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FINANCIAL MATHEMATICS

BY

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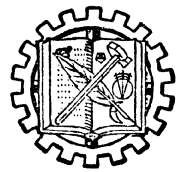

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PREFACE

This text is designed for a three-hour, one-year course for students who desire a knowledge of the mathematics of modern business and finance. While the vocational aspects of the subject should be especially attractive to students of commerce and business administration, yet an understanding of the topics that are considered—interest, discount, annuities, bond valuation, depreciation, insurance—may well be desirable information for the educated layman.

To live intelligently in this complex age requires more than a superficial knowledge of the topics to which we have just alluded, and it is palpably absurd to contend that the knowledge of interest, discount, bonds, and insurance that one acquires in school arithmetic is sufficient to understand modern finance. Try as one may, one cannot escape questions of finance. The real issue is: shall we deal with them with understanding and effectiveness or with superficiality and ineffectiveness?

While this text presupposes a knowledge of elementary algebra, we have listed for the student's convenience, page x, a page of important formulas from Miller and Richardson, *Algebra: Commercial—Statistical* that should be adequate for the well-prepared student. Although we make frequent reference to this Algebra in this text on Financial Mathematics, the necessary formulas are found in this reference list.

In the writing of this text the general student and not the pure mathematician has been kept constantly in mind. The text includes those techniques and artifices that many years of experience in teaching the subject have proved to be pedagogically fruitful. Some general features may be enumerated here: (1) The *illustrative examples* are numerous and are worked out in detail, many of them having been solved by more than one method in order that the student may compare the respective methods of attack. (2) *Line diagrams*, valuable in the analysis and presentation of problem material, have been given emphasis. (3) *Summaries* of important formulas occur at strategic points. (4) The *exercises* and *problems* are numerous, and they are purposely selected to show the applications of the theory to the many fields of activity. These exercises and problems are abundant, and no class will hope to do more than half of them. (5) Sets

of *review problems* are found at the ends of the chapters and the end of the book.

A few special features have also been included: (1) *Interest* and *discount* have been treated with unusual care, the similarities and differences having been pointed out with detail. (2) The treatment of *annuities* is pedagogical and logical. This treatment has been made purposely flexible so that, if it is desired, the applications may be made to depend upon *two* general formulas. No new formulas are developed for the solution of problems involving annuities due and deferred annuities, and these special annuities are analyzed in terms of ordinary annuities. (3) The discussion of probability and its application to insurance is more extended than that found in many texts.

In this edition we are including Answers to the exercises and problems.

While we have exercised great care in the preparation of this book, it is too much to expect that it is entirely free from errors. For the notification of such errors, we shall be truly grateful.

C. H. Richardson.

Bucknell University,
Lewisburg, Pennsylvania,
1946.

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* Miller and Richardson, *Algebra: Commercial—Statistical*, D. Van Nostrand Co., Inc., New York, N. Y.

CHAPTER I

SIMPLE INTEREST AND DISCOUNT

1. Interest.—*Interest is the sum received for the use of capital.* Ordinarily, the interest and capital are expressed in terms of money. The capital is referred to as the *principal.* To determine the proper amount of interest to be received for the use of a certain principal, we must know the *time* that the principal has been in use and the *rate of interest* that is being charged. The rate of interest is the rate per *unit of time* that the *lender* receives from the *borrower* for the use of the money. The *rate of interest* may also be defined as the *interest earned* by *one unit of principal* in *one unit of time.* The unit of time is almost invariably one year, and the unit of principal one dollar. The sum of the principal and interest is defined as the *amount.*

When interest is paid only on the principal lent, it is called *simple interest.* In case the interest is periodically added to the principal, and the interest in the following period is each time computed on this principal thus formed by adding the interest of the previous period, then we speak of the interest as being *compounded,* and the sum by which the original principal is increased at the end of the time is called the *compound interest.* In this chapter only simple interest calculations will be considered.

2. Simple interest relations.—Simple interest on any principal is obtained by multiplying together the numbers which stand for the principal, the rate, and the time in years.

If we let P = the principal,
i = the rate of interest (in decimal form),
n = the time (in years),
I = the interest,
and S = the amount,

it follows from the definitions of interest and amount that:

$$I = Pni, \tag{1}$$

and

$$S = P + I. \tag{2}$$

From relations (1) and (2), we get

$$S = P + Pni = P(1 + ni),$$

or,
$$P = \frac{S}{1 + ni}. \tag{3}$$

Relations (1) and (2) involve five letters (values). If we know any three of the values, the other two may be found by making use of these relations. Let us illustrate by examples.

Example 1. Find the interest on \$700 for 4 years at 5%. Find the amount.

Solution. Substituting in (1) the values, $P = 700$, $n = 4$, $i = 0.05$, we obtain

$$I = 700 \cdot 4 \cdot 0.05 = \$140.00, \text{ interest.}$$

And
$$S = 700 + 140 = \$840.00, \text{ amount.}$$

Example 2. A certain principal in 5 years, at 5%, amounts to \$625. Find the principal.

Solution. $S = 625$, $n = 5$, $i = 0.05$.

Substituting in (3), we have

$$P = \frac{625}{1 + (5 \cdot 0.05)} = \frac{625}{1.25} = \$500, \text{ principal.}$$

Example 3. Find the rate if \$500 earns \$45 interest in 18 months.

Solution. Here, $P = 500$, $I = 45$, $n = 1\frac{1}{2}$.

From relation (1) we have,

$$i = \frac{I}{Pn} = \frac{45}{500 \cdot \frac{3}{2}} = 0.06 = 6\%.$$

Example 4. In what time will \$300 earn \$81 interest at 6%?

Solution. Here, $P = 300$, $i = 0.06$, $I = 81$.
From relation (1) we have,

$$n = \frac{I}{Pi} = \frac{81}{300(0.06)} = 4\frac{1}{2} \text{ years.}$$

Exercises

1. Making use of relations (1) and (2), express S in terms of I, n, and i.

2. Find the interest on \$5,000 for 2½ years at 5%. Find the amount.

3. Find the simple interest on \$350 for 7 months at 6½%.

4. In what time will \$750 earn \$56.25 interest, if the rate is 5%?

5. At 4½%, what principal will amount to \$925 in 3½ years?

6. In what time will \$2,500 amount to \$2,981.25 at 3½%?

7. \$2,400 amounts to \$2,526 in 9 months. Find the rate.

8. What is the rate of interest when \$2,500 earns \$87.50 interest in 6 months?

9. What principal will earn \$300 interest in 16 months, at 5%?

10. In what time will \$305 amount to \$344.65 at 4% interest?

11. What is the rate when \$355 amounts to \$396.42 in 2 years and 4 months?

12. What sum must be placed at interest at 4% to amount to \$299.52 in 4 years and 3 months?

13. A building that cost \$7,500, rents for \$62.50 a month. If insurance and repairs amount to 1% each year, what is the net rate of interest earned on the investment?

14. If the interest on a certain sum for 4 months at 5% is \$7.54, what is the sum?

15. What principal in 2 years and 5 months, will amount to \$283.84, at 4½%?

16. At age 60 a person wishes to retire and invests his entire estate in bonds that pay 4% interest. This gives him a monthly income of \$87.50. What is the size of his estate?

3. Ordinary and exact interest.—Most of the problems considered in simple interest involve intervals of time measured in days or parts of a year. The general practice is to calculate the interest for a fractional part of a year on the basis of 360 days in a year (12 months of 30 days each). When 360 days is used as the basis for our calculations, we have what is called *ordinary simple interest.* When the exact number of days between two dates is counted and 365 days to a year is used as the basis of our calculations, we have what is known as *exact simple interest.*

If we let d = the time in days,
P = the principal,
i = the rate,
I_o = ordinary interest,
and I_e = exact interest, it follows that:

$$I_o = \frac{Pdi}{360}, \tag{4}$$

and

$$I_e = \frac{Pdi}{365}. \tag{5}$$

If we divide the members of (5) by the corresponding members of (4), we have

$$\frac{I_e}{I_o} = \frac{360}{365} = \frac{72}{73},$$

$$I_e = \frac{72}{73} I_o = I_o - \frac{1}{73} I_o. \tag{6}$$

We notice from (6) that the exact interest for any number of days is $72/73$ times the ordinary interest, or, in other words, exact interest is $I_o/73$ less than ordinary interest. Hence, *we may find the exact interest by first computing the ordinary interest and then diminishing it by $1/73$ of itself.*

Example. What is the ordinary interest on \$500 at 5% for 90 days? What is the exact interest?

Solution. Substituting in (4), we get

$$I_o = \frac{500 \cdot 90 \cdot 0.05}{360} = \$6.25.$$

$$6.25 \div 73 = 0.085+.$$

Hence,
$$I_e = 6.25 - 0.09 = \$6.16.$$

Thus the ordinary interest is \$6.25 and the exact interest is **\$6.16.**

The exact interest could have been computed by applying (5), but the method used above is usually shorter, as will be seen after the reading of Art. 5.

4. Methods of counting time.—In finding the time between two dates the exact number of days may be counted in each month, or the time may be first found in months and days and then reduced to days, using 30 days to a month.

Example 1. Find the time from March 5 to July 8.

Solution. By the first method the time is 125 days. By the second method we get 4 months and 3 days or 123 days.

Either of these methods of computing time may be used where ordinary interest is desired, but when exact interest is required the exact time must be employed. Use of the following table will greatly facilitate finding the **exact** number of days between two dates.

TABLE SHOWING THE NUMBER OF EACH DAY OF THE YEAR COUNTING FROM JANUARY 1

Day of month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29	..	88	119	149	180	210	241	272	302	333	363	29
30	30	..	89	120	150	181	211	242	273	303	334	364	30
31	31	..	90	...	151	...	212	243	...	304	...	365	31

NOTE.—For leap years the number of the day is one greater than the tabular number after February 28.

Example 2. Find the exact interest on \$450 from March 20 to August 10 at 7%.

Solution. The exact time is 143 days.

Substituting in (5), we have

$$I_e = \frac{450 \cdot 143 \cdot 0.07}{365} = \$12.34.$$

Example 3. Find the ordinary interest in the above exercise.

Solution. Either 143 days (exact time) or 4 months and 20 days (140 days) may be used for the time when computing the ordinary interest. Using 143 days and substituting in (4), we have

$$I_o = \frac{450 \cdot 143 \cdot 0.07}{360} = \$12.51.$$

Using 140 days and substituting in (4), we have

$$I_o = \frac{450 \cdot 140 \cdot 0.07}{360} = \$12.25.$$

Either \$12.51 or \$12.25 is considered the correct ordinary simple interest on the above amount from March 20 to August 10. The computation of *ordinary interest* for the *exact time* is said to be done by the *Bankers' Rule.*

Exercises

1. Find the ordinary and exact interest on the following:

a. \$300 for 65 days at 6%.
b. \$475.50 for 49 days at 5%.
c. \$58.40 for 115 days at 7%.
d. \$952.20 for 38 days at 4½%.

2. Find the ordinary and exact interest on \$2,400 at 8% from January 12 to April 6. Find the ordinary interest first and then use (6) to determine the exact interest.

3. Find the exact interest on \$350 from April 10 to September 5 at 7%.

4. Find the ordinary interest on \$850 from March 8 to October 5 at 6%.

5. How long will it take \$750 to yield \$6.78 exact interest at 6%?

6. How long will it take \$350 to yield \$3.65 ordinary interest at 5%?

7. The exact interest on \$450 for 70 days is \$7.77. What is the rate?

8. If the exact interest on a given principal is \$14.40, find the ordinary interest for the same period of time by making use of (6).

9. The ordinary interest on a certain sum is \$21.90. Find the exact interest for the same period of time.

10. What is the difference between the ordinary and exact interest on \$2,560 at 6% from May 5 to November 3?

11. The difference between the ordinary and exact interest on a certain sum is \$0.40. Find the exact interest on this sum.

5. The six per cent method of computing ordinary interest.—Ordinary simple interest may be easily computed by applying the methods of multiples and aliquot parts.

If we consider a year as composed of 12 months of 30 days each (360 days),

at 6%, the interest on \$1 for 1 year is \$0.06,
at 6%, the interest on \$1 for 2 mo. (60 days) is \$0.01,
at 6%, the interest on \$1 for 6 days is \$0.001.

That is, to find the interest on any sum of money at 6% for 6 days, point off three places in the principal sum; and for 60 days, point off two places in the principal sum.

By applying the above rule we may find the ordinary interest on any principal for any length of time at 6%. After the ordinary interest at 6% is found, it is easy to find it for any other rate. Also, by applying (6), Art. 3, the exact interest may be readily computed.

Example 1. What is the ordinary interest on \$3,754 for 80 days at 6%?

Solution.

\$37.54	= interest for	60	days	
12.51	" "	20	"	($\frac{1}{3}$·60 days)
\$50.05	" "	80	days	

Example 2. What is the ordinary simple interest on \$475.25 for 115 days at 6%?

Solution.

\$4.753	= interest for	60	days	
2.376	= " "	30	"	($\frac{1}{2}$·60 days)
1.584	= " "	20	"	($\frac{1}{3}$·60 days)
0.396	= " "	5	"	($\frac{1}{4}$·20 days)
\$9.11	= interest for 115 days.			

Example 3. Compute the ordinary interest on \$865 for 98 days at 8%.

Solution.

\$8.65	=	interest for 60 days at 6%
4.325	=	" " 30 " " " (Why?)
0.865	=	" " 6 " " "
0.288	=	" " 2 " " " (Why?)
\$14.128	=	interest for 98 days at 6%
4.709	=	" " " " " 2% ($\frac{1}{3}\cdot 6\%$)
\$18.84	=	interest for 98 days at 8%. (Why?)

Example 4. Find the simple interest on \$580 for 78 days at 4½%.

Solution.

\$5.80	=	interest for 60 days at 6%
1.45	=	" " 15 " " "
0.29	=	" " 3 " " "
\$7.54	=	interest for 78 days at 6%
1.885	=	" " " " " 1½% (Why?)
\$5.66	=	interest for 78 days at 4½%. (Why?)

Example 5. Find the exact simple interest on \$2,500 for 95 days at 7%.

Solution.

\$25.00	=	interest for 60 days at 6%
12.50	=	" " 30 " " "
2.08	=	" " 5 " " "
\$39.58	=	interest for 95 days at 6%
6.60	=	" " 95 " " 1%
\$46.18	=	ordinary interest for 95 days at 7% (Why?)
0.63	=	46.18 ÷ 73
\$45.55	=	exact interest for 95 days at 7%. (Why?)

Exercises

1. Find the interest at 6% on:
   \$900 for 50 days, \$365.50 for 99 days, \$750 for 70 days, \$870.20 for 126 days.
2. Solve 1, if the rate is 7½%.
3. Find the exact interest at 6% on:
   \$650 from March 3 to July 17
   \$800 from February 10, 1944, to May 5, 1944
   \$2,000 from August 10 to December 5.
4. Solve 3, if the rate is 8%.

5. A person borrowed \$250 from a bank on July 5 and signed a 7% note due November 20. On September 10 he paid the bank \$100. What was the balance (including interest) due on the note November 20? (Use exact time.)

6. Solve 5, if 30 days is counted to each month.

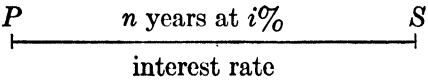
7. Solve 5, if exact interest is used.

8. What is the difference between the exact and ordinary interest on \$1,250 from March 10 to October 3 at 7%?

6. Present value and true discount.—In Art. 2 we found the relation between the principal, P, and the amount, S, to be expressed by the equations:

$$S = P(1 + ni) \quad \text{and} \quad P = \frac{S}{1 + ni}.$$

We may look upon P and S as equivalent values. That is, P, the value at the beginning of the period, is equivalent to S at the end of the period, and vice versa. The following line diagram emphasizes these ideas.



$$P = \frac{S}{1 + ni} \qquad S = P(1 + ni)$$

The quantity S is frequently called the *accumulated value* of P, and P is called the *present value* of S. Thus, the present value of a sum S due in n years is the principal P that will amount to S in n years. The quantity P is also called the discounted value of S due in n years. The difference between S and P, $S - P$, is called the *discount* on S as well as the interest on P. To distinguish it from Bank Discount (Art. 7) this discount on S at an interest rate $i\%$ is called the *true discount* on S. We thus have the several terms for P and S:

P	S
Principal	Amount
Present value of S	Accumulated value of P
Discounted value of S	Maturity value of P

$S - P$ = Interest on P at interest rate i

= Discount on S at interest rate i

Example 1. Find the present value of a debt of \$250 due in 6 months if the interest rate is 6%. Find the true discount.

Solution. Here, $S = 250$, $n = \frac{1}{2}$, and $i = 0.06$.
Substituting these values in formula (3), we get

$$P = \frac{250}{1 + \frac{1}{2}(0.06)} = \frac{250}{1.03} = \$242.72, \text{ present value.}$$

$$S - P = 250 - 242.72 = \$7.28, \text{ true discount.}$$

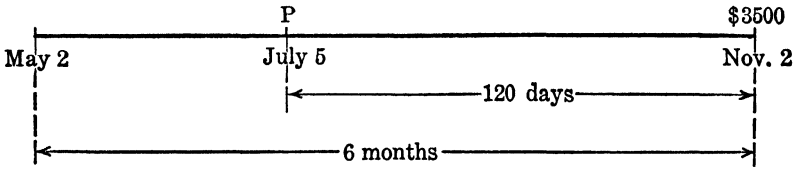
Example 2. A non-interest bearing note for \$3,500, dated May 2 was due in 6 months. Assuming an interest rate of 7½% find the value of the note as of July 5.

Solution.

May 2 + 6 months = November 2, due date.
From July 5 to November 2 = 120 days.
The present value of the maturity value as of July 5 (or for 120 days) is required and $S = 3{,}500$, $n = \frac{1}{3}$, and $i = 0.075$.

Hence, $$P = \frac{3{,}500}{1 + \frac{1}{3}(0.075)} = \frac{3{,}500}{1.025} = \$3{,}414.63.$$

The following line diagram exhibits graphically the important relationships of the example.



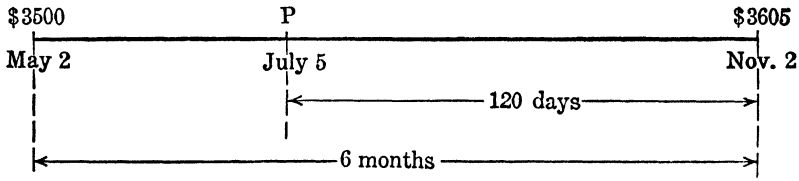
Example 3. On May 2, A loaned B \$3,500 for 6 months with interest at 6% and received from B a negotiable note. On July 5, A sold the note to C to whom money was worth 7½%. What did C pay A for the note?

Solution.

Interest on \$3,500 for 6 months at 6% = \$105.00.
\$3,500 + \$105.00 = \$3,605, maturity value.
May 2 + 6 months = November 2, maturity date.
From July 5 to November 2 = 120 days.
The present value of the maturity value as of July 5 (or for 120 days) is required and $S = 3{,}605$, $n = \frac{1}{3}$ and $i = 0.075$.

Hence, $$P = \frac{3{,}605}{1 + \frac{1}{3}(0.075)} = \frac{3{,}605}{1.025} = \$3{,}517.07,$$

the value of the note as of July 5.



The student will notice that in the solution of a problem of the above type we first find the maturity value of the note or debt and then find the present value of this maturity value as of the specified date.

Exercises

1. Accumulate (that is, find the accumulated value of) $2,000 for 2 years at 5% simple interest.

2. Accumulate $300 for 8 months at 6% simple interest.

3. At 6% simple interest find the present value of $6,000 due at the end of 8 months. What is the discount?

4. Discount (that is, find the discounted value of) $2,000 for 2 years at 5% simple interest.

5. Discount $300 for 8 months at 6% simple interest.

6. Draw graphs of the following functions using n as the horizontal axis and S as the vertical axis:

(a) $S = 100(1 + 0.06n) = 100 + 6n.$
(b) $S = 100(1 + 0.04n) = 100 + 4n.$

7. Mr. Smith buys a bill of goods from a manufacturer who asks him to pay $1,000 at the end of 60 days. If Mr. Smith wishes to pay immediately, what should the manufacturer be willing to accept if he is able to realize 6% on his investments?

8. Solve Exercise 7 under the assumption that the manufacturer can invest his money at 8%. Compare the results of Exercises 7 and 8 and note how the present value is affected by varying the interest rate.

9. I owe $1,500 due at the end of two years and am offered the privilege of paying a smaller sum immediately. At which simple interest rate, 5% or 6%, would my creditor prefer to compute the present value of my obligation?

10.

$1,000.00 Lewisburg, Penna.
June 1, 1944.

Six months after date I promise to pay X, or order, one thousand dollars together with interest from date at 7%.

Signed, Y.

(a) What is the maturity value of the note?
(b) If X sold the note to W, to whom money was worth 6%, four months after date, what did W pay X for the note?
(c) What rate of interest did X earn on the loan?

11. Solve Exercise 10 under the assumption that money was worth 8% to W.

7. Bank discount.—*Bank discount is simple interest, calculated on the maturity value of a note from the date of discount to the maturity date, and is paid in advance.* If a bank lends an individual \$100 on a six months' note, and the rate of discount is 8%, the banker gives the individual \$96 now and collects \$100 when the note becomes due. If one wishes to discount a note at a bank, the bank deducts from the maturity value of the note the interest (bank discount) on the maturity value from the date of discount to the date of maturity. The amount that is left after deducting the bank discount is known as the *proceeds.* The time from the date of discount to the maturity date is commonly known as the *term of discount.* An additional charge is usually made by the bank when discounting paper drawn on some out-of-town bank. This charge is known as *exchange.* The bank discount plus the exchange charge gives the bank's *total charge.* The maturity value minus the total charge gives the proceeds (when an exchange charge is made).

The terms *face of a note* and maturity value of a note need to be explained. The maturity value may or may not be the same as the face value. If the note bears no interest they are the same, but if the note bears interest the maturity value equals the face value increased by the interest on the note for the term of the note.

The discount, maturity value, rate of discount, proceeds (when no exchange charge is made), and the term of discount are commonly represented by the letters D, S, d, P, and n, respectively. From the definitions of bank discount and proceeds we may write

$$D = Snd, \tag{7}$$

and

$$P = S - D = S - Snd = S(1 - nd). \tag{8}$$

When applying formulas (7) and (8) we must express n in years and d in the decimal form.

The quantity P is frequently called the discounted value of S at the given rate of discount, and P is called the present value of S. S is also called the accumulated value of P. The difference between S and P, $S - P$, is called both the discount on S and the interest on P. In each instance

the calculation is at the discount rate d. The relations are pictured by the line diagram.

P — n years at $d\%$ — S

discount rate

$$P = S(1 - nd) \qquad S = \frac{P}{1 - nd}$$

$S - P$ = Interest on P at discount rate d

= Discount on S at discount rate d

Example 1. A six months' note, without interest, for \$375, dated May 6, was discounted August 1, at 6%. Find the proceeds.

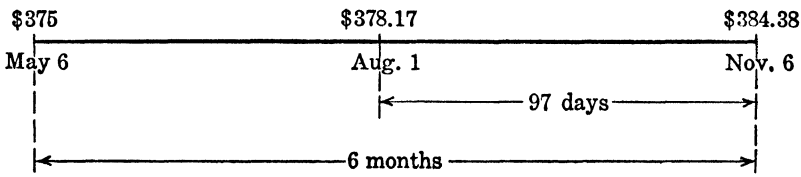
Solution.

May 6 + 6 mo. = Nov. 6, due date.
From August 1 to Nov. 6 = 97 days, term of discount.
Discount on \$375 for 97 days = \$6.07, bank discount.
\$375 − \$6.07 = \$368.93, proceeds.

Example 2. If the above note were a 5% interest-bearing note, what would be the proceeds?

Solution.

May 6 + 6 mo. = Nov. 6, due date.
From August 1 to Nov. 6 = 97 days, term of discount.
Interest on \$375 for 6 mo. at 5% = \$9.38.
\$375.00 + \$9.38 = \$384.38, maturity value.
Discount on \$384.38 for 97 days at 6% = \$6.21, bank discount.
\$384.38 − \$6.21 = \$378.17, proceeds.



Example 3. Solve Example 2, if ¼% of the maturity value were charged for exchange.

Solution.

May 6 + 6 mo. = Nov. 6, due date.
From August 1 to Nov. 6 = 97 days, term of discount.
Interest on \$375 for 6 mo. at 5% = \$9.38.

$375.00 + $9.38 = $384.38, maturity value.
Discount on $384.38 for 97 days at 6% = $6.21, bank discount.
¼% of $384.38 = $0.96, exchange charge.
$6.21 + $0.96 = $7.17, total charge made by the banker.
$384.38 − $7.17 = $377.21, proceeds.

Example 4.

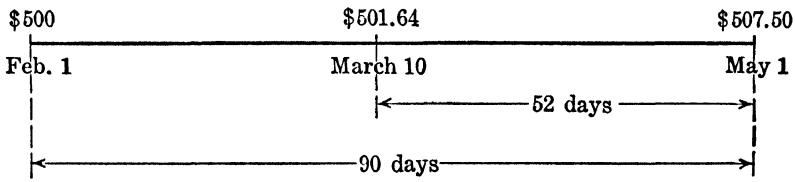
$500.00 Lewisburg, Penna.
February 1, 1944.

Ninety days after date I promise to pay X, or order, five hundred dollars together with interest from date at 6%.
Signed, Y.

On March 10, X sold the note to banker B who discounted the note at 8%. What proceeds did X receive for the note?

Solution.

90 days after Feb. 1, 1944 is May 1, 1944, the due date.
From March 10 to May 1 is 52 days, the term of discount.
The interest on $500 for 90 days at 6% = $7.50.
$500.00 + $7.50 = $507.50, the maturity value.
The discount on $507.50 for 52 days at 8% = $5.86, the bank discount.
$507.50 − $5.86 = $501.64, the proceeds.



In the solution of the above examples, certain fundamental facts have been used, which we now point out.

If the note is given for a certain number of months, the maturity (due) date is found by adding the number of months to the date of the note. This is illustrated in Example 1. Thus, if a note for six months, is dated May 6, it will be due on the corresponding (the 6th) day of the sixth month, or November 6. November 30, would have been the due date of this note, if it had been dated May 31. The correct date for three months after November 30, 1930 is Feb. 28, 1931 and the correct date for three months after November 30, 1931 is Feb. 29, 1932. What makes this difference?

If the term of the note is a fixed number of days, the due date is found by adding the number of days to the date of the note, using the exact number of days of the intervening months. Thus, 90 days after Feb. 1, 1932 is May 1, for the 28 days remaining in February + 31 days in March +30 days in April + 1 day in May = May 1. What is the correct date for 90 days after Feb. 1, 1931?

The term of discount is commonly found by counting the exact number of days between the date of discount and the due date. Thus, the term of discount in Example 1, is 97 days, being obtained as follows: 30 days remaining in August + 30 days in September + 31 days in October + 6 days in November = 97 days. The date of discount is excluded but the due date is included.

When February is an intervening month, use 28 days if no year date is given, but if it occurs in a leap year use 29 days.

These four examples illustrate all the fundamental facts that are used in discounting a note. They merit a careful study by the student.

Simple discount,* like simple interest, is seldom used in computations extending over a long period of time. In fact, the use of simple discount leads to absurd results in long-term transactions.

Illustration. At 6% discount, the present value of \$1,000 due at the end of 20 years is, using $P = S(1 - nd)$,

$$P = \$1{,}000[1 - 20(0.06)] = -\$200.$$

8. Summary and extension.—We have used two methods to accumulate P and to discount S. The first method was based upon the simple interest rate i and the second was based upon the simple discount rate d. The relationships that we have developed are the following:

At simple interest.	At simple discount.
$I = Pni$	$D = Snd$
$S = P + I$	$S = P + D$
$S = P(1 + ni)$	$S = \dfrac{P}{1 - nd}$
$P = \dfrac{S}{1 + ni}$	$P = S(1 - nd)$

Banks and individuals frequently lend money at a discount rate instead of an interest rate. There are two reasons why the creditor may

* Bank discount is frequently referred to as *simple discount.*

prefer to lend at a discount rate. First, the arithmetic is simplified when the maturity value is known, and second, a larger rate of return is obtained.

Thus, if I request a loan of \$100 from a bank for six months at 6% *discount*, the banker actually gives me \$97, collecting the *discount* of \$3 *in advance*, and takes my non-interest-bearing note for \$100. Note the simplicity of the arithmetic: $P = 100(1 - 0.06/2) = \$97$. Note also that the rate of return (the interest rate) is larger than 6%. For we have $P = \$97$, $n = \frac{1}{2}$, $S = \$100$, $i = (\quad)$. Using $S = P(1 + ni)$, we obtain

$$100 = 97\left(1 + \frac{i}{2}\right),$$

from which

$$i = 0.0619 = 6.19\%.$$

However, the banker should not be accused of unfair dealing if he quotes me the 6% *discount rate* or if he states that he charges 6% *in advance*. He should be criticised if he quotes an interest rate and then charges a discount rate. We shall return to the comparison of interest and discount rates in Art. 9.

Example 1. I desire \$900 as the proceeds of a 90 day loan from my banker B who charges 5% discount. What sum will I pay at the end of 90 days?

Solution. We have $P = \$900$, $n = \frac{1}{4}$, $d = 0.05$. From $P = S(1 - nd)$ we obtain

$$900 = S(1 - 0.05/4).$$

Solving, we find $S = \$911.392$.

Exercises

Find the proceeds of the following notes and drafts:

	Face	Time	Date of Paper	Rate of Interest	Date of Discount	Rate of Discount	Rate of Collection
1.	\$1,500	3 mo.	January 1		Jan. 25	6%	¼%
2.	380	90 days	March 10	5%	Apr. 20	6%	
3.	2,000	6 mo.	August 1	6%	Nov. 10	7%	⅛%
4.	575	4 mo.	May 10		Aug. 1	7%	¼%
5.	1,350	90 days	Feb. 1, 1928	6%	Mar. 7	8%	1/10%
6.	1,260	60 days	March 5	7%	April 1	6%	
7.	2,500	2 mo.	April 10		May 1	6%	1/10%

8. A \$2,500 6% interest-bearing note dated February 10, 1944 was due Sept. 1, 1944. It was discounted July 10 at 7½%. What were the proceeds?

9. A person wishes to receive \$250 cash from a bank whose discount rate is 6%. He gives the bank a note due in 4 months. What should be the face value of the note?

10. Solve formula (8) for n and d.

11. The proceeds on a \$400 non-interest-bearing note discounted 78 days before maturity were \$394.80. What was the rate of discount?

12. A bank will loan a customer \$1,000 for 90 days, discounting the note at 6%. For what amount should the note be drawn?

13. How long before maturity was a \$450 note discounted, if the proceeds were \$444.14, the discount rate being 7%?

14. A 90-day 6% note of \$5,000, dated June 15, payable at a Louisville bank, was discounted at a Chicago bank July 20, at 7%. If the exchange charge was \$1.00, find the proceeds.

15. A six months' note bearing 5% interest was dated March 7, 1935. It was discounted at 6% on July 15, the bank charging \$18.45 discount. Find the face of the note.

16. A man received \$882 as the proceeds of a 90-day non-interest-bearing note. The face of the note was \$900. What was the rate of discount.

17. A bank's discount rate is 7%. What should be the face of the note if the proceeds of a 6 months' loan are to be \$2,000?

18. A 4 months' note bearing 4½% interest, dated August 15, was discounted October 11, at 6%. The proceeds were \$791.33. Find the maturity value of the note. Find its face value.

19. A 90-day 7% note for \$1,200, dated April 1, was discounted June 10 at 6%. Find the proceeds.

20. How long before maturity was a \$500 6 months' 6% note discounted, if the proceeds were \$504.70, the discount rate being 8%?

21. The proceeds on a six months' 5% note, when discounted 87 days before maturity at 6% were \$1010.14. Find the face of the note.

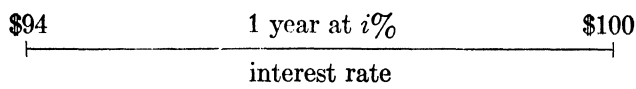
22. Find the present value of \$1,000 due at the end of 20 years if 5% discount rate is used.

9. Comparison of simple interest and simple discount rates.—In Art. 8 we gave brief mention to the relation of interest rate to discount rate. This relation is so important that we will consider the problem more thoroughly at this point. We shall approach the question through a series of examples.

Example 1. If \$100, due at the end of one year, is discounted at 6%, what is the corresponding rate of interest?

Solution. We have $S = \$100$, $n = 1$, $d = 0.06$. In order to find i, we will first find P. Using $P = S(1 - nd)$, we have

$$P = 100(1 - 0.06) = \$94.$$



Since $S - P$ is the interest on P, we may find i by using $I = Pni$. We have $I = \$6$, $n = 1$, $P = \$94$. Hence,

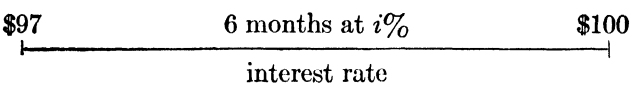
$$i = \tfrac{6}{94} = 0.06383 = 6.383\%$$

We might have employed the relation $S = P(1 + ni)$ to obtain the same result.

Example 2. If \$100, due at the end of 6 months, is discounted at 6%, what is the corresponding interest rate?

Solution. We have $S = \$100$, $n = \frac{1}{2}$, $d = 0.06$. From $P = S(1 - nd)$, we have

$$P = 100(1 - 0.06/2) = \$97.$$



Since $S - P$ is the interest on P, we may find i by using $I = Pni$. We have $I = \$3$, $n = \frac{1}{2}$, $P = \$97$. Hence

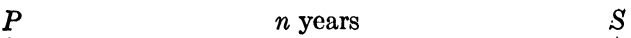
$$97(i/2) = 3,$$

and

$$i = 0.0619 = 6.19\%.$$

Thus we notice that the interest rates corresponding to a given discount rate vary with the term; the longer the term, the larger the interest rate.

In general, we say that, for a given term, an interest rate i and a corresponding discount rate d are equivalent if the present values of S at i and d are equal. Thus, if P is the present value of S due in n years,



we have

$$P = \frac{S}{1 + ni} \qquad \text{from (3),}$$

and $$P = S(1 - nd)$$ from (8).

Hence, $$\frac{S}{1 + ni} = S(1 - nd).$$

Solving we obtain

$$i = \frac{d}{1 - nd} \tag{9}$$

and $$d = \frac{i}{1 + ni}. \tag{10}$$

From (9) we observe that for a given d the values of i increase as n increases. From (10) we observe that for a given i the values of d decrease as n increases.

The student will also observe from (10) that $i/(1 + ni)$ is the present value of i due in n years. That is, $i/(1 + ni)$ *in advance* is equivalent to i at the end of the term. But $i/(1 + ni)$ equals d. Hence d is equal to i paid *in advance.* Thus, we say *discount is interest paid in advance.*

Exercises

1. Solve Example 1 by using formula (3).
2. Solve Example 2 by using formula (3).
3. Employing equation (9) complete the table:

d	.08	.08	.08	.08
n	1	½	¼	⅛
i				

4. Employing equation (10) complete the table:

i	.08	.08	.08	.08
n	1	½	¼	⅛
d				

5. A obtains \$780 from Bank B. For this loan he gives his note for \$800 due in 60 days. At what rate does Bank B discount the note? What rate of interest does A pay?

6. A note for \$800, dated June 15, due in 90 days and bearing interest at 6%, was sold on July 1 to a friend to whom money was worth 5%. What did the friend pay for the note?

7. If the note described in Exercise 6 were sold to Bank B on July 5 at a discount rate of 7%, what would Bank B pay for the note?

8. \$500.00 Pittsburgh, Penna.
May 15, 1945.

Ninety days after date I promise to pay John Jones, or order, five hundred dollars together with interest at 6% from date.

Signed, Wm. Smith.

(a) Thirty days after date Jones sold the note to Bank B who discounted it at 7%. What did Jones receive for the note?
(b) Would it have been to Jones' advantage to have sold the note to friend C, to whom money was worth 7%, rather than to Bank B?

9. \$1,000.00 Chicago, Ill.
May 15, 1945.

Six months after date I promise to pay Joe Brown, or order, one thousand dollars with interest from date at 5%.

Signed, Charles Paul.

(a) Two months after date Brown sold the note to Bank B who discounted it at 6%. What did Bank B pay for the note?
(b) Immediately after purchasing the note, Bank B sold the note to a Federal Reserve Bank at a re-discount rate of 4%. How much did Bank B gain on the transaction? [On transaction (b) use a 365-day year.]

10. Rates of interest corresponding to certain discount rates in the terms of settlement.—The subject of terms was discussed in *Alg.: Com.—Stat.*, p. 99.* An example will illustrate what is meant by the rates of interest corresponding to the rates of discounts of the terms of settlement.

Example 1. On an invoice of \$1,000, a merchant is offered the following terms: 5, 3/30, n/90. What is the interest rate corresponding to each of the rates of discount?

Solution.

I. If the buyer pays the account immediately, he receives a discount of \$50. That is, he settles the account for \$950 which means that he receives \$50 interest on \$950 for 90 days. We may determine the interest rate by substituting in $I = Pni$, thus obtaining:

$$i = \frac{I}{Pn} = \frac{50}{950(\frac{1}{4})} = \frac{50}{237.50}$$

$$= 0.2105 = 21.05\%.$$

II. If the buyer settles the account at the end of 30 days, he receives a discount of \$30. That is, the account is settled for \$970 which

* Miller and Richardson, *Algebra: Commercial—Statistical*, D. Van Nostrand Co.

means \$30 interest on \$970 for 60 days. We determine the interest rate as in I and find,

$$i = \frac{30}{970(\frac{1}{6})} = \frac{180}{970}$$

$$= 0.1855 = 18.6\%$$

The buyer may have his business so well organized that he knows about what his money is worth to him in the running of the business. He can then determine the best offer, in the terms of sale, to accept. An example will illustrate.

Example 2. Assuming that money is worth 20% to the merchant in his business, which is the best offer in Example 1?

Solution. To answer this question we must compare the present values of the separate offers. That is, which offer has the least present value assuming money worth 20%?

I. 5% discount on \$1,000 means a discount of \$50. Hence the present value of this offer is \$1,000 − \$50 = \$950.

II. 3% discount on \$1,000 means a discount of \$30 at the end of 30 days. Hence, \$970 is required to settle the account at the end of 30 days. Now, the present value of \$970 is

$$P = \frac{970}{1 + \frac{1}{12}(0.20)} = \frac{970}{1.0167} = \$954.06.$$

III. Here the present value of \$1,000 for 90 days at 20% is required.

Hence,
$$P = \frac{1,000}{1 + \frac{1}{4}(0.20)} = \frac{1,000}{1.05}$$

$$= \$952.38.$$

We notice that the 5% cash discount is the best offer (assuming money worth 20%) since it gives the least present value for the invoice.

Exercises

1. Determine the interest rates corresponding to bank discount rates of (a) 7% 90 days before maturity; (b) 7½% 60 days before maturity; (c) 6% 6 months before maturity; (d) 8% 4 months before maturity.

2. In discounting a 4 months' note a bank earns 9% interest. What rate of discount does it use?

3. What are the rates of discount corresponding to (a) 7% interest earned on a note discounted 90 days before maturity; (b) 8% interest earned on a note discounted 4 months before maturity; (c) 6% interest earned on a note discounted 6 months before maturity?

4. What rate of interest is earned on money used in discounting bills at a discount rate of 9% per annum?

5. What is the rate of discount at which a bank may as well employ its funds as to lend money at an interest rate of 8%?

6. A merchant has the privilege of 90 days credit or 3% off for cash. What rate of interest does he earn on his money if he pays cash?

7. A merchant bought a bill of goods amounting to \$2,500 and received the following terms: 4, 3/10, n/90. What is the interest rate corresponding to each of the rates of discount?

8. Assuming that money is worth 15% to the merchant in the conducting of his business, which is the best offer in Exercise 7? (See illustrative Example 2, Art. 10.)

9. On an invoice of \$4,200, a merchant is offered 60 days credit or a discount of 3% for cash. Not having the money to pay cash, he accepts the credit terms. What rate of interest does he pay on the net amount of the bill? How much would he have saved if he had borrowed the money at 7% and paid cash?

10. 7% interest was earned in discounting a note 90 days before maturity; 6% was earned in discounting a 4 months' note; and 5% was earned in discounting a 9 months' note. What were the corresponding discount rates?

11. Assuming money worth 20% in one's business, which one of following offers is the most advantageous to the buyer: 6, 5/30, n/4 mos.? (Assume an invoice of \$100.)

12. Solve Exercise 11, assuming money worth 18%.

13. A bank used a discount rate of 6% in discounting a 4 months' note. What rate of interest was earned on the transaction?

14. Assuming money worth 12%, which one of the following offers is the most advantageous to the buyer: 6, 4/30, n/4 mos.?

11. Exchanging debts.—When two or more debts (obligations) are to be compared we must know when each debt is due and then compare their values at some specified time. The value of a debt at a specified time depends upon the rate of interest that is used. Let us suppose that a debt of \$200 is due in 2 months and one of \$205 is due in 8 months. Assuming money worth 6%, compare their values now. The value of the first debt at this time is

$$\frac{200}{1 + \frac{1}{6}(0.06)} = \frac{200}{1.01} = \$198.02 \quad [(3), \text{Art. } 2]$$

and the value of the second debt at this time is

$$\frac{205}{1 + \frac{2}{3}(0.06)} = \frac{205}{1.04} = \$197.12.$$

Six months from now the first debt would be 4 months past due and should draw interest for that time. The second debt would not be due for 2 months and should be discounted for that time. Then their values 6 months from now would be

$$200[1 + \tfrac{1}{2}(0.06)] = 200(1.02) = \$204.00$$

and

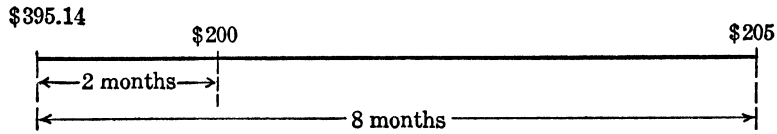
$$\frac{205}{1 + \tfrac{1}{6}(0.06)} = \frac{205}{1.01} = \$202.97.$$

We notice that the first debt has a greater value on both dates of comparison. If 6% is used the value of the first debt will always be greater than that of the second.

If 4% were used their values on the above dates would be \$198.67, \$199.67 and \$202.67, \$203.64; respectively. That is, if 4% interest is assumed the second debt has a greater value at all times.

If 6% interest is assumed, the sum of the values of the above debts at the present is \$395.14. This is shown by the equation

$$\frac{200}{1.01} + \frac{205}{1.04} = 395.14.$$



We say that the sum of the values of \$200 due in 2 months and \$205 due in 8 months is equal to \$395.14 due now, if money is assumed to be worth 6%. Also, the sum of the values of \$200 due in 2 months and \$205 due in 8 months is equal to the sum of the values of \$201.97 due in 3 months and \$201.97 due in 6 months, if 6% interest is assumed. This may be shown by comparing the two sets of debts on some common date. Suppose we take 8 months from now as a common date. Then

$$200(1.03) + 205 = 411.00$$

and

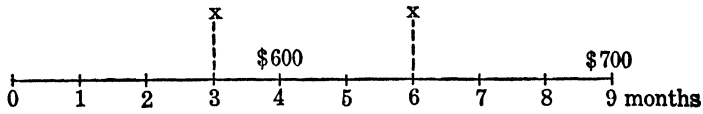
$$201.97(1.025) + 201.97(1.01) = 411.01.$$

Whenever the value of one set of obligations is equal to the value of another set of obligations on a common date, the one set may be exchanged for the other set, and the values of the two sets are said to be *equivalent.* The common date used for the date of comparison is usually known as the *focal date,* and the equality which exists, *on the focal date,* between the values

of the two sets of obligations is called an *equation of value.* An example will illustrate the meaning of focal date and equation of value.

Example 1. A person owes \$600 due in 4 months and \$700 due in 9 months. Find the equal payments necessary to equitably discharge the two debts, if made at the ends of 3 months and 6 months, respectively, assuming 6% simple interest.

Solution. We choose the end of 9 months for our focal date and set up the equation of value.*



Let x = the number of dollars in each of the equal payments.

The time from the date of making the first payment x until the focal date is 6 months and the payment will accumulate to

$$[1 + \tfrac{1}{2}(0.06)]x = (1.03)x \text{ on the focal date.}$$

The second payment is made 3 months before the focal date and it will accumulate to

$$[1 + \tfrac{1}{4}(0.06)]x = (1.015)x \text{ on the focal date.}$$

The \$600 debt is due in 4 months, just 5 months before the focal date, and will accumulate to

$$600[1 + \tfrac{5}{12}(0.06)] = 615.00 \text{ on the focal date.}$$

The \$700 debt is due on the focal date and will be worth \$700 on that date.

The equation of value becomes

$$(1.03)x + (1.015)x = 615 + 700,$$

$$(2.045)x = 1{,}315,$$

$$x = \$643.03, \text{ the amount of each of the equal payments.}$$

In setting up an equation of value, we assume that the equation is true for any focal date. That is, we assume that if the value of one set of debts is equal to the value of another set of debts on a given focal date, then the values are equal on any other focal date. If in the above prob-

* In the construction of the line diagram, (a) place at the respective points the *maturity* values, and (b) place the payments and the debts at different levels.

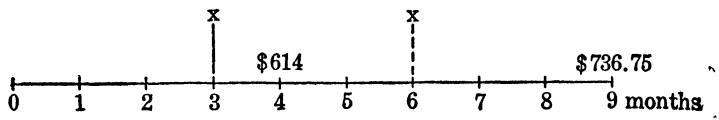
lem we had taken 3 months from now for the focal date, we would have obtained \$643.07 for the amount of one of the equal payments. Using 5 months from now as focal date we obtain \$643.02 as one of the equal payments. We notice that a change in the focal date changes the values of the payments, but this change is very slight and for short periods of time we may neglect the small differences caused by different choices of focal dates and choose the one that is most convenient. (In Art. 19 it will be shown that the amount x is independent of the focal date when the computations are based upon compound interest.) The last date occurring seems to be the most convenient, for then no discount is involved.

Example 2. Solve Example 1, assuming that the original debts bear 7% interest to maturity. Choose 9 months from now as the focal date.

Solution. \$600 at 7% amounts to \$614 in 4 months and on the focal date its amount is

$$614[1 + \tfrac{5}{12}(0.06)] = 614(1.025) = \$629.35.$$

\$700 at 7% amounts to \$736.75 in 9 months and on the focal date its amount is this maturity value (\$736.75).



The equation of value becomes

$$(1.03)x + (1.015)x = 629.35 + 736.75,$$

$$(2.045)x = 1{,}366.10,$$

$$x = \$668.02,$$ the amount of one of the equal payments.

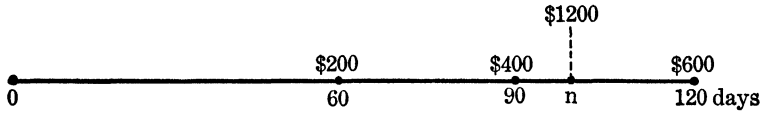
12. To find the date when the various sums (debts) due at different times may be paid in one sum.—A may owe B several sums (debts) due at different times and may desire to cancel all of them at one time by paying a single amount equal to the sum of the **maturity values** of the several debts. The problem, then, is to find a date when the single amount may be paid without loss to either A (debtor) or B (creditor). Evidently, this should be at a time when the total interest gained by the debtor on the sums past due would balance the total interest lost on the sums paid before they are due. The date to be found is known as the *equated date.*

The solutions of problems of this character may be effected by either

of two methods. We may base our procedure upon a *simple interest rate* i and choose the *latest date* mentioned in the problem as the focal date, or we may base our procedure upon a *simple discount rate* d and choose the *earliest date* mentioned in the problem as the focal date. If the former method is followed all sums will accumulate at i to the focal date whereas if the latter method is adopted all sums will be discounted at d to the focal date.

Example. A owes B the following debts: \$200 due in 60 days, \$400 due in 90 days, and \$600 due in 120 days. Find the time when these debts may be canceled by a single payment of their sum, \$1,200.

Solution. We have the debts and the payment as shown by the line diagram.



Let n days from now be the equated date.

We choose the focal date at the latest date, 120 days from now, and assume an interest rate i.

The first debt, \$200, will be at interest for 60 days and its value on the focal date is

$$200\left(1+\frac{60}{360}i\right).$$

The second debt, \$400, will be at interest for 30 days and its value on the focal date is

$$400\left(1+\frac{30}{360}i\right).$$

The third debt, \$600, due on the focal date, bears no interest and hence its value then is

$$600\left(1+\frac{0}{360}i\right).$$

The single payment, \$1,200, will be at interest $(120-n)$ days and thus its value on the focal date is

$$1{,}200\left(1+\frac{120-n}{360}i\right).$$

Expressing by an equation the fact that the value of the payment on the focal date is equal to the sum of the maturity values of the debts on that date, we have

$$1{,}200\left(1+\frac{120-n}{360}i\right)$$
$$=200\left(1+\frac{60}{360}i\right)+400\left(1+\frac{30}{360}i\right)+600\left(1+\frac{0}{360}i\right)$$

which reduces to

$$1{,}200\left(\frac{120-n}{360}i\right)=200\left(\frac{60}{360}i\right)+400\left(\frac{30}{360}i\right)+600\left(\frac{0}{360}i\right).$$

Note. The student should note that the last equation written above simply states that the interest on the payment equals the sum of the interest increments on the debts, all calculated from their due dates to the focal date.

Multiplying the last equation by 360 and dividing through by $100i$, we get

$$\begin{aligned} 12(120-n) &= 2(60)+4(30) \\ 1{,}440-12n &= 120+120 \\ -12n &= -1{,}200 \\ n &= 100. \end{aligned}$$

Hence, the \$1,200 may be paid 100 days from now and the equities be the same as if the debts were paid as originally scheduled.

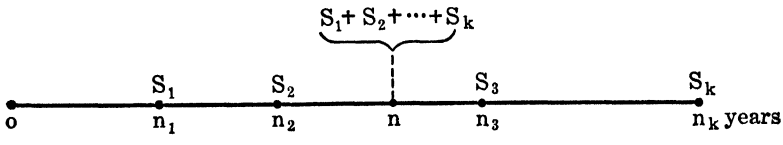
Note. The fact that the interest rate i divides out as a factor in solving the equation of value shows that the value of n is independent of i.

Exercise. Solve the preceding example by assuming a discount rate d and choosing (a) the earliest date, 60 days, as the focal date, and (b) the present or "now" as the focal date.

By following a line of reasoning similar to that used in solving the preceding example, we will solve the general problem.

Problem. Let $D_1, D_2, \cdots, D_k$ be k debts due in $n_1, n_2, \cdots, n_k$ years respectively, and let their maturity values be $S_1, S_2, \cdots, S_k$. We wish to find *the equated time*, that is, the time when the k debts may be settled by a single payment of $S_1 + S_2 + \cdots + S_k$.

Solution. We shall assume $n_1 < n_2 < n_3 < \cdots < n_k$, and we shall take the latest date, n_k, to be the focal date. Also we let n years from now be the equated time. The diagram gives us the picture.



Assuming an interest rate i, the accumulated values of S_1, S_2, etc., at n_k are $S_1[1 + (n_k - n_1)i]$, $S_2[1 + (n_k - n_2)i]$, etc., we then have the equation of value

$$[S_1 + S_2 + \cdots + S_k][1 + (n_k - n)i] =$$
$$S_1[1 + (n_k - n_1)i] + S_2[1 + (n_k - n_2)i] + \cdots + S_k[1 + (n_k - n_k)i].$$

Subtracting $S_1 + S_2 + \cdots + S_k$ from both sides of the equation we have

$$(S_1 + S_2 + \cdots + S_k)(n_k - n)i =$$
$$S_1(n_k - n_1)i + S_2(n_k - n_2)i + \cdots + S_k(n_k - n_k)i.$$

Note. This equation shows that the interest on the payment equals the sum of the interest increments on the maturity values, all calculated from their due dates to the focal date.

Solving for n we obtain

$$n = \frac{S_1n_1 + S_2n_2 + S_3n_3 + \cdots + S_kn_k}{S_1 + S_2 + S_3 + \cdots + S_k}. \tag{12}$$

If D_1, D_2, $\cdots$, D_k are not interest-bearing debts, $D_1 = S_1$, $D_2 = S_2$, $\cdots$, $D_k = S_k$, and equation (12) becomes

$$n = \frac{D_1n_1 + D_2n_2 + \cdots + D_kn_k}{D_1 + D_2 + \cdots + D_k}. \tag{12'}$$

If the debts involve short periods of time it is usually more convenient to express n, n_1, n_2, etc., in terms of either months or days.

Exercise. Derive formula (12) by assuming a discount rate d and choosing "now" as the focal date.

Exercise. The equated time has an interesting "teeterboard" property in that it is the "center of balance" when the maturity values are suspended as weights with lever arms measured from n. That is, let the lever arms be $\bar{n}_1 = n_1 - n$, $\bar{n}_2 = n_2 - n$, etc., respectively. Then,

$$S_1\bar{n}_1 + S_2\bar{n}_2 + S_3\bar{n}_3 + \cdots + S_k\bar{n}_k = 0.$$

13. To find the equated date of an account.—To find the equated date of an account means we must find *the date when the balance of the account can be paid without loss to either the debtor or the creditor.*

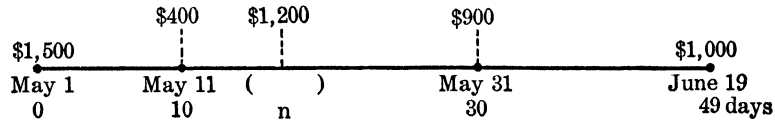
As in Art. 12, we assume that the sum of the values, as of the focal date, of all credits including the balance, is equal to the sum of the values on that date of all debits. Obviously, we may select the focal date in many ways. We may, for example, choose the earliest date mentioned in the problem as the focal date and *discount* all credits and debts to this

point. We shall illustrate this procedure in our discussion first by a specific example and then by the general problem.

Example. What is the equated date of the account?

1944	1944
May 1, Mdse., $1,500	May 11, Cash, $400
June 19, Mdse., $1,000	May 31, Cash, $900

Solution. The total of the debts is $2,500 and the total of the credits is $1,300. Our problem is to find the date when the balance, $1,200, can be paid without loss to either the debtor or the creditor. The line diagram gives us the picture.



We let the earliest date, May 1, be the focal date. Let n days from May 1 be the equated date. We assume a discount rate d and set up the equation of value.

$$400\left(1-\frac{10}{360}d\right)+1{,}200\left(1-\frac{n}{360}d\right)+900\left(1-\frac{30}{360}d\right)$$
$$=1{,}500\left(1-\frac{0}{360}d\right)+1{,}000\left(1-\frac{49}{360}d\right).$$

Subtracting 2,500 from both sides of the equation and multiplying by (-1), we get

$$400\left(\frac{10}{360}d\right)+1{,}200\left(\frac{n}{360}d\right)+900\left(\frac{30}{360}d\right)$$
$$=1{,}500\left(\frac{0}{360}d\right)+1{,}000\left(\frac{49}{360}d\right).$$

Note. This equation shows that the sum of the discounted values of the credits, as of May 1, equals the sum of the discounted values of the debts as of the same date. Further, since the last equation written above is divisible by d, the value of n is independent of the discount rate.

Multiplying the last equation by 360 and dividing by $100d$, we have

$$40+12n+270=490$$
$$12n=180$$
$$n=15 \text{ days}.$$

Thus the equated date is 15 days after May 1, or May 16.

Exercise. Solve the preceding example by assuming an interest rate i and choosing the latest date, June 19, as the focal date.

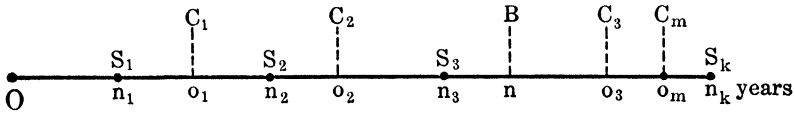
By following a line of reasoning similar to that used in solving the preceding example, we will solve the general problem.

Problem. Let $D_1, D_2, D_3, \cdots, D_k$ be k debts due in $n_1, n_2, n_3, \cdots, n_k$ years from now respectively, and let their maturity values be $S_1, S_2, S_3, \cdots, S_k$. Also, let $C_1, C_2, C_3, \cdots, C_m$ be m credits entered $o_1, o_2, o_3, \cdots, o_m$ years from now respectively. We wish to find the equated date of the account, that is, the date when the balance B,

$$B = (S_1 + S_2 + S_3 + \cdots + S_k) - (C_1 + C_2 + C_3 + \cdots + C_m),$$

can be paid without loss to either debtor or creditor.

Solution. We shall assume $n_1 < n_2 < n_3 < \cdots < n_k$ and $o_1 < o_2 < o_3 < \cdots < o_m$. For the sake of variety we shall take "now" to be the focal date. We assume a discount rate d and let n equal the number of years from now to the equated date. The line diagram gives us the picture.



By equating the sum of the credits, including the balance, discounted to the present, O, and the sum of the debts as of the same date, we have the equation of value

$$C_1(1-o_1d)+C_2(1-o_2d)+C_3(1-o_3d)+\cdots+C_m(1-o_md)+B(1-nd)$$
$$=S_1(1-n_1d)+S_2(1-n_2d)+S_3(1-n_3d)+\cdots+S_k(1-n_kd).$$

Subtracting $C_1 + C_2 + C_3 + \cdots + C_m + B$ from both sides of this equation, then multiplying by (-1), we get

$$C_1o_1d + C_2o_2d + C_3o_3d + \cdots + C_mo_md + Bnd$$
$$= S_1n_1d + S_2n_2d + S_3n_3d + \cdots + S_kn_kd.$$

Note. This last equation shows that the sum of the discounted values of the payments equals the sum of the discounted values of the debts, all discounted to the focal date, "now." Also, since every term of this equation contains the factor d, which may be divided out, the equated date is independent of d.

Dividing out d and solving for n, we get, replacing B by its value,

$$n = \frac{(S_1n_1+S_2n_2+S_3n_3+\cdots+S_kn_k)-(C_1o_1+C_2o_2+C_3o_3+\cdots+C_mo_m)}{(S_1+S_2+S_3+\cdots+S_k)-(C_1+C_2+C_3+\cdots+C_m)}. \tag{13}$$

If the debts are not interest-bearing, $S_1 = D_1$, $S_2 = D_2$, etc., in which case (13) becomes

$$n = \frac{(D_1n_1+D_2n_2+D_3n_3+\cdots+D_kn_k)-(C_1o_1+C_2o_2+C_3o_3+\cdots+C_mo_m)}{(D_1+D_2+D_3+\cdots+D_k)-(C_1+C_2+C_3+\cdots+C_m)}. \quad (13')$$

In practice we usually let the earliest date mentioned in the problem be "now," then $n_1 = 0$ and the first term in the numerator vanishes.

When accounts involve short periods of time, we usually express n, n_1, n_2, n_3, $\cdots$, n_k, o_1, o_2, o_3, $\cdots$, o_m, in months or days.

Note. *An account becomes interest-bearing on the equated date and the debtor should pay interest on the balance of the account from the equated date until the balance is paid.*

Exercise. Derive formula (13) by assuming an interest rate i and choosing the latest date, n_k, as the focal date.

Exercises

1. An obligation of \$500 is due in 3 months and another obligation of \$520 is due in 9 months. Assuming money worth 6% simple interest, compare the values of these obligations (a) now, (b) 6 months from now, (c) 12 months from now.

2. Solve Exercise 1, assuming money worth 9% simple interest.

3. A note for \$600 drawing 5% simple interest will be due in 5 months, and another note for \$600 drawing 6% interest will be due in 9 months. Assuming money worth 7% simple interest, compare the values of these obligations 7 months from now.

4. Solve Exercise 3, assuming money worth 8% simple interest.

5. *A* owes *B* \$500 due in 3 months, \$600 due in 5 months, and \$700 due in 8 months. Find the equal payments to be made at the end of 6 months and 12 months, respectively, which will equitably discharge the three debts if money is worth 5%.

6. Assuming 6% simple interest, find the equal payments that could be made in 3 months, 6 months, and 9 months, respectively to equitably discharge obligations of \$500 due in 2 months and \$800 due in 5 months.

7. Solve Exercise 5, assuming that the three debts draw 6% simple interest.

8. *A* owes *B* the following debts: \$700 due in 5 months at 7% interest, \$500 due in 6 months at 7% interest, and \$600 due in 9 months at 5% interest. Assuming money worth 6%, find the single payment that is necessary to equitably discharge the above debts 8 months from now.

9. Find the time when the following items may be paid in a single sum of \$3,000: \$1,500 due May 1, \$500 due June 12, \$800 due June 25, and \$200 due July 20.

10. Find the time when the following items may be paid in a single sum of \$2,300: \$500 due March 1, \$300 due April 10, \$800 due April 25, and \$700 due June 1.

11. Find the time when obligations of \$350 due in 2 months, \$600 due in 3 months, and \$850 due in 6 months may be settled by a single payment of \$1,800.

12. Find the time for settling in one payment of \$1,600 the following debts: \$200 due in 3 months, \$400 due in 5 months, \$300 due in 6 months, and \$700 due in 8 months.

13. Find the date when the following items may be paid in a single sum of $2,000:

Sept. 1, Mdse., 30 days, $400*
Sept. 27, Mdse., 60 days, $500
Nov. 9, Mdse., 2 months, $1,100

Check the correctness of the date by assuming 6% simple interest and showing that the interest on the past due items as of the equated date is the same as the interest from the equated date to the due dates of the items not yet due.

Find the time when the following accounts may be paid in single amounts:

14.

1941
January 2, Mdse., 30 da., $800
January 17, Mdse., 1 mo., $500
March 1, Mdse., 2 mo., $300
March 30, Mdse., net $400

15.

1941
July 1, Mdse., 60 da., $550
July 10, Mdse., 1 mo., $450
August 1, Mdse., 2 mo., $750
Sept. 1, Mdse., net, $350
Sept. 10, Mdse., 30 da., $400

Find the time when the balance of the following accounts may be paid in single amounts:

16.

1941	1941
April 1, Mdse., $700	April 20, Cash, $400
April 10, Mdse., $500	May 10, Cash, $300
July 1, Mdse., $800	May 31, Cash, $300

17.

1941	1941
July 1, Mdse., net, $575	July 10, Cash, $440
July 5, Mdse., 1 mo., $435	Aug. 1., Cash, $720
Aug. 1, Mdse., 60 da., $990	

18.

1944	1944
January 1, Balance, $1,900	Jan. 15, Cash, $1,560
January 20, Mdse., 1 mo., $1,450	Jan. 30, Note, † 2 mo., $1,200
March 10, Mdse., Net, $1,325	Feb. 1, Note, † 90 da., with interest, $500

19.

1944	1944
May 1, Balance, $500	May 15, Cash, $700
May 10, Mdse., 2 mo., $1,000	June 20, Cash, $1,000
June 7, Mdse., 30 days, $2,000	July 10, Cash, $400
July 1, Mdse., $600	

* When terms of credit are given on the different items, we must first find the due date of each item.

† When a note is given without interest, the time is figured to the due date of the note, but when the note bears interest the time is figured to the date that the note is given.

Review Problems *

1. A man derives an income of \$205 a year from some money invested at 4% and some at 5%. If the amounts of the respective investments were interchanged, he would receive \$200. How much has he in each investment?

2. A man has one sum invested at 4% and another invested at 5½%. His total annual interest is \$320. If both sums had been invested at 6%, the annual interest would have been \$390. Find the sums invested at each rate.

3. A man made three loans totaling \$15,000, the first at 4%, the second at 5% and the third at 6%, receiving for the whole \$770 per year. The interest on the second part is \$70 less than on the sum of the first and third parts. How was the money divided?

4. A man has three sums invested at 4%, 6%, and 7% respectively, the total interest received being \$280. If the three sums had been invested at 6%, 7% and 4% respectively, the total interest would have been \$305. How much was invested at each rate, if the sum invested at 4% was \$500 more than the sum invested at 7%?

5. One half of a man's property is invested at 4%, one third at 5%, and the rest at 6%. How much property has he if his income is \$560?

6. One man can do a piece of work in 10 days, another in 12 days, and a third in 15 days. How many days will it require all of them to do it when working together?

7. A certain tank can be filled by a supply pipe in 6 hours. It can be filled by another pipe in 8 hours and a third pipe can empty it in 12 hours. If all three pipes are running at the same time, how soon will it be filled?

8. How much cream that contains 32% butter fat should be added to 500 pounds of milk that contains 3% butter fat to produce a milk with 4% butter fat?

9. A merchant desires to mix coffee selling at 24 cents a pound with 80 pounds selling at 30 cents a pound and 60 pounds selling at 33 cents a pound to produce a mixture which he can sell at 28 cents a pound. How many pounds of the 24 cent coffee must he use?

10. How large a 6% interest-bearing note should be given April 1 to cancel a debt of \$1,200 due July 1?

11. What is the difference between the true and bank discount on a debt of \$1,000 due in 4 months, the interest rate and the discount rate being 7½%?

12. A note for \$2,500, bearing 5% interest, dated June 1 was due November 10. What should be paid for this note August 18, (a) if 6% simple interest is to be realized? (b) if 6% discount is to be realized?

13. A note of \$500, bearing 6% interest, is dated March 1. If it is due in 4 months, what would be its value May 1 at 4½%?

14. A merchant is offered a bill of goods invoiced at \$748.25 on 4 months' credit. As a settlement he gives his note with interest at 7½% for a sum which, at maturity, will cancel the debt. Find the face of the note.

15. On March 5, a bill of merchandise valued at \$3,000 was bought on 6 months' credit. On May 8, \$1,500 was paid on the account. On July 22 the present value of the balance of the debt was paid. Assuming money worth 6%, find the amount of the final payment.

* Many of these problems are review problems of algebra. For additional review problems in interest and discount, see end of this book.

16. A piece of property was offered for sale for \$2,900 cash or for \$3,000 due in 6 months without interest. If the cash offer was accepted, what rate of interest was realized?

17. The cash price of a certain article is \$90 and the price on 6 months' credit is \$95. How much better is the cash price for the purchaser, if money is worth 7%?

18. The present value at 5% of a debt due in 72 days is \$396.04. What is the amount of the debt?

19. Find the true discount on a debt of \$3,600 when paid 6 months before maturity, assuming 5% simple interest.

20. A father wishes to provide an educational fund of \$2,000 for his daughter when she reaches the age of 18. What sum should he invest at 4% simple interest on her thirteenth birthday in order that his wishes may be realized?

21. What cash payment on July 1 will cancel a debt of \$2,400 due December 8, if money is worth 8%?

22. A merchant buys a bill of goods from a jobber for \$1,500 on 4 months' credit. If the jobber can realize 6% simple interest on his money, what cash payment should he be willing to accept from the merchant?

23. A man borrows \$10,000. He agrees to pay \$1,000 at the end of each year for 10 years and 4% simple interest on all unpaid amounts. Find the total sum paid in discharging the debt.

24. Find the sum: $1 + (1.06) + (1.06)^2 + \cdots + (1.06)^9$.

25. Find the sum: $(1.03)^{-10} + (1.03)^{-9} + (1.03)^{-8} + \cdots + (1.03)^{-1}$.

26. Solve for n: $(1.05)^n = 6.325$.

27. Solve for n: $(1.045)^{-n} = 0.753$.

28. Find the rate of interest when, instead of paying \$100 cash for an article, the purchaser pays \$10 down and 10 monthly installments of \$10 each.

29. A man buys a bill of goods amounting to \$50. Instead of paying cash, he pays \$5 down and 5 monthly installments of \$10 each. Find the actual rate of interest paid.

30. On a cash bill for \$150, \$15 is paid down, followed by 10 monthly payments of \$15 each. Find the rate of interest paid.

31. The cash price of an article is C. Instead of paying cash the purchaser makes a down payment D followed by monthly installments of R at the end of each month for n months. Show that the interest rate i is given by the formula

$$i = \frac{24(nR + D - C)}{n(2C - 2D - nR + R)}$$

if all amounts are focalized at the time of the last payment.

32. (a) Using formula (12) show that R at the end of each month for n months is equivalent to nR at $(n + 1)/2$ months.

(b) Using the data of Exercise 31 and the conclusion of (a), focalizing all amounts at $(n + 1)/2$ months, show that

$$i = \frac{24(nR + D - C)}{(n + 1)(C - D)}.$$

(c) Note that $(nR + D - C)$ is the *total carrying charge* and $(C - D)$ is the *unpaid balance.*

CHAPTER II

COMPOUND INTEREST AND COMPOUND DISCOUNT

14. Compound interest.—Simple interest is calculated on the original principal only, and is proportional to the time. Its chief value is its application to short-term loans and investments. Long-term financial operations are usually performed under the assumption that the interest, when due, is added to the principal and the interest for the next period of time is calculated on the principal thus increased, and this process is continued with each succeeding accumulation of interest. Interest when so computed is said to be *compound.* Interest may be compounded annually, semi-annually, quarterly, or at some other regular interval. That is, interest is converted into principal at these regular intervals. The time elapsing between successive periods, when the interest is converted into principal, is commonly defined as the *conversion period.* For example, if the interest is converted into principal semi-annually, the conversion period is six months. The rate of interest is nearly always expressed on an annual basis and if nothing is specified as to the conversion period, it is commonly assumed to be one year. The final amount at the end of the time, after all of the interest has been converted into principal, is defined as the *compound amount.* Consequently, *the compound interest is equal to the compound amount minus the original principal.*

Example. Find the compound amount and compound interest on \$600 for four years at 5%, the interest being converted annually.

Solution. The interest for the first (conversion period) year is \$600(0.05) = \$30.00. When this is converted into principal, the amount at the end of the first year becomes \$630. The interest for the second year is \$630(0.05) = \$31.50, and when this is converted the principal becomes \$661.50. Continuing this process until the end of the fourth year, we find the compound amount to be \$729.30; and the compound interest for the given time is \$129.30, the difference between \$729.30 and \$600.

The solution of the above example can be written in the following form:

$$\begin{aligned}
\text{Interest for first year} &= \$600(0.05)\\
\text{Principal at end of first year} &= \$600 + \$600(0.05)\\
&= \$600(1 + 0.05) = \$600(1.05)\\
\text{Interest for second year} &= \$600(1.05)\ (0.05)\\
\text{Principal at end of second year} &= \$600(1.05) + \$600(1.05)\ (0.05)\\
&= \$600(1.05)\ (1.05)\\
&= \$600(1.05)^2\\
\text{Interest for third year} &= \$600(1.05)^2(0.05)\\
\text{Principal at end of third year} &= \$600(1.05)^2 + \$600(1.05)^2(0.05)\\
&= \$600(1.05)^2(1.05)\\
&= \$600(1.05)^3\\
\text{Interest for fourth year} &= \$600(1.05)^3(0.05)\\
\text{Principal at end of fourth year} &= \$600(1.05)^3 + \$600(1.05)^3(0.05)\\
&= \$600(1.05)^3(1.05)\\
&= \$600(1.05)^4\\
&= \$600(1.21550625)\\
&= \$729.30.
\end{aligned}$$

15. Compound interest formula.—If we let P be the original principal, i the **yearly rate of interest** and S the amount to which P will accumulate in n years and reason as in the illustrated example of Art. 14, we will obtain the compound interest formula.

The interest for the first year will be Pi and the principal at the end of the first year will be

$$P + Pi = P(1 + i).$$

The interest for the second year will be $Pi(1 + i)$ and the principal at the end of the second year will be

$$P(1 + i) + Pi(1 + i) = P(1 + i)^2.$$

By similar reasoning we find that the amount at the end of the third year is

$$P(1 + i)^2 + Pi(1 + i)^2 = P(1 + i)^3,$$

and in general the amount at the end of n years is $P(1 + i)^n$. Thus we have the formula

$$S = P(1 + i)^n. \tag{1}$$

This relation is easily visualized by the following line diagram:

P —— n years at $i\%$ —— $S = P(1 + i)^n$

compound interest

Example 1. Find the compound amount and compound interest on \$500 for 8 years at 6%, the interest being converted annually.

Solution. Here, $P = \$500$, $i = 0.06$, $n = 8$.

Substituting in (1) we have

$$S = 500(1.06)^8.$$

From Table III, $(1.06)^8 = 1.59384807$,

and $$S = 500(1.59384807) = \$796.92.$$

The compound interest is

$$\$796.92 - \$500.00 = \$296.92.$$

Example 2. Find the compound amount on \$850 for 12 years at 6¼%, the interest being converted annually.

Solution. Here, $P = \$850$, $i = 0.0625$, $n = 12$,

and $$S = 850(1.0625)^{12}.$$

We do not find the rate, 6¼%, in Table III, so we use logarithms to compute S.

$$\log 1.0625 = 0.02633$$

$$12 \log 1.0625 = 0.31596$$
$$\log 850 = 2.92942$$
$$\log S = 3.24538$$
$$S = \$1{,}759.50.$$

Using a table of seven place logarithms we find $S = \$1{,}759.41$, which is correct to six significant digits. When we use a table of five place logarithms for computing, our results will be accurate to four and never more than five significant digits.

When P, n, and i are given, the *amount* S computed by (1) is frequently called the *accumulated value* of P at the end of n years. Hence, to *accumulate* P for n years at $i\%$ we find the amount S by using (1). The quantity $(1 + i)$ is called the *accumulation factor*.

Similarly, when S, n, and i are given, the *principal* P is called the *discounted* value of S due at the end of n years. Hence, to *discount* S for n years at $i\%$ we find the principal P by using (1). The principal P is also called the *present value* of S.

Exercises

1. Find the amount of \$1,000 invested 15 years at 4%.

2. Find the amount of \$1,000 invested 12 years at 6%.

3. Accumulate \$500 for 15 years at 6%.

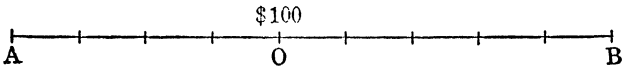
4. Discount \$800 for 20 years at 3%.

5. Find the difference between the amount of \$100 at simple interest and at compound interest for 5 years at 5%.

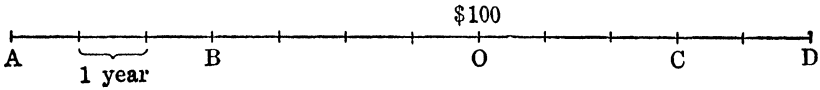
6. At the birth of a son a father deposited \$1,000 with a trust company that paid 4%, the fund accumulating until the son's twenty-first birthday. What amount did the son receive?

7. In the following line diagram each section represents 1 year. The point O denotes any given time. Any point to the right of O denotes a later time and any point to the left of O denotes an earlier time. Consider \$100 at O. Based upon $i = 4\%$, what is its value at B? at A?

Solution. At B the value is that of \$100 accumulated for 5 years, or $100(1 + .04)^5$. At A the value is that of \$100 discounted for 4 years or $100(1 + .04)^{-4}$.



8. In the following line diagram, based upon $i = 5\%$, find the values at A, B, C, and D of \$100 at O.



16. Nominal and effective rates of interest.—*The effective rate of interest is the actual interest earned on a principal of* \$1 *in one year.* When interest is converted into principal more than once a year, the actual interest earned (effective rate) is more than the quoted rate (nominal rate). Thus, if we have a nominal rate of 6% and the interest is converted semi-annually, the effective rate is by a method similar to that used in Art. 14,

$$(1.03)^2 - 1 = 0.0609 = 6.09\%.$$

Then, on a principal of \$10,000, a nominal rate of 6% convertible semi-annually gives in one year \$609.00 interest.

Similarly, if the rate is 6%, convertible quarterly, the effective rate is

$$(1.015)^4 - 1 = 0.06136 = 6.136\%.$$

If we let i stand for effective rate, j for nominal rate, and m for the number of conversions per year, then $\frac{j}{m}$ will be the interest on \$1 for one

conversion period. Hence, the amount of \$1 at the end of one year will be given by

$$\left(1 + \frac{j}{m}\right)^m \tag{2}$$

and the effective rate will be given by the equation

$$i = \left(1 + \frac{j}{m}\right)^m - 1. \tag{3}$$

We may also write

$$(1 + i) = \left(1 + \frac{j}{m}\right)^m. \tag{4}$$

If $\left(1 + \frac{j}{m}\right)^m$ be substituted for $(1 + i)$ in (1), we obtain the equation

$$S = P\left(1 + \frac{j}{m}\right)^{mn}. \tag{5}$$

This equation gives the amount of a principal P *at the end of* n *years at rate* j *convertible* m *times per year.* If $m = 1$, (5) reduces to (1). Hence we say that (5) is the general compound interest formula and (1) is a special case of (5).

From (4), we may easily find j in terms of m and i. Extracting the mth root of each member and transposing, we find

$$j = m\left[(1 + i)^{1/m} - 1\right]. \tag{4'}$$

Sometimes the nominal rate j is written with a subscript to show the frequency of conversion in a year. Thus j_m means that the nominal rate is j with m conversion periods in a year. We also find it convenient at times to use the symbol "j_m at i" to mean "the nominal rate j which converted m times a year yields the effective rate i." Values of j for given values of m and i are found in Table IX.

Example 1. Find the effective rate corresponding to a nominal rate of 5% when the interest is converted quarterly.

Solution. Here, $j = 0.05$ and $m = 4$.

Substituting in (3), we have

$$\begin{aligned} i &= (1.0125)^4 - 1 \\ &= (1.05094534) - 1 = 0.050945 \\ &= 5.0945\%. \end{aligned}$$

Example 2. Find the amount of \$750 for 15 years at 5% converted quarterly.

Solution. Here $P = \$750, j = 0.05, n = 15$ and $m = 4$. Substituting in (5) we have

$$S = 750(1.0125)^{60}.$$

From Table III, $(1.0125)^{60} = 2.10718135$,

and $$S = 750(2.10718135) = \$1{,}580.39.$$

Example 3. Find the compound amount of \$500 for 120 years at 3%.

Solution. Here $P = \$500, i = 0.03, n = 120$. We find no value of $(1 + i)^n$ in the table when $n = 120$, but we may apply the index law, $a^x \cdot a^y = a^{x+y}$.

Hence, $$(1.03)^{120} = (1.03)^{100} \cdot (1.03)^{20}$$

$$= (19.21863198)\ (1.80611123)$$

$$= 34.710987$$

and $$S = 500(34.710987) = \$17{,}355.49.$$

This example illustrates a method by which the table can be used when the time extends beyond the table limit.

Example 4. To what sum does \$5,000 amount in 7 years and 9 months at 4% converted semi-annually.

Solution. The given time contains 15 whole conversion periods and 3 months. Now, the compound amount at the end of the 15th period is

$$S = 5{,}000(1.02)^{15} = \$6{,}729.34.$$

The simple interest on \$6,729.34 for the remaining 3 months is

$$6{,}729.34 \times \tfrac{3}{12} \times 0.04 = \$67.29.$$

Hence, the amount at the end of 7 years and 9 months is

$$\$6{,}729.34 + \$67.29 = \$6{,}796.63.$$

The solution of Example 4 illustrates a plan that is usually used for finding the compound amount when the time is not a whole number of conversion periods. We may state the plan as follows:

I. *Find the compound amount for the whole number of conversion periods, using (5).*

II. *Find the simple interest on the resulting amount at the given rate for the remaining time.*

III. *Add the results of I and II.*

Exercises

1. Find the amount of \$800 invested for 8 years at 5%, convertible annually.

2. Solve Example 1, when the interest is converted (a) semi-annually, (b) quarterly. Use formula (5).

3. Find the compound interest on \$2,500 at 6½% for 8 years, if the interest is converted semi-annually.

4. A man pays \$1,000 for a 10 year bond that is to yield 5%, payable semi-annually. What will be the amount of the original investment at the end of 10 years if the dividends are immediately reinvested at 5%, payable semi-annually?

5. On January 1, 1928, \$1,500 was placed on time deposit at a certain bank. For 10 years the bank allowed 4% interest converted annually. During the next 4 years 3%, converted quarterly, was allowed, and on January 1, 1942 the interest rate allowed on such deposits was reduced to 2½%, converted semi-annually. What was the accumulated value of this original deposit as of January 1, 1945?

6. Find the effective rate equivalent to 6% nominal converted (a) semi-annually, (b) quarterly, (c) monthly.

7. A savings bank paid 5% compound interest on a certain deposit for 6 years and then 4% for the next 4 years. What single rate (equivalent rate) during the 10 years would have produced the same effect?

Solution.—Let i equal the equivalent rate.

$$\text{Then } (1+i)^{10} = (1.05)^6\,(1.04)^4.$$

$$\log 1.05 = 0.0211893$$
$$\log 1.04 = 0.0170333$$

$$6 \log 1.05 = 0.1271358$$
$$4 \log 1.04 = 0.0681332$$

$$10 \log (1+i) = 0.1952690$$
$$\log (1+i) = 0.0195269$$
$$(1+i) = 1.04599$$
$$i = 0.04599 = 4.599\%.$$

The value obtained for $(1 + i)$ is correct to six significant digits. A seven place table of logarithms was used here. When we use a table of seven place logarithms, we can be sure that our results are accurate to six significant digits.

8. What is the effective rate for 20 years equivalent to 6%, converted annually for the first 8 years; 5% converted semi-annually for the next 7 years; and 4%, converted quarterly for the last 5 years?

9. An individual has a sum of money to invest. He may buy saving certificates, paying 5½% convertible semi-annually, or deposit it in a building and loan association, which pays 5% convertible monthly. Assuming that the degree of safety of the two is the same, should he buy the certificates or deposit his money in the association?

10. Find the compound amount on \$750 for 8 years 9 months at 5% converted semi-annually.

11. Representing time along the horizontal axis and the computed values of S along the vertical axis, make graphs of $S = 100(1 + 0.04n)$ and $S = 100(1.04)^n$. Take for n the values 1, 5, 9, 13, 17, 21, 25 and use the same scale for both graphs.

12. Repeat Exercise 11, when the interest rate is 6%.

13. Accumulate \$2,000 for 12 years if the interest rate is 5% compounded monthly.

14. A house is offered for sale. The terms are \$4,000 cash, or \$6,000 at the end of 10 years without interest. If money is worth 4%, interest converted semi-annually, which method of settlement is to the advantage of the purchaser?

15. Find the effective rate equivalent to 7% converted (a) monthly, (b) quarterly, (c) semi-annually.

16. Find the nominal rate, converted quarterly, that will yield an effective rate of (a) 4%; (b) 5%; (c) 6%.

17. Present value at compound interest.—In Art. 6 we defined the present value P of a sum S, due in n years, from the standpoint of simple interest. The definition of present value will be the same here, except that compound interest is used in the place of simple interest. From the definition of present value, it follows that the present value P of a sum S may be obtained by solving equation (1), Art. 15 for P. Solving this equation for P, we have

$$P = \frac{S}{(1+i)^n} = S(1+i)^{-n} = Sv^n, \text{ where } v = \frac{1}{1+i}. \tag{6}$$

The number v is called the *discount factor.*

If the rate of interest is j, converted m times a year, we have from (5) Art. 16

$$P = \frac{S}{\left(1+\frac{j}{m}\right)^{mn}} = S\left(1+\frac{j}{m}\right)^{-mn}. \tag{7}$$

Compound discount is commonly defined as the future value S minus the present value P. If D stands for compound discount on S, we have

$$D = S - P. \tag{8}$$

Compare the above formula with (8), Art. 7.

Since P is defined as the principal that will accumulate to S, at compound interest, in n years, the difference $S - P$ also stands for the compound interest on P. Therefore, we may say that the compound discount on the accumulated value is the same as the compound interest on the present value for the given time at the specified interest rate.

Example 1. Find the present value and compound discount of \$4,000 due in 10 years at 5% converted annually.

Solution. Here, $S = \$4{,}000$, $i = 0.05$, and $n = 10$.
Substituting in (6), we have

$$P = 4{,}000(1.05)^{-10}.$$

From Table IV, $(1.05)^{-10} = 0.61391325$

and $$P = 4{,}000(0.61391325) = \$2{,}455.65.$$

Also, $$D = 4{,}000.00 - 2{,}455.65 = \$1{,}544.35.$$

Example 2. Find the present value of \$2,000 due in 8 years at 4¾% converted semi-annually.

Solution. Here, $S = \$2{,}000$, $j = 0.0475$, $m = 2$, and $n = 8$.
Substituting in (7), we have

$$P = 2{,}000(1.02375)^{-16}.$$

We do not find the rate, 2⅜%, in Table IV, so we use logarithms to compute S.

$$\log 1.02375 = 0.0101939$$
$$16 \log 1.02375 = 0.1631024$$

$$\log (1.02375)^{-16} = 9.8368976 - 10$$
$$\log 2{,}000 = 3.30103$$

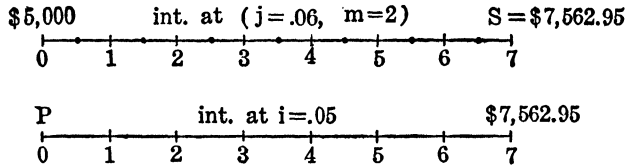
$$\log P = 3.13793$$
$$P = \$1{,}373.81.$$

Example 3. Find the present value of \$5,000 due in 7 years with interest at 6% converted semi-annually, assuming money worth 5%.

Solution. We first find the maturity value of the debt and then find the present value of this sum.

Hence, $S = 5{,}000(1.03)^{14}$
$= 5{,}000(1.51258972)$
$= \$7{,}562.95,$

and $P = S(1.05)^{-7}$
$= 7{,}562.95(1.05)^{-7}$
$= 7{,}562.95(0.71068133)$
$= \$5{,}374.86.$



This example illustrates a method for finding the present value of an interest-bearing debt.

Problems

1. What is the present value of a note of \$200 due in 6 years without interest, assuming money worth 6%?

2. Find the present value of \$3,000 due in 5 years, if the nominal rate is 5%, convertible semi-annually.

3. What sum of money invested now will amount to \$4,693.94 in 25 years if the nominal rate is 5¾%, convertible semi-annually?

4. A note of \$3,750 is due in 4½ years with interest at 6% payable semi-annually. Find its value 3 years before it is due, if at that time money is worth 5%.

5. What is the present value of a \$1,000 note due in 5 years with interest at 8% payable semi-annually, when money is worth 6%?

6. Compare the present values of non-interest-bearing debts of \$400 due in 3 years and \$450 due in 5 years, assuming money worth 6% converted semi-annually. Compare the values of these debts 2 years from now, assuming that money is still worth 6% converted semi-annually.

7. An investment certificate matures in 3 years for \$1,000. Its present cash value is \$860. If one desires his money to earn 5% annually, should he purchase the certificate?

8. A debt of \$4,500 will be due in 10 years. What sum must one deposit now in a trust fund, paying 4½% converted semi-annually, in order to pay the debt when it falls due?

9. What is the present value of \$300, due in 4 years and 3 months without interest, when money is worth 5%?

10. A father wishes, at the birth of his son, to set aside a sum that will accumulate to \$2,500 by the time the son is 21 years old. How much must be set aside, if it accumulates at 3% converted semi-annually?

11. Draw graphs of $P = \frac{S}{(1+i)^n}$ and $P = \frac{S}{1+ni}$ for integral values of n from 0 to 10. For convenience, take $S = 10$ and $i = 0.05$. Take values of n along the horizontal axis and corresponding values of P along the vertical axis, using the same scale and set of axes for both graphs. Use Table IV for finding the values of $P = \frac{S}{(1+i)^n}$.

12. If \$2,500 accumulates to \$3,700.61 in a certain time at a given rate, what is the present value of \$2,500 for the same time and rate?

13. Find the present value of a debt of \$250, due in 5 years 3 months and 15 days, if money is worth 5%.

14. An investment certificate matures in 7 years for \$500. If money is worth 4% for the first 3 years and 3½% thereafter, what is the present value of the certificate?

15. A man desires to sell a house and receives two offers. One is for \$2,500 cash and \$5,000 in 5 years. The other is for \$3,000 cash and \$4,000 to be paid in 3 years. On a 5% basis, which is the better offer for the owner of the house and what is the difference between the two offers?

16. An insurance company allows 3½% compound interest on all premiums paid one year or more in advance. A policy holder desires to pay in advance three annual premiums due in 1 year, 2 years, and 3 years respectively. How much must he pay the company now if each annual premium is \$21.97?

17. Making use of the binomial theorem (assuming n greater than 1) show that $(1+i)^n$ is greater than $(1+ni)$. Using Table III compare these values when $n = 5$ and $i = 0.06$.

18. Other problems solved by the compound interest formulas.— Formulas (1) and (5) each contain four letters (assuming m in (5) to be fixed). Any one of these letters can be expressed in terms of the other three. In Art. 16 we solved problems in which S was the unknown and in Art. 17 we solved for P. We shall now solve some problems when the value of n or j is required.

Example 1. In how many years will \$742.33 amount to \$1,000 if invested at 6%, converted quarterly?

Solution. From (5), Art. 16, we have

$$1{,}000 = 742.33(1.015)^{4n}.$$

Taking logarithms of both members of the above equation, we get

$$\log 1{,}000 = \log(742.33) + 4n \log(1.015).$$

Solving for n,

$$n = \frac{\log(1{,}000) - \log(742.33)}{4\log(1.015)} = \frac{3.00000 - 2.87060}{4(0.00647)}$$

$$= \frac{0.12940}{0.02588} = 5.$$

Hence, \$742.33 will amount to \$1,000 in 5 years, if the rate is 6% converted quarterly.

Example 2. How long will it take \$1,000 to amount to \$1,500 at 5% converted semi-annually?

Solution. Substituting in (5), Art. 16, we have

$$1{,}500 = 1{,}000(1.025)^{2n}.$$

The above equation reduces to

$$(1.025)^{2n} = 1.5.$$

From the 2½% column in Table III, we find that $(1.025)^{2n} = 1.4845\ 0562$ when $2n = 16$; and when $2n = 17$, $(1.025)^{2n} = 1.5216\ 1826$. The nearest time, then, is 16 semi-annual periods or 8 years. That is, \$1,000 amounts to \$1,484.51 in 8 years at 5% converted semi-annually. We now find the time required for \$1,484.51 to amount to \$1,500 at 5% simple interest. Here, P = \$1,484.51, I = \$15.49, and $i = 0.05$. We solve for n as in illustrated Example 4, Art. 70.

$$n = \frac{15.49}{(1{,}484.51)\ (0.05)} = \frac{15.49}{74.2255}$$

$$= 0.209 \text{ year (approximately), or 2 months and 15 days.}$$

Hence, we find that \$1,000 will amount to \$1,500 in 8 years 2 months and 15 days at 5% converted semi-annually.

Examples 1 and 2 illustrate methods for finding n, when S, P, and i are given.

Example 3. At what rate would \$2,500 amount to \$5,000 in 14 years if interest were converted semi-annually?

Solution. From (5), Art. 16, we have

$$5{,}000 = 2{,}500\left(1 + \frac{j}{2}\right)^{28}.$$

Taking logarithms of both members of the above equation, we get

$$\log 5{,}000 = \log 2{,}500 + 28 \log \left(1 + \frac{j}{2}\right),$$

$$\log \left(1 + \frac{j}{2}\right) = \frac{\log 5{,}000 - \log 2{,}500}{28}$$

$$= \frac{3.69897 - 3.39794}{28} = \frac{0.30103}{28}$$

$$= 0.01075.$$

$$\left(1 + \frac{j}{2}\right) = 1.025$$

$$\frac{j}{2} = 0.025$$

$$j = 0.05 = 5\%.$$

That is, the rate is 5% nominal, convertible semi-annually. From (4) Art. 16 we find the effective rate to be $i = 5.0625\%$.

Example 4. At what rate would \$1,500 amount to \$2,500 in 9 years, if the interest were converted annually?

Solution. From (1), Art. 15, we have

$$2{,}500 = 1{,}500(1 + i)^9.$$

Dividing the above equation through by 1,500, we get

$$(1 + i)^9 = 1.6667 \text{ (to 4 decimal places).}$$

In Table III we notice that when $i = 0.055$, $(1 + i)^9 = 1.6191$; when $i = 0.06$, $(1 + i)^9 = 1.6895$. Hence, i is a rate between 5½% and 6%.

By interpolation, we find

$$i = 0.055 + (0.005)\ (^{476}/_{704})$$

$$= 0.055 + 0.00338 = 0.05838.$$

Hence, the rate is 5.84% (approximately). The student should also solve this example by logarithms.

Examples 3 and 4 illustrate methods for finding the rate when S, P, and n are given.

Exercises

1. In what time will \$840 accumulate to \$2,500 at 5%, converted annually?

2. If \$1,000 is invested in securities and amounts to \$2,500 in 15 years, what is the average annual rate of increase?

3. At what rate must \$10,000 be invested to become \$35,000 in 25 years?

4. In how many years will \$400 amount to \$873.15 at 5% annually?

5. How long will it require any sum to double itself at effective rate i?

6. How long will it require a principal to double itself at (a) 5%, (b) 6%?

7. How long will it take \$1,500 to amount to \$5,000 at 6% converted quarterly?

8. At what rate will \$2,000 amount in 30 years to \$10,184.50 if the interest is converted semi-annually?

9. A will provides that \$15,000 be left to a boy to be held in trust until it amounts to \$25,000. When will the boy receive the fund if invested at 4% converted semi-annually?

10. A man invested \$1,500 in securities and re-invested the dividends from time to time and at the end of 10 years he found that his investments had accumulated to \$2,700. What was his average rate of interest?

19. Equation of value.—In Art. 11 the equation of value was defined and used in connection with simple interest. The *equation of value* used here will have the same meaning as in Art. 11. That is, it is the equation that expresses the equivalence of two sets of obligations on a common date (focal date). In Art. 11 we assumed, for convenience, that the equation of value is true for any focal date. However, this assumption is only approximately true, as was pointed out by a particular example. That is, when simple interest is used the equivalence of two sets of obligations actually depends upon the focal date selected. *The equivalence of two sets of sums, however, is independent of the focal date when the sums are accumulated* or *discounted by compound interest.* That is, if we have an equation of value for a certain focal date, we may obtain an equation of value for any other focal date by multiplying or dividing the first equation through by some power of $(1 + i)$ or of $(1 + j/m)$.

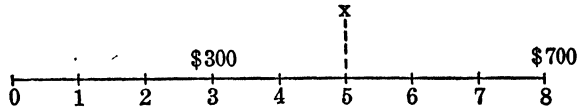
Example 1. A owes B the following debts: \$300 due in 3 years without interest and \$700 due in 8 years without interest. B agrees that A may settle the two obligations by making a single payment at the end of 5 years. If the two individuals agree upon 6% as a rate of interest, find the single payment.

Solution. Let x stand for the single payment, and choose 5 years from now as the focal date.

The \$300 debt is due 2 years before the focal date and amounts to $300(1.06)^2$ on the focal date.

The \$700 debt is due 3 years after the focal date and has a value of $700(1.06)^{-3}$ on the focal date.

The single payment x is to be made on the focal date and has a value of x on that date.



Then, for the equation of value, we have

$$\begin{aligned} x &= 300(1.06)^2 + 700(1.06)^{-3} \\ &= 300(1.12360000) + 700(0.83961928) \\ &= 337.08 + 587.73 \\ &= 924.81. \end{aligned}$$

Hence, the two debts may be discharged by a single payment of \$924.81 five years from now.

Had we assumed 8 years from now as focal date, our equation of value would have been

$$x(1.06)^3 = 300(1.06)^5 + 700.$$

Dividing the above equation through by $(1.06)^3$, we get

$$x = 300(1.06)^2 + 700(1.06)^{-3},$$

which is the equation of value obtained when 5 years from now is taken as the focal date. This is an illustration of the fact that an equation of value does not depend upon our choice of a focal date.

The student will observe that in the construction of the line diagram we place at the respective points the **maturity values** of the debts. Further, it should be observed that the payment and the debts are placed at different levels.

Example 2. Smith owes Jones \$500 due in 4 years with interest at 5% and \$700 due in 10 years with interest at 4½%. It is agreed that the two debts be settled by paying \$600 at the end of 3 years and the balance at the end of 8 years. Find the amount of the final payment, assuming an interest rate of 5½%.

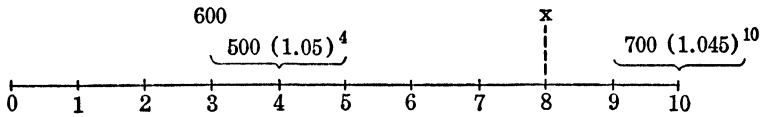
Solution. Let x stand for the final payment and choose 8 years from now as the focal date.

The maturity value of the \$500 debt is $500(1.05)^4$ and its value on the focal date is $500(1.05)^4(1.055)^4$.

The maturity value of the \$700 debt is $700(1.045)^{10}$ and its value on the focal date is $700(1.045)^{10}(1.055)^{-2}$.

The value of the \$600 payment is $600(1.055)^5$ on the focal date.

The value of the final payment is x on the focal date.



Expressing the fact that the value of the payments equals the value of the debts (on the focal date), our equation of value becomes

$$600(1.055)^5 + x = 500(1.05)^4(1.055)^4 + 700(1.045)^{10}(1.055)^{-2}.$$

Making use of Tables III and IV and performing the indicated multiplications, we have

$$784.176 + x = 752.900 + 976.688$$

and

$$x = 945.41.$$

Hence, the payment to be made 8 years from now is **\$945.41.**

20. Equated time.—In Art. 12 *equated time* was discussed and a formula (based upon simple interest) for finding this time was developed. Basing our discussion on compound interest, we shall now solve a particular example and then consider the general problem, thereby developing a formula.

Example 1. Find the time when debts of \$1,000 due in 3 years without interest and \$2,000 due in 5 years with interest at 5% may be settled by a single payment of \$3,000, assuming an interest rate of 6%.

Solution. Choose "now" as the focal date and let x stand for the time in years, measured from the focal date ("now"), until the single payment of \$3,000 should be made. Our equation of value becomes

$$3{,}000(1.06)^{-x} = 1{,}000(1.06)^{-3} + 2{,}000(1.05)^5(1.06)^{-5}$$

$$3{,}000(1.06)^{-x} = 1{,}000(0.83962) + 2{,}000(1.27628)\ (0.74726),$$

$$3{,}000(1.06)^{-x} = 839.62 + 1{,}907.43 = 2{,}747.05,$$

$$(1.06)^{-x} = \frac{2{,}747.05}{3{,}000.00},$$

$$(1.06)^{x} = \frac{3{,}000}{2{,}747.05},$$

$$x \log 1.06 = \log 3{,}000 - \log 2{,}747.05,$$

$$x = \frac{\log 3{,}000 - \log 2{,}747.05}{\log 1.06}$$

$$= \frac{3.47712 - 3.43886}{0.02531} = 1.51.$$

Hence, the two debts may be settled by a single sum of \$3,000 in 1 year, 6 months from "now."

Problem. Given that A owes B debts of $D_1, D_2, D_3, \cdots$ having maturity values of $S_1, S_2, S_3, \cdots$ and due in $n_1, n_2, n_3, \cdots$ years respectively. Assuming an interest rate of $i\%$, find the time when the debts may be settled by making a single payment of $S = S_1 + S_2 + S_3 + \cdots$.

Solution. Choose "now" as the focal date and let n stand for the time in years, measured from the focal date (now), until the single payment of S should be made.

Reasoning as in Example 1, the equation of value becomes

$$(S_1 + S_2 + S_3 + \cdots)(1 + i)^{-n} = S_1(1 + i)^{-n_1} + S_2(1 + i)^{-n_2} + S_3(1 + i)^{-n_3} + \cdots. \quad (9)$$

Solving the above equation for $(1 + i)^{-n}$, we get

$$(1 + i)^{-n} = \frac{S_1(1 + i)^{-n_1} + S_2(1 + i)^{-n_2} + S_3(1 + i)^{-n_3} + \cdots}{S_1 + S_2 + S_3 + \cdots}$$

and

$$(1 + i)^{n} = \frac{S_1 + S_2 + S_3 + \cdots}{S_1(1 + i)^{-n_1} + S_2(1 + i)^{-n_2} + S_3(1 + i)^{-n_3} + \cdots}.$$

Taking logarithms of both sides of the above equation and solving for n, we have

$$n = \frac{\log (S_1+S_2+S_3+\cdots) - \log [S_1(1+i)^{-n_1}+S_2(1+i)^{-n_2}+S_3(1+i)^{-n_3}+\cdots]}{\log (1 + i)}. \quad (10)$$

Formula (10) gives the exact value for the equated time. However, it is obviously very involved and is rather tedious to apply. We naturally seek a satisfactory approximation formula. We shall now proceed to find one.

If $(1 + i)^{-n}$ is expanded by the binomial theorem, we have

$$(1 + i)^{-n} = 1 - ni + \frac{n(n + 1)}{2} i^2 - \frac{n(n + 1)(n + 2)}{2 \cdot 3} i^3 + \cdots.$$

Neglecting all powers of i higher than the first gives $(1 - ni)$ as an approximate value of $(1 + i)^{-n}$.

Applying the binomial theorem to $(1 + i)^{-n_1}$, $(1 + i)^{-n_2}$, $\cdots$ and dropping powers of i higher than the first, we obtain $(1 - n_1 i)$, $(1 - n_2 i)$, $\cdots$ as approximate values of $(1 + i)^{-n_1}$, $(1 + i)^{-n_2}$, $\cdots$ respectively.

If in (9), $(1 + i)^{-n}$ and $(1 + i)^{-n_1}$, $(1 + i)^{-n_2}$, $\cdots$ are replaced by their approximate values, we get, on solving for n,

$$n = \frac{n_1 S_1 + n_2 S_2 + n_3 S_3 + \cdots}{S_1 + S_2 + S_3 + \cdots}. \tag{11}$$

Now, if the original debts, D_1, D_2, D_3, $\cdots$ are non-interest-bearing, S_1 S_2, S_3, $\cdots$, may be replaced by D_1, D_2, D_3, $\cdots$, respectively, and the above equation becomes

$$n = \frac{n_1 D_1 + n_2 D_2 + n_3 D_3 + \cdots}{D_1 + D_2 + D_3 + \cdots}. \tag{11'}$$

We notice that (11) is essentially the same as (12), Art. 12. When the periods of time involved are short and the debts, D_1, D_2, D_3, $\cdots$ do not draw interest, (11′) gives us a close approximation of the equated time. However, when the periods of time are short and the debts D_1, D_2, D_3, $\cdots$ draw interest (11) gives a good approximation to n.

Example 2. Find the equated time for paying in one sum debts of \$300 due in 3 years and \$150 due in 5 years.

Solution. Choosing "now" as focal date and substituting in (11′), we have

$$n = \frac{(300)3 + (150)5}{300 + 150} = 3.67 \text{ years}.$$

Assuming an interest rate of 6% and applying (10), we find

$$n = \frac{\log 450 - \log [300(1.06)^{-3} + 150(1.06)^{-5}]}{\log 1.06}$$

$$= \frac{2.65321 - 2.56118}{0.02531} = \frac{0.09203}{0.02531} = 3.64 \text{ years.}$$

We notice that the results by the two methods differ by only 0.03 of a year or about 11 days.

21. Compound discount at a discount rate.—In Art. 17 we defined the compound discount on the sum S as $S - P$, the difference between S and its present value P. The present value P has been found at the effective rate $i\%$ and at the nominal rate (j, m) to be

$$P = S(1 + i)^{-n} = S(1 + j/m)^{-mn}.$$

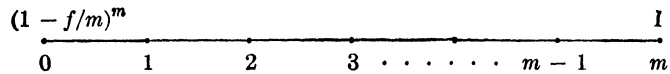
We may also find the present value P for a given discount rate. If the discount rate is d convertible annually, we have from (10) Art. 9 that $d = i/(1 + i)$ and $1 + i = 1/(1 - d)$. Hence we have

$$P = S(1 + i)^{-n} = S(1 - d)^n \tag{12}$$

as the present value of a sum S due in n years at the effective discount rate d. The compound discount on S is

$$D = S - P = S - S(1 - d)^n = S[1 - (1 - d)^n]. \tag{13}$$

If the discount is converted m times a year at the nominal rate f, the corresponding effective rate is the discount on \$1 in 1 year. We shall find the relation between d and f.



Consider \$1 due at the end of 1 year (m conversion periods). Its value at the end of the first discount period is $1 - f/m$. Its value at the end of the second discount period is $(1 - f/m)^2$, and at the end of the mth discount period, that is at the beginning of the year, is $(1 - f/m)^m$. But by Art. 7 its present value is $1 - d$. Therefore, we have

$$1 - d = (1 - f/m)^m \tag{14}$$

as the equation that expresses the relation between the nominal and

effective rates of discount. This is similar to (4) Art. 16, which shows the relation between the nominal and effective rates of interest.

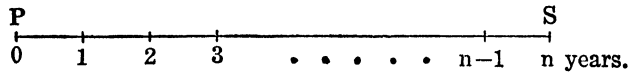
Further, we have upon substituting in (12)

$$P = S(1 - d)^n = S(1 - f/m)^{mn} \tag{15}$$

as the present value of a sum S due in n years discounted at a nominal rate of discount f convertible m times a year. Immediately we have the corresponding compound discount

$$D = S - P = S[1 - (1 - f/m)^{mn}]. \tag{16}$$

22. Summary of interest and discount.—Let P be the principal and



S be the accumulated value or amount of P at the end of n years. Then:

I. Simple interest and discount.

1. At simple interest rate i:

$$P = \frac{S}{1 + ni} \qquad S = P(1 + ni).$$

2. At simple discount rate d:

$$P = S(1 - nd) \qquad S = \frac{P}{1 - nd}.$$

In each case

3. $S - P$ = simple interest on P for n years.

= simple discount on S for n years.

Combining 1 and 2 we obtain

4. $$i = \frac{d}{1 - nd} \qquad d = \frac{i}{1 + ni}.$$

II. Compound interest and discount.

1. At effective rate of interest i:

$$P = S(1 + i)^{-n} \qquad S = P(1 + i)^n.$$

2. At nominal rate of interest (j,m):

$$P = S(1 + j/m)^{-mn} \quad S = P(1 + j/m)^{mn}.$$

3. At effective rate of discount d:

$$P = S(1 - d)^n \qquad S = P(1 - d)^{-n}.$$

4. At nominal rate of discount (f,m):

$$P = S(1 - f/m)^{mn} \qquad S = P(1 - f/m)^{-mn}.$$

Combining 1 and 2 we obtain

5. $$1 + i = (1 + j/m)^m.$$

Combining 3 and 4 we obtain

6. $$1 - d = (1 - f/m)^m.$$

In each case

7. $S - P$ = compound interest on P for n years.
= compound discount on S for n years.

Problems

1. A debt of \$1,500 is due without interest in 5 years. Assuming an interest rate of 5%, find the value of the debt (a) now, (b) in 3 years, (c) in 6 years.

2. Solve Problem 1, assuming that the debt draws 6% interest convertible semi-annually.

3. A debt of \$500, drawing 6% interest will be due in 4 years. Another debt of \$750, without interest will be due in 7 years. Assuming money worth 5%, compare the debts (a) now, (b) 4 years from now, (c) 6 years from now.

4. Set up the equation of value for Example 2, Art. 19, assuming now as the focal date and show that the equation is equivalent to the one used in the solution of the example.

5. A person is offered \$2,500 cash and \$1,500 at the end of each year for 2 years. He has a second offer of \$3,100 cash and \$800 at the end of each year for 3 years. Assuming that money is worth 6% to him, which offer should he accept?

6. A owes B debts of \$1,000 due at the end of each year for 3 years without interest. A desires to settle with B in full now and B agrees to accept settlement under the assumption that money is worth 5%. How much does A pay to B?

7. (a) In Problem 6 find the value of the debts 3 years from now, assuming 5% interest. (b) Also, find the present value of this result, assuming money worth 5%. (c) How does the result of (b) compare with the answer to Problem 6? Explain your results.

8. Smith owes Jones \$1,000 due in 2 years without interest. Smith desires to discharge his obligation to Jones by making equal payments at the end of each year for 3 years. They agree on an interest rate of 6%. Find the amount of each payment.

9. A man owes \$600 due in 4 years and \$1,000 due in 5 years. He desires to settle these debts by paying \$850 at the end of 3 years and the balance at the end of 6 years. Assuming money worth 6%, find the amount of the payment to be made at the end of 6 years.

10. Solve Problem 9, assuming that the debts draw 5% interest.

11. A man owes \$2,000 due in 2 years and \$3,000 due in 5 years, both debts with interest at 5%. Find the time when the two obligations may be paid in a single sum of \$5,000, if money is worth 6%, converted semi-annually.

12. *A* owes *B* \$200 due now, \$300 due in 2 years without interest, and \$500 due in 3 years with 4% interest. What sum will discharge the three obligations at the end of 1½ years if money is worth 6%, converted semi-annually?

13. There are three debts of \$500, \$1,000, and \$2,000 due in 3 years, 5 years and 7 years respectively, without interest. Find the time when the three obligations could be paid in a single sum of \$3,500, money being worth 5%.

14. Solve Problem 13, making use of the approximate formula, (11′).

15. Money being worth 6%, find the equated time for paying in one sum the following debts: \$400 due in 2 years, \$600 due in 3 years, \$800 due in 4 years and \$1,000 due in 5 years. Choose 2 years from now as focal date and set up an equation of value as in Example 1, Art. 12. Check the results by making use of the approximate formula.

16. Assuming money worth 5% show that \$500 now is equivalent to \$670.05 six years from now. Compare these two values on a 6% interest basis.

17. Show that:

$$\text{a.}\ \frac{j}{m} = \frac{\frac{f}{m}}{1-\frac{f}{m}}, \quad \text{b.}\ \frac{f}{m} = \frac{\frac{j}{m}}{1+\frac{j}{m}}. \tag{17}$$

18. Find the values of $(j, 2)$ and $(f, 2)$ that correspond to $i = 0.06$.

19. A money lender charges 3% a month paid in advance for loans. What is the corresponding nominal rate of interest? What is the effective rate?

20. I purchase from the Jones Lumber Company building material amounting to \$1,000. Their terms are "net 60 days, or 2% off for cash." What is the highest rate of interest I can afford to pay to borrow money so as to pay cash?

21. If a merchant's money invested in business yields him 2% a month, what discount rate can he afford to grant for the immediate payment of a bill on which he quotes "net 30 days"?

22. Find the nominal rate of interest convertible quarterly that is equivalent to $(j = .06, m = 2)$.

Hint. The two nominal rates are equivalent if they produce the same effective rate. Let i represent this common effective rate. Then $1 + i = (1 + .03)^2 = (1 + j/4)^4$.

23. Find the nominal rate of interest convertible semi-annually that is equivalent to $(j = .06, m = 4)$.

24. If \$2,350 amounts to \$3,500 in 4¾ years at the nominal rate $(j, 4)$, find j. Solve (a) by interpolation, and (b) by logarithms.

25. How long will it take a sum of money to double itself at (a) $i = .06$, (b) $(j = .06, m = 2)$, (c) $(j = .04, m = 2)$?

26. A man bought a house for \$4,000 and sold it in 8 years for \$7,000. What interest rate did he earn on his investment?

CHAPTER III

ANNUITIES CERTAIN

23. Definitions.—An *annuity* is a sequence of equal payments made at equal intervals of time. Strictly speaking, the word "annuity" implies yearly payments, but it is now understood to apply to all equal periodic payments, whether made annually, semi-annually, quarterly, monthly, weekly, or otherwise. Typical examples of annuities are: monthly rent on property, monthly wage of an individual, premiums for life insurance, dividends on bonds, and sinking funds.

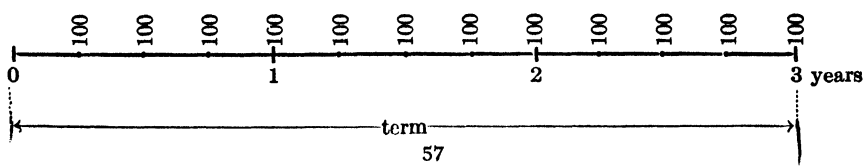
An *annuity certain* is one whose payments extend over a fixed number of years. A *contingent annuity* is one whose payments depend upon the happening of some event whose occurrence cannot be accurately foretold. The payments on a life insurance policy constitute a contingent annuity. In this chapter we shall be concerned entirely with annuities certain.

The time between successive payments is called the *payment period.* The time from the beginning of the first payment period to the end of the last payment period is called the *term* of the annuity.

Annuities certain may be classified into three groups: *Ordinary annuities, annuities due,* and *deferred annuities.* An *ordinary annuity* is one whose first payment is made at the *end* of the first payment period. If the first payment is made at the *beginning* of the first payment period, the annuity is called an *annuity due.* If the term of the annuity is not to begin until some time in the future, the annuity is called a *deferred annuity.*

The periodic payment into an annuity is frequently called the *periodic rent.* The sum of the payments of the annuity which occur in a year is called the *annual rent.*

Illustration. A sequence of payments of \$100 each, at the end of each quarter for 3 years, constitutes an annuity whose payment period is one-fourth of a year. The term begins immediately (one quarter before the first payment) and ends at the close of three years. The periodic rent is \$100 and the annual rent is 4(\$100), or \$400. This annuity is pictured in the line diagram.



There are four general cases of ordinary annuities to which we shall give especial consideration. They are briefly described by the outline:

A. Annuity payable annually.
   I. Interest at effective rate i.
   II. Interest at nominal rate (j, m).

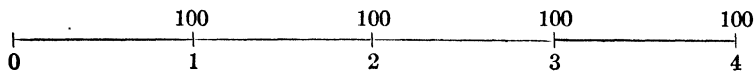
B. Annuity payable p times a year.
   I. Interest at effective rate i.
   II. Interest at nominal rate (j, m).

A. Annuity Payable Annually

24. Amount of an annuity.—*The sum to which the total number of payments of the annuity accumulate at the end of the term is called the amount, or the accumulated value, of the annuity.* We shall illustrate.

Example 1. \$100 is deposited in a savings bank at the end of each year for 4 years. If it accumulates at 5% converted annually, what is the total amount on deposit at the end of 4 years?

Solution. Consider the line diagram.



It is evident that the first payment will accumulate for 3 years. Hence its amount at the end of 4 years will be $\$100(1.05)^3$.

The second payment will accumulate for two years, and its amount will be $\$100(1.05)^2$, and so on.

Hence, the total amount at the end of 4 years will be given by

$$\$100(1.05)^3 + \$100(1.05)^2 + \$100(1.05) + \$100$$

or
$$\$100 + \$100(1.05) + \$100(1.05)^2 + \$100(1.05)^3. \qquad (1)$$

We may compute the above products by means of the compound interest formula; their sum will be the amount on deposit at the end of 4 years. However, we notice that (1) is a geometric progression, having 100 for the first term, (1.05) for the ratio, and 4 for the number of terms.

Therefore,
$$\text{Amount} = \frac{100[(1.05)^4 - 1]}{.05}. \qquad (2)$$

Evaluating (2) by means of Table III, we have

$$\frac{100[(1.05)^4 - 1]}{.05} = \frac{100(1.2155062 - 1)}{.05} = 431.01.$$

Hence, the amount of the above annuity is \$431.01.

The arithmetical solution of the above example may be tabulated as follows:

End of Year	Annual Deposit	Interest	Total Increase in Deposit	Total on Deposit
1	\$100.00	.....	\$100.00	\$100.00
2	100.00	\$ 5.00	105.00	205.00
3	100.00	10.25	110.25	315.25
4	100.00	15.76	115.76	431.01
Totals	\$400.00	\$31.01	\$431.01	

We shall now find the amount of an annuity of \$1 per annum for n years at an effective rate i. The symbol $s_{\overline{n}|i}$ is used to represent the amount of an annuity of 1 per annum payable annually for n years at the effective rate i. The first payment of 1 made at the end of the first year will be at interest for $n - 1$ years and will accumulate to $(1 + i)^{n-1}$.

The second payment of 1 will be at interest for $n - 2$ years and will accumulate to $(1 + i)^{n-2}$.

The third payment of 1 will be at interest for $n - 3$ years and will accumulate to $(1 + i)^{n-3}$, and so on.

The last payment will be a cash payment of 1. We have then

$$s_{\overline{n}|i} = (1 + i)^{n-1} + (1 + i)^{n-2} + (1 + i)^{n-3} + \ldots + (1 + i) + 1$$
$$= 1 + (1 + i) + (1 + i)^2 + \ldots + (1 + i)^{n-2} + (1 + i)^{n-1}. \qquad (3)$$

This is a geometric progression of n terms, having 1 for first term and $(1 + i)$ for ratio. Finding the sum (*Alg.: Com.—Stat.*,* Art. 60), we have †

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}. \qquad (4)$$

If the annual rent is R and if S represents the amount, we have

$$S = R \cdot s_{\overline{n}|i} = R\frac{(1 + i)^n - 1}{i}. \qquad (5)$$

* When it is not desired to emphasize the interest rate, this symbol is frequently written $s_{\overline{n}}$.

† See page x of this text for a list of formulas from *Alg.: Com.—Stat.*

Example 2. Find the amount of an annuity of \$200 per annum for 10 years at 5% converted annually.

Solution. Here, $R = \$200$, $n = 10$, and $i = 0.05$. Substituting in (5), we get

$$S = 200 \cdot s_{\overline{10}|.05} = 200 \frac{(1.05)^{10} - 1}{0.05}.$$

In Table V we find the amount of an annuity of 1 per period for n periods at rate i per period.

When $n = 10$ and $i = 0.05$, we find

$$s_{\overline{10}|.05} = \frac{(1.05)^{10} - 1}{0.05} = 12.57789254$$

and

$$S = 200(12.57789254) = 2515.58.$$

Hence, the amount of the annuity is \$2515.58.

Exercises

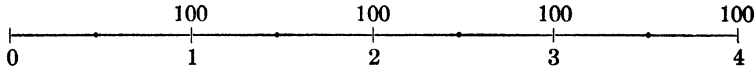
Find the amount of the following annuities:

1. \$300 per year for 10 years at 4% interest converted annually.
2. \$500 per year for 20 years at 5% converted annually.
3. \$200 per year for 6 years at 3% converted annually. Make a schedule showing the yearly increases and the amount of the annuity at the end of each year.
4. \$150 per year for 10 years at 6% converted annually.
5. In order to provide for the college education of his son, a father deposited \$100 at the end of each year for 18 years with a trust company that paid 4% effective. If the first deposit was made when the son was one year old, what was the accumulated value of all the deposits when the son was 18 years old?
6. A corporation sets aside \$3,700 annually in a depreciation fund which accumulates at 5%. What amount will be in the fund at the end of 15 years?
7. Write series (3) in the summation notation. (*Alg.: Com.—Stat.*, Art. 63.)
8. If \$1,000 is deposited at the end of each year for 10 years in a fund which is accumulated at 4% effective, what is the amount in the fund 4 years after the last deposit?
9. To create a fund of \$5,000 at the end of 10 years, what must a man deposit at the end of each year for the next 10 years if the deposits accumulate at 4% effective?
10. One man places \$4,000 at interest for 10 years; another deposits \$500 a year in the same bank for 10 years. Which has the greater sum at the end of the term if interest is at 4% effective?

Let us now find the amount of an annuity where the payments are made annually but the interest is converted more than once a year. We shall illustrate by an example.

Example 3. \$100 is deposited in a savings bank at the end of each year for 4 years. If it accumulates at 5% converted semi-annually, what is the total amount on deposit at the end of 4 years?

Solution. Consider the line diagram.



It is evident that the first deposit will accumulate for 3 years and at the end of 4 years, ((5), Art. 16), will amount to $\$100(1.025)^6$.

The second payment will amount to $\$100(1.025)^4$, and so on.

Hence, the total amount at the end of 4 years will be given by

$$\$100(1.025)^6 + \$100(1.025)^4 + \$100(1.025)^2 + \$100$$

or
$$\$100 + \$100(1.025)^2 + \$100(1.025)^4 + \$100(1.025)^6. \qquad (6)$$

We notice that (6) is a geometrical progression, having 100 for the first term, $(1.025)^2$ for ratio, and 4 for the number of terms. Substituting in (8) Art. 60, *Alg.*: *Com.—Stat.*, we have

$$S = \text{Amount} = \frac{100[(1.025)^8 - 1]}{(1.025)^2 - 1}. \qquad (7)$$

It is evident that Table V cannot be used here, but we may use Table III.

Thus,
$$S = \frac{100(1.21840290 - 1)}{1.05062500 - 1}$$

$$S = \frac{100(0.2184029)}{0.050625} = 431.41.$$

By writing (7) in the form

$$S = 100 \cdot \frac{(1.025)^8 - 1}{.025} \cdot \frac{.025}{(1.025)^2 - 1},$$

we can identify the last two terms in the product as $s_{\overline{8}|.025}$ and $1/s_{\overline{2}|.025}$. Then

$$S = 100. \quad s_{\overline{8}|.025} \cdot \frac{1}{s_{\overline{2}|.025}}$$

$$= 100(8.7361\ 1590) \cdot \frac{1}{2.025} = 431.41$$

as was obtained by the first method.

Hence, the amount of the annuity is \$431.41.

The arithmetical solution of the above example may be tabulated as follows:

End of Year	Annual Deposit	Interest	Total Increase in Deposit	Total on Deposit
½				
1	$100.00	.....	$100.00	$100.00
1½	.....	$2.50	2.50	102.50
2	100.00	2.56	102.56	205.06
2½	.....	5.13	5.13	210.19
3	100.00	5.25	105.25	315.44
3½	.....	7.89	7.89	323.33
4	100.00	8.08	108.08	431.41

We notice that the amount in Example 3 is 40 cents more than the amount in Example 1. This is due to the fact that the interest is converted semi-annually in Example 3 and only annually in Example 1.

If the interest is converted m times per year, we may substitute, [(4) Art. 16], $\left(1+\frac{j}{m}\right)^m$ for $(1+i)$ and $\left(1+\frac{j}{m}\right)^m - 1$ for i in (5) and obtain

$$S = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{m}-1}. \tag{8}$$

We can transform (8) into a form involving the annuity symbol $s_{\overline{n}|}$ by writing it in the form

$$S = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\frac{j}{m}} \cdot \frac{\frac{j}{m}}{\left(1+\frac{j}{m}\right)^{m}-1}$$

$$= R \cdot s_{\overline{mn}|\frac{j}{m}} \cdot \frac{1}{s_{\overline{m}|\frac{j}{m}}} \tag{8a}$$

Example 4. Find the amount of $200 per annum for 10 years at 5% converted quarterly.

Solution. Here, $R = \$200$, $n = 10$, $j = 0.05$, and $m = 4$. Substituting in (8a), we have

$$S = 200.\ s_{\overline{40}|.0125} \cdot \frac{1}{s_{\overline{4}|.0125}}$$

$$= 200(51.4895\ 5708) \cdot \frac{1}{4.0756\ 2695}$$

$$= 2{,}526.71$$

Hence the amount is \$2,526.71.

Why is the amount in Example 4 greater than the amount in Example 2?

Exercises

Find the amount of the following annuities:

1. \$300 per year for 8 years at 6% interest, converted semi-annually.

2. \$250 per year for 25 years at 5% converted quarterly.

3. \$500 per year for 5 years 4% converted semi-annually. Make a schedule showing the increases each six months and the amount of the annuity at the end of each six months.

4. \$600 per year for 30 years at 4½% converted semi-annually.

5. \$750 per year for 15 years at 4.2% converted semi-annually. (Hint: Use logarithms to evaluate $(1.021)^{30}$.)

6. On the first birthday of his son a father deposits \$100 in a savings bank paying 3½% interest, converted semi-annually. If he deposits a like amount on each birthday until the son is 21 years old, how much will be on deposit at that time?

7. A man deposits \$1,000 at the end of each year in a bank that pays 4% effective. Another man deposits \$1,000 at the end of each year in a bank that pays ($j = .035$, $m = 2$). At the end of 10 years how much more does the first man have than the second?

8. A man deposited \$1,000 a year in a bank. At the end of 15 years he had \$19,000.00 to his credit. What effective rate of interest did he receive? Solve by interpolation.

9. Solve Exercise 1 with the interest converted quarterly.

10. Solve Exercise 1 with the interest converted monthly.

11. Set up the series for the amount of an annuity of R at the end of each year for n years with interest at the nominal rate (j, m). Sum this series by (9) Art. 60, *Alg.: Com.—Stat.*, and thus obtain (8), Art. 24.

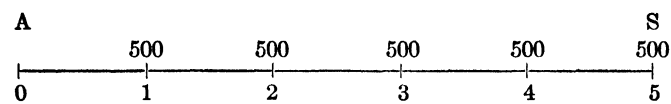
25. Present value of an annuity.—*The present value of an annuity is commonly defined as the sum of the present values of all the future payments.* Suppose an individual is to receive R dollars each year as an ordinary annuity and the payments are to last for n years. The individual may

do any one of three things with this annuity: (*a*) He may spend the payments as they are received; (*b*) accumulate the payments until the end of the last rent period (n years); (*c*) or sell the future payments to a bank (or similar institution) at the beginning of the first rent period.

If the same rate of interest is used to accumulate the payments as is used by the bank (or similar institution) in finding the present value of the future payments, it is evident that the sum (present value) paid to the individual by the bank at the beginning of the first rent period is equivalent to the present value of the sum to which the future payments will accumulate by the end of the last rent period. Consequently, we may also define the present value of an annuity as *that sum, which, placed at interest at a given rate at the beginning of the first rent period, will accumulate to the amount of the annuity by the end of the last rent period. Thus, it is the discounted value of S.*

Example 1. It is provided by contract that a young man receive \$500 one year from now and a like sum each year thereafter until 5 such payments in all have been received. Not wishing to wait to receive these payments as they come due, the young man sells the contract to a bank. If the bank desires to invest its funds at 6% interest compounded annually, how much does the young man receive now for his contract?

Solution.



The first payment is made one year from now and has a present value of $\$500(1.06)^{-1}$.

The second payment is due two years from now and has a present value of $\$500(1.06)^{-2}$, and so on until the last payment which has a present value of $\$500(1.06)^{-5}$. Summing up, we have

$$\text{Present value} = \$500(1.06)^{-1} + \$500(1.06)^{-2} + \ldots + \$500(1.06)^{-5}. \quad (9)$$

We notice that (9) is a geometrical progression having $500(1.06)^{-1}$ for the first term, $(1.06)^{-1}$ for ratio, and 5 for the number of terms. Substituting in (8), Art. 60, *Alg.: Com.—Stat.* we find

$$A = \text{Present value} = \frac{500(1.06)^{-1}[(1.06)^{-5} - 1]}{(1.06)^{-1} - 1}.$$

Multiplying the numerator and denominator of the above expression by (1.06),

$$A = \text{Present value} = \frac{500[(1.06)^{-5} - 1]}{1 - (1.06)}$$

$$A = 500\,\frac{1 - (1.06)^{-5}}{0.06}$$

$$A = 500(4.21236379) \qquad \text{[Table VI]}$$

$$A = \$2{,}106.182.$$

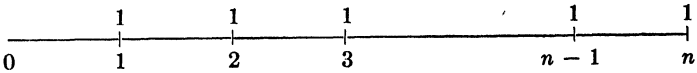
If the young man had waited to receive the payments as they became due and immediately invested them at 6% converted annually, his investments at the end of 5 years would have amounted to

$$S = 500\,\frac{(1.06)^5 - 1}{0.06} = \$2{,}818.546.$$

We notice that \$2,818.546 is the amount of \$2,106.182 for 5 years at 6%. For

$$\$2{,}106.182(1.06)^5 = 2{,}106.182(1.33822558) = \$2{,}818.546.$$

We shall now find the present value of an annuity of \$1 per annum for n years at the effective rate i. The symbol $a_{\overline{n}|i}$ or $a_{\overline{n}|}$ is used to represent the present value of this annuity. To find this value, we shall discount each payment to the beginning of the term.



The first payment of 1 made at the end of the first year when discounted to the present, by Art. 17, has the present value of $(1 + i)^{-1}$. Similarly, the second payment when discounted to the present has a present value of $(1 + i)^{-2}$. And so on for the other payments. We then have

$$a_{\overline{n}|i} = (1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3} + \cdots + (1 + i)^{-n} \qquad (10)$$

This is a geometric progression in which $a = (1 + i)^{-1}$, $r = (1 + i)^{-1}$, $l = (1 + i)^{-n}$. Finding the sum (*Alg.*: *Com.—Stat.*, Art. 60), we obtain

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}. \qquad (11)$$

The functions $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ are the two most important annuity functions. We frequently write them $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

Formula (11) may be easily derived from (5) Art. 24. For $a_{\overline{n}|i}$ is, by definition, the discounted value of $s_{\overline{n}|i}$. That is,

$$a_{\overline{n}|i} = s_{\overline{n}|i} \cdot (1+i)^{-n} = \frac{(1+i)^n - 1}{i} \cdot (1+i)^{-n} = \frac{1-(1+i)^{-n}}{i}.$$

If the annual rent is R, payable at the end of each year for n years, and if A represents the present value,

$$A = R \cdot a_{\overline{n}|i} = R\frac{1-(1+i)^{-n}}{i}. \tag{12}$$

If the interest is at the nominal rate (j, m), using the relation (4) Art. 16,

$$1 + i = (1 + j/m)^m,$$

we find

$$A = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{m}-1}, \tag{13}$$

which is easily reduced to

$$A = R \cdot a_{\overline{mn}|\frac{j}{m}} \cdot \frac{1}{s_{\overline{m}|\frac{j}{m}}}. \tag{13a}$$

26. Relation between $\frac{1}{a_{\overline{n}|}}$ and $\frac{1}{s_{\overline{n}|}}$.

We have $a_{\overline{n}|}(1+i)^n = s_{\overline{n}|}$ [Art. 25]

and $$(1+i)^n = \frac{s_{\overline{n}|}}{a_{\overline{n}|}}.$$

Substituting for $(1+i)^n$ in the equation

$$\frac{(1+i)^n - 1}{i} = s_{\overline{n}|}, \text{ we have}$$

$$\frac{s_{\overline{n}|} - a_{\overline{n}|}}{i a_{\overline{n}|}} = s_{\overline{n}|}.$$

Multiplying through by i and dividing through by $s_{\overline{n}|}$, we find

$$\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} = i,$$

or

$$\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i. \tag{14}$$

Table VII gives values for $\frac{1}{a_{\overline{n}|}}$. According to (14), values for $\frac{1}{s_{\overline{n}|}}$ are obtained by subtracting the rate i from the table values of $\frac{1}{a_{\overline{n}|}}$. Thus, to find $1/s_{\overline{20}|.04}$, we look up Table VII and obtain $1/a_{\overline{20}|.04} = 0.0735\ 8175$ Using relation (14), we find

$$\frac{1}{s_{\overline{20}|.04}} = 0.0735\ 8175 - .04 = 0.0335\ 8175.$$

27. Summary. Formulas of an ordinary annuity of annual rent R payable annually for n years.

I. Interest at effective rate i.

$$1.\quad S = R\frac{(1+i)^n - 1}{i} = R.\ s_{\overline{n}|i}.$$

$$2.\quad A = R\frac{1-(1+i)^{-n}}{i} = R.\ a_{\overline{n}|i}.$$

II. Interest at nominal rate (j, m).

$$1.\quad S = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^{m} - 1} = R.\, s_{\overline{mn}|\frac{j}{m}} \cdot \frac{1}{s_{\overline{m}|\frac{j}{m}}}$$

$$2.\quad A = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{m} - 1} = R.\ a_{\overline{mn}|\frac{j}{m}} \cdot \frac{1}{s_{\overline{m}|\frac{j}{m}}}$$

where

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}, \qquad a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i},$$

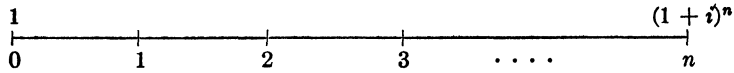
$$s_{\overline{n}|i} = (1+i)^n \cdot a_{\overline{n}|i}, \qquad a_{\overline{n}|i} = (1+i)^{-n} \cdot s_{\overline{n}|i},$$

$$\frac{1}{a_{\overline{n}|i}} - \frac{1}{s_{\overline{n}|i}} = i.$$

28. Other derivations of $a_{\overline{n}|}$ and $s_{\overline{n}|}$.—We have derived the formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$ by setting up series and then finding their sums by the formula for summing a geometric progression. It is of great value to derive the

formulas by a method called "direct reasoning" by some authorities, or "verbal interpretation" by other authorities.

Consider \$1 at 0. Its value at the end of n years is $(1 + i)^n$.



Also, from another point of view, \$1 at 0 will produce an annuity of i at the end of each year for n years and leave the original principal intact at the end of n years. For, at the end of the first year the amount is $(1 + i)$. Deposit the i into a separate account, and let the original principal \$1 again earn interest. It amounts to $(1 + i)$ at the end of the second year. We again deposit the i in the second account, and let the principal \$1 again earn interest. We continue this for n years. We thus find that \$1 at 0 is equivalent to an annuity of i for n years plus the original principal \$1 at n. In other words,

$$1 \text{ at } 0 = [\text{an annuity of } i \text{ for } n \text{ years}] + 1 \text{ at } n.$$

Let us now focalize all sums at the end of n years. Then

$$(1 + i)^n = i\, s_{\overline{n}|} + 1,$$

or, solving for $s_{\overline{n}|}$,

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

If we focalize all sums at the present, 0, we have

$$1 = i\, a_{\overline{n}|} + (1 + i)^{-n},$$

or,

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}.$$

Exercises

1. An individual is to receive an inheritance of \$1,000 at the end of each year for 15 years. If money is worth 5% effective, what is the present value of the inheritance?

2. Find the present value of an ordinary annuity of \$1,000 a year for 12 years at $(j = .05,\ m = 2)$.

3. How much money, if deposited with a trust company paying $(j = .04,\ m = 2)$, is sufficient to pay a person \$2,000 a year for 20 years, the first payment to be received 1 year from the date of deposit?

4. An article is listed for \$2,000 cash. A buyer wishes to purchase it in four equal annual installments, the first to be made 1 year from the date of purchase. If money is worth 6%, what is the amount of each installment?

5. A house was purchased for \$12,000, of which \$3,000 was cash. The balance was paid in 10 equal annual installments which began one year from the date of purchase. If money is worth ($j = .06$, $m = 2$), find the amount of each installment.

6. A house is offered for sale on the following terms: \$1,000 down, and \$500 at the end of each year for 10 years. If money is worth 6%, what is a fair cash price?

7. Prove: $s_{\overline{1}|} + s_{\overline{2}|} + s_{\overline{3}|} + \cdots + s_{\overline{n}|} = \dfrac{(1+i)\,s_{\overline{n}|} - n}{i}$.

8. Prove: $\displaystyle\sum_{x=1}^{n} a_{\overline{x}|} = \frac{n - a_{\overline{n}|}}{i}$.

9. Prove: $\displaystyle\sum_{x=1}^{n} a_{\overline{2x}|} = \frac{1}{i}\left[n - \frac{a_{\overline{2n}|}}{s_{\overline{2}|}}\right]$.

10. Evidently \$1 at 0 is equivalent to an annuity of $1/a_{\overline{n}|}$ at the end of each year for n years since the present value of the annuity is 1. Use this fact with Art. 28 to prove that

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i.$$

11. Show that $a_{\overline{m+n}|} = a_{\overline{m}|} + (1+i)^{-m}\, a_{\overline{n}|} = a_{\overline{n}|} + (1+i)^{-n} a_{\overline{m}|}$.

(a) by verbal interpretation. Draw line diagram.

(b) algebraically.

12. Find the value of $a_{\overline{120}|.04}$ by using the relation in Exercise 11.

13. Show that $s_{\overline{m+n}|} = (1+i)^n\, s_{\overline{m}|} + s_{\overline{n}|} = (1+i)^m s_{\overline{n}|} + s_{\overline{m}|}$.

14. Find the value of $s_{\overline{120}|.04}$ by using the relation in Exercise 13.

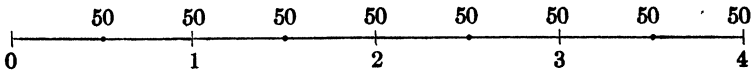
15. What do the formulas in Exercises 11 and 13 become if $m = 1$?

B. Annuity Payable p Times a Year

29. Amount of an annuity, where the annual rent, R, is payable in p equal installments.—In Art. 15, we derived the value of the compound amount of \$1 for n years, $(1+i)^n$, for integral values of n. We shall assume this relation to hold for fractional as well as for integral values of n. Consider

Example 1. \$50 is deposited in a savings bank at the end of every six months for 4 years. If it accumulates at 5% interest, converted annually, what is the total amount on deposit at the end of 4 years?

Solution.



The first deposit of \$50 is made at the end of six months and accumulates for $3\frac{1}{2}$ years. At the end of 4 years it will amount to $\$50(1.05)^{7/2}$.

The second deposit of \$50 is made at the end of the first year and will amount to $\$50(1.05)^3$ at the end of 4 years.

The third deposit of \$50 will amount to $\$50(1.05)^{5/2}$ at the end of 4 years, and so on.

Next to the last deposit will be at interest six months and will amount to $\$50(1.05)^{1/2}$ at the end of 4 years and the last deposit will be made at the end of 4 years and will draw no interest.

Hence, the total amount at the end of 4 years will be given by

$$\$50(1.05)^{7/2} + \$50(1.05)^3 + \ldots + \$50(1.05)^{1/2} + \$50;$$

or
$$\$50 + \$50(1.05)^{1/2} + \$50(1.05) + \ldots + \$50(1.05)^{7/2} \tag{15}$$

We notice that (15) is a geometrical progression having 50 for first term, $(1.05)^{1/2}$ for ratio, and 8 for the number of terms.

Substituting in (8), Art. 60, *Alg.: Com.—Stat.*, we have

$$S = \text{Amount} = \frac{50[(1.05)^4 - 1]}{(1.05)^{1/2} - 1}.$$

Using Table III and Table VIII, we have

$$S = \text{Amount} = \frac{50(1.21550625 - 1)}{1.02469508 - 1}$$

$$S = \frac{50(0.21550625)}{0.02469508} = 436.34.$$

Hence, the amount of the above annuity is \$436.34.

Let us now find the amount of an annuity of \$1 per annum, payable in p equal installments of $1/p$ at the end of every pth part of a year for n years at rate i, converted annually.

To assist him in following this discussion the student should draw a line diagram.

The amount of an annuity of \$1 per annum, payable in p equal installments at equal intervals during the year, will be denoted by the symbol, $s_{\overline{n}|}^{(p)}$. If the interest is converted annually, and i is the rate, $s_{\overline{n}|}^{(p)}$ can be expressed in terms of n, i, and p as follows: At the end of the first pth part of a year, $1/p$ is paid. This sum will remain at interest for $(n - 1/p)$ years and will amount to $1/p(1 + i)^{n-1/p}$.

The second installment of $1/p$ will be paid at the end of the second pth part of a year and will be at interest for $(n - 2/p)$ years, amounting to

$1/p(1+i)^{n-2/p}$ at the end of n years, and so on until np installments are paid.

Next to the last installment will be at interest for one pth part of a year and will amount to $1/p(1+i)^{1/p}$.

The last installment will be paid at the end of n years and will draw no interest. Adding all of these installments, beginning with the last one, we have

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} + \frac{1}{p}(1+i)^{1/p} + \frac{1}{p}(1+i)^{2/p} + \ldots + \frac{1}{p}(1+i)^{n-1/p}. \tag{16}$$

We notice that (16) is a geometrical progression having $1/p$ for first term, $(1+i)^{1/p}$ for ratio, and np for the number of terms. Substituting in (8), Art. 60, *Alg.: Com.—Stat.*, we have

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]}. \tag{17}$$

If the annual rent is R, we have

$$S = R\frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]}. \tag{18}$$

For convenience in evaluating, (18) may be written, (4′) Art. 16,

$$S = R\frac{(1+i)^n - 1}{i} \cdot \frac{i}{p[(1+i)^{1/p} - 1]}, \tag{19}$$

or

$$S = R(s_{\overline{n}|})\left(\frac{i}{j_p}\right). \tag{19a}$$

Table X gives values of $\frac{i}{j_p}$.

Example 2. Find the amount of an annuity of \$1,200 per year paid in quarterly installments of \$300 for 7 years if the interest rate is 5% converted annually.

Solution. Here, $R = \$1{,}200$, $n = 7$, $p = 4$, and $i = 0.05$. Substituting in (19), we have

$$S = 1{,}200\frac{(1.05)^7 - 1}{0.05} \cdot \frac{0.05}{4[(1.05)^{1/4} - 1]} = 1200.s_{\overline{7}|}\left(\frac{.05}{j_4}\right).$$

Using Table V and Table X, we have

$$S = 1{,}200(8.14200845)\ (1.01855942) = 9{,}951.74.$$

Hence, the amount of the above annuity is \$9,951.74.

Example 3. Find the amount of an annuity of \$200 per year paid in semi-annual installments for 10 years, the interest rate being 4.3% converted annually.

Solution. Here, $R = \$200$, $n = 10$, $p = 2$, and $i = 0.043$. The rate, 4.3%, is not given in our tables. We will evaluate by means of logarithms, using (18).

$$S = 200\,\frac{(1.043)^{10} - 1}{2[(1.043)^{\frac{1}{2}} - 1]}.$$

$$\log 1.043 = 0.0182843 \quad \text{(Table II.)}$$

$$10(\log 1.043) = 0.1828430$$

$$(1.043)^{10} = 1.5235 \quad \text{(Table I.)}$$

$$\tfrac{1}{2}(\log 1.043) = 0.0091422$$

$$(1.043)^{\frac{1}{2}} = 1.021274 \quad \text{(Table II.)}$$

Hence, $$S = \frac{200(1.5235 - 1)}{2(1.021274 - 1)} = \frac{100(0.5235)}{0.021274} = 2{,}460.75.$$

Consequently, the amount of the above annuity is \$2,460.75, and it is accurate to five significant digits. That is, the exact value is between \$2,460.75 and \$2,460.65.

Exercises

Find the amount of the following annuities:

1. \$300 per year paid in semi-annual installments for 10 years at 4% interest converted annually.

2. \$500 per year paid in quarterly installments for 20 years at 5% converted annually.

3. \$50 per month for 10 years at 4% interest converted annually.

4. \$250 at the end of every six months for 15 years at 4½% converted annually. Evaluate by logarithms, using (18), and then check the result by using Tables V and X.

5. \$100 quarterly for 12 years at 3¼% converted annually.

6. A young man saves \$50 a month and deposits it each month in a savings bank for 25 years. If the bank pays 3½% interest, converted annually, how much does he have on deposit at the end of the 25 years?

7. Solve Exercise 1, if it were paid in quarterly installments. Is the answer more or less than the answer of Exercise 1? Explain the difference.

8. Solve Exercise 2, if it were paid in semi-annual installments. Is the answer more or less than the answer of Exercise 2? Explain the difference.

9. \$100 is deposited in a savings bank at the end of every 3 months. If it accumulates at 3% converted annually, how much is on deposit at the end of 4 years? Solve fundamentally as a geometrical progression.

Let us now find the amount of an annuity paid in p equal installments each year where the interest is converted more than once a year. We will illustrate by an example.

Example 4. \$25 is deposited in a savings bank at the end of every three months for 4 years. If it accumulates at 5% interest, converted semi-annually, what is the total amount on deposit at the end of 4 years?

Solution. The first deposit of \$25 is made at the end of three months and is at interest for 3¾ years. At the end of 4 years it will amount to

$$\$25(1.025)^{15\frac{1}{2}}. \quad [(5), \text{Art. } 16.]$$

The second deposit of \$25 is made at the end of six months and is at interest for 3½ years. At the end of 4 years it will amount to

$$\$25(1.025)^{7}.$$

The third deposit of \$25 is made at the end of nine months and is at interest for 3¼ years. At the end of 4 years it will amount to

$$\$25(1.025)^{13\frac{1}{2}}, \text{ and so on.}$$

Next to the last deposit of \$25 will be at interest for ¼ year and will amount to

$$\$25(1.025)^{\frac{1}{2}}.$$

The last deposit of \$25 is made at the end of 4 years and draws no interest.

Hence, the total amount on deposit at the end of 4 years will be given by

$$\$25(1.025)^{15\frac{1}{2}} + \$25(1.025)^{7} + \cdots + \$25(1.025)^{\frac{1}{2}} + \$25$$

or
$$\$25 + \$25(1.025)^{\frac{1}{2}} + \$25(1.025) + \ldots + \$25(1.025)^{15\frac{1}{2}}. \tag{20}$$

We notice that (20) is a geometrical progression having 25 for first term, $(1.025)^{\frac{1}{2}}$ for ratio, and 16 for the number of terms. Substituting in (8), Art. 60, *Alg.: Com.—Stat.*, we have

$$S = \text{Amount} = \frac{25\{[(1.025)^{\frac{1}{2}}]^{16} - 1\}}{(1.025)^{\frac{1}{2}} - 1}$$

$$S = \frac{25[(1.025)^{8} - 1]}{(1.025)^{\frac{1}{2}} - 1}. \tag{21}$$

Using Table III and Table VIII, we have

$$S = \text{Amount} = \frac{25(1.21840290 - 1)}{1.01242284 - 1}$$

$$S = \frac{25(0.21840290)}{0.01242284} = 439.50.$$

Hence, the amount of the above annuity is \$439.50.

If the interest is converted m times per year, we may substitute $\left(1+\frac{j}{m}\right)^m$ for $(1+i)$ in (18) and obtain

$$S = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{p\left[\left(1+\frac{j}{m}\right)^{m/p}-1\right]}. \tag{22}$$

Let us consider further equation (21). Here $p = 4$, $m = 2$, and hence p/m is an integer. We may write (21) in the form

$$S = \frac{100}{2}\cdot\frac{[(1.025)^8-1]}{.025}\cdot\frac{.025}{2[(1.025)^{\frac{1}{2}}-1]}. \tag{21a}$$

$$= 50.s_{\overline{8}|}\cdot\frac{.025}{j_2}.$$

When $\frac{p}{m}$ is an integer, (22) becomes

$$S = \frac{R}{m}\cdot s_{\overline{mn}|\frac{j}{m}}\cdot\frac{\frac{j}{m}}{j_{\frac{p}{m}} \text{ at rate } \frac{j}{m}}. \tag{22a}$$

When p/m is an integer, (22) can easily be reduced to (22a) in which case we may apply Tables V and X. This transformation simplifies the arithmetical computation since S is expressed as a continued product.

Example 5. Find the amount of an annuity of \$600 per year paid in quarterly installments for 8 years, if the interest rate is 5% converted semi-annually.

Solution. Here, $R = \$600$, $n = 8$, $p = 4$, $m = 2$, and $j = 0.05$. Substituting in (22), we have

$$S = 600\frac{(1.025)^{16}-1}{4[(1.025)^{\frac{2}{4}}-1]}$$

$$= 300\frac{(1.025)^{16}-1}{2[(1.025)^{\frac{1}{2}}-1]}$$

$$= 300\frac{(1.025)^{16}-1}{0.025}\cdot\frac{0.025}{2[(1.025)^{\frac{1}{2}}-1]} = 300\cdot s_{\overline{16}|}\cdot\frac{.025}{j_2}.$$

Using Table V and Table X, we find

$$S = 300(19.38022483)\ (1.00621142) = \$5{,}850.18.$$

Example 6. Solve Example 5, if the interest is converted quarterly.

Solution. Here, $R = \$600$, $n = 8$, $m = p = 4$, and $j = 0.05$. Substituting directly in (22),

$$S = 150\,\frac{(1.0125)^{32} - 1}{0.0125} = 150\ s_{\overline{32}|.0125}$$

$$= 150(39.05044069) \qquad \text{(Table V)}$$

$$= \$5{,}857.57.$$

Example 7. Solve Example 5, if the payments are made semi-annually and the interest is converted quarterly.

Solution. Here, $R = \$600$, $n = 8$, $p = 2$, $m = 4$, and $j = 0.05$. Substituting in (22), we have

$$S = 600\,\frac{(1.0125)^{32} - 1}{2[(1.0125)^{\frac{1}{2}} - 1]}$$

$$= 300\,\frac{(1.0125)^{32} - 1}{(1.0125)^2 - 1}$$

$$= \frac{300(1.48813051 - 1)}{1.02515625 - 1} \qquad \text{(Table III)}$$

$$= \frac{300(0.48813051)}{0.02515625} = \$5{,}821.18.$$

In this example, m/p is an integer. When this is true we can write S as a product. Thus,

$$S = \frac{600}{2} \cdot \frac{(1.0125)^{32} - 1}{.0125} \cdot \frac{.0125}{(1.0125)^2 - 1} = 300 \cdot s_{\overline{32}|} \cdot \frac{1}{s_{\overline{2}|}}$$

in which Tables V and VII may be applied. In terms of annuity symbols, it is of the form

$$S = \frac{R}{p} \cdot s_{\overline{mn}|\frac{j}{m}} \cdot \frac{1}{s_{\overline{\frac{m}{p}}|\frac{j}{m}}}. \qquad (22b)$$

When m/p is an integer, (22) can easily be reduced to (22b).

Formula (22) is our most general formula for finding the amount of an annuity. The other forms (5), (8), and (18) are special cases of (22). Thus,

If $m = p = 1$, (22) reduces to

$$S = R\frac{(1+i)^n - 1}{i}. \tag{5}$$

If $p = 1$ and $m > 1$, (22) reduces to

$$S = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^{m} - 1}. \tag{8}$$

If $m = 1$ and $p > 1$, (22) reduces to

$$S = R\frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]}. \tag{18}$$

If $m = p$, (22) reduces to

$$S = \frac{R}{p}\frac{\left(1+\frac{j}{p}\right)^{np} - 1}{\frac{j}{p}}. \tag{23}$$

We observe that (23) is of the same form as (5), where n is replaced by np, i by j/p, and R by R/p.

In solving annuity problems the student should confine himself to the use of the fundamental formulas. Thus, if his problem requires that he find the amount of an annuity, he should use (5), (8), (18), or (22), and then effect the necessary transformation to reduce it to the annuity symbols that will entail the least amount of labor in obtaining the numerical result.

Exercises

1. A man deposits \$150 in a 4% savings bank at the end of every three months. If the interest is converted semi-annually, what amount will be to his credit at the end of 10 years?

2. Solve Exercise 1, with the interest converted (a) annually, (b) quarterly.

3. A man wishes to provide a fund for his retirement and begins at age 25 to deposit \$125 at the end of every three months with a trust company which allows 3% interest converted semi-annually. What will be the amount of the fund at age 60?

4. Solve Exercise 3, with the interest converted quarterly.

5. Fill out the following table for the amount of an annuity of \$300 per year for 12 years at 4%:

Annuity Payable	Interest Convertible		
	Annually	Semi-annually	Quarterly
Annually			
Semi-annually			
Quarterly			

6. Solve Exercise 5, with the rate of interest 5%.

7. Solve Exercise 5, with the rate of interest 4½ %.

8. Find the amount of an annuity of \$400 a year for 7 years at 7% interest converted semi-annually. Solve fundamentally as a geometrical progression using the principle of compound interest.

9. Solve Exercise 8, with the interest converted annually.

10. Solve Exercise 8, with the annuity payable in quarterly installments and the interest converted semi-annually.

11. \$250 is deposited at the end of every six months for 10 years in a fund paying 4% converted semi-annually. Then, \$150 is deposited at the end of every three months for 10 years and the interest rate is reduced to 3% converted quarterly. Find the total amount on deposit at the end of 20 years.

12. A man has \$2,500 invested in Government bonds which will mature in 15 years. These bonds bear 3% interest, payable January 1 and July 1. When these interest payments are received they are immediately deposited in a savings bank which allows 3½% interest converted semi-annually. To what amount will these interest payments accumulate by the end of 15 years?

13. A man begins at the age of thirty to save \$15 per month, and keeps all of his savings invested at an average rate of 4% effective. How much will he have as a retirement fund when he is sixty-five years old?

14. A man 25 years of age pays \$41.85 at the beginning of each year for 20 years for which he receives an insurance contract which will pay his estate \$1,000 in case of his death before 20 years and pay him \$1,000 cash, if living, at the end of 20 years. He also decides to deposit the same amount at the beginning of each year in a savings bank paying 3% interest. Compare the value of the two investments at the end of 20 years. On the basis of 3% interest what would you say his insurance protection cost for the 20 years?

15. A man deposits \$150 in a savings bank on his twenty-fifth birthday and a like amount every six months. If the bank pays 3% interest convertible semi-annually, how much does he have on deposit on his sixtieth birthday?

16. Solve Exercise 15, with the interest converted quarterly.

17. A man, age 25, pays \$24.03 a year in advance on a \$1,000, 20-payment life policy. If he should die at the end of 12 years, just before paying the 13th premium, how much

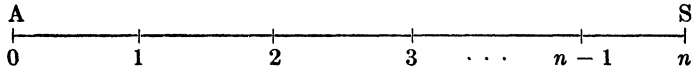
would his estate be increased by having taken the insurance instead of having deposited the \$24.03 each year in a savings bank paying 4% effective?

18. Find the amount of an annuity of \$200 a year payable in semi-annual installments for 7 years at 4% converted annually. Solve fundamentally as a geometrical progression.

19. Solve Exercise 18, with the interest converted quarterly.

20. Assume that R/p dollars is invested at the end of $1/pth$ of a year, at nominal rate j converted m times a year, and that a like amount is invested every pth part of a year until np such investments are made; sum up as a geometrical progression and thereby derive the formula (22).

30. Present value of an annuity of annual rent, R, payable in p equal installments.—In Art. 25 we considered the problem of finding the present value of an ordinary annuity with annual payments. We are now ready to consider the problem of finding the present value of an ordinary annuity of annual rent, R, with p payments a year.



We have found the amount S of such an annuity. When the interest is at the effective rate i, the amount S is given by (18); when the interest is at the nominal rate (j, m), the amount S is given by (22). If, as usual, A designates the present value of the annuity, evidently

$$A(1+i)^n = S = R\,\frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]} \tag{24}$$

if the interest is at the effective rate i, and

$$A\left(1+\frac{j}{m}\right)^{mn} = S = R\cdot\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{p\left[\left(1+\frac{j}{m}\right)^{m/p} - 1\right]} \tag{25}$$

if the interest is at the nominal rate (j, m).

Solving (24) and (25) respectively for A, we obtain

$$A = R\,\frac{1-(1+i)^{-n}}{p[(1+i)^{1/p} - 1]}, \tag{26}$$

and

$$A = R\,\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{p\left[\left(1+\frac{j}{m}\right)^{m/p} - 1\right]}. \tag{27}$$

We shall leave it as an exercise for the reader to show that (26) can be reduced to the form

$$A = R \cdot a_{\overline{n}|i} \cdot \frac{i}{j_p}.$$

and that (27) can be reduced to the forms

$$A = \frac{R}{m} \cdot a_{\overline{mn}|\frac{j}{m}} \cdot \frac{\frac{j}{m}}{j_{\frac{p}{m}} \text{ at rate } \frac{j}{m}}, \quad \frac{p}{m} \text{ an integer,} \tag{28a}$$

$$A = \frac{R}{p} \cdot a_{\overline{mn}|\frac{j}{m}} \cdot \frac{1}{s_{\overline{\frac{m}{p}}|\frac{j}{m}}}, \quad \frac{m}{p} \text{ an integer.} \tag{28b}$$

It is of great value to derive the fundamental formulas (26) and (27) by discounting each payment R/p to the present and finding the sum of the respective series. See p. 59. We shall set up the series and leave the details of summation and simplification to the reader.

If the interest is at the effective rate i, the present value is

$$A = \frac{R}{p}\left[(1+i)^{-1/p} + (1+i)^{-2/p} + \ldots + (1+i)^{-n}\right],$$

which simplifies to the value given in (26).

If the interest is at the nominal rate (j, m), the present value is

$$A = \frac{R}{p}\left[\left(1+\frac{j}{m}\right)^{-m/p} + \left(1+\frac{j}{m}\right)^{-2m/p} + \ldots + \left(1+\frac{j}{m}\right)^{-mn}\right],$$

which simplifies to the value given in (27).

Again we would advise the student, when solving annuity problems, to confine himself to the fundamental formulas. Thus, if his problem requires that he find the present value of an annuity, he should use (12), (13), (26), or (27), and then effect the necessary transformation to reduce it to the annuity symbols that will entail the least amount of labor in obtaining the numerical result.

31. Summary of ordinary annuity formulas.

S = The Amount of the Annuity.

A = The Present Value of the Annuity.

A. Annuity of annual rent R payable annually for n years.

I. At the effective rate i.

1. $$S = R\frac{(1+i)^n - 1}{i} = R \cdot s_{\overline{n}|} \tag{5}$$

2. $$A = R\frac{1-(1+i)^{-n}}{i} = R \cdot a_{\overline{n}|} \tag{12}$$

II. At the nominal rate (j, m).

1. $$S = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^{m} - 1}. \tag{8}$$

2. $$A = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{m} - 1}. \tag{13}$$

B. Annuity of annual rent R payable p times a year for n years.

I. At the effective rate i.

1. $$S = R\frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]}. \tag{18}$$

2. $$A = R\frac{1-(1+i)^{-n}}{p[(1+i)^{1/p} - 1]}. \tag{26}$$

II. At the nominal rate (j, m).

1. $$S = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{p\left[\left(1+\frac{j}{m}\right)^{m/p} - 1\right]}. \tag{22}$$

2. $$A = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{p\left[\left(1+\frac{j}{m}\right)^{m/p} - 1\right]}. \tag{27}$$

Example 1. Find the present value of an annuity of \$600 per year paid in quarterly installments for 8 years, if the interest rate is 5% converted semi-annually.

Solution. Here, $R = \$600$, $n = 8$, $p = 4$, $m = 2$, and $j = 0.05$. Substituting in (27), we have

$$A = 600\,\frac{1-(1.025)^{-16}}{4[(1.025)^{\frac{2}{4}}-1]} = 300\,\frac{1-(1.025)^{-16}}{2[(1.025)^{\frac{1}{2}}-1]}$$

$$= 300\,\frac{1-(1.025)^{-16}}{0.025}\cdot\frac{0.025}{2[(1.025)^{\frac{1}{2}}-1]}$$

$$= 300\cdot a_{\overline{16}|\cdot 025}\cdot\frac{.025}{j_2 \text{ at } .025}$$

$$= 300(13.05500266)\ (1.00621142) \quad \text{[Tables VI and X]}$$

$$= \$3{,}940.83.$$

Example 2. Solve Example 1, with the interest converted quarterly.

Solution. Here, $R = \$600$, $n = 8$, $m = p = 4$, and $j = 0.05$. Substituting in (27),

$$A = 150\,\frac{1-(1.0125)^{-32}}{0.0125} = 150\cdot a_{\overline{32}|\cdot 0125}$$

$$= 150(26.24127418) \qquad \text{[Table VI]}$$

$$= \$3{,}936.19.$$

Example 3. Solve Example 1, if the payments are made semi-annually and the interest is converted quarterly.

Solution. Here, $R = \$600$, $n = 8$, $p = 2$, $m = 4$, and $j = 0.05$. Substituting in (27), we have

$$A = 600\,\frac{1-(1.0125)^{-32}}{2[(1.0125)^{\frac{4}{2}}-1]}$$

$$= 300\,\frac{1-(1.0125)^{-32}}{(1.0125)^2-1}$$

$$= \frac{300(1-0.67198407)}{1.02515625-1} \quad \text{[Tables III and IV]}$$

$$= \frac{300(0.32801593)}{0.02515625} = \$3{,}911.74.$$

We may also solve this example by writing A in the form

$$A = 300\,\frac{1-(1.0125)^{-32}}{.0125}\cdot\frac{.0125}{(1.0125)^2-1}$$

and applying Tables VI and VII. We find

$$A = 300(26.2412\ 7418)\ (0.4968\ 9441)$$
$$= \$3911.74.$$

Exercises

1. Find the present value of an annuity of \$700 per year running for 15 years at 5% converted annually.

2. Solve Exercise 1, assuming that the interest is converted semi-annually.

3. A piece of property is purchased by paying \$1,000 cash and \$500 at the end of each year for 10 years without interest. What would be the equivalent price if it were all paid in cash at the date of purchase, assuming money is worth 5½%?

4. In order that his daughter may receive an income of \$800 payable at the end of each year for 5 years, a man buys such an annuity from an investment company. If the investment company allows 4% interest, converted annually, what sum does the man pay the company?

5. Solve Exercise 4, if the daughter is to receive \$400 at the end of each six months.

6. The beneficiary of a policy of insurance is offered a cash payment of \$10,000 or an annuity of \$750 for 20 years, the first payment to be made one year hence. Allowing interest at 3½% converted annually, which is the better option?

7. A building is leased for a term of 10 years at an annual rental of \$1,200 payable annually at the end of the year. Assuming an interest rate of 5.2% what cash payment would care for the lease for the entire term of 10 years?

8. Show that the results of Examples 5, 6, and 7 of Art. 29 are the compound amounts of the results of Examples 1, 2, and 3 respectively of Art. 31.

9. An individual made a contract with an insurance company to pay his family an annual income of \$4,000, payable in quarterly installments at the end of each quarter for 25 years. He paid for the contract in full at the time of purchase. Assuming money worth 4%, what did it cost?

10. Find the cost of the above annuity, with the interest converted quarterly.

11. Fill out the following table for the present value of an annuity of \$100 per year for 10 years, interest at 4%.

Annuity Payable	Interest Convertible		
	Annually	Semi-annually	Quarterly
Annually			
Semi-annually			
Quarterly			

12. Solve Exercise 11, with the rate 5%.

13. Derive formulas (22) and (27) from (18) and (26) respectively by using the relation $(1 + i) = (1 + j/m)^m$.

14. How much money, if deposited with a trust company paying ($j = .04$, $m = 2$), would be sufficient to provide a man with an income of \$100 a month for 25 years?

15. A house is sold "like paying rent" for \$50 a month for 12 years. What is the cash equivalent if money is worth 6%?

16. A coal mine is estimated to yield \$10,000 a year for the next 12 years. The mine is for sale. What is the present value of the total yield of the mine on a 5% basis?

17. State problems for which the following would give the answers:

$$\text{(a)}\quad S = 600.\ s_{\overline{48}|.015} \cdot \frac{1}{s_{\overline{6}|.015}}.$$

$$\text{(b)}\quad A = 200.\ a_{\overline{25}|.025} \cdot \frac{.025}{j_2 \text{ at } .025}.$$

18. A widow is to receive from a life insurance policy \$50 a month for 20 years. If money is worth 3%, what is a fair cash settlement?

19. A building and loan association accumulates its deposits at ($j = .06$, $m = 2$). If a man makes monthly deposits of \$35 each for 10 years, what sum should he have to his credit at the end of this time?

20. A man is offered a piece of property for \$10,000. He wishes to make a cash payment and semi-annual payments of \$500 for 10 years. What should be the cash payment if the seller discounts future payments at ($j = .06$, $m = 2$)?

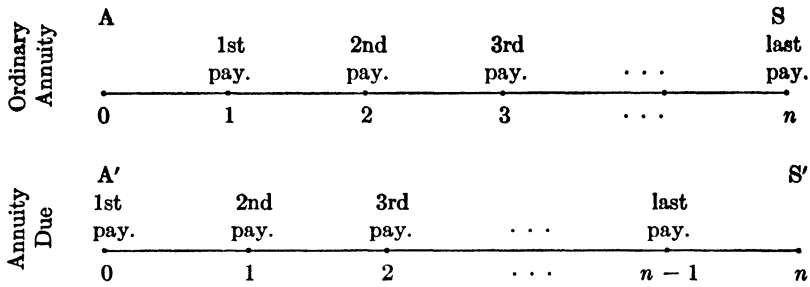
32. Annuities due.—In the previous sections of this chapter, we have been concerned with ordinary annuities,—that is, annuities in which the payments were made at the *ends* of the payment periods. An **annuity due** is one in which the payments are made at the *beginnings* of the payment periods.

The **term** of an annuity due extends from the beginning of the first payment period to the end of the last payment period. That is, it extends for one payment period after the last payment has been made. The **amount** of the annuity due is the value of the annuity at the end of the last payment period, that is, at the end of the term. The **present value** of an annuity due is the value of the annuity at the beginning of the term, or at the time of the initial payment. The present value includes the initial payment.

To solve problems involving annuities due it is neither necessary nor desirable that we invent a number of new formulas*. We can always analyze an annuity due problem in terms of ordinary annuities. It is important, however, that the student have a clear picture of the problem.

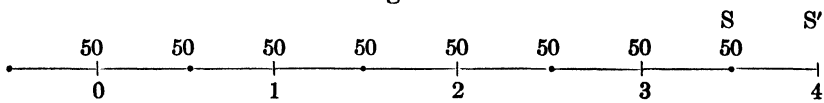
* The symbols, $s_{\overline{n}|}$ and $a_{\overline{n}|}$, in black roman type are frequently used to represent the amount and the present value of an annuity due of 1 per year for n years.

We submit the following line diagrams to assist the student in clearly understanding the similarities and the differences between ordinary annuities and annuities due.



Example 1. \$50 is deposited in a savings bank now and a like amount every six months until 8 such deposits in all have been made. How much is on deposit 4 years from now, if money accumulates at 5% converted annually?

Solution. Consider the line diagram.



First method. The amount of the annuity, S, just after the last deposit (at 3½ years) is that of an *ordinary* annuity with $R = 100$, $n = 4$, $p = 2$, $i = .05$. Using *BI*1, Art. 31, we find this amount to be

$$S = \frac{100\,[(1.05)^4 - 1]}{2\;\;[(1.05)^{\frac{1}{2}} - 1]}$$

$$= 100.\; s_{\overline{4}|.05} \cdot \frac{.05}{j_2 \text{ at } .05}$$

$$= 100(4.3101\;2500)\;(1.0123\;4754)$$

$$= 436.3344.$$

Now evidently S' is the value of S accumulated for ½ a year. Hence

$$S' = S(1.05)^{\frac{1}{2}} = 436.3344(1.0246\;9508)$$

$$= \$447.11.$$

Second method. If a deposit of \$50 had been made at the end of 4 years, the amount would have been that of an *ordinary* annuity with $R = 100$, $n = 4\frac{1}{2}$, $p = 2$, $i = .05$. Again using $BI1$, Art. 31, we find this amount to be

$$S'' = 50 \cdot \frac{(1.05)^{9/2} - 1}{(1.05)^{1/2} - 1}.$$

It is clear that the amount just before such a deposit was made, which is the amount S' that we are seeking, is

$$S' = S'' - 50 = 50 \cdot \frac{(1.05)^{9/2} - 1}{(1.05)^{1/2} - 1} - 50$$

$$= (1.05)^{1/2} \cdot \frac{50[(1.05)^4 - 1]}{(1.05)^{1/2} - 1} = (1.05)^{1/2} \cdot 100 \cdot s_{\overline{4}|} \cdot \frac{.05}{j_2}$$

$$= \$447.11.$$

Example 2. Find the amount of an annuity due of annual rent \$400 payable in quarterly installments for 8 years at 6% converted annually.

We shall leave it as an exercise for the student to show that the first method leads to the solution

$$S' = 400(1.06)^{1/4} \frac{(1.06)^8 - 1}{4[(1.06)^{1/4} - 1]}$$

$$= 400(1.06)^{1/4} \frac{(1.06)^8 - 1}{0.06} \cdot \frac{0.06}{4[(1.06)^{1/4} - 1]}$$

$$= 400(1.01467385)\ (9.89746791)\ (1.02222688) \qquad \text{[Tables VIII, V, X]}$$

$$= \$4{,}106.36.$$

Example 3. Solve Example 2 with the interest converted quarterly.

We shall leave it as an exercise for the student to show that the second method leads to the solution

$$S' = 100\left[\frac{(1.015)^{33} - 1}{0.015} - 1\right] = 100\,[s_{\overline{33}|.015} - 1]$$

$$= 100(42.29861233 - 1) \qquad \text{[Table V]}$$

$$= \$4{,}129.86.$$

Example 4. Solve Example 2 with the interest converted semi-annually. The application of the first method leads to the solution

$$S' = 400(1.03)^{\frac{1}{2}} \frac{(1.03)^{16} - 1}{4[(1.03)^{\frac{1}{2}} - 1]}$$

$$= 200(1.03)^{\frac{1}{2}} \frac{(1.03)^{16} - 1}{0.03} \cdot \frac{0.03}{2[(1.03)^{\frac{1}{2}} - 1]}$$

$$= 200(1.01488916)\ (20.1568813)\ (1.00744458) \qquad \text{[Tables VIII, V, X]}$$

$$= \$4{,}120.85.$$

Exercises

1. Set up a series for the accumulated values of the payments in Example 1 above, find the sum of the resulting geometric progression, and thus find S'.

2. Do the same for Example 3 above.

3. A man deposits \$150 in a savings bank on his twenty-fifth birthday and a like amount every six months. If the bank pays 3% interest convertible semi-annually, how much does he have on deposit on his sixtieth birthday?

4. Solve Exercise 3, with the interest converted quarterly.

5. A man, age 25, pays \$24.03 a year in advance on a \$1,000, 20-pay life policy. If he should die at the end of 12 years, just before paying the 13th premium, how much would his estate be increased by having taken the insurance instead of having deposited the \$24.03 each year in a savings bank paying 4% effective?

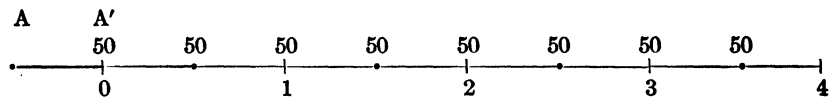
6. An insurance premium of \$48 is payable at the beginning of each year for 20 years. If the insurance company accumulates these payments at 5% converted semi-annually, find the amount of the payments at the end of the 20th year.

7. Find the amount of an annuity due of \$200 a year payable in semi-annual installments for 7 years at 4% converted annually. Solve fundamentally as a geometrical progression.

8. Solve Exercise 7, with the interest converted quarterly.

We have defined the present value of an annuity due to be the value of the annuity at the time of the initial payment. Consider the examples:

Example 1. An individual is to receive \$50 cash and a like sum every six months until 8 such payments in all have been made. What is the cash value of the payments, if money is worth 5% converted annually?



Solution. We shall solve this example by two methods.

First method. The payments constitute an *ordinary* annuity whose term begins 6 months before the present and ends at 3½ years. Its term

is therefore 4 years. We have for this annuity $R = \$100$, $n = 4$, $p = 2$, $i = .05$. Using $BI2$, Art. 31, we find

$$A = 100\,\frac{1 - (1.05)^{-4}}{2[(1.05)^{\frac{1}{2}} - 1]}$$

Evidently A', the required present value, is the value A accumulated ½ year at 5%. Hence

$$A' = (1.05)^{\frac{1}{2}}(100)\left[\frac{1 - (1.05)^{-4}}{0.05}\right]\left[\frac{0.05}{2[(1.05)^{\frac{1}{2}} - 1]}\right]$$

$$= 100(1.02469508)\ (3.54595050)\ (1.01234754)$$

$$= \$367.84.$$ [Tables VI, VIII, and X]

Second Method. If we disregard the first payment, the remaining 7 payments constitute an *ordinary* annuity whose term begins now. The value at 0 of this annuity is the present value of an ordinary annuity with $R = 100$, $n = 3½$, $p = 2$, $i = .05$. Using $BI2$, Art. 31, the value at 0 is

$$A'' = 50 \cdot \frac{1 - (1.05)^{-\frac{7}{2}}}{(1.05)^{\frac{1}{2}} - 1}.$$

Hence

$$A' = A'' + 50 = (1.05)^{\frac{1}{2}}(100)\left[\frac{1 - (1.05)^{-4}}{0.05}\right]\left[\frac{0.05}{2[(1.05)^{\frac{1}{2}} - 1]}\right]$$

$$= 100(1.02469508)\ (3.54595050)\ (1.01234754)$$

$$= \$367.84.$$ [Tables VI, VIII, and X]

Example 2. Find the present value of an annuity due of \$600 per year paid in quarterly installments for 8 years, if the interest rate is 5% converted semi-annually.

We shall leave it an exercise for the reader to show that the first method leads to the solution

$$A' = 600(1.025)^{\frac{1}{2}}\,\frac{1 - (1.025)^{-16}}{4[(1.025)^{\frac{1}{2}} - 1]}$$

$$= 300(1.025)^{\frac{1}{2}}\,\frac{1 - (1.025)^{-16}}{2[(1.025)^{\frac{1}{2}} - 1]}$$

$$= 300(1.025)^{\frac{1}{2}}\,\frac{1 - (1.025)^{-16}}{0.025} \cdot \frac{0.025}{2[(1.025)^{\frac{1}{2}} - 1]}$$

$$= 300(1.01242284)\ (13.05500266)\ (1.00621142)$$

$$= \$3{,}989.78.$$ [Tables VI, VIII, and X]

Example 3. Solve Example 2 with the interest converted quarterly.

We leave it an exercise for the reader to show that the second method leads to the solution

$$A' = 150\left[1 + \frac{1 - (1.0125)^{-31}}{0.0125}\right] = 150[1 + a_{\overline{31}|.0125}]$$

$$= 150(1 + 25.56929010) \quad \text{[Table VI]}$$

$$= 150(26.56929010) = \$3{,}985.39.$$

Example 4. Find the present value of an annuity due of \$600 per year paid in semi-annual installments for 8 years if the interest rate is 5% convertible quarterly.

An application of the first method leads to the solution

$$A' = 300(1.0125)^2 \cdot \frac{1 - (1.0125)^{-32}}{(1.0125)^2 - 1}$$

$$= 300(1.0125)^2 \cdot a_{\overline{32}|} \cdot \frac{1}{s_{\overline{2}|}} \text{ at } .0125$$

$$= \$4{,}010.15. \quad \text{[Tables III, VI and VII]}$$

Exercises

1. Set up a series for the present value of the payments in Example 1 above, find the sum of the resulting series, and thus find A'.

2. Do the same for Example 3 above.

3. A man leases a building for 4 years at a rental of \$100 a month payable in advance. Find the equivalent cash payment, if money is worth 5%.

4. A man pays \$500 cash and \$500 annually thereafter until 10 payments have been made on a house. Assuming money worth 6% converted semi-annually, what is the equivalent cash price?

5. An insurance policy provides that at the death of the insured the beneficiary is to receive \$1,200 per year for 10 years, the first payment being made at once. Assuming that money is worth 3½%, what is the value of a policy that will provide such a settlement?

6. Allowing interest at 5%, converted quarterly, what is the present cash value of a rental of \$2,000 per year, payable quarterly in advance for a period of 15 years?

7. Solve Exercise 6, with the payments made semi-annually.

8. A man deposits \$30 at the beginning of each month in a bank which pays 3% interest converted semi-annually. He makes these deposits for 120 months. What amount does he have to his credit at the end of the time.

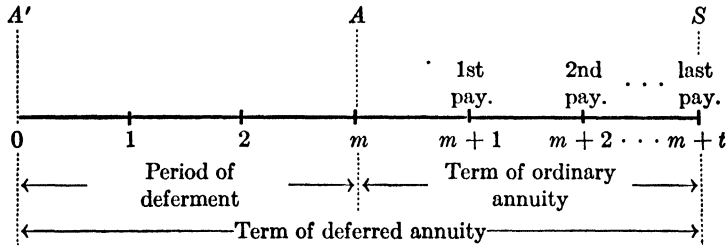
9. Prove that $a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.

10. Prove that $s_{\overline{n}|} = (1 + i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$.

33. Deferred annuities.—*A deferred annuity is one whose payments are to begin at the end of an assigned number of years or periods.* When we say that an annuity is *deferred m* payment periods, we mean that the annuity is "entered upon" at the end of m payment periods and that the first payment is made at the end of $(m + 1)$ payment periods. The m periods constitute the *period of deferment.*

The *amount* of a deferred annuity is the value of the annuity immediately after the last payment. The *present value* of a deferred annuity is the value of the annuity at the beginning of the period of deferment.

The following line diagram emphasizes the characteristics of a deferred annuity that continues t payment periods after being deferred m payment periods.

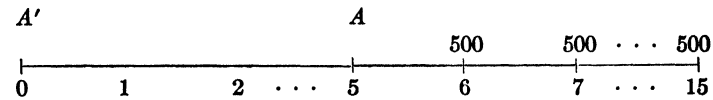


To solve problems involving deferred annuities, it is neither necessary nor desirable that we invent a number of new formulas.* Problems involving deferred annuities can always be analyzed in terms of ordinary annuities. We shall illustrate the methods of solution by a few examples.

Example 1. A young man is to receive \$500 at the end of 6 years and a like sum each year thereafter until he has received 10 payments in all. Assuming money worth 4% converted annually what is the present value of his future income?

We shall solve this problem by two methods.

Solution. First method. Consider the line diagram.



The value of the annuity at the end of 5 years is evidently

$$A = 500.\frac{1 - (1.04)^{-10}}{0.04} = 500 \cdot a_{\overline{10}|.04}.$$

* The symbols $m \mid s_{\overline{n}|}$ and $m \mid a_{\overline{n}|}$ are frequently used to represent the amount and the present value of an annuity of 1 per year for n years deferred m years.

The value at 0, which is the value we are seeking, is A discounted 5 years at 4%. Hence,

$$A' = (1.04)^{-5} A$$

$$= (1.04)^{-5}\left[500\,\frac{1-(1.04)^{-10}}{0.04}\right] = (1.04)^{-5}\cdot 500\cdot a_{\overline{10}|.04}$$

$$= 500(0.82192711)\ (8.11089578) \quad \text{[Tables IV and VI]}$$

$$= \$3{,}333.28.$$

Solution. Second method.

Imagine \$500 paid at the end of each year for the first five years. These payments together with the 10 given payments constitute an ordinary annuity of \$500 a year for 15 years. Its value at 0 is $500.\, a_{\overline{15}|.04}$. Now, if we subtract the value at 0 of the five imaginary payments, namely $500\ a_{\overline{5}|.04}$, we have

$$A' = 500\ a_{\overline{15}|.04} - 500\ a_{\overline{5}|.04} = 500\ [a_{\overline{15}|.04} - a_{\overline{5}|.04}]$$

$$= 500(11.1183\ 8743 - 4.4518\ 2233) \quad \text{[Table VI]}$$

$$= \$3{,}333.28.$$

The second method is much simpler from the standpoint of computation. The student, however, should become skilled in the use of both methods.

Example 2. Find the present value of an annuity of \$600 per year paid in quarterly installments for 8 years but deferred 5 years, assuming money worth 5% converted semi-annually.

Solution. We leave it as an exercise for the reader to show that the first method leads to

$$A' = 600(1.025)^{-10}\,\frac{1-(1.025)^{-16}}{4[(1.025)^{\frac{3}{4}}-1]}$$

$$= 300(1.025)^{-10}\,\frac{1-(1.025)^{-16}}{0.025}\cdot\frac{0.025}{2[(1.025)^{\frac{1}{2}}-1]}$$

$$= 300(0.78119840)\ (13.05500266)\ (1.00621142)$$

[Tables IV, VI and X]

$$= \$3{,}078.57.$$

Example 3. Solve Example 2 with the interest converted quarterly.

Solution. We shall leave it as an exercise for the reader to show that the second method leads to

$$A' = 150\left[\frac{1-(1.0125)^{-52}}{0.0125} - \frac{1-(1.0125)^{-20}}{0.0125}\right]$$

$$= 150\,[a_{\overline{52}|.0125} - a_{\overline{20}|.0125}]$$

$$= 150(38.06773431 - 17.59931613) \quad \text{[Table VI]}$$

$$= 150(20.46841818) = \$3{,}070.26.$$

Example 4. Find the present value of an annuity of \$600 a year paid in semi-annual installments for 8 years but deferred 5 years, assuming money is worth ($j = .05$, $m = 4$).

Solution. We leave it as an exercise for the reader to show that the first method leads to

$$A' = 600(1.0125)^{-20}\frac{1-(1.0125)^{-32}}{2[(1.0125)^{\frac{1}{2}} - 1]}$$

$$= 300(1.0125)^{-20}\frac{1-(1.0125)^{-32}}{(1.0125)^2 - 1}$$

$$= \frac{300(0.78000855)\,(1 - 0.67198407)}{1.02515625 - 1} \quad \text{[Tables III and IV]}$$

$$= \frac{300(0.78000855)\,(0.32801593)}{0.02515625} = \$3{,}051.19.$$

Remark **1.** It will be noted that we have given no examples that involve finding the *amount* of a deferred annuity. The amount of a deferred annuity is obviously the amount of an ordinary annuity to which we have already given much attention.

Remark **2.** The second method for evaluating the present value of a deferred annuity is preferable when $m = p$.

Exercises

1. If money is worth ($j = .04$, $m = 2$), find the present value of an annuity of \$1,000 a year, the first payment being due at the end of 8 years and the last at the end of 17 years.

2. Find the present value of an annuity of \$1,000 per year, payable in semi-annual installments, for 9 years but deferred 5 years assuming money worth 4% converted semi-annually. Solve by two methods.

3. Solve Exercise 2, with the interest converted annually.

4. Find the present value of an annuity of \$800 per annum paid in quarterly installments for 14 years, deferred 6 years, if money is worth 5% converted annually.

5. Solve Exercise 4, with the interest converted (a) semi-annually, (b) quarterly.

6. A will provides that a son, aged 15 years, is to receive \$1,000 when he reaches 25 and a like sum each year until he has received 15 payments in all. Assuming money worth 4% converted annually, what would be the inheritance tax of 5% on the son's share?

7. A geologist estimates that an oil well will produce a net annual income of \$50,000 for 10 years. Due to litigations the first income will not be available until the end of 4 years, but will come in at the end of each year thereafter until 10 full payments have been made. Assuming money worth 5½%, what is the present value of the well?

8. What sum should be set aside now to assure a person an income of \$150 at the end of each month for 20 years, if the income is deferred for 12 years, assuming money worth 4½% converted semi-annually?

9. A man offers to sell his farm for \$15,000 cash or \$7,500 cash and \$2,500 annually for 4 years, the first annual payment to be made at the end of 5 years. Assuming money worth 6%, what is the cash difference between the two offers?

10. What sum of money should a man set aside at the birth of his son in order to provide \$1,000 a year for 4 years to take care of the son's education, if the first installment is to be paid in 18 years? Assume 4% interest.

11. Prove: $m \mid a_{\overline{n}|i} = (1+i)^{-m} a_{\overline{n}|i}$.

12. Prove: $m \mid a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{m}|i}$.

34. Finding the interest rate of an annuity.—We may find the approximate interest rate of an annuity by the method of interpolation. This method will be sufficiently accurate for all practical purposes.

Example 1. At what rate, converted quarterly, will an annuity of \$100 per quarter amount to \$5,100 in 10 years?

Solution. Here, $R = \$400$, $m = p = 4$, $n = 10$, and $S = \$5{,}100$. Substituting in *B*III1, Art. 31, we have

$$5{,}100 = 100\,\frac{\left(1+\frac{j}{4}\right)^{40}-1}{\frac{j}{4}} = 100 s_{\overline{40}|\frac{j}{4}}.$$

Then $s_{\overline{40}|\frac{j}{4}} = 51.0000$. We now turn to Table V and follow $n = 40$ until we come to a value just less than 51.0000 and one just greater than 51.0000. We find the value 50.1668 corresponding to 1⅛% and the

value 51.4896 corresponding to 1¼%. Hence, the rate $j/4$ lies between 1⅛% and 1¼%. Interpolating, we have

$$\begin{array}{rrrr} & s_{\overline{40}|.0125} & = 51.4896 & \\ .00125 & x \quad s_{\overline{40}|\frac{j}{4}} & = 51.0000 & .8332 \quad 1.3228 \\ & s_{\overline{40}|.01125} & = 50.1668 & \end{array}$$

$$\frac{x}{.00125} = \frac{0.8332}{1.3228}, \quad x = 0.00079$$

$$\frac{j}{4} = 0.01125 + 0.00079 = 0.01204.$$

And $$j = 0.04816 = 4.816\% \text{ (approximately).}$$

This result may be checked by logarithms. We have

$$S = 100\,\frac{(1.01204)^{40} - 1}{0.01204}$$

$$\log 1.01204 = 0.0051977 \quad \text{(Table II)}$$

$$40 \log 1.01204 = 0.2079080$$

$$(1.01204)^{40} = 1.6140. \quad \text{(Table I)}$$

And $$S = 100\,\frac{(1.6140 - 1)}{0.01204}$$

$$= \frac{61.40}{0.01204} = \$5{,}099.67.$$

This result is only 33 cents less than the \$5,100 and the rate 4.816% is accurate enough. If 7 place logarithms had been used to find the antilogarithm of 0.2079080, our result would have been \$5,099.80 which differs from the \$5,100.00 by only 20 cents.

Example 2. The present value of an annuity of \$400 per annum for 20 years is \$5,000. Find the interest rate.

Solution. Here, $R = \$400$, $m = p = 1$, $n = 20$, and $A = \$5{,}000$. Substituting $AI2$, Art. 31, we have

$$5{,}000 = 400\,\frac{1 - (1 + i)^{-20}}{i} = 400\, a_{\overline{20}|}.$$

Then $$a_{\overline{20}|} = 12.5000.$$

We now turn to Table VI and follow $n = 20$, until we come to a value just greater than 12.5000 and one just less than 12.5000. We find the value

13.0079 corresponding to 4½% and 12.4622 corresponding to 5%. Hence, the rate i lies between 4½% and 5%.

Therefore,
$$i = 0.045 + \frac{0.5079}{0.5457}(0.005)$$
$$= 0.045 + 0.00465$$
$$= 0.04965 = 4.97\% \text{ (approximately).}$$

Example 3. A house is priced at \$2,500 cash or for \$50 a month in advance of 60 months. What is the effective rate of interest charged in the installment plan?

Solution. We have here an *annuity due* of 60 periods. Let us assume for convenience that the nominal rate is j converted 12 times a year. Then, $m = p = 12$, $R = \$600$, and $A' = \$2{,}500$. Substituting in $BII2$, Art. 31, we find

$$2{,}500 = 50\left(1 + \frac{1 - \left(1 + \frac{j}{12}\right)^{-59}}{\frac{j}{12}}\right)$$
$$= 50(1 + a_{\overline{59}|}).$$

Then
$$a_{\overline{59}|} = 49.0000.$$

Turning to Table VI, we find that when

$$\frac{j}{12} = \frac{7}{12}\%, \quad a_{\overline{59}|} = 49.7968;$$

when
$$\frac{j}{12} = \frac{3}{4}\%, \quad a_{\overline{59}|} = 47.5347.$$

Therefore,
$$\frac{j}{12} = 0.00583 + \frac{49.7968 - 49.0000}{49.7968 - 47.5347}(0.00167)$$
$$= 0.00583 + 0.00059 = 0.00642.$$

And
$$j = 0.07704 = 7.704\%. \quad \text{(Approximate nominal rate).}$$

Checking by logarithms as in Example 1, we find

$$A' = \$2{,}499.14,$$

which is 86 cents less than the \$2,500.00.

To find the effective rate, we have

$$(1 + i) = (1.00642)^{12} \qquad (4) \text{ Art. } 16.$$

$$\log 1.00642 = 0.0027792$$

$$12 \log 1.00642 = 0.0333504$$

$$(1.00642)^{12} = 1.07982 = (1 + i).$$

Therefore, $$i = 0.07982 = 7.982\%.$$

Exercises

1. At what rate of interest will an annuity of \$500 a year amount to \$25,000 in 25 years?

2. A house is offered for sale for \$6,000 cash or \$1,000 at the end of each year for the next 8 years. If the installment plan is used, what rate of interest is charged?

3. The cash price of an automobile is \$1,150. A man is allowed \$525 on his old car as a down payment. To care for the balance he pays \$57.20 at the end of each month for 12 months. What rate of interest is charged? Use simple interest. [See p. 34.]

4. A man deposits \$9,500 with a trust company now with the guarantee that he (or his heirs) is to receive \$1,000 each year for 25 years, the first \$1,000 to be paid at the end of 10 years. What effective rate of interest is the man allowed on his money?

35. The term of an annuity.—We illustrate by examples the method of finding the term of an annuity.

Example 1. In how many years will an annuity of \$400 per year amount to \$9,500, if the interest rate is 3½% converted annually?

Solution. Here, $R = \$400$, $S = \$9{,}500$, and $i = 0.035$. Substituting in $A11$, Art. 31, we have

$$9{,}500 = 400\,\frac{(1.035)^n - 1}{0.035} = 400 s_{\overline{n}|}.$$

And $$s_{\overline{n}|} = 9{,}500 \div 400 = 23.7500.$$

We now turn to Table V and follow down the 3½% column. We notice that when

$$n = 17, \quad s_{\overline{n}|} = 22.7050;$$

$$n = 18, \quad s_{\overline{n}|} = 24.4997.$$

It is evident that 18 payments of \$400 will amount to more than \$9,500. In fact, it will amount to \$9,799.88, which is \$299.88 more than is needed. Hence, \$400 per year for 17 years and

$$\$100.12, (\$400.00 - \$299.88)$$

at the end of 18 years will amount to exactly \$9,500.

Example 2. An individual buys a house for \$5,000 paying \$1,000 in cash. He agrees to pay the balance in installments of \$500 at the end of each year. How long will it take to pay the \$4,000 and interest at 6% converted annually?

Solution. Here, $A = \$4{,}000$, $R = \$500$, $m = p = 1$, and $i = 0.06$. Substituting in $A12$, Art. 31, we have

$$4{,}000 = 500\frac{1 - (1.06)^{-n}}{0.06} = 500a_{\overline{n}|}.$$

And $$a_{\overline{n}|} = 8.0000.$$

We now turn to Table VI and follow down the 6% column. We notice that when

$$n = 11,\ a_{\overline{n}|} = 7.8869$$

$$n = 12,\ a_{\overline{n}|} = 8.3838.$$

Hence, the present value of 11 payments is less than \$4,000 and the present value of 12 payments is more than \$4,000. Then, it is evident that the debtor must make 11 full payments of \$500 each and a 12th payment, at the end of 12 years, which is less than \$500.

If no payments were made, the original principal of \$4,000 would accumulate in 11 years to

$$4{,}000(1.06)^{11} = 4{,}000(1.89829856) = \$7{,}593.19.$$

However, if payments of \$500 are made regularly for 11 years, they will accumulate to

$$500\frac{(1.06)^{11} - 1}{0.06} = 500(14.97164264) = \$7{,}485.82$$

Hence, just after the 11th payment, the balance on the principal is \$7,593.19 − \$7,485.82 = \$107.37. That is, the debt could be cancelled by making an additional payment of \$107.37 along with the 11th regular payment. However, if the balance is not to be paid until the end of the 12th year, the payment would be \$107.37 plus interest on it for 1 year at 6%, or \$107.37 + \$6.44 = \$113.81. Then, 11 payments of \$500 and a partial payment of \$113.81 made at the end of 12 years will settle the debt.

Exercises

1. In how many years will an annuity of \$750 amount to \$10,000 if interest is at 6½%? Solve by interpolation.

2. Solve formula *AI*1, Art. 31, for n.

$$n = \frac{\log (iS + R) - \log R}{\log (1 + i)}.$$

3. Solve Exercise 1 by the formula given in Exercise 2.

4. A man borrows \$3,000 and desires to repay principal and interest in installments of \$400 at the end of each year. Find the number of full payments necessary and the size of the partial payment, if it is made 1 year after the last full payment is made, assuming an interest rate of 5%.

5. A man deposits \$15,000 in a trust fund with the agreement that he is to receive \$2,000 a year, beginning at the end of 10 years, until the fund is exhausted. If the trust company allows him 4% interest on his deposits, how many full payments of \$2,000 will be paid and what will be the fractional payment paid at the end of the next year?

36. Finding the periodic payment.—In the early sections of this chapter we solved the problems of finding the amount and the present value of an annuity under given conditions when the periodic payment was known. Our results were summarized in Art. 31.

We are now about to attack the inverse problem, that of finding the periodic payment under given conditions, when the amount or the present value of the annuity is known. The solution requires no new formulas. We must merely solve the equations of Art. 31 for R or R/p according as the annuity is payable annually or p times a year. Consider the following examples.

Example 1. A man buys a house for \$6,000 and pays \$1,000 in cash. The remainder with interest is to be paid in 40 equal quarterly payments, the first payment being due at the end of three months. Find the quarterly payment if the interest rate is 6% converted annually.

Solution. Here, $A = \$5{,}000$, $n = 10$, $p = 4$, $i = 0.06$. Substituting in Art. 31, *BI*2, and solving for R, we have

$$R = 5{,}000\,\frac{4[(1.06)^{1/4} - 1]}{1 - (1.06)^{-10}}$$

$$= \frac{5{,}000(0.05869538)}{1 - 0.55839478} \quad \text{[Tables IV and IX]}$$

$$= \$664.57, \text{ annual payment.}$$

Then, $\dfrac{R}{4} = \$166.14$, quarterly payment.

Example 2. Solve Example 1, with the interest converted semi-annually.

Solution. Using *B*II2, Art. 31, we have

$$R = 5{,}000\,\frac{4[(1.03)^{\frac{1}{2}} - 1]}{1 - (1.03)^{-20}}$$

$$= \frac{10{,}000\ 2[(1.03)^{\frac{1}{2}} - 1]}{1 - (1.03)^{-20}}$$

$$R = \frac{10{,}000(0.02977831)}{1 - 0.55367575} \quad \text{[Tables IV and IX]}$$

$$= \frac{10{,}000(0.02977831)}{0.44632425} = \$667.19.$$

Then, $\frac{R}{4} = \$166.80$, quarterly payment.

Example 3. Solve Example 1, with the interest converted quarterly.

Solution. Here, $m = p = 4$, and *B*II2, Art. 31 gives us

$$\frac{R}{4} = 5{,}000\,\frac{0.015}{1 - (1.015)^{-40}} = 5{,}000 \cdot \frac{1}{a_{\overline{40}|\cdot 015}}$$

$$= 5{,}000(0.03342710) \quad \text{[Table VII]}$$

$$= \$167.14, \text{ quarterly payment.}$$

Example 4. How much must be set aside semi-annually so as to have \$10,000 at the end of 10 years, interest being at the rate of 5% converted annually?

Solution. Here, $S = \$10{,}000$, $n = 10$, $p = 2$, $i = 0.05$. Substituting in *B*II1, Art. 31, we have

$$R = 10{,}000\,\frac{2[(1.05)^{\frac{1}{2}} - 1]}{(1.05)^{10} - 1}$$

$$= \frac{10{,}000(0.04939015)}{1.62889463 - 1} \quad \text{[Tables III and IX]}$$

$$= \$785.35, \text{ annual payment.}$$

Then, $\frac{R}{2} = \$392.67$, semi-annual payment.

Example 5. Solve Example 4, with the interest converted semi-annually.

Solution. Here, $m = p = 2$, the other conditions being the same as in Example 4. Substituting in *B*III1, Art. 31, we have

$$\frac{R}{2} = 10{,}000\,\frac{0.025}{(1.025)^{20} - 1} = 10{,}000 \cdot \frac{1}{s_{\overline{20}|.025}}$$

$$= 10{,}000(0.06414713 - 0.025) \quad \text{[Table VII]}$$

$$= 10{,}000(0.03914713) = \$391.47.$$

Example 6. Solve Example 4, with the interest converted quarterly.

Solution. Here, $m = 4$, the other conditions being the same as in Example 4. Substituting in *B*III1, Art. 31, we have

$$R = 10{,}000\,\frac{2[(1.0125)^{\frac{1}{2}} - 1}{(1.0125)^{40} - 1}$$

$$= 20{,}000\,\frac{(1.0125)^2 - 1}{0.0125} \cdot \frac{0.0125}{(1.0125)^{40} - 1}$$

$$= 20{,}000(2.01250000)\ (0.01942141) \quad \text{[Tables V and VII]}$$

$$= \$781.71, \text{ annual rent.}$$

Then, $\frac{R}{2} = \$390.86$, semi-annual payment.

Exercises

1. A man buys a farm for \$10,000. He pays \$5,000 cash and arranges to pay the balance with 5% interest converted semi-annually, by making equal payments at the end of each six months for 14 years. How much is the semi-annual payment?

2. How much must be set aside annually to accumulate to \$5,000 in 8 years, if money is worth 4½% converted semi-annually?

3. Solve Exercise 2, with the interest converted (a) annually, (b) quarterly.

4. In order to finance a school building costing \$100,000 a city issues 20-year bonds which pay 5% interest, payable semi-annually. How much must be deposited, at the end of each six months, in a sinking fund which accumulates at 4½% converted semi-annually, if the bonds are to be redeemed in full at the end of 20 years? What total semi-annual payment is necessary to pay the interest on the bonds and make the sinking fund payment?

5. Solve Exercise 4, with the interest on the sinking fund converted quarterly.

6. At the maturity of a \$20,000 endowment policy, the policyholder may take the full amount in cash or leave the full amount with the insurance company to be paid to him in 40 equal quarterly payments, the first payment to be made at the end of three months. If 4% interest, converted quarterly, is allowed on all money left with the company, how much is the quarterly payment?

7. A building is priced at \$25,000 cash. The owner agrees to accept \$5,000 cash and the balance, principal and interest, in equal annual payments for 15 years. If the interest rate is 7% effective, what is the annual payment?

8. Solve Exercise 7, with the interest converted quarterly.

37. Perpetuities and capitalized cost.—*An annuity whose payments continue forever is defined as a perpetuity.* It is evident that the *amount* of such an annuity increases indefinitely, but the present value is definite. The symbol A_∞, will denote the present value of a perpetuity of R dollars per annum, payable annually.

It is evident that the interest on A_∞ for one year at nominal rate (j, m) must equal R.

Hence,
$$A_\infty \left(1 + \frac{j}{m}\right)^m - A_\infty = R,$$

and
$$A_\infty = \frac{R}{\left(1 + \frac{j}{m}\right)^m - 1},$$

or
$$A_\infty = \frac{R}{\frac{j}{m}} \cdot \frac{\frac{j}{m}}{\left(1 + \frac{j}{m}\right)^m - 1} = \frac{Rm}{j} \cdot \frac{1}{s_{\overline{m}|\frac{j}{m}}}. \tag{29}$$

When $m = 1, j = i$, and (29) reduces to

$$A_\infty = \frac{R}{i}. \tag{29'}$$

Example 1. Find the present value of a perpetuity of \$500 per annum, if money is worth 4% converted annually.

Solution. Here, $R = \$500$, $i = 0.04$.

Then,
$$A_\infty = \frac{500}{0.04} = \$12{,}500. \quad \text{[Formula (29')]}$$

Example 2. Solve Example 1, with the interest converted quarterly.

Solution. Here, $R = \$500$, $j = 0.04$, and $m = 4$. Substituting in (29), we have

$$A_\infty = \frac{500}{0.01} \cdot \frac{0.01}{(1.01)^4 - 1} = 50{,}000 \frac{0.01}{(1.01)^4 - 1} = 50{,}000 \cdot \frac{1}{s_{\overline{4}|.01}}$$

$$= 50{,}000(0.24628109) \quad \text{[Table VII]}$$

$$= \$12{,}314.05.$$

There are times when a perpetuity must provide for payments at intervals longer than a conversion period. The symbol $A_{\infty, r}$, will denote the present value of a perpetuity of C dollars payable every r years.

It is evident that the compound interest on $A_{\infty, r}$, for r years at rate j converted m times a year must equal C.

Hence,
$$A_{\infty, r}\left(1 + \frac{j}{m}\right)^{mr} - A_{\infty, r} = C,$$

and
$$A_{\infty, r} = \frac{C}{\left(1 + \frac{j}{m}\right)^{mr} - 1},$$

or
$$A_{\infty, r} = \frac{C}{\frac{j}{m}} \cdot \frac{\frac{j}{m}}{\left(1 + \frac{j}{m}\right)^{mr} - 1} = \frac{Cm}{J} \cdot \frac{1}{s_{\overline{mr}|\frac{j}{m}}}. \tag{30}$$

If $m = 1$, $j = i$, and

$$A_{\infty, r} = \frac{C}{i} \cdot \frac{1}{s_{\overline{r}|i}} \tag{30'}$$

Example 3. What is the present value of a perpetuity of \$2,000 payable every 4 years, if money is worth 5% converted annually?

Solution. Here, $C = \$2{,}000$, $r = 4$, $i = 0.05$. Substituting in (30'), we have

$$A_{\infty, 4} = \frac{2{,}000}{0.05} \cdot \frac{0.05}{(1.05)^4 - 1} = 40{,}000 \cdot \frac{1}{s_{\overline{4}|.05}}$$

$$= 40{,}000(0.23201183) \quad \text{[Table VII]}$$

$$= \$9{,}280.47.$$

Example 4. Solve Example 3, with the interest converted semi-annually.

Solution. Here, $C = \$2{,}000$, $j = 0.05$, $m = 2$, and $r = 4$. We have

$$A_{\infty,4} = \frac{2{,}000}{0.025} \cdot \frac{0.025}{(1.025)^8 - 1} \quad \text{[Formula (30)]}$$

$$= 80{,}000 \frac{0.025}{(1.025)^8 - 1} = 80{,}000 \cdot \frac{1}{s_{\overline{8}|.025}}$$

$$= 80{,}000(0.11446735) = \$9{,}158.39. \quad \text{[Table VII]}$$

Example 5. A section of city pavement costs \$50,000. Its life is 25 years. Find the amount of money required to build it now and to replace it every 25 years, indefinitely, if money is worth 4% converted annually.

Solution. It is evident that the amount required to replace the pavement indefinitely is the present value of a perpetuity of \$50,000 payable every 25 years at 4%.

$$\therefore A_{\infty,25} = \frac{50{,}000}{0.04} \cdot \frac{0.04}{(1.04)^{25} - 1} \quad \text{[Formula (30')]}$$

$$= 1{,}250{,}000(0.02401196) \quad \text{[Table VII]}$$

$$= \$30{,}014.95.$$

Hence, the amount required to build the pavement plus the amount to replace it indefinitely equals

$$\$50{,}000 + \$30{,}014.95 = \$80{,}014.95.$$

This amount is called the *capitalized cost.* That is, *the capitalized cost is the first cost plus the present value of a perpetuity required to renew the project indefinitely.*

If we let K stand for the capitalized cost of an article whose first cost is C, and which must be renewed every r years at the cost C, we have

$$K = C + \frac{C}{\frac{j}{m}} \cdot \frac{\frac{j}{m}}{\left(1 + \frac{j}{m}\right)^{mr} - 1}$$

$$= C + \frac{C}{\left(1 + \frac{j}{m}\right)^{mr} - 1} = C \frac{\left(1 + \frac{j}{m}\right)^{mr}}{\left(1 + \frac{j}{m}\right)^{mr} - 1}$$

$$= \frac{C}{\frac{j}{m}} \frac{\frac{j}{m}}{1 - \left(1 + \frac{j}{m}\right)^{-mr}} = \frac{Cm}{j} \cdot \frac{1}{a_{\overline{mr}|\frac{j}{m}}}. \tag{31}$$

If $m = 1, j = i$,

then
$$K = \frac{C}{i} \cdot \frac{1}{a_{\overline{n}|i}} \tag{31'}$$

Example 6. An automobile costs \$1,000 and will last 7 years when it must be replaced at the same cost. Another automobile, which would serve the same purpose and would last 10 years, could be purchased. What could one afford to pay for the second automobile if it is to be as economical in the long run as the first, assuming money worth 5%?

Solution. When somebody says that a certain article is just as economical (cheap) in the long run as another article, he simply means that the two articles have the same capitalized cost.

The first automobile has a capitalized cost of

$$\frac{1{,}000}{0.05} \cdot \frac{0.05}{1 - (1.05)^{-7}}. \quad \text{[Formula (31')]}$$

If we let x stand for the cost of the second automobile, it will have a capitalized cost of

$$\frac{x}{0.05} \cdot \frac{0.05}{1 - (1.05)^{-10}}. \quad \text{[Formula (31')]}$$

Assuming that the two automobiles are equally economical, we have

$$\frac{x}{0.05} \cdot \frac{0.05}{1 - (1.05)^{-10}} = \frac{1{,}000}{0.05} \cdot \frac{0.05}{1 - (1.05)^{-7}}$$

and
$$x = 1{,}000 \frac{1 - (1.05)^{-10}}{0.05} \cdot \frac{0.05}{1 - (1.05)^{-7}}$$

$$= 1{,}000(7.72173493)\ (0.17281982) \quad \text{[Tables VI, VII]}$$

$$= \$1{,}334.47.$$

That is, one can afford to pay \$1,334.47 for the automobile that lasts 10 years, or \$334.47 more, for the additional 3 years of service.

We shall now find the additional cost w required to increase the life of a given article x years assuming money worth $i\%$.

Let C = original cost of an article to last n years. Its capitalized cost is

$$\frac{C}{i} \cdot \frac{i}{1 - (1 + i)^{-n}}.$$

Let $C + w$ = cost of an article to last $n + x$ years. Its capitalized cost is

$$\frac{C+w}{i} \cdot \frac{i}{1-(1+i)^{-(n+x)}}$$

Equating capitalized costs, we have

$$\frac{C+w}{i} \cdot \frac{i}{1-(1+i)^{-(n+x)}} = \frac{C}{i} \cdot \frac{i}{1-(1+i)^{-n}}.$$

The student may solve the above equation for w and get

$$w = C \frac{i}{(1+i)^x - 1} \cdot \frac{1-(1+i)^{-x}}{i}.$$

$$= C \frac{a_{\overline{x}|i}}{s_{\overline{n}|i}} \tag{32}$$

If the interest rate is j converted m times per year, (32) may be written

$$w = C \frac{\frac{j}{m}}{\left(1+\frac{j}{m}\right)^{mn} - 1} \cdot \frac{1-\left(1+\frac{j}{m}\right)^{-mx}}{\frac{j}{m}} \tag{32'}$$

Example 7. A cross tie costs \$1.00 and will last 10 years. The life of the tie can be extended to 18 years by treating with creosote. If money is worth 5%, how much could one afford to spend for the treatment?

Solution. Here, $C = \$1.00$, $n = 10$, $x = 8$, and $i = 0.05$. From (32) we have

$$w = 1.00 \cdot a_{\overline{8}|.05} \cdot \frac{1}{s_{\overline{10}|.05}}$$

$$= (0.07950458)\ (6.46321276) \qquad \text{[Tables VI, VII]}$$

$$= \$0.51.$$

That is, 51¢ could profitably be spent to treat the tie, if the service life would be extended 8 years.

Exercises

1. What amount would a railroad company be justified in expending per tie to extend the life of cross ties costing \$1.50 each from 12 to 20 years, money being worth 4%?

2. A hospital receives an annual income of \$120,000 as a perpetuity from a trust fund. What is the value of this perpetuity, money being worth 5% effective?

3. Solve Exercise 2, if the interest rate were 5% converted quarterly.

4. A railroad company has been paying a watchman \$1,600 a year to guard a crossing. The company decides to build an overhead crossing at a cost of \$22,000. If the overhead crossing must be rebuilt every 35 years at the same cost, how much does the company save by building it? Assume money worth 5%.

5. An office building is erected at a cost of \$100,000. It requires a watchman at an annual salary of \$1,500, and \$4,000 for repairs and renovation every 8 years. It must be rebuilt every 80 years at the original cost. How much money is required now to provide for its construction, maintenance, guarding and rebuilding, assuming money worth 3%? (Hint: Every 80 years when the building is rebuilt, the \$4,000 allowed for repairs and renovation is not needed. This amount may be applied on the \$100,000 for rebuilding, thereby reducing it to \$96,000.)

6. A state highway commission has a certain road graded and ready for surfacing. It may be graveled at a cost of \$2,000 per mile, or paved at a cost of \$10,000 per mile. It will cost \$200 per year to maintain the gravel road and it will need regraveling every 8 years at the original cost. The maintenance cost of the pavement is negligible and it will need repaving only every 40 years at the original cost. If the cost for clearing the road bed of the old paving is \$1,000, which type of road is more economical, assuming that the state can borrow money at 4%?

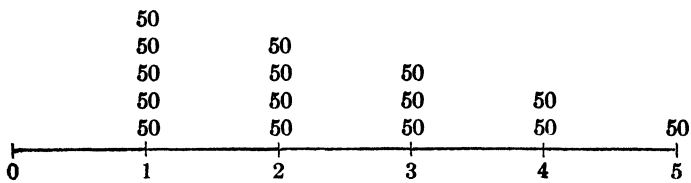
38. Increasing and decreasing annuities.—A sequence of periodic payments in which each payment exceeds by a fixed amount the preceding payment is called an *increasing annuity*. If each payment is less by a fixed amount than the preceding, the sequence is called a *decreasing annuity*.

Consider the following examples.

Example 1. Find the amount and the present value of a decreasing annuity with payments of \$250, \$200, \$150, \$100, \$50, at the ends of the next five years if money is worth 4%.

Solution. Here is the picture.

These payments are equivalent to the following five ordinary annuities, superimposed: (1) \$50 a year for 5 years; (2) \$50 a year for 4 years; (3) \$50 a year for 3 years; (4) \$50 a year for 2 years; (5) \$50 a year for 1 year. These annuities are exhibited in the following diagram.



To find the amount of a decreasing annuity, we first find its present value. The present value of the given decreasing annuity is

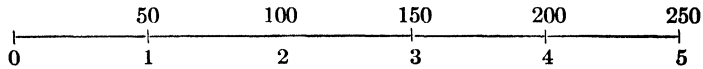
$$\begin{aligned} A &= 50\, a_{\overline{1}|} + 50\, a_{\overline{2}|} + 50\, a_{\overline{3}|} + 50 a_{\overline{4}|} + 50\, a_{\overline{5}|} \\ &= 50[a_{\overline{1}|} + a_{\overline{2}|} + a_{\overline{3}|} + a_{\overline{4}|} + a_{\overline{5}|}] \\ &= 50\left[\frac{5 - a_{\overline{5}|.04}}{.04}\right] \quad \text{Exercise 8, Art. 28} \\ &= 1250(5 - 4.4518\ 2233) \\ &= \$685.222. \end{aligned}$$

The amount of this annuity is clearly

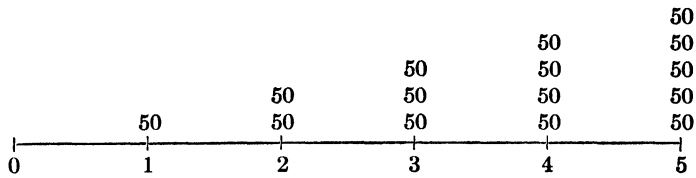
$$\begin{aligned} S &= A(1.04)^5 = 685.222(1.2166\ 5290) \\ &= \$833.68. \end{aligned}$$

Example 2. Find the amount and the present value of an increasing annuity with payments of \$50, \$100, \$150, \$200, \$250 at the ends of the next five years if money is worth 4%.

Solution. Here is the picture.



These payments are equivalent to five ordinary annuities that we exhibit in the following diagram.



The amount of this increasing annuity is

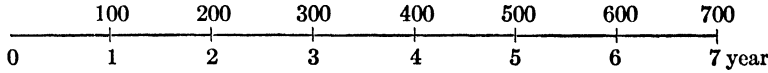
$$\begin{aligned} S &= 50\, s_{\overline{1}|} + 50\, s_{\overline{2}|} + 50\, s_{\overline{3}|} + 50\, s_{\overline{4}|} + 50\, s_{\overline{5}|} \\ &= 50[s_{\overline{1}|} + s_{\overline{2}|} + s_{\overline{3}|} + s_{\overline{4}|} + s_{\overline{5}|}] \\ &= 50\left[\frac{(1.04)\ s_{\overline{5}|.04} - 5}{.04}\right] \quad \text{Exercise 7, Art. 28} \\ &= 1250[(1.04)\ (5.4163\ 2256) - 5] \\ &= \$791.22. \end{aligned}$$

The present value of this increasing annuity is clearly

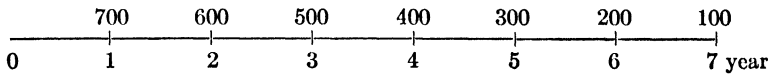
$$A = S(1.04)^{-5} = 791.22(0.8219\ 2711)$$
$$= \$650.33.$$

Exercises

1. If money is worth 5%, find the amount and the present value of the increasing annuity pictured on the diagram.



2. If money is worth 5%, find the amount and the present value of the decreasing annuity pictured on the diagram.



Problems

1. Show that formulas (12), (13), and (26) are special cases of formula (27). Follow method on page 76.

2. In order to accumulate \$20,000 in 14 years, how much must be deposited in a savings bank at the end of each year, if the interest is converted annually at 4%?

3. An automobile is bought for \$400 cash and \$62 a month for 15 months. What is the equivalent cash price if money is worth 7% converted monthly?

4. Find the rate of interest if an annuity of \$700 a year amounts in 15 years to \$15,000.

5. The proceeds of a \$5,000 insurance policy is to be paid in monthly installments of \$50 each. If money is worth 5% converted monthly, find the number of monthly payments. The first payment is made at the end of the first month.

6. A man buys a house for \$8,000, paying \$2,000 cash. He arranges for the balance, principal, and interest at 6%, to be paid in 60 monthly installments. Find the size of each installment if the interest is converted monthly.

7. A son is to receive \$1,000 a year for 12 years, the first payment being due 6 years hence. Find the present value of the son's share assuming 5% interest converted annually.

8. The beneficiary of an insurance policy is offered \$15,000 in cash or equal annual payments for 12 years, the first payment being due at once. Find the size of the annual payments if money is worth 4%.

9. A wooden bridge costs for construction \$22,500, and requires rebuilding every 20 years. How much additional money can be profitably expended for the erection of a concrete bridge instead, if money is worth 5% and the service life is extended to 40 years?

10. A building costs \$40,000 and has a life of 50 years. If it requires \$2,000 every 5 years for upkeep, what endowment should be provided at the time it is built to construct it, rebuild it every 50 years and provide for its upkeep? At the end of every 50 years the \$2,000 allowed for upkeep may be applied towards the reconstruction cost. Assume money worth 4%.

11. How much can a railroad company afford to pay to abolish a grade crossing which is guarded at a cost of \$1,000 per year, when money is worth 5% converted semi-annually?

12. A certain machine costs \$2,000 and must be replaced every 12 years at the same cost. A certain device may be added to the machine which will double its output, but the machine must then be replaced every 10 years. Assuming money worth 4%, what is the value of the device?

13. \$100 is deposited in a savings bank at the end of every six months for 10 years. During the first 6 years 3%, converted semi-annually, was allowed but during the last 4 years the rate was reduced to 2½%, converted semi-annually. Find the amount on deposit at the end of 10 years.

14. Derive formula (32), Art. 37.

15. An income of \$10,000 at the end of each year is equivalent to what income at the end of every 5 years, assuming money worth 5% converted semi-annually?

16. Solve Exercise 15, with the interest converted (a) annually, (b) quarterly.

17. A building has just been completed at a cost of \$250,000. It is estimated that \$2,500 will be needed at the end of every two years for repairs, and that every 15 years there must be renovation to the extent of \$10,000, and that the building will have a service life of 60 years with a salvage value of \$20,000. Find what equal annual amount should be set aside at 4% interest to cover repairs, renovations, and replacements. How should the \$2,500 repair fund and the \$10,000 renovation fund be used at the end of every 60 years?

18. A man borrowed \$10,000 with the understanding that it be repaid by 20 equal annual installments including principal and interest at 6% annually. Just after the 10th equal annual payment had been made the creditor agreed to reduce the principal by \$1,000 and reduce the rate to 4½%. Find the annual payment for the first 10 years and the annual payment for the last 10 years.

19. A mortgage for \$5,000 was given with the understanding that it might be repaid, principal and interest, by 15 equal annual payments. Find the annual payment if the interest rate was 7% for the first 8 years and 5% for the last 7 years.

20. A person pays \$12,500 into a trust fund now with the guarantee that he or his heirs will receive equal annual payments for 30 years, the first payment to be made at the end of 7 years. If the trust fund draws 4% interest, find the equal annual payment.

21. A perpetuity of \$25,000 a year is divided between a man's daughter and a university. The daughter receives the entire income until she has received as her share one half the present value of the perpetuity. Find the number of full payments she receives and the size of the last payment, if money is worth 5% converted annually.

22. A man pays \$6,000 into a trust fund and receives \$500 at the end of each year for 20 years. What rate of interest converted annually did he earn on his money?

23. At his son's birth a father set aside a sum sufficient to pay the boy \$1,000 a

year for 7 years, the first \$1,000 to be paid on his 18th birthday. What sum was set aside, if money was worth 4% converted semi-annually?

24. The amount of an annuity of \$800 per year is \$20,000 and the present value is \$9,235. Find the rate of interest.

25. A man buys a piano for \$300 and pays \$50 cash. The balance is to be paid for at \$12.50 at the end of each month for 24 months. What effective rate of interest does the purchaser pay? (Hint: Assume that the interest is converted monthly and find the nominal rate. Then find the effective rate.)

26. Is it economical to replace a machine which costs \$500 and lasts 8 years by one that costs \$650 and lasts 12 years? Assume that the annual running expense of each machine is the same and that money is worth 5%. Also assume that the two machines have the same output.

27. A person considers replacing a machine which costs \$400 and lasts 6 years by a machine which costs \$750 and answers the same purpose as the other machine. If the exchange is to be economical, how long should the new machine last? Assume that the annual running expense is the same for each machine and that money is worth 4%.

28. Derive formula (29′) by setting up a series and finding its sum. From (29′) derive (29).

29. Derive formula (30′) by setting up a series and finding its sum. From (30′) derive (30).

30. Derive formula (29′) from (5) by showing that $\lim_{n \to \infty} R\, a_{\overline{n}|i} = R/i$.

31. If money is worth ($j = .04$, $m = 2$), find the present value of the decreasing annuity: \$5,000, \$4,500, · · · \$500 payable semi-annually.

32. If money is worth ($j = .04$, $m = 2$), find the present value of the increasing annuity: \$500, \$1,000, · · · \$5,000 payable semi-annually.

33. State a problem for which the answer would be

$$1000\,\frac{(1.01)^{40} - 1}{(1.01)^4 - 1}.$$

34. State a problem for which the answer would be

$$500\,\frac{1 - (1.025)^{-30}}{(1.025)^2 - 1}.$$

35. State a problem for which the answer would be

$$500\,\frac{(1.02)^{24} - 1}{(1.02)^2 - 1}\,(1.02)^{10}.$$

36. In an increasing annuity, R is paid at the end of the first year, $2R$ at the end of the second year, and so on for n years. Show that

$$S = \frac{R}{i}\,[s_{\overline{n}|}(1 + i) - n],$$

$$A = \frac{R}{i}\,[(1 + i)a_{\overline{n}|} - n(1 + i)^{-n}].$$

37. In a decreasing annuity nR is paid at the end of the first year, $(n-1)R$ at the end of the second year, and so on for n years. Show that

$$S = \frac{R}{i}[n(1+i)^n - s_{\overline{n}|}],$$

$$A = \frac{R}{i}[n - a_{\overline{n}|}].$$

Review Problems *

1. \$1,000 Harrisburg, Pennsylvania, July 12, 1945.

Four months after date I promise to pay Joe Brown, or order, one thousand dollars with interest from date at 5%.

(Signed) JOHN JONES.

(*a*) Three months after date Brown sold the note to Bank B who discounted the note at 6% discount rate. What did Brown receive for the note?

(*b*) Immediately after purchasing the above note, Bank B sold the note to a Federal Reserve Bank at a re-discount rate of 4%. How much did Bank B gain on the transaction?

2. Same note as in Problem 1. Would it have been to Brown's advantage to have sold the note to friend C, to whom money was worth 6%, rather than to Bank B?

3. I bought a bill of lumber from the Jones Lumber Company who quoted the terms "net 60 days or 2% off for cash." What nominal rate of interest, j_6, could I afford to pay to borrow money to take advantage of the discount? What effective rate?

4. A note for \$1,000 with interest at $(j = .06, m = 2)$, and another for \$800 with interest at $(j = .05, m = 2)$, both due in 3 years, were purchased to net 7% effective. How much was paid for them?

5. A bank pays 4% interest on time deposits and loans money at 6% discount rate. What is the annual profit on time deposits amounting to \$100,000?

6. The Jones Lumber Company estimates that they can earn 3% a month on their money. If I buy a \$1,000 bill of lumber from them, what amount of discount can they afford to offer me to encourage immediate settlement in lieu of \$1,000 at the end of the month? What is the nominal rate of discount, f_{12}, that they can afford to offer?

7. A son is now 10 years old. The father wishes to provide now for the college and professional education of the son by depositing the proper amount with a trust company that pays $(j = .04, m = 2)$ on funds. It is estimated that the son will need \$1,000 a year for 7 years, the first payment to be made when the son is 18 years of age. Find the amount of the deposit.

8. A man owes a \$6,000 balance on a home. The balance is at $(j = .06, m = 2)$. The man agrees to pay the balance with payments of \$300 at the end of each half year. After how many payments will the balance be paid in full? What is the amount of the final partial payment?

9. A man at the age of 50 invests \$20,000 in an annuity payable to him if living (to his estate if he is dead) in equal monthly installments over a period of 15 years, the first installment to be due at the end of the first month after he reaches 65. On a 3½% basis, what is the monthly installment that he receives?

10. A man bought a refrigerator for \$250 paying \$50 down and the balance in 12 monthly installments of \$20 each. What rate of interest does the purchaser pay? [Use simple interest.]

* For additional review problems, see end of this book.

CHAPTER IV

SINKING FUNDS AND AMORTIZATION

39. Sinking funds.—When an obligation becomes due at some future date, it is usually desirable to provide for its payment by accumulating a fund with periodic contributions, together with interest earnings. *Such an accumulated fund is called a sinking fund.*

Example. A debt of \$6,000 is due in 5 years. A sinking fund is to be accumulated at 5%. What sum must be deposited in the sinking fund at the end of each year to care for the principal when due?

Solution. Here, $S = \$6{,}000$, $n = 5$, and $i = 0.05$. Since $m = p = 1$, we have from $A11$, Art. 31,

$$R = 6{,}000\,\frac{0.05}{(1.05)^5 - 1} = 6{,}000 \cdot \frac{1}{s_{\overline{5}|.05}}$$

$$= 6{,}000\ (0.18097480)\ \text{[Table VII]}$$

$$= \$1{,}085.85.$$

The amount in the sinking fund at any particular time may be shown by a schedule known as an *accumulation schedule.* The following is the schedule for the above problem:

Years	Annual Deposit	Interest on Fund	Total Annual Increase	Value of Fund at End of Each Year
1	\$1,085.85		\$1,085.85	\$1,085.85
2	1,085.85	\$ 54.29	1,140.14	2,225.99
3	1,085.85	111.30	1,197.15	3,423.14
4	1,085.85	171.16	1,257.01	4,680.15
5	1,085.85	234.01	1,319.86	6,000.01

40. Amortization.—Instead of leaving the entire principal of a debt standing for the term to be cancelled by a sinking fund, we may consider any payment over what is needed to pay interest on the principal to be

applied at once toward liquidation of the debt. As the debt is being paid off, a smaller amount goes towards the payment of interest, so that with a uniform payment per year, a greater amount goes towards the payment of principal. This method of extinguishing a debt is called the method of *amortization of principal.*

Example. Consider a debt of \$2,000 bearing 6% interest converted annually. It is desired to repay this in 8 equal annual installments, including interest. Find the annual installment.

Solution. Here, $A = \$2{,}000$, $n = 8$, $i = 0.06$, $m = p = 1$. Substituting in $A12$, Art. 31, we have

$$R = 2{,}000\,\frac{0.06}{1 - (1.06)^{-8}} = 2{,}000 \cdot \frac{1}{a_{\overline{8}|.06}}$$

$$= 2{,}000\ (0.16103594)\ \text{[Table VII]}$$

$$= \$322.07.$$

The interest for the first year will be \$120; hence \$202.07 of the first payment would be used for the reduction of principal, leaving \$1,797.93 due on principal at the beginning of the second year. The interest on this amount is \$107.88; hence, the principal is reduced by \$214.19, leaving \$1,583.74 due on principal at the beginning of the third year, and so on. This process may be continued by means of the following schedule known as an *amortization schedule:*

Year	Principal at Beginning of Year	Annual Payment	Interest at 6%	Principal Repaid
1	\$2,000.00	\$322.07	\$120.00	\$202.07
2	1,797.93	322.07	107.88	214.19
3	1,583.74	322.07	95.02	227.05
4	1,356.69	322.07	81.40	240.67
5	1,116.02	322.07	66.96	255.11
6	860.91	322.07	51.65	270.42
7	590.49	322.07	35.43	286.64
8	303.85	322.07	18.23	303.84
	\$9,609.63		\$576.57	\$1,999.99

Such a schedule gives us the amount remaining due on the principal at the beginning of any year during the amortization period. The principal

at the beginning of the last year should equal the last principal repaid, and the sum of the principals repaid should equal the original principal.

Exercises

1. Find the annual payment that will be necessary to amortize in 10 years a debt of \$2,500, bearing interest at 8% converted annually. Construct a schedule.

2. A mortgage of \$5,000 is due in 8 years. A man wishes to take care of this principal when due by depositing equal amounts at the end of each year in a sinking fund which pays 5% interest. Find the annual deposit and check by an accumulation schedule.

3. A man owes \$10,000 and agrees to pay it in 10 equal annual installments. Find the amount of each installment, allowing 6% for interest. Check by an amortization schedule.

4. A farmer buys a farm for \$10,000. He has \$6,000 to pay down and secures a federal farm loan for the balance to be amortized in 30 years at 5%. Find the annual payment and build up a schedule for the first 10 years.

5. In order to construct a filtering plant a city votes bonds for \$50,000 which bear 6% interest, payable semi-annually. A city ordinance requires that a sinking fund be established to retire the bonds when they mature in 15 years. What semi-annual deposit must be made into the sinking fund, if it accumulates at 4%, converted semi-annually? What is the total semi-annual expense for the city?

6. A mortgage for \$1,000 was given and it was agreed that it might be repaid, principal and interest, by 5 equal annual payments. Build up an amortization schedule if the interest rate is to be 5% for the first two years and 4% for the last three years.

41. Book value.—The book value of an indebtedness at any time may be defined as the difference between the original debt and the amount in the sinking fund at that time. Thus, in the example of Art. 39, we see that the book value of the debt at the end of the third year is \$2,576.86, (\$6,000 − \$3,423.14). If the debt is being amortized, then the book value of the debt at the beginning of any year is the outstanding principal at that time. Thus, in the example of Art. 40, we observe that the book value of the debt at the beginning of the fourth year (at the end of the third year just after the third payment has been made) is \$1,356.69. The subject of book value will be discussed further in connection with depreciation and valuation of bonds.

42. Amount in the sinking fund at any time.—To find the amount in the sinking fund at the end of k payment periods, $k < np$, we have only to find the accumulated value of an annuity of annual rent R for k payment periods by using the appropriate formula of Art. 31.

Example 1. Find the amount in the sinking fund of the Example of Art. 39, at the end of 4 years.

Solution. Here, $R = \$1{,}085.85$, $k = 4$, $m = p = 1$, and $i = 0.05$. Hence, using $AI1$, Art. 31, the amount is given by

$$S_{\overline{4}|} = 1{,}085.85\,\frac{(1.05)^4 - 1}{0.05} = 1{,}085.85 \cdot s_{\overline{4}|.05}$$

$$= 1{,}085.85\ (4.31012500)\ \text{[Table V]}$$

$$= \$4{,}680.15,$$

which checks with the amount given in the sinking fund schedule for the fourth year.

Example 2. A debt of \$3,000 is due in 12 years. A sinking fund is created by making equal annual payments. If the interest rate is 5% converted annually, find the annual payment and the amount in the sinking fund just after the eighth annual payment has been made.

Solution. Here, $S = \$3{,}000$, $n = 12$, $i = 0.05$, $k = 8$, $p = m = 1$.

$$R = 3{,}000\,\frac{0.05}{(1.05)^{12} - 1} = \$188.48,$$

and

$$S_{\overline{8}|} = 188.48\,\frac{(1.05)^8 - 1}{0.05} = \$1{,}799.82.$$

Hence, the amount in the sinking fund at the end of 8 years is **\$1,799.82.**

43. Amount remaining due after the *k*th payment has been made.—When loans are paid by the amortization process it is necessary at times to know the amount of indebtedness (book value) after a certain number of payments have been made. After k payments of R/p dollars have been made there remain $(np - k)$ payments and these remaining payments form an annuity whose *present value is exactly the amount due on the debt* after the kth payment has been made, and the debt could be cancelled by paying this present value.

Example 1. Find the amount of unpaid principal just after making the fifth payment in the Example of Art. 40.

Solution. Here, $R/p = \$322.07$, $n = 8$, $i = 0.06$, $m = p = 1$, and $k = 5$. We have three payments remaining. Hence

$$A_{\overline{3}|} = 322.07\,\frac{1 - (1.06)^{-3}}{0.06} \quad \text{[Formula } AI2\text{, Art. 31]}$$

$$= 322.07\ (2.67301195) \quad \text{[Table VI]}$$

$$= \$860.90.$$

This checks with the value given in the amortization schedule for the principal at the beginning of the 6th year (just after the fifth payment has been made).

Example 2. A debt of \$2,500 is to be amortized by 7 annual installments with interest at 6%. Find the amount unpaid on the principal just after making the fifth annual payment.

Solution. Here, $A = \$2{,}500$, $n = 7$, $k = 5$, $m = p = 1$, and $i = 0.06$. We have, using $A12$, Art. 31,

$$R = 2{,}500 \frac{0.06}{1 - (1.06)^{-7}} = \$447.84.$$

And

$$A_{\overline{2}|} = 447.84 \frac{1 - (1.06)^{-2}}{0.06}$$

$$= 447.84\ (1.83339267) = \$821.06.$$

Hence, the amount unpaid on the principal at the end of the fifth year or just at the beginning of the sixth year is \$821.06.

Exercises

1. A man has been paying off a debt of \$2,800 principal and interest in 20 equal quarterly payments with interest at 5% converted quarterly. At the time of the 13th payment what amount is necessary to make the payment that will extinguish the entire debt?

2. In order to pay a mortgage of \$5,000 due in 7 years, a man pays into a sinking fund equal amounts at the end of each month. If the sinking fund pays 6% interest converted monthly, how much has he accumulated at the end of 5 years?

3. A man owes \$4,000, which is to be paid, principal and interest, in 10 equal annual payments, the first payment falling due at the end of the first year. If the interest rate is 6%, find the balance due on the debt just after the 6th payment is made.

4. A building and loan association sells a house for \$7,500, collecting \$1,500 cash. It is agreed that the balance with interest is to be paid by making equal payments at the end of each month for 10 years. If the interest rate is 7%, converted monthly, find the monthly payment. What equity does the purchaser have in the house just after making the 50th payment? What is his equity after the 70th payment has been made?

5. A person owes a debt of \$8,000, bearing 5% interest, which must be paid by the end of 10 years but may be paid at the end of any year after the fourth. He pays into a sinking fund equal amounts at the end of each year, which will accumulate to \$8,000 at the end of 10 years. Just after making the 7th payment into the sinking fund, how much additional money would be required to pay the debt in full, if the sinking fund accumulates at 5%.

44. The amortization and sinking fund methods compared.—We shall make this comparison by discussing a problem.

Problem. Let us consider a debt of principal $A_{\overline{np}|}$ which is due in n years and draws interest at rate r payable p times a year.

Discussion. This debt may be amortized by making np equal payments direct to the creditor, or it may be cared for by the sinking fund method.

If the amortization method is used the periodic payment will be

$$R/p = A_{\overline{np}|} \frac{\frac{r}{p}}{1 - \left(1 + \frac{r}{p}\right)^{-np}}. \tag{1}$$

Formula (1) gives us the total periodic expense, if the method of amortization is used.

It is easily seen that, since $\frac{1}{a_{\overline{n}|}} = i + \frac{1}{s_{\overline{n}|}}$,

$$\frac{\frac{r}{p}}{1 - \left(1 + \frac{r}{p}\right)^{-np}} = \frac{r}{p} + \frac{\frac{r}{p}}{\left(1 + \frac{r}{p}\right)^{np} - 1},$$

and (1) may be written

$$R/p = A_{\overline{np}|}\left(\frac{r}{p}\right) + A_{\overline{np}|} \frac{\frac{r}{p}}{\left(1 + \frac{r}{p}\right)^{np} - 1}. \tag{2}$$

If the sinking fund method is used, the interest at rate r payable p times a year is paid direct to the creditor and a fund to care for the principal when it becomes due n years from now is created by depositing equal payments p times a year into a sinking fund which accumulates at rate j converted p times a year. If this method is used, the total expense per period will be the sum of the periodic interest and the periodic payment into the sinking fund and is given by

$$E = A_{\overline{np}|}\left(\frac{r}{p}\right) + A_{\overline{np}|} \frac{\frac{j}{p}}{\left(1 + \frac{j}{p}\right)^{np} - 1}. \tag{3}$$

Now, if the sinking fund rate is the same as the interest rate on the debt ($j = r$), then E of (3) is the same as R/p of (2). That is, when $j = r$, the periodic expense is the same by either plan, and the amortization method may be considered a special case of the sinking fund method where the creditor has charge of the sinking fund money and allows the same rate of interest on it that he charges on the debt.

If the sinking fund rate is less than the rate on the debt, that is, if $j < r$, then $\dfrac{1}{s_{\overline{np}|} \text{ at } j/p} > \dfrac{1}{s_{\overline{np}|} \text{ at } r/p}$ and E in (3) is greater than R/p in (2). That is, the sinking fund method is more expensive for the debtor than the amortization method.

If $j > r$, then $\dfrac{1}{s_{\overline{np}|} \text{ at } j/p} < \dfrac{1}{s_{\overline{np}|} \text{ at } r/p}$ and E is less than R/p. That is, the sinking fund method is less expensive for the debtor than the amortization method.

Example 1. A debt of \$10,000, with interest at 6%, payable semi-annually, is due in 10 years. Find the semi-annual expense if it is to be cared for by the amortization method.

Solution. Here, $A_{\overline{np}|} = \$10{,}000$, $r = 0.06$, $p = 2$, and $n = 10$. We have

$$R/2 = 10{,}000 \frac{0.03}{1 - (1.03)^{-20}} \quad \text{[Formula (1)]}$$

$$= 10{,}000\ (0.06721571) \quad \text{[Table VII]}$$

$$= \$672.16.$$

Example 2. Find the semi-annual expense in Example 1, if a sinking fund is accumulated at ($j = .05$, $p = 2$).

Solution. Here, $j = 0.05$ and the other conditions are the same. We have

$$E = 10{,}000(0.03) + 10{,}000 \frac{0.025}{(1.025)^{20} - 1} \quad \text{[Formula (3)]}$$

$$= 300.00 + 10{,}000(0.03914713) \quad \text{[Table VII]}$$

$$= 300.00 + 391.47 = \$691.47.$$

Example 3. Find the semi-annual expense in Example 1, if the sinking fund is accumulated at ($j = .06$, $p = 2$).

Solution. Here, $j = 0.06$ and the other conditions are the same. We have

$$E = 10{,}000(0.03) + 10{,}000\,\frac{0.03}{(1.03)^{20} - 1}$$

$$= 300.00 + 10{,}000(0.03721571)$$

$$= 300.00 + 372.16 = \$672.16.$$

Example 4. Find the semi-annual expense in Example 1, if the sinking fund is accumulated at ($j = .07, p = 2$).

Solution. Here, $j = 0.07$ and the other conditions are the same. We have

$$E = 10{,}000(0.03) + 10{,}000\,\frac{0.035}{(1.035)^{20} - 1}$$

$$= 300.00 + 10{,}000(0.03536108)$$

$$= 300.00 + 353.61 = \$653.61.$$

Compare the answers of Examples 1, 2, 3, and 4. Are the results consistent with the conclusions that we have already drawn?

Exercises

1. A man secures a \$15,000 loan with interest at 6½%, payable annually. He may take care of the loan (a) by paying the interest as it is due and paying the principal in full at the end of 10 years; or (b) by paying principal and interest in 10 equal annual installments. If a sinking fund can be accumulated at 5%, converted annually, which is the more economical method and by how much?

2. A debt of \$8,000 bears interest at 7%, payable semi-annually, and is due in 7 years. How much should be provided every six months to pay the interest and retire the debt when it is due, if deposits can be accumulated at 6%, converted semi-annually?

3. What would be the semi-annual expense in Exercise 2, if the debt could be retired by paying principal and interest in 14 equal semi-annual installments?

4. A debt of \$20,000 which bears interest at 5%, payable semi-annually, is to be paid in full in 20 years. The debtor has the privilege of paying the principal and interest in 40 equal semi-annual payments, or paying the interest semi-annually and paying the principal in full at the end of 20 years. Compare the two methods if a sinking fund may be created by making semi-annual payments which accumulate at (a) 4%, converted semi-annually; (b) 5%, converted semi-annually; (c) 6%, converted semi-annually.

45. Retirement of a bonded debt.—In the retirement of a debt which has been contracted by issuing bonds of given denominations, the periodic payments cannot be the same, because the payment on principal at the end of each period must be a multiple of the denomination (face value or

par value) of the bonds or their redemption value* (if not redeemed at par). By varying the number of bonds retired each time the payments can be made to differ from each other by an amount not greater than the redemption value of one bond. An example will make the method clear.

Example. Construct a schedule for the retirement, in 8 years, of a \$30,000 debt, consisting of bonds of \$100 face value, bearing interest at 6% payable annually, by making annual payments as nearly equal as possible.

Solution. If the annual payments were all equal, we would have

$$R = 30{,}000\,\frac{0.06}{1 - (1.06)^{-8}} = \$4{,}831.08.$$

The interest for the first year is \$1,800. Subtracting this amount from \$4,831.08 leaves \$3,031.08 available for the retirement of bonds. This will retire 30 bonds, for \$3,000 is the multiple of \$100 which is nearest to \$3,031.08. This makes a total payment (for interest and bonds retired) of \$4,800 for the first year. Subtracting the \$3,000 which has been paid on the principal from \$30,000 leaves \$27,000 as the principal at the beginning of the second year. The interest on this amount is \$1,620, which when subtracted from \$4,831.08 leaves \$3,211.08 to be used for retiring bonds the second year. This will retire 32 bonds, because \$3,200 is the multiple of \$100 which is nearest to \$3,211.08. Continuing this process, we obtain the following schedule:

Year	Unpaid Principal at Beginning of Year	Interest Due at End of Year	Number of Bonds Retired	Value of Bonds Retired	Annual Payment
1	\$30,000.00	\$1,800.00	30	\$3,000.00	\$4,800.00
2	27,000.00	1,620.00	32	3,200.00	4,820.00
3	23,800.00	1,428.00	34	3,400.00	4,828.00
4	20,400.00	1,224.00	36	3,600.00	4,824.00
5	16,800.00	1,008.00	38	3,800.00	4,808.00
6	13,000.00	780.00	41	4,100.00	4,880.00
7	8,900.00	534.00	43	4,300.00	4,834.00
8	4,600.00	276.00	46	4,600.00	4,876.00
Totals	\$144,500.00	\$8,670.00	300	\$30,000.00	\$38,670.00

* See Art. 54 for definitions.

As a check on the work of the schedule the interest on the total of the unpaid principals should equal the total of the interest due; and the sum of the totals in the third and fifth columns should equal the total in the sixth column.

We notice that the annual payment each year varies from the computed payment, \$4,831.08, by an amount less than \$50 (one-half the face of one bond).

Exercises

1. Solve the illustrative Example when the bonds have a \$500 face value.

2. A city borrows \$100,000 to erect a school building. The debt is in the form of bonds of face value \$1,000 bearing interest at 5% converted annually. The bonds are to be retired by 10 annual installments as nearly equal as possible. Set up a schedule showing the number of bonds retired each year.

Problems

1. Construct the amortization schedule for the repayment of a loan of \$10,000, principal and interest at 5% nominal, payable semi-annually, in ten semi-annual payments.

2. Construct an accumulation schedule for the accumulation of \$10,000 in 10 equal semi-annual installments at 6% interest, converted semi-annually.

3. A man deposits in a sinking fund equal quarterly payments sufficient to accumulate to \$5,000 in 5 years at 6% converted quarterly. What is the amount in the sinking fund just after the 9th quarterly payment has been made?

4. A debt of \$8,000 bearing 5% interest, converted quarterly, is arranged to be paid principal and interest in 30 equal quarterly payments. How much remains unpaid on the principal just after the 17th payment is made?

5. The cash price of a house is \$7,000. \$2,000 cash is paid and it is arranged to pay the balance by 70 equal monthly payments, including interest at 6%, converted monthly. Just after the 50th payment is made, what is the balance due on the principal?

6. A mortgage for \$7,500, bearing 6% interest payable semi-annually, is due in 12 years. A fund to care for the principal when it becomes due is established by making semi-annual payments into a sinking fund. (a) Find the semi-annual expense of the mortgage if the sinking fund accumulates at 5% semi-annually. (b) Find the semi-annual expense of the mortgage if it is amortized by equal semi-annual payments.

7. What is the book value of the debt in Problem 6 at the end of 7 years, (a) if the sinking fund method is used, (b) if the amortization method is used?

8. A man buys a house for \$5,500, paying \$1,500 cash. The balance with interest at 6% is to be cared for by paying \$700 at the end of each year as long as such a payment is necessary and then making a smaller payment at the end of the last year. Find the number of full payments and the amount of the final payment. What amount remains due just after making the 5th payment?

9. A city borrows \$100,000 at 5%. The debt is to be retired in 10 years by the accumulation of a sinking fund that is invested at 4% effective. What is the total annual expense to the city?

10. A county borrows \$50,000 to build a bridge. The debt is to be paid by amortization of the principal in 15 years at 5%. At the end of the tenth year what principal remains outstanding?

11. A fraternity chapter borrows \$60,000 at 6% to build a house. The debt is to be amortized in 25 years. What is the annual payment?

12. A fraternity chapter borrows \$60,000 at 6% to build a house. A sinking fund can be built up at 5%. What amount must be raised annually to pay this debt if the payments are to extend over 30 years?

Review Problems *

1. A well-known finance company requires payments of \$7.27 a month for 18 months for a loan of \$100. What rate of interest does the borrower pay?

2. The cash price of an automobile is \$995. An advertisement of a dealer stated, "If you want to buy on terms, pay a little more for the convenience, \$329 down and \$63 a month for 12 months." What rate of interest does one pay who purchases the car on the installment plan?

3. An automobile, cash price \$1,300, was purchased on the terms, \$507 down and \$57.50 a month for 18 months. What rate of interest was paid?

4. Solve $A = Ra_{\overline{n}|i}(1 + i)^{-m}$ (a) for m; (b) for n.

5. If C is the first cost and D is the renewal cost of an article whose life is r years, show that the capitalized cost, K, at the rate i is given by

$$K = C + \frac{D}{i} \cdot \frac{1}{s_{\overline{r}|i}}.$$

6. A machine costs \$2,500 new and must be replaced at the end of each 10 years. Find the capitalized cost if money is worth 5% and if the old machine has a salvage value of \$500.

7. A debt of \$10,000 with interest at ($j = .06$, $m = 12$) is to be amortized by payments of \$100 a month. After how many payments will the debt be paid in full? What is the final partial payment?

8. A \$10,000 bequest invested at 4% is to provide a scholarship of R at the end of each year for 25 years at which time the bequest is to be exhausted. Find R.

9. The Empire State Building was erected at a cost of \$52,000,000. If its estimated useful life is 100 years and its salvage value is to pay for its demolishing, what net annual income for 100 years would yield 5% on the investment?

10. If interest is at 5% for the first 10 years and 4% thereafter, what equal annual payments for 15 years will repay a \$10,000 loan?

11. Show (a) by verbal interpretation and (b) algebraically that

$$A(1 + i)^r = Rs_{\overline{r}|i} + Ra_{\overline{n-r}|i}$$

when $r \leq n$.

* For additional review problems, see end of this book.

CHAPTER V

DEPRECIATION

46. Definitions.—A building, a machine or any article of value into which capital has been invested will be referred to as an *asset.* These assets decrease in value due to use, action of the elements, lack of care, old age, and other causes. A part of this decrease in value may be taken care of by proper repairs, but repairs will not cause an asset to retain its original value. In fact, some assets will decrease in value whether they are used or not. This may be due to new inventions or decreases in the market prices or a combination of these and other causes. For example, an automobile will decrease in value even though it does not leave the floor of the showroom. (Why?) That part of the decrease in value of an asset which can not be cared for by repairs is commonly known as *depreciation.*

Good business principles demand that capital invested in an asset or a business consisting of several assets, should not be impaired. Hence, from the revenues of the asset or the business there should be set aside, periodically, certain sums, such that the accumulation of these sums at any time plus the value of the asset at that time shall equal its original value. The fund into which these periodical sums are set aside is known as a *depreciation reserve.* This depreciation reserve is usually retained in the business but is carried as a separate item on the books of the business. The object of the accounting for depreciation and the setting aside of a depreciation reserve is to recover only the capital originally invested in the asset. The accountant is not concerned with the replacement of the asset, whether lower or higher than the original cost. His chief concern is that the original capital be not impaired, for this is a fund that must be considered as belonging to the holders of the stock in the business.

These assets may never be replaced at any price for the company may go out of business. Then this accumulated value would be used to retire the capital stock. If the assets are replaced at a lower cost, then only a part of this accumulated value may be considered as used for the replacement. If the assets have to be replaced at a higher cost, then the differ-

ence between this cost and the accumulated value reserve must be met by increasing the original capital. Regardless of the way that depreciation is considered by the accountant, the mathematical principles involved in the treatment of the subject remain the same.

Although an asset may become obsolete or useless for the purpose for which it was intended originally, it may be of value for some other purpose. This value is commonly known as the *scrap value* or *trade in value* of the asset and the time it was in use up to the date it was replaced or discarded is known as its *useful life.* The original value minus the scrap value is defined as the *wearing value* or the *total depreciation* of the asset. At any time during the life of an asset its *book value* may be defined as the original value (or value when it became a part of the business) minus the value of the depreciation reserve. The amount by which the depreciation reserve increases any year is known as the *annual depreciation charge.*

47. Methods of treating depreciation.—There are many methods of treating depreciation. We shall treat four of the most common methods:

(a) The straight line method.
(b) The sinking fund method.
(c) The fixed percentage on decreasing value method.
(d) The unit cost method.

Some of the other methods used are the compound interest method, the service output method, the maintenance method, and so on.

48. The straight line method.—By this method the total depreciation (wearing value) is distributed equally over the life of the asset and the amounts in the depreciation reserve *do not earn interest.* If we let C stand for the original value (cost) of the asset, S stand for its scrap value, n stand for its useful (probable) life, W stand for its wearing value, and D stand for the annual depreciation charge to be made, it follows from the above definition of the straight line method that

$$D = \frac{W}{n}, \tag{1}$$

where $W = C - S$.

Example. A certain asset costs \$2,250. It is assumed that with proper care it will have a scrap value of \$170 after a useful life of 8 years. Using the straight line method, show by schedule and graph the value of the depreciation reserve and the book value of the asset at any time.

Solution. We have, $C = \$2{,}250$, $S = \$170$, $n = 8$, and $W = \$2{,}080$.

Therefore, $$D = \frac{2{,}080}{8} = \$260.$$

The value of the depreciation reserve at the end of the first year will be \$260 and this will increase each year by the constant amount, $D = \$260$, until at the end of 8 years it will contain \$2,080. The book value of the asset will decrease each year by the constant amount, $D = \$260$, until at the end of 8 years it will be \$170 (the scrap value).

The following schedule shows the book value of the asset and the amount in the depreciation reserve at any time.

SCHEDULE OF BOOK VALUE AND DEPRECIATION
STRAIGHT LINE METHOD

Age in Years	Book Value	Depreciation Charge	Total in Depreciation Reserve
0	\$2,250.00		
1	1,990.00	\$260.00	\$260.00
2	1,730.00	260.00	520.00
3	1,470.00	260.00	780.00
4	1,210.00	260.00	1,040.00
5	950.00	260.00	1,300.00
6	690.00	260.00	1,560.00
7	430.00	260.00	1,820.00
8	170.00	260.00	2,080.00

Observing the above schedule, we notice that the book value at the end of any year plus the total in the depreciation reserve at that time equals the original cost of the asset.

The changes in the book value and depreciation reserve may also be shown by graphs. [See Fig. 1.]

Observing the graphs for depreciation and book value, we notice that the ordinate for depreciation at any time plus the ordinate for book value at the same time equals the original value of the asset. We also observe that the graphs which represent the book value and depreciation reserve are straight lines. This suggests why this method is known as the straight line method.

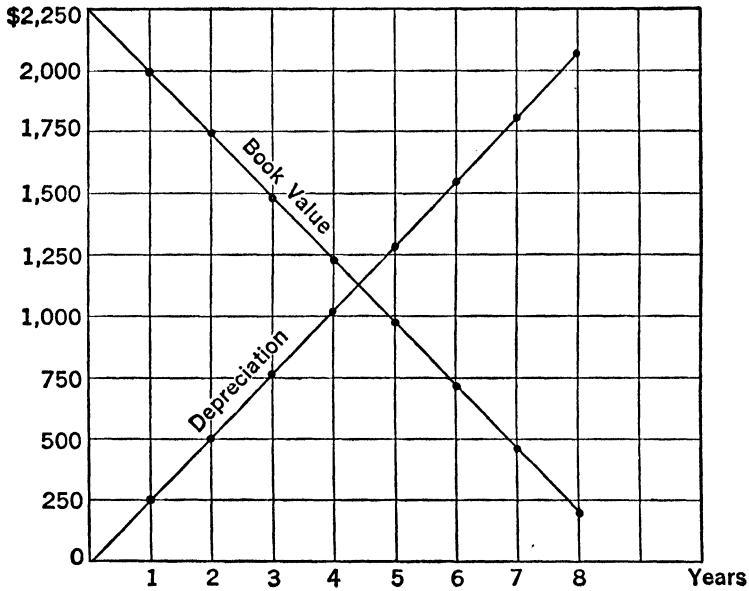


Fig. 1.—Graphical Representation of Book Value and Depreciation—Straight Line Method.

49. Fixed-percentage-on-decreasing-value method.—This method derives its name from the fact that the book value at the end of any year is obtained by decreasing the book value at the end of the preceding year by a fixed percentage. It is assumed that the book value is reduced from the original cost C to the scrap value S at the end of n years, and the amounts in the depreciation reserve do not earn interest.

Let C stand for the original cost of an asset and let x be the fixed percentage by which the book value is decreased each year.

During the first year the decrease in book value is Cx and consequently, the book value at the end of the first year is

$$C_1 = C - Cx = C(1 - x).$$

The book value at the end of the second year is

$$C_2 = C_1(1 - x) = C(1 - x)(1 - x) = C(1 - x)^2.$$

The book value at the end of the third year is

$$C_3 = C_2(1 - x) = C(1 - x)^2(1 - x) = C(1 - x)^3.$$

Continuing our reasoning we find the book value at the end of n years to be

$$C_n = C(1 - x)^n.$$

But the book value of the asset at the end of its useful life, n years, equals its scrap value S. Hence, we have*

$$\boldsymbol{C\,(1 - x)^n = S} \tag{2}$$

or

$$\log\,(1 - x) = \frac{\log S - \log C}{n}. \tag{3}$$

Using (3), the fixed percentage may be computed for any particular case.

If we let C_k represent the book value of the asset at the end of k years, we observe that

$$C_k = C(1 - x)^k, \tag{4}$$

and

$$\log C_k = \log C + k \log\,(1 - x) \tag{5}$$

We further observe that by using (3) and (5) and allowing k to assume all consecutive integers from 1 to n inclusive, we may compute, entirely by the use of logarithms, the successive book values of the asset. An example will illustrate the method.

Example. Find by the fixed percentage method the book values at the end of each year for a machine costing \$800, and having an estimated life of 8 years and a scrap value of \$80. Construct a schedule showing the book values and amount in the depreciation reserve at the end of each year.

Solution. Here, $C = \$800$, $S = \$80$, $n = 8$.

Using (3), we get

$$\log\,(1 - x) = \frac{\log 80 - \log 800}{8} = 9.87500 - 10.$$

Then using (5), we have

$$\log C_k = \log 800 + k(9.87500 - 10).$$

$$= 2.90309 + k(9.87500 - 10).$$

* It will be observed from (2) that, when $S = 0$, we have $x = 1$ for any assigned value of n. That is, the book value is reduced to zero at the end of 1 year, no matter what is the estimated value of n. This means that the method is impractical when S is zero. Even if the ratio of S to C is small, the depreciation charge is likely to be unreasonably large during the first years of operation.

Giving k all values from 1 to 8, we get

$$\log C_1 = 2.77809, \qquad C_1 = \$599.91.$$
$$\log C_2 = 2.65309, \qquad C_2 = 449.87.$$
$$\log C_3 = 2.52809, \qquad C_3 = 337.35.$$
$$\log C_4 = 2.40309, \qquad C_4 = 252.98.$$
$$\log C_5 = 2.27809, \qquad C_5 = 189.71.$$
$$\log C_6 = 2.15309, \qquad C_6 = 142.26.$$
$$\log C_7 = 2.02809, \qquad C_7 = 106.68.$$
$$\log C_8 = 1.90309, \qquad C_8 = 80.00.$$

The student will observe that the actual value of x (fixed percentage) was not needed in the above computations. Should we desire the value of x, we find that $1 - x$ is the antilogarithm of $9.87500 - 10$, or 0.7499. Hence, $x = 0.2501 = 25.01\%$.

Since the book value at the end of the first year is \$599.91, the depreciation charge for that year is

$$\$800.00 - \$599.91 = \$200.09.$$

The depreciation charge for the second year is

$$\$599.91 - \$449.87 = \$150.04$$

and the total in the depreciation reserve at the end of two years is

$$\$200.09 + \$150.04 = \$350.13.$$

The following schedule shows the book values and the amount in the depreciation reserve at the end of each year.

SCHEDULE OF BOOK VALUE AND DEPRECIATION
FIXED PERCENTAGE METHOD

Age in Years	Annual Depreciation	Total in Depreciation Reserve	Book Value
0	.....	.....	\$800.00
1	\$200.09	\$200.09	599.91
2	150.04	350.13	449.87
3	112.52	462.65	337.35
4	84.37	547.02	252.98
5	63.27	610.29	189.71
6	47.45	657.74	142.26
7	35.58	693.32	106.68
8	26.68	720.00	80.00

The changes in the book value and depreciation reserve may also be shown by graphs.

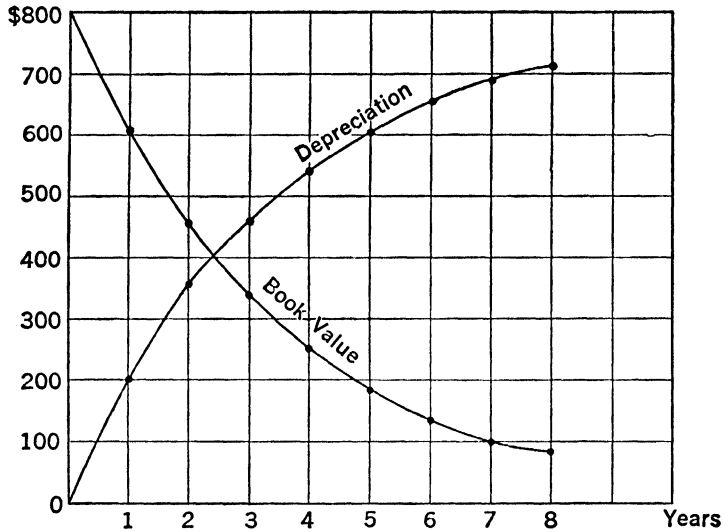


Fig. 2.—Graphical Representation of Book Value and Depreciation—Fixed Percentage Method.

50. The sinking fund method.—In the sinking fund method the total depreciation (wearing value) of the asset is provided for by accumulating a sinking fund at a given rate of compound interest. The annual payment into the sinking fund is the payment on an annuity which will have an amount equal to the total depreciation (wearing value) of the asset at the end of its useful life.

If C is the cost, S the scrap value, W the wearing value, and n the estimated useful life of the asset, we find, using $A11$, Art. 31, the annual payment into the sinking fund to be

$$R = W\frac{i}{(1+i)^n - 1} = \frac{W}{s_{\overline{n}|i}}, \tag{6}$$

where $W = C - S$.

By this method the depreciation charge for the first year is R and the amount in the depreciation reserve at the end of the first year is R. However, the depreciation charge increases each year and for any subsequent year it is R plus the interest on the amount in the depreciation reserve during that year.

Example. Assuming money worth $4\frac{1}{2}\%$, apply the sinking fund method to the Example discussed in Art. 49.

Solution. Here, $C = \$800$, $S = 80$, $n = 8$, $i = 0.045$, and $W = C - S = \$720$.

Using (6), we get

$$R = 720\frac{0.045}{(1.045)^8 - 1} = \$76.76.$$

The depreciation charge for the first year is $R = \$76.76$. Consequently, the amount in the depreciation reserve at the end of the first year is \$76.76 and the book value of the asset at that time is \$800.00 less \$76.76 or \$723.24. The depreciation charge for the second year is R, (\$76.76), plus the interest on \$76.76 (the amount in the depreciation reserve during the second year) at $4\frac{1}{2}\%$. Thus, the depreciation charge for the second year is \$76.76 + \$3.45 = \$80.21. Then, the amount in the depreciation reserve at the end of two years is \$76.76 plus \$80.21 or \$156.97 and the book value of the asset at that time is \$643.03. Values for subsequent years are found in a similar manner.

The following schedule will show the values for each year.

SCHEDULE OF BOOK VALUE AND DEPRECIATION
SINKING FUND METHOD

Age in Years	Annual Payment	Interest on Fund	Annual Depreciation Charge	Amount in Depreciation Reserve	Book Value of Asset
0	.....	.....	.....	.....	\$800.00
1	\$76.76	\$0.00	\$76.76	\$76.76	723.24
2	76.76	3.45	80.21	156.97	643.03
3	76.76	7.06	83.82	240.79	559.21
4	76.76	10.84	87.60	328.39	471.61
5	76.76	14.78	91.54	419.93	380.07
6	76.76	18.90	95.66	515.59	284.41
7	76.76	23.20	99.96	615.55	184.45
8	76.76	27.70	104.46	720.01	79.99

The above information is shown by means of graphs in Fig. 3.

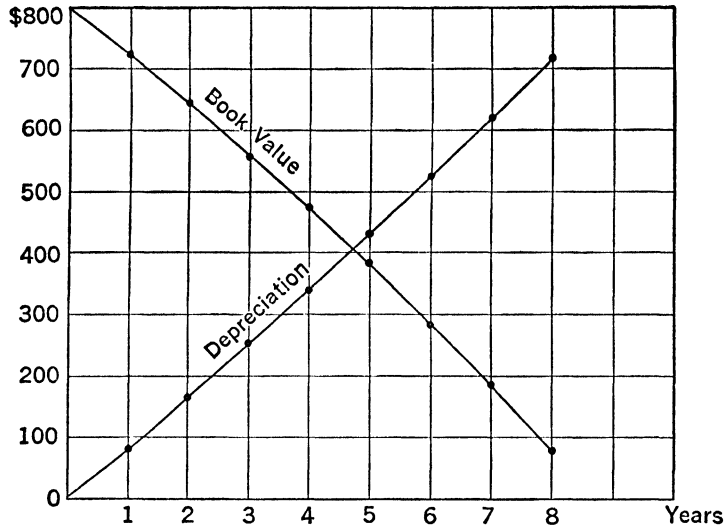


Fig. 3.—Graphical Representation of Book Value and Depreciation—Sinking Fund Method.

51. The unit cost method.—None of the three methods of depreciation already discussed takes into consideration the question of improvements in machinery. The unit cost method is based upon the principle that the value of the old machine should be decreased from year to year to such an extent that the net cost of a unit of output of the machine should be the same as the net cost of a unit of output of a new machine with which it could be replaced. The old machine should be so valued that its unit cost of production, after taking into account all charges for depreciation, repairs, interest, and operating expenses, is the same as that of a new machine. Let us illustrate by an example.

Example 1. Consider the replacement of a machine which costs \$300 a year to operate, costs \$100 a year for repairs, turns out 25 units of work per year and has a probable life of 5 years. A new machine costs \$2,500, costs \$400 a year to operate, costs \$100 a year for repairs, turns out 40 units of work per year, and has a probable life of 9 years. Find the value of the old machine, assuming money worth 4%.

Solution. Let x be the value of the old machine. The cost of repairs and operation on the old machine is \$400. $0.04x$ is the interest on the investment, and

$$x\,\frac{0.04}{(1.04)^5 - 1}$$

is the annual payment required to accumulate the value of the old machine in 5 years.

$$0.04x + x\frac{0.04}{(1.04)^5 - 1} = 0.22462711x.$$

Hence, the unit cost of production for the old machine is

$$\frac{400 + 0.22462711x}{25} = 16 + 0.0089851x.$$

Reasoning the same as above, we find the yearly cost for operating the new machine to be

$$400 + 100 + 2{,}500(0.04) + 2{,}500\frac{0.04}{(1.04)^9 - 1} = 836.232475.$$

Hence, the unit cost of production for the new machine is

$$\frac{836.232475}{40} = 20.905812.$$

According to the principle of the unit cost method, we have

$$16 + 0.0089851x = 20.905812,$$

and

$$x = \frac{4.905812}{0.008985} = \$546.00.$$

Hence, assuming money worth 4%, the value of the old machine as compared with the value of the new is \$546.00.

We shall now derive a formula for determining the value of the old machine as compared with the new machine. Let

C = the original cost of the new machine,
N = the estimated lifetime of the new machine,
O = the annual operating expense of the new machine not including repairs,
R = the annual cost of repairs for the new machine,
K = the annual rent of an annuity required to accumulate C in N years,
U = the number of units of output per year.

Let the corresponding letters o, r, k, and u denote the corresponding quantities for the old machine. Let c be the value of the old machine at

the time of making the comparison, and n the remaining lifetime of the old machine. Let i be the rate of interest.

The unit cost for the new machine is

$$\frac{O + R + K + Ci}{U}$$

and the unit cost of the old machine is

$$\frac{o + r + k + ci}{u}.$$

According to the principle of the unit cost method, we have

$$\frac{O + R + K + Ci}{U} = \frac{o + r + k + ci}{u}. \tag{7}$$

Since,
$$K = \frac{C}{s_{\overline{N}|}} \quad \text{and} \quad k = \frac{c}{s_{\overline{n}|}},$$

$$K + Ci = C\left(i + \frac{1}{s_{\overline{N}|}}\right) = \frac{C}{a_{\overline{N}|}} \qquad [(14), \text{Art. } 26]$$

and
$$k + ci = c\left(i + \frac{1}{s_{\overline{n}|}}\right) = \frac{c}{a_{\overline{n}|}}.$$

Then (7) becomes

$$\frac{O + R + \dfrac{C}{a_{\overline{N}|}}}{U} = \frac{o + r + \dfrac{c}{a_{\overline{n}|}}}{u}. \tag{8}$$

Solving (8) for c, we have

$$c = ua_{\overline{n}|}\left[\frac{O + R + \dfrac{C}{a_{\overline{N}|}}}{U} - \frac{o + r}{u}\right]. \tag{9}$$

If the number of units of output of the old and new machines are the same, $U = u$, (9) reduces to

$$c = a_{\overline{n}|}\left[O + R + \frac{C}{a_{\overline{N}|}} - o - r\right]. \tag{10}$$

If $O = o$, along with $U = u$, (10) reduces to

$$c = a_{\overline{n}|}\left(R + \frac{C}{a_{\overline{N}|}} - r\right). \tag{11}$$

If $O + R = o + r$, then (10) becomes

$$c = \frac{Ca_{\overline{n}|}}{a_{\overline{N}|}}. \tag{12}$$

Example 2. A machine having a remaining service life of 6 years turns out 30 units of work per year. Its operation costs \$300 per year, and repairs cost \$225 per year. A new machine, that turns out 40 units of work, costs \$1,000. It has a probable life of 10 years and will cost \$350 a year for operation and \$250 a year for repairs. Assuming money worth 5%, find the value of the old machine.

Solution. Here, $C = \$1{,}000$, $N = 10$, $O = \$350$, $R = \$250$, $U = 40$, $n = 6$, $o = \$300$, $r = \$225$, and $u = 30$.

$$\frac{1}{a_{\overline{N}|}} = \frac{1}{a_{\overline{10}|}} = 0.12950458,$$

$$a_{\overline{n}|} = a_{\overline{6}|} = 5.07569206.$$

Substituting in (9), we have

$$c = 30(5.07569206)\left[\frac{350 + 250 + 1{,}000(0.12950458)}{40} - \frac{300 + 225}{30}\right]$$

$$= 30(5.07569206)\,[18.23761 - 17.50000]$$

$$= 152.2708(0.7376) = \$112.31.$$

Exercises

1. A farmer pays \$235 for a binder. The best estimates show that it will have a life of 8 years and a scrap value of \$15. Find the annual depreciation charge by the straight line method and construct a schedule of depreciation.

2. A tractor costs \$1,200. It is estimated that with proper care it will have a life of 8 years with a scrap value of \$50 at the end of this time. Construct a depreciation schedule, using the sinking fund method and assuming 4% interest.

3. An automobile, costing \$950, has an estimated life of 5 years and a scrap value of \$50. Prepare a depreciation schedule using the fixed percentage method.

4. A machine costs \$5,000. The best estimates show that after 10 years of use its scrap value will be \$1,000. (a) Making use of the fixed percentage method, find the

book value of the machine at the ends of 7 and 8 years, respectively. (b) What is the depreciation charge for the 8th year?

5. Solve Exercise 4, making use of the sinking fund method and assuming an interest rate of 5%.

6. Solve Example 2 of Art. 51, if the new machine could turn out 45 units of work per year. Interpret the results.

7. How many units of work must be turned out by the new machine of Example 2, Art. 51, so that the old machine would not have any value?

8. From formula (9) derive a formula for the number of units a new machine should turn out in order to make the old machine worthless.

9. A machine having a probable life of 18 years has been in use for 8 years and turns out 200 units of work each year. The cost for operating is \$600 per year and repairs are \$400 per year. A new machine costs \$3,000 and has a probable life of 20 years and will turn out 200 units of work per year. It would cost \$500 per year to operate this machine and repairs would cost \$300 per year. Neither machine is supposed to have any salvage value. What is the value of the old machine on a 6% interest basis?

10. What output for the new machine in Exercise 9 would render the value of the old machine zero?

11. An asset costs \$1,000. It is estimated that with proper care it can be used for 8 years at which time it will have a value of \$50. Using the sinking fund method and assuming 4% interest, find the wearing value that remains at the end of 5 years. [Hint: The wearing value that remains at the end of any year equals the total wearing value minus the amount in the depreciation reserve at that time. Observing the schedule for the Example of Art. 50, we see that the wearing value that remains after 5 years of use is (\$720.00 − \$419.93) = \$300.07].

12. Solve Exercise 11, making use of the fixed percentage method.

13. Solve Exercise 11, making use of the straight line method.

52. Depreciation of mining property.—Investment in mines, oil wells, and timber tracts should yield not only interest on the investment, but additional income to provide for the restoration of the original capital when the asset is exhausted. The mining engineer can estimate the net annual return on the mine and the number of years before the mineral will be exhausted. From this net annual return, interest on the capital invested must be taken and also an annual payment to a depreciation reserve which shall accumulate to the original cost of the mine, less the salvage value, by the time it is exhausted.

An important problem in connection with mining property is, having given the net annual yield and the number of years this yield will continue, to determine the price that should be paid for the mines so that this net annual yield will provide a sufficient rate of interest on the investment and an annual payment to the depreciation reserve.

Assume that R is the net annual return and that this yield will continue for n years. Also assume that the rate of yield on the invested capital is to be r and the depreciation reserve is to be accumulated at rate i.

If we let P stand for the purchase price of the property, then the annual return on the capital invested would be Pr. Hence, the amount left from the net annual return, for the annual contribution to the depreciation reserve, would be $(R - Pr)$, and this must accumulate to $P - S$ in n years at rate i, where S is the salvage value.

Therefore, we have

$$P - S = (R - Pr)\frac{(1+i)^n - 1}{i} = (R - Pr)s_{\overline{n}|i}. \tag{12'}$$

When $S = 0$,

$$P = \frac{R}{r + \dfrac{i}{(1+i)^n - 1}} = \frac{R}{r + \dfrac{1}{s_{\overline{n}|i}}}. \tag{13}$$

Example. A mining engineer estimates that a copper mine will yield a net annual income of \$50,000 for the next 20 years. What price should be paid for the mine, if the depreciation reserve is to accumulate at 5%, if 10% is to be realized on the capital invested, and if $S = 0$?

Solution. We have, $R = \$50{,}000$, $n = 20$, $r = 10\%$, and $i = 5\%$. Making use of (13), we get

$$P = \frac{50{,}000}{0.10 + \dfrac{0.05}{(1.05)^{20} - 1}} = \frac{50{,}000}{0.10 + \dfrac{1}{s_{\overline{20}|.05}}}$$

$$= \frac{50{,}000.00}{0.10 + (0.03024259)} = \frac{50{,}000.00}{0.13024259}$$

$$= \$383{,}899, \text{ purchase price.}$$

This would give a return of \$38,389.90 on the invested capital and leave \$50,000 − \$38,389.90 = \$11,610.10 for the annual payment into the depreciation reserve. This annuity in 20 years at 5% will amount to \$383,899.

Exercises

1. An oil well which is yielding a net annual income of \$30,000 is for sale. The geologist estimates that this annual income will continue 10 years longer. What should be paid for the well, if the depreciation reserve is to accumulate at 4½%, and 8% is to be realized on the invested capital?

2. A gold mine is yielding a net annual income of \$100,000. Careful estimates show that the mine will continue to yield this net annual income for 25 years longer, at which time it will be exhausted. Find its value, if a return of 9% on the invested capital is desired and the depreciation reserve accumulates at 5%.

3. A 1,000 acre tract of timber land is for sale. It is estimated that the net annual income from the timber will be \$125,000 for the next 5 years, at which time the land will be worth \$25 per acre. How much per acre should be paid for the land, if the purchaser desires 10% on his investment and the depreciation reserve can be accumulated at 5%?

4. \$750,000 is paid for a mine which will be exhausted at the end of 25 years. What net annual income is required from the mine, if 8% is to be realized on the investment after the annual payments have been made into the depreciation reserve which accumulates at 4%?

53. Composite life of a plant.—We will consider that a manufacturing plant consists of several parts, each having a different probable life. By the composite life of a plant we mean a sort of average lifetime of the several parts, and we may define it more precisely as the *time required for the total of the equal annual payments to the depreciation reserves of the several parts to accumulate to the total wearing value of the plant.*

Let $W_1, W_2, W_3, \cdots, W_r$ be the wearing values of the several parts, with probable lives of $n_1, n_2, n_3, \cdots, n_r$ respectively, and let $W = W_1 + W_2 + W_3 + \cdots + W_r$ be the wearing value of the entire plant. Also let $D_1, D_2, D_3, \cdots, D_r$ be the annual payments to the depreciation reserves for the several parts and let $D = D_1 + D_2 + D_3 + \cdots + D_r$ be the depreciation for the whole plant.

Then by the straight line method, we have

$$n = \frac{W}{D} = \frac{W_1 + W_2 + W_3 + \cdots + W_r}{D_1 + D_2 + D_3 + \cdots + D_r},$$

or

$$n = \frac{W_1 + W_2 + W_3 + \cdots + W_r}{\frac{W_1}{n_1} + \frac{W_2}{n_2} + \frac{W_3}{n_3} + \cdots + \frac{W_r}{n_r}}. \tag{14}$$

Example 1. A plant consists of parts A, B, and C, having the following values, scrap values, and probable lives, respectively:

A	\$25,000	\$5,000	20 years
B	20,000	2,000	18 years
C	8,000	1,000	14 years

Find its composite life.

Solution. Here, $W_1 = \$20{,}000$, $W_2 = \$18{,}000$, $W_3 = \$7{,}000$, $n_1 = 20$, $n_2 = 18$, $n_3 = 14$. Using (14), we get

$$n = \frac{20{,}000 + 18{,}000 + 7{,}000}{\frac{20{,}000}{20} + \frac{18{,}000}{18} + \frac{7{,}000}{14}}$$

$$= \frac{45{,}000}{2{,}500} = 18.$$

Hence, the composite life is 18 years.

If the sinking fund method is used, we have

$$(D_1 + D_2 + \cdots + D_r)\, s_{\overline{n}|i} = (W_1 + W_2 + \cdots + W_r),$$

or

$$D\, s_{\overline{n}|i} = W, \tag{14'}$$

where $D_1 = W_1 \dfrac{1}{s_{\overline{n_1}|i}}$, and so on.

Solving (14') for n by the use of logarithms, we get

$$n = \frac{\log (Wi + D) - \log D}{\log (1 + i)}. \tag{15}$$

The value for n obtained from (15) gives us the composite life. We may also express (14') in the form

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i} = \frac{W}{D}, \tag{16}$$

and read the approximate value for n from Table V.

Example 2. Solve Example 1, using the sinking fund method and 5% interest.

Solution. Here, $W_1 = \$20{,}000$, $W_2 = \$18{,}000$, $W_3 = \$7{,}000$, $n_1 = 20$, $n_2 = 18$, $n_3 = 14$, $i = 0.05$.

Whence,

$$D_1 = 20{,}000 \frac{1}{s_{\overline{20}|.05}} = \$604.85,$$

$$D_2 = 18{,}000 \frac{1}{s_{\overline{18}|.05}} = \$639.83,$$

$$D_3 = 7{,}000 \frac{1}{s_{\overline{14}|.05}} = \$357.17,$$

$$D = \$1{,}601.85 \quad \text{and} \quad W = \$45{,}000.$$

Using (16), we get

$$s_{\overline{n}|.05} = \frac{(1.05)^n - 1}{0.05} = \frac{45{,}000}{1{,}601.85} = 28.0925.$$

From Table V, we notice that the nearest value of n is 18. In fact, when $n = 17$, the table value is 25.8404, and when $n = 18$, the table value is 28.1324. Hence, n is a little less than 18 and we say the composite life is approximately 18 years.

Using (15), we have

$$n = \frac{\log (3{,}851.85) - \log (1{,}601.85)}{\log (1.05)}$$

$$= \frac{3.58567 - 3.20462}{0.02119} = \frac{0.38105}{0.02119}$$

$$= 17.98, \text{ or approximately } 18.$$

Exercises

1. Allowing interest at 5%, find the composite life of the plant consisting of the following parts.

Parts	Original Cost	Scrap Value	Life
Building.....	$150,000	$40,000	25 years
Machinery....	75,000	25,000	25 years
Patterns....	15,000	.....	10 years
Tools.....	25,000	5,000	12 years

2. Solve Exercise 1, using the straight line method.

3. Allowing interest at 4%, find the composite life of the plant consisting of the following parts.

Parts	Cost	Scrap Value	Life
A	$200,000	$30,000	50 years
B	150,000	20,000	40 years
C	50,000	10,000	35 years
D	30,000	5,000	20 years
E	25,000	5,000	25 years

4. Solve Exercise 3 by the straight line method.

Problems

1. A church with a probable life of 75 years has just been completed at a cost of \$125,000. It is free of debt. For its replacement at the end of its probable life the congregation plans to make annual payments from their current funds into a sinking fund that will earn 4% effective. What is the annual payment?

2. The value of a machine decreases at a constant annual rate from the cost of \$1,200 to the scrap value of \$300 in 6 years. Find the annual rate of decrease, and the value of the machine at the ends of one, two, and three years.

3. The United States gross imports of crude rubber increased from 252,922 long tons in 1920 to 563,812 long tons in 1929. Find the annual rate of increase during this period, assuming that the annual rate of increase was constant.

4. A dormitory is planned at a cost of \$250,000. Its probable life is estimated to be 50 years at the end of which time its scrap value will be zero. To reconstruct the building at the end of its probable life, a sinking fund, into which semi-annual payments will be made, is to be created, the fund earning interest at ($j = .04$, $m = 2$). What is the semi-annual payment?

5. It is estimated that a quarry will yield \$15,000 per year for 8 years, at the end of which time it will be worthless. If a probable purchaser desires 8% on his investment and is able to accumulate a redemption fund at 4%, what should he pay for the quarry?

6. On a 3% basis find the annual charge for replacement of a plant, and its composite life, if the several parts are described by the table:

Part	Life in years	Cost	Scrap Value
A	40	\$200,000	\$10,000
B	25	50,000	3,000
C	15	20,000	1,000
D	10	10,000	1,000

7. A philanthropist wishes to donate a building to cost \$200,000 and to provide for its rebuilding every 50 years at the same cost. He also wishes to provide for its complete renovation every 10 years at a cost of \$20,000 and for annual repairs at a cost of \$2,000. What amount should he donate, if the sums can be invested at 4%?

8. In starting a transfer business it is planned to purchase 10 cabs annually for 5 years at a cost of \$1,000 per cab. On a 4% basis, what is the present value of these purchases if the first allotment is purchased immediately?

It is estimated that 5 years is the service life of these cabs. It is also planned to replace the worn out cabs by making annual payments at the end of each year into a sinking fund that earns 4% effective, R at the end of the first year, $2R$ at the end of the second year, $3R$ at the end of the third year, $4R$ at the end of the fourth year, $5R$ at the end of the fifth and later years. What is the annual payment into the sinking fund at the end of the first year? at the end of the second year? at the end of the fifth year? What is the amount in the sinking fund just after the first allotment for replacements? (See Art. 38.)

9. In starting a transfer business it is planned to purchase 10 cabs immediately, 8 cabs at the beginning of the second year, 6 at the beginning of the third year, 4 at the beginning of the fourth year and 2 at the beginning of the fifth year. On a 4% basis, what is the present value of these purchases if each cab costs \$1,000? (See Art. 38.)

10. Find the present value of the output of an oil well on the assumption that it will produce a net return of \$25,000 the first year, diminishing each year by \$5,000 until it is exhausted at the end of the fifth year. Use interest at 8% effective.

11. Show that the unit cost plan of appraisal of value gives the same result as the sinking fund method when the new and the old machines have the same output and the same annual expense charge for operation and upkeep.

Review Problems *

1. A quarry has sufficient stone to yield an income of \$20,000 a year for 5 years at the end of which time it will be exhausted. Find the value of the quarry if the investment is to yield 8% and the redemption fund is accumulated at 4%.

2. Telephone poles set in soil last 12 years, in concrete 20 years. If a telephone pole set in soil costs \$6, what can the company afford to pay to set the pole in concrete if money can be invested at 4%?

3. In computing the annual return at rate i on the capitalized cost, K, of an article, show that the return would be equivalent to allowing interest on the original investment, C, and allowing for depreciation by (6) Art. 50. (See Problem 5, page 121.)

4. A city incurs a debt of \$200,000 in constructing a high-school building. Which would be better: to pay the debt, principal and interest at 6½% in 20 annual installments, or to pay 6% interest each year on the debt and pay a fixed amount annually for 20 years into a sinking fund which accumulates at 4%?

5. A county borrows \$75,000 to build a bridge. The debt is to be paid by the amortization of the principal in 15 years at 6%. At the end of the tenth year what part of the debt is unpaid?

6. A man pays \$1,000 a year for 4 years and \$2,000 a year for four years on a debt of \$10,000 bearing interest at 6%. What part of the debt is unpaid at the end of 8 years?

7. A machine costing \$5,000 has an estimated life of 10 years and a scrap value of \$500. Find the constant rate at which it depreciates. What is its value at the end of the second year?

8. If W_r is the wearing value of a machine at the end of r years by the sinking fund method, show that

$$W_r = W \cdot \frac{a_{\overline{n-r}|}}{a_{\overline{n}|}}.$$

* For additional review problems, see end of this book.

CHAPTER VI

VALUATION OF BONDS

54. Definitions.—A *bond* may be defined as a certificate of ownership in a portion of a debt due from a city, corporation, government, or an individual. It is a promise to pay a stipulated sum on a given date, and to pay interest or *dividends* at a specified *dividend rate* and at definite intervals. The interval between dividend payments is usually a year, a half year, or a quarter year. The amount named in the bond is called the *face value* or *par value*. When the sum due is repaid as specified in the bond, the bond is surrendered to the debtor and it is said to be redeemed. The price at which a bond is redeemed is called the *redemption price*. It may be redeemed *at par*, *below par* or *above par*. When the redemption price of a bond is the same as the face value, it is said to be redeemed *at par*; if it is more than its face value it is said to be redeemed *at a premium*; and if it is less than its face value it is said to be redeemed *at a discount*.

55. Purchase price.—Bonds are usually bought to yield the purchaser a certain rate of interest on his investment. This rate may be very different from the rate of interest specified in the bond. To avoid confusion, we shall designate the rate of interest specified in the bond as the *dividend rate* and the rate of interest received by the purchaser, on his investment, as the *investment rate*. When an individual buys a bond he expects to receive the periodic dividends as they fall due from the date of purchase to the redemption date and also receive the redemption price when due. *It is clear then that the purchase price is really equal to the present value of the redemption price plus the present value of the annuity made from the periodic dividends, both figured at the investment rate.*

Example 1. Find the purchase price of a \$1,000, 4¼% bond, dividends payable annually, to be redeemed at par in 18 years when the investment rate is to be 6% annually.

Solution. Here, the redemption price is \$1,000, the dividend is \$42.50 annually. Denoting the purchase price by P, we get

$$P = 1{,}000(1.06)^{-18} + 42.50\,\frac{1 - (1.06)^{-18}}{0.06}$$

$$= 1{,}000(0.3503438) + 42.50(10.8276035)$$

$$= 350.34 + 460.17 = \$810.51.$$

Example 2. Find the purchase price of the above bond if it is to be redeemed at \$950.

Solution. $$P = 950(1.06)^{-18} + 42.50\,\frac{1 - (1.06)^{-18}}{0.06}$$

$$= 332.83 + 460.17 = \$793.00$$

If we let C = the redemption price,
(j, m) = nominal investment rate,
n = number of years before redemption,
R = the annual rent of the dividends,
p = the number of dividend payments each year,
and P = the purchase price,

we may write down the following general formula which will give the purchase price under all conditions.

$$P = C\left(1 + \frac{j}{m}\right)^{-mn} + R\,\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{p\left[\left(1 + \frac{j}{m}\right)^{m/p} - 1\right]}. \tag{1}$$

Now, if $m = p$ (that is, if the interest is converted at the same time that the dividends are paid), the above formula reduces to

$$P = C\left(1 + \frac{j}{p}\right)^{-np} + \frac{R}{p}\,\frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{\frac{j}{p}}. \tag{2}$$

In most cases formula (2) will apply.

When P is greater than C, the bond is *bought at a premium.* The difference, $(P - C)$, is the *premium.* Similarly, when P is less than C, the bond is *bought at a discount.* The difference, $(C - P)$, is the *discount.* When P equals C the bond is *bought at par.* The bond in Example 1 was bought at a discount of ($1,000 − $810.51), or $189.49.

Example 3. Find the purchase price of a $500, 6% bond, dividends payable semi-annually, to be redeemed at par in 20 years, when the investment rate is to be 5½% converted semi-annually.

Solution. Here, $C = \$500$, $n = 20$, $j = 5\frac{1}{2}\%$, $R = \$30$, $m = p = 2$. Using formula (2), we have

$$P = 500(1.0275)^{-40} + 15\frac{1 - (1.0275)^{-40}}{0.0275}$$

$$= 500(0.33785222) + 15(24.07810106)$$

$$= 168.926 + 361.172 = \$530.10$$

$$\text{Premium} = \$530.10 - \$500$$

$$= \$30.10.$$

Example 4. A $500, 5% bond, dividends payable semi-annually, is to be redeemed in 15 years at 104 (at 104% of the face). What should its purchase price be, if the investment rate is to be 6% converted semi-annually?

Solution. Since the bond is to be redeemed at 104, we have $C = \$520$. $n = 15$, $j = 6\%$, $R = \$25$, $m = p = 2$.

Making use of (2), we find

$$P = 520(1.03)^{-30} + 12.50\frac{1 - (1.03)^{-30}}{0.03}$$

$$= 520(0.41198676) + 12.50(19.60044135)$$

$$= 214.233 + 245.006 = \$459.24.$$

$$\text{Discount} = \$520 - \$459.24 \quad = \$60.76$$

If we let K equal the present value of the redemption price $= C\left(1+\frac{j}{p}\right)^{-np}$, and g equal the ratio of the annual rent of the dividends to the redemption price $= \frac{R}{C}$, formula (2) reduces to

$$P = K + \frac{g}{j}(C - K). \tag{3}$$

The student will notice that (3) does not require an annuity table for its evaluation. It was first established by Makeham, an English actuary.

CAUTION. Formula (2) was derived under the assumption $m = p$. Formula (3) was derived from (2). Therefore, (3) may be used only when $m = p$.

Exercises

Find the purchase price of each of the following:

1. A \$500, 6% bond, dividends payable semi-annually, redeemable in 10 years at par, the investment rate to be 5% convertible semi-annually.

2. A \$1,000, 5% bond, dividends payable semi-annually, redeemable in 12 years at 105, the investment rate to be 6% convertible semi-annually.

3. A \$10,000, 4% bond, dividends payable quarterly, redeemable in 20 years at 110, the investment rate to be 5% convertible quarterly.

4. A \$5,000, 7% bond, dividends payable annually, redeemable in 18 years at par, the investment rate to be 6% convertible annually.

5. A \$500, 5½% bond, dividends payable semi-annually, redeemable in 14 years at 102, the investment rate to be 6% convertible semi-annually.

6. Establish formula (3).

7. Use formula (3) to solve Example 3.

8. A \$2,000, 5% bond, dividends payable semi-annually, will be redeemed at 105 at the end of 10 years. Find the purchase price to yield 7% converted semi-annually.

9. Solve Exercise 8, with the yield rate (investment rate) 7% converted annually.

10. Should an investor, who wishes to make 6% (converted semi-annually) or more on his money, buy bonds at 88 which are to be redeemed in 10 years and bear 5% dividends payable semi-annually?

11. A \$5,000, 6% bond, dividends payable semi-annually, is to be redeemed in 16 years at 106. What should be paid for the bond if 5% (convertible annually) is to be realized on the investment?

56. Premium and discount.—If we subtract C from both members of formula (3) we will obtain the excess of purchase price over the redemption price. This result may be positive, negative, or zero. That is, the purchase price may be greater than the redemption price, less than the redemption price, or equal to the redemption price.

We have, if E is the excess,

$$E = P - C = K + \frac{g}{j}(C - K) - C$$

$$= \frac{g - j}{j}(C - K)$$

$$= \frac{g - j}{j}\left[C - C\left(1 + \frac{j}{p}\right)^{-np}\right]$$

$$= C\frac{g - j}{p} \cdot \frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{\frac{j}{p}}.$$

If we let k equal the excess of purchase price per unit of redemption price, it follows from the above equation that

$$k = \frac{g - j}{p} \cdot \frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{\frac{j}{p}}, \tag{4}$$

$$E = P - C = Ck, \quad \text{and} \quad P = C + Ck. \tag{5}$$

Example 1. A \$1,000, 6% semi-annual bond is to be redeemed in 10 years at \$1,050. Find the purchase price if the investment is to yield 5% semi-annually.

Solution. Here, $C = \$1{,}050$, $n = 10$, $j = 0.05$, $m = p = 2$, and $g = \frac{60}{1{,}050} = 0.057143$. Substituting in (4), we have

$$k = \frac{0.057143 - 0.05}{2} \cdot \frac{1 - (1.025)^{-20}}{0.025}$$

$$= (0.003571)(15.58916229)$$

$$= 0.055669.$$

And from (5), we get

$$E = P - C = 1{,}050(0.055669) = \$58.45.$$

Hence, the purchase price is \$58.45 more than the redemption price and

$$P = \$1{,}050 + \$58.45 = \$1{,}108.45.$$

In actual practice bonds are usually redeemed at par. Then C becomes the face value and $g = \dfrac{R}{C}$ becomes the actual dividend rate. Also, the value, k, obtained from (4) is the excess of purchase price per unit of face value, and the value, $P - C$, obtained from (5) is the premium or discount at which the bond is purchased. In fact, k is the premium or discount per unit of face value. It is evident that k is a premium when

$$g > j;$$

is a discount when

$$g < j;$$

is at par when

$$g = j.$$

Example 2. Solve Example 1, if the bond is to be redeemed at par.

Solution. Here, $C = \$1{,}000$, $n = 10$, $m = p = 2$, $j = 0.05$, and $g = 0.06$. We have

$$k = \frac{0.06 - 0.05}{2} \cdot \frac{1 - (1.025)^{-20}}{0.025} \quad \text{[Formula (4)]}$$

$$= (0.005)(15.58916229)$$

$$= 0.0779458.$$

And $E = P - C = 1{,}000(0.0779458) = \$77.95$

$$= \text{the premium.}$$

Hence, $P = \$1{,}000 + \$77.95 = \$1{,}077.95.$

Example 3. A \$500, 5% semi-annual bond is to be redeemed in 15 years at par. Find the purchase price if the investment is to yield 5½% semi-annually.

Solution. Here, $C = \$500$, $n = 15$, $m = p = 2$, $j = 0.055$, and $g = 0.05$. We have

$$k = \frac{0.05 - 0.055}{2} \cdot \frac{1 - (1.0275)^{-30}}{0.0275} \quad \text{[Formula (4)]}$$

$$= -(0.0025)(20.24930130) = -0.0506233.$$

And $E = P - C = 500(-0.0506233) = -\$25.31$

Hence, $P = \$500 - \$25.31 = \$474.69.$

That is, the discount is \$25.31 and the purchase price is \$474.69.

Exercises

Use formulas (4) and (5) in the solution of the following:

1. Find the purchase price of a \$1,000, 5% bond, dividends payable annually, redeemable in 20 years at par, if the investment rate is to be 5½% convertible annually.

2. Find the purchase price of a \$5,000, 4½% bond, dividends payable semi-annually, redeemable in 15 years at 102, if the investment rate is to be 4% convertible semi-annually.

3. What should be the purchase price of a \$10,000, 3½% bond, dividends payable semi-annually, redeemable in 35 years at par, if 4% (convertible semi-annually) is to be realized on the investment?

4. Find the purchase price of a \$500, 4½% bond, dividends payable quarterly, to be redeemed in 18 years at par, if the investment rate is to be 5% convertible quarterly.

5. What is the purchase price of a \$10,000, 6% bond, dividends payable semi-annually, redeemable in 30 years at 105, the investment rate to be 4½% convertible semi-annually?

6. What should be the purchase price of a \$1,000, 5% bond, dividends payable annually, to be redeemed in 10 years at 110, if the investment rate is to be 6% convertible annually?

7. Establish formula (4).

8. Use formulas (4) and (5) to solve Exercises 3 and 5, Art. 55.

9. Use formulas (4) and (5) to solve Exercise 10, Art. 55.

57. Amortization of premium and accumulation of discount.—When a bond is bought for more than the redemption value, provision should be made for restoring any excess of the original capital invested over the redemption price. The excess of interest on the bond over the interest required at the investment rate can and should be used for the gradual extinction of the excess book value* over the redemption price. The book value of a bond bought above redemption price thus diminishes at each interval until the redemption date, at which time its book value is equal to the redemption price. This amortization of the excess of purchase price over redemption price is called *amortization of the premium.*

When a bond is bought for less than the redemption price, we may think of it as having a periodically increasing book value, approaching the redemption price at maturity. The accumulation of the excess of redemption price over the purchase price is called *accumulation of the discount.* We shall illustrate by examples.

* The *book value* of a bond on a dividend date is the price P at which the bond would sell at a given investment rate.

Example 1. A \$1,000, 6% bond, dividends payable annually, redeemable in 6 years is bought to yield 5% annually. Find the purchase price and construct a schedule showing the amortization of the premium.

Solution. Here, $C = \$1{,}000$, $n = 6$, $j = 0.05$, $m = p = 1$, and $g = 0.06$.

Hence, $$k = 0.0507569.$$

$$\text{Premium} = P - C = \$50.76.$$

And $$P = \$1{,}050.76.$$

Now the book value of the bond at the date of purchase is \$1,050.76. At the end of the first year a \$60 dividend is paid on the bond. However, 5% on the book value for the first year is only \$52.54. This would leave a difference of \$60 − \$52.54 = \$7.46 for the amortization of premium for the first year. This would reduce the book value to \$1,043.30 for the second year. The interest on this amount at 5% is \$52.17. This leaves \$60 − \$52.17 = \$7.83 for the amortization of premium for the second year, and so on.

The following schedule shows the amount of amortization each year and the successive book values.

SCHEDULE OF AMORTIZATION—SCIENTIFIC METHOD

At End of Period	Dividend on Bond	Interest Earned on Book Value	Amortization of Premium	Book Value
0	.....	.....	.....	\$1,050.76
1	\$60.00	\$52.54	\$7.46	1,043.30
2	60.00	52.17	7.83	1,035.47
3	60.00	51.77	8.23	1,027.24
4	60.00	51.36	8.64	1,018.60
5	60.00	50.93	9.07	1,009.53
6	60.00	50.48	9.52	1,000.01
Total			\$50.75	

The amortization of the premium may also be cared for by the straight line method. By this method the premium is divided by the number of periods and the book value is decreased each period by this quotient. Thus, in the present problem we would have \$50.76 ÷ 6 = \$8.46. The following schedule illustrates the method.

SCHEDULE OF AMORTIZATION—STRAIGHT LINE METHOD

At End of Period	Dividend on Bond	Amortization	Book Value
0	.....	.....	$1,050.76
1	$60.00	$8.46	1,042.30
2	60.00	8.46	1,033.84
3	60.00	8.46	1,025.38
4	60.00	8.46	1,016.92
5	60.00	8.46	1,008.46
6	60.00	8.46	1,000.00

Example 2. A $10,000, 4% bond, dividends payable semi-annually, redeemable in 4 years, is bought to yield 5% semi-annually. Find the purchase price and construct a schedule showing accumulation of the discount.

Solution. Here, $C = \$10{,}000$, $n = 4$, $j = 0.05$, $m = p = 2$, and $g = 0.04$. Hence,

$$k = -0.0358506,$$

$$\text{Discount} = P - C = -\$358.51.$$

And

$$P = \$9{,}641.49.$$

The following schedule shows the accumulation of discount for each period and the book value for each period.

SCHEDULE OF ACCUMULATION—SCIENTIFIC METHOD

At End of Period	Dividend on Bond	Interest Earned on Book Value	Accumulation of Discount	Book Value
0	.....	.....	.....	$9,641.49
1	$200.00	$241.04	$41.04	9,682.53
2	200.00	242.06	42.06	9,724.59
3	200.00	243.11	43.11	9,767.70
4	200.00	244.19	44.19	9,811.89
5	200.00	245.28	45.28	9,857.17
6	200.00	246.43	46.43	9,903.60
7	200.00	247.59	47.59	9,951.19
8	200.00	248.78	48.78	9,999.97
Total			$358.48	

Exercises

1. A \$1,000, 5% bond, dividends payable semi-annually, redeemable in 7 years at par, is bought to yield 6% semi-annually. Construct an accumulation schedule.

2. A \$1,000, 5% bond, dividends payable annually, redeemable in 10 years, is bought to yield 4½% annually. Construct an amortization schedule.

3. Construct a schedule for the amortization of the premium of the bond in Exercise 1, Art. 55.

4. Construct an accumulation schedule for the bond of Exercise 6, Art. 56.

5. A \$500, 5% bond, pays dividends semi-annually and will be redeemed at 105 on January 1, 1946. It is bought on July 1, 1942, to yield 6% converted semi-annually. Find the purchase price and form a schedule showing the accumulation of the discount.

6. A \$5,000, 6% bond, paying semi-annual dividends will be redeemed at 110 on September 15, 1947. Find the price on September 15, 1942, to yield 5% converted semi-annually, and form a schedule showing the amortization of the premium.

58. Bonds purchased between dividend dates.—We shall consider two cases.

(a) When the bond is bought at a certain *quoted price and accrued interest* with no apparent regard for yield.

(b) When the bond is bought on a *strictly yield basis.*

By accrued interest in case (a) is meant accrued *simple* interest on the *face value* at the rate named in the bond. In other words, we mean the accrued dividend. We shall illustrate by an example.

Example 1. A bond of \$1,000 dated July 1, 1940, bearing 6% interest payable semi-annually, was purchased March 1, 1941, at 98.5 and accrued interest. What was paid for the bond?

Solution. The dividend dates are July 1, and Jan. 1. The *price quoted* on this bond is evidently \$985.00. Hence, the price paid on March 1 is \$985.00 plus the interest on \$1,000 from Jan. 1 to March 1 at 6%, or

$$\$985.00 + \$10.00 = \$995.00, \text{ purchase price.}$$

The student should observe that the purchase price is equal to the quoted price plus the dividend accrued from the last dividend date to the time of purchase.

When the bond is bought at a price to yield a given rate of interest on the investment, *the purchase price is equal to the value (purchase price) of the bond at the last dividend date (the one just before the date of purchase) plus the interest, at the investment rate, on this value, from the last dividend date to the date of purchase.* In practice, ordinary simple interest is used.

If P_0 stands for the purchase price at the last dividend date and d is the number of days from the last dividend date to the date of purchase, the purchase price may be defined by the formula

$$P = P_0 + \frac{P_0 dj}{360}. \tag{6}$$

Example 2. A bond of \$500 issued March 1, 1930, at 4% payable semi-annually and to be redeemed March 1, 1947, was purchased May 10, 1938, to realize 5% (converted semi-annually) on the investment. What should have been paid for the bond? Find the quoted price.

Solution. The time from March 1, 1938 (the last dividend date) to March 1, 1947 (the redemption date), is 9 years, and the purchase price as of the last dividend date is

$$P_0 = 500(1.025)^{-18} + 10\,\frac{1 - (1.025)^{-18}}{0.025} = \$464.12.$$

The time from March 1, 1938 (the last dividend date), to May 10, 1938 (the date of purchase), is 70 days.

Hence,
$$\frac{P_0 dj}{360} = \frac{(464.12)(70)(0.05)}{360} = \$4.51$$

and $P = 464.12 + 4.51 = \$468.63$, the purchase price on May 10, 1938.

Now, the quoted price as of May 10, 1938, is the purchase price as of that date minus the dividend accrued from March 1, 1938, to May 10, 1938. The accrued dividend is the ordinary simple interest on \$500 for 70 days at 4%, or \$3.89.

Hence, the quoted price is

$$\$468.63 - \$3.89 = \$464.74.$$

The student should observe the difference between purchase price and quoted price. Bonds are usually quoted on the market at a certain price plus accrued interest (at the dividend rate), guaranteed to yield a certain rate of interest on the investment. In the case of the above bond the quoted price as of May 10, 1938, would have been \$464.74 (or 92.95% of face) and accrued interest to yield 5% semi-annually on the investment if held to the date of redemption.

Exercises

1. A \$1,000, 6% bond, dividends payable semi-annually, dated January 1, 1942, was purchased September 10, 1944, at 97.5 and accrued interest. What was paid for the bond?

2. The bond described in Exercise 1 is to mature January 1, 1949. What should have been paid for it September 10, 1944, if purchased to yield 7% semi-annually?

3. At what price should a \$500, 6% semi-annual bond, dated April 1, 1939, and maturing April 1, 1946, be bought July 10, 1940, to yield 5½%, semi-annually, on the investment? Find the quoted price.

4. Should an investor, who wished to make 5% nominal, converted semi-annually, on his investment, have bought government bonds quoted at 89 on February 1, 1920? These bonds were redeemable November 15, 1942, and bore 4¼% interest, payable semi-annually.

5. On July 20, 1935, a man bought 5% semi-annual bonds, due October 1, 1945, on a 6% semi-annual basis. The interest dates were April 1 and October 1. What price did he pay? Find the quoted price for that date.

6. A \$1,000, 6% bond, dividends payable March 15 and September 15, is redeemable March 15, 1950. It was bought January 1, 1944, to yield 5½% converted semi-annually. Find the purchase price and the quoted price.

7. Find the quoted price for the bond of Exercise 6, as of July 5, 1947.

59. Annuity bonds.—An *annuity bond* is an interest-bearing bond, payable, principal and interest, in equal periodic payments or installments. It is evident that these equal periodic payments constitute an annuity whose present value is the face of the bond. The periodic payment can be found by using Art. 31. The purchase price at any date is the present value (figured at the investment rate) of the annuity composed of the periodic payments yet due. Let us illustrate by an example.

Example. At what price should a 4% annuity bond for \$5,000, payable in 8 equal annual payments, be purchased at the end of 3 years (just after the third payment has been made), if 5% (converted annually) is to be realized on the investment?

Solution. Using Art. 31, we find the periodic payment to be

$$R = 5{,}000\,\frac{0.04}{1-(1.04)^{-8}} = \$742.64.$$

The purchase price at the end of 3 years is equal to the present value of an annuity of \$742.64 for 5 years at 5% converted annually.

Hence,
$$P = 742.64\,\frac{1-(1.05)^{-5}}{0.05} = \$3{,}215.24.$$

60. Serial bonds.—When selling a set of bonds, a corporation may wish to redeem them in installments instead of redeeming all of the bonds on one date. When a bond issue is to be redeemed in several installments instead of all the bonds being redeemed on one date, the issue is known as a *serial issue* and the bonds of the issue are known as *serial bonds*. Evidently, the purchase price at any date is equal to the sum of the purchase prices of the installments yet to be redeemed.

Example. A city issues \$40,000 worth of 4% bonds, dividends payable semi-annually, to be redeemed by installments of \$4,000 in 2 years, \$6,000 in 4 years, \$8,000 in 6 years, \$10,000 in 8 years and \$12,000 in 10 years. An insurance company buys the entire issue on the date of issue so as to realize 5% (converted semi-annually) on the investment. What price was paid for the entire issue?

Solution. The purchase price of the entire issue is equal to the sum of the purchase prices of the five installments to be redeemed. Using (5), Art. 56, we have

$$4{,}000 - 4{,}000(0.005)\,\frac{1-(1.025)^{-4}}{0.025} = \$\ 3{,}924.76$$

$$6{,}000 - 6{,}000(0.005)\,\frac{1-(1.025)^{-8}}{0.025} = \$\ 5{,}784.90$$

$$8{,}000 - 8{,}000(0.005)\,\frac{1-(1.025)^{-12}}{0.025} = \$\ 7{,}589.69$$

$$10{,}000 - 10{,}000(0.005)\,\frac{1-(1.025)^{-16}}{0.025} = \$\ 9{,}347.25$$

$$12{,}000 - 12{,}000(0.005)\,\frac{1-(1.025)^{-20}}{0.025} = \$11{,}064.65$$

and $$P = \$37{,}711.25.$$

Hence, the purchase price of the issue is \$37,711.25.

Exercises

1. At what price should a 5% (payable semi-annually) annuity bond for \$10,000, payable in 26 equal semi-annual payments, be purchased at the end of 6 years, if 5½% (converted semi-annually) is to be realized on the investment?

2. A \$25,000 serial issue of 6% bonds, with semi-annual dividends, is to be redeemed by payments of \$5,000 at the end of 3, 4, 5, 6, and 7 years respectively. Find the purchase price of the entire issue, if bought now to realize 5% (converted semi-annually) on the investment. [Use (4) and (5) Art. 56.]

3. What is the purchase price of a bond of \$20,000 payable \$5,000 in 4 years, \$8,000 in 6 years, \$5,000 in 7 years, and \$2,000 in 9 years, with dividends at 5% semi-annually, if the purchaser is to receive 6%, converted semi-annually, on his investment?

4. Find the purchase price of a 10-year annuity bond for \$25,000, to be paid in semi-annual installments with interest at 6% converted semi-annually, if purchased at the end of 4 years to yield 5% converted semi-annually.

5. Find the purchase price on the date of issue of a \$2,000 bond bearing 4%, the principal and interest to be paid in 6 equal annual installments, if the purchaser is to realize 5% (convertible semi-annually) on his investment.

61. Use of bond tables.—Tables are available which give the purchase prices of bonds corresponding to given dividend rates, investment rates and times to maturity. These tables may be made as comprehensive as their purpose demands. The dividend rates may range from as low as 2 per cent to 8 or 9 per cent by intervals of ⅛ per cent. The investment rates may have about the same range, but with smaller intervals. The times to maturity may range from ¼, ½ or 1 year to 50 or 100 years by intervals of ¼, ½ or 1 year depending on whether or not the dividends are payable quarterly, semi-annually or annually. These tables may be arranged in various forms. The following is a brief portion of a bond table:

TABLE SHOWING PURCHASE PRICES OF A 4% BOND FOR \$1,000 WITH DIVIDENDS PAYABLE SEMI-ANNUALLY

Investment Rate Converted Semi-annually	Time to Maturity			
	5 Years	10 Years	15 Years	20 Years
2.00	\$1,094.71	\$1,180.46	\$1,258.08	\$1,328.35
2.50	1,070.09	1,131.99	1,186.67	1,234.95
3.00	1,046.11	1,085.84	1,120.08	1,149.58
3.50	1,022.75	1,041.88	1,057.97	1,071.49
4.00	1,000.00	1,000.00	1,000.00	1,000.00
4.50	977.83	960.09	945.89	934.52
5.00	956.24	920.05	895.35	874.49
5.50	935.20	885.71	848.14	819.41
6.00	915.70	851.23	804.00	768.85

Example. A \$500, 4% bond, dividends payable semi-annually, redeemable in 15 years at par, is bought to yield 5½% convertible semi-annually. Find its purchase price.

Solution. Observing the above table, we find the purchase price of a \$1,000 bond corresponding to the given dividend rate, investment rate and time to maturity is \$848.14. But we are considering a \$500 bond. Consequently, its purchase price is \$424.07.

Exercises

1. Consider a \$500 bond due in 20 years, and bearing semi-annual dividend coupons at 4% per annum. Find by the use of the above table the purchase price if the investment rate is to be 4½%. Check the result by calculations independent of the table.

2. Solve Exercise 1, if the investment rate is to be (*a*) 3%; (*b*) 3½%; (*c*) 5%; (*d*) 6%.

3. Consider a \$500, 4% bond, dividends payable semi-annually, which matures in 10 years. Using the above table and the method of interpolation find the approximate purchase price when the investment rate is to be (*a*) 3¾% (*b*) 5¼%. Check (*a*) by using formula (5), Art. 124 and logarithms.

4. Solve Exercise 3, if the investment rate is to be (*a*) 3¼%; (*b*) 4¾%.

62. Determining the investment rate when the purchase price of a bond is given.—At times the price of a bond is quoted on the market, guaranteed to yield a certain rate of interest on the investment, provided the bond is held until the date of maturity. At other times the price is quoted, but no investment rate is given. Before purchasing a bond at a certain price, the prospective buyer would naturally want to know (approximately at least) the rate of interest that would be realized by such an investment. Therefore, it is very important that we have a method of finding the investment rate when the purchase price is given. We shall discuss two methods: (a) when bond and annuity tables are available; (b) when no tables are available.

(a) When either bond or annuity tables are given the approximate investment rate may be found by the method of interpolation. We shall illustrate by examples.

Example 1. Find the rate of income realized on a 6% bond purchased for \$105, 10 years before maturity.

Solution. Since the bond is bought at a premium the investment rate will be less than the dividend rate. Let us try 5%.

Then, $$P = 100(1.05)^{-10} + 6\frac{1-(1.05)^{-10}}{0.05}$$

$$= 61.39 + 46.33 = \$107.72.$$

Evidently the investment rate is greater than 5%. Let us now try 5½%.

Then, $$P = 100(1.055)^{-10} + 6\frac{1-(1.055)^{-10}}{0.055}$$

$$= 58.54 + 45.23 = \$103.77.$$

We observe that the investment rate must lie between 5% and 5½%. Arranging the results thus obtained, we have

Cost	Investment Rate
107.72	5%
105.00	x%
103.77	5½%

Interpolating, we have

$$\frac{107.72 - 105.00}{107.72 - 103.77} = \frac{5 - x}{5 - 5\frac{1}{2}},$$

$$\frac{2.72}{3.95} = \frac{x - 5}{\frac{1}{2}},$$

$$3.95x = 21.11,$$

$$x = 5.344\%.$$

Example 2. Find the rate of income realized on a 4% semi-annual bond, purchased for \$94.50, 10 years before maturity.

Solution. Try 4½%. Then,

$$P = 100(1.0225)^{-20} + 2\frac{1 - (1.0225)^{-20}}{0.0225}$$

$$= \$96.01$$

The rate is evidently greater than 4½%. We shall now try 5%.

$$P = 100(1.025)^{-20} + 2\frac{1 - (1.025)^{-20}}{0.025}$$

$$= \$92.20.$$

We observe that the rate lies between 4½% and 5%.

Cost	Investment Rate
96.01	4½%
94.50	x%
92.20	5%

Interpolating, we have

$$\frac{96.01 - 94.50}{96.01 - 92.20} = \frac{4\frac{1}{2} - x}{4\frac{1}{2} - 5},$$

$$\frac{1.51}{3.81} = \frac{x - 4\frac{1}{2}}{\frac{1}{2}},$$

$$1.51 = 7.62x - 34.29,$$

$$7.62x = 35.80,$$

$$x = 4.7\%.$$

The student will observe that we find a rate that gives a purchase price a little larger than the given purchase price and then a rate which gives a purchase price a little smaller than the given purchase price. We then find the approximate rate by interpolation.

Example 3. A \$1,000, 4% bond, dividends payable semi-annually, was bought 20 years before maturity at \$850.25. Using the above bond table, Art. 61, find the approximate investment rate.

Solution.

When $j = 0.050$, $P = \$874.49$.

When $j = 0.055$, $P = \$819.41$.

Then,
$$j = 0.0500 + \frac{874.49 - 850.25}{874.49 - 819.41}(0.055 - 0.050)$$

$$= 0.0500 + \frac{24.24}{55.08}(0.005)$$

$$= 0.0500 + 0.0022 = 0.0522 = 5.22\%.$$

Exercises

1. Find the rate of income realized on a 5% semi-annual bond maturing in 18½ years when bought at \$103.35.

2. A \$1,000, 5% bond with semi-annual dividends, is redeemable at par at the end of 12 years. If it is quoted at \$1,075.60, what is the investment rate?

3. Find the effective rate realized by investing in 5% bonds with semi-annual dividends, redeemable at par, which are quoted at 84.2, 10 years before redemption.

4. A state bond bearing 5% interest, payable semi-annually, and redeemable in 8 years at par, was sold at 95. Find the yield rate.

5. On November 15, 1930, a certain United States Government bond sold at 90. If this bond is redeemable November 15, 1952, and bears 4% interest, payable semi-annually, find the yield rate on November 15, 1930.

6. A 4% bond, dividends payable semi-annually, was bought 15 years before maturity at 92.5. Using the bond table, find the approximate investment rate.

7. Using the bond table, find the approximate investment rate, when a 4% bond, dividends payable semi-annually, is bought 10 years before maturity at 106.3.

(b) When tables are not available the approximate investment rate may be found by solving formula (4), Art. 56 for j. This formula may be written

$$\frac{g - j}{k} = \frac{j}{1 - \left(1 + \frac{j}{p}\right)^{-np}}. \tag{7}$$

Expanding $\left(1+\frac{j}{p}\right)^{-np}$ by the binomial theorem and neglecting all terms that involve j^3 and higher powers of j, we get

$$\left(1+\frac{j}{p}\right)^{-np} = 1 - nj + \frac{np(np+1)}{2}\cdot\frac{j^2}{p^2},$$

and

$$\frac{g-j}{k} = \frac{j}{nj - \frac{n(np+1)}{2p}j^2} = \frac{1}{n - \frac{n(np+1)j}{2p}}.$$

Multiplying the above equation through by n and dividing out the right-hand member, we obtain

$$\frac{n(g-j)}{k} = 1 + \frac{np+1}{2p}j \text{ (approximately).}$$

Solving for j, we have

$$j = \frac{2p(ng-k)}{np(k+2)+k}, \tag{8}$$

which will give the approximate investment rate.

Example 4. Let us now apply formula (8) to Example 1, of Art. 62.

Solution. Here, $k = 0.05$, $n = 10$, $p = 1$, and $g = 0.06$.

Then,

$$j = \frac{2(0.60 - 0.05)}{10(2 + 0.05) + 0.05} = 0.05353,$$

and the approximate investment rate is 5.353%.

We notice that the result obtained by using formula (8) is approximately the same as that obtained by using annuity tables.

Ordinarily, (8) will give a result which is accurate enough. At least, it is accurate enough for the layman who might be interested in the purchasing of bonds. Naturally, bond houses and individuals dealing in bonds and quoting bond prices, to yield a certain rate of interest on the investment, would require a more accurate method. However, these people would have comprehensive bond and annuity tables available, by which the investment rate could be found to the required degree of accuracy.

Exercises

1. Apply formula (8) to Examples 2 and 3 of Art. 62.

2. Apply formula (8) to Exercises 1, 3, and 5 of Art. 62(a), page 158.

3. Apply formula (8) to Exercises 2, 4, and 6 of Art. 62(a), page 158.

4. A person bought a \$1,000, 5% bond, dividends payable semi-annually, 18 years before maturity for \$975. Find the investment rate by using annuity tables and then check the result by using formula (8).

5. A \$500, 3¼% Government bond, dividends payable June 15 and December 15, was bought June 15, 1945, for \$530. If this bond is to be redeemed December 15, 1956, find the investment rate as of June 15, 1945.

Problems

1. A \$1,000, 5% bond, dividends payable April 15 and October 15, maturing October 15, 1946, was bought April 15, 1943, to yield ($j = .06, m = 2$). Construct a schedule showing the accumulation of the discount.

2. A \$1,000, 6% bond, dividends payable semi-annually, maturing in 4 years, was bought to yield ($j = .05, m = 2$). Construct a schedule showing the amortization of the premium.

3. A \$300,000 issue of highway bonds bearing 4% interest, payable semi-annually, dated January 1, 1944, matures \$100,000 January 1, 1945, 1946 and 1947. What price should be paid for the issue to realize ($j = .03, m = 2$)?

4. A \$1,000 bond paying 5% semi-annually, redeemable at \$1,040 in 10 years, has been purchased for \$970. Find the investment rate.

5. A 4%, J. and J.,* bond is redeemable at par on January 1, 1952. Find the yield if it is purchased July 1, 1939, at 89.32.

6. A \$1,000, 6%, J. and J., bond is redeemable at par on July 1, 1950. Find the price to yield ($j = .05, m = 2$) on August 16, 1940.

7. Find the purchase price of a \$100, 5% bond, dividends payable semi-annually and redeemable at par in 10 years, to yield 6% effective.

8. Find the purchase price of a \$100, 4% bond, dividends payable semi-annually and redeemable in 20 years at 120, to yield 5% effective.

* That is, the dividends are payable January 1 and July 1.

CHAPTER VII

PROBABILITY AND ITS APPLICATION IN LIFE INSURANCE

63. The history of probabilities.—Aristotle (384–322 B. C.), the Greek philosopher, is credited with the first attempt to define the measure of a probability of an event. Aristotle says an event is probable when the majority, or at least the majority of the most intellectual persons, deem it likely to happen.

But the first real mathematical treatment of probability originated as isolated problems coming from games of chance. Cardan (1501–1576) and Galileo, two Italian mathematicians, solved many problems relating to the game of dice. Aside from his regular occupation as a mathematician, Cardan was also a professional gambler. As such he had evidently noticed that there was always more or less cheating going on in the gambling houses. This led him to write a little treatise on gambling in which he discussed some mathematical questions involved in the games of dice then played in the Italian gambling houses. The aim of this little book was to fortify the gamester against such cheating practices. Galileo was not a gambler, but was often consulted by a certain Italian nobleman on problems relating to the game of dice. As a result of these consultations and his investigations he has left a short memoir. Pascal (1623–1662) and Fermat (1601–1665), two great French mathematicians, were also consulted by professional gamblers and this led them to make their contributions to the subject of chance.

The Dutch physicist, Huyghens (1629–1695), and the German mathematician, Leibnitz (1646–1716), also wrote on chance. However, the first extensive treatise on the subject of chance was written by Jacob Bernoulli (1654–1705). In this treatment of the subject which was published in 1713, the author shows many applications of the new science to practical problems.

The first English treatise on probabilities was written by Abraham de Moivre (1667–1754). This was a remarkable treatment and may yet be read with profit. This book was translated into German by the Austrian mathematician, E. Czuber.

It was left for La Place (1749–1827), that great French mathematician, to leave the one really famous treatise on the theory of chance, "Théorie Analytique des Probabilitiés." Since the time of La Place many books and articles on the theory have been written by mathematicians in all lands.

The subject of probability has become so widespread in its applications that the best minds of the world have undertaken its further development. Today, the physicist, the chemist, the biologist, the statistician, the actuary, depend upon the results of the theory of probability for the development of their respective fields.

Probably the earliest writer on the application of the theory of probability to social phenomena was John Graunt (1620–1674) who, in 1662, published his "Observations on the London Bills of Mortality." The astronomer, Edmund Halley, published his Mortality Tables in 1693. Adolphe Quetelet (1796–1874) devoted his life to the applications of probabilities to scientific research, particularly to the study of populations.

Following the work of these investigators, life insurance organizations began to function. With the organization of the Equitable Society of London in 1762, life insurance was successfully placed on a scientific basis. The company employed the mathematician, Dr. Richard Price, to be the actuary to determine the premiums which should be charged. He drew up the Northampton Table of Mortality in 1783, and from this event insurance as a science may be said to date.

64. Meaning of *a priori* probability.—A box contains three white and four black balls. One ball is drawn at random and then replaced and this process is continued indefinitely. What proportion of the balls drawn will be black? Here there are seven balls to be drawn or we may say there are seven possibilities, and either of the seven balls is *equally likely* to be drawn or any one of the seven possibilities is *equally likely* to happen. Of the seven possibilities, any one of three would result in drawing a white ball and any one of four would result in drawing a black ball. We would say then that three possibilities of the seven are favorable to drawing a white ball and the other four possibilities are favorable to drawing a black ball. We put the above statement in another way by saying that in a single draw the probability of drawing a white ball is $\frac{3}{7}$ and the probability of drawing a black ball is $\frac{4}{7}$. This does not mean that out of only seven draws, exactly three would be white and four black. But it does mean that, if a single ball were drawn at random and were replaced and this process continued indefinitely, $\frac{3}{7}$ of the balls drawn would be white and $\frac{4}{7}$ would be black. Or the ratio of the number of white balls drawn to the number of black balls drawn would be as 3 to 4.

Reasoning similarly to the above led La Place to formulate the following *a priori* definition of probability:

If h is the number of possible ways that an event will happen and f is the number of possible ways that it will fail and all of the possibilities are *equally likely*, the probability that the event will happen is $p = \frac{h}{h+f}$ and the probability that it will fail is $q = \frac{f}{h+f}$.

It is evident, then, that the sum of the probability that an event will happen and the probability that it will fail is 1, the symbol for certainty.

In analyzing a number of possibilities we must be sure that each of them is *equally likely* to happen before we attempt to apply the above definition of probability.

Example: What is the probability that a man aged 25 and in good health will die before age 30? In this case we might reason thus: The event can happen in only one way and fail in only one way, and consequently the probability that he will die before age 30 is ½. But this reasoning is false for we are assuming that living five years and dying within five years are *equally likely* for a man now 25 years old. But this is not the actual experience. This example will be discussed in Art. 65.

Exercises

1. A bag contains 7 white and 5 black balls, and a ball is drawn at random. What is the probability (a) that the ball is white? (b) that the ball is black?

2. A deck of 52 cards contains 4 aces. If a card is drawn at random, what is the probability that it will be an ace?

3. A coin is tossed. What is the probability that it will fall head up?

4. If the probability of winning a game is ⅗, what is the probability of losing?

5. If the probability of a man living 10 years is 0.6, what is the probability of his dying within 10 years?

6. If a cubical die is tossed, what is the probability that it will fall with 6 up?

7. Two coins are tossed at random. What is the probability of obtaining (a) two heads? (b) one head and one tail?

8. Two cubical dice are tossed at random. Find the probability that the sum of the numbers is 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12.

9. A box contains 45 tickets numbered from 1 to 45. If a ticket is drawn at random, what is the probability that the number on it is (a) odd? (b) even? (c) divisible by 5? (d) larger than 35?

10. A coin and a cubical die are tossed simultaneously. Find the probability that they will fall with the coin head up and with a face on the die numbered less than 5.

11. Three coins are tossed. What is the probability of exactly two heads?

12. Which is the more likely to happen, a throw of 4 with one die or a throw of 8 with two dice?

13. A and B each throw two dice. If A throws 8, find the probability that B will throw a larger number.

65. Relative frequency. Empirical probability.—In the example and the exercises of Art. 64 the probabilities are derived in each case by an *a priori* determination of all the equally likely ways in which the event in question can happen. There are many classes of events in which the notion of probability is important although it is impossible to make an *a priori* determination of all the equally likely ways an event can happen or fail. In such cases we determine an approximate probability empirically by means of a large number of observations. Such determinations are necessary in the establishment of life insurance, pension systems, fire insurance, casualty insurance, and statistics.

If we have observed that an event has happened h times out of n possible ways, we call h/n the *relative frequency* of the event. When n is a large number, h/n may be considered a fair estimate of the probability derived from observation. Our confidence in the estimate increases as the number n of observed cases increases. If, as n increases indefinitely, the ratio h/n approaches a limiting value, this limiting value is the probability of the happening of the event. That is

$$p = \lim_{n \to \infty} \frac{h}{n}.$$

In statistical applications the limit of h/n cannot in general be determined, but satisfactory approximations to the limit may be found for many practical purposes.

We are now ready to solve the problem which was stated in Art. 64. The American Experience Table of Mortality shows that out of 89,032 men living at age 25, the number living at age 30 will be 85,441. Then the number dying before age 30 is 89,032 − 85,441 or 3,591. Hence the probability that a man aged 25 will die before age 30 is $\frac{3{,}591}{89{,}032} = .0403$. In this problem, n equals 89,032 and h equals 3,591.

We have previously stated that the value h/n is only an estimate, but it is accurate enough (when n is a large number) for many practical purposes. Life insurance companies use the American Experience Table of Mortality as a basis to determine the proper premiums to charge their policy holders.

Exercises

1. Among 10,000 people aged 30, 85 deaths occurred in a year. What was the relative frequency of deaths for this group?

2. Out of 10,000 children born in a city in a given year, 5,140 were boys and 4,860 were girls. What was the relative frequency of boy babies in the city that year?

3. A group of 10,000 college men was measured as to height. Of these, 1,800 were between 68 and 69 inches high. Estimate the relative frequency of height of college men between 68 and 69 inches.

66. Permutations. Number of permutations of things all different.— Each of the different ways that a number of things may be arranged is known as a *permutation* of those things. For example the different arrangements of the letters *abc* are: *abc, acb, bac, bca, cab, cba.* Here there are 3 different ways of selecting the first letter and after it has been selected in one of these ways there remain 2 ways of selecting the second letter. Then the first two letters may be selected in $3 \cdot 2$ or 6 ways. It is clear that we have no choice in the selection of the third letter and consequently the total number of permutations (or arrangements) of the three letters is 6. This example illustrates the following:

Fundamental Principle: *If one thing may be done in* p *ways and after it has been done in one of these ways, another thing may be done in* q *ways, then the two things together may be done in the order named in* pq *ways.*

It is evident that for each of the p ways of doing the first thing there are q ways of doing the second thing and the total number of ways of doing the two in succession is pq.

The above principle may be extended to three or more things.

Exercises

1. If 2 coins are tossed, in how many ways can they fall?

2. If 3 coins are tossed, in how many ways can they fall?

3. If 2 dice are thrown, in how many ways can they fall?

4. If 2 dice and 3 coins are tossed, in how many ways can they fall?

5. How many signals can be made by hoisting 3 flags if there are 9 different flags from which to choose?

6. In how many different ways can 3 positions be filled by selections from 15 different people?

7. How many four-digit numbers can be formed from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9?

Now suppose there are n things all different and we wish to find the number of permutations of these things taken r at a time, $n \geqq r$.

Since only r of the n things are to be used at a time, there are only r places to be filled. The first place may be filled by any one of the n things and the second place by any one of the $n - 1$ remaining things. Then the first and second places together may be filled in $n(n - 1)$ ways. The third place may be filled by any one of the $n - 2$ remaining things. Hence the first three places may be filled in $n(n - 1)(n - 2)$ ways. Reasoning in a similar way we see that after $r - 1$ places have been filled, there remain $n - (r - 1)$ things from which to fill the rth place. Applying the fundamental principle stated above we have

$$*_nP_r = n(n - 1)(n - 2) \cdots (n - r + 1). \tag{1}$$

When $r = n$, (1) becomes,

$$_nP_n = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1 = \dagger n!. \tag{2}$$

Exercises

1. A man has two suits of clothes, four shirts and three hats. In how many ways may he dress by changing suits, shirts and hats?

2. How many arrangements of the letters in the word "Mexico" can be made, using in each arrangement (a) 4 letters? (b) all the letters?

3. Four persons enter a street car in which there are 7 vacant seats. In how many ways may they be seated?

4. Three different positions in an office are to be filled and there are 15 applicants, each one being qualified to fill any one of the positions. In how many ways may the three positions be filled?

5. How many signals could be made from 5 different flags?

6. Find the number of permutations, P, of the letters $a\ a\ b\ b\ b$ taken 5 at a time. Hint: $P \cdot 2! \cdot 3! = 5!$.

7. If P represents the number of distinct permutations of n things, taken all at a time, when, of the n things, there are n_1 alike, n_2 others alike, n_3 others alike, etc., then:

$$P = \frac{n!}{n_1!n_2!n_3! \ldots}.$$

8. How many distinct permutations can be made of the letters of the word *attention* taken all at a time?

9. How many distinct permutations of the letters of the word *Mississippi* can be formed taking the letters all at a time?

* The symbol $_nP_r$ is used to denote the number of permutations of n things taken r at a time.

† $n!$ is a symbol which stands for the product of all the integers from 1 up to and including n, and is read "factorial n."

10. How many ways can ten balls be arranged in a line if 3 are white, 5 are red, and 2 are blue?

11. How many six-place numbers can be formed from the digits 1, 2, 3, 4, 5, 6, if 3 and 4 are always to occupy the middle two places?

12. In how many ways can 3 different algebras and 4 different geometries be arranged on a shelf so that the algebras are always together?

13. In how many ways can 10 boys stand in a row when:

(a) a given boy is at a given end?
(b) a given boy is at an end?
(c) two given boys are always together?
(d) two given boys are never together?

67. Combinations. Number of combinations of things all different.—*By a combination we mean a group of things without any regard for order of arrangement of the individuals within the group.* For example *abc, acb, bac, bca, cab, cba* are the same combination of the letters *abc*, but each arrangement is a different permutation.

By the number of combinations of n things taken r at a time is meant the number of different groups that may be formed from n individuals when r individuals are placed in each group. For example *ab*, *ac*, and *bc* are the different combinations of the letters *abc* when two letters are used at a time.

The symbol ${}_nC_r$ is universally used to stand for the number of combinations of n things taken r at a time. We will now derive an expression for ${}_nC_r$. For each one of the ${}_nC_r$ combinations there are $r!$ different permutations. And for all of the ${}_nC_r$ combinations there are ${}_nC_r \cdot r!$ permutations, which is the number of permutations of n things taken r at a time. Hence,

$$ {}_nC_r \cdot r! = {}_nP_r, $$

and

$$ {}_nC_r = \frac{{}_nP_r}{r!}. \tag{3} $$

Since

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1), $$

we have

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \tag{4} $$

$$ = \frac{n!}{r!(n-r)!} \tag{4'} $$

Exercises

1. Find the number of combinations of 10 things taken 7 at a time.

Solution. Here, $n = 10$ and $r = 7$.

Then, $$_{10}C_7 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 120.$$

2. How many committees of 5 can be selected from a group of 9 men?

3. Out of 8 Englishmen and 5 Americans how many committees of 3 Englishmen and 2 Americans can be chosen?

4. How many different sums can be made up from a cent, a nickel, a dime, a quarter, and a dollar?

5. An urn contains 5 white and 7 black balls. If 4 balls are drawn at random what is the probability that (a) all are black, (b) 2 are white and 2 are black?

Solution. (a) The total number of ways that 4 balls may be drawn from 12 balls is $_{12}C_4$ or 495 ways. And the number of ways that 4 black balls may be drawn is $_7C_4$ or 35 ways. Hence the probability of drawing 4 black balls is $^{35}/_{495}$ or $^{7}/_{99}$.

(*b*) Two white balls may be drawn in $_5C_2$ or 10 ways. And for each one of these 10 ways of drawing two white balls, two black balls may be drawn in $_7C_2$ or 21 ways. Then two white balls and two black balls may be drawn together in 10×21 or 210 ways (Fundamental Principle, Art. 66). Hence, the probability of drawing 2 white and 2 black balls is $^{210}/_{495}$ or $^{14}/_{33}$.

6. A bag contains 4 white, 6 black, and 7 red balls. If 4 balls are drawn at random, what is the probability that (a) all are black, (b) 2 black and 2 red, (c) 1 white, 1 black, and 2 red?

7. Prove that $_nC_r = {}_nC_{n-r}$.

8. Prove that the expansion of the binomial $(a + b)^n$ may be written

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots + b^n$$

$$= \sum_{r=0}^{r=n} {}_nC_ra^{n-r}b^r,$$

if we define $_nC_0$ to be 1.

9. How many straight lines are determined from 10 points, no 3 of which are in the same straight line?

10. How many different sums can be made from a cent, a nickel, a dime, a quarter, a half-dollar, and a dollar?

11. From 10 books, in how many ways can a selection of 6 be made: (a) when a specified book is always included? (b) when a specified book is always excluded?

12. Prove that $_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

13. Out of 6 different consonants and 4 different vowels, how many linear arrangements of letters, each containing 4 consonants and 3 vowels, can be formed?

14. A lodge has 50 members of whom 6 are physicians. In how many ways can a committee of 10 be chosen so as to contain at least 3 physicians?

15. In the equation of Exercise 8, make $a = b = 1$, and show that

$$ {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1. $$

16. Solve Exercise 4 above, using Exercise 15.

17. In how many ways can 7 men stand in line so that 2 particular men will not be together?

18. A committee of 7 is to be chosen from 8 Englishmen and 5 Americans. In how many ways can a committee be chosen if it is to contain: (a) just 4 Englishmen? (b) at least 4 Englishmen?

19. Prove: ${}_{n+2}C_{r+1} = {}_nC_{r+1} + 2 \cdot {}_nC_r + {}_nC_{r-1}$.

20. If ${}_nP_r = 110$ and ${}_nC_r = 55$, find n and r.

21. If ${}_nC_4 = {}_nC_2$, find n.

22. If ${}_nC_3 = 10/21({}_nC_5)$, find n.

23. If ${}_{2n}C_{n-1} = 91/24({}_{2n-2}C_n)$, find n.

24. Prove: ${}_nC_1 + 2 \cdot {}_nC_2 + 3 \cdot {}_nC_3 + \cdots + n \cdot {}_nC_n = n(2)^{n-1}$.

25. How many line-ups are possible in choosing a baseball nine of 5 seniors and 4 juniors from a squad of 8 seniors and 7 juniors, if any man can be used in any position?

68. Some elementary theorems in probability.—Sometimes it is convenient to consider an event as made up of simpler events. The given event is then said to be *compound.* Thus, the compound event may be made of simpler *mvtually exclusive events,* simpler *independent events,* or simpler *dependent events.*

A. Mutually Exclusive Events. Two or more events are said to be mutually exclusive when the occurrence of any one of them excludes the occurrence of any other. Thus, in the toss of a coin the appearance of heads and the appearance of tails are mutually exclusive. Also, if a bag contains white and black balls and a ball is drawn, the drawing of a white ball and the drawing of a black ball are mutually exclusive events.

Theorem. *Mutually exclusive events. If $p_1, p_2, \ldots, p_r$ are the separate probabilities of r mutually exclusive events, the probability that one of these events will happen on a particular occasion when all of them are in question is*

$$ P = p_1 + p_2 + p_3 + \ldots + p_r, \tag{5} $$

the sum of the separate probabilities.

This theorem follows from the definition of mutually exclusive events. For if $a_1, a_2, a_3, \cdots, a_r$, indicate the number of ways the separate events can happen, then the number of ways favorable to some event

is $a_1 + a_2 + a_3 + \cdots + a_r$. If m represents the total number of possibilities, favorable and unfavorable, then

$$P = \frac{a_1 + a_2 + a_3 + \cdots + a_r}{m} = \frac{a_1}{m} + \frac{a_2}{m} + \frac{a_3}{m} + \cdots + \frac{a_r}{m}$$
$$= p_1 + p_2 + p_3 + \cdots + p_r.$$

When two mutually exclusive events are in question, the probabilities are frequently called *either or* probabilities. Thus, if a die is thrown, the probability of *either* an ace *or* a deuce is $\frac{1}{6} + \frac{1}{6}$ or $\frac{1}{3}$.

B. Independent Events. Two or more events are *dependent* or *independent* according as the occurrence of any one of them does or does not affect the occurrence of the others. Thus, if A tosses a coin and B throws a die, the tossing of heads by A and the throwing of a deuce by B are independent events. However, if a bag contains a mixture of white and black balls and a ball is drawn and not returned to the bag, the probabilities in a second drawing will be dependent upon the first event.

Theorem. *Independent events. If $p_1, p_2, \ldots, p_r$ are the separate probabilities of r independent events, the probability that all of these events will happen together at a given trial is the product of their separate probabilities.*

Let $p_1 = a_1/m_1, p_2 = a_2/m_2, \ldots, p_r = a_r/m_r$ be the simple probabilities; where $a_1, a_2, \ldots, a_r$ are the ways favorable to the happening of the separate events; and $m_1, m_2, \ldots, m_r$ are the possible ways in which the separate events may happen or fail. By the Fundamental Principle, Art. 66, the number of ways favorable to the happening together of the r events is $a_1a_2 \ldots a_r$. And by applying the same principle we get $m_1m_2 \ldots m_r$ as the number of possible ways that the r events might happen or fail. Consequently,

$$P = \frac{a_1a_2 \cdots a_r}{m_1m_2 \cdots m_r}$$
$$= p_1p_2 \cdots p_r, \tag{6}$$

and the theorem is proved.

Corollary. *If $p_1, p_2, \ldots, p_r$ are the separate probabilities of r independent events, the probability that they will all fail on a given occasion is*

$$(1 - p_1)(1 - p_2) \ldots (1 - p_r), \tag{7}$$

and the probability that the first k events will succeed and the remainder fail is

$$p_1p_2 \ldots p_k(1 - p_{k+1}) \ldots (1 - p_r). \tag{8}$$

C. Dependent events. The following theorem for dependent events may be proved by a similar method to that used for independent events.

Theorem. *Dependent events. Let p_1 be the probability of a first event; let p_2 be the probability of a second event after the first has happened; let p_3 be the probability of a third event after the first two have happened; and so on. Then the probability that all of these events will occur in order is*

$$P = p_1 p_2 \ldots p_r. \tag{9}$$

Exercises

1. The probability that A will live 20 years is $\frac{1}{7}$, the probability that B will live 20 years is $\frac{1}{6}$, and the probability that C will live 20 years is $\frac{1}{5}$. What is the probability that all three will be living in 20 years?

Solution. We have here three independent events, where

$$p_1 = \tfrac{1}{7}, \quad p_2 = \tfrac{1}{6}, \quad \text{and} \quad p_3 = \tfrac{1}{5}.$$

Hence, $$P = (\tfrac{1}{7})(\tfrac{1}{6})(\tfrac{1}{5}) = \tfrac{1}{210}.$$

2. Find the probability of drawing 2 white balls in succession from a bag containing 4 white and 7 black balls, if the first ball drawn is not replaced before the second drawing is made.

Solution. We have here two dependent events. The probability that the first draw will be white is $\dfrac{4}{4+7} = \dfrac{4}{11}$; the probability that the second draw will be white is $\dfrac{3}{3+7} = \dfrac{3}{10}$.

Hence, $$p_1 = \tfrac{4}{11}, \quad p_2 = \tfrac{3}{10},$$

and $$P = (\tfrac{4}{11})(\tfrac{3}{10}) = \tfrac{6}{55}.$$

3. A and B, with others, are competitors in a race. The probability that A will win is $\frac{1}{4}$ and the probability that B will win is $\frac{1}{3}$. What is the probability that either A or B will win?

Solution. We have here two mutually exclusive events.

Hence, $$P = \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{12}.$$

4. Four coins are tossed at once. What is the probability that all will be heads?

5. A bag contains 3 white balls, 4 black balls and 5 red balls. One ball is drawn and not replaced, then a second ball is drawn and not replaced and then a third ball is drawn. What is the probability (a) that a ball of each color will be drawn, (b) that 2 blacks and 1 red will be drawn, (c) that all will be red?

6. Suppose that in Exercise 5 the balls are replaced after each draw. Then answer (a), (b) and (c).

7. Three men ages 28, 30 and 33 respectively form a partnership. What is the probability (a) that all three will be living at the end of 10 years, (b) that the first two

will be living, (c) that one only of the three will be living? Use the American Experience Table of Mortality, Table XI.

8. A man and wife are 29 and 25 years of age when they marry. What is the probability that they will both live to celebrate their golden wedding?

9. A, B, and C go bird-hunting. A has a record of 1 bird out of 2, B gets 2 out of 3, and C gets 3 out of 4. What is the probability that they will kill a bird at which all shoot simultaneously?

10. If the probability that A will die within a year is $\frac{2}{10}$ and the probability that B will die within a year is $\frac{3}{10}$, what is the probability that (a) both A and B will die within a year? (b) both A and B will live a year? (c) one life will fail within a year? (d) at least one life will fail within a year?

11. The probability that A will solve a problem is $\frac{1}{3}$ and that B will solve it is $\frac{2}{3}$. What is the probability that if A and B try the problem will be solved?

12. From a group of 6 men and 5 women, a committee of 5 is chosen by lot. What is the probability that it will consist of (a) all women? (b) all men? (c) 3 men and 2 women?

13. A committee of 7 is chosen from a group of 8 Englishmen and 5 Americans. What is the probability that it will contain (a) exactly 4 Englishmen? (b) at least 4 Englishmen?

14. From a lottery of 30 tickets marked 1, 2, . . . , 30, four tickets are drawn. What is the probability that the numbers 1 and 15 are among them?

15. From a pack of 52 cards, 3 cards are drawn at random. What is the probability that they are all clubs?

69. Mathematical expectation.—The *expected number of occurrences* of an event in n trials is defined to be np where p is the probability of occurrence of the event in a single trial.

Illustrations. If 100 coins are thrown or if one coin is thrown 100 times, theoretically, we " expect " 50 heads and 50 tails, for $n = 100$ and $p = \frac{1}{2}$.

If a die is rolled 36 times we " expect " an ace to turn up 6 times, for $n = 36$ and $p = \frac{1}{6}$.

If 0.008 is the probability of death within a year of a man aged 30, the " expected " number of deaths within a year among 10,000 men of this age would be 80, for $n = 10{,}000$ and $p = 0.008$.

If p is the probability of obtaining a sum of money, k, then pk represents the *mathematical expectation.*

Illustration. Suppose that 1,000 men, all aged 30, contribute to a fund with the understanding that each survivor will receive \$1,000 at age 60. The mortality tables show that approximately 678 will be alive. Hence, the expectation of each would be \$678. The fund must contain \$678,000 in order that each survivor receive \$1,000. Hence, neglecting interest, each of the 1,000 men will have to contribute \$678 to the fund.

70. Repeated trials.—When the probability that an event will happen in a single trial is known, it becomes a question of importance to determine the probability that the event will happen a specified number of times in a given number of trials.

To familiarize us with the method of proof of the general theorem of repeated trials, let us consider the

Example. What is the probability of throwing 2 aces in 4 throws of a die?

The conditions of the problem are met if in the first 2 throws we obtain aces and in the next 2 throws not-aces; or if in the first throw we get ace, the second throw not-ace, the third throw ace, and the fourth throw not-ace; and so on. We shall illustrate the possibilities symbolically as follows:

$$A_1A_2 --,\ A_1 - A_3 -,\ A_1 -- A_4,\ - A_2A_3 -,\ - A_2 - A_4,\ -- A_3A_4$$

Considering the first case, the probability of throwing an ace on any throw is $\frac{1}{6}$. The probability of not throwing an ace on any throw is $\frac{5}{6}$. Hence the probability of throwing an ace on the first and second throws and not throwing an ace on the two remaining throws is $(\frac{1}{6})^2(\frac{5}{6})^2$.

In the second case, the probability of events occurring as the symbol above indicates is $(\frac{1}{6})(\frac{5}{6})(\frac{1}{6})(\frac{5}{6}) = (\frac{1}{6})^2(\frac{5}{6})^2$.

The remaining cases may be treated in a similar manner, and in each instance the result for any specified set is $(\frac{1}{6})^2(\frac{5}{6})^2$. Now it is evident that the 2 aces can be selected from the 4 possible aces in ${}_4C_2 = 6$ ways. Since the 6 cases are mutually exclusive, the chance that one or the other of the specified cases occurs is $6(\frac{1}{6})^2(\frac{5}{6})^2 = {}^{150}\!/_{1296}$.

Let us now consider the important

Theorem of Repeated Trials. *If* p *is the probability of the success of an event in a single trial and* q *is the probability of its failure,* $(p + q = 1)$, *then the probability* P_r *that the event will succeed exactly* r *times in* n *trials is**

$$P_r = {}_nC_rp^rq^{n-r}. \tag{10}$$

For the probability that the event will succeed in each of r specified trials and will fail in the remaining $(n - r)$ trials is, by (6), p^rq^{n-r}. Further, it is possible for the r successes to occur out of n trials in ${}_nC_r$ different ways. These ways being mutually exclusive, by (5) the probability in question is $P_r = {}_nC_rp^rq^{n-r}$.

* It will be noted that (10) is the $(n - r + 1)$th term of the expansion $(p + q)^n$ and the $(r + 1)$th term of the expansion $(q + p)^n$.

The various probabilities are indicated in the following table:

VALUES OF P_r FOR VARIOUS VALUES OF r

r	P_r	*The Probability That in* n *Trials There Will Be*
n	p^n	n successes, 0 failures
$n-1$	${}_nC_1p^{n-1}q$	$n-1$ " , 1 "
$n-2$	${}_nC_2p^{n-2}q^2$	$n-2$ " , 2 "
.....	.....	....., ...
.....	.....	....., ...
$n-r$	${}_nC_rp^{n-r}q^r$	$n-r$ successes, r failures
.....	.....	..... " , ... "
r	${}_nC_rp^r\,q^{n-r}$	r " , $n-r$ "
.....	.....	..... " , ... "
2	${}_nC_2p^2q^{n-2}$	2 " , $n-2$ "
1	${}_nC_1pq^{n-1}$	1 " , $n-1$ "
0	q^n	0 " , n "
Total....	$(p+q)^n = 1$	

From the above table we have at once the following:

Corollary. *The probability that an event will succeed at least* r *times in* n *trials is* $P_r + P_{r+1} + \cdots + P_n$, *that is*:

$$\sum_r^n P_r = p^n + {}_nC_1p^{n-1}q + {}_nC_2p^{n-2}q^2 + \ldots + {}_nC_rp^rq^{n-r}. \qquad (11)$$

It will be noted that (11) consists of the first $(n-r+1)$ terms of the expansion $(p+q)^n$.

Example 1. An urn contains 12 white and 24 black balls. What is the probability that, in 10 drawings with replacements, exactly 6 white balls are drawn?

Solution. We have:

$$p = {}^{12}\!/_{36} = {}^{1}\!/_{3}, \quad q = {}^{24}\!/_{36} = {}^{2}\!/_{3},$$

$$n = 10, \quad r = 6, \quad n - r = 4.$$

Hence,

$$P_6 = {}_{10}C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^4 = \frac{3360}{3^{10}}.$$

Example 2. The American Experience Mortality Table states that for an individual aged 25 the probability of survival a year is $p = 0.992$ and the probability of death within a year is $q = 0.008$. Out of a group of 1,000 individuals aged 25, how many are expected to survive a year? What are some conclusions that may be drawn from the terms of the binomial expansion $(.992 + .008)^{1,000}$?

Solution. We have $n = 1{,}000$, $p = 0.992$, $q = 0.008$. By Art. 69, we expect $np = 1{,}000(0.992) = 992$ to survive the year, and $nq = 1{,}000(0.008) = 8$ to die within a year.

The terms of the expansion

$$(.992 + .008)^{1,000} = (.992)^{1,000} + 1{,}000(.992)^{999}(.008) + {}_{1,000}C_2(.992)^{998}(.008)^2 + \ldots + (.008)^{1,000}$$

give, by equation (10), the following probabilities:

$(.992)^{1,000}$ gives the probability that 1,000 will survive a year; $1{,}000(.992)^{999}(.008)^1$ gives the probability that 999 will live a year and 1 will die within a year, and so on.

Problems

1. If there are five routes from London to Cambridge, and three routes from Cambridge to Lincoln, how many ways are there of going from London to Lincoln going by the way of Cambridge?

2. Out of 20 boys and 25 girls, in how many ways can a couple be selected?

3. A committee of 5 is to be chosen from 15 Englishmen and 18 Americans. If the committee is to contain exactly 3 Americans and 2 Englishmen, in how many ways may it be chosen?

4. From 10 Democrats and 8 Republicans a committee of 3 is to be selected by lot. Find the probability that it will consist (a) of 2 Democrats and 1 Republican, (b) of 2 Republicans and 1 Democrat, (c) of 3 Democrats, (d) of 3 Republicans. What is the sum of the four answers?

5. Out of a party of 12 ladies and 15 gentlemen, in how many ways can 4 ladies and 4 gentlemen be selected for a dance?

6. In how many ways can 3 men choose hotels in a town where there are 6 hotels?

7. In how many ways can A, B, and C choose hotels in a town where there are 6 hotels, if (a) A and B refuse to stay at the same hotel, (b) they all stay at different hotels, (c) they all stay at the same hotel?

8. In how many ways can 7 books be arranged on a shelf, if 3 particular books are to be together?

9. How many signals can be made with 7 flags of different colors by arranging them on a mast (a) all together, (b) 4 at a time, (c) at least 1 at a time?

10. If the probability that A will die in 10 years is 0.2, that B will die in 10 years is 0.3, and that C will die in 10 years is 0.25, what is the probability that at the end of

10 years (a) all will be dead, (b) all will be living, (c) only two will be living, (d) at least two will be living?

11. If two dice are thrown, what is the probability of obtaining an odd number for the sum?

12. In tossing 10 coins, what is the probability of obtaining at least 8 heads?

13. A man whose batting average is $\frac{3}{10}$ will bat 4 times in a game. What is the probability that he will get 4 hits? 3 hits? 2 hits? at least 2 hits?

14. A machinist works 300 days in a year. If the probability of his meeting with an accident on any particular day is $\frac{1}{1000}$, what is the probability that he will entirely escape an accident for a year?

15. If it is known that 2 out of every 1,000 dwelling houses worth $5,000 burn annually, what is the risk assumed in insuring such a house for one year?

16. According to the American Experience Mortality Table out of 100,000 persons living at age 10 years, 91,914 are living at the age of 21 years. Each of 100 boys is now 10 years old. What is the probability that exactly 50 of them will live to be 21?

71. Meaning of mortality table.—If it were possible to trace a large number of persons, say 100,000, living at age 10 until the death of each occurred, and a record kept of the number living at each age x and the number dying between the ages x and $x + 1$, we would have a mortality table.

However, mortality tables are not constructed by observing a large number of individuals living at a certain age until the death of each, for it is evident that this method would not be practicable, but would be next to impossible, if not impossible. Mathematical methods have been devised for the construction of such tables, but the scope of this text does not permit the discussion of these methods.

Table XI is known as the American Experience Table of Mortality and is based upon the records of the Mutual Life Insurance Company of New York. It was first published in 1868 and is used for most life insurance written in the United States. It will be used in this book as a basis for all computations dealing with mortality statistics. It consists of five columns as follows: The first giving the ages running from 10 to 95, the different ages being denoted by x; the second giving the number living at the beginning of each age x and is denoted by l_x; the third giving the number dying between ages x and $x + 1$ and is denoted by d_x; the fourth giving the probability of dying in the year from age x to $x + 1$ and is denoted by q_x; and the fifth giving the probability of living a year from age x to age $x + 1$ and is denoted by p_x.

The American Experience Table, now 77 years old, is not expected to represent present-day experience. It is conservative in its estimates for insurance and thereby contributes a factor of safety to policies. Whatever added profit comes from its use is generally passed on to policy-

holders as dividends. It is now generally prescribed in the state laws as the standard for insurance evaluations.

While the American Experience Table furnishes a safe basis for insurance valuations, it is not at all suitable for the valuation of annuities. Annuities are paid to individuals during the years that they live, and computations based upon a table with mortality rates lower than the actual might easily cause a company to lose money. For the valuation of annuities, the American Experience Table is not legally prescribed so that the companies have been free to employ tables that more accurately represent the mortality they experience. The American Annuitants' Table is widely used for the valuation of annuities.

The American Experience Table and the American Annuitants' Table are " select " tables inasmuch as they show the mortality rates after the selection caused by medical examination. In 1915 the larger insurance companies of the United States cooperated in developing the American Men Mortality Table. It too is a " select " table.

Many mortality tables have been based upon the experience of the general population. Such a table includes many in poor health and others engaged in hazardous or unhealthy occupations. Since the rates of mortality in a table constructed from population records are higher than the rates of mortality of the select tables, such a table is unsuitable for life insurance valuations.

The *United States Life Tables** shows the rates of mortality among the general population in certain parts of the United States. For purposes of comparison, these tables are very enlightening, though they are inapplicable for insurance and annuity evaluations.

The following table shows the rates of mortality per 1,000 for a few ages according to the mortality tables that we have mentioned.

RATES OF MORTALITY PER 1,000

Age	American Experience	American Men	U. S. Life Table, 1910	American Annuitants'	
				Male	Female
30	8.43	4.46	6.51	4.99	4.52
35	8.95	4.78	8.04	6.00	5.27
40	9.79	5.84	9.39	7.51	6.39
45	11.16	7.94	11.52	9.78	8.07
50	13.78	11.58	14.37	13.15	10.56

* *United States Life Tables*, J. W. Glover, published by the Bureau of Census, Washington, D. C.

Exercises

1. What is the probability that a man aged 30 will live to be 65? What is the probability that the same man will die before reaching 65? What is the sum of the two probabilities?

2. Find the probability that a man aged 70 will live 10 years.

3. Suppose 100,000 lives age 10 were insured for one year by a company for $1,000 each, what would be the cost to each individual, neglecting the interest?

4. What would be the cost of $1,000 insurance for one year of an individual 30 years old, neglecting the interest, if based upon (a) the American Experience Table? (b) the American Men Table? (c) the United States Life Table?

5. Solve Exercise 4 for an individual aged 50?

72. Probabilities of life.—In Art. 71 we discussed the meaning of the mortality table and gave something concerning its history. We now derive some useful formulas based upon this table. We notice certain relations existing among the elements l_x, d_x, p_x and q_x of the table.

Since l_{x+1} denotes the number of people living at age $x + 1$ and l_x denotes the number living at age x, the probability, p_x, that a person age x will live one year is given by

$$p_x = \frac{l_{x+1}}{l_x}. \tag{12}$$

Since d_x stands for the number of people dying between the ages x and $x + 1$, the probability, q_x, that a person age x will die within a year is given by

$$q_x = \frac{d_x}{l_x}. \tag{13}$$

Since it is certain that a person age x will either live one year or die within the year, we have

$$p_x + q_x = 1. \tag{14}$$

From (12) and (13), we get

$$p_x + q_x = \frac{l_{x+1}}{l_x} + \frac{d_x}{l_x} = \frac{l_{x+1} + d_x}{l_x}.$$

Hence,

$$\frac{l_{x+1} + d_x}{l_x} = 1,$$

and

$$d_x = l_x - l_{x+1}. \tag{15}$$

The number of deaths between the ages x and $x + n$ is given by

$$l_x - l_{x+n} = d_x + d_{x+1} + \cdots + d_{x+n-1}. \tag{16}$$

When $(x + n)$ exceeds the oldest age in the table,

$l_{x+n} = 0$, and (16) becomes

$$l_x = d_x + d_{x+1} + \cdots \text{ to end of table.} \tag{17}$$

The probability that a person aged x will live n years is denoted by the symbol ${}_np_x$. Thus ${}_{15}p_{10}$ means the probability that a person aged 10 will live 15 years and is 89,032 ÷ 100,000 or 0.89032.

In general,

$${}_np_x = \frac{l_{x+n}}{l_x}. \tag{18}$$

The probability that a person aged x will die within n years is denoted by ${}_nq_x$. Since a person aged x will either live n years or die within that time, we have

$${}_np_x + {}_nq_x = 1, \text{ or}$$

$${}_nq_x = 1 - {}_np_x, \tag{19}$$

$$= 1 - \frac{l_{x+n}}{l_x},$$

$${}_nq_x = \frac{l_x - l_{x+n}}{l_x}. \tag{20}$$

The probability that a person aged x will die in the year after he reaches age $x + n$ is denoted by ${}_n|q_x$. This may be regarded as the compound event that consists of a person aged x living n years and one aged $x + n$ dying within that year. Thus we have

$${}_n|q_x = {}_np_x \cdot q_{x+n} \quad \text{(Art. 68)}$$

$$= \frac{l_{x+n}}{l_x} \cdot \frac{d_{x+n}}{l_{x+n}} = \frac{d_{x+n}}{l_x}. \tag{21}$$

Since

$$d_{x+n} = l_{x+n} - l_{x+n+1},$$

$$\frac{d_{x+n}}{l_x} = \frac{l_{x+n}}{l_x} - \frac{l_{x+n+1}}{l_x}$$

and

$${}_n|q_x = {}_np_x - {}_{n+1}p_x. \tag{22}$$

We observe from (22) that the probability that a person aged x will die in the year after reaching age $(x + n)$ is equal to the probability that a person aged x will live n years minus the probability that a person aged x will live $n + 1$ years.

The probability that a person aged x will live n years, and one aged y will die within that period is

$$ {}_np_x \cdot |\, {}_nq_y = {}_np_x(1 - {}_np_y). \quad [(6), \text{Art. } 68]. \tag{23} $$

Exercises

1. Verify from the table that $p_{15} = \dfrac{l_{16}}{l_{15}}$.

2. Verify that $q_{15} = \dfrac{d_{15}}{l_{15}}$. Does $p_{15} + q_{15} = 1$?

3. Verify that $l_{15} - l_{18} = d_{15} + d_{16} + d_{17}$.

4. Verify that $l_{90} = d_{90} + d_{91} + \ldots$ to end of table.

5. What is the probability that a person aged 20 will live 30 years and die within the next year?

6. Find the probability that a person aged 30 will live to be 65.

7. What is the probability that a person aged 25 will die within 10 years? What is the probability that he will die in the year after he reaches 35?

Problems

1. Find the probability that a man aged 40 will live to be 70.

2. What is the probability that three persons, each age 40, will all reach the age of 50? What is the probability that none will reach that age?

3. A boy 15 years old is to receive \$20,000 on attaining the age of 21. Neglecting interest, what is the value of the boy's expectation?

4. Show that the probability that at least one of two lives aged x and y, respectively, will survive n years is given by the expression ${}_np_x + {}_np_y - {}_np_x \cdot {}_np_y$. Hint: We have here three mutually exclusive events.

5. A father is 40 years old and his son is 15. What is the probability that both will live 10 years? What is the probability that at least one will live 10 years?

6. What is the probability that a person aged 40 will die in the year just after reaching 60?

7. If we assume that out of 10,000 automobiles of a certain class there are 70 thefts during the year, what would it cost an insurance company to insure 1,000 such cars against theft at \$700 each? What would be the premium on one such car? In this problem running expenses and interest on money are neglected.

8. Show that the probability that at least one of three lives x, y, z, respectively, will survive n years is given by the expression:

$$ {}_np_x \cdot {}_np_y \cdot {}_np_z - ({}_np_x \cdot {}_np_y + {}_np_y \cdot {}_np_z + {}_np_x \cdot {}_np_z) + {}_np_x + {}_np_y + {}_np_z. $$

9. A man 35 years of age and his wife 33 years of age are to receive \$10,000 at the end of 10 years if both are then living to receive it. Neglecting interest, what is the value of their expectation?

10. Two persons, A and B, are 42 and 45 years of age respectively. Find the probability (a) that both will survive 10 years, (b) that both will die within 10 years, (c) that A will survive 10 years and B will die during the time, (d) that B will survive 10 years and A will not survive. What is the sum of the four answers?

11. A man 50 years old will receive \$5,000 at the end of 10 years if he is alive. At 4% interest, find the present value of his expectation.

12. What is the probability that a man aged 50 will live 20 years longer?

13. Given two persons of ages x and y, express the probability that:

(a) both will live n years,
(b) both will die within n years,
(c) exactly one will live n years,
(d) exactly one will die within n years.

14. To what events do the following probabilities refer?

(a) $1 - {}_np_x \cdot {}_np_y$.
(b) $(1 - {}_np_x)(1 - {}_np_y)$.
(c) $1 - {}_n|q_x \cdot {}_n|q_y$.
(d) ${}_np_x \cdot p_{x+n}$.

15. Each of 7 boys is now 10 years old. What is the probability that (a) all seven will live to be 21 years old? (b) at least five of them will live to be 21?

16. Given 1,000 persons aged x, write expressions in terms of p_x and q_x for the following probabilities:

(a) that exactly 10 will die within a year.
(b) that not more than 10 will die within a year.

17. Prove: ${}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m} = {}_np_x \cdot {}_mp_{x+n}$.

18. Prove: ${}_5p_x = p_x \cdot p_{x+1} \cdot p_{x+2} \cdot p_{x+3} \cdot p_{x+4}$.

19. Translate the symbolic statement of Problem 18 into words.

20. Prove: ${}_np_x = p_x \cdot p_{x+1} \cdot p_{x+2} \cdot \cdots \cdot p_{x+n-1}$.

CHAPTER VIII

LIFE ANNUITIES

73. Pure endowments.—*A pure endowment is a sum of money payable to a person whose present age is x, at a specified future date, provided the person survives until that date.* We now find the cost of a pure endowment of \$1 to be paid at the end of n years to a person whose present age is x. The *symbol*, ${}_nE_x$, will represent the cost of such an endowment.

Suppose l_x individuals, all of age x, agree to contribute equally to a fund that will assure the payment of \$1 to each of the survivors at the end of n years. From the mortality table we see that out of the l_x individuals entering this agreement, l_{x+n} of them would be living at the end of n years. Consequently, the fund must contain l_{x+n} dollars at that time in order that each of the survivors receives \$1. The present value of this sum is

$$v^n \cdot l_{x+n},$$

where

$$v = \frac{1}{1+i} = (1+i)^{-1}.$$

The present value of the money contributed to the fund by the l_x individuals is

$${}_nE_x \cdot l_x.$$

If we equate the present value of the money contributed to the fund and the present value of the money received from the fund by the survivors, we have

$$l_x \cdot {}_nE_x = v^n \cdot l_{x+n}$$

and

$${}_nE_x = \frac{v^n l_{x+n}}{l_x}. \tag{1}$$

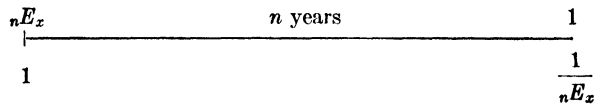
The preceding method of derivation is known as "the mutual fund" method. The formula may also be derived by using the notion of mathematical expectation.

It is clear that ${}_nE_x$ will be the *present value of the mathematical expectation,* which is the present value of \$1 due in n years multiplied by the probability that a person aged x will live n years. Consequently

$$ {}_nE_x = v^n \cdot {}_np_x = v^n \frac{l_{x+n}}{l_x}, $$

which is the same as (1).

It should be emphasized that ${}_nE_x$, the present value of \$1 payable in n years to a person aged x if he lives to receive it, is dependent upon the rate of interest i and the probability that (x) will live n years.* Since these two fundamental factors v^n and ${}_np_x$ are generally each less than unity, ${}_nE_x$ is generally less than unity. Further, considering both *interest* and *survivorship,* the quantity ${}_nE_x$ may be looked upon as a **discount** *factor* being the discounted value of 1 due in n years to (x). Similarly, the quantity $1/{}_nE_x$ may be looked upon as an *accumulation factor,* being the accumulated value at the end of n years of 1 due now to (x). The line diagram shows the equivalent values.



It is obvious that the present value A, of R payable in n years to (x), is given by

$$ A = R \cdot {}_nE_x. \tag{1'} $$

If the numerator and the denominator of (1) be multiplied by v^x, we get

$$ \frac{v^{x+n} l_{x+n}}{v^x l_x}, $$

and if we agree that the product $v^x l_x$ shall be denoted by the symbol D_x, (1) becomes

$$ {}_nE_x = \frac{D_{x+n}}{D_x}. \tag{2} $$

D_x is one of four symbols, called commutation symbols, that are used to facilitate insurance computations (see Table XII). This table is based on the American Experience Table of Mortality and a 3½% interest rate is used. There are other commutation tables based upon different tables of mortality and different rates of interest.

* We shall frequently use the symbol (x) to mean "a person aged x" or "a life aged x."

It will be observed as the theory develops that we rarely use the values given in the mortality table except to compute the values of the commutation symbols.

Unless otherwise specified, all computations in the numerical exercises will be based upon the American Experience Table of Mortality with 3½ per cent per annum as the interest rate.

Exercises

1. Find the present value (cost) of a pure endowment of \$5,000, due in 20 years and purchased at age 30, interest at 3½%.

Solution. Here, $x = 30$, $n = 20$, and

$$ {}_{20}E_{30} = \frac{D_{50}}{D_{30}} = \frac{12498.6}{30440.8} = 0.410587. \quad \text{[Formula (2) and Table XII]} $$

Hence,
$$ A = (5{,}000.00)\ {}_{20}E_{30} = 5{,}000\ (0.410587) $$
$$ = \$2{,}052.94. $$

2. Solve Exercise 1, with the rate of interest 3%.

3. An heir, aged 14, is to receive \$30,000 when he becomes 21. What is the present value of his expectation on a 4% basis?

4. Find the cost of a pure endowment of \$2,000, due in 10 years and purchased at age 35, interest at 3½%.

5. What pure endowment due at the end of 20 years could a person aged 45 purchase for \$5,000? Assume 3½% interest.

6. Solve Exercise 5, assuming 4% interest.

7. A boy aged 12 is to receive \$10,000 upon attaining age 21. Find the present value of the inheritance on a 4% basis.

8. A man aged 30 has \$10,000 that he wishes to invest with an insurance company that operates on a 3½% basis. He wishes the endowment to be payable to him when he attains the age of 50 years. What would be the amount of the investment at that time if he agrees to forfeit all rights in the event of death before he reaches age 50?

9. Two payments of \$5,000 each are to be received at the ends of 5 and 10 years respectively. Find the present value at 3½%

(a) if they are certain to be received;

(b) if they are to be received only if (25) is alive to receive them.

10. What pure endowment payable at age 65 could a man age 25 purchase with \$1,000 cash?

11. To what formula would the formula for ${}_nE_x$ reduce if (x) were sure to survive n years? To what would it reduce if money were unproductive?

12. Show that

(a) ${}_{m+n}E_x = {}_mE_x \cdot {}_nE_{x+m}$;

(b) ${}_nE_x = {}_1E_x \cdot {}_1E_{x+1} \cdot {}_1E_{x+2} \cdot \cdots {}_1E_{x+n-1}$.

74. Whole life annuity.—*A whole life annuity is a succession of equal periodic payments which continue during the entire life of the individual concerned.* It is evident that the cost of such an annuity depends upon the probability of living as well as upon the rate of interest.

The terms *payment interval, annual rent, term, ordinary, due, deferred,* have similar meanings in life annuities that they have in annuities certain. Unless otherwise specified, the words *life annuity* will be taken to mean *whole life annuity.*

75. Present value (cost) of a life annuity.—We now propose to find the present value of an ordinary life annuity of \$1 per annum payable to an individual, now aged x. The symbol, a_x, is used to denote the cost of such an annuity. We see that the present value of this annuity is merely the sum of the present values of pure endowments, payable at the ends of one, two, three, and so on, years. Consequently,

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \ldots \text{ to end of table}$$

$$= \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \frac{D_{x+3}}{D_x} + \ldots \text{ to end of table}$$

$$= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \ldots \text{ to end of table}}{D_x}$$

$$a_x = \frac{N_{x+1}}{D_x} \tag{3}$$

where

$$N_x = D_x + D_{x+1} + D_{x+2} + \ldots \text{ to end of table} \tag{4}$$

[See Table XII]

The symbol N_x (called " double bar N ") as defined above is that generally adopted in America. In actuarial parlance, it is frequently called the *American N.* The English textbooks use the single bar N which is defined by the equation

$$N_x = D_{x+1} + D_{x+2} + D_{x+3} + \ldots \text{ to end of table.}$$

In this book we shall use the " double bar " American N.

Exercises

1. What is the cost of a life annuity of \$600 per annum for a person aged 30, interest at 3½%?

Solution. From (3), Art. 75, we have

$$a_{30} = \frac{N_{31}}{D_{30}} = \frac{566362.9}{30440.8} = 18.60538. \qquad \text{[Table XII]}$$

Hence, the annuity has a cost (present value) of

$$600(18.60538) = \$11{,}163.23.$$

2. Find the present value of a life annuity to a person aged 60, the annual payment to be \$1,200.

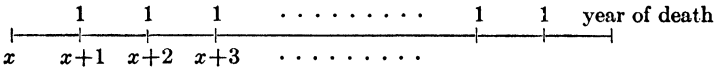
3. What annual life income could a person aged 50 purchase with \$10,000.

4. Derive the formula for a_x by the *mutual fund method.*

5. Show that $a_x = vp_x(1 + a_{x+1})$

(a) algebraically,

(b) by verbal interpretation or direct reasoning using the following line diagram:



6. A man aged 60 is promised a pension of \$600 at the end of each year as long as he lives. What is the present value of the pension?

7. The beneficiary, age 50, of a life insurance policy may receive \$25,000 cash or an ordinary life annuity of annual rent R. If she chooses the annuity, find R.

76. Life annuity due.—When the first payment under an annuity is made immediately, we have what is called an annuity due. The present value of an annuity due of \$1 per annum to a person aged x is denoted by $\mathbf{a}_x$. An annuity due differs from an ordinary annuity (Art. 75) only by an immediate payment. Consequently, we have*

$$\mathbf{a}_x = \mathbf{1} + a_x \qquad (5)$$

$$= 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + N_{x+1}}{D_x}$$

$$= \frac{D_x + D_{x+1} + D_{x+2} + \text{. . . to end of table}}{D_x}$$

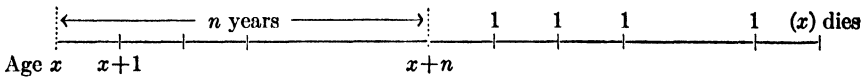
$$\mathbf{a}_x = \frac{N_x}{D_x}. \qquad (6)$$

77. Deferred life annuity.—When the first payment under an annuity is not made until some specified future date, and then only in case the individual, now aged x, is still living, we have what is called a deferred annuity. Since the first payment under an ordinary annuity is made at the end of one year, an annuity providing for first payment at the end

* Values of a_x and $\mathbf{a}_x$ may be found in Table XII.

of n years is said to be deferred $n - 1$ years. Then in an annuity deferred n years the first payment would not be made until the end of $n + 1$ years.

These payments are illustrated by the diagram

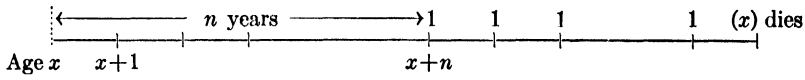


The present value of an annuity of \$1 per annum, deferred n years, payable to an individual now aged x, if he is then living is denoted by the symbol, ${}_n|a_x$. It is evident that the present value of such an annuity is merely the sum of the present values of pure endowments payable at the end of $n + 1$, $n + 2$, $n + 3$, and so on, years so long as the individual survives.

Consequently,

$$
\begin{aligned}
{}_n|a_x &= {}_{n+1}E_x + {}_{n+2}E_x + {}_{n+3}E_x + \ldots \text{ to end of table} \\
&= \frac{D_{x+n+1}}{D_x} + \frac{D_{x+n+2}}{D_x} + \ldots \\
{}_n|a_x &= \frac{N_{x+n+1}}{D_x}. \qquad (7)
\end{aligned}
$$

Let ${}_n|\mathrm{a}_x$ denote the present value of a *deferred whole life annuity due*, that is, a succession of \$1 payments to be made at the ends of n years, $n + 1$ years, and so on as long as (x) survives. These payments are illustrated by the following line diagram:



The value at age $x + n$ of these payments is a_{x+n}, and the value at age x, the present value, is $\mathrm{a}_{x+n} \cdot {}_nE_x$. Consequently

$$
{}_n|\ddot{a}_x = \mathrm{a}_{x+n} \cdot {}_nE_x = \frac{N_{x+n}}{D_{x+n}} \cdot \frac{D_{x+n}}{D_x} = \frac{N_{x+n}}{D_x}. \qquad (8)
$$

78. Temporary life annuity.—When the payments under a life annuity stop after a certain time although the individual be still living, we have what is called a *temporary annuity*. Such an annuity of \$1 per annum which ceases after n years is denoted by the symbol, $a_{x\overline{n}|}$. It is clear that the present value of a temporary annuity is equal to the sum of present

values of pure endowments of \$1 payable at the ends of 1, 2, 3, . . ., n years. Thus,

$$a_{x\overline{n}|} = {}_1E_x + {}_2E_x + \ldots + {}_nE_x$$

$$= \frac{D_{x+1} + D_{x+2} + \ldots + D_{x+n}}{D_x}$$

$$= \frac{D_{x+1} + D_{x+2} + \ldots \text{ to end of table}}{D_x} -$$

$$\frac{D_{x+n+1} + D_{x+n+2} + \ldots \text{ to end of table}}{D_x}$$

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \tag{9}$$

If the first of the n payments be made immediately and the last payment be made at the end of $n - 1$ years, we then have a temporary annuity due. Letting $a_{x\overline{n}|}$ represent the present value of such an annuity we get

$$a_{x\overline{n}|} = 1 + a_{x\overline{n-1}|}$$

$$= 1 + \frac{D_{x+1} + D_{x+2} + \ldots + D_{x+n-1}}{D_x}$$

$$= \frac{D_x + D_{x+1} + D_{x+2} + \ldots + D_{x+n-1}}{D_x}$$

$$a_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}. \tag{10}$$

Exercises

1. An insurance company accepts from a man, aged 30, \$85.89 per annum in advance for 10 years if living as payment for insurance. What would be the equivalent single premium based upon the American Experience Table of Mortality and 3½% interest?

2. A will provides that a son is to receive a life annuity of \$1,500 a year, the first payment to be made when the son attains the age of 60. What is the value of the son's share when he is 40 years old?

3. A man aged 50 pays \$10,000 for a life annuity whose first payment is to be made when he is 60 years old. What will be his annual income beginning at age 60?

4. A will provides that a son who is now 25 years old is to receive \$1,200 at the end of one year, and a like amount at the end of each year until 10 payments in all have been made. If each payment is contingent upon the son being alive, what is the value of his estate at age 25?

5. Make $n = 0$ in formula (7) and show that it reduces to formula (3). What does this mean?

6. Show that $a_x = a_{x\,\overline{n}|} + {}_n|a_x$

(a) algebraically,

(b) by direct reasoning with the aid of an appropriate line diagram.

7. Derive formulas (7) and (9) by the mutual fund method.

8. Derive formula (8) by finding the sum of appropriate pure endowments.

9. Draw line diagrams to illustrate the meaning of the following symbols:

$$a_{50},\ a_{25\,\overline{20}|},\ a_{25\,\overline{15}|},\ {}_{10}|\,a_{25}.$$

10. Prove $a_{x\,\overline{m+n}|} = a_{x\,\overline{m}|} + {}_mE_x \cdot a_{x+m\,\overline{n}|}$

(a) algebraically,

(b) by direct reasoning.

11 Prove the following identities:

(a) $a_{x\,\overline{n}|} = 1 + a_{x\,\overline{n-1}|}$,

(b) $a_x = a_{x\,\overline{n}|} + {}_n|\,a_x$.

12. A beneficiary, age 50, of a life insurance policy may receive \$25,000 cash or a temporary life annuity due for 15 years. If she chooses the annuity, find its amount.

79. Forborne temporary life annuity due.—An individual aged x may be entitled to a life annuity due of \$1 per annum, but forbears to draw it. Instead he requests that the unpaid installments be allowed to accumulate as pure endowments until he is aged $x + n$. Such an annuity is known as a *forborne temporary life annuity due.*

The problem here is to find the value of such an annuity, taken at age x, to the person at age $x + n$ if he is still alive. *This value is equal to the n-year pure endowment that the present value of a temporary life annuity due of* \$1 *per annum will buy.* The present value of a temporary life annuity due of \$1 per annum is

$$\frac{N_x - N_{x+n}}{D_x} \qquad \text{[(10) Art. 78]}$$

Since $\frac{D_{x+n}}{D_x}$ [(2) Art. 73] will buy an n-year pure endowment of \$1, \$1 will buy an n-year pure endowment of $\frac{D_x}{D_{x+n}}$, and consequently $\frac{N_x - N_{x+n}}{D_x}$ will buy an n-year pure endowment of *

$$ {}_nu_x = \frac{N_x - N_{x+n}}{D_x} \cdot \frac{D_x}{D_{x+n}} = \frac{N_x - N_{x+n}}{D_{x+n}}. \qquad (11)$$

* The symbol ${}_nu_x$ is customarily used to stand for the amount at age $x + n$ of the forborne temporary life annuity due of \$1 per annum. It is one of the most useful functions for the actuary.

It follows that R per annum payable in advance for n years as a temporary life annuity will buy an n-year pure endowment of

$$S = R \cdot {}_n u_x = R \frac{N_x - N_{x+n}}{D_{x+n}}. \tag{12}$$

Since $\frac{N_{x+n}}{D_{x+n}}$ is the cost of a life annuity due of \$1 per annum for an individual aged $x + n$, \$1 at age $x + n$ will buy a life annuity due of $\frac{D_{x+n}}{N_{x+n}}$ per annum, and $\frac{N_x - N_{x+n}}{D_{x+n}}$ at age $x + n$ will buy a life annuity due of

$$\frac{N_x - N_{x+n}}{D_{x+n}} \cdot \frac{D_{x+n}}{N_{x+n}} = \frac{N_x - N_{x+n}}{N_{x+n}}.$$

Hence, it follows that with \$1 per annum payable in advance by an individual now aged x, a life annuity due of $\frac{N_x - N_{x+n}}{N_{x+n}}$ per annum, beginning at age $x + n$, may be bought.

Then R dollars per annum payable in advance as a temporary life annuity by an individual now aged x, will buy a life annuity due of

$$R \frac{N_x - N_{x+n}}{N_{x+n}} \tag{13}$$

beginning at age $x + n$.

It may be shown that

$$K \frac{N_{x+n}}{N_x - N_{x+n}} \tag{14}$$

per annum payable in advance for n years by an individual now aged x, will buy him a life annuity due of K dollars per annum beginning when he is aged $x + n$. Here, an individual aged x is buying a regular life annuity of K dollars per annum, deferred $n - 1$ years, by paying $K \frac{N_{x+n}}{N_x - N_{x+n}}$ dollars annually in advance.

80. Summary of formulas of life annuities. Examples.

R = the annual payment,

(x) = the person of age x.

Pure Endowment: $A = R({}_nE_x) = R\dfrac{D_{x+n}}{D_x}.$ (1′)

Whole life annuity: $A = R(a_x) = R\dfrac{N_{x+1}}{D_x}.$ (3′)

Whole life annuity due: $A = R(\mathrm{a}_x) = R\dfrac{N_x}{D_x}.$ (6′)

Deferred life annuity: $A = R({}_n|\,a_x) = R\dfrac{N_{x+n+1}}{D_x}.$ (7′)

Deferred life annuity due: $A = R({}_n|\,\mathrm{a}_x) = R\dfrac{N_{x+n}}{D_x}.$ (8′)

Temporary life annuity: $A = R(a_{x\,\overline{n}|}) = R\dfrac{N_{x+1} - N_{x+n+1}}{D_x}.$ (9′)

Temporary life annuity due: $A = R(\mathrm{a}_{x\,\overline{n}|}) = R\dfrac{N_x - N_{x+n}}{D_x}.$ (10′)

Forborne temporary life annuity due: $S = R({}_nu_x) = R\dfrac{N_x - N_{x+n}}{D_{x+n}}.$ (12)

Example 1. A man aged 30 pays an insurance company \$1,000 annually, in advance, for 20 years for the purchase of a pure endowment. What will be the amount of the endowment if he lives to claim it?

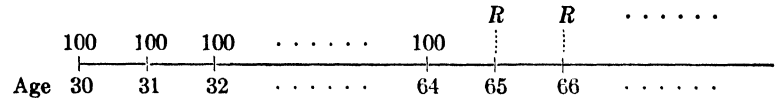
Solution. The annual payments constitute a forborne temporary life annuity due in which $x = 30$, $n = 20$, $R = 1{,}000$. Using (12), we find

$$S = 1{,}000\,\frac{N_{30} - N_{50}}{D_{50}} = 1{,}000\,\frac{596{,}804 - 181{,}663}{12{,}498.6}$$

$$= \$33{,}215.00.$$

Example 2. A man aged 30 pays an insurance company \$100 annually, in advance, for 35 years to purchase a life annuity, the first payment to be made when the annuitant reaches age 65. What is the annual rent of his annuity?

Solution. Consider the line diagram.



We shall choose age 65 as the focal time.

The value at age 65 of the payments is that of a forborne temporary life annuity due with $x = 30$, $n = 35$, $R = 100$. Using (12) we find

$$S = 100\frac{N_{30} - N_{65}}{D_{65}}.$$

The value of the benefit is that of a life annuity due on a life aged 65. Using (6′), the value of the benefit is

$$A = R\frac{N_{65}}{D_{65}}.$$

Therefore,

$$R \cdot \frac{N_{65}}{D_{65}} = 100\frac{N_{30} - N_{65}}{D_{65}},$$

and

$$R = 100\frac{N_{30} - N_{65}}{N_{65}} = 100\frac{596{,}804 - 48{,}616.4}{48{,}616.4}$$

$$R = \$1{,}127.58.$$

Exercises

1. In the settlement of an estate a man, aged 30, is to receive \$1,000 and a like amount at the end of each year. However, he requests that this annuity be forborne until he reaches the age of 60. What will be the amount of these forborne payments at that time on a 3½% interest basis?

2. A young man, aged 25, pays \$300 per annum in advance to accumulate as a pure endowment until age 60. What will be the amount of his endowment at age 60 on a 3½% basis? Suppose that at age 60 he does not take the amount of his endowment in cash, but instead purchases a life annuity due. What would be his annual income on a 3½% basis?

3. An individual now aged 30 desires to make provisions for his retirement at age 60. How much per annum, in advance, must he pay for the next 30 years to guarantee a life annuity due of \$3,000 per annum beginning at age 60?

4. A person whose present age is 25 desires to have a life income of \$1,500 beginning at age 60. How much must he invest annually in advance for the next 35 years to guarantee his desired income?

5. A man aged 50 pays an insurance company \$20,000 for a contract to pay him a life annuity with the first payment to be made at age 65. Find the annual payment of the annuity.

6. A corporation has promised to pay an employee, now aged 50, a pension of \$1,000 at the end of each year, starting with a payment on his 65th birthday. What is the present value of this expectancy?

7. A certain insurance policy on a life aged 30 calls for premiums of \$100 at the beginning of each year as long as he lives. Find the present value of the premiums.

8. A certain insurance policy on a life aged 30 calls for premiums of \$100 at the beginning of each year for 20 years. Find the present value of the premiums.

9. A man aged 30 wishes a life annuity of \$1,000 a year, the first payment to be made when he is 65 years old. To provide for this, he will pay R per year in advance for the next 20 years. Find R.

10. A man aged 55 is to receive a life annuity of \$1,000 a year, the first payment to be made immediately. He wishes to postpone the annuity so that the first payment will occur on his 65th birthday. What will be his annual income?

11. A certain life insurance policy matures when the policy-holder is aged 50 and gives him an option of \$10,000 in cash or a succession of equal payments for 10 years certain and as long thereafter as he may live. Should he die during the first ten years, the payments are to be continued to his heirs until a total of ten have been made. Find the annual payment under the optional plan.

Hint. The equation of value is $R(a_{\overline{10}|} + {}_{10}|a_{50}) = 10{,}000$.

12. Show by direct reasoning that the annual premium for n years, beginning at age x, for an annuity of 1 per year, beginning at age $x + n$, is given by $a_{x+n}/{}_nu_x$.

13. A boy of age 15 is left an estate of \$50,000 which is invested at 4% effective. He is to receive the income annually, if living, and at age 25 he is to receive the principal, if living. Find the present value of the inheritance.

14. How much must an individual now aged x invest at the beginning of each year for n years, if living, to secure an annuity of R dollars per annum payable for t years certain and as long thereafter as he may live?

Hint. Focalize at age $x + n$. Let y be the annual payment. The equation of value is

$$y({}_nu_x) = R(a_{\overline{t}|} + {}_t|a_{x+n}).$$

15. A person whose present age is 25 desires to have an income of \$1,000 a year for 10 years certain and as long thereafter as he may live, first payment at age 60. How much must he invest annually in advance for the next 35 years to guarantee this income?

81. Annuities payable m times a year.—Optional provisions are usually made in annuity contracts so that the periodical payments may be made m times a year. The symbol $a_x^{(m)}$ is used to denote the present value of a life annuity of \$1/m payable m times a year, and $a_x^{(m)}$ is used to denote the present value of a life annuity due of \$1/m payable m times a year. Theoretically, it follows from Art. 73, that

$$a_x^{(m)} = \frac{1}{m}\left[{}_{\frac{1}{m}}E_x + {}_{\frac{2}{m}}E_x + {}_{\frac{3}{m}}E_x + \cdots\right]. \tag{15}$$

It is apparent that (15) would involve considerable computation and besides the mortality table does not take into consideration fractional

parts of years. However, we may derive an approximate formula for $a_x^{(m)}$ which is accurate enough for most purposes.

The deferred annuity due may be written

$$_0|a_x = (1 + a_x) - 0$$

and

$$_1|a_x = (1 + a_x) - 1.$$

By simple proportion,

$$_{\frac{1}{m}}|a_x = (1 + a_x) - \frac{1}{m} = a_x + \frac{m-1}{m}$$

and, in general,

$$_{\frac{k}{m}}|a_x = (1 + a_x) - \frac{k}{m} = a_x + \frac{m-k}{m}.$$

Assume that we have m such annuities, where the first payments are to be made at the ends of $\frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \cdots, \frac{m}{m}$ of a year, respectively. Together they will provide \$1 at the end of each $\frac{1}{m}$th of a year. Hence,

$$ma_x^{(m)} = \left(a_x + \frac{m-1}{m}\right) + \left(a_x + \frac{m-2}{m}\right) + \cdots + \left(a_x + \frac{m-k}{m}\right) + \cdots + \left(a_x + \frac{m-m}{m}\right).$$

The right-hand side of the above equation is the sum of an arithmetical progression with a common difference of $-\frac{1}{m}$. Consequently

$$ma_x^{(m)} = ma_x + \frac{m(m-1)}{2m},$$

and

$$a_x^{(m)} = a_x + \frac{(m-1)}{2m} = \frac{N_{x+1}}{D_x} + \frac{(m-1)}{2m}. \tag{16}$$

If the first payment is made at once, we have

$$\mathrm{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)} = \frac{1}{m} + a_x + \frac{m-1}{2m}$$

$$\mathrm{a}_x^{(m)} = a_x + \frac{m+1}{2m}. \tag{17}$$

The student should observe the difference between (16) and (17).

If we let ${}_n|\,a_x^{(m)}$ stand for the present value of an annuity of \$1 deferred n years and payable in m installments a year, and reason as in Art. 73, we get

$$ {}_n|\,a_x^{(m)} = v^n \cdot {}_np_x \cdot a_{x+n}^{(m)} = v^n \frac{l_{x+n}}{l_x} \cdot a_{x+n}^{(m)} $$

$$ = \frac{D_{x+n}}{D_x} \cdot a_{x+n}^{(m)} $$

$$ {}_n|\,a_x^{(m)} = \frac{D_{x+n}}{D_x}\left(a_{x+n} + \frac{m-1}{2m}\right). \tag{18} $$

Also, if we let $a_{x\overline{n}|}^{(m)}$ stand for the present value of a temporary life annuity of \$1 payable in m installments a year and consider that a life annuity is made up of a temporary annuity and a deferred annuity, we get

$$ a_x^{(m)} = a_{x\overline{n}|}^{(m)} + {}_n|\,a_x^{(m)} $$

and

$$ a_{x\overline{n}|}^{(m)} = a_x^{(m)} - {}_n|\,a_x^{(m)} \tag{19} $$

$$ a_{x\overline{n}|}^{(m)} = a_x + \frac{m-1}{2m} - \frac{D_{x+n}}{D_x}\left(a_{x+n} + \frac{m-1}{2m}\right). \tag{20} $$

Exercises

1. What is the present value of a life annuity of \$100 payable at the end of every month to a person aged 30?

Solution. Here, $x = 30$ and $m = 12$. From (16), Art. 81, we have

$$ a_{30}^{(12)} = a_{30} + {}^{11}\!/_{24}. $$

$$ a_{30} = \frac{N_{31}}{D_{30}} = \frac{566{,}363}{30{,}440.8} = 18.6054. $$

and $$ (1{,}200)\, a_{30}^{(12)} = 1{,}200\ (18.6054 + 0.4583) = \$22{,}876.44. $$

2. Solve Exercise 1, with the annuity payable quarterly.

3. Find the cost of a temporary life annuity of \$600 per annum, payable in 12 monthly installments for 20 years, first payment due one month hence. Assume age 30.

4. Solve Exercise 3, with the annuity paid at the rate of \$300 at the end of every six months.

5. Find the cost of a life annuity due of \$1,000 per annum, payable in quarterly installments, for a person aged 40.

Problems

1. Show that the present value of an annuity of \$1, payable for n years certain and so long thereafter as the individual, now aged x, survives (first payment due one year hence) is given by

$$a_{\overline{n}|} + {}_n|a_x.$$

Also show that the present value of an annuity due of \$1, payable for n years certain and so long as an individual, now aged x, may live, is given by

$$1 + a_{\overline{n-1}|} + {}_{n-1}|a_x.$$

2. What is the value of an annuity of \$1,000 per annum payable at the end of each year for 10 years certain and so long thereafter as an individual, now aged 60, survives?

3. According to the terms of a will a person aged 30 is to receive a life income of \$6,000 per annum, first payment at once. An inheritance tax of 4% on the present value of the income must be paid at once. Find the present value of the income and the amount of the tax.

4. What would be the present value of the income of Problem 3 if payments of \$500 a month were made at the beginning of each month?

5. What would be the value of the annuity in Problem 2, if the payments were made at the end of each year for 20 years certain and for life thereafter?

6. A man carrying a \$20,000 life insurance policy arranges it so that the proceeds at his death shall be payable to his wife in annual installments for 10 years certain, first payment upon due proof of death. What would be the annual installment, assuming 3½% interest?

7. What would be the amount of the annual installments of Problem 6, if payable for 10 years certain and so long thereafter as the beneficiary shall survive, assuming that the beneficiary was 55 years of age at the death of the insured?

8. What would be the amount of the annual installments in the above problem, if payments were to be made throughout the life of the beneficiary?

9. What would be the amount of the annual installments in Problem 8, if payable for 10 years, each payment contingent upon the beneficiary being alive?

10. Assume that the proceeds in Problem 9 are to be paid monthly. What would be the monthly installment?

11. Show that

$$a_{x\,\overline{n}|}^{(m)} = a_x - \frac{m-1}{2m} - \frac{D_{x+n}}{D_x}\left[\frac{N_{x+n}}{D_{x+n}} - \frac{m-1}{2m}\right],$$

where $a_{x\,\overline{n}|}^{(m)}$ stands for the present value of a temporary annuity due of \$1 payable in m installments per annum.

12. A suit for damages due to the accidental death of a railroad employee 42 years old and earning \$175 a month was settled on the basis of three-fourths of the present value of the expected wages of \$175 a month during his after lifetime. What was the amount of the damages?

13. By the terms of a will, a son is bequeathed an estate of \$100,000 with the provision that he must pay his mother who is 60 years of age \$200 monthly as long as she lives? What is the value of the son's inheritance?

14. Prove: ${}_{m+n}u_x = {}_mu_x \cdot \dfrac{1}{{}_nE_{x+m}} + {}_nu_{x+m}$.

15. Prove:

(a) $\mathrm{a}_{x+1} = {}_1u_x(\mathrm{a}_x - 1)$,

(b) $\mathrm{a}_x = ({}_nu_x + \mathrm{a}_{x+n})\,{}_nE_x$,

(c) $a^{(m)}_{x\,\overline{n}|} = a_{x\,\overline{n}|} + \dfrac{m-1}{2m}(1 - {}_nE_x)$,

(d) $\mathrm{a}^{(m)}_{x\,\overline{n}|} = \mathrm{a}_{x\,\overline{n}|} - \dfrac{m-1}{2m}(1 - {}_nE_x)$.

16. A woman aged 30 offers \$20,000 to a benevolent organization if it will pay her 5% interest thereon at the beginning of each year for the remainder of her life. If the institution can purchase the desired annuity for her from an insurance company which operates on a 3½% basis, will it pay to accept the offer?

17. A man aged 65 is to receive a life annuity of \$1,000 a year, the first payment being due immediately. He desires to postpone the annuity so that the first annual payment will occur when he is aged 70. What will be the annual income from the new annuity?

18. A man aged 55 is entitled to a life annuity of \$1,000. He agrees to use it to purchase a 10-year pure endowment. What is the amount of the pure endowment?

19. A young man aged 25 is to receive as an inheritance a life income of \$100 a month, first payment immediately. An inheritance tax of 5% on the present value is levied. Find the amount of the tax.

20. A man aged 60 is granted a pension of \$1,000 a year for 10 years, first payment at once, and \$500 a year thereafter for the rest of his life. If all payments are contingent on his survival, find their present value.

21. Show that the present value of a perpetuity of \$1 per year, the first payment to be made at the end of the year in which (x) dies, is $1/i - a_x$. See Art. 37.

CHAPTER IX

LIFE INSURANCE, NET PREMIUMS (SINGLE AND ANNUAL)

82. Definitions.—A thorough mathematical treatment of life insurance involves many very complex problems. However, there are a few principles that are fundamental and it is these with which we wish to deal in this chapter. Life insurance is fundamentally sound only when a large group of individuals is considered. Each person contributes to a general fund from which the losses sustained by individuals of the group are paid. The organization that takes care of this fund and settles the claims for all losses is known as an *insurance company.* The deposit made to this fund by the individuals is called a *premium.* Since the payment of this premium by the individual insures a certain sum or *benefit* at his death,* he is spoken of as the *insured* and the person to whom the benefit is paid at the death of the insured is called the *beneficiary.* The agreement made between the insured and the company is called a *policy* and the insured is sometimes spoken of as the *policy-holder.*

The fundamental problem of life insurance is the determination of the *premium* that is to be charged the policy-holder in return for the *benefits* promised him by the policy. It is clear that the premium will depend upon the probability of dying and also upon the rate of interest on funds left with the company. That is, the premium requires a mortality table and an assumed rate of interest. The premium based upon these two items is called the *net premium.*

The American Experience Table of Mortality is the standard, in the United States, for the calculation of net premiums and for the valuation of policies. We shall in all our problems on life insurance assume this table and interest at $3\frac{1}{2}\%$. In computing the net premiums, we shall also assume that the benefits under the policy are paid at the ends of the years in which they fall due.

The insurance company has many expenses, in connection with the securing of policy-holders, such as advertising, commissions, salaries, office

* Certain insurance agreements specify the payment of an indemnity to the individual himself in case he is disabled by either accident or sickness. This is known as *accident and health insurance,* but we shall not attempt to treat it in this book.

supplies, et cetera, and consequently, must make a charge in addition to the net premium. The net premium plus this additional charge is called the *gross* or *office premium*. In this chapter we shall discuss only net premiums and leave gross premiums for another chapter. The premium may be *single*, or it may be paid annually, and this annual premium may sometimes be paid in semi-annual, quarterly or even monthly installments. All premiums are paid in advance.

83. Whole life policy.—*A whole life policy is one wherein the benefit is payable at death and at death only.* The net single premium on a whole life policy is the present value of this benefit. The symbol A_x will stand for the net single premium of a benefit of \$1 on (x).

Let us assume that each of l_x persons all of age x, buys a whole life policy of \$1. During the first year there will be d_x deaths, and consequently, at the end * of the first year the company will pay d_x dollars in benefits, and the present value of these benefits will be vd_x. There will be d_{x+1} deaths during the second year and the present value of the death benefits paid will be v^2d_{x+1}, and so on. The sum of the present values of all future benefits will be given by the expression

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots \text{ to end of table.}$$

Since l_x persons buy benefits of \$1 each, the present value of the total premiums paid to the company is $l_x \cdot A_x$.

Equating the present value of the total premiums paid and the present value of all future benefits, we have

$$l_x \cdot A_x = vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots \text{ to end of table.}$$

Solving the above equation for A_x, we get

$$A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots \text{ to end of table}}{l_x}. \qquad (1)$$

If both the numerator and the denominator of (1) be multiplied by v^x, we get

$$A_x = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + \cdots \text{ to end of table}}{v^x l_x}$$

$$= \frac{C_x + C_{x+1} + C_{x+2} + \cdots \text{ to end of table}}{D_x}$$

$$A_x = \frac{M_x}{D_x} \qquad (2)$$

* We assume that the death benefit is paid at the end of the year of death.

where

$$C_x = v^{x+1}d_x, \quad C_{x+1} = v^{x+2}d_{x+1}, \text{ and so on,}$$

and

$$M_x = C_x + C_{x+1} + C_{x+2} + \cdots \text{ to end of table.}$$

The expressions C_x and M_x are two new commutation symbols that are needed in this chapter. They are tabulated in Table XII.

If in (1) d_x be replaced by its equal $l_x - l_{x+1}$, and so on, we get

$$A_x = \frac{v(l_x - l_{x+1}) + v^2(l_{x+1} - l_{x+2}) + \cdots}{l_x}$$

$$= \frac{(vl_x + v^2l_{x+1} + \cdots)}{l_x} - \frac{(vl_{x+1} + v^2l_{x+2} + \cdots)}{l_x}$$

$$= v\left(1 + \frac{vl_{x+1} + v^2l_{x+2} + \cdots}{l_x}\right) - \left(\frac{vl_{x+1} + v^2l_{x+2} + \cdots}{l_x}\right)$$

$$= v(1 + {}_1E_x + {}_2E_x + \cdots) - ({}_1E_x + {}_2E_x + \cdots)$$

$$A_x = v(1 + a_x) - a_x. \quad \text{Art. 75.} \tag{3}$$

If (x) agrees to pay for the insurance of \$1 on his life in one installment in advance, the amount he must pay is A_x. Most people do not desire, or cannot afford, to purchase their insurance by a single payment, but prefer to distribute the cost throughout life or for a limited period. For the convenience of the insured, the policies commonly issued provide for the payment of premiums in equal annual payments. The corresponding net premiums are called *net level annual premiums.*

A common plan is to pay a level premium throughout the life of the insured. When this is the case the policy is called an *ordinary life insurance policy.*

We will denote the net annual premium of an ordinary life policy of \$1 by the symbol P_x. The payment of P_x, at the beginning of each year, for life forms a life annuity due and the present value of this annuity must be equivalent to the net single premium. Thus we have,

$$P_x \cdot \mathbf{a}_x = A_x. \tag{4}$$

Solving for P_x, we get

$$P_x = \frac{A_x}{\mathbf{a}_x} = \frac{M_x}{N_x}, \tag{5}$$

since

$$A_x = \frac{M_x}{D_x},$$

and

$$a_x = \frac{N_x}{D_x}. \quad [(6) \text{ Art. } 76]$$

Another common plan—probably the plan that occurs most frequently—is to pay for the insurance by paying the level premium for a limited number of years. When this is the case, the policy is called a *limited payment life policy*. The standard forms of limited payment policies are usually for ten, fifteen, twenty, or thirty annual payments, but other forms may be written.

Let us consider the n-payment life policy.

It is evident that the n annual premiums on the limited payment life policy form a temporary life annuity due. It is also clear that the present value of this annuity is equivalent to the net single premium A_x. Hence, if the net annual premium for a benefit of \$1 be denoted by ${}_nP_x$, we may write

$${}_nP_x \cdot a_{x\,\overline{n}|} = A_x.$$

Solving for ${}_nP_x$ and substituting for $a_{x\,\overline{n}|}$ and A_x, we have

$${}_nP_x = \frac{M_x}{N_x - N_{x+n}}. \tag{6}$$

Exercises

1. Use (1) Art. 83 to find the net single premium for a whole life policy to insure a person aged 91 for \$2,000.

2. Find the net single premium for a whole life policy of \$10,000 on a life aged 30.

3. Find the annual premium for an ordinary life policy of \$10,000 on a life aged 30.

4. Find the net annual premium on a 20-payment life policy of \$5,000 for a person aged 30.

5. Assuming that each of l_x persons, all of age x, buys an ordinary life policy of \$1, show from fundamental principles that

$$P_x(l_x + vl_{x+1} + v^2l_{x+2} + \cdots) = (vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots)$$

and thereby derive (5) Art. 83.

6. Show that $M_x = vN_x - N_{x+1}$.

7. Compare annual premiums on ordinary life policies of \$10,000 for ages 20 and 21 with those for ages 50 and 51. Note the annual change in cost for the two periods of life.

8. Find the net annual premium for a fifteen payment life policy of \$10,000 issued at age 50.

9. Find the net annual premium for a ten payment life policy of \$25,000 issued at age 55.

10. Find the net annual premium on a twenty payment life policy of \$5,000 for your age at nearest birthday.*

11. Compare annual premiums on twenty payment life policies of \$10,000 for ages 25 and 26 with those for ages 50 and 51. Note the annual change in cost for the two periods of life.

12. Find the net annual premium for a twenty-five payment life policy of \$10,000, issued at age 35.

13. Find the net annual premium for a thirty payment life policy of \$10,000, issued at age 35.

14. Using (10) Art. 9 with $n = 1$, and (3) Art. 83, show that $A_x = 1 - da_x$.

15. Show that $P_x = \dfrac{1}{a_x} - d$.

16. Give a verbal interpretation of the formula $A_x = v(1 + a_x) - a_x = va_x - a_x$.

17. Prove that $A_x = v - da_x$.

18. Let ${}_r|A_x$ denote the net single premium for an insurance of \$1 on (x) deferred r years (that is, the benefit is paid only if the insured dies after age $x + r$). Show that

$$ {}_r|A_x = \frac{M_{x+r}}{D_x}. $$

84. Term insurance.—*Term insurance is temporary insurance as it provides for the payment of the benefit only in case death occurs within a certain period of n years.* After n years the policy becomes void. The stated period may be any number of years, but usually term policies are for five years, ten years, fifteen years, and twenty years. The symbol $A^1_{x\overline{n}|}$ is usually used to denote the net single premium on a term policy of benefit \$1 for n years taken at age x.

If we assume that each of l_x persons, all of age x, buys a term policy of benefit \$1 for n years, the present value of the payments made by the company will be given by

$$ vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots + v^nd_{x+n-1}. $$

Since each of l_x persons buys a benefit of \$1, the present value of the premiums paid to the insurance company is $l_x \cdot A^1_{x\overline{n}|}$.

Equating the present value of the premiums paid to the company and the present value of the benefits paid by the company, we have

$$ l_x \cdot A^1_{x\overline{n}|} = vd_x + v^2d_{x+1} + \cdots + v^nd_{x+n-1} $$

and

$$ A^1_{x\overline{n}|} = \frac{vd_x + v^2d_{x+1} + \cdots + v^nd_{x+n-1}}{l_x}. \qquad (7) $$

* Your insurance age is that of your nearest birthday.

If both the numerator and the denominator of (7) be multiplied by v^x, we get

$$A^1_{x\,\overline{n}|} = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + \cdots \text{ to end of table}}{v^x l_x}$$

$$- \frac{v^{x+n+1}d_{x+n} + \cdots \text{ to end of table}}{v^x l_x}.$$

And

$$A^1_{x\,\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}. \tag{8}$$

When the term insurance is for one year only the net premium is called the *natural premium.* It is given by making $n = 1$ in (8). Thus,

$$A^1_{x\,\overline{1}|} = \frac{M_x - M_{x+1}}{D_x} = \frac{C_x}{D_x}. \tag{9}$$

The net annual premium for a term policy of \$1 for n years will be denoted by the symbol $P^1_{x\,\overline{n}|}$. It is evident that the annual premiums for a term policy constitute a temporary annuity due. This annuity is equivalent to the net single premium. Thus,

$$P^1_{x\,\overline{n}|} \cdot \mathrm{a}_{x\,\overline{n}|} = A^1_{x\,\overline{n}|}.$$

Solving for $P^1_{x\,\overline{n}|}$ and substituting for $\mathrm{a}_{x\,\overline{n}|}$ and $A^1_{x\,\overline{n}|}$, we get

$$P^1_{x\,\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \tag{10}$$

[(10) Art. 78] and (8) above.

Exercises

1. Find the net single premium for a term insurance of \$5,000 for 15 years for a man aged 30.

Solution. Here, $n = 15$ and $x = 30$. Using (8) Art. 84, we have

$$A^1_{30\,\overline{15}|} = \frac{M_{30} - M_{45}}{D_{30}} = \frac{10{,}259 - 7{,}192.81}{30{,}440.8} = \frac{3{,}066.19}{30{,}440.8} = 0.10072$$

and $5{,}000A^1_{30\,\overline{15}|} = 5{,}000(0.10072) = \$503.60$.

2. Find the net single premium for a term insurance of \$25,000 for 10 years for a man aged 40.

3. What are the natural premiums for ages 20, 25, 30, 35 and 40 for an insurance of \$1,000.

4. Find the net annual premium for a 20-year term policy of \$10,000 taken at age 35.

5. Show that the net annual premium on a k-payment n-year term policy of benefit \$1 ($k < n$) taken at age x is given by the expression

$$ {}_kP^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+k}}. \tag{11} $$

6. What is the net annual premium on a 20-payment 40-year term policy of \$1,000 for a man aged 20?

7. A person aged 25 buys a \$20,000 term policy which will terminate at age 65. Find the net annual premium.

8. Find the net annual premium on a 7-year term policy of \$5,000 taken at age 27.

85. Endowment insurance.—*In an endowment policy the company agrees to pay a certain sum in event of the death of the insured within a specified period, known as the endowment period, and also agrees to pay this sum at the end of the endowment period, provided the insured be living to receive the sum.* From the above definition it is evident that an endowment insurance of \$1 for n years may be considered as a term insurance of \$1 for n years plus an n-year pure endowment of \$1. (See Art. 73 and Art. 84.)

Thus, if we let the symbol $A_{x\overline{n}|}$ stand for the net single premium for an endowment of \$1 for n years, we have

$$ A_{x\overline{n}|} = A^1_{x\overline{n}|} + {}_nE_x $$

$$ = \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x} $$

$$ = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}, \tag{12} $$

since,

$$ A^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \qquad \text{[(8) Art. 84]} $$

and

$$ {}_nE_x = \frac{D_{x+n}}{D_x}. \qquad \text{[(2) Art. 73]} $$

We shall now find the net annual premium for an endowment of \$1 for n years, the premiums to be payable for k years. The symbol ${}_kP_{x\overline{n}|}$ will stand for the annual premium of such an endowment. It is clear that

these premiums constitute a temporary annuity due that is equivalent to the net single premium. Hence,

$$ {}_kP_{x\,\overline{n}|} \cdot a_{x\,\overline{k}|} = A_{x\,\overline{n}|}. $$

Solving for ${}_kP_{x\,\overline{n}|}$ and substituting for $a_{x\,\overline{k}|}$ and $A_{x\,\overline{n}|}$, we get

$$ {}_kP_{x\,\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+k}}. \tag{13} $$

If the number of annual payments is equal to the number of years in the endowment period, then $k = n$, and (13) becomes

$$ P_{x\,\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \tag{14} $$

Exercises

1. Find the net annual premium on a \$5,000 20-payment, 30-year endowment policy taken at age 25.

Solution. Here, $x = 25$, $n = 30$ and $k = 20$. Using (13), we have

$$ {}_{20}P_{25\,\overline{30}|} = \frac{M_{25} - M_{55} + D_{55}}{N_{25} - N_{45}} $$

$$ = \frac{11{,}631.1 - 5{,}510.54 + 9{,}733.40}{770{,}113 - 253{,}745} $$

$$ = \frac{15{,}853.96}{516{,}368} = 0.0307028, $$

and $$ (5{,}000)\ {}_{20}P_{25\,\overline{30}|} = 5{,}000(0.0307028) = \$153.51. $$

2. Find the net annual premium for a \$10,000 twenty payment endowment policy maturing at age 65, taken out at age 21.

3. Find the net annual premium on a \$25,000 15-year endowment policy, taken at age 55.

4. A person aged 22 buys a \$10,000 policy which endows at age 60. Find the net annual premium. The premiums are to be paid until age 60.

5. Find the net single premium on a \$10,000 10-year endowment policy, taken at age 50.

86. Annual premium payable by m equal installments.—In Art. 82 we mentioned the fact that the annual premium may be paid in semi-annual, quarterly or monthly installments.

We shall now find the total annual premium on an ordinary life insurance of \$1, when the premium is payable by m equal installments. The symbol $P_x^{(m)}$ will represent this total premium. It is evident that the premiums constitute an annuity due of $P_x^{(m)}$ per annum, payable in m equal installments of $P_x^{(m)}/m$ each, and the present value of this annuity must equal the net single premium for an insurance of \$1. Hence, we have

$$P_x^{(m)} \cdot \mathrm{a}_x^{(m)} = A_x.$$

Since,

$$\mathrm{a}_x^{(m)} = a_x + \frac{m+1}{2m} \quad [(17) \text{ Art. } 81]$$

we have

$$P_x^{(m)} = \frac{A_x}{a_x + \dfrac{m+1}{2m}}. \tag{15}$$

Example. Find the quarterly premium on an ordinary life policy of \$1,000 taken at age 30.

Solution. Here, $x = 30$ and $m = 4$. From (15), we have

$$P_{30}^{(4)} = \frac{A_{30}}{a_{30} + \frac{5}{8}}$$

$$= \frac{0.33702}{18.6054 + 0.6250} \quad [\text{Table XII}]$$

$$= \frac{0.33702}{19.2304} = 0.01752,$$

and

$$1{,}000 \cdot P_{30}^{(4)} = 1{,}000(0.01752) = \$17.52.$$

The quarterly premium is therefore $\frac{1}{4}$ (\$17.52) = \$4.38.

Making $m = 1$, 2, and 4 in (15), we get

$$P_x = \frac{A_x}{1 + a_x}, \quad P_x^{(2)} = \frac{A_x}{a_x + \frac{3}{4}}, \text{ and } P_x^{(4)} = \frac{A_x}{a_x + \frac{5}{8}}$$

respectively, which shows that twice the semi-annual premium is larger than the annual and four times the quarterly premium is larger than twice the semi-annual. This addition in premium takes account of two things

only: (1) the possibility that a part of the annual premium may be lost in the year of death; and (2) loss of interest on part of annual premium unpaid. On an annual basis the premium would be paid in full at the beginning of the year of death, while on a semi-annual or quarterly basis a part of the premium might remain unpaid at date of death, and the interest on that part of the premium that is not paid at the beginning of the year is lost annually.

However, in practice there is at least another element which is not provided for in this theoretical increase and that is the additional expense incurred in collecting premiums twice or four times a year instead of once. And then, too, it is the observation of most companies that the percentage of lapsed policies is greater when written on the semi-annual and quarterly basis than when written on the annual basis.

It is evident, then, that this theoretical increase is not sufficient to take care of the additional expenses incurred. To obtain the semi-annual premium many companies add 4% to the annual rate and then divide by 2 and to obtain the quarterly premium they add 6% to the annual rate and divide by 4.

We might derive formulas for the annual premiums on other types of policies, but, as indicated above, these formulas are not really used in practice.

Exercises

1. Find the total annual premium on an ordinary life policy \$1,000 taken at age 50, if the premiums are to be paid (a) semi-annually; (b) quarterly. Use formula (15) and then use the method that is used in practice by most companies and compare results.

2. Show that (15), Art. 86 can be written

$$P_x^{(m)} = \frac{M_x}{N_x - D_x\left(\frac{m-1}{2m}\right)}.$$

Make $m = 1$ and compare with (5), Art. 83.

3. Find the annual premium on an ordinary life policy of \$5,000 taken at age 25, if the premiums are to be paid (a) quarterly; (b) monthly.

87. Summary of formulas of life insurance premiums.—In this chapter we have discussed the "standard" policies and have derived the formulas for computing the net single and the net annual premiums under them. We summarize this information in the following table.

x = the age of the insured; F = the face of the policy.

Name of Policy	Policy Benefits	Premiums Paid	Single Premiums	Annual Premiums
Ordinary life.....	Whole life insurance	For life	$F\frac{M_x}{D_x}$	$F\frac{M_x}{N_x}$
n-payment life.....	Whole life insurance	For n years	$F\frac{M_x}{D_x}$	$F\frac{M_x}{N_x - N_{x+n}}$
n-year term.....	n-year term insurance	For n years	$F\frac{M_x - M_{x+n}}{D_x}$	$F\frac{M_x - M_{x+n}}{N_x - N_{x+n}}$
n-year k-payment endowment	(a) n-year pure endowment (b) n-year term insurance	For k years, $k \leqq n$	$F\frac{M_x - M_{x+n} + D_{x+n}}{D_x}$	$F\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+k}}$

88. Combined insurance and annuity policies.—The principles summarized in Art. 87 enable us to compute the premiums on the well-known standard policies. Today, combined insurance and annuity policies are frequently written, and we shall now illustrate the methods of computing the premiums for them. We shall merely need to apply the equation of value:

Present value of payments = Present value of benefits. (16)

Example 1. An insurance-annuity contract taken out by a life aged 25 provides for the following benefits:

(a) 10-year term insurance for \$5,000,

(b) a pure endowment of \$10,000 at the end of 10 years.

It is desired to pay for these benefits in 10 equal annual premiums in advance. What is the annual premium?

Solution. Let P be the required annual premium.

Present value of benefit (a) is $5{,}000\ (A^{1}_{25\,\overline{10}|}) = 5{,}000\,\frac{M_{25} - M_{35}}{D_{25}}$.

[(8) Art. 84]

Present value of benefit (b) is $10{,}000\ ({}_{10}E_{25}) = 10{,}000\,\frac{D_{35}}{D_{25}}$.

[(1′) Art. 80]

Present value of the payments is $P(a_{25\,\overline{10}|}) = P\frac{N_{25} - N_{35}}{D_{25}}$.

[(10′) Art. 80]

Hence, using (16), we have

$$P\frac{N_{25}-N_{35}}{D_{25}} = 5{,}000\frac{M_{25}-M_{35}}{D_{25}} + 10{,}000\frac{D_{35}}{D_{25}}.$$

$$P = \frac{5000(M_{25}-M_{35}) + 10{,}000\,D_{35}}{N_{25}-N_{35}}.$$

P = \$855.98. Table XII.

Example 2. An insurance-annuity contract taken out by a life aged 40 provides for the following benefits:

(a) a \$10,000 pure endowment payable at age 65,

(b) a \$10,000 20-payment life insurance,

(c) a life annuity of \$2,000 annually with the first payment at age 65.

If the premiums are to be paid annually in advance for 20 years, find the annual premium P. Set up in commutation symbols.

Solution.

Present value of benefit (a) is $10{,}000\ ({}_{25}E_{40}) = 10{,}000\frac{D_{65}}{D_{40}}$.
[(1′) Art. 80]

Present value of benefit (b) is $10{,}000\ (A_{40}) = 10{,}000\frac{M_{40}}{D_{40}}$.
[(2) Art. 83]

Present value of benefit (c) is $2{,}000\ ({}_{25}\,|\,a_{40}) = 2{,}000\frac{N_{65}}{D_{40}}$.
[(8′) Art. 80]

Present value of the payments is $P(a_{40\overline{20}|}) = P\frac{N_{40}-N_{60}}{D_{40}}$.
[(10′) Art. 80]

Hence, applying (16), we have

$$P\frac{N_{40}-N_{60}}{D_{40}} = 10{,}000\frac{D_{65}}{D_{40}} + 10{,}000\frac{M_{40}}{D_{40}} + 2{,}000\frac{N_{65}}{D_{40}}.$$

$$P = \frac{10{,}000\,(D_{65}+M_{40}) + 2{,}000\,N_{65}}{N_{40}-N_{60}}.$$

Problems

1. Find the net annual premium for an endowment policy for \$5,000 to mature at age 85 and taken at age 40.

2. For purposes of valuation, a policy for \$15,000 taken at age 35 provides that the insurance of the first year is term insurance, and that of subsequent years is a 14 payment life insurance on a life aged 36, so that the insurance is paid up in 15 payments in all. What is the first year premium and that of any subsequent year?

3. An insurance contract provides for the payment of \$1,000 at the death of the insured, and \$1,000 at the end of each year thereafter until 10 installments certain are paid. What is the net annual premium on such a contract for a person aged 40, if the policy is to become paid up in 20 payments?

4. What would be the net annual premium in Problem 3, if it were written on the ordinary life basis?

5. Assume that each of l_x persons, all of age x, buys an n-payment life policy of \$1; equate the present value of all premiums paid and all benefits received; and derive (6), Art. 83.

6. Reasoning as in Problem 5, derive (10), Art. 84.

7. Reasoning as in Problem 5, derive (14), Art. 85.

8. Prove that:

$$\text{(a) } A_x = \frac{P_x}{P_x + d}, \qquad \text{(b) } P_x = \frac{dA_x}{1 - A_x}.$$

9. Prove that:

$$\text{(a) } A^1_{x\,\overline{1}|} = v \cdot \frac{d_x}{l_x}, \qquad \text{(b) } A^1_{x\,\overline{n}|} = v a_{x\,\overline{n}|} - a_{x\,\overline{n}|}.$$

10. Show that

$$A_{x\,\overline{n}|} = v a_{x\,\overline{n}|} - a_{x\,\overline{n-1}|},$$

and interpret this formula verbally.

11. A 20-payment life insurance policy for \$1,000 issued to a life aged 30, for purposes of valuation, is treated as a one-year term policy at age 30 plus a 19-payment life policy at age 31. What is the net premium for the first year and the net level annual premium for the subsequent 19 payments?

12. For purposes of valuation, an ordinary life policy of \$1,000 issued to a life aged 30 is considered as a one-year term policy at age 30 and an ordinary life policy at age 31. What is the first net annual premium and the subsequent annual net level premiums?

13. A person aged 45 takes out a policy which promises \$10,000 if death occurs before age 65. If the insured is living at age 65, he is to receive \$1,000 annually as long as he lives, the first \$1,000 being paid when age 65 is reached. What is the net level annual premium if the policy is issued on a 20-payment basis?

14. A life insurance policy issued on a life aged 30 provides for the following benefits: In the event of death of the insured during the first 30 years the policy pays \$1,000, with a \$5,000 cash payment if the insured survives to age 60. If the policy is issued on a 20-payment net level basis, find the net premium.

Find the net periodic premium for each of the following policies.

Problem	Benefits of Policy	Age of Insured	Number of Annual Premiums
15.	(a) 10-year term insurance for \$10,000, (b) a pure endowment of \$20,000 at end of 20 years.	45	10
16.	(a) Whole life insurance of \$10,000, (b) a pure endowment at age 60 of \$10,000, (c) a life annuity of \$1,000 annually with first payment at age 65.	30	20
17.	(a) \$30,000 to beneficiary if death of insured occurs between ages 30 and 40, (b) \$25,000 to beneficiary if death of insured occurs between ages 40 and 50, (c) \$15,000 to beneficiary if death of insured occurs between ages 50 and 60.	30	20

Hint. The benefits under the policy in Problem 17 are the same as those under a policy providing for \$5,000 10-year term insurance, \$10,000 20-year term insurance, \$15,000 30-year term insurance, all issued to a life aged 30.

CHAPTER X

VALUATION OF POLICIES. RESERVES

89. Meaning of reserves.—Except at very low ages, the probability of dying in any year increases with increasing age. Consequently, the cost of insurance provided by the given policy, as indicated by the natural premium, increases with increasing age. The net level annual premium for the policy is larger than the natural premium during the early years of the policy and is therefore more than sufficient to cover the insurance, but, in the later years of the policy the net level premium is smaller than the natural premium and is therefore insufficient to cover the cost of the insurance.

To illustrate the above remarks, let us consider a numerical example. A man aged 35 takes out a $1,000 ordinary life policy. The net level annual premium under the American Experience 3½% Table is $19.91. The cost of insurance (natural premium) for the first policy year is $8.64, leaving a difference ($19.91 − $8.64) = $11.27. During the second year of the policy the cost of insurance (natural premium on a life aged 36) is $8.78, and thus the insured pays ($19.91 − $8.78) = $11.13 more than the expense due to mortality. This situation continues to age 57 when, and for later years, the net level premium $19.91 is insufficient to meet the cost of insurance, for, at age 57 the natural premium is $20.61. The following table compares the net level premium $19.91 with the increasing cost of insurance for an ordinary life policy of $1,000 on a life aged 35.

Attained Age	Natural Premium	Excess N.L.P.–N.P.	Attained Age	Natural Premium	Excess N.L.P.–N.P.
35	$8.64	$11.27	65	$38.77	−$18.86
40	9.46	10.45	70	59.90	− 39.99
45	10.79	9.12	75	91.18	− 71.27
50	13.32	6.59	80	139.58	−119.67
55	17.94	1.97	85	227.59	−207.68
60	25.79	−5.88	90	439.17	−419.26

It is evident that if an insurance company is to operate upon a solvent basis, it must accumulate a fund during the early policy years to meet the increased cost in the later policy years. These excesses of the net level premium over the natural premiums that appear in the early policy years are improved at interest and held by the company to meet the increased cost during the later policy years. The accumulation of these excesses results in a fund that is called the *reserve* or *the value of the policy.**

90. Computing reserves, Numerical illustration.—A glance at the American Experience Table of Mortality shows that of 100,000 persons alive at age 10 there remain 81,822 alive at age 35.

Let us assume that each of 81,822 persons, all aged 35, buys an ordinary life policy of $1,000. The total of the net annual premiums amounts to $1,629,076.02. This amount accumulates to $1,686,093.68 by the end of the first year. According to the table of mortality the death losses to be paid at the end of the first year amount to $732,000.00, leaving $954,093.68 in the reserve. This leaves a terminal reserve of $11.77 to each of the 81,090 survivors. The premiums received at the beginning of the second year amount to $1,614,501.90, which when added to $954,093.68 makes a total of $2,568,595.58, and so on. The following table is self explanatory.

TABLE SHOWING TERMINAL RESERVES ON AN ORDINARY LIFE POLICY FOR $1,000 ON THE LIFE OF AN INDIVIDUAL AGED 35 YEARS

Policy Year	Funds on Hand at Beginning of Year	Funds Accumulated at 3½%	Death Losses	Funds at End of Year	Amount to Credit of Each Survivor, Reserve
1	$1,629,076.02	$1,686,093.68	$732,000	$ 954,093.68	$11.77
2	2,568,595.58	2,658,496.43	737,000	1,921,496.43	23.91
3	3,521,324.66	3,644,571.02	742,000	2,902,571.02	36.46
4	4,487,625.03	4,644,692.94	749,000	3,895,692.94	49.40
5	5,465,835.36	5,657,139.60	756,000	4,901,139.60	62.75
..	.....	.....	.....	.....	.....

This illustrates what is known as the *retrospective method* of computing reserves because the reserve at the end of any policy year was determined exclusively from facts that belong to the *past history* of the policy.

* The reserve on any one policy at the end of any policy year is known as the terminal reserve for that year, or the policy value.

Exercises

1. The premium on a 5-year endowment insurance for \$1000 taken out at age 25 is \$183.56. Complete the following table and show that at the end of 5 years the fund is just sufficient to pay each survivor \$1000.00.

Policy Year	Funds on Hand at Beginning of Year	Funds Accumulated at 3½%	Death Losses	Funds at End of Year	Amount to Credit of Each Survivor, Reserve
1	\$16,342,713.92	\$16,914,708.91	\$718,000	\$16,196,708.91	\$183.40
2	32,407,626.75	33,541,893.69	718,000	32,823,893.69	374.72
3	48,903,015.45	50,614,620.99	718,000	49,896,620.99	574.33
4			718,000		
5			719,000		

2. The annual premium on a 10-payment life policy on a life aged 30 is \$40.6078. Prepare a table similar to that in Exercise 1 and thus compute the reserve on the policy at the end of each policy year.

91. Fackler's accumulation formula.—We will now develop a formula which expresses the terminal reserve of any policy year in terms of the reserve of the previous year. We will designate by ${}_rV_x$ the terminal reserve of the rth year on an insurance of \$1, and let P_x stand for the net annual premium. The reserve then at the beginning of the $(r+1)$th year will be ${}_rV_x + P_x$. This is called the initial reserve of the $(r+1)$th year. The aggregate reserve at the beginning of the $(r+1)$th year, for the l_{x+r} individuals insured, will be

$$l_{x+r}({}_rV_x + P_x).$$

This last amount will accumulate, by the end of the year, to

$$l_{x+r}({}_rV_x + P_x)(1+i).$$

Out of this amount the company will have to pay d_{x+r} as death claims for the year, leaving

$$l_{x+r}({}_rV_x + P_x)(1+i) - d_{x+r}$$

as the total reserve to the l_{x+r+1} surviving policy holders at the end of the $(r+1)$th year.

The terminal reserve then for the $(r + 1)$th year is

$$ {}_{(r+1)}V_x = \frac{l_{x+r}({}_rV_x + P_x)(1 + i) - d_{x+r}}{l_{x+r+1}} $$

$$ = \frac{(1 + i)l_{x+r}}{l_{x+r+1}}({}_rV_x + P_x) - \frac{d_{x+r}}{l_{x+r+1}}. \qquad (1) $$

If we now define the valuation factors (see Table XIII)

$$ u_x = \frac{(1 + i)l_x}{l_{x+1}} \quad \text{and} \quad k_x = \frac{d_x}{l_{x+1}}, $$

we have

$$ {}_{(r+1)}V_x = u_{x+r}({}_rV_x + P_x) - k_{x+r}. \qquad (2) $$

This formula is known as *Fackler's accumulation formula.* It will evidently work for any policy, for the factors u_{x+r} and k_{x+r} in no way depend upon the form of the policy. This formula is used very extensively by actuaries in preparing complete tables of terminal reserves. The valuation functions u_x and k_x are based upon the American Experience Table of Mortality and 3½% interest and are given in Table XIII.

To find the terminal reserve for the first policy year we make $r = 0$, and (2) becomes

$$ {}_1V_x = u_xP_x - k_x, \qquad (3) $$

for it is evident that ${}_0V_x = 0$.

Exercises

1. Show that $u_x = \frac{D_x}{D_{x+1}}$ and $k_x = \frac{C_x}{D_{x+1}}$ and verify the tabular values of u_x and k_x for the ages 20, 25, and 30 by making use of the C_x and D_x functions.

2. Making use of formulas (3) and (2) Art. 91, verify the reserves in the problem of Art. 90.

Solution. From (3) we have

$$ {}_1V_{35} = u_{35}P_{35} - k_{35}, \text{ and } P_{35} = 0.01991. $$

Hence $$ {}_1V_{35} = 1.044343\ (.01991) - 0.009027 $$

$$ = 0.011766. $$

Then, $$ 1{,}000\ {}_1V_{35} = \$11.77. $$

Also, $$ {}_2V_{35} = u_{36}\ ({}_1V_{35} + P_{35}) - k_{36}, \quad [(2) \text{ Art. } 91] $$

$$ = 1.044493\ (0.011766 + 0.01991) - 0.009172 $$

$$ = 0.023913. $$

Hence, $1{,}000\ {}_2V_{35} = \$23.91.$

${}_3V_{35} = ?$

3. Find the terminal reserve for each of the first five policy years on a ten payment life policy for $5,000 taken at age 25.

4. The terminal reserve at the end of the fifteenth policy year on a twenty-year endowment policy for $1,000 taken at age 25 is $665.59. Calculate the terminal reserves for the succeeding policy years until the policy matures.

5. The terminal reserve at the end of the tenth policy year on a fifteen payment life policy for $1,000 taken at age 30 is $272.96. Find the terminal reserve for the eleventh and twelfth years.

6. The terminal reserve at the end of the twenty-fifth policy year on an ordinary life policy for $1,000 taken at age 29 is $333.81. Find the terminal reserve for the twenty-sixth year.

92. Prospective method of valuation.—We now consider another method of valuation and derive a formula for determining the terminal reserve for any policy year independent of the reserve for the previous year. *At the end of the* rth *policy year the sum of the terminal reserve and the present value of the future premiums to be paid must equal the net single premium for a new policy on the life of the insured, who is now aged* $x + r$.

If we consider an ordinary life policy the present value of the future premiums to be paid would be $P_x a_{x+r}$ and the net single premium for a policy on the insured, now aged $x + r$, would be A_{x+r}. Again denoting the terminal reserve for the rth year by ${}_rV_x$, we obtain the relation,

$$ {}_rV_x + P_x a_{x+r} = A_{x+r}, \quad \text{[(5) Art. 76]} $$

and

$$ {}_rV_x = A_{x+r} - P_x a_{x+r}. \tag{4} $$

We see from equation (4) that *the* rth *year terminal reserve is equal to the net single premium for the attained age* $x + r$ *minus the present value of all future net annual premiums.* This definition of reserve will evidently hold for all forms of policies.

The value of ${}_rV_x$ may be expressed in terms of the commutation columns by remembering that

$$ A_{x+r} = \frac{M_{x+r}}{D_{x+r}}, \quad \text{[(2) Art. 83]} $$

$$ P_x = \frac{M_x}{N_x}, \quad \text{[(5) Art. 83]} $$

and

$$a_{x+r} = \frac{N_{x+r}}{D_{x+r}} \quad \text{[(6) Art. 76]}$$

Then

$$_rV_x = \frac{N_xM_{x+r} - M_xN_{x+r}}{N_xD_{x+r}} \tag{4'}$$

Replacing A_{x+r} by its equivalent $P_{x+r}(a_{x+r})$, equation (4) becomes

$$_rV_x = (a_{x+r})P_{x+r} - (a_{x+r})P_x,$$

or

$$_rV_x = (P_{x+r} - P_x)(a_{x+r}). \tag{5}$$

P_{x+r} is the net annual premium for an individual now aged $x + r$, but since he took his insurance at age x instead of waiting until age $x + r$, his annual saving in premium is $(P_{x+r} - P_x)$ and the present value of these annual savings is $(P_{x+r} - P_x)(a_{x+r})$ which is the policy reserve at the end of the rth year. Hence we have a verbal interpretation of the formula (5).

We will now derive an expression for the terminal reserve for the rth year on an n-payment life insurance of \$1. The symbol, $_{r:n}V_x$, will denote the rth year reserve for this policy. Immediately following equation (4), Art. 92, we defined reserve and said this definition would hold for all forms of policies. Here the net single premium for the attained age $x + r$ would be A_{x+r} and the present value of all future premiums would be given by

$$_nP_x \cdot a_{x+r\,\overline{n-r}|}$$

as they would constitute a temporary life annuity due, for $n - r$ years. Consequently, we may write

$$_{r:n}V_x = A_{x+r} - {}_nP_x \cdot a_{x+r\,\overline{n-r}|} \tag{6}$$

Denoting the rth year terminal reserve on a k-payment n-year endowment insurance of \$1 by $_{r:k}V_{x\,\overline{n}|}$ and following the same line of reasoning used in obtaining (6), we get,

$$_{r:k}V_{x\,\overline{n}|} = A_{x+r\,\overline{n-r}|} - {}_kP_{x\,\overline{n}|} \cdot a_{x+r\,\overline{k-r}|}. \tag{7}$$

When r is equal to or greater than k formula (7) becomes

$$_{r:k}V_{x\,\overline{n}|} = A_{x+r\,\overline{n-r}|}. \tag{8}$$

When the annual premiums are payable for the entire endowment period, $k = n$, and (7) reduces to

$$ {}_rV_{x\overline{n}|} = A_{x+r\,\overline{n-r}|} - P_{x\overline{n}|} \cdot \mathbf{a}_{x+r\,\overline{n-r}|}. \tag{9} $$

Exercises

1. Find the 20th year reserve on an ordinary life policy for \$5,000 taken at age 30.

Solution. Here, $r = 20$, $x = 30$. Then from (4) Art. 92, we have

$$ {}_{20}V_{30} = A_{50} - P_{30}\,(a_{50}). $$

But
$$ P_{30} = \frac{M_{30}}{N_{30}} = \frac{10{,}259}{596{,}804} = 0.01719. $$

Hence,
$$ {}_{20}V_{30} = 0.50849 - 0.01719(14.5346) $$
$$ = 0.25864, $$

and
$$ 5{,}000 \cdot {}_{20}V_{30} = \$1{,}293.20. $$

2. Find the terminal reserve of the 15th policy year on a 15-payment life policy of \$5,000 taken at age 35. Explain why this result equals the net single premium on a life policy taken at age 50.

3. Find the 20th year terminal reserve on a \$10,000 policy which is to mature as an endowment at age 65, if the policy was taken at age 30.

4. Find the 10th year reserve on a \$20,000, 20-year endowment policy taken at age 40.

5. Find the terminal reserve of the seventh policy year on a twenty payment life policy of \$2,500 taken at age 32.

6. Find the terminal reserve of the ninth policy year on an ordinary life policy of \$5,000 taken at age 40.

7. Verify the result for the third terminal reserve in Exercise 1, Art. 90.

8. Verify the result for the fifth terminal reserve of the illustrative problem in Art. 90.

9. Reduce formula (6) Art. 92 to commutation symbols.

93. Retrospective method of valuation.—In preceding sections we have alluded to the retrospective method of computing reserves. Fackler's accumulation formula, Art. 91, was developed from facts that pertain to the *past history* of the policy. It expresses the reserve of any policy year in terms of the reserve of the previous year, and is therefore very useful in preparing *complete* tables of terminal reserves. It cannot be used, however, for computing the reserve on a given policy for a *specified* policy year.

The problem of finding the reserve on a given policy for a specified policy

year was solved in Art. 92 by the prospective method. The thoughtful student will naturally enquire: "Can we develop formulas by the retrospective method for computing the reserves on given policies for specified policy years, and are the results consistent with those of Art. 92?"

We answer both questions in the affirmative.

From the retrospective point of view, the *r*th *terminal reserve for a given policy issued at age x is the accumulated value at age x + r of the past premiums less the accumulated value at age x + r of the past insurance benefits.* The past insurance benefits are those of an *r*-year term insurance on (x). That is,

$$\begin{pmatrix}\text{rth Terminal}\\ \text{reserve}\end{pmatrix} = \begin{pmatrix}\text{Value at age}\\ x+r\\ \text{of past premiums}\end{pmatrix} - \begin{pmatrix}\text{Value at age}\\ x+r\\ \text{of past benefits}\end{pmatrix}$$

Consider an ordinary life policy of \$1 on (x).

P_x = the net annual premium, and ${}_rV_x$ = the rth terminal reserve.

$$\begin{pmatrix}\text{Value at age } x+r\\ \text{of past premiums}\end{pmatrix} = P_{x:\overline{r}|}\ddot{s}_x = \frac{M_x}{N_x}\cdot\frac{N_x - N_{x+r}}{D_{x+r}}.$$

[(5) Art. 83] [(12) Art. 79]

$$\begin{pmatrix}\text{Value at age } x+r\\ \text{of past benefits}\end{pmatrix} = \frac{A^1_{x:\overline{r}|}}{{}_rE_x} = \frac{M_x - M_{x+r}}{D_x}\cdot\frac{D_x}{D_{x+r}} = \frac{M_x - M_{x+r}}{D_{x+r}}.$$

[(8) Art. 84] [(2) Art. 73]

Hence,

$${}_rV_x = \frac{M_x}{N_x}\cdot\frac{N_x - N_{x+r}}{D_{x+r}} - \frac{M_x - M_{x+r}}{D_{x+r}}.$$

$${}_rV_x = \frac{N_xM_{x+r} - M_xN_{x+r}}{N_xD_{x+r}}$$

which is the same as (4′) Art. 92.

Problems

1. How much does a person save by buying a \$10,000 ordinary life policy at age 25 instead of waiting until age 30? See formula (5), Art. 92.

2. Show that when $r = n$, the right-hand member of (6) Art. 92, reduces to A_{x+n} and explain the meaning of this result.

3. Derive formula (7), Art. 92.

4. To what does the right member of (9) reduce when $r = n$?

5. Express formula (6) in terms of the commutation symbols.

6. Express formula (9) in terms of the commutation symbols.

7. Making use of (3) and (5) Art. 83, show that

$$ {}_rV_x = \frac{a_x - a_{x+r}}{1 + a_x} = 1 - \frac{1 + a_{x+r}}{1 + a_x} = 1 - \frac{a_{x+r}}{a_x}. \tag{10} $$

8. Use formula (10) to find the twelfth year terminal reserve on a \$2,000 ordinary life policy taken at age 37.

9. (a) Show that ${}_{r:n}V_x = ({}_{n-r}P_{x+r} - {}_nP_x)\ (a_{x+r}\,\overline{n-r}|)$ and interpret the result.
(b) Derive a similar expression for the n-year endowment policy.

10. Build up a table of terminal reserves for the first 10 years on a 20-payment life policy of \$1,000 taken at age 30. Use (3) and (2), Art. 91 and check every 5 years by using (6), Art. 92.

11. Build up a table of terminal reserves for the first 10 years on an ordinary life policy of \$1,000 taken at age 33. Use (3) and (2) Art. 91 and check for the fifth and tenth years by using formula (10), Problem 7.

12. Build up a table of terminal reserves for the first 5 years on a 10 year endowment of \$1,000 taken at age 30. Use Fackler's formula and check the fifth year by using formula (9) Art. 92.

13. Solve Exercise 10, with the policy taken at age 40.

14. Solve Exercise 11, with the policy taken at age 38.

15. Develop a formula similar to (9), Art. 92, but for term insurance for a term of n years. Find the fifth year terminal reserve on a ten year term policy of \$1,000 issued at age 30.

16. Find the seventh year terminal reserve on a \$1,000, 15 year term policy issued at age 40.

CHAPTER XI

GROSS PREMIUMS, OTHER METHODS OF VALUATION, POLICY OPTIONS AND PROVISIONS, SURPLUS AND DIVIDENDS

94. Gross Premiums.—In Chapter IX a net premium was defined and we found the net premiums for a number of the standard policies. We saw that this net premium was large enough to take care of the yearly death claims and to build up a reserve sufficient to care for all future claims, but was not adequate to pay the running expenses of the company and provide against unforeseen contingencies.* Hence to care for these extra expenses a charge in addition to the net premium must be made. *This additional charge is sometimes spoken of as a loading, and the net premium plus this loading is called the gross premium.*

In Chapter IX we enumerated some of the expenses of the insurance company. To these we may add taxes imposed by state legislatures, medical expenses for the examination of new risks, expenses for collecting premiums, and many other minor ones.

We shall now discuss some of the methods used in arriving at a sufficient gross premium. At first thought it might seem reasonable to add a fixed amount to the net premium on each $1,000 insured regardless of age or kind of policy. This would give the same amount for expenses on an ordinary life policy for a young man, aged 25 say, as on a 20-year endowment policy for the same amount and age. The percentage of loading on the ordinary life policy would be about three times as large as that on the endowment policy, while as a matter of fact the expenses of each policy would be about the same percentage of the respective premium, for commissions are usually paid as a percentage of the premium, and taxes are charged in a like manner. Hence, we see that a constant amount added to a premium does not make adequate provisions and it is seldom used now without modification.

Sometimes loadings are effected by adding a fixed percentage of the net premium. Let us assume for the time being that this is 30%. Then the loading at age 25 on an ordinary life policy would be $4.53 and on a ten year endowment at age 65 it would be $32.75. It is evident that this method makes the loading very high for the older ages and thereby causes the premium to be unattractive to the applicant. As a matter of fact the

* The influenza epidemic of 1918 is an example of this.

$32.75 is more than is actually required to care for the expenses of the 10-year endowment taken at age 65. This method has its objections as well as the first method described.

Often a constant amount plus a fixed percentage of the net premium is added. This is a combination of the two methods described above. The constant gives an adequate amount for administration expenses as this depends more on the volume of insurance in force than on the amount of premiums, and the percentage provides for those expenses that are a certain percentage of the net premium.

If we add a constant $4 for each $1,000 of insurance and 15% of the net premium we get a premium that is very satisfactory. For example the net premium on an ordinary life policy of $1,000 at age 35 is $19.91. Adding $4.00 and 15%, we get $26.89 as our office premium.

Another plan is a modification of the percentage method. If 33⅓% be the percentage, ⅓ of the net premium is added to obtain the office premium on ordinary life. On limited payment life and endowment policies ⅙ of the net premium for the particular policy is added and then ⅙ of the net premium on an ordinary life for the same age. To illustrate:

Ordinary life, net rate, age 35	$19.91
⅓ of net rate	6.64
Gross premium	$26.55
20-year endowment, net rate, age 35	$40.11
⅙ of $40.11	6.68
⅙ of ordinary life rate	3.32
Gross premium	$50.01

If we let P_x' stand for the gross premium of an ordinary life policy of $1, and let r denote the rate of the percentage charge, and c the constant charge per $1,000 of insurance, we may express by the formula,

$$P_x' = P_x(1 + r) + \frac{c}{1{,}000}, \tag{1}$$

the ideas mentioned above. If the loading is a constant charge, r will be zero but if it is considered a percentage charge only, c will be zero. Formula (1) may be modified to apply to the different forms of policies. Nearly every company has its individual method of calculating gross premiums but all companies get about the same results.

95. Surplus and dividends.—The gross premium is divided into three parts. The *first part* is an amount sufficient to pay the death claims for

the year, where the number of deaths is based upon the American Experience Table of Mortality. The *second part* goes to build up the reserve. The *third part* is set aside to meet the expenses of the company.

As all new policy holders are selected by medical examination it is reasonable to expect that, under normal conditions, the actual number of deaths will be much smaller than the expected. Hence, a portion of the first part of the premium is not used for the current death claims, and is placed in a separate fund known as the *surplus*.

The reserve is figured on a 3½% interest basis, but the average interest earned by the funds of the company is usually considerably more than this. This additional interest is also added to the *surplus*.

After an insurance company has become well organized and its territory has been thoroughly developed its annual expenses are usually much less than the expected. Hence a portion of the third part of the premium is saved and added to the *surplus*.

Since the surplus comes from savings on the premiums, a part of it is refunded to the policy holders at the end of each year. These refunds are called *dividends*, but they are not dividends in the same sense as the interest on a bond. Most of these dividends come from savings on premiums and only a small amount comes from a larger interest earning on the reserve and other invested funds.

A large portion of this surplus must be held by the company for it is as essential for an insurance company to have an adequate surplus as it is for a trust company, a bank, or any other corporation. The surplus represents the difference between the assets and the liabilities, and a relatively large surplus is an indication of solvency.

96. Policy options.—In any standard life-insurance policy there is a nonforfeiture table giving the surrender or loan value, automatic extended insurance, and paid-up insurance at the end of each policy year beginning with the third.* In case the insured desires to quit paying any time after three annual payments have been made, he may surrender his policy and receive the cash value indicated in the table, or a paid-up policy for the amount indicated in the table. Or he may keep his policy and remain insured for the full face amount of the policy for the time stated in the table.

97. Surrender or loan value.—The surrender or loan value of a policy at the end of any policy year is the terminal reserve for that year less whatever charge (known as a surrender charge) the company makes for a surrender. This charge is a per cent of the terminal reserve and decreases

* Some companies begin the non-forfeiture table at the end of the second year.

each year. After 10 or 15 years there is usually no charge made upon surrender. The surrender value at the end of the tenth year on an ordinary life $1,000 policy, issued to a person age 25, is $89.43 less the surrender charge. Insurance laws allow companies to make a surrender charge. The companies, however, usually make a smaller charge than is allowed them by law.

We give a few reasons for this charge: *First*, the company is at an expense to secure a new policy holder in place of the one surrendered; *Second*, life insurance companies claim that the greatest number of lapses come from people who are in excellent health rather than from those in poor health. This would tend to increase the percentage of mortality and thereby decrease the surplus and dividends to policy holders. *Third*, if policy values were not subjected to a surrender charge, it is the belief that a large number of policy holders would either surrender their insurance or take the full loan value during hard times and thus cause financial loss to the company.*

98. Extended insurance.—Whenever the insured fails to pay his annual premium the company automatically extends his insurance for the full face of the policy unless he surrenders his policy and requests the surrender value or paid-up insurance. The length of time that the company can carry the insurance for the full amount, without further premiums, depends upon the surrender value of the policy at that time.

In order to find the time of extension we must solve the equation

$$\frac{M_{x+r} - M_{(x+r)+t}}{D_{x+r}} = {}_rV_x \quad [(8) \text{ Art. } 84] \tag{2}$$

for t. An example will show how this is done.

Example. The value at the end of the tenth year, of an ordinary life policy of $1,000, taken at age 25, is $89.43. Find the time of the automatic insurance.

Solution. Here, $x = 25$, $r = 10$, ${}_{10}V_{25} = 0.08943$, and

$$\frac{M_{35} - M_{35+t}}{D_{35}} = 0.08943$$

or

$$\begin{aligned} M_{35+t} &= M_{35} - (0.08943)D_{35} \\ &= 9{,}094.96 - (0.08943)(24{,}544.7) \\ &= 6{,}899.93. \end{aligned}$$

* For a more complete discussion of surrender values see " Notes on Life Insurance " by Fackler.

This value of M_{35+t} lies between M_{46} and M_{47}. By interpolation we find that $35 + t = 46$ years 9 months, approximately, or $t = 11$ years 9 months. Hence, the value \$89.43 is enough to buy a term policy of \$1,000 for 11 years and 9 months.

99. Paid-up insurance.—If at any time the insured surrenders his policy he may take a paid-up policy for the amount that his surrender value at that time will purchase for him at his attained age. For example, the value at the end of the tenth year, of an ordinary life policy of \$1,000 taken at age 25 is \$89.43. Find the paid-up insurance for that year. The insured is now age 35 and an insurance of \$1 will cost him

$$A_{35} = 0.37055.$$

Hence, he may buy for \$89.43 as much insurance as .37055 is contained in \$89.43, or approximately \$241.00.

The following is a non-forfeiture table for the first 10 years on an ordinary life policy for \$1,000 taken at age 25:

NON-FORFEITURE TABLE—\$1,000, ORDINARY LIFE, AGE 25

At End of	Cash or Surrender Value	Automatic Extension		Paid-up Insurance
		Years	Months	
3rd Year	\$23.70	3	1	\$73.00
4th "	32.16	4	2	97.00
5th "	40.91	5	5	121.00
6th "	49.98	6	7	146.00
7th "	59.35	7	10	170.00
8th "	69.04	9	2	194.00
9th "	79.07	10	5	218.00
10th "	89.43	11	9	241.00

In the above table the values are all based upon the full level net premium terminal reserves. In a standard policy these values would all be some smaller due to the surrender charge. Usually, only even dollars are published in non-forfeiture tables. If the preliminary term method or modified preliminary term methods of valuation are used,* all the values will be made somewhat smaller for the first few policy years.

* These methods are discussed in later sections.

We shall now outline a method for determining the surrender values, automatic extended insurance, and paid-up insurance for an endowment policy. The surrender values will be determined just as terminal reserves are determined (the surrender value is the terminal reserve less the surrender charge). The time for automatic extension must at no time extend beyond the date of maturity. Hence, only such a part of the surrender value will be used as is necessary to extend the insurance to the maturity date. The balance of the surrender value for that year will go to buy a pure endowment which will mature at the end of the endowment period. Let us consider a \$1,000, 20-year endowment for an individual aged 30.

The reserve (full level net premium method) for the fifth year is \$177.83. The cost of a 15-year paid-up term policy of \$1,000 for the attained age, 35, is \$111.61. This leaves (177.83 − 111.61) = \$66.22 with which to purchase a 15-year pure endowment. A pure endowment of \$1 will cost

$$_{15}E_{35} = 0.50922. \quad [(2) \text{ Art. } 73]$$

Hence, \$66.22 will buy as much pure endowment as 0.50922 is contained in 66.22, or \$130.00 (nearest dollar).

We now find the amount of the 15-year paid-up endowment that \$177.83 will buy. The cost of a \$1, 15-year paid-up endowment for age 35 is \$0.62083. Hence, \$177.83 will buy a paid-up endowment of

$$\frac{177.83}{0.62083} = \$286.00 \text{ (approximately)}.$$

The following is a non-forfeiture table for the first 10 years on a 20-year endowment of \$1,000 taken at age 30:

NON-FORFEITURE TABLE—\$1,000, 20-YEAR ENDOWMENT, AGE 30

At end of	Cash or Surrender Value	Automatic Extension		Pure Endowment	Paid-up Endowment
		Years	Months		
3rd Year	\$102.35	14	4	.....	\$175.00
4th "	139.32	16	no	\$47.00	231.00
5th "	177.83	15	"	130.00	286.00
6th "	217.95	14	"	208.00	341.00
7th "	259.74	13	"	282.00	394.00
8th "	303.29	12	"	353.00	447.00
9th "	348.67	11	"	421.00	498.00
10th "	395.98	10	"	491.00	554.00

In the above table the values are all based upon the full level net premium terminal reserves. However, these values would all be somewhat smaller due to the surrender charge.

In the event the policy holder paid only five premiums and then lapsed his policy, he could accept any one of the following options at the end of five years: Receive \$177.83 (less surrender charge) in cash, receive a paid-up 15-year term policy for \$1,000 and \$130 in cash at age 50, if living, or receive a paid-up endowment for \$286.00.

Exercises

1. Make a non-forfeiture table for the first 10 years of a \$1,000 ordinary life policy taken at age 40.

2. Make a non-forfeiture table for the first 10 years of a \$1,000 20-payment life policy taken at age 40.

3. Make a non-forfeiture table for the first five years of a \$1,000 20-year endowment policy taken at age 40.

4. Make a non-forfeiture table for the first 10 years of a \$1,000 policy taken at age 26, which is to endow at age 60.

5. A man who has attained the age of 35 surrenders his policy and chooses to elect the option which grants him extended insurance to the amount of \$5,000 for eight years. Find his surrender value.

6. A man who has attained the age of 35 surrenders his policy and elects the option of paid-up insurance. If his surrender value is \$5,000, find the amount of insurance he should receive.

7. A man aged 25 took out a convertible \$10,000 10-year term policy. At the end of 5 years he converted it into an ordinary life policy as of his attained age. How much ordinary life insurance did he obtain if all his reserve was used for that purpose?

8. A man aged 30 takes out an ordinary life policy for \$10,000. When he is 55 years of age, the company decides to go out of business. What sum is due him?

100. Preliminary term valuation.—In Chapter X we considered what is known as the *full level premium* method of valuation. *By this method the difference between the net annual premium and the natural premium for the first year is placed into the reserve.* It is clear that this leaves none of the net annual premium to care for the *first year's expenses* of the policy. The initial expenses of a policy are the greatest for they include an agent's commission, medical examiner's fee, taxes, etc. To illustrate the above remarks let us consider an ordinary life policy of \$1,000 taken at age 35. The net annual premium on this policy is \$19.91 and the office premium is \$26.55, leaving only \$6.64 to go towards initial expenses. The balance of the first year's expenses must come from the surplus. But this seems

unfair to the old policy holders as their contributions in the way of premiums have built up this surplus. It is perhaps fair that they should bear a small portion of the expenses of securing new business, but they should not pay so much as is required under the full level premium method of valuation. It is also evident that under this method it would be almost impossible for a new company to build up an adequate surplus.

A method known as a *preliminary term system* has been devised to meet the objections mentioned above, and we will now describe it. *Under this method all the first year premium is available for current mortality and expenses. The first year's insurance then is term insurance and the policy provides that it may be renewed at the end of the first year as a life or endowment policy at the same office premium. The net premium for the first year is the natural premium for the age when the policy was issued and the balance of the gross premium is considered as first year loading and is available for initial expenses. The net premium for the second and subsequent years is the net premium at an age one year older than when the policy was issued.*

Let us again consider the ordinary life policy of $1,000 taken at age 35. Here the office premium is $26.55 and since the natural premium for the first year is $8.65 there would be a first year loading of $17.90. The net annual premium for subsequent years would be $20.55 * which would leave $6.00 as a renewal loading. Had the policy been issued under the full level net premium system there would have been a uniform loading of $6.64.

A 20-payment life policy taken at age 35 would have a gross premium of $35.70. The first year natural premium would be $8.65, thus leaving a loading of $27.05 for initial expenses, and the net premium for the subsequent nineteen years would be the net premium on a 19-payment life policy as of age 36. This would be $28.89, thus resulting in a renewal loading of $6.81. Had this policy been issued under the full level net premium system there would have been a uniform loading of $8.31.

The preliminary term method when applied to ordinary life policies and limited payment life and endowment policies with long premium paying periods is sound in principle and is recognized by the best authorities. However, the system has some objections when it is applied to limited payment life and endowment policies of short premium paying periods. These objections will be discussed in Art. 101 and a remedy will be devised.

It is evident that, since the whole of the first year's gross premium is available for current mortality and expenses, there can be no terminal reserve set up until the end of the second year. It is also clear that this

* That is, the premium on a $1,000 ordinary life policy as of age 36.

reserve from year to year will be a little smaller than the full level net premium reserve until the policy matures.

Example 1. For an ordinary life policy of \$1,000 taken at age 30, find the terminal reserve for the first three policy years under the preliminary term system of valuation. Also find reserve for the twentieth year.

Solution. The insurance for the first year is term insurance and there is no first year reserve. To get the terminal reserve for the second year we make use of (3) Art. 91, letting $x = 31$. Then

$$
\begin{aligned}
{}_1V_{31} &= u_{31}P_{31} - k_{31} \\
&= 1.043884\ (0.01768) - 0.008583 \\
&= 0.00987,
\end{aligned}
$$

and $1{,}000\ {}_1V_{31} = 1{,}000\ (0.00987) = \$9.87$ (2nd year reserve).

Also,

$$
\begin{aligned}
{}_2V_{31} &= u_{32}({}_1V_{31} + P_{31}) - k_{32} \\
&= 1.043986\ (0.00987 + 0.01768) - 0.008682 \\
&= 0.020179,
\end{aligned}
$$

and $1{,}000\ {}_2V_{31} = 1{,}000\ (0.020179) = \$20.18$ (3rd year reserve).

The reserve for the 20th year will be the 19th year reserve for age 31. From (4), Art. 92, we get

$$
\begin{aligned}
{}_{19}V_{31} &= A_{50} - P_{31}\, a_{50} \\
&= 0.50849 - 0.01768\ (14.5346) \\
&= 0.25151,
\end{aligned}
$$

and

$$
\begin{aligned}
1{,}000\ {}_{19}V_{31} &= 1{,}000\ (0.25151) \\
&= \$251.51 \text{ (20th year reserve).}
\end{aligned}
$$

According to the full level premium method, the reserve for the third year would have been \$29.33 and that for the twentieth year would have been \$258.64. The student will observe that the difference between the reserves, for any particular year, according to the two methods decreases as the age of the policy increases. In fact, the reserves for the fortieth year differ by only \$3.64.

Example 2. For a 20-payment life policy of \$1,000 taken at age 30, find the terminal reserve for the first three policy years under the preliminary term system. Also find the reserve for the twentieth year.

Solution. The insurance for the first year is term insurance and there is no first year reserve. To get the terminal reserve for the second year we make use of (3) Art. 91, letting $x = 31$. Then

$$
\begin{aligned}
{}_1V_{31} &= u_{31} \cdot {}_{19}P_{31} - k_{31} \\
&= 1.043884\ (0.02601) - 0.008583 \\
&= 0.018568,
\end{aligned}
$$

and
$$
\begin{aligned}
1{,}000\ {}_1V_{31} &= 1{,}000\ (0.018568) \\
&= \$18.57 \text{ (2nd year reserve).}
\end{aligned}
$$

$$
\begin{aligned}
{}_2V_{31} &= u_{32}({}_1V_{31} + {}_{19}P_{31}) - k_{32} \\
&= 1.043986\ (0.018568 + 0.02601) - 0.008682 \\
&= 0.037859,
\end{aligned}
$$

and
$$
\begin{aligned}
1{,}000\ {}_2V_{31} &= 1{,}000\ (0.037859) \\
&= \$37.86 \text{ (3rd year reserve).}
\end{aligned}
$$

The reserve for the 20th year will be the 19th year reserve on a 19-payment life taken at age 31. From (6), Art. 92, we get

$$
\begin{aligned}
{}_{19:19}V_{31} &= A_{50} = 0.50849, \\
\text{and } 1{,}000\ {}_{19:19}V_{31} &= 1{,}000\ (0.50849) \\
&= \$508.49 \text{ (20th year reserve).}
\end{aligned}
$$

According to the full level premium method, the reserve for the third year would have been \$53.94 and that for the twentieth year would have been \$508.49. We observe that the difference in reserve by the two methods is \$16.08 at the end of the third year. However, at the end of 20 years there is no difference.

101. Modified preliminary term valuation.—In Art. 100 we mentioned the fact that the preliminary term method of valuation is objectionable when applied to limited payment life and endowment policies with short premium paying periods. This can best be illustrated by an example. Suppose we apply this method of valuation to a fifteen-payment endowment policy for $1,000 taken at age 35. The office premium is $67.92 and since the natural premium for the first year is $8.65 there would be a first year loading of $59.27. This is entirely too much for first year expenses. It is evident then that the preliminary term system should be modified when applied to short premium paying periods.

We found that in the case of the ordinary life policy taken at age 35 there was, according to the preliminary term system, a first year loading of $17.90 and this was adequate for initial expenses. Hence, if this amount is sufficient in the one case, it seems reasonable that the same amount, or but little more, should be adequate for limited payment and endowment policies of short premium paying periods. This then suggests a modification. *The ordinary life premium at any age forms the basis of the amount which can be used for first year expenses for limited payment and endowment policies taken at the same age.*

Another method of modification is that provided by the laws of Illinois, usually known as the " Illinois Standard." *Under the Illinois plan, twenty payment life policies and all other policies having premiums smaller than that of the twenty payment life policy for that age are valued on the preliminary term plan without any modification.* Then the twenty payment life premium forms the basis of the amount which can be used for first year expenses on all policies whose premiums are greater than that of the twenty payment life.*

The principles underlying the two methods of modification were recognized by the " Committee of Fifteen," composed of Insurance Commissioners and Governors, in 1906, and since that time the laws of many states have been amended so as to adopt the recommendations of this committee.

Some other states have other ways of modifying the preliminary term system, but the two modifications that we have here described will be sufficient for this discussion. We will now illustrate each of the above methods with an example.

Example 1. Find the terminal reserves for the first three years on a fifteen-year endowment policy of $1,000, issued at age 25, valued according

* This is spoken of as the *full preliminary term plan* to distinguish it from any one of the modified plans.

to the modified preliminary term system with the ordinary life as a basis of modification.

Solution. We shall base all our computations on an insurance of \$1 and then multiply by 1,000. The net premium for the first year is the natural premium plus a certain excess, e. The subsequent net annual premiums are the net ordinary life premiums for age 26, plus the same excess, e, required to mature the policy.

Neglecting e each year the value of the policy at the end of 15 years would be the full level net premium terminal reserve of the 14th policy year on an ordinary life policy of \$1 issued at age 26, or ${}_{14}V_{26}$. However, at the end of 15 years the policy must have a value of \$1. Hence, the excess payment of e each year must provide at maturity a pure endowment of

$$(1 - {}_{14}V_{26}).$$

This excess, e, is the annual payment on a forborne temporary annuity due at age 25 (Art. 79), that will accumulate in 15 years to

$$(1 - {}_{14}V_{26}).$$

Hence,
$$e\left(\frac{N_{25} - N_{40}}{D_{40}}\right) = (1 - {}_{14}V_{26}),$$

and
$$e = (1 - {}_{14}V_{26})\frac{D_{40}}{N_{25} - N_{40}}.$$

From (4), Art. 92, we get

$$\begin{aligned} {}_{14}V_{26} &= A_{40} - P_{26}\,a_{40} \\ &= 0.41003 - 0.01548\ (17.4461) \\ &= 0.13997, \end{aligned}$$

since,
$$P_{26} = 0.01548.$$

Then,
$$\begin{aligned} e &= (1 - 0.13997)\frac{19{,}727.4}{770{,}113 - 344{,}167} \\ &= 0.03983. \end{aligned}$$

The terminal reserve for the first year is

$$\begin{aligned} {}_1V_{25} &= u_{25}(e + A^1_{25\,\overline{1}|}) - k_{25} \quad [(2)\ \text{Art. } 91] \\ &= u_{25}\cdot e = 1.043415(0.03983) \\ &= 0.04156, \end{aligned}$$

since, $u_{25} \cdot A^1_{25\,\overline{1}|} = k_{25}$. [(9) Art. 84 and Exercise 1, Art. 91]

Then, $1{,}000\ {}_1V_{25} = 1{,}000(0.04156) = \$41.56$ (1st year reserve).

$$
\begin{aligned}
{}_2V_{25} &= u_{26}({}_1V_{25} + P_{26} + e) - k_{26} \\
&= 1.043415(0.04156 + 0.01548 + 0.03983) \\
&\quad - 0.008197 = 0.09288.
\end{aligned}
$$

Then, $1{,}000\ {}_2V_{25} = 1{,}000(0.09288) = \$92.88$ (2nd year reserve).

$$
\begin{aligned}
{}_3V_{25} &= u_{27}({}_2V_{25} + P_{26} + e) - k_{27} \\
&= 1.043554(0.09288 + 0.01548 + 0.03983) \\
&\quad - 0.008264 = 0.14638.
\end{aligned}
$$

Then, $1{,}000\,{}_3V_{25} = 1{,}000(0.14638) = \$146.38$ (3rd year reserve).

According to the full level premium method, the reserve for the first three years would be \$48.87, \$99.81, and \$152.90, respectively. We notice that the difference between the two methods for the first year is \$7.31 and for the third year the difference is \$6.52. There would be no difference for the fifteenth year.

Example 2. Find the terminal reserves for the first three years on a ten-year endowment policy of \$1,000, issued at age 25, valued according to the Illinois standard.

Solution. The net premium for the first year is the natural premium, $A^1_{25\,\overline{1}|}$, plus an excess e. The subsequent net annual premiums are the net premiums on a nineteen-payment life taken at age 26, plus the same excess e.

Neglecting e each year the value of the policy at the end of 10 years would be the full level net premium terminal reserve of the 9th policy year on a nineteen-payment life policy of \$1, issued at age 26, or ${}_{9:19}V_{26}$. However, at the end of 10 years the policy must have a value of \$1.

Hence, the excess payment of e each year must provide at maturity a pure endowment of $(1 - {}_{9:19}V_{26})$.

Therefore, $e\left(\dfrac{N_{25} - N_{35}}{D_{35}}\right) = (1 - {}_{9:19}V_{26})$

and $$e = (1 - {}_{9:19}V_{26})\,\frac{D_{35}}{N_{25} - N_{35}}.$$

From (6), Art. 92

$$_{9:19}V_{26} = A_{35} - {}_{19}P_{26} \cdot a_{35\,\overline{10}|}$$

$$= 0.17458,$$

since, $A_{35} = 0.37055,$

and $_{19}P_{26} = 0.02368,$ [(6) Art. 83]

$a_{35\,\overline{10}|} = 8.27575.$ [(10) Art. 76]

Then, $$e = (1 - 0.17458)\frac{24{,}544.7}{770{,}113 - 456{,}871}$$

$$= 0.06468.$$

Hence, $$_1V_{25} = u_{25} \cdot e = 1.043415(0.06468)$$

$$= 0.06749,$$

and $1{,}000\ {}_1V_{25} = 1{,}000(0.06749) = \$67.49$ (1st year reserve).

$$_2V_{25} = u_{26}({}_1V_{25} + {}_{19}P_{26} + e) - k_{26}$$

$$= 1.043484(0.06749 + 0.02368 + 0.06468)$$

$$- 0.008197 = 0.15443,$$

and $1{,}000\ {}_2V_{25} = 1{,}000(0.15443) = \$154.43$ (2nd year reserve).

$$_3V_{25} = u_{27}({}_2V_{25} + {}_{19}P_{26} + e) - k_{27}$$

$$= 1.043554(0.15443 + 0.02368 + 0.06468)$$

$$- 0.068264 = 0.24510,$$

and $1{,}000\ {}_3V_{25} = 1{,}000(0.24510) = \$245.10$ (3rd year reserve).

According to the full level premium method, the reserve for the first year would be \$82.08, for the second \$167.66, and for the third \$256.92. The difference between the two methods for the first year is \$14.59 and the difference for the third year is \$11.82.

Note.—" It should be noted that a modification of premiums and reserves is employed solely for the purpose of providing for large preliminary expenses in the first policy year, and does not in any way affect the yearly amount of gross premium actually paid to the

company by the policyholder. The modification is purely an internal transaction of the life insurance company, which releases a larger part of the gross premium for expenses in the first year and defers to a later date the setting up of a part of the reserve." *

102. Concluding remarks.—Before completing this elementary treatment of life insurance, we wish to emphasize the fact that we have attempted to give a mere introduction into a broad field. There are many topics that we have not touched. For the student who is interested in a further study of this important field, we suggest the following books:

Moir, Henry, *Life Assurance Primer*, The Spectator Company, New York City.

Menge, W. O., and Glover, J. W., *An Introduction to the Mathematics of Life Insurance*, The Macmillan Company, New York City.

Knight, Charles K., *Advanced Life Insurance*, John Wiley and Sons, New York City.

Spurgeon, E. F., *Life Contingencies*, The Macmillan Company, New York City.

Exercises

1. For a twenty payment life policy of \$1,000, taken at age 25, find the terminal reserve for the 15th policy year both under the level net premium system and under the preliminary term system of valuation.

2. Find the terminal reserve for the first three years on a 20-year endowment policy of \$1,000, issued at age 40, valued according to the modified preliminary term system with the ordinary life as a basis of modification.

3. Solve Exercise 2, using the Illinois Standard.

4. If the gross premium of a limited payment life policy of \$1 on (x) is found by increasing the net premium by a certain percentage r and adding to this a certain percentage s of the net ordinary life premium and further increasing this by a constant c, per \$1,000 insurance, show that the gross premium may be expressed by the formula

$$ {}_nP'_x = P_x \cdot s + {}_nP_x(1 + r) + \frac{c}{1{,}000}. \tag{3} $$

5. Making use of formula (3) find the office premium on a fifteen payment life policy of \$1,000 for the ages 20, 25, 30 and 35, where $r = 16\frac{2}{3}\%$, $s = 16\frac{2}{3}\%$, and $c = 50$ cents.

6. Making use of (1) find the office premiums on an ordinary life policy of \$1,000 for the ages 20, 25, 30 and 35, where $r = 33\frac{1}{3}\%$ and $c = 50$ cents.

* Menge, W. O. and Glover, J. W., *An Introduction to the Mathematics of Life Insurance*, 1935, p. 108.

7. The formula

$$P'_{x\,\overline{n}|} = P_x \cdot s + P_{x\,\overline{n}|}(1 + r) + \frac{c}{1{,}000} \tag{4}$$

gives the gross premium for an n-year endowment policy of \$1 on (x). Interpret the formula.

8. Making use of (4) find the office premium of a fifteen year endowment policy of \$1,000 for the ages 20, 25, 30 and 35, where $r = s = 16\frac{2}{3}\%$ and $c = 50$ cents.

Problems

1. By the terms of a will the income at 5% annually of a \$20,000 estate goes to a widow aged 50 during her lifetime. Find the value of her inheritance.

2. The will in Problem 1 requires that the residue of the estate shall go to a hospital when the widow dies. Find the value of this residue at the time the inheritance comes to the widow.

3. By the terms of a will the income at 5% annually of a \$20,000 estate goes to a son aged 25 for 10 years, or so long as he lives during the 10 years, after which the residue of the estate goes to a university. Find the present value of each legacy.

4. A widow aged 55 is to receive a life income of \$25,000 a year from her husband's estate. The inheritance tax law requires that the bequest be valued on a $3\frac{1}{2}\%$ basis. The law grants the widow an exemption of \$5,000, and on the remainder of the cash value of her inheritance a tax of 3% must be paid of the first \$50,000 over the exemption value, and 5% on the next \$50,000, then 10% on the cash value in excess of \$100,000. Find the inheritance tax on this bequest.

5. Under the Illinois Standard, the terminal reserve at the end of 25 years of a \$1,000, 15-payment life policy issued at age 35 is \$626.92. If the full amount of this reserve is allowed as cash surrender value, how much paid-up insurance will it purchase?

6. Under the full preliminary term valuation, the terminal reserve at the end of 25 years on a \$1,000 ordinary life policy issued at age 35 is \$400.25. If the full amount of this reserve is used to purchase extended insurance, how long is the extension?

7. Find the net first year and renewal premiums for an ordinary life policy of \$1,000 issued at age 25 according to the full preliminary term method.

8. Same as Problem 7 but for a 20-payment life policy.

9. Same as Problem 7 but for a 20-payment 20 year endowment policy.

REVIEW PROBLEMS

Percentage

1. A building worth \$15,000 is insured for \$12,000. For what per cent of its value is it insured?

2. A merchant fails, having liabilities of \$30,000, and resources of \$18,000. What per cent of his debts can he pay? He owes Joe Brown \$6,500. How much will Brown receive?

3. A manufacturer sells to a wholesaler at a profit of 20%. The wholesaler sells to the retailer at a 25% profit. The retailer sells to the consumer at a profit of 60%. If the consumer pays \$28.80, what is the cost to the manufacturer? To the wholesaler? To the retailer?

4. Which is better for the purchaser, a series of discounts of 30%, 20%, and 10%, or a single discount of 50%? What would be the difference on a bill of \$1,000?

5. A coat listed at \$100 is bought subject to discounts of 20%, 10%, and $8\frac{1}{3}\%$. (*a*) Find the net cost rate factor. (*b*) Find the net cost. (*c*) What single discount rate is equivalent to the given series of discounts? [*Alg.: Com.—Stat.*, p. 98.]

6. A coat cost a dealer \$66. He marked the coat so that he could "drop" the marked price 20% and still sell it so as to make a profit of 10% on the cost. What was the selling price? The marked price?

7. I can buy a living room suite for \$150, less $33\frac{1}{3}\%$ and 20%. From another dealer I can get the same suite for \$125, less 25% and $12\frac{1}{2}\%$. The terms in each case are "net 30 days or 2% off for cash." What is the least amount of cash for which I can purchase the suite?

8. A bill of goods is purchased subject to discounts of r_1 and r_2. Show that an equivalent single discount is their sum less their product.

9. Goods are bought subject to discounts of 25% and 20%. Find the marked price per dollar list if the goods are to be marked to realize a profit of $33\frac{1}{3}\%$.

10. At what price should goods costing \$432 be marked to make a profit of 25% of the cost after allowing a discount of 20%?

Simple Interest and Discount

11. A note for \$1,200 bearing interest at 5% and due in 8 months is sold to an investor to whom money is worth 6%. What does the investor pay for the note?

12. I purchased \$400 worth of lumber from a dealer who will allow me credit for 60 days. If I desire to pay immediately, what should he be willing to accept if he estimates that he earns 6% on his money?

13. A real estate dealer received two offers for a piece of property. Jones offered \$3,000 cash and \$5,000 in 6 months; Smith offered \$5,000 cash and \$3,000 in 1 year. Which was the better offer on a 6% basis?

14. The cash price of a washing machine is \$75. It is bought for \$10 down and \$10 a month for 7 months. What rate of interest is paid?

15. I borrow \$500 for six months from a bank that charges 6% in advance. For what amount do I make the note?

16. I owe \$500 due in 3 months and \$600 due in 12 months. I desire to pay these debts by making equal payments at the ends of six and nine months. On a 6% basis, find the equal payments. Choose 12 months as a focal date.

17. I owe William Brown \$500 due in 3 months with interest at 8% and \$800 due in 12 months without interest. We agree that I may liquidate these debts with equal payments at the ends of six and nine months on a 6% basis. Find the equal payments by focalizing at 12 months.

18. When could I liquidate the debts in Problem 16 by a single payment of \$1,100, the equities remaining the same?

19. When could I liquidate the debts in Problem 17 by a single payment of \$1,310, the equities remaining the same? Solve by setting up an equation of value with focal date at 12 months.

20. \$1,000 LOUISVILLE, KENTUCKY
February 12, 1945

Nine months after date I promise to pay Robert Brown, or order, one thousand dollars with interest at 7% from date.

Signed, GEORGE SANDERS.

(*a*) Five months after date, Brown sold the note to Bank B which operates on a 6% discount basis. What did Brown receive for the note?

(*b*) Bank B held the note for 1 month and then sold it to a Federal Reserve Bank which operates on a 4% discount basis. What did Bank B gain on the transaction?

Compound Interest and Discount

21. A man buys a house for \$6,000, pays \$2,000 cash, and gives a mortgage note at 6% for the balance. If he pays \$1,000 at the end of two years and \$1,000 at the end of 4 years, what will be the balance due at the end of 5 years?

22. I owe \$1,500. I arrange to pay \$$R$ at the end of 1 year, \$$2R$ at the end of 2 years and \$$3R$ at the end of 3 years. If money is worth 5% find R.

23. If ($j = .08$, $m = 12$), find i.

24. If a finance company charges 1% a month on loans, what is their effective earning?

25. I owe two sums: \$700 due in 6 months without interest and \$1,500 due in 18 months with interest at ($j = .06$, $m = 2$). On a ($j = .05$, $m = 2$) basis what amount will liquidate these debts at the end of 1 year?

26. A lot is priced at \$2,000 cash. A buyer purchased it with equal payments now and at the end of one year. On a 6% basis, what was the amount of the payments?

27. What sum payable in 2 years will discharge two debts, \$1,500 due in 3 years with interest at 5%, and \$2,000 due in four years with interest at 6%, money being worth 4%?

28. A merchant sells goods on the terms "net 90 days or 2% off for cash." Find the highest nominal rate of interest, j_4, at which a customer should borrow money in order to pay cash. Find the effective rate.

29. If $i = .06$, find d, j_4, and f_4.

30. If $d = .06$, find i, f_4, and j_4.

31. If $f_4 = .06$, find i, d, and j_4.

32. If $j_4 = .06$, find i, d, and f_4.

33. The Jones Lumber Co. estimates that money put into their business yields 1½% a month. Find the highest discount rate, $\frac{f_{12}}{12}$, they can afford to offer to encourage payment of a bill due in one month.

34. State a problem for which the answer would be the value of x determined by the equation:

$$7{,}860 = x(1.03)^{-2} + x(1.03)^{-4} + x.$$

35. State a problem for which the answer would be the value of x determined by the equation:

$$x(1.04)^2 + x(1.04) + x = 3{,}000(1.025)^6 + 2{,}000(1.04)^{-1}.$$

36. I can buy a piece of property for \$9,800 cash or for \$6,000 cash and payments of \$2,000 at the ends of 1 year and 2 years. Should I pay cash if I can invest money at 6%?

Annuities

37. A purchaser of a farm agreed to pay \$1,000 at the end of each year for 10 years. (*a*) What is the equivalent cash price if money is worth 5%? (*b*) At the end of 5 years, what must the purchaser pay if he desires to completely discharge his remaining liability on that date?

38. I owe \$6,000 due immediately. If money is worth ($j = .04$, $m = 4$), what equal quarterly payments will discharge the debt if the first payment occurs at the end of 3 years and the last at the end of 10 years?

39. A man buys a home for which the cash price is \$10,000. He pays \$1,200 down and agrees to pay the balance with interest at ($j = .05$, $m = 2$) by payments of \$1,200 at the end of each half-year as long as necessary with a final partial payment at the end of the last payment period. How many full payments are necessary? What is the final partial payment?

40. In Problem 39, find the principal outstanding just after the fifth payment of \$1,200.

41. Prove that $(1 + i)s_{\overline{n}|i} + 1 = s_{\overline{n+1}|i}$

(*a*) by verbal interpretation;

(*b*) algebraically.

42. A man buys a house of cash value \$25,000. He pays \$5,000 down and agrees to pay the balance with payments of \$1,000 at the beginning of each half-year for 14 years. Find the nominal rate j_2 and the effective rate i that the purchaser pays.

43. An annuity of \$100 a year amounts to \$3,492.58 in 20 years. Find i.

44. A man purchased a property paying \$3,000 down and \$500 at the end of each half-year for 10 years. If money was worth ($j = .07$, $m = 2$), what was the equivalent cash price?

45. A debt of \$10,000 is being amortized, principal and interest, by payments of \$1,000 at the end of each half-year. If interest is at ($j = .04$, $m = 2$), what is the final payment?

46. The sum of \$500 was paid annually into a fund for five years, and then \$800 a year was paid. If the funds accumulated at 4%, when did the total amount to \$12,000? Obtain the final payment.

47. The sum of \$100 was deposited at the end of each month for 8 years in a bank that paid 4% effective. What was the value of the account two years after the last deposit if no withdrawals were made?

48. A man deposited \$200 at the end of every quarter in a savings bank that paid 3½% effective. When did the account total \$10,000? What was the final partial payment?

49. A machine costs \$2,000 new and must be replaced at the end of 15 years at a cost of \$1,900. Find the capitalized cost if money can be invested at 4%.

50. Is it more profitable for a city to pay \$2 per square yard for paving that lasts five years than to pay \$3 per square yard for paving that lasts 8 years, money being worth 5%?

51. A lawn mower costs \$10 and will last 3 years. How much can one afford to pay for a better grade of mower that will last 5 years, money worth 4%?

Sinking Funds and Amortization

52. Find the annual payment necessary to amortize in 5 years a debt of \$1,000 which bears interest at 7%. Construct a schedule.

53. A corporation issues \$1,000,000, 6% bonds, dividends payable semi-annually. The dividends are paid as they fall due and the corporation makes semi-annual deposits into a sinking fund that will accumulate at $j_2 = .04$ to their face value in 15 years. Find the sinking fund deposit. Find the total semi-annual expense to the corporation.

54. A debt of \$100,000 bearing interest at 5% effective will be retired by a sinking fund at the end of 10 years that earns 4% effective. Find the total annual expense. At what rate of interest could the debtor just as well have agreed to amortize the debt?

55. Which will be better, to repay a debt of \$25,000, principal and interest at 5%, in 10 equal annual payments, or to pay 6% interest on the debt each year and accumulate a sinking fund of \$25,000 in 10 years at 4%?

56. A man purchases a house for \$12,000 paying one-half down. He arranges to pay \$1,500 per year principal and interest on the remaining amount until the debt is paid. How many payments of \$1,500 are made and what is the final payment at the end of the year of settlement if the debt bears interest at 6%?

57. At the end of two years what was the purchaser's equity in the house in Problem 56?

Depreciation

58. A dynamo costing \$5,000 has an estimated life of 10 years and a scrap value of \$200. Find the constant rate of depreciation. What is the book value of the machine at the end of 5 years?

59. What is the annual payment into the depreciation fund of the machine in Problem 58 if the fund increases at 4%? What is the book value of the machine at the end of 5 years?

60. A plant consists of three parts described by the table. Find the total annual depreciation charge on a 3% basis:

Part	Est. Life	Cost	Scrap Value
A	40	\$20,000	\$1,000
B	20	8,000	200
C	15	10,000	2,000

61. A Diesel engine costs \$50,000, lasts 20 years and has a salvage value of \$5,000. (*a*) Find the amount that should be in the sinking fund at the end of 10 years at 4½%. (*b*) What is the amount of depreciation during the eleventh year?

62. An old machine turns out annually 1,200 units at a cost of \$3,000 for operation and maintenance. It is estimated that at the end of 12 years it will have a salvage value of \$500. To replace the old machine by a new one would cost \$15,000, but 1,500 units could be turned out annually at an average annual cost of \$3,500 and this could be maintained for 25 years with a salvage value of \$1,000. On a 6% basis what is the value of the old machine?

63. What number of units output annually of the new equipment in Problem 62 would reduce the value of the old machine to \$4,000, all other data remaining the same?

64. What number of units output of the new machine in Problem 62 would render the old machine worthless?

Valuation of Bonds

65. Find the cost of a \$1,000, 5% J. and J. bond, redeemable at par in 10 years, if bought to yield ($j = .06$, $m = 2$).

66. Find the cost of a \$1,000 bond, redeemable in 8 years at 106, paying 6% convertible quarterly if bought to yield 8% effective.

67. Find the cost of the bond described in Problem 66 if bought to yield ($j = .08$, $m = 4$).

68. A \$10,000, 4% J. and J. bond, redeemable at par January 1, 1940, was bought July 1, 1936, to yield ($j = .06$, $m = 2$). Construct a schedule for the accumulation of the discount.

69. What was a fair price for the bond described in Problem 68 if bought on August 13, 1936?

70. A \$10,000, 7% J. and J. bond, was sold on June 1 at 102¼ and accrued interest. What was the selling price?

71. A \$1,000, 5% J. and J. bond, redeemable at par in 10 years was purchased for \$970. Find the yield rate, j_2.

Miscellaneous

72. If \$100 invested at 5% simple interest accumulates to the same amount as \$100 invested at 4% simple discount, find the time the investment runs.

73. Show that it takes three times as long for a principal P to quadruple itself at $i\%$ as it does to double itself.

74. Jones considers two offers for a piece of property. A offers \$3,000 cash and \$5,000 in 6 months. B offers \$5,000 cash and \$3,000 in 1 year. On a 5% simple interest basis, which is the better offer? Find the difference in the present values of the two offers.

75. If D_o and D_e denote ordinary and exact simple discounts on an amount S for n years at $d\%$, show that $D_e = D_o - D_o/73$. [Compare (6), page 4.]

76. How long will it take a principal P to double itself at the compound discount rate, $d\%$?

77. Prove: $\dfrac{1}{\mathbf{a}_{\overline{n}|i}} = \dfrac{1}{\mathbf{s}_{\overline{n}|i}} + d.$

78. Prove: $\mathbf{a}_{\overline{n}|i} = \dfrac{1 - v^n}{d}.$

79. If R_r denotes the amount in the depreciation fund at the end of r years under the S.F. plan, prove that $R_r = Rs_{\overline{r}|i}$.

80. If D_r denotes the depreciation charge during the rth year under the S.F. plan, prove that $D_r = R(1 + i)^{r-1}$. [See Exercise 79 above.]

81. If $a_{\overline{n}|i} = x$ and $s_{\overline{n}|i} = y$, prove that $i = (y - x)/xy$.

82. A debt D bearing interest at $i\%$ is being amortized by equal annual payments R. Show that the indebtedness remaining unpaid at the end of r years is $D - (R - Di)s_{\overline{r}|i}$.

83. Let $C = S$, and show that (2), page 142, can be reduced to form (12′), page 135. Explain how this can be true.

84. An alumnus, 50 years of age, proposes to give his college \$50,000 provided the college will pay him \$2,500 a year as long as he lives. If the college can borrow money at 4%, should it accept the proposition?

85. A note for \$3,000 with interest (compound) at 5%, due in 5 years, is discounted at the end of 2 years at discount rate of 4% compounded semi-annually. Find the proceeds and the discount.

86. A teacher provided for retirement by depositing \$300 a year with a trust company that granted him ($j = .04$, $m = 2$) interest rate. At the end of 25 years he retired and withdrew \$1,000 a year. For how many years could he enjoy this annuity?

87. It is estimated that a copper mine will produce \$30,000 a year for 18 years. If the investor desires to earn 12% on the investment and can earn 4% on the sinking fund, what can he afford to pay for the mine?

88. A timber tract is priced at \$1,000,000. It is estimated the tract will yield a net annual income of \$200,000 for 10 years and that the cleared land will be worth \$20,000. The lumber company wishes to earn 10% on the investment and can earn 4% on redemption funds. Is the tract a good buy?

89. Find the constant per cent by which the value of a machine is decreased if its cost is \$12,000, its scrap value \$2,000, and its estimated life 15 years.

90. Expand $(1 + j/m)^m$ by the binomial theorem, let m become infinite, and show that

$$\lim_{m \to \infty} \left(1 + \frac{j}{m}\right)^m = 1 + j + \frac{j^2}{2!} + \frac{j^3}{3!} + \cdots.$$

The series on the right is the infinite series expansion of e^j, where $e = 2.71828+$ and is called the base of the natural or Napierian logarithms. The series converges for all values of j. Thus, as m becomes infinite, $(1 + i)$ approaches e^j. (See page 139.)

When m becomes infinite, it is customary to replace j by δ. Thus, for continuous conversion we have

$$1 + i = e^\delta$$

$$\delta = \log_e (1 + i) = \frac{\log_{10} (1 + i)}{\log_{10} e} = \frac{\log_{10} (1 + i)}{.43429}.$$

The quantity δ is called the *force of interest.*

91. If $\delta = .06$, find i.

92. If $i = .06$, find δ.

93. Show that if the interest is converted continuously for n years, the accumulated value of S is

$$S = Pe^{n\delta}.$$

94. The population of Jacksonville increased continuously from 130,000 in 1930 to 173,000 in 1940. Find the continuous rate of increase. (Use results of Exercise 93 above.)

95. Proceed as in Exercise 90 and show that

$$\lim_{m \to \infty} \left(1 - \frac{f}{m}\right)^m = e^{-f}.$$

It is customary for continuous conversion of discount to replace f by δ'. Then we have

$$1 - d = e^{-\delta'}.$$

The quantity δ' is called *force of discount.*

96. Show that if the discount is converted continuously for n years, the discounted value of S is

$$P = Se^{-n\delta'}.$$

97. Find the amount of \$1,000 for 10 years at 4% nominal, converted continuously.

98. A machine depreciated continuously from a value of \$50,000 to a salvage value of \$10,000 in 20 years. Find the continuous rate of depreciation.

99. Jones bought a truck for \$2,000. Its estimated life was 5 years and its salvage value was \$500. Jones estimated the truck earned \$500 a year net. What did he earn on his investment if deposits for replacement earned 3%? (See page 135.)

100. A college invests \$400,000 in a dormitory. It is estimated that the college will derive \$25,000 net a year for 50 years at the end of which time the building will have a salvage value of \$100,000. What will the college earn on its investment if deposits for replacement earn 3%? (See page 135.)

Table I.—Common Logarithms of Numbers

To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
01	432	475	518	561	604	647	689	732	775	817
02	860	903	945	988	*030	*072	*115	*157	*199	*242
03	01 284	326	368	410	452	494	536	578	620	662
04	703	745	787	828	870	912	953	995	*036	*078
05	02 119	160	202	243	284	325	366	407	449	490
06	531	572	612	653	694	735	776	816	857	898
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302
08	03 342	383	423	463	503	543	583	623	663	703
09	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
11	532	571	610	650	689	727	766	805	844	883
12	922	961	999	*038	*077	*115	*154	*192	*231	*269
13	05 308	346	385	423	461	500	538	576	614	652
14	690	729	767	805	843	881	918	956	994	*032
15	06 070	108	145	183	221	258	296	333	371	408
16	446	483	521	558	595	633	670	707	744	781
17	819	856	893	930	967	*004	*041	*078	*115	*151
18	07 188	225	262	298	335	372	408	445	482	518
19	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
21	08 279	314	350	386	422	458	493	529	565	600
22	636	672	707	743	778	814	849	884	920	955
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
24	09 342	377	412	447	482	517	552	587	621	656
25	691	726	760	795	830	864	899	934	968	*003
26	10 037	072	106	140	175	209	243	278	312	346
27	380	415	449	483	517	551	585	619	653	687
28	721	755	789	823	857	890	924	958	992	*025
29	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
31	727	760	793	826	860	893	926	959	992	*024
32	12 057	090	123	156	189	222	254	287	320	352
33	385	418	450	483	516	548	581	613	646	678
34	710	743	775	808	840	872	905	937	969	*001
35	13 033	066	098	130	162	194	226	258	290	322
36	354	386	418	450	481	513	545	577	609	640
37	672	704	735	767	799	830	862	893	925	956
38	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
39	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
41	922	953	983	*014	*045	*076	*106	*137	*168	*198
42	15 229	259	290	320	351	381	412	442	473	503
43	534	564	594	625	655	685	715	746	776	806
44	836	866	897	927	957	987	*017	*047	*077	*107
45	16 137	167	197	227	256	286	316	346	376	406
46	435	465	495	524	554	584	613	643	673	702
47	732	761	791	820	850	879	909	938	967	997
48	17 026	056	085	114	143	173	202	231	260	289
49	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869

PP			
	44	**43**	**42**
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8
	41	**40**	**39**
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1
	38	**37**	**36**
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4
	35	**34**	**33**
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7
	32	**31**	**30**
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
51	898	926	955	984	*013	*041	*070	*099	*127	*156
52	18 184	213	241	270	298	327	355	384	412	441
53	469	498	526	554	583	611	639	667	696	724
54	752	780	808	837	865	893	921	949	977	*005
55	19 033	061	089	117	145	173	201	229	257	285
56	312	340	368	396	424	451	479	507	535	562
57	590	618	645	673	700	728	756	783	811	838
58	866	893	921	948	976	*003	*030	*058	*085	*112
59	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
61	683	710	737	763	790	817	844	871	898	925
62	952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21 219	245	272	299	325	352	378	405	431	458
64	484	511	537	564	590	617	643	669	696	722
65	748	775	801	827	854	880	906	932	958	985
66	22 011	037	063	089	115	141	167	194	220	246
67	272	298	324	350	376	401	427	453	479	505
68	531	557	583	608	634	660	686	712	737	763
69	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
71	300	325	350	376	401	426	452	477	502	528
72	553	578	603	629	654	679	704	729	754	779
73	805	830	855	880	905	930	955	980	*005	*030
74	24 055	080	105	130	155	180	204	229	254	279
75	304	329	353	378	403	428	452	477	502	527
76	551	576	601	625	650	674	699	724	748	773
77	797	822	846	871	895	920	944	969	993	*018
78	25 042	066	091	115	139	164	188	212	237	261
79	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
81	768	792	816	840	864	888	912	935	959	983
82	26 007	031	055	079	102	126	150	174	198	221
83	245	269	293	316	340	364	387	411	435	458
84	482	505	529	553	576	600	623	647	670	694
85	717	741	764	788	811	834	858	881	905	928
86	951	975	998	*021	*045	*068	*091	*114	*138	*161
87	27 184	207	231	254	277	300	323	346	370	393
88	416	439	462	485	508	531	554	577	600	623
89	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
91	28 103	126	149	171	194	217	240	262	285	307
92	330	353	375	398	421	443	466	488	511	533
93	556	578	601	623	646	668	691	713	735	758
94	780	803	825	847	870	892	914	937	959	981
95	29 003	026	048	070	092	115	137	159	181	203
96	226	248	270	292	314	336	358	380	403	425
97	447	469	491	513	535	557	579	601	623	645
98	667	688	710	732	754	776	798	820	842	863
99	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

PP		
	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

PP		
	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

PP	
	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

PP		
	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

PP		
	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

N	0	1	2	3	4	5	6	7	8	9	PP

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	750	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345	366
06	387	408	429	450	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675	695	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325	345	365	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985	*005	*025
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655	674	694	713	733	753	772	792	811
23	830	850	869	889	908	928	947	967	986	*005
24	35 025	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365	383	401	420	438	457
37	475	493	511	530	548	566	585	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525	543
43	561	578	596	614	632	650	668	686	703	721
44	739	757	775	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235	252
47	270	287	305	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

PP

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

Table I.—Common Logarithms of Numbers

To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
250	39 794	811	829	846	863	881	898	915	933	950	
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123	
52	40 140	157	175	192	209	226	243	261	278	295	
53	312	329	346	364	381	398	415	432	449	466	**18**
54	483	500	518	535	552	569	586	603	620	637	1 1.8
55	654	671	688	705	722	739	756	773	790	807	2 3.6
56	824	841	858	875	892	909	926	943	960	976	3 5.4
57	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	4 7.2 5 9.0 6 10.8
58	41 162	179	196	212	229	246	263	280	296	313	7 12.6
59	330	347	363	380	397	414	430	447	464	481	8 14.4 9 16.2
260	497	514	531	547	564	581	597	614	631	647	
61	664	681	697	714	731	747	764	780	797	814	
62	830	847	863	880	896	913	.929	946	963	979	
63	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	**17**
64	42 160	177	193	210	226	243	259	275	292	308	1 1.7
65	325	341	357	374	390	406	423	439	455	472	2 3.4
66	488	504	521	537	553	570	586	602	619	635	3 5.1 4 6.8 5 8.5
67	651	667	684	700	716	732	749	765	781	797	6 10.2
68	813	830	846	862	878	894	911	927	943	959	7 11.9
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120	8 13.6 9 15.3
270	43 136	152	169	185	201	217	233	249	265	281	
71	297	313	329	345	361	377	393	409	425	441	
72	457	473	489	505	521	537	553	569	584	600	
73	616	632	648	664	680	696	712	727	743	759	**16**
74	775	791	807	823	838	854	870	886	902	917	1 1.6
75	933	949	965	981	996	*012	*028	*044	*059	*075	2 3.2
76	44 091	107	122	138	154	170	185	201	217	232	3 4.8 4 6.4 5 8.0
77	248	264	279	295	311	326	342	358	373	389	6 9.6
78	404	420	436	451	467	483	498	514	529	545	7 11.2
79	560	576	592	607	623	638	654	669	685	700	8 12.8 9 14.4
280	716	731	747	762	778	793	809	824	840	855	
81	871	886	902	917	932	948	963	979	994	*010	
82	45 025	040	056	071	086	102	117	133	148	163	
83	179	194	209	225	240	255	271	286	301	317	**15**
84	332	347	362	378	393	408	423	439	454	469	1 1.5
85	484	500	515	530	545	561	576	591	606	621	2 3.0
86	637	652	667	682	697	712	728	743	758	773	3 4.5 4 6.0 5 7.5
87	788	803	818	834	849	864	879	894	909	924	6 9.0
88	939	954	969	984	*000	*015	*030	*045	*060	*075	7 10.5
89	46 090	105	120	135	150	165	180	195	210	225	8 12.0 9 13.5
290	240	255	270	285	300	315	330	345	359	374	
91	389	404	419	434	449	464	479	494	509	523	
92	538	553	568	583	598	613	627	642	657	672	
93	687	702	716	731	746	761	776	790	805	820	**14**
94	835	850	864	879	894	909	923	938	953	967	1 1.4
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114	2 2.8
96	47 129	144	159	173	188	202	217	232	246	261	3 4.2 4 5.6 5 7.0
97	276	290	305	319	334	349	363	378	392	407	6 8.4
98	422	436	451	465	480	494	509	524	538	553	7 9.8
99	567	582	596	611	625	640	654	669	683	698	8 11.2 9 12.6
300	712	727	741	756	770	784	799	813	828	842	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers

To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
300	47 712	727	741	756	770	784	799	813	828	842	
01	857	871	885	900	914	929	943	958	972	986	
02	48 001	015	029	044	058	073	087	101	116	130	
03	144	159	173	187	202	216	230	244	259	273	
04	287	302	316	330	344	359	373	387	401	416	
05	430	444	458	473	487	501	515	530	544	558	15
06	572	586	601	615	629	643	657	671	686	700	1 1.5
07	714	728	742	756	770	785	799	813	827	841	2 3.0
08	855	869	883	897	911	926	940	954	968	982	3 4.5
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122	4 6.0
310	49 136	150	164	178	192	206	220	234	248	262	5 7.5
11	276	290	304	318	332	346	360	374	388	402	6 9.0
12	415	429	443	457	471	485	499	513	527	541	7 10.5
13	554	568	582	596	610	624	638	651	665	679	8 12.0
14	693	707	721	734	748	762	776	790	803	817	9 13.5
15	831	845	859	872	886	900	914	927	941	955	
16	969	982	996	*010	*024	*037	*051	*065	*079	*092	
17	50 106	120	133	147	161	174	188	202	215	229	14
18	243	256	270	284	297	311	325	338	352	365	1 1.4
19	379	393	406	420	433	447	461	474	488	501	2 2.8
320	515	529	542	556	569	583	596	610	623	637	3 4.2
21	651	664	678	691	705	718	732	745	759	772	4 5.6
22	786	799	813	826	840	853	866	880	893	907	5 7.0
23	920	934	947	961	974	987	*001	*014	*028	*041	6 8.4
24	51 055	068	081	095	108	121	135	148	162	175	7 9.8
25	188	202	215	228	242	255	268	282	295	308	8 11.2
26	322	335	348	362	375	388	402	415	428	441	9 12.6
27	455	468	481	495	508	521	534	548	561	574	
28	587	601	614	627	640	654	667	680	693	706	
29	720	733	746	759	772	786	799	812	825	838	13
330	851	865	878	891	904	917	930	943	957	970	1 1.3
31	983	996	*009	*022	*035	*048	*061	*075	*088	*101	2 2.6
32	52 114	127	140	153	166	179	192	205	218	231	3 3.9
33	244	257	270	284	297	310	323	336	349	362	4 5.2
34	375	388	401	414	427	440	453	466	479	492	5 6.5
35	504	517	530	543	556	569	582	595	608	621	6 7.8
36	634	647	660	673	686	699	711	724	737	750	7 9.1
37	763	776	789	802	815	827	840	853	866	879	8 10.4
38	892	905	917	930	943	956	969	982	994	*007	9 11.7
39	53 020	033	046	058	071	084	097	110	122	135	
340	148	161	173	186	199	212	224	237	250	263	12
41	275	288	301	314	326	339	352	364	377	390	1 1.2
42	403	415	428	441	453	466	479	491	504	517	2 2.4
43	529	542	555	567	580	593	605	618	631	643	3 3.6
44	656	668	681	694	706	719	732	744	757	769	4 4.8
45	782	794	807	820	832	845	857	870	882	895	5 6.0
46	908	920	933	945	958	970	983	995	*008	*020	6 7.2
47	54 033	045	058	070	083	095	108	120	133	145	7 8.4
48	158	170	183	195	208	220	233	245	258	270	8 9.6
49	283	295	307	320	332	345	357	370	382	394	9 10.8
350	407	419	432	444	456	469	481	494	506	518	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
350	54 407	419	432	444	456	469	481	494	506	518	
51	531	543	555	568	580	593	605	617	630	642	
52	654	667	679	691	704	716	728	741	753	765	
53	777	790	802	814	827	839	851	864	876	888	
54	900	913	925	937	949	962	974	986	998	*011	
55	55 023	035	047	060	072	084	096	108	121	133	13
56	145	157	169	182	194	206	218	230	242	255	1 1.3
57	267	279	291	303	315	328	340	352	364	376	2 2.6
58	388	400	413	425	437	449	461	473	485	497	3 3.9
59	509	522	534	546	558	570	582	594	606	618	4 5.2
360	630	642	654	666	678	691	703	715	727	739	5 6.5 6 7.8 7 9.1 8 10.4 9 11.7
61	751	763	775	787	799	811	823	835	847	859	
62	871	883	895	907	919	931	943	955	967	979	
63	991	*003	*015	*027	*038	*050	*062	*074	*086	*098	
64	56 110	122	134	146	158	170	182	194	205	217	
65	229	241	253	265	277	289	301	312	324	336	
66	348	360	372	384	396	407	419	431	443	455	
67	467	478	490	502	514	526	538	549	561	573	12
68	585	597	608	620	632	644	656	667	679	691	1 1.2
69	703	714	726	738	750	761	773	785	797	808	2 2.4
370	820	832	844	855	867	879	891	902	914	926	3 3.6 4 4.8 5 6.0 6 7.2 7 8.4 8 9.6 9 10.8
71	937	949	961	972	984	996	*008	*019	*031	*043	
72	57 054	066	078	089	101	113	124	136	148	159	
73	171	183	194	206	217	229	241	252	264	276	
74	287	299	310	322	334	345	357	368	380	392	
75	403	415	426	438	449	461	473	484	496	507	
76	519	530	542	553	565	576	588	600	611	623	
77	634	646	657	669	680	692	703	715	726	738	
78	749	761	772	784	795	807	818	830	841	852	11
79	864	875	887	898	910	921	933	944	955	967	1 1.1
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081	2 2.2 3 3.3 4 4.4 5 5.5 6 6.6 7 7.7 8 8.8 9 9.9
81	58 092	104	115	127	138	149	161	172	184	195	
82	206	218	229	240	252	263	274	286	297	309	
83	320	331	343	354	365	377	388	399	410	422	
84	433	444	456	467	478	490	501	512	524	535	
85	546	557	569	580	591	602	614	625	636	647	
86	659	670	681	692	704	715	726	737	749	760	
87	771	782	794	805	816	827	838	850	861	872	
88	883	894	906	917	928	939	950	961	973	984	
89	995	*006	*017	*028	*040	*051	*062	*073	*084	*095	
390	59 106	118	129	140	151	162	173	184	195	207	10
91	218	229	240	251	262	273	284	295	306	318	1 1.0
92	329	340	351	362	373	384	395	406	417	428	2 2.0
93	439	450	461	472	483	494	506	517	528	539	3 3.0 4 4.0 5 5.0 6 6.0 7 7.0 8 8.0 9 9.0
94	550	561	572	583	594	605	616	627	638	649	
95	660	671	682	693	704	715	726	737	748	759	
96	770	780	791	802	813	824	835	846	857	868	
97	879	890	901	912	923	934	945	956	966	977	
98	988	999	*010	*021	*032	*043	*054	*065	*076	*086	
99	60 097	108	119	130	141	152	163	173	184	195	
400	206	217	228	239	249	260	271	282	293	304	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
400	60 206	217	228	239	249	260	271	282	293	304	
01	314	325	336	347	358	369	379	390	401	412	
02	423	433	444	455	466	477	487	498	509	520	
03	531	541	552	563	574	584	595	606	617	627	
04	638	649	660	670	681	692	703	713	724	735	
05	746	756	767	778	788	799	810	821	831	842	
06	853	863	874	885	895	906	917	927	938	949	
07	959	970	981	991	*002	*013	*023	*034	*045	*055	
08	61 066	077	087	098	109	119	130	140	151	162	11
09	172	183	194	204	215	225	236	247	257	268	1 1.1
410	278	289	300	310	321	331	342	352	363	374	2 2.2 3 3.3 4 4.4 5 5.5 6 6.6 7 7.7 8 8.8 9 9.9
11	384	395	405	416	426	437	448	458	469	479	
12	490	500	511	521	532	542	553	563	574	584	
13	595	606	616	627	637	648	658	669	679	690	
14	700	711	721	731	742	752	763	773	784	794	
15	805	815	826	836	847	857	868	878	888	899	
16	909	920	930	941	951	962	972	982	993	*003	
17	62 014	024	034	045	055	066	076	086	097	107	
18	118	128	138	149	159	170	180	190	201	211	
19	221	232	242	252	263	273	284	294	304	315	
420	325	335	346	356	366	377	387	397	408	418	
21	428	439	449	459	469	480	490	500	511	521	
22	531	542	552	562	572	583	593	603	613	624	
23	634	644	655	665	675	685	696	706	716	726	10
24	737	747	757	767	778	788	798	808	818	829	1 1.0 2 2.0 3 3.0 4 4.0 5 5.0 6 6.0 7 7.0 8 8.0 9 9.0
25	839	849	859	870	880	890	900	910	921	931	
26	941	951	961	972	982	992	*002	*012	*022	*033	
27	63 043	053	063	073	083	094	104	114	124	134	
28	144	155	165	175	185	195	205	215	225	236	
29	246	256	266	276	286	296	306	317	327	337	
430	347	357	367	377	387	397	407	417	428	438	
31	448	458	468	478	488	498	508	518	528	538	
32	548	558	568	579	589	599	609	619	629	639	
33	649	659	669	679	689	699	709	719	729	739	
34	749	759	769	779	789	799	809	819	829	839	
35	849	859	869	879	889	899	909	919	929	939	
36	949	959	969	979	988	998	*008	*018	*028	*038	
37	64 048	058	068	078	088	098	108	118	128	137	9
38	147	157	167	177	187	197	207	217	227	237	1 0.9 2 1.8 3 2.7 4 3.6 5 4.5 6 5.4 7 6.3 8 7.2 9 8.1
39	246	256	266	276	286	296	306	316	326	335	
440	345	355	365	375	385	395	404	414	424	434	
41	444	454	464	473	483	493	503	513	523	532	
42	542	552	562	572	582	591	601	611	621	631	
43	640	650	660	670	680	689	699	709	719	729	
44	738	748	758	768	777	787	797	807	816	826	
45	836	846	856	865	875	885	895	904	914	924	
46	933	943	953	963	972	982	992	*002	*011	*021	
47	65 031	040	050	060	070	079	089	099	108	118	
48	128	137	147	157	167	176	186	196	205	215	
49	225	234	244	254	263	273	283	292	302	312	
450	321	331	341	350	360	369	379	389	398	408	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
450	65 321	331	341	350	360	369	379	389	398	408	
51	418	427	437	447	456	466	475	485	495	504	
52	514	523	533	543	552	562	571	581	591	600	
53	610	619	629	639	648	658	667	677	686	696	
54	706	715	725	734	744	753	763	772	782	792	
55	801	811	820	830	839	849	858	868	877	887	
56	896	906	916	925	935	944	954	963	973	982	
57	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	
58	66 087	096	106	115	124	134	143	153	162	172	
59	181	191	200	210	219	229	238	247	257	266	
460	276	285	295	304	314	323	332	342	351	361	
61	370	380	389	398	408	417	427	436	445	455	
62	464	474	483	492	502	511	521	530	539	549	
63	558	567	577	586	596	605	614	624	633	642	
64	652	661	671	680	689	699	708	717	727	736	
65	745	755	764	773	783	792	801	811	820	829	
66	839	848	857	867	876	885	894	904	913	922	
67	932	941	950	960	969	978	987	997	*006	*015	
68	67 025	034	043	052	062	071	080	089	099	108	
69	117	127	136	145	154	164	173	182	191	201	
470	210	219	228	237	247	256	265	274	284	293	
71	302	311	321	330	339	348	357	367	376	385	
72	394	403	413	422	431	440	449	459	468	477	
73	486	495	504	514	523	532	541	550	560	569	
74	578	587	596	605	614	624	633	642	651	660	
75	669	679	688	697	706	715	724	733	742	752	
76	761	770	779	788	797	806	815	825	834	843	
77	852	861	870	879	888	897	906	916	925	934	
78	943	952	961	970	979	988	997	*006	*015	*024	
79	68 034	043	052	061	070	079	088	097	106	115	
480	124	133	142	151	160	169	178	187	196	205	
81	215	224	233	242	251	260	269	278	287	296	
82	305	314	323	332	341	350	359	368	377	386	
83	395	404	413	422	431	440	449	458	467	476	
84	485	494	502	511	520	529	538	547	556	565	
85	574	583	592	601	610	619	628	637	646	655	
86	664	673	681	690	699	708	717	726	735	744	
87	753	762	771	780	789	797	806	815	824	833	
88	842	851	860	869	878	886	895	904	913	922	
89	931	940	949	958	966	975	984	993	*002	*011	
490	69 020	028	037	046	055	064	073	082	090	099	
91	108	117	126	135	144	152	161	170	179	188	
92	197	205	214	223	232	241	249	258	267	276	
93	285	294	302	311	320	329	338	346	355	364	
94	373	381	390	399	408	417	425	434	443	452	
95	461	469	478	487	496	504	513	522	531	539	
96	548	557	566	574	583	592	601	609	618	627	
97	636	644	653	662	671	679	688	697	705	714	
98	723	732	740	749	758	767	775	784	793	801	
99	810	819	827	836	845	854	862	871	880	888	
500	897	906	914	923	932	940	949	958	966	975	
N	0	1	2	3	4	5	6	7	8	9	P P

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

TABLE I.—COMMON LOGARITHMS OF NUMBERS
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
500	69 897	906	914	923	932	940	949	958	966	975	
01	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
02	70 070	079	088	096	105	114	122	131	140	148	
03	157	165	174	183	191	200	209	217	226	234	
04	243	252	260	269	278	286	295	303	312	321	
05	329	338	346	355	364	372	381	389	398	406	
06	415	424	432	441	449	458	467	475	484	492	
07	501	509	518	526	535	544	552	561	569	578	
08	586	595	603	612	621	629	638	646	655	663	9
09	672	680	689	697	706	714	723	731	740	749	1 0.9
510	757	766	774	783	791	800	808	817	825	834	2 1.8
11	842	851	859	868	876	885	893	902	910	919	3 2.7
12	927	935	944	952	961	969	978	986	995	*003	4 3.6
13	71 012	020	029	037	046	054	063	071	079	088	5 4.5
14	096	105	113	122	130	139	147	155	164	172	6 5.4
15	181	189	198	206	214	223	231	240	248	257	7 6.3
16	265	273	282	290	299	307	315	324	332	341	8 7.2
17	349	357	366	374	383	391	399	408	416	425	9 8.1
18	433	441	450	458	466	475	483	492	500	508	
19	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	
21	684	692	700	709	717	725	734	742	750	759	
22	767	775	784	792	800	809	817	825	834	842	
23	850	858	867	875	883	892	900	908	917	925	8
24	933	941	950	958	966	975	983	991	999	*008	1 0.8
25	72 016	024	032	041	049	057	066	074	082	090	2 1.6
26	099	107	115	123	132	140	148	156	165	173	3 2.4
27	181	189	198	206	214	222	230	239	247	255	4 3.2
28	263	272	280	288	296	304	313	321	329	337	5 4.0
29	346	354	362	370	378	387	395	403	411	419	6 4.8
530	428	436	444	452	460	469	477	485	493	501	7 5.6
31	509	518	526	534	542	550	558	567	575	583	8 6.4
32	591	599	607	616	624	632	640	648	656	665	9 7.2
33	673	681	689	697	705	713	722	730	738	746	
34	754	762	770	779	787	795	803	811	819	827	
35	835	843	852	860	868	876	884	892	900	908	
36	916	925	933	941	949	957	965	973	981	989	
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	7
38	73 078	086	094	102	111	119	127	135	143	151	1 0.7
39	159	167	175	183	191	199	207	215	223	231	2 1.4
540	239	247	255	263	272	280	288	296	304	312	3 2.1
41	320	328	336	344	352	360	368	376	384	392	4 2.8
42	400	408	416	424	432	440	448	456	464	472	5 3.5
43	480	488	496	504	512	520	528	536	544	552	6 4.2
44	560	568	576	584	592	600	608	616	624	632	7 4.9
45	640	648	656	664	672	679	687	695	703	711	8 5.6
46	719	727	735	743	751	759	767	775	783	791	9 6.3
47	799	807	815	823	830	838	846	854	862	870	
48	878	886	894	902	910	918	926	933	941	949	
49	957	965	973	981	989	997	*005	*013	*020	*028	
550	74 036	044	052	060	068	076	084	092	099	107	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
550	74 036	044	052	060	068	076	084	092	099	107	
51	115	123	131	139	147	155	162	170	178	186	
52	194	202	210	218	225	233	241	249	257	265	
53	273	280	288	296	304	312	320	327	335	343	
54	351	359	367	374	382	390	398	406	414	421	
55	429	437	445	453	461	468	476	484	492	500	
56	507	515	523	531	539	547	554	562	570	578	
57	586	593	601	609	617	624	632	640	648	656	
58	663	671	679	687	695	702	710	718	726	733	
59	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	850	858	865	873	881	889	
61	896	904	912	920	927	935	943	950	958	966	
62	974	981	989	997	*005	*012	*020	*028	*035	*043	
63	75 051	059	066	074	082	089	097	105	113	120	
64	128	136	143	151	159	166	174	182	189	197	
65	205	213	220	228	236	243	251	259	266	274	
66	282	289	297	305	312	320	328	335	343	351	
67	358	366	374	381	389	397	404	412	420	427	
68	435	442	450	458	465	473	481	488	496	504	
69	511	519	526	534	542	549	557	565	572	580	
570	587	595	603	610	618	626	633	641	648	656	
71	664	671	679	686	694	702	709	717	724	732	
72	740	747	755	762	770	778	785	793	800	808	
73	815	823	831	838	846	853	861	868	876	884	
74	891	899	906	914	921	929	937	944	952	959	
75	967	974	982	989	997	*005	*012	*020	*027	*035	
76	76 042	050	057	065	072	080	087	095	103	110	
77	118	125	133	140	148	155	163	170	178	185	
78	193	200	208	215	223	230	238	245	253	260	
79	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	
81	418	425	433	440	448	455	462	470	477	485	
82	492	500	507	515	522	530	537	545	552	559	
83	567	574	582	589	597	604	612	619	626	634	
84	641	649	656	664	671	678	686	693	701	708	
85	716	723	730	738	745	753	760	768	775	782	
86	790	797	805	812	819	827	834	842	849	856	
87	864	871	879	886	893	901	908	916	923	930	
88	938	945	953	960	967	975	982	989	997	*004	
89	77 012	019	026	034	041	048	056	063	070	078	
590	085	093	100	107	115	122	129	137	144	151	
91	159	166	173	181	188	195	203	210	217	225	
92	232	240	247	254	262	269	276	283	291	298	
93	305	313	320	327	335	342	349	357	364	371	
94	379	386	393	401	408	415	422	430	437	444	
95	452	459	466	474	481	488	495	503	510	517	
96	525	532	539	546	554	561	568	576	583	590	
97	597	605	612	619	627	634	641	648	656	663	
98	670	677	685	692	699	706	714	721	728	735	
99	743	750	757	764	772	779	786	793	801	808	
600	815	822	830	837	844	851	859	866	873	880	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

8	
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

7	
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
600	77 815	822	830	837	844	851	859	866	873	880	
01	887	895	902	909	916	924	931	938	945	952	
02	960	967	974	981	988	996	*003	*010	*017	*025	
03	78 032	039	046	053	061	068	075	082	089	097	
04	104	111	118	125	132	140	147	154	161	168	
05	176	183	190	197	204	211	219	226	233	240	
06	247	254	262	269	276	283	290	297	305	312	
07	319	326	333	340	347	355	362	369	376	383	
08	390	398	405	412	419	426	433	440	447	455	8
09	462	469	476	483	490	497	504	512	519	526	1 0 8
610	533	540	547	554	561	569	576	583	590	597	2 1 6
11	604	611	618	625	633	640	647	654	661	668	3 2 4
12	675	682	689	696	704	711	718	725	732	739	4 3 2
13	746	753	760	767	774	781	789	796	803	810	5 4 0
14	817	824	831	838	845	852	859	866	873	880	6 4 8
15	888	895	902	909	916	923	930	937	944	951	7 5 6
16	958	965	972	979	986	993	*000	*007	*014	*021	8 6 4
17	79 029	036	043	050	057	064	071	078	085	092	9 7 2
18	099	106	113	120	127	134	141	148	155	162	
19	169	176	183	190	197	204	211	218	225	232	
620	239	246	253	260	267	274	281	288	295	302	
21	309	316	323	330	337	344	351	358	365	372	
22	379	386	393	400	407	414	421	428	435	442	
23	449	456	463	470	477	484	491	498	505	511	7
24	518	525	532	539	546	553	560	567	574	581	1 0 7
25	588	595	602	609	616	623	630	637	644	650	2 1 4
26	657	664	671	678	685	692	699	706	713	720	3 2 1
27	727	734	741	748	754	761	768	775	782	789	4 2 8
28	796	803	810	817	824	831	837	844	851	858	5 3 5
29	865	872	879	886	893	900	906	913	920	927	6 4 2
630	934	941	948	955	962	969	975	982	989	996	7 4 9
31	80 003	010	017	024	030	037	044	051	058	065	8 5 6
32	072	079	085	092	099	106	113	120	127	134	9 6 3
33	140	147	154	161	168	175	182	188	195	202	
34	209	216	223	229	236	243	250	257	264	271	
35	277	284	291	298	305	312	318	325	332	339	
36	346	353	359	366	373	380	387	393	400	407	
37	414	421	428	434	441	448	455	462	468	475	6
38	482	489	496	502	509	516	523	530	536	543	1 0 6
39	550	557	564	570	577	584	591	598	604	611	2 1 2
640	618	625	632	638	645	652	659	665	672	679	3 1 8
41	686	693	699	706	713	720	726	733	740	747	4 2 4
42	754	760	767	774	781	787	794	801	808	814	5 3 0
43	821	828	835	841	848	855	862	868	875	882	6 3 6
44	889	895	902	909	916	922	929	936	943	949	7 4 2
45	956	963	969	976	983	990	996	*003	*010	*017	8 4 8
46	81 023	030	037	043	050	057	064	070	077	084	9 5 4
47	090	097	104	111	117	124	131	137	144	151	
48	158	164	171	178	184	191	198	204	211	218	
49	224	231	238	245	251	258	265	271	278	285	
650	291	298	305	311	318	325	331	338	345	351	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers

To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
650	81 291	298	305	311	318	325	331	338	345	351	
51	358	365	371	378	385	391	398	405	411	418	
52	425	431	438	445	451	458	465	471	478	485	
53	491	498	505	511	518	525	531	538	544	551	
54	558	564	571	578	584	591	598	604	611	617	
55	624	631	637	644	651	657	664	671	677	684	
56	690	697	704	710	717	723	730	737	743	750	
57	757	763	770	776	783	790	796	803	809	816	
58	823	829	836	842	849	856	862	869	875	882	
59	889	895	902	908	915	921	928	935	941	948	
660	954	961	968	974	981	987	994	*000	*007	*014	
61	82 020	027	033	040	046	053	060	066	073	079	
62	086	092	099	105	112	119	125	132	138	145	
63	151	158	164	171	178	184	191	197	204	210	
64	217	223	230	236	243	249	256	263	269	276	
65	282	289	295	302	308	315	321	328	334	341	
66	347	354	360	367	373	380	387	393	400	406	
67	413	419	426	432	439	445	452	458	465	471	
68	478	484	491	497	504	510	517	523	530	536	
69	543	549	556	562	569	575	582	588	595	601	
670	607	614	620	627	633	640	646	653	659	666	
71	672	679	685	692	698	705	711	718	724	730	
72	737	743	750	756	763	769	776	782	789	795	
73	802	808	814	821	827	834	840	847	853	860	
74	866	872	879	885	892	898	905	911	918	924	
75	930	937	943	950	956	963	969	975	982	988	
76	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	
77	83 059	065	072	078	085	091	097	104	110	117	
78	123	129	136	142	149	155	161	168	174	181	
79	187	193	200	206	213	219	225	232	238	245	
680	251	257	264	270	276	283	289	296	302	308	
81	315	321	327	334	340	347	353	359	366	372	
82	378	385	391	398	404	410	417	423	429	436	
83	442	448	455	461	467	474	480	487	493	499	
84	506	512	518	525	531	537	544	550	556	563	
85	569	575	582	588	594	601	607	613	620	626	
86	632	639	645	651	658	664	670	677	683	689	
87	696	702	708	715	721	727	734	740	746	753	
88	759	765	771	778	784	790	797	803	809	816	
89	822	828	835	841	847	853	860	866	872	879	
690	885	891	897	904	910	916	923	929	935	942	
91	948	954	960	967	973	979	985	992	998	*004	
92	84 011	017	023	029	036	042	048	055	061	067	
93	073	080	086	092	098	105	111	117	123	130	
94	136	142	148	155	161	167	173	180	186	192	
95	198	205	211	217	223	230	236	242	248	255	
96	261	267	273	280	286	292	298	305	311	317	
97	323	330	336	342	348	354	361	367	373	379	
98	386	392	398	404	410	417	423	429	435	442	
99	448	454	460	466	473	479	485	491	497	504	
700	510	516	522	528	535	541	547	553	559	566	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

TABLE I.—COMMON LOGARITHMS OF NUMBERS

To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
700	84 510	516	522	528	535	541	547	553	559	566	
01	572	578	584	590	597	603	609	615	621	628	
02	634	640	646	652	658	665	671	677	683	689	
03	696	702	708	714	720	726	733	739	745	751	
04	757	763	770	776	782	788	794	800	807	813	
05	819	825	831	837	844	850	856	862	868	874	
06	880	887	893	899	905	911	917	924	930	936	
07	942	948	954	960	967	973	979	985	991	997	
08	85 003	009	016	022	028	034	040	046	052	058	
09	065	071	077	083	089	095	101	107	114	120	7
710	126	132	138	144	150	156	163	169	175	181	1 0.7
11	187	193	199	205	211	217	224	230	236	242	2 1.4
12	248	254	260	266	272	278	285	291	297	303	3 2.1
13	309	315	321	327	333	339	345	352	358	364	4 2.8
14	370	376	382	388	394	400	406	412	418	425	5 3.5
15	431	437	443	449	455	461	467	473	479	485	6 4.2
16	491	497	503	509	516	522	528	534	540	546	7 4.9
17	552	558	564	570	576	582	588	594	600	606	8 5.6
18	612	618	625	631	637	643	649	655	661	667	9 6.3
19	673	679	685	691	697	703	709	715	721	727	
720	733	739	745	751	757	763	769	775	781	788	
21	794	800	806	812	818	824	830	836	842	848	
22	854	860	866	872	878	884	890	896	902	908	
23	914	920	926	932	938	944	950	956	962	968	6
24	974	980	986	992	998	*004	*010	*016	*022	*028	1 0.6
25	86 034	040	046	052	058	064	070	076	082	088	2 1.2
26	094	100	106	112	118	124	130	136	141	147	3 1.8
27	153	159	165	171	177	183	189	195	201	207	4 2.4
28	213	219	225	231	237	243	249	255	261	267	5 3.0
29	273	279	285	291	297	303	308	314	320	326	6 3.6
730	332	338	344	350	356	362	368	374	380	386	7 4.2
31	392	398	404	410	415	421	427	433	439	445	8 4.8
32	451	457	463	469	475	481	487	493	499	504	9 5.4
33	510	516	522	528	534	540	546	552	558	564	
34	570	576	581	587	593	599	605	611	617	623	
35	629	635	641	646	652	658	664	670	676	682	
36	688	694	700	705	711	717	723	729	735	741	
37	747	753	759	764	770	776	782	788	794	800	5
38	806	812	817	823	829	835	841	847	853	859	1 0.5
39	864	870	876	882	888	894	900	906	911	917	2 1.0
740	923	929	935	941	947	953	958	964	970	976	3 1.5
41	982	988	994	999	*005	*011	*017	*023	*029	*035	4 2.0
42	87 040	046	052	058	064	070	075	081	087	093	5 2.5
43	099	105	111	116	122	128	134	140	146	151	6 3.0
44	157	163	169	175	181	186	192	198	204	210	7 3.5
45	216	221	227	233	239	245	251	256	262	268	8 4.0
46	274	280	286	291	297	303	309	315	320	326	9 4.5
47	332	338	344	349	355	361	367	373	379	384	
48	390	396	402	408	413	419	425	431	437	442	
49	448	454	460	466	471	477	483	489	495	500	
750	506	512	518	523	529	535	541	547	552	558	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
750	87 506	512	518	523	529	535	541	547	552	558	
51	564	570	576	581	587	593	599	604	610	616	
52	622	628	633	639	645	651	656	662	668	574	
53	679	685	691	697	703	708	714	720	726	731	
54	737	743	749	754	760	766	772	777	783	789	
55	795	800	806	812	818	823	829	835	841	846	
56	852	858	864	869	875	881	887	892	898	904	
57	910	915	921	927	933	938	944	950	955	961	
58	967	973	978	984	990	996	*001	*007	*013	*018	
59	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	
61	138	144	150	156	161	167	173	178	184	190	
62	195	201	207	213	218	224	230	235	241	247	
63	252	258	264	270	275	281	287	292	298	304	
64	309	315	321	326	332	338	343	349	355	360	
65	366	372	377	383	389	395	400	406	412	417	
66	423	429	434	440	446	451	457	463	468	474	
67	480	485	491	497	502	508	513	519	525	530	
68	536	542	547	553	559	564	570	576	581	587	
69	593	598	604	610	615	621	627	632	638	643	
770	649	655	660	666	672	677	683	689	694	700	
71	705	711	717	722	728	734	739	745	750	756	
72	762	767	773	779	784	790	795	801	807	812	
73	818	824	829	835	840	846	852	857	863	868	
74	874	880	885	891	897	902	908	913	919	925	
75	930	936	941	947	953	958	964	969	975	981	
76	986	992	997	*003	*009	*014	*020	*025	*031	*037	
77	89 042	048	053	059	064	070	076	081	087	092	
78	098	104	109	115	120	126	131	137	143	148	
79	154	159	165	170	176	182	187	193	198	204	
780	209	215	221	226	232	237	243	248	254	260	
81	265	271	276	282	287	293	298	304	310	315	
82	321	326	332	337	343	348	354	360	365	371	
83	376	382	387	393	398	404	409	415	421	426	
84	432	437	443	448	454	459	465	470	476	481	
85	487	492	498	504	509	515	520	526	531	537	
86	542	548	553	559	564	570	575	581	586	592	
87	597	603	609	614	620	625	631	636	642	647	
88	653	658	664	669	675	680	686	691	697	702	
89	708	713	719	724	730	735	741	746	752	757	
790	763	768	774	779	785	790	796	801	807	812	
91	818	823	829	834	840	845	851	856	862	867	
92	873	878	883	889	894	900	905	911	916	922	
93	927	933	938	944	949	955	960	966	971	977	
94	982	988	993	998	*004	*009	*015	*020	*026	*031	
95	90 037	042	048	053	059	064	069	075	080	086	
96	091	097	102	108	113	119	124	129	135	140	
97	146	151	157	162	168	173	179	184	189	195	
98	200	206	211	217	222	227	233	238	244	249	
99	255	260	266	271	276	282	287	293	298	304	
800	309	314	320	325	331	336	342	347	352	358	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
800	90 309	314	320	325	331	336	342	347	352	358	
01	363	369	374	380	385	390	396	401	407	412	
02	417	423	428	434	439	445	450	455	461	466	
03	472	477	482	488	493	499	504	509	515	520	
04	526	531	536	542	547	553	558	563	569	574	
05	580	585	590	596	601	607	612	617	623	628	
06	634	639	644	650	655	660	666	671	677	682	
07	687	693	698	703	709	714	720	725	730	736	
08	741	747	752	757	763	768	773	779	784	789	
09	795	800	806	811	816	822	827	832	838	843	
810	849	854	859	865	870	875	881	886	891	897	
11	902	907	913	918	924	929	934	940	945	950	
12	956	961	966	972	977	982	988	993	998	*004	
13	91 009	014	020	025	030	036	041	046	052	057	
14	062	068	073	078	084	089	094	100	105	110	
15	116	121	126	132	137	142	148	153	158	164	
16	169	174	180	185	190	196	201	206	212	217	
17	222	228	233	238	243	249	254	259	265	270	
18	275	281	286	291	297	302	307	312	318	323	
19	328	334	339	344	350	355	360	365	371	376	
820	381	387	392	397	403	408	413	418	424	429	
21	434	440	445	450	455	461	466	471	477	482	
22	487	492	498	503	508	514	519	524	529	535	
23	540	545	551	556	561	566	572	577	582	587	
24	593	598	603	609	614	619	624	630	635	640	
25	645	651	656	661	666	672	677	682	687	693	
26	698	703	709	714	719	724	730	735	740	745	
27	751	756	761	766	772	777	782	787	793	798	
28	803	808	814	819	824	829	834	840	845	850	
29	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	950	955	
31	960	965	971	976	981	986	991	997	*002	*007	
32	92 012	018	023	028	033	038	044	049	054	059	
33	065	070	075	080	085	091	096	101	106	111	
34	117	122	127	132	137	143	148	153	158	163	
35	169	174	179	184	189	195	200	205	210	215	
36	221	226	231	236	241	247	252	257	262	267	
37	273	278	283	288	293	298	304	309	314	319	
38	324	330	335	340	345	350	355	361	366	371	
39	373	381	387	392	397	402	407	412	418	423	
840	428	433	438	443	449	454	459	464	469	474	
41	480	485	490	495	500	505	511	516	521	526	
42	531	536	542	547	552	557	562	567	572	578	
43	583	588	593	598	603	609	614	619	624	629	
44	634	639	645	650	655	660	665	670	675	681	
45	686	691	696	701	706	711	716	722	727	732	
46	737	742	747	752	758	763	768	773	778	783	
47	788	793	799	804	809	814	819	824	829	834	
48	840	845	850	855	860	865	870	875	881	886	
49	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N	0	1	2	3	4	5	6	7	8	9	P P

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Table I.—Common Logarithms of Numbers

To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
850	92 942	947	952	957	962	967	973	978	983	988	
51	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
52	93 044	049	054	059	064	069	075	080	085	090	
53	095	100	105	110	115	120	125	131	136	141	
54	146	151	156	161	166	171	176	181	186	192	
55	197	202	207	212	217	222	227	232	237	242	
56	247	252	258	263	268	273	278	283	288	293	
57	298	303	308	313	318	323	328	334	339	344	
58	349	354	359	364	369	374	379	384	389	394	
59	399	404	409	414	420	425	430	435	440	445	
860	450	455	460	465	470	475	480	485	490	495	
61	500	505	510	515	520	526	531	536	541	546	
62	551	556	561	566	571	576	581	586	591	596	
63	601	606	611	616	621	626	631	636	641	646	
64	651	656	661	666	671	676	682	687	692	697	
65	702	707	712	717	722	727	732	737	742	747	
66	752	757	762	767	772	777	782	787	792	797	
67	802	807	812	817	822	827	832	837	842	847	
68	852	857	862	867	872	877	882	887	892	897	
69	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
71	94 002	007	012	017	022	027	032	037	042	047	
72	052	057	062	067	072	077	082	086	091	096	
73	101	106	111	116	121	126	131	136	141	146	
74	151	156	161	166	171	176	181	186	191	196	
75	201	206	211	216	221	226	231	236	240	245	
76	250	255	260	265	270	275	280	285	290	295	
77	300	305	310	315	320	325	330	335	340	345	
78	349	354	359	364	369	374	379	384	389	394	
79	399	404	409	414	419	424	429	433	438	443	
880	448	453	458	463	468	473	478	483	488	493	
81	498	503	507	512	517	522	527	532	537	542	
82	547	552	557	562	567	571	576	581	586	591	
83	596	601	606	611	616	621	626	630	635	640	
84	645	650	655	660	665	670	675	680	685	689	
85	694	699	704	709	714	719	724	729	734	738	
86	743	748	753	758	763	768	773	778	783	787	
87	792	797	802	807	812	817	822	827	832	836	
88	841	846	851	856	861	866	871	876	880	885	
89	890	895	900	905	910	915	919	924	929	934	
890	939	944	949	954	959	963	968	973	978	983	
91	988	993	998	*002	*007	*012	*017	*022	*027	*032	
92	95 036	041	046	051	056	061	066	071	075	080	
93	085	090	095	100	105	109	114	119	124	129	
94	134	139	143	148	153	158	163	168	173	177	
95	182	187	192	197	202	207	211	216	221	226	
96	231	236	240	245	250	255	260	265	270	274	
97	279	284	289	294	299	303	308	313	318	323	
98	328	332	337	342	347	352	357	361	366	371	
99	376	381	386	390	395	400	405	410	415	419	
900	424	429	434	439	444	448	453	458	463	468	
N	0	1	2	3	4	5	6	7	8	9	P P

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
900	95 424	429	434	439	444	448	453	458	463	468	
01	472	477	482	487	492	497	501	506	511	516	
02	521	525	530	535	540	545	550	554	559	564	
03	569	574	578	583	588	593	598	602	607	612	
04	617	622	626	631	636	641	646	650	655	660	
05	665	670	674	679	684	689	694	698	703	708	
06	713	718	722	727	732	737	742	746	751	756	
07	761	766	770	775	780	785	789	794	799	804	
08	809	813	818	823	828	832	837	842	847	852	
09	856	861	866	871	875	880	885	890	895	899	
910	904	909	914	918	923	928	933	938	942	947	
11	952	957	961	966	971	976	980	985	990	995	
12	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	
13	96 047	052	057	061	066	071	076	080	085	090	
14	095	099	104	109	114	118	123	128	133	137	
15	142	147	152	156	161	166	171	175	180	185	
16	190	194	199	204	209	213	218	223	227	232	
17	237	242	246	251	256	261	265	270	275	280	
18	284	289	294	298	303	308	313	317	322	327	
19	332	336	341	346	350	355	360	365	369	374	
920	379	384	388	393	398	402	407	412	417	421	
21	426	431	435	440	445	450	454	459	464	468	
22	473	478	483	487	492	497	501	506	511	515	
23	520	525	530	534	539	544	548	553	558	562	
24	567	572	577	581	586	591	595	600	605	609	
25	614	619	624	628	633	638	642	647	652	656	
26	661	666	670	675	680	685	689	694	699	703	
27	708	713	717	722	727	731	736	741	745	750	
28	755	759	764	769	774	778	783	788	792	797	
29	802	806	811	816	820	825	830	834	839	844	
930	848	453	858	862	867	872	876	881	886	890	
31	895	900	904	909	914	918	923	928	932	937	
32	942	946	951	956	960	965	970	974	979	984	
33	988	993	997	*002	*007	*011	*016	*021	*025	*030	
34	97 035	039	044	049	053	058	063	067	072	077	
35	081	086	090	095	100	104	109	114	118	123	
36	128	132	137	142	146	151	155	160	165	169	
37	174	179	183	188	192	197	202	206	211	216	
38	220	225	230	234	239	243	248	253	257	262	
39	267	271	276	280	285	290	294	299	304	308	
940	313	317	322	327	331	336	340	345	350	354	
41	359	364	368	373	377	382	387	391	396	400	
42	405	410	414	419	424	428	433	437	442	447	
43	451	456	460	465	470	474	479	483	488	493	
44	497	502	506	511	516	520	525	529	534	539	
45	543	548	552	557	562	566	571	575	580	585	
46	589	594	598	603	607	612	617	621	626	630	
47	635	640	644	649	653	658	663	667	672	676	
48	681	685	690	695	699	704	708	713	717	722	
49	727	731	736	740	745	749	754	759	763	768	
950	772	777	782	786	791	795	800	804	809	813	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Table I.—Common Logarithms of Numbers
To Five Decimal Places

N	0	1	2	3	4	5	6	7	8	9	P P
950	97 772	777	782	786	791	795	800	804	809	813	
51	818	823	827	832	836	841	845	850	855	859	
52	864	868	873	877	882	886	891	896	900	905	
53	909	914	918	923	928	932	937	941	946	950	
54	955	959	964	968	973	978	982	987	991	996	
55	98 000	005	009	014	019	023	028	032	037	041	
56	046	050	055	059	064	068	073	078	082	087	
57	091	096	100	105	109	114	118	123	127	132	
58	137	141	146	150	155	159	164	168	173	177	
59	182	186	191	195	200	204	209	214	218	223	
960	227	232	236	241	245	250	254	259	263	268	
61	272	277	281	286	290	295	299	304	308	313	
62	318	322	327	331	336	340	345	349	354	358	
63	363	367	372	376	381	385	390	394	399	403	5
64	408	412	417	421	426	430	435	439	444	448	1 0.5
65	453	457	462	466	471	475	480	484	489	493	2 1.0
66	498	502	507	511	516	520	525	529	534	538	3 1.5
67	543	547	552	556	561	565	570	574	579	583	4 2.0
68	588	592	597	601	605	610	614	619	623	628	5 2.5
69	632	637	641	646	650	655	659	664	668	673	6 3.0
970	677	682	686	691	695	700	704	709	713	717	7 3.5
71	722	726	731	735	740	744	749	753	758	762	8 4.0
72	767	771	776	780	784	789	793	798	802	807	9 4.5
73	811	816	820	825	829	834	838	843	847	851	
74	856	860	865	869	874	878	883	887	892	896	
75	900	905	909	914	918	923	927	932	936	941	
76	945	949	954	958	963	967	972	976	981	985	
77	989	994	998	*003	*007	*012	*016	*021	*025	*029	
78	99 034	038	043	047	052	056	061	065	069	074	
79	078	083	087	092	096	100	105	109	114	118	
980	123	127	131	136	140	145	149	154	158	162	
81	167	171	176	180	185	189	193	198	202	207	
82	211	216	220	224	229	233	238	242	247	251	
83	255	260	264	269	273	277	282	286	291	295	4
84	300	304	308	313	317	322	326	330	335	339	1 0.4
85	344	348	352	357	361	366	370	374	379	383	2 0.8
86	388	392	396	401	405	410	414	419	423	427	3 1.2
87	432	436	441	445	449	454	458	463	467	471	4 1.6
88	476	480	484	489	493	498	502	506	511	515	5 2.0
89	520	524	528	533	537	542	546	550	555	559	6 2.4
990	564	568	572	577	581	585	590	594	599	603	7 2.8
91	607	612	616	621	625	629	634	638	642	647	8 3.2
92	651	656	660	664	669	673	677	682	686	691	9 3.6
93	695	699	704	708	712	717	721	726	730	734	
94	739	743	747	752	756	760	765	769	774	778	
95	782	787	791	795	800	804	808	813	817	822	
96	826	830	835	839	843	848	852	856	861	865	
97	870	874	878	883	887	891	896	900	904	909	
98	913	917	922	926	930	935	939	944	948	952	
99	957	961	965	970	974	978	983	987	991	996	
1000	00 000	004	009	013	017	022	026	030	035	030	
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**P P**

TABLE II.—COMMON LOGARITHMS OF NUMBERS

From 1.00000 to 1.100000

To Seven Decimal Places

N	0	1	2	3	4	5	6	7	8	9
1000	000 0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001	4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002	8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001 3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004	7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002 1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006	5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003 0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008	4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009	8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004 3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011	7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005 1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013	6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006 0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015	4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016	8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007 3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018	7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008 1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020	6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009 0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022	4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023	8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
1024	010 3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025	7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011 1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027	5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028	9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012 4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030	8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013 2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032	6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014 1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034	5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035	9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015 3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037	7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016 1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039	6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017 0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041	4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042	8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018 2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044	7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019 1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046	5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047	9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020 3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049	7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N	0	1	2	3	4	5	6	7	8	9

TABLE II.—COMMON LOGARITHMS OF NUMBERS

From 1.00000 to 1.100000

To Seven Decimal Places

N	0	1	2	3	4	5	6	7	8	9
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051	6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022 0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053	4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054	8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023 2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056	6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024 0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058	4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059	8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025 3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061	7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026 1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063	5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064	9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027 3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066	7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028 1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068	5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069	9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029 3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071	7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030 1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073	5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031 0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075	4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076	8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032 2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078	6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033 0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080	4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081	8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034 2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083	6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035 0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085	4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086	8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036 2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088	6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037 0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090	4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091	8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038 2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093	6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039 0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095	4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096	8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040 2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098	6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099	9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041 3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N	0	1	2	3	4	5	6	7	8	9

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0041 6667	1.0050 0000	1.0058 3333	1.0075 0000	1.0100 0000
2	1.0083 5069	1.0100 2500	1.0117 0069	1.0150 5625	1.0201 0000
3	1.0125 5216	1.0150 7513	1.0176 0228	1.0226 6917	1.0303 0100
4	1.0167 7112	1.0201 5050	1.0235 3830	1.0303 3919	1.0406 0401
5	1.0210 0767	1.0252 5125	1.0295 0894	1.0380 6673	1.0510 1005
6	1.0252 6187	1.0303 7751	1.0355 1440	1.0458 5224	1.0615 2015
7	1.0295 3379	1.0355 2940	1.0415 5490	1.0536 9613	1.0721 3535
8	1.0338 2352	1.0407 0704	1.0476 3064	1.0615 9885	1.0828 5671
9	1.0381 3111	1.0459 1058	1.0537 4182	1.0695 6084	1.0936 8527
10	1.0424 5666	1.0511 4013	1.0598 8865	1.0775 8255	1.1046 2213
11	1.0468 0023	1.0563 9583	1.0660 7133	1.0856 6441	1.1156 6835
12	1.0511 6190	1.0616 7781	1.0722 9008	1.0938 0690	1.1268 2503
13	1.0555 4174	1.0669 8620	1.0785 4511	1.1020 1045	1.1380 9328
14	1.0599 3983	1.0723 2113	1.0848 3662	1.1102 7553	1.1494 7421
15	1.0643 5625	1.0776 8274	1.0911 6483	1.1186 0259	1.1609 6896
16	1.0687 9106	1.0830 7115	1.0975 2996	1.1269 9211	1.1725 7864
17	1.0732 4436	1.0884 8651	1.1039 3222	1.1354 4455	1.1843 0443
18	1.0777 1621	1.0939 2894	1.1103 7182	1.1439 6039	1.1961 4748
19	1.0822 0670	1.0993 9858	1.1168 4899	1.1525 4009	1.2081 0895
20	1.0867 1589	1.1048 9558	1.1233 6395	1.1611 8414	1.2201 9004
21	1.0912 4387	1.1104 2006	1.1299 1690	1.1698 9302	1.2323 9194
22	1.0957 9072	1.1159 7216	1.1365 0808	1.1786 6722	1.2447 1586
23	1.1003 5652	1.1215 5202	1.1431 3771	1.1875 0723	1.2571 6302
24	1.1049 4134	1.1271 5978	1.1498 0602	1.1964 1353	1.2697 3465
25	1.1095 4526	1.1327 9558	1.1565 1322	1.2053 8663	1.2824 3200
26	1.1141 6836	1.1384 5955	1.1632 5955	1.2144 2703	1.2952 5631
27	1.1188 1073	1.1441 5185	1.1700 4523	1.2235 3523	1.3082 0888
28	1.1234 7244	1.1498 7261	1.1768 7049	1.2327 1175	1.3212 9097
29	1.1281 5358	1.1556 2197	1.1837 3557	1.2419 5709	1.3345 0388
30	1.1328 5422	1.1614 0008	1.1906 4069	1.2512 7176	1.3478 4892
31	1.1375 7444	1.1672 0708	1.1975 8610	1.2606 5630	1.3613 2740
32	1.1423 1434	1.1730 4312	1.2045 7202	1.2701 1122	1.3749 4068
33	1.1470 7398	1.1789 0833	1.2115 9869	1.2796 3706	1.3886 9009
34	1.1518 5346	1.1848 0288	1.2186 6634	1.2892 3434	1.4025 7699
35	1.1566 5284	1.1907 2689	1.2257 7523	1.2989 0359	1.4166 0276
36	1.1614 7223	1.1966 8052	1.2329 2559	1.3086 4537	1.4307 6878
37	1.1663 1170	1.2026 6393	1.2401 1765	1.3184 6021	1.4450 7647
38	1.1711 7133	1.2086 7725	1.2473 5167	1.3283 4866	1.4595 2724
39	1.1760 5121	1.2147 2063	1.2546 2789	1.3383 1128	1.4741 2251
40	1.1809 5142	1.2207 9424	1.2619 4655	1.3483 4861	1.4888 6373
41	1.1858 7206	1.2268 9821	1.2693 0791	1.3584 6123	1.5037 5237
42	1.1908 1319	1.2330 3270	1.2767 1220	1.3686 4969	1.5187 8989
43	1.1957 7491	1.2391 9786	1.2841 5969	1.3789 1456	1.5339 7779
44	1.2007 5731	1.2453 9385	1.2916 5062	1.3892 5642	1.5493 1757
45	1.2057 6046	1.2516 2082	1.2991 8525	1.3996 7584	1.5648 1075
46	1.2107 8446	1.2578 7892	1.3067 6383	1.4101 7341	1.5804 5885
47	1.2158 2940	1.2641 6832	1.3143 8662	1.4207 4971	1.5962 6344
48	1.2208 9536	1.2704 8916	1.3220 5388	1.4314 0533	1.6122 2608
49	1.2259 8242	1.2768 4161	1.3297 6586	1.4421 4087	1.6283 4834
50	1.2310 9068	1.2832 2581	1.3375 2283	1.4529 5693	1.6446 3182

Table III.—Compound Amount of 1

$(1 + i)^n$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	1.2362 2002	1.2896 4194	1.3453 2504	1.4638 5411	1.6610 7814
52	1.2413 7114	1.2960 9015	1.3531 7277	1.4748 3301	1.6776 8892
53	1.2465 4352	1.3025 7060	1.3610 6628	1.4858 9426	1.6944 6581
54	1.2517 3745	1.3090 8346	1.3690 0583	1.4970 3847	1.7114 1047
55	1.2569 5302	1.3156 2887	1.3769 9170	1.5082 6626	1.7285 2457
56	1.2621 9033	1.3222 0702	1.3850 2415	1.5195 7825	1.7458 0982
57	1.2674 4946	1.3288 1805	1.3931 0346	1.5309 7509	1.7632 6792
58	1.2727 3050	1.3354 6214	1.4012 2990	1.5424 5740	1.7809 0060
59	1.2780 3354	1.3421 3946	1.4094 0374	1.5540 2583	1.7987 0960
60	1.2833 5868	1.3488 5015	1.4176 2526	1.5656 8103	1.8166 9670
61	1.2887 0601	1.3555 9440	1.4258 9474	1.5774 2363	1.8348 6367
62	1.2940 7561	1.3623 7238	1.4342 1246	1.5892 5431	1.8532 1230
63	1.2994 6760	1.3691 8424	1.4425 7870	1.6011 7372	1.8717 4443
64	1.3048 8204	1.3760 3016	1.4509 9374	1.6131 8252	1.8904 6187
65	1.3103 1905	1.3829 1031	1.4594 5787	1.6252 8139	1.9093 6649
66	1.3157 7872	1.3898 2486	1.4679 7138	1.6374 7100	1.9284 6015
67	1.3212 6113	1.3967 7399	1.4765 3454	1.6497 5203	1.9477 4475
68	1.3267 6638	1.4037 5785	1.4851 4766	1.6621 2517	1.9672 2220
69	1.3322 9458	1.4107 7664	1.4938 1102	1.6745 9111	1.9868 9442
70	1.3378 4580	1.4178 3053	1.5025 2492	1.6871 5055	2.0067 6337
71	1.3434 2016	1.4249 1968	1.5112 8965	1.6998 0418	2.0268 3100
72	1.3490 1774	1.4320 4428	1.5201 0550	1.7125 5271	2.0470 9931
73	1.3546 3865	1.4392 0450	1.5289 7279	1.7253 9685	2.0675 7031
74	1.3602 8298	1.4464 0052	1.5378 9179	1.7383 3733	2.0882 4601
75	1.3659 5082	1.4536 3252	1.5468 6283	1.7513 7486	2.1091 2847
76	1.3716 4229	1.4609 0069	1.5558 8620	1.7645 1017	2.1302 1975
77	1.3773 5746	1.4682 0519	1.5649 6220	1.7777 4400	2.1515 2195
78	1.3830 9645	1.4755 4622	1.5740 9115	1.7910 7708	2.1730 3717
79	1.3888 5935	1.4829 2395	1.5832 7334	1.8045 1015	2.1947 6754
80	1.3946 4627	1.4903 3857	1.5925 0910	1.8180 4398	2.2167 1522
81	1.4004 5729	1.4977 9026	1.6017 9874	1.8316 7931	2.2388 8237
82	1.4062 9253	1.5052 7921	1.6111 4257	1.8454 1691	2.2612 7119
83	1.4121 5209	1.5128 0561	1.6205 4090	1.8592 5753	2.2838 8390
84	1.4180 3605	1.5203 6964	1.6299 9405	1.8732 0196	2.3067 2274
85	1.4239 4454	1.5279 7148	1.6395 0235	1.8872 5098	2.3297 8997
86	1.4298 7764	1.5356 1134	1.6490 6612	1.9014 0536	2.3530 8787
87	1.4358 3546	1.5432 8940	1.6586 8567	1.9156 6590	2.3766 1875
88	1.4418 1811	1.5510 0585	1.6683 6134	1.9300 3339	2.4003 8494
89	1.4478 2568	1.5587 6087	1.6780 9344	1.9445 0865	2.4243 8879
90	1.4538 5829	1.5665 5468	1.6878 8232	1.9590 9246	2.4486 3267
91	1.4599 1603	1.5743 8745	1.6977 2830	1.9737 8565	2.4731 1900
92	1.4659 9902	1.5822 5939	1.7076 3172	1.9885 8905	2.4978 5019
93	1.4721 0735	1.5901 7069	1.7175 9290	2.0035 0346	2.5228 2869
94	1.4782 4113	1.5981 2154	1.7276 1219	2.0185 2974	2.5480 5698
95	1.4844 0047	1.6061 1215	1.7376 8993	2.0336 6871	2.5735 3755
96	1.4905 8547	1.6141 4271	1.7478 2646	2.0489 2123	2.5992 7293
97	1.4967 9624	1.6222 1342	1.7580 2211	2.0642 8814	2.6252 6565
98	1.5030 3289	1.6303 2449	1.7682 7724	2.0797 7030	2.6515 1831
99	1.5092 9553	1.6384 7611	1.7785 9219	2.0953 6858	2.6780 3349
100	1.5155 8426	1.6466 6849	1.7889 6731	2.1110 8384	2.7048 1383

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	1.5218 9919	1.6549 0183	1.7994 0295	2.1269 1697	2.7318 6197
102	1.5282 4044	1.6631 7634	1.8098 9947	2.1428 6885	2.7591 8059
103	1.5346 0811	1.6714 9223	1.8204 5722	2.1589 4036	2.7867 7239
104	1.5410 0231	1.6798 4969	1.8310 7655	2.1751 3242	2.8146 4012
105	1.5474 2315	1.6882 4894	1.8417 5783	2.1914 4591	2.8427 8652
106	1.5538 7075	1.6966 9018	1.8525 0142	2.2078 8175	2.8712 1438
107	1.5603 4521	1.7051 7363	1.8633 0768	2.2244 4087	2.8999 2653
108	1.5668 4665	1.7136 9950	1.8741 7697	2.2411 2417	2.9289 2579
109	1.5733 7518	1.7222 6800	1.8851 0967	2.2579 3260	2.9582 1505
110	1.5799 3091	1.7308 7934	1.8961 0614	2.2748 6710	2.9877 9720
111	1.5865 1395	1.7395 3373	1.9071 6676	2.2919 2860	3.0176 7517
112	1.5931 2443	1.7482 3140	1.9182 9190	2.3091 1807	3.0478 5192
113	1.5997 6245	1.7569 7256	1.9294 8194	2.3264 3645	3.0783 3044
114	1.6064 2812	1.7657 5742	1.9407 3725	2.3438 8472	3.1091 1375
115	1.6131 2157	1.7745 8621	1.9520 5832	2.3614 6386	3.1402 0489
116	1.6198 4291	1.7834 5914	1.9634 4522	2.3791 7484	3.1716 0693
117	1.6265 9226	1.7923 7644	1.9748 9865	2.3970 1865	3.2033 2300
118	1.6333 6973	1.8013 3832	1.9864 1890	2.4149 9629	3.2353 5623
119	1.6401 7543	1.8103 4501	1.9980 0634	2.4331 0876	3.2677 0980
120	1.6470 0950	1.8193 9673	2.0096 6138	2.4513 5708	3.3003 8689
121	1.6538 7204	1.8284 9372	2.0213 8440	2.4697 4226	3.3333 9076
122	1.6607 6317	1.8376 3619	2.0331 7581	2.4882 6532	3.3667 2467
123	1.6676 8302	1.8468 2437	2.0450 3600	2.5069 2731	3.4003 9192
124	1.6746 3170	1.8560 5849	2.0569 6538	2.5257 2927	3.4343 9584
125	1.6816 0933	1.8653 3878	2.0689 6434	2.5446 7224	3.4687 3980
126	1.6886 1603	1.8746 6548	2.0810 3330	2.5637 5728	3.5034 2719
127	1.6956 5193	1.8840 3880	2.0931 7266	2.5829 8546	3.5384 6147
128	1.7027 1715	1.8934 5900	2.1053 8284	2.6023 5785	3.5738 4608
129	1.7098 1181	1.9029 2629	2.1176 6424	2.6218 7553	3.6095 8454
130	1.7169 3602	1.9124 4092	2.1300 1728	2.6415 3960	3.6456 8039
131	1.7240 8992	1.9220 0313	2.1424 4238	2.6613 5115	3.6821 3719
132	1.7312 7363	1.9316 1314	2.1549 3996	2.6813 1128	3.7189 5856
133	1.7384 8727	1.9412 7121	2.1675 1044	2.7014 2112	3.7561 4815
134	1.7457 3097	1.9509 7757	2.1801 5425	2.7216 8177	3.7937 0963
135	1.7530 0485	1.9607 3245	2.1928 7182	2.7420 9439	3.8316 4673
136	1.7603 0903	1.9705 3612	2.2056 6357	2.7626 6009	3.8699 6319
137	1.7676 4365	1.9803 8880	2.2185 2994	2.7833 8005	3.9086 6282
138	1.7750 0884	1.9902 9074	2.2314 7137	2.8042 5540	3.9477 4945
139	1.7824 0471	2.0002 4219	2.2444 8828	2.8252 8731	3.9872 2695
140	1.7898 3139	2.0102 4340	2.2575 8113	2.8464 7697	4.0270 9922
141	1.7972 8902	2.0202 9462	2.2707 5036	2.8678 2554	4.0673 7021
142	1.8047 7773	2.0303 9609	2.2839 9640	2.8893 3424	4.1080 4391
143	1.8122 9763	2.0405 4808	2.2973 1971	2.9110 0424	4.1491 2435
144	1.8198 4887	2.0507 5082	2.3107 2074	2.9328 3677	4.1906 1559
145	1.8274 3158	2.0610 0457	2.3241 9995	2.9548 3305	4.2325 2175
146	1.8350 4588	2.0713 0959	2.3377 5778	2.9769 9430	4.2748 4697
147	1.8426 9190	2.0816 6614	2.3513 9470	2.9993 2175	4.3175 9544
148	1.8503 6978	2.0920 7447	2.3651 1117	3.0218 1667	4.3607 7139
149	1.8580 7966	2.1025 3484	2.3789 0765	3.0444 8029	4.4043 7910
150	1.8658 2166	2.1130 4752	2.3927 8461	3.0673 1389	4.4484 2290

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.0112 5000	1.0125 0000	1.0150 0000	1.0175 0000	1.0200 0000
2	1.0226 2656	1.0251 5625	1.0302 2500	1.0353 0625	1.0404 0000
3	1.0341 3111	1.0379 7070	1.0456 7838	1.0534 2411	1.0612 0800
4	1.0457 6509	1.0509 4534	1.0613 6355	1.0718 5903	1.0824 3216
5	1.0575 2994	1.0640 8215	1.0772 8400	1.0906 1656	1.1040 8080
6	1.0694 2716	1.0773 8318	1.0934 4326	1.1097 0235	1.1261 6242
7	1.0814 5821	1.0908 5047	1.1098 4491	1.1291 2215	1.1486 8567
8	1.0936 2462	1.1044 8610	1.1264 9259	1.1488 8178	1.1716 5938
9	1.1059 2789	1.1182 9218	1.1433 8998	1.1689 8721	1.1950 9257
10	1.1183 6958	1.1322 7083	1.1605 4083	1.1894 4449	1.2189 9442
11	1.1309 5124	1.1464 2422	1.1779 4894	1.2102 5977	1.2433 7431
12	1.1436 7444	1.1607 5452	1.1956 1817	1.2314 3931	1.2682 4179
13	1.1565 4078	1.1752 6395	1.2135 5244	1.2529 8950	1.2936 0663
14	1.1695 5186	1.1899 5475	1.2317 5573	1.2749 1682	1.3194 7876
15	1.1827 0932	1.2048 2918	1.2502 3207	1.2972 2786	1.3458 6834
16	1.1960 1480	1.2198 8955	1.2689 8555	1.3199 2935	1.3727 8571
17	1.2094 6997	1.2351 3817	1.2880 2033	1.3430 2811	1.4002 4142
18	1.2230 7650	1.2505 7739	1.3073 4064	1.3665 3111	1.4282 4625
19	1.2368 3611	1.2662 0961	1.3269 5075	1.3904 4540	1.4568 1117
20	1.2507 5052	1.2820 3723	1.3468 5501	1.4147 7820	1.4859 4740
21	1.2648 2146	1.2980 6270	1.3670 5783	1.4395 3681	1.5156 6634
22	1.2790 5071	1.3142 8848	1.3875 6370	1.4647 2871	1.5459 7967
23	1.2934 4003	1.3307 1709	1.4083 7715	1.4903 6146	1.5768 9926
24	1.3079 9123	1.3473 5105	1.4295 0281	1.5164 4279	1.6084 3725
25	1.3227 0613	1.3641 9294	1.4509 4535	1.5429 8054	1.6406 0599
26	1.3375 8657	1.3812 4535	1.4727 0953	1.5699 8269	1.6734 1811
27	1.3526 3442	1.3985 1092	1.4948 0018	1.5974 5739	1.7068 8648
28	1.3678 5156	1.4159 9230	1.5172 2218	1.6254 1290	1.7410 2421
29	1.3832 3989	1.4336 9221	1.5399 8051	1.6538 5762	1.7758 4469
30	1.3988 0134	1.4516 1336	1.5630 8022	1.6828 0013	1.8113 6158
31	1.4145 3785	1.4697 5853	1.5865 2642	1.7122 4913	1.8475 8882
32	1.4304 5140	1.4881 3051	1.6103 2432	1.7422 1349	1.8845 4059
33	1.4465 4398	1.5067 3214	1.6344 7918	1.7727 0223	1.9222 3140
34	1.4628 1760	1.5255 6629	1.6589 9637	1.8037 2452	1.9606 7603
35	1.4792 7430	1.5446 3587	1.6838 8132	1.8352 8970	1.9998 8955
36	1.4959 1613	1.5639 4382	1.7091 3954	1.8674 0727	2.0398 8734
37	1.5127 4519	1.5834 9312	1.7347 7663	1.9000 8689	2.0806 8509
38	1.5297 6357	1.6032 8678	1.7607 9828	1.9333 3841	2.1222 9879
39	1.5469 7341	1.6233 2787	1.7872 1025	1.9671 7184	2.1647 4477
40	1.5643 7687	1.6436 1946	1.8140 1841	2.0015 9734	2.2080 3966
41	1.5819 7611	1.6641 6471	1.8412 2868	2.0366 2530	2.2522 0046
42	1.5997 7334	1.6849 6677	1.8688 4712	2.0722 6624	2.2972 4447
43	1.6177 7079	1.7060 2885	1.8968 7982	2.1085 3090	2.3431 8936
44	1.6359 7071	1.7273 5421	1.9253 3302	2.1454 3019	2.3900 5314
45	1.6543 7538	1.7489 4614	1.9542 1301	2.1829 7522	2 4378 5421
46	1.6729 8710	1.7708 0797	1.9835 2621	2.2211 7728	2.4866 1129
47	1.6918 0821	1.7929 4306	2.0132 7910	2.2600 4789	2.5363 4351
48	1.7108 4105	1.8153 5485	2.0434 7829	2.2995 9872	2.5870 7039
49	1.7300 8801	1.8380 4679	2.0741 3046	2.3398 4170	2.6388 1179
50	1.7495 5150	1.8610 2237	2.1052 4242	2.3807 8893	2.6915 8803

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	1.7692 3395	1.8842 8515	2.1368 2106	2.4224 5274	2.7454 1979
52	1.7891 3784	1.9078 3872	2.1688 7337	2.4648 4566	2.8003 2819
53	1.8092 6564	1.9316 8670	2.2014 0647	2.5079 8046	2.8563 3475
54	1.8296 1988	1.9558 3279	2.2344 2757	2.5518 7012	2.9134 6144
55	1.8502 0310	1.9802 8070	2.2679 4398	2.5965 2785	2.9717 3067
56	1.8710 1788	2.0050 3420	2.3019 6314	2.6419 6708	3.0311 6529
57	1.8920 6684	2.0300 9713	2.3364 9259	2.6882 0151	3.0917 8859
58	1.9133 5259	2.0554 7335	2.3715 3998	2.7352 4503	3.1536 2436
59	1.9348 7780	2.0811 6676	2.4071 1308	2.7831 1182	3.2166 9685
60	1.9566 4518	2.1071 8135	2.4432 1978	2.8318 1628	3.2810 3079
61	1.9786 5744	2.1335 2111	2.4798 6807	2.8813 7306	3.3466 5140
62	2.0009 1733	2.1601 9013	2.5170 6609	2.9317 9709	3.4135 8443
63	2.0234 2765	2.1871 9250	2.5548 2208	2.9831 0354	3.4818 5612
64	2.0461 9121	2.2145 3241	2.5931 4442	3.0343 0785	3.5514 9324
65	2.6092 1087	2.2422 1407	2.6320 4158	3.0884 2574	3.6225 2311
66	2.0924 8949	2.2702 4174	2.6715 2221	3.1424 7319	3.6949 7357
67	2.1160 2999	2.2986 1976	2.7115 9504	3.1974 6647	3.7688 7304
68	2.1398 3533	2.3273 5251	2.7522 6896	3.2534 2213	3.8442 5050
69	2.1639 0848	2.3564 4442	2.7935 5300	3.3103 5702	3.9211 3551
70	2.1882 5245	2.3858 9997	2.8354 5629	3.3682 8827	3.9995 5822
71	2.2128 7029	2.4157 2372	2.8779 8814	3.4272 3331	4.0795 4939
72	2.2377 6508	2.4459 2027	2.9211 5796	3.4872 0990	4.1611 4038
73	2.2629 3994	2.4764 9427	2.9649 7533	3.5482 3607	4.2443 6318
74	2.2883 9801	2.5074 5045	3.0094 4996	3.6103 3020	4.3292 5045
75	2.3141 4249	2.5387 9358	3.0545 9171	3.6735 1098	4.4158 3546
76	2.3401 7659	2.5705 2850	3.1004 1059	3.7377 9742	4.5041 5216
77	2.3665 0358	2.6026 6011	3.1469 1674	3.8032 0888	4.5942 3521
78	2.3931 2675	2.6351 9336	3.1941 2050	3.8697 6503	4.6861 1991
79	2.4200 4942	2.6681 3327	3.2420 3230	3.9374 8592	4.7798 4231
80	2.4472 7498	2.7014 8494	3.2906 6279	4.0063 9192	4.8754 3916
81	2.4748 0682	2.7352 5350	3.3400 2273	4.0765 0378	4.9729 4794
82	2.5026 4840	2.7694 4417	3.3901 2307	4.1478 4260	5.0724 0690
83	2.5308 0319	2.8040 6222	3.4409 7492	4.2204 2984	5.1738 5504
84	2.5592 7473	2.8391 1300	3.4925 8954	4.2942 8737	5.2773 3214
85	2.5880 6657	2.8746 0191	3.5449 7838	4.3694 3740	5.3828 7878
86	2.6171 8232	2.9105 3444	3.5981 5306	4.4459 0255	5.4905 3636
87	2.6466 2562	2.9469 1612	3.6521 2535	4.5237 0584	5.6003 4708
88	2.6764 0016	2.9837 5257	3.7069 0723	4.6028 7070	5.7123 5402
89	2.7065 0966	3.0210 4948	3.7625 1084	4.6834 2093	5.8266 0110
90	2.7369 5789	3.0588 1260	3.8189 4851	4.7653 8080	5.9431 3313
91	2.7677 4367	3.0970 4775	3.8762 3273	4.8487 7496	6.0619 9579
92	2.7988 8584	3.1357 6085	3.9343 7622	4.9336 2853	6.1832 3570
93	2.8303 7331	3.1749 5786	3.9933 9187	5.0199 6703	6.3069 0042
94	2.8622 1501	3.2146 4483	4.0532 9275	5.1078 1645	6.4330 3843
95	2.8944 1492	3.2548 2789	4.1140 9214	5.1972 0324	6.5616 9920
96	2.9269 7709	3.2955 1324	4.1758 0352	5.2881 5429	6.6929 3318
97	2.9599 0559	3.3367 0716	4.2384 4057	5.3806 9699	6.8267 9184
98	2.9932 0452	3.3784 1600	4.3020 1718	5.4748 5919	6.9633 2768
99	3.0268 7807	3.4206 4620	4.3665 4744	5.5706 6923	7.1025 9423
100	3.0609 3045	3.4634 0427	4.4320 4565	5.6681 5594	7.2446 4612

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000	1.0350 0000
2	1.0455 0625	1.0506 2500	1.0557 5625	1.0609 0000	1.0712 2500
3	1.0690 3014	1.0768 9063	1.0847 8955	1.0927 2700	1.1087 1788
4	1.0930 8332	1.1038 1289	1.1146 2126	1.1255 0881	1.1475 2300
5	1.1176 7769	1.1314 0821	1.1452 7334	1.1592 7407	1.1876 8631
6	1.1428 2544	1.1596 9342	1.1767 6836	1.1940 5230	1.2292 5533
7	1.1685 3901	1.1886 8575	1.2091 2949	1.2298 7387	1.2722 7926
8	1.1948 3114	1.2184 0290	1.2423 8055	1.2667 7008	1.3168 0904
9	1.2217 1484	1.2488 6297	1.2765 4602	1.3047 7318	1.3628 9735
10	1.2492 0343	1.2800 8454	1.3116 5103	1.3439 1638	1.4105 9876
11	1.2773 1050	1.3120 8666	1.3477 2144	1.3842 3387	1.4599 6972
12	1.3060 4999	1.3448 8882	1.3847 8378	1.4257 6089	1.5110 6866
13	1.3354 3611	1.3785 1104	1.4228 6533	1.4685 3371	1.5639 5600
14	1.3654 8343	1.4129 7382	1.4619 9413	1.5125 8972	1.6186 9452
15	1.3962 0680	1.4482 9817	1.5021 9896	1.5579 6742	1.6753 4883
16	1.4276 2146	1.4845 0562	1.5435 0944	1.6047 0644	1.7339 8604
17	1.4597 4294	1.5216 1826	1.5859 5595	1.6528 4763	1.7946 7555
18	1.4925 8716	1.5596 5872	1.6295 6973	1.7024 3306	1.8574 8920
19	1.5261 7037	1.5986 5019	1.6743 8290	1.7535 0605	1.9225 0132
20	1.5605 0920	1.6386 1644	1.7204 2843	1.8061 1123	1.9897 8886
21	1.5956 2066	1.6795 8185	1.7677 4021	1.8602 9457	2.0594 3147
22	1.6315 2212	1.7215 7140	1.8163 5307	1.9161 0341	2.1315 1158
23	1.6682 3137	1.7646 1068	1.8663 0278	1.9735 8651	2.2061 1448
24	1.7057 6658	1.8087 2595	1.9176 2610	2.0327 9411	2.2833 2849
25	1.7441 4632	1.8539 4410	1.9703 6082	2.0937 7793	2.3632 4498
26	1.7833 8962	1.9002 9270	2.0245 4575	2.1565 9127	2.4459 5856
27	1.8235 1588	1.9478 0002	2.0802 2075	2.2212 8901	2.5315 6711
28	1.8645 4499	1.9964 9502	2.1374 2682	2.2879 2768	2.6201 7196
29	1.9064 9725	2.0464 0739	2.1962 0606	2.3565 6551	2.7118 7798
30	1.9493 9344	2.0975 6758	2.2566 0173	2.4272 6247	2.8067 9370
31	1.9932 5479	2.1500 0677	2.3186 5828	2.5000 8035	2.9050 3148
32	2.0381 0303	2.2037 5694	2.3824 2138	2.5750 8276	3.0067 0759
33	2.0839 6034	2.2588 5086	2.4479 3797	2.6523 3524	3.1119 4235
34	2.1308 4945	2.3153 2213	2.5152 5626	2.7319 0530	3.2208 6033
35	2.1787 9356	2.3732 0519	2.5844 2581	2.8138 6245	3.3335 9045
36	2.2278 1642	2.4325 3532	2.6554 9752	2.8982 7833	3.4502 6611
37	2.2779 4229	2.4933 4870	2.7285 2370	2.9852 2668	3.5710 2543
38	2.3291 9599	2.5556 8242	2.8035 5810	3.0747 8348	3.6960 1132
39	2.3816 0290	2.6195 7448	2.8806 5595	3.1670 2698	3.8253 7171
40	2.4351 8897	2.6850 6384	2.9598 7399	3.2620 3779	3.9592 5972
41	2.4899 8072	2.7521 9043	3.0412 7052	3.3598 9893	4.0978 3381
42	2.5460 0528	2.8209 9520	3.1249 0546	3.4606 9589	4.2412 5799
43	2.6032 9040	2.8915 2008	3.2108 4036	3.5645 1677	4.3897 0202
44	2.6618 6444	2.9638 0808	3.2991 3847	3.6714 5227	4.5433 4160
45	2.7217 5639	3.0379 0328	3.3898 6478	3.7815 9584	4.7023 5855
46	2.7829 9590	3.1138 5086	3.4830 8606	3.8950 4372	4.8669 4110
47	2.8456 1331	3.1916 9713	3.5788 7093	4.0118 9503	5.0372 8404
48	2.9096 3961	3.2714 8956	3.6772 8988	4.1322 5188	5.2135 8898
49	2.9751 0650	3.3532 7680	3.7784 1535	4.2562 1944	5.3960 6459
50	3.0420 4640	3.4371 0872	3.8823 2177	4.3839 0602	5.5849 2686

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	3.1104 9244	3.5230 3644	3.9890 8562	4.5154 2320	5.7803 9930
52	3.1804 7852	3.6111 1235	4.0987 8547	4.6508 8590	5.9827 1327
53	3.2520 3929	3.7013 9016	4.2115 0208	4.7904 1247	6.1921 0824
54	3.3252 1017	3.7939 2491	4.3273 1838	4.9341 2485	6.4088 3202
55	3.4000 2740	3.8887 7303	4.4463 1964	5.0821 4859	6.6331 4114
56	3.4765 2802	3.9859 9236	4.5685 9343	5.2346 1305	6.8653 0108
57	3.5547 4990	4.0856 4217	4.6942 2975	5.3916 5144	7.1055 8662
58	3.6347 3177	4.1877 8322	4.8233 2107	5.5534 0098	7.3542 8215
59	3.7165 1324	4.2924 7780	4.9559 6239	5.7200 0301	7.6116 8203
60	3.8001 3479	4.3997 8975	5.0922 5136	5.8916 0310	7.8780 9090
61	3.8856 3782	4.5097 8449	5.2322 8827	6.0683 5120	8.1538 2408
62	3.9730 6467	4.6225 2910	5.3761 7620	6.2504 0173	8.4392 0793
63	4.0624 5862	4.7380 9233	5.5240 2105	6.4379 1379	8.7345 8020
64	4 1538 6394	4.8565 4464	5.6759 3162	6.6310 5120	9.0402 9051
65	4.2473 2588	4.9779 5826	5.8320 1974	6.8299 8273	9.3567 0068
66	4.3428 9071	5.1024 0721	5.9924 0029	7.0348 8222	9.6841 8520
67	4.4406 0576	5.2299 6739	6.1571 9130	7.2459 2868	10.0231 3168
68	4.5405 1939	5.3607 1658	6.3265 1406	7.4633 0654	10.3739 4129
69	4.6426 8107	5.4947 3449	6.5004 9319	7.6872 0574	10.7370 2924
70	4.7471 4140	5.6321 0286	6.6792 5676	7.9178 2191	11.1128 2526
71	4.8539 5208	5.7729 0543	6.8629 3632	8.1553 5657	11.5017 7414
72	4.9631 6600	5.9172 2806	7.0516 6706	8.4000 1727	11.9043 3624
73	5.0748 3723	6.0651 5876	7.2455 8791	8.6520 1778	12.3209 8801
74	5.1890 2107	6.2167 8773	7.4448 4158	8.9115 7832	12.7522 2259
75	5.3057 7405	6.3722 0743	7.6495 7472	9.1789 2567	13.1985 5038
76	5.4251 5396	6.5315 1261	7.8599 3802	9.4542 9344	13.6604 9964
77	5.5472 1993	6.6948 0043	8.0760 8632	9.7379 2224	14.1386 1713
78	5.6720 3237	6.8621 7044	8.2981 7869	10.0300 5991	14.6334 6873
79	5.7996 5310	7.0337 2470	8.5263 7861	10.3309 6171	15.1456 4013
80	5.9301 4530	7.2095 6782	8.7608 5402	10.6408 9056	15.6757 3754
81	6.0635 7357	7.3898 0701	9.0017 7751	10.9601 1727	16.2243 8835
82	6.2000 0397	7.5745 5219	9.2493 2639	11.2889 2079	16.7922 4195
83	6.3395 0406	7.7639 1599	9.5036 8286	11.6275 8842	17.3799 7041
84	6.4821 4290	7.9580 1389	9.7650 3414	11.9764 1607	17.9882 6938
85	6.6279 9112	8.1569 6424	10.0335 7258	12.3357 0855	18.6178 5881
86	6.7771 2092	8.3608 8834	10.3094 9583	12.7057 7981	19.2694 8387
87	6.9296 0614	8.5699 1055	10.5930 0696	13.0869 5320	19.9439 1580
88	7.0855 2228	8.7841 5832	10.8843 1465	13.4795 6180	20.6419 5285
89	7.2449 4653	9.0037 6228	11.1836 3331	13.8839 4865	21.3644 2120
90	7.4079 5782	9.2288 5633	11.4911 8322	14.3004 6711	22.1121 7595
91	7.5746 3688	9.4595 7774	11.8071 9076	14.7294 8112	22.8861 0210
92	7.7450 6621	9.6960 6718	12.1318 8851	15.1713 6556	23.6871 1568
93	7.9193 3020	9.9384 6886	12.4655 1544	15.6265 0652	24.5161 6473
94	8.0975 1512	10.1869 3058	12.8083 1711	16.0953 0172	25.3742 3049
95	8.2797 0921	10.4416 0385	13.1605 4584	16.5781 6077	26.2623 2856
96	8.4660 0267	10.7026 4395	13.5224 6085	17.0755 0559	27.1815 1006
97	8.6564 8773	10.9702 1004	13.8943 2852	17.5877 7076	28.1328 6291
98	8.8512 5871	11.2444 6530	14.2764 2255	18.1154 0388	29.1175 1311
99	9.0504 1203	11.5255 7693	14.6690 2417	18.6588 6600	30.1366 2607
100	9.2540 4630	11.8137 1635	15.0724 2234	19.2186 3198	31.1914 0798

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000	1.0600 0000
2	1.0816 0000	1.0920 2500	1.1025 0000	1.1130 2500	1.1236 0000
3	1.1248 6400	1.1411 6613	1.1576 2500	1.1742 4138	1.1910 1600
4	1.1698 5856	1.1925 1860	1.2155 0625	1.2388 2465	1.2624 7696
5	1.2166 5290	1.2461 8194	1.2762 8156	1.3069 6001	1.3382 2558
6	1.2653 1902	1.3022 6012	1.3400 9564	1.3788 4281	1.4185 1911
7	1.3159 3178	1.3608 6183	1.4071 0042	1.4546 7916	1.5036 3026
8	1.3685 6905	1.4221 0061	1.4774 5544	1.5346 8651	1.5938 4807
9	1.4233 1181	1.4860 9514	1.5513 2822	1.6190 9427	1.6894 7896
10	1.4802 4428	1.5529 6942	1.6288 9463	1.7081 4446	1.7908 4770
11	1.5394 5406	1.6228 5305	1.7103 3936	1.8020 9240	1.8982 9856
12	1.6010 3222	1.6958 8143	1.7958 5633	1.9012 0749	2.0121 9647
13	1.6650 7351	1.7721 9610	1.8856 4914	2.0057 7390	2.1329 2826
14	1.7316 7645	1.8519 4492	1.9799 3160	2.1160 9146	2.2609 0396
15	1.8009 4351	1.9352 8244	2.0789 2818	2.2324 7649	2.3965 5819
16	1.8729 8125	2.0223 7015	2.1828 7459	2.3552 6270	2.5403 5168
17	1.9479 0050	2.1133 7681	2.2920 1832	2.4848 0215	2.6927 7279
18	2.0258 1652	2.2084 7877	2.4066 1923	2.6214 6627	2.8543 3915
19	2.1068 4918	2.3078 6031	2.5269 5020	2.7656 4691	3.0255 9950
20	2.1911 2314	2.4117 1402	2.6532 9771	2.9177 5749	3.2071 3547
21	2.2787 6807	2.5202 4116	2.7859 6259	3.0782 3415	3.3995 6360
22	2.3699 1879	2.6336 5201	2.9252 6072	3.2475 3703	3.6035 3742
23	2.4647 1554	2.7521 6635	3.0715 2376	3.4261 5157	3.8197 4966
24	2.5633 0416	2.8760 1383	3.2250 9994	3.6145 8990	4.0489 3464
25	2.6658 3633	3.0054 3446	3.3863 5494	3.8133 9235	4.2918 7072
26	2.7724 6978	3.1406 7901	3.5556 7269	4.0231 2893	4.5493 8296
27	2.8833 6858	3.2820 0956	3.7334 5632	4.2444 0102	4.8223 4594
28	2.9987 0332	3.4296 9999	3.9201 2914	4.4778 4307	5.1116 8670
29	3.1186 5145	3.5840 3649	4.1161 3560	4.7241 2444	5.4183 8790
30	3.2433 9751	3.7453 1813	4.3219 4238	4.9839 5129	5.7434 9117
31	3.3731 3341	3.9138 5745	4.5380 3949	5.2580 6861	6.0881 0064
32	3.5080 5875	4.0899 8104	4.7649 4147	5.5472 6238	6.4533 8668
33	3.6483 8110	4.2740 3018	5.0031 8854	5.8523 6181	6.8405 8988
34	3.7943 1634	4.4663 6154	5.2533 4797	6.1742 4171	7.2510 2528
35	3.9460 8899	4.6673 4781	5.5160 1537	6.5138 2501	7.6860 8679
36	4.1039 3255	4.8773 7846	5.7918 1614	6.8720 8538	8.1472 5200
37	4.2680 8986	5.0968 6049	6.0814 0694	7.2500 5008	8.6360 8712
38	4.4388 1345	5.3262 1921	6.3854 7729	7.6488 0283	9.1542 5235
39	4.6163 6599	5.5658 9908	6.7047 5115	8.0694 8699	9.7035 0749
40	4.8010 2063	5.8163 6454	7.0399 8871	8.5133 0877	10.2857 1794
41	4.9930 6145	6.0781 0094	7.3919 8815	8.9815 4076	10.9028 6101
42	5.1927 8391	6.3516 1548	7.7615 8756	9.4755 2550	11.5570 3267
43	5.4004 9527	6.6374 3818	8.1496 6693	9.9966 7940	12.2504 5463
44	5.6165 1508	6.9361 2290	8.5571 5028	10.5464 9677	12.9854 8191
45	5.8411 7568	7.2482 4843	8.9850 0779	11.1265 5409	13.7646 1083
46	6.0748 2271	7.5744 1961	9.4342 5818	11.7385 1456	14.5904 8748
47	6.3178 1562	7.9152 6849	9.9059 7109	12.3841 3287	15.4659 1673
48	6.5705 2824	8.2714 5557	10.4012 6965	13.0652 6017	16.3938 7173
49	6.8333 4937	8.6436 7107	10.9213 3313	13.7838 4948	17.3775 0403
50	7.1066 8335	9.0326 3627	11.4673 9979	14.5419 6120	18.4201 5427

TABLE III.—COMPOUND AMOUNT OF 1

$$(1 + i)^n$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
51	7.3909 5068	9.4391 0490	12.0407 6978	15.3417 6907	19.5253 6353
52	7.6865 8871	9.8638 6463	12.6428 0826	16.1855 6637	20.6968 8534
53	7.9940 5226	10.3077 3853	13.2749 4868	17.0757 7252	21.9386 9846
54	8.3138 1435	10.7715 8677	13.9386 9611	18.0149 4001	23.2550 2037
55	8.6463 6692	11.2563 0817	14.6356 3092	19.0057 6171	24 6503 2159
56	8.9922 2160	11.7628 4204	15.3674 1246	20.0510 7860	26.1293 4089
57	9.3519 1046	12.2921 6993	16.1357 8309	21.1538 8793	27.6971 0134
58	9.7259 8688	12.8453 1758	16.9425 7224	22.3173 5176	29.3589 2742
59	10.1150 2635	13.4233 5687	17.7897 0085	23.5448 0611	31.1204 6307
60	10.5196 2741	14.0274 0793	18.6791 8589	24.8397 7045	32.9876 9085
61	10.9404 1250	14.6586 4129	19.6131 4519	26.2059 5782	34.9669 5230
62	11.3780 2900	15.3182 8014	20.5938 0245	27.6472 8550	37.0649 6944
63	11.8331 5016	16.0076 0275	21.6234 9257	29.1678 8620	39.2888 6761
64	12.3064 7617	16.7279 4487	22.7046 6720	30.7721 1994	41.6461 9967
65	12.7987 3522	17.4807 0239	23.8399 0056	32.4645 8654	44.1449 7165
66	13.3106 8463	18.2673 3400	25.0318 9559	34.2501 3880	46.7936 6994
67	13.8431 1201	19.0893 6403	26.2834 9037	36.1338 9643	49.6012 9014
68	14.3968 3649	19.9183 8541	27.5976 6488	38.1212 6074	52.5773 6755
69	14.9727 0995	20.8460 6276	28.9775 4813	40.2179 3008	55.7320 0960
70	15.5716 1835	21.7841 3558	30.4264 2554	42.4299 1623	59.0759 3018
71	16.1944 8308	22.7644 2168	31.9477 4681	44.7635 6163	62.6204 8599
72	16.8422 6241	23.7888 2066	33.5451 3415	47.2255 5751	66.3777 1515
73	17.5159 5290	24.8593 1759	35.2223 9086	49.8229 6318	70.3603 7806
74	18.2165 9102	25.9779 8688	36.9835 1040	52.5632 2615	74.5820 0074
75	18.9452 5466	27.1469 9629	38.8326 8592	55,4542 0359	79.0569 2079
76	19.7030 6485	28.3686 1112	40.7743 2022	58.5041 8479	83.8003 3603
77	20.4911 8744	29.6451 9862	42.8130 3623	61.7219 1495	88.8283 5620
78	21.3108 3494	30.9792 3256	44.9536 8804	65.1166 2027	94.1580 5757
79	22.1632 6834	32.3732 9802	47.2013 7244	68.6980 3439	99.8075 4102
80	23.0497 9907	33.8300 9643	49.5614 4107	72.4764 2628	105.7959 9348
81	23.9717 9103	35.3524 5077	52.0395 1312	76.4626 2973	112.1437 5309
82	24.9306 6267	36.9433 1106	54.6414 8878	80.6680 7436	118.8723 7828
83	25.9278 8918	38.6057 6006	57.3735 6322	85.1048 1845	126.0047 2097
84	26.9650 0475	40.3430 1926	60.2422 4138	89.7855 8347	133.5650 0423
85	28.0436 0494	42.1584 5513	63.2543 5344	94.7237 9056	141.5789 0449
86	29.1653 4914	44.0555 8561	66.4170 7112	99.9335 9904	150.0736 3875
87	30.3319 6310	46.0380 8696	69.7379 2467	105.4299 4698	159.0780 5708
88	31.5452 4163	48.1098 0087	73.2248 2091	111.2285 9407	168.6227 4050
89	32.8070 5129	50.2747 4191	76.8860 6195	117.3461 6674	178.7401 0493
90	34.1193 3334	52.5371 0530	80.7303 6505	123.8002 0591	189.4645 1123
91	35.4841 0668	54.9012 7503	84.7668 8330	130.6092 1724	200.8323 8190
92	36.9034 7094	57.3718 3241	89.0052 2747	137.7927 2419	212.8823 2482
93	38.3796 0978	59.9535 6487	93.4554 8884	145.3713 2402	225.6552 6431
94	39.9147 9417	62.6514 7529	98.1282 6328	153.3667 4684	239.1945 8017
95	41.5113 8594	65.4707 9168	103.0346 7645	161.8019 1791	253.5462 5498
96	43.1718 4138	68.4169 7730	108.1864 1027	170.7010 2340	268.7590 3028
97	44.8987 1503	71.4957 4128	113.5957 3078	180.0895 7969	284.8845 7209
98	46.6946 6363	74.7130 4964	119.2755 1732	189.9945 0657	301.9776 4642
99	48.5624 5018	78.0751 3687	125.2392 9319	200.4442 0443	320.0963 0520
100	50.5049 4818	81.5885 1803	131.5012 5785	211.4686 3567	339.3020 8351

Table III.—Compound Amount of 1

$$(1 + i)^n$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000	1.0850 0000
2	1.1342 2500	1.1449 0000	1 1556 2500	1.1664 0000	1.1772 2500
3	1.2079 4963	1.2250 4300	1.2422 9688	1.2597 1200	1.2772 8913
4	1.2864 6635	1.3107 9601	1.3354 6914	1.3604 8896	1.3858 5870
5	1.3700 8666	1.4025 5173	1.4356 2933	1.4693 2808	1.5036 5669
6	1.4591 4230	1.5007 3035	1.5433 0153	1.5868 7432	1.6314 6751
7	1.5539 8655	1.6057 8148	1.6590 4914	1.7138 2427	1.7701 4225
8	1.6549 9567	1.7181 8618	1.7834 7783	1.8509 3021	1.9206 0434
9	1.7625 7039	1.8384 5921	1.9172 3866	1 9990 0463	2.0838 5571
10	1.8771 3747	1.9671 5136	2.0610 3156	2 1589 2500	2.2609 8344
11	1.9991 5140	2.1048 5195	2.2156 0893	2.3316 3900	2.4531 6703
12	2.1290 9624	2.2521 9159	2.3817 7960	2.5181 7012	2.6616 8623
13	2.2674 8750	2.4098 4500	2.5604 1307	2.7196 2373	2.8879 2956
14	2.4148 7418	2.5785 3415	2.7524 4405	2.9371 9362	3.1334 0357
15	2.5718 4101	2.7590 3154	2.9588 7735	3.1721 6911	3.3997 4288
16	2.7390 1067	2.9521 6375	3.1807 9315	3.4259 4264	3.6887 2102
17	2.9170 4637	3.1588 1521	3.4193 5264	3.7000 1805	4.0022 6231
18	3.1066 5438	3.3799 3228	3.6758 0409	3.9960 1950	4.3424 5461
19	3.3085 8691	3.6165 2754	3.9514 8940	4.3157 0106	4.7115 6325
20	3.5236 4506	3.8696 8446	4.2478 5110	4.6609 5714	5.1120 4612
21	3.7526 8199	4.1405 6237	4.5664 3993	5.0338 3372	5.5465 7005
22	3.9966 0632	4.4304 0174	4.9089 2293	5.4365 4041	6.0180 2850
23	4.2563 8573	4.7405 2986	5.2770 9215	5.8714 6365	6.5295 6092
24	4.5330 5081	5.0723 6695	5.6728 7406	6.3411 8074	7.0845 7360
25	4.8276 9911	5.4274 3264	6.0983 3961	6.8484 7520	7.6867 6236
26	5.1414 9955	5.8073 5292	6.5557 1508	7.3963 5321	8.3401 3716
27	5.4756 9702	6.2138 6763	7.0473 9371	7.9880 6147	9.0490 4881
28	5.8316 1733	6.6488 3836	7.5759 4824	8.6271 0639	9.8182 1796
29	6.2106 7245	7.1142 5705	8.1441 4436	9.3172 7490	10.6527 6649
30	6.6143 6616	7.6122 5504	8.7549 5519	10.0626 5689	11.5582 5164
31	7.0442 9996	8.1451 1290	9.4115 7683	10.8676 6944	12.5407 0303
32	7.5021 7946	8.7152 7080	10.1174 4509	11.7370 8300	13.6066 6279
33	7.9898 2113	9.3253 3975	10.8762 5347	12.6760 4964	14.7632 2913
34	8.5091 5950	9.9781 1354	11.6919 7248	13.6901 3361	16.0181 0360
35	9.0622 5487	10.6765 8148	12.5688 7042	14.7853 4429	17.3796 4241
36	9.6513 0143	11.4239 4219	13.5115 3570	15.9681 7184	18.8569 1201
37	10.2786 3603	12.2236 1814	14.5249 0088	17.2456 2558	20.4597 4953
38	10.9467 4737	13.0792 7141	15.6142 6844	18.6252 7563	22.1988 2824
39	11.6582 8595	13.9948 2041	16.7853 3858	20.1152 9768	24.0857 2865
40	12.4160 7453	14.9744 5784	18.0442 3897	21.7245 2150	26.1330 1558
41	13.2231 1938	16.0226 6989	19.3975 5689	23.4624 8322	28.3543 2190
42	14.0826 2214	17.1442 5678	20.8523 7366	25.3394 8187	30.7644 3927
43	14.9979 9258	18.3443 5475	22.4163 0168	27.3666 4042	33.3794 1660
44	15.9728 6209	19.6284 5959	24.0975 2431	29.5559 7166	36.2166 6702
45	17.0110 9813	21.0024 5176	25.9048 3863	31.9204 4939	39.2950 8371
46	18.1168 1951	22.4726 2338	27.8477 0153	34.4740 8534	42.6351 6583
47	19.2944 1278	24.0457 0702	29.9362 7915	37.2320 1217	46.2591 5492
48	20.5485 4961	25.7289 0651	32.1815 0008	40.2105 7314	50.1911 8309
49	21.8842 0533	27.5299 2997	34.5951 1259	43.4274 1899	54.4574 3365
50	23.3066 7868	29.4570 2506	37.1897 4603	46.9016 1251	59.0863 1551

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}.$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	0.9958 5062	0.9950 2488	0.9942 0050	0.9925 5583	0.9900 9901
2	0.9917 1846	0.9900 7450	0.9844 3463	0.9851 6708	0.9802 9605
3	0.9876 0345	0.9851 4876	0.9827 0220	0.9778 3333	0.9705 9015
4	0.9835 0551	0.9802 4752	0.9770 0302	0.9705 5417	0.9609 8034
5	0.9794 2457	0.9753 7067	0.9713 3688	0.9633 2920	0.9514 6569
6	0.9753 6057	0.9705 1808	0.9657 0361	0.9561 5802	0.9420 4524
7	0.9713 1343	0.9656 8963	0.9601 0301	0.9490 4022	0.9327 1805
8	0.9672 8308	0.9608 8520	0.9545 3489	0.9419 7540	0.9234 8322
9	0.9632 6946	0.9561 0468	0.9489 9907	0.9349 6318	0.9143 3982
10	0.9592 7249	0.9513 4794	0.9434 9534	0.9280 0315	0.9052 8695
11	0.9552 9211	0.9466 1489	0.9380 2354	0.9210 9494	0.8963 2372
12	0.9513 2824	0.9419 0534	0.9325 8347	0.9142 3815	0.8874 4923
13	0.9473 8082	0.9372 1924	0.9271 7495	0.9074 3241	0.8786 6260
14	0.9434 4978	0.9325 5646	0.9217 9780	0.9006 7733	0.8699 6297
15	0.9395 3505	0.9279 1688	0.9164 5183	0.8939 7254	0.8613 4947
16	0.9356 3656	0.9233 0037	0.9111 3686	0.8873 1766	0.8528 2126
17	0.9317 5425	0.9187 0684	0.9058 5272	0.8807 1231	0.8443 7749
18	0.9278 8805	0.9141 3616	0.9005 9923	0.8741 5614	0.8360 1731
19	0.9240 3789	0.9095 8822	0.8953 7620	0.8676 4878	0.8277 3992
20	0.9202 0371	0.9050 6290	0.8901 8346	0.8611 8985	0.8195 4447
21	0.9163 8544	0.9005 6010	0.8850 2084	0.8547 7901	0.8114 3017
22	0.9125 8301	0.8960 7971	0.8798 8816	0.8484 1589	0.8033 9621
23	0.9087 9636	0.8916 2160	0.8747 8525	0.8421 0014	0.7954 4179
24	0.9050 2542	0.8871 8567	0.8697 1193	0.8358 3140	0.7875 6613
25	0.9012 7012	0.8827 7181	0.8646 6803	0.8296 0933	0.7797 6844
26	0.8975 3041	0.8783 7991	0.8596 5339	0.8234 3358	0.7720 4796
27	0.8938 0622	0.8740 0986	0.8546 6782	0.8173 0380	0.7644 0392
28	0.8900 9748	0.8696 6155	0.8497 1118	0.8112 1966	0.7568 3557
29	0.8864 0413	0.8653 3488	0.8447 8327	0.8051 8080	0.7493 4215
30	0.8827 2610	0.8610 2973	0.8398 8395	0.7991 8690	0.7419 2292
31	0.8790 6334	0.8567 4600	0.8350 1304	0.7932 3762	0.7345 7715
32	0.8754 1577	0.8524 8358	0.8301 7038	0.7873 3262	0.7273 0411
33	0.8717 8334	0.8482 4237	0.8253 5581	0.7814 7158	0.7201 0307
34	0.8681 6599	0.8440 2226	0.8205 6915	0.7756 5418	0.7129 7334
35	0.8645 6364	0.8398 2314	0.8158 1026	0.7698 8008	0.7059 1420
36	0.8609 7624	0.8356 4492	0.8110 7897	0.7641 4896	0.6989 2495
37	0.8574 0372	0.8314 8748	0.8063 7511	0.7584 6051	0.6920 0490
38	0.8538 4603	0.8273 5073	0.8016 9854	0.7528 1440	0.6851 5337
39	0.8503 0310	0.8232 3455	0.7970 4908	0.7472 1032	0.6783 6967
40	0.8467 7487	0.8191 3886	0.7924 2660	0.7416 4796	0.6716 5314
41	0.8432 6128	0.8150 6354	0.7878 3092	0.7361 2701	0.6650 0311
42	0.8397 6227	0.8110 0850	0.7832 6189	0.7306 4716	0.6584 1892
43	0.8362 7778	0.8069 7363	0.7787 1936	0.7252 0809	0.6518 9992
44	0.8328 0775	0.8029 5884	0.7742 0317	0.7198 0952	0.6454 4546
45	0.8293 5211	0.7989 6402	0.7697 1318	0.7144 5114	0.6390 5492
46	0.8259 1082	0.7949 8907	0.7652 4923	0.7091 3264	0.6327 2764
47	0.8224 8380	0.7910 3390	0.7608 1116	0.7038 5374	0.6264 6301
48	0.8190 7100	0.7870 9841	0.7563 9884	0.6986 1414	0.6202 6041
49	0.8156 7237	0.7831 8250	0.7520 1210	0.6934 1353	0.6141 1921
50	0.8122 8784	0.7792 8607	0.7476 5080	0.6882 5165	0.6080 3882

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{2}{3}\%$	1%
51	0.8089 1735	0.7754 0902	0.7433 1480	0.6831 2819	0.6020 1864
52	0.8055 6084	0.7715 5127	0.7390 0394	0.6780 4286	0.5960 5806
53	0.8022 1827	0.7677 1270	0.7347 1809	0.6729 9540	0.5901 5649
54	0.7988 8956	0.7638 9324	0.7304 5709	0.6679 8551	0.5843 1336
55	0.7955 7467	0.7600 9277	0.7262 2080	0.6630 1291	0.5785 2808
56	0.7922 7353	0.7563 1122	0.7220 0908	0.6580 7733	0.5728 0008
57	0.7889 8608	0.7525 4847	0.7178 2179	0.6531 7849	0.5671 2879
58	0.7857 1228	0.7488 0445	0.7136 5878	0.6483 1612	0.5615 1365
59	0.7824 5207	0.7450 7906	0.7095 1991	0.6434 8995	0.5559 5411
60	0.7792 0538	0.7413 7220	0.7054 0505	0.6386 9970	0.5504 4962
61	0.7759 7216	0.7376 8378	0.7013 1405	0.6339 4511	0.5449 9962
62	0.7727 5236	0.7340 1371	0.6972 4678	0.6292 2592	0.5396 0358
63	0.7695 4591	0.7303 6190	0.6932 0310	0.6245 4185	0.5342 6097
64	0.7663 5278	0.7267 2826	0.6891 8286	0.6198 9266	0.5289 7126
65	0.7631 7289	0.7231 1269	0.6851 8594	0.6152 7807	0.5237 3392
66	0.7600 0620	0.7195 1512	0.6812 1221	0.6106 9784	0.5185 4844
67	0.7568 5265	0.7159 3544	0.6772 6151	0.6061 5170	0.5134 1429
68	0.7537 1218	0.7123 7357	0.6733 3373	0.6016 3940	0.5083 3099
69	0.7505 8474	0.7088 2943	0.6694 2873	0.5971 6070	0.5032 9801
70	0.7474 7028	0.7053 0291	0.6655 4638	0.5927 1533	0.4983 1486
71	0.7443 6874	0.7017 9394	0.6616 8654	0.5883 0306	0.4933 8105
72	0.7412 8008	0.6983 0243	0.6578 4909	0.5839 2363	0.4884 9609
73	0.7382 0423	0.6948 2829	0.6540 3389	0.5795 7681	0.4836 5949
74	0.7351 4114	0.6913 7143	0.6502 4082	0.5752 6234	0.4788 7078
75	0.7320 9076	0.6879 3177	0.6464 6975	0.5709 7999	0.4741 2949
76	0.7290 5304	0.6845 0923	0.6427 2054	0.5667 2952	0.4694 3514
77	0.7260 2792	0.6811 0371	0.6389 9308	0.5625 1069	0.4647 8726
78	0.7230 1536	0.6777 1513	0.6352 8724	0.5583 2326	0.4601 8541
79	0.7200 1529	0.6743 4342	0.6316 0289	0.5541 6701	0.4556 2912
80	0.7170 2768	0.6709 8847	0.6279 3991	0.5500 4170	0.4511 1794
81	0.7140 5246	0.6676 5022	0.6242 9817	0.5459 4710	0.4466 5142
82	0.7110 8959	0.6643 2858	0.6206 7755	0.5418 8297	0.4422 2913
83	0.7081 3901	0.6610 2346	0.6170 7793	0.5378 4911	0.4378 5063
84	0.7052 0067	0.6577 3479	0.6134 9919	0.5338 4527	0.4335 1547
85	0.7022 7453	0.6544 6248	0.6099 4120	0.5298 7123	0.4292 2324
86	0.6993 6052	0.6512 0644	0.6064 0384	0.5259 2678	0.4249 7350
87	0.6964 5861	0.6479 6661	0.6028 8700	0.5220 1169	0.4207 6585
88	0.6935 6874	0.6447 4290	0.5993 9056	0.5181 2575	0.4165 9985
89	0.6906 9086	0.6415 3522	0.5959 1439	0.5142 6873	0.4124 7510
90	0.6878 2493	0.6383 4350	0.5924 5838	0.5104 4043	0.4083 9119
91	0.6849 7088	0.6351 6766	0.5890 2242	0.5066 4063	0.4043 4771
92	0.6821 2868	0.6320 0763	0.5856 0638	0.5028 6911	0.4003 4427
93	0.6792 9827	0.6288 6331	0.5822 1015	0.4991 2567	0.3963 8046
94	0.6764 7960	0.6257 3464	0.5788 3363	0.4954 1009	0.3924 5590
95	0.6736 7263	0.6226 2153	0.5754 7668	0.4917 2217	0.3885 7020
96	0.6708 7731	0.6195 2391	0.5721 3920	0.4880 6171	0.3847 2297
97	0.6680 9350	0.6164 4170	0.5688 2108	0.4844 2850	0.3809 1383
98	0.6653 2141	0.6133 7483	0.5655 2220	0.4808 2233	0.3771 4241
99	0.6625 6074	0.6103 2321	0.5622 4245	0.4772 4301	0.3734 0832
100	0.6598 1153	0.6072 8678	0.5589 8172	0.4736 9033	0.3697 1121

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	0.6570 7372	0.6042 6545	0.5557 3991	0.4701 6410	0.3660 5071
102	0.6543 4727	0.6012 5015	0.5525 1689	0.4666 6412	0.3624 2644
103	0.6516 3214	0.5982 6781	0.5493 1257	0.4631 9019	0.3588 3806
104	0.6489 2827	0.5952 9136	0.5461 2683	0.4597 4213	0.3552 8521
105	0.6462 3562	0.5923 2971	0.5429 5957	0.4563 1973	0.3517 6753
106	0.6435 5415	0.5893 8279	0.5398 1067	0.4529 2281	0.3482 8469
107	0.6408 8380	0.5864 5054	0.5366 8004	0.4495 5117	0.3448 3632
108	0.6382 2453	0.5835 3288	0.5335 6756	0.4462 0464	0.3414 2210
109	0.6355 7630	0.5806 2973	0.5304 7313	0.4428 8302	0.3380 4168
110	0.6329 3905	0.5777 4102	0.5273 9665	0.4395 8612	0.3346 9474
111	0.6303 1275	0.5748 6669	0.5243 3801	0.4363 1377	0.3213 8093
112	0.6276 9734	0.5720 0666	0.5212 9711	0.4330 6577	0.3280 9993
113	0.6250 9279	0.5691 6085	0.5182 7385	0.4298 4196	0.3248 5141
114	0.6224 9904	0.5663 2921	0.5152 6812	0.4266 4124	0.3216 3506
115	0.6199 1606	0.5635 1165	0.5122 7982	0.4234 6615	0.3184 5056
116	0.6173 4379	0.5607 0811	0.5093 0885	0.4203 1379	0.3152 9758
117	0.6147 8220	0.5579 1852	0.5063 5512	0.4171 8491	0.3121 7582
118	0.6122 3123	0.5551 4280	0.5034 1851	0.4140 7931	0.3090 8497
119	0.6096 9086	0.5523 8090	0.5004 9893	0.4109 9683	0.3060 2473
120	0.6071 6102	0.5496 3273	0.4975 9629	0.4079 3730	0.3029 9478
121	0.6046 4168	0.5468 9824	0.4947 1047	0.4049 0055	0.2999 9483
122	0.6021 3279	0.5441 7736	0.4918 4140	0.4018 8640	0.2970 2459
123	0.5996 3431	0.5414 7001	0.4889 8896	0.3988 9469	0.2940 8375
124	0.5971 4620	0.5387 7612	0.4861 5307	0.3959 2525	0.2911 7203
125	0.5946 6842	0.5360 9565	0.4833 3363	0.3929 7792	0.2882 8914
126	0.5922 0091	0.5334 2850	0.4805 3053	0.3900 5252	0.2854 3479
127	0.5897 4365	0.5307 7463	0.4777 4369	0.3871 4891	0.2826 0870
128	0.5872 9658	0.5281 3396	0.4749 7302	0.3842 6691	0.2798 1060
129	0.5848 5966	0.5255 0643	0.4722 1841	0.3814 0636	0.2770 4019
130	0.5824 3286	0.5228 9197	0.4694 7978	0.3785 6711	0.2742 9722
131	0.5800 1613	0.5202 9052	0.4667 5703	0.3757 4899	0.2715 8141
132	0.5776 0942	0.5177 0201	0.4640 5007	0.3729 5185	0.2688 9248
133	0.5752 1270	0.5151 2637	0.4613 5881	0.3701 7553	0.2662 3018
134	0.5728 2593	0.5125 6356	0.4586 8316	0.3674 1988	0.2635 9424
135	0.5704 4906	0.5100 1349	0.4560 2303	0.3646 8475	0.2609 8439
136	0.5680 8205	0.5074 7611	0.4533 7832	0.3619 6997	0.2584 0039
137	0.5657 2486	0.5049 5135	0.4507 4895	0.3592 7541	0.2558 4197
138	0.5633 7745	0.5024 3916	0.4481 3483	0.3566 0090	0.2533 0888
139	0.5610 3979	0.4999 3946	0.4455 3587	0.3539 4630	0.2508 0087
140	0.5587 1182	0.4974 5220	0.4429 5198	0.3513 1147	0.2483 1770
141	0.5563 9351	0.4949 7731	0.4403 8308	0.3486 9625	0.2458 5911
142	0.5540 8483	0.4925 1474	0.4378 2908	0.3461 0049	0.2434 2486
143	0.5517 8572	0.4900 6442	0.4352 8989	0.3435 2406	0.2410 1471
144	0.5494 9615	0.4876 2628	0.4327 6542	0.3409 6681	0.2386 2843
145	0.5472 1609	0.4852 0028	0.4302 5560	0.3384 2860	0.2362 6577
146	0.5449 4548	0.4827 8635	0.4277 6033	0.3359 0928	0.2339 2650
147	0.5426 8429	0.4803 8443	0.4252 7953	0.3334 0871	0.2316 1040
148	0.5404 3249	0.4779 9446	0.4228 1312	0.3309 2676	0.2293 1723
149	0.5381 9003	0.4756 1637	0.4203 6102	0.3284 6329	0.2270 4676
150	0.5359 5688	0.4732 5012	0.4179 2313	0.3260 1815	0.2247 9877

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	0.9888 7515	0.9876 5432	0.9852 2167	0.9828 0098	0.9803 9216
2	0.9778 7407	0.9754 6106	0.9706 6175	0.9658 9777	0.9611 6878
3	0.9669 9537	0.9634 1833	0.9563 1699	0.9492 8528	0.9423 2233
4	0.9562 3770	0.9515 2428	0.9421 8423	0.9329 5851	0.9238 4543
5	0.9455 9970	0.9397 7706	0.9282 6033	0.9169 1254	0.9057 3081
6	0.9350 8005	0.9281 7488	0.9145 4219	0.9011 4254	0.8879 7138
7	0.9246 7743	0.9167 1593	0.9010 2679	0.8856 4378	0.8705 6018
8	0.9143 9054	0.9053 9845	0.8877 1112	0.8704 1157	0.8534 9037
9	0.9042 1808	0.8942 2069	0.8745 9224	0.8554 4135	0.8367 5527
10	0.8941 5881	0.8831 8093	0.8616 6723	0.8407 2860	0.8203 4830
11	0.8842 1142	0.8722 7746	0.8489 3323	0.8262 6889	0.8042 6304
12	0.8743 7470	0.8615 0860	0.8363 8742	0.8120 5788	0.7884 9318
13	0.8646 4742	0.8508 7269	0.8240 2702	0.7980 9128	0.7730 3253
14	0.8550 2835	0.8403 6809	0.8118 4928	0.7843 6490	0.7578 7502
15	0.8455 1629	0.8299 9318	0.7998 5150	0.7708 7459	0.7430 1473
16	0.8361 1005	0.8197 4635	0.7880 3104	0.7576 1631	0.7284 4581
17	0.8268 0846	0.8096 2602	0.7763 8526	0.7445 8605	0.7141 6256
18	0.8176 1034	0.7996 3064	0.7649 1159	0.7317 7990	0.7001 5937
19	0.8085 1455	0.7897 5866	0.7536 0747	0.7191 9401	0.6864 3076
20	0.7995 1995	0.7800 0855	0.7424 7042	0.7068 2458	0.6729 7133
21	0.7906 2542	0.7703 7881	0.7314 9795	0.6946 6789	0.6597 7582
22	0.7818 2983	0.7608 6796	0.7206 8763	0.6827 2028	0.6468 3904
23	0.7731 3210	0.7514 7453	0.7100 3708	0.6709 7817	0.6341 5592
24	0.7645 3112	0.7421 9707	0.6995 4392	0.6594 3800	0.6217 2149
25	0.7560 2583	0.7330 3414	0.6892 0583	0.6480 9632	0.6095 3087
26	0.7476 1516	0.7239 8434	0.6790 2052	0.6369 4970	0.5975 7928
27	0.7392 9806	0.7150 4626	0.6689 8574	0.6259 9479	0.5858 6204
28	0.7310 7348	0.7062 1853	0.6590 9925	0.6152 2829	0.5743 7455
29	0.7229 4040	0.6974 9978	0.6493 5887	0.6046 4697	0.5631 1231
30	0.7148 9780	0.6888 8867	0.6397 6243	0.5942 4764	0.5520 7089
31	0.7069 4467	0.6803 8387	0.6303 0781	0.5840 2716	0.5412 4597
32	0.6990 8002	0.6719 8407	0.6209 9292	0.5739 8247	0.5306 3330
33	0.6913 0287	0.6636 8797	0.6118 1568	0.5641 1053	0.5202 2873
34	0.6836 1223	0.6554 9429	0.6027 7407	0.5544 0839	0.5100 2817
35	0.6760 0715	0.6474 0177	0.5938 6608	0.5448 7311	0.5000 2761
36	0.6684 8667	0.6394 0916	0.5850 8974	0.5355 0183	0.4902 2315
37	0.6610 4986	0.6315 1522	0.5764 4309	0.5262 9172	0.4806 1093
38	0.6536 9578	0.6237 1873	0.5679 2423	0.5172 4002	0.4711 8719
39	0.6464 2352	0.6160 1850	0.5595 3126	0.5083 4400	0.4619 4822
40	0.6392 3216	0.6084 1334	0.5512 6232	0.4996 0098	0.4528 9042
41	0.6321 2080	0.6009 0206	0.5431 1559	0.4910 0834	0.4440 1021
42	0.6250 8855	0.5934 8352	0.5350 8925	0.4825 6348	0.4353 0413
43	0.6181 3454	0.5861 5656	0.5271 8153	0.4742 6386	0.4267 6875
44	0.6112 5789	0.5789 2006	0.5193 9067	0.4661 0699	0.4184 0074
45	0.6044 5774	0.5717 7290	0.5117 1494	0.4580 9040	0.4101 9680
46	0.5977 3324	0.5647 1397	0.5041 5265	0.4502 1170	0.4021 5373
47	0.5910 8355	0.5577 4219	0.4967 0212	0.4424 6850	0.3942 6836
48	0.5845 0784	0.5508 5649	0.4893 6170	0.4348 5848	0.3865 3761
49	0.5780 0528	0.5440 5579	0.4821 2975	0.4273 7934	0.3789 5844
50	0.5715 7506	0.5373 3905	0.4750 0468	0.4200 2883	0.3715 2788

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	0.5652 1637	0.5307 0524	0.4679 8491	0.4128 0475	0.3642 4302
52	0.5589 2843	0.5241 5332	0.4610 6887	0.4057 0492	0.3571 0100
53	0.5527 1044	0.5176 8229	0.4542 5505	0.3987 2719	0.3500 9902
54	0.5465 6162	0.5112 9115	0.4475 4192	0.3918 6947	0.3432 3433
55	0.5404 8120	0.5049 7892	0.4409 2800	0.3851 2970	0.3365 0425
56	0.5344 6843	0.4987 4461	0.4344 1182	0.3785 0585	0.3299 0613
57	0.5285 2256	0.4925 8727	0.4279 9194	0.3719 9592	0.3234 3738
58	0.5226 4282	0.4865 0594	0.4216 6694	0.3655 9796	0.3170 9547
59	0.5168 2850	0.4804 9970	0.4154 3541	0.3593 1003	0.3108 7791
60	0.5110 7887	0.4745 6760	0.4092 9597	0.3531 3025	0.3047 8227
61	0.5053 9319	0.4687 0874	0.4032 4726	0.3470 5676	0.2988 0614
62	0.4997 7077	0.4629 2222	0.3972 8794	0.3410 8772	0.2929 4720
63	0.4942 1090	0.4572 0713	0.3914 1669	0.3352 2135	0.2872 0314
64	0.4887 1288	0.4515 6259	0.3856 3221	0.3294 5587	0.2815 7170
65	0.4832 7602	0.4459 8775	0.3799 3321	0.3237 8956	0.2760 5069
66	0.4778 9965	0.4404 8173	0.3743 1843	0.3182 2069	0.2706 3793
67	0.4725 8309	0.4350 4368	0.3687 8663	0.3127 4761	0.2653 3130
68	0.4673 2568	0.4296 7277	0.3633 3658	0.3073 6866	0.2601 2873
69	0.4621 2675	0.4243 6817	0.3579 6708	0.3020 8222	0.2550 2817
70	0.4569 8566	0.4191 2905	0.3526 7692	0.2968 8670	0.2500 2761
71	0.4519 0177	0.4139 5462	0.3474 6495	0.2917 8054	0.2451 2511
72	0.4468 7443	0.4088 4407	0.3423 3000	0.2867 6221	0.2403 1874
73	0.4419 0302	0.4037 9661	0.3372 7093	0.2818 3018	0.2356 0661
74	0.4369 8692	0.3988 1147	0.3322 8663	0.2769 8298	0.2309 8687
75	0.4321 2551	0.3938 8787	0.3273 7599	0.2722 1914	0.2264 5771
76	0.4273 1818	0.3890 2506	0.3225 3793	0.2675 3724	0.2220 1737
77	0.4225 6433	0.3842 2228	0.3177 7136	0.2629 3586	0.2176 6408
78	0.4178 6337	0.3794 7879	0.3130 7523	0.2584 1362	0.2133 9616
79	0.4132 1470	0.3747 9387	0.3084 4850	0.2539 6916	0.2092 1192
80	0.4086 1775	0.3701 6679	0.3038 9015	0.2496 0114	0.2051 0973
81	0.4040 7194	0.3655 9683	0.2993 9916	0.2453 0825	0.2010 8797
82	0.3995 7670	0.3610 8329	0.2949 7454	0.2410 8919	0.1971 4507
83	0.3951 3148	0.3566 2547	0.2906 1531	0.2369 4269	0.1932 7948
84	0.3907 3570	0.3522 2268	0.2863 2050	0.2328 6751	0.1894 8968
85	0.3863 8882	0.3478 7426	0.2820 8917	0.2288 6242	0.1857 7420
86	0.3820 9031	0.3435 7951	0.2779 2036	0.2249 2621	0.1821 3157
87	0.3778 3961	0.3393 3779	0.2738 1316	0.2210 5770	0.1785 6036
88	0.3736 3621	0.3351 4843	0.2697 6666	0.2172 5572	0.1750 5918
89	0.3694 7956	0.3310 1080	0.2657 7997	0.2135 1914	0.1716 2665
90	0.3653 6916	0.3269 2425	0.2618 5218	0.2098 4682	0.1682 6142
91	0.3613 0448	0.3228 8814	0.2579 8245	0.2062 3766	0.1649 6217
92	0.3572 8503	0.3189 0187	0.2541 6990	0.2026 9057	0.1617 2762
93	0.3533 1029	0.3149 6481	0.2504 1369	0.1992 0450	0.1585 5649
94	0.3493 7976	0.3110 7636	0.2467 1300	0.1957 7837	0.1554 4754
95	0.3454 9297	0.3072 3591	0.2430 6699	0.1924 1118	0.1523 9955
96	0.3416 4941	0.3034 4287	0.2394 7487	0.1891 0190	0.1494 1132
97	0.3378 4861	0.2996 9666	0.2359 3583	0.1858 4953	0.1464 8169
98	0.3340 9010	0.2959 9670	0.2324 4909	0.1826 5310	0.1436 0950
99	0.3303 7340	0.2923 4242	0.2290 1389	0.1795 1165	0.1407 9363
100	0.3266 9805	0.2887 3326	0.2256 2944	0.1764 2422	0.1380 3297

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379	0.9661 8357
2	0.9564 7444	0.9518 1440	0.9471 8833	0.9425 9591	0.9335 1070
3	0.9354 2732	0.9285 9941	0.9218 3779	0.9151 4166	0.9019 4271
4	0.9148 4335	0.9059 5064	0.8971 6573	0.8884 8705	0.8714 4223
5	0.8947 1232	0.8838 5429	0.8731 5400	0.8626 0878	0.8419 7317
6	0.8750 2427	0.8622 9687	0.8497 8491	0.8374 8426	0.8135 0064
7	0.8557 6946	0.8412 6524	0.8270 4128	0.8130 9151	0.7859 9096
8	0.8369 3835	0.8207 4657	0.8049 0635	0.7894 0923	0.7594 1156
9	0.8185 2161	0.8007 2836	0.7833 6385	0.7664 1673	0.7337 3097
10	0.8005 1013	0.7811 9840	0.7623 9791	0.7440 9391	0.7089 1881
11	0.7828 9499	0.7621 4478	0.7419 9310	0.7224 2128	0.6849 4571
12	0.7656 6748	0.7435 5589	0.7221 3440	0.7013 7988	0.6617 8330
13	0.7488 1905	0.7254 2038	0.7028 0720	0.6809 5134	0.6394 0415
14	0.7323 4137	0.7077 2720	0.6839 9728	0.6611 1781	0.6177 8179
15	0.7162 2628	0.6904 6556	0.6656 9078	0.6418 6195	0.5968 9062
16	0.7004 6580	0.6736 2493	0.6478 7424	0.6231 6694	0.5767 0591
17	0.6850 5212	0.6571 9506	0.6305 3454	0.6050 1645	0.5572 0378
18	0.6699 7763	0.6411 6591	0.6136 5892	0.5873 9461	0.5383 6114
19	0.6552 3484	0.6255 2772	0.5972 3496	0.5702 8603	0.5201 5569
20	0.6408 1647	0.6102 7094	0.5812 5057	0.5536 7575	0.5025 6588
21	0.6267 1538	0.5953 8629	0.5656 9398	0.5375 4928	0.4855 7090
22	0.6129 2457	0.5808 6467	0.5505 5375	0.5218 9250	0.4691 5063
23	0.5994 3724	0.5666 9724	0.5358 1874	0.5066 9175	0.4532 8563
24	0.5862 4668	0.5528 7535	0.5214 7809	0.4919 3374	0.4379 5713
25	0.5733 4639	0.5393 9059	0.5075 2126	0.4776 0557	0.4231 4699
26	0.5607 2997	0.5262 3472	0.4939 3796	0.4636 9473	0.4088 3767
27	0.5483 9117	0.5133 9973	0.4807 1821	0.4501 8906	0.3950 1224
28	0.5363 2388	0.5008 7778	0.4678 5227	0.4370 7675	0.3816 5434
29	0.5245 2213	0.4886 6125	0.4553 3068	0.4243 4636	0.3687 4815
30	0.5129 8008	0.4767 4269	0.4431 4421	0.4119 8676	0.3562 7841
31	0.5016 9201	0.4651 1481	0.4312 8301	0.3999 8715	0.3442 3035
32	0.4906 5233	0.4537 7055	0.4197 4103	0.3883 3703	0.3325 8971
33	0.4798 5558	0.4427 0298	0.4085 0708	0.3770 2625	0.3213 4271
34	0.4692 9641	0.4319 0534	0.3975 7380	0.3660 4490	0.3104 7605
35	0.4589 6960	0.4213 7107	0.3869 3314	0.3553 8340	0.2999 7686
36	0.4488 7002	0.4110 9372	0.3765 7727	0.3450 3243	0.2898 3272
37	0.4389 9268	0.4010 6705	0.3664 9856	0.3349 8294	0.2800 3161
38	0.4293 3270	0.3912 8492	0.3566 8959	0.3252 2615	0.2705 6194
39	0.4198 8528	0.3817 4139	0.3471 4316	0.3157 5355	0.2614 1250
40	0.4106 4575	0.3724 3062	0.3378 5222	0.3065 5684	0.2525 7247
41	0.4016 0954	0.3633 4695	0.3288 0995	0.2976 2800	0.2440 3137
42	0.3927 7216	0.3544 8483	0.3200 0968	0.2889 5922	0.2357 7910
43	0.3841 2925	0.3458 3886	0.3114 4495	0.2805 4294	0.2278 0590
44	0.3756 7653	0.3374 0376	0.3031 0944	0.2723 7178	0.2201 0231
45	0.3674 0981	0.3291 7440	0.2949 9702	0.2644 3862	0.2126 5924
46	0.3593 2500	0.3211 4576	0.2871 0172	0.2567 3653	0.2054 6787
47	0.3514 1809	0.3133 1294	0.2794 1773	0.2492 5876	0.1985 1968
48	0.3436 8518	0.3056 7116	0.2719 3940	0.2419 9880	0.1918 0645
49	0.3361 2242	0.2982 1576	0.2646 6122	0.2349 5029	0.1853 2024
50	0.3287 2608	0.2909 4221	0.2575 7783	0.2281 0708	0.1790 5337

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	0.3214 9250	0.2838 4606	0.2506 8402	0.2214 6318	0.1729 9843
52	0.3144 1810	0.2769 2298	0.2439 7471	0.2150 1280	0.1671 4824
53	0.3074 9936	0.2701 6876	0.2374 4497	0.2087 5029	0.1614 9589
54	0.3007 3287	0.2635 7928	0.2310 9000	0.2026 7019	0.1560 3467
55	0.2941 1528	0.2571 5052	0.2249 0511	0.1967 6717	0.1507 5814
56	0.2876 4330	0.2508 7855	0.2188 8575	0.1910 3609	0.1456 6004
57	0.2813 1374	0.2447 5956	0.2130 2749	0.1854 7193	0.1407 3433
58	0.2751 2347	0.2387 8982	0.2073 2603	0.1800 6984	0.1359 7520
59	0.2690 6940	0.2329 6568	0.2017 7716	0.1748 2508	0.1313 7701
60	0.2631 4856	0.2272 8359	0.1963 7679	0.1697 3309	0.1269 3431
61	0.2573 5801	0.2217 4009	0.1911 2097	0.1647 8941	0.1226 4184
62	0.2516 9487	0.2163 3179	0.1860 0581	0.1599 8972	0.1184 9453
63	0.2461 5635	0.2110 5541	0.1810 2755	0.1553 2982	0.1144 8747
64	0.2407 3971	0.2059 0771	0.1761 8253	0.1508 0565	0.1106 1591
65	0.2354 4226	0.2008 8557	0.1714 6718	0.1464 1325	0.1068 7528
66	0.2302 6138	0.1959 8593	0.1668 7804	0.1421 4879	0.1032 6114
67	0.2251 9450	0.1912 0578	0.1624 1172	0.1380 0853	0.0997 6922
68	0.2202 3912	0.1865 4223	0.1580 6493	0.1339 8887	0.0963 9538
69	0.2153 9278	0.1819 9241	0.1538 3448	0.1300 8628	0.0931 3563
70	0.2106 5309	0.1775 5358	0.1497 1726	0.1262 9736	0.0899 8612
71	0.2060 1769	0.1732 2300	0.1457 1023	0.1226 1880	0.0869 4311
72	0.2014 8429	0.1689 9805	0.1418 1044	0.1190 4737	0.0840 0300
73	0.1970 5065	0.1648 7615	0.1380 1503	0.1155 7998	0.0811 6232
74	0.1927 1458	0.1608 5478	0.1343 2119	0.1122 1357	0.0784 1770
75	0.1884 7391	0.1569 3149	0.1307 2622	0.1089 4521	0.0757 6590
76	0.1843 2657	0.1531 0389	0.1272 2747	0.1057 7205	0.0732 0376
77	0.1802 7048	0.1493 6965	0.1238 2235	0.1026 9131	0.0707 2827
78	0.1763 0365	0.1457 2649	0.1205 0837	0.0997 0030	0.0683 3650
79	0.1724 2411	0.1421 7218	0.1172 8309	0.0967 9641	0.0660 2560
80	0.1686 2993	0.1387 0457	0.1141 4412	0.0939 7710	0.0637 9285
81	0.1649 1925	0.1353 2153	0.1110 8917	0.0912 3990	0.0616 3561
82	0.1612 9022	0.1320 2101	0.1081 1598	0.0885 8243	0.0595 5131
83	0.1577 4105	0.1288 0098	0.1052 2237	0.0860 0236	0.0575 3750
84	0.1542 6997	0.1256 5949	0.1024 0620	0.0834 9743	0.0555 9178
85	0.1508 7528	0.1225 9463	0.0996 6540	0.0810 6547	0.0537 1187
86	0.1475 5528	0.1196 0452	0.0969 9795	0.0787 0434	0.0518 9553
87	0.1443 0835	0.1166 8733	0.0944 0190	0.0764 1198	0.0501 4060
88	0.1411 3286	0.1138 4130	0.0918 7533	0.0741 8639	0.0484 4503
89	0.1380 2724	0.1110 6468	0.0894 1638	0.0720 2562	0.0468 0679
90	0.1349 8997	0.1083 5579	0.0870 2324	0.0699 2779	0.0452 2395
91	0.1320 1953	0.1057 1296	0.0846 9415	0.0678 9105	0.0436 9464
92	0.1291 1445	0.1031 3460	0.0824 2740	0.0659 1364	0.0422 1704
93	0.1262 7331	0.1006 1912	0.0802 2131	0.0639 9383	0.0407 8941
94	0.1234 9468	0.0981 6500	0.0780 7427	0.0621 2993	0.0394 1006
95	0.1207 7719	0.0957 7073	0.0759 8469	0.0603 2032	0.0380 7735
96	0.1181 1950	0.0934 3486	0.0739 5104	0.0585 6342	0.0367 8971
97	0.1155 2029	0.0911 5596	0.0719 7181	0.0568 5769	0.0355 4562
98	0.1129 7828	0.0889 3264	0.0700 4556	0.0552 0164	0.0343 4359
99	0.1104 9221	0.0867 6355	0.0681 7086	0.0535 9383	0.0331 8221
100	0.1080 6084	0.0846 4737	0.0663 4634	0.0520 3284	0.0320 6011

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730	0.9433 9623
2	0.9245 5621	0.9157 2995	0.9070 2948	0.8984 5242	0.8899 9644
3	0.8889 9636	0.8762 9660	0.8638 3760	0.8516 1366	0.8396 1928
4	0.8548 0419	0.8385 6134	0.8227 0247	0.8072 1674	0.7920 9366
5	0.8219 2711	0.8024 5105	0.7835 2617	0.7651 3435	0.7472 5817
6	0.7903 1453	0.7678 9574	0.7462 1540	0.7252 4583	0.7049 6054
7	0.7599 1781	0.7348 2846	0.7106 8133	0.6874 3681	0.6650 5711
8	0.7306 9021	0.7031 8513	0.6768 3936	0.6515 9887	0.6274 1237
9	0.7025 8674	0.6729 0443	0.6446 0892	0.6176 2926	0.5918 9846
10	0.6755 6417	0.6439 2768	0.6139 1325	0.5854 3058	0.5583 9478
11	0.6495 8093	0.6161 9874	0.5846 7929	0.5549 1050	0.5267 8753
12	0.6245 9705	0.5896 6386	0.5568 3742	0.5259 8152	0.4969 6936
13	0.6005 7409	0.5642 7164	0.5303 2135	0.4985 6068	0.4688 3902
14	0.5774 7508	0.5399 7286	0.5050 6795	0.4725 6937	0.4423 0096
15	0.5552 6450	0.5167 2044	0.4810 1710	0.4479 3305	0.4172 6506
16	0.5339 0818	0.4944 6932	0.4581 1152	0.4245 8109	0.3936 4628
17	0.5133 7325	0.4731 7639	0.4362 9669	0.4024 4653	0.3713 6442
18	0.4936 2812	0.4528 0037	0.4155 2065	0.3814 6590	0.3503 4379
19	0.4746 4242	0.4333 0179	0.3957 3396	0.3615 7906	0.3305 1301
20	0.4563 8695	0.4146 4286	0.3768 8948	0.3427 2896	0.3118 0473
21	0.4388 3360	0.3967 8743	0.3589 4236	0.3248 6158	0.2941 5540
22	0.4219 5539	0.3797 0089	0.3418 4987	0.3079 2567	0.2775 0510
23	0.4057 2633	0.3633 5013	0.3255 7131	0.2918 7267	0.2617 9726
24	0.3901 2147	0.3477 0347	0.3100 6791	0.2766 5656	0.2469 7855
25	0.3751 1680	0.3327 3060	0.2953 0277	0.2622 3370	0.2329 9863
26	0.3606 8923	0.3184 0248	0.2812 4073	0.2485 6275	0.2198 1003
27	0.3468 1657	0.3046 9137	0.2678 4832	0.2356 0450	0.2073 6795
28	0.3334 7747	0.2915 7069	0.2550 9364	0.2233 2181	0.1956 3014
29	0.3206 5141	0.2790 1502	0.2429 4632	0.2116 7944	0.1845 5674
30	0.3083 1867	0.2670 0002	0.2313 7745	0.2006 4402	0.1741 1013
31	0.2964 6026	0.2555 0241	0.2203 5947	0.1901 8390	0.1642 5484
32	0.2850 5794	0.2444 9991	0.2098 6617	0.1802 6910	0.1549 5740
33	0.2740 9417	0.2339 7121	0.1998 7254	0.1708 7119	0.1461 8622
34	0.2635 5209	0.2238 9589	0.1903 5480	0.1619 6321	0.1379 1153
35	0.2534 1547	0.2142 5444	0.1812 9029	0.1535 1963	0.1301 0522
36	0.2436 6872	0.2050 2817	0.1726 5741	0.1455 1624	0.1227 4077
37	0.2342 9685	0.1961 9921	0.1644 3563	0.1379 3008	0.1157 9318
38	0.2252 8543	0.1877 5044	0.1566 0536	0.1307 3941	0.1092 3885
39	0.2166 2061	0.1796 6549	0.1491 4797	0.1239 2362	0.1030 5552
40	0.2082 8904	0.1719 2870	0.1420 4568	0.1174 6314	0.0972 2219
41	0.2002 7793	0.1645 2507	0.1352 8160	0.1113 3947	0.0917 1905
42	0.1925 7493	0.1574 4026	0.1288 3962	0.1055 3504	0.0865 2740
43	0.1851 6820	0.1506 6054	0.1227 0440	0.1000 3322	0.0816 2962
44	0.1780 4635	0.1441 7276	0.1168 6133	0.0948 1822	0.0770 0908
45	0.1711 9841	0.1379 6437	0.1112 9651	0.0898 7509	0.0726 5007
46	0.1646 1386	0.1320 2332	0.1059 9668	0.0851 8965	0.0685 3781
47	0.1582 8256	0.1263 3810	0.1009 4921	0.0807 4849	0.0646 5831
48	0.1521 9476	0.1208 9771	0.0961 4211	0.0765 3885	0.0609 9840
49	0.1463 4112	0.1156 9158	0.0915 6391	0.0725 4867	0.0575 4566
50	0.1407 1262	0.1107 0965	0.0872 0373	0.0687 6652	0.0542 8836

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	4%	4½%	5%	5½%	6%
51	0.1353 0059	0.1059 4225	0.0830 5117	0.0651 8153	0.0512 1544
52	0.1300 9672	0.1013 8014	0.0790 9635	0.0617 8344	0.0483 1645
53	0.1250 9300	0.0970 1449	0.0753 2986	0.0585 6250	0.0455 8156
54	0.1202 8173	0.0928 3683	0.0717 4272	0.0555 0948	0.0430 0147
55	0.1156 5551	0.0888 3907	0.0683 2640	0.0526 1562	0.0405 6742
56	0.1112 0722	0.0850 1347	0.0650 7276	0.0498 7263	0.0382 7115
57	0.1069 3002	0.0813 5260	0.0619 7406	0.0472 7263	0.0361 0486
58	0.1028 1733	0.0778 4938	0.0590 2291	0.0448 0818	0.0340 6119
59	0.0988 6282	0.0744 9701	0.0562 1230	0.0424 7221	0.0321 3320
60	0.0950 6040	0.0712 8901	0.0535 3552	0.0402 5802	0.0303 1434
61	0.0914 0423	0.0682 1915	0.0509 8621	0.0381 5926	0.0285 9843
62	0.0878 8868	0.0652 8148	0.0485 5830	0.0361 6992	0.0269 7965
63	0.0845 0835	0.0624 7032	0.0462 4600	0.0342 8428	0.0254 5250
64	0.0812 5803	0.0597 8021	0.0440 4381	0.0324 9695	0.0240 1179
65	0.0781 3272	0.0572 0594	0.0419 4648	0.0308 0279	0.0226 5264
66	0.0751 2762	0.0547 4253	0.0399 4903	0.0291 9696	0.0213 7041
67	0.0722 3809	0.0523 8519	0.0380 4670	0.0276 7485	0.0201 6077
68	0.0694 5970	0.0501 2937	0.0362 3495	0.0262 3208	0.0190 1959
69	0.0667 8818	0.0479 7069	0.0345 0948	0.0248 6453	0.0179 4301
70	0.0642 1940	0.0459 0497	0.0328 6617	0.0235 6828	0.0169 2737
71	0.0617 4942	0.0439 2820	0.0313 0111	0.0223 3960	0.0159 6921
72	0.0593 7445	0.0420 3655	0.0298 1058	0.0211 7498	0.0150 6530
73	0.0570 9081	0.0402 2637	0.0283 9103	0.0200 7107	0.0142 1254
74	0.0548 9501	0.0384 9413	0.0270 3908	0.0190 2471	0.0134 0806
75	0.0527 8367	0.0368 3649	0.0257 5150	0.0180 3290	0.0126 4911
76	0.0507 5353	0.0352 5023	0.0245 2524	0.0170 9279	0.0119 3313
77	0.0488 0147	0.0337 3228	0.0233 5737	0.0162 0170	0.0112 5767
78	0.0469 2449	0.0322 7969	0.0222 4512	0.0153 5706	0.0106 2044
79	0.0451 1970	0.0308 8965	0.0211 8582	0.0145 5646	0.0100 1928
80	0.0433 8433	0.0295 5948	0.0201 7698	0.0137 9759	0.0094 5215
81	0.0417 1570	0.0282 8658	0.0192 1617	0.0130 7828	0.0089 1713
82	0.0401 1125	0.0270 6850	0.0183 0111	0.0123 9648	0.0084 1238
83	0.0385 6851	0.0259 0287	0.0174 2963	0.0117 5022	0.0079 3621
84	0.0370 8510	0.0247 8744	0.0165 9965	0.0111 3765	0.0074 8699
85	0.0356 5875	0.0237 2003	0.0158 0919	0.0105 5701	0.0070 6320
86	0.0342 8726	0.0226 9860	0.0150 5637	0.0100 0664	0.0066 6340
87	0.0329 6852	0.0217 2115	0.0143 3940	0.0094 8497	0.0062 8622
88	0.0317 0050	0.0207 8579	0.0136 5657	0.0089 9049	0.0059 3040
89	0.0304 8125	0.0198 9070	0.0130 0626	0.0085 2180	0.0055 9472
90	0.0293 0890	0.0190 3417	0.0123 8691	0.0080 7753	0.0052 7803
91	0.0281 8163	0.0182 1451	0.0117 9706	0.0076 5643	0.0049 7928
92	0.0270 9772	0.0174 3016	0.0112 3530	0.0072 5728	0.0046 9743
93	0.0260 5550	0.0166 7958	0.0107 0028	0.0068 7894	0.0044 3154
94	0.0250 5337	0.0159 6132	0.0101 9074	0.0065 2032	0.0041 8070
95	0.0240 8978	0.0152 7399	0.0097 0547	0.0061 8040	0.0039 4405
96	0.0231 6325	0.0146 1626	0.0092 4331	0.0058 5820	0.0037 2081
97	0.0222 7235	0.0139 8685	0.0088 0315	0.0055 5279	0.0035 1019
98	0.0214 1572	0.0133 8454	0.0083 8395	0.0052 6331	0.0033 1150
99	0.0205 9204	0.0128 0817	0.0079 8471	0.0049 8892	0.0031 2406
100	0.0198 0004	0.0122 5663	0.0076 0449	0.0047 2883	0.0029 4723

Table IV.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593	0.9216 5899
2	0.8816 5928	0.8734 3873	0.8653 3261	0.8573 3882	0.8494 5529
3	0.8278 4909	0.8162 9788	0.8049 6057	0.7938 3224	0.7829 0810
4	0.7773 2309	0.7628 9521	0.7488 0053	0.7350 2985	0.7215 7428
5	0.7298 8084	0.7129 8618	0.6965 5863	0.6805 8320	0.6650 4542
6	0.6853 3412	0.6663 4222	0.6479 6152	0.6301 6963	0.6129 4509
7	0.6435 0621	0.6227 4974	0.6027 5490	0.5834 9040	0.5649 2635
8	0.6042 3119	0.5820 0910	0.5607 0223	0.5402 6888	0.5206 6945
9	0.5673 5323	0.5439 3374	0.5215 8347	0.5002 4897	0.4798 7968
10	0.5327 2604	0.5083 4929	0.4851 9393	0.4631 9349	0.4422 8542
11	0.5002 1224	0.4750 9280	0.4513 4319	0.4288 8286	0.4076 3633
12	0.4696 8285	0.4440 1196	0.4198 5413	0.3971 1376	0.3757 0168
13	0.4410 1676	0.4149 6445	0.3905 6198	0.3676 9792	0.3462 6883
14	0.4141 0025	0.3878 1724	0.3633 1347	0.3404 6104	0.3191 4178
15	0.3888 2652	0.3624 4602	0.3379 6602	0.3152 4170	0.2941 3989
16	0.3650 9533	0.3387 3460	0.3143 8699	0.2918 9047	0.2710 9667
17	0.3428 1251	0.3165 7439	0.2924 5302	0.2702 6895	0.2498 5869
18	0.3218 8969	0.2958 6392	0.2720 4932	0.2502 4903	0.2302 8450
19	0.3022 4384	0.2765 0832	0.2530 6913	0.2317 1206	0.2122 4378
20	0.2837 9703	0.2584 1900	0.2354 1315	0.2145 4821	0.1956 1639
21	0.2664 7608	0.2415 1309	0.2189 8897	0.1986 5575	0.1802 9160
22	0.2502 1228	0.2257 1317	0.2037 1067	0.1839 4051	0.1661 6738
23	0.2349 4111	0.2109 4688	0.1894 9830	0.1703 1528	0.1531 4965
24	0.2206 0198	0.1971 4662	0.1762 7749	0.1576 9934	0.1411 5176
25	0.2071 3801	0.1842 4918	0.1639 7906	0.1460 1790	0.1300 9378
26	0.1944 9579	0.1721 9549	0.1525 3866	0.1352 0176	0.1199 0210
27	0.1826 2515	0.1609 3037	0.1418 9643	0.1251 8682	0.1105 0885
28	0.1714 7902	0.1504 0221	0.1319 9668	0.1159 1372	0.1018 5148
29	0.1610 1316	0.1405 6282	0.1227 8761	0.1073 2752	0.0938 7233
30	0.1511 8607	0.1313 6712	0.1142 2103	0.0993 7733	0.0865 1828
31	0.1419 5875	0.1227 7301	0.1062 5212	0.0920 1605	0.0797 4035
32	0.1332 9460	0.1147 4113	0.0988 3918	0.0852 0005	0.0734 9341
33	0.1251 5925	0.1072 3470	0.0919 4343	0.0788 8893	0.0677 3586
34	0.1175 2042	0.1002 1934	0.0855 2877	0.0730 4531	0.0624 2936
35	0.1103 4781	0.0936 6294	0.0795 6164	0.0676 3454	0.0575 3858
36	0.1036 1297	0.0875 3546	0.0740 1083	0 0626 2458	0.0530 3095
37	0.0972 8917	0.0818 0884	0.0688 4729	0.0579 8572	0.0488 7645
38	0.0913 5134	0.0764 5686	0.0640 4399	0 0536 9048	0.0450 4742
39	0.0857 7590	0.0714 5501	0.0595 7580	0.0497 1341	0.0415 1836
40	0.0805 4075	0.0667 8038	0.0554 1935	0.0460 3093	0.0382 6577
41	0.0756 2512	0.0624 1157	0.0515 5288	0.0426 2123	0.0352 6799
42	0.0710 0950	0.0583 2857	0.0479 5617	0.0394 6411	0.0325 0506
43	0.0666 7559	0.0545 1268	0.0446 1039	0.0365 4084	0.0299 5858
44	0.0626 0619	0.0509 4643	0.0414 9804	0.0338 3411	0.0276 1160
45	0.0587 8515	0.0476 1349	0.0386 0283	0.0313 2788	0.0254 4848
46	0.0551 9733	0.0444 9859	0.0359 0961	0.0290 0730	0.0234 5482
47	0.0518 2848	0.0415 8747	0.0334 0428	0.0268 5861	0.0216 1734
48	0.0486 6524	0.0388 6679	0.0310 7375	0.0248 6908	0.0199 2382
49	0.0456 9506	0.0363 2410	0.0289 0582	0.0230 2693	0.0183 6297
50	0.0429 0616	0.0339 4776	0.0268 8913	0.0213 2123	0.0169 2439

TABLE V.—AMOUNT OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0041 6667	2.0050 0000	2.0058 3333	2.0075 0000	2.0100 0000
3	3.0125 1736	3.0150 2500	3.0175 3403	3.0225 5625	3.0301 0000
4	4.0250 6952	4.0301 0013	4.0351 3631	4.0452 2542	4.0604 0100
5	5.0418 4064	5.0502 5063	5.0586 7460	5.0755 6461	5.1010 0501
6	6.0628 4831	6.0755 0188	6.0881 8354	6.1136 3135	6.1520 1506
7	7.0881 1018	7.1058 7939	7.1236 9794	7.1594 8358	7.2135 3521
8	8.1176 4397	8.1414 0879	8.1652 5284	8.2131 7971	8.2856 7056
9	9.1514 6749	9.1821 1583	9.2128 8349	9.2747 7856	9.3685 2727
10	10.1895 9860	10.2280 2641	10.2666 2531	10.3443 3940	10.4622 1254
11	11.2320 5526	11.2791 6654	11.3265 1396	11.4219 2194	11.5668 3467
12	12.2788 5549	12.3355 6237	12.3925 8529	12.5075 8636	12.6825 0301
13	13.3300 1739	13.3972 4018	13.4648 7537	13.6013 9325	13.8093 2804
14	14.3855 5913	14.4642 2639	14.5434 2048	14.7034 0370	14.9474 2132
15	15.4454 9896	15.5365 4752	15.6282 5710	15.8136 7923	16.0968 9354
16	16.5098 5520	16.6142 3026	16.7194 2193	16.9322 8183	17.2578 6449
17	17.5786 4627	17.6973 0141	17.8169 5189	18.0592 7394	18.4304 4314
18	18.6518 9063	18.7857 8791	18.9208 8411	19.1947 1849	19.6147 4757
19	19.7296 0684	19.8797 1685	20.0312 5593	20.3386 7888	20.8108 9504
20	20.8118 1353	20.9791 1544	21.1481 0493	21.4912 1897	22.0190 0399
21	21.8985 2942	22.0840 1101	22.2714 6887	22.6524 0312	23.2391 9403
22	22.9897 7330	23.1944 3107	23.4013 8577	23.8222 9611	24.4715 8598
23	24.0855 6402	24.3104 0322	24.5378 9386	25.0009 6336	25.7163 0183
24	25.1859 2054	25.4319 5524	25.6810 3157	26.1884 7059	26.9734 6485
25	26.2908 6187	26.5591 1502	26.8308 3759	27.3848 8412	28.2431 9950
26	27.4004 0713	27.6919 1059	27.9873 5081	28.5902 7075	29.5256 3150
27	28.5145 7549	28.8303 7015	29.1506 1035	29.8046 9778	30.8208 8781
28	29.6333 8622	29.9745 2200	30.3206 5558	31.0282 3301	32.1290 9669
29	30.7568 5867	31.1243 9461	31.4975 2607	32.2609 4476	33.4503 8766
30	31.8850 1224	32.2800 1658	32.6812 6164	33.5029 0184	34.7848 9153
31	33.0178 6646	33.4414 1666	33.8719 0233	34.7541 7361	36.1327 4045
32	34.1554 4090	34.6086 2375	35.0694 8843	36.0148 2991	37.4940 6785
33	35.2977 5524	35.7816 6686	36.2740 6045	37.2849 4113	38.8690 0853
34	36.4448 2922	36.9605 7520	37.4856 5913	38.5645 7819	40.2576 9862
35	37.5966 8268	38.1453 7807	38.7043 2548	39.8538 1253	41.6602 7560
36	38.7533 3552	39.3361 0496	39.9301 0071	41.1527 1612	43.0768 7836
37	39.9148 0775	40.5327 8549	41.1630 2630	42.4613 6149	44.5076 4714
38	41.0811 1945	41.7354 4942	42.4031 4395	43.7798 2170	45.9527 2361
39	42.2522 9078	42.9441 2666	43.6504 9562	45.1081 7037	47.4122 5085
40	43.4283 4199	44.1588 4730	44.9051 2352	46.4464 8164	48.8863 7336
41	44.6092 9342	45.3796 4153	46.1670 7007	47.7948 3026	50.3752 3709
42	45.7951 6548	46.6065 3974	47.4363 7798	49.1532 9148	51.8789 8946
43	46.9859 7866	47.8395 7244	48.7130 9018	50.5219 4117	53.3977 7936
44	48.1817 5358	49.0787 7030	49.9972 4988	51.9008 5573	54.9317 5715
45	49.3825 1088	50.3241 6415	51.2889 0050	53.2901 1215	56.4810 7472
46	50.5882 7134	51.5757 8497	52.5880 8575	54.6897 8799	58.0458 8547
47	51.7990 5581	52.8336 6390	53.8948 4959	56.0999 6140	59.6263 4432
48	53.0148 8521	54.0978 3222	55.2092 3621	57.5207 1111	61.2226 0777
49	54.2357 8056	55.3683 2138	56.5312 9009	58.9521 1644	62.8348 3385
50	55.4617 6298	56.6451 6299	57.8610 5595	60.3942 5732	64.4631 8218

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0041 6667	2.0050 0000	2.0058 3333	2.0075 0000	2.0100 0000
3	3.0125 1736	3.0150 2500	3.0175 4167	3.0225 5625	3.0301 0000
4	4.0252 0868	4.0301 0013	4.0350 0174	4.0451 1266	4.0604 0100
5	5.0423 0955	5.0503 7563	5.0584 4523	5.0751 2516	5.1010 0401
6	6.0639 8204	6.0759 3751	6.0879 1284	6.1126 6329	6.1520 1005
7	7.0903 0116	7.1067 8220	7.1233 9284	7.1578 1172	7.2135 3015
8	8.1213 3491	8.1430 1611	8.1648 1284	8.2106 1781	8.2856 5545
9	9.1571 5433	9.1846 5119	9.2122 0284	9.2711 4294	9.3685 1201
10	10.1978 3350	10.2317 4444	10.2656 8284	10.3395 0226	10.4622 1713
11	11.2434 4956	11.2843 3066	11.3252 4284	11.4157 1477	11.5668 3930
12	12.2940 8268	12.3424 0231	12.3908 9284	12.4999 0364	12.6825 0769
13	13.3497 1612	13.4060 7132	13.4626 3284	13.5921 1615	13.8093 3277
14	14.4104 3624	14.4753 3467	14.5404 1284	14.6924 2153	14.9474 2610
15	15.4762 3253	15.5502 0134	15.6243 4284	15.8008 1662	16.0968 9036
16	16.5471 9763	16.6307 0234	16.7144 2284	16.9173 0421	17.2578 5926
17	17.6233 2737	17.7169 1085	17.8107 5284	18.0419 0298	18.4304 3785
18	18.7047 1979	18.8079 4241	18.9134 3284	19.1747 2849	19.6147 4223
19	19.7914 7614	19.9041 8612	20.0225 6284	20.3148 0412	20.8108 8965
20	20.8836 0091	21.0067 1555	21.1381 4284	21.4623 2310	22.0190 0855
21	21.9811 0183	22.1167 2913	22.2602 7284	22.6171 7382	23.2391 9864
22	23.0840 8990	23.2342 4077	23.3889 5284	23.7794 4394	24.4715 9063
23	24.1925 7943	24.3593 5997	24.5242 8284	24.9492 0832	25.7163 0654
24	25.3065 8804	25.4921 0177	25.6663 6284	26.1264 5838	26.9734 6961
25	26.4261 3669	26.6325 8728	26.8152 0284	27.3112 9212	28.2431 9431
26	27.5512 4966	27.7809 3422	27.9708 1284	28.5037 7466	29.5256 2625
27	28.6819 5457	28.9372 5689	29.1332 0284	29.7038 7828	30.8208 8251
28	29.8183 0241	30.1016 7692	30.3024 8284	30.9116 7247	32.1290 9133
29	30.9603 2757	31.2742 0330	31.4787 6284	32.1272 2384	33.4503 8224
30	32.1080 6787	32.4548 5232	32.6621 6284	33.3505 9616	34.7848 8606
31	33.2615 6456	33.6437 4759	33.8527 0284	34.5818 4037	36.1327 3492
32	34.4208 5234	34.8409 0008	35.0504 0284	35.8210 2457	37.4940 6227
33	35.5859 7939	36.0463 2808	36.2553 8284	37.0682 1405	38.8690 0289
34	36.7569 8739	37.2600 4722	37.4676 6284	38.3235 7127	40.2576 9292
35	37.9338 2154	38.4821 7046	38.6873 6284	39.5871 5587	41.6602 6985
36	39.1165 3054	39.7127 1811	39.9145 0284	40.8590 3468	43.0768 7255
37	40.3051 5665	40.9517 0770	41.1491 0284	42.1392 7170	44.5076 4128
38	41.4997 4574	42.1992 5424	42.3911 8284	43.4279 3811	45.9527 1769
39	42.7003 4730	43.4554 7051	43.6408 6284	44.7250 9228	47.4122 4486
40	43.9069 1450	44.7203 7686	44.8981 6284	46.0307 9981	48.8863 6731
41	45.1195 0419	45.9940 9174	46.1631 0284	47.3451 2356	50.3752 3098
42	46.3381 6693	47.2766 3120	47.4357 0284	48.6681 2367	51.8789 8329
43	47.5629 6703	48.5681 0936	48.7160 0284	49.9998 6769	53.3977 7312
44	48.7939 6254	49.8686 4791	50.0040 2284	51.3404 2047	54.9317 5085
45	50.0312 1537	51.1783 6615	51.3000 0284	52.6898 4427	56.4810 6836
46	51.2747 8134	52.4973 8148	52.6038 6284	54.0482 0875	58.0458 7904
47	52.5247 2030	53.8258 1904	53.9156 3284	55.4155 8104	59.6263 3783
48	53.7810 9617	55.1637 0214	55.2353 3284	56.7920 2566	61.2225 9121
49	55.0439 9700	56.5111 5165	56.5630 8284	58.1776 1454	62.8348 1712
50	56.3134 8503	57.8682 8691	57.8988 0284	59.5724 1706	64.4631 6529
51	56.6928 5366	57.9283 8880	59.1985 7877	61.8472 1424	66.1078 1401
52	57.9290 7388	59.2180 3075	60.5439 0381	63.3110 6835	67.7688 9215
53	59.1704 4503	60.5141 2090	61.8970 7659	64.7859 0136	69.4465 8107
54	60.4169 8855	61.8166 9150	63.2581 4287	66.2717 9562	71.1410 4688
55	61.6687 2600	63.1257 7496	64.6271 4870	67.7688 3409	72.8524 5735
56	62.9256 7902	64.4414 0384	66.0041 4040	69.2771 0035	74.5809 8192
57	64.1878 6935	65.7636 1086	67.3891 6455	70.7966 7860	76.3267 9174
58	65.4553 1881	67.0924 2891	68.7822 6801	72.3276 5369	78.0900 5966
59	66.7280 4930	68.4278 9105	70.1834 9791	73.8701 1109	79.8709 6025
60	68.0060 8284	69.7700 3051	71.5929 0165	75.4241 3693	81.6696 6986
61	69.2894 4152	71.1188 8066	73.0105 2691	76.9898 1795	83.4863 6655
62	70.5781 4753	72.4744 7507	74.4364 2165	78.5672 4159	85.3212 3022
63	71.8722 2314	73.8368 4744	75.8706 3411	80.1564 9590	87.1744 4252
64	73.1716 9074	75.2060 3168	77.3132 1281	81.7576 6962	89.0461 8695
65	74.4765 7278	76.5820 6184	78.7642 0655	83.3708 5214	90.9366 4882
66	75.7868 9184	77.9649 7215	80.2236 6442	84.9961 3353	92.8460 1531
67	77.1026 7055	79.3547 9701	81.6916 3579	86.6336 0453	94.7744 7546
68	78.4239 3168	80.7515 7099	83.1681 7034	88.2833 5657	96.7222 2021
69	79.7506 9806	82.1553 2885	84.6533 1800	89.9454 8174	98.6894 4242
70	81.0829 9264	83.5661 0549	86.1471 2902	91.6200 7285	100.6763 3684
71	82.4208 3844	84.9839 3602	87.6496 5394	93.3072 2340	102.6831 0021
72	83.7642 5860	86.4088 5570	89.1609 4359	95.0070 2758	104.7099 3121
73	85.1132 7634	87.8408 9998	90.6810 4909	96.7195 8028	106.7570 3052
74	86.4679 1500	89.2801 0418	92.2100 2188	98.4449 7714	108.8246 0083
75	87.8281 9797	90.7265 0500	93.7479 1367	100.1833 1446	110.9128 4684
76	89.1941 4880	92.1801 3752	95.2947 7650	101.9346 8932	113.0219 7530
77	90.5657 9109	93.6410 3821	96.8506 6270	103.6991 9949	115.1521 9506
78	91.9431 4855	95.1092 4340	98.4156 2490	105.4769 4349	117.3037 1701
79	93.3262 4500	96.5847 8962	99.9897 1604	107.2680 2056	119.4767 5418
80	94.7151 0436	98.0677 1357	101.5729 8938	109.0725 3072	121.6715 2172
81	96.1097 5062	99.5580 5214	103.1654 9849	110.8905 7470	123.8882 3694
82	97.5102 0792	101.0558 4240	104.7672 9723	112.7222 5401	126.1271 1931
83	98.9165 0045	102.5611 2161	106.3784 3980	114.5676 7091	128.3883 9050
84	100.3286 5254	104.0739 2722	107.9989 8070	116.4269 2845	130.6722 7440
85	101.7466 8859	105.5942 9685	109.6289 7475	118.3001 3041	132.9789 9715
86	103.1706 3312	107.1222 6834	111.2684 7710	120.1873 8139	135.3087 8712
87	104.6005 1076	108.6578 7968	112.9175 4322	122.0887 8675	137.6618 7499
88	106.0363 4622	110.2011 6908	114.5762 2889	124.0044 5265	140.0384 9374
89	107.4781 6433	111.7521 7492	116.2445 9022	125.9344 8604	142.4388 7868
90	108.9259 9002	113.3109 3580	117.9226 8367	127.8789 9469	144.8632 6746
91	110.3798 4831	114.8774 9048	119.6105 6599	129.8380 8715	147.3119 0014
92	111.8397 6434	116.4518 7793	121.3082 9429	131.8118 7280	149.7850 1914
93	113.3057 6336	118.0341 3732	123.0159 2601	133.8004 6185	152.2828 6933
94	114.7778 7071	119.6243 0800	124.7335 1891	135.8039 6531	154.8056 9803
95	116.2561 1184	121.2224 2954	126.4611 3110	137.8224 9505	157.3537 5501
96	117.7405 1230	122.8285 4169	128.1988 2103	139.8561 6377	159.9272 9256
97	119.2310 9777	124.4426 8440	129.9466 4749	141.9050 8499	162.5265 6548
98	120.7278 9401	126.0648 9782	131.7046 6960	143.9693 7313	165.1518 3114
99	122.2309 2690	127.6952 2231	133.4729 4684	146.0491 4343	167.8033 4945
100	123.7402 2243	129.3336 9842	135.2515 3903	148.1445 1201	170.4813 8294

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	125.2558 0669	130.9803 6692	137.0405 0634	150.2555 9585	173.1861 9677
102	126.7777 0589	132.6352 6875	138.8399 0929	152.3825 1281	175.9180 5874
103	128.3059 4633	134.2984 4509	140.6498 0876	154.5253 8166	178.6772 3933
104	129.8405 5444	135.9699 3732	142.4702 6598	156.6843 2202	181.4640 1172
105	131.3815 5675	137.6497 8701	144.3013 4253	158.8594 5444	184.2786 5184
106	132.9289 7990	139.3380 3594	146.1431 0036	161.0509 0035	187.1214 3836
107	134.4828 5065	141.0347 2612	147.9956 0178	163.2587 8210	189.9926 5274
108	136.0431 9586	142.7398 9975	149.8589 0946	165.4832 2296	192.8925 7927
109	137.6100 4251	144.4535 9925	151.7330 8643	167.7243 4714	195.8215 0506
110	139.1834 1769	146.1758 6725	153.6181 9610	169.9822 7974	198.7797 2011
111	140.7633 4860	147.9067 4658	155.5143 0225	172.2571 4684	201.7675 1731
112	142.3498 6255	149.6462 8032	157.4214 6901	174.5490 7544	204.7851 9248
113	143.9429 8698	151.3945 1172	159.3397 6091	176.8581 9351	207.8330 4441
114	145.5427 4942	153.1514 8428	161.2692 4285	179.1846 2996	210.9113 7485
115	147.1491 7754	154.9172 4170	163.2099 8010	181.5285 1468	214.0204 8860
116	148.7622 9912	156.6918 2791	165.1620 3832	183.8899 7854	217.1606 9349
117	150.3821 4203	158.4752 8704	167.1254 8354	186.2691 5338	220.3323 0042
118	152.0087 3429	160.2676 6348	169.1003 8219	188.6661 7203	223.5356 2343
119	153.6421 0401	162.0690 0180	171.0868 0109	191.0811 6832	226.7709 7966
120	155.2822 7945	163.8793 4681	173.0848 0743	193.5142 7708	230.0386 8946
121	156.9292 8895	165.6987 4354	175.0944 6881	195.9656 3416	233.3390 7635
122	158.5831 6098	167.5272 3726	177.1158 5321	198.4353 7642	236.6724 6712
123	160.2439 2415	169.3648 7344	179.1490 2902	200.9236 4174	240.0391 9179
124	161.9116 0717	171.2116 9781	181.1940 6502	203.4305 6905	243.4395 8370
125	163.5862 3887	173.0677 5630	183.2510 3040	205.9562 9832	246.8739 7954
126	165.2678 4819	174.9330 9508	185.3199 9474	208.5009 7056	250.3427 1934
127	166.9564 6423	176.8077 6056	187.4010 2805	211.0647 2784	253.8461 4653
128	168.6521 1616	178.6917 9936	189.4942 0071	213.6477 1330	257.3846 0800
129	170.3548 3331	180.5852 5836	191.5995 8355	216.2500 7115	260.9584 5408
130	172.0646 4512	182.4881 8465	193.7172 4778	218.8719 4668	264.5680 3862
131	173.7815 8114	184.4006 2557	195.8472 6506	221.5134 8628	268.2137 1900
132	175.5056 7106	186.3226 2870	197.9897 0744	224.1748 3743	271.8958 5619
133	177.2369 4469	188.2542 4184	200.1446 4740	226.8561 4871	275.6148 1475
134	178.9754 3196	190.1955 1305	202.3121 5785	229.5575 6982	279.3709 6290
135	180.7211 6293	192.1464 9062	204.4923 1210	232.2792 5160	283.1646 7253
136	182.4741 6777	194.1072 2307	206.6851 8392	235.0213 4598	286.9963 1926
137	184.2344 7681	196.0777 5919	208.8908 4749	237.7840 0608	290.8662 8245
138	186.0021 2046	198.0581 4798	211.1093 7744	240.5673 8612	294.7749 4527
139	187.7771 2929	200.0484 3872	213.3408 4881	243.3716 4152	298.7226 9473
140	189.5595 3400	202.0486 8092	215.5853 3709	246.1969 2883	302.7099 2167
141	191.3493 6539	204.0589 2432	217.8429 1822	249.0434 0580	306.7370 2089
142	193.1466 5441	206.0792 1894	220.1136 6858	251.9112 3134	310.8043 9110
143	194.9514 3214	208.1096 1504	222.3976 6498	254.8005 6558	314.9124 3501
144	196.7637 2977	210.1501 6311	224.6949 8469	257.7115 6982	319.0615 5936
145	198.5835 7865	212.2009 1393	227.0057 0544	260.6444 0659	323.2521 7495
146	200.4110 1023	214.2619 1850	229.3299 0538	263.5992 3964	327.4846 9670
147	202.2460 5610	216.3332 2809	231.6676 6317	266.5762 3394	331.7595 4367
148	204.0887 4800	218.4148 9423	234.0190 5787	269.5755 5569	336.0771 3911
149	205.9391 1779	220.5069 6870	236.3841 6904	272.5973 7236	340.4379 1050
150	207.7971 9744	222.6095 0354	238.7630 7669	275.6418 5265	344.8422 8960

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0112 5000	2.0125 0000	2.0150 0000	2.0175 0000	2.0200 0000
3	3.0338 7656	3.0376 5625	3.0452 2500	3.0528 0625	3.0604 0000
4	4.0680 0767	4.0756 2695	4.0909 0338	4.1062 3036	4.1216 0800
5	5.1137 7276	5.1265 7229	5.1522 6693	5.1780 8938	5.2040 4016
6	6.1713 0270	6.1906 5444	6.2295 5093	6 2687 0596	6.3081 2096
7	7.2407 2986	7.2680 3762	7.3229 9419	7 3784 0831	7.4342 8338
8	8.3221 8807	8.3588 8809	8.4328 3911	8.5075 3045	8.5829 6905
9	9.4158 1269	9.4633 7420	9.5593 3169	9.6564 1224	9.7546 2843
10	10.5217 4058	10.5816 6637	10.7027 2167	10.8253 9045	10.9497 2100
11	11.6401 1016	11.7139 3720	11.8632 6249	12.0148 4394	12.1687 1542
12	12.7710 6140	12.8603 6142	13.0412 1143	13.2251 0371	13.4120 8973
13	13.9147 3584	14.0211 1594	14.2368 2960	14.4565 4303	14.6803 3152
14	15.0712 7662	15.1963 7988	15.4503 8205	15.7095 3253	15.9739 3815
15	16.2408 2848	16.3863 3463	16.6821 3778	16.9844 4935	17.2934 1692
16	17.4235 3780	17.5911 6382	17.9323 6984	18.2816 7721	18.6392 8525
17	18.6195 5260	18.8110 5336	19.2013 5539	19.6016 0656	20.0120 7096
18	19.8290 2257	20.0461 9153	20.4893 7572	20.9446 3468	21.4123 1238
19	21.0520 9907	21.2967 6893	21.7967 1636	22.3111 6578	22.8405 5863
20	22.2889 3519	22.5629 7854	23.1236 6710	23.7016 1119	24.2973 6980
21	23.5396 8571	23.8450 1577	24.4705 2211	25.1163 8938	25.7833 1719
22	24.8045 0717	25.1430 7847	25.8375 7994	26.5559 2620	27.2989 8354
23	26.0835 5788	26.4573 6695	27.2251 4364	28.0206 5490	28.8449 6321
24	27.3769 9790	27.7880 8403	28.6335 2080	29.5110 1637	30.4218 6247
25	28.6849 8913	29.1354 3508	30.0630 2361	31.0274 5915	32.0302 9972
26	30.0076 9526	30.4996 2802	31.5139 6896	32.5704 3969	33.6709 0572
27	31.3452 8183	31.8808 7337	32.9866 7850	34.1404 2238	35.3443 2383
28	32.6979 1625	33.2793 8429	34.4814 7867	35.7378 7977	37.0512 1031
29	34.0657 6781	34.6953 7659	35.9987 0085	37.3632 9267	38.7922 3451
30	35.4490 0769	36.1290 6880	37.5386 8137	39.0171 5029	40.5680 7921
31	36.8478 0903	37.5806 8216	39.1017 6159	40.6999 5042	42.3794 4079
32	38.2623 4688	39.0504 4069	40.6882 8801	42.4121 9955	44.2270 2961
33	39.6927 9829	40.5385 7120	42.2986 1233	44.1544 1305	46.1115 7020
34	41.1393 4227	42.0453 0334	43.9330 9152	45.9271 1527	48.0338 0160
35	42.6021 5987	43.5708 6963	45.5920 8789	47.7308 3979	49.9944 7763
36	44.0814 3417	45.1155 0550	47.2759 6921	49.5661 2949	51.9943 6719
37	45.5773 5030	46.6794 4932	48.9851 0874	51.4335 3675	54.0342 5453
38	47.0900 9549	48.2926 4243	50.7198 8538	53.3336 2365	56.1149 3962
39	48.6198 5906	49.8862 2921	52.4806 8366	55.2669 6206	58.2372 3841
40	50.1668 3248	51.4895 5708	54.2678 9391	57.2341 3390	60.4019 8318
41	51.7312 0934	53.1331 7654	56.0819 1232	59.2357 3124	62.6100 2284
42	53.3131 8545	54.7973 4125	57.9231 4100	61.2723 5654	64.8622 2330
43	54.9129 5879	56.4823 0801	59.7919 8812	63.3446 2278	67.1594 6777
44	56.5307 2957	58.1883 3687	61.6888 6794	65.4531 5367	69.5026 5712
45	58.1667 0028	59.9156 9108	63.6142 0096	67.5985 8386	71.8927 1027
46	59.8210 7566	61.6646 3721	65.5684 1398	69.7815 5908	74.3305 6447
47	61.4940 6276	63.4354 4518	67.5519 4018	72.0027 3637	76.8171 7576
48	63.1858 7097	65.2283 8824	69.5652 1929	74.2627 8425	79.3535 1927
49	64.8967 1201	67.0437 4310	71.6086 9758	76.5623 8298	81.9405 8966
50	66.6268 0002	68.8817 8989	73.6828 2804	78.9022 2468	84.5794 0145

TABLE V.—AMOUNT OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	68.3763 5152	70.7428 1226	75.7880 7046	81.2830 1361	87.2709 8948
52	70.1455 8548	72.6270 9741	77.9248 9152	83.7054 6635	90.0164 0027
53	71.9347 2332	74.5349 3613	80.0937 6489	86.1703 1201	92.8167 3746
54	73.7439 8895	76.4666 2283	82.2951 7136	88.6782 9247	95.6730 7221
55	75.5736 0883	78.4224 5562	84.5295 9893	91.2301 6259	98.5865 3365
56	77.4238 1193	80.4027 3631	86.7975 4292	93.8266 9043	101.5582 6432
57	79.2948 2981	82.4077 7052	89.0995 0606	96.4686 5752	104.5894 2961
58	81.1868 9665	84.4378 6765	91.4359 9865	99.1568 5902	107.6812 1820
59	83.1002 4923	86.4933 4099	93.8075 3863	101.8921 0405	110.8348 4257
60	85.0351 2704	88.5745 0776	96.2146 5171	104.6752 1588	114.0515 3942
61	86.9917 7222	90.6816 8910	98.6578 7149	107.5070 3215	117.3325 7021
62	88.9704 2966	92.8152 1022	101.1377 3956	110.3884 0522	120.6792 2161
63	90.9713 4699	94.9754 0034	103.6548 0565	113.3202 0231	124.0928 0604
64	91.9947 7464	97.1625 9285	106.2096 2774	116.3033 0585	127.5746 6216
65	95.0409 6586	99.3771 2526	108.8027 7215	119.3386 1370	131.1261 5541
66	97.1101 7672	101.6193 3933	111.4348 1374	122.4270 3944	134.7486 7852
67	99.2026 6621	103.8895 8107	114.1063 3594	125.5695 1263	138.4436 5209
68	101.3186 9621	106.1882 0083	116.8179 3098	128.7669 7910	142.2125 2513
69	103.4585 3154	108.5155 5334	119.5701 9995	132.0204 0124	146.0567 7563
70	105.6224 4002	110.8719 9776	122.3637 5295	135.3307 5826	149.9779 1114
71	107.8106 9247	113.2578 9773	125.1992 0924	138.6990 4653	153.9774 6937
72	110.0235 6276	115.6736 2145	128.0771 9738	142.1262 7984	158.0570 1875
73	112.2613 2784	118.1195 4172	130.9983 5534	145.6134 8974	162.2181 5913
74	114.5242 6778	120.5960 3599	133.9633 3067	149.1617 2581	166.4625 2231
75	116.8126 6579	123.1034 8644	136.9727 8063	152.7720 5601	170.7917 7276
76	119.1268 0828	125.6422 8002	140.0273 7234	156.4455 6699	175.2076 0821
77	121.4669 8487	128.2128 0852	143.1277 8292	160.1833 6441	179.7117 6038
78	123.8334 8845	130.8154 6863	146.2746 9967	163.9865 7329	184.3059 9558
79	126.2266 1520	133.4506 6199	149.4688 2016	167.8563 3832	188.9921 1549
80	128.6466 6462	136.1187 9526	152.7108 5247	171.7938 2424	193.7719 5780
81	131.0939 3960	138.8202 8020	156.0015 1525	175.8002 1617	198.6473 9696
82	133.5687 4642	141.5555 3370	159.3415 3798	179.8767 1995	203.6203 4490
83	136.0713 9481	144.3249 7787	162.7316 6105	184.0245 6255	208.6927 5180
84	138.6021 9801	147.1290 4010	166.1726 3597	188.2449 9239	213.8666 0683
85	141.1614 7273	149.9681 5310	169.6652 2551	192.5392 7976	219.1439 3897
86	143.7495 3930	152.8427 5501	173.2102 0389	196.9087 1716	224.5268 1775
87	146.3667 2162	155.7532 8945	176.8083 5695	201.3546 1971	230.0173 5411
88	149.0133 4724	158.7002 0557	180.4604 8230	205.8783 2555	235.6177 0119
89	151.6897 4739	161.6839 5814	184.1673 8954	210.4811 9625	241.3300 5521
90	154.3962 5705	164.7050 0762	187.9299 0038	215.1646 1718	247.1566 5632
91	157.1332 1494	167.7638 2021	191.7488 4889	219.9299 9798	253.0997 8944
92	159.9009 6361	170.8608 6796	195.6250 8162	224.7787 7295	259.1617 8523
93	162.6998 4945	173.9966 2881	199.5594 5784	229.7124 0148	265.3450 2094
94	165.5302 2276	177.1715 8667	203.5528 4971	234.7323 6850	271.6519 2135
95	168.3924 3776	180.3862 3151	207.6061 4246	239.8401 8495	278.0849 5978
96	171.2868 5269	183.6410 5940	211.7202 3459	245.0373 8819	284.6466 5898
97	174.2138 2978	186.9365 7264	215.8960 3811	250.3255 4248	291.3395 9216
98	177.1737 3537	190.2732 7980	220.1344 7868	255.7062 3947	298.1663 8400
99	180.1669 3989	193.6516 9580	224.4364 9586	261.1810 9866	305.1297 1168
100	183.1938 1796	197.0723 4200	228.8030 4330	266.7517 6789	312.2323 0591

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0225 0000	2.0250 0000	2.0275 0000	2.0300 0000	2.0350 0000
3	3.0680 0625	3.0756 2500	3.0832 5625	3.0909 0000	3.1062 2500
4	4.1370 3639	4.1525 1563	4.1680 4580	4.1836 2700	4.2149 4288
5	5.2301 1971	5.2563 2852	5.2826 6706	5.3091 3581	5.3624 6588
6	6.3477 9740	6.3877 3673	6.4279 4040	6.4684 0988	6.5501 5218
7	7.4906 2284	7.5474 3015	7.6047 0876	7.6624 6218	7.7794 0751
8	8.6591 6186	8.7361 1590	8.8138 3825	8.8923 3605	9.0516 8677
9	9.8539 9300	9.9545 1880	10.0562 1880	10.1591 0613	10.3684 9581
10	11.0757 0784	11.2033 8177	11.3327 6482	11.4638 7931	11.7313 9316
11	12.3249 1127	12.4834 6631	12.6444 1585	12.8077 9569	13.1419 9192
12	13.6022 2177	13.7955 5297	13.9921 3729	14.1920 2956	14.6019 6164
13	14.9082 7176	15.1404 4179	15.3769 2107	15.6177 9045	16.1130 3030
14	16.2437 0788	16.5189 5284	16.7997 8639	17.0863 2416	17.6769 8636
15	17.6091 9130	17.9319 2666	18.2617 8052	18.5989 1389	19.2956 8088
16	19.0053 9811	19.3802 2483	19.7639 7948	20.1568 8130	20.9710 2971
17	20.4330 1957	20.8647 3045	21.3074 8892	21.7615 8774	22.7050 1575
18	21.8927 6251	22.3863 4871	22.8934 4487	23.4144 3537	24.4996 9130
19	23.3853 4066	23.9460 0743	24.5230 1460	25.1168 6844	26.3571 8050
20	24.9115 2003	25.5446 5761	26.1973 9750	26.8703 7449	28.2796 8181
21	26.4720 2923	27.1832 7405	27.9178 2593	28.6764 8572	30.2694 7068
22	28.0676 4989	28.8628 5590	29.6855 6615	30.5367 8030	32.3289 0215
23	29.6991 7201	30.5844 2730	31.5019 1921	32.4528 8370	34.4604 1373
24	31.3674 0338	32.3490 3798	33.3682 2199	34.4264 7022	36.6665 2821
25	33.0731 6996	34.1577 6393	35.2858 4810	36.4592 6432	38.9498 5669
26	34.8173 1628	36.0117 0803	37.2562 0892	38.5530 4225	41.3131 0168
27	36.6007 0590	37.9120 0073	39.2807 5467	40.7096 3352	43.7590 6024
28	38.4242 2178	39.8598 0075	41.3609 7542	42.9309 2252	46.2906 2734
29	40.2887 6677	41.8562 9577	43.4984 0224	45.2188 5020	48.9107 9930
30	42.1952 6402	43.9027 0316	45.6946 0830	47.5754 1571	51.6226 7728
31	44.1446 5746	46.0002 7074	47.9512 1003	50.0026 7818	54.4294 7098
32	46.1379 1226	48.1502 7751	50.2698 6831	52.5027 5852	57.3345 0247
33	48.1760 1528	50.3540 3445	52.6522 8969	55.0778 4128	60.3412 1005
34	50.2599 7563	52.6128 8531	55.1002 2765	57.7301 7652	63.4531 5240
35	52.3908 2508	54.9282 0744	57.6154 8391	60.4620 8181	66.6740 1274
36	54.5696 1864	57.3014 1263	60.1999 0972	63.2759 4427	70.0076 0318
37	56.7974 3506	59.7339 4794	62.8554 0724	66.1742 2259	73.4578 6930
38	59.0753 7735	62.2272 9664	65.5839 3094	69.1594 4927	77.0288 9472
39	61.4045 7334	64.7829 7906	68.3874 8904	72.2342 3275	80.7249 0604
40	63.7861 7624	67.4025 5354	71.2681 4499	75.4012 5973	84.5502 7775
41	66.2213 6521	70.0876 1737	74.2280 1898	78.6632 9753	88.5095 3747
42	68.7113 4592	72.8398 0781	77.2692 8950	82.0231 9645	92.6073 7128
43	71.2573 5121	75.6608 0300	80.3941 9496	85.4838 9234	96.8486 2928
44	73.8606 4161	78.5523 2308	83.6050 3532	89.0484 0911	101.2383 3130
45	76.5225 0605	81.5161 3116	86.9041 7379	92.7198 6139	105.7816 7290
46	79.2442 6243	84.5540 3443	90.2940 3857	96.5014 5723	110.4840 3145
47	82.0272 5834	87.6678 8530	93.7771 2463	100.3965 0095	115.3509 7255
48	84.8728 7165	90.8595 8243	97.3559 9556	104.4083 9598	120.3882 5659
49	87.7825 1126	94.1310 7199	101.0332 8544	108.5406 4785	125.6018 4557
50	90.7576 1776	97.4843 4879	104.8117 0079	112.7968 6729	130.9979 1016

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	93.7996 6416	100.9214 5751	108.6940 2256	117.1807 7331	136.5828 3702
52	96.9101 5661	104.4444 9395	112.6831 0818	121.6961 9651	142.3632 3631
53	100.0906 3513	108.0556 0629	116.7818 9365	126.3470 8240	148.3459 4958
54	103.3426 7442	111.7569 9645	120.9933 9573	131.1374 9488	154.5380 5782
55	106.6678 8460	115.5509 2136	125.3207 1411	136.0716 1972	160.9468 8984
56	110.0679 1200	119.4396 9440	129.7670 3375	141.1537 6831	167.5800 3099
57	113.5444 4002	123.4256 8676	134.3356 2718	146.3883 8136	174.4453 3207
58	117.0991 8992	127.5113 2893	139.0298 5692	151.7800 3280	181.5509 1869
59	120.7339 2169	131.6991 1215	143.8531 7799	157.3334 3379	188.9052 0085
60	124.4504 3493	135.9915 8995	148.8091 4038	163.0534 3680	196.5168 8288
61	128.2505 6972	140.3913 7970	153.9013 9174	168.9450 3991	204.3949 7378
62	132.1362 0754	144.9011 6419	159.1336 8002	175.0133 9110	212.5487 9786
63	136.1092 7221	149.5236 9330	164.5098 5622	181.2637 9284	220.9880 0579
64	140.1717 3083	154.2617 8563	170.0338 7726	187.7017 0662	229.7225 8599
65	144.3255 9477	159.1183 3027	175.7098 0889	194.3327 5782	238.7628 7650
66	148.5729 2066	164.0962 8853	181.5418 2863	201.1627 4055	248.1195 7718
67	152.9158 1137	169.1986 9574	187.5342 2892	208.1976 2277	257.8037 6238
68	157.3564 1713	174.4286 6314	193.6914 2021	215.4435 5145	267.8268 9406
69	161.8969 3651	179.7893 7971	200.0179 3427	222.9068 5800	278.2008 3535
70	166.5396 1758	185.2841 1421	206.5184 2746	230.5940 6374	288.9378 6459
71	171.2867 5898	190.9162 1706	213.1976 8422	238.5118 8565	300.0506 8985
72	176.1407 1106	196.6891 2249	220.0606 2054	246.6672 4222	311.5524 6400
73	181.1038 7705	202.6063 5055	227.1122 8760	255.0672 5949	323.4568 0024
74	186.1787 1429	208.6715 0931	234.3578 7551	263.7192 7727	335.7777 8824
75	191.3677 3536	214.8882 9705	241.8027 1709	272.6308 5559	348.5300 1083
76	196.6735 0941	221.2605 0447	249.4522 9181	281.8097 8126	361.7285 6121
77	202.0986 6337	227.7920 1709	257.3122 2983	291.2640 7469	375.3890 6085
78	207.6458 8329	234.4868 1751	265.3883 1615	301.0019 9693	389.5276 7798
79	213.3179 1567	241.3489 8795	273.6864 9485	311.0320 5684	404.1611 4671
80	219.1175 6877	248.3827 1265	282.2128 7345	321.3630 1855	419.3067 8685
81	225.0477 1407	255.5922 8047	290.9737 2747	332.0039 0910	434.9825 2439
82	231.1112 8763	262.9820 8748	299.9755 0498	342.9640 2638	451.2069 1274
83	237.3112 9160	270.5566 3966	309.2248 3137	354.2529 4717	467.9991 5469
84	243.6507 9567	278.3205 5566	318.7285 1423	365.8805 3558	485.3791 2510
85	250.1329 3857	286.2785 6955	328.4935 4837	377.8569 5165	503.3673 9448
86	256.7609 2969	294.4355 3379	338.5271 2095	390.1926 6020	521.9852 5329
87	263.5380 5060	302.7964 2213	348.8366 1678	402.8984 4001	541.2547 3715
88	270.4676 5674	311.3663 3268	359.4296 2374	415.9853 9321	561.1986 5295
89	277.5531 7902	320.1504 9100	370.3139 3839	429.4649 5500	581.8406 0581
90	284.7981 2555	329.1542 5328	381.4975 7170	443.3489 0365	603.2050 2701
91	292.2060 8337	338.3831 0961	392.9887 5492	457.6493 7076	625.3172 0295
92	299.7807 2025	347.8426 8735	404.7959 4568	472.3788 5189	648.2033 0506
93	307.5257 8645	357.5387 5453	416.9278 3418	487.5502 1744	671.8904 2073
94	315.4451 1665	367.4772 2339	429.3933 4962	503.1767 2397	696.4065 8546
95	323.5426 3177	377.6641 5398	442.2016 6674	519.2720 2569	721.7803 1595
96	331.8223 4099	388.1057 5783	455.3622 1257	535.8501 8645	748.0431 4451
97	340.2883 4366	398.8084 0177	468.8846 7342	552.9256 9205	775.2246 5457
98	348.9448 3139	409.7786 1182	482.7790 0194	570.5134 6281	803.3575 1748
99	357.7960 9010	421.0230 7711	497.0554 2449	588.6288 6669	832.4750 2059
100	366.8465 0213	432.5486 5404	511.7244 4867	607.2877 3270	862.6116 5666

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0400 0000	2.0450 0000	2.0500 0000	2.0550 0000	2.0600 0000
3	3.1216 0000	3.1370 2500	3.1525 0000	3.1680 2500	3.1836 0000
4	4.2464 6400	4.2781 9113	4.3101 2500	4.3422 6638	4.3746 1600
5	5.4163 2256	5.4707 0973	5.5256 3125	5.5810 9103	5.6370 9296
6	6.6329 7546	6.7168 9166	6.8019 1281	6.8880 5103	6.9753 1854
7	7.8982 9448	8.0191 5179	8.1420 0845	8.2668 9384	8.3938 3765
8	9.2142 2626	9.3800 1362	9.5491 0888	9.7215 7300	9.8974 6791
9	10.5827 9531	10.8021 1423	11.0265 6432	11.2562 5951	11.4913 1598
10	12.0061 0712	12.2882 0937	12.5778 9254	12.8753 5379	13.1807 9494
11	13.4863 5141	13.8411 7879	14.2067 8716	14.5834 9825	14.9716 4264
12	15.0258 0546	15.4650 3184	15.9171 2652	16.3853 9065	16.8699 4120
13	16.6268 3768	17.1599 1327	17.7129 8285	18.2867 9814	18.8821 3767
14	18.2919 1119	18.9321 0937	19.5986 3199	20.2925 7203	21.0150 6593
15	20.0235 8764	20.7840 5429	21.5785 6359	22.4086 6350	23.2759 6988
16	21.8245 3114	22.7193 3673	23.6574 9177	24.6411 3999	25.6725 2808
17	23.6975 1239	24.7417 0689	25.8403 6636	26.9964 0269	28.2128 7976
18	25.6454 1288	26.8550 8370	28.1323 8467	29.4812 0483	30.9056 5255
19	27.6712 2940	29.0635 6246	30.5390 0391	32.1026 7110	33.7599 9170
20	29.7780 7858	31.3714 2277	33.0659 5410	34.8683 1801	36.7855 9120
21	31.9692 0172	33.7831 3680	35.7192 5181	37.7860 7550	39.9927 2668
22	34.2479 6979	36.3033 7795	38.5052 1440	40.8643 0965	43.3922 9028
23	36.6178 8858	38.9370 2996	41.4304 7512	44.1118 4669	46.9958 2769
24	39.0826 0412	41.6891 9631	44.5019 9887	47.5379 9825	50.8155 7735
25	41.6459 0829	44.5652 1015	47.7270 9882	51.1525 8816	54.8645 1200
26	44.3117 4462	47.5706 4460	51.1134 5376	54.9659 8051	59.1563 8272
27	47.0842 1440	50.7113 2361	54.6691 2645	58.9891 0943	63.7057 6568
28	49.9675 8298	53.9933 3317	58.4025 8277	63.2335 1045	68.5281 1162
29	52.9662 8630	57.4230 3316	62.3227 1191	67.7113 5353	73.6397 9832
30	56.0849 3775	61.0070 6966	66.4388 4750	72.4354 7797	79.0581 8622
31	59.3283 3526	64.7523 8779	70.7607 8988	77.4194 2926	84.8016 7739
32	62.7014 6867	68.6662 4524	75.2988 2937	82.6774 9787	90.8897 7803
33	66.2095 2742	72.7562 2628	80.0637 7084	88.2247 6025	97.3431 6471
34	69.8579 0851	77.0302 5646	85.0669 5938	94.0771 2207	104.1837 5460
35	73.6522 2486	81.4966 1800	90.3203 0735	100.2513 6378	111.4347 7987
36	77.5983 1385	86.1639 6581	95.8363 2272	106.7651 8879	119.1208 6666
37	81.7022 4640	91.0413 4427	101.6281 3886	113.6372 7417	127.2681 1866
38	85.9703 3626	96.1382 0476	107.7095 4580	120.8873 2425	135.9042 0578
39	90.4091 4971	101.4644 2398	114.0950 2309	128.5361 2708	145.0584 5813
40	95.0255 1570	107.0303 2306	120.7997 7424	136.6056 1407	154.7619 6562
41	99.8265 3633	112.8466 8760	127.8397 6295	145.1189 2285	165.0476 8356
42	104.8195 9778	118.9247 8854	135.2317 5110	154.1004 6360	175.9505 4457
43	110.0123 8169	125.2764 0402	142.9933 3866	163.5759 8910	187.5075 7724
44	115.4128 7696	131.9138 4220	151.1430 0559	173.5726 6850	199.7580 3188
45	121.0293 9204	138.8499 6510	159.7001 5587	184.1191 6527	212.7435 1379
46	126.8705 6772	146.0982 1353	168.6851 6366	195.2457 1936	226.5081 2462
47	132.9453 9043	153.6726 3314	178.1194 2185	206.9842 3392	241.0986 1210
48	139.2632 0604	161.5879 0163	188.0253 9294	219.3683 6679	256.5645 2882
49	145.8337 3429	169.8593 5720	198.4266 6259	232.4336 2696	272.9584 0055
50	152.6670 8366	178.5030 2828	209.3479 9572	246.2174 7645	290.3359 0458

TABLE V.—AMOUNT OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	4%	4½%	5%	5½%	6%
51	159.7737 6700	187.5356 6455	220.8153 9550	260.7594 3765	308.7560 5886
52	167.1647 1768	196.9747 6946	232.8561 6528	276.1012 0672	328.2814 2239
53	174.8513 0639	206.8386 3408	245.4989 7354	292.2867 7309	348.9783 0773
54	182.8453 5865	217.1463 7262	258.7739 2222	309.3625 4561	370.9170 0620
55	191.1591 7299	227.9179 5938	272.7126 1833	327.3774 8562	394.1720 2657
56	199.8055 3991	239.1742 6756	287.3482 4924	346.3832 4733	418.8223 4816
57	208.7977 6151	250.9371 0960	302.7156 6171	366.4343 2593	444.9516 8905
58	218.1496 7197	263.2292 7953	318.8514 4479	387.5882 1386	472.6487 9040
59	227.8756 5885	276.0745 9711	335.7940 1703	409.9055 6562	502.0077 1782
60	237.9906 8520	289.4979 5398	353.5837 1788	433.4503 7173	533.1281 8089
61	248.5103 1261	303.5253 6190	372.2629 0378	458.2901 4217	566.1158 7174
62	259.4507 2511	318.1840 0319	391.8760 4897	484.4960 9999	601.0828 2405
63	270.8287 5412	333.5022 8333	412.4698 5141	512.1433 8549	638.1477 9349
64	282.6619 0428	349.5098 8608	434.0933 4398	541.3112 7170	677.4366 6110
65	294.9683 8045	366.2378 3096	456.7980 1118	572.0833 9164	719.0828 6076
66	307.7671 1567	383.7185 3335	480.6379 1174	604.5479 7818	763.2278 3241
67	321.0778 0030	401.9858 6735	505.6698 0733	638.7981 1698	810.0215 0236
68	334.9209 1231	421.0752 3138	531.9532 9770	674.9320 1311	859.6227 9250
69	349.3177 4880	441.0236 1679	559.5509 6258	713.0532 7415	912.2001 6005
70	364.2904 5876	461.8696 7955	588.5285 1071	753.2712 0423	967.9321 6965
71	379.8620 7711	483.6538 1513	618.9549 3625	795.7011 2046	1027.0080 9983
72	396.0565 6019	506.4182 3681	650.9026 8306	840.4646 8209	1089.6285 8582
73	412.8988 2260	530.2070 5747	684.4478 1721	887.6902 3960	1156.0063 0097
74	430.4147 7550	555.0663 7505	719.6702 0807	937.5132 0278	1226.3666 7903
75	448.6313 6652	581.0443 6193	756.6537 1848	990.0764 2893	1300.9486 7977
76	467.5766 2118	608.1913 5822	795.4864 0440	1045.5306 3252	1380.0056 0055
77	487.2796 8603	636.5599 6934	836.2607 2462	1104.0348 1731	1463.8059 3659
78	507.7708 7347	666.2051 6796	879.0737 6085	1165.7567 3226	1552.6342 9278
79	529.0817 0841	697.1844 0052	924.0274 4889	1230.8733 5254	1646.7923 5035
80	551.2449 7675	729.5576 9854	971.2288 2134	1299.5713 8693	1746.5998 9137
81	574.2947 7582	763.3877 9497	1020.7902 6240	1372.0478 1321	1852.3958 8485
82	598.2665 6685	798.7402 4575	1072.8297 7552	1448.5104 4294	1964.5396 3794
83	623.1972 2952	835.6835 5680	1127.4712 6430	1529.1785 1730	2083.4120 1622
84	649.1251 1870	874.2893 1686	1184.8448 2752	1614.2833 3575	2209.4167 3719
85	676.0901 2345	914.6323 3612	1245.0870 6889	1704.0689 1921	2342.9817 4142
86	704.1337 2839	956.7907 9125	1308.3414 2234	1798.7927 0977	2484.5606 4591
87	733.2990 7753	1000.8463 7685	1374.7584 9345	1898.7263 0881	2634.6342 8466
88	763.6310 4063	1046.8844 6381	1444.4964 1812	2004.1562 5579	2793.7123 4174
89	795.1762 8225	1094.9942 6468	1517.7212 3903	2115.3848 4986	2962.3350 8225
90	827.9833 3354	1145.2690 0659	1594.6073 0098	2232.7310 1660	3141.0751 8718
91	862.1026 6688	1197.8061 1189	1675.3376 6603	2356.5312 2252	3330.5396 9841
92	897.5867 7356	1252.7073 8692	1760.1045 4933	2487.1404 3976	3531.3720 8032
93	934.4902 4450	1310.0792 1933	1849.1097 7680	2624.9331 6394	3744.2544 0514
94	972.8698 5428	1370.0327 8420	1942.5652 6564	2770.3044 8796	3969.9096 6944
95	1012.7846 4845	1432.6842 5949	2040.6935 2892	2923.6712 3480	4209.1042 4961
96	1054.2960 3439	1498.1550 5117	2143.7282 0537	3085.4731 5271	4462.6505 0459
97	1097.4678 7577	1566.5720 2847	2251.9146 1564	3256.1741 7611	4731.4095 3486
98	1142.3665 9080	1638.0677 6976	2365.5103 4642	3436.2637 5580	5016.2941 0696
99	1189.0612 5443	1712.7808 1939	2484.7858 6374	3626.2582 6237	5318.2717 5337
100	1237.6237 0461	1790.8559 5627	2610.0251 5693	3826.7024 6680	5638.3680 5857

Table V.—Amount of Annuity of 1 per Period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0650 0000	2.0700 0000	2.0750 0000	2.0800 0000	2.0850 0000
3	3.1992 2500	3.2149 0000	3.2306 2500	3.2464 0000	3.2622 2500
4	4.4071 7463	4.4399 4300	4.4729 2188	4.5061 1200	4.5395 1413
5	5.6936 4098	5.7507 3901	5.8083 9102	5.8666 0096	5.9253 7283
6	7.0637 2764	7.1532 9074	7.2440 2034	7.3359 2904	7.4290 2952
7	8.5228 6994	8.6540 2109	8.7873 2187	8.9228 0336	9.0604 9702
8	10.0768 5648	10.2598 0257	10.4463 7101	10.6366 2763	10.8306 3927
9	11.7318 5215	11.9779 8875	12.2298 4883	12.4875 5784	12.7512 4361
10	13.4944 2254	13.8164 4796	14.1470 8750	14.4865 6247	14.8350 9932
11	15.3715 6001	15.7835 9932	16.2081 1906	16.6454 8746	17.0960 8276
12	17.3707 1141	17.8884 5127	18.4237 2799	18.9771 2646	19.5492 4979
13	19.4998 0765	20.1406 4286	20.8055 0759	21.4952 9658	22.2109 3603
14	21.7672 9515	22.5504 8786	23.3659 2066	24.2149 2030	25.0988 6559
15	24.1821 6933	25.1290 2201	26.1183 6470	27.1521 1393	28.2322 6916
16	26.7540 1034	27.8880 5355	29.0772 4206	30.3242 8304	31.6320 1204
17	29.4930 2101	30.8402 1730	32.2580 3521	33.7502 2569	35.3207 3306
18	32.4100 6738	33.9990 3251	35.6773 8785	37.4502 4374	39.3229 9538
19	35.5167 2176	37.3789 6479	39.3531 9194	41.4462 6324	43.6654 4998
20	38.8253 0867	40.9954 9232	43.3046 8134	45.7619 6430	48.3770 1323
21	42.3489 5373	44.8651 7673	47.5525 3244	50.4229 2144	53.4890 5936
22	46.1016 3573	49.0057 3916	52.1189 7237	55.4567 5516	59.0356 2940
23	50.0982 4205	53.4361 4090	57.0278 9530	60.8932 9557	65.0536 5790
24	54.3546 2778	58.1766 7076	62.3049 8744	66.7647 5922	71.5832 1882
25	58.8876 7859	63.2490 3772	67.9778 6150	73.1059 3995	78.6677 9242
26	63.7153 7769	68.6764 7036	74.0762 0112	79.9544 1515	86.3545 5478
27	68.8568 7725	74.4838 2328	80.6319 1620	87.3507 6836	94.6946 9193
28	74.3325 7427	80.6976 9091	87.6793 0991	95.3388 2983	103.7437 4075
29	80.1641 9159	87.3465 2927	95.2552 5816	103.9659 3622	113.5619 5871
30	86.3748 6405	94.4607 8632	103.3994 0252	113.2832 1111	124.2147 2520
31	92.9892 3021	102.0730 4137	112.1543 5771	123.3458 6800	135.7729 7684
32	100.0335 3017	110.2181 5426	121.5659 3454	134.2135 3744	148.3136 7987
33	107.5357 0963	118.9334 2506	131.6833 7963	145.9506 2044	161.9203 4266
34	115.5255 3076	128.2587 6481	142.5596 3310	158.6266 7007	176.6835 7179
35	124.0346 9026	138.2368 7835	154.2516 0558	172.3168 0368	192.7016 7539
36	133.0969 4513	148.9134 5984	166.8204 7600	187.1021 4797	210.0813 1780
37	142.7482 4656	160.3374 0202	180.3320 1170	203.0703 1981	228.9382 2981
38	153.0268 8259	172.5010 2017	194.8569 1258	220.3159 4540	249.3979 7935
39	163.9736 2995	185.6402 9158	210.4711 8102	238.9412 2103	271.5968 0759
40	175.6319 1590	199.6351 1199	227.2565 1960	259.0565 1871	295.6825 3624
41	188.0479 9044	214.6095 6983	245.3007 5857	280.7810 4021	321.8155 5182
42	201.2711 0981	230.6322 3972	264.6983 1546	304.2435 2342	350.1698 7372
43	215.3537 3195	247.7764 9650	285.5506 8912	329.5830 0530	380.9343 1299
44	230.3517 2453	266.1208 5125	307.9669 9080	356.9496 4572	414.3137 2059
45	246.3245 8662	285.7493 1084	332.0645 1511	386.5056 1738	450.5303 9661
46	263.3356 8475	306.7517 6260	357.9693 5375	418.4260 6677	489.8254 8032
47	281.4525 0426	329.2243 8598	385.8170 5528	452.9001 5211	532.4606 4615
48	300.7469 1704	353.2700 9300	415.7533 3442	490.1321 6428	578.7198 0107
49	321.2954 6665	378.9989 9951	447.9348 3451	530.3427 3742	628.9109 8416
50	343.1796 7198	406.5289 2947	482.5299 4709	573.7701 5642	683.3684 1782

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	0.9958 5062	0.9950 2488	0.9942 0050	0.9925 5583	0.9900 9901
2	1.9875 6908	1.9850 9938	1.9826 3513	1.9777 2291	1.9703 9506
3	2.9751 7253	2.9702 4814	2.9653 3733	2.9555 5624	2.9409 8521
4	3.9586 7804	3.9504 9566	3.9423 4034	3.9261 1041	3.9019 6555
5	4.9381 0261	4.9258 6633	4.9136 7723	4.8894 3961	4.8534 3124
6	5.9134 6318	5.8963 8441	5.8793 8084	5.8455 9763	5.7954 7647
7	6.8847 7661	6.8620 7404	6.8394 8385	6.7946 3785	6.7281 9453
8	7.8520 5969	7.8229 5924	7.7940 1875	7.7366 1325	7.6516 7775
9	8.8153 2915	8.7790 6392	8.7430 1781	8.6715 7642	8.5660 1758
10	9.7746 0164	9.7304 1186	9.6865 1315	9.5995 7958	9.4713 0453
11	10.7298 9374	10.6770 2673	10.6245 3669	10.5206 7452	10.3676 2825
12	11.6812 2198	11.6189 3207	11.5571 2016	11.4349 1267	11.2550 7747
13	12.6286 0280	12.5561 5131	12.4842 9511	12.3423 4508	12.1337 4007
14	13.5720 5257	13.4887 0777	13.4060 9291	13.2430 2242	13.0037 0304
15	14.5115 8762	14.4166 2465	14.3225 4473	14.1369 9495	13.8650 5252
16	15.4472 2418	15.3399 2502	15.2336 8160	15.0243 1261	14.7178 7378
17	16.3789 7843	16.2586 3186	16.1395 3432	15.9050 2492	15.5622 5127
18	17.3068 6648	17.1727 6802	17.0401 3354	16.7791 8107	16.3982 6858
19	18.2309 0438	18.0823 5624	17.9355 0974	17.6468 2984	17.2260 0850
20	19.1511 0809	18.9874 1915	18.8256 9320	18.5080 1969	18.0455 5297
21	20.0674 9352	19.8879 7925	19.7107 1404	19.3627 9870	18.8569 8313
22	20 9800 7653	20.7840 5896	20.5906 0220	20.2112 1459	19.6603 7934
23	21.8888 7289	21.6756 8055	21.4653 8745	21.0533 1473	20.4558 2113
24	22.7938 9831	22.5628 6622	22.3350 9938	21.8891 4614	21.2433 8726
25	23.6951 6843	23.4456 3803	23.1997 6741	22.7187 5547	22.0231 5570
26	24.5926 9884	24.3240 1794	24.0594 2079	23.5421 8905	22.7952 0366
27	25.4865 0506	25.1980 2780	24.9140 8862	24.3594 9286	23.5596 0759
28	26.3766 0254	26.0676 8936	25.7637 9979	25.1707 1251	24.3164 4316
29	27.2630 0668	26.9330 2423	26.6085 8307	25.9758 9331	25.0657 8530
30	28.1457 3278	27.7940 5397	27.4484 6702	26.7750 8021	25.8077 0822
31	29.0247 9612	28.6507 9997	28.2834 8006	27.5683 1783	26.5422 8537
32	29.9002 1189	29.5032 8355	29.1136 5044	28.3556 5045	27.2695 8947
33	30.7719 9524	30.3515 2592	29.9390 0625	29.1371 2203	27.9896 9255
34	31.6401 6122	31.1955 4818	30.7595 7540	29.9127 7621	28.7026 6589
35	32.5047 2486	32.0353 7132	31.5753 8566	30.6826 5629	29.4085 8009
36	33.3657 0109	32.8710 1624	32.3864 6463	31.4468 0525	30.1075 0504
37	34.2231 0481	33.7025 0372	33.1928 3974	32.2052 6576	30.7995 0994
38	35.0769 5084	34.5298 5445	33.9945 3828	32.9580 8016	31.4846 6330
39	35.9272 5394	35.3530 8900	34.7915 8736	33.7052 9048	32.1630 3298
40	36.7740 2881	36.1722 2786	35.5840 1396	34.4469 3844	32.8346 8611
41	37.6172 9009	36.9872 9141	36.3718 4487	35.1830 6545	33.4996 8922
42	38.4570 5236	37.7982 9991	37.1551 0676	35.9137 1260	34.1581 0814
43	39.2933 3013	38.6052 7354	37.9338 2612	36.6389 2070	34.8100 0806
44	40.1261 3788	39.4082 3238	38.7080 2929	37.3587 3022	35.4554 5352
45	40.9554 8999	40.2071 9640	39.4777 4248	38.0731 8136	36.0945 0844
46	41.7814 0081	41.0021 8547	40.2429 9170	38.7823 1401	36.7272 3608
47	42.6038 8461	41.7932 1937	41.0038 0287	39.4861 6774	37.3536 9909
48	43.4229 5562	42.5803 1778	41.7602 0170	40.1847 8189	37.9739 5949
49	44.2386 2799	43.3635 0028	42.5122 1380	40.8781 9542	38.5880 7871
50	45.0509 1582	44.1427 8635	43.2598 6460	41.5664 4707	39.1961 1753

TABLE VI.—PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	45.8598 3317	44.9181 9537	44.0031 7940	42.2495 7525	39.7981 3617
52	46.6653 9401	45.6897 4664	44.7421 8335	42.9276 1812	40.3941 9423
53	47.4676 1228	46.4574 5934	45.4769 0144	43.6006 1351	40.9843 5072
54	48.2665 0184	47.2213 5258	46.2073 5853	44.2685 9902	41.5686 6408
55	49.0620 7651	47.9814 4535	46.9335 7933	44.9316 1193	42.1471 9216
56	49.8543 5003	48.7377 5657	47.6555 8841	45.5896 8926	42.7199 9224
57	50.6433 3612	49.4903 0505	48.3734 1020	46.2428 6776	43.2871 2102
58	51.4290 4840	50.2391 0950	49.0870 6898	46.8911 8388	43.8486 3468
59	52.2115 0046	50.9841 8855	49.7965 8889	47.5346 7382	44.4045 8879
60	52.9907 0584	51.7255 6075	50.5019 9394	48.1733 7352	44.9550 3841
61	53.7666 7800	52.4632 4453	51.2033 0800	48.8073 1863	45.5000 3803
62	54.5394 3035	53.1972 5824	51.9005 5478	49.4365 4455	46.0396 4161
63	55.3089 7627	53.9276 2014	52.5937 5787	50.0610 8640	46.5739 0258
64	56.0753 2905	54.6543 4839	53.2829 4073	50.6809 7906	47.1028 7385
65	56.8385 0194	55.3774 6109	53.9681 2668	51.2962 5713	47.6266 0777
66	57.5985 0814	56.0969 7621	54.6493 3888	51.9069 5497	48.1451 5621
67	58.3553 6078	56.8129 1165	55.3266 0040	52.5131 0667	48.6585 7050
68	59.1090 7296	57.5252 8522	55.9999 3413	53.1147 4607	49.1669 0149
69	59.8596 5770	58.2341 1465	56.6693 6287	53.7119 0677	49.6701 9949
70	60.6071 2798	58.9394 1756	57.3349 0925	54.3046 2210	50.1685 1435
71	61.3514 9672	59.6412 1151	57.9965 9579	54.8929 2516	50.6618 9539
72	62.0927 7680	60.3395 1394	58.6544 4488	55.4768 4880	51.1503 9148
73	62.8309 8103	61.0343 4222	59.3084 7877	56.0564 2561	51.6340 5097
74	63.5661 2216	61.7257 1366	59.9587 1959	56.6316 8795	52.1129 2175
75	64.2982 1292	62.4136 4543	60.6051 8934	57.2026 6794	52.5870 5124
76	65.0272 6596	63.0981 5466	61.2479 0988	57.7693 9746	53.0564 8637
77	65.7532 9388	63.7792 5836	61.8869 0297	58.3319 0815	53.5212 7364
78	66.4763 0924	64.4569 7350	62.5221 9021	58.8902 3141	53.9814 5905
79	67.1963 2453	65.1313 1691	63.1537 9310	59.4443 9842	54.4370 8817
80	67.9133 5221	65.8023 0538	63.7817 3301	59.9944 4012	54.8882 0611
81	68.6274 0467	66.4699 5561	64.4060 3118	60.5403 8722	55.3348 5753
82	69.3384 9426	67.1342 8419	65.0267 0874	61.0822 7019	55.7770 8666
83	70.0466 3326	67.7953 0765	65.6437 8667	61.6201 1930	56.2149 3729
84	70.7518 3393	68.4530 4244	66.2572 8585	62.1539 6456	56.6484 5276
85	71.4541 0846	69.1075 0491	66.8672 2705	62.6838 3579	57.0776 7600
86	72.1534 6898	69.7587 1135	67.4736 3089	63.2097 6257	57.5026 4951
87	72.8499 2759	70.4066 7796	68.0765 1789	63.7317 7427	57.9234 1535
88	73.5434 9633	71.0514 2086	68.6759 0845	64.2499 0002	58.3400 1520
89	74.2341 8720	71.6929 5608	69.2718 2283	64.7641 6875	58.7524 9030
90	74.9220 1212	72.3312 9958	69.8642 8121	65.2746 0918	59.1608 8148
91	75.6069 8300	72.9664 6725	70.4533 0363	65.7812 4981	59.5652 2919
92	76.2891 1168	73.5984 7487	71.0389 1001	66.2841 1892	59.9655 7346
93	76.9684 0995	74.2273 3818	71.6211 2017	66.7832 4458	60.3619 5392
94	77.6448 8055	74.8530 7282	72.1999 5379	67.2786 5467	60.7544 0982
95	78.3185 6218	75.4756 9434	72.7754 3047	67.7703 7685	61.1429 8002
96	78.9894 3950	76.0952 1825	73.3475 6967	68.2584 3856	61.5277 0299
97	79.6575 3308	76.7116 5995	73.9163 9075	68.7428 6705	61.9086 1682
98	80.3228 5450	77.3250 3478	74.4819 1294	69.2236 8938	62.2857 5923
99	80.9854 1524	77.9353 5799	75.0441 5539	69.7009 3239	62.6591 6755
100	81.6452 2677	78.5426 4477	75.6031 3712	70.1746 2272	63.0288 7877

TABLE VI.—PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	82.3023 0049	79.1469 1021	76.1588 7702	70.6447 8682	63.3949 2947
102	82.9566 4777	79.7481 6937	76.7113 9392	71.1114 5094	63.7573 5591
103	83.6082 7991	80.3464 3718	77.2607 0648	71.5746 4113	64.1161 9397
104	84.2572 0818	80.9417 2854	77.8068 3331	72.0343 8325	64.4714 7918
105	84.9034 4381	81.5340 5825	78.3497 9288	72.4907 0298	64.8232 4671
106	85.5469 9795	82.1234 4104	78.8896 0355	72.9436 2579	65.1715 3140
107	86.1878 8175	82.7098 9158	79.4262 8359	73.3931 7696	65.5163 6772
108	86.8261 0628	83.2934 2446	79.9598 5115	73.8393 8160	65.8577 8983
109	87.4616 8258	83.8740 5419	80.4903 2428	74.2822 6461	66.1958 3151
110	88.0946 2163	84.4517 9522	81.0177 2093	74.7218 5073	66.5305 2625
111	88.7249 3437	85.0266 6191	81.5420 5895	75.1581 6450	66.8619 0718
112	89.3526 3171	85.5986 6856	82.0633 5606	75.5912 3027	67.1900 0710
113	89.9777 2450	86.1678 2942	82.5816 2991	76.0210 7223	67.5148 5852
114	90.6002 2354	86.7341 5862	83.0968 9803	76.4477 1437	67.8364 9358
115	91.2201 3959	87.2976 7027	83.6091 7785	76.8711 8052	68.1549 4414
116	91.8374 8338	87.8583 7838	84.1184 3671	77.2914 9431	68.4702 4172
117	92.4522 6558	88.4162 9690	84.6248 4182	77.7086 7922	68.7824 1755
118	93.0644 9681	88.9714 3970	85.1282 6033	78.1227 5853	69.0915 0252
119	93.6741 8767	89.5238 2059	85.6287 5926	78.5337 5536	69.3975 2725
120	94.2813 4869	90.0734 5333	86.1263 5554	78.9416 9267	69.7005 2203
121	94.8859 9036	90.6203 5157	86.6210 6602	79.3465 9322	70.0005 1686
122	95.4881 2315	91.1645 2892	87.1129 0742	79.7484 7962	70.2975 4145
123	96.0877 5747	91.7059 9893	87.6018 9638	80.1473 7432	70.5916 2520
124	96.6849 0367	92.2447 7505	88.0880 4946	80.5432 9957	70.8827 9722
125	97.2795 7209	92.7808 7070	88.5713 8308	80.9362 7749	71.1710 8636
126	97.8717 7301	93.3142 9920	89.0519 1361	81.3263 3001	71.4565 2115
127	98.4615 1666	93.8450 7384	89.5296 5731	81.7134 7892	71.7391 2985
128	99.0488 1324	94.3732 0780	90.0046 3032	82.0977 4583	72.0189 4045
129	99.6336 7290	94.8987 1422	90.4768 4873	82.4791 5219	72.2959 8064
130	100.2161 0576	95.4216 0619	90.9463 2851	82.8577 1929	72.5702 7786
131	100.7961 2189	95.9418 9671	91.4130 8554	83.2334 6828	72.8418 5927
132	101.3737 3131	96.4595 9872	91.8771 3561	83.6064 2013	73.1107 5175
133	101.9489 4401	96.9747 2509	92.3384 9442	83.9765 9566	73.3769 8193
134	102.5217 6994	97.4872 8865	92.7971 7758	84.3440 1554	73.6405 7617
135	103.0922 1899	97.9973 0214	93.2532 0060	84.7087 0029	73.9015 6056
136	103.6603 0104	98.5047 7825	93.7065 7892	85.0706 7026	74.1599 6095
137	104.2260 2590	99.0097 2960	94.1573 2787	85.4299 4567	74.4158 0293
138	104.7894 0335	99.5121 6875	94.6054 6270	85.7865 4657	74.6691 1181
139	105.3504 4314	100.0121 0821	95.0509 9857	86.1404 9288	74.9199 1268
140	105.9091 5496	100.5095 6041	95.4939 5056	86.4918 0434	75.1682 3038
141	106.4655 4847	101.0045 3772	95.9343 3364	86.8405 0059	75.4140 8948
142	107.0196 3330	101.4970 5246	96.3721 6272	87.1866 0108	75.6575 1434
143	107.5714 1902	101.9871 1688	96.8074 5261	87.5301 2514	75.8985 2905
144	108.1209 1517	102.4747 4316	97.2402 1804	87.8710 9195	76.1371 5747
145	108.6681 3126	102.9599 4344	97.6704 7364	88.2095 2055	76.3734 2324
146	109.2130 7674	103.4427 2979	98.0982 3397	88.5454 2982	76.6073 4974
147	109.7557 6103	103.9231 1422	98.5235 1350	88.8788 3854	76.8389 6014
148	110.2961 9353	104.4011 0868	98.9463 2663	89.2097 6530	77.0682 7737
149	110.8343 8356	104.8767 2505	99.3666 8765	89.5382 2858	77.2953 2413
150	111.3703 4044	105:3499 7518	99.7846 1078	89.8642 4673	77.5201 2290

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	0.9888 7515	0.9876 5432	0.9852 2167	0.9828 0098	0.9803 9216
2	1.9667 4923	1.9631 1538	1.9558 8342	1.9486 9875	1.9415 6094
3	2.9337 4460	2.9265 3371	2.9122 0042	2.8979 8403	2.8838 8327
4	3.8899 8230	3.8780 5798	3.8543 8465	3.8309 4254	3.8077 2870
5	4.8355 8200	4.8178 3504	4.7826 4497	4.7478 5508	4.7134 5951
6	5.7706 6205	5.7460 0992	5.6971 8717	5.6489 9762	5.6014 3089
7	6.6953 3948	6.6627 2585	6.5982 1396	6.5346 4139	6.4719 9107
8	7.6097 3002	7.5681 2429	7.4859 2508	7.4050 5297	7.3254 8144
9	8.5139 4810	8.4623 4498	8.3605 1732	8.2604 9432	8.1622 3671
10	9.4081 0690	9.3455 2591	9.2221 8455	9.1012 2291	8.9825 8501
11	10.2923 1832	10.2178 0337	10.0711 1779	9.9274 9181	9.7868 4805
12	11.1666 9302	11.0793 1197	10.9075 0521	10.7395 4969	10.5753 4122
13	12.0313 4044	11.9301 8466	11.7315 3222	11.5376 4097	11.3483 7375
14	12.8863 6880	12.7705 5275	12.5433 8150	12.3220 0587	12.1062 4877
15	13.7318 8509	13.6005 4592	13.3432 3301	13.0928 8046	12.8492 6350
16	14.5679 9514	14.4202 9227	14.1312 6405	13.8504 9677	13.5777 0931
17	15.3948 0360	15.2299 1829	14.9076 4931	14.5950 8282	14.2918 7188
18	16.2124 1395	16.0295 4893	15.6725 6089	15.3268 6272	14.9920 3125
19	17.0209 2850	16.8193 0759	16.4261 6837	16.0460 5673	15.6784 6201
20	17.8204 4845	17.5993 1613	17.1686 3879	16.7528 8130	16.3514 3334
21	18.6110 7387	18.3696 9495	17.9001 3673	17.4475 4919	17.0112 0916
22	19.3929 0371	19.1305 6291	18.6208 2437	18.1302 6948	17.6580 4820
23	20.1660 3580	19.8820 3744	19.3308 6145	18.8012 4764	18.2922 0412
24	20.9305 6693	20.6242 3451	20.0304 0537	19.4606 8565	18.9139 2560
25	21.6865 9276	21.3572 6865	20.7196 1120	20.1087 8196	19.5234 5647
26	22.4342 0792	22.0812 5299	21.3986 3172	20.7457 3166	20.1210 3576
27	23.1735 0598	22.7962 9925	22.0676 1746	21.3717 2644	20.7068 9780
28	23.9045 7946	23.5025 1778	22.7267 1671	21.9869 5474	21.2812 7236
29	24.6275 1986	24.2000 1756	23.3760 7558	22.5916 0171	21.8443 8466
30	25.3424 1766	24.8889 0623	24.0158 3801	23.1858 4934	22.3964 5555
31	26.0493 6233	25.5692 9010	24.6461 4582	23.7698 7650	22.9377 0152
32	26.7484 4236	26.2412 7418	25.2671 3874	24.3438 5897	23.4683 3482
33	27.4397 4522	26.9049 6215	25.8789 5442	24.9079 6951	23.9885 6355
34	28.1233 5745	27.5604 5644	26.4817 2849	25.4623 7789	24.4985 9172
35	28.7993 6460	28.2078 5822	27.0755 9458	26.0072 5100	24.9986 1933
36	29.4678 5127	28.8472 6737	27.6606 8431	26.5427 5283	25.4888 4248
37	30.1289 0114	29.4787 8259	28.2371 2740	27.0690 4455	25.9694 5341
38	30.7825 9692	30.1025 0133	28.8050 5163	27.5862 8457	26.4406 4060
39	31.4290 2044	30.7185 1983	29.3645 8288	28.0946 2857	26.9025 8883
40	32.0682 5260	31.3269 3316	29.9158 4520	28.5942 2955	27.3554 7924
41	32.7903 7340	31.9278 3522	30.4589 6079	29.0852 3789	27.7994 8945
42	33.3254 6195	32.5213 1874	30.9940 5004	29.5678 0135	28.2347 9358
43	33.9435 9649	33.1074 7530	31.5212 3157	30.0420 6522	28.6615 6233
44	34.5548 5438	33.6863 9536	32.0406 2223	30.5081 7221	29.0799 6307
45	35.1593 1212	34.2581 6825	32.5523 3718	30.9662 6261	29.4901 5937
46	35.7570 4536	34.8228 8222	33.0564 8983	31.4164 7431	29.8923 1360
47	36.3481 2891	35.3806 2442	33.5531 9195	31.8589 4281	30.2865 8196
48	36.9326 3674	35.9314 8091	34.0425 5365	32.2938 0129	30.6731 1957
49	37.5106 4202	36.4755 3670	34.5246 8339	32.7211 8063	31.0520 7801
50	38.0822 1708	37.0128 7574	34.9996 8807	33.1412 0946	31.4236 0589

TABLE VI.—PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	38.6474 3345	37.5435 8099	35.4676 7298	33.5540 1421	31.7878 4892
52	39.2063 6188	38.0677 3431	35.9287 4185	33.9597 1913	32.1449 4992
53	39.7590 7232	38.5854 1660	36.3829 9690	34.3584 4633	32.4950 4894
54	40.3056 3394	39.0967 0776	36.8305 3882	34.7503 1579	32.8382 8327
55	40.8461 1514	39.6016 8667	37.2714 6681	35.1354 4550	33.1747 8752
56	41.3805 8358	40.1004 3128	37.7058 7863	35.5139 5135	33.5046 9365
57	41.9091 0613	40.5930 1855	38.1338 7058	35.8859 4727	33.8281 3103
58	42.4317 4896	41.0795 2449	38.5555 3751	36.2515 4523	34.1452 2650
59	42.9485 7746	41.5600 2419	38.9709 7292	36.6108 5526	34.4561 0441
60	43.4596 5633	42.0345 9179	39.3802 6889	36.9639 8552	34.7608 8668
61	43.9650 4952	42.5033 0054	39.7835 1614	37.3110 4228	35.0596 9282
62	44.4648 2029	42.9662 2275	40.1808 0408	37.6521 3000	35.3526 4002
63	44.9590 3119	43.4234 2988	40.5722 2077	37.9873 5135	35.6398 4316
64	45.4477 4407	43.8749 9247	40.9578 5298	38.3168 0723	35.9214 1486
65	45.9310 2009	44.3209 8022	41.3377 8618	38.6405 9678	36.1974 6555
66	46.4089 1975	44.7614 6195	41.7121 0461	38.9588 1748	36.4681 0348
67	46.8815 0284	45.1965 0563	42.0808 9125	39.2715 6509	36.7334 3478
68	47.3488 2852	45.6261 7840	42.4442 2783	39.5789 3375	36.9935 6351
69	47.8109 5527	46.0505 4656	42.8021 9490	39.8810 1597	37.2485 9168
70	48.2679 4094	46.4696 7562	43.1548 7183	40.1779 0267	37.4986 1929
71	48.7198 4270	46.8836 3024	43.5023 3678	40.4696 8321	37.7437 4441
72	49.1667 1714	47.2924 7431	43.8446 6677	40.7564 4542	37.9840 6314
73	49.6086 2016	47.6962 7093	44.1819 3771	41.0382 7560	38.2196 6975
74	50.0456 0708	48.0950 8240	44.5142 2434	41.3152 5857	38.4506 5662
75	50.4777 3259	48.4889 7027	44.8416 0034	41.5874 7771	38.6771 1433
76	50.9050 5077	48.8779 9533	45.1641 3826	41.8550 1495	38.8991 3170
77	51.3276 1510	49.2622 1761	45.4819 0962	42.1179 5081	39.1167 9578
78	51.7454 7847	49.6416 9640	45.7949 8485	42.3763 6443	39.3301 9194
79	52.1586 9317	50.0164 9027	46.1034 3335	42.6303 3359	39.5394 0386
80	52.5673 1092	50.3866 5706	46.4073 2349	42.8799 3474	39.7445 1359
81	52.9713 8286	50.7522 5389	46.7067 2265	43.1252 4298	39.9456 0156
82	53.3709 5957	51.1133 3717	47.0016 9720	43.3663 3217	40.1427 4663
83	53.7660 9104	51.4699 6264	47.2923 1251	43.6032 7486	40.3360 2611
84	54.1568 2674	51.8221 8532	47.5786 3301	43.8361 4237	40.5255 1579
85	54.5432 1557	52.1700 5958	47.8607 2218	44.0650 0479	40.7112 8999
86	54.9253 0588	52.5136 3909	48.1386 4254	44.2899 3099	40.8934 2156
87	55.3031 4549	52.8529 7688	48.4124 5571	44.5109 8869	41.0719 8192
88	55.6767 8169	53.1881 2531	48.6822 2237	44.7282 4441	41.2470 4110
89	56.0462 6126	53.5191 3611	48.9480 0234	44.9417 6355	41.4186 6774
90	56.4116 3041	53.8460 6035	49.2098 5452	45.1516 1037	41.5869 2916
91	56.7729 3490	54.1689 4850	49.4678 3696	45.3578 4803	41.7518 9133
92	57.1302 1992	54.4878 5037	49.7220 0686	45.5605 3860	41.9136 1895
93	57.4835 3021	54.8028 1518	49.9724 2055	45.7597 4310	42.0721 7545
94	57.8329 0997	55.1138 9154	50.2191 3355	45.9555 2147	42.2276 2299
95	58.1784 0294	55.4211 2744	50.4622 0054	46.1479 3265	42.3800 2254
96	58.5200 5235	55.7245 7031	50.7016 7541	46.3370 3455	42.5294 3386
97	58.8579 0096	56.0242 6698	50.9376 1124	46.5228 8408	42.6759 1555
98	59.1919 9106	56.3202 6368	51.1700 6034	46.7055 3718	42.8195 2505
99	59.5223 6446	56.6126 0610	51.3990 7422	46.8850 4882	42.9603 1867
100	59.8490 6251	56.9013 3936	51.6247 0367	47.0614 7304	43.0983 5164

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379	0.9661 8357
2	1.9344 6955	1.9274 2415	1.9204 2434	1.9134 6970	1.8996 9428
3	2.8698 9687	2.8560 2356	2.8422 6213	2.8286 1135	2.8016 3698
4	3.7847 4021	3.7619 7421	3.7394 2787	3.7170 9840	3.6730 7921
5	4.6794 5253	4.6458 2850	4.6125 8186	4.5797 0719	4.5150 5238
6	5.5544 7680	5.5081 2536	5.4623 6678	5.4171 9144	5.3285 5302
7	6.4102 4626	6.3493 9060	6.2894 0806	6.2302 8296	6.1145 4398
8	7.2471 8461	7.1701 3717	7.0943 1441	7.0196 9219	6.8739 5554
9	8.0657 0622	7.9708 6553	7.8776 7826	7.7861 0892	7.6076 8651
10	8.8662 1635	8.7520 6393	8.6400 7616	8.5302 0284	8.3166 0532
11	9.6491 1134	9.5142 0871	9.3820 6926	9.2526 2411	9.0015 5104
12	10.4147 7882	10.2577 6460	10.1042 0366	9.9540 0399	9.6633 3433
13	11.1635 9787	10.9831 8497	10.8070 1086	10.6349 5533	10.3027 3849
14	11.8959 3924	11.6909 1217	11.4910 0814	11.2960 7314	10.9205 2028
15	12.6121 6551	12.3813 7773	12.1566 9892	11.9379 3509	11.5174 1090
16	13.3126 3131	13.0550 0266	12.8045 7315	12.5611 0203	12.0941 1681
17	13.9976 8343	13.7121 9772	13.4351 0769	13.1661 1847	12.6513 2059
18	14.6676 6106	14.3533 6363	14.0487 6661	13.7535 1308	13.1896 8173
19	15.3228 9590	14.9788 9134	14.6460 0157	14.3237 9911	13.7098 3742
20	15.9637 1237	15.5891 6229	15.2272 5213	14.8774 7486	14.2124 0330
21	16.5904 2775	16.1845 4857	15.7929 4612	15.4150 2414	14.6979 7420
22	17.2033 5232	16.7654 1324	16.3434 9987	15.9369 1664	15.1671 2484
23	17.8027 8955	17.3321 1048	16.8793 1861	16.4436 0839	15.6204 1047
24	18.3890 3624	17.8849 8583	17.4007 9670	16.9355 4212	16.0583 6760
25	18.9623 8263	18.4243 7642	17.9083 1795	17.4131 4769	16.4815 1459
26	19.5231 1260	18.9506 1114	18.4022 5592	17.8768 4242	16.8903 5226
27	20.0715 0376	19.4640 1087	18.8829 7413	18.3270 3147	17.2853 6451
28	20.6078 2764	19.9648 8866	19.3508 2640	18.7641 0823	17.6670 1885
29	21.1323 4977	20.4535 4991	19.8061 5708	19.1884 5459	18.0357 6700
30	21.6453 2985	20.9302 9259	20.2493 0130	19.6004 4135	18.3920 4541
31	22.1470 2186	21.3954 0741	20.6805 8520	20.0004 2849	18.7362 7576
32	22.6376 7419	21.8491 7796	21.1003 2623	20.3887 6553	19.0688 6547
33	23.1175 2977	22.2918 8094	21.5088 3332	20.7657 9178	19.3902 0818
34	23.5868 2618	22.7237 8628	21.9064 0712	21.1318 3668	19.7006 8423
35	24.0457 9577	23.1451 5734	22.2933 4026	21.4872 2007	20.0006 6110
36	24.4946 6579	23.5562 5107	22.6699 1753	21.8322 5250	20.2904 9381
37	24.9336 5848	23.9573 1812	23.0364 1609	22.1672 3544	20.5705 2542
38	25.3629 9118	24.3486 0304	23.3931 0568	22.4924 6159	20.8410 8736
39	25.7828 7646	24.7303 4443	23.7402 4884	22.8082 1513	21.1024 9987
40	26.1935 2221	25.1027 7505	24.0781 0106	23.1147 7197	21.3550 7234
41	26.5951 3174	25.4661 2200	24.4069 1101	23.4123 9997	21.5991 0371
42	26.9879 0390	25.8206 0683	24.7269 2069	23.7013 5920	21.8348 8281
43	27.3720 3316	26.1664 4569	25.0383 6563	23.9819 0213	22.0626 8870
44	27.7477 0969	26.5038 4945	25.3414 7507	24.2542 7392	22.2827 9102
45	28.1151 1950	26.8330 2386	25.6364 7209	24.5187 1254	22.4954 5026
46	28.4744 4450	27.1541 6962	25.9235 7381	24.7754 4907	22.7009 1813
47	28.8258 6259	27.4674 8255	26.2029 9154	25.0247 0783	22.8994 3780
48	29.1695 4777	27.7731 5371	26.4749 3094	25.2667 0664	23.0912 4425
49	29.5056 7019	28.0713 6947	26.7395 9215	25.5016 5693	23.2765 6450
50	29.8343 9627	28.3623 1168	26.9971 6998	25.7297 6401	23.4556 1757

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	30.1558 8877	28.6461 5774	27.2478 5400	25.9512 2719	23.6286 1630
52	30.4703 0687	28.9230 8072	27.4918 2871	26.1662 3999	23.7957 6454
53	30.7778 0623	29.1932 4948	27.7292 7368	26.3749 9028	23.9572 6043
54	31.0785 3910	29.4568 2876	27.9603 6368	26.5776 6047	24.1132 9510
55	31.3726 5438	29.7139 7928	28.1852 6879	26.7744 2764	24.2640 5323
56	31.6602 9768	29.9648 5784	28.4041 5454	26.9654 6373	24.4097 1327
57	31.9416 1142	30.2096 1740	28.6171 8203	27.1509 3566	24.5504 4760
58	32.2167 3489	30.4484 0722	28.8245 0806	27.3310 0549	24.6864 2281
59	32.4858 0429	30.6813 7290	29.0262 8522	27.5058 3058	24.8177 9981
60	32.7489 5285	30.9086 5649	29.2226 6201	27.6755 6367	24.9447 3412
61	33.0063 1086	31.1303 9657	29.4137 8298	27.8403 5307	25.0673 7596
62	33.2580 0573	31.3467 2836	29.5997 8879	28.0003 4279	25.1858 7049
63	33.5041 6208	31.5577 8377	29.7808 1634	28.1556 7261	25.3003 5796
64	33.7449 0179	31.7636 9148	29.9569 9887	28.3064 7826	25.4109 7388
65	33.9803 4405	31.9645 7705	30.1284 6605	28.4528 9152	25.5178 4916
66	34.2106 0543	32.1605 6298	30.2953 4409	28.5950 4031	25.6211 1030
67	34.4357 9993	32.3517 6876	30.4577 5581	28.7330 4884	25.7208 7951
68	34.6560 3905	32.5383 1099	30.6158 2074	28.8670 3771	25.8172 7489
69	34.8714 3183	32.7203 0340	30.7696 5522	28.9971 2399	25.9104 1052
70	35.0820 8492	32.8978 5698	30.9193 7247	29.1234 2135	26.0003 9664
71	35.2881 0261	33.0710 7998	31.0650 8270	29.2460 4015	26.0873 3975
72	35.4895 8691	33.2400 7803	31.2068 9314	29.3650 8752	26.1713 4275
73	35.6866 3756	33.4049 5417	31.3449 0816	29.4806 6750	26.2525 0508
74	35.8793 5214	33.5658 0895	31.4792 2936	29.5928 8106	26.3309 2278
75	36.0678 2605	33.7227 4044	31.6099 5558	29.7018 2628	26.4066 8868
76	36.2521 5262	33.8758 4433	31.7371 8304	29.8075 9833	26.4798 9244
77	36.4324 2310	34.0252 1398	31.8610 0540	29.9102 8964	26.5506 2072
78	36.6087 2675	34.1709 4047	31.9815 1377	30.0099 8994	26.6189 5721
79	36.7811 5085	34.3131 1265	32.0987 9685	30.1067 8635	26.6849 8281
80	36.9497 8079	34.4518 1722	32.2129 4098	30.2007 6345	26.7487 7567
81	37.1147 0004	34.5871 3875	32.3240 3015	30.2920 0335	26.8104 1127
82	37.2759 9026	34.7191 5976	32.4321 4613	30.3805 8577	26.8699 6258
83	37.4337 3130	34.8479 6074	32.5373 6850	30.4665 8813	26.9275 0008
84	37.5880 0127	34.9736 2023	32.6397 7469	30.5500 8556	26.9830 9186
85	37.7388 7655	35.0962 1486	32.7394 4009	30.6311 5103	27.0368 0373
86	37.8864 3183	35.2158 1938	32.8364 3804	30.7098 5537	27.0886 9926
87	38.0307 4018	35.3325 0671	32.9308 3994	30.7862 6735	27.1388 3986
88	38.1718 7304	35.4463 4801	33.0227 1527	30.8604 5374	27.1872 8489
89	38.3099 0028	35.5574 1269	33.1121 3165	30.9324 7936	27.2340 9168
90	38.4448 9025	35.6657 6848	33.1991 5489	31.0024 0714	27.2793 1564
91	38.5769 0978	35.7714 8144	33.2838 4905	31.0702 9820	27.3230 1028
92	38.7060 2423	35.8746 1604	33.3662 7644	31.1362 1184	27.3652 2732
93	38.8322 9754	35.9752 3516	33.4464 9776	31.2002 0567	27.4060 1673
94	38.9557 9221	36.0734 0016	33.5245 7202	31.2623 3560	27.4454 2680
95	39.0765 6940	36.1691 7089	33.6005 5671	31.3226 5592	27.4835 0415
96	39.1946 8890	36.2626 0574	33.6745 0775	31.3812 1934	27.5202 9387
97	39.3102 0920	36.3537 6170	33.7464 7956	31.4380 7703	27.5558 3948
98	39.4231 8748	36.4426 9434	33.8165 2512	31.4932 7867	27.5901 8308
99	39.5336 7968	36.5294 5790	33.8846 9598	31.5468 7250	27.6233 6529
100	39.6417 4052	36.6141 0526	33.9510 4232	31.5989 0534	27.6554 2540

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730	0.9433 9623
2	1.8860 9467	1.8726 6775	1.8594 1043	1.8463 1971	1.8333 9267
3	2.7750 9103	2.7489 6435	2.7232 4803	2.6979 3338	2.6730 1195
4	3.6298 9522	3.5875 2570	3.5459 5050	3.5051 5012	3.4651 0561
5	4.4518 2233	4.3899 7674	4.3294 7667	4.2702 8448	4.2123 6379
6	5.2421 3686	5.1578 7248	5.0756 9206	4.9955 3031	4.9173 2433
7	6.0020 5467	5.8927 0094	5.7863 7340	5.6829 6712	5.5823 8144
8	6.7327 4487	6.5958 8607	6.4632 1276	6.3345 6599	6.2097 9381
9	7.4353 3161	7.2687 9050	7.1078 2168	6.9521 9525	6.8016 9227
10	8.1108 9578	7.9127 1818	7.7217 3493	7.5376 2583	7.3600 8705
11	8.7604 7671	8.5289 1692	8.3064 1422	8.0925 3633	7.8868 7458
12	9.3850 7376	9.1185 8078	8.8632 5164	8.6185 1785	8.3838 4394
13	9.9856 4785	9.6828 5242	9.3935 7299	9.1170 7853	8.8526 8296
14	10.5631 2293	10.2228 2528	9.8986 4094	9.5896 4790	9.2949 8393
15	11.1183 8743	10.7395 4573	10.3796 5804	10.0375 8094	9.7122 4899
16	11.6522 9561	11.2340 1505	10.8377 6956	10.4621 6203	10.1058 9527
17	12.1656 6885	11.7071 9143	11.2740 6625	10.8646 0856	10.4772 5969
18	12.6592 9697	12.1599 9180	11.6895 8690	11.2460 7447	10.8276 0348
19	13.1339 3940	12.5932 9359	12.0853 2086	11.6076 5352	11.1581 1649
20	13.5903 2634	13.0079 3645	12.4622 1034	11.9503 8249	11.4699 2122
21	14.0291 5995	13.4047 2388	12.8211 5271	12.2752 4406	11.7640 7662
22	14.4511 1533	13.7844 2476	13.1630 0258	12.5831 6973	12.0415 8172
23	14.8568 4167	14.1477 7489	13.4885 7388	12.8750 4240	12.3033 7898
24	15.2469 6314	14.4954 7837	13.7986 4179	13.1516 9895	12.5503 5753
25	15.6220 7994	14.8282 0896	14.0939 4457	13.4139 3266	12.7833 5616
26	15.9827 6918	15.1466 1145	14.3751 8530	13.6624 9541	13.0031 6619
27	16.3295 8575	15.4513 0282	14.6430 3362	13.8980 9991	13.2105 3414
28	16.6630 6322	15.7428 7351	14.8981 2726	14.1214 2172	13.4061 6428
29	16.9837 1463	16.0218 8853	15.1410 7358	14.3331 0116	13.5907 2102
30	17.2920 3330	16.2888 8854	15.3724 5103	14.5337 4517	13.7648 3115
31	17.5884 9356	16.5443 9095	15.5928 1050	14.7239 2907	13.9290 8599
32	17.8735 5150	16.7888 9086	15.8026 7667	14.9041 9817	14.0840 4339
33	18.1476 4567	17.0228 6207	16.0025 4921	15.0750 6936	14.2302 2961
34	18.4111 9776	17.2467 5796	16.1929 0401	15.2370 3257	14.3681 4114
35	18.6646 1323	17.4610 1240	16.3741 9429	15.3905 5220	14.4982 4636
36	18.9082 8195	17.6660 4058	16.5468 5171	15.5360 6843	14.6209 8713
37	19.1425 7880	17.8622 3979	16.7112 8734	15.6739 9851	14.7367 8031
38	19.3678 6423	18.0499 9023	16.8678 9271	15.8047 3793	14.8460 1916
39	19.5844 8484	18.2296 5572	17.0170 4067	15.9286 6154	14.9490 7468
40	19.7927 7388	18.4015 8442	17.1590 8635	16.0461 2469	15.0462 9687
41	19.9930 5181	18.5661 0949	17.2943 6796	16.1574 6416	15.1380 1592
42	20.1856 2674	18.7235 4975	17.4232 0758	16.2629 9920	15.2245 4332
43	20.3707 9494	18.8742 1029	17.5459 1198	16.3630 3242	15.3061 7294
44	20.5488 4129	19.0183 8305	17.6627 7331	16.4578 5063	15.3831 8202
45	20.7200 3970	19.1563 4742	17.7740 6982	16.5477 2572	15.4558 3209
46	20.8846 5356	19.2883 7074	17.8800 6650	16.6329 1537	15.5243 6990
47	21.0429 3612	19.4147 0884	17.9810 1571	16.7136 6386	15.5890 2821
48	21.1951 3088	19.5356 0654	18.0771 5782	16.7902 0271	15.6500 2661
49	21.3414 7200	19.6512 9813	18.1687 2173	16.8627 5139	15.7075 7227
50	21.4821 8462	19.7620 0778	18.2559 2546	16.9315 1790	15.7618 6064

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	4%	4½%	5%	5½%	6%
51	21.6174 8521	19.8679 5003	18.3389 7663	16.9966 9943	15.8130 7607
52	21.7475 8193	19.9693 3017	18.4180 7298	17.0584 8287	15.8613 9252
53	21.8726 7493	20.0663 4466	18.4934 0284	17 1170 4538	15.9069 7408
54	21.9929 5667	20.1591 8149	18.5651 4556	17 1725 5486	15.9499 7554
55	22.1086 1218	20.2480 2057	18.6334 7196	17 2251 7048	15.9905 4297
56	22.2189 1940	20.3330 3404	18.6985 4473	17.2750 4311	16.0288 1412
57	22.3267 4943	20.4143 8664	18.7605 1879	17.3223 1575	16.0649 1898
58	22.4295 6676	20.4922 3602	18.8195 4170	17.3671 2393	16.0989 8017
59	22.5284 2957	20.5667 3303	18.8757 5400	17 4095 9614	16.1311 1337
60	22.6234 8997	20.6380 2204	18.9292 8952	17 4498 5416	16.1614 2771
61	22.7148 9421	20.7062 4118	18.9802 7574	17 4880 1343	16.1900 2614
62	22.8027 8289	20.7715 2266	19.0288 3404	17.5241 8334	16.2170 0579
63	22.8872 9124	20.8339 9298	19.0750 8003	17 5584 6762	16.2424 5829
64	22.9685 4927	20.8937 7319	19.1191 2384	17.5909 6457	16.2664 7009
65	23.0466 8199	20.9509 7913	19 1610 7033	17 6217 6737	16.2891 2272
66	23.1218 0961	21.0057 2165	19.2010 1936	17.6509 6433	16.3104 9314
67	23.1940 4770	21.0581 0684	19.2390 6606	17.6786 3917	16.3306 5390
68	23.2635 0740	21 1082 3621	19.2753 0101	17 7048 7125	16.3496 7349
69	23.3302 9558	21.1562 0690	19.3098 1048	17 7297 3579	16.3676 1650
70	23.3945 1498	21.2021 1187	19.3426 7665	17 7533 0406	16.3845 4387
71	23.4562 6440	21.2460 4007	19.3739 7776	17 7756 4366	16.4005 1308
72	23.5156 3885	21.2880 7662	19.4037 8834	17 7968 1864	16.4155 7838
73	23.5727 2966	21.3283 0298	19.4321 7937	17.8168 8970	16.4297 9093
74	23.6276 2468	21.3667 9711	19.4592 1845	17.8359 1441	16.4431 9899
75	23.6804 0834	21.4036 3360	19.4849 6995	17 8539 4731	16.4558 4810
76	23.7311 6187	21.4388 8383	19.5094 9519	17 8710 4010	16.4677 8123
77	23.7799 6333	21.4726 1611	19.5328 5257	17.8872 4180	16.4790 3889
78	23.8268 8782	21.5048 9579	19.5550 9768	17.9025 9887	16.4896 5933
79	23.8720 0752	21.5357 8545	19.5762 8351	17.9171 5532	16.4996 7862
80	23.9153 9185	21.5653 4493	19.5964 6048	17 9309 5291	16.5091 3077
81	23.9571 0754	21.5936 3151	19.6156 7665	17.9440 3120	16.5180 4790
82	23.9972 1879	21.6207 0001	19.6339 7776	17 9564 2768	16.5264 6028
83	24.0357 8730	21.6466 0288	19.6514 0739	17 9681 7789	16.5343 9649
84	24.0728 7240	21.6713 9032	19.6680 0704	17.9793 1554	16.5418 8348
85	24.1085 3116	21.6951 1035	19.6838 1623	17 9898 7255	16.5489 4668
86	24.1428 1842	21.7178 0895	19.6988 7260	17.9998 7919	16.5556 1008
87	24.1757 8694	21.7395 3009	19.7132 1200	18.0093 6416	16.5618 9630
88	24.2074 8745	21.7603 1588	19.7268 6857	18.0183 5466	16.5678 2670
89	24.2379 6870	21.7802 0658	19.7398 7483	18.0268 7645	16.5734 2141
90	24.2672 7759	21.7992 4075	19.7522 6174	18.0349 5398	16.5786 9944
91	24.2954 5923	21.8174 5526	19.7640 5880	18.0426 1041	16.5836 7872
92	24.3225 5695	21.8348 8542	19.7752 9410	18.0498 6769	16.5883 7615
93	24.3486 1245	21.8515 6499	19.7859 9438	18.0567 4662	16.5928 0769
94	24.3736 6582	21.8675 2631	19.7961 8512	18.0632 6694	16.5969 8839
95	24.3977 5559	21.8828 0030	19.8058 9059	18.0694 4734	16.6009 3244
96	24.4209 1884	21.8974 1655	19.8151 3390	18.0753 0553	16.6046 5325
97	24.4431 9119	21.9114 0340	19.8239 3705	18.0808 5833	16.6081 6344
98	24.4646 0692	21.9247 8794	19.8323 2100	18.0861 2164	16.6114 7494
99	24.4851 9896	21.9375 9612	19.8403 0571	18.0911 1055	16.6145 9900
100	24.5049 9900	21.9498 5274	19.8479 1020	18.0958 3939	16.6175 4623

Table VI.—Present Value of Annuity of 1 per Period

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593	0.9216 5899
2	1.8206 2642	1.8080 1817	1.7955 6517	1.7832 6475	1.7711 1427
3	2.6484 7551	2.6243 1604	2.6005 2574	2.5770 9699	2.5540 2237
4	3.4257 9860	3.3872 1126	3.3493 2627	3.3121 2684	3.2755 9666
5	4.1556 7944	4.1001 9744	4.0458 8490	3.9927 1004	3.9406 4208
6	4.8410 1356	4.7665 3966	4.6938 4642	4.6228 7966	4.5535 8717
7	5.4845 1977	5.3892 8940	5.2966 0132	5.2063 7006	5.1185 1352
8	6.0887 5096	5.9712 9851	5.8573 0355	5.7466 3894	5.6391 8297
9	6.6561 0419	6.5152 3225	6.3788 8703	6.2468 8791	6.1190 6264
10	7 1888 3022	7.0235 8154	6.8640 8096	6.7100 8140	6.5613 4806
11	7.6890 4246	7.4986 7434	7.3154 2415	7.1389 6426	6.9689 8439
12	8.1587 2532	7.9426 8630	7.7352 7827	7.5360 7802	7.3446 8607
13	8.5997 4208	8.3576 5074	8.1258 4026	7.9037 7594	7.6909 5490
14	9.0138 4233	8.7454 6799	8.4891 5373	8.2442 3698	8.0100 9668
15	9.4026 6885	9.1079 1401	8.8271 1974	8.5594 7869	8.3042 3658
16	9.7677 6418	9.4466 4860	9.1415 0674	8.8513 6916	8.5753 3325
17	10.1105 7670	9.7632 2299	9.4339 5976	9.1216 3811	8.8251 9194
18	10.4324 6638	10.0590 8691	9.7060 0908	9.3718 8714	9.0554 7644
19	10.7347 1022	10.3355 9524	9.9590 7821	9.6035 9920	9.2677 2022
20	11.0185 0725	10.5940 1425	10.1944 9136	9.8181 4741	9.4633 3661
21	11.2849 8333	10.8355 2733	10.4134 8033	10.0168 0316	9.6436 2821
22	11.5351 9562	11.0612 4050	10.6171 9101	10.2007 4366	9.8097 9559
23	11.7701 3673	11.2721 8738	10.8066 8931	10.3710 5895	9.9629 4524
24	11.9907 3871	11.4693 3400	10.9829 6680	10.5287 5828	10.1040 9700
25	12.1978 7672	11.6535 8318	11.1469 4586	10.6747 7619	10.2341 9078
26	12.3923 7251	11.8257 7867	11.2994 8452	10.8099 7795	10.3540 9288
27	12.5749 9766	11.9867 0904	11.4413 8095	10.9351 6477	10.4646 0174
28	12.7464 7668	12.1371 1125	11.5733 7763	11.0510 7849	10.5664 5321
29	12.9074 8984	12.2776 7407	11.6961 6524	11.1584 0601	10.6603 2554
30	13.0586 7591	12.4090 4118	11.8103 8627	11.2577 8334	10.7468 4382
31	13.2006 3465	12.5318 1419	11.9166 3839	11.3497 9939	10.8265 8416
32	13.3339 2925	12.6465 5532	12.0154 7757	11.4349 9944	10.9000 7757
33	13.4590 8850	12.7537 9002	12.1074 2099	11.5138 8837	10.9678 1343
34	13.5766 0892	12.8540 0936	12.1929 4976	11.5869 3367	11.0302 4279
35	13.6869 5673	12.9476 7230	12.2725 1141	11.6545 6822	11.0877 8137
36	13.7905 6970	13.0352 0776	12.3465 2224	11.7171 9279	11.1408 1233
37	13.8878 5887	13.1170 1660	12.4153 6953	11.7751 7851	11.1896 8878
38	13.9792 1021	13.1934 7345	12.4794 1351	11.8288 6899	11.2347 3620
39	14.0649 8611	13.2649 2846	12.5389 8931	11.8785 8240	11.2762 5457
40	14.1455 2687	13.3317 0884	12.5944 0866	11.9246 1333	11.3145 2034
41	14.2211 5199	13.3941 2041	12.6459 6155	11.9672 3457	11.3497 8833
42	14.2921 6149	13.4524 4898	12.6939 1772	12.0066 9867	11.3822 9339
43	14.3588 3708	13.5069 6167	12.7885 2811	12.0432 3951	11.4122 5197
44	14.4214 4327	13.5579 0810	12.7800 2615	12.0770 7362	11.4398 6357
45	14.4802 2842	13.6055 2159	12.8186 2898	12.1084 0150	11.4653 1205
46	14.5354 2575	13.6500 2018	12.8545 3858	12.1374 0880	11.4887 6686
47	14.5872 5422	13.6916 0764	12.8879 4287	12.1642 6741	11.5103 8420
48	14.6359 1946	13.7304 7443	12.9190 1662	12.1891 3649	11.5303 0802
49	14.6816 1451	13.7667 9853	12.9479 2244	12.2121 6341	11.5486 7099
50	14.7245 2067	13.8007 4629	12.9748 1157	12.2334 8464	11.5655 9538

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.0041 6667	1.0050 0000	1.0058 3333	1.0075 0000	1.0100 0000
2	0.5031 2717	0.5037 5312	0.5043 7924	0.5056 3200	0.5075 1244
3	0.3361 1496	0.3366 7221	0.3372 2976	0.3383 4579	0.3400 2211
4	0.2526 0958	0.2531 3279	0.2536 5644	0.2547 0501	0.2562 8109
5	0.2025 0693	0.2030 0997	0.2035 1357	0.2045 2242	0.2060 3980
6	0.1691 0564	0.1695 9546	0.1700 8594	0.1710 6891	0.1725 4837
7	0.1452 4800	0.1457 2854	0.1462 0986	0.1471 7488	0.1486 2828
8	0.1273 5512	0.1278 2886	0.1283 0351	0.1292 5552	0.1306 9029
9	0.1134 3876	0.1139 0736	0.1143 7698	0.1153 1929	0.1167 4037
10	0.1023 0596	0.1027 7057	0.1032 3632	0.1041 7123	0.1055 8208
11	0.0931 9757	0.0936 5903	0.0941 2175	0.0950 5094	0.0964 5408
12	0.0856 0748	0.0860 6643	0.0865 2675	0.0874 5148	0.0888 4879
13	0.0791 8532	0.0796 4224	0.0801 0064	0.0810 2188	0.0824 1482
14	0.0736 8082	0.0741 3609	0.0745 9295	0.0755 1146	0.0769 0117
15	0.0689 1045	0.0693 6436	0.0698 1999	0.0707 3639	0.0721 2378
16	0.0647 3655	0.0651 8937	0.0656 4401	0.0665 5879	0.0679 4460
17	0.0610 5387	0.0615 0579	0.0619 5966	0.0628 7321	0.0642 5806
18	0.0577 8053	0.0582 3173	0.0586 8499	0.0595 9766	0.0609 8205
19	0.0548 5191	0.0553 0253	0.0557 5532	0.0566 6740	0.0580 5175
20	0.0522 1630	0.0526 6645	0.0531 1889	0.0540 3063	0.0554 1532
21	0.0498 3183	0.0502 8163	0.0507 3383	0.0516 4543	0.0530 3075
22	0.0476 6427	0.0481 1380	0.0485 6585	0.0494 7748	0.0508 6371
23	0.0456 8531	0.0461 3465	0.0465 8663	0.0474 9846	0.0488 8584
24	0.0438 7139	0.0443 2061	0.0447 7258	0.0456 8474	0.0470 7347
25	0.0422 0270	0.0426 5186	0.0431 0388	0.0440 1650	0.0454 0675
26	0.0406 6247	0.0411 1163	0.0415 6376	0.0424 7693	0.0438 6888
27	0.0392 3645	0.0396 8565	0.0401 3793	0.0410 5176	0.0424 4553
28	0.0379 1239	0.0383 6167	0.0388 1415	0.0397 2871	0.0411 2444
29	0.0366 7974	0.0371 2914	0.0375 8186	0.0384 9723	0.0398 9502
30	0.0355 2936	0.0359 7892	0.0364 3191	0.0373 4816	0.0387 4811
31	0.0344 5330	0.0349 0304	0.0353 5633	0.0362 7352	0.0376 7573
32	0.0334 4458	0.0338 9453	0.0343 4815	0.0352 6634	0.0366 7089
33	0.0324 9708	0.0329 4727	0.0334 0124	0.0343 2048	0.0357 2744
34	0.0316 0540	0.0320 5586	0.0325 1020	0.0334 3053	0.0348 3997
35	0.0307 6476	0.0312 1550	0.0316 7024	0.0325 9170	0.0340 0368
36	0.0299 7090	0.0304 2194	0.0308 7710	0.0317 9973	0.0332 1431
37	0.0292 2003	0.0296 7139	0.0301 2698	0.0310 5082	0.0324 6805
38	0.0285 0875	0.0289 6045	0.0294 1649	0.0303 4157	0.0317 6150
39	0.0278 3402	0.0282 8607	0.0287 4258	0.0296 6893	0.0310 9160
40	0.0271 9310	0.0276 4552	0.0281 0251	0.0290 3016	0.0304 5560
41	0.0263 8352	0.0270 3631	0.0274 9379	0.0284 2276	0.0298 5102
42	0.0260 0303	0.0264 5622	0.0269 1420	0.0278 4452	0.0292 7563
43	0.0254 4961	0.0259 0320	0.0263 6170	0.0272 9338	0.0287 2737
44	0.0249 2141	0.0253 7541	0.0258 3443	0.0267 6751	0.0282 0441
45	0.0244 1675	0.0248 7117	0.0253 3073	0.0262 6521	0.0277 0505
46	0.0239 3409	0.0243 8894	0.0248 4905	0.0257 8495	0.0272 2775
47	0.0234 7204	0.0239 2733	0.0243 8798	0.0253 2532	0.0267 7111
48	0.0230 2929	0.0234 8503	0.0239 4624	0.0248 8504	0.0263 3384
49	0.0226 0468	0.0230 6087	0.0235 2265	0.0244 6292	0.0259 1474
50	0.0221 9711	0.0226 5376	0.0231 1611	0.0240 5787	0.0255 1273

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	0.0218 0557	0.0222 6269	0.0227 2563	0.0236 6888	0.0251 2680
52	0.0214 2916	0.0218 8675	0.0223 5027	0.0232 9503	0.0247 5603
53	0.0210 6700	0.0215 2507	0.0219 8919	0.0229 3546	0.0243 9956
54	0.0207 1830	0.0211 7686	0.0216 4157	0.0225 8938	0.0240 5658
55	0.0203 8234	0.0208 4139	0.0213 0671	0.0222 5605	0.0237 2637
56	0.0200 5843	0.0205 1797	0.0209 8390	0.0219 3478	0.0234 0823
57	0.0197 4593	0.0202 0598	0.0206 7251	0.0216 2496	0.0231 0156
58	0.0194 4426	0.0199 0481	0.0203 7196	0.0213 2597	0.0228 0573
59	0.0191 5287	0.0196 1392	0.0200 8170	0.0210 3727	0.0225 2020
60	0.0188 7123	0.0193 3280	0.0198 0120	0.0207 5836	0.0222 4445
61	0.0185 9888	0.0190 6096	0.0195 2999	0.0204 8873	0.0219 7800
62	0.0183 3536	0.0187 9796	0.0192 6762	0.0202 2795	0.0217 2041
63	0.0180 8025	0.0185 4337	0.0190 1366	0.0199 7560	0.0214 7125
64	0.0178 3315	0.0182 9681	0.0187 6773	0.0197 3127	0.0212 3013
65	0.0175 9371	0.0180 5789	0.0185 2946	0.0194 9460	0.0209 9667
66	0.0173 6156	0.0178 2627	0.0182 9848	0.0192 6524	0.0207 7052
67	0.0171 3639	0.0176 0163	0.0180 7449	0.0190 4286	0.0205 5136
68	0.0169 1788	0.0173 8366	0.0178 5716	0.0188 2716	0.0203 3888
69	0.0167 0574	0.0171 7206	0.0176 4622	0.0186 1785	0.0201 3280
70	0.0164 9971	0.0169 6657	0.0174 4138	0.0184 1464	0.0199 3282
71	0.0162 9952	0.0167 6693	0.0172 4239	0.0182 1728	0.0197 3870
72	0.0161 0493	0.0165 7289	0.0170 4901	0.0180 2554	0.0195 5019
73	0.0159 1572	0.0163 8422	0.0168 6100	0.0178 3917	0.0193 6706
74	0.0157 3165	0.0162 0070	0.0166 7814	0.0176 5796	0.0191 8910
75	0.0155 5253	0.0160 2214	0.0165 0024	0.0174 8170	0.0190 1609
76	0.0153 7816	0.0158 4832	0.0163 2709	0.0173 1020	0.0188 4784
77	0.0152 0836	0.0156 7908	0.0161 5851	0.0171 4328	0.0186 8416
78	0.0150 4295	0.0155 1423	0.0159 9432	0.0169 8074	0.0185 2488
79	0.0148 8177	0.0153 5360	0.0158 3436	0.0168 2244	0.0183 6984
80	0.0147 2464	0.0151 9704	0.0156 7847	0.0166 6821	0.0182 1885
81	0.0145 7144	0.0150 4439	0.0155 2650	0.0165 1790	0.0180 7180
82	0.0144 2200	0.0148 9552	0.0153 7830	0.0163 7136	0.0179 2851
83	0.0142 7620	0.0147 5028	0.0152 3373	0.0162 2847	0.0177 8886
84	0.0141 3391	0.0146 0855	0.0150 9268	0.0160 8908	0.0176 5273
85	0.0139 9500	0.0144 7021	0.0149 5501	0.0159 5308	0.0175 1998
86	0.0138 5935	0.0143 3513	0.0148 2060	0.0158 2034	0.0173 9050
87	0.0137 2685	0.0142 0320	0.0146 8935	0.0156 9076	0.0172 6417
89	0.0135 9740	0.0140 7431	0.0145 6115	0.0155 6423	0.0171 4089
89	0.0134 7088	0.0139 4837	0.0144 3588	0.0154 4064	0.0170 2056
90	0.0133 4721	0.0138 2527	0.0143 1347	0.0153 1989	0.0169 0306
91	0.0132 2629	0.0137 0493	0.0141 9380	0.0152 0190	0.0167 8832
92	0.0131 0803	0.0135 8724	0.0140 7679	0.0150 8657	0.0166 7624
93	0.0129 9234	0.0134 7213	0.0139 6236	0.0149 7382	0.0165 6673
94	0.0128 7915	0.0133 5950	0.0138 5042	0.0148 6356	0.0164 5971
95	0.0127 6837	0.0132 4930	0.0137 4090	0.0147 5571	0.0163 5511
96	0.0126 5992	0.0131 4143	0.0136 3372	0.0146 5020	0.0162 5284
97	0.0125 5374	0.0130 3583	0.0135 2880	0.0145 4696	0.0161 5284
98	0.0124 4976	0.0129 3242	0.0134 2608	0.0144 4592	0.0160 5503
99	0.0123 4790	0.0128 3115	0.0133 2549	0.0143 4701	0.0159 5936
100	0.0122 4811	0.0127 3194	0.0132 2696	0.0142 5017	0.0158 6574

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
101	0.0121 5033	0.0126 3473	0.0131 3045	0.0141 5533	0.0157 7413
102	0.0120 5449	0.0125 3947	0.0130 3587	0.0140 6243	0.0156 8446
103	0.0119 6054	0.0124 4611	0.0129 4319	0.0139 7143	0.0155 9668
104	0.0118 6842	0.0123 5457	0.0128 5234	0.0138 8226	0.0155 1073
105	0.0117 7809	0.0122 6481	0.0127 6238	0.0137 9487	0.0154 2656
106	0.0116 8948	0.0121 7679	0.0126 7594	0.0137 0922	0.0153 4412
107	0.0116 0256	0.0120 9045	0.0125 9029	0.0136 2524	0.0152 6336
108	0.0115 1727	0.0120 0575	0.0125 0628	0.0135 4291	0.0151 8423
109	0.0114 3358	0.0119 2264	0.0124 2385	0.0134 6217	0.0151 0669
110	0.0113 5143	0.0118 4107	0.0123 4298	0.0133 8296	0.0150 3069
111	0.0112 7079	0.0117 6102	0.0122 6361	0.0133 0527	0.0149 5620
112	0.0111 9161	0.0116 8242	0.0121 8571	0.0132 2905	0.0148 8317
113	0.0111 1386	0.0116 0526	0.0121 0923	0.0131 5425	0.0148 1156
114	0.0110 3750	0.0115 2948	0.0120 3414	0.0130 8084	0.0147 4133
115	0.0109 6249	0.0114 5506	0.0119 6041	0.0130 0878	0.0146 7245
116	0.0108 8880	0.0113 8195	0.0118 8799	0.0129 3803	0.0146 0488
117	0.0108 1639	0.0113 1013	0.0118 1686	0.0128 6857	0.0145 3860
118	0.0107 4524	0.0112 3956	0.0117 4698	0.0128 0037	0.0144 7356
119	0.0106 7530	0.0111 7021	0.0116 7832	0.0127 3338	0.0144 0973
120	0.0106 0655	0.0111 0205	0.0116 1085	0.0126 6758	0.0143 4709
121	0.0105 3896	0.0110 3505	0.0115 4454	0.0126 0294	0.0142 8561
122	0.0104 7251	0.0109 6918	0.0114 7936	0.0125 3942	0.0142 2525
123	0.0104 0715	0.0109 0441	0.0114 1528	0.0124 7702	0.0141 6599
124	0.0103 4288	0.0108 4072	0.0113 5228	0.0124 1568	0.0141 0780
125	0.0102 7965	0.0107 7808	0.0112 9033	0.0123 5540	0.0140 5065
126	0.0102 1745	0.0107 1647	0.0112 2940	0.0122 9614	0.0139 9452
127	0.0101 5625	0.0106 5586	0.0111 6948	0.0122 3788	0.0139 3939
128	0.0100 9603	0.0105 9623	0.0111 1054	0.0121 8060	0.0138 8524
129	0.0100 3677	0.0105 3755	0.0110 5255	0.0121 2428	0.0138 3203
130	0.0099 7844	0.0104 7981	0.0109 9550	0.0120 6888	0.0137 7975
131	0.0099 2102	0.0104 2298	0.0109 3935	0.0120 1440	0.0137 2837
132	0.0098 6449	0.0103 6704	0.0108 8410	0.0119 6080	0.0136 7788
133	0.0098 0883	0.0103 1197	0.0108 2972	0.0119 0808	0.0136 2825
134	0.0097 5403	0.0102 5775	0.0107 7619	0.0118 5621	0.0135 7947
135	0.0097 0005	0.0102 0436	0.0107 2349	0.0118 0516	0.0135 3151
136	0.0096 4689	0.0101 5179	0.0106 7161	0.0117 5493	0.0134 8437
137	0.0095 9453	0.0101 0002	0.0106 2052	0.0117 0550	0.0134 3801
138	0.0095 4295	0.0100 4902	0.0105 7021	0.0116 5684	0.0133 9242
139	0.0094 9213	0.0099 9879	0.0105 2067	0.0116 0894	0.0133 4759
140	0.0094 4205	0.0099 4930	0.0104 7187	0.0115 6179	0.0133 0349
141	0.0093 9271	0.0099 0055	0.0104 2380	0.0115 1536	0.0132 6012
142	0.0093 4408	0.0098 5250	0.0103 7644	0.0114 6965	0.0132 1746
143	0.0092 9615	0.0098 0616	0.0103 2978	0.0114 2464	0.0131 7549
144	0.0092 4890	0.0097 5850	0.0102 8381	0.0113 8031	0.0131 3419
145	0.0092 0233	0.0097 1252	0.0102 3851	0.0113 3664	0.0130 9356
146	0.0091 5641	0.0096 6719	0.0101 9386	0.0112 9364	0.0130 5358
147	0.0091 1114	0.0096 2250	0.0101 4986	0.0112 5127	0.0130 1423
148	0.0090 6650	0.0095 7844	0.0101 0649	0.0112 0953	0.0129 7551
149	0.0090 2247	0.0095 3500	0.0100 6373	0.0111 6841	0.0129 3739
150	0.0089 7905	0.0094 9217	0.0100 2159	0.0111 2790	0.0128 9988

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.0112 5000	1.0125 0000	1.0150 0000	1.0175 0000	1.0200 0000
2	0.5084 5323	0.5093 9441	0.5112 7792	0.5131 6295	0.5150 4950
3	0.3408 6130	0.3417 0117	0.3433 8296	0.3450 6746	0.3467 5467
4	0.2570 7058	0.2578 6102	0.2594 4478	0.2610 3237	0.2626 2375
5	0.2068 0034	0.2075 6211	0.2090 8932	0.2106 2142	0.2121 5839
6	0.1732 9034	0.1740 3381	0.1755 2521	0.1770 2256	0.1785 2581
7	0.1493 5762	0.1500 8872	0.1515 5616	0.1530 3059	0.1545 1196
8	0.1314 1071	0.1321 3314	0.1335 8402	0.1350 4292	0.1365 0980
9	0.1174 5432	0.1181 7055	0.1196 0982	0.1210 5813	0.1225 1544
10	0.1062 9131	0.1070 0307	0.1034 3418	0.1098 7534	0.1113 2653
11	0.0971 5984	0.0978 6839	0.0992 9384	0.1007 3038	0.1021 7794
12	0.0895 5203	0.0902 5831	0.0916 7999	0.0931 1377	0.0945 5960
13	0.0831 1626	0.0838 2100	0.0852 4036	0.0866 7283	0.0881 1835
14	0.0776 0138	0.0783 0515	0.0797 2332	0.0811 5562	0.0826 0197
15	0.0728 2321	0.0735 2646	0.0749 4436	0.0763 7739	0.0778 2547
16	0.0686 4363	0.0693 4672	0.0707 6508	0.0721 9958	0.0736 5013
17	0.0649 5698	0.0656 6023	0.0670 7966	0.0685 1623	0.0699 6984
18	0.0616 8113	0.0623 8479	0.0638 0578	0.0652 4492	0.0667 0210
19	0.0587 5120	0.0594 5548	0.0608 7847	0.0623 2061	0.0637 8177
20	0.0561 1531	0.0568 2039	0.0582 4574	0.0596 9122	0.0611 5672
21	0.0537 3145	0.0544 3748	0.0558 6550	0.0573 1464	0.0587 8477
22	0.0515 6525	0.0522 7238	0.0537 0331	0.0551 5638	0.0566 3140
23	0.0495 8833	0.0502 9666	0.0517 3075	0.0581 8796	0.0546 6810
24	0.0477 7701	0.0484 8665	0.0499 2410	0.0513 8565	0.0528 7110
25	0.0461 1144	0.0468 2247	0.0482 6345	0.0497 2952	0.0512 2044
26	0.0445 7479	0.0452 8729	0.0467 3196	0.0482 0269	0.0496 9923
27	0.0431 5273	0.0438 6677	0.0453 1527	0.0467 9079	0.0482 9309
28	0.0418 3299	0.0425 4863	0.0440 0108	0.0454 8151	0.0469 8967
29	0.0406 0498	0.0413 2228	0.0427 7878	0.0442 6424	0.0457 7836
30	0.0394 5953	0.0401 7854	0.0416 3919	0.0431 2975	0.0446 4992
31	0.0383 8866	0.0391 0942	0.0405 7430	0.0420 7005	0.0435 9635
32	0.0373 8535	0.0381 0791	0.0395 7710	0.0410 7812	0.0426 1061
33	0.0364 4349	0.0371 6786	0.0386 4144	0.0401 4779	0.0416 8653
34	0.0355 5763	0.0362 8387	0.0377 6189	0.0392 7363	0.0408 1867
35	0.0347 2299	0.0354 5111	0.0369 3363	0.0384 5082	0.0400 0221
36	0.0339 3529	0.0346 6533	0.0361 5240	0.0376 7507	0.0392 3285
37	0.0331 9072	0.0339 2270	0.0354 1437	0.0369 4257	0.0385 0678
38	0.0324 8589	0.0332 1983	0.0347 1613	0.0362 4990	0.0378 2057
39	0.0318 1773	0.0325 5365	0.0340 5463	0.0355 9399	0.0371 7114
40	0.0311 8349	0.0319 2141	0.0334 2710	0.0349 7209	0.0365 5575
41	0.0305 8069	0.0313 2063	0.0328 3106	0.0343 8170	0.0359 7188
42	0.0300 0709	0.0307 4906	0.0322 6426	0.0338 2057	0.0354 1729
43	0.0294 6064	0.0302 0466	0.0317 2465	0.0332 8666	0.0348 8993
44	0.0289 3949	0.0296 8557	0.0312 1038	0.0327 7810	0.0343 8794
45	0.0284 4197	0.0291 9012	0.0307 1976	0.0322 9321	0.0339 0962
46	0.0279 6652	0.0287 1675	0.0302 5125	0.0318 3043	0.0334 5342
47	0.0275 1173	0.0282 6406	0.0298 0342	0.0313 8836	0.0330 1792
48	0.0270 7632	0.0278 3075	0.0293 7500	0.0309 6569	0.0326 0184
49	0.0266 5910	0.0274 1563	0.0289 6478	0.0305 6124	0.0322 0396
50	0.0262 5898	0.0270 1763	0.0285 7168	0.0301 7391	0.0318 2321

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	0.0258 7494	0.0266 3571	0.0281 9469	0.0298 0269	0.0314 5856
52	0.0255 0606	0.0262 6897	0.0278 3287	0.0294 4665	0.0311 0909
53	0.0251 5149	0.0259 1653	0.0274 8537	0.0291 0492	0.0307 7392
54	0.0248 1043	0.0255 7760	0.0271 5138	0.0287 7672	0.0304 5226
55	0.0244 8213	0.0252 5145	0.0268 3018	0.0284 6129	0.0301 4337
56	0.0241 6592	0.0249 3739	0.0265 2106	0.0281 5795	0.0298 4656
57	0.0238 6116	0.0246 3478	0.0262 2341	0.0278 6606	0.0295 6120
58	0.0235 6726	0.0243 4303	0.0259 3661	0.0275 8503	0.0292 8667
59	0.0232 8366	0.0240 6158	0.0256 6012	0.0273 1430	0.0290 2243
60	0.0230 0985	0.0237 8993	0.0253 9343	0.0270 5336	0.0287 6797
61	0.0227 4534	0.0235 2758	0.0251 3604	0.0268 0172	0.0285 2278
62	0.0224 8969	0.0232 7410	0.0248 8751	0.0265 5892	0.0282 8643
63	0.0222 4247	0.0230 2904	0.0246 4741	0.0263 2455	0.0280 5848
64	0.0220 0329	0.0227 9203	0.0244 1534	0.0260 9821	0.0278 3855
65	0.0217 7178	0.0225 6268	0.0241 9094	0.0258 7952	0.0276 2624
66	0.0215 4758	0.0223 4065	0.0239 7386	0.0256 6813	0.0274 2122
67	0.0213 3037	0.0221 2560	0.0237 6376	0.0254 6372	0.0272 2316
68	0.0211 1985	0.0219 1724	0.0235 6033	0.0252 6596	0.0270 3173
69	0.0209 1571	0.0217 1527	0.0233 6329	0.0250 7459	0.0268 4665
70	0.0207 1769	0.0215 1941	0.0231 7235	0.0248 8930	0.0266 6765
71	0.0205 2552	0.0213 2941	0.0229 8727	0.0247 0985	0.0264 9446
72	0.0203 3896	0.0211 4501	0.0228 0779	0.0245 3600	0.0263 2683
73	0.0201 5779	0.0209 6600	0.0226 3368	0.0243 6750	0.0261 6454
74	0.0199 8177	0.0207 9215	0.0224 6473	0.0242 0413	0.0260 0736
75	0.0198 1072	0.0206 2325	0.0223 0072	0.0240 4570	0.0258 5508
76	0.0196 4442	0.0204 5910	0.0221 4146	0.0238 9200	0.0257 0751
77	0.0194 8269	0.0202 9953	0.0219 8676	0.0237 4284	0.0255 6447
78	0.0193 2536	0.0201 4435	0.0218 3645	0.0235 9806	0.0254 2576
79	0.0191 7226	0.0199 9341	0.0216 9036	0.0234 5748	0.0252 9123
80	0.0190 2323	0.0198 4652	0.0215 4832	0.0233 2093	0.0251 6071
81	0.0188 7812	0.0197 0356	0.0214 1019	0.0231 8828	0.0250 3405
82	0.0187 3678	0.0195 6437	0.0212 7583	0.0230 5936	0.0249 1110
83	0.0185 9908	0.0194 2881	0.0211 4509	0.0229 3406	0.0247 9173
84	0.0184 6489	0.0192 9675	0.0210 1784	0.0228 1223	0.0246 7581
85	0.0183 3409	0.0191 6808	0.0208 9396	0.0226 9375	0.0245 6321
86	0.0182 0654	0.0190 4267	0.0207 7333	0.0225 7850	0.0244 5381
87	0.0180 8215	0.0189 2041	0.0206 5584	0.0224 6636	0.0243 4750
88	0.0179 6081	0.0188 0119	0.0205 4138	0.0223 5724	0.0242 4416
89	0.0178 4240	0.0186 8490	0.0204 2984	0.0222 5102	0.0241 4370
90	0.0177 2684	0.0185 7146	0.0203 2113	0.0221 4760	0.0240 4602
91	0.0176 1403	0.0184 6076	0.0202 1516	0.0220 4690	0.0239 5101
92	0.0175 0387	0.0183 5271	0.0201 1182	0.0219 4882	0.0238 5859
93	0.0173 9629	0.0182 4724	0.0200 1104	0.0218 5327	0.0237 6868
94	0.0172 9119	0.0181 4425	0.0199 1273	0.0217 6017	0.0236 8118
95	0.0171 8851	0.0180 4366	0.0198 1681	0.0216 6944	0.0235 9602
96	0.0170 8816	0.0179 4540	0.0197 2321	0.0215 8101	0.0235 1313
97	0.0169 9007	0.0178 4941	0.0196 3186	0.0214 9480	0.0234 3242
98	0.0168 9418	0.0177 5560	0.0195 4268	0.0214 1074	0.0233 5383
99	0.0168 0041	0.0176 6391	0.0194 5560	0.0213 2876	0.0232 7729
100	0.0167 0870	0.0175 7428	0.0193 7057	0.0212 4880	0.0232 0274

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
1	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000	1.0350 0000
2	0.5169 3758	0.5188 2716	0.5207 1825	0.5226 1084	0.5264 0049
3	0.3484 4458	0.3501 3717	0.3518 3243	0.3535 3036	0.3569 3418
4	0.2642 1893	0.2658 1788	0.2674 2059	0.2690 2705	0.2722 5114
5	0.2137 0021	0.2152 4686	0.2167 9832	0.2183 5457	0.2214 8137
6	0.1800 3496	0.1815 4997	0.1830 7083	0.1845 9750	0.1876 6821
7	0.1560 0025	0.1574 9543	0.1589 9747	0.1605 0635	0.1635 4449
8	0.1379 8462	0.1394 6735	0.1409 5795	0.1424 5639	0.1454 7665
9	0.1239 8170	0.1254 5689	0.1269 4095	0.1284 3386	0.1314 4601
10	0.1127 8768	0.1142 5876	0.1157 3972	0.1172 3051	0.1202 4137
11	0.1036 3649	0.1051 0596	0.1065 8629	0.1080 7745	0.1110 9197
12	0.0960 1740	0.0974 8713	0.0989 6871	0.1004 6209	0.1034 8395
13	0.0895 7686	0.0910 4827	0.0925 3252	0.0940 2954	0.0970 6157
14	0.0840 6230	0.0855 3653	0.0870 2457	0.0885 2634	0.0915 7073
15	0.0792 8852	0.0807 6646	0.0822 5917	0.0837 6658	0.0868 2507
16	0.0751 1663	0.0765 9899	0.0780 9710	0.0796 1085	0.0826 8483
17	0.0714 4039	0.0729 2777	0.0744 3186	0.0759 5253	0.0790 4313
18	0.0681 7720	0.0696 7008	0.0711 8063	0.0727 0870	0.0758 1684
19	0.0652 6182	0.0667 6062	0.0682 7802	0.0698 1388	0.0729 4033
20	0.0626 4207	0.0641 4713	0.0656 7173	0.0672 1571	0.0703 6108
21	0.0602 7572	0.0617 8733	0.0633 1941	0.0648 7178	0.0680 3659
22	0.0581 2821	0.0596 4661	0.0611 8640	0.0627 4739	0.0659 3207
23	0.0561 7097	0.0576 9638	0.0592 4410	0.0608 1390	0.0640 1880
24	0.0543 8023	0.0559 1282	0.0574 6863	0.0590 4742	0.0622 7283
25	0.0527 3599	0.0542 7592	0.0558 3997	0.0574 2787	0.0606 7404
26	0.0512 2134	0.0527 6875	0.0543 4116	0.0559 3829	0.0592 0540
27	0.0498 2188	0.0513 7687	0.0529 5776	0.0545 6421	0.0578 5241
28	0.0485 2525	0.0500 8793	0.0516 7738	0.0532 9323	0.0566 0265
29	0.0473 2081	0.0488 9127	0.0504 8935	0.0521 1467	0.0554 4538
30	0.0461 9934	0.0477 7764	0.0493 8442	0.0510 1926	0.0543 7133
31	0.0451 5280	0.0467 3900	0.0483 5453	0.0499 9893	0.0533 7240
32	0.0441 7415	0.0457 6831	0.0473 9263	0.0490 4662	0.0524 4150
33	0.0432 5722	0.0448 5938	0.0464 9253	0.0481 5612	0.0515 7242
34	0.0423 9655	0.0440 0675	0.0456 4875	0.0473 2196	0.0507 5966
35	0.0415 8731	0.0432 0558	0.0448 5645	0.0465 3929	0.0499 9835
36	0.0408 2522	0.0424 5158	0.0441 1132	0.0458 0379	0.0492 8416
37	0.0401 0643	0.0417 4090	0.0434 0953	0.0451 1162	0.0486 1325
38	0.0394 2753	0.0410 7012	0.0427 4764	0.0444 5934	0.0479 8214
39	0.0387 8543	0.0404 3615	0.0421 2256	0.0438 4385	0.0473 8775
40	0.0381 7738	0.0398 3623	0.0415 3151	0.0432 6238	0.0468 2728
41	0.0376 0087	0.0392 6786	0.0409 7200	0.0427 1241	0.0462 9822
42	0.0370 5364	0.0387 2876	0.0404 4175	0.0421 9167	0.0457 9828
43	0.0365 3364	0.0382 1688	0.0399 3871	0.0416 9811	0.0453 2539
44	0.0360 3901	0.0377 3037	0.0394 6100	0.0412 2935	0.0448 7768
45	0.0355 6805	0.0372 6752	0.0390 0693	0.0407 8518	0.0444 5343
46	0.0351 1921	0.0368 2676	0.0385 7493	0.0403 6254	0.0440 5108
47	0.0346 9107	0.0364 0669	0.0381 6358	0.0399 6051	0.0436 6919
48	0.0342 8233	0.0360 0599	0.0377 7158	0.0395 7777	0.0433 0646
49	0.0338 9179	0.0356 2348	0.0373 9773	0.0392 1314	0.0429 6167
50	0.0335 1836	0.0352 5806	0.0370 4092	0.0388 6550	0.0426 3371

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
51	0.0331 6102	0.0349 0870	0.0367 0014	0.0385 3382	0.0423 2156
52	0.0328 1884	0.0345 7446	0.0363 7444	0.0382 1718	0.0420 2429
53	0.0324 9094	0.0342 5449	0.0360 6297	0.0379 1471	0.0417 4100
54	0.0321 7654	0.0339 4799	0.0357 6491	0.0376 2558	0.0414 7090
55	0.0318 7489	0.0336 5419	0.0354 7953	0.0373 4907	0.0412 1323
56	0.0315 8530	0.0333 7243	0.0352 0612	0.0370 8447	0.0409 6730
57	0.0313 0712	0.0331 0204	0.0349 4404	0.0368 3114	0.0407 3245
58	0.0310 3977	0.0328 4244	0.0346 9270	0.0365 8848	0.0405 0810
59	0.0307 8268	0.0325 9307	0.0344 5153	0.0363 5593	0.0402 9366
60	0.0305 3533	0.0323 5340	0.0342 2002	0.0361 3296	0.0400 8862
61	0.0302 9724	0.0321 2294	0.0339 9767	0.0359 1908	0.0398 9249
62	0.0300 6795	0.0319 0126	0.0337 8402	0.0357 1385	0.0397 0480
63	0.0298 4704	0.0316 8790	0.0335 7866	0.0355 1682	0.0395 2513
64	0.0296 3411	0.0314 8249	0.0333 8118	0.0353 2760	0.0393 5308
65	0.0294 2878	0.0312 8463	0.0331 9120	0.0351 4581	0.0391 8826
66	0.0292 3070	0.0310 9398	0.0330 0837	0.0349 7110	0.0390 3031
67	0.0290 3955	0.0309 1021	0.0328 3236	0.0348 0313	0.0388 7892
68	0.0288 5500	0.0307 3300	0.0326 6285	0.0346 4159	0.0387 3375
69	0.0286 7677	0.0305 6206	0.0324 9955	0.0344 8618	0.0385 9453
70	0.0285 0458	0.0303 9712	0.0323 4218	0.0343 3663	0.0384 6095
71	0.0283 3816	0.0302 3790	0.0321 9048	0.0341 9266	0.0383 3277
72	0.0281 7728	0.0300 8417	0.0320 4420	0.0340 5404	0.0382 0973
73	0.0280 2169	0.0299 3568	0.0319 0311	0.0339 2053	0.0380 9160
74	0.0278 7118	0.0297 9222	0.0317 6698	0.0337 9191	0.0379 7816
75	0.0277 2554	0.0296 5358	0.0316 3560	0.0336 6796	0.0378 6919
76	0.0275 8457	0.0295 1956	0.0315 0878	0.0335 4849	0.0377 6450
77	0.0274 4808	0.0293 8997	0.0313 8633	0.0334 3331	0.0376 6390
78	0.0273 1589	0.0292 6463	0.0312 6806	0.0333 2224	0.0375 6721
79	0.0271 8784	0.0291 4338	0.0311 5382	0.0332 1510	0.0374 7426
80	0.0270 6376	0.0290 2605	0.0310 4342	0.0331 1175	0.0373 8489
81	0.0269 4350	0.0289 1248	0.0309 3674	0.0330 1201	0.0372 9894
82	0.0268 2692	0.0288 0254	0.0308 3361	0.0329 1576	0.0372 1628
83	0.0267 1387	0.0286 9608	0.0307 3389	0.0328 2284	0.0371 3676
84	0.0266 0423	0.0285 9298	0.0306 3747	0.0327 3313	0.0370 6025
85	0.0264 9787	0.0284 9310	0.0305 4420	0.0326 4650	0.0369 8662
86	0.0263 9467	0.0283 9633	0.0304 5397	0.0325 6284	0.0369 1576
87	0.0262 9452	0.0283 0255	0.0303 6667	0.0324 8202	0.0368 4756
88	0.0261 9730	0.0282 1165	0.0302 8219	0.0324 0393	0.0367 8190
89	0.0261 0291	0.0281 2353	0.0302 0041	0.0323 2848	0.0367 1868
90	0.0260 1126	0.0280 3809	0.0301 2125	0.0322 5556	0.0366 5781
91	0.0259 2224	0.0279 5523	0.0300 4460	0.0321 8508	0.0365 9919
92	0.0258 3577	0.0278 7486	0.0299 7038	0.0321 1694	0.0365 4273
93	0.0257 5176	0.0277 9690	0.0298 9850	0.0320 5107	0.0364 8834
94	0.0256 7012	0.0277 2126	0.0298 2887	0.0319 8737	0.0364 3594
95	0.0255 9078	0.0276 4786	0.0297 6141	0.0319 2577	0.0363 8546
96	0.0255 1366	0.0275 7662	0.0296 9605	0.0318 6619	0.0363 3682
97	0.0254 3868	0.0275 0747	0.0296 3272	0.0318 0856	0.0362 8995
98	0.0253 6578	0.0274 4034	0.0295 7134	0.0317 5281	0.0362 4478
99	0.0252 9489	0.0273 7517	0.0295 1185	0.0316 9886	0.0362 0124
100	0.0252 2594	0.0273 1188	0.0294 5418	0.0316 4667	0.0361 5927

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
1	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000	1.0600 0000
2	0.5301 9608	0.5339 9756	0.5378 0488	0.5416 1800	0.5454 3689
3	0.3603 4854	0.3637 7336	0.3672 0856	0.3706 5407	0.3741 0981
4	0.2754 9005	0.2787 4365	0.2820 1183	0.2852 9449	0.2885 9149
5	0.2246 2711	0.2277 9164	0.2309 7480	0.2341 7644	0.2373 9640
6	0.1907 6190	0.1938 7839	0.1970 1747	0.2001 7895	0.2033 6263
7	0.1666 0961	0.1697 0147	0.1728 1982	0.1759 6442	0.1791 3502
8	0.1485 2783	0.1516 0965	0.1547 2181	0.1578 6401	0.1610 3594
9	0.1344 9299	0.1375 7447	0.1406 9008	0.1438 3946	0.1470 2224
10	0.1232 9094	0.1263 7882	0.1295 0458	0.1326 6777	0.1358 6796
11	0.1141 4904	0.1172 4818	0.1203 8889	0.1235 7065	0.1267 9294
12	0.1065 5217	0.1096 6619	0.1128 2541	0.1160 2923	0.1192 7703
13	0.1001 4373	0.1032 7535	0.1064 5577	0.1096 8426	0.1129 6011
14	0.0946 6897	0.0978 2032	0.1010 2397	0.1042 7912	0.1075 8491
15	0.0899 4110	0.0931 1381	0.0963 4229	0.0996 2560	0.1029 6276
16	0.0858 2000	0.0890 1537	0.0922 6991	0.0955 8254	0.0989 5214
17	0.0821 9852	0.0854 1758	0.0886 9914	0.0920 4197	0.0954 4480
18	0.0789 9333	0.0822 3690	0.0855 4622	0.0889 1992	0.0923 5654
19	0.0761 3862	0.0794 0734	0.0827 4501	0.0861 5006	0.0896 2086
20	0.0735 8175	0.0768 7614	0.0802 4259	0.0836 7933	0.0871 8456
21	0.0712 8011	0.0746 0057	0.0779 9611	0.0814 6478	0.0850 0455
22	0.0691 9881	0.0725 4565	0.0759 7051	0.0794 7123	0.0830 4557
23	0.0673 0906	0.0706 8249	0.0741 3682	0.0776 6965	0.0812 7848
24	0.0655 8683	0.0689 8703	0.0724 7090	0.0760 3580	0.0796 7900
25	0.0640 1196	0.0674 3903	0.0709 5246	0.0745 4935	0.0782 2672
26	0.0625 6738	0.0660 2137	0.0695 6432	0.0731 9307	0.0769 0435
27	0.0612 3854	0.0647 1946	0.0682 9186	0.0719 5228	0.0756 9717
28	0.0600 1298	0.0635 2081	0.0671 2253	0.0708 1440	0.0745 9255
29	0.0588 7993	0.0624 1461	0.0660 4551	0.0697 6857	0.0735 7961
30	0.0578 3010	0.0613 9154	0.0650 5144	0.0688 0539	0.0726 4891
31	0.0568 5535	0.0604 4345	0.0641 3212	0.0679 1665	0.0717 9222
32	0.0559 4859	0.0595 6320	0.0632 8042	0.0670 9519	0.0710 0234
33	0.0551 0357	0.0587 4453	0.0624 9004	0.0663 3469	0.0702 7293
34	0.0543 1477	0.0579 8191	0.0617 5545	0.0656 2958	0.0695 9843
35	0.0535 7732	0.0572 7045	0.0610 7171	0.0649 7493	0.0689 7386
36	0.0528 8688	0.0566 0578	0.0604 3446	0.0643 6635	0.0683 9483
37	0.0522 3957	0.0559 8402	0.0598 3979	0.0637 9993	0.0678 5743
38	0.0516 3192	0.0554 0169	0.0592 8423	0.0632 7217	0.0673 5812
39	0.0510 6083	0.0548 5567	0.0587 6462	0.0627 7991	0.0668 9377
40	0.0505 2349	0.0543 4315	0.0582 7816	0.0623 2034	0.0664 6154
41	0.0500 1738	0.0538 6158	0.0578 2229	0.0618 9090	0.0660 5886
42	0.0495 4020	0.0534 0868	0.0573 9471	0.0614 8927	0.0656 8342
43	0.0490 8989	0.0529 8235	0.0569 9333	0.0611 1337	0.0653 3312
44	0.0486 6454	0.0525 8071	0.0566 1625	0.0607 6128	0.0650 0606
45	0.0482 6246	0.0522 0202	0.0562 6173	0.0604 3127	0.0647 0050
46	0.0478 8205	0.0518 4471	0.0559 2820	0.0601 2175	0.0644 1485
47	0.0475 2189	0.0515 0734	0.0556 1421	0.0598 3129	0.0641 4768
48	0.0471 8065	0.0511 8858	0.0553 1843	0.0595 5854	0.0638 9766
49	0.0468 5712	0.0508 8722	0.0550 3965	0.0593 0230	0.0636 6356
50	0.0465 5020	0.0506 0215	0.0547 7674	0.0590 6145	0.0634 4429

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	4%	$4\frac{1}{2}$%	5%	$5\frac{1}{2}$%	6%
51	0.0462 5885	0.0503 3232	0.0545 2867	0.0588 3495	0.0632 3880
52	0.0459 8212	0.0500 7679	0.0542 9450	0.0586 2186	0.0630 4617
53	0.0457 1915	0.0498 3469	0.0540 7334	0.0584 2130	0.0628 6551
54	0.0454 6910	0.0496 0519	0.0538 6438	0.0582 3245	0.0626 9602
55	0.0452 3124	0.0493 8754	0.0536 6686	0.0580 5458	0.0625 3696
56	0.0450 0487	0.0491 8105	0.0534 8010	0.0578 8698	0.0623 8765
57	0.0447 8932	0.0489 8506	0.0533 0343	0.0577 2900	0.0622 4744
58	0.0445 8401	0.0487 9897	0.0531 3626	0.0575 8006	0.0621 1574
59	0.0443 8836	0.0486 2221	0.0529 7802	0.0574 3959	0.0619 9200
60	0.0442 0185	0.0484 5426	0.0528 2818	0.0573 0707	0.0618 7572
61	0.0440 2398	0.0482 9462	0.0526 8627	0.0571 8202	0.0617 6642
62	0.0438 5430	0.0481 4284	0.0525 5183	0.0570 6400	0.0616 6366
63	0.0436 9237	0.0479 9848	0.0524 2442	0.0569 5258	0.0615 6704
64	0.0435 3780	0.0478 6115	0.0523 0365	0.0568 4737	0.0614 7615
65	0.0433 9019	0.0477 3047	0.0521 8915	0.0567 4800	0.0613 9066
66	0.0432 4921	0.0476 0608	0.0520 8057	0.0566 5413	0.0613 1022
67	0.0431 1451	0.0474 8765	0.0519 7757	0.0565 6544	0.0612 3454
68	0.0429 8578	0.0473 7487	0.0518 7986	0.0564 8163	0.0611 6330
69	0.0428 6272	0.0472 6745	0.0517 8715	0.0564 0242	0.0610 9625
70	0.0427 4506	0.0471 6511	0.0516 9915	0.0563 2754	0.0610 3313
71	0.0426 3253	0.0470 6759	0.0516 1563	0.0562 5675	0.0609 7370
72	0.0425 2489	0.0469 7465	0.0515 3633	0.0561 8982	0.0609 1774
73	0.0424 2190	0.0468 8606	0.0514 6103	0.0561 2652	0.0608 6505
74	0.0423 2334	0.0468 0159	0.0513 8953	0.0560 6665	0.0608 1542
75	0.0422 2900	0.0467 2104	0.0513 2161	0.0560 1002	0.0607 6867
76	0.0421 3869	0.0466 4422	0.0512 5709	0.0559 5645	0.0607 2463
77	0.0420 5221	0.0465 7094	0.0511 9580	0.0559 0577	0.0606 8315
78	0.0419 6939	0.0465 0104	0.0511 3756	0.0558 5781	0.0606 4407
79	0.0418 9007	0.0464 3434	0.0510 8222	0.0558 1243	0.0606 0724
80	0.0418 1408	0.0463 7069	0.0510 2962	0.0557 6948	0.0605 7254
81	0.0417 4127	0.0463 0995	0.0509 7963	0.0557 2884	0.0605 3984
82	0.0416 7150	0.0462 5197	0.0509 3211	0.0556 9036	0.0605 0903
83	0.0416 0463	0.0461 9663	0.0508 8694	0.0556 5395	0.0604 7998
84	0.0415 4054	0.0461 4379	0.0508 4399	0.0556 1947	0.0604 5261
85	0.0414 7909	0.0460 9334	0.0508 0316	0.0555 8683	0.0604 2681
86	0.0414 2018	0.0460 4516	0.0507 6433	0.0555 5593	0.0604 0249
87	0.0413 6370	0.0459 9915	0.0507 2740	0.0555 2667	0.0603 7956
88	0.0413 0953	0.0459 5522	0.0506 9228	0.0554 9896	0.0603 5795
89	0.0412 5758	0.0459 1325	0.0506 5888	0.0554 7273	0.0603 3757
90	0.0412 0775	0.0458 7316	0.0506 2711	0.0554 4788	0.0603 1836
91	0.0411 5995	0.0458 3486	0.0505 9689	0.0554 2435	0.0603 0025
92	0.0411 1410	0.0457 9827	0.0505 6815	0.0554 0207	0.0602 8318
93	0.0410 7010	0.0457 6331	0.0505 4080	0.0553 8096	0.0602 6708
94	0.0410 2789	0.0457 2991	0.0505 1478	0.0553 6097	0.0602 5190
95	0.0409 8738	0.0456 9799	0.0504 9003	0.0553 4204	0.0602 3758
96	0.0409 4850	0.0456 6749	0.0504 6648	0.0553 2410	0.0602 2408
97	0.0409 1119	0.0456 3834	0.0504 4407	0.0553 0711	0.0602 1135
98	0.0408 7538	0.0456 1048	0.0504 2274	0.0552 9101	0.0601 9935
99	0.0408 4100	0.0455 8385	0.0504 0245	0.0552 7577	0.0601 8803
100	0.0408 0800	0.0455 5839	0.0503 8314	0.0552 6132	0.0601 7736

Table VII.—Periodical Payment of Annuity Whose Present Value is 1

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
1	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000	1.0850 0000
2	0.5492 6150	0.5530 9179	0.5569 2771	0.5607 6923	0.5646 1631
3	0.3775 7570	0.3810 5166	0.3845 3763	0.3880 3351	0.3915 3925
4	0.2919 0274	0.2952 2812	0.2985 6751	0.3019 2080	0.3052 8789
5	0.2406 3454	0.2438 9069	0.2471 6472	0.2504 5645	0.2537 6575
6	0.2065 6831	0.2097 9580	0.2130 4489	0.2163 1539	0.2196 0708
7	0.1823 3137	0.1855 5322	0.1888 0032	0.1920 7240	0.1953 6922
8	0.1642 3730	0.1674 6776	0.1707 2702	0.1740 1476	0.1773 3065
9	0.1502 3803	0.1534 8647	0.1567 6716	0.1600 7971	0.1634 2372
10	0.1391 0469	0.1423 7750	0.1456 8593	0.1490 2949	0.1524 0771
11	0.1300 5521	0.1333 5690	0.1366 9747	0.1400 7634	0.1434 9293
12	0.1225 6817	0.1259 0199	0.1292 7783	0.1326 9502	0.1361 5286
13	0.1162 8256	0.1196 5085	0.1230 6420	0.1265 2181	0.1300 2287
14	0.1109 4048	0.1143 4494	0.1177 9737	0.1212 9685	0.1248 4244
15	0.1063 5278	0.1097 9462	0.1132 8724	0.1168 2954	0.1204 2046
16	0.1023 7757	0.1058 5765	0.1093 9116	0.1129 7687	0.1166 1354
17	0.0989 0633	0.1024 2519	0.1060 0003	0.1096 2943	0.1133 1198
18	0.0958 5461	0.0994 1260	0.1030 2896	0.1067 0210	0.1104 3041
19	0.0931 5575	0.0967 5301	0.1004 1090	0.1041 2763	0.1079 0140
20	0.0907 5640	0.0943 9293	0.0980 9219	0.1018 5221	0.1056 7097
21	0.0886 1333	0.0922 8900	0.0960 2937	0.0998 3225	0.1036 9541
22	0.0866 9120	0.0904 0577	0.0941 8687	0.0980 3207	0.1019 3892
23	0.0849 6078	0.0887 1393	0.0925 3528	0.0964 2217	0.1003 7193
24	0.0833 9770	0.0871 8902	0.0910 5008	0.0949 7796	0.0989 6975
25	0.0819 8148	0.0858 1052	0.0897 1067	0.0936 7878	0.0977 1168
26	0.0806 9480	0.0845 6103	0.0884 9961	0.0925 0713	0.0965 8016
27	0.0795 2288	0.0834 2573	0.0874 0204	0.0914 4809	0.0955 6025
28	0.0784 5305	0.0823 9193	0.0864 0520	0.0904 8891	0.0946 3914
29	0.0774 7440	0.0814 4865	0.0854 9811	0.0896 1854	0.0938 0577
30	0.0765 7744	0.0805 8640	0.0846 7124	0.0888 2743	0.0930 5058
31	0.0757 5393	0.0797 9691	0.0839 1628	0.0881 0728	0.0923 6524
32	0.0749 9665	0.0790 7292	0.0832 2599	0.0874 5081	0.0917 4247
33	0.0742 9924	0.0784 0807	0.0825 9397	0.0868 5163	0.0911 7588
34	0.0736 5610	0.0777 9674	0.0820 1461	0.0863 0411	0.0906 5984
35	0.0730 6226	0.0772 3396	0.0814 8291	0.0858 0326	0.0901 8937
36	0.0725 1332	0.0767 1531	0.0809 9447	0.0853 4467	0.0897 6006
37	0.0720 0534	0.0762 3685	0.0805 4533	0.0849 2440	0.0893 6799
38	0.0715 3480	0.0757 9505	0.0801 3197	0.0845 3894	0.0890 0966
39	0.0710 9854	0.0753 8676	0.0797 5124	0.0841 8513	0.0886 8193
40	0.0706 9373	0.0750 0914	0.0794 0031	0.0838 6016	0.0883 8201
41	0.0703 1779	0.0746 5962	0.0790 7663	0.0835 6149	0.0881 0737
42	0.0699 6842	0.0743 3591	0.0787 7789	0.0832 8684	0.0878 5576
43	0.0696 4352	0.0740 3590	0.0785 0201	0.0830 3414	0.0876 2512
44	0.0693 4119	0.0737 5769	0.0782 4710	0.0828 0152	0.0874 1363
45	0.0690 5968	0.0734 9957	0.0780 1146	0.0825 8728	0.0872 1961
46	0.0687 9743	0.0732 5996	0.0777 9353	0.0823 8991	0.0870 4154
47	0.0685 5300	0.0730 3744	0.0775 9190	0.0822 0799	0.0868 7807
48	0.0683 2506	0.0728 3070	0.0774 0527	0.0820 4027	0.0867 2795
49	0.0681 1240	0.0726 3853	0.0772 3247	0.0818 8557	0.0865 9005
50	0.0679 1393	0.0724 5985	0.0770 7241	0.0817 4286	0.0864 6334

Table VIII.—Compound Amount of 1 for Fractional Periods

$$(1+i)^{1/p}$$

p	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
2	1.0020 8117	1.0024 9688	1.0029 1243	1.0037 4299	1.0049 8756
3	1.0013 8696	1.0016 6390	1.0019 4068	1.0024 9378	1.0033 2228
4	1.0010 4004	1.0012 4766	1.0014 5515	1.0018 6975	1.0024 9068
6	1.0006 9324	1.0008 3160	1.0009 6987	1.0012 4611	1.0016 5977
12	1.0003 4656	1.0004 1571	1.0004 8482	1.0006 2286	1.0008 2954
13	1.0003 1990	1.0003 8373	1.0004 4751	1.0005 7494	1.0007 6570
26	1.0001 5994	1.0001 9185	1.0002 2373	1.0002 8743	1.0003 8276

p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
2	1.0056 0927	1.0062 3059	1.0074 7208	1.0087 1205	1.0099 5050
3	1.0037 3602	1.0041 4943	1.0049 7521	1.0057 9963	1.0066 2271
4	1.0028 0081	1.0031 1046	1.0037 2909	1.0043 4658	1.0049 6293
6	1.0018 6627	1.0020 7257	1.0024 8452	1.0028 9562	1.0033 0589
12	1.0009 3270	1.0010 3575	1.0012 4149	1.0014 4677	1.0016 5158
13	1.0008 6092	1.0009 5604	1.0011 4594	1.0013 3540	1.0015 2444
26	1.0004 3037	1.0004 7790	1.0005 7280	1.0006 6748	1.0007 6193

p	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
2	1.0111 8742	1.0124 2284	1.0136 5675	1.0148 8916	1.0173 4950
3	1.0074 4444	1.0082 6484	1.0090 8390	1.0099 0163	1.0115 3314
4	1.0055 7815	1.0061 9225	1.0068 0522	1.0074 1707	1.0086 3745
6	1.0037 1532	1.0041 2392	1.0045 3168	1.0049 3862	1.0057 5004
12	1.0018 5594	1.0020 5984	1.0022 6328	1.0024 6627	1.0028 7090
26	1.0008 5616	1.0009 5017	1.0010 4396	1.0011 3752	1.0013 2401
52	1.0004 2799	1.0004 7497	1.0005 2184	1.0005 6860	1.0006 6179

p	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$	6%
2	1.0198 0390	1.0222 5242	1.0246 9508	1.0271 3193	1.0295 6302
3	1.0131 5941	1.0147 8046	1.0163 9636	1.0180 0713	1.0196 1282
4	1.0098 5341	1.0110 6499	1.0122 7224	1.0134 7518	1.0146 7385
6	1.0065 5820	1.0073 6312	1.0081 6485	1.0089 6340	1.0097 5880
12	1.0032 7374	1.0036 7481	1.0040 7412	1.0044 7170	1.0048 6755
26	1.0015 0963	1.0016 9439	1.0018 7831	1.0020 6138	1.0022 4363
52	1.0007 5453	1.0008 4684	1.0009 3871	1.0010 3016	1.0011 2118

p	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
2	1.0319 8837	1.0344 0804	1.0368 2207	1.0392 3048	1.0416 3333
3	1.0212 1347	1.0228 0912	1.0243 9981	1.0259 8557	1.0275 6644
4	1.0158 6828	1.0170 5853	1.0182 4460	1.0194 2655	1.0206 0440
6	1.0105 5107	1.0113 4026	1.0121 2638	1.0129 0946	1.0136 8952
12	1.0052 6169	1.0056 5415	1.0060 4492	1.0064 3403	1.0068 2149
26	1.0024 2504	1.0026 0564	1.0027 8544	1.0029 6443	1.0031 4262
52	1.0012 1179	1.0013 0197	1.0013 9175	1.0014 8112	1.0015 7008

Table IX.—Nominal Rate j which if Converted p Times per Year Gives Effective Rate i

$$j_p = p[(1 + i)^{1/p} - 1]$$

p	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
2	.0041 6234	.0049 9377	.0058 2485	.0074 8599	.0099 7512
3	.0041 6089	.0049 9169	.0058 2203	.0074 8133	.0099 6685
4	.0041 6017	.0049 9065	.0058 2062	.0074 7900	.0099 6272
6	.0041 5945	.0049 8962	.0058 1921	.0074 7667	.0099 5859
12	.0041 5873	.0049 8858	.0058 1780	.0074 7434	.0099 5446
13	.0041 5868	.0049 8850	.0058 1769	.0074 7416	.0099 5414
26	.0041 5834	.0049 8802	.0058 1704	.0074 7309	.0099 5224

p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
2	.0112 1854	.0124 6118	.0149 4417	.0174 2410	.0199 0099
3	.0112 0807	.0124 4828	.0149 2562	.0173 9890	.0198 6813
4	.0112 0285	.0124 4183	.0149 1636	.0173 8631	.0198 5173
6	.0111 9763	.0124 3539	.0149 0710	.0173 7374	.0198 3534
12	.0111 9241	.0124 2895	.0148 9785	.0173 6119	.0198 1898
13	.0111 9200	.0124 2846	.0148 9714	.0173 6022	.0198 1772
26	.0111 8960	.0124 2549	.0148 9288	.0173 5443	.0198 1017

p	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
2	.0223 7484	.0248 4567	.0273 1349	.0297 7831	.0346 9899
3	.0223 3333	.0247 9451	.0272 5170	.0297 0490	.0345 9943
4	.0223 1261	.0247 6899	.0272 2087	.0296 6829	.0345 4978
6	.0222 9192	.0247 4349	.0271 9009	.0296 3173	.0345 0024
12	.0222 7125	.0247 1804	.0271 5936	.0295 9524	.0344 5078
26	.0222 6013	.0247 0434	.0271 4283	.0295 7561	.0344 2420
52	.0222 5537	.0246 9848	.0271 3575	.0295 6721	.0344 1281

p	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$	6%
2	.0396 0781	.0445 0483	.0493 9015	.0542 3386	.0591 2603
3	.0394 7821	.0443 4138	.0491 8907	.0540 2139	.0588 3847
4	.0394 1363	.0442 5996	.0490 8894	.0539 0070	.0586 9538
6	.0393 4918	.0441 7874	.0489 8908	.0537 8036	.0585 5277
12	.0392 8488	.0440 9771	.0488 8949	.0536 6039	.0584 1061
26	.0392 5031	.0440 5417	.0488 3597	.0535 9593	.0583 3425
52	.0392 3551	.0440 3552	.0488 1306	.0535 6834	.0583 0157

p	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
2	.0639 7674	.0688 1609	.0736 4414	.0784 6097	.0832 6667
3	.0636 4042	.0684 2737	.0731 9942	.0779 5670	.0826 9933
4	.0634 7314	.0682 3410	.0729 7840	.0777 0619	.0824 1758
6	.0633 0644	.0680 4156	.0727 5827	.0774 5674	.0821 3712
12	.0631 4033	.0678 4974	.0725 3903	.0772 0836	.0818 5792
26	.0630 5113	.0677 4676	.0724 2134	.0770 7506	.0817 0811
52	.0630 1295	.0677 0268	.0723 7098	.0770 1802	.0816 4401

Table X.—The Value of the Conversion Factor

$$\frac{i}{j_p} = \frac{i}{p[(1+i)^{1/p} - 1]}$$

p	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
2	1.0010 4058	1.0012 4844	1.0014 5621	1.0018 7150	1.0024 9378
3	1.0013 8761	1.0016 6482	1.0019 4193	1.0024 9585	1.0033 2596
4	1.0015 6115	1.0018 7305	1.0021 8485	1.0028 0812	1.0037 4223
6	1.0017 3471	1.0020 8131	1.0024 2781	1.0031 2046	1.0041 5861
12	1.0019 0829	1.0022 8960	1.0026 7080	1.0034 3286	1.0045 7510
13	1.0019 2164	1.0023 0563	1.0026 8950	1.0034 5690	1.0046 0714
26	1.0020 0176	1.0024 2182	1.0028 0166	1.0036 0111	1.0047 9941
p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
2	1.0028 0463	1.0031 1529	1.0037 3604	1.0043 6176	1.0049 7525
3	1.0037 4068	1.0041 5516	1.0049 8346	1.0058 1084	1.0066 3733
4	1.0042 0892	1.0046 7537	1.0056 0755	1.0065 3878	1.0074 6856
6	1.0046 7730	1.0051 9575	1.0062 3191	1.0072 6707	1.0083 0125
12	1.0051 4583	1.0057 1632	1.0068 5652	1.0079 9571	1.0091 3389
13	1.0051 8188	1.0057 5637	1.0069 0458	1.0080 5177	1.0091 9796
26	1.0053 9818	1.0059 9669	1.0071 9296	1.0083 8820	1.0095 8243
p	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%	$3\frac{1}{2}\%$
2	1.0055 9371	1.0062 1142	1.0068 2837	1.0074 4458	1.0086 7475
3	1.0074 6292	1.0082 8761	1.0091 1141	1.0099 3431	1.0115 7748
4	1.0083 9839	1.0093 2677	1.0102 5422	1.0111 8072	1.0130 3094
6	1.0093 3444	1.0103 6665	1.0113 9789	1.0124 2816	1.0144 8578
12	1.0102 7107	1.0114 0725	1.0125 4243	1.0136 7662	1.0159 4203
26	1.0107 7565	1.0119 6786	1.0131 5908	1.0143 4929	1.0167 2674
52	1.0109 9195	1.0122 0819	1.0134 2343	1.0146 3757	1.0170 6316
p	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$	6%
2	1.0099 0195	1.0111 2621	1.0123 4754	1.0135 6596	1.0147 8151
3	1.0132 1713	1.0148 5328	1.0164 8597	1.0181 1522	1.0197 4104
4	1.0148 7744	1.0167 2026	1.0185 5942	1.0203 9495	1.0222 2688
6	1.0165 3957	1.0185 8953	1.0206 3570	1.0226 7810	1.0247 1676
12	1.0182 0351	1.0204 6109	1.0227 1479	1.0249 6465	1.0272 1070
26	1.0191 0023	1.0214 6980	1.0238 3548	1.0261 9729	1.0285 5526
52	1.0194 8470	1.0219 6231	1.0243 1602	1.0267 2586	1.0291 3186
p	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
2	1.0159 9419	1.0172 0402	1.0184 1103	1.0196 1524	1.0208 1667
3	1.0213 6348	1.0229 8254	1.0245 9826	1.0262 1065	1.0278 1974
4	1.0240 5523	1.0258 8002	1.0277 0129	1.0295 1904	1.0313 3332
6	1.0267 5172	1.0287 8298	1.0308 1059	1.0328 3456	1.0348 5492
12	1.0294 5294	1.0316 9143	1.0339 2617	1.0361 5721	1.0383 8455
26	1.0309 0941	1.0332 5978	1.0356 0640	1.0379 4927	1.0402 8845
52	1.0315 3404	1.0339 3242	1.0363 2705	1.0387 1794	1.0411 0511

TABLE XI.—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x	Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x
10	100,000	749	0.007 490	0.992 510	53	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1,199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60,779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1,516	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	61	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	64	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1,980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2,158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2,391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2,487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2,505	0.080 178	0.919 822
31	84,721	721	0.008 610	0.991 490	74	28,738	2,501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2,476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2,291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2,196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 766	80	14,474	2,091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1,964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 534
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1,001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE XII.—COMMUTATION COLUMNS, SINGLE PREMIUMS, AND ANNUITIES DUE. AMERICAN EXPERIENCE TABLE, 3½ PER CENT

Age x	D_x	N_x	C_x	M_x	$\ddot{a}_x = 1 + a_x$	A_x
10	70891.9	1575 535	513.02	17612.9	22.2245	0.24845
11	67981.5	1504 643	493.69	17099.9	22.1331	0.25154
12	65189.0	1436 662	475.08	16606.2	22.0384	0.25474
13	62509.4	1371 473	457.16	16131.1	21.9403	0.25806
14	59938.4	1308 963	439.91	15674.0	21.8385	0.26151
15	57471.6	1249 025	423.88	15234.1	21.7329	0.26508
16	55104.2	1191 553	407.87	14810.2	21.6236	0.26877
17	52832.9	1136 449	392.47	14402.3	21.5102	0.27261
18	50653.9	1083 616	378.15	14009.8	21.3926	0.27659
19	48562.8	1032 962	364.36	13631.7	21.2707	0.28071
20	46556.2	984 400	351.07	13267.3	21.1443	0.28497
21	44630.8	937 843	338.73	12916.3	21.0134	0.28940
22	42782.8	893 213	326.82	12577.5	20.8779	0.29399
23	41009.2	850 430	315.33	12250.7	20.7375	0.29873
24	39307.1	809 421	304.24	11935.4	20.5922	0.30365
25	37673.6	770 113	293.55	11631.1	20.4417	0.30873
26	36106.1	732 440	283.62	11337.6	20.2858	0.31401
27	34601.5	696 334	274.03	11054.0	20.1244	0.31947
28	33157.4	661 732	264.76	10779.9	19.9573	0.32512
29	31771.3	628 575	256.16	10515.2	19.7843	0.33097
30	30440.8	596 804	247.85	10259.0	19.6054	0.33702
31	29163.5	566 363	239.797	10011.2	19.4202	0.34328
32	27937.5	537 199	232.331	9771.38	19.2286	0.34976
33	26760.5	509 262	225.406	9539.04	19.0304	0.35646
34	25630.1	482 501	218.683	9313.64	18.8256	0.36339
35	24544.7	456 871	212.157	9094.96	18.6138	0.37055
36	23502.5	432 326	206.383	8882.80	18.3949	0.37795
37	22501.4	408 824	200.757	8676.42	18.1688	0.38560
38	21539.7	386 323	195.798	8475.66	17.9354	0.39349
39	20615.5	364 783	190.945	8279.86	17.6946	0.40163
40	19727.4	344 167	186.684	8088.92	17.4461	0.41003
41	18873.6	324 440	182.493	7902.23	17.1901	0.41869
42	18052.9	305 566	178.828	7719.74	16.9262	0.42762
43	17263.6	287 513	175.421	7540.91	16.6543	0.43681
44	16504.4	270 250	172.680	7365.49	16.3744	0.44628
45	15773.6	253 745	170.127	7192.81	16.0867	0.45600
46	15070.0	237 972	168.345	7022.68	15.7911	0.46600
47	14392.1	222 902	166.872	6854.34	15.4878	0.47626
48	13738.5	208 510	166.047	6687.47	15.1770	0.48677
49	13107.9	194 771	165.983	6521.42	14.8591	0.49752
50	12498.6	181 663	166.424	6355.44	14.5346	0.50849
51	11909.6	169 165	167.316	6189.01	14.2041	0.51967
52	11339.5	157 252	168.601	6021.70	13.8679	0.53104

TABLE XII.—COMMUTATION COLUMNS, SINGLE PREMIUMS, AND ANNUITIES DUE. AMERICAN EXPERIENCE TABLE, 3½ PER CENT

Age x	D_x	N_x	C_x	M_x	$a_x = 1 + a_x$	A_x
53	10787.4	145916.	170.234	5853.10	13.5264	0.54258
54	10252.4	135128.	172.317	5682.86	13.1801	0.55430
55	9733.40	124876.	174.646	5510.54	12.8296	0.56615
56	9229.60	115142.	177.325	5335.90	12.4753	0.57813
57	8740.17	105912.8	180.168	5158.57	12.1179	0.59022
58	8264.44	97172.6	183.139	4978.40	11.7579	0.60239
59	7801.82	88908.2	186.340	4795.27	11.3958	0.61463
60	7351.65	81106.4	189.604	4608.93	11.0324	0.62692
61	6913.44	73754.7	192.909	4419.32	10.6683	0.63924
62	6486.75	66841.3	196.117	4226.41	10.3043	0.65155
63	6071.27	60354.5	199.109	4030.30	9.9410	0.66383
64	5666.85	54283.3	201.887	3831.19	9.5791	0.67607
65	5273.33	48616.4	204.457	3629.30	9.2193	0.68824
66	4890.55	43343.1	206.522	3424.84	8.8626	0.70030
67	4518.65	38452.5	208.022	3218.32	8.5097	0.71223
68	4157.82	33933.9	208.903	3010.30	8.1615	0.72401
69	3808.32	29776.1	208.858	2801.40	7.8187	0.73560
70	3470.67	25967.7	207.881	2592.54	7.4820	0.74698
71	3145.43	22497.1	205.639	2384.66	7.1523	0.75813
72	2833.42	19351.6	201.851	2179.02	6.8298	0.76904
73	2535.75	16518.2	196.436	1977.17	6.5141	0.77972
74	2253.57	13982.5	189.491	1780.73	6.2046	0.79018
75	1987.87	11728.9	181.253	1591.24	5.9002	0.80048
76	1739.39	9741.02	171.940	1409.99	5.6002	0.81062
77	1508.63	8001.63	161.889	1238.05	5.3039	0.82064
78	1295.73	6493.00	151.2646	1076.158	5.0111	0.83054
79	1100.647	5197.27	140.0891	924.894	4.7220	0.84032
80	923.338	4096.62	128.8801	784.805	4.4368	0.84997
81	763.234	3173.29	116.9588	655.924	4.1577	0.85940
82	620.465	2410.05	104.4881	538.966	3.8843	0.86865
83	494.995	1789.59	91.6152	434.478	3.6154	0.87774
84	386.641	1294.59	78.9565	342.862	3.3483	0.88677
85	294.610	907.95	67.0490	263.906	3.0819	0.89578
86	217.598	613.34	55.8566	196.857	2.8187	0.90468
87	154.383	395.74	45.1992	141.000	2.5634	0.91332
88	103.963	241.36	34.82426	95.8011	2.3216	0.92149
89	65.6231	137.398	25.09929	60.9768	2.0937	0.92920
90	38.3047	71.775	16.82244	35.8775	1.8738	0.93664
91	20.18692	33.4700	10.385393	19.05509	1.6580	0.94393
92	9.11888	13.2831	5.588150	8.66970	1.4567	0.95074
93	3.22236	4.16420	2.285484	3.08155	1.2923	0.95630
94	0.827611	0.94184	0.685393	0.79576	1.1380	0.96152
95	0.114232	0.114232	0.110369	0.110369	1.0000	0.96618

3½ Per Cent

$$u_x = \frac{D_x}{D_{x+1}} \qquad k_x = \frac{C_x}{D_{x+1}}$$

Age x	u_x	k_x	Age x	u_x	k_x
10	1.042 811	0.007 546	53	1.052 185	0.016 604
11	1.042 838	0.007 573	54	1.053 323	0.017 704
12	1.042 866	0.007 600	55	1.054 585	0.018 922
13	1.042 894	0.007 627	56	1.055 999	0.020 289
14	1.042 922	0.007 654	57	1.057 563	0.021 800
15	1.042 962	0.007 692	58	1.059 296	0.023 474
16	1.042 990	0.007 720	59	1.061 234	0.025 347
17	1.043 019	0.007 748	60	1.063 385	0.027 425
18	1.043 059	0.007 787	61	1.065 780	0.029 739
19	1.043 100	0.007 826	62	1.068 433	0.032 303
20	1.043 141	0.007 866	63	1.071 365	0.035 136
21	1.043 195	0.007 917	64	1.074 625	0.038 285
22	1.043 248	0.007 969	65	1.078 270	0.041 807
23	1.043 303	0.008 022	66	1.082 304	0.045 704
24	1.043 358	0.008 076	67	1.086 782	0.050 031
25	1.043 415	0.008 130	68	1.091 774	0.054 855
26	1.043 484	0.008 197	69	1.097 284	0.060 178
27	1.043 554	0.008 264	70	1.103 403	0.066 090
28	1.043 625	0.008 333	71	1.110 117	0.072 576
29	1.043 710	0.008 415	72	1.117 388	0.079 602
30	1.043 796	0.008 498	73	1.125 218	0.087 167
31	1.043 884	0.008 583	74	1.133 660	0.095 323
32	1.043 986	0.008 682	75	1.142 852	0.104 204
33	1.044 102	0.008 795	76	1.152 960	0.113 971
34	1.044 221	0.008 910	77	1.164 314	0.124 941
35	1.044 343	0.009 027	78	1.177 243	0.137 433
36	1.044 493	0.009 172	79	1.192 031	0.151 720
37	1.044 647	0.009 320	80	1.209 771	0.168 861
38	1.044 830	0.009 498	81	1.230 099	0.188 502
39	1.045 018	0.009 679	82	1.253 477	0.211 089
40	1.045 238	0.009 891	83	1.280 245	0.236 952
41	1.045 463	0.010 109	84	1.312 384	0.268 004
42	1.045 721	0.010 359	85	1.353 917	0.308 133
43	1.046 001	0.010 629	86	1.409 469	0.361 806
44	1.046 331	0.010 947	87	1.484 979	0.434 762
45	1.046 684	0.011 289	88	1.584 244	0.530 671
46	1.047 106	0.011 697	89	1.713 188	0.655 254
47	1.047 571	0.012 146	90	1.897 500	0.833 333
48	1.048 111	0.012 668	91	2.213 750	1.138 889
49	1.048 745	0.013 280	92	2.829 873	1.734 177
50	1.049 463	0.013 974	93	3.893 571	2.761 905
51	1.050 272	0.014 755	94	7.245 000	6.000 000
52	1.051 177	0.015 629	95		

ANSWERS
TO
EXERCISES AND PROBLEMS

Chapter I

Page 3

2. I = \$625.00; S = \$5,625.00.
3. \$13.27.
4. 1½ years.
5. \$799.14.
8. 7%.
11. 5%.
14. \$452.40.
6. 5½ years.
9. \$4,500.00.
12. \$256.00.
15. \$256.00.
7. 7%.
10. 3¼ years.
13. 9%.
16. \$26,250.00.

Pages 6–7

1. (a) I_o = \$3.25; I_e = \$3.21.
(b) I_o = \$3.24; I_e = \$3.19.
(c) I_o = \$1.31; I_e = \$1.29.
(d) I_o = \$4.52; I_e = \$4.46.
2. I_o = \$44.80; I_e = \$44.19.
3. \$9.93.
4. \$29.89.
5. 55 days.
6. 75 days.
7. 9%.
8. \$14.60.
9. \$21.60.
10. \$1.06.
11. \$28.80.

Pages 8–9

1. (a) \$7.50.
(b) \$6.04.
(c) \$8.75.
(d) \$18.27.
2. (a) \$9.38.
(b) \$7.54.
(c) \$10.94.
(d) \$22.84.
3. (a) \$14.53.
(b) \$11.05.
(c) \$38.47.
4. (a) \$19.64.
(b) \$14.93.
(c) \$52.00.
5. \$155.33.
6. \$155.20.
7. \$153.20.
8. \$0.69.

Pages 11–12

1. \$2,200.00.
2. \$312.00.
3. P = \$5,769.23; Disc. = \$230.77.
4. \$1,818.18.
5. \$288.46.
7. \$990.10.
8. \$986.84.
9. 5%.
10. (a) \$1,035.00.
(b) \$1,024.75.
(c) 7.4%.
11. (b) \$1,021.38.
(c) 6.41%.

Pages 16–17

1. \$1,479.75.
2. \$381.61.
3. \$2,024.17.
4. \$569.09.
5. \$1,352.13.
6. \$1,267.69.
7. \$2,480.83.
8. \$2,556.46.
9. \$255.10.
11. 6%.
12. \$1,015.23.
13. 67 days.
14. \$5,019.73.
15. \$2,000.00.
16. 8%.

Pages 16–17—*Continued*

17. $2,072.54.
18. S = $800.00; Face = $788.18.
19. $1,216.93.
20. ¼ year.
21. $1,000.00.
22. 0.

Pages 19–20

1. i = 6.383%.
2. i = 6.185%.
3. i = .0869; .0833; .0816; .0808.
4. d = .0741; .0769; .0784; .0792.
5. d = 15%; i = 15.4%.

6. $803.74.
7. $800.95.
8. (a) $501.58.
(b) $501.65.
9. (a) $1,004.50.
(b) $6.83.

Pages 21–22

1. (a) .0712.
(b) .0759.
(c) .0619.
(d) .0822.
2. 8.74%.
3. (a) .0688.
(b) .0779.
(c) .0583.
4. 9.89%.
5. 7.41%.
6. i = 12.4% or 3.1% per 90 days.
7. 16⅔%; 13.92%.
8. 4% cash discount is best.
9. 18.56%; $78.47 at end of 60 days.
10. 6.88%; 5.88%; 4.82%.
11. 5/30 is best.
12. 5/30 is best.
13. 6.12%.
14. 6% cash discount is best.

Pages 31–32

1. (a) $492.61; $497.61.
(b) 507.50; 512.32.
(c) 522.50; 527.80.
2. (a) $489.00; $487.12.
(b) 511.25; 508.56.
(c) 533.75; 531.70.
3. $619.65; $619.77.
4. $620.67; $618.75.

5. $912.66, F.D. at 8 mo.; $912.55, F.D. at 12 mo.

6. $437.93.
7. $938.08, F.D. at 12 mo.

8. $1,873.22, F.D. at 9 mo.; $1,873.31, F.D. at 8 mo.

9. May 28.
10. April 22.
11. 4 mo. 7 days.
12. 6¼ months.
13. Dec. 9.
14. March 2.
15. Sept. 12.
16. May 11.
17. Oct. 3.
18. Jan. 15.
19. July 16.

Pages 33–34

1. $2,000.00; $2,500.00.
2. $2,500.00; $4,000.00.
3. $3,000.00; $7,000.00; $5,000.00.
4. $1,000.00; $1,500.00; $2,500.00.
5. $12,000.00.

6. 4 days.
7. 4⅘ hours.
8. 17.857 lbs.
9. 115 lbs.

Pages 33–34—*Continued*

10. \$1,182.27 for 3 mos.; \$1,182.07 for exact days.

11. B.D. = \$25.00; T.D. = \$24.39.
12. (a) \$2,520.96.
(b) \$2,520.46.
13. \$506.11.
14. \$730.00.
15. \$1,459.06.
16. 6.89%.
17. \$1.79.
18. \$400.00.
19. \$87.80.
20. \$1,666.67.
21. \$2,317.60.
22. \$1,470.59.
23. \$12,200.00.
24. 13.18.
25. 8.5302.
26. 37.8 yrs.
27. 6.45.

28. 26⅔% if all amts. are focalized at 10 mos.
29. 48% if all amts. are focalized at 5 mos.
30. 26⅔% if all amts. are focalized at 10 mos.

Chapter II

Page 38

1. \$1,800.94.
2. \$2,012.20.
3. \$1,198.28.
4. \$442.94.
5. \$2.63.
6. \$2,278.77.

Pages 41–42

1. \$1,181.96.
2. (a) \$1,187.60.
(b) \$1,190.50.
3. \$1,670.40.
4. \$1,638.62.
5. \$2,695.97.
6. (a) 6.09%.
(b) 6.136%.
(c) 6.168%.
8. 5.18%.
9. $i_1 = 5.58\%$; $i_2 = 5.12\%$.
10. \$1,155.48.
13. \$3,639.70.
14. Better to pay cash.
15. (a) 7.23%.
(b) 7.19%.
(c) 7.12%.
16. (a) 3.94136%.
(b) 4.90889%.
(c) 5.86954%.

Pages 44–45

1. \$140.99.
2. \$2,343.60.
3. \$1,137.75.
4. \$4,226.67.
5. \$1,106.12.
6. (a) \$334.99 and \$334.84.
(b) \$377.04 and \$376.87.
7. Yes.
8. \$2,883.67.
9. \$243.76.
10. \$1,337.72.
12. \$1,688.91.
13. \$193.07.
14. \$387.35.
15. $P_1 = \$6{,}417.63$; $P_2 = \$6{,}455.35$.
16. \$61.55.

Page 48

1. 22.35 years.
2. 6.3%.
3. 5.14%.
4. 16.
5. $\dfrac{.30103}{\log (1 + i)}$.
6. (a) 14.2.
(b) 11.9.
7. 20.2 years.
8. $j_2 = 5.5\%$.
9. 12.9 years.
10. 6.054%.

Pages 55–56

1. (a) \$1,175.29.
   (b) \$1,360.54.
   (c) \$1,575.00.
2. (a) \$1,579.49.
   (b) \$1,828.46.
   (c) \$2,116.67.
3. For the \$500 debt:
   (a) \$519.32.
   (b) \$631.24.
   (c) \$695.94.
   For the \$750 debt:
   (a) \$533.01.
   (b) \$647.88.
   (c) \$714.29.
5. $P_1 = \$5{,}250.09$; $P_2 = \$5{,}238.41$.
6. \$2,723.25.
7. (a) \$3,152.50.
   (b) \$2,723.25.
8. \$332.96.
9. \$721.80.
10. \$1,159.94.
11. 0.66 years.
12. \$1,024.51.
13. 5.81 years.
14. 5.86 years.
15. $3\frac{6}{7}$ years.
16. \$709.26.
18. $j_2 = 5.91\%$; $f_2 = 5.74\%$.
19. 44.13%.
20. $j_6 = 12.24\%$; $i = 12.89\%$.
21. $f_{12} = 23.53\%$.
22. $j_4 = 5.955664\%$.
23. 6.045%.
24. (a) 8.48%.
    (b) 8.48%.
25. (a) 11.89 years.
    (b) 11.72 years.
    (c) 17.5 years.
26. 7.25%.

Chapter III

Page 60

1. \$3,601.83.
2. \$16,532.98.
3. \$1,293.68.
4. \$1,977.12.
5. \$2,564.54.
6. \$79,840.69.
7. $\sum_{x=1}^{n} (1+i)^{x-1}$.
8. \$14,045.45.
9. \$416.45.
10. $S_1 = \$5{,}920.98$; $S_2 = \$6{,}003.05$.

Page 63

1. \$2,978.85.
2. \$12,088.47.
3. \$2,710.33.
4. \$36,919.78.
5. \$15,303.59.
6. \$3,037.04.
7. $S_1 = \$12{,}006.11$; $S_2 = \$11{,}748.01$.
8. 3.2878%.
9. \$2,983.81.
10. \$2,987.18.

Pages 68–69

1. \$10,379.66.
2. \$8,832.09.
3. \$27,084.63.
4. \$577.18.
5. \$1,228.03.
6. \$4,680.04.

Page 72

1. \$3,637.50.
2. \$16,839.82.
3. \$7,334.80.
4. \$10,507.65.
5. \$5,825.65.
6. \$23,742.48.
7. \$3,655.42.
8. \$16,737.12.
9. \$1,692.16.

Pages 76–78

1. $7,325.48.
2. (a) $7,310.84.
   (b) $7,332.96.
3. $30,705.23.
4. $30,774.62.

5.

Annuity Payable	Annually	Interest Convertible Semi-ann.	Quarterly
Annually	$4,507.74	$4,518.10	$4,523.39
Semi-ann.	4,552.38	4,563.28	4,568.85
Quarterly	4,574.80	4,585.98	4,591.70

6.

Annuity Payable	Annually	Interest Convertible Semi-ann.	Quarterly
Annually	$4,775.14	$4,792.45	$4,801.35
Semi-ann.	4,834.10	4,852.36	4,861.74
Quarterly	4,863.76	4,882.50	4,892.13

7.

Annuity Payable	Annually	Interest Convertible Semi-ann.	Quarterly
Annually	$4,639.51	$4,652.77	$4,659.72
Semi-ann.	4,691.13	4,705.11	4,712.43
Quarterly	4,717.08	4,731.64	4,738.94

8. $3,474.59.
9. $3,461.61.
10. $3,566.07.
11. $15,157.30.
12. $1,463.14.
13. $13,498.73.
14. $158.26.
15. $18,779.88 if payment at age 60 is included.
16. $18,822.76 if payment at age 60 is included.
17. $624.49.
18. $1,595.30.
19. $1,598.46.

Pages 82–83

1. $7,265.76.
2. $7,235.16.
3. $4,768.81.
4. $3,561.46.
5. $3,596.72.
6. $10,659.30.
7. $9,177.71 by interpolation.
   $9,176.77 by logarithms.
9. $63,417.98.
10. $63,028.88.

11.

Annuity Payable	Annually	Interest Convertible Semi-ann.	Quarterly
Annually	$811.09	$809.48	$808.66
Semi-ann.	819.12	817.57	816.78
Quarterly	823.16	821.64	820.87

12.

Annuity Payable	Annually	Interest Convertible Semi-ann.	Quarterly
Annually	$772.17	$769.84	$768.64
Semi-ann.	781.71	779.46	778.31
Quarterly	786.50	784.30	783.17

14. $19,010.68.
15. $5,167.18.
16. $88,632.52.
18. $9,048.57.
19. $5,712.91.
20. $2,561.26.

Page 86

1. $447.11.
2. $4,129.86.
3. See 15, p. 77.
4. See 16, p. 77.
5. $624.49.
6. $1,678.57.
7. $1,626.89.
8. $1,630.59.

Page 88

1. $367.84.
2. $3,985.39.
3. $4,369.52.
4. $3,887.56.
5. $10,329.22.
6. $21,280.01.
7. $21,412.19.
8. $4,198.60.

Pages 91–92

1. $6,134.82.
2. $6,149.34.
3. $6,171.81.
4. $6,018.89.
5. (a) $5,974.89.
   (b) $5,952.48.
6. $A' = \$7{,}811.63$; Tax = $390.58.
7. $320,957.26
8. $13,949.28.
9. $638.28.
10. $1,863.49.

Page 95

1. 5.33%.
2. 6.88%.
3. 19.7% with F.D. at 12 mo.
4. 4.76%.

Page 97

1. 9 full payments with a partial payment at end of 10 years.
4. 9 full payments; $255.53 at end of 10 years.
5. 14 full payments; $402.39 at end of 24 years.

Pages 99–100

1. $250.44.
2. $532.09.
3. (a) $533.05.
   (b) $531.59.
4. $1,567.74; $4,067.74.
5. $1,563.39; $4,063.39.
6. $609.11.
7. $2,195.89.
8. $2,221.75.

Pages 104–105

1. 0.67.
2. $2,400,000.
3. $2,355,465.79.
4. $5,128.45.
5. $174,951.78.
6. $1,010.21.

Pages 107–109

2. $1,093.38.
3. $1,288.00.
4. 4.905%.
5. 130.
6. $116.
7. $6,944.59.
8. $1,536.81.
9. $8,480.01.
10. $55,454.05.
11. $19,753.09.
12. $1,456.93.
13. $2,276.27.
15. $55,325.34.
16. (a) $55,256.31.
    (b) $55,360.76.
17. $2,638.80.
18. $871.85; $684.58.
19. $535.39.
20. $914.67.
21. 14; $5,267.97.
22. 5.45%.
23. $3,056.70.
24. 4.66%.
25. 19.75%.
26. Yes.
27. 14 years.
31. $25,435.38.
32. $23,968.84.

Page 110

1. (a) \$1,011.59.
   (b) \$1.69.
2. Yes, by 2 cents.
3. $j_6 = 12.24\%$; $i = 12.88\%$.
4. \$1,732.02.
5. \$2,382.98.
6. \$29.13; 34.95%.
7. \$4,542.09.
8. \$299.68.
9. \$238.63.
10. 44⅑% using simple interest.

Chapter IV

Page 113

1. \$372.57.
2. \$523.61.
3. \$1,358.68.
4. \$260.21.
5. \$1,232.50; \$2732.50.
6. \$228.49.

Page 115

1. \$1,219.14.
2. \$3,351.75.
3. \$1,883.18.
4. \$69.67; \$6,037.46; \$8,255.66.
5. \$2,821.36.

Page 118

1. \$81 a year in favor of (b).
2. \$748.21.
3. \$732.57.
4. (a) \$796.72 and \$831.12.
   (b) \$796.72 and \$796.72.
   (c) \$796.72 and \$765.25.

Page 120

1. \$1,142.59.
2. \$872.31.
3. \$2,067.01.
4. \$321.43; \$3,834.72.
5. \$1,610.70.
6. (a) \$456.85.
   (b) \$442.86.
7. (a) \$3,670.08.
   (b) \$3,777.69.
8. 7; \$147.15; \$1,406.93.
9. \$13,329.09.
10. \$20,855.57.
11. \$4,693.60.
12. \$4,503.09.

Page 121

Problems

1. 53.8% by simple interest theory.
2. 28.2% by simple interest theory.
3. 53% by simple interest theory.
4. (a) $m = \dfrac{\log R + \log a_{\overline{n}|i} - \log A}{\log (1 + i)}$

   (b) $n = \dfrac{\log R - \log [R - Ai(1 + i)^m]}{\log (1 + i)}$

5. \$2,821.36.
6. \$5,680.18.
7. 138; \$97.58.
8. \$640.12.
9. \$2,619,923.28.
10. \$956.50.

Chapter V

Page 133

1. \$27.50.
2. \$124.81.
3. 44.5%, rate of depreciation.
4. (a) \$1,620.66 and \$1,379.73.
   (b) \$240.93.
5. R = \$318.02.
   (a) \$2,410.68 and \$1,963.19.
   (b) \$447.49.
6. −\$196.25.
7. 42− units.
9. \$453.04.
10. 213−.
11. \$391.58.
12. \$103.76.
13. \$356.25.

Pages 135–136

1. \$185,898.00.
2. \$901,286.91.
3. \$460.98.
4. \$78,008.97.

Page 138

1. 20.2 years.
2. 20.4 years.
3. 38.6 years.
4. 39.11 years.

Pages 139–140

1. \$278.63.
2. 20.63%; \$952.44; \$755.95; \$600.00.
3. 9.32%.
4. \$800.69.
5. \$79,563.85.
6. \$5,615.60; 30 years.
7. \$316,956.82.
8. \$46,298.95; R = \$1,846.27; Amt. in S.F. = \$19,216.09.
9. \$28,505.24.
10. \$62,955.62.

Page 140

1. \$75,578.04.
2. \$8.69.
4. Amortization plan better by \$565.07 per year.
5. \$40,250.97.
6. \$1,666.40.
7. 20.57%; \$3,154.56.

Chapter VI

Page 144

1. \$538.97.
2. \$939.92.
3. \$9,110.50.
4. \$5,541.38.
5. \$480.92.
8. \$1,766.01.
9. \$1,781.97.
10. Yes; P = \$92.56.
11. \$5,719.47.

Page 147

1. \$940.25.
2. \$5,335.16.
3. \$9,062.53.
4. \$470.44.
5. \$12,587.75.
6. \$982.24.

Page 150

1. P = \$943.52.
2. P = \$1,039.56.
3. P = \$538.97.
4. P = \$982.24.
5. P = \$504.75.
6. P = \$5,609.40.

Page 152

1. \$986.83.
2. P_0 = \$961.96;
   P = \$975.24.
3. P_0 = \$512.63;
   P = \$520.46;
   Q.P. = \$512.13.
4. Yes; P = \$90.75.
5. P_0 = \$92.29;
   P = \$93.98;
   Q.P. = \$92.45.
6. P_0 = \$1,027.02;
   P = \$1,043.96;
   Q.P. = \$1,025.96.
7. P_0 = \$1,013.65;
   P = \$1,031.00;
   Q.P. = \$1,012.33.

Pages 153–154

1. \$6,063.69.
2. \$26,084.46.
3. \$19,006.41.
4. \$17,237.05.
5. \$1,932.61.

Page 155

1. \$467.26.
2. (a) \$574.79.
   (b) \$535.75.
   (c) \$437.25.
   (d) \$384.43.
3. (a) \$510.47.
   (b) \$451.44.
4. (a) \$531.93.
   (b) \$470.04.

Page 158

1. 0.0473.
2. 0.04195.
3. 0.0739.
4. 0.0579.
5. 0.0474.
6. 0.0471.
7. 0.0326.

Page 160

Exercises

1. 0.0469; 0.0517.
2. 0.0474; 0.0718; 0.0469.
3. 0.0420; 0.0577; 0.0468.
4. 0.0521.
5. 0.0367.

Page 160

Problems

1. \$968.85.
2. \$1,035.85.
3. \$305,753.73.
4. By interpolation 0.0571.
   By formula 0.0568.
5. 0.0517.
6. P_0 = \$1,043.76;
   P = \$1,050.43.
7. \$93.18.
8. \$95.69.

Chapter VII

Pages 163–164

1. (a) $7/12$; (b) $5/12$.
2. $1/13$.
3. $1/2$.
4. $2/5$.
5. 0.4.
6. $1/6$.
7. (a) $1/4$; (b) $1/2$.
8. $1/36$; $1/18$; $1/12$; $1/9$; $5/36$; $1/6$; $5/36$; $1/9$; $1/12$; $1/18$; $1/36$.
9. (a) $23/45$.
   (b) $22/45$.
   (c) $1/5$.
   (d) $2/9$.
10. $1/3$.
11. $3/8$.
12. Former.
13. $5/18$.

Page 165 (Top)

1. 0.0085.
2. 0.514.
3. 0.18.

Page 165 (Bottom)

1. 4.
2. 8.
3. 36.
4. 288.
5. 504.
6. 2,730.
7. 3,024.

Pages 166–167

1. 24.
2. (a) 360.
   (b) 720.
3. 840.
4. 2,730.
5. 325.
6. 10.
8. 30,240.
9. 34,650.
10. 2,520.
11. 48.
12. 720.
13. (a) 362,880.
    (b) 725,760.
    (c) 725,760.
    (d) 2,903,040.

Pages 168–169

2. 126
3. 560.
4. 31.
6. (a) $3/476$.
   (b) $9/68$.
   (c) $126/595$.
9. 45.
10. 63.
11. (a) 126.
    (b) 84.
13. 302,400.
14. 878,948,939.
16. 31.
17. 3,600.
18. (a) 700.
    (b) 1,408.
20. $n = 11$, $r = 2$.
21. 6.
22. 10.
23. 7.
25. 711,244,800.

Pages 171–172

4. $1/16$.
5. (a) $1/22$.
   (b) $1/22$.
   (c) $1/22$.
6. (a) $5/144$.
   (b) $5/108$.
   (c) $125/1728$.
7. (a) 0.7624.
   (b) 0.8378.
   (c) 0.0205.
8. 0.0570.
9. $23/24$.

Pages 171–172—*Continued*

10. (a) 0.06.
(b) 0.56.
(c) 0.38.
(d) 0.44.

11. $\frac{7}{9}$.

12. (a) $\frac{1}{462}$.
(b) $\frac{1}{77}$.
(c) $\frac{100}{231}$.

13. (a) $\frac{175}{429}$.
(b) $\frac{32}{39}$.

14. $\frac{2}{145}$.

15. $\frac{11}{850}$.

Pages 175–176

1. 15.
2. 500.
3. 85,680.
4. (a) $\frac{45}{102}$.
(b) $\frac{35}{102}$.
(c) $\frac{15}{102}$.
(d) $\frac{7}{102}$.
5. 675,675.
6. 216.
7. (a) 180.
(b) 120.
(c) 6.
8. 720.
9. (a) 5,040.
(b) 840.
(c) 13,699.
10. (a) 0.015.
(b) 0.42.
(c) 0.425.
(d) 0.845.
11. $\frac{1}{2}$.
12. $\frac{56}{1024}$.
13. 0.0081; 0.0756; 0.2646; 0.3483.
14. 0.743.
15. \$10.
16. ${}_{100}C_{50}(.91914)^{50}(.08086)^{50}$.

Page 178

1. 0.5775; 0.4225; 1.
2. 0.3753.
3. \$7.49.
4. (a) \$8.43.
(b) \$4.46.
(c) \$6.51.
5. (a) \$13.78.
(b) \$11.58.
(c) \$14.37.

Page 180

Exercises

5. 0.0104.
6. 0.5775.
7. 0.08098; 0.00822.

Page 180

Problems

1. 0.4938.
2. 0.7138; 0.001201.
3. \$19,092.07.
5. 0.8264; 0.9920.
6. 0.01979.
7. \$4,900; \$4.90.
9. \$8,249.20.
10. (a) 0.77124.
(b) 0.01477.
(c) 0.11479.
(d) 0.09920.
11. \$2,802.61.
12. 0.55253.
13. (a) ${}_np_x \cdot {}_np_y$.
(b) $(1 - {}_np_x)(1 - {}_np_y)$.
(c) ${}_np_x + {}_np_y - 2{}_np_x \cdot {}_np_y$.
(d) Same as (c).
15. (a) 0.5542.
(b) 0.9856.
16. (a) ${}_{1000}C_{10}\, p_x{}^{990} q_x{}^{10}$.
(b) $\sum_{r=0}^{10} {}_{1000}C_r\, p_x{}^{1000-r} q_x{}^r$.

Chapter VIII

Page 184

2. \$2,261.72.
3. \$21,597.29.
4. \$1,285.30.
5. \$14,956.01.
6. \$16,469.28.
7. \$6,555.76.
8. \$24,355.37.
9. (a) \$7,754.46.
(b) \$7,297.62.
10. \$7,144.18.

Page 186

2. \$12,038.88.
3. \$738.84.
6. \$6,019.44.
7. \$1,847.10.

Page 188

1. \$712.83.
2. \$6,167.04.
3. \$1,541.01.
4. \$9,559.39.
12. \$2,348.54.

Page 192

1. \$70,147.19.
2. \$28,116.41; \$2,548.53.
3. \$471.83.
4. \$176.57.
5. \$5,141.72.
6. \$3,889.75.
7. \$1,960.54.
8. \$1,363.77.
9. \$117.11.
10. \$2,568.60.
11. \$662.39.
13. \$48,752.88.
15. \$129.53.

Page 195

2. \$7,592.16.
3. \$7,991.04.
4. \$7,917.36.
5. \$17,071.10.

Page 196

2. \$11,376.75.
3. A = \$117,632.40; Tax = \$4,705.30.
4. \$114,882.40.
5. \$14,644.05.
6. \$2,323.50.
7. \$1,470.32.
8. \$1,558.90.
9. \$2,552.70.
10. \$218.59, first payment immediately
12. \$25,805.64.
13. \$74,822.32.
16. Yes.
17. \$1,872.19.
18. \$21,834.77.
19. \$1,199.00.
20. \$9,266.29.

Page 201

1. \$1,887.86.
2. \$3,370.15.
3. \$171.90.
4. \$123.56.
7. \$134.78; \$137.72; \$349.85; \$365.86.
8. \$477.69.
9. \$1,806.51.
11. \$225.25; \$229.29; \$408.20; \$421.97.
12. \$242.04.
13. \$222.78.

Page 203

2. $2,196.79.
3. $7.54; $7.79; $8.14; $8.64; $9.46.
4. $107.97.
6. $13.52.
7. $221.81.
8. $40.69.

Page 205

2. $237.37. **3.** $1,614.80. **4.** $188.65. **5.** $7,279.34.

Page 207

1. (a) $35.60; $36.38.
(b) $35.91; $37.08.
3. (a) $76.93.
(b) $77.25.

Page 210

1. $118.27.
2. $129.66; $531.56.
3. $286.48.
4. $218.97.
11. $8.14; $26.02.
12. $8.14; $17.68.
13. $410.73.
14. $102.15.
15. $948.94.
16. $541.32.
17. $324.32.

Page 214

1. $183.40; $374.72; $574.33; $782.62; $1,000.00.
2. $33.89; $69.19; $105.94; $144.22; $184.10; $225.64; $268.93; $314.04; $361.05; $410.06.

Page 216

3. $153.07; $312.47; $478.50; $651.47; $831.64.
4. $726.72; $790.57; $857.29; $927.04; $1,000.00.
5. $306.67; $341.77.
6. $351.53.

Page 218

2. $2,542.46.
3. $4,088.92.
4. $7,933.18.
5. $356.85.
6. $741.45.

Pages 219–220

1. $409.15.
4. 1.
8. $364.33.
10. $17.30; $35.27; $53.94; $73.32; $93.46; $114.39; $136.11; $158.69; $182.12; $206.47.
11. $10.76; $21.89; $33.39; $45.27; $57.54; $70.19; $83.25; $96.70; $110.57; $124.85.
12. $81.97; $167.47; $256.64; $349.66; $446.72.
13. $22.25; $45.30; $69.17; $93.88; $119.46; $145.93; $173.31; $201.62; $230.88; $261.10.

Pages 219–220—*Continued*

14. $13.42; $27.28; $41.55; $56.27; $71.42; $87.03; $103.07; $119.56; $136.46; $153.79

15. $1.55. 16. $13.29.

Page 227

1.

At end of	Reserve	Automatic Extension		Paid-up Insurance
		Years	Months	
1st year	$ 14.67	1	6	$ 35.00
2nd "	29.81	3	1	70.00
3rd "	45.39	4	7	104.00
4th "	61.43	6	0	138.00
5th "	77.92	7	4	171.00
6th "	94.86	8	6	204.00
7th "	112.25	9	7	236.00
8th "	130.07	10	5	267.00
9th "	148.29	11	2	298.00
10th "	166.88	11	9	328.00

2.

At end of	Reserve	Automatic Extension		Paid-up Insurance
		Years	Months	
1st year	$ 22.25	2	4	$ 54.00
2nd "	45.30	4	9	106.00
3rd "	69.17	7	1	158.00
4th "	93.88	9	3	210.00
5th "	119.46	11	2	262.00
6th "	145.93	12	10	313.00
7th "	173.31	14	4	364.00
8th "	201.62	15	7	414.00
9th "	230.88	16	8	464.00
10th "	261.10	17	7	513.00

Page 227—*Continued*

3.

At end of	Reserve	Automatic Extension		Pure Endowment	Paid-up Insurance
		Years	Months		
1st year	$ 33.15	3	6	.....	$ 58.78
2nd "	67.59	7	2	.....	116.63
3rd "	103.38	10	7	.....	173.55
4th "	140.58	13	8	.....	229.54
5th "	179.23	16	3	$33.08	284.54

5. $316.58. **6.** $13,493.46. **7.** $30.29. **8.** $3,456.10.

Page 235

1. $306.79, Net Level Reserve; $301.48, F.P.T. Reserve.

2. $18.75; $53.71; $90.03.
3. $10.90; $46.14; $82.75.
5. $32.09; $34.88; $38.26; $42.37.
6. $18.47; $20.64; $23.42; $27.04.
8. $66.26; $66.75; $67.44; $68.42.

Page 236

1. $13,534.60.
2. $10,169.80.
3. (a) $7,966.16.
(b) $14,376.56.
4. $23,074.00.
5. $1,000.00.
6. 14 yrs. 7 mos. **7.** $7.79; $15.48. **8.** $7.79; $23.68. **9.** $7.79; $41.61.

Review Problems

Pages 237–243

1. 80%.
2. 60%; $3,900.
3. $12; $14.40; $18.
4. Single discount; $4.
5. 66%; $66; 34%.
6. $72.60; $90.75.
7. $79.40.
9. 0.80.
10. $675.
11. $1,192.31.
12. $396.04.
13. Jones' offer by $24.18.
14. 24.49%.
15. $515.46.
16. $548.90.
17. $651.81.
18. $7\frac{10}{11}$ months.
19. $8\frac{1}{2}$ months.
20. (a) $1,031.45.
(b) $5.35.
21. $3,101.89.
22. $279.76.
23. 8.347%.
24. 12.68%.
25. $2,316.61.
26. $1,029.12.
27. $4,004.13.
28. 8.16%; 8.41%.

Pages 237–243—*Continued*

29. $d = 5.66\%$, $j_4 = 5.87\%$, $f_4 = 5.78\%$.
30. $i = 6.38\%$, $j_4 = 6.24\%$, $f_4 = 6.14\%$.
31. $i = 6.23\%$, $d = 5.86\%$, $j_4 = 6.09\%$.
32. $i = 6.14\%$, $d = 5.78\%$, $f_4 = 5.91\%$.
33. 1.4778%.
36. Yes, and save \$133.21.
37. (a) \$7,721.73.
(b) \$7,052.25.
38. \$350.36.
39. 8; \$244.57.
40. \$3,648.80.
42. $j_2 = 5.40\%$; $i = 5.47\%$.
43. $i = 5.51$.
44. \$10,106.20.
45. \$270.33.
46. $n = 14$; \$479.20.
47. \$12,177.03.
48. 41 full payments; \$125.90.
49. \$3,997.64.
50. Yes.
51. \$16.04.
52. $R = \$243.89$.
53. \$24,649.90; \$54,649.90.
54. \$13,329.09; 5.6%.
55. 1st method better by \$344.66 a year.
56. 4 full payments; \$1,073.71 at end of 5 years.
57. \$8,348.40.

58. 27.522%; \$1,000.
59. $R = \$399.80$.
B.V. = \$2,834.56.
60. \$972.40.
61. (a) \$17,626.51.
(b) \$2,227.60.
62. \$6,319.55.
63. 1,620 units.
64. 1,862 units.

65. \$925.61.
66. \$927.66.
67. \$914.51.
68. $P = \$9,376.97$.
69. \$9,444.17.
70. \$10,518.61 by Bankers' Rule.
71. $j_2 = 5.3914\%$.

Miscellaneous

72. 5 years.
74. B's offer.
76. $\dfrac{\log 0.5}{\log (1 - d)}$.
84. Yes. About 41 yrs. to exhaust principal.
85. \$3,391.75; \$437.09.
86. 18 + years.

87. \$188,687.20.
88. Yes.
89. 11.26%.
91. .06184.
92. .05827.
94. .0285.
97. \$1,491.83.
98. .0805.
99. .094.
100. .053.

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