangle ADF = \triangle ADE + \triangle DEF = \triangle ADE + \triangle DBE$  (II. 5.)  $= \triangle ABE = \frac{1}{2}\triangle ABC$  for  $AE = EC$ . This construction fails if  $AD < BD$ . In such a case,  $BC$  must be bisected instead of  $AC$ .

2. Draw  $BE \parallel$  to  $DA$  to meet  $CA$  at  $E$ . Join  $DE$ .  $\triangle CDE = \triangle ADC + \triangle EDA = \triangle ADC + \triangle ADB$  (II. 5.)  $= \triangle ABC$ .

3. Let  $AB$  be the given str. line. Draw another str. line  $ACDE$  making  $AC = CD = DE$ . Join  $BE$  and draw  $DG$  and  $CF \parallel$  to  $BE$  to meet  $AB$  at  $G$  and  $F$ . Draw  $FH, GK \parallel$  to  $AE$  to meet  $GD, BE$ .  $FD$  and  $GE$  are parms.  $\therefore FH = CD = AC$ , and  $GK = DE = CD$  (II. 2.)  $\therefore$  from  $\triangle s$   $ACF, FHG, AF = FG$  (I. 20. and 16.). Similarly  $BG = FG = AF$ .

4. Use the method of the above exercise.

5. Let the two medians  $BD, CE$  of  $\triangle ABC$  meet at  $G$ . Produce  $AG$  to  $H$ , making  $GH = GA$ .  $E, G$  are the mid. pts. of  $AB$  and  $AH \therefore EG$  is  $\parallel$  to  $BH$  (Exercises xx. 1). Similarly  $GD$  is  $\parallel$  to  $CH \therefore BGCH$  is a parm. Also its diagonals bisect one another  $\therefore AG$  passes through the mid. pt. of  $BC$ .

6. Let  $AO, BO$  bisectors of angles of  $\triangle ABC$  meet at  $O$ . Draw  $OL$  perp. to  $BC, OM$  to  $AC, ON$  to  $AB$ . From  $\triangle s$   $AON, AOM, ON = OM$  (I. 16.). Similarly from  $\triangle s$   $BON, BOL, ON = OL$  (I. 16.)  $\therefore OM = OL \therefore$  from  $\triangle s$   $COM, COL, \angle OCM = \angle OCL$  (I. 17.).

7. Let  $D, E, F$  be the mid. pts. of the sides  $BC, CA, AB$  of  $\triangle ABC$ . Let the perps. at  $D$  and  $E$  meet at  $O$ . Join  $OF$ . [The perps.  $DO, EO$  must meet, for if they were parallel  $CB$  and  $CA$  would also be parallel.] From  $\triangle s$   $BDO, CDO, BO = CO$  (I. 4.). From  $\triangle s$   $AOE, COE, AO = CO$  (I. 4.)  $\therefore AO = BO \therefore$  from  $\triangle s$   $AFO, BFO, \angle AFO = \angle BFO = \text{a rt. } \angle$  (I. 7.).

8. (1) Let  $AD$  be less than  $BE$ . Draw  $HCK \parallel$  to  $DFE$  to meet  $DA$  produced at  $H$  and  $EB$  at  $K$ .  $\angle BCK = \angle ACH$  (I. 3.).  $\angle BKC = \angle CHA$  (I. 20.)  $\therefore$  from  $\triangle s$   $BCK, ACH, BK = AH \therefore AD + BE = (HD - HA) + (KE + BK) = HD + KE = 2CF$  (II. 2.). (2) Let  $DFE$  cut  $AB$  between  $A$  and  $C$ . Draw  $HCK \parallel$  to  $DFE$  to meet  $AD$  produced in  $H$ , and  $BE$  in  $K$ . As in the above  $BK = AH \therefore BE - AD = BK + KE - (AH - DH) = EK + DH = 2CF$  (II. 2.).

9. Let AB be the given str. line. At A and B make  $\angle$ s DAB, ABD each equal to half a rt.  $\angle$ . Bisect  $\angle$ ABD by BC meeting AD at C. Draw CE perp. to CA to meet AB at E. E is the reqd. pt. For,  $\angle$ AEC =  $\frac{1}{2}$  a rt.  $\angle$  (I. 22.) =  $\angle$ ECB +  $\angle$ EBC (I. 22.)  $\therefore \angle$ ECB =  $\frac{1}{4}$  rt.  $\angle$  =  $\angle$ EBC  $\therefore$  EB = EC  $\therefore$  AE<sup>2</sup> = 2EC<sup>2</sup> (II. 11.) = 2BE<sup>2</sup>.

10. See Example 6, Exercises xxii.

### MENSURATION EXAMPLES. EXERCISES XXIV.

1. For Part 1, see Example 9, Exercises xxii. (2) FG<sup>2</sup> = GC<sup>2</sup> + CF<sup>2</sup> = 2  $\therefore$  FG =  $\sqrt{2}$  = EH. EF<sup>2</sup> = EB<sup>2</sup> + BF<sup>2</sup> = 8  $\therefore$  EF =  $2\sqrt{2}$  = HG  $\therefore$  the perimeter =  $6\sqrt{2}$  = 6(1.41421) = 8.49 in. approx.

2. If the edge is straight, it will coincide with the line drawn. For Part 2, see Example 15, Exercises A.

3. The centre of the rolling circle is always at a distance of 3 in. from the centre of the fixed circle  $\therefore$  the locus is a circle of rad. 3 in. centre at the centre of the fixed circle.

4. Draw a str. line AB 5 cms. long, and at A draw AD perp. to AB, making AD = 8 cms. Through D draw DCE  $\parallel$  to AB, and bisect  $\angle$ BAD by AC meeting DCE at C. Thro. B draw BE  $\parallel$  to AC to meet DCE at E. Altitude of parm. = 8 cms.  $\therefore$  its area =  $8 \times AB = 40$  sq. cms., and  $\angle$ BAC =  $\frac{1}{2}$  a rt.  $\angle$   $\therefore$  ABEC is the parm. reqd. By measurement AC = 11.3 cms.

5. Draw AB 10 cms. long and BC at rt.  $\angle$ s to it 4 cms. long. AC<sup>2</sup> = AB<sup>2</sup> + BC<sup>2</sup> = 100 + 16 cms.  $\therefore$  AC =  $\sqrt{116}$  = 10.77 cms.  $\therefore$  the dist. reqd. = 11 kilometres, to the nearest kilometre.

6. Let AB, CD be the chords, O the centre of the circle, OE perp. to AB, OF perp. to CD, so that OE =  $\frac{3r}{5}$ , and OF =  $\frac{4r}{5}$ , where r = the radius of the circle.  $\angle$ OAB =  $\angle$ OBA (I. 5.)  $\therefore$  from  $\triangle$ s OAE, OBE, AE = EB (I. 16.). Similarly, CF = FD  $\therefore$  AB<sup>2</sup> = 4AE<sup>2</sup> = 4(AO<sup>2</sup> - EO<sup>2</sup>) = 4( $r^2 - \frac{9r^2}{25}$ )  $\therefore$  AB =  $\frac{8 \cdot r}{5}$ . CD<sup>2</sup> = 4CF<sup>2</sup> = 4(CO<sup>2</sup> - OF<sup>2</sup>) = 4( $r^2 - \frac{16r^2}{25}$ )  $\therefore$  CD =  $\frac{6r}{5}$   $\therefore$   $\frac{3 \cdot AB}{CD} = \frac{24}{8} = 3 \therefore$  the shorter chord is contained 4 times in 3 times the longer chord.

7. Draw AB, AC at rt.  $\angle$ s to one another and each equal to 2 in. By measurement  $BC = 2.83$  in.

8. Bisect AB by taking  $AC = 3$  in. Draw CD at rt.  $\angle$ s to AB by I. 28. and cut off  $CD = 3$  in. By measurement  $AD = 4.24$  in. Also by II. 11.  $AD^2 = AC^2 + CD^2 = 18 \therefore AD = 3\sqrt{2} = 3(1.41421) = 4.24$  in. approx.

9. Draw the  $\triangle$  by the method of I. 25. Draw CD perp. to the base AB. By measurement,  $CD = 2.83$  in. Or,  $\angle CAB = \angle CBA$  (I. 5.)  $\therefore$  from  $\triangle$ s ADC, BDC,  $AD = DB \therefore CD^2 = CA^2 - AD^2$  (II. 11.)  $= 9 - 1 = 8 \therefore CD = \sqrt{8} = 2.83$  in. to the nearest hundredth of an inch.

10. Draw AB 4 in. long, and make  $\angle BAC = 45^\circ$ , cut off  $AC = 4$  in. Draw  $CD \parallel$  to AB, and  $BD \parallel$  to AC. Also draw CE perp. to AB. ABDC is the rhombus, and CE its altitude. By measurement,  $CE = 2.83$  in.  $\therefore$  the area of the rhombus  $= 4 \times 2.83 = 11.32$  sq. in. *N.B.*—We can only *measure* to the nearest hundredth of an inch. By calculation it will be found that  $CE = 2.82842$  more exactly. Whence the area  $= 11.31$  sq. in. to the nearest hundredth. The error in the measurement of CE is multiplied by 4 in finding the area.

11. By measurement the altitude of the  $\triangle = 3.46$  in.  $\therefore$  the area  $= \frac{1}{2} \times 4 \times 3.46 = 6.92$  sq. in. (This result is inaccurate for the reason given in the previous example.)

12. If  $x$  is the third side,  $5^2 = x^2 + 4^2$  (II. 11.). Whence  $x = 3$  in.  $\therefore$  the area  $= \frac{1}{2}$  alt.  $\times$  base  $= \frac{1}{2} \times 3 \times 4 = 6$  sq. in.

13. Let  $Ox, Oy$  be the perp<sup>r</sup> str. lines. Mark points 2 units from  $Ox$  and 1 unit from  $Oy$ , 4 units from  $Ox$  and 2 units from  $Oy$ , and so on. The locus will then be seen to be a str. line thro. O.

14. Mark on the squared paper a rectangle whose sides are 3 units and 4 units long. We see then that the area of the rect. consists of 3 rows of square units, each row consisting of 4 sq. units  $\therefore$  the area  $= 12$  sq. units.

15. By I. 23. the locus is a str. line bisecting at rt.  $\angle$ s the str. line joining the given pts. To test the locus with circles, take centres on the locus.

16. Draw AB, BC at rt.  $\angle$ s to one another and each 3 cms. long. Draw AD at rt.  $\angle$ s to AB. With centre C and rad.

6 cms. describe a circle cutting AD in D, taking D on the same side of AB as the pt. C. ABCD is the reqd. quadr.

**17.** Draw AB 5 cms. long, and AC at rt.  $\angle$ s to it 4 cms. long. Draw CDE  $\parallel$  to AB. With centre A and rad. 5 cms. describe a circle cutting CDE at D. Draw BE  $\parallel$  to AD meeting CD at E. DABE is the reqd. rhombus.

### **MENSURATION PROBLEMS. EXERCISES XXV.**

**1.** Area = base  $\times$  alt. =  $24 \times 2$  sq. in. = 48 sq. in.

**2.** Area = breadth  $\times$  length =  $18 \times 14\frac{1}{2}$  sq. ft. =  $\frac{1}{9}^8 \times \frac{2}{2}^9$  sq. yds. = 29 sq. yds.

**3.** Area =  $\frac{1}{2}$  base  $\times$  alt. =  $12 \times 14$  sq. in. = 168 sq. in.

**4.** Area =  $9^2 + 12^2$  (II. 11.) =  $3^2(3^2 + 4^2) = 15^2 = 225$  sq. in.

**5.** Let  $x$  be the side reqd.  $x^2 + 48^2 = 52^2$  (II. 11.)  $\therefore x^2 = (52 - 48)(52 + 48) = 4 \times 100 \therefore x = 20$  in.

**6.** Draw two str. lines AB, AC, each 2 in. long, at rt.  $\angle$ s to one another.  $BC^2 = 8$  (II. 11.)  $\therefore BC = \sqrt{8} = 2.83$  in. by measurement.

**7.** Draw AB 2 cms. long, AC at rt.  $\angle$ s to it, 1 cm. long.  $BC^2 = 4 + 1$  (II. 11.)  $\therefore BC = \sqrt{5} = 2.24$  cms. by measurement.

**8.** Draw AB 3 inches long, and BC at rt.  $\angle$ s to it. With centre A and rad. 4 inches describe a circle meeting BC at C.  $BC^2 = AC^2 - AB^2$  (II. 11.) =  $16 - 9 = 7 \therefore BC = \sqrt{7} = 2.65$  inches.

**9.**  $AC^2 = 24^2 + 32^2$  (II. 11.) =  $8^2(3^2 + 4^2) = 8^2 \times 5^2 \therefore AC = 40$  ft.  $\therefore$  10 ft. of the ladder projects beyond the top of the wall.

**10.** Draw AB, BC at rt.  $\angle$ s to one another making AB = BC = 5 cms.  $AC^2 = AB^2 + BC^2$  (II. 11.) =  $2 \cdot AB^2$ . By measurement, AC = 7.07 cms.

**11.** If AB represents 72 ft., AC 30 ft., then BC represents 78 ft. Also  $BC^2 - AB^2 = 78^2 - 72^2 = 6 \times 150 = 30^2 = AC^2 \therefore \angle CAB$  is a rt.  $\angle$ . (II. 12.)

**12.** 4 in. = 10.2 cms. nearly.

**13.** 6 cms. = 2.36 in.

**14.** If ABC be the  $\triangle$  such that AB = 6 cms.  $\angle ABC = 60^\circ$ ,  $\angle ACB =$  a rt.  $\angle$ , produce BC to D making CD = CB. Join AD.

From  $\triangle s$  ACD, ACB, AD=AB, and  $\angle ADC = \angle ABC = 60^\circ$  (I. 4.)  $\therefore \angle BAD = 60^\circ$  (I. 22.)  $\therefore \triangle ABD$  is equilateral (I. 6.) and  $BC = \frac{1}{2}BD = \frac{1}{2}BA = 3$  cms. By measurement,  $AC = 5.2$  cms.

15. Let ABCD be the quadl. such that  $AC = 30$ , and  $BD = 40$  ft. Let AC, BD cut at rt.  $\angle s$  at O. Area of figure  $= \triangle ABD + \triangle BCD = \frac{1}{2}AO \cdot BD + \frac{1}{2}CO \cdot BD = \frac{1}{2}(AO + CO)BD = \frac{1}{2} \cdot 30 \times 40 = 600$  sq. ft.

16. Let ABC be the  $\triangle$  such that  $AB = 4$  in.,  $BC = 6$  in., and  $\angle ABC = 30^\circ$ . Draw AD perp. to BC. By measurement, or as in Example 14 above,  $AD = 2$  in.  $\therefore$  area of  $\triangle = \frac{1}{2}AD \cdot BC = 6$  sq. in.

17. If  $x$  be a side of the square,  $x^2 = 15$  acres = 150 sq. chains  $\therefore x = \sqrt{150}$  chains = 12.25 chains approx. = 12 chains 25 links.

18. If  $x$  be a side of the sq. then  $2x^2 = 36$  (II. 11.)  $\therefore$  area of square  $= x^2$  sq. ft. = 18 sq. ft.

19. If  $x$  ft. be the length of the other side of the rect.  $x^2 = 8^2 - 5^2 = 39$  sq. ft.  $\therefore$  area of rect.  $= 5 \times \sqrt{39} = 5 \times 6.245 = 31$  sq. ft. (to the nearest sq. ft.).

20. Area of hall  $= 8 \times 12 \times 144$  sq. in. Area of each tile  $= 9$  sq. in.  $\therefore$  no. of tiles  $= \frac{8 \times 12 \times 144}{9} = 1536$ .

21. Let  $x$  ft. be the length of the hall.  $8x =$  area of hall in sq. ft.  $= \frac{1920 \times 16}{144} = \frac{640}{3}$  ft.  $\therefore x = \frac{80}{3}$  ft. = 26 ft. 8 in.

22. Let  $x$  yds. be the length of the other side.  $\frac{1}{2}x \times 22 =$  area of  $\triangle = 1210$   $\therefore x = 110$  yds.

23. Let  $x$  ft. be the length of the other side.  $x^2 = 39^2 - 15^2$  (II. 11.)  $= 3^2(13^2 - 5^2) = 3^2(13 - 5)(13 + 5)$ ,  $3^2 \times 144$   $\therefore x = 36$  ft.  $\therefore$  area of  $\triangle = \frac{1}{2}x \times 15 = 270$  sq. ft.

24. Draw AD perp. to BC.  $\angle CAD = 45^\circ$  (I. 22.)  $\therefore 2AD^2 = AC^2$  (II. 11.)  $= 25$ .  $\therefore AD = \frac{5\sqrt{2}}{2} = \frac{5}{2}(1.41421) = 3.54$  ft. approx. Area of  $\triangle = \frac{1}{2}AD \times BC = \frac{1}{2} \times \frac{5\sqrt{2}}{2} \times 12 = 15\sqrt{2}$  sq. ft. = 21.21 sq. ft. approx.

25. Let  $x$  ft. be the reqd. length of the perp.  $\frac{1}{2}x \times 14 =$  area of  $\triangle = \frac{1}{2} \times 6 \times 12$   $\therefore x = \frac{6 \times 12}{14} = 5\frac{1}{7}$  ft.

26. Construct the  $\triangle$  as in I. 25. By measurement, the perp. on the 5 cm. side  $= 5.4$  cms. Area  $= \frac{1}{2} \times 5.4 \times 5 = 13.5$  sq. cms.

27. Let ABCD be the rhombus having  $\angle ADC = 45^\circ$ . Draw AE perp. to CD.  $\angle DAE = 45^\circ$  (I. 22.)  $\therefore AE = DE$  (I. 6.)  $\therefore 2AE^2 = AD^2 = 36$ , and  $AE = 3\sqrt{2}$  ft. Area of rhombus  $= 6 \times 3\sqrt{2}$  sq. ft.  $= 18(1.41421) = 25$  sq. ft. (to the nearest sq. ft.).

28. If  $x$  ft. be the length of the diagonal  $x^2 = 2 \times 18^2$  (II. 11.)  $\therefore x = 18\sqrt{2} = 25.46$  ft. to the nearest hundredth of a foot.

29. Dist. reqd.  $= \sqrt{30^2 + 40^2}$ . (II. 11.)  $= 50$  miles.

30. If  $x$  links be the length of the diagonal,  $x^2 = 2 \times 410^2$  (II. 11.).  $x = 410\sqrt{2} = 410(1.41421) = 580$  links (to the nearest link)  $= 5$  chains 80 links.

31. If  $x$  ft. be the dist. reqd.  $x^2 = 31^2 - 23^2$  (II. 11.)  $= (31 - 23)(31 + 23) = 8 \times 54$   $\therefore x = 12\sqrt{3}$  ft.  $= 20.78$  ft. (to two dec. places).

32. Let AB ( $=x$  ft.) be the height of the tower, BC its shadow. As in Example 14 above,  $AC = 2AB = 2x$  ft.  $\therefore 4x^2 = x^2 + 200^2$  (II. 11.).  $3x^2 = 200^2$ ,  $x = \frac{200}{3}\sqrt{3} = 115.47$  ft. (to two dec. places).

33. Draw ABC horizontally so that  $AB = BC = 2$  inches. Draw BD, 1 in. long, perp. to ABC, to represent the boy; and CE also perp. to ABC. Produce AD to meet CE at E. CE is the lamp-post. Draw DF  $\parallel$  to BC to meet DE at F.  $FC = BD$  (II. 2.) 1 in. and  $DF = BC = AB$ . From  $\triangle$ s EFD, DBA,  $EF = BD = 1$  in.  $\therefore EC = 2$  in.  $\therefore$  the post is 10 ft. high.

34. Let  $x$  be the hypotenuse, then  $x^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 + 2m^2n^2 + n^4$   $\therefore x = m^2 + n^2$  cms. When  $m = n + 1$ ,  $m^2 + n^2 - 2mn = \overline{m - n}^2 = 1$ . When  $m = 13$  and  $n = 12$ , the sides are respectively 25, 312, 313 cms.

35. Let ABCD be the trapezium, such that  $AB = 9$ ,  $BC = 10$ ,  $CD = 30$ ,  $DA = 17$  ft. Let  $x$  be the perp. dist. between AB and CD. Draw AE, BF perp. to CD.  $DE + EF + CF = 30$   $\therefore \sqrt{17^2 - x^2} + 9 + \sqrt{100 - x^2} = 30$  (II. 11. and 2.)  $\therefore \sqrt{17^2 - x^2} = 21 - \sqrt{100 - x^2}$ ,  $289 - x^2 = 441 + 100 - x^2 - 42\sqrt{100 - x^2}$ , whence  $\sqrt{100 - x^2} = 6$  and  $x = 8$  ft.  $\therefore$  area of trapezium  $= \frac{1}{2}(9 + 30)8 = 156$  sq. ft.

36. Make DQ equal to BP. Join AQ, BD.  $DQ = BP$   $\therefore \triangle ADQ = \triangle PBQ$ .  $CQ = AP$   $\therefore \triangle APQ = \triangle BQC$   $\therefore$  fig. ADQP  $= \frac{1}{2}$  parm. ABCD.

**37.** (1) Let ABCD be the sq., E the mid. point of BC. Area of  $\triangle DCE = \frac{1}{2}DC \cdot CE = 9$  sq. in. Area of ABED =  $36 - 9 = 27$  sq. in. (2) Join AC cutting DE at O. Take F the mid. pt. of AD and join FB, cutting AC at P. FP is  $\parallel$  to DO (II. I.) and AF = FD  $\therefore$  AP = PO. Similarly CO = OP  $\therefore$  AO = 2OC  $\therefore$   $\triangle AOD = 2\triangle DOC$ , i.e.  $\triangle DOC = \frac{1}{3}\triangle ADC = \frac{1}{3}$  of sq. = 6 sq. in.  $\therefore \triangle AOD = 2\triangle DOC = 12$  sq. in. and  $\triangle COE = \triangle DCE - \triangle DOC = 3$  sq. in. and the remaining part AOEB = 15 sq. in.

**38.** Let ABCD be the trapezium, such that AB = 20, BC = 13, CD = 34, DA = 15 yds. Draw AE, and BF perp. to CD, and let  $x$  yds. be the perp. dist. between AB and CD, so that AE = BF =  $x$ . DE + EF + CF = 34  $\therefore \sqrt{15^2 - x^2} + 20 + \sqrt{13^2 - x^2} = 34$  (II. 11. and 2.)  $\therefore \sqrt{15^2 - x^2} = 14 - \sqrt{13^2 - x^2}$ ,  $225 - x^2 = 196 - 28\sqrt{13^2 - x^2} + 169 - x^2$   $\therefore 28\sqrt{13^2 - x^2} = 140$   $\therefore 169 - x^2 = 25$   $\therefore x = 12$   $\therefore$  area of trapezium =  $\frac{1}{2}(AB + CD)AE = \frac{1}{2} \cdot 54 \times 12 = 324$  sq. yds.

\* \* \* The additional Exercises, 39-81, will be found on pages 174-177.

### MISCELLANEOUS EXERCISES XXVI.

**1.**  $a + b > c$  in all  $\triangle$ s (I. 12.)  $a + b = c$  is untrue for all  $\triangle$ s, for the same reason.  $a + b < c$  is untrue for all  $\triangle$ s, for the same reason.  $a^2 + b^2 = c^2$  is true for right  $\angle$ d  $\triangle$ s, when  $c$  is the side opp. the rt.  $\angle$  (II. 11.).

**2.** If ABCD is the reqd. quadr. and E the mid. pt. of BC, DE is perp. to BC for DB = DC. (I. 7.). Thro. D draw FDH  $\parallel$  to BC, forming the rectangle BCHF.  $\angle ABD =$  complement of  $\angle DBC = \angle ACB$ .  $\therefore$  from  $\triangle$ s ABC, DFB, BA = FD (I. 16.) = BE =  $\frac{1}{2}BC$ . Hence the following construction. Draw any str. line BC, and BA at rt.  $\angle$ s to it, making BA =  $\frac{1}{2}BC$ . Join AC. Draw BO perp. to AC and produce it to D making BD = CA. ABCD is the reqd. quadr.

**3.** Let AB be gr. than AC, so that  $\angle ACB > \angle ABC$   $\therefore \angle BAF$ , the complement of  $\angle ABC > \angle CAF$  the complement of  $\angle ACB$   $\therefore \angle BAF > \frac{1}{2}\angle BAC$   $\therefore$  D falls between B and F. Perps. from D to AB, AC are equal (I. 24.)  $\therefore \triangle ABD > \triangle ADC$ , i.e.  $> \frac{1}{2}\triangle ABC$   $\therefore BD > BE$   $\therefore$  E the mid. pt. of BC lies between B and D  $\therefore$  AE, AD, AF are in order of magnitude (Exercises xviii. 3.).

**4.** Let ABC be the equilateral  $\triangle$ , BL, CN the altitudes of the parms. on AB and AC. Draw BK perp. to BC and equal to

**BL + CN.** Draw  $EKF \parallel$  to  $BC$ , and any  $\parallel$ s  $BE, CF$  completing the parm.  $BEFC$ .  $BEFC$  is the parm. reqd. (II. 4.).

**5.** Let  $ABC$  be the  $\triangle$ ,  $DE$  the given perimeter. From  $DE$  cut off  $DF = BC$ . Bisect  $FE$  at  $G$ . Bisect  $BC$  at  $L$ . Thro.  $A$  draw  $AKH \parallel$  to  $BC$ . With centre  $L$  and rad.  $FG$  describe a circle cutting  $AKH$  at  $K$ . Join  $LK$  and complete the parm.  $BLKH$ . The perimeter of this parm.  $= 2 \cdot BL + 2 \cdot KL = DE$ . Also parm.  $BL = 2\triangle ABL = \triangle ABC \therefore HBLK$  is the parm. reqd.

**6.** Let  $A$  be the given pt.,  $BC$  the given line. Draw  $AD$  perp. to  $BC$ , and produce it to  $E$  making  $DE = DA$ . With centre  $D$  and rad.  $DA$  or  $DE$  describe a circle cutting  $BC$  at  $F$  and  $G$ .  $AFEG$  is the reqd sq.  $\angle GAD = \frac{1}{2}$  a rt.  $\angle = \angle DEF$  (I. 22.)  $\therefore AG$  is  $\parallel$  to  $FE$ . Similarly  $AF$  is  $\parallel$  to  $GE \therefore AFEG$  is a parm. Also  $AG = AF$  (I. 4.) and  $\angle GAF =$  a rt.  $\angle \therefore AFEG$  is a sq.

**7.** Let  $\triangle ADE$  be described on the same side of  $AD$  as the pt.  $B$ .  $\angle EAD = \frac{2}{3}$  of a rt.  $\angle$  (I. 22.)  $= \angle BAC \therefore \angle EAB = \angle DAC \therefore$  from  $\triangle$ s  $EAB, DAC, EB = CD$  (I. 4.). If  $\triangle ADE$  be described on the opp. side of  $AD$  to the pt.  $B$ ,  $CE$  will be equal to  $BD$ .

**8.** Let  $M$  and  $N$  be on the same side of  $PQ$ . Draw  $MC, NC$  to meet on  $PQ$  at  $C$ , so that  $\angle BCP = \angle NCQ$  (Exercises xviii., Example 2). Thro.  $L$  draw  $BLA \parallel$  to  $NC$  to meet  $CM$  at  $B$ , and  $PQ$  at  $A$ . Thro.  $A$  draw  $AD \parallel$  to  $CB$  to meet  $NC$  at  $D$ .  $\angle BAC = \angle NCQ$  (I. 20.)  $= \angle BCA \therefore BA = BC$ . Also, by construction,  $ABCD$  is a parm.  $\therefore CD = BA = BC = AD \therefore ABCD$  is also a rhombus, and the fig. reqd.

**9.** Produce  $AB$  to  $E$  making  $BE$  equal to the given line. Join  $DE$ , and draw  $DX$  making  $\angle EDX = \angle DEB$ . Draw  $BF \parallel$  to  $DE$  to meet  $DX$  in  $F$ .  $\angle XFB = \angle XDE$  (I. 20.)  $= \angle XED = \angle XBF$  (I. 20.)  $\therefore BX = FX$  (I. 6.). Also  $XD = XE$  (I. 6.)  $\therefore DF = BE \therefore DX - BX = DX - FX = DF = BE \therefore X$  is the pt. reqd.

**10.** Let  $DPE$  meet  $AB$  in  $D$  and  $AC$  produced in  $E$ . Draw  $CF \parallel$  to  $DB$  to meet  $PE$  at  $F$ .  $\angle PCE > \angle ABC$  (I. 8.)  $> \angle PCF$  (I. 20.)  $\therefore F$  falls between  $P$  and  $E$ .  $\triangle DPB = \triangle FPC$  (I. 3. 20. 16.)  $\therefore$  adding fig.  $ADPC$  to each  $\triangle ABC = \text{fig. } DFCA < \triangle ADE$ .

**11.** Let  $\triangle ABC$  have  $AB > AC$ . With centre  $A$  and rad. equal to  $\frac{1}{2}(AB + AC)$  describe a circle cutting  $BC$  at  $P$ .  $2AP = AB + AC \therefore AP - AC = AB - AP \therefore P$  is the pt. reqd.

12. Let the lines EP, FP bisecting AD and BC at rt.  $\angle$ s meet at P. From  $\triangle$ s AEP, DEP, PA = PD (I. 4.). From  $\triangle$ s BFP, CFP, BP = CP (I. 4.)  $\therefore \triangle$ s APB, DPC are equal in all respects (I. 7.). By bisecting AC and BD at rt.  $\angle$ s, another pt. Q may be found satisfying the reqd. conditions.

13. Draw AD perp. to BC in the equilateral  $\triangle$ ABC. BD = DC (I. 16.)  $\therefore$  AB = 2BD.  $BD^2 + AD^2 = AB^2$  (II. 11.) =  $4BD^2$   $\therefore$   $AD^2 = 3BD^2$ .

14. Let ABCD be the given sq. P the given pt. in AB, AP being gr. than PB. Join BD.  $\triangle$ ABD =  $\frac{1}{2}$  the sq. Draw PF as in Exercises xxiii. 1, so that PF bisects the  $\triangle$ ADB.  $\triangle$ APF =  $\frac{1}{2}\triangle$ ABD =  $\frac{1}{4}$  of the sq. In DC make DG = BP. Join PG. By superposition fig. APGD = fig. PBCG =  $\frac{1}{2}$  sq. ABCD  $\therefore$  since  $\triangle$ APF =  $\frac{1}{4}$  of the sq., fig. FPGD =  $\frac{1}{4}$  of the sq. From GC ( $>$ GD) cut off GH =  $\frac{1}{2}$ DC.  $\triangle$ PGH =  $\frac{1}{4}$  of the sq. (II. 9.)  $\therefore$  PF, PG, PH are the reqd. lines.

15. In  $\triangle$ ABC let AD be drawn perp. to BC, and take any pt. P in BC.  $PB^2 = PD^2 + BD^2$  (II. 11.).  $PC^2 = PD^2 + CD^2$  (II. 11.)  $\therefore$   $PB^2 \approx PC^2 = BD^2 \approx CD^2$ .

15 (2).  $PB^2 - PC^2 = BD^2 - CD^2 = BA^2 - CA^2$ . This is only true when P lies in AD, as may be seen by drawing a perp. QH to BC from a point Q outside AD. ( $BQ^2 - CQ^2 = BH^2 - CH^2$  which is not equal to  $BD^2 - CD^2$ .) Let P be the intersection of the altitudes AD, BE. Then  $PB^2 - PC^2 = AB^2 - AC^2$ , and  $PA^2 - PC^2 = AB^2 - BC^2$   $\therefore$  by subtraction  $PB^2 - PA^2 = BC^2 - AC^2$   $\therefore$  P must lie in the altitude CF.

16. CA = CD  $\therefore \angle$ CAD =  $\angle$ CDA. CA = CE  $\therefore \angle$ CAE =  $\angle$ CEA  $\therefore \angle$ DAE =  $\angle$ ADE +  $\angle$ AED  $\therefore \angle$ DAE = a rt.  $\angle$  (I. 22.).

17. If  $x^\circ$  be the smallest angle.  $6x = 180$  (I. 22.)  $\therefore x = 30^\circ$   $\therefore$  we have to describe a  $\triangle$  whose angles are  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$ . Describe an equilateral  $\triangle$ ABC, and draw AD perp. to BC.  $\triangle$ s ADB, ADC both satisfy the reqd. conditions.

18. Join EC, EB.  $\triangle$ FCB =  $\triangle$ FCA (II. 5.) =  $\triangle$ ECA (II. 5.) =  $\triangle$ EBA (II. 5.).

19. Reduce the given fig. to a  $\triangle$ ABC as in (II. 15.). Bisect BC at D, and BD at E. Draw EF perp. to BD to meet AF  $\parallel$  to BC at F. Produce FE to G making EG = EF. BFDG is the reqd. rhombus. For  $\triangle$ BFD =  $\frac{1}{2}\triangle$ ABC (II. 6.). Also BF = FD

from  $\triangle s$  BEF, DEF (I. 4.). Similarly  $BG = DG$ , and  $\triangle BFD = \triangle BGD$  (II. 5.). Also  $BF = DG$  from  $\triangle s$  BEF, DEG (I. 4.). Similarly  $FD = BG$ .

20. Let ABC be the  $\triangle$  so that  $\angle ACB = 2\angle ABC$ , and let AD be perp. to BC. Make  $\angle ABF$  equal to  $\angle ABC$ , and draw AF  $\parallel$  to BC to meet BF at F. Draw FH perp. to BC. From  $\triangle s$  BHF, CDA,  $BF = CA$ , and  $BH = CD$ . Also  $\angle FAB = \angle ABC$  (I. 20.)  $= \angle FBA$   $\therefore$   $FA = FB$  (I. 6.)  $\therefore BD - CD = BD - BH = HD = AF = BF = AC$ .

21. Let ABCD, PQRS be the quadls., E being the mid. pt. of AB and PQ, F of BC and QR, G of CD and RS, H of DA and SP. Also let P, Q, R fall without ABCD, and S within. Join AP, BQ, CR, SD. Then by I. 3. and 4.  $\triangle AHP = \triangle SHD$ .  $\triangle BEQ = \triangle AEP$ ,  $\triangle BFQ = \triangle CFR$ ,  $\triangle CGR = \triangle SGD$ . But fig. PAEQFCRGDH  $= ABCD + \triangle AHP + \triangle QBE + \triangle QBF + \triangle CRG$  and also  $= PQRS + \triangle EAP + \triangle SHD + \triangle GSD + \triangle FCR$   $\therefore ABCD = PQRS$ .

Another method. Let ABCD be a quadl. having E, F, G, H the mid. pts. of AB, BC, CD, DA.  $\triangle AEH = \frac{1}{4}\triangle ABD$ ,  $\triangle CGF = \frac{1}{4}\triangle CDB$  (Exercises XX. 1, 2.)  $\therefore \triangle AEH + \triangle CGF = \frac{1}{4}ABCD$ . Similarly  $\triangle BFE + \triangle DHG = \frac{1}{4}ABCD$   $\therefore$  the  $\triangle s$  exterior to the parm. EFGH  $= \frac{1}{2}ABCD$   $\therefore ABCD =$  twice parm. EFGH. Similarly any other quadl. PQRS which has E, F, G, H for the mid. pts. of its sides  $=$  twice parm. EFGH  $\therefore ABCD = PQRS$ .

22. Produce QP to S making  $PS = PQ$ . Join PS.  $QR^2 = QP^2 + PR^2 = 4QP^2$  (II. 11.)  $\therefore QR = 2.PQ$ . From  $\triangle s$  RPQ, RPS,  $SR = QR$ ,  $SP = QP$  and  $\angle QRP = \angle SRP$  (I. 4.)  $\therefore QRS$  is an equilateral  $\triangle$  and  $\angle RQP = 60^\circ = 2\angle QRP$ .

23. Take X in DB nearer to B than D; and let FXHB, GXED by the parms. about DB, F lying in AB, G in AD, E in CD, H in CB.  $\frac{1}{2}$  parm. ABCD  $+ \triangle ACX =$  fig. ADCX  $=$  parm. GE  $+ \triangle AGX + \triangle ECX =$  parm. GE  $+ \text{complement AX}$  (1)  $\therefore$  parm. GE  $+ \text{complement AX} - \triangle ACX = \frac{1}{2}$  parm. ABCD. Also  $\frac{1}{2}$  parm. ABCD  $= \triangle ACB =$  parm. FH  $+ \triangle ACX + \triangle AFX + \triangle CHX =$  parm. FH  $+ \triangle ACX + \text{complement AX}$   $\therefore$  from (1) parm. GE  $-$  parm. FH  $= 2\triangle ACX$ .

24. Let ABCD be the quadl.  $\triangle ABC = \frac{1}{2}$  quadl.  $= \triangle DBC$   $\therefore AD$  is  $\parallel$  to BC (II. 7.). Similarly AB is  $\parallel$  to DC  $\therefore ABCD$  is a parm.

25. Thro. E draw FEG  $\parallel$  to AB to meet AD at F and BC at G.  $\triangle DEF = \triangle CEG$  (I. 20. and 16.)  $\therefore$  parm. ABGF  $=$  fig. ABCD. Also  $\triangle AEB = \frac{1}{2}$  parm. ABGF (II. 9.)  $= \frac{1}{2}$  fig. ABCD.

**26.** Let ABCD be the larger sq., AEFG the smaller, BAE being a str. line, and G lying in AD. From AB cut off AH = GD. Join FH and cut along this line. Join CH and cut along this line also. Place  $\triangle HBC$  so that H coincides with F and HB with FG ( $HB = FG$ ) and BC falls along GD,  $\triangle HBC$  occupying the position FGK.  $DK = GK - GD = CB - BH = AB - AH = HB = FE$ .  $\therefore \triangle KDC = \triangle FEH$  in all respects. Also  $FK^2 = HB^2 + BC^2 = FE^2 + EH^2 = FH^2$ .  $\therefore FK = FH$ .  $\angle FHE =$  complement of  $\angle HFE =$  complement of  $\angle CHB$ .  $\therefore FHC$  is a rt.  $\angle$ . In like manner  $\angle FKC$  is a rt.  $\angle$  and  $\angle KCH$ .  $\therefore KFHC$  is a sq. equal to the two given squares.

**27.** Let EFKB HDGF be parms. about BD a diagonal of parm. ABCD, E lying in AB, K in BC, G in CD, H in DA. Join HE, GK.  $\triangle HEG = \frac{1}{2}$  parm. ADGE (II. 9.) =  $\frac{1}{2}$  parm. HDCK (II. 10.) =  $\triangle HKG$  (II. 9.)  $\therefore EK$  is  $\parallel$  to HG (II. 7.).

**28.** Let ABCD be the rect., O the pt. within it. Draw MON perp. to AD and BC, meeting AD at N and BC at M.  $AO^2 + OC^2 = (AN^2 + ON^2) + (OM^2 + CM^2)$  (II. 11.) =  $BM^2 + OM^2 + ON^2 + DN^2$  (II. 2.) =  $OB^2 + OD^2$  (II. 11.).

**29.** Let AB be  $>$  AC. Draw BH and CF perp. to DME. From  $\triangle s$  BMH, CMF,  $BH = CF$  (I. 3. and 16.). Also  $\angle HDB = \angle ADE = \angle AED$  (I. 6.)  $\therefore$  from  $\triangle s$  HDB, EFC,  $BD = CE$  (I. 16.).

**30.** Let ABCD be the sq. and let the given side of the rect. be  $>$  AB. Produce AD to E, so that AE = the given side. Draw EHF  $\parallel$  to AB to meet BC produced in F. Join AF meeting CD at G. Draw HGK  $\parallel$  to EA or FB. Rect. EAKH = DK + complement EG = DK + complement GB (II. 10.) = DB = sq. ABCD. The same construction holds when AE  $<$  AB.

**31.** Let ABCD be the parm. E the mid. pt. of BC, F the mid. pt. of DA. Let ED and BF meet AC at G and H. Draw HK  $\parallel$  to AD or BC to meet DE at K.  $HK = FD$  (II. 2.) = AF.  $\therefore$  from  $\triangle s$  HKG, AFH,  $AH = HG$  (I. 20. and 16.). Also from  $\triangle s$  HGK CGE,  $HG = GC$  (I. 20. and 16.)  $\therefore AC$  is trisected at G and H.

**32.** Take O any pt. within the  $\triangle ABC$ .  $OB + OC < AB + AC$ ,  $OA + OB < CA + CB$ ,  $OC + OA < BC + BA$  (I. 13.)  $\therefore$  adding,  $2(OA + OB + OC) < 2(AB + BC + CA)$ .  $\therefore OA + OB + OC < AB + BC + CA$ . Also  $OB + OC > BC$ ,  $OC + OA > AC$ ,  $OA + OB > AB$  (I. 12.)  $\therefore$  add-

ing  $2(OA + OB + OC) > AB + BC + CA$ , *i.e.*  $OA + OB + OC > \frac{1}{2}(AB + BC + CA)$ .

**33.** Let  $AB, AC$  be the given lines. In  $AB$  take  $DE$  equal to the given length. Draw  $DF$  perp. to  $AB$ , and  $EF$  perp. to  $AC$ , these lines meeting at  $F$ . Draw  $FG \parallel$  to  $AB$  to meet  $AC$  at  $G$ . Draw  $GK$  perp. to  $AB$ , and  $GH$  perp. to  $AC$  to meet  $AB$  at  $H$ . In  $\triangle s$   $GKH, FDE, GK = FD$  (II. 2.).  $\angle GKH = \angle FDE$  and  $\angle GHK = \angle FBD$  (I. 20.)  $\therefore KH = DE \therefore GK, GH$  are the reqd. lines.

**34.** Let  $ABCD$  be the given sq. on  $AB$ , on the same side of it as the pts.  $C, D$ , describe an equilateral  $\triangle ABE$ . Bisect  $\angle EAB$  by  $AF$ , meeting  $BC$  at  $F$ .  $\angle FAB = \frac{1}{2}\angle EAB = 30^\circ$  (I. 22.)  $\therefore \angle AFB = 60^\circ \therefore AFB$  is half the equilateral  $\triangle$  on  $AF \therefore FB = \frac{1}{2}AF \therefore AB^2 = AF^2 - FB^2$  (II. 11.)  $= 3FB^2 \therefore$  the sq. on  $FB$  is the sq. reqd.

**35.** Draw  $DE$  perp. to  $AB$ .  $\angle EBD = \frac{1}{2}$  a rt.  $\angle \therefore \angle BDE = \frac{1}{2}$  a rt.  $\angle$  (I. 22.)  $\therefore ED = EB$  (I. 6.). From  $\triangle s$   $ADE, ADC, DE = DC$  and  $AE = AC$  (I. 26.)  $\therefore AB = AE + EB = AC + ED = AC + CD \therefore CD = AB - AC$ .

**36.** Let  $AB, AC$  be the given str. lines, and  $BD \parallel$  to the third given line. Cut off  $BD$  equal to the given length. Draw  $DE \parallel$  to  $AB$  to meet  $AC$  at  $E$ . Draw  $EF \parallel$  to  $DB$  to meet  $AB$  at  $F$ .  $EF = DB =$  the given length; and it is in the reqd. direction. The problem is impossible (1) when the given direction is  $\parallel$  to  $AB$  or  $AC$ ; (2) when  $DE$  does not meet  $AC$ , *i.e.* when  $AB$  and  $AC$  are  $\parallel$ .

**37.** Let  $AC$  be  $< AB$ , so that  $\angle ACB > \angle ABC$  (I. 10.). Produce  $BA$  to  $D$  and let  $AE$  bisect  $\angle DAC$ . Draw  $AF \parallel$  to  $BC$ , so that  $F$  falls within  $\angle DAC$ .  $\angle DAC = \angle B + \angle C > 2\angle B \therefore \angle DAE > \angle B > \angle DAF$  (I. 20.)  $\therefore AE$  meets  $BC$  produced beyond  $C$ . Similarly if  $AC > AB$ ,  $AE$  will meet  $CB$  produced beyond  $B$ .

**38.** With the fig. of II. 10, let parm.  $FH =$  parm.  $KG$ . Since the complements are equal,  $GK + GH = \frac{1}{2}$  parm.  $ABCD$ , *i.e.* parm.  $HBCK = \frac{1}{2}$  parm.  $ABCD =$  parm.  $AHKD \therefore AH = HB \therefore$  parm.  $HG =$  parm.  $FH =$  parm.  $GK =$  parm.  $FK$ .

**39.** Let  $ABCD$  be the given parm. Produce  $AB$  to  $E$  making  $BE = AB$ . Make  $\angle EAK$  equal to the given  $\angle$  and let  $CD$  meet  $AK$  at  $K$ . Join  $KE$ .  $\triangle AKE = 2\triangle ABK$  (II 6.)  $=$  parm.  $ABCD$  (II. 9.) and has  $\angle KAE$  equal to the given angle.

**40.** Let ABCD be the given rect. AB being  $>$  AD. On AB describe the sq. ABFE, AE passing thro. D, and BF thro. C. Bisect AE at O and with centre O and rad. OA or OE describe a circle to meet CD produced at H. Join AH, HE, HO.  $\angle OAH = \angle OHA$ , and  $\angle OEH = \angle OHE \therefore \angle AHE = \angle HAE + \angle HEA \therefore \angle AHE = \text{a rt. } \angle$  (I. 22.)  $\therefore$  as in (II. 11) the sq. on AH = rect. ABCD.

**41.** If ABCD is the given parm. draw AF, BE perp. to CD meeting it in E and F. Rect. ABEF = parm. ABCD (II. 3.). Then use the preceding exercise.

**42.** Let ABCD be a parm. and O a pt. within it. Let EOF be  $\parallel$  to AD or CB, and GOH  $\parallel$  to CD or AB. Also let parm. DO = parm. BO. Join AO, OC.  $\triangle AGO = \triangle AFO$  and  $\triangle EOC = \triangle OHC$ . Also parm. DO = parm. BO  $\therefore \triangle AGO + \text{parm. DO} + \triangle EOC = \frac{1}{2}$  parm. ABCD =  $\triangle DAC \therefore$  AOC must be a str. line.

**43.** Draw EPF  $\parallel$  to AB and CD meeting AD at E, and BC at F. Draw GPH  $\parallel$  to AD and BC meeting AB at G and CD at H.  $\triangle APB = \triangle AGP + \triangle PGB = \frac{1}{2}$  parm.  $EG + \frac{1}{2}$  parm. GF (II. 9.) =  $\frac{1}{2}$  parm.  $HF + \frac{1}{2}$  parm. GF (II. 10.) =  $\triangle CPF + \triangle FPB$  (II. 9.) =  $\triangle CPB$ .

**44.** Let ABCD be the given parm. It is reqd. to describe a parm. equal to ABCD, having an angle equal to  $\angle DAB$ , and one side equal to the given line P. Let P be  $>$  AD. From AD produced cut off AE = P. Complete parm. BAEF. Join AF cutting CD at G. Draw HGK  $\parallel$  to AE or BF meeting EF at H, and AB at K. Parm. EK = parm. EG + parm. DK = parm. GB + parm. DK (II. 10.) = parm. ABCD  $\therefore$  EAKH is the reqd. parm.

**45.** Let ABCD be the given quadl. Join DB. Draw EAF, GCK  $\parallel$  to DB, and any two parallels thro. D and B to form the parm. EGKF.  $EGKF = EDBF + DGKB = 2\triangle ADB + 2\triangle DCB$  (II. 9.) = 2 fig. ABCD.

**46.** Let ABC be the given  $\triangle$ . Bisect BC at D and join AD. Draw AE  $\parallel$  to BC and BE  $\parallel$  to AD to meet at E. With centre B and rad. equal to  $\frac{1}{2}(AB + AC)$  describe a circle cutting AE at H. Draw DK  $\parallel$  to BH. Parm. HBDK = parm. EBDA (II. 3.) =  $2\triangle ABD$  (II. 9.) =  $\triangle ABC$  (II. 6.). Also its perimeter =  $2HB + 2BD = AB + AC + BC \therefore$  HBDK is the reqd. parm.

**47.** With the fig. of II. 10. take P the mid. pt. of AC and drawn MPN  $\parallel$  to AB and CD, meeting AD at M and BC at N.

Also draw  $RPS \parallel$  to  $AD$  and  $BC$ , meeting  $AB$  at  $R$ , and  $CD$  at  $S$ . Let  $HEK$  meet  $MP$  at  $L$ ,  $E$  lying between  $A$  and  $P$ ,  $L$  lying between  $P$  and  $M$ . But  $PM = PN \therefore PN > LM \therefore$  parm.  $PG >$  parm.  $EM$ . In parm.  $ARPM$ , complement  $ER =$  complement  $EM \therefore$  adding  $RG$  to each, parm.  $HG =$  parm.  $RG +$  parm.  $EM <$  parm.  $RG +$  parm.  $PG <$  parm.  $RBNP$ . Thus we see that the complement  $HG$  is greatest when  $E$  lies at  $P$  the mid. pt. of  $AC$ .

**48.** Join  $GD$ ,  $BF$ .  $\triangle GDB = \frac{1}{2}$  rect.  $ABDE$  (II. 9.)  $= \frac{1}{2}$  rect.  $ACFG = \triangle GBF$  (II. 9.)  $\therefore DF$  is  $\parallel$  to  $GB$  (II. 7 ).

**49.** Let  $ABCD$  be the quadl. Join  $BD$ . Of the  $\triangle$ s  $ABD$ ,  $ACD$  let  $ABD$  be the smaller. Draw  $AE \parallel$  to  $BD$  to meet  $CB$  produced in  $E$ . Join  $DE$ . Bisect  $CE$  at  $F$  and join  $DF$ .  $\triangle EBD = \triangle ABD$  (II. 5.)  $\therefore$  adding  $BCD$  to each,  $\triangle EDC =$  quadl.  $ABCD \therefore \triangle DFC = \frac{1}{2} \triangle EDC$  (II. 6.)  $= \frac{1}{2}$  quadl.  $ABCD \therefore DF$  bisects the quadl.

**50.** Produce  $GF$  to meet  $CB$  at  $H$ . Join  $AH$ , meeting  $EF$  at  $K$ . Draw  $LKM \parallel$  to  $GF$  or  $AEB$  cutting  $AG$  at  $L$ , and  $CH$  at  $M$ . Parm.  $KB =$  parm.  $GK$  (II. 10 )  $\therefore$  adding  $LE$  to each, parm.  $LB =$  parm.  $GE \therefore$  rect.  $LDCM =$  rect.  $ABCD +$  rect.  $AEFG$ .

**51.**  $\triangle FHG + \triangle FKG = \frac{1}{2}$  parm.  $AG + \frac{1}{2}$  parm.  $CF$  (II. 9.)  $= \frac{1}{2}$  parm.  $ABCD = \triangle ABC$  (II. 2.)

**52.** Let  $ABC$  be the given  $\triangle$ . Bisect  $BC$  at  $D$ , and draw  $DE$ ,  $CF$  perp. to  $BDC$  to meet  $EFH$ ,  $\parallel$  to  $BC$ , at  $E$  and  $F$ . Along  $DC$  make  $DG$  equal to the given str. line, and draw  $GH$  perp. to  $DG$ . Join  $DH$ , cutting  $CF$  at  $K$ . Draw  $NKM \parallel$  to  $EF$  or  $BC$ , meeting  $DE$  at  $N$  and  $GH$  at  $M$ . Rect.  $NDGM =$  rect.  $NC +$  rect.  $CM =$  rect.  $NC +$  rect.  $EK$  (II. 10.)  $=$  rect.  $EDCF = 2\triangle ADC$  (II. 9.)  $= \triangle ABC$ .

**53.** Let  $ABCD$  be the given parm. Draw  $EF \parallel$  to  $AB$  and at the given perpendicular distance from  $AB$ . Let  $EF$  cut  $AD$  at  $E$ . Join  $BE$  and produce it to meet  $CD$  produced at  $G$ . Draw  $GKH \parallel$  to  $AD$  or  $BC$  to meet  $FE$  and  $BA$  produced at  $H$  and  $K$ . Parm.  $KBFH =$  parm.  $AF +$  parm.  $KE =$  parm.  $AF +$  parm.  $EC$  (II. 10.)  $=$  parm.  $ABCD$ . Also  $KBEH$  is equiangular to  $ABCD$ , and is therefore the parm. reqd.

**54.** Let  $ABCD$  be the given quadl. Join  $AC$ , and draw  $DE \parallel$  to  $AC$  to meet  $BA$  produced at  $E$ . Join  $EC$ . Draw  $EF \parallel$  to  $BC$ , and  $CF \parallel$  to  $EB$  to meet at  $F$ . Bisect  $BE$  at  $G$  and draw  $GH$

|| to BC to meet CF at H. Parm. GBCH =  $\frac{1}{2}$  parm. EBCF (II. 4.) =  $\triangle EBC$  (II. 2.) =  $\triangle ABC + \triangle EAC = \triangle ABC + \triangle ACD$  (II. 5.) = quadl. ABCD.

**55.** Draw AHK perp. to GB, meeting GB at H and EC at K. In  $\triangle s$  GAB, CAE, AG = AC, AB = AE,  $\angle$  at A is common  $\therefore \angle ACE = \angle AGB$  (I. 4.) = complement of  $\angle GAH$  (I. 22.) =  $\angle KAC$   $\therefore CK = AK$  (I. 6.). Also  $\angle KEA =$  complement of  $\angle KCA$  (I. 22.) = complement of  $\angle KAC = \angle KAE$   $\therefore KE = KA$  (I. 6.)  $\therefore KE = KC$ .

**56.** Let ABC be the  $\triangle$  such that AB = 2. AC. Bisect AB at D and draw DE perp. to AB to meet BC at E. Join AE. From  $\triangle s$  ADE, BDE,  $\angle EAD = \angle EBD$  (I. 4)  $\therefore \angle AEC = 2\angle ABE$  (I. 22.). Also in the rt.  $\angle d.$   $\triangle AED$ , AE > AD  $\therefore AE > AC$   $\therefore \angle ACE > \angle AEC$ , i.e.  $\angle ACB > 2\angle ABC$ .

**57.** Let ABC be an isos.  $\triangle$  having AB = AC, and  $\angle BAC = 30^\circ$ . On the same side of AC as the pt. B describe an equilateral  $\triangle AEC$ . Let EC meet AB in F  $\angle EAF = 30^\circ$ . From  $\triangle s$  AFE, AFC, AF is perp. to EC and bisects it (I. 4.). In the rt.  $\angle d.$   $\triangle BFC$ , BC > CF, i.e. BC >  $\frac{1}{2}CE$ , i.e. BC >  $\frac{1}{2}AC$ .

**58.** Let PB produced meet QR at S, and QB produced meet CP at T.  $\angle QBR =$  a rt.  $\angle \therefore \angle TBR =$  a rt.  $\angle \therefore \angle TBC = \frac{1}{2}$  a rt.  $\angle = \angle TCB \therefore BT$  is perp. to CT (I. 22.)  $\therefore$  from  $\triangle s$  BTC, BRC, BT = BR. Also AQTP is a rectangle  $\therefore PT = AQ = QB \therefore$  in  $\triangle s$  PTB, QBR, BT = BR, PT = QB and  $\angle PTB =$  a rt.  $\angle = \angle QBR \therefore PB = QR$  and  $\angle BPT = \angle BQR$  (I. 4.)  $\therefore$  in  $\triangle s$  PTB, QSB,  $\angle BPT = \angle BQS$ ,  $\angle PBT = \angle QBS$  (I. 3)  $\therefore \angle BTP = \angle BSQ \therefore \angle BSQ =$  a rt.  $\angle$ , i.e. PB is at rt.  $\angle s$  to QR.

**59.** Let AC meet HK at E.  $\angle AHB = \angle B$  (I. 5.) =  $\angle D = \angle AKD$  (I. 5.). Also AB = AD  $\therefore$  from  $\triangle s$  ABH, ADK, BH = DK (I. 16.)  $\therefore CH = CK \therefore \angle KHC = 90^\circ - \frac{1}{2}\angle C \therefore \angle B + 60^\circ + 90^\circ - \frac{1}{2}\angle C = \angle AHB + \angle AHK + \angle KHC = 180^\circ \therefore \angle B - \frac{1}{2}\angle C = 30^\circ$ . But  $\angle B + \angle C = 180^\circ \therefore \frac{3}{2}\angle C = 150^\circ \therefore \angle C = 100^\circ = \frac{10}{9}$  of a rt.  $\angle$ .

## EXERCISES XXVII.

**1.** Let AB, CD be equal chords, E the centre.  $\angle AEB = \angle CED$  by I. 7.

**2.** If AB > CD,  $\angle AEB > \angle CED$  by I. 15,

3. If ABC are pts. of intersection of str. line and circle, D the centre,  $\angle DAB = \angle DCB$  (I. 5.)  $= \angle DBC$  (I. 5.), which is impossible by I. 8.

4. Proved in III. 4.

5. The same as 4, since the diagonals of a parm. bisect each other.

6. The diagonals AC, BD may be proved to pass through centre E.  $\angle EAB = \angle EBA$  (I. 5.),  $\angle ECB = \angle EBC$  (I. 15.)  $\therefore$  in  $\triangle ABC$  one angle = the sum of the other two  $\therefore \angle ABC$  is a rt.  $\angle$ .

7. Since the chords are  $\parallel$ , a perp. to one is perp. to all  $\therefore$  it bisects all, and is the locus of their mid. points.

8. Let AB, AC be the chds., DA the radius; DE, DF perps. to the chds. Then  $AE = AF$  (I. 17.)  $\therefore AB = AC$ . The converse is proved by I. 7.

9. Since the join of the mid. points is perp. to the first chord, it passes through the centre; and since it joins the centre to the mid. point of the 2nd chord, it is perp. to that (III. 3.).

10. Let ABCD be the line, EF the perp. from the common centre.  $AF = FD$ , and  $BF = FC$  (III. 3.)  $\therefore AB = CD$ .

11. Half chord = 12  $\therefore$  distance  $= \sqrt{15^2 - 12^2} = 9$  (II. 11.).

12. Let the perps. drawn from the centre E meet the chds. AB, CD in F, G. Draw EH  $\parallel$  to the chds.  $\angle HEF = \text{alt. } \angle EFB = \text{a rt. } \angle$ .  $\angle HEG = \text{alt. } \angle EGD = \text{a rt. } \angle \therefore FEG$  is a str. line and is the join of the mid. pts.

13. Let CAB be the line, DF, EG perps. from centres.  $DE = FG$  (II. 2.)  $= \frac{1}{2} CB$  (III. 3.).

14. The line of centres is perp. to AB (III. 2.)  $\therefore$  it is perp. to CF (say at H)  $\therefore CH = HF$  and  $DH = HE$  (III. 3.)  $\therefore CD = EF$ .

15. Let AEB, CED be the chds., F the centre. Let EF meet the circle in K, L. Let FG, FH be perp. to AB, CD. In  $\triangle s$  FAG, FCH,  $FG = FH$  (I. 17.)  $\therefore$  in  $\triangle s$  FEG, FEH,  $\angle FEG = \angle FEH$  (I. 7.)  $\therefore KL$  is one of the bisectors. But the bisectors of supplementary  $\angle s$  are at rt.  $\angle s \therefore$  the other bisector is perp. to KL, and is therefore bisected by it (III. 3.).

16. Let AB, AD be the equal lines, FC, FE perps. from centre. In  $\triangle$ s ABF, ADF,  $\angle BAF = \angle DAF$  (I. 7.)  $\therefore FC = FE$  (I. 16.).

17. In  $\triangle$ s ABD, ACE,  $\angle ABD = \angle ACE$  (I. 5.),  $\angle ADB = \angle AEC$  (I. 5.) and  $AB = AC \therefore BD = EC$  (I. 16.).

18. Let AB be common chd., CED line of centres.  $CE = \frac{1}{2}CD = \frac{1}{2}$  radius (diagonals of rhombus bisect each other at rt.  $\angle$ s)  $\therefore AE^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2 \therefore AB^2 = 3r^2$ .

19. Let ABC, DEF be the parallels. GH, KL perps. to them through the centres.  $GK = \frac{1}{2}AC$ , and  $HL = \frac{1}{2}DF$  (III. 3.). But  $GK = HL$  (II. 2.)  $\therefore AC = DF$ .

### EXERCISES XXVIII.

1. Let D, E be the centres, then AED is a st. line.  $\angle ECA = \angle EAC$  (I. 5.)  $= \angle DBA$  (I. 5.)  $\therefore DB$  is  $\parallel$  to  $EC$  (I. 19.).

2. The join of centres DE passes through A.  $\angle ECA = \angle EAC$  (I. 5.)  $= \angle DAB$  (I. 3.)  $= \angle DBA$  (I. 5.)  $\therefore DB$  is  $\parallel$  to  $EC$  (I. 18.).

3. BC passes through the pt. of contact F. AB, AC produced go through the pts. of contact D, E. Perimeter of  $ABC = AB + BF + AC + CF = AB + BD + AC + CE = 2AD = a$  constant.

4. Let OP, OQ be tangents, C the centre. From  $\triangle$ s OPC, OQC by I. 17.,  $OP = OQ$ .

5. Let C be centre. Produce CP to S so that  $PS = CP$ . The identically equal  $\triangle$ s CPO, SPO make up an equilateral  $\triangle$ .  $\angle POC = \frac{1}{2}\angle COS = 30^\circ$ . Similarly  $\angle COQ = 30^\circ \therefore \angle POQ = 60^\circ$ . But the  $\angle$ s OPQ, OQP are equal  $\therefore$  each of them is  $60^\circ \therefore \triangle POQ$  is equilateral.

6. Let ADB touching at D meet the two radii in A, B. Let AE, BF be tangents, C the centre. By using I. 17. for the  $\triangle$ s FCB, DCB, we show that  $\angle FCD = 2\angle BCD$ . Similarly  $\angle ECD = 2\angle ACD \therefore \angle FCD + \angle DCE = 2\angle ACB = 2$  rt. angles  $\therefore FC = CE$  are in a st. line. But the  $\angle$ s E, F are rt.  $\angle$ s.  $\therefore AE$  is  $\parallel$  to  $BF$ .

7. Take centre C.  $\angle CQA = \angle CAQ$  (I. 5.)  $= \angle QAP$  (hyp.)  $\therefore$  by I. 18. AP is  $\parallel$  to CQ, and consequently perp. to tangent at Q.

8. (1) Let A be the given point in which all touch the given st. line AB. Then DAC perp. to AB contains all the centres (III. 5., Cor. 2).

(2) If E be the given point in which a given circle, whose centre is F, is touched by a number of circles, the centres must all lie in FE (produced if necessary) III. 6.

### EXERCISES XXIX.

1. Let AB, CD be the tangents, BE, ED radii to the pts. of contact. Let EF be parallel to AB and CD.  $\angle FEB = \text{alt.}$   $\angle EBA = \text{a rt. } \angle$  (I. 20.). Similarly  $\angle FED$  is a rt.  $\angle \therefore$  BED is a straight line, i.e. the pts. of contact are the ends of a diameter.

2. The distance of each chord from the common centre is the radius of the inner circle  $\therefore$  the chords are equidistant from centre, and consequently equal (III. 10.).

3. Let BAC, DAE be the chords, F the centre, FG, FH perps. to BC, DE. From  $\triangle$ s FAG, FAH,  $FG = FH$  (I. 16.)  $\therefore$  chord BC = chord DE (III. 10.).

4. Let PT be such a tangent, P the pt. of contact, C the centre.  $CT^2 = CP^2 + PT^2 = \text{a constant} + \text{a constant} \therefore$  CT is of constant length, and the locus of T is a circle with centre C.

5. Let TP, TQ be tangents; OP, OQ radii.  $\angle OPQ = \angle OQP$  (I. 5.)  $\therefore$  their complements are equal, i.e.  $\angle TPQ = \angle TQP$ .

6. In  $\triangle$ s TOP, TOQ,  $\angle TOP = \angle TOQ$  (I. 17.).

7. Let AP, BQ be parallel tangents, touching at A, B; PQ a third line touching at R. Draw radii CA, CB, CR. CA, CB are perp. to parallel str. lines  $\therefore$  they are in a str. line.  $\angle PCR = \frac{1}{2}\angle ACR$  (I. 17.). Similarly  $\angle QCR = \frac{1}{2}\angle BCR \therefore \angle PCQ = \frac{1}{2}(\angle ACR + \angle BCR) = \text{a rt. angle}$ .

8. Let AB, BC, CD, DA touch at E, F, G, H. As in xxviii. 4. tangents are equal  $\therefore AE + EB + DG + GC = AH + BF + DH + CF$ , i.e.  $AB + DC = AD + BC$ .

9. With the same figure as in 8, O being the centre,  $\angle AOE = \angle AOH$  (I. 7.). Similarly with the other angles  $\therefore \angle AOE + \angle EOB + \angle COG + \angle GOD = \angle AOH + \angle BOF + \angle FOC + \angle DOH$ , i.e.  $\angle AOB + \angle COD = \angle AOD + \angle BOC$ ,

10. Let A be the centre of the outer, B of the inner circle. Let DBAC be the diameter of the inner circle which passes through A. Of all lines drawn from A to the inner circle AD is greatest, AC is least (III. 7. or 8.)  $\therefore$  the tangent drawn at D is the least chord and that at C is the greatest chord of the outer circle (III. 10.).

11. Let A be the given point, B the centre. Draw chords CAD, EAF each making an  $\angle 45^\circ$  with AB. Draw BG, BH perp. to the chords. These perps. are equal (I. 16.)  $\therefore$  CD = EF (III. 10.). Also  $\angle DAF$  is a rt. angle.

12. Let A, B be centres; D, E opposite ends of  $\parallel$  diameters DAF, EBG; C the pt. of contact. Join DC, EC. By I. 5. and I. 22.  $\angle ACD = \frac{1}{2} \angle FAC$  and  $\angle ECB = \frac{1}{2} \angle GBC$ . But  $\angle FAC = \text{alt. } \angle CBG$  (I. 20.)  $\therefore \angle ACD = \angle ECB = \text{supplement of } \angle ACE$  since ACB is a str. line  $\therefore$  DC and CE are in a str. line.

13. Let CT the common tangent meet AB in T. The tangents are equal  $\therefore \angle TCA = \angle TAC$  and  $\angle TCB = \angle TBC$  (I. 5.). In  $\triangle ACB$  one angle = the sum of the other two  $\therefore$  ACB is a rt. angle.

### EXERCISES XXX.

1.  $\angle CAE = \angle BDE$  (III. 12.).  $\angle ACE = \angle DBE$  (III. 12.).  $\angle AEC = \angle BED$  (I. 3.).

2. Let ABC be the triangle; P, Q, R pts. on the arcs BC, CA, AB respectively. The  $\angle$ s P, Q, R are the supplements of  $\angle$ s CAB, ABC, BCA (III. 13.)  $\therefore$  the sum of  $\angle$ s P, Q, R, CAB, ABC, BCA = 6 rt. angles  $\therefore$  sum of  $\angle$ s P, Q, R = 4 rt. angles.

3. Let the pentagon be ABCDE, P a point in arc AB. The ext.  $\angle$  of a regular pentagon =  $\frac{4}{5}$  rt.  $\angle$   $\therefore \angle EBA = \frac{2}{5}$  rt.  $\angle$  (I. 5. and I. 22.).  $\angle APE = \angle ABE$  (III. 12.) =  $\frac{2}{5}$  rt.  $\angle$ .  $\angle EPD = \angle ECD$  (III. 12.) =  $\frac{2}{5}$  rt.  $\angle$ .  $\angle APB = \text{supplement of } \angle AEB = \frac{8}{5}$  rt.  $\angle$  = the sum of the  $\angle$ s subtended by AE, ED, DC, CB.

4. In  $\triangle ABC$  let D, E, F be the feet of perps., M, Q, R the mid. pts. of sides. In the rt. angled  $\triangle BEC$  the mid. point of hypotenuse is equidistant from the vertices  $\therefore \angle MEB = \angle MBE$  (I. 5.). Similarly  $\angle REB = \angle RBE$   $\therefore \angle REM = \angle RBM = \angle RQM$  (II. 2.)  $\therefore$  M, Q, R, E are concyclic (III. 13.). Similarly D, F lie on the circle MQR.

5. With the same figure as in the previous question, let N be the mid. point of OP, and NS parallel to OM, and therefore perp. to BC.  $\triangle SMN = \triangle SON$  (II. 5.)  $= \triangle SPN$  (II. 6.)  $= \triangle SDN$  (II. 5.)  $\therefore MS = SD$  (from the area of a  $\triangle$ ). Thus N lies on the perpendicular bisector of the chord MD. Similarly N lies on the perpendicular bisector of the chord QE  $\therefore$  N is the centre of the circle.

6. Let AB be the given base, C one position of the vertex, P another position. Let the circle through A, B, C cut AP in Q. Join BQ.  $\angle AQB = \angle ACB$  (III. 13.)  $= \angle APB$  (Hyp.). But this is impossible by I. 8.; P must lie on the arc ACB, i.e. the locus is the arc of segment which is on the given base, and which contains the given angle.

7. Let ABCD be a cyclic quadl., AB produced to E.  $\angle CBE =$  supplement of  $\angle CBA$  (I. 1.)  $= \angle ADC$  (III. 13.).

8.  $\angle EBC = 180^\circ - \angle ABC = \angle ADC$ .  $\angle E$  is common  $\therefore$  the  $\triangle$ s are equiangular to each other.

9. The angles at the point are together  $360^\circ \therefore$  each is  $120^\circ \therefore$  in order that there may be such a point each angle of the  $\triangle$  must be less than  $120^\circ$  (I. 13.).

10.  $\angle BAC + \angle BCA = 90^\circ = \angle DAC + \angle DCA \therefore \angle BAC - \angle DAC = \angle DCA - \angle BCA$ .

(2) Let BD, AC meet at E.  $\angle BCA + \angle DAC = \angle BCA + \angle DBC$  (III. 12.)  $= \angle BEA$  (I. 22.)  $= 60^\circ$ . But  $\angle BCA + \angle DCA = 90^\circ \therefore$  by subtraction  $\angle DCA - \angle DAC = 30^\circ$ .

11. Let A, B, C, D be the centres, E, F, G, H the pts. of contact, E lying on AB and so on.  $\angle AEH = \angle AHE$  (I. 5.)  $= \alpha$  say.  $\angle BEF = \angle BFE = \beta$ ,  $\angle CFG = \angle CGF = \gamma$ ,  $\angle DGH = \angle DHG = \delta$ . In  $\triangle AEH$   $2\alpha = 180 - A$ , and similarly for  $\beta$ , etc. But  $A + B + C + D = 360^\circ$  (I. 22., Cor.)  $\therefore \alpha + \beta + \gamma + \delta = 180^\circ$ . Now  $\alpha + \beta + HEF = 180^\circ$ , and  $\gamma + \delta + HGF = 180^\circ \therefore$  by addition  $\angle HEF + \angle HGF = 180^\circ$ , i.e. the figure EFGH is cyclic.

12. Let the quadl. be ABCD, the intersection of diagonals E. The  $\angle$ s subtended at centre by BC, AD are  $2\angle BDC$ ,  $2\angle DCA$  (III. 11.) The sum of these  $= 2\angle DEA$  (I. 22.)  $= 180^\circ$ .

13.  $\angle EFG = \frac{1}{2}B + \frac{1}{2}C$  (I. 22.),  $\angle EHG = \frac{1}{2}A + \frac{1}{2}D$  (I. 22.)  $\therefore \angle EFG + \angle EHG = \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = 180^\circ$  (I. 22., Cor.).

14. Suppose AB, DC meet in P; AD, BC in Q; and the circles meet in R.  $\angle PRC = 180^\circ - \angle PBC$  (III. 13.)  $= \angle CBA$ . Similarly  $\angle QRC = \angle CDA$ .  $\therefore \angle PRC + \angle QRC = \angle CBA + \angle CDA = 180^\circ$  (III. 13.)  $\therefore$  P, R, Q are in a str. line.

15. Complete the circle of which APB is a segment. Produce QB to meet the circle in R.  $\angle Q = \angle BPQ$  (I. 5.) = supplement of  $\angle APB = \angle ARB$  (III. 13.)  $\therefore$  if the figure were folded along AB, the arc ARB would pass through the point Q  $\therefore$  the locus of Q is an arc of an equal circle.

16.  $\angle ADB = 90^\circ = \angle AEB$   $\therefore$  the circle through B, D, A passes through E.  $\angle DEC =$  supplement of  $\angle DEA = \angle ABD$  (III. 13.).

17. As in Question 16, a circle will go round ABDE  $\therefore \angle ADE = \angle ABE$  (III. 12.).

18.  $\angle C + \angle ABE = 180^\circ$  (III. 13.),  $\angle D + \angle ABF = 180^\circ$  (III. 13.)  $\therefore \angle C + \angle D + 180^\circ = 360^\circ$  (I. 1.)  $\therefore \angle C + \angle D = 180^\circ$   $\therefore$  CE is  $\parallel$  to DF (I. 19.).

19. The  $\triangle$ s ADE, BDE are equilateral  $\therefore \angle ADB = 60^\circ + 60^\circ = 120^\circ$ ;  $\angle BCA = \frac{1}{2} \angle ADB$  (III. 11.)  $= 60^\circ$ . Similarly  $\angle BPA = 60^\circ$   $\therefore \triangle BCP$  is equilateral.

20.  $\angle$ s in the same segt. of a circle are equal  $\therefore$  since the circles are equal, and  $AC = AD$ ,  $\angle$  in segt. CBA of circle CBA  $= \angle$  in segt. DBA of circle DBA  $\therefore \angle ABC = \angle ABD$   $\therefore$  BDC is a str. line.

21. Let C be the centre,  $2\angle C = \angle O =$  a constant, and each of the sides CP, CQ = the constant radius  $\therefore$  the base PQ is of constant length.

22. Let ABCD be the quadl., CD produced to E, F the intersection of the bisectors.  $\angle EDF = \frac{1}{2} \angle EDA = \frac{1}{2}$  int. oppte.  $\angle ABC = \angle CBF$   $\therefore \angle CBF + \angle CDF = \angle EDF + \angle CDF = 180^\circ$   $\therefore$  F lies on the circle BCD.

23. Take any point F on the circumference not in the arc BDC.  $\angle EDB = 180^\circ - \angle BDC = \angle BFC = \frac{1}{2} \angle BAC$  (III. 11.).

24. Let CB meet DE in H. The  $\angle$ s at G, E being rt.  $\angle$ s, the quadl. GFED is cyclic. Similarly for CFBD  $\therefore \angle DEG = \angle DFG$  (III. 12.)  $= 90^\circ - \angle GDF = \angle FDB = \angle FCB$  (III. 12.)  $= \angle EHB$  (I. 20.), (since FC is  $\parallel$  to ED)  $\therefore$  EG is  $\parallel$  to BC.

25.  $\angle DRC = 180^\circ - \angle P - \angle Q$  (I. 22.)  $= 180^\circ - \angle CAP - \angle DAQ$  (I. 5.)  $= \angle CAD = \angle CBD$  (I. 7.)  $\therefore$  D, B, R, C are concyclic.

**26.** Let ABCD be a cyclic quadr. Let a str. line meet AB, DC, AD, BC in F, G, H, K respectively. By hypothesis  $\angle DHK = \angle CKH$ . But  $\angle DHK = \angle HGD + \angle GDH$  (I. 22.)  $= \angle HGD + \angle B$ . Also  $\angle CKH = \angle BFG + \angle B$  (I. 22.)  $\therefore \angle BFG = \angle HGD$ .

**27.** In the quadr. ABDC,  $\angle A$  is constant (AB, AC being given tangents). Also  $\angle BDC$  is constant (III. 12.)  $\therefore$  the sum of the other two  $\angle$ s is constant (I. 22. Cor.).

### EXERCISES XXXI.

**1.** Let AB, DC be parallel. Then  $\angle BAC = \text{alt. } \angle ACD$  (I. 20.)  $\therefore$  arc BC = arc AD (III. 14.)  $\therefore$  chord BC = chord AD (III. 15.). Also the whole arc ADC = whole arc DCB  $\therefore$  chord AC = chord DB (III. 15.).

**2.** Let tangent at C be  $\parallel$  to AB. Let CDE be the radius cutting AB at D. CE is perp. to the tangent and therefore to AB  $\therefore$  D is the mid. point of AB. From I. 4. AC = CB  $\therefore$  arc AC = arc CB (III. 16.).

**3.** Let AEB, CED be any chords containing a constant  $\angle$ . The sum of the angles at B and C  $= \angle E =$  a constant  $\therefore$  the sum of the arcs AC, BD is constant.

**4.** With the same figure the  $\angle E = \angle B + \angle C$  at circumference = an  $\angle$  at the circumference standing on the sum of arcs AC, DB = an  $\angle$  at the centre on half the sum of the arcs.

**5.** Let AB, CD intersect at an external pt. E. Join BC.  $\angle E = \angle ABC - \angle BCD$  (I. 22.) = an angle at circumference standing on an arc AC - BD = an angle at centre standing on half that arc.

**6.**  $\angle ECD = \angle EAB$  (III. 15.) = alt.  $\angle EFD$  (I. 20.)  $\therefore$  CE = EF (I. 6.).

**7.** Arc BD = arc DC, since the  $\angle$ s at A are equal;  $\therefore$  chord BD = chord DC (III. 15.) = DE (*Hyp.*)  $\therefore \angle DBE = \angle DEB$  (I. 5.)  $= \frac{1}{2} \angle A + \angle ABE$  (I. 22.)  $= \angle DAC + \angle ABE = \angle DBC + \angle ABE$  (III. 12.)  $\therefore \angle CBE = \angle ABE$ .

**8.**  $\angle C$  is constant (III. 12.), since A and B are fixed points. Similarly  $\angle D$  is constant. The 3rd angle of the  $\triangle CBD$  must be constant.

**9.** The  $\angle$ s APQ, AQP are constant (III. 12.)  $\therefore$  by I. 22.  $\angle RAQ$  is constant  $\therefore$  the arc RQ, and consequently the chord RQ, is constant (III. 14., 15.).

10. In  $\triangle CED$ ,  $\angle CAB = \angle E = \angle EAC = \angle CAB$  (Hyp.);  $\angle CDE = 180^\circ - \angle CDA = \angle ABC$ . Also  $CE = CA \therefore DE = AB$  (I. 16.).

11.  $AB = CD \therefore$  arc  $AB =$  arc  $CD$  (III. 16.)  $\therefore \angle ACB = \angle DAC$  (III. 15.)  $\therefore AD$  is  $\parallel$  to  $BC$  (I. 18.). Also (by addition of arc  $AD$ ) the arc  $DAB =$  arc  $CDA \therefore$  chord  $DB =$  chord  $CA$  (III. 15.).

### EXERCISES XXXII.

NOTE.—It follows from III. 17. that a chord which subtends a right angle at the circumference is a diameter.

1. On the hypotenuse  $AB$  of  $\triangle ABC$  let a circle be described. It must pass through  $C$ . For if it cut  $AC$  at  $D$ , the  $\angle ADB$  would be a rt. angle (III. 17.) and so equal to  $\angle ACB$ , which is impossible (I. 8.).

2. As in Question 1,  $C$  must lie on the circle whose diameter is  $AB$ .

3. Join  $CB$ . Arc  $AC +$  arc  $BD =$  arc subtended by the sum of the  $\angle$ s  $C$  and  $B$  at the circumference  $=$  arc subtended by a rt.  $\angle$  (I. 22.)  $=$  a semicircle. It might also be proved by drawing  $AF \parallel$  to  $CD$ , and proving arc  $FD =$  arc  $AC$  (I. 20. and III. 14.).

4.  $OQP$  is a rt.  $\angle$  (III. 17.)  $\therefore PQ$  is a tangent (III. 5.).

5.  $CBA, ABD$  are rt.  $\angle$ s (III. 17.)  $\therefore C, B, D$  are in a str. line (I. 2.).

6. Draw  $CF$  perp. to tangent at  $A$ , and  $CG \parallel$  to  $FA$ . Join  $AD$ .  $\triangle$ s  $BGC, BDA$  are identically equal (I. 16.)  $\therefore BG = BD$ .  $CD = CB - BD = AB - BG = AG = CF$  (II. 2.).

7. Since the arcs are equal, the chords are equal (III. 15.)  $\therefore$  they are equidistant from the centre (III. 10.)  $\therefore$  they touch a concentric circle.

8. Let  $BC$  be the diameter,  $A$  the point of contact.  $\angle PAQ = \angle BAC = 90^\circ$  (III. 17.)  $\therefore PQ$  is a diameter.

9. Chord  $AB =$  chord  $AE$  (radii)  $\therefore$  arc  $AB =$  arc  $AE$  (III. 16.) arc  $AD =$  arcs  $AE$  and  $ED =$  arc  $EC \therefore$  chord  $AD =$  chord  $EC$  (III. 15.). Also since arc  $AE =$  arc  $DC$ ,  $\angle EDA =$  alt.  $\angle DAC$  (III. 15.)  $\therefore ED$  is  $\parallel$  to  $AC$ .

10. The mid. pt. of the hypotenuse of a right-angled  $\triangle$  is equidistant from the vertices  $\therefore QD = QC \therefore \angle QDC = \angle C$  (I. 5.)

$= \angle QRP$  (II. 2.)  $\therefore \angle QRP + \angle QDP = \angle QDC + \angle QDP = 2$  rt.  $\angle$ s  
 $\therefore$  QRPD is cyclic.

**11.** In  $\triangle AFB$ , G is the intersection of the perps. AE, BD  
 $\therefore$  G is the orthocentre  $\therefore$  FG is perp. to AB.

Or, let FG meet AB at H. In the cyclic quadr. DGFE,  $\angle DEG = \angle DFG$  (III. 12.). And in the cyclic quadr. ADEB, ext.  $\angle DEF =$  int. oppte.  $\angle DAB$ . But  $\angle BHF = \angle DFG + \angle DAH$  (I. 22.)  $\therefore$   
 $\angle BHF = \angle DEG + \angle DEF = a$  rt.  $\angle$ .

**12.** Let C be the centre of the circle Q, CD a diameter of the circle P. Let A, B be the pts. of intersection of the circle.  $\angle$ s CAD, CBD are right  $\angle$ s (III. 17.)  $\therefore$  AD, BD are the tangents at A, B (III. 5.) i.e. the tangents at A, B meet at D.

**13.** Let AC be the diameter of smaller, AB of larger circle : AQP the chord. The  $\angle$  at Q is a rt.  $\angle$  (III. 17.)  $\therefore$  Q is the mid. pt. of AP (III. 3.).

**14.** Let DOC be perp. to the fixed diameter AB.  $\angle OPA = \angle OAP = 90^\circ - \angle APN = \angle BPN \therefore$  the bisector of  $\angle OPN$  is the bisector of  $\angle APB \therefore$  it passes through C the mid. pt. of the arc AB (III. 14.). If P were on the other side of AB, the bisector would pass through D.

### **EXERCISES XXXIII.**

**1.** Let ACB be the arc, C the mid. point, AE the tangent, CE perp. to AE.  $CA = CB$  (III. 15.)  $\therefore \angle CAB = \angle CBA = \angle CAE$  (III. 18.)  $\therefore CD = CE$  (I. 16.).

**2.** Let ABCD be the circle, A the point, BC the chord, AD the diameter. Join BD.  $\angle ABD = 90^\circ$  (III. 17.)  $\therefore$  complement of  $\angle ABC = \angle CBD = \angle CAD$  (III. 12.) Also  $\angle ACB = \angle ADB$  (III. 12.) = complement of  $\angle BAD$ .

**3.**  $\angle ATP + 2\angle BPT = \angle ATP + \angle BPT + \angle PAB$  (III. 18.)  $= \angle ABP + \angle PAB$  (I. 22.)  $= 90^\circ$  (III. 17.).

**4.**  $\angle BAD = \angle C$  (III. 18.)  $= 60^\circ$ . Similarly  $\angle ABD = 60^\circ \therefore$   $\triangle ABD$  is equilateral.

**5.**  $\angle APC = \angle CAP$  (I. 5.)  $= \frac{1}{2} \angle PCD$  (I. 22.)  $= 30^\circ$ . Similarly  $\angle B = 30^\circ \therefore \angle APC = \angle B \therefore$  AP touches the circle BCP (III. 18.).

**6.** Draw AD perp. to BC. The semicircle on AB passes thro. D (III. 17.). Similarly the semicircle on AC passes thro. D.

7. Draw a str. line FAG touching the circle ABC at A.  $\angle DAG = \angle FAB$  (I. 3.)  $= \angle ACB$  (III. 18.)  $= \angle AED$  (I. 20.)  $\therefore$  circle EAD touches FG at A  $\therefore$  the circles touch each other at A.

8. Let the circles ABC, AED touch at A, FAG being the common tangent. Let BAD, CAE be the lines.  $\angle CBA = \angle CAG$  (III. 18.)  $= \angle FAE$  (I. 3.)  $= \angle ADE$  (III. 18.)  $\therefore$  BC is  $\parallel$  to ED (I. 18.).

9. With internal contact,  $\angle CBA = \angle CAG$  (III. 18.)  $= \angle ADE$  (III. 18.)  $\therefore$  BC is  $\parallel$  to ED (I. 19.).

10.  $\angle TPS = \angle SPR$  (III. 15.)  $= \angle PRQ$  (I. 5.)  $\therefore$  PT is a tangent (III. 18.).

11. Draw str. line FAG to touch the circle ABC at A.  $\angle FAB = \angle ACB$  (III. 18.)  $= \angle AED$  (I. 20.)  $\therefore$  circle ADE touches FG at A (III. 18.), and so touches the circle ABC.

12. The circles touch at A, EAF being the common tangent, BC the chord touching at D. AC cuts the inner circle at H.  $\angle BAD = \angle EAD - \angle EAB = \angle AHD - \angle C$  (III. 18.)  $= \angle HDC$  (I. 22.)  $= \angle DAH$  (III. 18.).

### EXERCISES XXXIV.

1. Draw two circles with the given radius and with the given points for centres. The intersection gives the centre of the required circle.

2. Draw two perp. diameters and join their ends. I. 4. proves the sides equal. III. 17. proves the  $\angle$ s right  $\angle$ s.

3. Draw two perp. diameters. Draw the bisectors of the angles between them. The 8 arcs thus obtained are equal (III. 14.). Hence the 8 chords are equal (III. 15.). III. 15. will prove the angles of the octagon all equal.

4. Draw the chord at rt.  $\angle$ s to the join of the given point to the centre. Prove by III. 3.

5. With centre A on the circle, and with radius equal to the given length, describe a circle cutting the given circle in B. From C the centre of the given circle draw CD perp. to AB. With centre C and radius CD describe a circle, and from the given point O draw a tangent to this circle. This tangent can be proved to be the required line (III. 10.).

6. Let  $O$  be the point,  $C$  the centre. Describe a circle with centre  $C$  and radius equal to the given distance. A tangent from  $O$  to this circle is the required line.

7. Draw a diameter perp. to the given line: the tangents at the ends of this diameter are  $\parallel$  to the given line.

8. Draw a diameter  $\parallel$  to the given line: the tangents at the ends of this diameter are perp. to the given line.

9. Join the centre  $C$  to the given pt.  $A$ . Draw a chord  $BA$  perp. to  $CA$ . Draw  $CG$  perp. to any other chord  $EAF$ .  $CGA$  is a rt.  $\angle \therefore CAG < \text{a rt. } \angle$  (I. 9.)  $\therefore CA > CG$  (I. 11.)  $\therefore BD < EF$  (III. 11.).

10. Let  $O$  be the given point,  $A$  the centre. Draw the diameter  $BC$  perp. to  $OA$ . The circle with radius  $OB$  is the one required (I. 4.).

11. Let  $A$  be the given point,  $CBD$  the given line,  $B$  the point of contact.  $BE$  drawn perp. to  $CD$  must contain the reqd. centre. Make  $\angle BAF$  equal to  $\angle ABE$ , and let  $AF$  meet  $BE$  in  $F$ .  $AF = BF$  (I. 6.)  $\therefore$  the circle with radius  $FA$  or  $FB$  is the one required.

12. The centre is at the intersection of the diagonals, and the radius is half of either diagonal.

13. From the centre  $A$  draw  $AB$  perp. to the given str. line  $BC$ . Cut off  $BC$  equal to half the given length. Through  $C$  draw  $CDE \parallel$  to  $BA$  to meet the circle in  $D$  and  $E$ . Draw a chord  $DG \parallel$  to  $CB$ , cutting  $AB$  at  $F$ .  $DG = 2DF$  (III. 3.)  $= 2CB$  (II. 2.) = the given length. Another chord could be similarly drawn from  $E$ .

14. Let  $AB, CD$  be  $\parallel$ , and  $AC$  the third line. Bisect the angles  $A$  and  $C$  by  $AE, CE$ . Draw  $EF, EG, EH$  perp. to  $AC, CD, AB$ .  $EF = EH$  in  $\triangle AFE, AHE$  (I. 16.). Similarly  $EF = EG$ .  $\therefore E$  is the required centre.

15. Draw str. lines  $\parallel$  to the given str. lines at a distance from them equal to the given radius. The intersection of these gives the reqd. centre. There are 4 solutions.

16. From the given centre  $A$  draw  $AB$  perp. to the given str. line  $BC$ . Cut off  $BC$  equal to half the given length. Then  $AC$  is the radius of the reqd. circle.

17. Let  $ABC$  be the circle,  $O$  the centre. With centre  $A$  and radius  $AO$  describe a circle cutting the given circle in  $D, E$ . Produce  $AO$  to meet the circle in  $F$ .  $DEF$  is the reqd.  $\triangle$ . The  $\triangle AOD$  is equilateral,  $\therefore \angle DOA = 60^\circ$ . Thus each of the  $\angle$ s  $DOE, EOF, FOD$  is  $120^\circ \therefore \triangle DEF$  is equilateral (III. 14. and 15.).

18. Draw 6 radii, so that each of the 6 angles at the centre is  $60^\circ$  (as in Question 17.). The sides of the hexagon are equal (III. 14. and 15.). The angles are proved equal as in Ex. L., 3.

19. Bisect each of the angles at the centre in Question 18.

20. On the given base describe a segment containing the given angle (III. 23.). Find where the arc is cut by a line drawn  $\parallel$  to the base at the distance of the given altitude.

21. Let  $ABC$  be the goal line,  $AB$  the goal,  $CDE$  the line in which the ball is taken out. On  $AB$  describe a segment of a circle  $ABDE$  containing the given  $\angle$ .  $D$  or  $E$  is the reqd. point.

(2) Bisect  $AB$  at  $F$ , and draw  $FG \parallel$  to  $CD$ . With centre  $B$  and radius  $FC$ , cut  $FG$  at  $G$ . The circle with centre  $G$  and radius  $FC$  will touch  $CD$  at some point  $H$ ; and  $AHB$  may be proved to be the maximum subtended  $\angle$  (I. 8.).

22. See Question 5.

23. Make  $\angle BAC$  equal to the given vertical  $\angle$ . Make  $AB, AC$  each equal to half the given perimeter. Make  $ABE, ACE$  rt.  $\angle$ s. Describe a circle  $BDC$  with centre  $E$ . Describe a circle with centre  $A$  and radius the given altitude. Draw  $DF$  to touch both these circles (Exercises xxxvi. 1.), and to meet  $AB, AC$  at  $H, K$ . The perimeter of the  $\triangle AHK = AH + HK + AK + KD = AH + HB + AK + KC$  (tangents equal)  $= AB + AC =$  given perimeter. Also  $AF =$  given altitude: and  $A =$  given vertical  $\angle$ .

24. Let  $AB$  be the given side,  $BAC$  the given  $\angle$ . With centre  $A$  and the given altitude for radius, describe a circle; and from  $B$  draw  $BDC$  touching it at  $D$ .  $ABC$  is the reqd.  $\triangle$ .

25. Bisect  $AB$  the common chord by  $DCE$  perp. to it. With centre  $A$  and radius equal to each of the given radii in turn, cut  $DCE$  at  $D, E$ . These are the required centres.

26. Let A, B be the centres,  $r_1, r_2$  the radii of the given circles,  $r$  the radius of the reqd. circle. With centre A and radius  $r_1 + r$ , describe a circle. With centre B and radius  $r_2 + r$ , describe a circle. Let C be a pt. of intersection. The circle whose centre is C and radius  $r$  is the one required. Another solution may be obtained by using  $r - r_1, r - r_2$  for AC, BC provided  $r$  is large enough. In the 1st case AB must be less than  $2r + r_1 + r_2$ , in the 2nd case AB must be less than  $2r - r_1 - r_2$ .

27. Let A be the first given point, B the given point of contact on the circle with centre C. Let DE which bisects AB at rt.  $\angle$ s meet CB at E. The reqd. centre is E, radius EA. A may be internal or external.

28. Let A be the first given point, B the given pt. in the line BC. Draw DE bisecting AB at rt.  $\angle$ s, and let it meet in E the perp. to BC drawn from B. The required centre is E.

29. Let A be a common point; B, C centres. Bisect BC at D. Draw EAF perp. to DA. Draw CG, BH perp. to EF; CM, DK perp. to DA, BH. In  $\triangle$ s BKD, DMC,  $KD = MC$  (I. 16.)  $\therefore HA = AG$  (II. 2.)  $\therefore 2HA = 2AG$ , i.e.  $FA = AE$ .

30. Let A be the pt. where the given str. line is met by the bisector of the  $\angle$  between the two given lines BEC, CFD. Draw AB, AD perp. to BC, CD; and mark off BE, DF each equal half the given length. AE or AF is the reqd. radius.

31. Any circle described with its centre at the incentre of the  $\triangle$  has this property, since the chords cut off are equidistant from the centre, and therefore equal.

32. In the second circle place a chord of the reqd. length. Draw a perp. to this chord from the centre. With the perp. for radius describe a concentric circle. Draw (Ex. xxxvi. 1.) a common tangent to this and the first circle.

33. In the given circles place chords of the given lengths. Draw perps. from the centres. Describe circles with these perps. as radii. Draw a common tangent to these two circles. [Impossible when the given chords are greater than the corresponding diameters of the given circles.]

34. Cut off a segment containing an angle of  $60^\circ$ . The other segment contains an angle of  $120^\circ$ .

35. Cut off a segment containing an angle of  $30^\circ$ .

36. Let A, B be the given points. On AB as diameter describe a circle cutting the given str. line in C, D.

37. Let A be the given pt., BC the given chord, O the centre. On OA describe a semicircle cutting BC at D. Then ADE is the required chord. OD is perp. to AE (III. 17.)  $\therefore AD = DE$  (III. 3.).

38. Let A, B be the centres. Draw any radii AC, BD. Draw CE, DF perp. to these, and equal to the given lengths. Join AE, BF. Describe circles with centres A, B and radii AE, BF, and let them intersect at G. The tangents GH, GK are of the required lengths. For  $GH = EC$ , and  $GK = FD$  (I. 17.).

39. Draw two radii AB, AC including an angle supplementary to the given  $\angle$ . Let the tangents at B, C meet at D. The point D is the one required. For B, C being rt.  $\angle$ s the angle CDB is the supplement of  $\angle CAB$ .

(2)  $\angle DAB = \frac{1}{2}\angle CAB = a$  constant; and AB is of constant length  $\therefore AD$  is constant and the locus of D is a concentric circle.

40. Draw radii AB, AC, including a rt.  $\angle$ . BC is the required chord: for  $\angle ABC = 45^\circ \therefore \angle BAD = 45^\circ \therefore AD = DB = \frac{1}{2}BC$ , where AD is perp. to BC.

41. Let A, B be the given points. On AB describe a segment containing the given  $\angle$ . The pt. or pts. in which this meets the given circle will give the required pt.

42. Let A be the given pt. Take any pt. B on the outer circle. With centre B and radius the given length, cut the inner circle at C. Produce BC to meet the circles in D, E. From the common centre O draw a perp. OF to BE. With centre O and radius OF describe a circle. Draw a tangent from A to this circle. This tangent is the line required (III. 10.).

43. Let A be the point. Cut off a segment BDC containing an  $\angle$  equal to the given  $\angle$  (III. 24.). From the centre E draw to BC a perp. EF. Describe a concentric circle with radius EF. Draw  $\overset{\text{small arc}}{\text{AGHD}}$  a tangent to this circle at H.  $GD = BC$ , since  $\overset{\text{small arc}}{EH} = \overset{\text{small arc}}{EF}$  (III. 10.)  $\therefore \text{arc } GD = \text{arc } BC$  (III. 16.)  $\therefore \angle GCD = \angle BDC$  (III. 15.).

44. Through A, one pt. of intersection, draw any such line CD. From the centres E, F draw perps. EG, FH. Draw EK  $\parallel$  to CD.  $CD = 2GH$  (III. 3.). Also  $GH = EK < EF$  (I. 11.)  $\therefore CD < 2EF$  unless CD is  $\parallel$  to EF  $\therefore$  the maximum position is  $\parallel$  to EF.

45. Draw BC perp. to AB to meet the circle in C. The circle on AC as diameter touches the given circle (III. 6.). Draw any line AEF to meet the circles.  $\angle ACB = \angle AEB$  (III. 12.)  $> \angle AFB$  (I. 8.)  $\therefore$  C is the point required.

46. Let A be the centre of the given circle, BC the given str. line, C the given pt. In BC produced make CD equal to the radius of the given circle. Draw EB bisecting AD at rt.  $\angle$ s.  $AB = BD$  (I. 4.), and  $AF = CD$  (cons.)  $\therefore BF = BC$ , and the circle described with centre B is the one required.

### EXERCISES XXXV.

1. Let ABC be an equilateral  $\triangle$ , I the incentre. In  $\triangle$ s IFB, IDC,  $\angle IBF = 30^\circ = \angle ICD$ ,  $\angle F = 90^\circ = \angle D$ , and  $IF = ID$  (radii)  $\therefore IB = IC =$  similarly IA.

2. Let DEF be a circumscribing equilat.  $\triangle$ , ABC an inscribed equilat.  $\triangle$ , DE touching the circle at C, EF at A, FD at B.  $\angle FBA = \angle FAB = \angle ACB$  (III. 18.)  $= 60^\circ \therefore \triangle FAB$  is equilat.  $\therefore FA = AB$ . Similarly  $AE = AC = AB \therefore FE = 2AB$ .

3. Let  $\angle ACB = \angle DFE \therefore$  arc  $AB =$  arc  $DE \therefore$  chord  $AB =$  chord  $DE$  (III. 15.)  $\therefore$  the  $\triangle$ s are equal in all respects (I. 16.).

4. AB, AC are produced to D, E. Bisect  $\angle$ s CBD, BCE by BF, CF. From F draw perps. to the sides. These are equal (I. 16.)  $\therefore$  the circle with any of these three for radius is the one required.

5. PC is perp. to AC (III. 17.), and BT is perp. to AC (hyp.)  $\therefore PC$  is  $\parallel$  to BT. Similarly PB is  $\parallel$  to CT.

6. EB, BD are the bisectors of supplementary  $\angle$ s.  $\angle EBD = \frac{1}{2}$  of 2 rt.  $\angle$ s = a rt.  $\angle$ . Similarly  $\angle ECD$  is a rt.  $\angle \therefore$  ED is the diameter of a circle through B and C.

7. Let I be the incentre of  $\triangle ABC$  which has a rt.  $\angle$  at A. ID, IE, IF perp. to BC, CA, AB. Tangents are equal  $\therefore BF = BD$ , and  $CE = CD$ , also  $FA = IE$  (II. 2.)  $= r$ , and  $EA = IF = r \therefore BF + FA + CE + EA = BC + 2r$ , i.e.  $AB + AC =$  the sum of the diameters.

8. In the cyclic quadr.  $ODCE$   $\angle ODE = \angle OCE$  (III. 12.)  $= 90^\circ - \angle BAC$ . Similarly  $\angle ODF = \angle OBF = 90^\circ - \angle BAC \therefore DO$  bisects  $\angle EDF$ . Similarly for  $EO \therefore O$  is the incentre of  $\triangle DEF$ .

9. Draw  $DE, DF, DG$  perp. to  $BC, CA, AB$ .  $DE = DG$  (I. 16.). Similarly  $DE = DF \therefore DG = DF \therefore$  in  $\triangle s$   $DAG, DAF, \angle DAG = \angle DAF$  (I. 17.)  $\therefore DA$  bisects  $\angle BAC$  and consequently contains the incentre.

10. Let  $ABC$  be the equilat.  $\triangle$ .  $\angle BOA = 2\angle C$  (III. 11.)  $= 120^\circ \therefore \angle BOD = 60^\circ$ .  $\angle ADB = \angle ACB$  (III. 12.)  $= 60^\circ \therefore \triangle BOD$  is equilateral.

### EXERCISES XXXVI.

4. Proved in Ex. xxxv. 8.

5. Let  $ABC$  be the  $\triangle$ ,  $AGD, BGE, CGF$  the medians. Let  $BC$  be trisected at  $H, K$ . Let  $L, N$  be the mid. pts. of  $AG, BH$ .  $\triangle ABN = \triangle ANH = \triangle AHD$  (II. 6.)  $\therefore \triangle ABH = \frac{2}{3}\triangle ABD$ . Similarly  $\triangle AGB = \frac{2}{3}\triangle ABD \therefore \triangle ABH = \triangle AGB \therefore GH$  is  $\parallel$  to  $AB$  (II. 7.). Similarly  $GK$  is  $\parallel$  to  $AC \therefore \angle HGK = \angle BAC = a$  constant  $\therefore$  the locus of  $G$  is the arc of a segment on  $HK$ .

6. Let  $A$  be the fixed pt.,  $BC$  the given str. line. Draw  $AB$  perp. to  $BC$  and produce  $AB$  to  $D$ , making  $BD$  equal to  $AB$ .

Let  $O$  be the centre of one of the circles. Then  $OD = OA$  (I. 4.)  $\therefore D$  lies on the circle. Similarly  $D$  lies on each of the circles.

8. Let the circle touch  $BC$  at  $D$ .  $AF = AE$  (tangents), and similarly  $BF = BD, CE = CD \therefore AF + AE = AB + BD + CD + AC = 2s \therefore s = AF = AE$ .

9.  $BD = BF = AF - AB = s - c$ ; similarly  $CD = s - b$ .

10. Proved in Ex. xxxv. 5.

11.  $L$  is the mid. pt. of  $BC$  (III. 3.). But the mid. pt. of one diagonal of a parm. is the mid. pt. of the other  $\therefore L$  is the mid. pt. of  $PT \therefore OL = \frac{1}{2}AT$  (Ex. xx. 1.).

12. Let the altitudes be  $AD, BE, CF$ ; the orthocentre  $T$ .  $\angle BTD = 90^\circ - \angle TBD = \angle BCE$  (from  $\triangle BCE$ ). Similarly  $\angle DTC = \angle ABD \therefore \angle BTC = \angle C + \angle B = 180^\circ - A = a$  constant  $\therefore$  the locus is the arc of a segment on  $BC$ .

**EXERCISES XXXVII.**

1. The str. line perp. to the given str. line at the given pt. contains all the centres (III. 5. Cor. 2.).

2. Join the mid. pts. to the centre. These joins are perp. to the chords (III. 3) and are therefore equal (III. 10.)  $\therefore$  the locus is a concentric circle.

3. Join the fixed pt. to the common centre. This line subtends a rt.  $\angle$  at a pt. of contact  $\therefore$  the locus of the pts. of contact is a circle whose diameter is this line (III. 17.).

4. Let A be the fixed pt., C the centre of the circle, CP perp. to a chord. P is the mid. pt. of the chord, and the locus of P is the circle whose diameter is AC (III. 17.).

5. Produce BC to E making CE = BC. A, C are the mid. pts. of BD, BE  $\therefore$  AC is  $\parallel$  to DE  $\therefore \angle BDE = \angle BAC =$  a constant. BE = 2BC = a constant  $\therefore$  the locus of D is the arc of a segment of a circle on BE.

6. Let A, B be the fixed pts. P the pt. of contact of the circles. Draw the common tangent at P meeting AB at T. Tangent TA = TP = TB  $\therefore$  P lies on the circle whose diameter is AB.

7. Let E be the intersection of AD, BC.  $\angle AEB = \angle CAD + \angle ACB = 90^\circ +$  a constant  $\angle$  (III. 17. 12.)  $\therefore$  the locus of E is the arc of a segment on AB.

8. Let BC be the given base, A the vertex, O the circumcentre.  $\angle BOC = 2\angle A =$  a constant  $\therefore$  the locus of O is the arc of a segment on BC.

9. Let I be the incentre, BC the given base, A the vertex. Let AI be produced to D.  $\angle BIC = \angle BID + \angle DIC = \frac{A}{2} + \frac{B}{2} + \frac{A}{2} + \frac{C}{2}$  (I. 22)  $= \frac{A}{2} + 90^\circ =$  a constant  $\therefore$  the locus of I is the arc of a segment on BC.

10. Let BC be the base, K the excentre.  $\angle BKC = 180^\circ - \angle KBC - \angle KCB = 180^\circ - \left(90^\circ - \frac{B}{2}\right) - \left(90^\circ - \frac{C}{2}\right) = \frac{B+C}{2} = 90^\circ - \frac{A}{2} =$  a constant  $\therefore$  the locus of K is the arc of a segment on BC. (It is the remaining arc of the circle mentioned in the preceding.)

11. Let the diagonals intersect at E. Bisect AB at F. E is the mid. pt. of AC, F of AB  $\therefore$  EF is  $\parallel$  to CB  $\therefore \angle AEF = \angle ACB = a$  constant. Also  $AF = \frac{1}{2}AB = a$  constant  $\therefore$  the locus of E is the arc of a segment on AF.

12. Let AB be the edge of the ruler sliding on CA, CB; D its mid. pt.  $CD = \frac{1}{2}AB$  (Ex. xviii. 9. or III. 17.) = a constant  $\therefore$  the locus of D is a circle with centre C.

13. Let the bisectors meet at E.  $\angle EAC + \angle ECA = \frac{1}{2}(\angle BAC + \angle ACD) = 90^\circ$  (I. 20.)  $\therefore \angle AEC = 90^\circ \therefore$  E lies on the circle whose diameter is AC.

14. Let R be the pt. of intersection.  $\angle P = \angle PAC, \angle Q = \angle QAD$  (I. 5.).  $\angle CRD = 180^\circ - \angle P - \angle Q$  (I. 22)  $= 180^\circ - \angle PAC - \angle QAD = \angle CAD = \angle CBD \therefore$  R lies on a circle through C, B, D.

15. Let APB be the segment on AB, Q the centre of the circle of which this is a segment. Let APB be on the side of AB remote from C the centre of the given circle. AB is the common chord of the two circles  $\therefore$  CQ bisects AB at rt.  $\angle$ s  $\therefore \angle CQA = \frac{1}{2}\angle AQB = \angle APB$  (III. 11.) = a constant  $\therefore$  Q is on a circle through A and C. By drawing AB in different positions it is found that two circles are obtained.

### EXERCISES XXXVIII.

1.  $r = \sqrt{26^2 - 24^2} = 10$ .
2.  $\text{Tangent} = \sqrt{37^2 - 35^2} = \sqrt{72 \times 2} = 12$ .
3. The two radii and the chord form an equilat.  $\triangle \therefore r = 6$ .
4.  $\text{Distance} = \sqrt{65^2 - 63^2} = 16$ .
5. Distance of 1st chord fr. centre  $= \sqrt{85^2 - 36^2} = 77$ . Distance of 2nd chord fr. centre  $= \sqrt{85^2 - 51^2} = 68 \therefore$  distance between chords  $= 77 \pm 68 = 145$  or 9
6.  $\text{Chord} = 2\sqrt{13^2 - 5^2} = 24$ .
7.  $\text{Chord} = 2\sqrt{27^2 - 12^2} = 6\sqrt{65} = 48.37$  decimetres; chord of half arc  $= \sqrt{9 \times 65 + 15^2} = 9\sqrt{10} = 28.46$  decimetres.
8.  $32^2 + (r - 8)^2 = r^2 \therefore 16r = 8^2 + 32^2 \therefore r = 68$ .
9. Triangle formed by centres has base 20, sides  $10 + r$ ,  $10 + r$ , and altitude  $20 - r \therefore (20 - r)^2 + 10^2 = (10 + r)^2 \therefore 400 - 40r = 20r \therefore r = \frac{20}{3} = 6\frac{2}{3}$ .

10. Chord  $= 2\sqrt{25^2 - 24^2} = 14$ .

11.  $r = 4$ .

12. Chord  $= 4 \cdot 8$ .

13. Distance  $= \sqrt{35^2 - 28^2} = 21$ .

14. Distance  $= \sqrt{70^2 - 24^2} = \sqrt{4324} = 65 \cdot 76$ .

15. The mid. point of chord is the centre of circle. Since 3 equal str. lines are drawn fr. it to circumf.  $\therefore r = 5$ .

16. Distance  $= \sqrt{2 \cdot 6^2 - 2 \cdot 4^2} = 1$ .

17. Distance of 1st chord fr. centre  $= \sqrt{5^2 - 3^2} = 4$ . Distance of 1st chord fr. centre  $= \sqrt{5^2 - 4^2} = 3 \therefore$  distance apart  $= 4 \pm 3 = 7$  or 1.

18.  $3 \cdot 57$  each.

19. Distance  $= 2 \cdot 1$ .

20. Distance  $= 5 \cdot 74$ .

21.  $13 \cdot 86$  cms. or  $5 \cdot 2$  inches.

22. Let O be the centre, A a pt. on the circumference. Cut the circumference at B, C by a circle with centre A and radius AO. Let AO, BO, CO meet the circle in D, E, F. AOB, AOC are equilat.  $\triangle$ s by construction  $\therefore \angle BOC = 120^\circ \therefore \angle COE = 60^\circ \therefore$  by I. 3. all the angles at O are equal  $\therefore$  the 6 arcs are all equal. The  $\angle$ s of the hexagon are equal; for each stands on two-thirds of the circumference.

23.  $3 \cdot 75$ .

24.  $r = \frac{ab}{\sqrt{4a^2 - b^2}}$ . [In rt.-angled  $\triangle$  formed by radius, tangent, and line joining the external pt. to the centre  $\frac{b^2}{4} =$  product of segments of hypotenuse  $= \sqrt{r^2 - \frac{b^2}{4}} \sqrt{a^2 - \frac{b^2}{4}} \therefore$  by squaring we get  $r$ .

25.  $6 \cdot 3$ .

26.  $QC = b - c$ . Let  $OA = x$ . Then  $OD = x - a$ .  $OQ = OB - QB = OB - QC = x - b + c \therefore$  by II. 11.  $(x - b + c)^2 - (x - a)^2 = c^2 \therefore (c + a - b)(2x - a - b + c) = c^2 \therefore 2x(c + a - b) + c^2 - 2bc + b^2 - a^2 = c^2 \therefore x = \frac{a^2 + 2bc - b^2}{2(c + a - b)}$ . When  $b = 5$ , and  $a = c = 3$ ,  $OA = 7$ , and  $QC = 2$ .

27. Let A, B be the posts, T the tree. Sets of sufficient measurements, (1) lengths of AT and BT, (2) AT and  $\angle$ BAT, (3) BT and  $\angle$ ABT, (4)  $\angle$ sABT and BAT.

28.  $60^\circ + \angle BCA < 180^\circ$  (I. 9.)  $\therefore \angle BCA < 120^\circ$ . The least distance of C from AB is the perpendicular, *i.e.*  $AC = 4\sqrt{3} = 6.93$  cms.

29. Each  $\angle = 34\frac{1}{2}^\circ$  approximately. Prove III. 12.

30. Let  $PA = a$ ,  $PB = b$ ,  $PQ = x$ .  $x^2 + a^2 + x^2 + b^2 = AQ^2 + BQ^2 = AB^2 = (a + b)^2 \therefore x^2 = ab$ . Thus rect.  $PA \cdot PB = PQ^2 \therefore$  the maximum is when Q is as far as possible from P, *i.e.* when  $PA = PB$ .

(2) The semiperimeter is given, viz. 1000 yds. By the property just proved the rectangle is a maximum when the adjacent sides are equal  $\therefore$  each side = 500 yds.

31. Make  $PA = 11$ ,  $PB = 7$ , and let APB be a str. line. Draw a semicircle on AB, and draw PQ perp. to AB to meet the circumference in Q.  $QP^2 = PA \cdot PB = 77$ .  $QP = \sqrt{77} = 8.78$ .

32. Describe an equilateral  $\triangle AOB$ .  $\angle AOB = 60^\circ$ . Describe a circle with centre O and radius OA. The larger segment is the one required. For the  $\angle$  in it =  $\frac{1}{2} \angle AOB$  (III. 11.) =  $30^\circ$ .

33.  $180^\circ$  in each case.

34. Let P be the external point, PT a tangent.  $TP^2 = (\frac{13}{4})^2 - (\frac{5}{4})^2 = 3^2$ .

35. 11.3 cms.

36. Draw AB eastwards of length 10, BC north-east. Let  $BC = 4$ . Draw CD perp. to AB produced.  $AD = 10 + 2\sqrt{2}$ ,  $CD = 2\sqrt{2} \therefore AC^2 = (10 + 2\sqrt{2})^2 + (2\sqrt{2})^2 = 116 + 40\sqrt{2} = 116 + 40 \times 1.4142 = 172.568 \therefore AC = 13.13$ .

37. The 4 nearest, the 1 farthest.

38.  $\angle AOB = 48\frac{1}{2}^\circ$  approx. Each of the others =  $24\frac{1}{4}^\circ$ .

39. PQ becomes the tangent at A.

40.  $\angle OAB = 90^\circ - \frac{1}{2} \angle AOB = 75^\circ, 80^\circ, 85^\circ, 87\frac{1}{2}^\circ, 89\frac{1}{2}^\circ, 89\frac{3}{4}^\circ, 89^\circ 59\frac{1}{2}'$ . When  $\angle AOB$  becomes zero, the chord becomes a tangent and the  $\angle OAB$  becomes  $90^\circ$ . Thus the tangent at A is perp. to OA.

41. On AB describe an equilat.  $\triangle$  remote from C, and on AC one remote from B. The intersection of the circumcircles of these  $\triangle$ s will give the point required. Prove by III. 13.

42. Let AB be the given str. line. O the centre of the circle. Draw OD perp. to AB. From DB cut off DE equal to .4 inch. From E draw a perp. to AB, meeting the circle at P. Draw a chord PRQ  $\parallel$  to BA, cutting OD at R. PQ is bisected by the perp. OR (III. 3.), and PR = DE (II. 2.)  $\therefore$  PQ = .8 inch.

43. Draw str. lines  $\parallel$  to the given str. lines, and at a distance .7 inch from them. The intersections of these give the reqd. centres. There are 4 positions.

44. In each circle place a chord of length 1 inch. Draw the perps. to these from the respective centres. With these perps. for radii describe two circles. Draw the 4 common tangents to these inner circles (Ex. xxxvi. 1). These are the lines required; for we have drawn them at such a distance from the centres that the intercepted chords = the 1 inch chords previously drawn (III. 10.).

45. Whichever side of the inscribed  $\triangle$  we take, the altitude must be the greatest possible  $\therefore$  the vertex must be at the mid. point of the arc; *i.e.* the  $\triangle$  must be isosceles whichever side we take for base  $\therefore$  it must be equilateral. The radius =  $\frac{2}{3}$  of a median =  $\frac{2}{3}$  of a side  $\times \frac{\sqrt{3}}{2} \therefore$  a side =  $3\sqrt{3} \therefore$  perimeter =  $9\sqrt{3}$  cms.

### EXERCISES XXXIX.

1. Circumference =  $\frac{44}{7}$  of radius.
2. Diameter = circumf.  $\times \frac{1}{\pi} = 77 \times \frac{7}{22} = \frac{49}{2} = 24\frac{1}{2}$ .
3. Distance =  $\frac{22}{7} \times 6 \times 6000 = \frac{792000}{7} = 113143$  feet nearly.
4. Number of laps =  $\frac{1760}{\frac{44}{7} \text{ of } 80} = \frac{1760 \times 7}{44 \times 80} = 3\frac{1}{2}$ .
5. Distance =  $\frac{22}{7} \times 32 \times 2000$  inches =  $\frac{22}{7} \times \frac{32 \times 2000}{1760 \times 36}$  miles =  $3\frac{11}{81}$ .
6. Distance in 1 minute =  $\frac{22}{7} \times 7 \times 240 = 22 \times 240$  feet.  
Distance in 1 hour =  $22 \times 240 \times 60$  feet = 60 miles.

7. Perimeter of hexagon =  $6r$ . Circumf. of circle =  $2\pi r$ .  
 Ratios =  $\frac{3}{\pi}$ .

8. Number of revolutions = distance  $\div$  circumference of wheel =  $5 \times \frac{2^2}{7} \times 448 \times 12 \div (\frac{2^2}{7} \text{ of } 32) = \frac{2 \cdot 2 \cdot 4 \cdot 0 \times 1 \cdot 2}{3 \cdot 2} = 840$ .

### EXERCISES XL.

1. Arc =  $\frac{1}{6}$  of circumference =  $4\frac{1}{6}$ .
2. Arc =  $\frac{5 \cdot 4}{3 \cdot 6 \cdot 0}$  of circumference =  $\frac{5 \cdot 4}{3 \cdot 6 \cdot 0} \times \frac{2 \cdot 2}{7} \times 14 = 6 \cdot 6$ .
3. Arc =  $\frac{7 \cdot 5}{3 \cdot 6 \cdot 0} \times \frac{4 \cdot 4}{7} \times 21 = 27 \cdot 5$ .
4. The  $\triangle$  formed by chord and radii is equilateral  $\therefore$  arc =  $\frac{1}{6}$  circumference =  $\frac{1}{6} \times 3 \cdot 1416 \times 170 = 89 \cdot 012 = 89$  to nearest inch.
6. Arc  $\frac{22\frac{1}{2}}{360} \times \frac{4 \cdot 4}{7} \times 1760 = \frac{1}{16} \times \frac{4 \cdot 4}{7} \times 1760 = \frac{4 \cdot 4 \times 1 \cdot 1 \cdot 0}{7} = 691 \cdot 4$  yds.
7. Distance =  $\frac{6 \cdot 3}{3 \cdot 6 \cdot 0} \times 3 \cdot 1416 \times 7925 \cdot 6 = 435 \cdot 7336$ .
8. Arc = 300, circumference =  $7925 \cdot 6 \times 3 \cdot 14159$ . Difference = angle subtended at centre =  $\frac{300}{7 \cdot 9 \cdot 2 \cdot 5 \cdot 6 \times 3 \cdot 1 \cdot 4 \cdot 1 \cdot 5 \cdot 9}$  of  $360^\circ = 4 \cdot 34^\circ$ .
9. Let ACB be the arc, CDO the bisecting radius, CD the height.  $OD = r - 4$ ,  $AO = r$ ,  $AD = 8 \therefore r^2 = (r - 4)^2 + 8^2 \therefore 8r = 16 + 64 = 80 \therefore r = 10$ .
10. Let ACB be the arc, AC half the arc, CD the height, O the centre.  $CD = \sqrt{AC^2 - AD^2} = \sqrt{3 \cdot 9^2 - 3 \cdot 6^2} = 1 \cdot 5 \therefore DO = r - 1 \cdot 5 \therefore$  in  $\triangle ADO$   $r^2 = (r - 1 \cdot 5)^2 + 3 \cdot 6^2 \therefore 3r = 1 \cdot 5^2 + 3 \cdot 6^2 = 15 \cdot 21 \therefore r = 5 \cdot 07$ .

### EXERCISES XLI.

1. Area =  $\frac{2 \cdot 2}{7} \times \frac{7}{2} \times \frac{7}{2} = 38 \cdot 5$  sq. ft.
2. Area =  $3 \cdot 1416 \times \frac{2 \cdot 1}{2} \times \frac{2 \cdot 1}{2} = 345 \cdot 3614$  sq. ft. = 345 sq. ft. 52 sq. in.
3.  $\pi r^2 = 3850 \therefore r^2 = 3850 \times \frac{7}{2 \cdot 2} = 35^2$ . Circumference =  $2\pi r = \frac{4 \cdot 4}{7} \times 35 = 220$ .

4. Area of track = total area - area of grass =  $\pi(301^2 - 294^2)$   
 $= \frac{\pi}{7} \times (301 - 294)(301 + 294) = \frac{\pi}{7} \times 7 \times 595 = 13090$ .

5. Divide the  $\triangle$  into 3  $\triangle$ s by joining the incentre to the vertices. The areas of these 3  $\triangle$ s are  $\frac{1}{2}r \cdot 5$ ,  $\frac{1}{2}r \cdot 4$ ,  $\frac{1}{2}r \cdot 3$ .  $6r$  = the sum of these = area of whole  $\triangle = \frac{1}{2} \times 3 \times 4 = 6 \therefore r = 1$   
 $\therefore$  area of circle =  $\pi = 3\frac{1}{7}$  sq. ft.

6.  $\frac{22}{7}r^2 = 260 \cdot 26 \therefore r^2 = 260 \cdot 26 \times \frac{7}{22} = 82 \cdot 81 \therefore r = 9 \cdot 1$ .

7. The triangle formed by joining the centres has each side 2 ft.  $\therefore$  its area =  $\sqrt{3} = 1 \cdot 732$  sq. ft. Each of the 3 sectors is  $\frac{1}{6}$  of a circle  $\therefore$  together their area =  $\frac{1}{2}\pi r^2 = 1 \cdot 5708 \therefore$  remaining area =  $\cdot 1612$  sq. ft.

8.  $2\pi r = 1 \therefore r = \frac{1}{2\pi} = \cdot 1592$  ft. Area =  $\pi r^2 = 2\pi r \times \frac{1}{2}r = \cdot 0796$  sq. ft. =  $11 \cdot 46$  sq. in.

9. Area of square = 200 (II. 14.); area of circle =  $100\pi = 314 \cdot 16$ . Difference =  $114 \cdot 16$ .

10. If a  $\triangle$  has angles  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  it is one-half of an equilateral  $\triangle$ , and it can be proved by II. 11. that the sides are  $2a$ ,  $a$ ,  $a\sqrt{3}$ . The side opposite to the  $60^\circ$  = the side opposite to the  $30^\circ \times \sqrt{3}$ . The  $\frac{1}{2}$  side of the equilat.  $\triangle = \frac{\text{perimeter}}{6} = \frac{p}{6}$   
 $\therefore$  the radius of the incircle =  $\frac{p}{6\sqrt{3}}$ . The  $\frac{1}{2}$  side of the hexagon =  $\frac{p}{12} \therefore$  the radius =  $\frac{p\sqrt{3}}{12} \therefore$  the radii are as 2 to 3  $\therefore$  the areas are as 4 to 9.

11. Innermost circle =  $\frac{\pi r^2}{n+1} \therefore$  radius =  $\frac{r}{\sqrt{n+1}}$ . Area of 2nd circle =  $\frac{2\pi r^2}{n+1} \therefore$  radius =  $\frac{r\sqrt{2}}{\sqrt{n+1}}$ . Area of 3rd circle =  $\frac{3\pi r^2}{n+1} \therefore$  radius =  $\frac{r\sqrt{3}}{\sqrt{n+1}}$ , and so on.

12. Innermost circle =  $\frac{\pi r^2}{3} \therefore$  radius =  $\frac{r}{\sqrt{3}} = \frac{\sqrt{3}}{3}$  ft. =  $4\sqrt{3}$  inches =  $6 \cdot 93$ . Area of 2nd circle =  $\frac{2\pi r^2}{3} \therefore$  radius =  $\frac{r\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$  ft. =  $4\sqrt{6}$  inches =  $9 \cdot 8$ .

## EXERCISES XLII.

1.  $CE^2 = CB^2 - EB^2 = CB^2 - AD^2 = AB^2 + AC^2 - AD^2 = DF^2 + CD^2 = CF^2$ .

2. Let OD cut AC at E.  $\angle ODA = \angle OAD$  (I. 5.)  $= 90^\circ - \angle BAD = 90^\circ - \angle DAE \therefore \angle OEA = 90^\circ$  (I. 22.).  $\angle BAD = \angle$  in alt. segment (III. 19.)  $= \frac{1}{2} \angle AOD$  (III. 12.)  $\therefore \angle AOD = \angle BAC$ .

3. Let E be the centre, EF perp. to AB.  $FA = \frac{1}{2}AB$  (III. 3.)  $=$  a constant.  $\angle AEF = \frac{1}{2} \angle AEB = \angle AOB$  (III. 11.)  $=$  a constant  $\therefore$  AE is of constant length (I. 16.)  $\therefore$  EO is also of constant length.

4.  $\angle ASR + \angle ARS = 180^\circ - A =$  a constant  $\therefore$  arc RPQ + arc PQS  $=$  a constant (III. 14.)

5. Let E, F be the centres, EG, FH perps. to AB, AC (BDA being an obtuse  $\angle$ ).  $\angle GEA = \frac{1}{2} \angle BEA =$  supplement of  $\angle BDA$  (III. 11. and 13.)  $= \angle ADC = \frac{1}{2} \angle AFC$  (III. 11.)  $= \angle AFH$ .  $AG = \frac{1}{2}AB = \frac{1}{2}AC = AH \therefore EG = FH$  (I. 16.).

6.  $\frac{1}{2}(A - C) = 90^\circ - \angle AEH - (90^\circ - \angle CFG) = \angle CFG - \angle AEH = \angle FEG - \angle EFH$  (III. 18.)  $= \angle$  subtending arc FG  $- \angle$  subtending arc EH  $= \angle$  subtending arc EFG  $- \angle$  subtending arc FEH  $= \angle EHG - \angle FGH$ .

7. Let PAB be the diameter of the larger, PA of the smaller circle. Let the smaller circle roll into a new position in which Q is the point of contact, P' the new position of P. Join AQ. Let R be the centre in its new position the arc PQ = arc P'Q, for one has rolled on the other. But the radius of the circle is half that of the other  $\therefore$  arc P'Q subtends at R twice the angle which the arc PQ subtends at A (p. 198)  $\therefore \angle PAQ = \frac{1}{2} \angle P'RQ = \angle P'AQ \therefore$  P' lies on PAB  $\therefore$  as the circle moves the point P traces out the diameter PAB.

8. Let AB be the given str. line. O the centre of the given circle. Make  $\angle COD$  equal to twice the given  $\angle$ , and at C, D draw tangents CE, DE. With centre O and rad. OE, describe a circle cutting AB at F. F is the reqd. pt., for tangents from F will include an  $\angle$  equal to  $\angle CED$ ; the chd. of contact will be equal to CD and will  $\therefore$  subtend an  $\angle$  at the circumference equal to the given  $\angle$ . The problem is impossible if OE is less than the perp. from O upon the given line.

9. If  $ABC$  be the  $\triangle$ ,  $I$  the incentre,  $L, M$  the excentres opposite to  $A, B$  respectively,  $\angle IBL = \frac{1}{2}(B + 180^\circ - B) = 90^\circ$ . Similarly  $\angle ICL = 90^\circ \therefore$  the circle whose diameter is  $IL$  passes through  $B$  and  $C$ . Similarly  $\angle LAM = 90^\circ$ , and  $LBM$  has been proved to be  $90^\circ \therefore$  the circle whose diameter is  $LM$  passes through  $A$  and  $B$ .

10. Let  $A, B$  be the pts. Describe a circle on  $AB$  as diameter. Cut this at  $C$  by a circle with centre  $B$  and radius the given length.  $AC$  will be the radius of the required circle,  $BC$  the tangent.

11. Let  $L$  be mid. pt. of  $AC$ . Then  $FL = LC$  (Ex. xviii. 9.)  $\therefore \angle LFC = \angle LCF = 90^\circ - A = \angle FBP$  (Ex. xxxvi. 3.)  $\therefore FL$  is a tangent (III. 18.). Similarly for  $DL$ . Also  $BP$  is a diameter of this circle since  $D, F$  are rt.  $\angle$ s  $\therefore$  tangents of  $B, D$  are perp. to  $BD \therefore$  parallel to  $AC$ .

12. Let  $AC$  meet the circle at  $D$ . In the circle  $DPBA$   $\angle DAP = \angle DBP$ , and any increase of  $\angle DAP$  is accompanied by an equal increase of  $\angle DPB$  (III. 12.)  $\therefore$  rate of revolution of  $BP$  = that of  $AP$ . Also by III. 11. rate of revolution of  $CP$  = twice that of  $AP$ .

13. Let  $A, B$  be the centres. In each circle place a chord equal to the given length. Draw  $AC, BD$  perp. to these chords. Draw circles with centres  $A, B$  and radii  $AC, BD$ . Draw a common tangent to these two circles (Ex. xxxvi. 1.) This can be proved to be the required line (III. 10.).

14. Let the incircles of  $\triangle$ s  $ABD, ACD$  touch  $AD$  at  $E, F$ .  $2DE = BD + AD + AB - 2AB = BD + AD - AB$  (Ex. xxxvi. 7.)  $2DF = CD + AD - AC \therefore 2DE - 2DF = BD - CD - AB + AC = 0$  (Ex. xxxvi. 7.)  $\therefore$  the two circles touch  $AD$  at the same pt.  $\therefore$  they touch each other.

15. Draw  $PO$  perp. to  $L$  and produce it to  $R$  so that  $OR = PO$ . Draw  $RQT$  touching the circle and cutting  $L$  at  $Q$ .  $\angle PQO = \angle RQO$  (I. 4.).

16.  $\angle C'OA' = \angle COA$  (I. 3.)  $\therefore C'A' = CA$  (III. 14. 15.). Similarly for the other sides  $\therefore$  the  $\triangle$ s are equal in all respects (I. 7.).

17. Let  $I, I'$  be the incentre and excentre. These pts. are on the bisector of the  $\angle A$ . Let  $IF, I'K$  be perp. to  $AB$ . Draw  $IL \parallel$  to  $AB$ .  $\angle LI I' = 45^\circ \therefore \angle LI' I = 45^\circ \therefore LI' = LI$ , i.e.  $I'K - IF = AK - AF = s - (s - a) = BC$  (Ex. xxxvi. 7.).

18. Let A, B be the centres of smaller and larger circles. Let AB meet the smaller at D, longer circle at E. Let FHG be the common chord, A, E, H, D, B being in one straight line.  $AH = \sqrt{13^2 - 12^2} = 5 \therefore HD = 8$ .  $HB = \sqrt{15^2 - 12^2} = 9 \therefore BD = 1$ .  $\therefore DE = 14$ .

19. Let A be the centre of one circle, C the point of contact, B the centre of the other circle. Mark two points D on AC at a distance from C equal to the radius of the second circle. Join DB. Make  $\angle DBE$  equal to  $\angle BDE$ , the point E being in AC. Let BE meet the second circle in F.  $EF = EC$  (I. 6.)  $\therefore$  a circle described with centre E and radius EC will be the one required (III. 6.). The two positions of D will give two solutions.

20. Draw LM, LN perp. to AC, AB. Suppose AB greater than AC. Since L is on the bisector of LA,  $LM = LN$ .  $\triangle LAC = \frac{1}{2} LM \cdot AC < \frac{1}{2} LN \cdot AB \therefore \triangle LAC < \triangle LAB \therefore LC < LB \therefore L$  lies between D and C.  $\angle C > \angle B \therefore \angle PAC < \angle PAB \therefore \angle PAC < \frac{1}{2} A \therefore P$  lies between L and C  $\therefore L$  lies between P and D.

21. See Ex. L. 15.

22. Draw BC, BD perp. to AL, AM.  $\angle BLC = 180^\circ - \angle ALB = \angle BMD$  (III. 13.)  $\therefore LC = MD$  (I. 16.)  $\therefore AL + AM = AC + AD = 2AC = \text{a constant}$ .

23. Take centre C. Let AM meet the circle in R. Draw CTS perp. to BR meeting QP in T. CTS bisects QP and BR (III. 3.). Also  $TN = SB$  (II. 2.)  $= SR = TM \therefore QN = PM$ .

24. Let O be the centre. In the  $\triangle s$  POR, QOT,  $PO = QO$ ,  $OR = OT \therefore PR = QT$  and  $\angle OPR = \angle OQT$  (I. 17.)  $\therefore PR$  and  $QT$  are equal and parallel  $\therefore PRQT$  is a parm.

25. Arcs AD, BC together = semicircumference  $\therefore$  the  $\angle s$  subtended by them at the circumference = a rt.  $\angle$ , i.e.  $\angle EBD + \angle EDB = \text{a rt. } \angle \therefore \angle AED$  is a rt.  $\angle$ .

26. Distance from vertex to orthocentre = twice distance from circumcentre to base of any  $\triangle$  (Ex. xxxvi. 11.)  $\therefore BR = CQ$ . Also BR is  $\parallel$  to CQ, since both are perp. to AD  $\therefore RQ$  is equal and  $\parallel$  to BC. Similarly for the other sides  $\therefore PQRS$  is equal in all respects to ABCD.

27. Let the bisectors meet in T.  $\angle CPB + \angle CQD = \angle BCD - \angle PBC + \angle BCD - \angle CDQ = 2 \angle BCD - \angle ADC - \angle ABC = 2 \angle BCD - 180^\circ \therefore \angle CPT + \angle CQT = \angle BCD - 90^\circ$ . But  $\angle CPQ + \angle CQP = 180^\circ - \angle BCD \therefore$  by addition  $\angle TPQ + \angle TQP = 90^\circ \therefore T = 90^\circ$ .

28. In  $\triangle ADC$ , AE is perp. to DC, and CE is perp. to AD  $\therefore$  DE is perp. to AC (Ex. xxxvi. 3.)

29. Draw diameter AD; produce it to E so that  $DE = AD$ . Join EC. BD is the join of mid. pts. of AC, AD  $\therefore$  BD is  $\parallel$  to CE. But  $\angle ABD$  is a rt.  $\angle$  (III. 17.)  $\therefore$  C is a rt. angle  $\therefore$  C lies on a circle whose diameter is AE.

30. Produce AP to meet the circumcircle of ABC in Q.  $\angle BCQ = \angle BAQ$  (III. 12.)  $= 90^\circ - B = \angle PCB$ . Similarly  $\angle CBQ = \angle CBP \therefore \triangle$ s PCB, QCB are equal in all respects (I. 16.)  $\therefore$  circumcircle of  $\triangle$  PCB = circumcircle of  $\triangle$  QCB = circle ABC. Similarly for the others

31. Let ABCD be the quadl., AO, BO the bisectors of  $\angle$ s A, B. Draw OE, OF, OH perp. to AB, BC, AD. Then it can be proved by I. 16. that  $FB = BE$ ,  $HA = AE$ , and  $OE = OF = OH$ . Draw OG perp. to CD. Suppose OG gr. than OF. Then by II. 11.  $CG < CF$  and  $GD < DH \therefore AB + CD < BC + AD$ . Similarly OG is not less than OF  $\therefore$  a circle with centre O passes through E, F, G, H and touches the sides of ABCD.

32. Let OA, OB be fixed radii, P any pt. on the circle, PQ, PR the perps. Let these meet the circle in S, T. R, Q are mid. pts. of PT, PS  $\therefore RQ = \frac{1}{2}TS$ . But  $\angle P = 180^\circ - \angle O =$  a constant  $\therefore$  chord TS is of constant length (III. 14. 15.).

33. Let P be the point, OA, OB the fixed lines. Let the perps PR, PQ be produced to T, S, making  $RT = PR$  and  $QS = PQ$ . The circle whose centre is O and radius OP passes through T, S. Also  $TS = 2RQ =$  a constant, and the  $\angle TPS =$  supplement of  $O =$  a constant  $\therefore$  the radius of the circumcircle of TPS is constant, and as it has a fixed centre O, the circle is fixed.

## GRAPHS.

## EXERCISES XLIII. a.

2. (a) mid. pt. (0, 0) the origin; (b) mid. pt. (3, 0) a pt. on the axis of  $x$ ; (c) mid. pt. (2, 2); (d) mid. pt. (-4, 4).

3. The pts. all lie on a line  $\parallel$  to the axis of  $Y$ .

4. If A and B are the pts. A lies on OY, B on OX.  $\triangle OAB = \frac{1}{2} OA \times OB = 12$  sq. units.

5. The fig. is a rect. whose sides are 6 and 8 units long. Its area  $= 6 \times 8 = 48$  sq. units.

6. The ordinates of the first two points are 0 and 12. The abscissae of the other two points are 1.5 and 3.5.

7. Area  $= 18$  sq. units.

## EXERCISES XLIII. b.

1. (a) A str. line  $\parallel$  to OY, and at a distance 4 from it on the positive side; (b) a str. line  $\parallel$  to OX, and at a distance 5 from it on the positive side; (c) a str. line  $\parallel$  to OY, and at a distance 2 from it on the negative side; (d) a str. line  $\parallel$  to OX, and at a distance 3 from it on the negative side.

2. (a) A str. line thro. the origin, and thro. the point (5, 15); (b) a str. line thro. the origin, and thro. the point (5, -10).

3. (a) A str. line thro. the origin, and thro. the point (10, 5); (b) a str. line thro. the origin, and thro. the point (10, -5).

4. A str. line  $\parallel$  to OX, and at a distance 4 from it on the negative side.

5.  $y = x + 2$  is a str. line thro. the pts. (0, 2) and (5, 7).

6.  $y = x - 2$  is a str. line thro. the pts. (0, -2) and (7, 5).

7. A str. line thro. the pts. (0, 5) and (5, 10).
8. A str. line thro. the pts. (0, 6) and (5, 11).
9. A str. line thro. the pts. (0, 1) and (5, 11).
10.  $y = 2x + 3$  is a str. line thro. the pts. (0, 3) and (5, 13).
11.  $y = 4 - 3x$  is a str. line thro. the pts. (0, 4) and (-5, 19).
12.  $y = 5 - 6x$  is a str. line thro. the pts. (0, 5) and (1, -1).
13. A str. line thro. the pts. (0, 6) and (1, 8)
14. A str. line thro. the pts (0, 3) and (4, 0).
15. A str. line thro. the pts. (0, -3) and (4, 0).
16.  $y = \frac{3x-5}{6}$  is a str. line thro. the pts.  $(1, -\frac{1}{3})$  and  $(\frac{5}{3}, 0)$ .
17.  $y = \frac{5-3x}{6}$  is a str. line thro. the pts.  $(1, \frac{1}{3})$  and  $(3, -\frac{2}{3})$ .
18. A str. line thro. the pts. (5, 3) and (-3, -3).
19. A str. line thro. the pts. (2, 0) and (0, -3).
20. The first line passes thro. the pts. (0, -3), (-5, -13).  
The second line passes thro. the pts. (0, 7), (14, 0). If these  
lines are drawn it will be seen that they cut at the pt. (4, 5)  
 $\therefore x = 4, y = 5$  is the reqd. solution.
21. The first line passes thro. the pts. (19, 2), (-2, 8).  
The second line passes thro. the pts. (7, 1), (17, 7). If these  
are drawn they will be seen to intersect at the pt. (12, 4).
22. The first line passes thro. the pts. (34, 2), (16, 12).  
The second line passes thro. the pts. (1, -22), (5, 4). If these  
are drawn they will be seen to intersect at the pt. (7, 17).
23. The first line passes thro. the pts. (0, 0) and (3, 4).  
The second line passes thro. the pts (0, 21) and (21, 0). If  
these are drawn they will be seen to intersect at the pt.  
(9, 12).
24. The first line passes thro. the pts. (4, -5), (6, 9). The  
second line passes thro. the pts. (-7, 1), (17, 3). They will  
be seen to intersect at the pt. (5, 2).

**25.** The first line passes thro. the pts. (15, 0), (0, 15). The second line passes thro. the pts. (5, 0), (0, -5). They will be seen to intersect at the pt. (10, 5).

**26.** When these pts. are plotted, it will be seen that they all lie on the str. line represented by the equation  $y = 3x$ .

**27. First method.** When these pts. are plotted, it will be seen that they lie in a str. line. If the equation of this line is  $ax + by = 1$ , (0, -5) is on the line  $\therefore -5b = 1 \therefore b = -\frac{1}{5}$ , (3, 1) is on the line  $\therefore 3a + b = 1$ ,  $a = \frac{2}{5} \therefore$  the equation reqd. is  $y + 5 = 2x$ .

*Second method.* If  $(x, y)$  is any pt. on the line, it will be seen from similar  $\triangle$ s that  $\frac{y+5}{x} = 2 \therefore y + 5 = 2x$  is the reqd. equation.

**28.** When the pts. are plotted, it will be seen that they lie in a str. line. If its equation is  $ax + by = 1$ , (0, 4) satisfy the equation  $\therefore 4b = 1$ ,  $b = \frac{1}{4}$ , (2, 10) satisfy the equation  $\therefore 2a + 10b = 1$ , whence  $a = -\frac{3}{4} \therefore y - 4 = 3x$  is the reqd. equation.

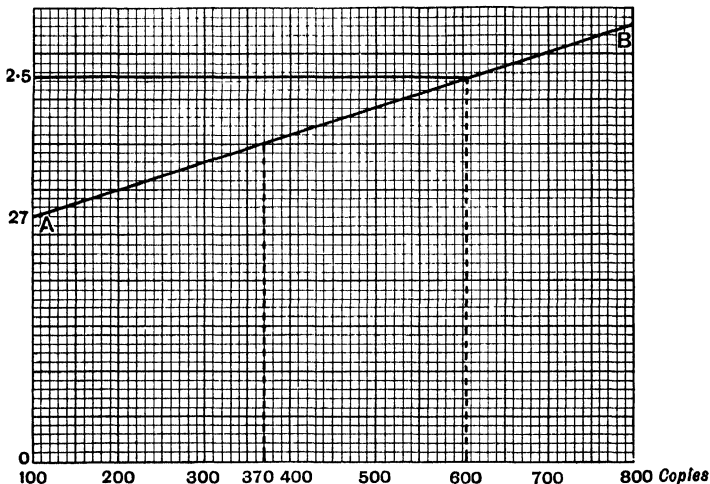
**29.** If  $x$  in. =  $y$  cms.,  $\frac{x}{10} = \frac{y}{25.4}$ . Taking an inch as unit for both  $x$  and  $y$  values, mark the pt. A whose co-ors. are (10, 25.4). Join OA. OA is the graph of  $\frac{x}{10} = \frac{y}{25.4}$ . Take the pt. P on OA whose ordinate is 5.6. Its abscissa will be found to be 2.2 nearly  $\therefore 5.6$  cms. = 2.2 in. nearly. Take the pt. Q on OA whose abscissa is 4.9. Its ordinate will be found to be 11.45 nearly  $\therefore 4.9$  in. = 11.45 cms. nearly.

**30.** If  $x$  cms. =  $y$  inches,  $\frac{x}{10} = \frac{y}{3.9}$ . Taking an inch as unit for both  $x$  and  $y$  values, mark the pt. A whose co-ors. are (10, 3.9). Join OA. OA is the graph of  $\frac{x}{10} = \frac{y}{3.9}$ . Take the pt. P on OA whose ordinate is 3.6. Estimating the second dec. place, the ordinate will be found to be 9.23  $\therefore 3.6$  in. = 9.23 cms. Take the pt. Q on OA whose abscissa is 8.6 cms. Estimating the second dec. place, the ordinate will be found to be 3.35  $\therefore 8.6$  cms. = 3.35 in.

**31.** Plot the pt. A whose co-ors. are (100, 69). Join OA. OA is the graph whose ordinates correspond to the marks on the paper of max. 69, and whose abscissae correspond to the marks on the paper of max. 100. The abscissae of the pts. whose ordinates are 60, 54, 46, 35, 32, 29, 27, 26, 25, 12 are the marks reqd. These will be found to be (to the nearest integer) 87, 78, 67, 51, 47, 42, 39, 38, 36, 17.

**32.** 50 articles cost 58 pence. Plot the pt. A whose co-ors. are (58, 50). Join OA. OA is the graph whose abscissae give the price in pence of the number of articles corresponding to its ordinates. The abscissa of the pt. whose ordinate is 23 will be found to be  $26\cdot5$   $\therefore$  23 things cost  $26\cdot5$  pence = 2s.  $2\frac{1}{2}$ d. The ordinate of the pt. whose abscissa is 36 will be found to be just over 31  $\therefore$  only 31 articles can be obtained for 3s.

**33.** On paper ruled in inches and tenths of an inch, take OA on a vertical line equal to 2·7 inches, one-tenth of an inch representing one shilling. 800 copies cost  $27 + 7 \times 3 = 48$  shillings. Taking an inch horizontally to represent 100 copies,



mark the pt. B whose abscissa is 8 in. (800 copies) and ordinate 4·8 in. (48 shillings). Join AB. The ordinates in the diagram

(which is reduced in printing) give the price in shillings of the number of copies, as shown in the abscissa line. Thus 370 copies cost 35·1s. = 35s. 1d. approx., and for £2. 2s. 6d. we get 615 copies (to the nearest five).

**34.** Taking one-tenth of an inch horizontally to represent one week, and one-tenth of an inch vertically to represent £1, plot the pt. (52, 120), A. Join OA. The ordinates of pts. on OA give the wages corresponding to the number of weeks represented by the abscissae. The ordinate corresponding to the abscissa 23 will be found to be 53 approx.  $\therefore$  the clerk's wages for 23 weeks = £53.

**35.** Take the ordinate AM of any pt. A on OP, then if PN is the ordinate of P,  $\frac{AM}{OM} = \frac{PN}{ON} = \frac{8\ 66}{10} = \cdot 866 \therefore AM = 0\cdot 866$  of OM, the dist. of A from OY. The ordinate of the pt. whose abscissa is 3 is 2·60  $\therefore 0\cdot 866$  of 3 = 2·60. The ordinate of the pt. whose abscissa is 6·5 is 5·63  $\therefore 0\cdot 866$  of 6·5 = 5·63. The ordinate of the pt. whose abscissa is 4·8 is 4·16  $\therefore 0\cdot 866$  of 4·8 = 4·16. To find  $\frac{1}{0\cdot 866}$  of 5, we must read off the abscissa of the pt. whose ordinate is 5. For if Q is that pt. and QK is perp. to OY,  $\frac{QK}{OK} = \frac{ON}{PN}$ ; i.e.  $\frac{QK}{5} = \frac{10}{8\ 66} \therefore QK = \frac{1}{0\cdot 866}$  of 5. Its value will be found to be 5·77, estimating the second dec. place.

**36.** If  $y$ £ is the cost of  $x$  copies,  $y = \frac{x}{10} + 100$ .  $\frac{x}{10} + 100$  is the reqd. expression. When  $x = 0$ ,  $y = 100$ ; plot the pt. (0, 1 in.), for 1 in. = £100. When  $x = 5000$ ,  $y = 600$ ; plot the pt. (5 in., 6 in.), for 5000 copies are represented by 5 inches, and £600 = 6 inches. Join these points by a str. line. The ordinate of any pt. on it gives the price of the no. of copies represented by the corresponding abscissa. 2500 copies = 2·5 inches. The ordinate whose abscissa is 2·5 in. is found to be 3·5 inches =  $3\cdot 5 \times 100 = \text{£}350 \therefore$  2500 copies cost £350. £525 = 5·25 in. The abscissa of the pt. whose ordinate is 5·25 in. is found to be 4·25 in. =  $4\cdot 25 \times 1000 = 4250$  copies  $\therefore$  4250 copies can be obtained for £525.

**37.** Writing  $x$  instead of  $t$ , we have to draw the graph of  $y = 4 + 3x$ . The co-ors. of any pt. on this line give us corre-

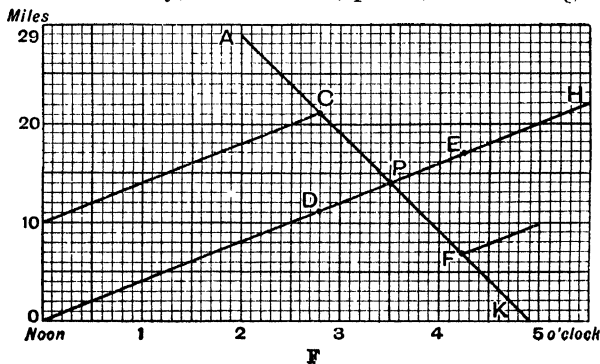
sponding times and velocities. The line passes thro. the pts. (0, 4), (5, 19). Read off the ordinate of the pt. whose abscissa is 3. This is 13. Read off the ordinate of the pt. whose abscissa is 4.5. This is 17.5. Read off the abscissa of the pt. whose ordinate is 11.5. This is 2.5  $\therefore$  13 and 17.5 ft. per sec. are the velocities reqd and 2.5 secs. the time reqd.

38. If  $y$  kilogrammes =  $x$  lbs.,  $\frac{y}{1} = \frac{x}{2.2}$  or  $\frac{y}{5} = \frac{x}{11}$ . Drawing the graph of this equation [a str. line thro. the origin, and thro. the pt. (11, 5)], its ordinates and abscissae give us corresponding numbers of kilogrammes and lbs. From the graph, when  $y = 25$ ,  $x = 55$   $\therefore$  25 kilogrammes = 55 lbs. Similarly, 38 kilogrammes = 84 lbs nearly, 32.5 lbs. = 14.8 kilogrammes, and 38 lbs. = 17.3 kilogrammes.

39. As in the preceding, if  $x$  c. ins. =  $y$  c. cms. we must draw the graph of  $\frac{x}{10} = \frac{y}{164}$  or  $\frac{x}{5} = \frac{y}{82}$ . The co-ors. of pts. on this line give us corresponding numbers of c. ins. and c. cms. 80 c. cms. = 49 c. in. nearly, 40 c. cms. = 24.5 c. in. nearly, 2.5 c. ins. = 41 c. cms.

40. If  $x^\circ$  Reaumur =  $y^\circ$  Fahr.  $\frac{x}{80} = \frac{y-32}{212-32}$ , i.e.  $9x = 4y - 128$ . The graph of this equation is the graph reqd. (see Art. 10, p. 212).  $60^\circ$  R. =  $167^\circ$  F.,  $43^\circ$  F. =  $5^\circ$  R. nearly.

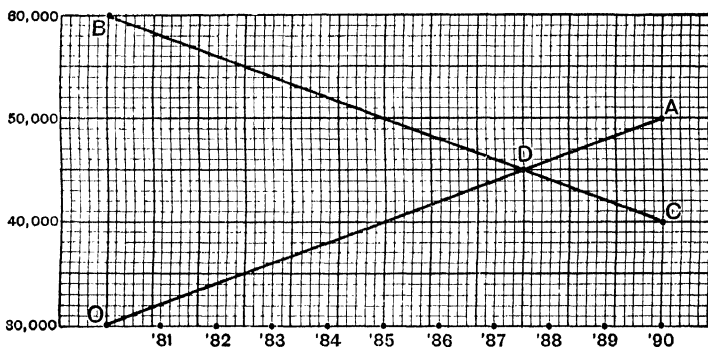
41. Taking 10 units to an hour horizontally, and one unit to a mile vertically, as in Art. 11, p. 214, OH is the graph of



the walker, and AK the graph of the rider. They meet at P, in 3.5 hrs. after noon, viz. at 3.30 p.m. They are 10 m. apart when they are at D and C respectively, i.e. in 2.8 hrs. after noon, or at 2.48 p.m. They are also 10 m. apart when they are at E and F respectively, in 4.2 hrs. after noon, i.e. at 4.12 p.m.

**42.** Plot the pt. C whose co-ors. are (15, 100). Join OA. OA is the graph of A's motion, the  $x$  values denoting seconds, the  $y$  values yards. In OX take OD equal to 3 units. Take also a pt. such that its vertical distance from D is 100, and its horizontal distance from D 12 secs. This will be seen to be the pt. C. Join DC. DC is the graph of B's motion. We thus see that B overtakes A at C, i.e. in 15 secs. from A's start and 100 yds. from the starting-point.

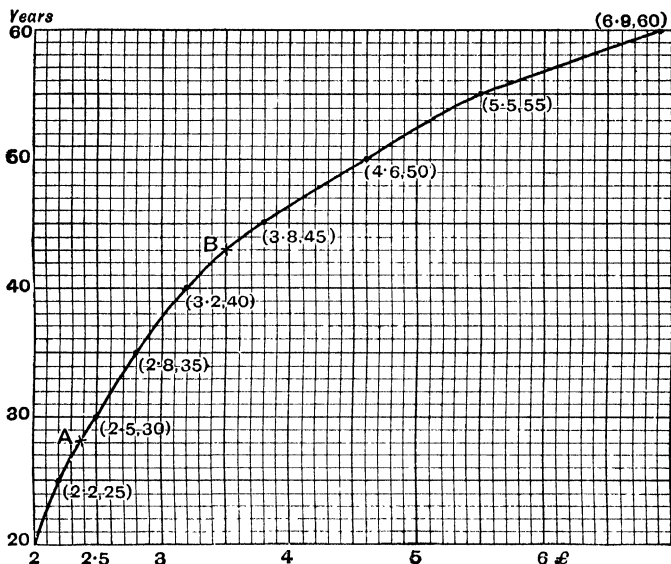
**43.** Measuring the years along horizontal lines, 6 units to a year, and the populations along vertical lines, 10 units to 10,000, OA in the diagram is the graph showing the growth of



popn. in the first town, and BC the graph showing the popn. in the second town. At D, where OA and BC meet, the popns. are equal, i.e. at the end of June, '88.

**44.** Join the pts. (0, 33), (100, 88). This line is the graph reqd. The abscissae give the scaled marks, the unreduced marks being shown by the ordinates. The scaled marks reqd. are 58, 38, 29.

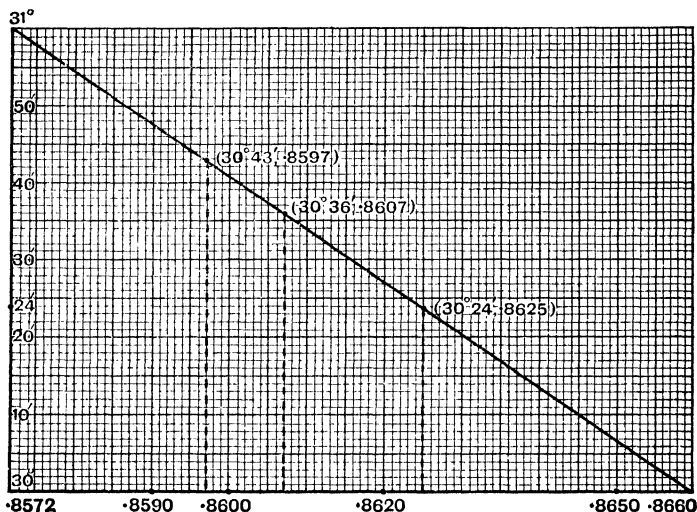
45. The diagram shows the pts.  $(2, 20)$ ,  $(2.2, 25)$   $(2.5, 30)$  ... Joining these by an even curve, we have the graph reqd. The abscissae of the pts., whose ordinates are 28 and 43, give the premiums reqd. £24, £35 to the nearest pound.



46. Take an inch (or a centimetre) horizontally to represent  $\cdot 002$ , and an inch vertically to represent  $\cdot 1$ . Plot the pts.  $(17, 1.2304)$ ,  $(18, 1.2553)$  and join them. This line is the graph reqd. [N.B. The pt.  $(18, 1.2553)$  lies 10 inches vertically above  $(17, 1.2304)$  and  $\frac{2.4}{2} = 12.45$  inches horizontally from it.] From the graph we see that the abscissa 1.2395 corresponds to the ordinate 17.36  $\therefore \log 17.36 = 1.2395$ . In the same way  $\log 17.68 = 1.2474$ . Also the ordinate corresponding to the abscissa 1.2350 is 17.18  $\therefore 17.18$  is the number whose log is 1.2350. In the same way 17.82 is the number whose log is 1.2508. [The above results are not absolutely true, for the intermediate logs are only approximately proportional to the difference in the numbers.]

47. Measure sines horizontally and degrees vertically. Take one inch horizontally to represent  $\cdot 001$ , and one inch vertically to represent 10 minutes. Plot the pts.  $(50^\circ, \cdot 7660)$ ,  $(51^\circ, \cdot 7771)$  and join them. [*N.B.* The second pt. lies 6 in. vertically above the first, and 11.1 inches horizontally from it.] The line joining these two pts. is the graph reqd. From it we read off  $\sin 50^\circ 15' = \cdot 7688$ ,  $\sin 50^\circ 48' = \cdot 7749$ ,  $\cdot 7683 = \sin 50^\circ 12'$ , and  $\cdot 7729 = \sin 50^\circ 37'$ .

48. As the angle increases the cosine diminishes. The diagram now explains itself.



49. Plot the points  $(22^\circ, \cdot 4040)$ ,  $(23^\circ, \cdot 4245)$  with the same units as in Example 47, and read off the reqd. values.  $\tan 22^\circ 44' = \cdot 4190$ ,  $\tan 22^\circ 54' = \cdot 4224$ ,  $\cdot 4122 = \tan 22^\circ 24'$ ,  $\cdot 4204 = \tan 22^\circ 48'$ .

50. Use half an inch horizontally to represent one million, and half an inch vertically to represent 10 years. Plot the pts.  $(8.9, 1801)$ ,  $(10.2, 1811)$ , etc. Join these point to point. It will be seen that the abscissa corresponding to 1837 is 15.1  $\therefore$  15.1 millions was the popn. in 1837. The year corresponding to the abscissa 24 is 1875.

**51.** Use paper ruled in cms. and mms., and take one cm. horizontally to denote a year, and one cm. vertically to denote  $\cdot 1$  of a million £. For the first graph plot the pts. (1884, 2·51), (1885, 2·47), etc. For the second graph plot the pts. (1884, 1·29), (1885, 1·35), etc. It will be seen from the two papers that, usually, as the total expenditure diminishes the salaries of officials increase.

**52.** Use paper ruled in inches and tenths of inches; take half an inch horizontally to denote a month, and one inch vertically to denote an inch of rainfall. Plot the pts. estimating the second dec. place.

**53.** Using paper ruled in inches and tenths of an inch (or in cms. and mms.), take one-tenth horizontally to denote a month, and one-tenth vertically to denote a penny. Plot the pts. (1891, 45), (1892, 40), etc., and join them by an even curve. This is the reqd. graph. Price of silver on May 1st, 1895, 30·4 pence.

**54.** Take one-tenth of an inch horizontally to represent one foot, and one inch vertically to represent one second. Plot the pts. (2, 1), (6, 2), (12, 3), (20, 4), and so on. [*N.B.* The total space in 4 secs. =  $2 + 4 + 6 + 8 = 20$ .] Join the pts. by an even curve and we have the graph. The abscissa corresponding to the 4·2 ordinate is 22 nearly  $\therefore$  the body describes 22 ft. in 4·2 secs. In the same way we see that the body describes 62 ft. (nearly) in 7·4 secs. The ordinate corresponding to the abscissa 15 is 3·4  $\therefore$  the body takes 3·4 secs. to describe 15 feet.

**55.** Take one inch horizontally to represent one inch of the string, and one inch vertically to represent one pound. Plot the pts. (7·7, ·6), (8·0, 1·2), (8·4, 2·0), (8·8, 2·8), (9·0, 3·2). When we examine these we see they lie in a straight line. This line is the graph reqd. The ordinate corresponding to the abscissa 10 is 5·2  $\therefore$  5·2 lbs. will stretch the string to 10 inches. The abscissa corresponding to the ordinate 2·25 is 8·5  $\therefore$  when the wt. is 2·25 lbs. the stretched length is 8·5 inches. The unstretched length is 7·4 in.

**56.** Use paper ruled in cms. and mms.; take 1 mm. horizontally to represent one degree, and 1 mm. vertically to represent  $\cdot 01$  of a radian. Plot the pts. (0, 0), (15, ·26), etc. Join them.

The graph will be a str. line. From it we see that 40 degrees =  $\cdot 7$  radians, 70 degrees =  $1\cdot 22$  radians,  $\cdot 64$  radians =  $37^\circ$ , and  $\cdot 86$  radians =  $49^\circ$ .

**57.** Take one-tenth of an inch horizontally to denote  $10^\circ$ , and one-tenth of an inch vertically to represent  $\cdot 1$ . Estimating carefully the second dec. place, plot the pts.  $(0, 0)$ ,  $(15, \cdot 26)$ ,  $(30, \cdot 5)$ , etc. To obtain the graph from  $90^\circ$  to  $180^\circ$ , use the facts that  $\sin 120^\circ = \sin 60^\circ$ ,  $\sin 135^\circ = \sin 45^\circ$ ,  $\sin 150^\circ = \sin 30^\circ$ ,  $\sin 180^\circ = 0$ . The graph from  $0^\circ$  to  $-180^\circ$  may be similarly obtained, remembering that  $\sin(-30^\circ) = -\sin 30^\circ$ , and so on.

**58.** Use the same units as in Example 57, and we obtain the graph in a similar manner.

**59.** With the same units as in the preceding two examples plot the points. We shall see that the vertical line through the  $90^\circ$  pt. is an asymptote to the graph (see Art. 22.).

### EXERCISES XLIV. a.

1. A circle, centre at the origin, rad. 6 units (Art. 12).
2. A point. The origin.
3. A circle, centre at the origin, rad. 7 units (Art. 12).
4. A circle, centre at the origin, rad. 9 units (Art. 12).
5. The equation may be written  $(x+4)^2 + (y-4)^2 = 32$   
 $\therefore$  the graph is a circle, centre at the pt.  $(-4, 4)$ , rad.  $4\sqrt{2}$   
 (Art. 14).
6. The equation may be written  $(x-4)^2 + (y-3)^2 = 25$   
 $\therefore$  the graph is a circle, centre at the pt.  $(4, 3)$ , rad. 5  
 (Art. 14).
7. A circle, centre at  $(3, 4)$ , rad. 6 (Art. 14).
8. A circle, centre at  $(1, 2)$ , rad. 6 (Art. 14).
9. A circle, centre at  $(-2, 3)$ , rad. 5 (Art. 14).
10. A circle, centre  $(3, -3)$ , rad. 4 (Art. 14).
11.  $y = \sqrt{15 - 2x - x^2} \therefore y^2 = 15 - 2x - x^2$ ;  $x^2 + 2x + y^2 = 15$ ;  
 $(x+1)^2 + y^2 = 16 \therefore$  the graph is a circle, centre at  $(-1, 0)$ ,  
 rad. 4 (Art. 14).

12.  $y = \sqrt{21 + 4x - x^2}$ ;  $y^2 = 21 + 4x - x^2$ ;  $x^2 - 4x + y^2 = 21$ ;  $(x-2)^2 + y^2 = 25$   $\therefore$  the graph is a circle, centre (2, 0), rad. 5 (Art. 14).

13.  $y = \sqrt{15 + 2x - x^2}$ ;  $y^2 = 15 + 2x - x^2$ ;  $x^2 - 2x + y^2 = 15$ ;  $(x-1)^2 + y^2 = 16$   $\therefore$  the graph is a circle, centre at (1, 0), rad. 4 (Art. 14).

14.  $y = \sqrt{14x - x^2 - 13}$ ;  $y^2 = 14x - x^2 - 13$ ;  $x^2 - 14x + y^2 = -13$ ;  $(x-7)^2 + y^2 = 36$   $\therefore$  the graph is a circle, centre at (7, 0), rad. 6 (Art. 14).

15. The graph of  $x^2 + y^2 = 36$  is a circle, centre at the origin, rad. 6 (Art. 12).  $x^2 + y^2 - 8x - 20 = 0$  may be written  $(x-4)^2 + y^2 = 36$   $\therefore$  its graph is a circle, centre at (4, 0), rad. 6 (Art. 14). Drawing these circles, they will be seen to meet at the pts (2, 5.66), (2, -5.66)  $\therefore x=2$ ,  $y=5.66$ , and  $x=2$ ,  $y=-5.66$  are the reqd. solutions. [Half an inch or one inch should be taken as the unit.]

16. When

|       |         |         |         |         |     |
|-------|---------|---------|---------|---------|-----|
| $y=0$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | ... |
| $x=0$ | 2       | 4.5     | 8       | 12.5    | ... |

Points on the graph are given by the above table. Joining them, we have a parabola (see Art. 16).

17.  $4x = y^2 + 8$ . When

|       |         |         |         |         |     |
|-------|---------|---------|---------|---------|-----|
| $y=0$ | $\pm 1$ | $\pm 2$ | $\pm 4$ | $\pm 6$ | ... |
| $x=2$ | 2.25    | 3       | 6       | 11      | ... |

Points on the graph are given by the above table. Joining them, we have a parabola.

18.  $y = 2\sqrt{x-4}$ ,  $y^2 = 4(x-4)$ ,  $4x = y^2 + 16$ . When

|       |         |         |         |         |     |
|-------|---------|---------|---------|---------|-----|
| $y=0$ | $\pm 1$ | $\pm 2$ | $\pm 4$ | $\pm 8$ | ... |
| $x=4$ | 4.25    | 5       | 8       | 20      | ... |

Points on the graph are given by the above table. Joining them we have a parabola.

19.  $y = 4\sqrt{x+4}$ ,  $y^2 = 16(x+4)$ ,  $16x = y^2 - 64$ ,  $x = \frac{y^2 - 64}{16}$ .

When

|          |         |         |         |         |          |     |
|----------|---------|---------|---------|---------|----------|-----|
| $y=0$    | $\pm 2$ | $\pm 4$ | $\pm 6$ | $\pm 8$ | $\pm 10$ | ... |
| $x = -4$ | $-3.75$ | $-3$    | $-1.75$ | $0$     | $2.25$   | ... |

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit for the  $x$  values.]

20. When

|           |     |       |     |        |     |     |
|-----------|-----|-------|-----|--------|-----|-----|
| $y=0$     | $1$ | $2$   | $3$ | $4$    | $5$ | ..  |
| $x = .25$ | $0$ | $.25$ | $1$ | $2.25$ | $4$ | ... |

When

|          |        |      |        |      |     |
|----------|--------|------|--------|------|-----|
| $y = -1$ | $-2$   | $-3$ | $-4$   | $-5$ | ... |
| $x = 1$  | $2.25$ | $4$  | $6.25$ | $9$  | ... |

Points on the graph are given by the above tables. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit for the  $x$  values.]

21. When

|       |         |         |         |         |         |     |
|-------|---------|---------|---------|---------|---------|-----|
| $x=0$ | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | ... |
| $y=0$ | $.25$   | $1$     | $2.25$  | $4$     | $6.25$  | ... |

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit.]

22. When

|       |         |         |         |         |         |     |
|-------|---------|---------|---------|---------|---------|-----|
| $x=0$ | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | ... |
| $y=2$ | $2.25$  | $3$     | $4.25$  | $6$     | $8.25$  | ... |

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit.]

**23** When

|       |         |         |         |         |         |     |
|-------|---------|---------|---------|---------|---------|-----|
| $y=0$ | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | ... |
| $x=0$ | $-.25$  | $-1$    | $-2.25$ | $-4$    | $-6.25$ | ... |

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch will be a suitable unit.]

**24.** When

|       |         |         |         |         |         |     |
|-------|---------|---------|---------|---------|---------|-----|
| $y=0$ | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | ... |
| $x=2$ | $1.75$  | $1$     | $-.25$  | $-2$    | $-4.25$ | ... |

Points on the graph are given by the above table. Joining them, we have a parabola. [Four tenths of an inch will be a suitable unit.]

**25.** When

|      |     |         |         |         |         |         |         |     |
|------|-----|---------|---------|---------|---------|---------|---------|-----|
| $x=$ | $0$ | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 4$ | $\pm 5$ | $\pm 6$ | ... |
| $y=$ | $2$ | $1.75$  | $1$     | $-.25$  | $-2$    | $-4.25$ | $-7$    | ..  |

Points on the graph are given by the above table. Joining them, we have a parabola. [Four-tenths of an inch is a suitable unit.]

**26.**  $4y=(x-1)^2$ . When

|         |     |       |     |        |     |        |     |
|---------|-----|-------|-----|--------|-----|--------|-----|
| $x=0$   | $1$ | $2$   | $3$ | $4$    | $5$ | $6$    | ... |
| $y=.25$ | $0$ | $.25$ | $1$ | $2.25$ | $4$ | $6.25$ | ... |

When

|        |        |      |        |      |     |
|--------|--------|------|--------|------|-----|
| $x=-1$ | $-2$   | $-3$ | $-4$   | $-5$ | ... |
| $y=1$  | $2.25$ | $4$  | $6.25$ | $9$  | ... |

Points on the graphs are given by the above tables. Joining them, we have a parabola.

**27.** When

|             |   |    |     |     |     |     |     |
|-------------|---|----|-----|-----|-----|-----|-----|
| $x =$       | 0 | 1  | 2   | 3   | 4   | 5   | 6   |
| $x^2 =$     | 0 | 1  | 4   | 9   | 16  | 25  | 36  |
| $-6x + 1 =$ | 1 | -5 | -11 | -17 | -23 | -29 | -35 |
| $y =$       | 1 | -4 | -7  | -8  | -7  | -4  | 1   |

Plot the pts. (0, 1), (1, -4), (2, -7), (3, -8), (4, -7), (5, -4), (6, 1), and join them by an even curve.  $x^2 - 6x + 1 = 0$  when  $y = 0$ , *i.e.* when the graph cuts the axis of  $x$ . From the graph we see that at these pts.  $x = 5.8$ , or  $\cdot 2 \therefore 5.8$  and  $\cdot 2$  are approximate roots of the equation. Also we see that -8 is the least value of  $y \therefore$  the minimum value of  $x^2 - 6x + 1$  is -8. [Use an inch for the  $x$  unit, and half an inch for the  $y$  unit.]

**28.** When

|                        |    |    |     |     |    |    |
|------------------------|----|----|-----|-----|----|----|
| $x =$                  | -2 | -1 | 0   | 1   | 2  | 3  |
| $4x^2 =$               | 16 | 4  | 0   | 4   | 16 | 36 |
| $4x + 15 =$            | 7  | 11 | 15  | 19  | 23 | 27 |
| $y = 4x^2 - 4x - 15 =$ | 9  | -7 | -15 | -15 | -7 | 9  |

Plot the pts. (-2, 9), (-1, -7), (0, -15), (1, -15), (2, -7), (3, 9). Join them by an even curve. This gives the reqd. portion of the graph. The roots of  $4x^2 - 4x - 15 = 0$  are given by the  $x$  values when the graph cuts the axis of  $x$ . From the graph we see that these are -1.5, 2.5. [Use an inch as the  $x$  unit, and one-tenth of an inch as the  $y$  unit.] From the symmetry of the graph we see that  $y$  is a minimum when  $x = .5$ , *i.e.* when  $y = 4(\cdot 5)^2 - 4 \times \cdot 5 - 15 = -16$ .

**29.** The graph of  $x^2 + y^2 = 25$  is a circle, rad. 5. Describe it, using half-inch units. (0, -5), (4, 7) are points on the graph

of  $y = 3x - 5$ . Join them by a str. line. This is the graph of  $y = 3x - 5$ . We see that the str. line and circle cut at the pts.  $(0, -5)$ ,  $(3, 4)$ .  $\therefore x = 0$  or  $3$ ,  $y = -5$  or  $4$  are the roots reqd.

**30.** Use a centimetre as the  $x$  unit, and half a centimetre as the  $y$  unit. Trace the graph of  $y = x^2$  (Art. 16), and the graph of  $8y - 10x - 75 = 0$ . [ $(-7.5, 0)$ ,  $(-3.5, 5)$  are pts. on this line.] The abscissae of the pts. where the graphs meet give the roots of the equation (Art. 18). These will be found to be  $-2.5$  and  $3.75$ .

**31.**  $x^2 - 6x + 5 = 0$ , i.e.  $(x - 1)(x - 5) = 0$ ;  $x - 1 = 0$ , or  $x - 5 = 0$   $\therefore$  the graph is two str. lines  $\parallel$  to  $OY$  and distant 1 and 5 units from it on the positive side.

**32.**  $y^2 + 5y + 6 = 0$ ;  $(y + 2)(y + 3) = 0$ ;  $y + 2 = 0$ , or  $y + 3 = 0$   $\therefore$  the graph is two str. lines  $\parallel$  to  $OX$  and distant 2 and 3 units from it on the negative side.

**33.**  $x^2 + x - 6 = 0$ ;  $(x + 3)(x - 2) = 0$ ;  $x + 3 = 0$ , or  $x - 2 = 0$   $\therefore$  the graph is two str. lines  $\parallel$  to  $OY$ ; the first at a distance 3 units from it on the negative side, the other at a distance 2 units from it on the positive side.

**34.**  $y^2 - 3y - 28 = 0$ ;  $(y - 7)(y + 4) = 0$ ;  $y - 7 = 0$ , or  $y + 4 = 0$   $\therefore$  the graph is two str. lines  $\parallel$  to  $OX$ ; the first at a distance 7 units from it on the positive side, the other at a distance 4 units from it on the negative side.

**35.** The given equation may be written  $(x + 2y)(x + 3y) = 0$   $\therefore x + 2y = 0$ , or  $x + 3y = 0$ ; i.e. the graph is two str. lines through the origin (Art. 15). The first passes thro. the pt.  $(6, -3)$ . The second passes thro. the pt.  $(6, -2)$ .

**36.** The given equation may be written  $(2x + y)^2 = 0$   $\therefore$  the graph is two *coincident* str. lines, each represented by the equation  $2x + y = 0$ . This line is thro. the origin and thro. the pt.  $(4, -8)$ .

**37.** Draw the graph of  $y = x^2$ , using an inch as the  $x$  unit, one-tenth of an inch as the  $y$  unit (Art. 16). With the same units draw the graph of  $y = 3x + 6$ . [It passes thro.  $(0, 6)$  and  $(-2, 0)$ ]. The abscissae of the two pts. where these graphs meet give us the roots reqd. They are seen to be  $4.4$  and  $-1.4$

**38.** Draw the graphs of  $y = x^2$  and  $y + 4x = 8$  with the same units as in the preceding example. The roots, given by the abscissae of the pts. where these meet, will be seen to be  $-1.46$  and  $-5.46$ .

**39.** With the same units, draw the graph of  $y = x^2$ , and the graph of  $y - x - 20 = 0$ . [ $(-6, 14)$ ,  $(0, 20)$  are pts. on the str. line.] The abscissae of the pts. where these graphs meet give the reqd. roots. They will be found to be  $-4$  and  $5$ . From the graph we see that  $x^2 - x - 20$  is negative as long as the pt. whose abscissa in  $x$  lies between the pts. where the graphs meet, *i.e.* as long as  $x$  is between  $-4$  and  $5$  (Art. 18).

**40.** Use one centimetre for  $x$  and  $y$  units, and draw the graphs of  $x^2 + y^2 = 25$  and  $x - 2y + 2 = 0$ . The pts. where these graphs meet will be found to be  $(4, 3)$  and  $(-4.8, -1.4)$ .  $\therefore x = 4$  or  $-4.8$ ,  $y = 3$  or  $-1.4$  are the reqd. roots.

**41.** Use one inch for the  $x$  unit, and one-tenth of an inch for the  $y$  unit. Trace the graphs of  $y = x^2$  and  $x + 6 - 2y = 0$ . The abscissae of the pts. where these graphs meet give the reqd. roots. They will be seen to be  $2$  and  $-1.5$ . As in Art. 16, we see that the expression  $x + 6 - 2x^2$  is positive as long as  $x$  lies between  $2$  and  $-1.5$ .

**42.** Trace the graph of  $y = 2x^2 - x - 6$ . When

|                 |    |   |    |    |
|-----------------|----|---|----|----|
| $x=0$           | 1  | 2 | 3  | 4  |
| $2x^2=0$        | 2  | 8 | 18 | 32 |
| $x+6=6$         | 7  | 8 | 9  | 10 |
| $y=2x^2-x-6=-6$ | -5 | 0 | 9  | 22 |

When

|               |    |    |    |
|---------------|----|----|----|
| $x=$          | -1 | -2 | -3 |
| $2x^2=$       | 2  | 8  | 18 |
| $x+6=$        | 5  | 4  | 3  |
| $y=2x^2-x-6=$ | -3 | 4  | 15 |

Plotting the points  $(0, -6)$ ,  $(1, -5)$ ,  $(2, 0)$ ,  $(3, 9)$ ,  $(4, 32)$ ,  $(-1, -3)$ ,  $(-2, 4)$ ,  $(-3, 15)$ , and joining them, we have the graph. From the figure we see that the least value of  $y$ , i.e. of  $2x^2 - x - 6$ , is  $-6.1$  approx.

**43.** Use the formula  $s = ut + \frac{1}{2}ft^2$ . In 1 second the first particle rises  $128 - 16 = 112$  ft. In 2 seconds the first particle rises  $256 - 64 = 192$  ft. In 3 seconds the first particle rises  $384 - 144 = 240$  ft. In 4 seconds the first particle rises  $512 - 256 = 256$  ft. In 1 second the second particle falls 16 ft. In 2 seconds the second particle falls  $16 \times 4 = 64$  ft. In 3 seconds the second particle falls  $16 \times 9 = 144$  ft. In 4 seconds the second particle falls  $16 \times 16 = 256$  ft. Take A as the starting pt. of the first, and B, 256 units vertically above it, as the starting pt. of the second particle. Measuring the seconds of time horizontally from A and B, and using a fairly large time unit, say 2 inches, plot the pts.  $(1, 112)$ ,  $(2, 192)$ ,  $(3, 240)$ ,  $(4, 256)$  for the first particle, and join them by an even curve. The ordinates must be measured upwards from A to B. Plot the pts.  $(1, 16)$ ,  $(2, 64)$ ,  $(3, 144)$ ,  $(4, 256)$  for the second particle, measuring the times horizontally, and the distances *vertically downwards* from B. The time of the pt. where these graphs meet gives us the time of the meeting of the particles. It will be found to be 2 secs. To find when they are 160 ft. apart, mark off a length of 160 units on a straight edge of paper, and move it  $\parallel$  to AB until the vertical dist. between pts. on the graphs is equal to this marked distance. Read off the time from the graph. It will be found to be  $\frac{3}{4}$  sec. or  $3\frac{1}{4}$  sec.

**44.** Use the formula  $v^2 = 2gs$ . When

|       |    |    |    |    |    |    |    |    |     |     |     |
|-------|----|----|----|----|----|----|----|----|-----|-----|-----|
| $v=8$ | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80  | 88  | 96  |
| $s=1$ | 4  | 9  | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 |

Use one mm for both  $v$  and  $s$  units. Measure  $s$  vertically downwards and  $v$  horizontally. Plot the pts.  $(8, 1)$ ,  $(16, 4)$ ,  $(24, 9)$ , etc., and join them by an even curve. This is the graph reqd. When  $s=124$ , we see that  $v=89$  approx.  $\therefore$  the velocity of the body when it has fallen 124 ft. is 89 ft. per sec. approx.

**45.** Use the formula  $s = \frac{1}{2}ft^2$  for the motion of the second particle. When

|       |   |    |    |    |     |     |     |       |
|-------|---|----|----|----|-----|-----|-----|-------|
| $t =$ | 1 | 2  | 3  | 4  | 5   | 6   | ... | secs. |
| $s =$ | 4 | 16 | 36 | 64 | 100 | 144 | ... | feet. |

Plot the pts. (1, 4), (2, 16), etc., measuring times horizontally with an inch unit, and spaces vertically upwards with one-tenth of an inch as unit. Join them, and we have the graph of the second particle. Take a point 48 units vertically above the starting pt. of the second particle, and measuring lines horizontally as before, and spaces vertically *downwards*, draw the graph of the first particle. This is a str. line, thro. the pts (1, 4), (2, 8), (3, 12), etc. The time given by the point where the graphs meet gives us the time of meeting. It will be seen to be 3 secs. To find when they are 33 ft. apart, thro. a pt. 33 units vertically below the starting pt. of the first particle, draw a str. line  $\parallel$  to its graph. The line of the pt. where this meets the graph of the second particle gives us the time reqd. It will be found to be 1.5 secs.

### EXERCISES XLIV. b.

**1.** Beginning at the bottom of the sheet from the left, mark the inches horizontally 5, 5.2, 5.4, etc., and vertically 25, 27, 29, etc. Mark the points (5, 25), (6, 36). Mark also (5.2, 5.2<sup>2</sup>), (5.4, 5.4<sup>2</sup>), (5.6, 5.6<sup>2</sup>), *i.e.* (5.2, 27.04), (5.4, 29.16), (5.6, 31.36). Connect these points by a smooth curve, and read off the abscissa corresponding to the ordinate whose square root is wanted. Thus  $\sqrt{31} = 5.57$ ,  $\sqrt{28} = 5.29$ ,  $\sqrt{29.6} = 5.44$ ,  $\sqrt{31.3} = 5.6$ .

**2.** Mark the inches horizontally 5, 5.2, 5.4, etc., and vertically 125, 135, 145, etc. Mark the points (5, 125), (6, 216). Mark also (5.4, 5.4<sup>3</sup>), (5.6, 5.6<sup>3</sup>), *i.e.* (5.4, 157.46), (5.6, 175.6). Connect these points by a smooth curve. Read off the abscissae corresponding to ordinates 144, 198. Thus  $\sqrt[3]{144} = 5.24$ ,  $\sqrt[3]{198} = 5.83$ .

3. For the graph of  $y = x^3$  we have

|       |             |   |       |
|-------|-------------|---|-------|
| $x=0$ | $\cdot 5$   | 1 | 1·5   |
| $y=0$ | $\cdot 125$ | 1 | 3·375 |

Take 2 inches for the unit, and, by means of the values given above, draw the graph from (0, 0) to (1·5, 3·375). The graph of  $y = 2x - 1$  is a str. line passing through (2, 3) and ( $\cdot 5$ , 0). The values of  $x$  at the intersections are  $\cdot 62$  and 1. These are the positive values of  $x$  which make  $x^3$  equal to  $2x - 1$ , i.e. they are positive solutions of  $x^3 - 2x + 1 = 0$ .

4. Mark at the inches horizontally 6, 6·2, 6·4, etc., and vertically 38, 40, 42, etc., beginning from an intersection near the bottom left-hand corner. Mark the points (6, 36), (7, 49). Mark also (6·3, 6·3<sup>2</sup>), (6·6, 6·6<sup>2</sup>), i.e. (6·3, 39·69), (6·6, 43·56). Connect these points by a smooth curve and read off the abscissa corresponding to any ordinate whose sq. rt. is required. Thus  $\sqrt{39\cdot 4} = 6\cdot 28$ , and  $\sqrt{46\cdot 7} = 6\cdot 83$ .

5. It passes through (1, 1), (2,  $\frac{1}{2}$ ), (3,  $\frac{1}{3}$ ), etc., ( $\frac{1}{2}$ , 2), ( $\frac{1}{3}$ , 3), etc., (-1, -1), (-2,  $-\frac{1}{2}$ ), (-3,  $-\frac{1}{3}$ ), etc., ( $-\frac{1}{2}$ , -2), ( $-\frac{1}{3}$ , -3), etc. A rectangular hyperbola with the axes of  $x$  and  $y$  for asymptotes. (See § 22.)

6. A rectangular hyperbola passing through the points (2, 2), (4, 1), (5,  $\cdot 8$ ), etc., (1, 4), ( $\cdot 8$ , 5), etc., (-2, -2), (-1, -4), (- $\cdot 8$ , -5), etc., (-4, -1), (-5, - $\cdot 8$ ), etc. (See § 22.)

7. and 8. Determine points as in Example 6. A rectangular hyperbola in each case.

9. When  $y = 0, \pm 1, \pm 2, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8$ ;  $x = \pm 1, \pm \cdot 99, \pm \cdot 97, \pm \cdot 87, \pm \cdot 78, \pm \cdot 66, \pm \cdot 48, 0$ . Ellipse, centre (0, 0), semi-axes along the axes of  $x, y$ , and equal to 1 and 8.

10. Ellipse, centre (0, 0), semi-axes along the axes of  $x, y$ , and equal to 5 and 1.

11. When  $y = 0, \pm 2, \pm 4, \pm 6, \pm 8$ , etc.,  $x = \pm 8, \pm 8\cdot 2, \pm 8\cdot 9, \pm 10, \pm 11\cdot 3$ , etc. A rectangular hyperbola, centre (0, 0).

12. As Question 11.

**13.** When  $x = \pm 1, \pm 1.5, \pm 2$ , etc.,  $y = 0, \pm 8.9, \pm 13.9$ , etc. Hyperbola, centre (0, 0).

**14.**  $\frac{x^2}{9} + \frac{y^2}{16} = 1$ . Ellipse, centre (0, 0), semi-axes 3 along the axis of  $x$ , 4 along the axis of  $y$ .

**15.** Hyperbola

**16.** Hyperbola } as in 13.

**17.**  $6x^2 - 5xy + y^2 = 0$ , i.e.  $(3x - y)(2x - y) = 0$ . The equation is satisfied by every point on the str. line  $3x - y = 0$ , and every point on the str. line  $2x - y = 0$   $\therefore$  the graph is 2 str. lines passing through (0, 0), one of them going through (1, 3), the other through (1, 2).

**18.** The two str lines  $2y + x = 0$ ,  $2y - x = 0$ .

**19.** The equation is satisfied by every point along the str. line  $x = 0$ , and every pt. along the str. line  $y = 0$ : i.e. the graph is the two axes.

**20.**  $x^2 + y^2 = 0$ . This is not satisfied by any point except (0, 0)  $\therefore$  it represents the origin. In fact it is a circle whose centre is (0, 0) and radius zero.

**21.** Two str. lines, one  $\parallel$  to the axis of  $y$  at a distance 3 from it, the other  $\parallel$  to the axis of  $x$  at a distance 4 from it.

**22.** This is satisfied only by  $x - 3 = 0$ , and  $y - 4 = 0$  simultaneously. It represents the point (3, 4).

**23.** A rectangular hyperbola.

When

|          |   |          |     |      |
|----------|---|----------|-----|------|
| $x = 3$  | 2 | 1        | 0   | - 1  |
| $y = .5$ | 1 | $\infty$ | - 1 | - .5 |

etc.

**24.** When

|          |   |   |   |   |   |     |
|----------|---|---|---|---|---|-----|
| $x = 5$  | 4 | 3 | 2 | 1 | 0 | - 1 |
| $y = 12$ | 6 | 2 | 0 | 0 | 2 | 6   |

a parabola with vertex at (1.5, - .25).

## 25. When

|       |     |     |    |   |   |   |   |    |
|-------|-----|-----|----|---|---|---|---|----|
| $x =$ | -1  | 0   | 1  | 2 | 3 | 4 | 5 | 6  |
| $y =$ | -75 | -32 | -9 | 0 | 1 | 0 | 3 | 16 |

The curve cuts the axis of  $y$  where  $y = -32$ , rises steeply through  $(1, -9)$  to  $(3, 1)$ , cutting the axis of  $x$  at  $(2, 0)$ , bends down towards the axis of  $x$ , which it touches at  $(4, 0)$  and rises again steeply to an infinite distance. It also goes to infinity in a negative direction. By the form  $y = (x-2)(x-4)(x-4)$ , it is clear that the axis of  $x$  is cut at  $(2, 0)$ , and cut in 2 coincident pts. (*i.e.* touched) at  $(4, 0)$ .

26. When  $x = 0, .5, 1, 1.5, 2, 3, 4, y = 0, .25, 2, 3.38, 16, 54, 128$ . These points enable us to draw the graph in the 1st quadrant, and the rest is in the 3rd quadrant symmetrically situated to this part, as may be seen by putting  $-x, -y$ , for  $x, y$ .

27. In the figure of § 17, if we were to change the sign of every abscissa without altering the ordinate we should get the graph of  $y = -x^3$ . In fact the graph of  $y = -x^3$  may be seen by holding up to the light the graph of  $y = x^3$ , looking at it through the paper from the back.

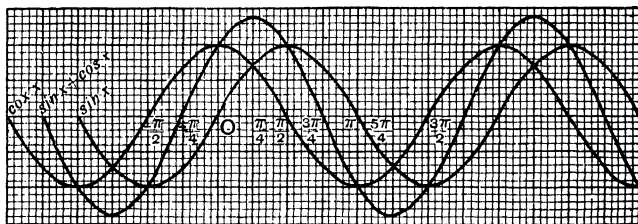
28. Any value of  $y$  gives two equal and opposite values of  $x \therefore$  the curve is symmetrical with regard to the axis of  $y$ , but not with regard to the axis of  $x$ . No negative values of  $y$  are possible. Thus the curve lies on the upper side of the axis of  $x$ , touching it at the origin. Points  $(0, 0), (1, 1), (\frac{1}{2}, \frac{1}{8}), (\frac{3}{2}, \frac{27}{8}), (2, 16)$  enable us to draw it.

29.  $y = (x-1)(x-2)^2$ . When  $x = -2, -1, 0, 1, \frac{4}{3}, \frac{3}{2}, 2, 3, 4, 5, y = -48, -18, -2, 0, \frac{4}{27}, \frac{1}{8}, 0, 2, 12, 36$ , the graph ascends steeply from  $(-2, -48)$  to  $(1, 0)$ , where it cuts the axis of  $x$ , then by values of  $x$  near to  $\frac{4}{3}$  it will be found to turn downwards at the point  $(\frac{4}{3}, \frac{4}{27})$ . It touches the axis of  $x$  at  $(2, 0)$  and rises to an infinite distance as indicated by the points given.

30.  $y = x^3 - 4x + 1$ . When  $x = -3, -2, -1, 0, .5, 1, 1.5, 2, 3, y = -14, 1, 4, 1, -.88, -2, .375, 1, 16$ . The diagram drawn by means of these pts. shows that the function vanishes for

three values of  $x$  lying respectively between  $-3$  and  $-2$ , between  $0$  and  $1$ , and between  $1$  and  $2$ .

**31.** The unit for  $x$  may be taken the same as that for  $y$ ; but for this question it is more convenient to mark the inches along the axis of  $x$  as  $\frac{\pi}{2}$ ,  $\pi$ ,  $\frac{3\pi}{2}$ , etc.; those on the axis of  $y$  as  $1$ ,  $2$ , etc. The pts.  $(0, 0)$ ,  $(\frac{\pi}{6}, \frac{1}{2})$ ,  $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ ,  $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ ,  $(\frac{\pi}{2}, 1)$ , etc., lie on the graph of  $\sin x$ ; and by continuing in this way we get the graph, which extends similarly on both sides of the origin. By moving the curve back through one inch ( $\frac{\pi}{2}$ ) along the axis of  $x$  we get the graph of  $\cos x$ . By taking for each value of  $x$  an ordinate equal to the algebraic sum of the corresponding ordinates of  $\sin x$  and  $\cos x$  we get the graph of  $\sin x + \cos x$ . This curve cuts the axis of  $x$  at  $-\frac{\pi}{4}$ ,  $\frac{3\pi}{4}$ ,  $\frac{7\pi}{4}$ , etc. Thus the general solution of the equation  $\sin x + \cos x = 0$  is  $x = n\pi - \frac{\pi}{4}$ .



**32.** When

| $x=0^\circ$                    | $1^\circ$ | $2^\circ$ | $3^\circ$ | $4^\circ$ | $5^\circ$ | $5.8^\circ$ | $6^\circ$ | $7^\circ$ | $8^\circ$ | $9^\circ$ |
|--------------------------------|-----------|-----------|-----------|-----------|-----------|-------------|-----------|-----------|-----------|-----------|
| $\tan 10x - 2 \tan 9x + 1 = 1$ | .86       | .72       | .56       | .39       | .19       | .02         | -.02      | -.18      | .52       | $\infty$  |

From these the graph may be drawn. It crosses the axis of  $x$  where  $x = 5.9$  approximately  $\therefore \tan 10x - 2 \tan 9x + 1$  vanishes when  $x = 5.9^\circ$ . The graph crosses also at  $7.6^\circ$   $\therefore$  the expression vanishes when  $x = 7.6^\circ$ .

**33.** Beginning at the bottom left-hand point, mark the vertical inches as  $39^\circ$ ,  $41^\circ$ ,  $43^\circ$ , etc., and the horizontal inches 0, 4 p.m., 8 a.m., etc. Join the points indicated, and read off the temperature for 3 p.m. Result  $52.1^\circ$ .

**34.** Take the tenths of an inch along the axis of  $x$  to represent minutes of time, those along the axis of  $y$  to represent minute divisions on the clock face. The long hand travels 60 divisions in 60 minutes  $\therefore$  the line joining (0, 0) to (60, 60) will represent its motion. The short hand starts 5 divisions ahead (at 1 o'clock), but goes only 5 divisions in 60 minutes  $\therefore$  the line joining (0, 5) to (60, 10) represents its motion. The intersection of these shows (by the abscissa) how many minutes after 1 they are together. A line drawn parallel to the 2nd graph and at a vertical distance 30 from it [*i.e.* a line joining (0, 35) to (60, 40)] will, by its intersection with the 1st graph, show the time at which the hands are opposite. A line drawn parallel to the graph of the short hand at a vertical distance 10 from it will, by intersecting the graph of the long hand, show at how many minutes past 1 o'clock the hands are 10 divisions apart. Similarly for 25 divisions and for 15 divisions apart (*i.e.* for hands at right angles). Results to the nearest minute: (a), (1) 5 minutes past 1, (2) 38, (3) 16, (4) 22, (5) 33. The same method will do for the times between 4 and 5 o'clock, but the 2nd graph in this case is the str. line from (0, 20) to (60, 25) since the hour hand has a start of 20 divisions. (b) Results, (1) 22 minutes past 4, (2) 55, (3) 11 and 33, (4) 5 and 38, (5) 49. (c) Results, (1) 27 minutes past 5, (2) at 6 o'clock only, (3) 16 and 38, (4) 11 and 44, (5) 55. (d) Results, (1) 44, (2) 11, (3) 33 and 55, (4) 27, (5) 16.

**35.** Take the directrix for axis of  $y$ , O the origin; and along the axis of  $x$  mark off OS 1 inch. On the axis of  $x$  take any point N. With centre S and radius ON describe a circle cutting at P (above and below the axis) the ordinate through N. P is a point on the parabola. Similarly any number of points may be found.

**36.** Take the directrix for axis of  $y$ , O the origin; and along the axis of  $x$  mark off OS 2 inches. Join the points (0, 0) and (10, 7), and let this line cut the ordinate through

any pt.  $N$  (on the axis of  $x$ ) at  $Q$ . With centre  $S$  and radius  $NQ$  describe a circle cutting the ordinate through  $N$  at  $P$ .  $SP = NQ = ON \times .7 = e \cdot ON$ , where  $e = \text{eccentricity}$   $\therefore P$  is a point on the conic. Similarly other points may be found.

**37.** Take  $X$  for origin, the directrix for axis of  $y$ . Let  $XS$  be 1 inch. Join  $(0, 0)$  to  $(1, 1.5)$  and produce this line to meet at  $Q$  the ordinate through any point  $N$  on the axis of  $x$ . With centre  $S$  and radius  $NQ$  describe a circle cutting at  $P$  the ordinate through  $N$ .  $SP = NQ = ON \times 1.5 = e \cdot ON$ , where  $e = \text{eccentricity}$   $\therefore P$  is a point on the required hyperbola. Similarly other points may be found.

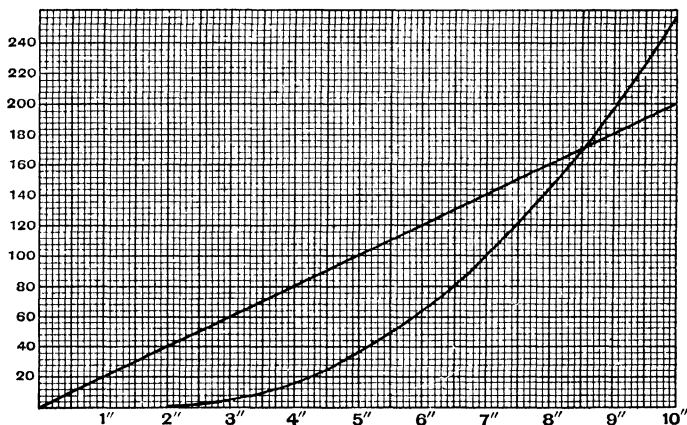
**38.** Take the axis of  $y$  for directrix,  $S$  the focus on the axis of  $x$ ,  $X$  the origin. Join  $(0, 0)$  to  $(1, 1)$  and let this line cut at  $Q$  the ordinate through any pt.  $N$  on the axis of  $x$ . With centre  $S$  and radius  $XQ$  describe a circle cutting the ordinate  $NQ$  at  $P$ .  $SP = XN \cdot \sqrt{2} = e \cdot XN$  [since the eccentricity of a rectangular hyperbola is  $\sqrt{2}$ ]  $\therefore P$  is a point on a rectangular hyperbola whose focus is  $S$  and directrix the axis of  $y$ . Similarly for other points. For a rectangular hyperbola with its asymptotes on the axes of  $x$  and  $y$ , see § 22.

**39.** Take each horizontal tenth of an inch to represent £1 of capital, and each vertical *inch* to represent £1 of interest. Join the origin to the point  $(100, 3)$ . The interest on £57 is shown by the ordinate corresponding to the abscissa 57. Interest = £1.7 = £1. 14s. Interest on £34 is represented by the ordinate whose abscissa is 34. Interest = £1.02 = £1 to the nearest shilling.

**40.** Take  $A$  for origin,  $AB$  8 inches vertically to represent the chain  $AB$ . Take  $BD$  6.4 inches horizontally to represent the weight of the chain. The tension at any point  $P$  varies as  $BP$ ; for the tension at  $P = \text{weight of chain below } P$ . Mark the vertical inches 1, 2, 3, 4, etc.; and at these points draw horizontal distances .1, .4, .9, 1.6, etc. Connecting by a curve the points thus obtained, we get the required graph. At 3 ft. 6 in. from the lower end the tension =  $12\frac{1}{4}$  lbs. weight. At 6 ft. 3 in. from the lower end the tension = 39 lbs. nearly.

**41.** Mark the horizontal inches as seconds 0, 1, 2, etc., and the vertical ones as 0, 40 feet, 80 feet, etc. Join  $(0, 0)$  to

(10, 200). This represents the motion of the 1st particle. For the 2nd particle  $s=4(t-2)^2$ , since it does not start till after 2 seconds. When  $t=3, 4, 5, 6, 7, 8, 9, 10, s=4, 16, 36, 64, 100, 144, 196, 256$ . The graph is a curve through these points. At the intersection we have  $t=8.53$ . By joining (7.5, 140) to (8.5, 160), and (8.5, 180) to (9.5, 200), and observing where these cut the curved graph, we get the times when they are 10 feet apart, viz. 8.23 secs., and 8.82 secs. At the end of the 4th second they are 64 feet apart.



42. Done in the text.

## EXERCISES XLV.

1.  $(2a)^2 = 4a^2$ , also from a figure.
2.  $(3a)^2 = 9a^2$ , also from a figure.
3. Let CB be a side of the larger square. In CB mark off CD equal to a side of the smaller square. Produce BC to A making CA = CB. The rect. AD . DB =  $CB^2 - CD^2$ .
4. Let ABC be a  $\triangle$  right-angled at A. By II. 11  $AB^2 = BC^2 - AC^2 = (BC + AC)(BC - AC)$ .
5. Let ABC be a  $\triangle$  right-angled at A. Let AD be perp. to BC. By II. 11.  $BC^2 = AB^2 + AC^2 = AD^2 + BD^2 + AD^2 + DC^2 = BD^2 + DC^2 + 2AD^2$ . By IV. 4.  $BC^2 = BD^2 + CD^2 + 2BD . DC \therefore AD^2 = BD . DC$ .
6. Let ABC be a  $\triangle$  in which AB = AC. Let D be any pt. of BC, E the mid. pt. of BC.  $AB^2 - AD^2 = BE^2 + AE^2 - (DE^2 + AE^2) = BE^2 - DE^2 = BD . DC$  (II. 5).
7. Proved in the course of the proof of II. 11.; or thus,  $AB^2 = BD^2 + AD^2 = BD^2 + BD . DC$  (Question 5) =  $BD . BC$  (IV. 3.).
8. Let AB be divided equally at C, unequally at D.  $AD^2 - DB^2 = (AD - DB)(AD + DB) = (AC + CD - AC - CD) AB = 2CD . AB$ .
9. Let AB be the greater; in it cut off BC equal to the less. By IV. 7.  $AB^2 + BC^2 = 2AB . BC + AC^2$ , i.e.  $AB^2 + BC^2$  is not less than  $2AB . BC$ .
10. Let ABC be the  $\triangle$  right-angled at A, D any pt. in BC, E the mid. pt.  $\angle B = 45^\circ = \angle BAE \therefore AE = BE$ . By II. 8.  $BD^2 + DC^2 = 2BE^2 + 2DE^2 = 2AE^2 + 2DE^2 = 2AD^2$ .
11. Let PR, QS intersect at T. The diagonals of a rhombus bisect each other at rt.  $\angle$ s  $\therefore$  PR bisects QS at rt.  $\angle$ s  $\therefore$  RTP passes through O, since OS = OQ.  $OP . OR = OT^2 - PT^2$  (IV. 6.) =  $OT^2 + TS^2 - PT^2 - TS^2 = OS^2 - SP^2$ .

12. If  $x, y$  be the length sides of the rect,  $a$  that of the square,  $xy = a^2$  (hyp.). The perimeter of the rectangle =  $2(x + y)$ , that of the square =  $4a$ .  $(x + y)^2 = (x - y)^2 + 4xy = (x - y)^2 + 4a^2 \therefore (x + y)^2 > 4a^2 \therefore x + y > 2a$ .

13. Let  $x, y$  be the lengths of the two parts,  $a$  of the whole line.  $2x^2 + 2y^2 = (x + y)^2 + (x - y)^2 = a^2 + (x - y)^2 \therefore x^2 + y^2$  is a minimum when  $x - y = 0$ .

14. Draw PE, PF perp. to AB, BC.  $\angle EAP = 45^\circ = \angle EPA \therefore AE = EP = BF$ . Similarly  $EB = FC$ .  $AP^2 - BP^2 = AE^2 - EB^2$  (II. 11.) =  $BF^2 - FC^2 = BP^2 - PC^2 \therefore AP^2 + PC^2 = 2BP^2 = \text{sq. on the diagonal of the sq. on BP}$ .

15.  $AB^2 = AD^2 + BD^2 + 2AD \cdot DB$  (II. 4.) =  $AD^2 + BD^2 + 2CD^2$  (hyp.) =  $AC^2 + BC^2$  (II. 11.)  $\therefore ACB$  is a rt.  $\angle$ .

16.  $AC^2 - AE^2 = AD^2 + CD^2 - AD^2 - DE^2 = CD^2 - DE^2 = (CD + DE)(CD - DE) = 4DF \cdot CF$ .

17. By IV. 9.  $BE^2 + DE^2 = 2OE^2 + 2OD^2$ . But  $DE^2 = AD^2 = AO^2 + OD^2 = 2OD^2 \therefore BE^2 = 2OE^2 = \text{sq. on diagonal of sq. on OE}$ .

### EXERCISES XLVI.

1.  $12^2 > 6^2 + 8^2 \therefore$  the  $\angle$  opposite to the side 12 is obtuse.

2.  $13^2 < 9^2 + 12^2 \therefore$  the  $\angle$  opposite to 13 is acute; and this is the greatest  $\angle$  (I. 10.)  $\therefore$  the angles are all acute.

3. AC, BD are equal and bisect each other at E.  $PA^2 + PC^2 = 2PE^2 + 2AE^2$  (IV. 12.) =  $2PE^2 + 2BE^2 = PB^2 + PD^2$  (IV. 12.).

4.  $AB^2 + 2AC \cdot CE = AC^2 + BC^2$  (IV. 11.).  $AC^2 + 2AB \cdot BF = AB^2 + BC^2$  (IV. 11.)  $\therefore$  by adding and removing the common parts from both sides  $2AC \cdot CE + 2AB \cdot BF = 2BC^2$ .

5. Let AD, BE, CF be the medians. Produce AO to H making OH = AO. By Ex. xx. 1. OBHC is a parm.  $\therefore OD = \frac{1}{2}OH = \frac{1}{2}OA$ . Similarly for OE, OF.  $2OB^2 + 2OC^2 = 4BD^2 + 4OD^2$  (IV. 12.) =  $BC^2 + OA^2$ .  $2OC^2 + 2OA^2 = CA^2 + OB^2$ .  $2OA^2 + 2OB^2 = AB^2 + OC^2 \therefore$  by addition  $3(OA^2 + OB^2 + OC^2) = AB^2 + BC^2 + CA^2$ .

6. Let AB = 8 in., BC = 9 in.,  $\angle ABC = 60^\circ$ . Draw AD perp. to BC. ABD is half the equilateral  $\triangle$  on AB  $\therefore BD = 4$  in. and  $AD = 4\sqrt{3}$  in.  $\therefore AC^2 = AD^2 + CD^2 = 48 + 25 = 73$ .  $AC = \sqrt{73} = 8.54$  in.

7. Let  $AB = 6$  in.,  $BC = 8$  in.,  $\angle ABC = 120^\circ$ . Draw  $AD$  perp. to  $BC$  produced.  $ABD$  is half the equilateral  $\triangle$  on  $AB \therefore DB = \frac{AB}{2} = 3$  in. and  $AD = 3\sqrt{3} \therefore AC^2 = AD^2 + DC^2 = 27 + 121 = 148$  and  $AC = \sqrt{148} = 12.17$  in.

8. Let  $AB = 6$ ,  $BC = 8$ , and  $AC = 10$  cms. Also let  $AD$ ,  $BE$ ,  $CF$  be the medians.  $\angle ABC = \text{a rt. } \angle$  (II. 12.)  $\therefore BE = \frac{AC}{2} = 5$  cms.  $AD^2 = AB^2 + BD^2 = 36 + 16 = 52 \therefore AD = 7.2$  cms.  $CF^2 = CB^2 + BF^2 = 64 + 9 = 73 \therefore CF = 8.5$  cms.

9. Using IV. 12. the lengths of the medians are found to be 9.8, 9.2, and 6 cms.

10. Let  $AD$  be perp. to  $BC$ ,  $BD = \frac{3}{5}BC = 6$  cms.,  $CD = \frac{2}{5}BC = 4$  cms.  $\therefore AD^2 = AB^2 - BD^2 = 64 - 36 = 28 \therefore AC^2 = AD^2 + CD^2 = 28 + 16 = 44$  and  $AC = \sqrt{44} = 6.63$  cms.

11. Let  $AD$ ,  $BE$ ,  $CF$  be the medians,  $O$  their intersection. As in Question 5,  $2OB^2 + 2OC^2 = BC^2 + OA^2$ . But  $OA = \frac{2}{3}x$ ,  $OB = \frac{2}{3}y$ ,  $OC = \frac{2}{3}z \therefore \frac{8}{9}y^2 + \frac{8}{9}z^2 = BC^2 + \frac{4}{9}x^2$ .

12.  $9BA^2 = 8x^2 + 8y^2 - 4z^2 = 200 + 392 - 16 = 576 \therefore BA^2 = 64 \therefore BA = 8$ .

13.  $AC^2 = BC^2 + BA^2 - 2BD \cdot BC$  (IV. 11.),  $\angle B$  being acute.  $AC^2 = BC^2 + BA^2 - 2BF \cdot BA$  (IV. 11.),  $\angle B$  being acute  $\therefore BD \cdot BC = BF \cdot BA$ . IV. 10. would be used if  $B$  were obtuse.

14. Let  $A$ ,  $B$  be the fixed pts.,  $C$  the mid. pt. of  $AB$ ,  $P$  the moving pt.  $2CP^2 + 2AC^2 = AP^2 + BP^2$  (IV. 12.) = a constant (hyp.).  $AC$  is constant  $\therefore CP$  is constant in length  $\therefore$  the locus is a circle whose centre is  $C$ .

15. Let  $O$  be the mid. pt. of  $BC$ . Then  $AB^2 + AC^2 = 2OB^2 + 2OA^2$  (IV. 12.) = a constant.

16. Let  $AD$ ,  $BE$ ,  $CF$  be the medians.  $4AD^2 + 4BD^2 = 2AB^2 + 2CA^2 \therefore 4AD^2 = 2AB^2 + 2CA^2 - BC^2$ .  $4BE^2 = 2BC^2 + 2AB^2 - CA^2$ .  $4CF^2 = 2CA^2 + 2BC^2 - AB^2 \therefore 4(AD^2 + BE^2 + CF^2) = 3(AB^2 + BC^2 + CA^2)$ .

17. Let  $ABCD$  be a parm. Then  $AC$ ,  $BD$  bisect each other at  $E$ .  $AB^2 + BC^2 + DC^2 + DA^2 = 2AB^2 + 2BC^2 = 4AE^2 + 4BE^2$  (IV. 12.) =  $AC^2 + BD^2$

18.  $PA^2 + PC^2 = 2AO^2 + 2OP^2$ .  $PB^2 + PD^2 = 2BO^2 + 2OP^2 \therefore PA^2 + PB^2 + PC^2 + PD^2 = 2AO^2 + 2BO^2 + 4OP^2 = a \text{ constant}$ .

19. Let ABC be an isosceles  $\triangle$  on base BC, PBC another  $\triangle$  such that AP is  $\parallel$  to BC. The perp. AD bisects BC.  $AB^2 + AC^2 = 2BD^2 + 2AD^2$ .  $PB^2 + PC^2 = 2BD^2 + 2PD^2$  (IV. 12.). But  $AD < PD$  (I. 11.)  $\therefore AB^2 + AC^2 < PB^2 + PC^2$ .

20. Let ABCD be a quadl., E, F the mid. pts. of AC, BD.  $(AB^2 + BC^2) + (CD^2 + DA^2) = 2AE^2 + 2BE^2 + 2CE^2 + 2DE^2$  (IV. 12.)  $= 4AE^2 + 2BE^2 + 2DE^2 = 4AE^2 + 4BF^2 + 4EF^2$  (EF being the median of  $\triangle BED$ )  $= AC^2 + BD^2 + 4EF^2$ .

21. Let  $AB = 7$ ,  $BC = 5$ ,  $CA = 8$  in. Draw BD perp. to AC.  $AB^2 = AC^2 + BC^2 - 2 \cdot CD \cdot CA$  (IV. 11.), i.e.  $49 = 64 + 25 - 16CD$   $\therefore CD = \frac{5}{2}$  in.  $= \frac{1}{2}CB \therefore \angle ACB = 60^\circ$ , for  $\triangle DCB$  is half an equilateral  $\triangle$ .

22. Let O be the intersection of BD, AC.  $AB^2 - AE^2 = BO^2 + AO^2 - EO^2 - AO^2 = BO^2 - EO^2 = BE \cdot ED$  (IV. 5.). If E divides BD externally, use IV. 6.

23.  $2HM^2 + 2AM^2 - 2KL^2 - 2AL^2 = HK^2 + AH^2 - HK^2 - AK^2$  (IV. 12.)  $\therefore 2(HM^2 - KL^2) = AH^2 - AK^2 + \frac{1}{2}AH^2 - \frac{1}{2}AK^2 \therefore 8(HM^2 - KL^2) = 6(AH^2 - AK^2) = 3(2AH^2 + 2DH^2 - 2AK^2 - 2DK^2) = 3(AB^2 + AD^2 - AC^2 - AD^2)$  (IV. 12.)  $= 3(AB^2 - AC^2)$ .

24.  $AB^2 = AC^2 + BC^2 - 2AC \cdot CF$  (IV. 11.)  $\therefore BC^2 = 2AC \cdot CF =$  twice figure ECFG.

25. Let ABCD be a quadl., AD and BC subtending obtuse  $\angle$ s at E the intersection of diagonals.  $AD^2 > AE^2 + ED^2$ ,  $BC^2 > BE^2 + EC^2$  (IV. 11.)  $\therefore AD^2 + BC^2 > AE^2 + EB^2 + EC^2 + ED^2$ . Similarly  $AC^2 + BD^2 < AE^2 + EB^2 + EC^2 + ED^2$ .

26.  $\angle AOB$  is gr. than  $\angle D$  (I. 8.)  $\therefore \angle AOB$  is obtuse  $\therefore AB^2 > AO^2 + BO^2$ . Similarly  $BC^2 > BO^2 + CO^2$  and  $CA^2 > CO^2 + OA^2 \therefore AB^2 + BC^2 + CA^2 > 2(AO^2 + BO^2 + CO^2)$ .

27.  $4BE^2 + 4AE^2 - 4CF^2 - 4AF^2 = 2AB^2 + 2BC^2 - 2AC^2 - 2BC^2$  (IV. 12.)  $\therefore 4BE^2 - 4CF^2 + AC^2 - AB^2 = 2AB^2 - 2AC^2 \therefore 4(BE^2 - CF^2) = 3(AB^2 - AC^2)$ .

28. Let AB be the given base. Since the area is given, the altitude is given. Draw AD perp. to AB and equal to the given altitude. Let APB be such a  $\triangle$ . Then  $AP^2 + BP^2 = \frac{1}{2}AB^2 + 2EP^2$  (E being the mid. pt. of AB). Now  $AP^2 + BP^2$  is given and AB

is given  $\therefore$  EP is known. Describe a circle with centre E and radius equal to this known value of EP. The point where this cuts the parallel to AB drawn through D is the required vertex.

### EXERCISES XLVII.

1.  $DO \cdot CO = AO \cdot BO$  (IV. 13.), *i.e.*  $DO = \frac{1 \cdot 5}{7} = 2\frac{1}{7}$  in.
2. If  $r =$  radius,  $AO \cdot OB + OD^2 = r^2 \therefore r^2 = 21 + 4 = 25$ , and  $r = 5$  in. We see that AB is a diameter, *i.e.* D lies at the mid. pt. of AB.
3. Let  $CO = x$  in. so that  $DO = 9 - x$  in.  $AO \cdot OB = CO \cdot OD$  (IV. 13.)  $\therefore 20 = x(9 - x)$ , whence  $x = 5$  or 4 in., and  $DO = 4$  or 5 in.
4.  $OA \cdot OB = OC \cdot OD$  (IV. 14. Cor.)  $\therefore OB = \frac{7 \times 10}{5} = 14$  in.  $\therefore AB = 9$  in.
5. If OC is the tangent,  $OC^2 = OA \cdot OB$  (IV. 14.)  $= 4 \times 9 = 36$   $\therefore OC = 6$  in.
6. If OC is a tangent,  $OC^2 = OA \cdot OB$  (IV. 14.)  $= 36 \therefore OC = 6$  cm.  $\therefore CD^2 = OD^2 - OC^2 = 64 - 36 = 28$ .  $CD = \sqrt{28} = 5.29$  cm.
7. On AB, on the same side as the pt. C, describe an equilateral  $\triangle AEB$ . E is the centre reqd. By measurement, the intercept on  $OC = 7.15$  cm.
8. If DE is perp. to AB,  $DE^2 = OD^2 - OE^2 = 8^2 - 7^2 = 15$   $\therefore DE = 3.87$  cm.
9. Let O be the centre of the wheel, OC the vertl. rad., AB the face of the brick, so that  $CA = 12$ , and  $AB = 4$  in. Produce AB to meet the circumference again at D. Draw OE perp. to BD.  $AB \cdot AD = AC^2$  (IV. 14.)  $\therefore AD = \frac{144}{4} = 36 \therefore DE = \frac{1}{2}BD = 16$ ,  $OC = AE = 4 + 16 = 20$ .
10. Let BA be the common chord, meeting in E the common tangent CD.  $CE^2 = BE \cdot EA = ED^2$  (IV. 14.).
11. Let C be any point in the common chord BA produced, CD, CE tangents to the two circles.  $CD^2 = BC \cdot CA = CE^2$  (IV. 14.).
12. Let AB, AC be tangents. Draw ADE cutting the circle at D, E.  $AB^2 = AD \cdot AE = AC^2$  (IV. 14.).

13. Sq. on tangent  $CP=AC$ .  $CB=a$  constant, since A, B, C are fixed points  $\therefore$  locus of P is a circle whose centre is C.

14. Let AP be such a line drawn from a fixed pt. A. Let Q be its mid. pt., C the centre of circle. Bisect AC at D.  $DQ = \frac{1}{2}PC$  (Ex. xx. 1.) = a constant  $\therefore$  the locus of Q is a circle with centre at the fixed pt. D.

15. Proved in the last line but four of II. 11. Or thus:  $\angle DBO = 90^\circ - \angle DOB = \angle BAD \therefore BO$  touches the circumcircle of  $\triangle BDA \therefore OD \cdot OA = OB^2$  (IV. 14.).

16. Let AB cut CD at rt.  $\angle$ s in E. The centre O lies in AB (III. 3. Cor.).  $AE \cdot EB = AO^2 - OE^2$  (IV. 5.)  $= CO^2 - OE^2 = CE^2 = CE \cdot ED$ .

17.  $CD \cdot CE = CB \cdot CA$  (IV. 14. Cor.), i.e.  $x(x+c) = b(a+b)$ .

18. Let D be the mid. pt. of BC.  $BE \cdot BA = BD^2 = \frac{1}{4}BC^2 = \frac{1}{4}BA^2 \therefore BE = \frac{1}{4}BA \therefore EA = \frac{3}{4}BA$ .

19.  $AP \cdot AQ = AC \cdot AB \therefore PCBQ$  is a cyclic quadl.  $\therefore \angle CPQ + \angle CBQ = 180^\circ$  (IV. 14.). But  $CPQ = 90^\circ$  (IV. 18.)  $\therefore \angle CBQ = 90^\circ \therefore Q$  lies on a str. line through B perp. to AB.

20. Let E, F be the mid. pts. of CD, AB. EF bisects AB at rt.  $\angle$ s (II. 1. 2.)  $\therefore EF$  contains the centre of the circle  $\therefore DE$  is a tangent  $\therefore DA \cdot DO = DE^2 = \frac{1}{4}DA^2 \therefore DO = \frac{1}{4}DA$ .

21. Let AEB, CED be chords.  $AB^2 - CD^2 = (AE + EB)^2 - (CE + ED)^2 = (AE - EB)^2 + 4AE \cdot EB - (CE - ED)^2 - 4CE \cdot ED = (AE - EB)^2 - (CE - ED)^2$  (IV. 13.).

22. The circle whose diameter is AB passes through P, Q, since the  $\angle$ s P, Q are rt.  $\angle$ s  $\therefore AO \cdot OP = BO \cdot OQ$  (IV. 13. or 14.).

23. Let AB, CD be the  $\parallel$  chords cut by PQ in R, S. Let T be the mid. pt. of PQ.  $PT^2 - TR^2 = PR \cdot RQ$  (IV. 5.)  $= AR \cdot RB$  (IV. 13.)  $= CS \cdot SD$  (hyp.)  $= QS \cdot SP$  (IV. 13.)  $= QT^2 - TS^2$ . But  $PT = QT \therefore TR = TS \therefore$  the mid. pt. of PQ is the mid. pt. of RS and therefore lies on the line  $\parallel$  to AB and CD and equidistant from them.

24.  $\angle AEB = \angle EAB$  (I. 5.)  $= \frac{1}{2}\angle EBC = \frac{1}{2}\angle ECB$  (I. 5.)  $= \angle EDC \therefore AE$  touches the circle through E, B, D (III. 18.).

25. Proved on page 271.

## EXERCISES XLVIII.

1.  $AB^2 + BH^2 = 2AB \cdot BH + AH^2$  (IV. 7.)  $= 2AH^2 + AH^2$  (hyp.).
2.  $\triangle BAF = \triangle DAH$  in all respects (I. 4.)  $\therefore \angle LBH = \angle ADH = 90^\circ - \angle AHD = 90^\circ - \angle LHB$  (I. 3.)  $\therefore HLB$  is a rt.  $\angle$ .
3.  $\triangle ACK = \frac{1}{2}$  fig. HC (II. 9.)  $= \frac{1}{2}$  fig. FH (IV. 16.)  $= \triangle AFK$  (II. 9.)  $\therefore FC$  is  $\parallel$  to  $AK$  (II. 7.).  $\triangle AGK = \frac{1}{2}$  fig. FK (II. 9.)  $= \frac{1}{2}$  fig. AC (IV. 16.)  $= \triangle ABK$  (II. 9.)  $\therefore GB$  is  $\parallel$  to  $AK$  (II. 7.).
4.  $\angle EOD = \angle HOB$  (I. 3.)  $= 90^\circ - \angle EBL$  (Question 2)  $= 90^\circ - \angle EFB$  (I. 5.)  $= \angle FDO$   $\therefore EO = ED$  (I. 6.)  $= EA$   $\therefore \angle AOD$  is a rt.  $\angle$  (III. 18.).
5. Let  $AB$  be divided at  $H$ .  $AH \cdot HB = AB \cdot BH - BH^2 = AH^2 - BH^2 = (AH + BH)(AH - BH)$ .
6. Let  $AH = x$ , then  $HB = 3 - x$   $\therefore x^2 = 3(3 - x)$   $\therefore x^2 + 3x = 9$   $\therefore x = \frac{\sqrt{45} - 3}{2} = \frac{3}{2}(\sqrt{5} - 1) = 1.85$ .  $3 - x = 1.15$ .
7. Let  $AH = x$ , then  $BH = x - 3$   $\therefore x(x - 3) = 9$   $\therefore x = \frac{3}{2}(\sqrt{5} + 1) = \frac{3}{2} \times 3.236 = 4.85$   $\therefore x - 3 = 1.85$ .
8.  $AB \cdot AL = AH^2$  (hyp.)  $\therefore AL^2 + AL \cdot LB = AH^2$   $\therefore AL^2 = AH^2 - AL \cdot LB = AH^2 - AL \cdot AH = AH \cdot LH$ .

## EXERCISES XLIX.

1. Let  $ABC$  be a rt.  $\angle$ . Construct a triangle  $ABD$  having the  $\angle$ s  $B, D$  each double of  $\angle A$ . Bisect  $\angle ABD$  by  $BE$ . Bisect  $\angle$ s  $ABE, EBD$ .  $\angle ABD = \frac{4}{5}$  of a rt.  $\angle$   $\therefore \angle DBC = \frac{1}{5}$  of a rt.  $\angle$ .
2.  $\angle BCD = \angle B = \frac{4}{5}$  of a rt.  $\angle$ .  $\angle ACD =$  supplement of  $\angle BCD = \frac{6}{5}$  of a rt.  $\angle = 3\angle A$ .
3.  $AC$  and  $CD$  each subtend  $\frac{2}{5}$  of a rt.  $\angle$  at circumference, or  $\frac{1}{5}$  of 4 rt.  $\angle$ s at the centre of the smaller circle  $\therefore$  they are sides of an inscribed regular pentagon.
4.  $\angle AED =$  supplement of  $\angle ACD$  (III. 13.)  $= \angle BCD = \frac{4}{5}$  of a rt.  $\angle$ .  $\angle ADE = \angle AED$  (I. 5.)  $= \frac{4}{5}$  of a rt.  $\angle$   $\therefore \angle DAE = \frac{2}{5}$  of a rt.  $\angle$ .
5.  $\triangle ABD = \triangle ADE$  in all respects by the last example  $\therefore$  circumcircle of  $\triangle ABD =$  circumcircle of  $\triangle ADE =$  circumcircle of  $\triangle ACD$ .
6.  $\angle DAE = 36^\circ$  (Example 4)  $= \angle CDA$   $\therefore CD$  is  $\parallel$  to  $AEK$ .  $\angle BDC = 36^\circ = \angle DAE = \angle DCE$  (III. 12.)  $\therefore BDK$  is  $\parallel$  to  $CE$ .

7. In a  $\triangle$  having each of the base-angles double of the vertical  $\angle$  divide the vertical angle into 4 equal parts. Each of these is  $\frac{1}{10}$  of a rt.  $\angle$ .

8. The  $\angle$  at the centre  $= 2\angle CBD$  (III. 11.)  $= 144^\circ = 180^\circ - A$ .  $\therefore$  the centre lies on the arc CD. Also the centre lies on the str. line bisecting CD at rt.  $\angle$ s  $\therefore$  it lies at the mid. pt. of arc CD.

9. Draw a  $\triangle ABC$  in which  $\angle B = \angle C = 2\angle A$ . Draw an equilat.  $\triangle ABD$  on the same side of AB.  $\angle DBC = (\frac{4}{5} - \frac{2}{3})$  of a rt.  $\angle = \frac{2}{15}$  of a rt. angle. Half this angle is the one required.

10.  $\angle BAE = \angle BAD + \angle DAE = (\frac{2}{5} + \frac{2}{5})$  of a rt.  $\angle \therefore$  BE subtends at the centre  $\frac{1}{5}$  of 4 rt.  $\angle$ s  $\therefore$  BE is the side of a regular inscribed pentagon.

11. BD subtends  $\frac{2}{5}$  of a rt.  $\angle$  at the centre  $\therefore$  BD is the side of a regular inscribed decagon.

12. In the figure of IV. 17. draw DF perp. to BC. DF bisects BC, since DBC is isosceles. Let  $AC = x$ ,  $BC = a - x$ ; then  $BD = x$ .  $x = \frac{\sqrt{5}-1}{2} \cdot a$ ,  $a - x = a - \frac{\sqrt{5}-1}{2} \cdot a = \frac{3-\sqrt{5}}{2} \cdot a$ .  $BF = \frac{1}{2}BC = \frac{3-\sqrt{5}}{4} \cdot a$ .  $AF = a - BF = \frac{\sqrt{5}+1}{4} \cdot a \therefore AF \cdot FB = \frac{(\sqrt{5}+1)(3-\sqrt{5})}{16} a^2 = \frac{\sqrt{5}-1}{8} a^2 = \frac{1}{4}ax$ .

13. Let A be the centre, BC a side of the inscribed decagon. ABC is a  $\triangle$  with each base-angle double of  $\angle A \therefore BC = \frac{r}{2}(\sqrt{5}-1) \dots$  page 255.

## EXERCISES L.

2. The bisectors are concurrent (Example 1)  $\therefore$  they are equal (I. 6.).

4. Draw two perpendicular diameters and join their ends. The sides are equal (I. 4.). Any angle is a rt.  $\angle$  (III. 17.).

5. Draw two perp. diameters and draw tangents at their ends. The sides of the quadrilateral formed by these tangents are all equal; for each = a diameter (II. 2.). The angles are rt.  $\angle$ s; for each = an angle at the centre (II. 2.).

6. Let ABCD be the square, E, F, G, H the mid. pts. of AB, BC, CD, DA. Let EG, FH intersect at O. The figures formed

are rectangular parms. (II. 1.)  $\therefore$  EO, FO, GO, HO are all equal, since each is equal to half a side of the square. The circle whose centre is O and radius EO is the one required (III. 5.).

7. The diagonals of a square are equal, and bisect each other  $\therefore$  the circle whose centre is their intersection and radius half a diagonal is the one required.

8. By drawing radii OA, OB, OC, OD, OE including  $\angle$ s of  $72^\circ$  we obtain 5 equal arcs. Draw tangents LAF, FBG, GCH, HDK, KEL. The quadrilateral AFBO is divided by FO into two  $\Delta$ s equal in all respects (I. 17.)  $\therefore \angle FOB = 36^\circ$ . Similarly  $\angle GOB = 36^\circ \therefore \Delta$ s FBO, GBO are equal in all respects  $\therefore FG = 2FB = 2FA = FL$ . Similarly all the sides of the pentagon are equal. Also each  $\angle$  = supplement of  $\angle$  at centre =  $108^\circ \therefore$  the pentagon is regular.

9. Bisect the angles. The bisectors are concurrent and equal (Ex. I. 1. 2.)  $\therefore$  the perps. from the point of concurrence to the sides are equal (I. 16.). With any one of these perps. as radius the circle may be described.

10. Bisect the angles. The bisectors are concurrent and equal (Ex. I. 1. 2.). With any one of these as radius the circle may be described.

11. See Ex. xxxviii. 22.

12. The vertical  $\angle$  of a  $\Delta$  whose base-angles are each twice the vertical  $\angle = 36^\circ$ . The angle of an equilat.  $\Delta = 60^\circ$ . The difference =  $24^\circ$ . Place an angle of  $24^\circ$  at the centre. The chord subtended is one side of the regular quindecagon. Place equal chords consecutively in the circle, and the required figure is described. The figure is equilateral by construction. It is also equiangular: for each  $\angle$  subtends  $\frac{1}{15}$  of the circumference.

13. EF may be cut off on either side of E  $\therefore$  there are 2 solutions.

16. Let A be the given pt., BC the given str. line which is to contain the centre. Draw AD perp. to BC, produce AD to E making  $DE = AD$ . By III. 1. the circle must pass through E as well as A  $\therefore$  the problem is the same as the one in Question 13.

17. On  $OP$  as diameter describe a circle. Place in it a chord  $OC$  equal to a side of the given square. Describe a circle with centre  $P$  and radius  $PC$  cutting the given line in  $A, B$ .  $OA \cdot OB = OC^2$  (IV. 14.).

19. Draw a str. line  $OAB$  making  $OA, OB$  equal to the sides of the given rectangle. Describe any circle through  $A, B$ . Draw a tangent  $OC$ .  $OC$  is a side of the required square (IV. 14.).

20. Let  $AB$  be a side of the given square. Draw a circle touching  $AB$  at  $B$ . With centre  $A$  and radius  $AC$  equal to the given side of the rectangle cut the circle at  $C$ . Let  $AC$  cut the first circle at  $D$ . The rect. contained by  $AC, AD = AB^2$  and is therefore the required rectangle.

21. Draw any circle through the given pts.  $A, B$  cutting the given circle at  $C, D$ . Let  $AB, CD$  meet at  $E$ . Through  $E$  draw a diameter  $EFG$  of the given circle. The circle described through  $ABF$  must pass through  $G$  (IV. 14. Cor.).

22. The intersection of the common chords is the point. Prove by IV. 14.

23. Describe a circle about  $ABC$ . Draw the tangent at  $A$  meeting  $BC$  produced at  $D$ .  $AD^2 = BD \cdot DC$  (IV. 14.).

24. Let  $h$  be the height of flagstaff  $AB$ ,  $k$  that of the tower  $BC$ . Let  $D$  be the point of contact of the horizontal through  $C$  and a circle through  $A, B$ . Let  $E$  be any other pt. in  $CD$ . Join  $AE$  cutting the circle at  $F$ .  $\angle ADB = \angle AFB$  (III. 12.)  $> \angle AEB$  (I. 8.). Thus  $D$  is the required point; and  $CD^2 = BC \cdot CA$  (IV. 14.)  $\therefore CD = \sqrt{k(k+h)}$ .

25. Let  $AB$  be the given str. line. On  $AB$  as diameter describe a circle. Take centre  $C$ , and at any pt.  $P$  draw a tangent  $PQ$  equal to a side of the given square. Produce  $AB$  to  $R$ , making  $CR = CQ$ . Draw a tangent  $RS$ .  $AR \cdot RB = RS^2$  (IV. 14.)  $= PQ^2$  (I. 17) = the given square.

26. Let  $AB$  be the given str. line. On  $AB$  describe a semicircle  $ADB$ . Draw a str line  $\parallel$  to  $AB$  at a distance equal to a side of the given square, and let one of the points of intersection with the semicircle be  $D$ . Draw  $DE$  perp. to  $AB$ .  $AE \cdot EB = DE^2$  (IV. 13.)  $\therefore E$  is the required point. If the side of the given square is gr. than  $\frac{1}{2}AB$ , the problem is impossible.

**27.** Let AB be a diamr. of the circle. Make  $\triangle ADC$  having its sides 3, 4, and 5 cms. long, the 5 cm. side AC lying along AB. Produce AD to meet the circle at E. AEB is the  $\triangle$  reqd. (I. 19. 22.).

**28.** With centre B and radius equal to a side of the given square describe a circle. Draw from A a tangent AP. In AB cut off AC equal to AP.  $AB^2 - AC^2 = AB^2 - AP^2 = BP^2 =$  the given square.

**29.** Let O be the centre. Draw OD a radius perp. to OA, OE, OF perp. to OB, OC. The  $\triangle DEF$  is the  $\triangle ABC$  turned through  $90^\circ$  without any alteration of size or shape.

**30.** Bisect AB at O. Draw OE perp. to AB. Make  $\angle s$  CAF, ACF each  $45^\circ$ . With centre A and radius AF describe a circle cutting OE at E. In OB cut off OD equal to OE.  $AD^2 + DB^2 = 2AO^2 + 2OD^2$  (IV. 8.)  $= 2AO^2 + 2OE^2 = 2AE^2$  (II. 11.)  $= AF^2 + FC^2 = AC^2$ .  $2AC^2 = 2AD^2 + 2DB^2 = (AD + DB)^2 + (AD - DB)^2 \therefore 2AC^2$  is a minimum when  $AD = DB \therefore$  the least value of  $2AC^2$  is  $4AD^2$  when  $AD = \frac{1}{2}AB$ , i.e.  $2AC^2$  must not be less than  $AB^2$ .

**31.** Let AB be the str. line bisected at C. Take D any pt. in AB.  $AD \cdot DB = AC^2 - CD^2 \therefore AD \cdot DB$  is a maximum when CD is a minimum, i.e. when D is at C.

**32.** Let p be length of perp. Then  $p \times 26 =$  twice area  $=$  side  $24 \times$  perp.  $= 240 \therefore p = \frac{120}{13} = 9\frac{3}{13}$ .

### EXERCISES LI.

**1.** Let A be a pt. of intersection, AC, AD tangents. Since AD touches one circle and DAC is a rt.  $\angle$ , AC passes through the centre of this circle: but AC is a tangent to the other circle  $\therefore$  a tangent to one circle passes through the centre of the other.

**2.** Let A be the given point of intersection. The centres of all the circles must lie on the tangent at A to the given circle (Question 1).

**3.** Let A be the given pt, B the given pt. of intersection with the circle. Draw BC touching the given circle. Make an  $\angle BAC$  equal to  $\angle ABC$ .  $AC = CB$  (I. 6.)  $\therefore$  the circle with centre C and radius CA passes through A and cuts the given

circle orthogonally at B, since CB is a tangent to one circle and radius of the other.

4. The centres of the circumcircles are at F, E, the mid. pts. of AB, AC.  $\angle FDA = \angle FAD$  (I. 5.) and  $\angle EDA = \angle EAD \therefore$  whole  $\angle FDE = \angle FAE = 90^\circ$ , i.e. the radii are at rt.  $\angle$ s  $\therefore$  the tangents at D are at rt.  $\angle$ s.

5. Let the circle whose radius is PA cut at D a circle through B, C.  $PB \cdot PC = PA^2$  (IV. 14.)  $= PD^2 \therefore$  PD touches the circle DBC. But PD is a radius of the circle DA  $\therefore$  the circles cut orthogonally.

6. Let P, Q be the points of contact. Let a circle through T, U meet the first circle at R.  $CR^2 = CP^2 = CU \cdot CT$  (Ex. xlv. 7.)  $\therefore$  CR touches the circle TUR  $\therefore$  the circles cut orthogonally.

### EXERCISES LII.

1. Let PQ be a common tangent meeting the radical axis in T.  $TP = TQ$  (property of radical axis).

2. Let AB, AC be tangents drawn from a pt. A on the radical axis. Let D, E be the centres.  $AC = AB$  since A is on the radical axis  $\therefore$  the circle with centre A and radius AB goes through C. Also it is touched by BD, CE, since the  $\angle$ s B, C are rt.  $\angle$ s;  $\therefore$  it cuts both circles orthogonally.

3. Draw tangents from the radical centre, and use any of these as radius.

4. Let A, B, C be the pts. of contact, E, F the centres of the circles which touch at A; D the other centre. Let the tangents at A, B meet at T. T lies on the bisector of  $\angle AFB$  (from  $\triangle TAF$ ,  $TBF$ ). Similarly any other pair of tangents meet on the bisector of an  $\angle$  of the  $\triangle DEF$ . But the bisectors are concurrent  $\therefore$  the tangents are concurrent.

Nos. 5 and 6 are particular cases of No. 3. In No. 5 two of the circles are of infinitely small radius; in No. 6 one circle is so.



### EXERCISES LIII.

1. Let the diagonals of the quadr. ABCD meet at O.  $\frac{\triangle AOD}{\triangle AOB} = \frac{DO}{BO} = \frac{\triangle COD}{\triangle COB}$  (V. 1.).

2. Let ABCD be a trapezium having AB  $\parallel$  to CD. Let the diagonals AC, BD meet at O. AB is  $\parallel$  to the base CD of  $\triangle DOC$ .  $\therefore \frac{CO}{OA} = \frac{DO}{OB}$  (V. 2. Cor.).

3. From D a pt. in the base BC let DE, DF  $\parallel$  to AB and AC respectively meet AC at E and AB at F. Let AD, EF meet at O. AO = OD (II. 2.)  $\therefore$  the locus of O is a str. line  $\parallel$  to BC and bisecting the sides AB, AC (V. 2.).

4. Let DEF be the mid. pts. of the sides BC, CA, AB of  $\triangle ABC$ . Join BE, CF.  $\triangle BFE = \triangle AFE$  (II. 6.) =  $\triangle EFC$  (II. 6.)  $\therefore$  EF is  $\parallel$  to BC (II. 7.). Similarly DF is  $\parallel$  to CA  $\therefore$  DFEC is a parm.  $\therefore EF = CD = \frac{1}{2}BC$ . Similarly  $DE = \frac{1}{2}AB$ , and  $DF = \frac{1}{2}AC$ .

5. Let the three  $\parallel$  str. lines AB, CD, EF cut off intercepts AC, CE, BD, DF on the str. lines ACE, BDF. Join E, B cutting CD at G.  $\frac{AC}{CE} = \frac{BG}{GE}$  (V. 2) =  $\frac{BD}{DF}$  (V. 2.).

6.  $\frac{BG}{BD} = \frac{BE}{BA}$  (V. 2.) =  $\frac{BF}{BC}$  (V. 2.)  $\therefore$  GF is  $\parallel$  to CD (V. 2.).

7. Draw PD  $\parallel$  to AB to meet BC at D. Produce BD to C making DC = BD. Join CP and produce it to meet BA at A.  $\frac{CP}{PA} = \frac{CD}{DB}$  (V. 2.)  $\therefore CP = PA$ .

8. Draw DE  $\parallel$  to BC to meet AC at E. In AC produced take CF = CE. Join DF meeting BC at H.  $\frac{DH}{HF} = \frac{EC}{CF}$  (V. 2.)  $\therefore DH = HF$   $\therefore$  DHF is the line reqd.

9.  $\triangle DBE = \triangle DCE$  (II. 5.)  $\therefore \triangle DBF = \triangle ECF$ . Also  $\frac{\triangle ADF}{\triangle BDF} = \frac{AD}{DB}$  (V. 1.)  $= \frac{AE}{EC}$  (V. 2.)  $= \frac{\triangle AEF}{\triangle ECF}$  (V. 1.)  $\therefore \triangle ADF = \triangle AEF$ .

10. Join AO.  $\triangle ANO = \frac{1}{2}$  parm. ANOM  $= \triangle AMO = \triangle OMN$  (II. 2.)  $\therefore \frac{\triangle BNO}{\triangle OMN} = \frac{\triangle BNO}{\triangle AON} = \frac{BN}{AN}$  (V. 1.)  $= \frac{BO}{OC}$  (V. 2.)  $= \frac{AM}{CM}$  (V. 2.)  $= \frac{\triangle AMO}{\triangle CMO}$  (V. 1.)  $= \frac{\triangle OMN}{\triangle CMO}$ .

11.  $\frac{\triangle BED}{\triangle AED} = \frac{BE}{EA}$  (V. 1.)  $= \frac{1}{2} \therefore \triangle BED = \frac{1}{2} \triangle AED$ .  $\frac{\triangle CED}{\triangle AED} = \frac{CD}{DA}$  (V. 1.)  $= \frac{2}{1} \therefore \triangle CED = 2 \triangle AED \therefore \triangle CED = 4 \triangle BED \therefore \frac{\triangle BED}{\triangle CED} = \frac{1}{4}$ .

12. Let AH meet DE at M. Join BM cutting FG at O.  $\frac{\triangle FAH}{\triangle ADH} = \frac{AF}{AD}$  (V. 1.)  $= \frac{BH}{BD}$  (V. 2.)  $= \frac{BO}{BM}$  (V. 2.)  $= \frac{\triangle HOB}{\triangle HBM}$ . Also  $\triangle DAB = \triangle MAB$  (II. 5.)  $\therefore \triangle ADH = \triangle HBM \therefore$  from the above  $\triangle FAH = \triangle HOB$ . And they are between the same parallels  $\therefore FH = HO$ . Thus if  $HO = HF$ , BO produced meets AH on DE. In the same way, since  $KO = KG$ , we can prove BO produced meets CK on DE  $\therefore$  AH and CK meet on DE.

13.  $\frac{FK}{DK} = \frac{AK}{CK}$  (V. 2.)  $= \frac{CL}{AL} = \frac{GL}{DL}$  (V. 2.)  $\therefore FG$  is  $\parallel$  to AC (V. 2.).

14. AC is  $\parallel$  to BD (V. 2.). Let BO be gr. than AO, so that DO is gr. than CO.  $\frac{OB}{OA} = \frac{OD}{OC} \therefore \frac{OB - OA}{OA} = \frac{OD - OC}{OC}$ , i.e.  $\frac{2OP}{OA} = \frac{2OQ}{OC} \therefore PQ$  is  $\parallel$  to AC and  $\therefore$  also to BD (V. 2.).

15. With the figure of II. 10.  $\frac{\text{parm. KG}}{\text{parm. HG}} = \frac{KE}{HE}$  (V. 1.)  $= \frac{\text{parm. FK}}{\text{parm. FH}}$  (V. 1.)  $= \frac{\text{parm. HG}}{\text{parm. FH}}$ .

16. Let the medians AD, BE, of  $\triangle ABC$  meet at O. Join OC.  $\triangle BEC = \frac{1}{2} \triangle ABC = \triangle ADC \therefore \triangle BOD = \triangle AOE = \triangle COE$  (II. 6.). But  $\triangle COD = \triangle BOD$  (II. 6.)  $\therefore \triangle BOC = 2 \triangle COE \therefore BO = 2OE$  (V. 1.). Similarly it may be shown that CF divides BE in the ratio of 2 to 1  $\therefore$  the medians are concurrent.

17.  $\frac{\triangle AEB}{\triangle ADB} = \frac{AE}{AD} = \frac{1}{3}$  (V. 1.)  $\therefore \triangle AEB = \frac{1}{3} \triangle ADB = \frac{1}{6}$  parm. ABCD (II. 2.).

18.  $\frac{\triangle AED}{\triangle ABD} = \frac{DE}{DB}$  (V. 1.)  $= \frac{3}{7} \therefore \triangle AED = \frac{3}{7} \triangle ABD = \frac{3}{14}$  parm. ABCD  $\therefore \triangle AED : \text{parm. ABCD} :: 3 : 14$ .

### EXERCISES LIV.

1. Let AD bisect  $\angle BAC$  and base BC of the  $\triangle ABC$ .  $\frac{BA}{AC} = \frac{BD}{DC} = 1$  (V. 3.)  $\therefore BA = AC$ .

2.  $\frac{AE}{EB} = \frac{AD}{BD}$  (V. 3.)  $= \frac{AD}{DC} = \frac{AF}{FC}$  (V. 3.)  $\therefore EF$  is  $\parallel$  to BC (V. 2.).

3. Let AD, BE, the bisectors of  $\angle$ s A and B of  $\triangle ABC$  meet at G.  $\frac{GA}{GD} = \frac{BA}{BD}$  from  $\triangle ABD$  (V. 3.). Also since AD bisects  $\angle A$

$\frac{BA}{AC} = \frac{BD}{DC}$  (V. 3.)  $\therefore \frac{BA}{BD} = \frac{AC}{DC} \therefore \frac{GA}{GD} = \frac{BA}{BD} = \frac{AC}{DC} = \frac{BA + AC}{BD + CD} = \frac{BA + AC}{BC}$ . In the same way it may be shown that the bisector

CF of  $\angle C$  divides AD in the same ratio  $\therefore AD, BE, CF$  are concurrent.

4. AO, BO, CO bisect the  $\angle$ s A, B, C. Hence, as in the preceding example, it may be proved that  $\frac{AO}{OD} = \frac{BA + AC}{BC}$ .

5. Let the bisectors of  $\angle$ s A and C meet BD at E. Also let the bisector of  $\angle D$  meet AC at F.  $\frac{DA}{AB} = \frac{DE}{EB}$  (V. 3.)  $= \frac{DC}{CB}$  (V. 3.)

$\therefore \frac{DA}{DC} = \frac{AB}{CB}$ . Also  $\frac{AF}{CF} = \frac{AD}{CD}$  (V. 3.)  $= \frac{AB}{CB} \therefore BF$  bisects  $\angle B$  (V. 3.)

$\therefore$  the bisectors of  $\angle$ s B and D meet on AC.

6. AB bisects CD at rt.  $\angle$ s (III. 3.)  $\therefore$  arc BC = arc BD  $\therefore \angle BGC = \angle BGD \therefore \frac{CG}{GD} = \frac{CE}{ED}$  (V. 3.). Similarly  $\frac{CF}{FD} = \frac{CE}{ED} \therefore \frac{CG}{GD} = \frac{CF}{FD}$ .

7. Let OA and OB meet the inner circle at E and F, and let GOH be the common tangent at O to the circles, G lying on the same side of OC as the pt. A.  $\angle OFC = \text{supplement of } \angle OEC = \angle AEC$ .  $\angle OCF = \angle HOF$  (III. 18.)  $= \angle EAC$  (III. 18.)  $\therefore \angle FOC = \angle ECA$  (I. 22.)  $= \angle COE$  (III. 18.)  $\therefore \frac{AO}{OB} = \frac{AC}{BC}$  (V. 3.).

8. AD the altitude of the isos.  $\triangle ABC$  bisects the vertical  $\angle$  at A  $\therefore$  O the centre of the incircle lies in AD. Also BO and CO bisect the  $\angle$ s at B and C. From  $\triangle ABD$   $\frac{OD}{OA} = \frac{BD}{BA}$  (V. 3.)  
 $\therefore \frac{OD}{OA + OD} = \frac{BD}{BA + BD} \therefore \frac{OD}{AD} = \frac{2BD}{2(BA + BD)} = \frac{\text{base}}{\text{perimeter}}$ .

9. Let AD meet the base at O. Produce BD to meet AC produced at F. Bisect DF at E and join OE. From  $\triangle$ s BAD, FAD,  $BD = FD$  (I. 16.)  $\therefore \frac{BE}{EF} = \frac{3}{1} = \frac{BA}{AC} = \frac{BO}{OC}$  (V. 3.)  $\therefore$  OE is  $\parallel$  to CF (V. 2.)  $\therefore \frac{DO}{AO} = \frac{DE}{FE}$  (V. 2.)  $\therefore DO = AO$ .

10.  $AE = \frac{4}{7}AB$  and  $AF = \frac{2}{7}AD \therefore \frac{AE}{AF} = \frac{4}{2} = \frac{2}{1}$  and  $\frac{EG}{FG} = \frac{AE}{AF}$  (V. 3.)  $= \frac{2}{1}$ .

11. Let the pt. O fall within BD.  $\frac{BD}{CD} = \frac{BA}{AC}$  (V. 3.)  $\therefore \frac{BD - CD}{BD + CD} = \frac{BA - AC}{BA + AC}$ , i.e.  $\frac{2OD}{2OB} = \frac{BA - AC}{BA + AC} \therefore \frac{OD}{OB} = \frac{BA - AC}{BA + AC}$ .

### EXERCISES LV.

1. Let AO bisect  $\angle A$  and BO bisect the extr.  $\angle$  at B of  $\triangle ABC$ . Join CO, and let AO meet BC at F.  $\frac{AB}{BF} = \frac{AO}{OF}$  (V. 4.). Also  $\frac{AB}{AC} = \frac{BF}{CF}$  (V. 3.)  $\therefore \frac{AB}{BF} = \frac{AC}{CF} \therefore \frac{AC}{CF} = \frac{AO}{OF} \therefore$  CO bisects the extr.  $\angle$  at C, which proves the proposition.

2. Produce DP to E and AP to F. PA bisects ext.  $\angle$  of  $\triangle CPD \therefore \frac{AD}{AC} = \frac{PD}{PC}$  (V. 4.).  $\angle APB = \text{a rt. } \angle$  (III. 17.)  $\therefore \angle BPD = \text{complement of } \angle FPD = \text{complement of } \angle EPA = \text{complement}$

of  $\angle APC = \angle CPB$ , i.e.  $PB$  bisects  $\angle CPD \therefore \frac{PD}{PC} = \frac{BD}{BC}$  (V. 3.)  $\therefore \frac{AD}{AC} = \frac{BD}{BC} \therefore \frac{AC}{BC} = \frac{AD}{BD}$ .

3. Let the bisectors of the int. and ext.  $\angle$ s at  $P$  meet  $AB$  at  $C$  and  $D$ .  $\frac{AC}{CB} = \frac{AP}{PB}$  (V. 3.)  $\therefore C$  is a fixed pt.  $\frac{AD}{DB} = \frac{AP}{PB}$  (V. 4.)  $\therefore D$  is also a fixed pt. Also  $CPD$  is a rt.  $\angle \therefore$  the locus of  $P$  is a circle on  $CD$  as diameter.

4.  $\angle OAB = \angle ODC$ ,  $\angle BOA = \angle COD \therefore \angle OBA = \angle OCD \therefore \angle OBC = \angle OCB \therefore OB = OC \therefore AB = CD$  (I. 4.).  $OA$  bisects ext.  $\angle$  of  $\triangle BOD \therefore \frac{AD}{AB} = \frac{OD}{OB} = \frac{OA}{OC} = \frac{AB}{BC}$  (V. 3.).

5. Let  $AD$ , the bisector of  $\angle A$  of  $\triangle BAC$ , meet the base  $BC$  at  $D$ ; and let  $AE$ , the bisector of the ext.  $\angle$  at  $A$ , meet  $BC$  produced at  $E$ .  $\angle DAE =$  a rt.  $\angle$ .  $\frac{EB}{EC} = \frac{AB}{AC}$  (V. 4.)  $= \frac{BD}{BC} \therefore E$  is a fixed pt.  $\therefore$  the locus of  $A$  is a circle on  $DE$  as diameter.

### EXERCISES LVI.

1. Take  $D, E, F$  the mid. pts. of the sides  $BC, CA, AB$  of  $\triangle ABC$ .  $\frac{AF}{FB} = 1 = \frac{AE}{EC} \therefore EF$  is  $\parallel$  to  $BC$  (V. 2.). Similarly  $DE$  is  $\parallel$  to  $AB$  and  $DF$  to  $CA$ . Also  $\triangle AEF$  is equiangular to  $\triangle ABC$  (I. 20.) and  $\therefore$  similar to it (V. 5.), and in the same way  $\triangle CED, \triangle BFD$  are similar to  $\triangle ABC$ . Also  $FEDB$  is a parm.  $\therefore \triangle FED = \triangle BDF$  in all respects (II. 2. and I. 4.). Similarly each of the  $\triangle$ s  $AFE, EDC$  is equal to  $\triangle DEF$  in all respects  $\therefore \triangle AEF = \triangle BDF = \triangle CDE = \triangle DEF$ , and each is similar to  $\triangle ABC$ .

2. With the fig. of IV. 13.  $\angle CEA = \angle BED$  (I. 3.),  $\angle ECA = \angle EBD$  in the same segment  $\therefore \triangle$ s  $CEA, BED$  are equiangular, and  $\therefore$  similar (V. 5.)  $\therefore \frac{CE}{EA} = \frac{BE}{ED} \therefore$  rect.  $CE \cdot ED =$  rect.  $BE \cdot EA$ .

3. With the fig. of IV. 14,  $\angle OCA = \angle OBC$  in alternate segment, and  $\angle COA$  is common to  $\triangle$ s  $OCA, OBC \therefore$  these  $\triangle$ s are equiangular  $\therefore \frac{OA}{OC} = \frac{OC}{OB}$  (V. 5.)  $\therefore$  rect.  $OA \cdot OB = OC^2$ .

4.  $\angle OBC = \angle ODA$  in the same segment, and  $\angle O$  is common to  $\triangle s$  BCO, DAO  $\therefore$  these  $\triangle s$  are equiangular  $\therefore \frac{OB}{OC} = \frac{OD}{OA}$  (V. 5.)  $\therefore$  rect. OA . OB = rect. OC . OD.

5. Let AD, BE two medians of  $\triangle ABC$  cut at G.  $\frac{CE}{EA} = \frac{CD}{DB}$   
 $\therefore$  DE is  $\parallel$  to AB (V. 2.)  $\therefore \triangle CDE$  is equiangular to  $\triangle CBA$ ,  
 and  $\therefore$  similar (V. 5.)  $\therefore \frac{DE}{AB} = \frac{DC}{BC} = \frac{1}{2}$ . Also  $\triangle s$  DGE, AGB are  
 equiangular (I. 20.) and  $\therefore$  similar (V. 5.)  $\therefore \frac{DG}{AG} = \frac{DE}{AB} = \frac{1}{2}$ , which  
 proves the proposition.

6. Let AB be the man CD the post, AE the shadow of the man, so that DBE is a str. line. Let AE =  $x$ .  $\triangle s$  EAB, ECD are equiangular  $\therefore \frac{EA}{AB} = \frac{EC}{CD}$ , i.e.  $\frac{x}{6} = \frac{x+4}{12} \therefore x = 4$  ft.

7. Let AB be the man, CD the post, AE the shadow of the man (= 10 ft.). Let AC =  $x$ .  $\triangle s$  EAB, ECD are equiangular,  
 $\therefore \frac{EA}{AB} = \frac{EC}{CD}$ , i.e.  $\frac{10}{6} = \frac{x+10}{12} \therefore x = 10$  ft.

8. Let AB be the pole, AC its shadow; DE the height of the house, DF its shadow.  $\triangle s$  CAB, FDE are equiangular,  
 $\therefore \frac{DE}{DF} = \frac{BA}{AC}$ , i.e.  $\frac{DE}{60} = \frac{10}{20} \therefore DE = 30$  ft.

9. Let  $\triangle ABC$  be such that BC = 5, AB = 9, AC = 7 cms.; also let DEF be a similar  $\triangle$  having EF = 3 cms.  $\frac{DE}{EF} = \frac{AB}{BC}$ , i.e.  
 $\frac{DE}{3} = \frac{9}{5} \therefore DE = 5.4$  cms.  $\frac{DF}{EF} = \frac{AC}{BC}$ , i.e.  $\frac{DF}{3} = \frac{7}{5} \therefore DF = 4.2$  cms.

10. Let AB =  $x$ , and AC =  $y$  ft. As in lvi. 4,  $\triangle s$  OAC, ODB are similar  $\therefore \frac{OB}{OD} = \frac{OC}{OA}$ , i.e.  $\frac{x+5}{12} = \frac{4}{5} \therefore x = 4.6$  ft. Also  $\frac{AC}{AO} = \frac{BD}{DO}$ , i.e.  $\frac{y}{5} = \frac{7}{12} \therefore y = 2$  ft. 11 in.

11. Let EF be the crease cutting AB at E and CD at F, and AC at O. When the folding is done,  $\angle EOA$  coincides with  $\angle EOC$  and is  $\therefore$  equal to it.  $\therefore \angle EOA = \angle EOC = a$  rt.  $\angle$ .  $\triangle EAO$

is similar to  $\triangle CAB \therefore \frac{EO}{AO} = \frac{BC}{BA} = \frac{6}{8} \therefore EO = \frac{3}{4}AO = \frac{3}{8}AC = \frac{3}{8}\sqrt{6^2 + 8^2} = \frac{15}{4}$  ft.  $\therefore EF = \frac{15}{2} = 7\frac{1}{2}$  ft.

12. Let EOF be the crease, cutting AB at E, AD at O, AC at F. As in the previous example, EOF is perp. to AOD, and  $AO = OD \therefore EOF$  is  $\parallel$  to BC, and since  $AO = OD$ ,  $AE = EB$ , and  $AF = CF \therefore EF = \frac{1}{2}BC$ .

13. Let OE be the crease, cutting AB at O and AC at E.  $AO = OB$ , and OE is perp. to AOB as in the previous examples  $\therefore \triangle s AOE, ACB$  are equiangular  $\therefore \frac{OE}{AO} = \frac{BC}{AC}$ , i.e.  $\frac{OE}{\frac{13}{2}} = \frac{5}{12} \therefore OE = \frac{65}{24} = 2.71$  in. nearly.

14. Let DE be the line  $\parallel$  to BC cutting AB at D and AC at E.  $\triangle s ADE, ABC$  are similar  $\therefore \frac{DE}{BC} = \frac{AD}{AB}$ , i.e.  $\frac{DE}{9} = \frac{4}{7} \therefore DE = 5\frac{1}{7}$  in.

15. Let ABCD be a trapezium, having AB parallel to CD and equal to 2CD. Let the diagonals AC, DB meet at O.  $\triangle s DOC, BOA$  are equiangular (I. 20.)  $\therefore \frac{AO}{OC} = \frac{BO}{OD} = \frac{AB}{CD} = 2$ , i.e.  $AO = 2OC$ , and  $BO = 2OD$ , which proves the proposition.

16. Let OAB, OCD, OEF intercept on the  $\parallel$  str. lines ACE, BDF, portions AC, CE, and BD, DF.  $\frac{AC}{BD} = \frac{OC}{OD}$  (for  $\triangle s OCA, ODB$  are equiangular)  $= \frac{CE}{DF}$  (for  $\triangle s OCE, ODF$  are equiangular).

17. Let C, D be the centres of the circles, and AB a common tangent meet CD produced at O. In  $\triangle s OAC, OBD$ ,  $\angle OBD = a$  rt.  $\angle = \angle OAC$   $\angle O$  is common  $\therefore$  the  $\triangle s$  are equiangular  $\therefore \frac{OC}{OD} = \frac{AC}{BD}$ . Similarly if a common tangent EF divide CD internally at P,  $\triangle s CEP, DFP$  are equiangular  $\therefore \frac{CP}{PD} = \frac{CE}{DF}$ .

18. Let AD be the median to the base BC of  $\triangle ABC$ . Draw  $EF \parallel$  to BC, to meet the median at G.  $\frac{EG}{BD} = \frac{AG}{AD}$  for  $\triangle s AGE, ADB$  are equiangular (I. 20.)  $= \frac{GF}{CD}$  for  $\triangle s AGF, ADC$  are equiangular  $\therefore EG = GF$ .

**19.** Let  $\angle ADB$  be obtuse, so that  $\angle ADC$  is acute. Let  $P$  be the centre of the circum-circle to  $\triangle ADB$ , and  $Q$  that of the circum-circle to  $\triangle ADC$ .  $P$  lies in  $PE$  which bisects  $AB$  at rt.  $\angle$ s.  $Q$  lies in  $QF$  which bisects  $AC$  at rt.  $\angle$ s. Also  $\angle APB = 2$  supplement of  $\angle ADB$  (III. 11.)  $= 2\angle ADC = \angle AQC$  (III. 11.)  $\therefore \angle BPE = \angle AQF$ . Also  $\angle PEB = a$  rt.  $\angle = \angle QFA \therefore \triangle$ s  $PBE$ ,  $QAF$  are equiangular  $\therefore \frac{PB}{QA} = \frac{BE}{FA} = \frac{AB}{AC}$ .

**20.** Let the perpendiculars  $AM$ ,  $BN$  on a str. line  $DE$  be in a constant ratio. Let  $DE$  cut  $AB$  at  $O$ .  $\triangle$ s  $AMO$ ,  $BNO$  are equiangular  $\therefore \frac{AO}{BO} = \frac{AM}{BN} = a$  constant  $\therefore O$  is a fixed pt., i.e.  $DE$  passes thro. a fixed pt.  $O$  in  $AB$ .

**21.**  $\frac{BO}{OE} = \frac{AO}{OD} \therefore DE$  is  $\parallel$  to  $AB$  (V. 2.)  $\therefore \triangle$ s  $AOB$ ,  $DOE$  are equiangular (I. 20.)  $\therefore \frac{AB}{DE} = \frac{BO}{OE} = \frac{2}{1}$ . Also  $\triangle$ s  $CDE$ ,  $CBA$  are equiangular (I. 20.)  $\therefore \frac{CD}{CB} = \frac{CE}{CA} = \frac{DE}{AB} = \frac{1}{2} \therefore CB$  and  $CA$  are bisected at  $D$  and  $E$ .

**22.**  $\triangle$ s  $QAO$ ,  $QCD$  are equiangular (I. 20.)  $\therefore \frac{QO}{QD} = \frac{AO}{CD}$  (V. 5.)  $= \frac{BO}{CD} = \frac{PO}{PD}$  for  $\triangle$ s  $PBO$ ,  $PCD$  are equiangular (I. 20.)  $\therefore \frac{QO}{OP} = \frac{QD}{PD}$ .

**23.** Let  $DA$  produced meet  $EC$  at  $F$ . From  $\triangle$ s  $BAD$ ,  $CAF$ ,  $AD = AF$  (I. 20. and 16.).  $\triangle$ s  $BOE$ ,  $AOD$  are equiangular (I. 20.)  $\therefore \frac{BO}{OA} = \frac{BE}{DA}$  (V. 5.)  $= \frac{DF}{DA}$  (for  $BEFD$  is a parm.)  $= 2 \therefore BO = 2OA$   $\therefore O$  is a pt. of trisection of  $AB$ .

**24.** Let  $AB$  be  $> AC$ , and let  $CN$  meet  $AB$  at  $E$ . From  $\triangle$ s  $ANC$ ,  $ANE$ ,  $CN = NE$  and  $AC = AE \therefore \frac{CN}{NE} = \frac{CO}{OB} \therefore ON$  is  $\parallel$  to  $BE$  (V. 2.). Also  $\frac{ON}{BE} = \frac{CO}{CB} = \frac{1}{2} \therefore ON = \frac{1}{2} BE = \frac{1}{2} (AB - AC)$  for  $AC = AE$ .

**25.** Let  $P$  be the centre of circle  $ABC$ ,  $Q$  that of circle  $ABD$ . Let  $Q$  fall on the opp. side of  $BD$  to the pt.  $A$ , so that  $P$  falls

on the same side of BC as the pt. A.  $\angle BQD = 2$  supplement of  $\angle BAD$  (III. 12.)  $= 2\angle CAB = \angle CPB \therefore \angle PCB + \angle PBC = \angle QDB + \angle QBD$  (I. 22.), *i.e.*  $\angle PCB = \angle PBC = \angle QDB = \angle QBD \therefore \triangle s$  PCB, QDB are equiangular  $\therefore \frac{BC}{BD} = \frac{PC}{QB} = \frac{\text{diameter of circle ABC}}{\text{diameter of circle ABD}}$ .

26.  $\angle EAC = \frac{1}{2}(180^\circ - A)$  and  $\angle ECA = \frac{1}{2}(180^\circ - C) \therefore \angle AEC = 180^\circ - \frac{1}{2}(180^\circ - A) - \frac{1}{2}(180^\circ - C) = \frac{A + C}{2} = \frac{180^\circ - B}{2} = \angle FBA$ .

Also  $\angle F$  is common to  $\triangle s$  FAB, FDE  $\therefore$  the  $\triangle s$  are equiangular (I. 22.)  $\therefore \frac{FA}{FB} = \frac{FD}{FE}$ , *i.e.* FA and FB are inversely proportional to FE and FD.

27. Let N be the mid. pt. of BC. NE bisects the arc BC of the circum-circle of  $\triangle ABC$ . Also since  $\angle BAE = \angle CAE$ , ADE also bisects this arc  $\therefore$  E lies on the circum-circle at the mid. pt. of arc BC. In  $\triangle s$  ABE, BDE,  $\angle BEA$  is common.  $\angle BAE = \angle EAC = \angle DBE$  in the same segment  $\therefore$  the  $\triangle s$  are equiangular  $\therefore \frac{AE}{EB} = \frac{EB}{ED} \therefore \text{rect. AE} \cdot \text{ED} = \text{BE}^2$ .

28.  $\triangle s$  FED, AEB are equiangular (I. 20.)  $\therefore \frac{FD}{AB} = \frac{DE}{EB} = \frac{3}{1} \therefore FD = 3AB$ , *i.e.*  $FC + CD = 3AB \therefore FC = 2AB$  (II. 2.).

### EXERCISES LVII.

1.  $\frac{AO}{OD} = \frac{CO}{OB} \therefore \text{rect. AO} \cdot \text{OB} = \text{rect. CO} \cdot \text{OD} \therefore A, B, C, D$  are concyclic (IV. 13.).

2.  $\frac{OA}{OC} = \frac{OD}{OB} \therefore \text{rect. OA} \cdot \text{OB} = \text{rect. OC} \cdot \text{OD}$ . If the circum-circle of  $\triangle BAC$  does not pass thro. D, let it cut OC again at E. Then  $\text{rect. OE} \cdot \text{OC} = \text{rect. OA} \cdot \text{OB}$  (IV. 14. Cor.)  $= \text{rect. OC} \cdot \text{OD} \therefore OD = OE$ , *i.e.* D must coincide with E.

3.  $\frac{BD}{DA} = \frac{DA}{DC}$  and  $\angle BDA = \angle CDA \therefore \triangle BDA$  is similar to  $\triangle ADC$  (V. 7.)  $\therefore \angle BAC = \angle BAD + \angle DAC = \angle DCA + \angle ABD \therefore \angle BAC = \text{a rt. } \angle$  (I. 22.).

4.  $\triangle s$  ADE, ABC are similar by V. 7.  $\therefore \angle DAC = \angle DAE \therefore$  AE falls along AC, *i.e.* AEC is a str. line.

5. Draw BE perp. to AC.  $\frac{AC}{CD} = \frac{AB}{BC} \therefore \text{rect. AC} \cdot \text{BC} = \text{rect. AB} \cdot \text{CD}$ . But  $\frac{1}{2}AB \cdot \text{CD} = \text{area of } \triangle ABC = \frac{1}{2}AC \cdot \text{BE} \therefore \frac{1}{2}AC \cdot \text{BE} = \frac{1}{2}AC \cdot \text{BC} \therefore \text{BE} = \text{BC}$ . But BE is perp. to AC  $\therefore$  BE must coincide with BC, *i.e.*  $\angle ACB = \text{a rt. } \angle \therefore \angle BCD = \text{complement of } \angle ACD = \angle CAB$ .

6.  $\text{BC} = \text{BD} \therefore \angle BCD = \angle BDC \therefore \angle ACB = \angle BDE$ . Hence in  $\triangle s$  ACB, BDE,  $\frac{AC}{CB} = \frac{BD}{DE}$  and  $\angle ACB = \angle BDE \therefore \triangle s$  ACB, BDE are similar (V. 7.)  $\therefore \angle CBA = \angle DEB$ , and  $\angle A$  is common to  $\triangle s$  ACB, ABE  $\therefore \triangle s$  ACB, ABE are equiangular (I. 22.) and  $\therefore$  similar (V. 5.).

7.  $\frac{AE}{AC} = \frac{DF}{DA}$  (hyp.)  $\therefore \frac{AE}{AB} = \frac{DF}{DB}$ . Also  $\angle EAB = \angle BDF \therefore \triangle s$  AEB, DFB are similar (V. 7.)  $\therefore \angle EBA = \angle FBD \therefore$  adding  $\angle ABF$  to each,  $\angle EBF = \angle ABD$ . Also since  $\triangle s$  AEB, DFB are similar,  $\frac{AB}{EB} = \frac{BD}{BF}$ , *i.e.*  $\frac{AB}{BD} = \frac{EB}{BF}$  and  $\angle ABD = \angle EBF \therefore \triangle s$  EBF, ABD are similar.

8. Let the lines thro. A and B meet in Q. Join CQ.  $\triangle s$  AQB, DPE have their sides respectively parallel and are  $\therefore$  similar  $\therefore \frac{BQ}{EP} = \frac{AB}{DE} = \frac{2}{1} = \frac{BC}{EF} \therefore \frac{BQ}{BC} = \frac{EP}{EF}$ . Also QB and BC are respectively  $\parallel$  to EP and EF  $\therefore \angle QBC = \angle PEF$ , and as proved above,  $\frac{BQ}{BC} = \frac{EP}{EF} \therefore \triangle QBC$  is similar to  $\triangle PEF$  (V. 7.)  $\therefore QC$  is  $\parallel$  to PF, which proves the proposition.

9. Take O the centre of the circle on which the pt. P lies, and let C be one of the cutting pts. of the circles. Rect. OA. OB = OC<sup>2</sup> (IV. 14.) (for OC is a tangent to the circle) ABC = OP<sup>2</sup>  $\therefore \frac{OA}{OP} = \frac{OP}{OB}$  and  $\angle O$  is common to  $\triangle s$  PAO, BPO  $\therefore$  these  $\triangle s$  are similar (V. 7.)  $\therefore \frac{PA}{PB} = \frac{PO}{OB} = \text{a constant ratio}$ .

10. In  $\triangle$ s DBC, PAC,  $\angle DBC = \angle PAC$  and  $\frac{DB}{BC} = \frac{PA}{AC} \therefore$  the  $\triangle$ s are similar  $\therefore \angle DCB = \angle PCA \therefore$  adding  $\angle BCP$  to each,  $\angle DCP = \angle BCA = \frac{1}{2}$  a rt.  $\angle$ . Also since  $\triangle$ s DCB, PCA are similar,  $\frac{DC}{PC} = \frac{BC}{CA}$  and the included  $\angle$ s DCB, PCA are equal  $\therefore \triangle DPC$  is similar to  $\triangle BAC \therefore \angle DPC = \angle BAC =$  a rt.  $\angle \therefore \angle PDC = \frac{1}{2}$  a rt.  $\angle$  (I. 22.)  $\therefore PD = PC$ .

11. Let DE produced meet AC at F. Draw BH  $\parallel$  to AC to meet DE at H. From similar  $\triangle$ s DBH, DAF,  $\frac{BH}{AF} = \frac{DB}{DA}$  (V. 5.)  $= \frac{BE}{EC}$  (hyp.)  $= \frac{BH}{CF}$  from similar  $\triangle$ s HEB, FEC  $\therefore AF = CF$ .

12. In  $\triangle$ s ABC, DBA,  $\angle B$  is common, and  $\frac{BD}{BA} = \frac{BA}{BC} \therefore$  the  $\triangle$ s are similar (V. 7.)  $\therefore \frac{BD}{AD} = \frac{BA}{AC}$ , and  $\angle BDA = \angle CAB$ . In like manner  $\triangle$ s AEC, BAC are similar, and  $\therefore \frac{AE}{EC} = \frac{BA}{AC}$  and  $\angle AEC = \angle BAC \therefore \angle AEC = \angle BDA \therefore \angle ADE = \angle AED \therefore AE = AD \therefore \frac{BD}{AD} = \frac{BA}{AE} = \frac{AD}{EC}$ .

13. Rect. OA . OB =  $OC^2$  (IV. 14.) =  $OD^2 \therefore \frac{OA}{OD} = \frac{OD}{OB}$ . Also  $\angle BOD$  is common to  $\triangle$ s DOA, BOD  $\therefore$  these  $\triangle$ s are similar (V. 7.)  $\therefore \angle ODA = \angle OBD = \angle CEA$  (in same segment)  $\therefore EF$  is  $\parallel$  to  $OD$  (I. 20.).

14. Rect. AO . BO =  $OC^2$  (hyp.) =  $OP^2 \therefore \frac{AO}{OP} = \frac{OP}{OB}$  and  $\angle O$  is common to  $\triangle$ s AOP, POB  $\therefore$  these  $\triangle$ s are similar (V. 7.)  $\therefore \angle BPO = \angle OAP$ . Also  $\angle CPA + \angle PAC = \angle OCP$  (I. 22.) =  $\angle OPC = \angle BPC + \angle BPO \therefore \angle CPA = \angle BPC$ .

15. Let the line thro. C meet AD at G and AE at H. From similar  $\triangle$ s CDG, BDA,  $\frac{CG}{BA} = \frac{CD}{DB} = \frac{CA}{AB}$  (V. 3.) =  $\frac{EC}{EB}$  (V. 4.) =  $\frac{CH}{AB}$  from similar  $\triangle$ s EBA, ECH  $\therefore CG = CH$ .

**EXERCISES LVIII.**

1. Let  $ABC$  be an isos.  $\triangle$  having  $AB=AC$ , and let  $AD$  be perp. to the base  $BC$ . Produce  $AD$  to meet the circum-circle in  $E$ . Join  $BE$ . The centre of the circum-circle lies in  $AD$   $\therefore$   $ABE$  is a rt.  $\angle$ . Also  $AD$  is perp. to  $BC$   $\therefore$   $\triangle$ s  $ABE$ ,  $ADB$  are similar (V. 9.)  $\therefore \frac{EA}{AB} = \frac{AB}{AD}$ .

2. Let  $ABC$  be the  $\triangle$  rt.  $\angle$ d. at  $C$ , so that  $AB=10$ , and  $AC=7$ . Draw  $CD$  perp. to  $AB$ .  $\triangle$ s  $ACB$ ,  $ADC$  are similar (V. 9.)  $\therefore \frac{AD}{AC} = \frac{AC}{AB}$   $\therefore AD = \frac{49}{10} = 4.9$ , and  $BD = 10 - 4.9 = 5.1$ .

**EXERCISES LIX.**

1.  $ACD$  is a str. line (I. 2).  $\triangle ABC = \triangle DCE$   $\therefore$  rect.  $AC \cdot CB =$  rect.  $DC \cdot CE$   $\therefore \frac{AC}{CD} = \frac{CE}{CB}$   $\therefore \frac{AC}{AC+CD} = \frac{CE}{CE+CB}$ , i.e.  $\frac{AC}{AD} = \frac{CE}{EB}$ .  
From similar  $\triangle$ s  $AFC$ ,  $ABD$ ,  $\frac{CF}{DB} = \frac{CA}{DA}$  (V. 5.)  $= \frac{CE}{EB} = \frac{CG}{DB}$  from similar  $\triangle$ s  $ECG$ ,  $EBD$   $\therefore CF = CG$ .

2. Join  $AD$ ,  $BE$ .  $\angle ADC = \angle ACD = \angle ECB = \angle BEC$   $\therefore$   $\triangle$ s  $DAC$ ,  $EBC$  are equiangular  $\therefore \frac{AC}{CD} = \frac{BC}{CE}$  (V. 5.)  $\therefore$  rect.  $AC \cdot CE =$  rect.  $BC \cdot CD$   $\therefore \triangle ACE = \triangle DCB$  (V. 10.).

3. Rect.  $OD \cdot OC =$  rect.  $OP \cdot OQ$  (IV. 14. Cor.)  $=$  rect.  $OA \cdot OB$  (IV. 14. Cor.)  $\therefore$  rect.  $OD \cdot OC$  is constant for all directions of  $OPQ$ , and  $C$  is a fixed pt.  $\therefore D$  is a fixed pt. Also since rect.  $OA \cdot OB =$  rect.  $OC \cdot OD$ ,  $\frac{OB}{OC} = \frac{OD}{OA}$ , i.e.  $\frac{OC+CB}{OC} = \frac{OA+AD}{OA}$   $\therefore \frac{CB}{OC} = \frac{AD}{OA}$ , i.e.  $\frac{OA}{AD} = \frac{OC}{CP}$ .

4. From similar  $\triangle$ s  $GBA$ ,  $ABC$ ,  $\frac{BG}{BA} = \frac{BA}{BC}$  (V. 9.)  $\therefore$  rect.  $BG \cdot BC = BA^2$ , i.e. rect.  $BG \cdot BF =$  rect.  $BE \cdot BA$ . Also  $\angle FBG = \frac{2}{3}$  of a rt.  $\angle = \angle ABE$   $\therefore \triangle BFG = \triangle BEA$ . In like manner,  $\triangle FGC = \triangle CDA$ .

5. Draw  $CD$  perp. to the hypotenuse  $AB$  of the  $\triangle$ , and let  $BC = 2CA$ . The circles on diameters  $BC$ ,  $CA$  each pass thro.  $D$ .  $\therefore CD$  is their common chord. Also rect.  $CD \cdot AB = 2$  area of  $\triangle ABC = \text{rect. } CA \cdot CB = 2CA^2 \therefore \frac{CD}{AB} = \frac{2CA^2}{AB^2} = \frac{2CA^2}{BC^2 + CA^2} = \frac{2}{5}$ , for  $BC^2 = 4CA^2 \therefore CD = \frac{2}{5}AB$ .

6. In  $\triangle s$   $ABC$ ,  $CBD$   $\angle B$  is common,  $\angle ACB = \angle BDC \therefore$  the  $\triangle s$  are equiangular  $\therefore \frac{BA}{BC} = \frac{BC}{BD} \therefore \text{rect. } BA \cdot BD = BC^2 \therefore BC$  touches the circum-circle of  $\triangle ADC$  (IV. 15.).

7. If  $AB$  is the common chord,  $OP \cdot OQ = OA \cdot OB$  (IV. 13.)  $= OR \cdot OS$  (IV. 13)  $\therefore \frac{OP}{OR} = \frac{OS}{OQ}$ .

8.  $\angle AEC = \text{a rt. } \angle = \angle DOC \therefore DOEC$  are concyclic  $\therefore \text{rect. } AD \cdot AE = \text{rect. } AO \cdot AC = 2AO^2 = AB^2 \therefore AB$  touches the circum-circle of  $\triangle BDE$  (IV. 15.).

9. Draw  $AK$  to touch the circle at  $K$ . Rect.  $AC \cdot AB = AK^2$  (hyp.)  $= \text{rect. } AE \cdot AD$  (IV. 14.)  $\therefore C, B, D, E$  are concyclic  $\therefore \angle CBE = \angle CDE$  (in the same segment)  $= \angle DFE$  (III. 18.)  $\therefore DF$  is  $\parallel$  to  $AB$  (I. 18.).

10. In  $\triangle s$   $ABD$ ,  $BOD$ ,  $\angle OBD = \angle CAD$  (in same segment)  $= \angle BAD$ , and  $\angle BDO$  is common  $\therefore$  the  $\triangle s$  are equiangular  $\therefore \frac{AD}{DB} = \frac{DB}{DO} \therefore DB^2 = \text{rect. } AD \cdot DO$ .

11.  $\frac{\triangle AOC}{\triangle BOD} = \frac{\text{rect. } AO \cdot OC}{\text{rect. } BO \cdot OD}$  (V. 10)  $= \frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$ .  $\frac{\triangle AOD}{\triangle BOC} = \frac{\text{rect. } AO \cdot OD}{\text{rect. } OB \cdot OC} = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$ .

12.  $\triangle AOC = \triangle BOD \therefore \text{rect. } AO \cdot OC = \text{rect. } DO \cdot OB \therefore 4 \times 5 = 2OD \therefore OD = 10$ .

### EXERCISES LX.

1. Let  $A$  be the area reqd.  $\frac{A}{75} = \frac{3^2}{5^2}$  (V. 11.)  $\therefore A = 27$  sq. in.

2.  $\triangle s ADE, ABC$  are similar  $\therefore \frac{\triangle ADE}{\triangle ABC} = \frac{AD^2}{AB^2}$  (V. 11.)  $= \frac{3^2}{8^2} = \frac{9}{64}$ .

3. Let DE be the reqd. line, meeting AB produced at D and AC produced at E.  $\frac{AB^2}{AD^2} = \frac{\triangle ABC}{\triangle ADE} = \frac{1}{4} \therefore AD = 2AB.$

4. If DE is the reqd. line, meeting AB and AC produced at D and E,  $\frac{AB^2}{AD^2} = \frac{\triangle ABC}{\triangle ADE}$  (V. 11.)  $= \frac{1}{9} \therefore AD = 3AB.$

$$5. \frac{\triangle ADE}{\triangle ABC} = \frac{AD^2}{AB^2} \text{ (V. 11.)} = \frac{1}{4^2} = \frac{1}{16}.$$

$$6. \frac{AD^2}{AB^2} = \frac{\triangle ADE}{\triangle ABC} \text{ (V. 11.)} = \frac{1}{9} \therefore \frac{AD}{AB} = \frac{1}{3}, \text{ i.e. } \frac{AD}{AD + DB} = \frac{1}{3} \therefore$$

$$\frac{AD}{BD} = \frac{1}{2}.$$

7.  $\angle BDA = \text{a rt. } \angle = \angle BEA \therefore B, D, E, A \text{ are concyclic } \therefore \angle DEC = \text{supplement of } \angle AED = \angle ABD$  (III. 13.). Also  $\angle C$  is common to  $\triangle s DEC, ABC \therefore$  the  $\triangle s$  are equiangular (I. 22.)  $\therefore \frac{\triangle CDE}{\triangle ABC} = \frac{CD^2}{CA^2}$  (V. 11.).

8.  $\frac{\triangle BDF}{\triangle FAD} = \frac{BF}{FA}$  (V. 1.)  $= \frac{BH}{HD}$  (V. 2.). But  $\triangle AFD = \frac{1}{2}HD \cdot AD \therefore \triangle BDF = \frac{1}{2}BH \cdot AD.$  Similarly  $\triangle AED = \frac{1}{2}DG \cdot AD$  and  $\triangle DEC = \frac{1}{2}CG \cdot AD \therefore$  quadr. AFDE  $= \frac{1}{2}HG \cdot AD \therefore$  the three figures are proportional to BH, HG, GC.

9. From similar  $\triangle s$  BFD, BAC,  $\frac{BF}{FD} = \frac{BA}{AC} \therefore \frac{BF}{AE} = \frac{BA}{AC} \dots (1).$   
Also  $\frac{AE}{EC} = \frac{BD}{DC}$  (V. 2.)  $= \frac{BA}{AC}$  (V. 3.)  $\therefore$  from (1)  $\frac{BF}{EC} = \frac{BA^2}{AC^2}.$

10. If ABCD is a sq.,  $\frac{\triangle \text{ on } AB}{\text{similar } \triangle \text{ on } BD} = \frac{AB^2}{BD^2}$  (V. 11.)  $= \frac{AB^2}{2AB^2}$  (II. 11.)  $\therefore \triangle \text{ on } AB = \frac{1}{2} \text{ similar } \triangle \text{ on } BD.$

11. Let DEF be the equilateral  $\triangle$  formed by the perpendiculars, A lying in EF, B in DF, C in DE.  $\triangle DBC$  is half the equilateral  $\triangle$  on CD  $\therefore BD = \frac{1}{2}CD.$  Also  $CE = DB \therefore DE = 3DB.$  Also from  $\triangle DBC, DC^2 = DB^2 + BC^2$  (II. 11.)  $\therefore 4DB^2 = DB^2 + BC^2 \therefore 3DB^2 = BC^2.$   
 $\frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{DE^2} = \frac{BC^2}{9DB^2} = \frac{3DB^2}{9DB^2} = \frac{1}{3}.$

12. If the  $\triangle BDC$  were folded about  $BC$ , the pt.  $D$  would coincide with  $G$  the centroid of  $\triangle BAC \therefore \triangle BDC = \frac{1}{3} \triangle BAC$ .

13. Let  $DE$  drawn  $\parallel$  to  $AB$  meet  $AC$  at  $E$ .  $\frac{\triangle DEC}{\triangle ABC} = \frac{CD^2}{CB^2}$   
(V. 11.)  $= \frac{CD^2}{2CD^2} = \frac{1}{2}$ .

14.  $\angle EBD =$  supplement of  $\angle ABD = \angle ACD$  (III. 13.). Also  $\angle E$  is common to  $\triangle s$   $EBD, ECA \therefore \triangle s$   $EBD, ECA$  are equiangular  
 $\therefore \frac{\triangle EAC}{\triangle EBD} = \frac{AC^2}{BD^2}$  (V. 11.).

15. Let  $BAC$  be a  $\triangle$  rt.  $\angle$  d. at  $A$ , and let  $AD$  be drawn perp. to  $BC$ .  $\frac{BD}{DC} = \frac{\triangle ABD}{\triangle CAD}$  (V. 1.)  $= \frac{AB^2}{AC^2}$  (V. 11.), for  $\triangle s$   $ABD, CAD$  are similar (V. 9.).

16. In  $\triangle s$   $ANC, BAC$ ,  $\angle C$  is common,  $\angle ANC = \angle BAC \therefore$  the  $\angle s$  are equiangular. In like manner,  $\triangle s$   $AMB, CAB$  are equiangular  $\therefore \triangle s$   $AMB, ANC$  are equiangular. Also  $\frac{BM}{NC} = \frac{\triangle AMB}{\triangle ANC}$   
(V. 1.)  $= \frac{AB^2}{AC^2}$  (V. 11.).

17.  $OL$  and  $OM$  are respectively perp. to  $AB$  and  $BC$ . Join  $OB$  cutting  $AM$  at  $P$ . From  $\triangle s$   $LOB, MOB$ ,  $\angle LOB = \angle MOB$  (1. 7.)  $\therefore$  from  $\triangle s$   $LOP, MOP$ ,  $\angle LPO = \angle MPO =$  a rt.  $\angle \therefore$   
 $\frac{\triangle LBM}{\triangle OLM} = \frac{BP}{OP}$  (V. 1. Cor.)  $= \frac{\triangle LPB}{\triangle LPO}$  (V. 1.)  $= \frac{BL^2}{OL^2}$  (V. 11.), (for  $\triangle s$   $LPB, LPO$  are similar)  $= \frac{(\text{side of fig.})^2}{(\text{diamr. of in-circle})^2}$ .

18. The rt.  $\angle$  d.  $\triangle s$   $OPN, OTC$  have the  $\angle O$  common  $\therefore$  the  $\triangle s$  are similar  $\therefore \frac{\triangle OPN}{\triangle OTC} = \frac{OP^2}{OT^2}$  (V. 11.)  $= \frac{OP^2}{OP \cdot OQ}$  (IV. 14.)  $= \frac{OP}{OQ}$ .

19. Join  $AC$ .  $\frac{\triangle BFG}{\triangle BAC} = \frac{BF^2}{BA^2}$  (V. 11.)  $= \frac{1}{9} \therefore \triangle BFG = \frac{1}{9} \triangle BAC$ . In like manner,  $\triangle DML = \frac{1}{9} \triangle DAC \therefore \triangle BFG + \triangle DML = \frac{1}{9}$  fig.  $ABCD$ . In like manner,  $\triangle AEN + \triangle CHK = \frac{1}{9}$  fig.  $ABCD \therefore$  fig.  $EFGHKL MN = \frac{7}{9}$  of fig.  $ABCD$ .

20. Let O be the centre of the circle, AB a side of the inscribed hexagon; EAF, EBG sides of the circum-hexagon. Join EO meeting AB at H. As in Question 12,  $\triangle AEB = \frac{1}{3} \triangle AOB$   $\therefore$  quadr. OAEB  $= \frac{4}{3} \triangle AOB$   $\therefore$  the circumscribed hexagon  $= \frac{4}{3}$  of the inscribed.

21. EA = EC  $\therefore \angle EAC = \angle ECA = \angle ABC$  (III. 18.)  $= \angle OCB$   $\therefore$   $\triangle$ s EAC, OCB are equiangular  $\therefore \frac{\triangle ECA}{\triangle OCB} = \frac{AC^2}{BC^2}$  (V. 11.)  $= \frac{\triangle ADC}{\triangle CDB}$  (V. 11.) for these  $\triangle$ s are similar (V. 9.)  $= \frac{AD}{BD}$  (V. 1.).

### EXERCISES LXI.

1. Let EG meet BC at K.  $\triangle$ s CKG, HAE, EKB are equiangular (I. 2. and 20.). Also CG, AE, EB are corresponding sides in these  $\triangle$ s. And AE = CF and EB = FG  $\therefore$  by V. 15.  $\triangle EBK = \triangle AEH + \triangle CKG$ . Adding DAEK to each, parm. ABCD  $= \triangle DHG$ .

### EXERCISES LXII.

1. Let the circles ACB, ADB intersect in A and B, P and Q being their centres. Let PQ meet the circles in C and D between P and Q. Draw the diameter EQF perp. to PQ.  $\angle QAP = \angle QBP = a$  rt.  $\angle$   $\therefore \angle$ s AQP, APQ, EQA, BQP are each equal to half a rt.  $\angle$   $\therefore$  half the circle ADB  $= 4$  sector AQD  $= 2$  sector AQB  $=$  sector AQB + sector APB  $=$  fig. AQB + area common to the circles  $= PA^2 +$  area common to both circles.

### EXERCISES LXIII.

1.  $AD^2 = BD \cdot DC = AD \cdot DE$  (IV. 13.)  $\therefore DA = DE$ . Also in  $\triangle$ s EDC, ECA,  $\angle E$  is common,  $\angle ECD = \angle BAE$  (in same segment)  $= \angle EAC$   $\therefore$  the  $\triangle$ s are equiangular  $\therefore \frac{ED}{EC} = \frac{EC}{EA}$   $\therefore EC^2 = ED \cdot EA$   $\therefore 2EC^2 = 2 \cdot ED \cdot EA = EA^2$ .

2. AD bisects  $\angle A$  (V. 3.). As in the previous exercise  $BE^2 = AE \cdot ED$ . Also as in V. 17.,  $AB \cdot AC = AE \cdot AD$   $\therefore AB \cdot AC + BE^2 = AE \cdot AD + AE \cdot ED = AE^2$ .

3. If P is the reqd. pt., and PM, PN be drawn perp. to AC and BD, and if X is the diameter of the circle,  $PA \cdot PC = X \cdot PM$

and PB.  $PD = X$ .  $PN \therefore PM = PN$ . Hence if AC and BD meet at O we bisect either of the  $\angle$ s AOB, AOD, any one of the four points where these lines meet the circumference will satisfy the reqd. condition.

4. By V. 19.  $PB \cdot AC + PC \cdot AB = PA \cdot BC \therefore PB + PC = PA$ .

5. Let the bisector of  $\angle A$  in  $\triangle ABC$  meet BC at D and the circum-circle at E. As in Example 1 above  $EC^2 = ED \cdot EA = 2ED^2$  for  $EA = 2 \cdot ED$ . Also from the similar  $\triangle$ s EDC, ECA,  $\frac{CA}{CE} = \frac{DC}{DE} \therefore \frac{CA^2}{CE^2} = \frac{CE^2}{DE^2} = 2$ , i.e.  $CA^2 = 2CD^2$ . In like manner,  $BA^2 = 2BD^2$ .

6. Let AB be the given base, P any position of the vertex, and let PM be perp. to AB. The rect.  $PA \cdot PB \propto \frac{1}{2}PM \cdot AB$ . Let X be the diameter of the circum-circle of  $\triangle PAB$ . Then by V. 18.  $PA \cdot PB = X \cdot PM \therefore X \cdot PM \propto \frac{1}{2}PM \cdot AB \therefore X$  is constant  $\therefore$  the locus of P is a circle which passes thro. the pts. A and B.

7. Let ABCD be the quadl., and let its diagonals meet at O. Draw BM, DN perp. to AC.  $BA \cdot BC = X \cdot BM$  where X is the diameter of the circum-circle (V. 18). Also  $DA \cdot DC = X \cdot DN$  by the same prop.  $\therefore \frac{BA \cdot BC}{DA \cdot DC} = \frac{BM}{DN} = \frac{OB}{OD}$  from the similar  $\triangle$ s BOM, DON.

8. Let EB produced meet the circum-circle of  $\triangle ABC$  at F. Join AF.  $\angle AFB = \text{supplement of } \angle ACB = \angle BCE$ .  $\angle ABF = \angle DBE = \angle CBE \therefore \triangle$ s ABF, EBC are equiangular  $\therefore \frac{AB}{BF} = \frac{BE}{BC}$  (V. 5.)  $\therefore AB \cdot BC = BE \cdot BF \therefore AB \cdot BC + BE^2 = BE^2 + BE \cdot BF = BE \cdot EF$  (IV. 3.)  $= CE \cdot EA$  (IV. 14. Cor.).

9. Let ABCD be the quadl., AC being a diameter of the circum-circle, and AC bisecting BD at E. AC is at rt.  $\angle$ s to BD (III. 3.).  $AB \cdot CD + AD \cdot BC = AC \cdot BD$  (V. 19.)  $= AC \cdot BE + AC \cdot DE = 2\triangle ABC + 2\triangle ADC = 2 \text{ fig. } ABCD$ .

10. Draw AM perp. to BC. Let X be the diameter of the circum-circle of  $\triangle ABD$ , and Y that of the circum-circle of  $\triangle ACD$ .  $AB \cdot AD = X \cdot AM$  (V. 18.).  $AC \cdot AD = Y \cdot AM$  (V. 18.)  $\therefore \frac{X}{Y} = \frac{AB}{AC} = \frac{AD}{CD}$  from the similar  $\triangle$ s ABD, CAD.

## EXERCISES LXIV.

1. Let AB be the given str. line. Draw any other str. line AC, and from it cut off equal parts AD, DE, EF, FG, GH. Join BH, and draw DP, EQ, FR, GS  $\parallel$  to BH to meet AB at P, Q, R, S. By V. 21. AB is divided into 5 equal parts at P, Q, R, S.

2. Let AB be the given line. Draw any other line AH from it, and cut off equal parts AD, DE, EF, FG, GH. Join EB and draw HP  $\parallel$  to EB to meet AB produced in P.  $\frac{AP}{PB} = \frac{AH}{HE}$  (V. 2.)  $= \frac{5}{3}$ .

3. Let EF be the given line, ABCD the given rect. To EF AB, BC, find a fourth proportional FG as in V. 23.  $\frac{EF}{AB} = \frac{BC}{FG} \therefore EF \cdot FG = AB \cdot BC \therefore$  the rect. contained by EF and FG is the reqd. rect.

4. If ABCD is the given rect., to AB and BC find a mean proportional EF as in V. 24.  $EF^2 = AB \cdot BC \therefore$  the sq. on EF is the reqd. sq.

5.  $\angle ABC =$  supplement of  $\angle CBE = \angle CDE$ , and  $\angle A$  is common to  $\triangle s$  ABC, ADE  $\therefore \frac{AE}{AD} = \frac{AC}{AB} = \frac{4}{3}$ . Also  $\frac{AD}{AB} = \frac{10}{3} \therefore \frac{AE}{AB} = \frac{40}{9}$ , i.e.  $\frac{AB + BE}{AB} = \frac{40}{9} \therefore \frac{BE}{BA} = \frac{31}{9}$ .

6. Draw AB 3.6 in. long, and in AB produced make BC 1 in. long. On AC describe a semi-circle ADC, and draw BD perp. to AC to meet the circle at D.  $BD^2 = AB \cdot BC$  (V. 9. Cor.)  $= 3.6$  sq. in.  $\therefore$  the sq. on BD is the reqd. sq. By measurement  $BD = 1.90$  in.

7. Draw AB 2.4 in. long, and draw AC perp. to AB and 1 in. long. Join BC. Draw CD perp. to BC to meet BA pro-

duced in D.  $\triangle$ s CAB, DAC are similar (V. 9.)  $\therefore \frac{AB}{AC} = \frac{AC}{AD} \therefore AB \cdot AD = AC^2 = 1 \text{ sq. in.} \therefore$  the rect. contained by AB and AD is the reqd. rect.

8. Draw lines 3·7, 1·7, and 2·9 in. long. Find AB a fourth proportional to these. Then  $\frac{3\cdot7}{1\cdot7} = \frac{2\cdot9}{AB} \therefore AB \times 3\cdot7 = 1\cdot7 \times 2\cdot9 \therefore$  the rect. whose sides are AB and the line 3·7 in. long is the reqd. rect.

9. See Book V., Prop. 28.

10. If ABC is the given  $\triangle$ , bisect BC at D, and draw BF, DE perp. to BC. Also draw AEF  $\parallel$  to BC. Rect. FBDE =  $\triangle$ ABC (IV. 9.). To BD and DE find a mean proportional X (V. 24.). Then BD · DE =  $X^2 \therefore$  the sq. on X is the reqd. sq.

11. Use the method of Example 6 above. By measurement the side of the sq. will be found to be 1·67 in.

12. Describe, by the method of Example 6 above, a sq. whose area is 3·6 sq. in. If X be its side,  $X^2 = 3\cdot6 \therefore X = \sqrt{3\cdot6}$ . By measurement X will be found to be 1·90 in. long  $\therefore \sqrt{3\cdot6} = 1\cdot90$ .

13. Describe the sq. by the method of Example 10 above. By measurement its side will be found to be 1·49 in.

14. Let ABC be the given  $\triangle$ . Trisect AB at D and E. (V. 20.). Draw DH  $\parallel$  to BC to meet AC at H.  $\triangle$ s ADH, ABC are equiangular (I. 20. and 22.)  $\therefore \frac{\triangle ADH}{\triangle ABC} = \frac{AD^2}{AB^2} = \frac{1}{9} \therefore$  ADH is the reqd.  $\triangle$ .

15. Let ABC be the given  $\triangle$ . Take  $AD = \frac{1}{4}AB$  (V. 20.). Draw DE  $\parallel$  to BC to meet AC at E. As in the preceding example  $\frac{\triangle ADE}{\triangle ABC} = \frac{AD^2}{AB^2} = \frac{1}{16} \therefore$  ADE is the reqd.  $\triangle$ .

16. Produce AB to D, making BD equal to BA. Draw DE  $\parallel$  to AC and CE  $\parallel$  to AD. Parm. ADEC =  $2\triangle ADC = 4\triangle ABC \therefore$  ADEC is the reqd. parm.

17. From the external pt. A draw any str. line cutting the circle at B and C. Take X a mean proportional to AB, AC. With centre A and rad. X describe a circle cutting the given circle at D.  $AC \cdot AB = X^2 = AD^2 \therefore AD$  is a tangent to the given circle (IV. 15.).

18. Draw AB 3.12 in. long, and BC at rt.  $\angle$ s to it 1.28 in. long, and complete the rect. ABCD. To AB and BC find a mean proportional BE as in V. 24.  $BE^2 = AB \cdot BC$  and the sq. on BE is the sq. reqd. By measurement  $BE = 2$  in., and the diagonal of the sq. on  $BE = 2.82$  in.

19. Draw  $\angle BAC = 30^\circ$  with a protractor, or by bisecting the  $\angle$  of an equilateral  $\triangle$ . Make  $AB = 4$  in. and  $AC = 5$  in. Join CB. Produce AB to D making  $AD = 4.5$  in. Join CD, and draw  $BE \parallel$  to  $DC$ , to meet AC at E. Join DE.  $\triangle BEC = \triangle BED$  (II. 5.)  $\therefore \triangle ADE = \triangle ABC$ . Thro. E draw  $EF \parallel$  to  $AD$ . With centre A and rad. 3.25 in. describe a circle cutting EF at F. Join AF, DF.  $\triangle ADF = \triangle AED$  (II. 5.)  $= \triangle ABC \therefore ADF$  is the  $\triangle$  reqd.

20. Use the method of Example 10 above.

21. By V. 22. find E a third proportional to A and B. Then  $\frac{A}{E} = \frac{A^2}{B^2}$  (V. 12., Cor. 2.). To A, E, and C find a fourth proportional D, as in V. 23. Then  $\frac{C}{D} = \frac{A}{E} = \frac{A^2}{B^2} \therefore D$  is the reqd. line.

22. Let AB be the given line. With centre B and any rad describe a circle; and with centre A and rad. double the first describe a second circle, cutting the first at C. Join AC, BC, and bisect  $\angle ACB$  by AD meeting AB at D.  $\frac{AD}{DB} = \frac{AC}{CB}$  (V. 3.)  $= \frac{2}{1}$   $\therefore D$  is a point of trisection of AB. Bisecting AD we have the other pt. of trisection.

23. Let AB be the given str. line. Draw any other str. line AC, and in it make  $AD = 5$  cms. and  $DE = 3$  cms. Join EB, and draw  $DF \parallel$  to EB to cut AB at F.  $\frac{AF}{FB} = \frac{AD}{DE}$  (V. 2.)  $= \frac{5}{3}$ .

**24.** Let  $AB$  be the given line. With any unit greater than one-fifth of  $AB$ , describe a circle with centre  $A$  and rad. 3 units. Also with centre  $B$  and rad. 2 units describe a circle cutting the first at  $C$ . Join  $AC$ ,  $BC$ , and produce  $AC$  to  $D$ . Bisect  $\angle BCD$  by  $CE$  meeting  $AB$  produced at  $E$ .  $\frac{AE}{EB} = \frac{AC}{CB}$  (V. 4.)  $= \frac{3}{2}$ .

**25.** Let  $ABC$  be the  $\triangle$ . Bisect  $AB$  at  $D$ , and draw  $DE$  perp. to  $AB$ , making  $DE = DA = DB$ . With centre  $A$  and rad.  $AE$  describe a circle cutting  $AB$  at  $F$ . Draw  $FG \parallel$  to  $BC$ , meeting  $AC$  at  $G$ . From  $\triangle s$   $BDE$ ,  $ADE$ ,  $\angle BED = \angle EBD = \angle EAD = \angle DEA = \frac{1}{2}$  a rt.  $\angle$ , and  $BE = AE \therefore BA^2 = 2EA^2$  (II. 11.)  $= 2AF^2 \therefore \frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AB^2} = \frac{1}{2} \therefore \triangle AFG = \frac{1}{2} \triangle ABC$ .

**26.** Let  $ABC$  be the  $\triangle$ . On  $AB$  describe the equilateral  $\triangle ABD$ . Bisect  $\angle ABD$  by  $BE$ , and draw  $AE$  perp. to  $BA$  to meet this line at  $E$ . With centre  $A$  and rad.  $AE$  describe a circle cutting  $AB$  at  $F$ . Draw  $FG \parallel$  to  $BC$  to meet  $AC$  at  $G$ .  $\angle ABE = 30^\circ$  and  $\angle BAE =$  a rt.  $\angle \therefore \triangle BEA$  is half the equilateral  $\triangle$  on  $BE \therefore BE = 2EA \therefore AB^2 = BE^2 - EA^2 = 3EA^2$ .  $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AB^2} = \frac{AE^2}{AB^2} = \frac{1}{3} \therefore \triangle AFG = \frac{1}{3} \triangle ABC$ . Again, at  $F$  draw  $FH$  perp. to  $AB$  and equal to  $FA$ . With centre  $A$  and rad.  $AH$  describe a circle cutting  $AB$  at  $K$ . Draw  $KM \parallel$  to  $BC$  to meet  $AC$  at  $M$ .  $\frac{\triangle AKM}{\triangle AFG} = \frac{AK^2}{AF^2} = \frac{AH^2}{AF^2} = 2 \therefore \triangle AKM = 2\triangle AFG = \frac{2}{3} \triangle ABC \therefore FG$  and  $KM$  trisect triangle  $ABC$ .

**27.** Draw  $AB$  so that the sq. on  $AB =$  the given sum of sqs. On  $AB$  as diamr. describe a circle, centre  $O$ , and draw  $OD$  perp. to  $AB$  to meet the circle at  $D$ . Divide  $AB$  in the given ratio at  $C$ . Join  $DC$ , and produce it to meet the circle at  $E$ . Join  $AE$ ,  $BE$ .  $\angle AEC = \frac{1}{2} \angle AOD = \frac{1}{2}$  a rt.  $\angle$ . In like manner  $\angle DEB = \frac{1}{2}$  a rt.  $\angle \therefore AE^2 + BE^2 = AB^2 =$  the given sum of sqs. Also  $\frac{AE}{BE} = \frac{AC}{CB}$  (V. 3.)  $=$  the given ratio  $\therefore AE$  and  $EB$  are the reqd. lines.

*Second method.* Let  $a$  and  $b$  denote the reqd. lines, so that  $\frac{a}{b} = \frac{AC}{CB}$ . Draw  $CD$  perp. to  $AC$ , making  $CD = CB$ . Join  $AD$ .

$\frac{b}{a} = \frac{CB}{CA} \therefore \frac{a^2 + b^2}{a^2} = \frac{CB^2 + CA^2}{CA^2} = \frac{CD^2 + CA^2}{CA^2} = \frac{DA^2}{CA^2} \therefore \frac{AB^2}{a^2} = \frac{DA^2}{CA^2}$   
 $\therefore \frac{DA}{CA} = \frac{AB}{a} \therefore a$  is a fourth proportional to  $DA$ ,  $CA$ ,  $AB$ , and this can be found by V. 23. A fourth proportional to  $AC$ ,  $CB$ ,  $a$  gives  $b$ .

**28.** Let  $a, b, c$  denote the reqd. lines. Let  $AD$  be a str. line such that  $a^2 + b^2 + c^2 = AD^2$ . Divide  $AD$  at  $B$  and  $C$  in the given ratios, so that  $a : b : c :: AB : BC : CD$ . Draw  $BE$  perp. to  $BA$  and equal to  $BC$ . Join  $AE$ . Draw  $EF$  perp. to  $AE$  and equal to  $CD$ . Join  $AF$ .  $\frac{b}{a} = \frac{BC}{AB} \therefore \frac{b^2 + a^2}{a^2} = \frac{BC^2 + AB^2}{AB^2} = \frac{AE^2}{AB^2}$ .  
 Also  $\frac{c}{a} = \frac{CD}{AB} \therefore \frac{c^2}{a^2} = \frac{EF^2}{AB^2} \therefore \frac{a^2 + b^2 + c^2}{a^2} = \frac{AE^2 + EF^2}{AB^2} = \frac{AF^2}{AB^2} \therefore \frac{AD^2}{a^2} = \frac{AF^2}{AB^2}$  or  $\frac{AF}{AB} = \frac{AD}{a} \therefore a$  is a fourth proportional to  $AF$ ,  $AB$ ,  $AD$ , and this is found as in V. 23.  $b$  and  $c$  can then be found by the method of the preceding example.

**29.** Let  $a$  and  $b$  denote the reqd. lines,  $a$  being the greater. Take  $AB$  such that  $AB^2 = a^2 - b^2 =$  the given difference. Divide  $AB$  at  $C$  in the given ratio, so that  $\frac{a}{b} = \frac{AC}{CB}$ . On  $AC$  describe a semicircle  $ADC$ , and place a chord  $CD$  in it equal to  $CB$ . Join  $AD$ .  $\frac{b}{a} = \frac{CB}{CA} \therefore \frac{a^2 - b^2}{a^2} = \frac{CA^2 - CB^2}{CA^2} = \frac{CA^2 - CD^2}{CA^2} = \frac{AD^2}{CA^2} \therefore \frac{AB^2}{a^2} = \frac{AD^2}{CA^2}$   
 $\frac{AD}{CA} = \frac{AB}{a} \therefore$  a fourth proportional to  $AD$ ,  $CA$ ,  $AB$  gives us the line  $a$ , and this can be found by V. 23. A fourth proportional to  $AC$ ,  $CB$ ,  $a$  gives us the line  $b$ .

**30.** Let  $ABC$  be the given  $\triangle$ . Draw  $BD$  perp. to  $BA$  and equal to it. With centre  $A$  and rad.  $AD$  describe a circle meeting  $AB$  produced at  $E$ . Draw  $EF \parallel$  to  $BC$  to meet  $AC$  produced at  $F$ .  $\triangle AEF, ABC$  are equiangular (I. 20.) and  $\therefore$  similar (V. 5.)  $\therefore \frac{\triangle AEF}{\triangle ABC} = \frac{AE^2}{AB^2} = \frac{AD^2}{AB^2} = 2 \therefore \triangle AEF$  is similar to  $\triangle ABC$  and twice its area.

31. Let  $ABC$  be the given  $\triangle$ . Produce  $BA$  to  $D$  making  $AD=AB$ , and on  $BD$  describe an equilateral  $\triangle BDE$ . Join  $AE$ . With centre  $A$  and rad.  $AE$  describe a circle cutting  $AB$  produced at  $F$ . Draw  $FG \parallel$  to  $BC$  to meet  $AC$  produced at  $G$ .  $AE$  is perp. to  $BD$  (I. 7.)  $\therefore EA^2 = EB^2 - BA^2 = 3BA^2$ ;  $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AB^2} = \frac{AE^2}{AB^2} = 3$ . Also the  $\triangle$ s are similar by construction  $\therefore AFG$  is the reqd.  $\triangle$ .

32. Let  $ABC$  be the given  $\triangle$ . Draw  $BD$  perp. to  $BA$  and equal to  $2BA$ . With centre  $A$  and rad.  $AD$ , describe a circle cutting  $AB$  produced at  $E$ . Draw  $EF \parallel$  to  $BC$  to meet  $AC$  produced at  $F$ .  $\frac{\triangle AEF}{\triangle ABC} = \frac{AE^2}{AB^2} = \frac{AD^2}{AB^2} = \frac{5AB^2}{AB^2}$  (II. 11.) = 5. Also the  $\triangle$ s are similar by construction  $\therefore AEF$  is the reqd.  $\triangle$ .

33. Let  $ABC$  be the given  $\triangle$ . To  $AB$  and  $AC$  find a mean proportional  $X$ , and from  $AB$  and  $AC$  (produced if necessary) cut off  $AD=AE=X$ . Join  $DE$ . Rect.  $AB \cdot AC = X^2 = \text{rect. } AD \cdot AE$   $\therefore \triangle$ s  $ADE, ABC$  are equal in area (V. 10.). Also  $ADE$  is isosceles, and is  $\therefore$  the reqd.  $\triangle$ .

34. Join  $OA$ , and divide it at  $F$ , so that  $\frac{OF}{FA}$  = the given ratio. Draw  $FD \parallel$  to  $AC$  to meet  $AB$  at  $D$ . Join  $OD$  and produce it to meet  $AC$  at  $E$ .  $\frac{OD}{DE} = \frac{OF}{FA}$  (V. 2.) = the given ratio.

35. Describe the circum-circle, and from it cut off a segment  $ACB$  containing an angle equal to the given vertical  $\angle$  (III. 25.). Bisect arc  $AEB$ , on the opp. side of  $AB$  to the pt.  $C$ , at  $E$ . Divide  $AB$  at  $D$  in the given ratio of the sides. Join  $ED$  and produce it to meet the circle at  $C$ . Join  $AC, CB$ . Arc  $AE = \text{arc } BE$   $\therefore \angle ACE = \angle BCE$   $\therefore \frac{AC}{CB} = \frac{AD}{DB}$  = the given ratio.  $\therefore ACB$  is the  $\triangle$  reqd.

36. Let  $\triangle AOB$  be gr. than  $\triangle COD$ . Take a pt.  $F$  in  $OA$  such that  $OF$  is a fourth proportional to  $OB, OC$ , and  $OD$ . Then  $\frac{OB}{OC} = \frac{OD}{OF}$   $\therefore OB \cdot OF = OC \cdot OD$   $\therefore \triangle BOF = \triangle COD$  (V. 10.). Thro.  $F$  draw  $FE \parallel$  to  $OB$  to meet  $AB$  at  $E$ . Join  $OE$ .  $\triangle BEO = \triangle BOF$  (II. 5.) =  $\triangle COD$   $\therefore BEO$  is the reqd.  $\triangle$ .

**37.** Make  $AD = 1.5$  in. so that  $DB = .9$  in.  $\frac{AD}{DB} = \frac{15}{9} = \frac{5}{3}$ . In  $AB$  produced make  $BE = 3.6$  in.  $\frac{BE}{EA} = \frac{3.6}{6} = \frac{3}{5}$ . Then as in Exercises lv. 3, we see that the circle on  $DE$  as diameter is the locus.

**38.** Let  $\frac{AC}{CB}$  be the given ratio,  $ACB$  being a str. line. On  $AB$  describe a semicircle and draw  $CD$  perp. to  $AB$  to meet it at  $D$ . Join  $DA$ ,  $DB$ . In  $CD$  make  $CE$  equal to a side of the given sq., and draw  $EF$ ,  $EG \parallel$  to  $DA$ ,  $DB$  respectively to meet  $AB$  in  $F$  and  $G$ .  $AC \cdot CB = CD^2$  (V. 9.). Also  $\triangle s$   $ECF$ ,  $ACD$  are similar and  $\triangle ECG$ ,  $DCB \therefore \frac{CE}{CF} = \frac{CD}{CA}$  and  $\frac{CE}{CG} = \frac{CD}{CB} \therefore \frac{CE^2}{CF \cdot CG} = \frac{CD^2}{CA \cdot CB} = 1 \therefore CE^2 = CF \cdot CG$ . Also  $\frac{CF}{CA} = \frac{CE}{CD}$  (V. 2.)  $= \frac{CG}{CB} \therefore \frac{CF}{CG} = \frac{CA}{CB} \therefore$  the rect.  $CF \cdot CG$  in the reqd. rect.

**39.** Let  $AB$  be a side of the given equilateral  $\triangle$ . Divide  $AB$  at  $C$  in the given ratio. On  $AB$  describe a semi-circle, and draw  $CD$  perp. to  $AB$  to meet it at  $D$ . Join  $AD$ ,  $BD$ . Equilateral  $\triangle$  on  $AD$   $\frac{AD^2}{BC \cdot BA} = \frac{AC \cdot AB}{BC}$ . Also the sum of these  $\triangle s$  = equilateral  $\triangle$  on  $AB$  (V. 15.)  $\therefore$  the equilateral  $\triangle s$  on  $AD$ ,  $BD$  are the reqd.  $\triangle s$ .

**40.** On  $AB$ , 3.8 in. long, describe a semi-circle  $ABC$ . Draw  $BD$ , 1.3 in. long, at rt.  $\angle s$  to  $BA$ . Draw  $DC \parallel$  to  $BA$  to meet the semi-circle at  $C$ , and draw  $CE$  perp. to  $BA$ .  $\angle ACB$  is a rt.  $\angle \therefore AE \cdot EB = CE^2$  (V. 9.)  $= BD^2 \therefore AE$ ,  $EB$  are the reqd. segments. By measurement,  $AE = 3.28$  in. and  $BE = .52$  in.

**41.** Let  $ABCDE$  be the given figure,  $ab$  ( $\parallel$  to  $AB$ ) the given str. line. Using the method of I. 31. (1st method), draw  $bc \parallel$  to  $BC$ ,  $ac \parallel$  to  $AC$ ,  $cd \parallel$  to  $CD$ ,  $ad \parallel$  to  $AD$ ,  $de \parallel$  to  $DE$ ,  $ae \parallel$  to  $AE$ ,  $abcde$  is similar to  $ABCDE$  (V. 25.).

**42.** Let  $D$  be the given pt. in  $AB$ ,  $BD$  being gr. than  $DA$ . Take  $E$  in  $BC$  such that  $BE = \frac{1}{3}BC$  (V. 20.). Join  $DE$  and draw  $AF \parallel$  to  $DE$  to meet  $AB$  in  $F$ . Join  $DF$ .  $\triangle DEF = \triangle DAE \therefore$

$\triangle DFB = \triangle AEB = \frac{1}{3}\triangle ABC$ . If  $CF > FB$ , from  $FC$  cut off  $FG = FB$ .  
 $\triangle DFG = \triangle DFB$  (II. 6.)  $= \frac{1}{3}\triangle ABC$  and the problem is solved. If  
 $CF < FB$ , as in the first part, find a point  $H$  in  $AC$  such that  
 $\triangle AHD = \frac{1}{3}\triangle ABC$ . The pt.  $F$  might also be found as follows.  
 To  $BD$ ,  $AB$ ,  $BE$  find a fourth proportional  $BF$  (V. 23.). Then  
 $\frac{BD}{AB} = \frac{BE}{BF} \therefore BD \cdot BF = AB \cdot BE \therefore \triangle BDF = \triangle ABE$  (V. 10.)  $=$   
 $\frac{1}{3}\triangle ABC$ .

**43.** Let  $ABC$  be the  $\triangle$ ,  $P$  the given pt. within it. Join  $PA$ ,  
 $PB$ ,  $PC$ . Draw  $AD \parallel$  to  $PB$  to meet  $CB$  produced at  $D$ ; and  
 draw  $AE \parallel$  to  $PC$  to meet  $BC$  produced at  $E$ . Join  $PD$ ,  $PE$ .  
 $\triangle ABP = \triangle BPD$  and  $\triangle APC = \triangle PCE$  (II. 5.)  $\therefore \triangle DPE = \triangle ABC$ .  
 Divide  $DE$  into four equal parts,  $DL$ ,  $LM$ ,  $MN$ ,  $NE$  (V. 21.).  
 Let  $M$  fall within  $BC$ ,  $L$  and  $N$  without it. Join  $PM$ . Draw  
 $LQ \parallel$  to  $PB$  to meet  $AB$  at  $Q$ , and  $NR \parallel$  to  $CP$  to meet  $AC$  at  $R$ .  
 Join  $PQ$ ,  $PR$ .  $\triangle AQP + \triangle BQP = \triangle APB = \triangle PBD$  (II. 5.)  $= \triangle PBL$   
 $+ \triangle PLD$ . But  $\triangle BQP = \triangle BPL$  (II. 5.)  $\therefore \triangle AQP = \triangle PLD =$   
 $\frac{1}{4}\triangle PDE = \frac{1}{4}\triangle ABC$ . In like manner,  $\triangle APR = \frac{1}{4}\triangle ABC$ . Also  
 quadl.  $MPQB = \triangle PBM + \triangle PQB = \triangle PBM + \triangle PBL$  (II. 5.)  $= \triangle LMP$   
 $= \frac{1}{4}\triangle PDE = \frac{1}{4}\triangle ABC$ . In like manner, quadl.  $PMCR = \frac{1}{4}\triangle ABC$   
 $\therefore$  the str. lines  $PQ$ ,  $PM$ ,  $PR$  divide the  $\triangle$  into four equal  
 parts.

**44.** Let  $ABC$  be the  $\triangle$ . Draw  $AD$  perp. to  $BC$ , and bisect  $BC$  at  
 $E$ ,  $E$  falling between  $D$  and  $B$ . Let  $BGH$  be the reqd.  $\triangle$ ,  $GH$   
 being perp. to  $BC$  at  $G$ .  $2\triangle BGH = \frac{BG \cdot GH}{\frac{1}{2}BC \cdot AD} = \frac{BG \cdot GH}{BE \cdot AD} = \frac{BG^2}{BE \cdot BD}$ ,  
 for by similar  $\triangle$ s,  $\frac{GH}{AD} = \frac{BG}{BD} \therefore BG^2 = BE \cdot BD$ . Hence the  
 following construction. To  $BE$  and  $BD$  take a mean propor-  
 tional  $BG$  along  $BC$ . Draw  $GH$  perp. to  $BC$  to meet  $AB$  at  $H$ .  
 $\triangle BGH = \frac{1}{2}\triangle ABC$ .

**45.** Let  $ABCD$  be the quadl.,  $O$  the given pt. in  $AB$ . Join  
 $OD$ ,  $OC$ . Draw  $AE \parallel$  to  $OD$  to meet  $CD$  produced at  $E$ . Draw  
 $BF \parallel$  to  $OC$  to meet  $DC$  produced at  $F$ . Join  $OE$ ,  $OF$ .  $\triangle OEF$   
 $=$  fig.  $ABCD$  (II. 5.). Bisect  $EF$  at  $G$ . Join  $OG$ . Fig.  $ADGO$   
 $= \triangle OGD + \triangle OAD = \triangle OGD + \triangle OED$  (II. 5.)  $= \triangle EOG = \frac{1}{2}\triangle OEF$   
 $= \frac{1}{2}$  fig.  $ABCD \therefore OG$  bisects the quadl.

**46.** Let O be the given pt. in the side AB of the quadl. ABCD. Reduce the quadl. to a triangle OEF as in the preceding example. Divide EF into three equal parts EG, GH, HF (V. 21.). Join OG, OH. As in the preceding, fig. DGOA =  $\triangle$ EOG =  $\frac{1}{3}$  quadl. ABCD. In like manner  $\triangle$ OGH = fig. OHCB =  $\frac{1}{3}$  quadl. ABCD  $\therefore$  OG, OH divide the quadl. into three equal parts.

**47.** Let ABCD be the sq. Join AC, BD. Draw AE perp. to AC, and make  $\angle ACE$  equal to  $30^\circ$ . With centre A and rad. AE describe a circle cutting AD at F. Draw FG  $\parallel$  to BD, to meet AB at G.  $EC = 2AE$ .  $EC^2 - AE^2 = AC^2 \therefore 3AE^2 = AC^2 = 2AB^2 \therefore \frac{AE^2}{AB^2} = \frac{2}{3}$ . Also  $\frac{\triangle AFG}{\triangle ADB} = \frac{AF^2}{AD^2} = \frac{AE^2}{AB^2} = \frac{2}{3} \therefore \triangle AFG = \frac{2}{3} \triangle ADB = \frac{1}{3}$  sq. ABCD. Cutting off DH from DC equal to DF, and drawing HK  $\parallel$  to DB,  $\triangle CHK = \frac{1}{3}$  sq. ABCD  $\therefore$  GF and HK,  $\parallel$  to BD, trisect the sq.

**48.** In AB 5 cms. long take a pt. C such that  $\frac{CB}{CA} = \frac{3}{5}$  (V. 21.). On AB describe a segment of a circle containing an angle of  $60^\circ$  and use the method of Example 35 above.

**49.** Draw  $\angle BAC = 45^\circ$ , and take  $\frac{BA}{AC} = \frac{3}{5}$ . Join BC. Draw AD perp. to BC and from it, produced if necessary, cut off AE = 6.5 cms. Draw FEG perp. to AE, meeting AC, AB at F and G. By similar  $\triangle$ s  $\frac{AC}{AF} = \frac{AD}{AE} = \frac{AB}{AG} \therefore \frac{AF}{AG} = \frac{AC}{AB} = \frac{5}{3}$ . Also the alt. of  $\triangle AFG = 6.5$  cms.  $\therefore$  AFG is the  $\triangle$  reqd.

**50.** Draw AB = 3.7 cms. and from it produced cut off AC = 8.6 cms. Draw AE perp. to AB and equal to 1 cm. Take X a fourth proportional to AB, AE, and AC, and from AE produced cut off AF = X. Draw FHK  $\parallel$  to AB, and bisect  $\angle CAF$  by AH meeting FK at H. Draw BK  $\parallel$  to AH. ABKH is the reqd. parm. Draw BG perp. to AB to meet FHK at G. Parm. ABKH = rect. ABGF (II. 3.) = AB . AF = AB . X = AE . AC = 8.6 sq. cms.

**51.** Make  $\angle ACB = 41^\circ$ , and cut off AC = 3.1 in. With centre A and rad. 2.7 in. describe a circle cutting BC at B. Make  $\angle BCD = 63^\circ$  and let CD meet BA produced at D. To BA

and BD find a mean proportional  $X$  (V. 24.), and from BD cut off  $BE=X$ . Draw  $EF \parallel$  to  $CD$  to meet  $CB$  at  $F$ .  
 $\frac{\triangle EBF}{\triangle BCD} = \frac{BE^2}{BD^2} = \frac{BA \cdot BD}{BD^2} = \frac{BA}{BD} = \frac{\triangle ABC}{\triangle CDB} \therefore \triangle EBF = \triangle ABC$  and  
 $\angle EFB = \angle DCB = 63^\circ \therefore EBF$  is the reqd.  $\triangle$ .

52. Draw perp. diameters  $AOB$ ,  $COD \therefore \angle ACB = \angle$  in semi-circle = a rt.  $\angle$ . Similarly  $\angle$ s at  $A$ ,  $B$ ,  $D$  are rt.  $\angle$ s. Also from  $\triangle$ s  $AOC$ ,  $BOC$ ,  $AC=BC$  (I. 4.)  $\therefore ADBC$  is a sq. With centre  $D$  and rad.  $DO$  describe a circle cutting the first circle in  $E$  and  $F$ .  $C$ ,  $E$ ,  $F$  are alternate pts. of a regular hexagon in the circle.  $\therefore CEF$  is an equilateral  $\triangle$ . To  $EF$  and  $AD$  take a third proportional  $X$  (V. 22.).  $\frac{EF}{X} = \frac{EF^2}{AD^2}$  (V. 12. Cor. 2.) =  $\frac{\text{sq. on side of equilateral } \triangle}{\text{sq. in the circle}}$ . By measurement the ratio of  $X$  to  $EF$  will be found to be  $2:3$ .

53. Let  $AB$ ,  $BC$  be the given str. lines,  $O$  the given pt. Draw  $OD \parallel$  to  $CB$  to meet  $BA$  at  $D$ . In  $BD$  produced make  $DA = 2BD$ . Join  $AO$  and produce it to meet  $BC$  at  $C$ .  $\frac{AO}{OC} = \frac{AD}{DB}$  (V. 2.) =  $\frac{2}{1} \therefore AOC$  is the reqd. line.

54. Let  $ABCD$  be the given sq. Make  $\angle BAE 60^\circ$ , the pt.  $E$  being in  $BC$  produced. From  $BE$  cut off  $BF$  a mean proportional to  $BC$  and  $BE$ . Draw  $FG \parallel$  to  $EA$  to meet  $AB$  at  $G$ . In  $AB$  produced make  $BH = BF$ . Join  $FH$ ,  $AC$ .  $\frac{\triangle GFB}{\triangle AEB} = \frac{BF^2}{BE^2} = \frac{BC}{BE} = \frac{\triangle ABC}{\triangle AEB}$  (V. 1.)  $\therefore \triangle GFB = \triangle ABC \therefore \triangle GFH = 2\triangle ABC = \text{square } ABCD$ . Also  $\triangle GFH$  is equilateral, since  $\angle H = \angle BGF = 60^\circ$ .

55. Let  $ABCD$  be the sq. Take  $BE = \frac{1}{3}BA$  (V. 20.). Make  $\angle PBE = 60^\circ$ ,  $P$  falling in  $CD$ . Join  $PE$ .  $\triangle PBE = \frac{1}{6}$  sq.  $ABCD$ . To  $BE$  and  $BP$  take a mean proportional  $FG$  (V. 24.). On  $FG$  describe an equilateral  $\triangle FOG$ .  $\triangle FOG = \triangle PBE$  (V. 10.). With centre  $O$  and rad.  $OF$  or  $OG$  describe a circle, and round its circumference set off chords  $GH$ ,  $HK$ ,  $KL$ ,  $LM$  each equal to its radius.  $FGHKLM$  is a regular hexagon and is equal to six times  $\triangle FOG$ , and is therefore equal to the sq.

56. Let  $ABC$  be the equilateral triangle. Bisect  $AC$  at  $D$ . From  $AC$  cut off  $AF$  a mean proportional to  $AD$ ,  $AC$ , and draw

FG  $\parallel$  to CB to meet AB in G. Draw FH, GH perp. to AF, AG; FK, AK perp. to FG, AG; AL, GL perp. to FA, FG. The hexagon ALGHFK =  $2\triangle AFG = \triangle ABC$ , for  $\frac{\triangle AFG}{\triangle ABC} = \frac{AF^2}{AC^2} = \frac{AC \cdot AD}{AC^2} = \frac{AD}{AC} = \frac{1}{2}$ .

57. Let O be the pt. within the circle. Draw any chd EOF and take OD a mean proportional to OE and OF (V. 24.). If  $x$  and  $y$  are the segments of the reqd. chord,  $x \cdot y = OE \cdot OF = OD^2 \dots (1)$ . Take a line  $p$  such that  $\frac{p}{OD} =$  the given ratio  $= \frac{x}{y} \dots (2)$ . From (1)  $\frac{x}{OD} = \frac{OD}{y} = \frac{p}{x} \therefore x^2 = p \cdot OD \dots (3)$ . Hence the following construction. Take  $p$  (as above) such that the given ratio  $= \frac{p}{OD}$  (V. 23.) then one segment [by (3)] is a mean proportional between  $p$  and OD.

58. On AB the given base describe a segment of a circle containing an angle equal to the given vertical angle. Draw AC a diameter. Let P, Q be the sides of the given rectangle. To AC, P and Q take a fourth proportional X. At A draw AD perp. to AB and equal to X. Draw DE  $\parallel$  to AB to meet the circle at E. Draw EF perp. to AB. Rect. AE . EB = rect. AC . EF (V. 18.) = AC . X = P . Q. Also  $\angle AEB =$  the given angle  $\therefore$  AEF is the reqd.  $\triangle$ .

59. Draw AB 3 in. long, and with centre A and rad. 2.3 in. describe a circle. Draw BC perp. to AB  $\frac{2}{3}$  in. long, and draw CD  $\parallel$  to BA to meet the circle at D. Join DB. Draw DE perp. to AB. Area of  $\triangle ADB = \frac{1}{2} AB \cdot DE = \frac{1}{2} \times 3 \times \frac{2}{3} = 1$  sq. in. By measurement DB = 1.04 in. If we take the other pt. where CD meets the circle, we obtain a second  $\triangle$  satisfying the given conditions.

60. Draw a str. line ABCD, such that AB = 1 in., BC = 3 in., BD = 4.7 in. On AC and AD (on the same side of the line) draw semi-circles. Draw BEF perp. to ABC to meet the circles at E and F. Join EA, EC. Draw FG  $\parallel$  to EA and FH  $\parallel$  to EC to meet ABC at G and H.  $\frac{BG}{BA} = \frac{BF}{BE} = \frac{BH}{BC} \therefore \frac{BG}{BH} = \frac{BA}{BC} = \frac{1}{3}$ . Also BG . BH = BF<sup>2</sup> (for  $\angle GFH = \angle AEC =$  a rt.  $\angle$ ) = BA . BD = 4.7 sq. in.  $\therefore$  the rect. contained by BG and BH is the reqd. rect.

**61.** Draw a str. line AB 3.28 in. long, and AC at rt.  $\angle$ s to it 2 in. long. Join BC. Make  $\angle BAD = 60^\circ$ , and draw CD  $\parallel$  to AB. To AB and AD find a mean proportional X, and from AD and AB cut off AF = AE = X. Join EF.  $\angle FAE = 60^\circ$  and  $\angle AFE = \angle AEF \therefore \triangle AEF$  is equilateral. Also  $\frac{\triangle AEF}{\triangle ADB} = \frac{AE \cdot AF}{AD \cdot AB}$  (V. 10.)  $\therefore \triangle AEF = \triangle ADB = \triangle ACB = \frac{1}{2} \times 2 \times 3.28 = 3.28$  sq. in.

**62.** Draw the diagonals AOC, BOD of the quadl. ABCD. On OA describe an equilateral  $\triangle OEA$ , and draw EF perp. to OA. From OA cut off OG = EF. Draw GH  $\parallel$  to AD to meet OD at H, HK  $\parallel$  to DC to meet OC at K, KL  $\parallel$  to CB to meet OB at L, and join LG.  $\frac{\triangle OGH}{\triangle ODA} = \frac{OG^2}{OA^2} = \frac{EF^2}{OA^2} = \frac{EF^2}{OE^2} = \frac{3}{4}$  for  $OF = \frac{1}{2} OA = \frac{1}{2} OE$ . Also since  $\frac{OG}{OA} = \frac{OH}{OD} = \frac{OK}{OC} = \frac{OL}{OB}$ ,  $\triangle$ s OHK, OCD,  $\triangle$ s OLK, OCB, and  $\triangle$ s OLG, OBA are in the same ratio. Also these pairs of  $\triangle$ s are similar  $\therefore$  the quadls. FHKL, ABCD are similar  $\therefore$  FHKL is the reqd. quadl.

**63.** Describe the circum-circle. Let AE be a diameter of it. Make  $\angle EAF$  equal to the given difference of angles, and make AF equal to the side AD of a rectangle on AE equal to the given rectangle contd. by the sides. Through F draw a perp. to AF to meet the circumference in B, C. Join AB, AC.  $AB \cdot AC = AE \cdot AF$  (V. 18.) = the given rectangle.  $\angle C - \angle B = 90^\circ - B - (90^\circ - C) = \angle BAF - \angle ECB = \angle BAF - \angle EAB$  (III. 12.) =  $\angle EAF$  = the given angle.

**64.** If BAC is the reqd.  $\triangle$  on given base BC, AB being gr. than AC, let the bisector of the ext.  $\angle CAE$  meet BC produced at O.  $\frac{BO}{OC} = \frac{BA}{AC}$  = the given ratio. Also  $\angle AOC = C - \frac{1}{2} \angle CAE = C - \frac{1}{2}(B + C) = \frac{C - B}{2} = \frac{1}{2}$  the given diff. of base angles. Hence the following construction. Let  $\frac{a}{b}$  = the given ratio,  $a > b$ . Take a line X such that  $\frac{X}{BC} = \frac{b}{a - b}$  and in BC produced make CO = X.  $\frac{CO}{BC} = \frac{b}{a - b} \therefore \frac{CO}{BO} = \frac{b}{a}$ . Thro. O draw OA making  $\angle BOA$

$= \frac{1}{2}$  given diff. of base angles. Draw CD perp. to OA and produce it to E, making DE = DC. Join BE cutting ODA at A. Join AC. In  $\triangle$ s ADC, ADE,  $\angle CAD = \angle DAE$  (I. 4.)  $\therefore \frac{BA}{AC} = \frac{BO}{OC}$  (V. 4.)  $= \frac{a}{b}$ . Also  $\angle C - \angle B = 2\angle AOC = \text{given } \angle$ , as above;  $\therefore$  ABC is the  $\triangle$  reqd.

### EXERCISES LXV.

1.  $\triangle$ s ABE, BCF are equal in all respects (I. 4.)  $\therefore \angle BEG = \angle BFC$   $\therefore \triangle$ s BEG, BFC are equiangular  $\therefore \frac{GE}{GB} = \frac{CF}{BC} = \frac{1}{2}$ . Also  $\triangle$ s BGE, AGB are equiangular  $\therefore \frac{AG}{BG} = \frac{BG}{GE} = 2 \therefore AG = 2BG = 4GE \therefore GE = \frac{1}{5}AE$ . From  $\triangle$ s AHB, CHF,  $\frac{BH}{HF} = \frac{AB}{FC} = 2 \therefore BH = 2HF \therefore BH = \frac{2}{3}BF = \frac{2}{3}AE = \frac{10}{3}GE$ , i.e.  $BG + GH = \frac{10}{3}GE \therefore GH = \frac{10}{3}GE - BG = \frac{10}{3}GE - 2GE = \frac{4}{3}GE$ , or  $GE = \frac{3}{4}GH$ .

2.  $\frac{BD}{EC} = \frac{\triangle ABD}{\triangle EAC}$  (V. 1.)  $= \frac{BA \cdot AD}{AC \cdot AE}$  (V. 10.)  $\therefore \frac{BA}{AC} = \frac{AE \cdot BD}{AD \cdot EC}$   
 ... (1). Also  $\frac{BE}{CD} = \frac{\triangle ABE}{\triangle ACD}$  (V. 1.)  $= \frac{BA \cdot AE}{CA \cdot AD} \therefore \frac{BA}{AC} = \frac{BE \cdot AD}{AE \cdot CD}$   
 $\therefore$  from (1)  $\frac{BA^2}{AC^2} = \frac{BD \cdot BE}{CD \cdot CE}$ .

3. Let the line joining A to the mid. pt. of the base BC meet CD at O. AO bisects  $\angle BAC$  (I. 7.)  $\therefore \frac{DO}{CO} = \frac{DA}{CA}$  (V. 3.)  $= \frac{DA}{AB} = \frac{BD}{AD}$ , for  $AB \cdot BD = AD^2 \therefore$  CD is divided in the same ratio as AB, i.e. it is divided in extreme and mean ratio.

The solutions of 4-8 will be found on page 178.

### EXERCISES LXVI.

1. and 2. Done in the text-book.

3. Let A and B be the fixed pts. and P one position of the moving pt. Let the internal and external bisectors of the  $\angle P$  meet AB and AB produced at C and D.  $\frac{AC}{CB} = \frac{AP}{PB}$  (V. 3.)  $= a$

constant  $\therefore C$  is a fixed point for all positions of  $P$ . Similarly by V. 4.  $D$  is a fixed pt.  $\therefore$  the locus of  $P$  is a circle on  $CD$  as diameter, for  $\angle CPD$  is a rt.  $\angle$ .

4. Let  $AP$  be the given str. line. Draw  $OA$  perp. to  $AP$ , and divide it at  $B$  so that  $\frac{OB}{BA} = \frac{2}{1}$ . Draw  $OP$ , and divide it at  $Q$  so that  $\frac{OQ}{QP} = \frac{2}{1}$ . Join  $BQ$ .  $Q$  is a pt. on the locus.  $\frac{OQ}{QP} = \frac{OB}{BA} \therefore QB$  is  $\parallel$  to  $AP \therefore \angle OBQ$  is a rt.  $\angle$ . Also  $B$  is a fixed pt.  $\therefore$  the str. line  $QB$  is the locus reqd., viz. a str. line  $\parallel$  to the given str. line.

5. Draw the str. line  $AB$  to meet the given  $\parallel$  str. lines,  $CD$ ,  $EF$  at  $B$  and  $A$ . Bisect  $AB$  at  $P$ , and draw  $MPN$  perp. to  $CD$  and  $EF$ . From  $\triangle s$   $APN$ ,  $BPM$ ,  $PN = PM$ , and  $MN$  is constant  $\therefore$  the locus of  $P$  is a str. line  $\parallel$  to the given str. lines and equidistant from them.

6. If  $AB$  be the given base,  $P$  any position of the vertex on the given line  $PE$ ,  $Q$  the intersn. of the medians lies in  $DP$  such that  $\frac{DQ}{QP} = \frac{1}{2}$ , when  $D$  is the mid. pt. of  $AB$ . Thus we see, as in Example 4 above, that the locus of  $Q$  is a str. line  $\parallel$  to the given str. line.

7. On the given base  $BC$  describe a segment,  $BAC$ , of a circle containing an angle equal to the given vertical angle (III. 23.). Then the arc is the locus of the vertex of the  $\triangle$  (III. 12.). Bisect  $BC$  at  $D$ . Join  $AD$ . Then the intersn. of the medians lies at  $P$  in  $DA$ , where  $\frac{DP}{PA} = \frac{1}{2}$ . Draw  $PE \parallel$  to  $AB$ ,

and  $PF \parallel$  to  $AC$  to meet  $BC$  at  $E$  and  $F$ .  $\frac{DE}{EB} = \frac{DP}{PA} = \frac{2}{1} \therefore E$  is a fixed pt. In like manner,  $F$  is a fixed pt. Also  $\angle EPF = \angle BAC$  (I. 20.) = the given  $\angle \therefore$  the locus of  $P$  is an arc of a circle on  $EF$ , containing an angle equal to the given angle.

8. The base and area being constant, the altitude of the  $\triangle$  is constant, i.e. the vertex lies on a fixed str. line  $\parallel$  to the base. The locus can now be found as in Example 4 above.

9. (1) When the given str. lines are  $\parallel$ , the locus is a str. line  $\parallel$  to them.

(2) When they intersect, let AB, AC be the given lines. Let P be any pt. on the locus, and draw PM, PN perp. to AB and AC. In AP take any pt. Q and draw QR, QS perp. to AB and AC. By similar  $\triangle$ s  $\frac{QS}{PN} = \frac{QA}{PA} = \frac{QR}{PM} \therefore \frac{QS}{QR} = \frac{PN}{PM} \therefore Q$  is a pt. on the locus  $\therefore$  the locus is the str. line AP. The line AP is however only part of the locus, for in the same way it may be shown that pts. on a str. line thro. A and within the  $\angle$  formed by AB and AC produced satisfy the given condition.

10. Describe any circle PAD passing thro. the pts. A and D. Also thro. P, B, C, describe another circle cutting the former circle again at Q. Join QP and produce it to meet AD at O. Rect. OA . OD = rect. OP . OQ = rect. OB . OC (IV. 14.). To OA and OD take a mean proportional OE in the line OABCD. With centre O and rad. OE describe a circle ERS and take any pt. R on it. OA . OD = OE<sup>2</sup> = OR<sup>2</sup>  $\therefore \frac{OA}{OR} = \frac{OR}{OD} \therefore \triangle$ s OAR, ORD are similar  $\therefore \angle ORA = \angle ODR$ . Also since OB . OC = OA . OD = OE<sup>2</sup> we can prove in the same way that  $\angle ORB = \angle OCR$ . But  $\angle ARB = \angle ORB - \angle ORA$  and  $\angle CRD = \angle OCR - \angle ODR \therefore \angle ARB = \angle CRD \therefore$  the locus of pts. at which AB and CD subtend equal  $\angle$ s is the circle ERS. By lxxv. 2. it can be shewn to be the same question as lxxvi. 3.

11. Let ABC be the  $\triangle$ ; P, Q, R the given str. lines. It is reqd. to find a pt. O such that the perps. from O on the sides BC, CA, AB are proportional to P, Q, R. On the same side of BC as A, draw BD perp. to BC and equal to P. On the same side of AC as B draw CE perp. to AC and equal to Q. Draw DF and EF  $\parallel$  to BC and AC, meeting at F. The perps. from F on BC and AC are equal to P and Q respectively  $\therefore$  the pt. reqd. lies in CF. In the same way a line BG may be found in which the reqd. pt. lies  $\therefore$  it lies at the pt. where CF and BG meet.

12. Let A be the fixed angular pt., B any position of the angular pt. in the given line EF, and C the vertex whose locus it is reqd. to find. If the circum-circle of  $\triangle ABC$  meet EF at G,

$\angle AGB =$  supplement of  $\angle C$  and is  $\therefore$  constant. Also  $A$  is a fixed pt.  $\therefore AG$  is a fixed line. Join  $GC$ .  $\angle BGC = \angle BAC$  in the same segment  $\therefore \angle BGC$  is fixed  $\therefore$  the locus of  $C$  is the fixed str. line  $GC$ .

### EXERCISES LXVII.

1. Let  $AB$  be the polar of  $P$ ,  $CD$  the polar of  $Q$ . Let  $AB$ ,  $CD$  meet at  $O$ . The polar of  $P$  passes thro.  $O \therefore$  the polar of  $O$  passes thro  $P$ . In like manner the polar of  $O$  passes thro.  $Q \therefore PQ$  is the polar of  $O$ .

2. With the fig. of Example 1 above,  $O$  is the pole of  $PQ$ .

3. With the fig on p. 372, let  $P$  be the fixed pt.  $Q$  is the pole of the str. line  $ABP$ . It is reqd. to find the locus of  $Q$ . This is proved on p. 372 to be a str. line at rt.  $\angle$ s to  $OP$ .

4. If  $P$  and  $Q$  be the pts. and  $O$  the centre of the circle,  $OP$  and  $OQ$  are respectively perp. to the polars of  $P$  and  $Q \therefore$  the  $\angle POQ =$  the  $\angle$  between the polars of  $P$  and  $Q$ .

### EXERCISES LXVIII.

1. Let  $PB$ ,  $P'B'$  be two  $\parallel$  radii of the circles (see fig. on p. 376).  $\angle OBP = \angle OB'P'$  and  $\angle O$  is common to  $\triangle$ s  $OPB$ ,  $OB'P'$   $\therefore \frac{OP}{OP'} = \frac{PB}{P'B'}$ , i.e.  $PP'$  is divided externally in the ratio of the radii  $\therefore O$  is a centre of similitude. In like manner, if  $PB$ ,  $P'B'$  are drawn in opp. directions, it will be seen that  $BB'$  divides  $PP'$  internally in the ratio of the radii, and  $\therefore$  passes thro. a centre of similitude.

2. This is best proved by means of the prop. on Transversals on p. 379. Let  $A$ ,  $B$ ,  $C$  be the centres of the circles  $R_1$ ,  $R_2$ ,  $R_3$  their radii. Let  $P$  be the internal c. of similitude of  $A$  and  $B$ ,  $Q$  the internal c. of similitude of  $C$  and  $A$ ,  $R$  the external c. of similitude of  $B$  and  $C$ .  $\frac{AP}{PB} = \frac{R_1}{R_2}$ ,  $\frac{CQ}{AQ} = \frac{R_3}{R_1}$ , and  $\frac{BR}{CR} = \frac{R_2}{R_3} \therefore \frac{AP \cdot BR \cdot CQ}{BP \cdot CR \cdot AQ} = 1 \therefore P, Q, R$  are collinear (prop. on p. 379). In like manner, the rest of the prop. may be proved.

3. Let a circle (centre A) touch the circles whose centres are C and D at P and W. Also let a circle (centre B) touch the circles whose centres are C and D at Q and V. Take the case where P and W lie on one side of CD and Q and V on the opp. side, all the contacts being external. Let QP meet circle A again at R. Join CA (passing thro. P), CB (passing thro. Q), and AR. From the isos.  $\triangle CPQ$ ,  $\triangle ARP$  we see that CQ is  $\parallel$  to AR, i.e. QB and AR are  $\parallel$   $\therefore$  QR passes thro. the external c. of similitude of circles A and B. In like manner, VW passes thro. the same c. of similitude. Let O be this centre. Draw OEF a common tangent to circles A and B.  $OP \cdot OQ = OE \cdot OF = OW \cdot OV$  (p. 376)  $\therefore$  O is a pt. on the radical axis of circles C and D. In the same way we may prove the other cases of the prop.

4. Let A and B be the centres of the fixed circles. Take the case where the variable circle of centre P touches them both externally. Join PA, PB passing thro. the pts. of contact, C, D. Let DC meet the circle (centre A) again at E. Join AE. From the isos.  $\triangle PCD$ ,  $\triangle ACE$  we see that PD is  $\parallel$  to AE, i.e. BD is  $\parallel$  to AE  $\therefore$  DE passes thro. the external c. of similitude of the fixed circles. In the same way the other cases may be proved.

### EXERCISES LXIX.

1. Take ABC so that AB and AC are equal to the given str. lines. On BC as diameter describe a circle, and from A draw tangents AD, AE to it. If the chd. of contact DE meets BC at F, AF will be the H.M. reqd. Join BD, DC.  $\angle ADB = \angle BCD$  (in alt. segt.)  $= \angle BDF$  (V. 9.)  $\therefore$  DB bisects the intr.  $\angle ADF$  of  $\triangle ADF$ . BDC is a rt.  $\angle$   $\therefore$  DC bisects the extr.  $\angle$  of  $\triangle ADF$   $\therefore \frac{AC}{CF} = \frac{AB}{BF}$   $\therefore$  AC, AF, AB are in H.M. (p. 384).

2. With the same figure  $AB + AC = 2AB + BC = 2AB + 2BO = 2AO$   $\therefore$  AO is the A.M. between AB and AC. Also  $AD^2 = AB \cdot AC$  (IV. 14.)  $\therefore$  AD is the G.M. between AB and AC. From similar  $\triangle s$  ADF, AOD  $\frac{AO}{AD} = \frac{AD}{AF}$ , i.e. the A.M., G.M., and H.M. are in continued proportion.

3. From an external pt. O draw OA, OB, tangents to, and OPRQ cutting at P, Q the circle whose centre is D. Join AB cutting OD at C and OQ at R.  $OP \cdot OQ = OA^2$  (IV. 14.)  $= OC \cdot OD$  for  $\angle ACO$  is a rt.  $\angle \therefore P, Q, D, C$  are concyclic  $\therefore \angle PCO =$  supplement of  $\angle PCD = \angle PQD = \angle DPQ = \angle DCQ \therefore CO$  bisects the extr.  $\angle$  of  $\triangle PCQ$ . Also  $\angle ACO$  is a rt.  $\angle \therefore CA$  bisects the intr.  $\angle PCQ \therefore \frac{OQ}{OP} = \frac{CQ}{CP}$  (V. 4.)  $= \frac{QR}{RP}$ , i.e. PQ is divided internally and externally in the same ratio at R and O  $\therefore OQ$  is cut harmonically at P and R.

4. Let the transversal DPEQ meet BC at P, CE at E, and AC produced at Q. From the similar  $\triangle s$  QEC, QDA,  $\frac{QD}{QE} = \frac{AD}{CE} = \frac{DB}{PE} = \frac{DP}{PE}$  from the similar  $\triangle s$  BPD, CPE  $\therefore DE$  is divided internally at P, and externally at Q in the same ratio  $\therefore$  the pts. D and E are harmonically conjugate with respect to the pts. P and Q.

5. From the similar  $\triangle s$  AEC, AOD,  $\frac{EC}{OD} = \frac{AC}{AD} = \frac{BC}{BD}$  (hyp.)  $= \frac{CF}{OD}$  from the similar  $\triangle s$  CBF, DBO  $\therefore CE = CF$ .

6. Draw  $A'C'B'D'$  any transversal cutting OA, OC, OB, OD at A', C', B', D'. Draw  $E'C'F' \parallel$  to OD cutting OA, OB at E' and F'. As in Example 5 above,  $E'C' = C'F'$ . Thus in  $\triangle OE'F'$  the transversal  $A'C'B'D'$  is drawn through C' the mid. pt. of  $E'F'$   $\therefore$  as in Example 4 above,  $A'B'$  is divided harmonically at C' and D'.

7. To CA and CB take a mean proportional X, and from CBA cut off  $CE = CF = X$ .  $CA \cdot CB = CE^2 \therefore \frac{CA}{CE} = \frac{CE}{CB} \therefore \frac{CA + CE}{CE + CB} = \frac{AF}{AE} = \frac{BF}{BE} \therefore AB$  is divided harmonically at E and F in such a manner that  $CE = CF$ .

8. Let OC and OD bisect internally and externally the  $\angle AOB$ . Draw any transversal ACBD.  $\frac{AD}{DB} = \frac{AO}{OB}$  (V. 4.)  $= \frac{AC}{CB} \therefore AB$  is divided harmonically at C and D.

## EXERCISES LXX.

1. Let A be the fixed pt., P one of the pts. in the plane, AB perp. to the plane.  $BP^2 = AP^2 - AB^2$  (II. 11.) = a constant, since AP is of constant length;  $\therefore$  the locus of P is a circle whose centre is the fixed pt. B.

2. Let AB be the fixed str. line, P any position of the moving pt. If P move only in the plane PAB its locus is a str. line  $PQ \parallel$  to AB  $\therefore$  the locus of all the positions of P is the surface generated by allowing PQ to move without altering its distance from AB, i.e. a right cylinder whose axis is AB (VII. Def. 14.).

3. Let A, B be the given pts., P a position of the moving pt. If P remain in the plane PAB its locus is the line bisecting AB at rt.  $\angle$ s (I. 23.). The complete locus is found by revolving the figure about AB, and is therefore the plane bisecting AB at rt.  $\angle$ s.

4. Let A, B be the fixed pts., P the moving pt. Draw PN perp. to AB.  $AN^2 - BN^2 = AP^2 - BP^2$  (II. 11.) = a constant  $\therefore$  N is a fixed pt.  $\therefore$  the locus of P is a plane cutting AB at rt.  $\angle$ s.

5. Let A, B be the fixed pts., P the moving pt., C the mid. pt. of AB.  $2CP^2 + 2AC^2 = AP^2 + BP^2$  = a constant  $\therefore$  CP = a constant  $\therefore$  the locus of P is a sphere whose centre is C.

6. Let A, B, C be the three given points, P the moving pt., PD perp. to the plane ABC.  $PA^2 = PB^2 = PC^2$  (hyp.)  $\therefore AD^2 = BD^2 = CD^2$  (II. 11.)  $\therefore$  D is the circum-centre of the  $\triangle ABC$   $\therefore$  the locus of P is the perp. to the plane ABC drawn through the circum-centre. There is no such point if A, B, C are collinear.

7. Cut off equal lengths OA, OB, OC. The reqd. line is the perp. OP from O to the plane ABC. For the right-angled  $\triangle$ s OAP, OBP, OCP are equal in all respects (I. 17.)  $\therefore$  OP is equally inclined to OA, OB, OC.

8. Let ABDC be a skew quadl., E, F, G, H the mid. pts of AB, AC, DB, DC. EF is  $\parallel$  to BC and equal to  $\frac{1}{2}BC$  (Ex. xx. 1.); so also is GH.  $\therefore$  EF is equal and  $\parallel$  to GH.  $\therefore$  EFGH is a parm. (II. 1.).

9. As in the last question EF, GH are  $\parallel$   $\therefore$  they are in one plane  $\therefore$  EH and GF are concurrent.

### EXERCISES LXXI.

1. Let A, B be two points, E the mid. pt. of AB; C, F, D their projections on the plane. Let GEH be  $\parallel$  to CD meeting CA, DB at G, H.  $\triangle GAE = \triangle HBE$  in all respects (I. 16.)  $\therefore GA = BH \therefore AC + BD = GC + HD = 2EF$  (II. 2.).

2. Let A, B, C be three pts., G their centroid, D the mid. pt. of BC. Let H, K, L, N, M be the projections of these 5 pts. on the plane. Let PGQ be  $\parallel$  to HM meeting HA, MD in P, Q. From similar  $\triangle$ s PGA, QGD,  $\frac{PA}{DQ} = \frac{AG}{GD} = \frac{2}{1}$ . As in Question 1,  $BK + CL = 2DM = 2DQ + 2QM = PA + 2GN \therefore AH + BK + CL = PH + 2GN = 3GN$ .

3. AP is perp. to the first plane,  $\therefore$  the plane APQ which contains AP is perp. to the first plane (VI. 16.). Similarly the plane APQ is perp. to the second plane,  $\therefore$  the plane APQ is perp. to the common section of the other two planes (VI. 17.).

4. Let PQR, SQR be intersecting planes. Let ABC, DEF be  $\parallel$  planes meeting PQR in AB, DE, and SQR in BC, EF. AB is  $\parallel$  to DE, and BC is  $\parallel$  to EF (VI. 14.)  $\therefore \angle ABC = \angle DEF$  (VI. 10.).

5. Let a str. line PQ meet a plane XY in Q. Let QR be the projection of PQ. PQR is the minimum  $\angle$ . For in XY draw any other line QS making it equal to QR. PR is perp. to the plane  $\therefore$  PRS is a rt.  $\angle \therefore PS > PR \therefore$  in  $\triangle$ s PQS, PQR,  $\angle PQS$  is gr. than  $\angle PQR$  (I. 15.).

6. Let PO be equally inclined to three str. lines OA, OB, OC which are in a plane. Let PQ be perp. to the plane, Q lying within the  $\angle AOB$ . Make OA equal to OB. In  $\triangle$ s POA, POB,  $PA = PB$  (I. 4.)  $\therefore$  in the right-angled  $\triangle$ s PQA, PQB,  $QA = QB$  (I. 17.)  $\therefore$  in  $\triangle$ s AOQ, BOQ,  $\angle AOQ = \angle BOQ$  (I. 7.)  $\therefore Q$  lies on the bisector of the  $\angle AOB$ . Similarly Q lies on the bisector of the  $\angle AOC \therefore Q$  lies at the common pt., viz. O, i.e. PO is perp. to the plane.

7. Let AB, CD be  $\parallel$  str. lines, EF, GH their projections on a plane XY. Let AK, CL be  $\parallel$  to EF, GH. AE is  $\parallel$  to CG (VI. 6.), AB is  $\parallel$  to CD (hyp.)  $\therefore$  the planes BAE, DCG are  $\parallel$  (VI. 13.)  $\therefore$  their intersections with XY, viz EF, GH are  $\parallel$  (VI. 14.). The

$\triangle$ s BAK, DCL are equiangular to each other (VI. 10.)  $\therefore$   
 $\frac{AB}{CD} = \frac{AK}{CL} = \frac{EF}{GH}$ .

8. Let ABCD be the base, O the vertex, OE the vertical height.  $OA^2 = OE^2 + AE^2 = 2AE^2$ , similarly  $OB^2 = 2AE^2$ ,  $AB^2 = AE^2 + EB^2 = 2AE^2 \therefore OA = OB = AB$ .

### EXERCISES LXXII.

1. Let AB be the line L. BC its projection on Q. Let BD be the trace of Q on P, and let AG, CF be  $\parallel$  to BD. AC is perp. to the plane Q  $\therefore$  ACF is a rt.  $\angle$   $\therefore$  CAG is a rt.  $\angle$  (I. 20.). AB is perp. to the plane P  $\therefore$  ABD is a rt.  $\angle$   $\therefore$  BAG is a rt.  $\angle$  (I. 20.)  $\therefore$  AG is perp. to the plane ABC (VI. 4.)  $\therefore$  DB is perp. to the plane ABC (VI. 8.)  $\therefore$  DBC is a rt.  $\angle$ .

2. Let AB be  $\parallel$  to a str. line CD, which is in a plane P. AB and CD are in one plane  $\therefore$  if AB met the plane P in a point R, R would lie in the plane ABDC  $\therefore$  R would lie in the intersection of the planes, *i.e.* in CD  $\therefore$  AB and CD would not be parallel.

3. Let the planes be P, Q, R. If the lines of intersection are not  $\parallel$ , let AB, AC be the intersections of P with Q, and P with R. Then the pt. A lies on R  $\therefore$  the intersection of Q with R passes through A.

4. Let A be the given pt., BC the given str. line, P a plane containing BC; AE perp. to the plane P. Let AD be perp. to BC. Join DE. D and A are fixed points, and AED a rt.  $\angle$   $\therefore$  the locus of E is a circle whose diameter is AD; for E must lie in a plane through A perp. to BC (Question 1).

5. Let A be the centre. Join OA, and draw QB  $\parallel$  to PA meeting OA at B. Then  $\frac{OB}{BA} = \frac{OQ}{QP} = a$  given ratio  $\therefore$  B is a fixed pt. Also  $\frac{QB}{PA} = \frac{OQ}{OP} = a$  given ratio  $\therefore$  QB = a constant  $\therefore$  as P traces a circle round A, Q traces a circle round B, QB and PA remaining parallel.

6. The same as 4.

7. Let  $AB, CD$  be  $\parallel$  str. lines; let  $BE, DF$  be their projections.  $AE$  is  $\parallel$  to  $CF$  (VI. 6.),  $AB$  is  $\parallel$  to  $CD$  (hyp.)  $\therefore$  plane  $BAE$  is  $\parallel$  to plane  $DCF$  (VI. 13.)  $\therefore BE$  is  $\parallel$  to  $DF$  (VI. 14.)  $\therefore \angle ABE = \angle CDF$  (VI. 10.).

8. Let  $P$  be any position of the pt.,  $PQ$  perp. to  $BC$  the intersection of the planes. Let  $R, S$  be the projections of  $P$  on the planes.  $PR = PS$  (hyp.)  $\therefore PQ$  bisects  $\angle SQR$  (I. 17.)  $\therefore$  the locus of  $P$  is a plane bisecting the angle between the given planes.

9. The locus is the str. line which is the intersection of planes bisecting the  $\angle$ s between two pairs of the given planes (see the preceding).

10. If the 4 planes form a tetrahedron, the pts. are the centres of spheres touching the 4 faces. A sphere may touch all internally, or one externally, the rest internally. 5 points.

### EXERCISES LXXIII.

1. Let  $ABCD$  be a skew quadr.,  $BD$  a diagonal.  $\angle ABC$  is less than  $\angle ABD + \angle CBD$  (VI. 18.).  $\angle CDA$  is less than  $\angle BDA + \angle BDC$  (VI. 18.)  $\therefore \angle ABC + \angle BCD + \angle CDA + \angle DAB$  is less than the sum of the  $\angle$ s of the  $\triangle$ s  $ABD, CBD$ , i.e. less than 4 rt.  $\angle$ s.

2. Let  $BAC, CAD, DAB$  be the containing  $\angle$ s,  $AP$  any line.  $\angle PAB + \angle PAC > \angle BAC$  (VI. 18.).  $\angle PAC + \angle PAD > \angle CAD$ ,  $\angle PAD + \angle PAB > \angle DAB$   $\therefore 2(\angle PAB + \angle PAC + \angle PAD) > \angle BAC + \angle CAD + \angle DAB$ .

3. Let  $ABC$  be a  $\triangle$ ,  $DEF$  an inscribed  $\triangle$ ,  $P$  any pt. outside the plane of  $ABC$ .  $\angle BPF + \angle BPD > \angle FPD$  (VI. 18.).  $\angle CPD + \angle CPE > \angle DPE$ ,  $\angle EPA + \angle APF > \angle EPF$   $\therefore$  by addition the result follows.

4. Let a plane  $BCD$  cut  $AO$  at rt.  $\angle$ s in  $D$ . Let the plane bisecting the dihedral  $\angle$  (i.e. the angle  $BDC$ ) meet  $BC$  at  $E$ .  $BO = CO$ , and  $BD = CD$  (I. 16.)  $\therefore$  in  $\triangle$ s  $BDE, CDE$ ,  $BE = CE$  (I. 4.)  $\therefore$  in  $\triangle$ s  $BOE, COE$ ,  $\angle BEO = \angle CEO$  (I. 7.), and the  $\angle$ s at  $E$  are rt.  $\angle$ s.  $\therefore CE$  is perp. to  $EO$  and  $ED$ .  $\therefore$  the plane  $CBO$  which contains  $CE$  is perp. to the plane  $DEO$  (VI. 16.).

5. Let  $AB, AC$  be equally inclined to a plane  $BCD$ . Let  $D$  be the projection of  $A$ . In  $\triangle$ s  $ABD, ACD$ ,  $\angle ABD = \angle ACD$  (hyp.),

the angles at D are equal, and AD is common  $\therefore AB = AC$  (I. 16.)  
 $\therefore \angle ABC = \angle ACB$  (I. 5.).

6. Let AOE, BOF be the perpendiculars. The plane AOB contains AE and is therefore perp. to plane BCD (VI. 16.); it also contains BF and therefore is perp. to plane ACD  $\therefore$  plane AOB is perp. to the line CD (VI. 17.).

7. Let O be the pt., AB, CD, EF be  $\parallel$  str. lines. Let OP be a str. line  $\parallel$  to AB. Then OP must be  $\parallel$  to CD, EF etc. ... OP is  $\parallel$  to AB  $\therefore$  the plane OAB passes through P. Similarly the plane OCD passes through P, and so on  $\therefore$  the planes have a common line OP  $\therefore$  if any plane cuts these planes, the lines of intersection all pass through some point on OP.

8. Let ABC be a  $\triangle$  right-angled at C, P a pt. equidistant from A, B, C; D the mid. pt. of AB.  $AD = BD = CD$  (Ex. xviii. 9.)  $\therefore \triangle$ s PDA, PDB, PDC are equal in all respects (I. 7.)  $\therefore \angle PDC = \angle PDA = \angle PDB = 90^\circ$   $\therefore$  PD is perp. to plane ABC (VI. 4.).

9. In the figure of VI. 20. draw LP  $\parallel$  to AB. Then LK is perp. to the plane DLP (VI. 4.) and to the plane HKB  $\therefore$  these planes are parallel, and one contains CD, the other AB.

10. Let E, F, G, H be the mid pts. of AB, CD, CB, AD. FG, HE are  $\parallel$  to BD and each  $= \frac{1}{2}BD$  (Ex. xx. 1.)  $\therefore$  FEHG is a parm.  $\therefore$  FE is in the same plane as EG, EH. GE is  $\parallel$  to CA and therefore perp. to AB. Similarly EH is perp. to AB  $\therefore$  AB is perp. to the plane HEG (VI. 4.)  $\therefore$  AB is perp. to EF.

11. Let AHD, BHE be perp. to BC, CA; HP perp. to the plane ABC. The plane PBE is perp. to plane ABC (VI. 16.), and AC is drawn in the plane ABC perp. to the common section  $\therefore$  AC is perp. to the plane PBE. But it follows from VI. 8. that if a str. line is perp. to a plane it is perp. to any str. line in that plane  $\therefore$  AC is perp. to BP.

[NOTE.—In the figure of VI. 8. AB is  $\parallel$  to CD, and CD is perp. to EF  $\therefore$  AB is perp. to EF.]

12. Let OA, OB, OC meet a str. line in A, B, C. The lines OA, OC, AC are in one plane (VI. 2.), and B lies in AC  $\therefore$  the lines OA, OC, OB are in one plane.

13. Let MR be  $\parallel$  to NO. Then MR is perp. to PQ (VI. 8.). But MO is perp. to PQ  $\therefore$  PQ is perp. to the plane RMO (VI. 4.).

But the points R, M, N, O are in one plane since RM is  $\parallel$  to ON  
 $\therefore$  PQ is perp. to MN.

14. Let A, B be the given pts. Draw BD perp. to the given plane and produce it to C making  $DC = BD$ . Join AC meeting the plane at P. In the plane take any other point Q. Join PB, AQ, PD, QD, QC, QB.  $PB = PC$  and  $QB = QC$  (I. 4.).  $AQ + QB = AQ + QC > APC$  (I. 12.) *i.e.*  $> AP + PB \therefore$  the position found by joining AC is the one required.

15. Let PM, PN, PR perp. to OA, OB, OC be all equal; the angles POM, PON, POR are equal (I. 17), *i.e.* the locus of P is a str. line through O equally inclined to the 3 str. lines.

16. Let AB, CD be two str. lines not in a plane. If AC, BD were parallel, the pts. A, C, D, B would be in a plane  $\therefore$  AB, CD would be in a plane.

17. Proved in VII. 22.

18. In the figure of VI. 15., let the planes PQ, TX be  $\parallel$ .  
 $\frac{AE}{EB} = \frac{AH}{HD}$  (hyp.)  $\therefore$  EH is  $\parallel$  to BD and therefore  $\parallel$  to the plane TX (Ex. lxxii. 2.). Similarly HF is  $\parallel$  to the plane PQ and therefore to TX  $\therefore$  the plane RS is  $\parallel$  to the plane TX (VI. 13.).

19. By VI. 14. AB is  $\parallel$  to  $ab \therefore ABba$  is a parm., and so is  $BCcb$ , etc.  $\therefore$  AB, BC, etc., are respectively equal to  $ab$ ,  $bc$ , etc. ... Also  $\angle ABC = \angle abc$  (VI. 10.); and similarly for the other  $\angle$ s.

20. Produce CO to meet AB at F. The plane CPF which contains PO is perp. to the plane ABC (VI. 16.). Also it contains PC and is therefore perp. to the plane ABP  $\therefore$  it is perp. to the line AB (VI. 17.)  $\therefore$  COF is perp. to AB. Similarly AO is perp. to BC  $\therefore$  O is the orthocentre of  $\triangle ABC$ .

21. In question 20 it is proved that the plane PCF is perp. to AB  $\therefore$  PF is perp. to AB  $\therefore$  the feet of the perps. from P coincide with the feet of the perps. from the vertices to the sides of  $\triangle ABC$ . From the cyclic quadl. DOEC,  $\angle EDO = \angle ECO = 90^\circ - \angle CAB$ . Similarly  $\angle FDO = 90^\circ - \angle CAB \therefore$  DA is the internal bisector of the  $\angle EDF \therefore$  BC, which is at rt.  $\angle$ s to it, is the external bisector.

22. Let BAC, EDF, HGK be the 3  $\angle$ s. Make AB, AC, DE, DF, GH, GK all equal. Construct a  $\triangle LMN$  with its sides MN, NL, LM equal to BC, EF, HK. Take O the circumcentre of LMN, and

draw  $OP$  perp. to the plane  $LMN$ . With centre  $L$  and radius equal to  $AB$  cut  $OP$  at  $P$ , and join  $LP$ ,  $MP$ ,  $NP$ . From  $\triangle s$   $LOP$ ,  $MOP$ ,  $NOP$  by I. 4.  $LP = MP = NP$   $\therefore$  by I. 7. we can prove that the  $\angle s$  at  $P$  are equal to the given  $\angle s$ .

**23.** Let  $BAC$ ,  $CAD$ ,  $DAB$  form a trihedral  $\angle$ ,  $AP$  being any line drawn within the trihedral  $\angle$  to meet the plane  $BCD$  at  $P$ . Produce  $CP$  to meet  $BD$  at  $E$ .  $\angle CAD + \angle DAB = \angle CAD + \angle DAE + \angle EAB > \angle CAE + \angle EAB$  (VI. 18.). Similarly  $\angle CAE + \angle EAB > \angle CAP + \angle BAP$   $\therefore \angle CAD + \angle DAB > \angle CAP + \angle BAP$ . Similarly  $\angle DAB + \angle BAC > \angle DAP + \angle CAP$  and  $\angle BAC + \angle CAD > \angle BAP + \angle DAP$   $\therefore$  by adding and dividing by 2,  $\angle BAC + \angle CAD + \angle DAB > \angle BAP + \angle CAP + \angle DAP$ . The rest has been proved in question 2.

**24.** Let the planes intersect in  $AO$ . Let  $AB$  be a str. line perp. to  $AO$ ,  $AC$  its projection;  $AD$  a line in the first plane not perp. to  $AO$ ,  $AE$  its projection,  $BD$  being  $\parallel$  to  $AO$ . In the right-angled  $\triangle s$   $ABD$ ,  $ACE$ ,  $AB < AD$ ,  $AC < AE$ , and  $BC = DE$  (II. 2.)  $\therefore$  by superposing the right-angled  $\triangle s$   $ABC$ ,  $ADE$  we can prove  $\angle BAC$  gr. than  $\angle DAE$ .

**25.** Let  $AD$  be the common section of the given planes,  $AB$ ,  $AC$  str. lines drawn in the planes perp. to  $AD$ . Draw  $BC$  perp. to  $AC$ . At  $B$  in the plane  $BAC$  make  $\angle CBE$  equal to the complement of the given  $\angle$ ,  $BE$  meeting  $CA$  at  $E$ . Draw a circle with centre  $C$  and radius  $CE$  cutting  $AD$  at  $D$ . The right-angled  $\triangle s$   $BCE$ ,  $BCD$  are equal in all respects (I. 4.)  $\therefore \angle CDB = \angle CEB =$  the given  $\angle$   $\therefore$   $CDB$  is the required plane. Impossible if the given  $\angle$  is gr. than the  $\angle$  between the planes.

**26.** Let  $O$  be the given pt.,  $AB$ ,  $CD$  the given str. lines. Let the plane  $OAB$  cut  $CD$  at  $E$ .  $OE$  is the required line.

**27.**  $OA^2 + OB^2 = AB^2$  (II. 11)  $= AC^2$  (hyp.)  $= OA^2 + OC^2$  (II. 11.)  $\therefore OB = OC$ . Similarly  $OB = OA$ .

### EXERCISES LXXIV.

**2.** Let  $P$ ,  $Q$ ,  $R$  be the centres of three adjacent faces,  $N$  the mid. pt. of the common edge of the first two faces,  $a$  the length of any edge.  $PQ^2 = \frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2}$ . Similarly  $QR^2 = \frac{a^2}{2}$ , and  $RP^2 = \frac{a^2}{2}$   $\therefore PQR$  is an equilateral  $\triangle$  whose side is  $\frac{1}{2}a\sqrt{2}$   $\therefore$  the

new faces formed are equal equilateral  $\triangle$ s, and there are 8 of them  $\therefore$  the figure is a regular octahedron.

3. Let ABCD, CDEF be adjacent faces.  $AF^2 = AB^2 + BF^2 = AB^2 + BC^2 + CF^2$  (II. 11.)  $= 3AB^2$ .

4. Let ABCD, CDEF, EFGH be consecutive faces of a parallelepiped. Thus ADEH, BCFG are opposite faces. HE is equal and  $\parallel$  to BC  $\therefore$  HBCE is a parm. (II. 1.)  $\therefore$  BE, HC bisect each other. HDCG is a parm.  $\therefore$  DG, HC bisect each other. Thus we may prove that all the diagonals pass through the mid. pt. of HC and are bisected there.

5. Let ABCD, BCFE, EFGH be consecutive faces, so that AD, BC, FE, GH are parallel. Bisect at K, L, M, N, P, Q the sides BF, BC, CD, DH, HG, GF, which do not meet the diagonal AE.  $AM = AQ = EQ = EM$  (I. 4.)  $\therefore$  AMEQ is a rhombus  $\therefore$  MQ, AE bisect each other at rt.  $\angle$ s at O. Similarly KN, AE bisect each other at rt.  $\angle$ s at O, and LP, AE bisect each other at rt.  $\angle$ s at O  $\therefore$  MQ, KN, LP are in one plane perp. to AE (VI. 5.).  $KQ = \frac{1}{2}BG = \frac{1}{2}FH = QP$  (Ex. xx. 1.). Also  $KQ = \frac{1}{2}BG$  and is  $\parallel$  to BG  $\therefore$   $KQ = \frac{1}{2}CH$  and is  $\parallel$  to CH. But  $MN = \frac{1}{2}CH$  and is  $\parallel$  to CH  $\therefore$  KQ is equal and  $\parallel$  to MN. Similarly all the opposite sides of KLMNPQ are equal and parallel; and all the sides are equal, for each side = half the diagonal of a face  $\therefore$  the figure is a regular hexagon.

6. Let A be the vertex, AH perp. to the plane BCDE,  $a$  the length of an edge.  $AH = \sqrt{BA^2 - BH^2} = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$ . When  $a = 10$  cms.,  $AH = 5\sqrt{2} = 5 \times 1.4142 = 7$  (to nearest cm.).

7. Let P, Q, R be points on edges which meet in A.  $RP^2 + PQ^2 = AP^2 + AR^2 + AQ^2 + AP^2 = RQ^2 + 2AP^2 > RQ^2 \therefore$  RPQ is an acute angle. Similarly the other  $\angle$ s of the  $\triangle PQR$  are acute.

8. Let A, B be the fixed points, P one position of the moving point. Let C, D divide AB internally and externally in the given ratio. Then C, D are fixed points; also CP, PD are the bisectors of internal and external  $\angle$ s between AP, PB (V. 3. 4.)  $\therefore \angle CPD$  is a rt.  $\angle \therefore$  P is at a constant distance from the mid. pt. of CD  $\therefore$  the locus is a sphere with centre at the mid. pt. of CD.

9. Draw OM perp. to the plane. In OM take a point N such  $ON \cdot OM =$  the given constant  $= OQ \cdot OP \therefore PQNM$  is a cyclic quadl.  $\therefore \angle OQN = \angle PMN = 90^\circ \therefore Q$  is at a constant distance from the mid. pt. of ON  $\therefore$  the locus of  $Q$  is a sphere, i.e. *the inverse of a plane is a sphere* (see page 369).

10. Draw AO from vertex A perp. to the face BCD. Let AEO be a plane perp. to BC meeting it at E. In the right-angled  $\triangle AOE$ ,  $AE > OE \therefore \frac{1}{2}AE \cdot BC > \frac{1}{2}OE \cdot BC$ , i.e.  $\triangle ABC > \triangle OBC$ . Similarly  $\triangle ACD > \triangle OCD$ , and  $\triangle ADB > \triangle ODB \therefore$  the three faces which meet at A are together gr. than  $\triangle BCD$ .

11. Bisect BC at E. Join AE, DE.  $\triangle ABC$  is isosceles  $\therefore AE$  is perp. to BC.  $CD = AB$  (hyp.)  $= AC$  (hyp.)  $= BD \therefore \triangle DBC$  is isosceles  $\therefore DE$  is perp. to BC  $\therefore BC$  is perp. to the plane ADE (VI. 4.) and therefore to AD.

12. Let ABCD be a tetrahedron. Let a plane through AB perp. to CD meet CD at E. Then BE, AE are perp. to DC and so contain the orthocentres P, Q of  $\triangle s$  BDC, ADC  $\therefore ABP$  is the plane required.

13. Proved in VII. 1.

14. If B be joined to P, Q, R the vertices of the base, the tetrahedron is divided into 3 equal parts. BC is the altitude and ARQ the base of one of these parts. But the base ARQ = the base PQR of the whole tetrahedron  $\therefore$  altitude  $BC = \frac{1}{3}$  of altitude AB. Also  $PB = \frac{a\sqrt{3}}{3}$  where  $a =$  length of edge (VII. 5.) and  $BD \cdot AP = 2\triangle ABP = AB \cdot PB \therefore \frac{BD}{AB} = \frac{PB}{AP} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \therefore BD^2 = \frac{1}{3}AB^2 \therefore BC^2 : BD^2 : BA^2 = \frac{1}{9}BA^2 : \frac{1}{3}BA^2 : BA^2 = 1 : 3 : 9$ .

15. Let AB, AC, AD, etc., be the edges, AB being the greatest, and let AO be perp. to the base.  $BO^2 = BA^2 - AO^2 > CA^2 - AO^2 \therefore BO > CO$   $\therefore$  by turning the  $\triangle AOC$  round AO into the plane AOB we can see that the  $\angle ABO < \angle ACO$  (I. 8.) Similarly  $\angle ABO < \angle ADO$ , and so on.

16. In the tetrahedron ABCD let E, F be the mid. pts. of BC, AD. Let  $a$  be the length of an edge. AE, DE are equal and are perp. to BC  $\therefore$  their plane is perp. to BC  $\therefore EF$  is

perp. to BC. But EF is perp. to AD since EAD is an isosceles

△. Also  $EF^2 = AE^2 - AF^2 = AB^2 - BE^2 - AF^2 = a^2 - \frac{a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2} \therefore EF = \frac{1}{2}a\sqrt{2} = \frac{1}{2}$  diagonal of the sq. on  $a$ .

17. Let BD, CD, CA, BA, edges of a tetrahedron, be bisected at E, F, G, H. Let P, Q, R, S be points in which the same edges are cut by a plane parallel to AD and BC. HE is  $\parallel$  to AD and is half of AD (Ex. xx. 1.). Similarly for GF.  $\therefore$  HE is equal and  $\parallel$  to GF. Similarly EF, HG are  $\parallel$  to BC and each  $= \frac{1}{2}BC \therefore$  HEFG is a parm. SR is in a plane with BC and does not meet

BC (hyp.)  $\therefore$  SR is  $\parallel$  to BC  $\therefore \frac{SR}{BC} = \frac{AS}{AB}$  (V. 5.)  $= \frac{DP}{DB}$  (V. 2.)  $= \frac{PQ}{BC} \therefore SR = PQ$ . Similarly  $SP = RQ \therefore$  SRQP is a parm. Again

$\frac{SR}{BC} = \frac{AS}{AB}$  and  $\frac{SP}{AD} = \frac{SB}{AB} \therefore \frac{SR \cdot SP}{BC \cdot AD} = \frac{AS \cdot SB}{AB^2}$ . Similarly  $\frac{HG \cdot HE}{BC \cdot AD} = \frac{AH \cdot HB}{AB^2} \therefore \frac{SR \cdot SP}{HG \cdot HE} = \frac{AS \cdot SB}{AH \cdot HB} = \frac{AH^2 - SH^2}{AH^2} \therefore \frac{\text{parm. SQ}}{\text{parm. HF}}$   
 $=$  a ratio less than 1, i.e. the parm. HF is the maximum.

18. Let ABCD be a regular tetrahedron. Let PQRS be a plane section and let it be a parallelogram. Let P, Q, R, S lie on BD, CD, CA, BA. Since the plane SQ cuts the planes ABD, ACD in  $\parallel$  lines, it is  $\parallel$  to their common section AD. Similarly it is  $\parallel$  to BC  $\therefore \triangle ASR$  is equilateral  $\therefore SR = AS$ . Similarly  $SP = SB \therefore$  the perimeter of parm. SQ = 2AB.

19. Let ABCD be a tetrahedron, and let a sphere touch its edges BA, BC, BD, CD, CA, AD in pts. E, F, G, H, K, L.  $AD + BC = AL + DL + BF + FC = AE + DH + BE + HC$  (tangents to a sphere)  $= AB + DC =$  (similarly)  $AC + BD$ .

20. Let E, F, G, H, K, L be the mid. pts. of AD, BC, CD, AB, AC, BD. HF is equal  $\frac{1}{2}AC$  and is  $\parallel$  to AC. EG is equal  $\frac{1}{2}AC$  and is  $\parallel$  to AC  $\therefore$  HF is equal and  $\parallel$  to EG  $\therefore$  HFGE is a parm.  $\therefore$  HG passes through the mid. pt. of EF and is bisected there (II. 2. Cor. 3.). Similarly LK passes through the mid. pt. of EF and is bisected there.

21. See Question 5.

22. Let ABCD be such a tetrahedron. Draw DE perp. to BC meeting BC in E. Join AE. Draw BF perp. to DC meeting DC in F and DE in O. BC is perp. to ED and DA  $\therefore$  BC is perp.

to the plane ADE (VI. 4. and 8.)  $\therefore$  the plane ADE is perp. to the plane DBC. Similarly the plane ABF is perp. to the plane DBC  $\therefore$  the intersection AO is perp. to the plane DBC (VI. 17.). But O is the orthocentre of DBC. Similarly the perp. from each vertex to the opposite face is the line joining the vertex to the orthocentre of that face. Draw BP, DR perp. to AF, AE. BP must intersect AO; for they lie in the plane AFB. DR must intersect AO; for they lie in the plane AED  $\therefore$  the three perps. AO, BP, DR intersect each other. But they are not in one plane  $\therefore$  they must be concurrent. Similarly the fourth perp. CQ meets BP on AO, and meets DR on AO  $\therefore$  the 4 perps. are concurrent.

**23.** Let ABCD be any tetrahedron, E, F, G, H, K, L the mid. pts. of BC, AD, AB, CD, AC, BD.  $4EF^2 + AD^2 = 2AE^2 + 2DE^2$  (IV. 12.)  $= AB^2 + AC^2 - 2BE^2 + DB^2 + DC^2 - 2CE^2$  (IV. 12.)  $= AB^2 + AC^2 + DB^2 + DC^2 - BC^2$   $\therefore 4EF^2 + AD^2 + BC^2 = AB^2 + AC^2 + DB^2 + CD^2$ . Similarly  $4GH^2 + AB^2 + CD^2 = AD^2 + BC^2 + AC^2 + BD^2$ , and  $4KL^2 + AC^2 + BD^2 = AB^2 + BC^2 + AD^2 + CD^2$   $\therefore$  by addition  $4(EF^2 + GH^2 + KL^2) =$  the sum of the sqs. on the six edges.

**24.** If  $a$  be the length of an edge,  $A = \frac{a^2\sqrt{3}}{4}$   $\therefore a^2 = \frac{4A\sqrt{3}}{3}$ .

But the altitude  $= \frac{a\sqrt{6}}{3}$  (VII. 5.)  $\therefore$  Volume  $= \frac{1}{3} \cdot \frac{a\sqrt{6}}{3} \cdot A = \frac{A\sqrt{6}}{9} \cdot \frac{2A^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}}{\sqrt{3}} = \frac{2^{\frac{3}{2}} \cdot 3^{\frac{1}{2}} A^{\frac{3}{2}}}{9}$ .

**25.** In the tetrahedron ABCD, let AB be at rt.  $\angle$ s to CD, and AC at rt.  $\angle$ s to BC. Let E, F, G, H, K, L be the mid. pts. of BA, BD, BC, CD, CA, AD. EGHK is a parm. with its sides  $\parallel$  to AC, BD  $\therefore$  it is a rectangle  $\therefore$  diagonal EH = GK. Similarly FGKL is a rectangle  $\therefore$  diagonal GL = FK  $\therefore$  in the parm. EFHK the diagonals EH, FK are equal  $\therefore$  it is a rectangle. But its sides are  $\parallel$  to AD, BC  $\therefore$  AD, BC are at rt.  $\angle$ s. Also  $AD^2 + BC^2 = 4FE^2 + 4EK^2 = 4FK^2 = 4FG^2 + 4GK^2 = DC^2 + BA^2$ . Similarly for  $AC^2 + BD^2$ .

**26.** Each of these planes contains one of the three joins of the mid. pts. of opposite edges. But there is one point common to all these joins (Question 20)  $\therefore$  this point is common to all the six planes.

27. Take a plane through PQ a diagonal of a regular octahedron whose edge is  $a$ , and let it cut an edge at rt.  $\angle$ s in R. Draw QS perp. to PR produced. PRQ is the dihedral  $\angle$  of the octahedron.  $PR = RQ = \frac{a\sqrt{3}}{2}$ ; RT (the perp. from R to PQ)  $= \frac{a}{2} \therefore \frac{1}{2}PQ = \sqrt{\frac{3a^2}{4} - \frac{a^2}{4}} = \frac{a}{\sqrt{2}} \therefore PQ = a\sqrt{2}$ . By similar  $\triangle$ s  $\frac{SQ}{PQ} = \frac{RT}{PR} = \frac{1}{\sqrt{3}} \therefore SQ = \frac{a\sqrt{2}}{\sqrt{3}} = \frac{a\sqrt{6}}{3} \therefore \triangle QRS$  is equal in all respects to  $\triangle AEH$  in VII. 5.  $\therefore \angle AEH = \angle QRS =$  supplement of  $\angle PRQ$ , i.e. the dihedral  $\angle$ s of the regular tetrahedron and octahedron are supplementary.

### EXERCISES LXXV.

1. Length reqd.  $= \sqrt[3]{2000} = x$ .  $\log x = \frac{1}{3}(3.30103) = 1.1003$ ,  $x = 12.60$  ft. = 12 ft. 7.2 in.

2. Area of end  $= \frac{1}{2} \times \frac{\sqrt{3}}{2}$  sq. ft.  $\therefore$  vol.  $= \frac{6}{2} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} = 3(.866025) = 2.598$  c. ft.

3. Let  $a - b$ ,  $a$ ,  $a + b$  inches be the lengths of the sides.  $3a = 15 \therefore a = 5$ ,  $(a - b)^3 + a^3 + (a + b)^3 = 495 \therefore 375 + 6 \times 5b^2 = 495$ ,  $b^2 = 4$ ,  $b = 2$ . The sides are 3, 5, 7 inches respectively. The vols. are 27, 125, 343 cub. in.

4. Area reqd.  $= \frac{\text{vol.}}{\text{length}} = \frac{1}{3000}$  sq. ft.  $= \frac{144}{3000}$  sq. in. = .048 sq. in.

5. Wt. of 1st. bar  $= 96 \times 18$  lbs. Let  $l$  = length,  $d$  = thickness reqd. in inches.  $2 \times 9 \times d =$  vol. of 2 in. of 2nd bar  $= 27 \therefore d = 1\frac{1}{2}$  in. Also  $l \times 9 \times \frac{3}{2} =$  vol. of whole bar  $= 96 \times 18 \therefore l = \frac{96 \times 18 \times 2}{3 \times 9} = 10$  ft. 8 in.

6. Let the water rise  $x$  ft.  $x \times 12 \times \frac{7}{2} = 195$ ,  $x = \frac{65}{14}$  ft.  $= \frac{32.5}{7} = 4.6428$  ft. = 4 ft. 8 in.

7.  $BD = \sqrt{7^2 + 24^2} = 25$  in.  $\therefore CD = \sqrt{25^2 - 20^2} = 15$  in. Vol. of prism  $= 18 \times ABCD = 18 \times \frac{1}{2} [7 \times 24 + 15 \times 20]$  c. in.  $= 9 \times 12 [14 + 25] = 12 \times 351 = 4212$  c. in. Area of ends  $= 7 \times 24 + 20 \times 15$  sq. in. Area of faces  $= 18(7 + 24 + 15 + 20) \dots$  Total area  $= 7 \times 24 + 20 \times 15 + 18 \times 66$  sq. in.  $= 12 [14 + 25 + 99] = 12 \times 138$  sq. in.  $= 11\frac{1}{2}$  sq. ft.

8. Let  $x$  = dist. between the  $\parallel$  sides of the trapezium.  
 $x^2 = 13^2 - 5^2 = 12^2 \therefore x = 12 \therefore$  area of base  $= \frac{1}{2}(19 + 9)12$   
 $= 6 \times 28$  sq. in. Vol.  $= \frac{7 \times 8 \times 7}{6} \times 28$  c. ft.  $= 8\frac{1}{6}$  c. ft.

9. Vol. contained by side walls  $= 85 \times 9 \times 14$  c. ft. Vol. cont. by roof  $= \frac{1}{2} \times 7 \times 85 \times 9 \therefore$  total volume  $= 85 \times 9 \times 7 \times \frac{5}{2}$  c. ft.  $= 765 \times 7 \times \frac{5}{2} = \frac{53550}{4} = 13387\frac{1}{2}$  c. ft.

10. Area of hexagon  $= 3 \times 4 \times \frac{\sqrt{3}}{2}$  sq. ft. Vol. of prism  $= 6 \times 3 \times 2 \times \sqrt{3}$  c. ft.  $= 6 \times 6(1.73205) \dots = 6 \times 10.3923 \dots = 62.354$  c. ft.

11. Vol.  $= 9\pi \times 5 = \frac{9}{2}(31.416)$   $[\pi r^2 h] = 9(15.708) = 141.372$  c. ft.  $= 141$  to the nearest c. ft.

12. Suppose the last man uses  $x$  in. of the rad., the preceding man  $y$  in. and the first man  $z$  in.,  $\pi x^2 = \frac{1}{3}\pi(\frac{5}{2})^2$ ,  $x = \frac{5}{2} \cdot \frac{\sqrt{3}}{3}$  ft.  $= \frac{17.3205}{12}$  ft.  $= 17.32$  in.;  $\pi(x+y)^2 = \frac{2}{3}\pi(\frac{5}{2})^2$ ,  $x+y = \frac{5}{2} \cdot \frac{\sqrt{6}}{3} = \frac{24.49}{12}$  ft.  $= 24.49$  in.;  $\therefore y = 7.17$  in.,  $z = 30 - x - y = 5.51$  in.

13. Let  $x$  inches = a side of the cube. If ABCDEF is a right section of the tube, AC must be a diagonal of a face of the cube;  $\therefore AC = x\sqrt{2} = 2 \times 6 \sin 60^\circ = 6\sqrt{3}$ ,  $x = 3\sqrt{6}$  in.  $= 3(2.449) = 7.35$  in. approx. A model greatly simplifies this question.

14. Let  $h$  = ht.,  $r$  = rad. of base,  $2\pi r = 10$ ,  $\pi r^2 h = 600$ ,  $h = \frac{600}{25} \cdot \pi = 24 \times 3.1416 = 4 \times 18.8496 = 75$  ft. to the nearest foot.

15. Vol  $= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot 6$  c. ft.  $= \frac{3}{2}(1.73205) = 3(866025) = 2.598$  c. ft. Number of gallons  $= \frac{3\sqrt{3}}{2} \times \frac{1000}{16} \times \frac{1}{10} = \frac{2.598075 \times 100}{16} = \frac{64.9518}{4} \dots = 16.24$  gallons nearly.

16. Let  $r$  be the radius reqd.  $\pi r^2 \times 11 = \pi 36 \times 12 + \pi 18^2 \times 6$ ,  $r^2 = \frac{1}{11}(18 \times 12) \times (2 + 9) = 18 \times 12$ ,  $r = 6\sqrt{6} = 6 \times 2.449 = 14.7$  in. nearly.

17. Let  $x$  be the ht. and  $3x$  length of the slope of the part dug out.  $9x^2 = x^2 + 70^2 \therefore 8x^2 = 70^2 \therefore x = \frac{70}{2\sqrt{2}} = \frac{70}{4}\sqrt{2}$ . Vol.  $= 70 \times \frac{1}{2} \times 70x = \frac{70^3\sqrt{2}}{8}$  c. ft.  $= \frac{70^3}{8}(1.41421) = \frac{70^3}{8}(98.9947) = \frac{70}{8} \times 6929.629 = \frac{485074.03}{8} = 60634$  c. ft.

18. Let ABCD be a section of the cutting, AB being the base, BC the sloping side. Draw BE perp. to CD, CE = EB tan  $30^\circ = \frac{8\sqrt{3}}{3}$ . Vol. = area of trapezium ABCD  $\times 1000$  c. metres. Vol.

$$= \frac{8}{2}(9 \cdot 4 + 9 \cdot 4 + \frac{8\sqrt{3}}{3})1000 = 8(9400 + \frac{4000\sqrt{3}}{3}) = 75200 + \frac{32000\sqrt{3}}{3}$$

$$\times 8 = 75200 + 2309 \cdot 4 \times 8 = 75200 + 18475 \cdot 2 = 93675 \cdot 2 \text{ c. metres.}$$

### EXERCISES LXXVI.

1. Area of end  $CD = 5 \times 6$  sq. ft. Perp. from B on AC  $= \frac{1 \cdot 2 \times \sqrt{3}}{2} = 6\sqrt{3}$ . Vol.  $= 5 \times 6 \times 6\sqrt{3} = 3 \times 6(17 \cdot 3205) = 3(103 \cdot 9230) = 311 \cdot 769$  c. ft.

2 Vol.  $= \frac{2 \cdot 5}{1 \cdot 2} \times \frac{1 \cdot 3}{2}$  c. ft.  $= \frac{3 \cdot 2 \cdot 5}{4} = 13$  c. ft. 936 c. in.

3. Let AB be the length of pipe. Draw AC vertically and BC horizontally. Vol.  $= AC \times$  area of horizl. section  $= 4 \times \frac{1 \cdot 3 \times 1 \cdot 2}{\sqrt{2}}$  c. in.  $= 2 \times 13 \times 12 \times 1 \cdot 41421 = 2 \times 12 \times 18 \cdot 38473 = 2 \times 220 \cdot 61676 = 441 \cdot 233$  c. in.

4. Let  $r$  = rad. of circum-circle of the pentagon,  $a$  = a side of the pentagon,  $h$  = dist. of incentre from a side of the pentagon.  $r^2 = \frac{2a^2}{5 - \sqrt{5}}$  (p. 255)  $= \frac{32(5 + \sqrt{5})}{20} = \frac{8}{5}(5 + \sqrt{5})$ ,  $h^2 = r^2 - \frac{a^2}{4} = \frac{8}{5}(5 + \sqrt{5}) - 4 = 4 + \frac{8\sqrt{5}}{5} = 7 \cdot 578 \therefore h = 2 \cdot 752$ . Area of ends  $= \frac{2}{5} \cdot 5 \cdot ha = 20 \times 2 \cdot 752 = 55 \cdot 04$ . Area of sides  $= 5 \times 4 \times 7 = 140$  sq. ft. Total area  $= 195 \cdot 04$  sq. ft.

5. Taking the length as 10 in. and the altitude as 8 in. Base  $\times$  alt. = rt. sectn.  $\times$  length  $\therefore$  base  $= \frac{3 \cdot 40 \times 10}{8} = 425$  sq. in.  $= 2$  sq. ft. 137 sq. in.

### EXERCISES LXXVII.

1. Let ABCD be the tetrahedron. Draw CHE perp. to BD and let AH be the alt. of the tetrahedron.  $CH = \frac{2}{3}CE = \frac{2}{3} \cdot \frac{a\sqrt{3}}{2}$

$$\therefore AH = \sqrt{l^2 - \frac{a^2}{3}}. \text{ Vol.} = \frac{1}{3}AH \times \frac{a^2}{2} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{12} \sqrt{l^2 - \frac{a^2}{3}} \cdot a^2.$$

$$2. \text{ Alt.} = \sqrt{l^2 - \frac{a^2}{2}}. \text{ Vol.} = \frac{a^2}{3} \sqrt{l^2 - \frac{a^2}{2}}.$$

3. Let  $l$  = alt. of each slant face,  $h$  = alt. of pyramid.  $l = \sqrt{h^2 + \frac{a^2}{4}} \therefore 2A = al = a\sqrt{h^2 + \frac{a^2}{4}}$ ,  $4A^2 = a^2(h^2 + \frac{a^2}{4})$ ,  $h^2 = \frac{4A^2}{a^2} - \frac{a^2}{4}$ ,  $h = \sqrt{\frac{4A^2}{a^2} - \frac{a^2}{4}}$ . Vol.  $= \frac{1}{3}a^2h = \frac{a^2}{3} \sqrt{\frac{4A^2}{a^2} - \frac{a^2}{4}}$ .

4. Let  $h$  = alt. of pyramid. Rad. of circum-circle of base  $= a \therefore h^2 = l^2 - a^2 \therefore h = \sqrt{l^2 - a^2}$ . Vol.  $= \frac{1}{3} \sqrt{l^2 - a^2} \cdot 3 \cdot a^2 \frac{\sqrt{3}}{2} = \frac{a^2}{2} \sqrt{3(l^2 - a^2)}$ .

5. Let  $x$  = rad. of circum-circle of base.  $a^2 = 2x^2 - 2x^2 \frac{1}{\sqrt{2}}$   
(IV. 11.)  $= x^2(2 - \sqrt{2})$ ,  $x^2 = \frac{a^2}{2 - \sqrt{2}} = \frac{a^2(2 + \sqrt{2})}{2}$ . Vol.  $\frac{h}{3} \times 8 \times \frac{1}{2} x^2 \frac{1}{\sqrt{2}}$   
 $= \frac{8h}{6\sqrt{2}} \frac{a^2}{2} (2 + \sqrt{2}) = \frac{\sqrt{2}}{3} a^2 h (2 + \sqrt{2})$ ,  $l^2 = h^2 + x^2 = h^2 + \frac{a^2}{2} (2 + \sqrt{2})$ .

6. As in the preceding,  $x^2 = \frac{a^2}{2} (2 + \sqrt{2}) \therefore h^2 = l^2 - x^2 = l^2 - \frac{a^2}{2} (2 + \sqrt{2})$ . Vol.  $= \frac{\sqrt{2}}{3} a^2 (2 + \sqrt{2}) \sqrt{l^2 - \frac{a^2}{2} (2 + \sqrt{2})}$ .

7. Let  $h$  = altitude of the pyramid. Area of base  $= 3 \cdot a^2 \frac{\sqrt{3}}{2}$   
 $\therefore v = \frac{h}{3} \cdot 3a^2 \frac{\sqrt{3}}{2}$ ,  $h = \frac{2v}{a^2 \sqrt{3}}$ ,  $l = \sqrt{h^2 + a^2} = \sqrt{\frac{4v^2}{3a^4} + a^2}$ .

8. If  $x$  = edge of base,  $\frac{1}{3} \cdot h \cdot x^2 = v \therefore x^2 = \frac{3v}{h}$ ,  $l = \sqrt{h^2 + \frac{x^2}{2}}$   
 $= \sqrt{h^2 + \frac{3v}{2h}}$ .

9. Let  $h$  = alt. of pyramid.  $h^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$ ,  $h = \frac{a}{\sqrt{2}}$ . Vol.  $= \frac{1}{3} h a^2 = \frac{a^3}{3\sqrt{2}}$ .

10. Let ABCD be the base, and O the vertex of the pyramid. Join DB, AC, cutting at G. OG =  $h$ . Let NMLK be a face of the cube, N lying in OA, M in OB, L in BG, K in AG. Also let  $x$  = edge of cube.  $\frac{x}{h} = \frac{NK}{OG} = \frac{AK}{AG} = \frac{AG - KG}{AG} = 1 - \frac{KG}{AG} = 1 - \frac{x}{a}$ .  $x = \frac{ah}{a+h}$ .

11. Area of PQR  $= \frac{1}{4} \triangle BDC$  (V. 11)  $= \frac{1}{4} \cdot \frac{1}{2} \frac{a^2 \sqrt{3}}{2} = \frac{a^2 \sqrt{3}}{16}$ . Perp. from O on PQR  $= \frac{1}{2} h = \frac{\sqrt{6}a}{6}$  (VII. 5.)  $\therefore$  vol. OPQR  $= \frac{1}{3} \frac{a^2 \sqrt{3}}{16} \times \frac{\sqrt{6}a}{6} = \frac{\sqrt{2}a^3}{96}$ .

12. Let B, C, D be the mid. pts. of the edges which are conterminous at A. Vol. =  $\frac{1}{3} AC \times \triangle DAB = \frac{a}{6} \times \frac{1}{2} \frac{a^2}{4} = \frac{a^3}{48}$ .

13. If O is the centre, and POQ a diagonal of the base, draw from an angular pt. A of the top face AN perp. to POQ.  $OQ = \frac{a}{\sqrt{2}}$ ,  $ON = \frac{b}{\sqrt{2}}$ . Let  $h$  = alt. of frustum;  $h^2 = c^2 - \left(\frac{a-b}{\sqrt{2}}\right)^2 = c^2 - \frac{(a-b)^2}{2}$ ;  $\therefore$  vol. =  $\frac{1}{3} \sqrt{c^2 - \frac{(a-b)^2}{2}} \sqrt{a^2 + ab + b^2}$ .

14. Thro. A and A' ends of the top ridge draw vertl. planes APQ, A'P'Q' perp. to the longer sides and cutting them in P, Q, and P', Q'. Let APQ cut off pyramid APCBQ from the end of the bank. Draw AN perp. to PQ, and AM perp. to BC. MN bisects BC and PQ. Also  $\angle AMN = \angle APN$  (hyp.)  $\therefore$  from  $\triangle$ s AMN, APN,  $MN = PN = b$   $\therefore$  vol. =  $2 \times$  pyramid APCBQ + triangular prism between A and A'. Vol. =  $\frac{2}{3} h \cdot 2b^2 + h \cdot b(2a - 2b) = \frac{2hb}{3} [2b + 3a - 3b] = \frac{2hb}{3} (3a - b)$ .

15.  $\frac{\text{The pyramid cut off}}{\text{The whole pyramid}} = \frac{1}{2} \therefore \frac{\text{alt. of pyramid cut off}}{\text{alt. of whole pyramid}} = \frac{1}{\sqrt[3]{2}}$  (VII. 14.)  $\therefore$  the plane must divide the altitude (measured from the vertex) in the ratio of  $1 : \sqrt[3]{2} - 1$ .

16. With the same construction as in Example 14 above,  $h = 10 \tan 40^\circ$ . Vol. =  $\frac{2}{3} \cdot 10 \tan 40^\circ \times 10 \times 20 + \frac{1}{2} 10 \tan 40^\circ \times 20 \times 80 = 4000 \tan 40^\circ \left(\frac{1}{3} + 2\right) = 4000 \times \frac{7}{3} \times .8391 = \frac{4000}{3} \times 5.8737 = 4000 \times 1.9579 = 7831.6$  c. ft.

17. Let ABCD be the pyramid, BCD being an equilateral  $\triangle$ , and the  $\angle$ s at A rt.  $\angle$ s. Let  $p_1, p_2, p_3$  be the perps. from any pt. in BCD on the other faces.  $\triangle$ s ACB, ACD, ABD are equal in area. And  $p_1 \times \triangle ACB + p_2 \times \triangle ABD + p_3 \times \triangle ACD = 3$  vol. of pyramid  $\therefore p_1 + p_2 + p_3 = \frac{3 \text{ vol.}}{\triangle ACB}$ , which is constant.

18. Sum of perps.  $\times$  area of any face = 3 vol. of whole figure.

19. Alt. =  $\frac{\sqrt{6}a}{3}$  (VII. 3.)  $\therefore$  vol. =  $\frac{1}{3} \cdot \frac{\sqrt{6}a}{3} \cdot \frac{1}{2} \frac{a^2 \sqrt{3}}{2} = \frac{a^3 \sqrt{2}}{12}$ .

**20.** If OA, OB, OC, OD are four conterminous edges, ABCD is a sq. Draw ON perp. to DB. Let  $ON = h$ ,  $DN = \frac{a\sqrt{2}}{2}$   $\therefore h^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$ ,  $h = \frac{a}{\sqrt{2}}$ . Vol. =  $\frac{2}{3}h \cdot a^2 = \frac{\sqrt{2}a^3}{3}$ .

**21.** As in the preceding, diagonal =  $2h = \frac{2a}{\sqrt{2}} = \sqrt{2}a$ .

**22.** Let OABCD be the pyramid on the sq. base ABCD. Draw OE perp. to BC, and let  $OE = l$ . Then  $l^2 = 8^2 + 6^2 = 10^2$   $\therefore$  area of face =  $\frac{1}{2} \times 12 \times 10 = 60$  sq. ft.

**23.** Let OABCD be the pyramid on a sq. base ABCD. Draw OE perp. to BC, and let  $OE = l$ , and  $h =$  alt. of pyramid. Then  $l^2 = 16^2 - 6^2 = 2^2(8^2 - 9) = 2^2(55)$ ,  $h^2 = l^2 - 6^2 = 2^2(55 - 3^2) = 2^2 \times 46$ ,  $h = 2\sqrt{46} = 2 \times 6.782 = 13.564$  ft. Vol. =  $\frac{1}{3}h \cdot 144 = 48h = 12 \times 54.256 = 651.072$  c. ft.

**24.** If O is the vertex of the pyramid, AB one edge of the base, P the centre of the base and PN perp. to AB.  $PN = 3\sqrt{3}$  in. from  $\triangle APB$   $\therefore ON^2 = 9^2 + (3\sqrt{3})^2$   $\therefore ON = \sqrt{108} = 6\sqrt{3}$ . Slant surface =  $\frac{6}{2} \times 6 \times 6\sqrt{3} = 187.06$  sq. in.

**25.** Let  $h =$  alt. of pyramid. Taking a slant edge as 20 in. long,  $h^2 = 20^2 - 12^2 = 4^2 \cdot 4^2$ ,  $h = 16$ . Vol. =  $\frac{6}{3}h \cdot \frac{1 \cdot 2^2 \cdot \sqrt{3}}{2} = 72\sqrt{3}h = 1152\sqrt{3} = 1995.26$  c. in.

**26.** Vol. of whole pyramid =  $\frac{1}{3} \times 7^2 \times 8$   $\therefore$  if  $v =$  vol. of pyramid cut off,  $\frac{3v}{7^2 \times 8} = \frac{2^3}{8^3} = \frac{1}{64}$ .  $v = \frac{49}{24}$  c. in. Vol. of frustum =  $\frac{7^2 \times 8}{3} - \frac{49}{24} = \frac{7(448-7)}{24} = \frac{7 \times 441}{24} = \frac{7 \times 147}{8} = \frac{1029}{8} = 128.625$  c. in.

**27.** Let  $2x$  be any diagonal.  $2x^2 = 36$   $\therefore x = 3\sqrt{2}$ . Vol. =  $\frac{2}{3} \cdot 3\sqrt{2} \cdot 36 = 72\sqrt{2} = 72(1.41421) = 8(12.72789) = 101.823$  c. in.

**28.** Vol. of pyramid =  $\frac{1}{3} \times (\frac{9}{2})^2 \times \frac{15}{2}$  c. ft. Vol. of frustum =  $\frac{61}{3} \left[ (\frac{15}{2})^2 + \frac{15}{2} \cdot \frac{9}{2} + \frac{81}{4} \right] = \frac{61}{3 \times 4} (225 + 135 + 81) = \frac{61}{3 \times 4} (441) = \frac{61 \times 147}{4} = \frac{8967}{4}$   $\therefore$  total vol. =  $2241.75 + \frac{81 \times 5}{8} = 2241.75 + 50.625 = 2292.375$  c. ft. Weight =  $\frac{2292.375 \times 178}{2240}$  tons = 173.975 tons, nearly.

**29.** Surface =  $\frac{4}{2} 4^2 \frac{\sqrt{3}}{2} = 16\sqrt{3} = 16(1.73205) = 4(6.92820) = 27.713$  sq. in.

30. If  $l = a$  slant edge,  $l^2 = (\frac{1}{2})^2 + (5\sqrt{2} - \frac{5\sqrt{2}}{2})^2 = (\frac{5}{2})^2(9 + 2)$   
 $\therefore l = \frac{5}{2}\sqrt{11} = 8.29$  in. Let  $x = \text{alt. of a face}$ ,  $l^2 = x^2 + (\frac{5}{2})^2$   
 $\therefore x^2 = \frac{2.5}{4}(11 - 1) = \frac{2.5}{4} \cdot 10$ . Area of slant face  $= \frac{1}{2}(10 + 5)\frac{5\sqrt{10}}{2}$   
 $= \frac{7.5\sqrt{10}}{4} = 59.29$  c. in.

31. (1) Let the plane CDEF cut off the angular prism ACDEFB. Let  $a = \text{edge of cube}$ ,  $AC = b$ ,  $AD = c$ ,  $BF = d$ ,  $BE = e$ .  
 Vol. cut off = a frustum of a pyramid  $= \frac{1}{3}a \left[ \frac{1}{2}bc + \frac{1}{2}\sqrt{bcde} + \frac{de}{2} \right]$   
 $= \frac{a}{6}[bc + \sqrt{bcde} + de]$ .

(2) Let ABCD be a face, EFGH the cutting plane, AE, DF, CG BH being parallel edges. Vol. = Pyramid FGCBH + pyr. FABCD + pyr. FABHE  $= a \times \text{area GCBH} + \text{ED area ABCD} + a \times \text{area ABHE}$   
 $= \frac{a}{3} \left[ \frac{a(d+e)}{2} + c \cdot a + \frac{a(b+d)}{2} \right] = \frac{a^2}{6}(b + 2c + 2d + e) =$   
 $\frac{a^2}{2}(b + e) = \frac{a^2}{2}(c + d)$ .

### EXERCISES LXXVIII.

1. Cone cut off  $= \frac{1}{8}$  of whole cone (VII. 14.)  $\therefore \text{frustum} = \frac{7}{8}$   
 of whole cone  $= \frac{7}{8} \cdot \frac{1}{3} \cdot \pi r^2 h = \frac{7\pi r^2 h}{24}$ .

2. Alt.  $= \frac{\sqrt{6}a}{3}$  (VII. 5.)  $\therefore \text{vol. of frustum} = \frac{7}{8}$  of tetra-  
 hedron (VII. 14.)  $= \frac{7}{8} \cdot \frac{1}{3} \cdot \frac{\sqrt{6}a}{3} \cdot \frac{1}{2} \cdot \frac{a^2\sqrt{3}}{2} = \frac{7\sqrt{2}a^3}{96}$ .

3.  $\frac{\text{Vol. of cone cut off}}{\text{Vol. of whole cone}} = \frac{2^3}{5^3}$  (VII. 14.)  $\therefore \text{vol. of cone cut}$   
 off  $= \frac{2^3}{5^3} \cdot \frac{1}{3} \cdot \pi r^2 h = \frac{8\pi r^2 h}{375}$ .

4.  $\frac{\text{Surface of cone cut off}}{\text{Surface of whole cone}} = \frac{2^2}{5^2} = \frac{4}{25}$   $\therefore \text{curved surface of}$   
 frustum cut off  $= \frac{2}{5} \cdot \frac{1}{5} \text{ curved surface of whole cone (V. 12.)}$   
 $= \frac{2}{25} \pi r l = \frac{2}{25} \pi r \sqrt{h^2 + r^2}$ .

5. Vol. generated is two cones on a common base of rad.  $h$ .  
 If  $x$  is the alt. of one cone,  $a - x$  is the alt. of the other  
 $\therefore \text{vol.} = \frac{1}{3}\pi h^2(a - x) + \frac{1}{3}\pi h^2 x = \frac{1}{3}\pi h^2 a$ .

6. Alt. of  $\triangle = \frac{2A}{a} \therefore$  as in the preceding example, vol.  
 $= \frac{1}{3}\pi\left(\frac{2A}{a}\right)^2 \cdot a = \frac{4}{3}\pi\frac{A^2}{a}.$

7. Let  $h$  = the perp. from the rt.  $\angle$  upon the hypotenuse.  
 $h = \frac{bc}{a} \therefore$  as in the preceding, vol. generated  $= \frac{1}{3}\pi h^2 a = \frac{1}{3}\pi \frac{b^2 c^2}{a}.$

8. The vol. generated is a frustum of a cone, radii of ends  $b$  and  $c$ ; alt.  $a \therefore$  vol.  $= \frac{\pi a}{3}[b^2 + bc + c^2]$  (VII. 13.).

9. Alt. of cone  $= \frac{r}{\sqrt{3}} \therefore$  vol.  $= \frac{1}{3}\pi r^2 h = \frac{\pi r^3}{3\sqrt{3}}.$

10. Alt. of cone  $= \sqrt{l^2 - r^2} \therefore$  vol.  $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}.$

11. Let  $x$  = alt. of frustum,  $l$  = length of its slant side.  $l = \sqrt{241}$  by VII. 18.  $241 = x^2 + 16$ ,  $x = 15$  ft.

12. Radius of base = 1 in.  $\therefore$  length of slant side = cosec  $15^\circ$  in.  
 $\therefore$  surface  $= \pi r l = \frac{\pi}{\sin 15^\circ} = \pi \frac{2\sqrt{2}}{\sqrt{3}-1} = \pi \sqrt{2}(\sqrt{3}+1) = \pi(\sqrt{6}+\sqrt{2})$   
 $= 12.14$  sq. in.

13. If  $r$  = rad. of base and  $l$  = slant side,  $\pi r^2 = 180$  sq. ft.  
 Area of canvas  $= \pi r l = \pi r \sqrt{r^2 + \left(\frac{15}{2}\right)^2} = \frac{22}{7} \cdot \sqrt{\frac{180 \times 7}{22} + \left(\frac{180 \times 7}{22} + \frac{15^2}{4}\right)}$   
 $= \frac{15 \cdot 22 \cdot 6}{7} \cdot \sqrt{\frac{7}{44} \cdot \frac{(56+55)}{22}} = \frac{15 \cdot 6}{7} \sqrt{388.5} = \frac{90}{7} \times 19.71 = 253.4$  sq. ft.

14. If  $r$  = rad. of base,  $r = \frac{14}{\sqrt{3}} = \frac{14\sqrt{3}}{3}$  in. Vol. of whole cone  
 $= \frac{1}{3}\pi\left(\frac{14}{3}\right)^2 \cdot 14 = \frac{1}{3} \times \frac{22}{7} \times \frac{14^3}{3} = \frac{1}{3} \times \frac{176 \times 7^2}{3} = \frac{1}{3} \times \frac{1232 \times 7}{3} = \frac{1}{3} \times \frac{8624}{3}$   
 $= \frac{1}{3} 2874.67$  c. in. Vol. of cone cut off  $= \frac{1}{8}\left(\frac{2874.67}{3}\right) = \frac{359.33}{3}$   
 c. in.  $= 119.78$  c. in. Vol. of frustum  $= \frac{2515.34}{3}$  c. in.  $= 838.45$  c. in.  
 Slant surface of whole cone  $= \pi r l = \frac{22}{7} \cdot \frac{14\sqrt{3}}{3} \cdot \sqrt{\frac{14^2}{3} + 14^2} = \frac{22}{7} \cdot \frac{14^2}{3} \cdot 2 = 14 \times \frac{88}{3} = \frac{1232}{3}$  sq. in. Slant surface of cone cut off  $= \frac{1}{12}(1232) = 102.67$  sq. in. Slant surface of frustum cut off  $= 308$  sq. in.

15. Outside curved surface  $= 2\pi \cdot \frac{5}{2} \cdot 4 = 20\pi$  sq. in. Outside plane surface  $= \pi\left(\frac{5}{2}\right)^2 = \frac{25\pi}{4}$  sq. in. Inside surface =

$$\pi \frac{5}{2} \sqrt{16 + \frac{25}{4}} = \pi \frac{5}{4} \sqrt{89} = \pi \frac{10}{8} \times 9.434 = \frac{11}{7} \times \frac{1}{4} \times 94.34 = \frac{1037.74}{7 \times 4} \\ = \frac{259.44}{7} = 37.06 \text{ sq. in.} \quad \text{Whole surface} = \frac{105\pi}{4} + 37.06 = \\ \frac{15 \times 11}{2} + 37.06 = 82.5 + 37.06 = 119.56 \text{ sq. in.}$$

$$16. \text{ Vol.} = \frac{\pi}{3} (5\sqrt{3})^2 10 = \frac{750\pi}{3} = \frac{1500 \times 11}{3 \times 7} = \frac{16500}{3 \times 7} = 786 \text{ c. in}$$

17. Proved in VII. 18.

18.  $r$  = rad. of whole cone,  $r'$  = rad. of cone cut off;  $h$  = ht. of whole cone,  $h'$  = ht. of cone cut off;  $l$  = length of slant side,  $l'$  = length of slant side cut off;  $\pi r'l' + \pi r'^2 = \pi rl - \pi r'l' + \pi r'^2 + \pi r^2$ ;  $2r'l' = rl + r^2$ ;  $\frac{l'}{l} = \frac{r'}{r}$   $\therefore \frac{2l'^2 r}{l} = rl + r^2$ ;  $\frac{l'^2}{l^2} = \frac{r+l}{2l}$ ;  $\frac{l'}{l} = \frac{\sqrt{r+l}}{\sqrt{2l}}$ . Reqd. ratio =  $\frac{l'}{l-l'} = \frac{\sqrt{r+l}}{\sqrt{2l} - \sqrt{r+l}}$ .

### EXERCISES LXXIX.

1. By symmetry rad. =  $\frac{1}{4}$  alt. =  $\frac{\sqrt{6}a}{12}$

2. Rad. =  $\frac{3}{4}$  alt. =  $\frac{3}{4} \frac{\sqrt{6}a}{3} = \frac{a\sqrt{6}}{4}$ .

3. Rad. of section =  $\sqrt{10^2 - 6^2} = 8$  ft. Area =  $\pi 64 = 64(3.1416) = 201.0624$  sq. ft.

4. Area of cap. =  $2\pi \times 10 \times 2$  (VII. 23. Cor.). Area of circ. sectn. =  $\pi 6^2$ . Total area =  $\pi(40 + 36) = 76\pi = 238.76$  sq. ft.

5. Let  $x$  = rad. of the common section.  $10x = 8 \times 6$ . Area of section =  $\pi x^2 = \frac{\pi \times (48)^2}{100} = 72.38$  sq. ft.

6. Let  $r$  = rad. of sphere.  $4\pi r^2 = 1000 \therefore$  area of section =  $\pi(r^2 - 25) = 250 - 25\pi = 171.46$  sq. ft.

7. Let  $x$  = rad. of inner surface.  $\frac{4}{3}\pi x^3 \frac{6.6}{10} = [\frac{4}{3}\pi(6^3 - x^3)] \frac{4.2}{10}$ ,  $x^3(66 + 42) = 6^3 \times 42$ ,  $x^3 = \frac{6^3 \times 4.2}{18} = 6 \times 14$ .  $\log x = \frac{1}{3}(1.9243) = .6414$ ,  $x = 4.379$  in. Thickness of iron = 1.62 in.

8. Wt. =  $\frac{4}{3}\pi(11^3 - 9^3) \frac{7.776}{1728}$  ozs. =  $3.1416(1331 - 729) 6 = 3.1416 \times 602 \times 6 = 709.22$  lbs.

9. Let  $a$  = edge of cube,  $r$  = rad. of sphere.  $V_1$  = vol. of cube,  $V_2$  = vol. of sphere.  $6a^2 = 4\pi r^2$ ,  $\frac{a^2}{r^2} = \frac{4\pi}{6}$ .  $\frac{V_1}{V_2} = \frac{a^3}{\frac{4}{3}\pi r^3} = \frac{3}{4\pi} \cdot \left(\frac{2\pi}{3}\right)^{\frac{3}{2}} = \frac{3}{4} \left(\frac{8\pi}{27}\right)^{\frac{1}{2}} = \frac{3}{4} (\cdot 9308)^{\frac{1}{2}} = \frac{3}{4} (\cdot 9648) = \cdot 72$  nearly, *i.e.*  $\frac{V_1}{V_2} = \frac{72}{100}$ .

10. Vol. of iron =  $\frac{4}{3}\pi[5^3 - 4^3] = \frac{4}{3}\pi(125 - 64) = \frac{4 \times 61\pi}{3}$  c. in.  
 Wt. =  $\frac{4 \times 61 \times \pi}{3} \times \frac{1000}{1728} \times \frac{721}{100}$  ozs. =  $\frac{6710 \times 103}{3 \times 216}$  ozs. = 66.66 lbs.

11. (1) Let  $x$  in. be the thickness of gold.  $\frac{4}{3}\pi[2+x]^3 - 2^3] = \frac{4}{3}\pi 2^3$ ,  $[2+x]^3 = 2 \times 2^3$ ,  $x = 2[\sqrt[3]{2} - 1] = 2(\cdot 26) = \cdot 52$  in.  $[\log 2^{\frac{1}{3}} = \frac{1}{3}(\cdot 3010) = \cdot 1003 = \log 1\cdot 260]$ .  
 (2)  $4\pi(2+x)^2 = 4\pi \cdot 2^2 \times 2$ ,  $2+x = 2\sqrt{2}$ ,  $x = 2(\sqrt{2} - 1) = 2(\cdot 414) = \cdot 83$  in.

12. Let  $O$  be the centre of the earth,  $B$  the pt. at an alt. of 2000 ft.,  $BA$ ,  $BD$  tangents to the centre. Let  $OB$  meet  $AD$  at  $C$ .  $OC \cdot OB = OA^2$ . Hence if  $h$  = alt. of the visible segment.  $4000 - h$   

$$= \frac{4000^2}{4000 + \frac{2000}{5280}}$$

$$h = 4000 - \frac{4000^2}{4000 + \frac{2000}{5280}} = \frac{4000 \times 5280 + 2000}{2\pi \cdot 4000h}$$

$$= \frac{4000}{10561} \cdot \text{Fraction reqd.} = \frac{2\pi \cdot 4000h}{4\pi 4000^2} = \frac{h}{8000} = \frac{1}{21122}$$

13. Let  $d$  = diamr. reqd. Vol. melted down =  $1^3 - \frac{4}{3}\pi(\frac{1}{2})^3$   
 $\therefore \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = 1 - \frac{4}{3} \times \frac{\pi}{8}$ ,  $\frac{d^3}{8} = \frac{3}{4\pi} - \frac{1}{8}$ ,  $d^3 = \frac{6}{\pi} - 1 = 6 \times \cdot 31831 - 1 = \cdot 90986$ .  $\log d = \frac{1}{3}(\bar{1}\cdot 9590) = \bar{1}\cdot 9863 = \log \cdot 9690$ ,  $d = \cdot 969$  ft. = 11.63 in.

14. Let  $2r$  = the diameter reqd.  $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 7^3 - \frac{4}{3}\pi 5^3 \therefore r^3 = 7^3 - 5^3 = 343 - 125 = 218$ .  $\log r = \frac{1}{3}(2\cdot 3385) = \cdot 7795 = \log 6\cdot 019$ ,  $r = 6\cdot 019$  in., diam. = 12 in.

15. If  $r$  = rad. of the sphere,  $r = 6 \tan 30^\circ = 2\sqrt{3}$  in. Surface =  $4\pi r^2 = 48\pi = 48(3\cdot 1416) = 4(37\cdot 6992) = 150\cdot 80$  sq. in.

16. Let  $r$  = rad. of sphere.  $4\pi r^2 = \pi(\frac{5}{2})^2 \therefore r = \frac{5}{4}$  ft. Vol. =  $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi \frac{125}{64} = \frac{125}{3 \times 16} \cdot \pi = \frac{3141\cdot 6}{3 \times 16 \times 8} = \frac{1047\cdot 2}{16 \times 8} = \frac{130\cdot 9}{16} = \frac{32\cdot 7}{4} = 8\cdot 2$  c. ft.

17. Let  $x$  = dist. reqd.,  $h$  = ht. of visible segment,  $O$  the centre of the sphere,  $D$  the pt. of observation,  $DA$  and  $DE$  tangents to the sphere. Also let  $OD$  cut the sphere at  $C$ , and  $AE$  at  $B$ .  $2\pi 5h = \frac{3}{8} 4 \cdot \pi \cdot 25 \therefore h = \frac{15}{4}$  ft.,  $OB = \frac{5}{4}$  ft.  $OB \cdot OD = OA^2 \therefore (\frac{5}{4})(5+x) = 25$ ,  $5+x=20$ ,  $x=15$  ft.

18. Let  $r$  = rad. reqd.  $\frac{\pi r^2}{100} = \frac{4}{3} \pi \left(\frac{1}{2}\right)^3$ ,  $r^2 = \frac{100}{6} = 16.67$ ,  $r = 4.08$  in.

19. Let  $O$  be the centre of the earth,  $AD$  a diamr. of the ice field, and let the bisector  $OC$  of the  $\angle AOD$  meet  $AD$  at  $B$ .  $BC = 4000 - OB = 4000 - 4000 \cos 5^\circ = 4000(\cdot 003805) \therefore$  area reqd.  $= 2 \times \frac{2}{7} \times 4000^2 \times \cdot 003805 = \frac{2 \cdot 678720}{7} = 382674 \frac{2}{7}$  sq. miles.

20. Let  $a$  = an edge of the cube.  $a^3 = \frac{4}{3} \pi [5^3 - 3^3] = \frac{4}{3} \pi \cdot 98 = \frac{4}{3} \times 98 \times 3.1416 = 410.51024$  c. ft.  $\log a = \frac{1}{3} (2.6133) = .8711 = \log 7.432$ ,  $a = 7.43$  ft.

21. Let  $r$  = rad. of rim of umbrella,  $O$  the centre of the sphere of which the umbrella is a segment,  $h$  the ht. of the segment.  $2\pi \cdot \frac{7}{2} h = \frac{44}{3} \therefore h = \frac{44}{3} \cdot \frac{1}{2 \cdot 2} = \frac{2}{3}$  ft. Draw  $ON$  perp. to the circle formed by the rim.  $ON = \frac{7}{2} - \frac{2}{3} = \frac{17}{6}$  ft. Area reqd.  $= \pi r^2 = \pi \left[ \left(\frac{7}{2}\right)^2 - \left(\frac{17}{6}\right)^2 \right] = \frac{22}{8} [49 - \frac{289}{9}] = 13.27$  sq. ft.

22. Let  $O$  be the centre of the sphere,  $A, B, C$  angular pts. of one end of the prism,  $OG$  perp. to  $ABC$ ,  $AGN$  perp. to  $BC$ .  $AG = \frac{2}{3} AN = \frac{2}{3}$ .  $AC \sin 60^\circ = \frac{\sqrt{3}}{3}$  ft.  $OG^2 = AO^2 - AG^2 = 1 - \frac{1}{3} = \frac{2}{3}$ .  $OG = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$  ft. Ht. of prism  $= \frac{2\sqrt{6}}{3}$  ft. Vol. of prism  $\frac{1}{2} \cdot \frac{\sqrt{3}}{2} \times 2\sqrt{6} = \frac{\sqrt{2}}{2}$  c. ft.  $= 1221.88$  c. in.

23. Alt. of tetrahedron  $= \frac{10\sqrt{6}}{3}$  cms. Vol. of tetra.  $= \frac{1}{3} \frac{1000}{2} \frac{\sqrt{3}}{2} \times \frac{10\sqrt{6}}{3}$  c. cms.  $= \frac{10000\sqrt{2}}{12}$ . Let  $n$  be the no. of bullets.  $n \times \frac{4}{3} \pi \left(\frac{1}{2}\right)^3 = \frac{1000\sqrt{2}}{12}$ ,  $n = \frac{1000\sqrt{2}}{2\pi} = \frac{1414.21}{2} \times .31831 = 225$ .

24. Let  $h$  = ht. of cone.  $\frac{1}{3} \pi r^2 h = \frac{2}{3} \pi r^3 \therefore h = 2r$ .

25. Mass  $= \frac{4}{3} \pi [6^3 - 4^3] \frac{594}{1728}$  lbs.  $= \pi 19 \times \frac{28}{9} = 3.1416 \times \frac{19 \times 28}{9} = 186$  lbs.



### ADDITIONAL EXERCISES XVII.

**29.** Draw  $AB$  5 cms. Make  $\angle BAD = 62^\circ$ . Make  $AD$  11 cms. From  $C$  the mid. point of  $BD$  draw  $CE$  perp. to  $BD$  to meet  $AD$  at  $E$ . The  $\triangle ABE$  is the one reqd. For its perimeter =  $DE + EA + AB = 16$  cms.  $BE = 5.45$  cms.;  $EA = 5.55$  cms. approximately.

**30.** Draw  $AB$  5 cms. Cut off  $AC$  equal to 4 cms. Draw  $CD$  4 cms. perp. to  $AB$ . Join  $AD$ ,  $BD$ .  $AD = 5.66$  cms.;  $BD = 4.12$  cms.

**31.** (a) The  $\angle$  of a regular hexagon =  $120^\circ \therefore$  3 regular hexagons fit round a point without leaving any gap. Thus any number of equal regular hexagons can be fitted together to form a pavement.

• (b) The  $\angle$  of a regular octagon =  $135^\circ \therefore$  two regular octagons and a square fit round a point. Fit together two equal regular octagons with a common side. One square with a side of the same length will fill the gap. Two other equal regular octagons may be applied to the remaining sides of the square, and so on.

**32.** The sum of the other two  $\angle$ s =  $165^\circ$ . Make any  $\angle BAC$ , say  $70^\circ$ . Make  $\angle ABC = 95^\circ$ . Then  $\angle C = 180^\circ - 70^\circ - 95^\circ = 15^\circ$ .

**33.** (a) Draw perps. to two sides from their mid. points. The intersection of these is the point equidistant from all the vertices (I. 23.). Distance 1.78 cms.

(b) Bisect two of the angles (or two exterior angles). The intersection of the bisectors is a point equidistant from all the sides (I. 24.).  $r = .82$  cms.

**34.** From  $D$ , the mid. point of  $AB$  draw  $DH$  perp. to  $AB$ . Any point in  $DH$  is equidistant from  $A$  and  $B$  (I. 23.). With centre  $C$  and radius 2 cms. describe a circle cutting  $DH$  in  $E$ ,  $F$ . These are the reqd. points.

**35.** Through  $E$ , the mid. point of  $BC$  draw a perp.  $DEF$ . Cut this perp at  $D$ ,  $F$  by a circle with centre  $A$  and radius 2 inches.  $D$ ,  $F$  are the required points. By measurement  $ED = EF = 1$  inch.  $\angle BAD = 30^\circ$ .

36. Heights in feet 10·35, 20, 28·28, 34·64. The increase of height is not proportional to the increase of angle of elevation.

37.  $x = 6$ ,  $y = 4$ ,  $z = 3$ . Construct by I. 25.

38. Draw AB to represent 70 horizontal feet. Draw perps. DB, ACE. Let DB, AC each represent 20 feet. Make  $\angle CDE$   $56\frac{1}{2}^\circ$ . AE represents the flagstaff. Height 125·76 feet.

39. Draw AB vertical to represent 100 feet. Draw AD, BC horizontal. Make an angle DAC  $20^\circ$ . BC represents the breadth of river, 274·75 feet.

### ADDITIONAL EXERCISES XXV.

39. The reqd. error = 3·4 feet.

40. (1) Ht. of tower = 93·3 yards.

(2) Find a pt. on the ground where the tree subtends  $45^\circ$ . The dist. of this pt. from the tree = the ht. of the tree.

41. 16·1 yards.

42. The diagonals bisect one another at rt.  $\angle$ s

43. Draw AB equal to 65 mms.  $\angle ABC$ ,  $70^\circ$ . Draw AD  $\parallel$  to BC, and with centre B and rad. 85 mms. describe an arc cutting AD at D. Draw DC  $\parallel$  to AB. ABCD is the reqd. parm.

44. Take AB 12 cms. long and draw CD  $\parallel$  to it, at a dist. of 3 cms. from it. With centres A and B and rad. 6 cms. describe arcs cutting CD at C and D. ACDB is the trapezium.

45. Make  $\triangle AOB$  such that AB = 33 mms., AO = 30 mms. and BO = 38 mms. Produce AO, BO to C and D, making OC equal to OA, and OD equal to OB. ABCD is the parm.

46. Draw lines  $\parallel$  to those containing the  $\angle 65^\circ$ , and at distances of 1 in. and 2 in. from them. Their pt. of intersection is the reqd. pt.

47. Reqd. distance = 28·85 ft.

48. Let ABCDEF be the hexagon. Draw GAH  $\parallel$  to CE to meet EF at H and CB at G. Rect. GHEC = hexagon in area.

From EH produced cut off  $EK = EC$ . Bisect EK at O, describe circle KPE, centre O, cutting HG at P.  $EP^2 = EH \cdot EK$  (proved in last line but four of II. 11.)  $= EC \cdot EH$  = the hexagon. Area = 2.6 sq. in.

**49.** If ABCD is the sq., bisect AD at E, BC at F. Produce EF to H, making  $FH = FE$ . AHD is the  $\triangle$  reqd.

**50.** (1) Seven rectangles each 12 sq. cms. in area.

(2) Twelve. (3) 84.

**51.** By measurement, the longer diagonal = 7.39 in. Diagonal of sq. = 5.66. Reqd. shortening = 1.73 in. Area of rhombus =  $8\sqrt{2} = 11.31$ . Increase of area = 4.69 sq. in.

**52.** Correct area = 288 sq. ft.  $\therefore$  correct side = 16.97 ft. He must shorten each side by .03 feet.

**53.** Let the pole BC break at O, and just miss the window D, in the house AD.  $OC = OD$ . Let  $OC = x$  and EO be  $\parallel$  to AB. From  $\triangle DEO$ ,  $(x - 20)^2 + 40^2 = x^2$  (II. 11.) whence  $x = 50$  feet. This problem might also be solved on sqd. paper by making  $\angle CDO = \angle DCO$ .

**54.** Let ABCDEF be the hexagon, and draw HFG  $\parallel$  to EA to meet DE at H and BA at G. As in problem 48 above, rect. HGBD = the hexagon. Produce BG to K, making  $GK = GB$ . KHB is the reqd.  $\triangle$ .  $BK = 6$  in.

**55.** If ABCD is the trapezium, AB being the largest side, draw DE perp. to AB.  $AE = 1$  cm.  $\therefore DE = \sqrt{AD^2 - AE^2} = \sqrt{8} = 2.83$  cms.

**56.** If  $x$  is one side of the rect.  $6 - x$  is the other side  $\therefore x(6 - x) = \frac{27}{4}$  whence  $x = 4\frac{1}{2}$  or  $1\frac{1}{2}$  approx.  $\therefore 4\frac{1}{2}, 4\frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2}$  in. are the parts.

**57.** The  $\triangle$  formed will be half an equilateral  $\triangle$ . Ht. = 10 feet. Horizontal dist. = 17.32 ft.

**58.** If  $x$  and  $x + 24$  be the sides,  $x(x + 24) = 640$  whence  $x = 16$   $\therefore$  the sides are 16 and 40 cms.

**59.** Describe ABC an equilat.  $\triangle$  on a base 8. Bisect BC at D, AB at E.  $AD = 4\sqrt{3}$ ,  $AE = 4$ ,  $\angle EAD = 30^\circ$   $\therefore$  we have to find a  $\triangle$ , rt  $\angle$ d. and isos. and equal in area to EAD. Take F the mid. pt. of BD.  $\triangle AFD = \triangle AED$ . From DA cut off  $DH = DF$ .

On AD describe a semicircle, and let HK perp. to AHD meet it at K.  $DK^2 = DH \cdot DA$  as in problem 48 above  $= 2\triangle ADF = 2\triangle AED$ . But if  $x$  is the reqd. hypotenuse,  $\triangle AED = \frac{x^2}{4} \therefore \frac{x^2}{4} = DK^2 \therefore x = 2DK$ .

60. Area = a rect.  $3\frac{1}{2}$  ft. by  $2\pi \times \frac{5.90}{4} = 2750$  sq. ft. taking  $\pi = \frac{22}{7}$ .

61. If  $x$  and  $2x$  are the sides,  $x^2 = 98$ ,  $x = 7\sqrt{2}$ . Hypotenuse  $= x\sqrt{5} = 7\sqrt{10} = 22.13$  cms. If  $p$  is the perp. reqd.  $\frac{p}{2} \times 7\sqrt{10} = 98$ ,  $p = \frac{28}{\sqrt{10}} = \frac{28\sqrt{10}}{10} = 8.85$  cms.

62. Draw a str. line ABC, making  $AB = 2.5$  cms.  $AC = 3.6$  cms. On AC describe a semicircle, and let BD, perp. to ABC, meet it at D. AD is a side of the reqd. sq. For  $\triangle ADC$  is rt.  $\angle d$ . (Ex. xviii. 9.) and  $AD^2 = AB \cdot AC$  (II. 11. last line but four). The diagonal  $= 4.2$  cms.

63. Alt.  $= 2\sqrt{3}$  (II. 11.)  $= 3.46$  cms..

64. Length of each side  $= 1.008$  of the given line. Diff.  $= .008$  of the given line. Fraction  $= \frac{1}{128}$  about.

65. Area  $= \frac{1}{2} \times 84 \times (37 + 41) = 42 \times 78 = 3276$  sq. ft.

66. Let ABCD be the fig.  $AB = 28$ ,  $BC = 25$ ,  $CD = 3$ ,  $AD = 30$  cms. Draw DE, CF perp. to AB. Let  $DE (= CF) = x$ ,  $28 = AE + EF + FB = \sqrt{30^2 - x^2} + 3 + \sqrt{25^2 - x^2}$ . Whence  $x = 24$  cms. Area  $= \frac{1}{2}(28 + 3)24 = 31 \times 12 = 372$  sq. cms.

67. The two  $\triangle s$  formed are equal in all respects. Width of road  $= 18 + 24 = 42$  feet. Length of ladder  $= \sqrt{18^2 + 24^2} = 6\sqrt{3^2 + 4^2} = 30$  ft.

68. Length of median  $= 13$ .

69. Area  $= 6$  times an equilat.  $\triangle$  of sides 16  $= 665.1$  sq. units.

70. (a) Area  $= 13,403$  sq. yds. } See p. 98.  
(b) Area  $= 136,350$  sq. yds. }

71. The angles are equal to those of  $\triangle ABC$ . If we turn ABC thro. a rt.  $\angle$ , its sides become  $\parallel$  to those of the new  $\triangle$ .

72.  $PM + PN = 4$  in.  $= OA = OB$ , for  $\triangle s$  AMP, BNP are isos. and rt.  $\angle d$ .

73. Draw AE perp. to BD. AE bisects BD.  $BD = 16\sqrt{4^2 + 3^2} = 80 \therefore BE = 40 \therefore AE = \sqrt{96^2 - 40^2} = 8\sqrt{144 - 25} = 8\sqrt{119} = 87.28$ . Area of ABCD  $= \frac{1}{2}BC \cdot CD + \frac{1}{2}AE \cdot BD = 32 \times 48 + 40 \times 87.28 = 5027$  sq. yds.

74. PQ = 5405 yds.

75. Complete the rect. CBFH. Join HB, and produce it to meet DA in K. Draw KLM  $\parallel$  to ABF to meet CB, HF in L, M. BLMF is the reqd. fig. by II. 10.

76. If the duck enters the water at A, descends to B, and emerges at C. ABC is an isos.  $\triangle$ . Draw BD perp. to AC. ABD is half an equilat.  $\triangle \therefore AC = 2AD = 2 \times 9\sqrt{3} = 31.18$  feet.  $BD = \frac{1}{2}AB = 9$  feet.

77. Let ABC be the  $\triangle$ ,  $\angle B$  being a rt.  $\angle$ , ACDE the road. Draw AF, CG perp. to ED.  $\triangle$ s AFE, CGD are isos. and rt.  $\triangle$ d.  $\therefore AE = 11.5 \times \sqrt{2} \therefore$  area of ground  $= \frac{1}{2}BE \cdot BD = \frac{1}{2}(150 + 11.5 \times \sqrt{2})^2 = 13821.76$  sq. metres  $\therefore$  cost = £6911 to the nearest pound.

78. If A is the top of the tower, B the observer's eye, and BC perp. to the tower. ABC is half an equilat.  $\triangle \therefore AC = 200\sqrt{3} = 346.4$  ft.  $\therefore$  ht. of tower = 351.4 feet.

79. Let ABCD be the sq., EF one of the  $\parallel$  lines cutting AB at E. Draw EM perp. to BD.  $BE = EM\sqrt{2} = \sqrt{2}$  cms.  $\therefore AE = 12 - \sqrt{2}$ ,  $\triangle AEF = \frac{1}{2}(12 - \sqrt{2})^2 = 56.03$  sq. cms. Area of middle portion  $= 144 - 112.06 = 31.94$  sq. cms.

80. The area is made up of a  $\triangle$  and four trapeziums. Total area  $= \frac{3}{2} + \frac{5}{2} + \frac{6}{2} + \frac{8\frac{1}{2}}{2} + \frac{7\frac{1}{2}}{2} = 15$  sq. in.

81. If we suppose the vertical surface of the tower unwrapped until it lies in a plane, the rope will then run in a str. line along a diagonal of the rectangle thus formed. Length of rope  $= \sqrt{48^2 + 20^2} = 52$  feet.

**EXERCISES LXV.** (*Continued.*)

4. Draw any regular hexagon, reduce it to a  $\Delta$ , and following the construction in V. 26. obtain the line GM, so that  $GM^2$  = the area of the hexagon.

On a str. line PQ, 2.36 in. long, describe a semicircle, and from PQ cut off PN equal to 1 in. Draw NR perp. to PQ to meet the circle in R.  $PR^2 = PN \cdot PQ = 2.36$ .

Now proceed as in V. 26., making  $AB : AB' = GM : PR$ .

The proof is the same as in V. 26.

5. and 6. Similar to 4, but starting with any regular pentagon and octagon respectively.

7. Use the method of V. 26. A, taking  $n = 3$ .

8. As in V. 26. A, we have to get  $Ab$  such that  $\frac{Ab}{AB} = \frac{1}{\sqrt{5}}$ .

Hence on FG (5 half-inches long) describe a semicircle, from FG cut off FH equal to 1 half-inch, and draw HK perp. to FG to meet the circumference in K.  $FK^2 = FH \cdot FG = 5$ .  $\therefore FK = \sqrt{5}$ .

$\therefore \frac{FH}{FK} = \frac{1}{\sqrt{5}}$ , and we then find  $Ab$  such that  $\frac{Ab}{AB} = \frac{FH}{FK}$ .

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