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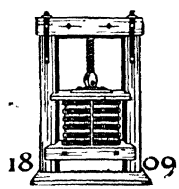
With Applications to Hydraulics, Aeronautics, Meteorology and other Subjects

BY

LUDWIG PRANDTL

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Authorized Translation



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AUTHOR'S PREFACE

This book is a translation of the 1949 edition of my *Führer durch die Strömungslehre*. The German title may, perhaps, be translated as "A Guide to the Dynamics of Fluids". It indicates that the book is intended to be a guide to the reader, to the beginner and the advanced student as well as to the expert in an adjoining field of research. To achieve this, complex mathematical analysis has been avoided as far as possible, and in general formulæ take second place, the principal object being the awakening of clear, intuitive apprehension. The British publisher offered some objections to a literal translation of the German title. However, the one finally chosen attempts to distinguish this book from the normal run of textbooks with their numerous formulæ.

The reader will find that the body of the text is divided into parts in large type and other parts in small type, and it is important at this point to stress the meaning of this distinction. Briefly, the parts in large type give an introduction, and the paragraphs in small type give more detailed information for advanced readers and experts. In particular, the large type of the first three chapters is in itself an introductory course for beginners, while in the fourth and fifth chapters the parts in larger type assume that the reader, having studied the first three, is able to understand the somewhat more advanced problems with which it is necessary to deal. Throughout, the text in small type is addressed to those readers who have a mathematical interest and to those who wish to obtain a deeper insight into the subject of fluid dynamics. Finally, footnotes contain numerous references to original papers and textbooks for those readers who are already familiar with the problems of fluid dynamics and wish to refer to original sources for further detail. (It should be understood here that the list of references is in no way complete and that only those references have been included which it is thought will be of the greatest assistance to the reader.)

A list of the most important textbooks and reference books is given at the end of the volume.

As a matter of interest let me mention in brief the origin of the German book, which arose step by step from two articles in Vol. IV of the *Handwörterbuch der Naturwissenschaften* (Gustav Fischer, Jena,

1913) on the motion of liquids and gases. These appeared again in more detailed form in Vol. I, 2 of the *Lehrbuch der Physik* by Müller-Pouillet in 1929. These articles, together with others from the same source, were translated and published by Blackie & Son in 1930 under the title of *The Physics of Solids and Fluids*. My book, in its first independent form, was published by Vieweg und Sohn, Brunswick, in 1931, under the title *Abriss der Strömungslehre* (" Short Introduction to Fluid Dynamics "); a second edition followed in 1935. A considerably enlarged and revised edition, which was intended to smooth the way for advanced students to the special literature concerned with the individual sections, was brought out in 1942. A second edition appeared in 1944, and in 1949 a third edition followed, of which this book is a translation. It was somewhat enlarged by dealing with several important papers by foreign scientists which reached us in the post-war years.

In order to complete the references to papers published in Great Britain and the U.S.A. during and after the 1939–45 war, an appendix has been added, which was prepared by Mr. H. B. Squire, assisted by Mr. H. Charnock, Mr. S. P. Hutton and Mr. E. G. Richardson, to whom I offer my thanks. A key to the use of this appendix is given on p. 432.

I should also like to express my thanks to a great number of colleagues for all the help and advice they rendered when I was preparing the earlier editions, and to say that their suggestions are still alive in this volume.

The translation is the work of Miss W. M. Deans.

The text of the present edition has been carefully checked for errors by my colleague Dr. H. Merbt, Göttingen; Prof. H. B. Squire was our adviser on English technical expressions. My thanks are due to both these gentlemen.

Acknowledgment is also made to Messrs. Blackie & Son for the care they have taken with both the translation and the production of my book.

LUDWIG PRANDTL.

GÖTTINGEN
1952

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CHAPTER I

Equilibrium of Liquids and Gases

1. Properties of Liquids.

Liquids are distinguished from solids by the fact that their particles are readily displaced. Whereas forces of finite (and sometimes very considerable) magnitude are required to deform a solid, no force at all is required to alter the shape of a liquid, provided only that sufficient time is allowed for the change of shape to take place. Liquids as well as solids offer resistance to rapid change of shape, but this resistance disappears very quickly once the motion has ceased. In the case of liquids this property of resisting deformation is called *viscosity*. Viscosity will be discussed in detail in Chapter III, § 1, p. 98. In addition to ordinary mobile liquids, there also exist very viscous liquids, whose resistance to deformation is very considerable (but vanishes when the liquid is at rest). Between such liquids and (amorphous) solids there exist all degrees of gradation: e.g. when glass is heated it passes through all these intermediate stages, while asphalt and similar substances exhibit such gradations at ordinary temperatures.

Experiment.—If a barrel of asphalt is upset, it takes a few days or several weeks to empty itself, according to the temperature. The material which flows out forms a flat cake on which, although it is really continuing to flow farther, people can walk about without any visible effect; but marks will be seen if people remain standing on it for some time. If the mass is struck with a hammer it fractures like glass.

In the study of the equilibrium of liquids we are concerned only with liquids at rest, or moving sufficiently slowly to be considered so; hence we may without loss of accuracy take the resistance to deformation as zero, and immediately obtain the following *definition of the liquid state.*

A liquid in equilibrium offers no resistance to change of shape.

According to the conceptions of the kinetic theory of matter, the smallest existing particles of any body (the molecules) are in continual motion: it is the kinetic energy of this motion that appears in the form of heat. From this point of view liquids are distinguished from solids by the fact that the particles do not, as in the case of the latter, oscillate about fixed positions, but change places

more or less frequently.* If there are stresses in a liquid, those changes of place are favoured which bring about a yielding in the direction in which there are differences of stress. This yielding causes the differences of stress to vanish more or less rapidly when the liquid is at rest, but, while the shape of the liquid is changing, it gives rise to stresses which are greater the more rapid the change of shape. The gradual softening of amorphous bodies with rise of temperature may be imagined to take place as follows. If we heat a solid body, i.e. if we increase the energy of the molecular motions, what happens at first is that a few particles which at the moment happen to have particularly large amplitudes exchange places; on further heating more and more exchanges take place, and finally they become general throughout the body. In crystalline solids the passage from the solid state to the liquid as a rule takes place discontinuously by way of melting, i.e. it is due to the breaking down of the regular structure.

Another property of liquids is their great resistance to change of volume. It is quite impossible to force a litre of water into a half-litre vessel, and if this quantity of water is put into a two-litre vessel, it only half-fills it, even if the air is pumped away from above the water. Water, however, is not absolutely incompressible; under high pressures it can be compressed to a noticeable extent (under a pressure of about 1000 atmospheres † its volume is diminished by 5 per cent). Similar relationships hold for other liquids.

2. Theory of Stresses.

We shall now consider the nature of the stresses in a liquid in equilibrium more closely. Perhaps it is not unnecessary to remark that the general theorems dealing with forces on a body in equilibrium may also be applied to liquid bodies. To justify this statement, a special "principle of solidification", founded on the following considerations, is often brought into use. "The equilibrium of any system free to move cannot be affected by any movable parts subsequently becoming solidified; that is, any portion of a liquid in equilibrium may without disturbing the equilibrium be imagined as solidified, and theorems on the equilibrium of rigid bodies can be applied to the solidified part of the liquid." ‡ This digression by way of rigid bodies, however, is not absolutely necessary. General theorems on mechanical equilibrium have repeatedly been deduced with the help of the "rigid body". These theorems may, however, equally well be applied to a "system of masses at rest" in which the individual masses are free to move (although on account of the system being in equilibrium this freedom is not utilized). So long as we are dealing with a case of actual rest,

* For a more detailed account, see M. von Laue, *Forsch. u. Fortschr.*, Vol. 21/23, p. 53 (1947).

† For units of pressure, see § 3, p. 6; §§ 8, 9, pp. 20–2.

‡ Here, of course, we are not thinking of a physical solidification involving change of volume, crystal formation, &c., but of an ideal solidification without any displacement or change of volume.

both points of view are equally justified; but if we are dealing with a case of motion the principle of solidification is apt to lead us into difficulties, as here nothing is rigid. In view of subsequent applications to hydrodynamics we now proceed to give a short account of the essential principles of the other method, which is commonly used in elasticity also.

We begin by assuming that all forces are mutual actions between masses. If, for example, the mass m_1 attracts another mass m_2 with a force F, m_2 acts on m_1 with the same force F, attracting m_1 towards it; that is, the two forces act in opposite directions (Newton's law of action and reaction). In a *system of masses*, which may be marked off in any way from other masses, we distinguish two kinds of forces: *internal forces*, which act between two masses belonging to the system and hence invariably occur in *opposed pairs*, and *external forces*, which act between any one mass belonging to the system and an external mass, and thus occur only *once* in the system. If we combine all the forces acting on all the masses of the system (vectorially or by components), the internal forces always cancel each other in pairs, so that the external forces alone figure in the result. In order that the system should be in equilibrium it is necessary that the resultant of the forces acting on each individual mass (the vector sum or the three component sums) should *vanish*. If we combine all these forces for all the masses of the system, there remains, by what we have said above, only the resultant of all the external forces, and as the resultant of the forces acting on each individual mass vanishes, owing to there being equilibrium, the *resultant of the external forces acting on the system vanishes*. This theorem, which assumes nothing about the system of masses except that it is in equilibrium, turns out extremely useful in all manner of applications. If we work in rectangular Cartesian co-ordinates, the theorem comprises the three equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

where X, Y, Z are the components of the external forces along the x-, y-, and z-axes.

There is a completely analogous theorem for the couples formed by the external forces. For equilibrium these must vanish also.

In both elastic solids and liquids we are concerned with the stresses within the body. These obviously consist of the internal forces acting between the smallest particles of the body. In general we content ourselves with statements about the average state in a region which contains a very large number of particles, for it would be much too confusing to picture all the individual forces between the particles, which in addition have their own vigorous thermal motions. But how can we get hold of the internal forces, seeing that our theorems merely give statements about external forces? The answer is: "We

must turn the internal forces into external forces!" This can be done as follows. We imagine a body cut into two portions, and we take one of the portions (I in fig. 1.1) as our system of masses; then all the forces exerted by the particles of portion II on the particles of portion I, which were hitherto internal forces, become external forces. If the whole body is under external compressive stress (indicated by two arrows in fig. 1.1), there are also internal stresses, and if we imagine the section completed we shall find that forces are exerted across the plane of the cut by the particles to the right on the particles to the left. If we add up all these forces, i.e. combine them to form a resultant, the latter is in equilibrium with the forces acting on portion I (cf. fig. 1.2). This gives a unique expression for the resultant of the forces acting across the section.*

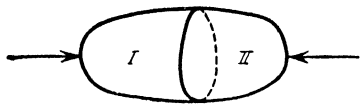


Fig. 1.1

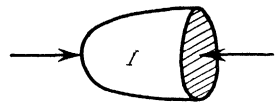


Fig. 1.2

By " stress " we now mean the *force per unit area acting across a section.* In the above example we obviously obtain the average stress over the section if we divide the total force by the area of the section. Hence we see in addition that the " stress on an area " is a vector just as a force is.

Our principle of turning internal forces into external ones by means of an imaginary section (which in what follows will be called the " principle of section " for short) permits of extensive application; by taking a number of sections we isolate an element (parallelepiped, prism, tetrahedron, &c.) from the interior of the body in which we wish to investigate the stresses, and study its equilibrium. In the simplest case all the forces acting on the element are compressive forces. From the equilibrium of such elements we can deduce various important theorems about stresses, one of which may be given here as an example, together with its proof. " If the stress vector is given for three planes of section which together form a solid angle, the stress vector is known for any other plane of section." *Proof*: We cut the solid angle by the fourth plane, in which the stress is to be found, so that a tetrahedron is formed as in fig. 1.3. The forces 1, 2, and 3 are obtained by multiplying the given stress vectors by the areas of the corresponding triangles. There is only *one* direction and magnitude for force 4 such that it is in equilibrium with forces 1, 2, and 3; this

* Of course we might equally well have begun by considering portion II; we should thereby have obtained an equal and opposite force (namely the force exerted by portion I on portion II).

force, divided by the area of the corresponding triangle, is the stress required. For purposes of calculation it is convenient to take the planes 1, 2, and 3 as co-ordinate planes (see fig. 1.3).

Further details of the theory of stresses will be found in textbooks on elasticity. Here we may merely mention that the " stress ", which represents the totality of stress vectors for all possible planes of section through a point, may be related to an ellipsoid; the stress components can accordingly be given in the form of a " tensor ". By the theorem given above, the stress at any point (with the corresponding ellipsoid) is given when the stress vectors for three planes of section forming a solid angle are given. Corresponding to the three principal axes of an ellipsoid, there exist for every stress three sections, mutually at right angles, such that the associated stress vector is perpendicular to the section. The three stresses distinguished in this way are called *principal stresses* and the corresponding directions *principal directions*.

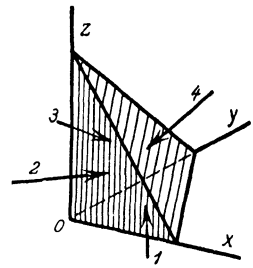


Fig. 1.3

3. **Pressure in a Liquid.**

The stresses in a liquid in equilibrium are particularly simple. Resistance to deformation, i.e. to displacement of the particles relative to one another, is related to friction. If there is no friction between two solid bodies in contact the pressure must always be normal to the surface of contact of the bodies (so that no work is done when either body slides along the surface of contact). Similarly, the absence of resistance to deformation in a liquid is expressed by the fact that in the interior of the liquid the *stress*, which in this case is usually called the *pressure*, *is everywhere normal to the surface on which it acts*. We may right away take this fact as a *definition* of the liquid state; this definition is completely equivalent to the definition given on p. 1.

From this property of liquid pressure we may immediately deduce another by considering a simple case of equilibrium. In imagination we isolate a small triangular prism from the liquid. Let the top and bottom surfaces of the prism be perpendicular to the edges of the prism (of course we could also think of the prism as being solidified in its place in the liquid and investigate the equilibrium of the forces exerted on the prism by the rest of the liquid). The pressures on the end surfaces are equal in magnitude and opposite in direction and are therefore in equilibrium by themselves, so that they need not be considered further. Since the forces on the lateral surfaces are normal to these surfaces, they lie in a plane perpendicular to the edges of the prism. Fig. 1.4 shows an end-on view of the prism and the lateral forces

acting on it, fig. 1.5 the triangle that the forces must form in order that there may be equilibrium. As the sides of the triangle in fig. 1.5 are respectively perpendicular to those in fig. 1.4, the triangles are equiangular and therefore similar. Hence it follows that the three pressures are proportional to the corresponding sides of the prism. Now if we want to obtain the pressures per unit area the total pressures must be divided by the areas of the corresponding faces of the prism. The faces of the prism are all of the same height, so that their areas are proportional to their bases, the sides of the end of the prism, i.e. proportional to the pressures acting on them. Hence it follows that the *pressure per unit area*, which we shall without more ado call the *pressure*, is the same on all three faces of the prism. As the prism was chosen quite arbitrarily, we conclude that *at any one point in the liquid the pressure is the same in all directions* (more accurately, for all possible positions of an imaginary plane of section). In this case the stress ellipsoid is a sphere. Thus all that is required to determine a system of pressures of this kind (known as hydrostatic pressure) is a single numerical value, namely that of the pressure p. By what we have said above, p obviously denotes the force exerted on unit area. A number of widely differing units of pressure are available, varying according to the units of force and area chosen. In practice it is convenient to use kilograms weight per square centimetre (kg. wt./cm.^{2} or Kg./cm.^{2}) * or kilograms weight per square metre (Kg./m.^{2}), or, in British units, pounds weight per square inch (Lb./in.^{2}). For some other units of pressure see §§ 8, 9 (pp. 20–2).

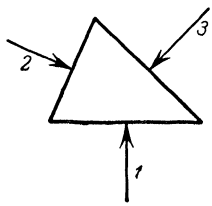


Fig. 1.4

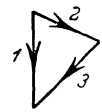


Fig. 1.5

4. Distribution of Pressure in a Liquid, Gravity being neglected.

Every liquid has weight. In many cases, however, especially when we are dealing with high pressures, it is unnecessary to take account of gravity effects. This greatly simplifies the discussion. We again start from a prism in equilibrium; in this case, however, we take a prism of elongated form and consider its equilibrium with respect to displacements along its axis. To begin with we assume that the pressure varies from place to place within the liquid. Let the area of the cross-section of the prism (equal to the area of the end faces, which as before we take perpendicular to the axis) be a (fig. 1.6). We take this area so small that variations of pressure within it can be neglected. If the pressure on one end of the prism is p_1 and on the other p_2, we

* We here adopt the system of denoting the gravitational units by making the first letter a capital, so that, for example, a pound mass is lb. and a pound force is Lb.

have the forces ap_1 and ap_2 acting in opposite directions along the axis of the prism. But by our fundamental assumption all the pressures on the side faces of the prism are normal to these faces, and therefore also normal to the axis of the prism, so that they contribute nothing to the component of force parallel to the axis of the prism, no matter how the pressure is distributed over them. That is, it is necessary for equilibrium that the individual forces in the direction in question, ap_1 and ap_2, should be in equilibrium by themselves, so that we must have

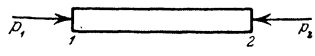


Fig. 1.6

$$ap_1 = ap_2, \text{ or } p_1 = p_2;$$

as the position of the prism is perfectly arbitrary, this means that in the absence of gravity (and any other externally impressed forces) *the pressure is the same at all points in the liquid.*

If the liquid happens to occupy a narrow or winding space, so that it is impossible to describe a prism in the liquid between two arbitrary points, we are still free to repeat the argument as often as we like, passing from a point 1 to another point 2, from the latter to a point 3 in another direction, and so on, till the required end-point n is reached. From $p_1 = p_2$, $p_2 = p_3$, &c., we then get $p_1 = p_n$. Another even more elegant argument is as follows. Imagine a capacious vessel, large enough to contain the vessel under consideration and completely filled with liquid, and suppose that subsequently, after equilibrium has been reached, all the liquid except that occupying the inner vessel is solidified. According to our solidification principle (p. 2, § 2), this does not alter the configuration of forces at all; that is, in equilibrium the pressure is the same everywhere, no matter how narrow the space is.

Note.—In extremely narrow spaces a very considerable time may elapse before equilibrium is reached after the pressure of the liquid has been changed, e.g. by an external load. In plastic potter's clay (which consists of very fine solid particles with the interstices between them filled with water), for example, the period of time may amount to several days, or, in the case of whole strata of clay in the ground, to years *; during this time the water flows from places where the pressure is high to places where the pressure is low (cf. Chapter III, § 9, p. 150), and at the same time there is elastic yielding of the solid framework.

We may sum up the above paragraphs as follows. *The pressure in a liquid in equilibrium is everywhere normal to the surface on which it acts and—in the absence of gravity or other body-forces—is the same at all places and in all directions.*

What is true of the pressure in the interior of the liquid is also true of the pressure on the walls of the vessel containing the liquid. In order to assure ourselves of this we take a plane section through the liquid just in front of the wall—or at some little distance from it—and a cylindrical surface perpendicular to this plane, as in fig. 1.7. By

* See e.g. Terzaghi, *Erdbaumechanik* (Vienna, 1924).

considering the equilibrium of the portion of water enclosed in this way we obtain the force (F), or more accurately the component of force acting on the wall in the direction *normal* to the plane of section; this force * is equal to ap. This way of looking at the matter has the advantage of at once making the fact clear that even large irregularities in the wall do not affect the result in the least.

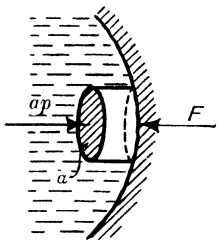


Fig. 1.7

5. Properties of Gases.

Gases are distinguished from liquids by the fact that by the application of suitable pressures they can be compressed into a very small space, and, on the other hand, if they are given more space than they occupy in their ordinary condition they fill the space uniformly, and their pressure falls below that of the surrounding air. Otherwise their behaviour closely resembles that of liquids; in their case also the resistance to deformation vanishes when they are at rest, and, moreover, when internal displacements occur they exhibit a definite viscosity.† Thus, provided no changes of volume are involved, the behaviour of a gas differs qualitatively in no way from that of a liquid which fills the same space and has no free surface.

The most important gas in practice is atmospheric air. The other gases behave in essentially the same way. As we shall explain in more detail later on, atmospheric air at the surface of the earth is subject to a pressure which, broadly speaking, has a constant value of about 1 Kg./cm.^{2} The average atmospheric pressure at sea-level is taken as equal to that of a column of mercury 760 mm. high or 1·0332 Kg./cm.^{2} (see also p. 23). At higher levels the atmospheric pressure is smaller; cf. § 7, p. 14.

Various instruments are used to measure the pressure of the atmosphere (or other gases). If they indicate differences of pressure, they are called *pressure gauges* or *manometers*; if they indicate the absolute pressure of the gas surrounding them, they are called *barometers*. Columns of liquid can be used for both purposes (cf. §§ 8, 9, p. 18), but other types of apparatus are also much used, in which a spring is acted on by the pressure to be measured. For example, in order to measure the absolute pressure of the air, we may use a small metal box exhausted of air, with a flexible lid connected to a strong spring in such a way that the tension of the spring prevents the lid giving way under the external atmospheric pressure. The movements of the lid are re-

* In fig. 1.7 the force F is shown as being exerted by the wall on the portion of liquid under consideration. The pressure exerted on the wall by the liquid is in the opposite direction.

† See the note at the end of Chap. III, § 4, p. 119.

corded on a greatly magnified scale by a pointer, which records the change of pressure if the apparatus is moved to a place where the atmospheric pressure is different, or if the atmospheric pressure is changing. This instrument is known as the *aneroid barometer.*

The law in accordance with which the pressure of a gas changes for a given change of volume was first discovered by Robert Boyle in 1662 (and hence is called Boyle's law) and again independently by Mariotte in 1679. It states that the pressure (at one and the same temperature) is inversely proportional to the volume. That is, if a quantity of gas is compressed to half its volume its pressure is doubled, and conversely, if its volume is doubled its pressure is halved. This law is expressed by the formula

$$pV = p_1V_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p_1 is the original pressure, V_1 the original volume, and p, V are any pair of corresponding values of these quantities.

The volume of a gas also varies greatly with the temperature. In 1802 Gay-Lussac found that provided the pressure is kept constant the expansion for a rise in temperature of 1° C. is always 1/273 of the volume at 0° C. This, moreover, is true—apart from small deviations * —for all gases and all temperatures. The property may be expressed by the formula

$$V = V_0(1 + \alpha\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where V_0 is the volume at 0° C., θ the temperature on the Celsius scale, and α (= 1/273) the " coefficient of expansion ".† This is called Charles' law or Gay-Lussac's law; at medium pressures it holds fairly accurately for air and other gases which are not easily liquefied.

As formula (2) turns out to be independent of the pressure at the time in question, except that the pressure must be constant for the two states compared in the formula, we can combine formula (2) with formula (1). We thereby obtain the following formula, which applies to all temperatures and pressures:

$$pV = p_0V_0(1 + \alpha\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here p_0 means a fixed, but arbitrary, initial pressure and V_0 the volume at the initial pressure p_0 and the temperature 0° C. Equation (3) is sometimes called the combined Boyle's and Charles' law. It is also called the " equation of state ", because it connects the three quantities pressure, volume, and temperature. In particular, as actual gases exhibit certain deviations from it, it is called the " equation of state of a perfect gas ". These deviations are fairly small for gases at

* See e.g. J. K. Roberts, *Heat and Thermodynamics,* Fourth Edition, p. 17 (Blackie & Son, Ltd.).

† More accurately 1/273·2.

ordinary temperatures, but they become very marked if the gases are strongly compressed, especially if the temperature is lowered so far that the gases liquefy.

Here we can give only a brief account of one of these deviations.* Equation (1) means that if the pressure were immensely large the volume of the gas would become exceedingly small, and the formula enables us to calculate at what pressure the gas would reach the density of water, of gold, &c. This, however, is not so. There is a limiting volume beyond which the gas cannot be compressed, no matter how high the pressure is; in this state the molecules of the gas are packed as tightly together as is possible. We can take account of this fact by altering equation (3) to

$$p(V - V') = p_0(V_0 - V')(1 + \alpha\theta),$$

where V' denotes the small limiting volume. For every finite value of p, V obviously turns out somewhat larger than V', while for volumes V which are large in comparison with V' the results given by the above equation are practically the same as those given by equation (1).

When a gas is compressed, heat is produced; thus proper observations of Boyle's law (which is valid for constant temperature only) can only be made if sufficient time is allowed during, or at least after, the compression for the gas to give up the heat generated and take on the temperature of its surroundings. The same is true for the cooling which occurs on expansion. If the gas is not given time to come to the same temperature as its surroundings the pressure must obviously increase faster than the volume decreases. We learn from thermodynamics that if none of the heat generated is given up at all, i.e. if the compression or expansion is extremely rapid, we get the formula

$$pV^\gamma = p_1V_1^\gamma, \quad \ldots\ldots\ldots \quad (4)$$

instead of equation (1), where $\gamma = c_p/c_v$, i.e. the ratio of the specific heat at constant pressure to the specific heat at constant volume.† For dry atmospheric air $\gamma = 1{\cdot}405$. A compression or expansion according to the law of equation (4) is called an *adiabatic* compression or expansion,‡ in contradistinction to a compression or expansion in accordance with equation (1), which is called *isothermal*.

The magnitude of the heating effect associated with adiabatic compression may be calculated from equations (3) and (4): adiabatic expansion of course gives rise to a corresponding cooling effect.

Note.—The properties of gases which we have sketched in this section are very well explained by the assumption that the molecules of the gas keep darting about very rapidly, continually colliding with other molecules and with the wall of the containing vessel. The pressure of the gas is the resultant effect of these

* Fuller details are given in Roberts, *Heat and Thermodynamics*, Fourth Edition, Chapter IV.

† *Ibid.*, pp. 193–4. ‡ *Ibid.*, pp. 174.

collisions, and the temperature is synonymous with the kinetic energy of the particles. The latter does actually increase on compression, as the velocities of the particles are increased by their elastic reflection from the walls of the vessel advancing to meet them.*

6. Equilibrium of a Heavy Liquid.

The effect of gravity on any mass m is that the mass is attracted towards the centre of the earth † with a force of magnitude mg, where g is the acceleration of a body falling freely and in medium latitudes is numerically equal to about 981 cm./sec.^{2} The force mg is called the *weight* of the mass m. As quantities of liquid are often measured by their volume it is convenient to give a special name to the *mass of unit volume*. It is called the *density* and is denoted by ρ, so that a quantity of liquid of volume V and density ρ has mass ρV and weight $g\rho V$. The product $g\rho$ is the *weight of unit volume* and is often denoted by w. This must not be confused with the *specific gravity*; the latter means the weight of unit volume of the substance when the weight of unit volume of water is taken as unity, or what is the same thing, the density of the substance relative to water as unity. The weight of unit volume only coincides with the specific gravity when the units are chosen in a special way (centimetres and grams weight; decimetres and kilograms weight; metres and metric tons); taking metres and kilograms weight, for example, w_W, the weight per unit volume of water, is equal to 1000 Kg./m.^{3}

As the attraction g is not exactly the same at all points of the earth's surface the weight per unit volume also differs to some extent from place to place. For this reason physicists prefer the concept of density, which is independent of gravity. In hydrostatic calculations, however, it is convenient to replace the product $g\rho$ by the weight of unit volume w.

The fundamental problem of hydrostatics, i.e. of the theory of the equilibrium of heavy liquids, *is the determination of the distribution of pressure* (of the " pressure field ") *in a homogeneous heavy liquid.*

We again consider the equilibrium of a prism (marked off in the liquid in any way) with respect to displacements in the direction of its axis, and begin with one like that in fig. 1.6 (p. 7), in which the axis is horizontal, i.e. at right angles to the direction of gravity (the direction of the plumb line). Thus the weight of the prism has no component in the direction of the axis, so that all the relevant considerations of § 4 (p. 6) may be applied here. We again have $p_1 = p_2$:

* For further details see e.g. the account of the kinetic theory of gases in Roberts, *Heat and Thermodynamics*, Chapter III.

† This is not quite accurate: it would be true only if the earth did not rotate. In reality what we call gravity is the combined effect of the earth's attraction and the centrifugal force; the direction of the plumb line at places in the northern hemisphere cuts the earth's axis somewhat to the south of the centre of the earth.

repeating the argument for adjacent prisms with their axes horizontal, we find that at every point of a horizontal plane the pressure has the same value.

The relationship between different horizontal planes is now obtained by considering the equilibrium with respect to upward and downward displacements of a prism or cylinder with its axis vertical. Here we find that the weight of the prism plays a part in the attainment of equilibrium. Acting downwards we have the weight $W = wV = wah$, and also (as shown in fig. 1.8) $p_1 a$, the pressure on the upper end surface: acting upwards we have $p_2 a$, the pressure on the lower end surface. Hence for equilibrium we must have

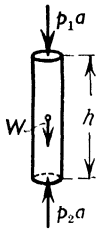


Fig. 1.8

$$wah + p_1 a = p_2 a,$$

so that

$$p_2 - p_1 = wh. \quad . \quad . \quad . \quad . \quad (5)$$

(The difference of pressure between the points 1 and 2 is equal to the weight of the intervening vertical column of liquid of unit cross-section.) Repeated application of this argument yields the following result. *The pressure continually increases with increasing depth in such a way that for every unit distance traversed it rises by an amount equal to w, while it remains constant for every horizontal plane.*

If we set up a system of rectangular axes with the z-axis pointing vertically upwards, and if p_0 is the pressure in the horizontal plane $z = 0$, the pressure at any other point, p, is given by

$$p = p_0 - wz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As may be shown by means of the principle of solidification explained above (cf. p. 2), this relationship holds not only for large vessels filled with liquid, but also for any system of tubes, for the interstices in a mass of gravel or sand, &c. All that we assume is that the liquid is homogeneous, continuous, and at rest.

The principle of solidification also provides a very simple solution of the problem of finding the force on a body submerged in a liquid. We may first imagine the body as replaced by liquid. It is clear that the new portion of liquid must have the same shape as the body, and the same weight per unit volume as the rest of the liquid, and that it will be held in equilibrium by the pressures on its surface. The resultant of these pressures must be vertically upwards and pass through the centre of gravity of the new portion of liquid. In magnitude this upward thrust is equal to the product of V, the volume of liquid displaced, and w, the weight per unit volume of the liquid. If we now imagine the new portion of liquid solidified, none of these relationships is altered, not even if another body of the same shape is substituted.

This principle, known from its discoverer as *Archimedes' principle*, may also be expressed as follows: A body submerged in a liquid loses weight by an amount equal to the weight of the liquid which it displaces. If we weigh a body first in a liquid and then in air—which also exerts a small upward thrust—we have

$$\text{upward thrust in liquid} - \text{upward thrust in air} = V(w_{\text{liquid}} - w_{\text{air}}).$$

Thus if we know the weights per unit volume of the liquid and of air we can find V, or if we know V, we can find the weight per unit volume of the liquid; the weight per unit volume of air can be calculated as in § 7 (p. 14).

If we are dealing with a heterogeneous liquid (a liquid varying in temperature from place to place, a salt solution varying in concentration from place to place, &c.), the deductions from the prism with its axis horizontal may be applied without change, so that in this case also the pressure is the same in every horizontal plane. We now select two such horizontal planes at a (not too large) distance (h) apart (fig. 1.9), the upper one being under the pressure p_1 and the lower one under the pressure p_2, and consider two vertical prisms of height h, the average weight per unit volume of liquid being w_1 in the left-hand prism and w_2 in the right-hand prism.

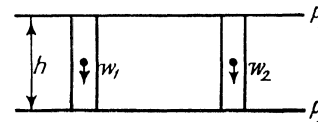


Fig. 1.9

For equilibrium it is necessary that on the left $p_2 - p_1$ should be equal to $w_1 h$ and on the right $p_2 - p_1$ should be equal to $w_2 h$. But these relationships cannot co-exist unless $w_1 = w_2$. Otherwise there would not be equilibrium, and the liquid would be set in motion. We can make our argument more refined by taking the height h very small and repeating the argument for any number of pairs of neighbouring horizontal planes. In this way we obtain the following result. *A heavy heterogeneous liquid cannot be in equilibrium unless the density is constant in every horizontal layer.* The question of the equilibrium of two non-miscible liquids of different density, one above the other, is then settled. By our theorem, it is necessary for equilibrium that *the surface of separation should be a horizontal plane.* We can of course obtain the same result by applying the argument illustrated by fig. 1.9 directly to two homogeneous layers of liquid, one above the other, the surface of separation being in the first instance taken as unknown except for the fact that it lies between our two horizontal planes.

As regards the stability of such arrangements of liquids in layers, we have to remark in addition that the less dense liquid must always

be *above* the more dense. The reverse arrangement is unstable; the least disturbance upsets it completely.

For the proof of this we briefly remark that we may again start from fig. 1.9 by taking a disturbed (e.g. a slightly sloping) surface of separation between the two horizontal planes and calculating the differences of pressure which thereby arise. In the stable case these tend to diminish the inclination of the surface, in the unstable case to increase it.

Exactly analogous relationships hold in the case where the density varies continuously. If the density decreases upwards at all points of the liquid, there is stability. In contradistinction to the stably-stratified heterogeneous liquid, the homogeneous liquid represents the case of neutral equilibrium, in which any parts may be displaced in any way without causing forces disturbing equilibrium to arise.

As regards the distribution of pressure in the heterogeneous liquid, equation (5) (p. 12) holds in its differential form

$$dp = -w\,dz \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

for every layer in which the density is sufficiently uniform. If w is given as a function of z, we obtain the relationship

$$p = p_0 - \int_0^z w\,dz \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

by integration.

7. Equilibrium of a Heavy Gas.

The conditions of equilibrium of a mass of heavy gas are in all essentials the same as for the equilibrium of a liquid, so that the relationships in the last section may be applied. In many cases, e.g. when the mass of gas does not extend over too great a range of height, the weight per unit volume of the gas may be taken as constant from point to point. This enables us to use equations (5) and (6) of the previous section (p. 12), i.e. the gas may simply be regarded as a homogeneous liquid. For greater ranges of height (e.g. several kilometres) this is no longer permissible. In this case the differences of pressure are so large that the densities above and below are markedly different, owing to the compressibility of the gas. Differences of temperature frequently play a part as well. Here, therefore, we must work with the formula for heterogeneous liquids: in this case, however, it is more convenient to write the formula in another form, as w is not in the first instance known as a function of the *height* z but is determined by the *pressure* p. We divide equation (5*a*) by w and integrate, obtaining

$$\int_p^{p_0} \frac{dp}{w} = z. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From this integral we obtain various results, corresponding to various distributions of temperature. Here we shall confine ourselves to the most important case, that of *constant temperature*. The weight of unit volume, being inversely proportional to the volume of a definite quantity of gas, is proportional to the pressure, by Boyle's law; that is,

$$w = w_0 \frac{p}{p_0},$$

so that

$$\int_p^{p_0} \frac{dp}{w} = \frac{p_0}{w_0} \int_p^{p_0} \frac{dp}{p} = \frac{p_0}{w_0} \log_e \frac{p_0}{p}. \quad . \quad . \quad . \quad . \quad (9)$$

Here, as we readily see from equation (5) (p. 12), p_0/w_0 is the height of a column of fluid of *constant* weight per unit volume w_0, at the lower end of which the pressure is p_0 and at the upper end of which the pressure is zero. This height is called the *height of the homogeneous atmosphere*. In relation to the actual atmosphere it merely represents a convenient mathematical quantity. As an example we shall calculate its numerical value. To do this we require to know the numerical value of w_0; this can be obtained by weighing an evacuated vessel, provided with a tap, on a sensitive balance, and then weighing again after the tap is opened and sufficient time has elapsed for the temperature to become steady.* As the vessel was empty at first and is now full of air, it is necessarily heavier by W, the weight of the air contained in it. If we now find V, the volume of the vessel, say by evacuating the vessel again, opening the tap under water, and again weighing the vessel full of water, we get the value of w_0 ($= W/V$, corresponding to p_0, the pressure at the ground level. For any other pressure p_0, w_0 may be calculated by proportion. For the sake of simplicity we shall assume that p_0 is exactly equal to 1 Kg./cm.^{2} Then for air of medium humidity at temperature θ we have, by Charles' law,

$$w = \frac{1{\cdot}245}{1 + \alpha\theta} \text{ Kg./m.}^3$$

(For a pressure p_0 of one atmosphere (1·0332 Kg./cm.^{2}) we should have to insert the number 1·286 instead of 1·245 for air of medium humidity, and 1·293 for artificially dried air.)

In examples in dynamics the density $\rho = w/g$ is used as a measure of mass inertia. For air at room temperature we may take the average value 1·23 Kg./m.^{3} for w. Taking $g = 9{\cdot}81$ m./sec.^{2}, we accordingly have the average value 0·125 Kg.-sec.^{2}/m.^{4} for ρ.

* The air in the vessel is at first warmed, owing to the work done by the outer air as it rushes into the vessel.

Now to calculate p_0/w_0 in equation (9) we have to express p_0 in the same system of units as w_0. 1 Kg./cm.^{2} = 10,000 Kg./m.^{2}, so that

$$\frac{p_0}{w_0} = \frac{10{,}000}{1{\cdot}245}(1 + \alpha\theta) = 8030(1 + \alpha\theta).$$

The dimensions of p_0/w_0 are Kg./m.^{2} ÷ Kg./m.^{3} = m. Thus the height of the homogeneous atmosphere is $8030(1 + \alpha\theta)$ metres for air of medium humidity (independent of the pressure, but dependent on the temperature). We denote this height by H_0. Applying equation (8) (p. 14) twice, we obtain

$$z_1 = H_0 \log_e \frac{p_0}{p_1};\ z_2 = H_0 \log_e \frac{p_0}{p_2},$$

i.e.
$$z_1 - z_2 = H_0 \log_e \frac{p_2}{p_1}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is Laplace's law, or the *barometric formula* (so called because the pressures are measured by the barometer; cf. p. 21). Solving equation (10) for p, we get the relationship between pressure and height:

$$p = p_1 e^{-(z-z_1)/H_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Arguing as we did in connexion with fig. 1.8 (p. 12), we readily see that the weight of a column of air with a base of area a and extending from the point z upwards to the limits of the atmosphere is equal to ap. It follows at once that p is equal to the weight of the column of air of unit cross-section above z. Formula (11) is graphically represented in fig. 1.10. The pressure falls continually, but at a decreasing rate as the height increases, becoming zero at an infinite height. The fall of pressure in the free atmosphere with increasing height is easily measured by carrying an instrument for measuring pressure (a barometer) up a tower or mountain. It can even be detected quite easily in a house of more than one storey. If the temperature of the air is observed at the same time, the observed differences of pressure enable us to determine the difference of height very accurately. This is the usual method for registering the height of aeroplanes. (If the difference of height is known the method can of course be used to determine the average density of the intervening stratum of air.)

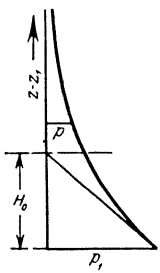


Fig. 1.10.—Distribution of pressure in an atmosphere in which the temperature is constant.

If the temperature is not constant throughout the mass of air the barometric formula can still be applied by dividing the total height into sections within which the differences of temperature are no longer

large. The atmospheric height H_0 for each section is then calculated for the mean value of the temperature in the section.

We have still to settle the question when the equilibrium of a stratified mass of gas is *stable* and when it is not. It is not sufficient that the upper layers should be less dense than the lower ones. The reason is that if a mass of gas moves upwards or downwards it comes to a region where the pressure is different and its density is thereby altered. The correct answer to our question is as follows. Stability exists if a portion of gas which is forced upwards is denser under the new pressure than its new surroundings are, and if a portion of gas which is forced downwards is less dense under the new pressure than its new surroundings are; for under these circumstances the portion of gas will tend to return to its former position. There exists a certain arrangement of layers (temperature distribution) in a mass of gas which in this respect is analogous to the homogeneous liquid, i.e. with this distribution the mass of gas is in neutral equilibrium. In order that this may be so, every portion of gas which is removed to any other place must after displacement fit into its new surroundings as if it had hitherto belonged there. If the pressure is changed without the gas being given any opportunity to exchange heat with its surroundings, the portion of gas behaves adiabatically. Now if the stratification is such that pressure and density behave in accordance with the adiabatic change of state (cf. § 5, equation (4), p. 10), so that p is proportional to w^ν (w being inversely proportional to the volume) every portion of gas forced to move upwards or downwards does actually come to a neighbourhood where the temperature is the same as that which it has itself reached owing to its own adiabatic change of state, so that it has no opportunity whatever to exchange heat. It can be shown that this *adiabatic stratification* has this in common with the homogeneous liquid, that it arises from thorough mixing of a gas which was originally stratified arbitrarily, just as e.g. a salt solution originally heterogeneous is made homogeneous by stirring.

In atmospheric air the adiabatic stratification is distinguished by the fact that the temperature falls by 1° C. for a rise of about 100 metres. A smaller fall of temperature than this means stability, so that if the temperature increases with height the stability is all the greater. A sharper decrease of temperature than 1° per 100 metres does not as a rule occur in free air, as this corresponds to unstable conditions. It is found, however, in the neighbourhood of the ground when the latter is hotter than the air. Then, of course, the air is not in equilibrium but is full of vertical currents in both directions.

The fall of pressure in the adiabatically stratified atmosphere can be calculated very easily by substituting

$$w = w_0 \left(\frac{p}{p_0}\right)^{1/\nu},$$

from equation (4), p. 10, in equation (8), p. 14. Integration then gives

$$z = \frac{\gamma H_0}{\gamma - 1}\left\{1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}, \text{ i.e. } p = p_0\left(1 - \frac{\gamma - 1}{\gamma}\frac{z}{H_0}\right)^{\gamma/(\gamma-1)}.$$

From this the temperature relation given above is obtained as follows: Using the equation of state $p/\rho = RT$, in which $\rho = w/g$ is the density, T ($= 273{\cdot}2° + \theta$) is the absolute temperature and R the gas constant, and remembering that $p_0/w_0 = H_0$, we have

$$\frac{RT}{g} = \frac{p}{w} = H_0 - \frac{\gamma - 1}{\gamma} z,$$

so that

$$\frac{dz}{dT} = -\frac{\gamma}{\gamma - 1}\frac{R}{g}.$$

For air of average humidity $R/g = 29{\cdot}4$ metres per degree; that is, $dz/dT = -102$ metres per degree.

Note.—If we replace γ in the above formulæ by another number n, we obtain a convenient interpolation formula which serves to describe the states of stratification which actually occur in the atmosphere, and which are known as *polytropic* (cf. Chap. V, § 14, p. 391). Obviously n is always less than γ for stable stratifications.

8. The Mutual Action of Atmospheric Pressure and Liquid Pressure. Manometers depending on a Liquid.

The difference between the pressure of the air in a vessel and the external atmospheric pressure, provided it is not too large, can readily be measured by means of a U-tube partly filled with liquid (cf. fig. 1.11). Neglecting the weight of the air itself, we get the following results. At A the pressure in the liquid is equal to p_1, the pressure of the air in the vessel. At the same height in the other limb of the U-tube, i.e. at B, the pressure is the same (the vessels being in communication). Let the free surface of the liquid in the latter limb be at C; the pressure there is equal to p_0, the atmospheric pressure. Putting h for the height BC, we have (§ 6, p. 12)

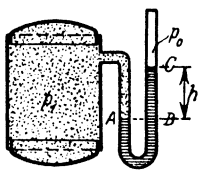


Fig. 1.11.—Hydrostatic measurement of pressure (U-tube).

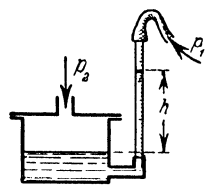


Fig. 1.12.—Manometer

$$p_1 = p_0 + wh.$$

Thus a U-tube filled with liquid is very well suited to the measurement of such differences of pressure. In practice various forms are used. In order to avoid having to take readings at two places (A and C in fig. 1.11) one of the limbs is frequently made in the form of a wide vessel, in which the change of level is very small (fig. 1.12). In this case both

openings must be exposed to atmospheric pressure in order to find the zero reading. For very small differences of pressure we employ a more refined method for reading the height of the liquid (we use a kind of cathetometer, or observe a magnified image of a scale floating on the surface of the liquid (A. Betz)), or we may use inclined glass tubes, as in Töpler's pressure-level (fig. 1.13) or Krell's micromanometer (fig. 1.14).

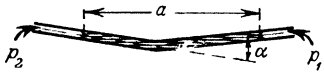


Fig. 1.13.—Töpler's pressure-level

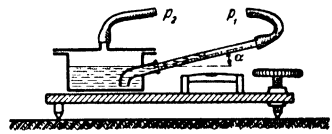


Fig. 1.14.—Micromanometer

In both, displacement of the liquid in the tube through a distance x corresponds to a change of level of $h = x \sin\alpha$, and hence to a difference of pressure $p_2 - p_1 = wx \sin\alpha$. It is not easy, however, to determine the small inclination α with sufficient accuracy, and unless one actually tests them one cannot rely on the tubes being absolutely straight. Hence for very accurate measurements the instruments should be calibrated.(1)

In Krell's micromanometer this can be done by first measuring d, the diameter of the wide vessel, which is turned so as to be accurately cylindrical. The level fixed to the base of the instrument is then carefully adjusted, suitable weighed-out quantities of liquid are added to the wide vessel several times in succession, and the corresponding heights of the liquid in the narrow tube are read off on the scale. The added liquid spreads out over the wide vessel and exerts a pressure at the original surface level equal to its weight divided by $\pi d^2/4$, the area of the circle. Thus the equilibrium of the original liquid is the same as if an equivalent pressure were exerted by the air. The fact that the added fluid really mixes with that already in the vessel makes no difference to this argument. A calibration curve or table can then be constructed from corresponding values of the pressure and the level of the liquid.

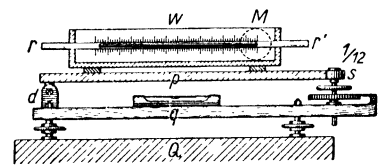


Fig. 1.15.—Töpler's pressure-level

Töpler's pressure-level may be made into an extremely sensitive instrument for measuring very small differences of pressure by adopting certain refinements due to Töpler himself.* The tube rr' in fig. 1.15 contains a thread of non-volatile, mobile liquid (xylene) about 25 cm. long, and is placed in a glass-walled water-bath W, immediately in front of a millimetre scale. The bath itself is supported on a metal plate p, the slope of which can be altered by means of the screw s, the angle through which the screw is turned being read off on an indicator. As the pitch of the screw is known, the inclination of the plate p can be found. A microscope M attached to the plate p enables the movements of the right-hand meniscus of the liquid to be observed. If the zero reading of the instrument (the reading when the pressures at r and r' are the same) has been taken, any difference of pressure

* *Wiedem. Ann.*, Vol. 34, p. 790 (1888); Vol. 56, p. 609 (1895).

[1] References of this type are to the Appendix on p. 432.

which may arise can be measured by the shift of the meniscus, or by the angle through which the screw must be turned in order to bring the meniscus back to its place. The calibration of the instrument for the first method of observation is carried out by observing the displacement of the meniscus caused by turning the screw in the absence of any pressure difference, and finding the relation between the displacement of the meniscus and the angle of inclination (ϕ) caused by the motion of the screw. If the distance between the two ends of the thread of liquid is a, the difference of height effective from the hydrostatic point of view is $h = a \sin \phi$. With a particularly sensitive instrument (with the angle α (fig. 1.13) between the two limbs very small) Töpler was able to detect differences of pressure of 10^{-8} Kg./cm.^{2} (about 10^{-8} atmosphere). (The bend in the tube is not shown in fig. 1.15; it is to be imagined as being in the neighbourhood of the letter W.)

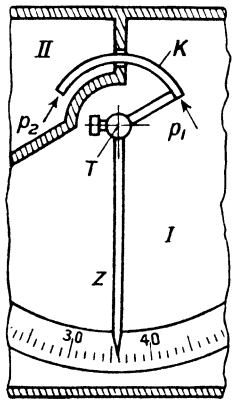


Fig. 1.16. — Reichardt's instrument for measuring small differences of pressure.

T, torsional suspension; K, plunger; Z, pointer.

Other forms of hydrostatic apparatus for the measurement of very small differences of pressure have since been developed.* Types of apparatus not involving the use of a liquid have also been devised; of these, that due to H. Reichardt † is capable of measuring extremely small differences of pressure (10^{-4} mm. water or less). A system consisting of a pointer Z and a plunger K bent into the arc of a circle is suspended from a platinum wire as in a galvanometer. The plunger passes, with a play of about 0·1 mm., through the partition separating two spaces I and II, to which the pressures p_1 and p_2 which are to be measured are transferred through tubes. These tubes must be wide, in order that the unavoidable flow through the space round the plunger may not give rise to an appreciable fall in pressure in the tubes.

The use of manometers depending on liquids has given rise to a special kind of pressure unit frequently used in practice. It is natural to measure the pressure by the height of the column of liquid itself: hence, as the various liquids must be distinguished from each other on account of their different densities, we speak of " millimetres of mercury ", " feet of water ", &c. Water, however, is a very unsuitable liquid to use when accurate readings are desired, as it is inclined to cling irregularly to the walls of the glass tube. Liquids which dissolve grease (alcohol, toluene, xylene, &c.) are very much better. For larger differences of pressure mercury is very suitable; when pure it gives a very clear reading, provided the glass tube is not too narrow. As the specific gravity of mercury is 13·6 (more accurately 13·595 at 0° C.), 1 mm. of mercury = 13·6 Kg./m.^{2} = ·00136 Kg./cm.^{2} Conversely 1 Kg./cm.^{2} = 735·5 mm. of mercury. For practical reasons the unit 1 millimetre of mercury is a great favourite with physicists.

* See e.g. the article on micromanometers by A. Betz in *Handbuch der Experimentalphysik*, Vol. IV, 1, p. 511 *et seq.*

† *Zeitschr. f. Instrumentenkunde*, Vol. 55, p. 24 (1935); a short account is given in *VDI-Zeitschr.*, Vol. 79, p. 1503 (1935).

The name " 1 tor " has recently been suggested for it, in honour of Torricelli.*

For the sake of completeness we shall briefly refer to *manometers depending on the action of a spring*, which are largely used to measure high pressures. The commonest form is the Bourdon gauge (fig. 1.17), the essential part of which is a metal tube (*b* in fig. 1.17) of elliptical cross-section bent in the form of a circle. Such tubes have the property of expanding when there is an excess of pressure inside, owing to the fact that the elliptical cross-section tends to approach the circular form, so that the tube tends to straighten out. The tube *b* is soldered on at *a*, while the other end *c* is free to move. Its movements are transmitted by means of the intermediate member *f* to the lever *d*, to which the pointer is attached. Instruments of this kind are made for pressures up to several hundred atmospheres. Like all instruments depending on springs the Bourdon gauge requires to be calibrated; at moderate pressures this can be done by means of a column of mercury.

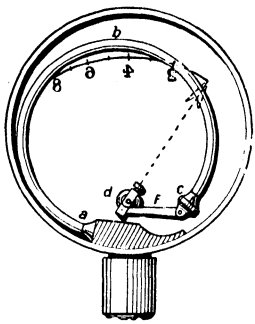


Fig. 1.17.—The Bourdon gauge

9. **Reduced Pressure. The Barometer.**

If some of the air is pumped out of the vessel in fig. 1.11 (p. 18), so that the pressure inside is reduced below the outer atmospheric pressure, the liquid will stand higher in limb A of the U-tube than in limb B; it is " sucked up ". A somewhat different arrangement of the same experiment is shown in fig. 1.18.

In the case of fig. 1.11 we have an " excess of pressure " in the vessel, in the case of fig. 1.18 a " reduction of pressure ". By the numerical value of the excess or reduction of pressure we mean the difference between the pressure in the vessel and that of the atmosphere, which is measured by *h*, the height of the column of liquid in fig. 1.11 or 1.18.

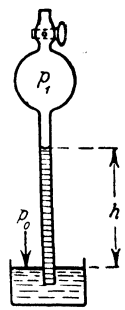


Fig. 1.18

From the physical point of view the question now arises how high a liquid can be raised in this way. In olden times the rise of a liquid was " explained " by saying that " Nature abhors a vacuum ", and people were quite well satisfied with this expression, although in reality it merely stated a fact and explained nothing. No one carried out any experiments to discover whether or not there was any limit to Nature's

* For other units, see pp. 22–3.

abhorrence of a vacuum. It was the misfortune of some Florentine pump-makers, who had made a water pump with the suction valve more than thirty-three feet above the level of the water and found that despite all their efforts the water refused to rise to this height, that impelled Galileo to investigate the matter. It was Galileo's pupil Torricelli, however, who first reached the real truth by means of an experiment with mercury which he induced his friend Viviani to make in 1643.

From our point of view there is no difficulty in answering the question raised above. The sucking-up of the liquid merely means that the pressure on it is less than that of the atmosphere outside. The pressure in the vessel of fig. 1.18 cannot be reduced below its value when all the air is pumped out of the vessel, which is equal to zero. Thus the column of liquid cannot ascend beyond a height h corresponding to the full atmospheric pressure $p_0 (h = p_0/w)$. Now Viviani's experiment, which we mentioned above, was as follows. He took a glass tube 4 ft. (120 cm.) long, with a bulb blown on one end, the other end being left open. He filled it completely with mercury through the open end, closed this end with his finger, turned the tube upside down, and placed the end in a shallow dish of mercury. If he then removed his finger from the end of the tube, the mercury sank to a level about 30 in. (75 cm.) above that of the mercury in the dish, leaving an empty space at the top of the tube. Torricelli rightly inferred from this that the column of mercury was kept in equilibrium by the pressure of the air outside, or, we may say, by the weight of a column of air of the same cross-section reaching up to the limits of the atmosphere. Torricelli noticed in addition that the column of mercury was not always of the same height, and concluded that the pressure of the atmosphere is to some extent variable—a fact which in meteorology is extremely important. Torricelli also came to the conclusion that the atmospheric pressure should be less at the top of a mountain than in a valley, so that the column of mercury should be shorter at the higher elevation. Some years later this was verified by Perrier at Pascal's instigation, by measuring the heights of the column of mercury on the Puy de Dôme (3200 ft.) and at its foot; he found a difference of three inches in the direction predicted. The universally applied name of *barometer* for an instrument for measuring the pressure of the atmosphere is also due to Pascal. As the etymology of the word suggests, the instrument measures the weight of the column of air situated above the place where it is.

Another unit of pressure, which is derived from the barometer, is the " atmosphere ". The average height of the barometer at sea-level is about 760 mm. (30 in.) of mercury. It has been agreed to regard this as the normal barometric height when the temperature of the mercury is 0° C., and the corresponding pressure is called " 1 atmosphere "

As the specific gravity of mercury at 0° C. is 13·595, i.e. 1 cubic centimetre weighs 13·595 grams, a mercury column 76 cm. long corresponds to a pressure of

$$76 \times 13{\cdot}595 = 1033{\cdot}2 \text{ Gm./cm.}^2 = 1{\cdot}0332 \text{ Kg./cm.}^2$$

This pressure * corresponds to a column of water 10·332 metres (34 feet) high (the " height of the water barometer "). The height of the suction valve of a pump above the level of the water which it is desired to raise must therefore be less than the above (in view of losses due to the clearance and leakages it should not exceed 20 to 24 feet).

As the preceding definition of the atmosphere involves the earth's attraction, which varies from point to point of the earth's surface, it is necessary when high accuracy is required to adopt a definite value of the acceleration due to gravity for use in the definition of the unit of pressure. The mean value of gravity is taken as 980·665 cm./sec.^{2}, i.e. the value corresponding to sea-level in latitude 45°. If gravity has a different value (g), the standard atmospheric pressure is given by $(1{\cdot}0332 \times 980{\cdot}665)/g$ local Kilograms per square centimetre. In order to avoid this somewhat arbitrary definition a pressure unit derived from the c.g.s. system has been suggested, namely, one million dynes per square centimetre, which is called " 1 bar ". Assuming gravity to have its mean value, a bar corresponds to a column of mercury 750·06 mm. high. The thousandth part of this, called a *millibar*, is the pressure unit employed by meteorologists. One atmosphere = 1013·2 mb.; 1 Kg./cm.^{2} = 980·66 mb.

10. Equilibrium of Liquids in Other Fields of Force.

In our discussion in §§ 6–9 (pp. 11–23) we took the existence of a uniform gravitational field for granted, i.e. we assumed that the earth's acceleration is everywhere the same in magnitude and direction. For regions of moderate extent this is sufficiently accurate for most applications. If, however, we wish to consider a region of the earth which is no longer small compared with the earth's radius, we must take account of the fact that the earth's acceleration varies both in magnitude and in direction. In the case of a liquid at rest with respect to a uniformly rotating vessel we must add the acceleration away from the centre to that due to gravity if we are to deal with the rotating liquid as with one at rest. We shall therefore give a perfectly general discussion of the equilibrium of a homogeneous or heterogeneous liquid in any field of force in which the force on unit mass (i.e. the acceleration) varies from point to point in magnitude and direction.

Starting from the account given in § 6 (p. 11), we readily establish

* [Equal to 14·7 Lb. per square inch in British units.]

the fact that, as before, the pressure at any point cannot vary in any direction normal to the direction of the force at that point (we consider the equilibrium of a small prism like that in fig. 1.6, p. 7, with its axis perpendicular to the direction of the force). We can further prove that the pressure must increase in the direction of the force, and in such a way that

$$dp = g\rho\, dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(from the equilibrium of a small prism of height dh like that in fig. 1.8, p. 12, with its axis in the direction of the force; dp is the rise of pressure and g the strength of the field of force). It immediately follows from the former of these statements that if we consider all the directions at right angles to the direction of the force at any point, the pressure on the element of surface at right angles to the force must be constant. In the case where the adjacent surface elements can be combined to form a finite surface, i.e. if there exist surfaces which are everywhere normal to the lines of force (such surfaces we shall call *normal surfaces*), it follows from the above that the pressure is constant along every normal surface. If a field of force has no normal surfaces, it is impossible for a liquid to be in equilibrium in that field of force.*

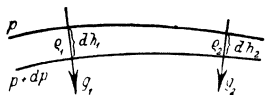


Fig. 1.19

We now assume that the field of force does possess normal surfaces, and we consider two such normal surfaces, corresponding to pressures p and $p + dp$. We select two positions, 1 and 2 (see fig. 1.19), and assume, in virtue of the relationship given above, that dp is equal on the one side to $g_1\rho_1 dh_1$ and on the other to $g_2\rho_2 dh_2$. In the case where ρ is either constant or a function of p (homogeneous liquid or homogeneous gas; cf. §§ 6, 7, p. 11) we finally have $\rho_1 = \rho_2$, since $p_1 = p_2$, and $g_1 dh_1 = g_2 dh_2$. Now $g\,dh$ is the work done by the force in passing from one normal surface to the other. As the above discussion shows, this work is the same for all paths between the two normal surfaces, which is the criterion that a field of force should have a potential. Thus the normal surfaces are equipotential surfaces. If we introduce U, the potential at a point A, by the formula

$$dU = -g\,dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

* In this case a curve drawn perpendicular to the lines of force returns to the line of force from which it started, not at its original starting-point, but at a point higher up or lower down the line of force, according to the sense in which the curve is described. An example of a vector field of this kind may be obtained, e.g. by joining two parallel planes by a number of straight lines perpendicular to the planes and hence parallel to each other, rotating one of the planes with respect to the other through a small angle about one of the straight lines as axis, so that the straight lines become twisted up, and taking the latter as the lines of force.

(the negative sign arising from the fact that in (12) dh was taken as positive in the direction of g), we see that we may write

$$dp = -\rho dU, \text{ or } dU = -\frac{dp}{\rho}, \quad . \quad . \quad . \quad (14)$$

whence it follows that

$$U_A - U_B = \int_A^B \frac{dp}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the case assumed here (namely that of a homogeneous liquid or homogeneous gas), the right-hand side can be integrated and we then get the pressure directly as a function of the potential. We may therefore summarize our results as follows:

In the case of a homogeneous liquid or homogeneous gas, equilibrium is impossible unless the field of force has a potential. The equipotential surfaces, which are at right angles to the force, are at the same time surfaces of constant pressure. The pressure increases in the direction of the force in such a way that $dp = -\rho dU$.

In heterogeneous liquid it is possible to imagine a case where, although $g_1 dh_1 + g_2 dh_2$, the density is distributed in such a way that $\rho_1 g_1 dh_1$ is everywhere equal to $\rho_2 g_2 dh_2$. Equilibrium would then exist; it is easy to see, however, that the equilibrium would be unstable, for in order to destroy it we need only displace the liquid along the normal surface (which requires no expenditure of work), thus altering the distribution of the density. If, then, we are to confine ourselves to stable conditions, we need only consider fields of force which have a potential. If, on the other hand, $g_1 dh_1 = g_2 dh_2$, we must also have $\rho_1 = \rho_2$ for equilibrium. We can therefore make the following statement:

Stable stratification of a heterogeneous liquid is possible only when the field of force has a potential. The equipotential surfaces are at the same time surfaces of equal pressure and of equal density.

Thus here again we may apply formulæ (14) and (15). The conditions for the stability of the stratification are the same as those discussed in §§ 6, 7 (p. 11) for a uniform gravitational field.

The fields of force which occur in physics (apart from certain forces arising from electric currents flowing in magnetic fields) almost always have a potential, so that the above requirement that a potential should exist scarcely involves any practical limitation. On the other hand, the requirement that the density should be constant over any equipotential surface is of practical importance. This condition may be infringed, e.g. by local heating of the liquid or gas, resulting in a decrease of the density at the points in question. Equilibrium is no longer possible in such a case; the heated liquid begins to move, thereby setting the neighbouring liquid in motion, and the whole mass

does not come to rest until the warmest parts form a layer above the rest.

The free surface of a liquid and the boundary between two non-miscible liquids differing in density are always equipotential surfaces; hence the term *level surface* (from the free surface or level of an imaginary liquid) is sometimes used for equipotential surface. In geodesy the surface of the sea is the standard level surface to which all measurements of height are referred.

The application of the laws stated above may be illustrated by a simple example.

The equilibrium of a heavy homogeneous liquid at rest relative to its containing vessel, which is revolving uniformly about a vertical axis.—We proceed to write down the expression for the potential, which is formed by the addition of two parts, one due to gravity and the other due to centrifugal force.

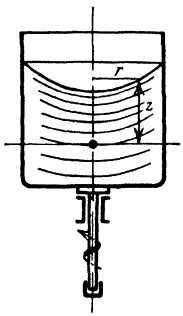


Fig. 1.20.—Liquid in a rotating vessel

If we take cylindrical co-ordinates r and z (fig. 1.20), the positive direction of z being taken upwards, the part of the potential due to gravity is

$$U_1 = U_0 + gz,$$

where g is the acceleration of gravity and U_0 an arbitrary initial value. To obtain the part due to centrifugal force we begin by taking the acceleration $\omega^2 r$, where ω is the common angular velocity of the vessel and the liquid, and integrating it along its own direction, i.e. in the direction of r. Hence the second part of the potential comes to

$$U_2 = -\frac{\omega^2 r^2}{2}.$$

Altogether, then, we have

$$U = U_1 + U_2 = U_0 + gz - \frac{\omega^2 r^2}{2};$$

putting U constant, we get the equipotential surfaces, namely

$$z = \text{const.} + \frac{\omega^2 r^2}{2g}.$$

Thus the free surfaces and all the surfaces of equal pressure are paraboloids with the same parameter g/ω^2.

The actual pressure is again given by $p = p_0 - \rho U$, i.e. putting $g\rho = w$,

$$p = \text{const.} + w\left(-z + \frac{\omega^2 r^2}{2g}\right).$$

11. **Surface Tension (Capillarity).**

(*a*) *General Principles.*

The free surface of a liquid exhibits a tendency to contract. This can be explained by assuming that the surface is in a state of tension like that of a thin uniformly stretched skin. We may suppose that

each particle of liquid near the surface is pulled towards the interior of the liquid by the attractions of the neighbouring particles, so that no more particles remain at the surface than are absolutely necessary to form the surface.* The same state of affairs is found at the surface of separation of two liquids which do not mix. We speak of *surface tension* and *capillary forces*; the name *capillarity* for the whole group of phenomena arises from the particularly striking behaviour of liquid in narrow tubes (capillary tubes, from Latin *capillus*, a hair) which is described below.

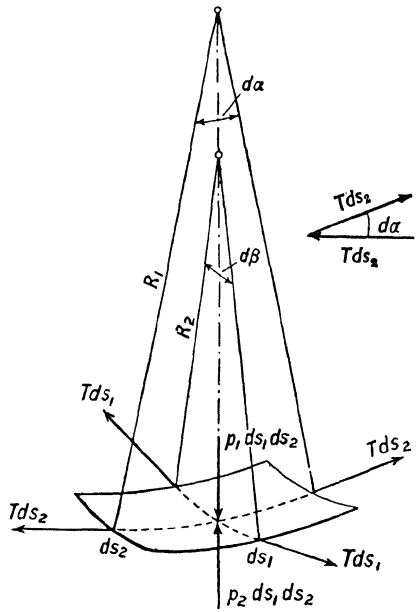


Fig. 1.21

Surface tension does not appear at a plane surface, for there the forces are in equilibrium. If the surface is curved, surface tension gives rise to pressure differences, which serve to maintain equilibrium. If we consider a small rectangle in a curved surface with sides ds_1 and ds_2 (fig. 1.21), the pressure difference $p_1 - p_2$ gives rise to a force $(p_1 - p_2)ds_1 ds_2$ on the area $ds_1 ds_2$. If the surface tension is T per unit length, it gives rise to two forces $T ds_1$ on the opposite sides ds_1 of the rectangle and two forces $T ds_2$ on the opposite sides ds_2. If the angle between the two latter forces is $d\alpha = ds_1/R_1$, their resultant is $T ds_2 \,.\, d\alpha = T ds_2 \,.\, ds_1/R_1$; similarly, if the angle between the two former forces is $d\beta = ds_2/R_2$, their resultant is $T ds_1 \,.\, ds_2/R_2$. Since the three forces are in equilibrium and the factor $ds_1 ds_2$ common to them all can be omitted, we have

$$p_1 - p_2 = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad \ldots \quad (16)$$

where, as we see from the above argument, R_1, R_2 are the radii of curvature of the curves of intersection of the surface with two planes at right angles to one another and to the tangent plane.†

* It can be shown by calculation that this gives rise to a tension like that in a stretched membrane; see a paper by the author in *Ann. d. Physik* (6), Vol. 1, p. 59 (1947).

† From equation (16) we may deduce the geometrical theorem that the sum $1/R_1 + 1/R_2$ is independent of the direction of ds_1, since the left-hand side of equation (16) is certainly independent of that direction.

(*b*) *Heavy Liquid.*

Since the pressure in a heavy liquid in equilibrium is given by $p = p_0 - wz$, where z is measured vertically upwards and w is the weight per unit volume, we have, at the surface separating two liquids with weight per unit volume w_1 and w_2,

$$p_1 = p_0 - w_1 z \quad \text{and} \quad p_2 = p_0 - w_2 z,$$

so that equation (16) gives

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{w_2 - w_1}{T} z \qquad \ldots\ldots \quad (17)$$

as the law of curvature of the boundary surface. Two examples of these surfaces are given in fig. 1.22. A value for T can be obtained by measuring the forms which are produced; a better method, however, is explained in (*c*) below.

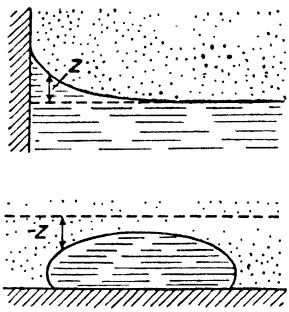


Fig. 1.22.—Capillary surfaces of heavy liquids

From equation (17) we see that in the case where the weights of unit volume of the two liquids are very nearly the same, the various surfaces are magnified n times (i.e. R_1, R_2 and z are multiplied by n) but remain geometrically similar when $(w_2 - w_1)/T$ is reduced to $1/n^2$ of its previous value. If $w_2 = w_1$ the effect of gravity vanishes; such surfaces are known as *minimal surfaces*. If as $w_2 \to w_1$ the plane $z = 0$ is allowed to move to infinity,

$$1/R_1 + 1/R_2 = \text{const.};$$

this gives minimal surfaces with a given volume, of which the sphere is the simplest case. These minimal surfaces can be beautifully exhibited by blowing soap bubbles. Spherical soap bubbles have an excess pressure inside of

$$p_1 - p_2 = 4T/R,$$

since here there are two surfaces of the soap solution in contact with air, i.e. $2T$ replaces T in equation (16).

(*c*) *Angle of Contact.*

If the boundaries of three liquids meet along a straight line, they must meet at certain definite angles in order that the three surface tensions T_{12}, T_{13} and T_{23} may be in equilibrium, as may be seen from the triangle formed by the three forces (fig. 1.23). It may happen that T_{13} exceeds the sum of T_{12} and T_{23}; equilibrium is then impossible. This occurs e.g. when substance 1 is air, substance 2 mineral oil, and substance 3 water. Substance 2 then spreads over the whole surface, though possibly in a very thin layer, as may be seen in the spreading of drops of lubricating oil over wet streets. If, however, substance 2

is melted fat, when exposed between water and air it takes the form of thin lenses (circular drops of fat floating on soup); fig. 1.23 is drawn for this case. If one of the three substances is a solid, the only equilibrium in question is that of the components of the forces in the direction parallel to the solid

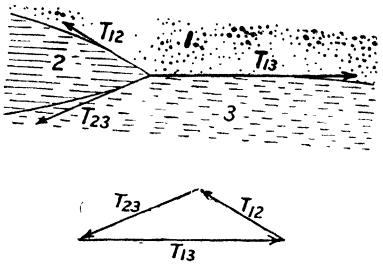


Fig. 1.23.– Equilibrium of three surface tensions

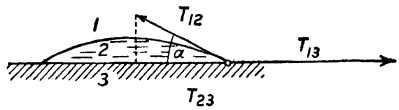


Fig. 1.24.—Angle of contact at a solid surface

surface, the only direction in which freedom to move still exists. Assuming equivalent surface tensions along the solid surface, we have

$$T_{12} \cos\alpha + T_{23} = T_{13},$$

where α is the angle of contact (see fig. 1.24), that is,

$$\cos\alpha = \frac{T_{13} - T_{23}}{T_{12}}. \qquad . \quad . \quad . \quad . \quad . \quad (18)$$

If T_{12}, the surface tension at the boundary of the two liquids 1 and 2, is already known and α is found by observation, the difference $(T_{13} - T_{23})$ may be obtained; T_{13} and T_{23}, on the other hand, cannot be determined singly. The difference may even be negative; then the angle α is greater than $\frac{1}{2}\pi$, as is the case e.g. for air, mercury, and glass. A drop of mercury is shown in the lower part of fig. 1.22. Of course we can also have $T_{13} - T_{23} > T_{12}$; in this case the whole of the solid surface is covered by the liquid 2, as is notably the case with petrol.

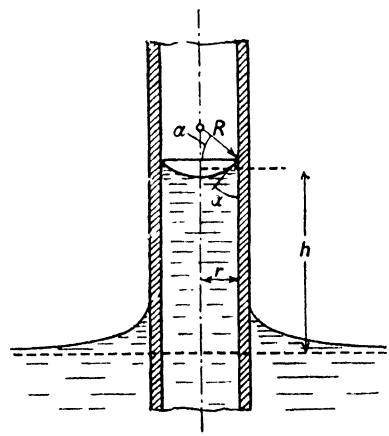


Fig. 1.25.—Capillary rise of liquid in a fine tube

In a narrow tube a liquid rises to a considerable height above the level of the liquid outside. If r is the internal radius of the tube, and if for simplicity we regard the surface of the liquid as a spherical cap (the smaller r is compared with h, the difference of level, the truer this is), then we see from

fig. 1.25 that R, the radius of the sphere, is equal to $r/\cos\alpha$, where α is the angle of contact. Thus by equation (17)

$$h = \frac{2T_{12}}{w_2 - w_1} \cdot \frac{\cos\alpha}{r}. \qquad \ldots\ldots \quad (19)$$

Hence h may be very large when r is very small. (This explains the suction effect of blotting-paper, clay with very fine pores, and so on.)

It is possible to eliminate $\cos\alpha$ from equation (19) by using equation (18) and multiplying both sides by πr^2; we thus obtain the very easily interpreted equation

$$(w_2 - w_1)\pi r^2 h = (T_{13} - T_{23})2\pi r.$$

(Weight − upward thrust = resultant pull on wall of tube.)

If the " pull " is negative, i.e. $\alpha > \frac{1}{2}\pi$, as in the case of mercury, h is actually negative (i.e. fig. 1.25 is turned upside down). If the surface is wetted T_{12} takes the place of $T_{13} - T_{23}$, that is, $\cos\alpha = 1$, $\alpha = 0$. This gives the maximum value of h. By measuring h and r we then have

$$T_{12} = \tfrac{1}{2}(w_2 - w_1)hr.$$

Another method of finding surface tension is by measurements of capillary waves (see Chapter II, § 15, p. 86).

Values of T at 20° C.

Water/air:	$T = 72{\cdot}5$ dynes/cm.
Oil/air:	$T = 25$–30 dynes/cm.
Mercury/air:	$T = 472$ dynes/cm.

CHAPTER II

Kinematics: Dynamics of Frictionless Fluids

Preliminary Remarks.

The flow of liquids and the flow of gases have so much in common that they are best discussed together. Gases, of course, are much more compressible than liquids; but the question which concerns us here is whether in any particular fluid motion compression takes place to a noticeable extent, or not. Compression is considerable when the changes of pressure are considerable. If, however, the gas is flowing with a small or moderate velocity, or if the mass of gas does not extend very far in the vertical direction, the changes of pressure remain small compared with the average pressure, and the volume changes are then so small that they are usually omitted entirely in order to simplify the calculations. Under such circumstances the flow of a gas differs in no way from that of an incompressible liquid. If a change of volume of one per cent may be regarded as negligible, we may apply the formulæ relating to the flow of incompressible liquid to the flow of air at ordinary temperature, provided the rate of flow does not exceed about 50 m./sec. or the height 100 metres (cf. § 4(*b*), p. 44, and Chapter I, § 7, p. 14). For a velocity of 150 m./sec. the changes of volume amount to about 10 per cent. When the velocity of flow becomes of the order of the velocity of sound (about 340 m./sec.) the changes of volume become very considerable, causing marked changes in the configuration of the flow. For rates of flow greater than the velocity of sound, in fact, the character of the flow is absolutely different from that of ordinary liquids.

Most of the sections in this chapter and the next deal only with cases of flow where the changes of volume are negligible. So in order to avoid being continually compelled to speak of "liquids and gases" we shall hereafter use the word "fluid" to denote both liquids and gases; extending this phraseology, we shall refer to gases as "compressible fluids". The special features of fluid motions where the volume changes are large will be discussed in Chapter IV (p. 258).

1. KINEMATICS

1. Methods of Representing the Motion.

If for every instant of time we were given the position of every particle of a fluid, we should be able to obtain a complete picture of the flow of the fluid. The change in the position of a particle as time goes on would then give us its velocity and acceleration. To work this out mathematically we begin by introducing a special system of axes to distinguish the separate particles from each other; this system of axes is rigidly fixed to the particle of fluid in question, but is movable in space. We may do this, e.g. by considering a family of surfaces $a = \text{const.}$, where a in any initial position is given as a function of the space co-ordinates x, y, z. If we take two further families of surfaces $b = \text{const.}$ and $c = \text{const.}$, chosen so that an a-surface, a b-surface, and a c-surface always cut in one point only, then the particle of fluid at the point of intersection is completely determined by the values of a, b, and c. We now stipulate that the particle of fluid shall permanently retain the " fluid co-ordinates " a, b, c, which comes to this, that every surface $a = \text{const.}$ and likewise every surface $b = \text{const.}$ or $c = \text{const.}$ must always consist of the same particles of fluid. The original choice of the fluid co-ordinates is arbitrary, and is determined by considerations of convenience only; e.g. we may even take the ordinary co-ordinates of the particle in any initial position or state of rest as fluid co-ordinates.

The problem of finding the motion, i.e. of finding the changes of position of all the particles, consists in expressing the co-ordinates (x, y, z) of the particle at any instant as functions of the time and of the fluid co-ordinates (a, b, c) of the particle: that is,

$$\left.\begin{aligned} x &= F_1(a, b, c, t) \\ y &= F_2(a, b, c, t) \\ z &= F_3(a, b, c, t) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

In order to describe the condition of the moving fluid completely we still require a statement about the pressure p, and another about the density ρ if it is variable. The actual working out of this plan to a final numerical result, however, is possible only in a few special cases. As it is really not so very essential to know the fate of each individual particle, we usually content ourselves with a simpler method of representing the motion, which merely aims at describing the motion at every point and at every time, without considering the whereabouts of individual particles. If the flow we are dealing with remains the same as time goes on (" steady flow ") we have only to state the magni-

tude and direction of the velocity at every point of the space in which the flow is taking place, with the addition of the corresponding data relating to the pressure and if necessary the density. If, however, the flow is not steady such data are necessary for every instant of time. The data are usually put in mathematical form by expressing u, v, w, the three components of the velocity referred to rectangular axes (and if necessary the pressure p and density ρ also), as functions of the space co-ordinates x, y, z, and of the time t: that is,

$$\left.\begin{aligned} u &= f_1(x, y, z, t) \\ v &= f_2(x, y, z, t) \\ w &= f_3(x, y, z, t) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The set of formulæ (1) is generally called after Lagrange, and the set (2) after Euler, although both were used by Euler.

If in any particular case it is desirable to trace out the *path of a particle* of fluid, we have to integrate the three simultaneous equations $dx = u\,dt$, $dy = v\,dt$, $dz = w\,dt$. As the three constants of integration may at once be interpreted as fluid co-ordinates, this brings us back to the method of representation (1).

In order to obtain a clearer idea of the state of flow of a fluid at a particular instant, we introduce the idea of the so-called *stream-lines* or *lines of flow*, the tangents to which are at every point in the direction of the velocity, so that stream-lines are perfectly analogous to " lines of force " in a field of force.

The differential equations of the stream-lines are

$$dx : dy : dz = u : v : w.$$

When the motion is steady the stream-lines coincide with the paths of the particles. When the motion varies with time, on the other hand, they do not by any means do so, as the stream-lines give a picture of the directions of the velocity which occur *simultaneously* at *one* given instant, whereas the paths of the particles give a picture of the directions of the velocity of *one* particle at *successive* instants of time.

Here we would remind the reader that the stream-lines of one and the same fluid motion, and also the paths of the particles, differ entirely in appearance if the system of reference is changed, e.g. if in the case of the motion of a solid body in a fluid the observer is at one time at rest relative to the undisturbed fluid, and at another time moves with the body, so that for him the body is then at rest with the fluid flowing past it.

The stream-lines can be made visible by distributing small particles over the surface or throughout the interior of the fluid so that they share in the motion of the fluid. If a short exposure is given, each particle makes a short line on the photographic plate. If the particles

have been strewn thickly enough, these lines combine in the photograph to form a diagram of the stream-lines (the paths of the fluid particles would be obtained by strewing the particles more thinly and making a long exposure). The two illustrations show simultaneous

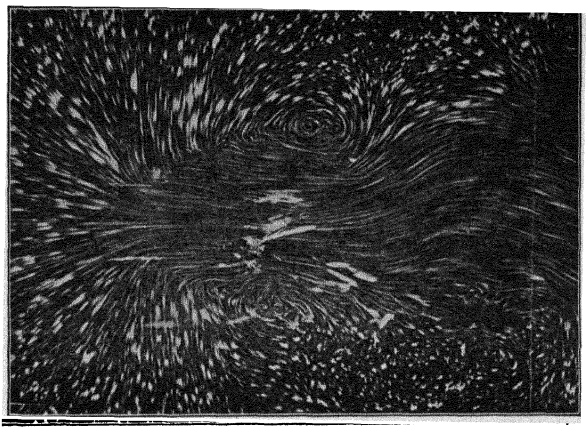


Fig. 2.1.—Flow round a moving plate, after Ahlborn. Camera at rest. The path of the plate is shown by the traces left by its edges.

photographs of the motion of a plate in liquid at rest, the systems of reference being different in the two cases: fig. 2.1 is taken by a camera at rest and fig. 2.2 by a camera moving with the plate. These photographs (due to Professor F. Ahlborn,* Hamburg) were obtained by

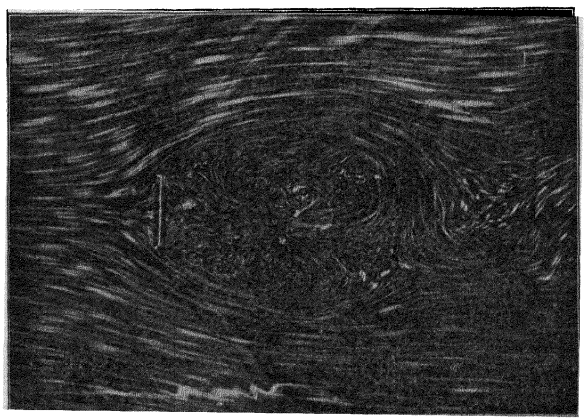


Fig. 2.2.—Flow round a moving plate, after Ahlborn. Camera moving with the plate

* *Jahrb. d. Schiffbautechn. Ges.*, Vol. 10, p. 370 (1909).

sprinkling the fluid with lycopodium powder. The lighting was obtained from a mixture of flashlight powder and time-exposure powder.

If we draw the stream-lines through all the points of a small closed curve, they form a tube of any length, provided the " velocity field " is continuous everywhere. Such a tube has the peculiar property that the fluid within it at the instant in question flows as if it were in a solid tube (for, by definition, it flows in the direction of the stream-lines; if fluid were to pass through the wall of the tube it would mean that there was a component of velocity oblique to the wall, i.e. oblique to the stream-lines, which is contrary to the definition of a stream-line). Such tubes we call *stream-tubes*. In steady motion the stream-tubes are permanent and the particles of fluid within them flow on permanently as if they were contained in a solid tube. In non-steady motion, on the other hand, the particles which at a subsequent instant of time are connected by a stream-tube are in general different from the particles which were so previously. If we imagine the whole space filled by the fluid to be divided up into such stream-tubes, we obtain a very clear picture of the motion of the fluid. In a large number of the simpler problems, in particular in the flow in pipes and water-courses, it is permissible to regard the whole space filled by the moving liquid as a single stream-tube. We then neglect differences of velocity within a cross-section and merely endeavour to find an expression for the mean velocity of flow across it. Engineers in particular use this method extensively for practical calculations.

2. **Continuity.**

In the actual motion of liquids and gases matter is neither created nor destroyed. Hence we have to see that the velocity fields which we are considering do not contravene the principle of the conservation of matter. In the case of steady motion it is very easy to express this condition provided the course of the stream-lines is sufficiently well known. We consider a single stream-tube and have merely to express the fact that the mass of fluid flowing across each cross-section in unit time is the same. (If the mass were not the same for any two cross-sections, the mass contained in the stream-tube between the two cross-sections would have to increase or decrease continually, which would contradict our assumption that the motion is steady.) If a is the cross-section of the tube at a definite point, q the mean velocity * across this section, and ρ the density there, the volume of liquid crossing the section in unit time is aq, and the mass crossing in unit time is ρaq. For continuity, then, ρaq must have the same value for all

* The letter q is used instead of v for the velocity of flow, because in calculations of the flow of gases, where the volume changes are considerable, the volume occupied by unit mass of gas is generally denoted by v, in accordance with the notation of thermodynamics, which is also involved in these problems.

cross-sections of the same stream-tube. It also follows that in steady flow a stream-tube can never come to an end in the interior of a fluid. It may extend from one boundary of the space under consideration to another, or it may return into itself.

If we are dealing with a case where there is no change of volume, e.g. the flow of an incompressible fluid, the above relationship for the mass flowing across a section also holds for the volume, and as a greater volume can never in this case flow across one section of a stream-tube than across any other the limitation to steady motion now disappears. Thus in general we have

$$aq = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for *fluid motion when there are no volume changes*, i.e. the velocity is inversely proportional to the cross-section of the stream-tube. If the whole space filled by the fluid is divided into single stream-tubes such that the total quantities of fluid flowing through them in unit time are the same, many stream-tubes will be crowded together at places where the velocity is large, i.e. where the cross-section of the tube is small, and the tubes will spread out correspondingly wider where the velocity is small. The number of stream-tubes passing through unit area at any point is proportional to the velocity at that point. Thus in fluid motion where there are no volume changes the stream-line diagram not only indicates the direction of flow at any point but also its magnitude (by the density of the stream-lines).

The relationships just mentioned are particularly obvious in cases where the whole flow may be treated as a single stream-tube. Here the cross-sections of the stream-tube are the same as the given cross-sections of the flow, and the mean velocity at any point is given by

$$aq = Q,$$

where Q denotes the volume flowing past that point in unit time.

For a fluid motion in which volume changes occur we likewise have

$$\rho aq = M,$$

where M is the mass of fluid flowing past a point in unit time. Here, however, the density ρ can, as a rule, only be determined in conjunction with the pressure, so that the velocity cannot be obtained from considerations of continuity alone (cf. Chapter IV, p. 258).

When we are dealing in the above way with steady flow with no volume changes, we have only *one* independent variable, namely the distance (measured along the centre-line of the tube) between the cross-section under consideration and any origin. In such a case we speak of " one-dimensional treatment ", in contradistinction to the three-dimensional treatment in which the variations in space of the velocity

and the other quantities are fully taken into account. One-dimensional problems relating to water form what is known as *hydraulics*, whereas problems involving more dimensions than one are referred to as *hydrodynamics* *. The term *aerodynamics* is generally used for problems arising from flight and other topics involving the flow of air.

We express the condition of continuity in three dimensions (in which u, v, w, the three components of the velocity referred to rectangular axes, are in general given as functions of the space coordinates x, y, z) by requiring that the amount of fluid flowing into a small parallelepiped with sides dx, dy, dz should equal the amount flowing out. In the case where there are no volume changes, which we shall alone consider here, this is expressed mathematically as follows (cf. fig. 2.3†). The quantity of fluid which u, the x-component of the velocity, causes to flow into the parallelepiped per second in the direction of the x-axis is $u\,dy\,dz$, while the quantity flowing out on the opposite side, where u is replaced by $u+\frac{\partial u}{\partial x}dx$, is $\left(u+\frac{\partial u}{\partial x}dx\right)dy\,dz$. Thus the excess of fluid flowing out is $\frac{\partial u}{\partial x}dx\,dy\,dz$. Similar expressions are obtained for the y- and z-directions. The condition that, taken all over, as much fluid flows in as flows out is therefore expressed by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the *equation of continuity* of hydrodynamics.

If the fluid is bounded anywhere by a solid body or by another fluid, the principle of continuity requires that there should be no gap and that the fluids should not mix. In order that these contingencies may be avoided the components of the velocity at right angles to the boundary surface must obviously be the same on either side of the boundary surface. If we are considering bodies at rest amongst moving fluid, or solid boundary walls, the component of the fluid velocity at right angles to the surface of the body or wall must vanish there. As far as continuity is concerned, the component parallel to the wall may have any value we please.

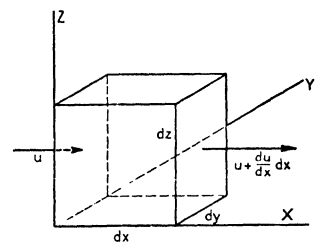


Fig. 2.3

* There is no general agreement on this definition.

† The figures 3–6, 8, 9, 11, 17, 25–8, 32–44, 50–52, 54, 55, 59, 61, 64–67 in the present chapter and 1, 2, 4, 13, 44, 48–53, 58, 61, 62, 99, 131–33 in the following one are taken from the author's paper *Über Flüssigkeitsbewegung*, in the *Handwörterbuch der Naturwissenschaften*, Vol. IV, with the kind permission of the publisher, Gustav Fischer, Jena.

2. DYNAMICS OF FRICTIONLESS FLUID

3. Forces in Moving Fluid. Bernoulli's Theorem.

A fluid at rest is acted on and kept in equilibrium by two kinds of forces, *gravity* (and other body-forces) and *differences of pressure*. These are also present in the case of fluid in motion; in addition, there is the *friction* of the fluid, which is to be regarded as a resistance to change of shape. In the present chapter we shall neglect friction, but it will be discussed in detail in the next chapter. As it happens, the fluids which are most important in actual practice (water, air, &c.) have a very small viscosity, so that under ordinary circumstances the frictional resistance which they exhibit is very small, and may justifiably be ignored: this neglect of viscosity, moreover, is compulsory, because it is only in the absence of frictional resistance that the relationships between the forces become simple enough for us to obtain clear and conclusive results. Hence it is customary to deduce the fundamental laws of fluid motion with the aid of the ideal conception of *frictionless fluid*, and only subsequently to inquire what deviations from the ideal state are caused by the presence of friction. In what follows, then, we assume that the fluid is "ideal", and in addition we shall assume that it is incompressible as well as frictionless, so that there are no volume changes.

In order to deduce the dynamical relationship between the pressure and the body-force on the one hand and the nature of the motion on the other, we have to start from Newton's law that Force = Mass $\times$ Acceleration, which is the foundation of all dynamics. We begin by considering how we may calculate the motion occurring at any one instant at various points along a stream-tube: for this we require to find the component of the acceleration in the direction of motion, the so-called longitudinal acceleration, which we shall begin by evaluating. Let s be the length of arc measured along the stream-line, t the time, q the velocity. When s is changed by an amount ds and t by dt simultaneously, the change in the velocity is then *

$$dq = \frac{\partial q}{\partial s} ds + \frac{\partial q}{\partial t} dt.$$

When the fluid is accelerated the change of velocity is the same as that of a *single* particle. Hence we must have $ds = q\,dt$, so that the longitudinal acceleration is given by

$$\frac{dq}{dt} = q \frac{\partial q}{\partial s} + \frac{\partial q}{\partial t}. \qquad \ldots\ldots (5)$$

* $\partial q/\partial t$ is the partial differential coefficient (s remaining constant), dq/dt the total differential coefficient (the particle remaining the same).

Here $q\partial q/\partial s$ denotes the part of the acceleration due to the particle coming to a place where the velocity is different, and $\partial q/\partial t$ the part due to the change in the flow with time at any one place. If the flow is steady, the second term is equal to zero. The first part may also be written in the form $\frac{\partial}{\partial s}(\frac{1}{2}q^2)$.

In order to apply the relationship Force = Mass × Acceleration, we again isolate a small cylinder of cross-section da and length ds from the moving fluid, just as we did before when investigating the conditions of equilibrium of a fluid (Chapter I, § 6, p. 12). The axis of the cylinder is taken along the direction of flow (see fig. 2.4). The mass of the cylinder is $\rho da ds$.

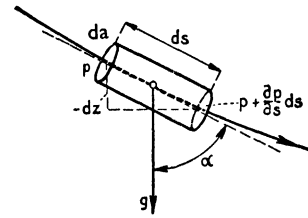


Fig. 2.4

Now provided the motion is regarded as frictionless the forces acting on the element are as follows. First we have a *difference of pressure.* Let the pressure on the up-stream end of the cylinder be p; the total force on the end surface due to the pressure is then $p\,da$. At the down-stream end the pressure has the slightly different value $p + \frac{\partial p}{\partial s}ds$, so that the resultant of the two pressures is

$$p\,da - \left(p + \frac{\partial p}{\partial s}ds\right)da = -\frac{\partial p}{\partial s}ds\,da.$$

Suppose that the fluid is also acted on by a body-force (e.g. gravity) and let the force on unit mass be g. If the direction of this force makes an angle α with the direction of the flow, the component acting on the mass $\rho da ds$ in the direction of its motion is $\rho da ds \,.\, g\cos\alpha$. Each term in the equation Force = Mass × Acceleration now contains the factor $da ds$, which we may therefore omit (i.e. the arbitrarily chosen volume of the element makes no difference to the final result). Dividing by ρ, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} + g\cos\alpha = \frac{\partial}{\partial s}\left(\tfrac{1}{2}q^2\right) + \frac{\partial q}{\partial t}. \quad . \quad . \quad . \quad (6)$$

The body-force is ordinarily that of gravity alone; g is then constant in magnitude and direction and if z is the vertical co-ordinate we may write $-\partial z/\partial s$ for $\cos\alpha$ (fig. 2.5, p. 40).

If the motion we are dealing with is steady (i.e. $\partial q/\partial t = 0$), and if we further assume that the density ρ is constant, all the terms are differential coefficients with respect to s, so that in this case we may

integrate equation (6) *along the stream-line.* From

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{\partial z}{\partial s} + \frac{\partial}{\partial s}\left(\tfrac{1}{2}q^2\right) = 0$$

we obtain

$$\frac{p}{\rho} + gz + \tfrac{1}{2}q^2 = \text{const.} \qquad \ldots\ldots (7)$$

This result, which is called *Bernoulli's theorem,** is the most important relationship used in problems of one-dimensional flow, but is also of fundamental importance throughout the subject of hydrodynamics. It is an expression of the conservation of energy, for the terms each represent an energy per unit mass, the first the work done by the pressure, the second the potential energy due to gravity, and the third the kinetic energy. If we divide all the terms of equation (7) by g they become of the dimensions of a length and may be taken to mean heights. If we put $w = g\rho$, as we did in the last chapter, Bernoulli's theorem takes the form

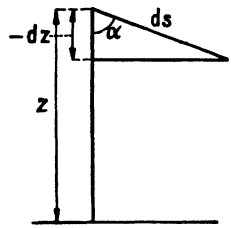


Fig. 2.5

$$\frac{p}{w} + z + \frac{q^2}{2g} = \text{const.} \qquad \ldots\ldots (7a)$$

frequently used by engineers. By Chapter I, § 6 (p. 11), p/w denotes the height of a column of fluid, the weight of which gives rise to the pressure p; this height is therefore called the *pressure head*; z is the height of the point in question above any arbitrarily fixed horizontal plane, and may be called the *potential head*, while $q^2/2g$ is the height through which a body must fall freely in order to attain the velocity q, and hence may be called the *velocity head.* According to Bernoulli's theorem, therefore, the sum of the pressure head, the potential head, and the velocity head is constant along a stream-line. The value of the constant may, however, vary from one stream-line to another, in particular, if the stream-lines have different origins. If all the stream-lines come from a space in which the relationships are static (i.e. in which the fluid is at rest or in uniform motion in a straight line), the constant is the same for all the stream-lines, i.e. in this case Bernoulli's theorem holds for all the stream-lines and throughout the space. (By Chapter I, § 6, p. 11, $p/w + z = \text{const.}$ in a fluid at rest, which agrees with Bernoulli's theorem when q is zero or constant.) We may mention that the special kind of flow described here is identical with the steady irrotational motion discussed later (p. 58).

* After Daniel Bernoulli (1738).

The integration may also be carried out for other body-forces, provided the forces have a potential (U), as $g\cos\alpha$ may then be put equal to $-\partial U/\partial s$. If we are dealing with a compressible fluid, the integration is likewise possible provided the fluid is homogeneous, i.e. provided the density depends on the pressure only. Then $\int dp/\rho = \mathbf{P}(p)$, a function* of the pressure, so that we may write $\frac{1}{\rho}\frac{\partial p}{\partial s} = \frac{\partial \mathbf{P}}{\partial s}$. Integration with respect to s then gives the general form of Bernoulli's theorem for steady motion:

$$\mathbf{P} + U + \tfrac{1}{2}q^2 = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

Mathematical Appendix

In the mathematical theory of fluid motion we usually start with u, v, w, the three components of the velocity referred to rectangular axes. We then obtain, as the analogue of equation (5) (p. 38),

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (8)$$

for the component of acceleration in the direction of the x-axis, and corresponding formulæ for dv/dt and dw/dt. The fall of pressure has the three components $-\partial p/\partial x$, $-\partial p/\partial y$, $-\partial p/\partial z$ per unit volume. Let the body-force (hitherto g) have the three components X, Y, Z; then the components of the force acting on unit volume are ρX, ρY, ρZ, since the mass of unit volume is ρ. Hence, considering the mass contained in unit volume, we obtain the three equations

$$\left.\begin{aligned} \rho\frac{du}{dt} &= \rho X - \frac{\partial p}{\partial x} \\ \rho\frac{dv}{dt} &= \rho Y - \frac{\partial p}{\partial y} \\ \rho\frac{dw}{dt} &= \rho Z - \frac{\partial p}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

as the expression of the (vector) relationship "Mass × Acceleration = Sum of the Forces". These equations, in which the accelerations may further be replaced by the expressions in the equations (8), are called *Euler's hydrodynamical equations*. In order to illustrate their use we shall deduce Bernoulli's theorem for a single stream-line from them.

We multiply the equations (9) by dx, dy, dz respectively, at the same time assuming that

$$dx : dy : dz = u : v : w,$$

which means that dx, dy, dz are to be the components of an element of the stream-line. We may then replace $v\,dx$ by $u\,dy$, $w\,dx$ by $u\,dz$, &c.

Hence we may write

$$\frac{du}{dt}dx = \frac{\partial u}{\partial t}dx + u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right).$$

* $\mathbf{P}$ denotes capital ρ.

The expression in brackets means the change in u as we move along the stream-line, which we call du for short; that is,

$$\frac{du}{dt}dx = \frac{\partial u}{\partial t}dx + u\,du.$$

If we assume for simplicity that the motion is steady, so that $\partial u/\partial t$, &c., are equal to zero, the left-hand side obtained by multiplying the equations (9) by dx, dy, dz respectively, adding, and dividing by ρ is

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = u\,du + v\,dv + w\,dw = d\left(\frac{u^2 + v^2 + w^2}{2}\right).$$

If the body-forces have a potential U, i.e. if

$$X = -\frac{\partial U}{\partial x}, \quad Y = -\frac{\partial U}{\partial y}, \quad Z = -\frac{\partial U}{\partial z},$$

the corresponding terms obtained from the equations (9) are

$$-\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = -dU.$$

Similarly we have $$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) = \frac{1}{\rho}dp,$$

so that, if we confine ourselves to points on a single stream-line, we obtain

$$d\left(\frac{u^2 + v^2 + w^2}{2}\right) + dU + \frac{dp}{\rho} = 0,$$

which comes to the same as equation (7*b*) (p. 41).

4. Deductions from Bernoulli's Theorem.

A large number of problems may be solved very simply by means of Bernoulli's theorem. Here we shall discuss three particularly important examples.

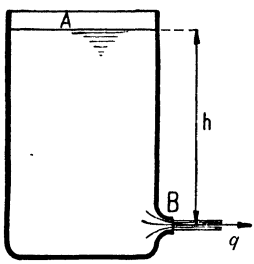


Fig. 2.6.—The efflux from a vessel

(*a*) *The Efflux of Liquid from a Vessel as a result of Gravity.*—If we trace back the stream-lines in the vessel in fig. 2.6 from the outflow B, we find that they lead to the free surface of the water at A; the latter slowly falls as the mass of water does so. The particles of water at A are acted on by the atmospheric pressure p_0, and so are those in the free jet * at B (see below). If the area of the water surface is large compared with that of the lower opening, the velocity at A is so small that its square may be neglected in comparison with that of the velocity at B. Hence, if z_A, z_B are the heights of A and B above a standard level,

$$\frac{p_0}{\rho} + gz_B + \tfrac{1}{2}q_B^2 = \frac{p_0}{\rho} + gz_A + 0$$

* Here we neglect the weight of the air, which is permissible unless accuracy to the third decimal place is required.

by Bernoulli's theorem, so that

$$\frac{q_B^2}{2g} = z_A - z_B = h,$$

or

$$q_B = \sqrt{(2gh)}. \quad \ldots\ldots\ldots \quad (10)$$

Thus the velocity of a particle of water flowing out at B is the same as it would have been if the particle had fallen freely through the height h. What happens in actual fact is that the particles at A are pulled down a little into a new position, the particles previously occupying that position are likewise drawn down further, and so on, and the amounts of work done by these particles (which, thanks to the internal mechanism of the liquid, are all transferred to the particle which actually flows out) add up to exactly the amount of work which would be done by a single particle of the size of the one which flows out in falling through the whole height. The relationship expressed by equation (10) is called *Torricelli's theorem*.

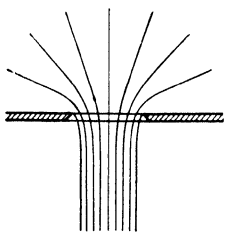


Fig. 2.7.—Flow through an opening in a straight wall

The cross-section of the jet does not in general coincide with that of the orifice. For example, in the case of a jet emerging from a thin-walled circular opening the cross-section of the jet is about 0·61 to 0·64 of the cross-section of the hole. This phenomenon, which is called *contraction*,* is due to the fact that the fluid inside the vessel flows radially towards the hole and when it reaches the edge cannot immediately turn from the radial direction into the direction of the axis of the jet. A case of this kind is shown in fig. 2.7. In a rounded-off opening (fig. 2.8), however, where the change of direction of the stream-tubes is completed within the mouthpiece, the ratio mentioned above (which is called the coefficient of contraction) is almost equal to 1. Q, the quantity of fluid flowing through an orifice of cross-section a per second, is given by $Q = \alpha a\sqrt{(2gh)}$, where α is the coefficient of contraction. For a non-circular thin-walled opening, α usually differs only slightly from its value for a circular opening, but the form of the jet in such a case is generally fairly complicated. For example, the jet from a square hole is transformed into one with a slender cross-shaped section, that from a rectangular hole into a ribbon perpendicular to the longer side of the rectangle.

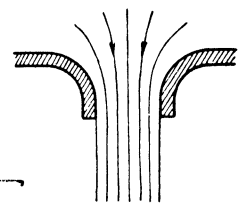


Fig. 2.8.—Flow through a rounded-off opening

* [The section at which the jet becomes parallel is referred to as the *vena contracta*.]

(*b*) *The Efflux from a Vessel as a result of Excess Pressure inside.*—Let p_1 be the pressure within the vessel in fig. 2.9 and p_0 the atmospheric pressure outside. For a horizontal stream-line $z_A = z_B$, so that if we again regard the velocity at A as negligibly small we have

$$\frac{p_0}{\rho} + \tfrac{1}{2}q^2 = \frac{p_1}{\rho} + 0$$

by Bernoulli's theorem, i.e.

$$q = \sqrt{\frac{2(p_1 - p_0)}{\rho}} = \sqrt{\frac{2g(p_1 - p_0)}{w}}. \quad . \quad . \quad (11)$$

If we denote the height $(p_1 - p_0)/w$, i.e. the height of a column of liquid with a weight per unit volume w and a difference of pressure between its upper and lower ends of $(p_1 - p_0)$, by h, we again have $q = \sqrt{(2gh)}$, by formula (11). Formula (11) enables us to estimate the value of the velocity up to which it is still permissible to treat a gas as an incompressible fluid. The limiting velocity q_1, of course, depends on the size of density variation regarded as permissible, so that the higher the degree of accuracy demanded, the lower is the limiting velocity. If for example we allow $\Delta\rho/\rho$, the variation of density, to reach a value of 1 per cent, i.e. 0·01, we obtain $\Delta p \sim \gamma p \Delta\rho/\rho$, since $\Delta p/p \sim \gamma\Delta\rho/\rho$, from the equation $pV^\gamma = \text{const.}$* or $p = \text{const.}\ \rho^\gamma$; for air at ordinary pressure this comes to about $1{\cdot}405 \times 10{,}332 \times 0{\cdot}01 = 144{\cdot}5$ Kg./m.^{2} Taking 0·125 $\frac{\text{Kg.-sec.}^2}{\text{m.}^4}$ as an average value for ρ, we obtain

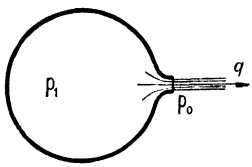


Fig. 2.9

$$q_1 = \sqrt{(2\Delta p/\rho)} = \sqrt{2320} = \text{about } 48 \text{ m./sec.}$$

If we allow density variations of 10 per cent, our approximate formula yields a velocity $\sqrt{10}$ times greater than before, i.e. about 150 m./sec. The effects of the density variation are twofold: from the kinematic point of view the cross-sections of the stream-tubes are altered, and from the dynamical point of view the extent of the pressure change associated with an acceleration is altered [cf. § 3, equation (6), p. 39].

(*c*) *The Damming-up of Fluid in front of an Obstacle.*—If an obstacle is situated in a fluid flowing uniformly with the velocity q_0, the fluid is dammed up immediately in front of the obstacle and divides in all directions in order to flow round it (fig. 2.10). At the centre of the dammed-up region, the "stagnation point", the fluid comes entirely to rest. Thus if p_1 is the pressure at the stagnation point and p_0

* Cf. Chapter I, § 5, p. 10.

the pressure at the same depth in undisturbed fluid, and we apply Bernoulli's theorem to the stream-line passing through the stagnation point, we have

$$\frac{p_1}{\rho} + 0 = \frac{p_0}{\rho} + \tfrac{1}{2}q_0^2,$$

that is,

$$p_1 = p_0 + \tfrac{1}{2}\rho q_0^2.$$

The extra pressure $p_1 - p_0 = \frac{1}{2}\rho q_0^2$ is called the *kinetic pressure*, *dynamic pressure*, or *stagnation pressure*. Observations of this rise of pressure may be used to measure velocities of flow. If a body is moved with the velocity v through air (or fluid) at rest, the flow (referred to a system of co-ordinates moving along with it) is just like that described above, the velocity q_0 being equal and opposite to v. Hence in this case

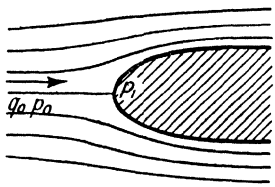


Fig. 2.10.—Flow in the neighbourhood of an obstacle

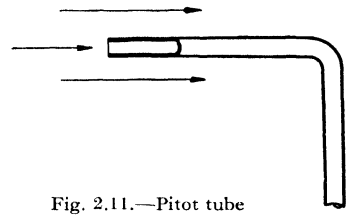


Fig. 2.11.—Pitot tube

also there is a rise of pressure $\rho v^2/2$. If there is a hole in the obstacle at the stagnation point, the pressure p_1 is propagated through it into the interior and may be transmitted to a measuring instrument. A simple bent tube (fig. 2.11), called a Pitot tube after its inventor, is a sufficient "obstacle" for the purpose of measuring the pressure p_1 $(= p + \frac{1}{2}\rho q^2)$ in a fluid in motion.

To every point of the moving fluid there corresponds a value of the pressure p_1 which would be indicated by a Pitot tube at that point, as well as the original pressure p there, which would be indicated by a pressure gauge moving with the fluid. Engineers call p the *static pressure* and p_1 the *total pressure* or *Pitot head*, i.e. the total pressure is the sum of the static pressure and the dynamic pressure. Using Bernoulli's theorem

$$\frac{p}{\rho} + gz + \tfrac{1}{2}q^2 = \text{const.}$$

and introducing the total pressure $p_1 = p + \frac{1}{2}\rho q^2$, we have

$$\frac{p_1}{\rho} + gz = \text{const., or } p_1 + wz = \text{const.},$$

so that p_1 is distributed according to the laws of hydrostatics, i.e.

if all the stream-lines have the same constant, p_1 is constant in every horizontal plane.

In order to utilize the above relationships to find the velocity of flow we have to determine the static pressure p as well as p_1. The former is far more difficult to ascertain than the latter, as if we insert a measuring instrument the static pressure is disturbed at the very point where we wish to measure it. For the carrying out of such pressure measurements see § 7 (p. 53).

5. Further Discussion of Pressure Distribution in Moving Fluids.

The following discussion is not confined to the case of frictionless fluid but also holds (with slight alterations in certain circumstances) for fluids in which the friction is fairly large. In the first paragraph, however, the fluid is assumed to be incompressible and of constant density.

(*a*) The pressure in a fluid of this kind may be expressed as the sum of two terms, one of which represents the pressure which would exist if the fluid were at rest. Let p' denote this " equilibrium pressure ". Obviously $p' = \text{const.} - wz$. If for the actual pressure in the moving fluid we put $p = p' + p^*$, p^* represents the difference between the pressure when the fluid is in motion and that when it is at rest. If it is permissible to apply Bernoulli's theorem, i.e. $p + wz + \frac{1}{2}\rho q^2 = \text{const.}$, it follows from the above expression for p' that $p^* + \frac{1}{2}\rho q^2 = \text{const.}$ That is, p^* is distributed as if the fluid, though possessing inertia, were unaffected by gravity; the height z has no effect on p^*. This result, which is closely connected with the fact that every particle of a heavy fluid is kept up by the thrust exerted on it by neighbouring particles, may obviously be extended to fluids with friction also. In what follows, therefore, we shall always ignore the effects of gravity when we are dealing with motion under water or in the air, which comes to this, that we always consider the pressure difference p^* instead of the pressure p. We shall, however, write p for p^* for the sake of simplicity.

If in the case of the flow of air or water the pressure is measured by externally situated pressure-indicators at rest, which are connected by pipes or rubber tubes with the point at which the pressure is to be measured, the effect of gravity on the fluid in the tubes is such that the pressure indicated by the instrument is independent of the height of the point at which the pressure is to be measured. Thus the pressure shown by the instrument is of the nature of p^*. If the pressure is registered by means of a Pitot tube (see above) facing the flow, the instrument at rest indicates a constant pressure along a stream-line. In the case where all the stream-lines have the same constant the pressure indicated is the same for the whole region.

(*b*) Bernoulli's theorem is concerned with the pressures *along* a stream-line. By considering the transverse acceleration instead of the

longitudinal acceleration we similarly obtain an expression for the differences of pressure in a direction at right angles to that of the flow. As is well known, the transverse acceleration is in the direction of the principal normal of the path and is of magnitude q^2/r, where r is the radius of curvature of the latter. If we consider the forces acting on a small prism with its axis in the direction of the principal normal, we find that the component in the direction of r is

$$\frac{q^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial s'}, \qquad \ldots\ldots\ldots \quad (12)$$

where ds' is an element of arc in the direction of the principal normal and p is taken as meaning the same as p^* in the preceding sub-section. The formula is an expression for the centrifugal effect when the fluid flows in curved lines. The pressure rises as we pass from the concave side of the stream-line to the convex by an amount equal to $\rho q^2/r$ per unit length. This, then, is the relationship between neighbouring stream-tubes. The most important fact thus established is that *when the flow is along straight lines* ($r = \infty$) *no difference of pressure can exist in a direction transverse to the direction of flow.* When the flow is along curved lines we obtain a particularly simple result in the special case already discussed where the constant in Bernoulli's theorem has the same value for all the stream-lines. Here, by differentiating $\int \frac{dp}{\rho} + \frac{1}{2}q^2 = \text{const.}$ (cf. equation (7*b*), p. 41) with respect to s', we obtain another expression for $\frac{1}{\rho}\frac{\partial p}{\partial s'}$, namely $-q\frac{\partial q}{\partial s'}$. Equating this to the expression previously obtained, we have

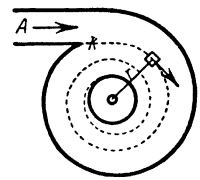


Fig. 2.12.—Spiral volute chamber

$$\frac{\partial q}{\partial s'} + \frac{q}{r} = 0. \qquad \ldots\ldots\ldots \quad (13)$$

Anticipating results obtained later (§ 8, p. 56), we may mention that this expresses the fact that in the flow along curved lines the individual fluid elements experience no rotation (*irrotational motion*).†

As an example we shall consider the motion in the spiral volute chamber of a centrifugal pump (fig. 2.12). All the stream-lines are initially parallel at A and the velocity in all the stream-tubes may be taken to be the same there, so that (the pressure being constant for parallel flow) Bernoulli's constant is the same for all the stream-lines.

† It is easy to show that the "circulation" along a rectangle formed from two radial elements of length ds' and two arcs of stream-lines vanishes if equation (13) holds. The proof is given in § 8 (p. 56).

The radii of curvature of the individual stream-lines may be taken as approximately equal to the radius r drawn outwards from the centre O, and the element of arc ds' may be put equal to dr. Then, by what we have said above, $dq/dr + q/r = 0$, or $dq/q = -dr/r$, or, if we integrate, $\log q = \log C - \log r$, i.e. $q = C/r$, where C is the constant of integration. Thus the velocity increases considerably towards the centre.* The pressure is given by $p = \text{const.} - \rho C^2/2r^2$, by Bernoulli's theorem. If the fluid flows out at radius r_1, and the external pressure is p_0, the pressure at any other point is given by

$$p = p_0 + \tfrac{1}{2}\rho C^2\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right),$$

so that if the radius of the hole where the fluid flows out is fairly small, a very considerable excess of pressure may exist at A.

(c) When the motion of the fluid is not steady an additional pressure field due to the change of the flow has to be added to the pressures hitherto considered. Here we shall confine our attention to the longitudinal acceleration, in which, by equation (5) (p. 38), there appears the extra term $\partial q/\partial t$ (the change of velocity at a fixed point as time goes on). If we carry out the calculations which led us to Bernoulli's theorem with the complete form of equation (6) (p. 39), we have the additional term $\int_0^s \frac{\partial q}{\partial t}\,ds$ on the left-hand side of equation (7) (p. 40). If, for example, we are dealing with a tube of constant cross-section, in which the velocity is the same for all cross-sections (as we assume that the fluid is frictionless, we shall take the velocity to be constant over each cross-section), $\partial q/\partial t$ does not vary from point to point, so that the integral may be put equal to $\frac{dq}{dt} \cdot s$.

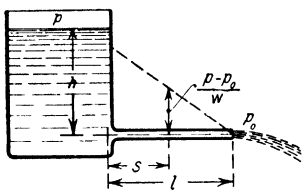


Fig. 2.13.—Fluid beginning to flow out of a vessel

Example.—Fluid begins to flow out of a vessel by a side tube of length l (fig. 2.13). Along the axis of the tube (which is assumed to be horizontal) we have

$$\frac{p}{\rho} + \tfrac{1}{2}q^2 + \frac{dq}{dt} \cdot s = \text{const.} = \frac{p_0}{\rho} + gh.$$

As long as dq/dt differs from zero, the fall of pressure along the tube

* The radial component of the velocity is likewise proportional to $1/r$ on account of continuity when the spiral chamber is all at the same height, so that the angle between the stream-lines and the radius is always the same; the stream-lines are therefore logarithmic spirals.

is proportional to s. If the pressure at the end of the tube ($s = l$) is p_0, we have

$$\frac{p_0}{\rho} + \tfrac{1}{2}q^2 + \frac{dq}{dt} \cdot l = \frac{p_0}{\rho} + gh,$$

so that

$$\frac{dq}{dt} = \frac{1}{l}(gh - \tfrac{1}{2}q^2).$$

Initially ($q = 0$) we have the simple relationship $dq/dt = gh/l$. As q increases, dq/dt continually decreases and subsequently becomes zero, i.e. the motion becomes steady, with $q = \sqrt{(2gh)}$. Here we need not discuss the exact law according to which q increases with time; it is not difficult to work out. We may find an approximate value for the time taken to reach the steady state by assuming that the acceleration dq/dt is constant until $q_1 = \sqrt{(2gh)}$ is reached and inserting q_1/T instead of dq/dt in the formula above. This gives

$$T = \frac{q_1 l}{gh} = \frac{2l}{q_1}.$$

Another simple example of non-steady motion of a fluid is the oscillation of a column of fluid in a bent tube under the influence of gravity (fig. 2.14). Let the tube have a constant cross-section, and let the length of the column of liquid, measured along the axis of the tube, be l. Let the displacement of the fluid from the equilibrium

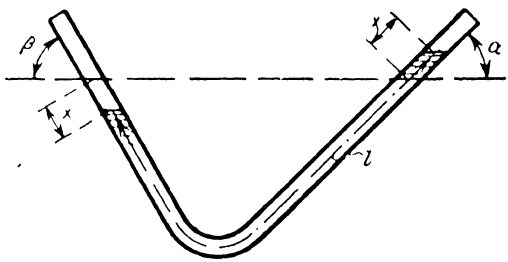


Fig. 2.14.—Oscillations of a column of water

position at a certain time be x, measured along the axis of the tube (by the principle of continuity it is the same at both ends and also for any intermediate position). The velocity is the same everywhere, namely $q = dx/dt$, so that $q\partial q/\partial s = 0$, and hence the acceleration is simply d^2x/dt^2. The one end is raised by $h_1 = x \sin\alpha$, the other lowered by $h_2 = x \sin\beta$, so that the difference of height is $h_1 + h_2 = x(\sin\alpha + \sin\beta)$.

At both ends the pressure is p_0, so that if we apply the general form of Bernoulli's theorem to the two ends we obtain

$$gx(\sin\alpha + \sin\beta) + l\frac{d^2x}{dt^2} = 0.$$

The solution of this differential equation, which is the same as that for simple harmonic oscillations, is $x = A\cos(\omega t + \gamma)$, where $\omega = \sqrt{\{g(\sin\alpha + \sin\beta)/l\}}$. This gives the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left\{\frac{l}{g(\sin\alpha + \sin\beta)}\right\}}.$$

For a vertical U-tube ($\sin\alpha = \sin\beta = 1$), $T = 2\pi\sqrt{(l/2g)}$, i.e. the period is the same as that of a pendulum half as long as the column of fluid.

6. Flowing together of Two Streams of Fluid. Surfaces of Discontinuity. Eddy Formation.

If two streams of fluid of differing origin unite at the back of a sharp edge (fig. 2.15), the constant of Bernoulli's theorem is in general not the same for the two currents. As there is equality of pressure along the surface separating the two streams ("surface of discontinuity") the velocity must differ in magnitude on the two sides. But even when the Bernoulli constant of both streams is the same, the direction of the flow may be different on the two sides. Thus in the cases considered the velocity varies abruptly in the surface of discontinuity. In the first case there is a longitudinal discontinuity in the velocity, in the second a transverse one. Such surfaces of discontinuity may frequently be observed, although, being unstable, they do not last long in their original form. There is a strong tendency for accidental indentations to increase rapidly in size, so that the differences of velocity increase in some places and diminish in others. The result of this is that the surface of discontinuity breaks down into a large number of eddies, or vortices, usually irregular. We shall describe this phenomenon somewhat more closely, as it has important bearings on our ideas about the motion of actual fluids. Owing to fluctuations in the flow the surface of separation may take on a slightly wavy form as shown in fig. 2.16. The waves advance with a velocity which is equal to the mean of the two velocities and is indicated by a dotted line in fig. 2.15. The system of co-ordinates used in fig. 2.16 moves along with this average velocity, so that the

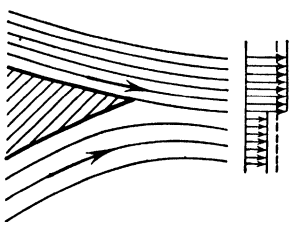


Fig. 2.15.—Two streams of fluid flowing together

wave crests and troughs remain at rest with respect to it. With respect to this system of co-ordinates the upper fluid is flowing towards the right and the lower to the left. If we now investigate the distribution of pressure using the results of last section, both Bernoulli's theorem and the equation for the transverse increase in pressure (equation (12), p. 47) lead to the conclusion that, the motion being assumed steady, there is an excess of pressure at the crests of each fluid and a defect of pressure at the troughs (shown by + and — in fig. 2.16). This distribution of pressure, however, clearly indicates that the motion cannot possibly be steady: on the contrary, the fluid in the regions of excess pressure will tend to move towards the neighbouring regions of reduced pressure, which obviously means that the undulation becomes more marked. The subsequent fate of such a surface of discontinuity is depicted in fig. 2.17. In the end it breaks down into separate eddies. As in an actual case the initial small disturbances do not usually form a regular undulation but are distributed irregularly, the final state as a rule also consists of an irregular medley of large and small eddies.*

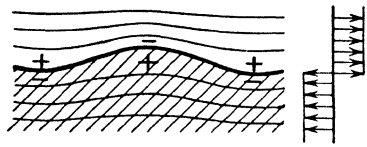
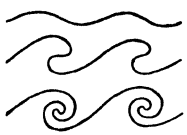
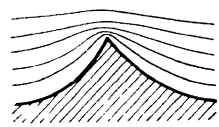


Fig. 2.16

In this connexion we may discuss another kind of surface of discontinuity giving rise to eddies (we may mention in passing that whenever a surface of discontinuity is newly formed the phenomena almost invariably resemble those in the case described below). If a fluid flows over a sharp edge, the flow about the edge is initially as shown in fig. 2.18. The velocity at the actual edge is very large and in the case of frictionless fluid would theoretically be infinitely great. Further observation shows that the velocity at the edge speedily diminishes owing to the formation of an eddy. We may explain this behaviour by adducing the special principle that the fluid seeks to avoid infinite velocities and forms surfaces of discontinuity instead. Later on we shall realize that this principle really depends on the existence of friction in the fluid, for no matter how small the friction may be, it is not without effect in the neighbourhood of solid walls. If


Fig. 2.17. — Eddies arising from waves in a surface of discontinuity


Fig. 2.18

* The irregular fluttering of flags in the wind arises in a similar way. The distribution of pressure in fig. 2.16 remains the same if the direction of the lower stream of fluid is reversed, i.e. if it is moving in the same direction as the upper one. A slight bellying of the fabric of the flag therefore has a tendency to become more marked. (As the swellings move on with the wind to some extent, the actual process is somewhat more complicated.)

for the moment we assume that there is an eddy behind the sharp edge which causes the fluid to move in a circle and approach the edge from behind, the conditions that the two streams should flow together at the sharp edge are fulfilled and a surface of discontinuity is formed there (fig. 2.19); the surface of discontinuity is rolled up by the eddy and thereby feeds it afresh, so that the eddy grows. In reality the eddy and the surface of discontinuity form one entity, and grow together from minute beginnings (fig. 2.20). Subsequently the eddy detaches itself, and the surface of discontinuity, which is always being added to at the sharp edge, breaks down into separate eddies, as described above. Very similar phenomena take place at the edge of a circular hole in a plane wall. The front edge of the surface of discontinuity rolls up and forms a vortex ring, which is propagated onwards, forming a sharply bounded jet of fluid (fig. 2.21). Very fine vortex rings can be produced by filling a box with a flexible back wall and a circular hole in the front wall with smoke and striking the back wall. As in this case the outward flow very soon comes to an end, no jet is formed, but only a ring, which continues to move onwards by itself, and as it carries smoke with it, is made visible in the form of a smoke-ring. These vortex rings are very stable and last until almost their entire stock of energy has been used up by friction.

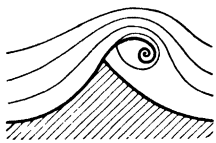


Fig. 2.19

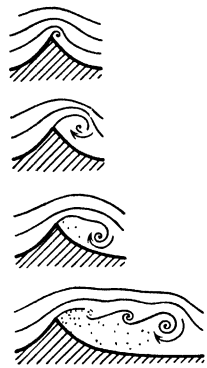


Fig. 2.20.—Formation and destruction of a surface of discontinuity

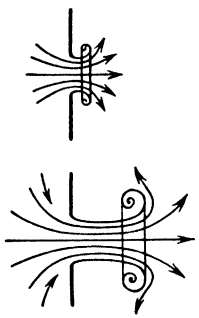


Fig. 2.21.—Formation of a jet

Transverse discontinuities in velocity occur, e.g. when fluid flows together behind a plate inclined to the direction of the flow at a small angle. On the " pressure side " the stream-lines spread out to right and left as a result of the excess of pressure formed there and are bent together by the partial vacuum on the " suction side ". Thus at the rear edge of the plate—except at its centre—the flow has a component towards the side edge on the pressure side and a component towards the middle on the suction side; in the case of steady flow the magnitude of the velocity must always be the same on both sides of the surface of discontinuity, on account of the continuity of the pressure and the common origin of all the stream-lines, so that here the discontinuity of the velocity is normal to the main stream. We know by observation that these surfaces of discontinuity curl up, starting from

the side edges, and thus give rise to two eddies, which are found all along the path of the plate. Fig. 2.22 illustrates these phenomena; it shows the shape of the surface of discontinuity at various distances behind the plate. The study of such phenomena has become of great importance in connexion with the behaviour of aeroplane wings (see Chapter III, §§ 16, 17, p. 196). Eddies of this kind may be made visible, e.g. by producing puffs of cigarette smoke in still air and cutting through the smoke with the end of a ruler held so that its plane is inclined at a small angle to the direction in which it is moving.

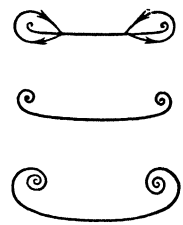


Fig. 2.22. — Surfaces of discontinuity behind a plate oblique to the direction of flow

7. Further Details about Surfaces of Discontinuity. The Measurement of Pressure.

In a fluid where the friction is small enough to be neglected in a first approximation we may take it as a definite rule, by what we have said above, that a surface of discontinuity is always formed when fluid flows over a sharp edge. If the sharp edge forms a closed curve, and if the fluid flows through the inside of the closed curve, as in the case of orifices, suddenly widening tubes, &c., a jet is formed like those described in § 4(*a*) (p. 42); this happens with water flowing out under water just as with air flowing out into air. In these cases, however, the jet is obliterated after a certain distance by the eddies arising from the surface of discontinuity. If, on the other hand, the fluid flows round the outside of the curved edge, as e.g. in the flow past a disc at right angles to the stream, a region of "dead water" is formed at the back. This region, however, is not completely at rest but is more or less filled with eddies (cf. fig. 2.2, p. 34); the velocity is nevertheless considerably smaller than it is outside. Helmholtz * was the first to study such cases theoretically. He used the method of conformal representation (§ 9(*c*), p. 64) to discuss, e.g. the form of a jet emerging from a slit in a plane wall in the absence of gravity. For Kirchhoff's treatment of a flow with dead water see Chapter III, § 14 (p. 181).

Under favourable conditions surfaces of discontinuity are formed at rounded surfaces also (cf. Chapter III, § 6, p. 135). Hence the flow at rounded-off edges is usually similar to that at sharp edges.

The case illustrated in figs. 2.23, 2.24 is of special interest. Initially the flow must be as shown in fig. 2.23, and eddies and surfaces of discontinuity are formed at the sharp edges; but when these disappear the final motion—at least if we assume that the distance between the two edges is sufficiently small to prevent instability—is as shown in fig. 2.24. In the hollow (slit) the fluid is practically at rest. The pressure

* *Ostwalds Klassiker*, No. 79.

there is obviously equal to the pressure in the moving fluid, since the pressure is constant in the region at rest, and must pass continuously into that of the moving fluid in the surface of discontinuity. If we connect a pressure gauge to the inside of the slit by a pipe or rubber tube, we can thereby measure the pressure in the moving fluid. Instead of a slit, a hole of any shape, e.g. of circular cross-section, may be used. The edges of the hole or slit must be well smoothed; there must on no account be any sharp ridge projecting into the moving fluid, as the surface of discontinuity would become curved and the pressure in it would then diverge considerably from that of the neighbouring parts of the fluid. The edges of the hole may be slightly rounded off. A convenient arrangement for recording the pressure at the wall of a tube is shown in fig. 2.25. We may apply the same fundamental principle to measure the pressure in the interior of the moving fluid by using a very thin disc with a hole in the middle, soldered on to the end of a thin tube (Ser's disc, fig. 2.26). This apparatus, however, is very sensitive to deviations of the direction of the flow out of the plane of the disc. It is better to use an instrument like that in fig. 2.27, which gives the pressure fairly accurately for deviations up to about 5°. When placed obliquely the pressure indicated by it is too low.

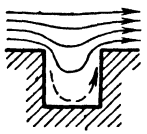


Fig. 2.23

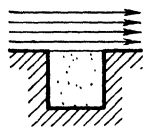


Fig. 2.24

Flow past a hollow (slit)

By combining measurements of this kind with the measurements of total pressure discussed in § 4(*c*) (fig. 2.11, p. 45), we are now in a position to find the kinetic pressure (dynamic pressure) $\frac{1}{2}\rho q^2$, from which the velocity q can be calculated if the density ρ is known. If we take metre, kilogram, and second as our basis of units, a fairly accurate value for ρ or w/g in the case of atmospheric air is $\frac{1}{8}$ Kg.-sec.^{2}/m.^{4}, which, as we easily see, gives $q^2/16$ (Kg./m.^{2}) for the dynamic pressure. For water $\rho = w/g = 102$ in the same units. In the case of water, however, it is more usual to give pressure heads than pressures. If the head of water (in metres) is h, $q = \sqrt{(2gh)}$ as before.

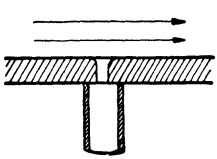


Fig. 2.25.—Perforation for measuring the pressure on a surface

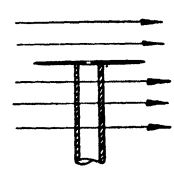


Fig. 2.26.—Ser's disc

The static tube in fig. 2.27 may be combined with the Pitot tube (fig. 2.11, p. 45) to form a single instrument which is very convenient for measuring velocities of flow. Fig. 2.28 shows a form of combined Pitot and static tube distinguished by its great insensitivity towards deviations of the instrument out of the direction of flow of the fluid. For measurements of the velocity of air flow it is usually

employed in conjunction with a micromanometer (cf. Chapter I, § 8, p. 18).

The measurement of pressure through a perforation permits of many other useful applications. Thus the pressure distribution on the surface of a body over which fluid is flowing (e.g. the surface of a model airship or aeroplane wing) can be studied by making a series of perforations and connecting them successively with one side of a pressure

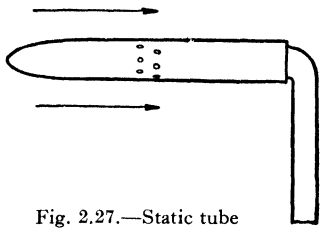


Fig. 2.27.—Static tube

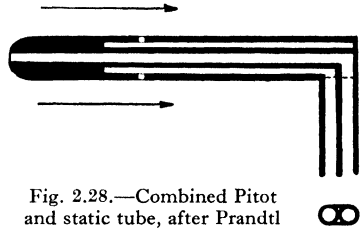


Fig. 2.28.—Combined Pitot and static tube, after Prandtl

gauge, the opposing pressure being kept constant (or we may use a number of pressure gauges working simultaneously (multi-tube manometer), on the principle of communicating vessels, making the pressure distribution immediately visible). Fig. 2.29 shows a familiar old experiment on the pressure distribution in a tube which narrows and widens again, which is used to demonstrate the truth of Bernoulli's theorem. The pressure at the end of the tube may be suitably regulated by means of the throttle valve there. When the valve is fully open there is a marked fall of pressure at *b*; this may be visibly demonstrated by

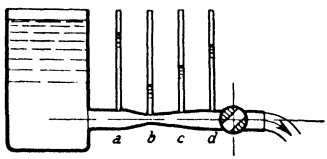


Fig. 2.29.—Diminution of pressure at a narrowed part of a tube

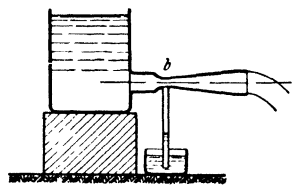


Fig. 2.30

means of a side-tube leading downwards so as to dip into a dish of mercury (fig. 2.30). Owing to friction, the increase of pressure behind the narrowest part of the tube is somewhat less than that indicated by theory; qualitatively, however, the phenomena are in accordance with Bernoulli's theorem and in the narrowest part actually agree with theory very well, provided the formation of eddies is prevented by having the tube sufficiently rounded off. Such pressure differences produced at narrowed parts of tubes may be used to measure quantities of liquid (cf. Chapter III, § 12, p. 170). For other pressure distributions see figs. 3.73, 3.74, p. 192; fig. 3.85, p. 201.

8. More Detailed Discussion of the Motion of a Homogeneous Frictionless Fluid. Irrotational Motion.

The mathematical theory of hydrodynamics reaches beyond the ground covered in the preceding sections; as we indicated on p. 32, its real object is to enable us to find the velocity at every point of space, instead of being content with average values as we have generally done hitherto. A good deal has been accomplished in this direction for the homogeneous frictionless fluid, but an understanding of the methods involved would require mathematics more advanced than that assumed here. We must therefore content ourselves with some of the more general principles relating to the behaviour of such fluids in motion and with a few simple examples. To these we shall prefix a theorem due to Thomson (Lord Kelvin), which in the first instance we shall state without proof. Before giving the theorem we must explain some of the ideas involved.

(1) By "fluid lines" and "fluid surfaces" we mean lines and surfaces which always consist of the same particles of fluid.

(2) By the line-integral of the velocity along a given curve taken between the points A and B, we mean the integral of the product of the line-element ds and the component of velocity in the direction of ds, i.e.

$$L = \int_A^B q \cos\alpha \, ds = \int_A^B \boldsymbol{q} \, d\boldsymbol{s},$$

where α is the angle between q and ds.* In the case of non-steady motion, it is to be understood that these line-integrals are formed for a momentary state of the velocity distribution.

(3) The value of the line-integral along a *closed* curve is called the *circulation* (which we shall denote by Γ), i.e. if we use the symbol $\oint$ to denote a line-integral taken round a closed curve,

$$\Gamma = \oint \boldsymbol{q} \, d\boldsymbol{s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

With these definitions Thomson's theorem may be stated as follows: *In a homogeneous frictionless fluid the circulation along a closed fluid line remains constant as time goes on.*

Many important deductions may be made from this theorem. The first is as follows:

If the motion starts from a state of rest, the circulation at the beginning, i.e. before the motion is set up, is certainly equal to zero for *every* closed "fluid line", and hence remains permanently equal to zero for that curve. If, however, the line-integral round any

* $\boldsymbol{q}\,d\boldsymbol{s}$ is the scalar product of the vectors $\boldsymbol{q}$ and $d\boldsymbol{s}$: see e.g. Wilson, *Modern Physics*, Third Edition, p. 401 (Blackie & Son, Ltd.), or Gans, *Vector Analysis*, p. 17 (Blackie & Son, Ltd.).

closed curve in the region is equal to zero, the line-integral taken from one point A to another point B is independent of the path, no matter how the latter is chosen inside the region. For it is obviously possible to return from B to A along the path previously traversed (which exactly cancels the A to B contribution to the integral, the direction of ds being reversed) and go back to B by another route. The whole is the integral from A to B, plus an integral taken round a closed curve, which is equal to zero, i.e. the total is equal to the integral from A to B, as was to be proved. If we regard the point A as fixed, to every point B there corresponds a numerical value of $\int_A^B \boldsymbol{q}\,d\boldsymbol{s}$. We denote this by Φ and call it the potential at the point B. If we proceed from B to another point C at distance ds, we have

$$\int_A^C = \int_A^B + \boldsymbol{q}\,d\boldsymbol{s}, \quad \text{or} \quad \Phi_C = \Phi_B + q\,ds\cos\alpha = \Phi_B + q\,dh \qquad (15)$$

(where dh means the projection of ds in the direction of q), since in forming the integral $\int_A^C$ the path of integration may be taken to pass through B. If $\alpha = 90°$, $\cos\alpha = 0$, so that $\Phi_C = \Phi_B$. Thus if $\Phi_C = \Phi_B$ the element $ds = \overrightarrow{BC}$ is always at right angles to the direction of q. All the points for which $\Phi = \Phi_B$ together make up a surface passing through the point B.* It follows from the above that the tangent plane to this surface at the point B is at right angles to the velocity vector $\boldsymbol{q}$ at the point B. In general, therefore, the *stream-lines*, which of course are everywhere in the direction of the velocity, are everywhere *normal* to the surfaces $\Phi = \text{const.}$

Putting $\Phi_C - \Phi_B = d\Phi$ in equation (15), we get on the one hand

$$\frac{\partial\Phi}{\partial s} = q\cos\alpha \qquad (16)$$

for every value of α, and on the other hand

$$\frac{d\Phi}{dh} = q, \qquad (17)$$

where dh, by what we have said above, is normal to the surface $\Phi = \text{const.}$ In vector notation we write

$$\boldsymbol{q} = \text{grad}\,\Phi, \qquad (18)$$

thereby combining the statement (17) with the fact that $\boldsymbol{q}$ is at right angles to the surfaces $\Phi = \text{const.}$, i.e. the velocity both in magnitude and in direction is equal to the maximum increase (gradient) of Φ.

The geometrical conceptions of the potential and the gradient

* This surface divides the region where $\Phi > \Phi_B$ from the region where $\Phi < \Phi_B$.

stated here are in complete agreement with that of the potential of a force, from which the name potential is taken. The gradient of the force potential, however, is the intensity of a field, whereas that of our potential is a velocity. The latter potential is therefore called the *velocity potential.*

If we make use of these conceptions, it follows from what we have said above that *every motion of a homogeneous frictionless fluid which is set up from rest possesses a potential.* Such motions we shall call *irrotational,* as they are distinguished by the fact that the elements of the fluid are not subject to rotation; for the circulation round a closed curve serves as a measure of the rotation, and, as we have already said, the circulation round a closed curve is in our present case equal to zero.

If, as an example of a contrary state of affairs, we consider a fluid which is rotating with angular velocity ω like a rigid body, the velocity of points on the circumference of a circle of radius r is ωr along the circumference, provided that the centre of the circle is taken as the origin of co-ordinates.* The line-integral round the circumference is therefore given by $\Gamma = 2\pi r \,.\, \omega r = 2\pi r^2\omega$. Dividing by a $(= \pi r^2)$, the area of the circle, $\Gamma/a = 2\omega$, so that Γ/a is a good measure of the rotation. If the position of the area a is perfectly general, i.e. if it makes an angle α with the axis of rotation, we readily obtain the result $\Gamma/a = 2\omega \sin\alpha$. The maximum value of Γ/a therefore occurs when the area a is at right angles to the axis of rotation.

In irrotational motion the circulation is zero for any curve entirely within the region of flow, so that there are no rotations anywhere in this region: hence the name "irrotational". It was believed that this necessarily involved the fact that an eddy or vortex could never arise when a homogeneous frictionless fluid is set in motion from a state of rest. If, however, we consider in greater detail the phenomena involved in the formation of a surface of discontinuity like that described in § 6 (p. 50), it turns out that all the "fluid lines" drawn in the interior of the fluid at rest are displaced and deformed in such a way as to avoid the surface of discontinuity, so that none of them penetrate it. Thomson's theorem therefore tells us *nothing* about the *mutual relationships* between the regions on either side of the surface of discontinuity. It follows that Thomson's theorem is in no way contradicted by the fact that surfaces of discontinuity and hence eddies may arise at sharp edges in fluid which is practically frictionless.

Note.—In actual fluids, which always possess some degree of viscosity, the *surface* of discontinuity is replaced by a *layer* of discontinuity, which, however, is often very thin. The particles in the layer of discontinuity always come from the immediate neighbourhood of the surface of the solid body, where the effects of friction are no longer negligible, even if the viscosity is extremely small. Viscosity effects must therefore be taken into account in a more accurate investi-

* As a motion of translation makes no contribution to the circulation, in calculating the latter such motions may be ignored.

gation of the internal phenomena occurring in layers of discontinuity. In the study of the external phenomena, however, it is sufficient in most cases to consider the *surface* of discontinuity instead of the *layer* of discontinuity. For frictional effects see Chapter III, §§ 1–6 (p. 98).

By considering the fall of pressure at right angles to a stream-line, we deduced (p. 47) that for fluid motions in which the constant of Bernoulli's theorem has the same value for all the stream-lines in a region,

$$\frac{\partial q}{\partial s'} + \frac{q}{r} = 0,$$

where r is the radius of curvature of the stream-line. Now the circulation round a small quadrilateral consisting of two stream-lines and two normals to them (cf. fig. 2.31) gives

$$qr\,d\phi - \left(q + \frac{\partial q}{\partial s'}ds'\right)(r + ds')d\phi = -ds'\,d\phi\left(r\frac{\partial q}{\partial s'} + q + \frac{\partial q}{\partial s'}ds'\right),$$

as the normals contribute nothing. The last term in the bracket, being a small term of higher order, may be omitted, and the remaining expression is equal to zero, by the previous statement; the above motion, therefore, is a motion for which the circulation is zero for every small closed curve, i.e. the motion is irrotational. Conversely, for every case of steady irrotational motion, Bernoulli's theorem is true independently of the stream-line (for the proof, see the end of the mathematical appendix below).

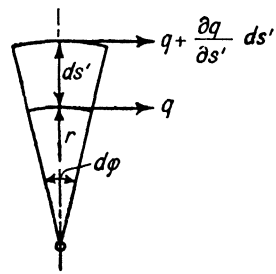


Fig. 2.31

Mathematical Appendix

(*a*) *Proof of Thomson's Theorem.*—If u, v, w are the components of the velocity, the line-integral of the velocity is $\int(u\,dx + v\,dy + w\,dz)$, as any interval ds may be replaced by the three intervals dx, dy, dz, and u, v, w are the components of the velocity in the directions of dx, dy, dz, respectively. We now have to find the change of the integral with time, assuming that the curve round which it is taken always consists of the same particles of fluid. We shall represent this differentiation by d/dt (D/Dt is also used). We begin by evaluating $\frac{d}{dt}\int u\,dx$, for which we may at once write $\int \frac{du}{dt}dx + \int u\frac{d}{dt}(dx)$. In the first term we may substitute $X - \frac{1}{\rho}\frac{\partial p}{\partial x}$ for $\frac{du}{dt}$, by Euler's equations (§ 3, p. 41). In order to evaluate the second term, we note that dx/dt is obviously equal to u for a definite particle. Thus $\frac{d}{dt}(x + dx) = u + du$, i.e. $\frac{d}{dt}(dx) = du$, where du is the difference

at any fixed instant between the u-components of the particles of the curve with x-co-ordinates $x + dx$ and x. Applying the same process to the two other terms of the line-integral, we therefore obtain

$$\frac{d}{dt}\int (u\,dx + v\,dy + w\,dz) = \int (X\,dx + Y\,dy + Z\,dz) - \int \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) + \int (u\,du + v\,dv + w\,dw).$$

We now assume that the body-force X, Y, Z has a potential U, i.e. we assume that

$$X = -\frac{\partial U}{\partial x}, \quad Y = -\frac{\partial U}{\partial y}, \quad Z = -\frac{\partial U}{\partial z}.$$

Further, we assume that the density depends on the pressure only ("homogeneous liquid"). The above expressions are then all integrable. The change with time of the line-integral along a fluid line (see p. 56), between two points A and B, which always coincide with the particles of fluid initially associated with them, is therefore

$$\frac{d}{dt}\int_A^B (u\,dx + v\,dy + w\,dz) = U_A - U_B + \mathbf{P}_A - \mathbf{P}_B + \left(\frac{u^2 + v^2 + w^2}{2}\right)_B - \left(\frac{u^2 + v^2 + w^2}{2}\right)_A,$$

where as in § 3 (p. 41) we have put $\int dp/\rho = \mathbf{P}$. It may be noted that here the sign of the square of the velocity is opposite to that occurring in Bernoulli's theorem.

If we are dealing with a closed curve, A and B coincide and the right-hand side becomes equal to zero. This proves the theorem. The importance of the assumptions involved is clearly brought out. The first, that the field of force must have a potential, was not mentioned in our previous statement of the theorem, as it was obvious from the discussion that no body-force was under consideration at all. We previously hinted at the second, that of homogeneity, which is much more important. Thomson's theorem does not hold for a non-homogeneous fluid. The above proof indicates how the change in time of the line-integral may be evaluated, e.g. in the case of irregular heating of a mass of air.

(*b*) *Remarks on Irrotational Motion.*—The angular velocity ω has three components (the rotations about the axes of co-ordinates):

$$\omega_x = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),$$

$$\omega_y = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),$$

$$\omega_z = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

If these rotations are all to vanish, we must have $\partial v/\partial x = \partial u/\partial y$, and so on. If we now introduce the velocity potential Φ, i.e. put $u = \partial\Phi/\partial x$, $v = \partial\Phi/\partial y$, $w = \partial\Phi/\partial z$, these relationships are identically true (the one written down gives $\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial x}\right)$, which, as is well known, always holds good if Φ is a

continuous function of several variables). Using $\partial v/\partial x = \partial u/\partial y$, &c., we may transform equation (8) of § 3 (p. 41) as follows:

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} \\ &= \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2+v^2+w^2}{2}\right).\end{aligned}$$

If we now multiply the three Eulerian equations by dx, dy, dz, respectively, and add, the terms of the resulting equation are all integrable without any restriction on the path of integration, and we obtain

$$\frac{\partial \Phi}{\partial t} + \frac{u^2+v^2+w^2}{2} + \mathbf{P} + U = \text{const.} \quad . \ . \ . \quad (18a)$$

As the integration is performed with the time remaining constant, the constant on the right-hand side may have different values at different times (e.g. the pressure may vary owing to outside influences), so that it is more correct to replace the constant by $f(t)$. The expression $\partial\Phi/\partial t$ is obtained by using the facts that

$$\Phi = \int (u\,dx + v\,dy + w\,dz) \quad \text{and} \quad \int \frac{\partial u}{\partial t} dx = \frac{\partial}{\partial t}\int u\,dx, \text{ &c.}$$

For steady flow the above equation becomes the ordinary expression of Bernoulli's theorem.

9. Further Details of Irrotational Motion.

Formulæ giving the relationship between the velocity potential Φ and the velocity components u, v, w may be deduced from equation (16) of § 8 (p. 57), putting ds equal to dx, dy, dz respectively: we have

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}. \quad . \ . \ . \quad (19)$$

Substituting in the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(equation (4), p. 37), we obtain

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \quad . \ . \ . \ . \quad (20)$$

This equation, known as Laplace's equation, likewise occurs in connexion with electrostatic potentials (and elsewhere) * and holds in

* Also e.g. in the case of flow with friction between parallel plates close to each other: this phenomenon is often utilized in order to exhibit the stream-lines of irrotational motion. Under suitable conditions the results are the same as for irrotational motion, although the forces acting are quite different; cf. Chapter III, § 9 (p. 150).

that case for those parts of the field where no charges are situated (and the dielectric constant does not vary). Hence the solutions of equation (20) already known from electrostatics may be used here also, e.g. the solution for a point-charge or an electric doublet (dipole). In this connexion the fact that the sum or difference of two solutions gives another solution, as is obvious from the linear nature of equation (20), is of great practical importance. In this " superposition " of two potentials the velocities are compounded according to the parallelogram law.

Some Examples of Irrotational Motion.

(*a*) *The Flow against a Fixed Plane.*—One of the simplest distributions of potential we can assume is given by

$$\Phi = (ax^2 + by^2 + cz^2)/2.$$

$a + b + c$ is then equal to zero, by equation (20). If we desire the distribution to be symmetrical about the z-axis, we may put $b = a$, $c = -2a$, which satisfies equation (20), and we have

$$\Phi = \frac{a}{2}(x^2 + y^2 - 2z^2),$$

which gives $u = ax$, $v = ay$, $w = -2az$. The stream-lines in the yz-plane ($x = 0$) are given by the differential equation

$$\frac{dz}{dy} = \frac{w}{v} = -\frac{2z}{y},$$

which is satisfied by

$$\log_e z = \text{const.} - 2 \log_e y$$

or

$$y^2 z = \text{const.}$$

(the so-called cubical hyperbola, see fig. 2.32).

If the motion is steady, i.e. if a is constant as time goes on, the pressure is given by

$$p = \text{const.} - \frac{\rho}{2}(u^2 + v^2 + w^2)$$

$$= \text{const.} - \frac{\rho a^2}{2}(x^2 + y^2 + 4z^2).$$

That is, the pressure is a maximum at $x = y = z = 0$, and the equipotential surfaces are ellipsoids with their principal axes in the ratio $1 : 1 : \frac{1}{2}$ (cf. fig. 2.32).

(*b*) *Sources and Sinks.*—According to the remarks made in connexion with equation (20), the known solutions for electrostatic potential are also solutions for possible irrotational motions of fluid

provided that the boundary conditions for the latter happen to be realizable. Even so simple an electrostatic field as that of a point charge gives an important case of fluid motion, that due to a " source " or " sink ". The potential is $\Phi = \pm c/r$, where r denotes the distance from a point O. It is therefore constant over the surfaces of spheres described with O as centre, and the velocity, being normal to the equipotential surfaces, is purely radial and of magnitude $|c|/r^2$. The quantity of fluid flowing in unit time through the sphere of radius r (area of surface $4\pi r^2$) is given by $Q = 4\pi r^2 . c/r^2 = 4\pi c$. In the case of a " source " this quantity appears at the point O every second and in the case of a " sink " disappears there. This cannot be exactly realized physically, but fluid may, for example, be sucked away through a narrow tube at O, and we may then expect to obtain a flow approximately of the form described above in the neighbourhood of the tube (only approximately, as disturbing effects due to the finite volume of the tube, &c., enter in).

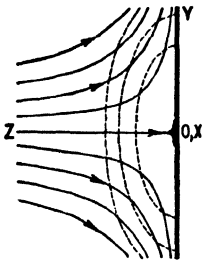


Fig. 2.32.—Flow against a fixed plane

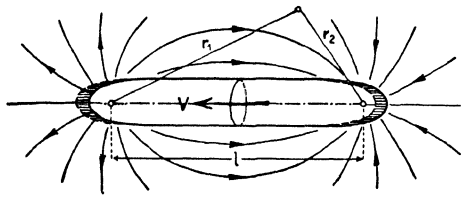


Fig. 2.33.—Irrotational flow round a moving body (system of co-ordinates at rest)

Another very useful application of sources and sinks is as follows. If a bar-shaped body moves onwards with the velocity V in the direction of its long axis, fluid is continually compressed at the front end and flows together in the space left behind (cf. fig. 2.33). Thus the motion in the neighbourhood of the front of the body is as if there were a source there, and that near the rear of the body as if there were a sink there. In fact the fluid motion is accurately represented by the formula

$$\Phi = c\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

(strictly speaking, for this to be true the ends of the bar must have a definite and well-rounded shape, but even when they are of a different shape the equation forms a useful approximation). Q, the " strength " of the source or sink, is to be taken as equal to aV, where a denotes the cross-section of the bar, so that $c = aV/4\pi$. If we consider this motion (which in the first instance, owing to the continual advance of the bar and the distribution of velocity surrounding it, is not a steady

one) from the point of view of a system of co-ordinates moving with the body, we get a steady motion in which the body is at rest and the stream of fluid flows past it. This flow is represented mathematically by the potential $\Phi' = \Phi + Vx$; its stream-lines are shown in fig. 2.34, and the distribution of pressure over the surface of the body (calculated from Bernoulli's equation) is shown below.*

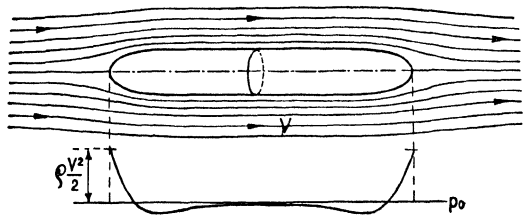


Fig. 2.34.—Flow round a moving body (system of co-ordinates moving with the body). Below, the distribution of pressure

If we bring the source and sink closer and closer together and increase their strengths in proportion as we diminish the distance between them we obtain a "doublet" or "dipole" in the limit, and the flow shown in fig. 2.34 passes into that round a sphere (cf. fig. 2.35). Here the potential is $\Phi = Vx(1 + a^3/2r^3)$, where a is the radius of the sphere. (It is worth noting that, owing to frictional effects, the actual flow round a sphere appears somewhat different.)

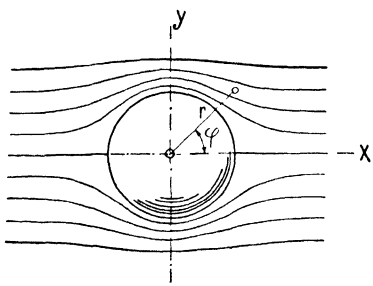


Fig. 2.35.—Irrotational flow round a sphere

(c) *Two-dimensional Motion.*—If the motion is such that all the stream-lines are plane curves in planes exactly parallel to one another, and that the flow is the same at every point on a straight line perpendicular to the set of parallel planes, we speak of *two-dimensional motion.* If we take one of these planes as the xy-plane, this means that the last of the three velocity components u, v, w is equal to zero and the two former are functions of x and y alone. The mathematical theory of such fluid motions has been very extensively developed, as here a particularly powerful method is available. It is found, in fact, that both the real and the imaginary parts of any analytic function of the complex variable $x + iy$ represent potentials satisfying equation (20) (p. 61).

* The flow about other slender bodies which are surfaces of revolution can be represented by a continuous distribution of sources along the axis; cf. footnote, p. 192.

Let the complex variable be $z = x + iy$, and let the function be $F(z)$, its real part Φ, and its imaginary part Ψ, i.e. $F = \Phi + i\Psi$.

Then $$\frac{\partial F}{\partial x} = \frac{dF}{dz}\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial F}{\partial y} = \frac{dF}{dz}\frac{\partial z}{\partial y},$$

but as $$\frac{\partial z}{\partial x} = 1 \quad \text{and} \quad \frac{\partial z}{\partial y} = i,$$

we have $$\frac{\partial F}{\partial x} = \frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y} = \frac{1}{i}\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial y}.$$

Equating real and imaginary parts and remembering that $1/i = -i$, we have

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} (= u) \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} (= v). \quad . \quad . \quad (a)$$

Hence $$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0.$$

As $\partial^2\Phi/\partial z^2$ is identically equal to zero, this means that the equation of continuity [(20), p. 61] is satisfied. Moreover, Ψ likewise satisfies the equation $\partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 = 0$, so that Ψ may also be taken to represent a velocity potential. From the equations (a) it is easy to see that at every point the two directions of flow are at right angles to one another and the two velocities of flow are equal.* Hence the equipotential lines of one flow are the stream-lines of the other (the velocity being always normal to the equipotential surface). The function which is constant along a stream-line is called the stream function. If Φ is the velocity potential, Ψ is the stream function. The stream function has another physical meaning, as the difference between the values of the function at two different points represents the volume flowing in a layer of unit depth between the two points in unit time.

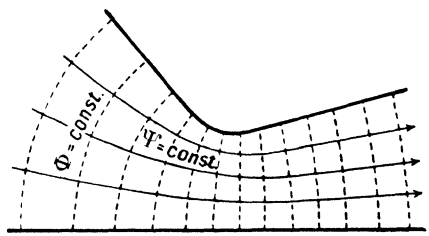


Fig. 2.36.—Graphical construction for Φ and Ψ

These properties of equipotential lines and stream-lines form the basis of a graphical method for obtaining the two sets of lines forming a network of small squares for given boundary conditions. We begin with a rough sketch

* The directions of the gradients, α and β, are given by $\tan\alpha = \frac{\partial\Phi}{\partial y} \Big/ \frac{\partial\Phi}{\partial x} = v/u$ and $\tan\beta = \frac{\partial\Psi}{\partial y} \Big/ \frac{\partial\Psi}{\partial x} = -u/v$, i.e. $\tan\beta = -1/\tan\alpha$. In both cases the value of the gradient is $\sqrt{(u^2 + v^2)}$.

of the stream-lines, draw a system of lines orthogonal to them, and improve the sketch until the network is sufficiently square everywhere. Criteria for this are equality in length of the lines joining the mid-points of opposite sides, and orthogonality of the two families of diagonal curves, drawn through the corners of the squares, which satisfy the equations $\Phi + \Psi = \text{const.}$, $\Psi - \Phi = \text{const.}$ (Figs. 2.37, 2.38, 2.41–3, 2.49 and 2.59 were obtained in this way.) Fig. 2.36 should be studied as an introductory example.

The following may be adduced as simple examples of flow in two dimensions. Two-dimensional flow against a fixed plane is represented by the function $F = \frac{1}{2}az^2$:

$$\Phi + i\Psi = \tfrac{1}{2}a(x^2 + 2ixy - y^2),$$

so that

$$\Phi = \tfrac{1}{2}a(x^2 - y^2),$$

$$\Psi = axy.$$

The stream-lines $\Psi = \text{const.}$ are rectangular hyperbolas, and the velocities are

$$u = \partial\Phi/\partial x = ax; \quad v = \partial\Phi/\partial y = -ay.$$

The flow due to a two-dimensional source is obtained by putting $F = b\log z$: $\log z = \log r + i\phi$, where r, ϕ are polar co-ordinates (so that $\Phi = \text{const.}$ on the circles $r = \text{const.}$, $\Psi = \text{const.}$ on the straight lines $\phi = \text{const.}$).

As a third example we shall discuss the flow in the angle formed by two plane boundaries meeting at an angle α. If the origin is at the point of intersection and the x-axis along one boundary, the function F is given by az^n/n, where $n = \pi/\alpha$. Introducing polar co-ordinates, we have $z = x + iy = r(\cos\phi + i\sin\phi)$, and by a well-known theorem

$$z^n = r^n(\cos n\phi + i\sin n\phi).$$

Hence the stream function is $\Psi = \dfrac{ar^n}{n}\sin n\phi$, so that $\Psi = 0$ for $\phi = 0$, π/n, $2\pi/n$, &c., i.e. for $\phi = 0$, α, 2α, ... The form of the stream-lines for different values of α is shown in fig. 2.37. For $\alpha < \pi$ the velocity at the origin is zero, for $\alpha > \pi$ it is infinite. The passage to the limit $\alpha = 0$, i.e. to a flow between parallel boundaries, gives

$$F = a'e^{\mu z} = a'e^{\mu x}(\cos\mu y + i\sin\mu y),$$

h, the distance between the two boundaries, being equal to π/μ. The flow

$$F' = a'e^{-i\mu z} = a'e^{\mu y}(\cos\mu x - i\sin\mu x),$$

which is the same flow turned through a right angle, serves to represent wave phenomena; cf. fig. 2.59 (p. 87).

The flow about a circular cylinder of radius r_1 is given by $F = V(z + r_1^2/z)$. According to the above, $\Psi = V \sin\phi(r - r_1^2/r)$. This vanishes at points on the x-axis, where $\sin\phi = 0$, and on the circle of radius r_1, where $r - r_1^2/r = 0$. The stream-line diagram closely resembles fig. 2.35.

The number of examples may be extended indefinitely. There are also a number of special methods for finding suitable solutions. For example, the complex relationship $z = f(\zeta)$, where $\zeta = \xi + i\eta$ is another complex number, is such that to every pair of values ξ, η there corresponds one pair of values x, y. Thus to every point of the

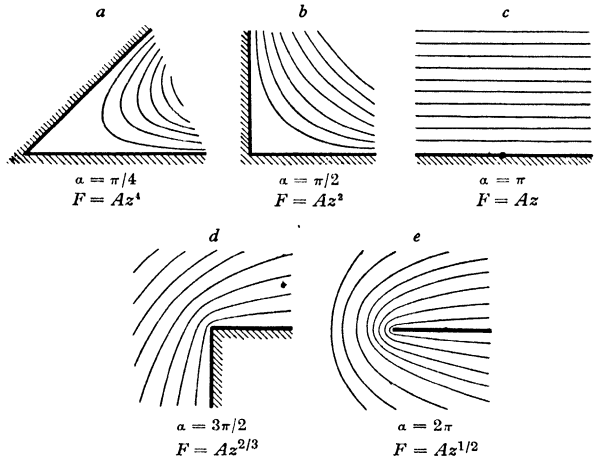


Fig. 2.37.—Cases of flow for which $F = Az^n$

$\xi\eta$-plane there corresponds one point of the xy-plane; one plane is said to be *represented* on the other. (A line corresponds to a line, and the intersection of two lines corresponds to the intersection of the corresponding lines.) In fact, relationships similar to those of equations (*a*) (p. 65) hold. An orthogonal network becomes an orthogonal network (but in general a curvilinear one) and the scale of representation is the same in both directions, so that infinitesimal elements of area are represented by geometrically similar elements. This kind of representation is accordingly known as *conformal representation*.*(2) The above examples of flow in two dimensions at once become examples of conformal representation if we write ξ, η for Φ, Ψ. The last example shows among other things how half the $\Phi\Psi$-plane may be represented on a region which is bounded by two portions of the x-axis and a semicircle of radius r_1 joining their ends.

* See A. Betz, *Konforme Abbildung* (Springer, Berlin, Göttingen, Heidelberg, 1948).

Now comes an additional fact which is of great importance in hydrodynamics. If F is an analytic function of z and z an analytic function of ζ, F is also an analytic function of ζ, i.e. $F = \Phi + i\Psi$ gives a possible flow in the ζ-plane. Hence if we have a possible flow in the xy-plane, any conformal representation of the xy-plane on the $\xi\eta$-plane will give a flow in the $\xi\eta$-plane.

Various methods are known, e.g. for the representation of the region outside a wing-like profile on the region outside a circle. It is then possible to deduce a flow about the wing or aerofoil from the flow about the circle, and so on.*(3)

As an easy calculation shows, $dF/dz = u - iv$. If we call this quantity (the complex quantity conjugate to the complex velocity $u + iv$) w, $w = dF/dz$ is likewise an analytic function of z or of F, so that the relationship between the $\Phi\Psi$-plane and the uv-plane is again that of conformal representation. In some cases statements may be made about the velocities which are sufficient to determine the region of flow completely in the w-plane. If, for example, a jet flows out through a slit between plane walls, the direction of the boundary stream-line is given so long as it flows along a wall. The direction is not known for the boundary of the free jet, but the magnitude of the velocity is constant when the pressure is constant, in view of Bernoulli's equation. The region of flow is accordingly marked off (cf. fig. 2.38 *a* and *b*) and it is merely necessary to take proper note of the singularities which occur in order to obtain F as a function of w. If w is then expressed as $w(F)$, a function of F, we have $dF/dz = w(F)$ or $z = \int dF/w(F)$. Finally by separating real and imaginary parts we obtain x and y for every value of Φ and Ψ, i.e. the stream-line diagram.

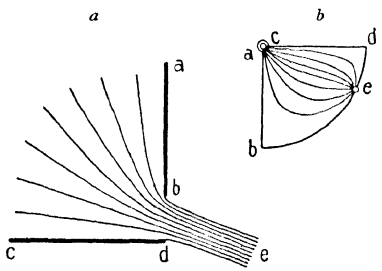


Fig. 2.38 *a*, *b*.—Flow from a slit: stream-lines and corresponding velocity diagram

This short survey may suffice to give the reader an idea of the use of the complex variable.

10. Irrotational Motion with Circulation; the Lift on an Aerofoil; the Magnus Effect.

Although in all irrotational motions the circulation in every *small* region vanishes, there are cases where, taken all over, circulation does exist. For this to occur it is of course necessary that the region in

* Details and extensive references to the literature are to be found in H. Schmidt, *Die Aerodynamik des Fluges* (Berlin and Leipzig, 1929), and W. Müller, *Mathematische Strömungslehre* (Berlin, 1928).

which the flow takes place should be multiply-connected, i.e. should be such that curves exist which cannot be contracted to zero by continuous alteration without passing outside the region.* If the circulation along such a curve is Γ, then, provided the motion is irrotational (i.e. provided the circulation vanishes in every simply-connected part of the region), it is easy to show that the circulation along any other curve which may be formed from the first by continuous deformation without passing outside the region is likewise equal to Γ. Thus the velocity potential, which, of course, we defined as a line-integral taken between the point in question and a fixed point, is " many-valued " for a flow of this kind; it decreases by the amount Γ for every revolution.

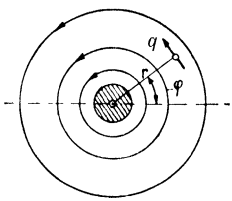


Fig. 2.39.—Irrotational flow with circulation

The simplest case of two-dimensional flow of this kind is given by the potential $\Phi = -c\phi$, where ϕ denotes the argument (cf. fig. 2.39). This potential, which satisfies equation (20), p. 61,† decreases by $2\pi c$ for each revolution ($\phi_2 = \phi_1 + 2\pi$); $2\pi c$ therefore represents the circulation Γ. Here the equipotential surfaces are planes through the axis, so that the stream-lines are circles. The velocity q ($d\Phi/ds$) is equal to c/r, since $ds = r\,d\phi$, so that the flow is the same as in the example considered in § 5(*b*) (p. 46). For $r = 0$ the formula would give $q = \infty$, so that the flow only has a physical meaning if it is bounded by a central core of finite diameter (shown shaded in fig. 2.39). This core may consist either of a solid body, or of fluid in rotation (in which no velocity potential exists), or, finally, of some other (less dense) fluid which takes no part in the rotation, e.g. if the circulating fluid is water, the core may consist of air (free vortex). Owing to the action of gravity the surface of the free vortex takes the form shown in fig. 2.40, the equation of which, by Bernoulli's theorem, is

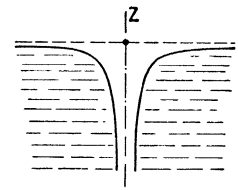


Fig. 2.40.—Free vortex

$$z = z_0 - q^2/2g = z_0 - c^2/2gr^2.$$

Such funnels are often to be seen in running water, during the emptying of a bath, and so on. In all such cases the moving fluid must previously have possessed circulation.

A different application of irrotational motion with circulation is made in the theory of the lift on an aerofoil. The flow round an aerofoil (fig. 2.41) may be represented by " superposing " the " ordinary

* A room with a pillar in the middle and the space surrounding a ring are examples of doubly-connected regions.

† In complex notation $F = -ic \log z$.

irrotational flow" (fig. 2.42) without circulation and the circulatory flow of fig. 2.43. Thus the resulting flow also exhibits circulation, which is very closely related to the occurrence of a lifting force: even without calculation it is easy to see that the circulatory flow acts with the irrotational flow of fig. 2.42 above the aerofoil and against it below. By Bernoulli's theorem this means that the pressure above the aerofoil

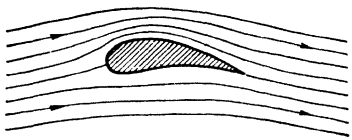


Fig. 2.41.—Flow round an aerofoil

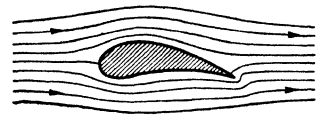


Fig. 2.42.—Simple irrotational flow round an aerofoil

is diminished and that below the aerofoil is increased, i.e. there is a lifting force. Kutta and Joukowski independently found by theoretical calculations that this force is accurately proportional to the circulation Γ, and in fact is equal to $\rho\Gamma V$ per unit length, where V is the velocity of the aerofoil relative to the fluid. A proof of this theorem will be given in § 13(*a*) (p. 83).

According to Thomson's theorem, no circulation can arise in the case of motion starting from rest, even in multiply-connected space, for in the state of rest the circulation along every line is zero, and must remain zero. In actual fact the circulation usually results from the formation of a surface of discontinuity. In the spiral volute chamber of fig. 2.12 (p. 47), for example, an eddy like that shown in fig. 2.19 (p. 52) is formed at the sharp edge K when the motion begins. The eddy subsequently flows away at O and during the rest of the flow only its circulation remains.

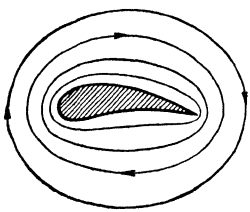


Fig. 2.43.—Circulatory flow round an aerofoil

The circumstances are very similar in the case of an aerofoil. When the motion begins, a surface of discontinuity like that in fig. 2.44 is formed at the rear edge of the aerofoil, and the fluid originally flows past it as in fig. 2.42. Subsequently the eddy arising from the surface of discontinuity moves on and leaves the aerofoil with a circulation equal and opposite to its own. (The fluid lines (p. 56) which enclose both the aerofoil and the eddy still have zero circulation, as required by Thomson's theorem.)



Fig. 2.44.—An aerofoil beginning to move

Note.—In order that the space round the aerofoil may be doubly-connected the aerofoil must either be bounded laterally by two parallel walls, or must be assumed to extend to infinity both ways. Neither is true of an actual aerofoil. The circulation round the aerofoil, which exists in the latter case also and is

necessary for the existence of a lifting force, is then due to a surface of discontinuity of the kind described in § 6 above (p. 50), with a transverse discontinuity in the velocity (for more details see Chapter III, § 15, p. 188).

A circulation similar to that round an aerofoil is also produced (owing to frictional effects: see Chapter III, § 7, p. 142), when fluid flows transversely across a circular cylinder rotating about its axis, and in this case gives rise to a transverse force of magnitude $\rho\Gamma V$ per unit length in a direction at right angles to the flow. A force of this kind also occurs with triangular or rectangular prisms rotating about their long axes, with bullets, and so on. The direction of the force is always from the side where the circulation and the flow oppose each other towards that where they reinforce each other. This phenomenon is known as the *Magnus effect* after its discoverer Magnus (1852).

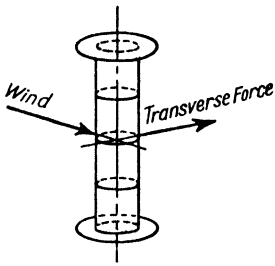


Fig. 2.45.—Rotating cylinder

The impulse to study this group of phenomena came from the behaviour of bullets, which frequently receive unintentional rotations about transverse axes as they leave the barrel. These lateral deviations may also be observed in the flight of " cut " tennis and golf balls through the air, and may be very considerable. The phenomenon was employed by Flettner on a large scale in order to drive a ship by means of the wind, using rapidly rotating vertical cylinders instead of sails (rotor ship). It is advantageous to add projecting discs at the ends (cf. fig. 2.45), as otherwise the air at the ends of the cylinders which is not moving round with them is forced into the low-pressure region at the suction side, where it tends to disturb the flow.*

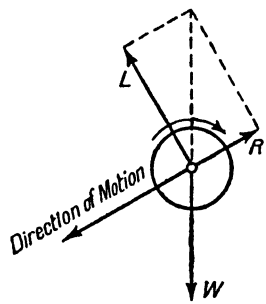


Fig. 2.46

Experiment.—A rotating cylinder driven by a small electric motor is placed on a small truck running on rails. If air is blown on to the cylinder in a direction at right angles to the rails by means of a small screw fan (table fan), the truck runs along the rails. If the fan is turned so that the wind makes a different angle with the rails, we can study the behaviour of the cylinder-sail under various courses. It is possible to make the truck run almost in the teeth of the wind. If the direction of revolution is reversed the truck runs in the opposite direction.—When a light cylinder in rapid

* Experiments with ships of this type were satisfactory in themselves, but the ordinary motor-ship was superior from the economic point of view, so that the Flettner drive did not catch on.

rotation with its axis horizontal is let drop it does not fall vertically but glides down smoothly. In addition to the lift (L) at right angles to its path it is subject to a resistance or drag (R) in the direction of its path; in the most favourable case, however (that of an elongated cylinder with discs at the ends), the drag is considerably smaller than the lift. The resultant of these two forces balances the weight of the cylinder (fig. 2.46) and so hinders its fall.

11. Rotational Motion of a Frictionless Fluid. Vortex Filaments.

In studying the rotational motion of a homogeneous frictionless fluid we may again take Thomson's theorem on the constancy of the circulation along a fluid line (p. 56) as our guide. From this theorem and the geometrical properties of the vector representing the rotation we may deduce theorems which bring out certain important geometrical and mechanical relationships. These are Helmholtz's well-known theorems on vortex motion, which he discovered by another method, starting from electrodynamical considerations.* The results which follow from these theorems, however, are complicated unless the region in which the fluid is in rotation has a thread-like form and the rest of the fluid is free from rotation. In such a case we speak of a *vortex filament*. The most important theorems about vortex filaments may, however, be deduced from the behaviour of the surrounding irrotational motion, so that we need not go further into the mechanics of rotating fluid here.

When we were discussing irrotational motion with circulation in § 10 (p. 68)—we are dealing with a similar case here—we mentioned that the region in which this irrotational motion takes place is multiply-connected and that the circulation for all curves which may be deformed into one another without crossing the boundary of the region must be the same. For vortex filaments this means that they must form a closed ring or else must extend on either side to the boundary of the fluid, and that the magnitude of the circulation round them at any given instant must be the same at all points. These properties, which are closely related to the geometrical properties of irrotational motion, may also be explained by considering the closed curve shown in fig. 2.47. The line is chosen in such a way that it can be contracted to zero without touching the region of the vortex filament, so that the circulation about it is certainly zero. But it consists of two portions which run in opposite directions along one and the same line AB, so that their contributions cancel each other, and of two other portions which encircle the vortex filament at A and B. As the total circulation is zero, the

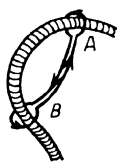


Fig. 2.47

* H. von Helmholtz, *Crelles Journ.*, Vol. 55, p. 25 (1858): *Ostwalds Klassiker*, No. 79.

contributions of these latter must also be equal and opposite. The curve encircles the vortex filament at A in the direction opposite to that in which it does so at B, so that the circulation taken in the same direction is of the same magnitude at A as at B. If the vortex filament came to an end at any point in the fluid, one of the loops could be slipped over the end of the filament, leaving the other in position round the filament. The former would then have given up its circulation while the latter would have retained it. The circulation round the whole curve would thus be changed although it had been displaced only within an irrotational field of flow. This is clearly impossible. Hence we have the theorem: *A vortex filament cannot end at any point within the fluid, and the circulation about it is everywhere the same.* To this purely geometrical statement we may add the dynamical statement following from Thomson's theorem, that *the circulation about the vortex filament remains unaltered as time goes on.*

If we consider the behaviour of very small closed fluid lines whose circulation always remains equal to zero so long as they remain in the region where a velocity potential exists, but in general differs from zero in the interior of the vortex filament, it follows from Thomson's theorem that the circulation does not change as time goes on. From this it follows immediately that *the vortex filament always consists of the same particles of fluid.* As the momentum and energy of the vortex filament itself are small compared with the corresponding quantities for the irrotational flow surrounding it, the vortex filament for the most part moves in a way prescribed by the motion of the surrounding fluid [cf. example (*a*) below]. The irrotational motion can of course be geometrically related to the circulation about the axis of the vortex filament; this is very often convenient for purposes of calculation. The motion of each element of the vortex filament then appears as due to all the other elements of the vortex filament and the whole irrotational flow as due to the vortex filament. This, however, is merely to be regarded as a geometrical relationship; as regards the energy, the external motion plays the predominating part.

The formulæ by which the velocity field may be calculated from the geometrical configuration of the vortex filaments are, moreover, in complete agreement with the electrodynamic relationships between an electrical conductor and a magnetic field as expressed in Biot and Savart's law. Here the vortex filament corresponds to the conductor, and the circulation and velocity field to the current and its magnetic field respectively. The current density corresponds to the strength of the rotation. Like the circulation, the current is the same at all points of the conductor, so that the current density is inversely proportional to the cross-section of the conductor. In the vortex filament we have $\Gamma = 2\omega a$, by the paragraph in small type on p. 58

(with $\sin\alpha = 1$), so that a decrease in a means an increase in ω and conversely. This is also true for all time. If a part of the vortex filament is stretched, its angular velocity ω increases in inverse proportion to its cross-section. The length of a section of the vortex filament also increases in inverse proportion to a, since the volume of course remains the same. Thus *the angular velocity alters in direct proportion to the length of the section.*

The above gives the gist of Helmholtz's theorems.

Examples.—(*a*) *Straight Parallel Vortex Filaments in a Fluid otherwise devoid of Circulation.*—A vortex filament with circulation Γ (of strength Γ) is surrounded by a velocity field as in the irrotational motion with circulation described in § 10, p. 69. The velocity is in a direction at right angles to the axis of the vortex and to the radius vector (r) drawn from the axis of the vortex to the position of the particle in question, and is of magnitude $\Gamma/2\pi r$. If there are several vortex filaments, the separate velocity fields are superposed and each vortex filament takes part in the motion which the others produce at the place where it is situated. Thus if there are two parallel vortex filaments, the vortex filaments rotate about an axis coinciding with the line of the resultant force which would occur if the vortex filaments were replaced by forces proportional to their respective strengths; the forces are to be taken in the same direction if the vortices are rotating in the same direction and in opposite directions if the vortices are rotating in opposite directions. In the one case the axis of rotation lies between the vortex filaments, in the other outside them. If two vortices of equal strength are rotating in opposite directions (a " vortex pair ", corresponding to a couple), there results a motion in a straight line at right angles to the line joining the two vortices, the velocity being $\Gamma/2\pi d$, where d is the distance between the axes of the two vortices. From the point of view of a system of co-ordinates at rest relative to the undisturbed fluid the stream-lines of a vortex pair are as shown in fig. 2.48, whereas with respect to a system of reference moving with the axes of the vortices they are as shown in fig. 2.49. The region shaded in the latter figure remains permanently in the neighbourhood of the vortex filaments. If the latter are produced in a region where the fluid is coloured [see (*d*), p. 76], a quantity of coloured fluid corresponding to the region shaded in fig. 2.49 moves into the uncoloured liquid.

Note.—For the above relationships to hold accurately the parallel vortex filaments would theoretically have to extend to infinity both ways or, in the case of a fluid bounded by two parallel boundaries at right angles to the axes of the vortices, from one wall to the other. In the latter case, however, there would be disturbances owing to friction at the boundaries. The bottom of a vessel and the free surface of the fluid may also be taken as boundaries.

(*b*) *Circular " vortex rings ".*—Here the velocity of advance is in-

creased owing to the curvature of the filament; the more concentrated the rotating core of the vortex, the greater is this increase in velocity. (This phenomenon may be deduced correctly from Helmholtz's laws as an effect due to neighbouring elements of the vortex, but may also be explained dynamically as an effect due to the stresses arising in the vortex ring owing to the centrifugal force of the rotating fluid and tending to make the vortex ring smaller: cf. the remarks on the Kutta-Joukowski theorem on p. 70.) Owing to the increased velocity of advance the fluid carried along by the vortex ring is usually ring-shaped also (e.g. smoke rings), in contradistinction to

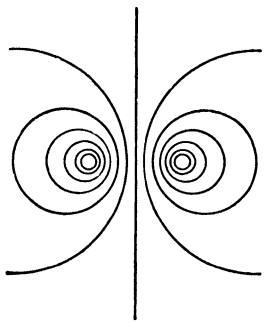


Fig. 2.48.—Stream-lines of a vortex pair; co-ordinate sytem at rest

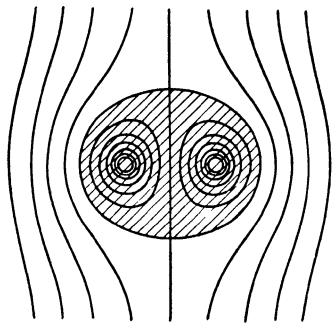


Fig. 2.49.—Stream-lines of a vortex pair; co-ordinate system moving with the axes of the vortices

fig. 2.49. Two vortex rings with a common axis, moving in the same direction, have the following mutual effects on each other. The front one expands while the rear one contracts, with the result that the velocity of the former decreases and that of the latter increases. The rear one finally slips through the front one and the process begins again with the rôles reversed. Two equal vortex rings with a common axis moving towards each other increase in diameter more and more rapidly as they approach, their motion being thereby slowed down so much that they do not come into actual contact. In this case the plane half-way between the two vortex rings behaves like a solid boundary, so that the same phenomena occur when a vortex ring impinges on a wall.

For the numerical calculation of the velocity fields of vortex rings of any shape we may use formulæ which have been developed for the calculation of the magnetic fields due to electrical circuits; the velocity of flow corresponds to the strength of the magnetic field and the strength of the vortex to the current strength. For details see e.g. Prandtl-Tietjens, Vol. I, pp. 187–192. A simple account is given by A. Betz.*

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 8, p. 149 (1928).

(*c*) *Surfaces of Discontinuity.*—The surfaces of discontinuity described in § 6 (p. 50), may also be thought of as consisting of a number of vortex filaments arranged to form a sheet in such a way that the axes of the vortex filaments are at right angles to the direction of the vector representing the discontinuity in the velocity. That a set of parallel vortex filaments actually does give a flow which at some distance from the vortex filaments is the same as the flow when a surface of discontinuity is present, is shown by the diagrammatic sketch in fig. 2.50. Conversely, surfaces of discontinuity are unstable and readily break down into isolated vortices (cf. fig. 2.17, p. 51).

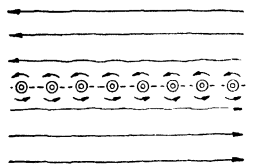


Fig. 2.50.—Vortex sheet

(*d*) *The Production of Vortices.*—Vortices, together with the circulatory irrotational motion associated with them, usually arise from surfaces of discontinuity. All irrotational motions are due to the effects of pressure exerted on the fluid by some part of its boundaries or free surface or by bodies in its interior. Circulatory motion chiefly occurs when one part of an area in the interior of the fluid has been subjected to pressure for a time while the adjacent parts have not. This is exemplified by the production of a vortex ring in fig. 2.21 (p. 52), where the wall round the hole offers resistance to the pressure acting on it from the left, and the hole does not. Another important example is the motion of an aeroplane wing, in which the strip of air over which the wing passes is temporarily subjected to the weight of the aeroplane, whereas the region outside this strip is subjected to no such pressure. As we have already stated on p. 53 in connexion with fig. 2.22, the surface of discontinuity behind the wing gives rise to two vortices starting from the extremities of the wing. A vortex like that in fig. 2.44 is produced at the point where the motion began. This "starting vortex" and the lateral vortices together form one (usually rather diffused) vortex filament. The wing itself, with its circulation, which is exactly the same as that round the vortex filament, makes the whole arrangement into a kind of vortex ring, part of which, however, is formed by a solid body. From the kinematical point of view, the matter may be regarded as follows: part of the vortex filament is situated in the interior of the solid body, in this case producing only irrotational motion with circulation outside. From the dynamical point of view, however, we have to assume that Helmholtz's dynamical theorem does not hold for this part of the vortex filament, owing to its motion with the solid body, and, in particular, that new particles of fluid are continually brought into the lateral vortex filaments from its ends. The quantitative investigation of these vortex motions has yielded important conclusions about the behaviour of aerofoils; cf. especially Chapter III, §§ 15, 16 (p. 188).

12. Momentum Theorems for Steady Motion.

The fundamental theorems of general dynamics which give the rates of change of momentum and of angular momentum in terms of the applied forces and couples apply, in particular, to the steady motion of fluids, and also to motions which, though not steady, may on the average be regarded as steady. The value of these theorems lies in the fact that they contain statements about conditions *along the boundaries* of a region only, so that they may be used to draw conclusions about phenomena which have not been completely mastered in detail.

By the momentum of a mass we mean the product of its mass and its velocity (momentum is a vector and therefore has three components, like the velocity). *The change of the momentum with time is equal to the resultant force acting on the mass.* In Chapter I, § 2 (p. 3), it was shown that in summing over all the masses of a mechanical system all the internal forces of the system cancel out, by the principle of action and reaction, and there remain only the external forces (due to masses not considered to belong to the system).

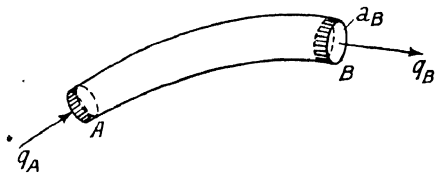


Fig. 2.51

Now the change of momentum of a mass of fluid which is bounded in any way and is in *steady* motion is entirely due to the fact that the boundaries of the fluid are displaced by the motion; every particle of the fluid in the interior is replaced by another which has taken up its velocity. What happens at the boundaries may be shown in the case of a stream-tube. It is an essential feature of our theorem that every individual mass belonging to the system shall remain in the system and that no new masses shall enter it. Thus the boundary surfaces drawn for the purpose of applying the theorem are displaced with the flow; they are "fluid surfaces". Hence in the case of our stream-tube the mass $dm_1 = \rho a_A q_A dt$ vanishes at A in fig. 2.51 during the time dt, and the mass $dm_2 = \rho a_B q_B dt$ appears at B. By continuity $dm_1 = dm_2 = dm$. In time dt the stream-tube contributes the positive amount $q_B dm$ at B to the total change of momentum, i.e. $q_B dm/dt$ in unit time, or $\rho a_B q_B^2$ (in the direction of q_B), and similarly at A the negative amount $-q_A dm/dt$, or $-\rho a_A q_A^2$ (in the direction opposite to q_A). The vector sum of these changes of momentum per unit time is now equated to the resultant of the external forces acting on the mass of fluid in question. Instead of the changes of momentum we may consider the "reactions" to which they give rise, i.e. the forces of the same magnitude but in the opposite directions. The vector sum of these reactions

is then obviously in equilibrium with the forces acting on the mass of fluid.* Thus to the flow shown in fig. 2.51 there corresponds a reaction at A of the magnitude given above *in* the direction of the entering flow and a reaction at B in a direction *opposite* to that of the flow emerging from the stream-tube.

This statement completes the transition from the fluid boundary surfaces of the mass of fluid in question to a surface fixed in space, across which changes of momentum (or their reactions) and pressures are transmitted. In order to apply the momentum theorems correctly, the mass of liquid under consideration must be surrounded by a suitable closed surface, which we shall call the " control surface " (marked by a dotted line in some of the following figures), and the reactions arising from all the stream-tubes entering or leaving it must be calculated. By the laws of statics, the forces of reaction, together with all the external forces acting on the liquid, must then form a system in equilibrium, i.e. the resultant of the forces and also the total moment of the forces about each co-ordinate axis must vanish.† Frequently, however, we are interested in the equation for one component only.

Note.—In the case of non-steady motion we have to add a contribution due to the change of momentum in the interior of the fluid. If, as is frequently the case with turbulent motion, the non-steady motion has a constant average momentum, the contributions from the interior of the fluid cancel out on the average, so that the momentum theorems may be used as in the case of steady motion, except that care must be taken in the formation of the averages in the control surface: cf. § 14 (p. 84).

Examples.—(*a*) *Reaction of a Fluid flowing in a Curved Channel.*—Suppose that the fluid flows in with velocity q_1 and pressure p_1 (see fig. 2.52), then by the above discussion the momentum carried across the area a_1 is $\rho a_1 q_1^2$; it is equivalent to a force exerted by the inflowing liquid in its own direction. In addition there is a force $p_1 a_1$ in the same direction due to the pressure. There is a corresponding force $a_2(\rho q_2^2 + p_2)$ at the outflow end of the channel, in a direction opposite to the velocity there (i.e. the force is always directed towards the interior of the control surface). The actual force exerted by the pressure of the fluid on the wall of the channel is given by the resultant of these two forces.

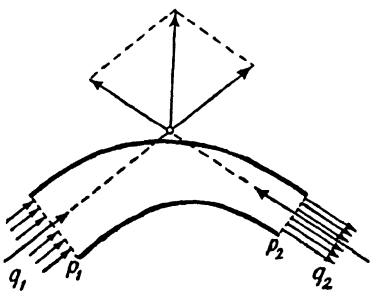


Fig. 2.52

* The argument is exactly the same as that in the introduction of the " force of inertia " in d'Alembert's principle in the dynamics of rigid bodies.

† Engineers like to consider the forces exerted by the fluid on the walls of the vessel, rather than the forces acting on the fluid; cf. Examples (*a*) and (*b*).

(*b*) *Reaction due to the Efflux of a Jet.*—A jet which flows through a hole from a region where the pressure is p_1 to a region where the pressure is p_2 carries with it momentum to the amount $I = \rho a_j q^2$ per second, where a_j is the cross-section of the jet. As $q = \sqrt{\{2(p_1 - p_2)/\rho\}}$ (cf. § 4(*b*), p. 44), we have $I = 2a_j(p_1 - p_2)$, i.e. I is equal to twice the force which would be exerted by a difference of pressure $p_1 - p_2$ on a piston with the same cross-section as the jet. As a pressure corresponding to this momentum must exist, it follows that owing to the suppression of excess pressure at the hole and the fall of pressure in its neighbourhood owing to the flow of the fluid towards the hole, there is a reduction of pressure on the wall, as compared with the case of the closed vessel, corresponding to double the cross-section of the jet. This reduction

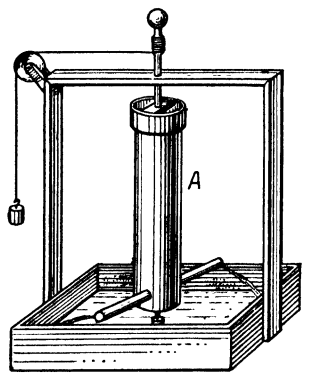


Fig. 2.53.—Segner's turbine

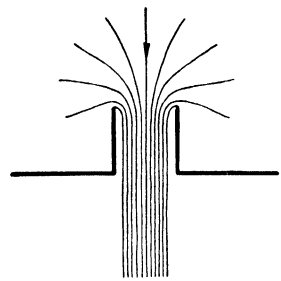


Fig. 2.54.—Borda's mouthpiece

of pressure manifests itself as a back pressure, the " reaction of the jet ", which is easily demonstrated by placing a vessel with a hole in its side on a truck which moves readily. The vessel then begins to move in the direction opposite to that of the issuing jet. A related experiment is that of Segner's turbine, sometimes called Barker's mill (fig. 2.53), which after the above requires no special explanation. The issuing water may be made to raise a weight, as in the figure, or to do any other piece of work. Water wheels of this kind were formerly constructed on a large scale also.

In a particular case, namely that of *Borda's mouthpiece* (fig. 2.54), the so-called coefficient of contraction, i.e. the ratio of the cross-section of the jet to that of the hole, can be calculated from the momentum. For as the whole excess pressure is here exerted on all the walls subject to pressures with components in the direction of the jet, the loss of excess pressure over the cross-section of the hole (a) must be equal to the momentum of the jet, so that

$$a(p_1 - p_2) = 2a_j(p_1 - p_2), \text{ or } a_j/a = 1/2.$$

(*c*) *Sudden Enlargement of a Pipe.*—If a stream of fluid with velocity q_1 passes from a cylindrical pipe into a wider pipe which is also cylindrical, the jet, being unstable (cf. § 6, p. 51), will mix with the surrounding fluid and after the mixing will flow on almost uniformly with an average velocity q_2. Here the momentum theorem enables us to calculate $p_2 - p_1$, the increase of pressure due to the mixing, without being acquainted with the details of the process. In the fluid at rest surrounding the earlier part of the jet the pressure is the same as in the jet itself [cf. Chapter III, § 5(*a*)]. For the control surface drawn in fig. 2.55 we have

$$\frac{dm}{dt}(q_1 - q_2) = a_2(p_2 - p_1),$$

as only the two end surfaces contribute to the force, or, substituting $\rho a_2 q_2$ for dm/dt,

$$p_2 - p_1 = \rho q_2(q_1 - q_2).$$

In a gradually widening pipe we should have $p_2' - p_1 = \frac{1}{2}\rho(q_1^2 - q_2^2)$ by Bernoulli's theorem, so that the sudden widening causes a loss of pressure $p_2' - p_2$, which, as may easily be verified, is of magnitude $\frac{1}{2}\rho(q_1 - q_2)^2$. As this formula is identical with that for the loss of kinetic energy at the impact of two inelastic bodies, the expression "loss due to impact" is frequently used in connexion with the sudden enlargement of a pipe, although the phenomenon has nothing to do with impact, the only thing that it has in common with impact being that the velocities are redistributed.

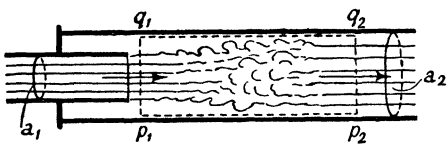


Fig. 2.55.—Sudden enlargement of a pipe

(*d*) *Floating of Heavy Bodies in Air.*—In order that a heavy body may be kept floating in air at rest, it is necessary that new masses of air should continuously be accelerated downwards. If q is their final velocity, which for the sake of simplicity may be regarded as uniform, and if $dm/dt = M$ is the mass set in motion per unit time, the force required, P, is equal to the momentum Mq, provided there are no appreciable differences of pressure in the masses moving downwards. This argument is true to a close approximation, e.g. for a hovering helicopter if the helicopter is sufficiently far from the ground. A current of air moving vertically downward with momentum Mq is then formed. If its path is sufficiently long the current of air will mix with air at rest from the surroundings and will thereby have its motion retarded. The momentum, however, remains unaltered, as the mass in motion is correspondingly increased. The weight of the

helicopter is finally transferred to the ground by the current of air impinging on the ground and losing its momentum.

In the case of the aeroplane the downward moving masses of air consist of the system of vortices left behind (cf. § 11, p. 76); but a pressure field also plays a part here, so that the relationships are more complicated. It depends on the form of the control surface whether the equivalent of the upward force is obtained in the form of momentum or of pressure. In this case also the pressure on the ground below the aeroplane is raised, the weight of the aeroplane being thereby transferred to the ground.

13. Further Examples on the Momentum Theorems.

(*a*) *A Row of Blades. The Kutta-Joukowski Theorem.*—In order to study the reaction of the blades of a turbine or propeller, &c., on the fluid streaming past them it is usual to begin by considering the simpler case of a row of blades consisting of a number of identical blades arranged parallel to each other and of infinite span (p. 198). Here the momentum theorems for the force components parallel and perpendicular to the plane of the row of blades, together with Bernoulli's theorem and the equation of continuity, enable us to draw very valuable conclusions about the relationship between the forces on the blades and the velocities of flow. Fig. 2.56 shows a row of blades and the flow past them as it appears to an observer at rest with respect to the blades. The row of blades shown corresponds to the case of a propeller: in the case of a turbine the blades are bent the other way and the force components are in the reverse direction, but the following calculations hold for either case.

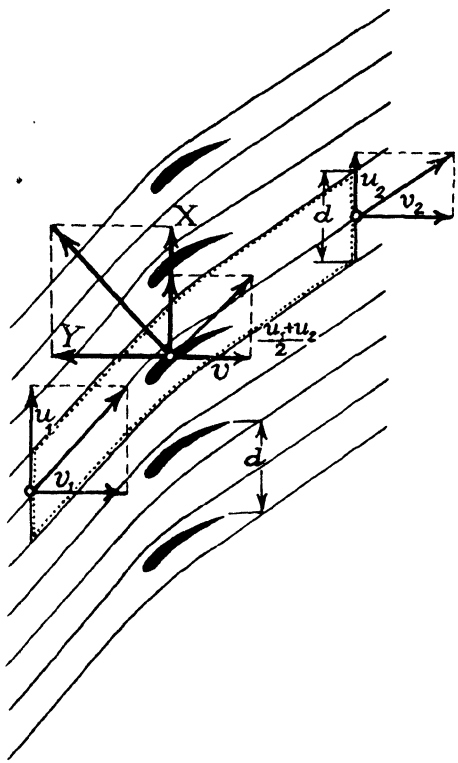


Fig. 2.56.—A row of blades

Let the components of velocity parallel and perpendicular to the row of blades be u and v, and let the corresponding forces on a blade per

unit length be X and Y (the directions shown in fig. 2.56 being taken as positive). The suffix 1 refers to inflow and the suffix 2 to outflow.

If we assume that no losses occur, we have irrotational flow with circulation round the blades. The momentum theorem makes use of the fact that under such conditions the velocity is practically constant at some distance in front of the row of blades, and also at some distance behind them. No further details need be known about what happens between the blades so long as it is assured that no large eddies are formed there (these may occur if the blades are unsuitably shaped). To begin with, continuity requires that

$$Q = v_1 d = v_2 d$$

(d being the distance between any pair of adjacent blades and Q the quantity of fluid passing per second through an area enclosed between them, of unit depth and parallel to the line of the blades). Thus v_1 is actually equal to v_2, and we shall henceforth write v for either velocity. From Bernoulli's equation (q^2, the square of the resultant velocity, being equal to $u^2 + v^2$) we have

$$p_1 + \tfrac{1}{2}\rho({u_1}^2 + v^2) = p_2 + \tfrac{1}{2}\rho({u_2}^2 + v^2)$$

or

$$p_2 - p_1 = \tfrac{1}{2}\rho({u_1}^2 - {u_2}^2). \quad . \quad . \quad . \quad . \quad (21)$$

In order to apply the momentum theorem we form a control surface with an outline (the heavily dotted line in fig. 2.56) consisting of two similarly situated stream-lines, at a distance apart equal to the distance between two adjacent blades, and two straight lines of length d parallel to the plane of the row of blades. Let the depth of all the faces of the control surface normal to the plane of the paper be unity. Nothing flows through the two stream-line surfaces, and owing to their identical properties the distribution of pressure is the same on both, so that they contribute nothing either to the momentum or to the resultant pressure. Thus we have only to calculate the contributions of the surfaces parallel to the plane of the row of blades. The mass flowing across them per second is $\rho Q = \rho v d$, so that

$$X = 0 + \rho v d(u_1 - u_2), \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$Y = d(p_2 - p_1) + 0. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

It is convenient to introduce the circulation about a blade into these formulæ. In order to calculate it, we again use the dotted outline. To begin with, the two stream-lines are traversed in opposite directions, so that their contributions are equal and opposite. The two straight pieces contribute $u_1 d$ and $-u_2 d$, so that

$$\Gamma = d(u_1 - u_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Using equation (21) and remembering that

$$u_1^2 - u_2^2 = (u_1 - u_2)(u_1 + u_2),$$

we obtain

$$X = \rho\Gamma v, \quad \ldots\ldots\ldots \quad (25)$$

$$Y = \rho\Gamma\frac{u_1 + u_2}{2}, \quad \ldots\ldots\ldots \quad (26)$$

from (22) and (23).

The fact that the ratio Y/X is equal to $(u_1 + u_2)/2v$ means, as is easily seen by considering the corresponding pair of similar triangles in fig. 2.56, that the resultant of X and Y is at right angles to the resultant velocity formed from $(u_1 + u_2)/2$ and v. If we call the resultant force R and the resultant average velocity q_m, we have

$$R = \rho\Gamma q_m, \quad \ldots\ldots\ldots \quad (27)$$

as easily follows from (25) or (26) by similar triangles.

We shall now put equation (27) to a special use. If d, the distance between each pair of neighbouring blades, is made to increase more and more, the circulation $\Gamma = d(u_1 - u_2)$ being kept constant, the difference $(u_1 - u_2)$ becomes smaller and smaller. In the limit, when d is infinitely great, it becomes zero, and the velocities in front of and behind the one blade left in a finite position are then identical, provided the distance from the blade is sufficiently great, and we may now replace the resultant average velocity q_m by q_∞, which represents the undisturbed velocity at an infinite distance. Changing the system of reference, we get the case of a blade moving with velocity q_∞ in a fluid which is at rest at infinity. Denoting this velocity by V, we obtain a force at right angles to V of magnitude

$$R = \rho\Gamma V$$

per unit length, by equation (27). Hence for a portion of blade or aerofoil of length l we obtain a transverse force or lift (L) of magnitude

$$L = \rho\Gamma V l. \quad \ldots\ldots\ldots \quad (28)$$

This is the Kutta-Joukowski theorem. It may of course be proved in other ways; for example, Joukowski proved it by applying the momentum theorem to a control surface consisting of a circular cylinder of very large radius with its axis along that of the aerofoil. In this case we get L half in the form of momentum and the remainder in the form of a resultant of pressures. The chief importance of this theorem is that it enables us to find the circulation associated with a given lift, which serves to determine the vortices left behind the aerofoil.

(*b*) *Moment of Momentum; Euler's Turbine Theorem.*—We may form moments of momenta analogous to moments of forces and for these we have a theorem analogous to that for the rate of change of momentum, namely: *The time rate of change of moment of momentum is equal to the resultant moment of the forces.* For the steady flow of a fluid it becomes (by arguments used earlier) the theorem that the moments of the external forces and of the "reactions" of the fluid are equal.

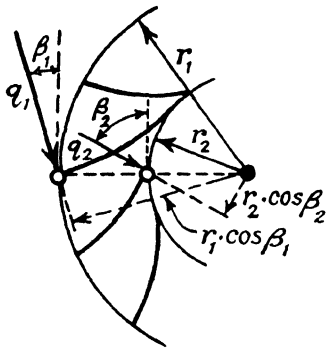


Fig. 2.57

As an example we shall deduce Euler's turbine equation. Let the mass of water flowing through a turbine per second be M (fig. 2.57). Let the absolute velocity of the entering fluid be q_1, its angle with the direction of motion of the wheel β_1, and the radius at which it enters r_1. The water flows through the revolving wheel in a direction approximately determined by the blades. The relative velocity of outflow at radius r_2 combined with the peripheral velocity of the wheel there gives the absolute velocity of outflow, q_2 in the direction β_2. Then no matter what is happening inside the wheel, the turning moment or torque exerted by the water on the turbine * is equal to

$$M(q_1 r_1 \cos\beta_1 - q_2 r_2 \cos\beta_2).$$

The most favourable working conditions, then, are when the water flows out in a radial direction, i.e. when $\cos\beta_2 = 0$ (because in that case the lost kinetic energy of the outflowing water is least). The amount of work supplied in this most favourable case is given by the product of the torque and ω, the angular velocity of the wheel, i.e. $Mr_1\omega q_1 \cos\beta_1$.

Note.—If the same theorem is applied to a circulatory fluid motion in which there is no turbine, we obviously get

$$q_1 r_1 \cos\beta_1 = q_2 r_2 \cos\beta_2,$$

or, if all the angles β are so small that $\cos\beta$ may be put equal to 1, qr = const., a result which we have already obtained in another way in § 5(*b*) (p. 47).

14. Momentum Theorem for Flow where the Velocity fluctuates.

In many practical applications it happens that the velocity in the region occupied by the flow is subject to considerable and usually irregular fluctuations, but a time-average of the velocity independent of the interval of time chosen can be stated for any point. Then the

* Here the moment of momentum per unit mass is taken as the product of q and $r\cos\beta$; it is equally justifiable to regard it as the product of $q\cos\beta$, the tangential component of q, and the radius r.

average values at different points taken together again represent a steady flow and we speak of a varying or "turbulent" flow, which, however, is "steady on the average". The phenomenon discussed in § 12(*c*) (p. 80) is an example of this. If the mean values of the velocity components are denoted by u_m, v_m, w_m, and the temporary deviations from these by u', v', w', so that the actual velocities are $u = u_m + u'$, $v = v_m + v'$, $w = w_m + w'$, the mean values of u', v', w' are zero by definition; this, however, is not true for mean values of squares and products of u', v', w'.

Now in applying the momentum theorem to such motions it is not sufficient to apply it to the steady average motion, but the mean value must be formed from the separate momenta (or reactions) crossing the fixed control surface. We shall work this out in two cases.

(1) x-component of the momentum for a portion of the control surface at right angles to the x-axis. The mass $\rho u\,dt$ passes through unit area during the time-interval dt, moving with the x-component of the velocity, namely u, so that the momentum transferred in time dt is $\rho u^2 dt$. The total effect over a longer time T is $\frac{1}{T}\int_0^T \rho u^2 dt$ per second, or more shortly, if we denote mean values by a suffix m, $\rho(u^2)_m$. Now $u^2 = (u_m + u')^2 = (u_m)^2 + 2u_m u' + u'^2$. In forming the mean value of u^2 it is to be noted that u_m already represents the mean value of u and that the mean value of u' is zero. Hence

$$\rho(u^2)_m = \rho(u_m)^2 + \rho(u'^2)_m. \qquad . \quad . \quad . \quad . \quad (29)$$

Thus we have to add to the momentum $\rho(u_m)^2$ due to the steady average motion the momentum due to the mean value of the square of the deviation (calculated, as always, as a reaction in the direction from outside to inside, i.e. having the same effect as an additional pressure).

(2) x-component of the momentum for a portion of the control surface at right angles to the y-axis. Here the mass passing through unit area in time dt is $\rho v\,dt$; the x-component of the velocity is again u, so that the x-component of the momentum transferred is $\rho uv\,dt$. In this case the mean value of the momentum per second is $\rho(uv)_m$. Since $u = u_m + u'$, $v = v_m + v'$, we have

$$(uv)_m = u_m v_m + u_m v'_m + u'_m v_m + (u'v')_m,$$

or, owing to the middle terms vanishing,

$$\rho(uv)_m = \rho u_m v_m + \rho(u'v')_m. \qquad . \quad . \quad . \quad . \quad (30)$$

Thus in this case also a term due to the variation has to be added to the momentum $(\rho u_m v_m)$ of the steady average motion. The mean value of the product of u' and v' of course need not vanish, as e.g. positive values of u' may for the most part coincide with positive values of

v' and negative values of u' for the most part with negative values of v', so that the product is predominantly positive in both cases. If, on the other hand, positive values of u' are associated with negative values of v', and vice versa, negative products predominate. The reaction corresponding to $\rho(u'v')_m$ which is exerted on unit area is a force in the direction of the x-axis acting on an area perpendicular to the y-axis, i.e. a *shearing stress*. Taking the direction of the reaction into account, this " apparent shearing stress " is given by

$$\tau' = -\rho(u'v')_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(3) We have still to investigate whether the mass of fluid in the interior of the control surface contributes anything to the momentum. It is easy to see, however, that the change of velocity in time in one and the same place, if it is not to vary on the average, must contain equally large positive and negative contributions, so that the interior on the average contributes nothing. On the whole, then, it appears that in motions which are steady on the average the fluctuations in the velocity give rise to amounts of momentum whose mean values represent stresses of the kind occurring in very viscous fluids (cf. Chapter III, § 1, p. 98). Applications of this discussion will be made in the section on turbulent flow (Chapter III, § 4, p. 111).

15. Waves on the Surface of a Liquid.

If, as is usually permissible, we neglect the mass of the portions of air set in motion by the liquid as compared with that of the liquid, the condition to be satisfied at the free surface is simply that the pressure there must be equal to the atmospheric pressure p_0. We know by observation that in the simplest cases of wave motion the individual particles of the surface of the water describe paths which are approximately circular. With a system of co-ordinates advancing with the waves at a velocity equal to the velocity of propagation of the crests and troughs we obviously obtain a steady motion to which Bernoulli's theorem may be applied (see fig. 2.58). If c is the velocity of propagation of the waves, r the radius of the circular path described by a particle in the surface, and T the time taken to describe it (i.e. if the velocity of the particle in its circular path is $2\pi r/T$), the velocity of flow, if we adopt the system of co-ordinates mentioned above, will be

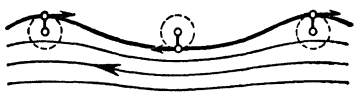


Fig. 2.58.—Wave motion

$$q_1 = c - \frac{2\pi r}{T} \text{ for a crest, and}$$

$$q_2 = c + \frac{2\pi r}{T} \text{ for a trough.}$$

The difference of height h is equal to $2r$, and the pressure is the same at all points of the free surface, hence

$$q_2^2 - q_1^2 = 2gh = 4gr:$$

the left-hand side is equal to $8\pi cr/T$, so that

$$c = \frac{gT}{2\pi}, \quad \ldots\ldots\ldots \quad (32)$$

which does not contain r, i.e. the velocity of the waves is independent of the height to which the crests rise. If we are given the wave-length λ instead of the time of oscillation T, we have available the further fact that the advance of the crests and troughs at the rate c over a distance equal to the wave-length occupies a time equal to the period of oscillation, i.e.

$$\lambda = cT. \quad \ldots\ldots\ldots \quad (33)$$

Eliminating T from (32) and (33), we have

$$c = \sqrt{\frac{g\lambda}{2\pi}}. \quad \ldots\ldots\ldots \quad (34)$$

Thus in water waves, in contradistinction to ordinary sound waves, the velocity of the waves depends to a great extent on the wave-length. Long waves advance more rapidly than short ones. Waves may be superposed on each other without much disturbance. Short waves

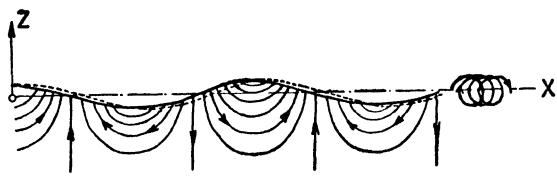


Fig. 2.59.—Stream-lines of wave motion

ride on the long waves, but tend to get left behind. The stream-lines of wave motion referred to a system of co-ordinates at rest relative to the undisturbed water are shown in fig. 2.59. As we see from the stream-lines, the motion of the water dies away very rapidly below the surface, at the rate $e^{-2\pi(z_1-z)/\lambda}$; that is, at a depth equal to the wave-length it is only about 1/500 of that at the surface.

Notes.—(1) Waves on the surface of a liquid form an irrotational motion (§ 8, p. 56). For waves of small amplitude the velocity potential, by a formula given on p. 66, is given by

$$\Phi = a_1 e^{\mu z} \cos \mu(x - ct),$$

where $\mu = 2\pi/\lambda$. For finite amplitudes the cosine is replaced by a Fourier series; the amplitudes of the various terms are obtained from the condition that the pressure must be the same at all points of the surface.

More accurate theory shows that formula (34) holds only for waves of small amplitude and is then independent of the amplitude; for waves of large amplitude, however, the velocity of the waves is somewhat greater. Moreover, the paths of the particles of water are no longer closed; on the contrary, the water advances more in the crest than it retires in the trough (see the small diagram on the right-hand side of fig. 2.59). Thus there is an actual forward movement of the water in the wave. According to calculations by Stokes (1847) the highest possible steady waves exhibit crests with an angle of 120°. If more energy is supplied to the waves they begin to break.*

(2) In the case of small wave-lengths surface tension (capillarity) plays a part as well as gravity. As it tends to flatten out the wavy surface of the liquid, it has the effect of increasing the velocity of propagation of the waves. If T is the surface tension, we have

$$c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}\right)}. \quad \ldots\ldots\ldots \quad (35)$$

With long waves the expression reduces practically to the first term, whereas with very short waves the second term predominates. The velocity has a minimum value $c_1 = \sqrt[4]{(4gT/\rho)}$ for the wave-length $\lambda_1 = 2\pi\sqrt{(T/g\rho)}$. For water ($\rho = 1$ gm./cm.^{3}, $T = 72$ dynes/cm. $= 72$ gm./sec.^{2}), $\lambda_1 = 1{\cdot}72$ cm. and $c_1 = 23{\cdot}3$ cm./sec. Waves whose length is greater than λ_1 are called gravity waves, waves whose length is less than λ_1 capillary waves.

We must of course distinguish between the velocity with which the tops of the waves advance, the so-called *phase-velocity* c, and the velocity of propagation of a group of waves, the so-called *group-velocity* U. This problem is most conveniently studied by superposing two waves of the same amplitude but differing slightly in wave-length. The line of argument, which holds not only for water waves but quite generally for any kind of waves in which the phase-velocity depends on the wave-length, i.e. for waves which exhibit "dispersion", starts with a simple sine wave

$$y = A \sin(\mu x - \nu t).$$

If x is increased by $2\pi/\mu$, or t by $2\pi/\nu$, the sine takes the same value as before, so that $\lambda = 2\pi/\mu$ is the wave-length and $T = 2\pi/\nu$ the period of oscillation. If $\mu x - \nu t = \text{const.}$, i.e. $x = \text{const.} + \nu t/\mu$, the argument of the sine function remains constant in time, i.e. y is constant in time, which means that the whole wave-form is displaced towards the right with the velocity $c = \nu/\mu$. On this we superpose a second wave y', with the same amplitude but slightly differing values of μ and ν which we shall denote by μ' and ν', i.e. $y' = A \sin(\mu' x - \nu' t)$. Then the effect of superposing the two waves is given by

$$y + y' = A[\sin(\mu x - \nu t) + \sin(\mu' x - \nu' t)].$$

* For the mathematical theory, see H. Lamb, *Hydrodynamics*, § 246.

At places where both oscillations are in the same direction the amplitude is equal to $2A$, while at places where they are in opposite directions the amplitude is zero. The term " beats " is applied to this phenomenon. Using the well-known formula

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta),$$

we have

$$y + y' = 2A\sin\{\tfrac{1}{2}(\mu + \mu')x - \tfrac{1}{2}(\nu + \nu')t\}\cos\{\tfrac{1}{2}(\mu - \mu')x - \tfrac{1}{2}(\nu - \nu')t\}.$$

In this expression the sine factor represents a wave in which the coefficients are equal to the mean of μ, μ' and ν, ν' respectively, and the remaining factor, which varies only very slowly when $\mu - \mu'$, $\nu - \nu'$ are small, may be regarded as a varying amplitude (see fig. 2.60). The " wave-group " ends wherever the cosine becomes equal to zero. The velocity of advance of these points, the group-velocity U, is therefore equal to $(\nu - \nu')/(\mu - \mu')$, by what we have said above, or with sufficient accuracy for long groups (slow beats)



Fig. 2.60.—Beats

$$U = \frac{d\nu}{d\mu}. \qquad \ldots\ldots\ldots \quad (36)$$

From the fact that no energy can travel past the nodes, we may draw the conclusion (capable of rigorous proof also for single trains of waves) that the rate of transfer of energy is identical with the group velocity.

For water waves (gravity waves) we have

$$\nu = \frac{2\pi}{T} = \frac{g}{c}$$

by equation (32); but by equation (34)

$$c = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{g}{\mu}},$$

so that $\nu = \sqrt{(g\mu)}$, i.e.

$$U = \frac{d\nu}{d\mu} = \tfrac{1}{2}\sqrt{\frac{g}{\mu}} = \tfrac{1}{2}c. \qquad \ldots\ldots \quad (37)$$

Thus the groups of waves advance with the velocity $\frac{1}{2}c$, or in other words, the crests of the waves advance through the group, moving at a rate equal to twice the group-velocity: new waves are continually formed at the rear of the group and disappear at the front. This is

very well shown in the case of the waves produced when a stone is thrown into water at rest.

The waves set up by ships represent another type of wave-group. We obtain a configuration very like the waves produced by ships merely by calculating the form of the waves arising from a local pressure disturbance advancing with constant velocity c over the surface of deep water at rest. According to the calculations of Lord Kelvin, Ekman, and others, the wave system is like that in fig. 2.61. (The continuous lines in this figure represent wave crests.) This system of waves advances with the pressure disturbance. By equation (34), p. 87, the wave-length of these transverse waves is $\lambda = 2\pi c^2/g$. The length of the wave-group is equal to half the distance traversed by the pressure disturbance.

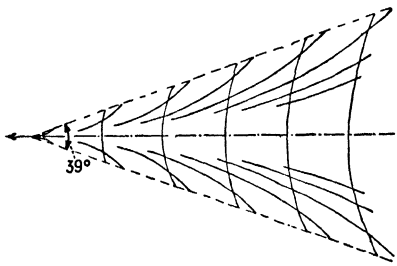


Fig. 2.61.—System of waves arising from a pressure disturbance moving uniformly over the surface of water

A system of waves of this type is set up by the bow of a ship and a similar one by the stern, and these two interfere with one another.

As may easily be shown by a calculation analogous to that given above for gravity waves, the group-velocity of capillary waves exceeds the wave-velocity (in the limiting case of very small waves it is equal to one and a half times the latter). Thus in the case of a pressure disturbance advancing with constant velocity the group moves ahead of the point where it is produced. In actual fact it is found that if a fishing-line or some similar obstacle at rest dips into running water whose velocity exceeds 23·3 cm./sec., capillary waves are formed upstream and gravity waves downstream; the latter are approximately as shown in fig. 2.61, while the former spread out in circular arcs above the obstacle. If the velocity is less than 23·3 cm./sec., no waves are formed.

If two fluids of differing density are superposed, wave motions may also take place in the boundary surface between them. For two fluids of density ρ_1 and ρ_2 at rest the theoretical phase-velocity is given by

$$c = \sqrt{\left(\frac{g\lambda}{2\pi}\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} + \frac{2\pi T}{\lambda(\rho_1 + \rho_2)}\right)}. \quad . \quad . \quad . \quad (38)$$

If the upper fluid is flowing over the lower one with velocity q_1, theory predicts that long waves alone are stable, shorter waves being unstable, as was shown in § 6 (p. 50) for the motion of two streams of fluid along a layer of discontinuity. In certain cases this leads to a mixing of the two liquids in the intermediate region, which causes the flow to become stable once more. As the velocity q_1 increases, the boundary between

instability and stability is displaced towards the longer wave-lengths. Waves of the kind described above may also arise at the boundary between two layers of air of differing density, such as may be present in the atmosphere; such waves are often made visible by the formation of long parallel bands of cloud (Helmholtz).

The application of the theory of frictionless fluid to the case of the waves produced on the surface of water by the wind proved unsatisfactory in several respects. For example, W. Thomson (Lord Kelvin) * made calculations which indicated that waves could not be produced by wind velocities of less than 6·4 m./sec., and such waves would have a minimum velocity of about 23·3 cm./sec. and a wave-length of 1·72 cm. (p. 88); higher winds would of course produce longer waves. Actually, according to wind-tunnel experiments by Sir Thomas Stanton,† wind velocities of about 2 metres per second give rise to undulations of wave-length about 3 cm., and wind velocities of 2·5 metres per second and upwards give rise to gravity waves of increasing wave-length. An explanation of similar observations was given by Jeffreys,‡ to the effect that as a result of friction the distribution of pressure over the surface of the wave becomes unsymmetrical, and hence the wind does work on every wave crest, provided the wind velocity is greater than the phase-velocity of the waves. By measurements of the pressure distribution over models of water waves Motzfeld § found that the resistance which the air offers to the waves is approximately proportional to the 1·5th power of the inclination of the surface of the wave to the horizontal (measured at the point of inflexion), and also proportional to the square of the difference between the wind velocity and the phase-velocity. On this basis, Motzfeld calculated that, taking the damping of the waves by the viscosity of the water into account, the inclination of the surface of the wave for a given wind velocity q_w tends towards a fixed limit, which is greatest for waves for which $c = \frac{1}{3}q_w$. Then from equation (34) the corresponding wave-length is given by

$$\lambda_1 = \frac{2\pi}{9}\frac{q_w^2}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Note.—The results of recent theoretical calculations by W. Wuest || on the first occurrence of waves are in good agreement with those of Stanton's experiments.

Two informative papers have recently appeared on the problem of water

* W. Thomson, *Phil. Mag.* (4), Vol. 42, p. 368 (1871).

† Communicated after Stanton's death by Miss D. Marshall and R. Haugton, *Proc. Roy. Soc.* (A), Vol. 137, p. 283 (1932).

‡ H. Jeffreys, *Proc. Roy. Soc.* (A), Vol. 107, p. 189 (1925); Vol. 110, p. 241 (1926).

§ H. Motzfeld, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 17, pp. 193–212 (1937).

|| *Zeitschr. f. angew Math. u. Mech.*, Vol. 29, p. 239 (1949).

waves and wind friction on the surface of water. The first, by C. Neumann,* gives a systematic account of a number of observations by others. The second, by H. U. Roll,† describes his own observations on waves and wind profiles in shoaling water. The discussion of wind friction in the first paper is based on measurements of the piling-up of water by the wind in bays and large inland lakes by E. Palmén ‡ and others. The second paper makes use of very careful measurements of the lowest 2 metres of the wind profile, using the formulæ for the flow past rough plates in Chapter III, § 5 (*b*) and (*d*), pp. 124, 129. Both authors find that the roughness of the established wind pattern on the water is surprisingly small. According to Neumann, the effect of the wind pressure on the wave peaks is that the wave-length λ and hence also the rate of advance c increases until the latter is nearly equal to the wind velocity, in the case of a storm to nearly 9/10 of q_{air}, the " wind velocity at the height of the anemometer " (in Palmén's work this was 6 metres). By measurement, λ was equal to $4{\cdot}9\, q_{\text{air}}^2/g$.

The formulæ given above apply to waves in deep water. The state of affairs is altered if the depth of the water is small compared with the wave-length (the case of " long waves "). For depths equal to half a wave-length, however, the former formulæ are still sufficiently accurate. In the case of smaller depths the water particles move in elliptical paths and the relationships between wave-length and velocity of propagation are less simple. For very small depths, or on the other hand for very great wave-lengths, where the particles of water in the surface mainly move backwards and forwards with a very moderate rise and fall, simple relationships are once more obtained. We may again consider periodic waves of approximate sine form and neglect the effect of vertical accelerations on the pressure distribution in the very flat elliptical paths; the distribution of the pressure in each vertical line is then purely statical, the differences of level practically giving rise to horizontal accelerations only. Here, however, we shall follow out a still simpler calculation based on the behaviour of a slight elevation on the surface (fig. 2.62). This discussion is closely related to that of the propagation of pressure in a compressible medium given in Chapter IV, § 1 (p. 259). We assume that an elevation of the surface such that the depth of water measured from a plane bed increases from h_1 to h_2 is propagated towards the right with velocity c (fig. 2.62). Let the water be at rest before the arrival of the elevation, and after the raising of the level let the velocity of the water towards the right be q.

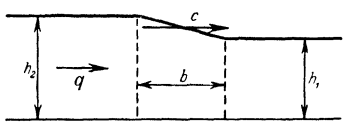


Fig. 2.62.—An elevation travelling over the surface of water

This velocity is required to raise the level from h_1 to h_2 in the

* *Zeitschr. f. Meteorologie*, Vol. 2, p. 193 (1948).

† *Ann. d. Meteorologie*, Vol. 1, p. 139 (1948).

‡ *Ann. d. Hydrographie*, Vol. 60, p. 435 (1932); Vol. 61, p. 43 (1933).

transitional region of breadth b by pushing the water together from the sides. If for simplicity we assume that the surface in the transitional region has a constant slope equal to $(h_2 - h_1)/b$, the rate at which the level rises in the region b is given by $v = c(h_2 - h_1)/b$, provided the velocity q is small enough to be neglected in comparison with the velocity of propagation c.

If we consider a region of unit depth at right angles to the plane of the paper in fig. 2.62, continuity requires that

$$h_2 q = bv$$

or

$$h_2 q = c(h_2 - h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This equation does not contain the breadth b, so that the latter quantity has no effect on the final result; equation (40) would still hold if the profile of the elevation were not rectilinear. The elevation could then be broken up into a number of elevations with rectilinear profiles. On adding the equations of continuity for the individual elevations we should again obtain $(h_2 - h_1)$ on the right, and q on the left from the individual velocity differences, provided, of course, that the differences between the various values of h_2 are negligible. From equation (40), moreover, it follows that when q is small $(h_2 - h_1)$ must also be small; that is, equation (40) holds only for slight elevations, so that the proviso just mentioned seems quite permissible.

In addition to the kinematical relationship (40) we must also have a dynamical one. The latter is obtained as follows. The mass of water of breadth b is moving with acceleration, for its particles begin at the right-hand edge with velocity zero and reach velocity q at the left-hand edge. The time taken by the elevation to pass a certain particle is obviously $\tau = b/c$; the acceleration of the particle is therefore $q/\tau = qc/b$. The mass of a volume of water of breadth b and unit depth at right angles to the plane of fig. 2.62 is $\rho b h_m$, where h_m is the mean level. The pressures at the same height on the two sides of the mass of water differ by $w(h_2 - h_1)$, where $w = g\rho$. Thus the total horizontal force on the mass of water (with a permissible neglect of small quantities) is $h_m w(h_2 - h_1)$. Hence the equation Force = Mass $\times$ Acceleration gives

$$qc = g(h_2 - h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The breadth b is again absent from the equation, and it may in fact be shown that equation (41) also holds for an elevation with a different profile, provided only that $(h_2 - h_1)$ is small compared with h_1 and h_2.

For simplicity we now substitute h_m for h_2 on the left-hand side of equation (40), which is also permissible in view of the quantities which have already been neglected. If we now divide (41) by (40), we have

$$c^2 = gh_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

A succession of elevations and depressions obviously gives rise to waves. By equation (42) the velocity of propagation of these waves is independent of the form of the wave (i.e. as in the case of sound waves, there is no dispersion). Long waves in shallow water, then, are propagated with the velocity $c = \sqrt{(gh)}$.

The following points may also be mentioned in connexion with fig. 2·62. If there are several slight elevations following one another, each giving rise to a fresh rise in the level, the velocity $\sqrt{(gh)}$ of the later elevations exceeds that of the earlier. What is more important, however, is the fact that the later elevations are moving in a mass of water which is itself moving forwards with velocity q. The later elevations will accordingly catch up on the earlier ones, thus giving rise to an elevation of greater amplitude. These considerations may also be applied to the shape of the individual elevation. Thus the elevation of the shape given in fig. 2.62 may be regarded as a succession of a large number of very small elevations occupying the interval b. Our argument then shows that this interval will become shorter and shorter, until a steep step is formed. This is actually observed. For the same reason the crests of waves in shallow water travel faster than the troughs and finally topple over (breakers).

The theory of elevations of finite height may be discussed by means of the momentum theorem in a way entirely analogous to that used in the case of flow through a suddenly widening pipe [§ 12(*c*), p. 80]. It is then convenient to consider the problem from the point of view of a system of reference moving with the elevation, so that the phenomenon is steady. The velocity of an elevation of finite height exceeds that of a slight elevation. Here again there is a loss of kinetic energy, the equivalent of which appears in the foaming of the breaking waves.

16. **The Flow of Water in Open Water-courses.**

Between the nature of the flow of water in a water-course and the velocity of propagation of long waves there exists a relationship similar to that between the motion of a gas and the velocity of sound (see Chapter IV, §§ 2, 4, pp. 264, 271). If the velocity of flow is less than the velocity of propagation of slight elevations, the result of damming up the flow in any way (by means of a weir, dam, &c.) is to raise the level at points upstream. If, however, the velocity of flow exceeds the velocity of propagation of slight elevations, a permanent elevation of finite height is produced at the weir or a short distance above it, but farther upstream the flow is in no way affected by the obstacle. Irregularities of any kind along the edge of the channel give rise to small oblique waves closely resembling the oblique shock waves mentioned on p. 275. The two kinds of flow in a water-course may be referred to as "tranquil flow" * and "shooting",† according as the

* [Ger. *Strömen.*] † [Ger. *Schiessen.*]

velocity of flow is less or greater than the wave-velocity of long waves.

If Q_1, the amount of water flowing per unit breadth, is given, and we consider the depths of water corresponding to a set of increasing velocities, we obtain the following results (see fig. 2.63). The amount by which the water-level is lowered below the level in a state of rest is $h = q^2/2g$. The local depth of water required to enable a quantity of water Q_1 to flow per unit breadth is $d = Q_1/q$, so that the depth of the corresponding point of the channel below the level of the water at rest is

$$z = h + d = \frac{q^2}{2g} + \frac{Q_1}{q}.$$

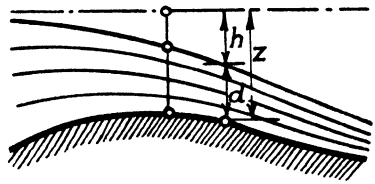


Fig. 2.63.—Flow over the crest of a weir

If we now let q take all values from 0 to ∞, z is at first infinite, on account of the second term, and returns to infinity again in virtue of the first term. z must therefore have a minimum value somewhere in between, just as in the case of the cross-section of the stream-tube in the gaseous flow discussed on p. 265. For this minimum we must have

$$\frac{q}{g} - \frac{Q_1}{q^2} = 0, \text{ or } q = \sqrt[3]{(Q_1 g)},$$

by differentiation, so that

$$h_1 = \tfrac{1}{2}\sqrt[3]{\frac{Q_1^2}{g}} \text{ and } d = \sqrt[3]{\frac{Q_1^2}{g}} = 2h_1.$$

This makes $q = \sqrt{(gd)}$, i.e. equal to the velocity of an elevation on water of depth d. Hence the result of this calculation is as follows. If water flows over a low-crested weir, d, the depth of the water over the highest point of the weir is equal to two-thirds of z, the depth of the weir-crest below the level of the water at rest, and the velocity there is $\sqrt{(\frac{2}{3}gz)}$. Hence the discharge is given by

$$Q_1 = qd = \tfrac{2}{3}z\sqrt{(\tfrac{2}{3}gz)}. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Below the weir-crest the motion is a " shooting " one which usually passes into " tranquil flow " by way of an abrupt change of level in the surface.

With more sharply curved weirs we may no longer assume that the velocity of flow is the same all over the cross-section, but qualitatively the behaviour remains like that described here; see also § 12(*e*) of the following chapter (p. 172).

The formulæ just discussed may be given a much wider application.

We consider a water-course whose bed has a slight slope but is otherwise of any shape (fig. 2.64), and for various levels of the water at rest (the dotted lines) we plot the depths d (two for each point and level) corresponding to a fixed value of Q, the quantity flowing per second. We thus obtain the water surfaces shown in the figure. It is only the curve I–IV, which passes through the double point and corresponds to the lowest possible level of the water at rest, that gives a flow of the type shown in fig. 2.63. The lines I–II and III–IV, corresponding to higher levels of the water, are also found in practice (cf. figs. 2.65, 2.66). The lines shown by dashes in fig. 2.64, corresponding to lower levels of the water at rest, may also be realized in practice, at least in their upper parts, namely, in the rear of an elevation; here, of course, there is a loss of energy (fig. 2.67).

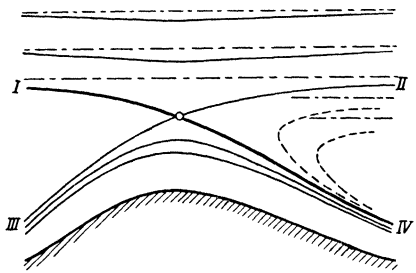
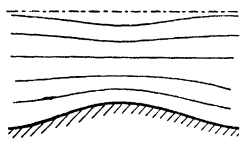
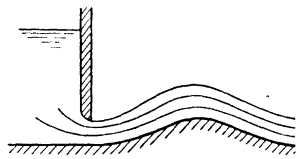
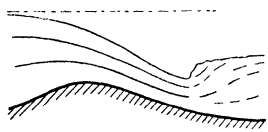


Fig. 2.64

In the case shown in fig. 2.65 the velocity is less than the velocity of a slight elevation, and there is a lowering of the level over the top of the hump in the bed. In the case shown in fig. 2.66 the velocity is greater than the velocity of a slight elevation; here the water is heaped up over the hump in the bed, the depth over the latter being increased.


Fig. 2.65


Fig. 2.66


Fig. 2.67

To explain fig. 2.67 it should be pointed out that the velocity of the flow between the crest of the weir and the abrupt change of level is greater than that of the corresponding slight elevation, and that of the subsequent flow is less than this value. As any disturbance of the flow can only be propagated with the velocity of a slight elevation, the flow of the "shooting" type cannot be altered by a change in the level of the bed, so that the change finally takes place very abruptly. For

related phenomena in open channels involving frictional effects, see Chapter III, § 12(*f*), p. 173.

It may also be mentioned here that in the above argument the effect of vertical accelerations was neglected. In " shooting " flow quantitative corrections of only moderate magnitude result when vertical accelerations are taken into consideration, but in " tranquil flow " a change in type occurs, in that standing waves usually occur downstream from the disturbance. Their wave-length follows from equation (34), § 15 (p. 87), the local velocity of flow taking the place of the velocity of propagation *c*. Calculations on phenomena of this type were made by J. Boussinesq.*

* " Essai sur la théorie des eaux courantes ", *Mém. Savants Étrang.*, Vol. 23 (1877); see also P. Forchheimer, *Hydraulik*, 3rd ed., p. 230 (Teubner, Leipzig, 1930).

CHAPTER III

Motion of Viscous Fluids: Turbulence: Fluid Resistance: Practical Applications

1. Viscosity (Internal Friction).

All actual fluids possess a definite "viscosity", which is made manifest by their offering internal friction to change of shape. As examples of particularly viscous liquids we may mention honey, glycerine, and thick oils. In order to understand the real nature of viscosity we shall begin by considering the following simple example. Let one of two parallel plates separated by fluid move in its own plane with velocity V, while the other is at rest (fig. 3.1). Owing to friction the flow set up in the fluid is such that the layers close to the plates have the same velocity as the plates ("stick" to the plates), whereas the intermediate layers slip past each other with velocities (u) proportional to their distance from the plate at rest:

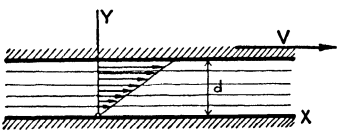


Fig. 3.1

$$u = V\frac{y}{d}.$$

The fact that the fluid has friction is shown by the existence of a force which resists the motion of the upper plate and is of magnitude $\tau = \mu V/d$ per unit area of the plate. A more general statement is as follows. The shearing stress produced by the layers slipping over each other is

$$\tau = X_y = \mu\frac{du}{dy}. \qquad \ldots\ldots\ldots \quad (1)$$

(X_y denotes the force per unit area in the direction of the x-axis on an element of area whose normal drawn outwards is parallel to the y-axis.) The quantity μ is called the *coefficient of viscosity* or for short the *viscosity*.* Equation (1) was in effect given by Newton and hence is often referred to as *Newton's law*.

* The letter μ is commonly used by writers on hydrodynamics, whereas physicists generally denote the viscosity by η (dimensions gm. cm.$^{-1}$ sec.$^{-1}$ in c.g.s. units).

A knowledge of these facts is sufficient to enable us to discuss some other simple cases where the motion also consists of the slipping of layers of fluid over other layers (so-called *laminar* or *streamline flow*). The motion of fluid with friction (viscous fluid) in a straight tube of circular section is an example of this. The difference of pressure

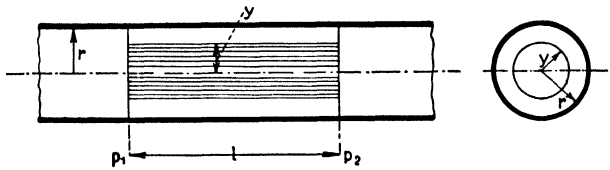


Fig. 3.2

$p_1 - p_2$ exerts the force $(p_1 - p_2)\pi y^2$ on a cylindrical mass of liquid of radius y (fig. 3.2). The opposing force consists of the friction on the curved surface $2\pi yl$, equal to τ per unit length, so that the total force is $2\pi yl\tau$. Equating the two forces, we have *

$$-\tau = \frac{p_1 - p_2}{l}\frac{y}{2}. \qquad \ldots\ldots\ (2)$$

From equation (1), $du/dy = \tau/\mu$; integrating and determining the constant of integration so that the outermost layer of fluid sticks to the wall of the tube, i.e. is at rest, we have

$$u = \frac{p_1 - p_2}{4\mu l}(r^2 - y^2), \qquad \ldots\ldots\ (3)$$

where r is the radius of the tube; the velocity distribution is therefore represented by a paraboloid of revolution (see fig. 3.13, p. 124). We can now evaluate the discharge, or quantity of fluid passing in unit time; it is given by

$$Q = \int_0^r 2\pi y\, dy\,.\, u = \frac{\pi r^4}{8\mu}\frac{p_1 - p_2}{l}. \qquad \ldots\ (4)$$

This formula is of fundamental importance in the investigation of the law of frictional resistance, as it has been found to agree very accurately with the results of experiment; it also provides the best way of determining the coefficient of viscosity, μ. The result that the discharge is proportional to the fall of pressure per unit length and to the fourth power of the radius of the pipe was proved experi-

* The sign is minus, because here τ is in the direction opposite to that in the first example.

mentally by G. Hagen in 1839,* and was re-discovered independently by Poiseuille.† The paper by Hagen (who was an engineer) was apparently ignored by physicists, and the law is usually named after its second discoverer, Poiseuille, although it would really be more correct to call it the Hagen-Poiseuille law.‡ We would emphasize at this early stage that it is only in narrow tubes that the Hagen-Poiseuille law is true for all velocities attainable in practice. In wider tubes it must be replaced by another law for large velocities. This deviation is in no way due to any inaccuracy in the law of friction, which, on the contrary, like the fact that a layer of fluid adheres to the wall, has been accurately verified by experiments in narrow tubes for the great majority of fluids.

Note.—From the point of view of the kinetic theory of gases viscosity is to be regarded as due to a process of exchange of momentum between neighbouring layers of fluid with differing velocities, caused by the motions of the molecules themselves. According to this, the equalization of the velocities of adjacent layers is to be regarded as a sort of diffusion process (diffusion of momentum) and may be dealt with by means of the relationships deduced in last chapter [§ 14 (2), p. 85]. Here the shearing stress is again given by $\rho(u'v')_m$, but the velocities u', v' are no longer fluctuations due to turbulence, but molecular velocities [cf. also the remark on the viscosity of gases in § 4(*d*) (p. 117)]. In highly rarefied gases, where the "mean free path" is no longer negligibly small compared with the dimensions of the containing vessel, it has been found that slipping appears to take place along the wall, owing to the fact that the molecules approaching the wall on the average bring with them a finite velocity component parallel to the wall, whereas the molecules rebounding from the wall exhibit an irregular distribution such that the average tangential component is equal to zero. The average value of the tangential velocity of all the particles therefore differs from zero. In gases at ordinary pressures the mean free path is so short that this phenomenon is not noticeable.

In liquids which form drops viscosity arises in another way. Here the molecules are so closely packed that as a rule they merely oscillate to and fro in the small spaces available to them and only occasionally exchange places. Such exchanges in general occur at random. Under the influence of a shearing stress (which may be thought of as an elastic stress resulting from molecular forces) exchanges of place along the line of the shearing stress are favoured, so that the total effect is one of slipping. The greater the freedom to exchange places the less the viscosity of the liquid.

It may be mentioned that in addition to ordinary liquids in which the slip $\partial u/\partial y$ is strictly proportional to the shearing stress τ, there are also abnormal liquids for which this proportional relationship does not hold.

In the first place, we have the intermediate stages between solids and liquids mentioned on p. 1, in which the number of exchanges of place are sharply restricted. For these Maxwell gave a set of equations based on the stresses in elastic bodies but with a "relaxation term" added; that is, all the stresses at any

* G. Hagen, *Poggendorffs Annalen*, Vol. 46, p. 423 (1839).

† Poiseuille, *Comptes Rendus*, Vol. 11 (1840); Vol. 12 (1841); *Mém. des Savants Étrangers*, Vol. 9 (1846).

‡ See W. Ostwald, *Kolloid-Zeitschr.*, Vol. 36, p. 99 (1925).

instant diminish with time, but any change of shape brings new elastic stresses into existence.*

Another type, differing from these Maxwellian liquids, consists of pastes and plastic masses, in which motion does not occur until a certain "yield point" τ_0 is reached; in the simplest case † $\tau = \tau_0 + \mu\, du/dy$.

A third group, which has been much studied, is that of the "colloidal liquids", which are mostly solutions of submicroscopic threadlike configurations or networks. Many of them show so-called "thixotropy"; that is, when at rest they are viscous, or even solid, but when stirred become thin, which indicates a breakdown of the threads or network, the latter being re-formed when the liquid is allowed to come to rest. The study of all these properties, which are of importance in industrial chemistry as characteristic properties of substances, is known as *rheology*.‡

According to the general theory of viscous fluids, the change in shape of the individual elements of fluid gives rise to stresses analogous to those in elastic bodies, but with the difference that in the former case the pressures are proportional not to the change in shape but to the rate of change of shape. By the results of the theory of elasticity, the formulæ for the nine stress components (three for each of the

* In the simple case shown in fig. 3.1 we have, to the first order and provided $T\,du/dy$ is small, the differential equation

$$\frac{\partial \tau}{\partial t} = G\frac{du}{dy} - \frac{\tau}{T},$$

where G is the shear modulus and T the time constant of the relaxation. With $du/dy =$ const. and $\tau = 0$ for $t = 0$, this gives

$$\tau = GT\frac{du}{dy}(1 - e^{-t/T}).$$

For $t \to \infty$ this again tends to Newton's law (1), with $\mu = GT$. For a discussion of the effect of quantities of higher order in $T\,du/dy$, see H. Fromm, *Zeitschr. f. Math. u. Mech.*, Vol. 25/27, p. 146 (1947) where among other information will be found the asymptotic formula

$$\tau \sim GT\frac{du}{dy}\Big/\left\{1 + \left(T\frac{du}{dy}\right)^2\right\}.$$

For a more detailed account, see the same journal, Vol. 28, p. 43 (1948).

† See the author's paper "Ein Gedankenmodell zur kinetischen Theorie des festen Körpers", *Zeitschr. f. Math. u. Mech.*, Vol. 8, p. 85 (1928), in which elastic hysteresis, relaxation and after-working, as well as the law of force for a constant rate of change of shape $du/dy = c$, are traced through all stages of softening and explained by the thermal motions of the molecules (see fig. 19 on p. 101 of the paper). An approximate formula (curve 4 in fig. 18 of the paper) is $\tau = G\log[cT + \sqrt{\{1 + (cT)^2\}}]$, where T is the time constant. For very large values of cT this gives a law for plastic metals ($\tau = a + b\log c$) which is in good agreement with experiment, and for small values the law for a viscous liquid.

‡ The following books contain information on rheology: E. C. Bingham, *Fluidity and Plasticity* (New York, 1922). R. Houwink, *Elastizität, Plastizität und Struktur der Materie* (Dresden, 1938). G. W. Scott Blair, *An Introduction to Industrial Rheology* (London, 1938), German translation by H. Kauffmann (Dresden, 1940). W. Philippoff, *Viscosität der Kolloide* (Dresden, 1942). M. Reiner, *Ten Lectures on Theoretical Rheology* (Lewis and Co., London, 1943). G. W. Scott Blair, *A Survey of General and Applied Rheology* (London, 1943).

three planes at right angles to the axes of co-ordinates) are therefore

$$\left.\begin{aligned} X_x &= 2\mu\frac{\partial u}{\partial x} & X_y = Y_x &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\ Y_y &= 2\mu\frac{\partial v}{\partial y} & Y_z = Z_y &= \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\\ Z_z &= 2\mu\frac{\partial w}{\partial z} & Z_x = X_z &= \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

If these stresses have the same value at all points of a region (as is the case in homogeneous strain), they are mutually in equilibrium by themselves. If, however (as in a more general type of deformation), they vary from point to point, this variation from place to place in general gives rise to forces which we may take to have the components X', Y', Z' (per unit volume). According to the theory of elasticity,

$$X' = \frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

with similar equations for Y' and Z'.

Substituting the values of X_x, &c., from equation (5) in equation (6), we obtain

$$X' = \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right); \quad . \quad . \quad (7)$$

the second part of this expression vanishes if the flow takes place without change of volume. Similar expressions hold for Y' and Z'.

In viscous fluids forces like X', &c., have to be added to the forces due to pressure differences (and possibly body-forces) discussed in § 3 of the previous chapter (p. 41); together with these they determine the acceleration of the particle of fluid.

By adding the expressions (7) to the right-hand side of Euler's equations [see the mathematical appendix to Chapter II, § 3 (p. 41)], we obtain the Navier-Stokes differential equations for viscous fluid.

Thus if the flow takes place without change of volume these equations become

$$\left.\begin{aligned} \rho\frac{du}{dt} &= \rho X-\frac{\partial p}{\partial x}+\mu\Delta u,\\ \rho\frac{dv}{dt} &= \rho Y-\frac{\partial p}{\partial y}+\mu\Delta v,\\ \rho\frac{dw}{dt} &= \rho Z-\frac{\partial p}{\partial z}+\mu\Delta w,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where the symbol Δ is as usual employed to denote $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and, as before, du/dt means

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

and so on.

In cases of flow where, as in that discussed above, the u-component is the predominating one and alters most rapidly in the y-direction, the most important of the stresses is X_y [τ in equation (1)], so that the chief part of the force X' is $\partial X_y/\partial y$. By equation (5) the latter is chiefly represented by $\mu\partial^2 u/\partial y^2$, so that this is the chief term occurring in the equation along with the fall of pressure $(-\partial p/\partial x)$ and the "force of inertia" * $-\rho du/dt$ (cf., for example, Chap. II, § 3, p. 38). We shall, however, go no further into the detailed investigations here, as great difficulties are in general encountered in working them out. Instead we shall discuss dynamical similarity, which will be of great use in enabling us to gain a general view of the subject.

2. **Dynamical Similarity: Reynolds' Number.**

The question we discuss here is as follows. If the external conditions are geometrically similar (pipes of geometrically similar shape, geometrically similar bodies surrounded by fluid, &c.), under what conditions is the flow in the two cases geometrically similar also? To begin with we may give the following answer: In the cases under comparison the ratio of the three forces due to fall of pressure, friction, and inertia must be the same at corresponding points. As the three forces are subject to the condition of equilibrium. only two of them need be considered further, and we shall select the forces of friction and inertia, as the fall of pressure—at least in cases of flow where the volume remains unchanged—is not in itself sufficiently distinctive of the flow. The various cases of flow which are geometrically similar to one another will now be distinguished by characteristic lengths l_1, l_2, &c. (e.g. the diameter or length of a body, the internal breadth of a pipe), and characteristic velocities v_1, v_2, &c. (e.g. the velocity of motion of a body or the mean velocity over a fixed cross-section of a pipe). The densities and viscosities may be different (ρ_1, ρ_2, &c., and μ_1, μ_2, &c., respectively). If the flow is similar in the two cases, the components of the forces of inertia, one of which is

$$-\rho\frac{du}{dt} = -\rho\left(u\frac{\partial u}{\partial x} + \ldots\right),$$

(see above), must be proportional to the products $\rho v^2/l$, as in virtue

* By "force of inertia" we mean the resistance of an inert mass to acceleration, so that it is equal to —Mass × Acceleration.

of the assumed similarity the u-values at corresponding points, and hence also small differences of them, must be proportional to the characteristic velocities v_1, v_2, and the lengths x and y and also small differences of them must be proportional to the characteristic lengths l_1, l_2. The forces of friction, on the other hand, are proportional to $\mu v/l^2$, like the expression $\mu \partial^2 u/\partial y^2$ ($\partial^2 u$ means a small difference of velocity of the second order and is therefore proportional to the velocity v, whereas ∂y^2 is the square of a small difference in length and is proportional to l^2, when similarly situated adjacent points are taken). Our postulate of dynamical similarity therefore means that in the cases compared $\rho v^2/l$ and $\mu v/l^2$ are in a fixed ratio. Owing to its meaning (the ratio of two forces per unit volume), this ratio must be a pure number. In symbols it is

$$\frac{\rho v^2}{l} \div \frac{\mu v}{l^2} = \frac{\rho v l}{\mu},$$

so that two systems 1 and 2 may be expected to exhibit dynamical similarity if

$$\frac{\rho_1 v_1 l_1}{\mu_1} = \frac{\rho_2 v_2 l_2}{\mu_2} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This number, which gives the ratio of the forces of inertia and the forces of viscosity, is called *Reynolds' number*, in honour of Osborne Reynolds, the discoverer of this law of similarity.* The ratio μ/ρ (viscosity/density) is called the *kinematic viscosity*, and is denoted by ν. As may be seen from the fact that Reynolds' number is a pure number, ν has the dimensions L^2/T.

The nature of any particular case of flow of a viscous fluid may be determined from the value of the corresponding Reynolds' number $\mathscr{R} = \rho v l/\mu = v l/\nu$. A small value of Reynolds' number means that viscous forces predominate, a large value that forces of inertia predominate. As we shall show in what follows, the laws of motion of fluids and the laws of resistance to motion are very different for these two cases. As the structure of formula (9) shows, the magnitudes of the spatial dimensions and the velocities play a decisive part in addition to the viscosity. If the space dimensions are very minute the laws which hold for small values of vl/ν will hold for all velocities which occur in practice, whereas if they are large the laws will hold only for very small velocities or very viscous fluids. It is also interesting to note that if similarity is maintained (the value of $\mathscr{R}$ being kept the same when the external conditions are geometrically similar), the differences of pressure arising as a result of the motion are proportional to ρv^2 or, by equation (9), to $\mu v/l$.

* O. Reynolds, *Phil. Trans.* (1883); *Scientific Papers*, Vol. II, p. 51.

We append some values for the kinematic viscosity ν.

Water at 0° C.	$\nu = 0{\cdot}0178$ cm.^{2}/sec.
,, 20°	$\nu = 0{\cdot}0100$,,
,, 50°	$\nu = 0{\cdot}0056$,,
,, 100°	$\nu = 0{\cdot}0030$,,
Mercury at 0°	$\nu = 0{\cdot}00125$,,
,, 100°	$\nu = 0{\cdot}00091$,,
Glycerine at 20°	$\nu = 6{\cdot}8$,,
Air at 0° and 760 mm. ..	$\nu = 0{\cdot}133$,,
,, 100° and 760 mm. ..	$\nu = 0{\cdot}245$,,
,, 0° and 7·6 mm. ..	$\nu = 13{\cdot}3$,,
,, 0° and 76,000 mm. ..	$\nu = 0{\cdot}00133$,,

The unit of viscosity 1 gm./cm.-sec. has been called 1 *poise* (in honour of Poiseuille), and the unit of kinematic viscosity 1 cm.^{2}/sec. 1 *stoke* (in honour of Stokes).*

3. Motion of Bodies in Viscous Fluids. Stokes' Formula. Boundary Layers.

The mathematical treatment of the motion of bodies in a viscous fluid has turned out to be so difficult that so far only limiting cases have proved amenable to discussion, namely, the case where the viscosity is very great, i.e. Reynolds' number is very small, and the case where the viscosity is very small, i.e. Reynolds' number is very large.

(*a*) *Viscosity markedly Predominant.*—This happens with very viscous fluids (e.g. lubricating oil) on one hand, and, on the other hand, with ordinary fluids when the dimensions concerned are very minute; in such cases all inertia effects may be neglected, as they are small compared with the effects of viscosity, and we have to assume that each particle is in equilibrium under the action of the pressure gradient and the force of friction. By our discussion on p. 104, the forces of friction per unit volume are proportional to $\mu v/l^2$ for geometrically similar cases, and owing to the existence of equilibrium the pressures are similarly related, so that here geometrical similarity necessarily involves dynamical similarity. The volume is proportional to l^3, so that the total forces of resistance must be proportional to μvl. In a few cases where the shape of the body is simple it has been found possible to calculate the motion and hence to find the resistance to the

* The unit of shearing stress τ in the c.g.s. system is 1 dyne/cm.^{2} = 1 Gm./cm.-sec.^{2} Since the unit of du/dy = (cm./sec.)/cm. = 1/sec., it follows from equation (1), p. 98, that that of μ is 1 gm./cm.-sec. The unit of ρ is 1 gm./cm.^{3}, hence that of $\nu = \mu/\rho$ is 1 cm.^{2}/sec., as was given by the law of dynamical similarity above. In the cm.-Kg.-sec. system the unit of shearing stress τ is 1 Kg./cm.^{2}, which is 981,000 times the previous unit, and so is that of μ, namely 1 Kg.-sec./cm.^{2} Since the unit of ρ is 1 Kg.-sec.^{2}/cm.^{4}, that of ν is again 1 cm.^{2}/sec., as must necessarily be the case.

latter. A very well-known case is Stokes' solution for the motion of a sphere,* the resistance of which is

$$R = 6\pi\mu vr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where v is the velocity and r the radius. This is important in connexion with the fall of small drops of liquid. The resistance must be equal to the weight less the upward thrust, i.e.

$$6\pi\mu vr = \frac{4\pi}{3}(\rho_1 - \rho_2)gr^3,$$

where ρ_1 is the density of the drop and ρ_2 the density of the surrounding fluid. This gives

$$v = \frac{2}{9}\frac{\rho_1 - \rho_2}{\mu} gr^2 \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

for the rate of fall of the drops. The formula holds only for values of Reynolds' number which are small compared with unity. (For water-droplets in air $v = 1{\cdot}3 \times 10^6 r^2$, r being in centimetres; the formula holds for droplets with a radius less than 1/100 mm. Such droplets are found in fine mist.)

This whole group of fluid motions may be referred to as " creeping motion ".

(*b*) *Reynolds' Number very large, or the Friction very small.*—The characteristic feature of this limiting case is that frictional effects take a back place, i.e. we should merely have the motion of frictionless fluid discussed in §§ 3–11 of last chapter (p. 38) if it were not for the fact that fluid sticks to the boundary, a condition which cannot be fulfilled in the motion of frictionless fluid. Now closer investigation shows that though fluid with only slight friction behaves just like frictionless fluid at places where there is no boundary, a thin " boundary layer " is formed at the walls as a result of friction, and that in this layer the value of the velocity varies from that corresponding to frictionless fluid motion to that involved by the fact that fluid adheres to the boundary. The smaller the viscosity, the thinner is this boundary layer. The frictional forces per unit volume are then very small in the interior of the fluid, whereas in the boundary layer they are of the same order of magnitude as the forces of inertia, as here, of course, the velocities are changed by finite amounts. The distribution of the velocity in a boundary layer is shown in fig. 3.3. If the dimensions of the body in the direction of flow are of the order of magnitude l,

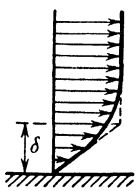


Fig. 3.3.—Distribution of velocity in the neighbourhood of a boundary

* *Trans. Cambridge Phil. Soc.*, Vol. 9, p. 8 (1850); *Math. and Phys. Papers*, Vol. III, p. 55.

and the thickness of the boundary layer at the end of the distance l is of the order of magnitude δ, the force of friction $\mu\partial^2 u/\partial y^2$ (y being at right angles to the surface) is of the order $\mu v/\delta^2$, and the force of inertia is as before of the order $\rho v^2/l$. These two expressions are of the same order of magnitude * if

$$\delta \sim \sqrt{\frac{\mu l}{\rho v}}. \quad \ldots\ldots\ldots \quad (11)$$

An estimate leading to the same result is obtained from the following momentum considerations for the flow along a flat plate. Let the length of the plate be l, its breadth b, the velocity of flow q, and the approximate thickness of the boundary layer δ (fig. 3.4); then the mass moving into the boundary layer per second $\sim \rho bq\delta$. This mass, which arrived with the velocity q, loses a definite fraction of its velocity in the boundary layer; the corresponding change of momentum is Mass $\times$ Loss of velocity and is therefore proportional to $\rho bq^2\delta$. The force exerted on the fluid by the friction at the boundary, which by equation (1) (p. 98) may be taken to vary as $\mu lbq/\delta$, must be equal to this change of momentum. From the proportionality of the two expressions it follows that as before

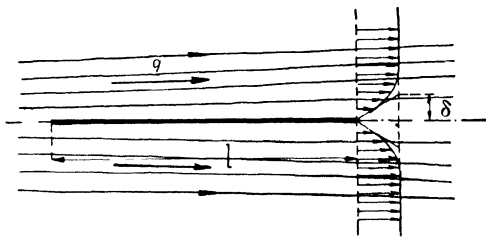


Fig. 3.4.—Flow along a plate

$$\delta \sim \sqrt{\frac{\mu l}{\rho q}} \text{ or } \sqrt{\frac{\nu l}{q}}.$$

Thus the ratio δ/l is proportional to $\sqrt{(\nu/ql)}$. If we put $ql/\nu = \mathscr{R}_l$, $q\delta/\nu = \mathscr{R}_\delta$, we have $\delta/l \sim 1/\sqrt{\mathscr{R}_l}$ and hence $\mathscr{R}_\delta \sim \sqrt{\mathscr{R}_l}$. This relationship is likewise true for all boundary layers in steady flow. We may also introduce the time which a single particle takes to flow past the body. For particles which do not lie too close to the surface, this length of time (t) is of the order l/q, so that we may also write

$$\delta \sim \sqrt{(\nu t)}. \quad \ldots\ldots\ldots \quad (12)$$

This formula may also be applied to motions which have only started from rest within a short time t, and then shows us that at the beginning of the motion the boundary layer increases in proportion to the square root of the time.

Thus we find that any body moving in fluid with slight friction

* The sign $\sim$ means "is proportional to".

drags a thin layer of fluid along with it. The same is true when fluid flows through short pipes, &c. With longer pipes, the boundary layer will increase as the square root of the distance from the point of inflow, by equation (11), and hence if the pipe is long enough will fill the whole of it, i.e. the effects of friction will extend over the whole cross-section of the pipe.* As a matter of fact, this spreading-out in many cases goes on much more rapidly than one would expect from formula (11), owing to a mixing process taking place, which is called *turbulence* (cf. next section). In cases of flow like that shown in fig. 3.3 there are shearing stresses on the boundary, which give rise to frictional resistance. For the plate of fig. 3.4 we obtain the following results. Here the shearing stress $\tau = \mu \partial u/\partial y$ is approximately given by

$$\tau \sim \mu \frac{q}{\delta} \sim \sqrt{\frac{\mu \rho q^3}{l}};$$

if b is the breadth of the plate, so that the total area is $2bl$, the resistance is therefore

$$R \sim 2bl\tau = b\sqrt{(\mu \rho l q^3)} \times \text{constant} \quad . \quad . \quad . \quad (13)$$

(cf. the detailed account on p. 193).

Experiment.—If a suitable body (a plate, cylinder, sphere, or the like) is placed in a not too rapid current of water, and dye is applied to a point of its surface by means of a tapering glass tube, a thin layer of dye will remain on the surface of the body for a considerable time after the removal of the glass tube, thus clearly showing that the layer of water next the surface of the body tends to cling to it.

Mathematical Appendix.—It has been found possible to work out an exact† theory of flow in boundary layers. In the case where the viscosity almost vanishes, the passage from the layer adhering to the wall to the undisturbed flow takes place, as mentioned above, within a very thin layer. Hence it is permissible to neglect the differences of pressure at different points of a normal to the wall within this thin layer. Further, the increase in velocity per unit length along this normal is very considerable, since the velocity rises from zero to ordinary values within a very short distance. Hence in the expression for the frictional force per unit volume (the x-axis being taken in the direction of flow and the y-axis along the normal) there remains, of the terms on the right-hand side of equation (7), p. 102, only the term $\mu \partial^2 u/\partial y^2$ of the order $\rho u \partial u/\partial x$, all the other terms being negligible in comparison with it.

In the case of flow in two dimensions the effect of moderate curvature of the layer may be neglected (i.e. the x co-ordinate may be taken as identical with the arc of the stream-line along the boundary). We then obtain the system of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

* Cf. the detailed account on p. 166.

† I.e. exact in the sense of tending to a limit as the viscosity vanishes.

Since the boundary layer is so thin, the variation in pressure along the normal is small within the layer; hence as an approximation p may be regarded as a function of x and t, determined by the external flow. Along the boundary, where $u = 0$, $v = 0$, the left-hand side of (14) vanishes, so that here

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y\ 0} = \frac{1}{\mu}\frac{\partial p}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Thus if the pressure falls in the direction of flow ($\partial p/\partial x$ negative) the velocity profile is convex, whereas if the pressure rises ($\partial p/\partial x$ positive) it is concave in the neighbourhood of the boundary, and therefore has a point of inflexion (cf. fig. 3.18, p. 136).

The equations may be solved in series consisting of sums of powers of x (or for motion starting from rest, powers of t) multiplied by power series in y, which are made to approach the given velocity potential $U(x)$ or $U(x, t)$ by asymptotic expansion.† For the flow along a thin plate as in fig. 3.4 with $U =$ const. the numerical factor in equation (11) works out at 3·012 if x is the distance from the front edge and δ is as shown in fig. 3.3. That is, only a small error is made if we put

$$\delta \approx 3\sqrt{(\nu x/U)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For the boundary layer of the flow against a fixed plane on p. 62 ($U = ax$) we have the corresponding rule-of-thumb formula

$$\delta \approx 0{\cdot}8\sqrt{(\nu/a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The thickness of the layer in which noticeable differences in velocity occur as a result of viscosity is given by the distance δ_1 from the boundary for which $u = 0{\cdot}99U$. For the boundary layer of the plate δ_1 is found to be about $5\sqrt{(\nu x/U)}$, for the flow against a fixed plane $\delta_1 \approx 2{\cdot}4\sqrt{(\nu/a)}$.

Note.—In addition to δ, δ_1, the practical measures of the boundary layer used here, there is another with a more strictly scientific basis, the *displacement thickness*

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy,$$

where U is the velocity of irrotational motion. The name arises from the fact that owing to the existence of the boundary layer the external flow is displaced from the boundary through a distance δ^* as compared with the frictionless case. In the cases of flow along a plate or against a fixed plane, δ^* is respectively $1{\cdot}73\sqrt{(Ux/\nu)}$ or $0{\cdot}637\sqrt{(\nu/a)}$.

(*c*) An excellent general survey of recent literature on methods of calculation in connexion with laminar flow in boundary layers is given by W. Tollmien in the *F.I.A.T. Review of Aerodynamics* (see also Görtler's article on viscous fluids in the *F.I.A.T. Review of Applied Mathematics*). A systematic account of the sub-

† See e.g. H. Blasius, "Über Grenzschichten in Flüssigkeiten bei sehr kleiner Reibung", *Zeitschr. f. Math. u. Phys.*, Vol. 56, p. 1 (1908), who was the first to give solutions of problems of steady and non-steady flow; instructive accounts are given by Tollmien, *Handb. d. Exp.-Phys.*, Vol. IV, 1, p. 239; see also the author's article in Durand, *Aerodynamic Theory*, Vol. III, p. 80 *et seq.* More recent calculations are given by Howarth, *Rep. and Mem.*, 1632 (1935) and *Proc. Roy. Soc.* (A), Vol. 164, p. 547 (1938); Falkner and Skan, *Rep. and Mem.*, 1314 [*Phil. Mag.* (7), Vol. 12, p. 865 (1931)]; Hartree, *Proc. Camb. Phil. Soc.*, Vol. 33, p. 223 (1937), and others; see also Goldstein, *Modern Developments in Fluid Dynamics*, Vol. I, p. 116 *et seq.* For numerical methods of evaluation, see e.g. H. Görtler, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 19, p. 129 (1939).

ject is to be found in Vol. I of *Modern Developments in Fluid Dynamics* by S. Goldstein and others[4] (see especially pp. 116–90).

Following the lines of the earliest papers, the calculations were first carried out by means of power series in y combined with asymptotic expansions for the approximation to irrotational motion. Special mention may be made of the calculations of Howarth and Hartree, who give very valuable tables of numerical values, the former for the velocity potential $U = a + bx$, the latter for $U = ax^m$ (flow round wedge-shaped bodies of various angles). Both papers, as well as the older calculations, show that for various increases or decreases in pressure it is possible to obtain an approximation to the solution in the form of a series $u/U = f(y/\delta, \lambda)$ involving only one parameter. A suitable parameter is

$$\lambda = \frac{\delta^2}{\nu}\frac{\partial U}{\partial x},$$

where $U(x)$ is the velocity of the associated irrotational motion. It is usual to represent the profile by polynomials of the fourth degree,* or more recently the Hartree profile† has been used. The conditions to be satisfied are the "integral condition" due to Th. von Kármán‡ (a theorem on the loss of momentum for the element dx), and the differential equation (14) for the point $y = 0$, i.e. equation (16). Subsequently it was found § to be an improvement to replace δ by the *momentum thickness*

$$\theta = \int_0^\infty \frac{u(U-u)}{U^2}\,dy.$$

The calculation leads to an ordinary differential equation of the first order for $z = \delta^2/\nu$ (or θ^2/ν) which can be solved numerically or graphically. An empirical function which occurs in the equation depends on the one-parameter family of profiles chosen. On the whole, the results obtained by these methods are satisfactory.

Where greater accuracy is required it is possible to use methods of successive approximation in which, starting with a given profile $u_0 = u(x_0, y)$, a profile $u_1 = u(x_0 + \Delta x, y)$ is calculated for small values of Δx, the process being repeated as often as desired.|| The differential equation (14) is of the parabolic type, so that all the numerical methods converge only when the velocity of flow is positive at every point of the profile. Hence at the boundary, at which the adhesion condition $u = 0$ must be satisfied, one might suspect the risk of singularities occurring if the initial profile does not satisfy certain conditions, e.g. equation (16)· For this reason no method is applicable beyond the so-called separation point of the flow (p. 136), where in addition to $u_{(y=0)} = 0$, we have $(\partial u/\partial y)_{(y=0)} = 0$ (cf. § 6, p. 136). The calculation can be completely mechanized if we use finite differences in the y direction as well as in the x direction; it is then possible to study the effect of any arbitrary change in the distribution of pressure or of the initial profile.¶

* K. Pohlhausen, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 252 (1921).

† A. Walz, *LGL-Bericht*, 141, p. 8 (1941).

‡ Th. von Kármán, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 235 (1921); see also Durand, Vol. III, p. 102 *et seq.* or *Handb. d. Exp.-Phys.*, Vol. IV, 1, p. 283.

§ H. Holstein and T. Bohlen, *LGL-Bericht*, S10, on a premium award of 1940, p. 5. For a new method by A. Walz, using an energy theorem by K. Wieghardt, see *Ing.-Arch.*, Vol. 16, p. 243 (1948).

|| L. Prandtl, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 18, p. 77 (1938); see also H. Görtler, ibid., Vol. 19, p. 129 (1939).

¶ H. Görtler, *UM* 6015 (1944) and *Ing.-Arch.*, Vol. 16, p. 173 (1948); K. Schröder, *FB* 1741 (1943).

For a theorem by which boundary-layer problems for solids of revolution subjected to symmetrical flow can be reduced to problems of two-dimensional boundary-layer flow, see W. Mangler, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 28, p. 97 (1948). For flow along a plate with slight corrugations, see H. Görtler, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 25/27, p. 223 (1947); Vol. 28, p. 13 (1948), especially the summary on p. 21.

4. **Turbulence.***(5)

(*a*) When a viscous fluid flows through long straight tubes it is found that when the velocity is fairly high the Hagen-Poiseuille law given by formula (4), p. 99 (according to which the fall of pressure is proportional to the quantity of fluid flowing through), is replaced by another law, in which the fall of pressure is much greater and in fact nearly proportional to the square of the quantity of fluid. At the same time it is found that the flow, which in the region where the Hagen-Poiseuille law holds is smooth and rectilinear, becomes full of irregular eddying motions at higher velocities. This may be seen very clearly in the case of fluid flowing through glass tubes if a little coloured fluid is added by means of a small dropping-tube. With low velocities this forms a straight clean-cut filament, whereas with larger velocities the filament breaks off at a greater or smaller distance from the inlet and the colour thenceforth extends almost uniformly throughout the fluid.† The smooth motion is called " laminar " or " stream-line " motion, and the eddying motion " turbulent " or " sinuous " motion. Making use of the ideas of similarity discussed in § 2 (p. 103), Osborne Reynolds was enabled to predict that if the behaviour of tubes of varying width and of various liquids is investigated the transition from laminar to turbulent flow will always take place for the same numerical value of $\bar{q}d/\nu$ ($\bar{q}$ being the average velocity and d the diameter of the tube).‡ When the conditions at the inflow are sufficiently uniform this actually happens. For example, if a tube is joined to a smooth-walled vessel by a sharp edge, the " critical value " of Reynolds' number $\bar{q}d/\nu$ is equal to 2800.§ If the inlet is well rounded off and the fluid contents of the vessel are brought practically to rest, $\bar{q}d/\nu$ may rise to 40,000 or more, while if the inflow is very irregular it may fall to about 2320.§

* A more mathematical discussion of this subject is given in W. Tollmien's article in *Handb. d. Exp.-Physik.*, Vol. IV, 1, p. 289 *et seq.*; see also *Aerodynamic Theory*, Vol. III, p. 119 *et seq.*; Goldstein, *Modern Developments in Fluid Dynamics*, Vol. I, p. 191 *et seq.*; and Th. von Kármán, *Proc. IV International Congress for Applied Mechanics* (Cambridge, 1934), p. 54. For the latest German literature, see the author's article in the *F.I.A.T. Review, Aerodynamics*, Chapter III.

† This experiment was first carried out by Reynolds (1883).

‡ Engineers usually insert the diameter (d) and not the radius (r) of the tube in their formulæ. As $d = 2r$, $\bar{q}d/\nu$ is of course equal to $2\bar{q}r/\nu$. The two kinds of Reynolds' number must be carefully distinguished. Here the former usage is adopted.

§ L. Schiller, *Forschungsheft des Ver. Deutsch. Ing.*, 248, p. 16 (1922).

(*b*) The phenomenon of turbulence is not confined to flow in pipes and channels but is also found in the boundary layers described in § 3, p. 105. Here Reynolds' number qd/ν is obviously replaced by the number $q_1\delta/\nu$, where δ is the thickness of the boundary layer and q_1 the velocity of flow outside the boundary layer. Since for plates and similar objects the boundary layer is very thin near the edge exposed to the flow [cf. equation (11), § 3], the flow, unlike that in a pipe, remains laminar for a little way and only becomes turbulent further along, where $q_1\delta/\nu$ passes a certain critical value. For bevel-edged plates of length l towed through water at rest Reynolds' number $q_1 l/\nu$ has been found to be about 500,000. With a slender rounded-off front edge and a current of air particularly free of turbulence, the value of 3,000,000 has been reached in a wind tunnel at the National Bureau of Standards in Washington.

For the flat plate Blasius' formula gives

$$\frac{q_1\delta}{\nu} = 3\sqrt{\left(\frac{q_1 l}{\nu}\right)} \text{ approximately}$$

[cf. equation (17), p. 109]. The above critical values of $q_1 l/\nu$ therefore give $q_1\delta/\nu \approx 2100$, or 5200. Making a comparison with the pipe by putting, say, $d = 4\delta$, and $\bar{q} = \frac{1}{2}q_1$, we have $qd/\nu = 4200$ or 10,400, values which are quite feasible if the disturbances on entering the pipe are moderate or small. Further information on the sudden change of laminar flow in the boundary layer into turbulent flow will be found at the end of § 6 (p. 135).

(*c*) From the theoretical point of view there are two different problems to be considered: (1) the causes which lead to turbulence, and (2) the phenomena of flow where turbulence has already set in.

During the last fifty years and more a great deal of ingenuity, especially on the mathematical side, has been devoted to answering the question how turbulence arises, but for long the results obtained were unsatisfactory. It is only recently that a really satisfactory picture of the phenomenon has been obtained. The following is a brief sketch of the development of the theory from its beginnings.

O. Reynolds * pointed out that in certain forms of disturbance, energy is transferred from the main flow to the turbulent flow, but that on the other hand the disturbing motions can always be damped down if the viscosity is sufficiently great, so that the persistence of laminar motion for small values of Reynolds' number becomes intelligible. As regards the explanation of how turbulence arises, however, there is no proof that in the course of a definite flow phenomenon, more energy passes in the long run from the main flow to the sub-

* O. Reynolds, *Phil. Trans.* (1895); *Scientific Papers*, Vol. II, p. 535: see also H. A. Lorentz, *Abh. über Theor. Physik*, Vol. I, p. 43 (Leipzig, 1907; English translation published by Macmillan).

sidiary than vice versa. The detailed investigation of turbulence therefore became of great importance.

The following gives some indication of the main stages of the roundabout way by which the present state of this problem has been reached.* The first studies of the dynamics of small disturbances superposed on a given fluid motion were made by Lord Rayleigh,† on the flow of *frictionless fluid* between parallel boundaries, the velocity distribution being polygonal. Rayleigh was able to show that convex profiles are stable with respect to any small disturbance, but profiles with one or more re-entrant angles are unstable, that is, any deviation from the given flow increases more and more as time goes on. A. Sommerfeld‡ was the first to study the differential equation for perturbations of a *viscous fluid* and motions of the type shown in fig. 3.1 (p. 98), but found that there was stability for all values of Reynolds' number, that is, all types of disturbance tended to die away. In this he was supported by his fellow-worker L. Hopf.§

The author ‖ demonstrated the first case of instability in a profile without re-entrant angles and showed that it was a consequence of the secondary boundary layer which is formed in the immediate neighbourhood of the boundary during oscillations of a boundary layer, when Reynolds' number is large. The amplification of the oscillations arises from the fact that the periodic thickening and thinning of the secondary boundary layer set up a velocity field at right angles to the boundary; this velocity field extends a considerable distance into the fluid and, owing to its phase displacement, energy is transferred away from the main flow into the oscillation. O. Tietjens¶ made the corresponding calculations, for simplicity taking viscosity into account in the secondary boundary layer only. No critical value of Reynolds' number was obtained.

Further investigation showed that a second layer behaving in a special way is formed when the velocity of the main flow agrees with the phase-velocity of the oscillation. In the absence of friction this would lead to singularities in the motion of the particles, as here they are subject to the same pressure gradient for a very long time. If viscosity is postulated in this second layer also, however, the disturbance is free from singularities. With viscosity there is a second place where there is phase displacement of longitudinal motion. This phase displacement has a damping effect, and this, in conjunction with the amplification due to the secondary boundary layer, now gives a critical value for Reynolds' number. Here we have only hinted at the difficulties of the calculation; to have carried it out is the notable achievement of W. Tollmien.**(6)

* A detailed account of the whole position as it is at present and of the stages leading up to it is given in W. Tollmien's article in the Aerodynamics volume (Chapter II, " Laminare Grenzschichten ", § 3) of the *F.I.A.T. Review of Physics*. Reference should also be made to an earlier comprehensive report with bibliography by H. Schlichting, *Naturwissenschaften*, Vol. 22, p. 376 (1934).

† *Scientific Papers*, Vol. I, p. 474 (1880); Vol. III, p. 17 (1887); Vol. IV, p. 203 (1895).

‡ *Atti* IV *Congresso Internat. dei Matematici* (Rome, 1908), p. 116.

§ *Ann. d. Physik*, Vol. 44, p. 1 (1914).

‖ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 431 (1921); *Phys. Zeitschr.*, Vol. 23, p. 19 (1922).

¶ *Diss., Göttingen* (1922); *Zeitschr. f. angew. Math. u. Mech.*, Vol. 5, p. 200 (1925).

** " Über die Entstehung der Turbulenz ", *Göttinger Nachr., Math.-Phys. Kl.* (1929), p. 21. See also *Aachener Vorträge*, p. 18 (1929), and supplementary calculations by H. Schlichting, *Göttinger Nachr. Math.-Phys. Kl.* (1933), p. 181, and *Zeitschr. f. angew. Math. u. Mech.*, Vol. 13, p. 171 (1933); *Göttinger Nachr.*, Neue Folge, Fachgr. I, Vol. 1, p. 47 (1935), where there is a detailed bibliography; see also the more refined mathematical theory given by Tollmien, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 25/27, pp. 33, 70 (1947).

The unstable oscillations calculated by Tollmien, which are of a very long-drawn-out form, still do not represent turbulence, which is a motion with a much shorter wave-length. In practice, however, it is found that the actual turbulence regularly sets in at a point only a short distance from where instability is predicted by calculation. For years unsuccessful attempts were made to demonstrate the existence of the waves experimentally. Recently, however, they have been demonstrated beyond question by G. B. Schubauer and H. K. Skramstad † in a wind tunnel of the National Bureau of Standards in Washington which has a particularly low natural turbulence of its own. Wave-length, frequency, and amplification of the amplitude agree so well with the calculated values that the theory must be regarded as proved in every particular. Fig. 3.5 gives an idea of the nature of the results of Tollmien's calculations. The " neutral stability curve " (the boundary between the unstable region and the stable region) is obtained by plotting $\alpha\delta^*$ against $U\delta^*/\nu = \mathscr{R}_{\delta^*}$, where δ^* is the displacement thickness (p. 109), $\alpha = 2\pi/\lambda$, and λ is the wave-length of the disturbance. The numbers give the amplification of the amplitudes of accidental disturbances at the lower boundary of the unstable region as they cross this region, from data given by H. Schlichting.‡

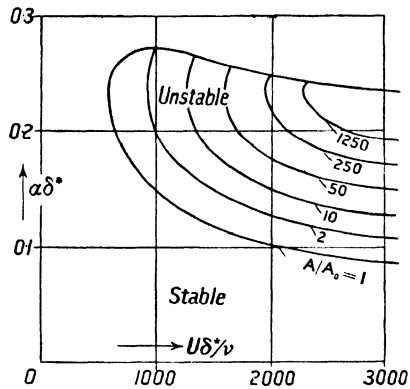


Fig. 3.5.—Neutral stability curve of the boundary layer of a plate

We may briefly mention H. Schlichting's successful applications § of Tollmien's method of calculation to the occurrence of turbulence in the boundary layer inside a rotating circular cylinder and to that occurring in the flow past a plate where there is thermal stratification. For the latter problem, see also Chapter V, § 12(*d*), p. 381.

It will be seen from the above that as a rule turbulence arises from a boundary layer. According to theory the established Hagen-Poiseuille flow in a pipe (fig. 3.13, p. 124) is stable even for high values of Reynolds' number.|| In reality, however, boundary layers are present in this case also in the initial period, and even later at the entrance to the pipe (fig. 3.47, p. 166). Any irregularities that may occur from time to time likewise give rise to the formation of boundary layers and hence, if Reynolds' number is sufficiently large, to turbulence; and once turbulence sets in, it spreads (see below).

† Secret report of National Advisory Council for Aeronautics (April, 1943) now made available; see also *Journ. Aeronautical Sciences*, Vol. 14, p. 69 (1947).

‡ *Naturwissenschaften*, Vol. 22, p. 376 (1934).

§ *Göttinger Nachr., Math.-Phys. Kl.* (1932), p. 160; *Zeitschr. f. angew. Math. u. Mech.*, Vol. 15, p. 313 (1935).

|| See e.g. J. Pretsch, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 21, p. 204 (1941), where the initial flow is also discussed and references to the literature are given.

In another paper Tollmien * showed that S-shaped velocity profiles, i.e. those having a point of inflexion, have a particularly strong tendency to instability, and calculated the rate at which disturbances are amplified in such cases. The more S-shaped the velocity profile,

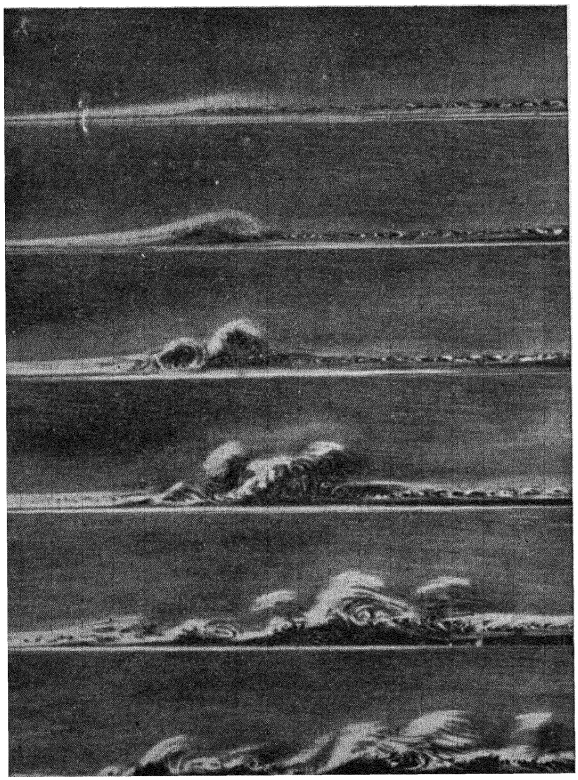


Fig. 3.6.—Propagation of turbulence

the smaller the critical value of Reynolds' number. As such profiles occur when the pressure rises in the direction of the flow,† a rise of pressure in this direction favours the occurrence of turbulence.

Continuing the theoretical work, of which a brief account has been given above, and which relates specially to the boundary layer of a plate, with p = const.,

* "Ein allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen", *Göttinger Nachr., Math.-Phys. Kl.*, I, 1, p. 79 (1935).

† Cf. what was said on p. 109 about the curvature of the profile in the neighbourhood of the boundary.

H. Schlichting and A. Ulrich * and J. Pretsch † independently applied Tollmien's methods of calculation to boundary layers with a rise or fall of pressure and calculated the critical values of $q\delta/\nu$ in terms of the " profile parameter " of the laminar profile. As might be expected, the critical values of $q\delta/\nu$ for boundary

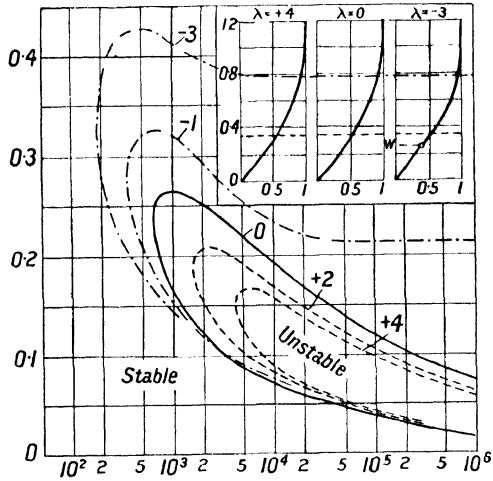


Fig. 3.7.—Neutral stability curves

——— flow past a plate
– – – – – flow with a fall of pressure
–·–·–·– flow with a rise of pressure

layers with a rise in pressure were markedly less than for the profile of the boundary layer of a plate, and for boundary layers with a fall in pressure markedly greater. Fig. 3.7 shows the regions of instability for various values of the profile parameter λ as given by Schlichting and Ulrich.

Since for a given aerofoil profile set at a given angle of incidence in a wind stream it is possible to find the pressure distribution and hence, by means of one-parameter boundary-layer methods, the behaviour of the profile parameter of the

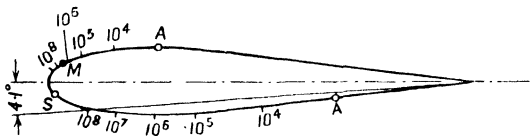


Fig. 3.8.—Aerofoil profile with critical values of Reynolds' number, after Schlichting

boundary layer and the thickness of the boundary layer, and, on the other hand, since the critical value of $q\delta/\nu$ corresponding to any value of the profile parameter can be found from fig. 3.7, it is possible to mark at every point of the outline of an

* *LGL-Ber.*, S. 10, p. 75 (1940); *Jahrb. d. Deutsch. Luftfahrt-Forschung*, Part I, p. 8 (1942).

† *Jahrb. d. Deutsch. Luftfahrt-Forschung*, Part I, p. 58 (1941); Part I, p. 54 (1942).

aerofoil profile the critical value of Reynolds' number qb/ν, which marks the limit of stability for that point when air flows past the aerofoil [b is the breadth of the aerofoil (fig. 3.80, p. 198)]. In this way it is possible to determine the properties of an aerofoil in ideal conditions purely by calculation, without the use of a wind tunnel.* Fig. 3.8 gives an example of this sort of calculation.†

It is possible that the passage from the Tollmien oscillations to turbulence occurs as a result of the oscillations of increased amplitude giving rise locally to S-shaped velocity profiles, which bring about a rapid collapse into eddies. According to observation, turbulence arising from any sort of disturbance always spreads of its own accord, new eddies continually forming upstream as the eddies first formed move away downstream. The process is illustrated in fig. 3.6, which was taken by a cine-camera moving with the group of eddies. The initial disturbance was caused by sucking out a little fluid through a slit in the wall (on the left in the first photograph) for a short time. So far, neither a theoretical explanation of the spread of turbulence nor an exact theory of the fully established phenomenon has been forthcoming.

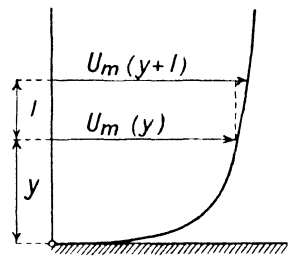


Fig. 3.9

(*d*) The following sub-section applies to turbulence which is already fully established and to the resulting increase of friction.

In § 14 of the last chapter it was shown that apparent stresses arise when there are variations of the velocity, owing to the momenta arising from these variations, e.g. a shearing stress $\tau = -\rho(u'v')_m$, where u', v' are the deviations of u, v, the components of the velocity, from their average values u_m, v_m. In order to obtain a formula of practical use we must attempt to express u', v', the fluctuations of velocity, by other quantities related to the distribution of mean velocity. Clearly it is essential to have a length which may be interpreted as the diameter of the balls of fluid which move as a whole and also as the path traversed by these balls relative to the rest of the fluid before they lose their individuality again by mixing with the turbulent fluid by which they are surrounded. We cannot say *a priori* that these two lengths are exactly the same; we may expect, however, that they will, broadly speaking, be proportional to each other. We now assume that the flow is such that the velocity varies in a direction at right angles to the stream-lines. If now (see fig. 3.9) a ball of fluid is displaced from a position y where the mean velocity is $u_m(y)$ through a distance l in a direction at right angles to the flow, the difference between its old and

* Examples of such calculations will be found in the paper by Schlichting and Ulrich (loc. cit.) and that by K. Bussmann, *Forschungs-Bericht* 1632 (1942).

† In fig. 3.8, S is the stagnation point, M the point of maximum velocity.

its new velocity is $u_m(y+l) - u_m(y)$, for which we may as a first approximation write $l\partial u_m/\partial y$. This gives the fluctuation u', at least as regards its order of magnitude. The value of v' is found from the assumption that two balls of fluid, which enter the layer in question from opposite sides and subsequently move on after each other, approach or recede from one another with relative velocity $2l\partial u_m/\partial y$, and that this gives rise to transverse velocities of the same order of magnitude. Hence v' must also be of the order $l\partial u_m/\partial y$. In forming the average value $(u'v')_m$ we have still to consider the signs of the corresponding u- and v-components. It is easy to see, however, that in crossing a control surface (p. 78) parallel to the boundary the parts moving away from the boundary are too slow and those moving towards the boundary too quick, so that negative values of u' correspond to positive values of v' and positive values of u' to negative values of v'. Thus the product $u'v'$ is negative in both cases, and hence the apparent shearing stress is positive and of the order $\rho(l\partial u_m/\partial y)^2$. If we arbitrarily take the unknown factor of proportionality as unity, we merely make a slight change in the meaning of l. To make the formula accurately express the fact that a positive shearing stress corresponds to positive values of $\partial u_m/\partial y$ and a negative shearing stress to negative values of $\partial u_m/\partial y$, we must write *

$$\tau' = \rho l^2 \left|\frac{\partial u_m}{\partial y}\right| \frac{\partial u_m}{\partial y}. \qquad \ldots\ldots \quad (19)$$

From this formula, which, as we have already mentioned, is only an approximate expression, we infer that with change of velocity the apparent stresses due to turbulent mixing are proportional to the square of the velocity. In actual fact all hydraulic resistances behave like this in the main. The length l, which we shall now call the "mixture length", bears a definite relationship to the mean free path in the kinetic theory of gases. In the latter the transference of momentum due to molecular motion is discussed in a way similar to our present account of the transference of momentum by the large-scale ("molar") motion of balls of fluid. As in the present case, the deviations from the mean velocity of particles moving upwards or downwards are given by $u' = \mp l_1 \partial u_m/\partial y$; the transverse velocity v', however, is not proportional to u' but equal to the molecular velocity. Thus the shearing stresses due to molecular motion (the viscosity stresses) are proportional to the first power of $\partial u_m/\partial y$, and so on. In gases the mean free path l_1 is inversely proportional to the density, so that here the factor representing the viscosity, in which the factor ρl_1 occurs, is independent of the density.

* $|\,a\,|$, the "absolute value" of a, means that the numerical value of a is to be inserted with the positive sign, no matter whether a is positive or negative.

If we put $A = \rho l^2 \mid \partial u_m/\partial y \mid$ in equation (19), p. 118, and seek to interpret A, we obtain an equation $\tau' = A\,\partial u_m/\partial y$ which is of the same type as the equation for the viscous shearing stress, $\tau = \mu\,\partial u/\partial y$, and A has the dimensions of viscosity. A, however, is generally larger by several powers of ten; it is known as the "Austausch" or "exchange coefficient".*[7] Another important difference from ordinary viscosity is that A usually varies very much from point to point; for example, it tends to zero as the boundary wall is approached.

(*e*) The effects of turbulent mixing include not only the increased apparent viscosity due to transfer of momentum but also the transfer by convection of all the properties of the moving matter (heat content, quantity of admixed matter, and so on). If e.g. the quantity of admixed matter varies from point to point, more material will be carried away with the portions of fluid coming from regions rich in it than will be brought in by portions of fluid coming from regions poor in it. Thus on the average the material will move from rich regions to poor regions. In the case of temperature differences this means a sort of turbulent heat conduction; in the case of differences in concentration, a sort of turbulent diffusion. Thus, since the quantity of heat contained in unit mass is $c_p\theta$, where θ is the temperature and c_p the specific heat at constant pressure, the quantity of heat flowing across unit area per unit of time is given by

$$Q = -c_p A_Q \frac{\partial \theta_m}{\partial y} = -c_p \rho l^2 \left| \frac{\partial u_m}{\partial y} \right| \frac{\partial \theta_m}{\partial y}.$$

That is, $c_p A_Q$ is the eddy conductivity. In the case of chemical or mechanical admixture of concentration c, the mass of admixed substance transferred across unit area in unit time is given by

$$M = -A_M \frac{\partial c_m}{\partial y}.$$

There remains the question whether A_Q and A_M agree numerically with A^τ, seeing that the mechanism of propagation of a property of matter or of an admixed substance is not entirely identical with that of transfer of momentum. Here we must distinguish between turbulence due to friction at a boundary wall and turbulence arising from a free surface of discontinuity. In so far as the latter is concerned, simultaneous measurements of the velocity profile and the temperature profile in the wake of a heated bar,† made by A. Fage and V. M. Falkner at the suggestion of G. I. Taylor,‡ definitely gave the value 2 for the ratio A_Q/A_τ. This value was also found in more recent experiments by H. Reichardt § on the mixing of a jet of air with air at a different temperature. For turbulent

* The name is due, I believe, to W. Schmidt, *Der Massenaustausch in freier Luft und verwandte Erscheinungen* (Hamburg, 1925); but J. Boussinesq used the same idea in his *Théorie de l'écoulement tourbillonant* (Paris, 1897), writing $\rho\epsilon$ instead of A.

† Cf. § 14(*c*), p. 186.

‡ *Proc. Roy. Soc.* (A), Vol. 135, p. 702 (1932).

§ *VDI-Forschungsheft* 414 (1942); see especially fig. 6, p. 12.

layers arising from friction along a boundary wall, A_Q has hitherto usually been taken as equal to A_τ and no deviation from existing experimental results on heat transfer has thereby been established. Nevertheless, H. Reichardt * was able to show by his theoretical investigations on the laws of heat transfer in turbulent boundary layers that the simultaneous measurements of velocity profiles and temperature profiles by F. Éliás † for plates and by H. Lorenz ‡ for pipes give $A_Q/A_\tau = 1{\cdot}4$ and $1{\cdot}5$ respectively (see also p. 406). What is true of A_Q presumably also applies to A_M.(8)

The difference in behaviour in the two types of turbulence is connected with the fact that in turbulence due to friction at a boundary, eddies with their axes parallel to the direction of flow apparently predominate, whereas in free turbulence eddies with their axes at right angles to the line of flow predominate. If only eddies of the first kind are present, the velocities of translation make no contribution to the mean velocity; in the other case they make a very considerable contribution, so that the distributions of mean velocity and of mean temperature or concentration exhibit well-marked differences (the mean velocity distribution being more sharply pointed).§ That the heat exchange is much more marked than the momentum exchange in the case of free turbulence was also shown by experiments of R. Gran Olsson ‖ on the smoothing out of temperature and velocity distributions in the rear of lattices of heated rods, in which the temperature differences vanished much more rapidly than the velocity differences.

(*f*) *Energy Relations in Turbulent Flow.*—Work is done on an element of the fluid by the apparent stresses and the corresponding pressure differences; this work serves to maintain the turbulent motion within the element (and possibly also in the surrounding region, if less energy is available there). In the very simple case considered here the work done on unit volume per second is $\tau' \partial u_m/\partial y$. This work enables the eddying motion to maintain itself against the resistance which it meets. The forward motion of the individual ball of turbulence we started with, relative to its surroundings, is itself a turbulent motion, and hence gives rise to a turbulence of the second order with smaller balls of turbulence, and these in turn produce turbulence of the third order. This process goes on until the final elements are so small that they do not become turbulent; in them, as a result of viscosity, what is left of the kinetic energy is transformed into heat.

The amount of energy transformed into heat in unit volume in unit time is given by the dissipation function D, which is built up ¶ of the mean values of the squares and products of the partial differential coefficients of u', v', w' with respect to x, y, z. These mean values arise chiefly from the smaller eddies. In order to evaluate them G. I. Taylor has introduced a length λ such that, for example,

$$\left\{\left(\frac{\partial u'}{\partial y}\right)^2\right\}_m = \frac{(u'^2)_m}{\lambda^2} \times \text{constant};$$

see § 5(*g*), p. 133, where a strict definition of λ is given, and also references to the literature. If for brevity we write u' instead of $\sqrt{\{(u')^2\}_m}$, we may put

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 297 (1940); see especially p. 327.

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 9, p. 434 (1929) (*Abh. d. Aerodyn. Inst. Aachen*, No. 9 (1930)).

‡ *Zeitschr. f. techn. Phys.*, Vol. 15, p. 376 (1934).

§ A theory of these phenomena is given by G. I. Taylor, loc. cit., p. 685.

‖ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 16, p. 257 (1936).

¶ In the case of isotropic turbulence (§ 5(*g*), p. 133), D is given by the simple expression $D = 7{\cdot}5\mu\{(\partial u'/\partial y)^2\}_m$, according to G. I. Taylor. For other forms of turbulence (wall turbulence, free turbulence) the numerical value has still to be found.

$D = \mu(u'/\lambda)^2 \times$ constant. Since $u' = l\left|\frac{\partial u_m}{\partial y}\right|$ [see (d)], we can put u'/l as an approximation for $\left|\frac{\partial u_m}{\partial y}\right|$ and replace $|\tau|$ by $\rho u'^2$, in the equation $D = \tau'(\partial u_m/\partial y)$. We thus have

$$\mu(u'/\lambda)^2 \times \text{constant} = \rho u'^3/l,$$

whence

$$\lambda = \sqrt{(\nu l/u')} \times \text{constant}.$$

If $\mathscr{R}_l = u'l/\nu$ is introduced as Reynolds' number for the motion of an eddy we have $\lambda \sim l/\sqrt{\mathscr{R}_l}$.

5. Further Study of Turbulence.

The "mixture length" (l) in turbulent motion in general varies from place to place. Regarding its magnitude no general theory is yet available, although in a number of particular cases it has been found possible to make assumptions leading to results in good agreement with experiment. In many cases it is permissible to neglect the actual shearing stresses arising from the viscosity in comparison with the apparent shearing stresses, and also—a more far-reaching assumption—to leave the effect of viscosity on the magnitude of l out of account. In this case, therefore, we have to deal, so to speak, with the turbulence of an ideal fluid with zero viscosity. If Reynolds' number is sufficiently large this point of view is certainly justified.

Here we shall discuss two cases in more detail, the so-called free turbulence and the turbulence which arises along a smooth boundary [(a) and (b)]; the effect of viscosity in the latter case is then discussed in (c). Subsection (d) deals with the flow along a rough boundary and (e) with the flow past a plate. Subsection (f) deals with stratified fluid and flow round curves, and finally (g) and (h) with turbulence in wind tunnels.

(a) In cases like the mixing of a free jet having a sufficiently high value of Reynolds' number with the fluid at rest surrounding it, it seems plausible to take the mixture length for every cross-section as being proportional to the breadth of the jet there ($l = \alpha b$).

By b we may, for example, mean half the base of a parabolic or paraboloidal distribution of velocity, in which the maximum velocity and quantity of fluid moving coincide with those of the actual flow considered. Some such assumption is necessary, as the actual flow passes diffusely into the external fluid without any perceptible boundary. Making an assumption of this kind we get values for α round about 0·125.

Observation shows that free jets in a sufficiently large space full of fluid at rest spread out in such a way that except in the immediate neighbourhood of the outflow the breadth of the jet is very accurately proportional to the distance from the point of outflow, the velocity diminishing with increasing distance. Throughout the jet the pres-

sure is nearly the same as in the surrounding fluid.* The decrease in velocity with increase of distance from the point of outflow is therefore due to the frictional stresses alone. The decrease in velocity does not take place in such a way that the same quantity of fluid flows across all cross-sections; this is not the case, as during the advance of the flow fresh masses of fluid at rest are continually being drawn in and carried along. On the other hand, the momentum of the jet, $I = \rho \iint u^2 da$, is constant on account of the constant pressure (cf. § 12 of last chapter, p. 77). We have $I = \rho u_1^2 \pi b^2 \times$ constant, where u_1 is the maximum velocity in the cross-section. Thus it follows from the fact that I is constant that u_1 is proportional to $1/b$, i.e. to $1/x$. The flow is somewhat as shown in fig. 3.10.

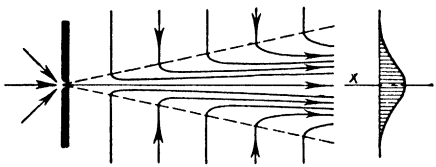


Fig. 3.10.—Mean stream-lines of a jet of air spreading out

Formula (19) of § 4 is in very good agreement with this relationship between b and u_1. Starting from this formula we can not only deduce their behaviour, but also the distribution of velocity in detail, only the constant α or one corresponding to it remaining arbitrary.† The observed behaviour of a jet when spreading out, which we have described above, thus furnishes a proof of the reliability of formula (19). We shall briefly indicate how an approximation yielding the most important relationships may be obtained without actually carrying out the calculations.

As before we put $l = \alpha b$. By the momentum theorem we have $u_1 b = \text{const.} = C$. If in the expression for the shearing stress, $\tau' = -\rho l^2 (\partial u/\partial y)^2$ (the minus sign holding for positive values of y), we replace $\partial u/\partial y$ by the approximate value $-2u_1/b$, we get

$$\tau'_m = -4\rho l^2 \left(\frac{u_1}{b}\right)^2 = -4\alpha^2 \rho u_1^2$$

as the mean value over the cross-section. By formula (6) (p. 102) the force per unit volume is $\partial\tau'/\partial y$; starting from the middle of the jet, τ' increases from zero to a maximum and then decreases to zero again, so that $\partial\tau'/\partial y$ is first negative, then positive. In the first region the main flow is retarded, while in the second the air carried along is accelerated. For the central flow we may take $\partial\tau'/\partial y$

* In discussions of the ordinary liquid spray one sometimes finds the assumption that there is a rise of pressure in the air-jet as the velocity decreases, by Bernoulli's theorem, and that the pressure at the point of outflow is therefore reduced, thus causing fluid to be sucked up. This is incorrect: Bernoulli's theorem is true only when frictional stresses are absent, which is certainly not the case here. On the contrary, the suction is due to the flow round the edge of the tube which projects into the jet at right angles. In the spreading jet the pressure is practically the same as in the surrounding air at rest.

† W. Tollmien, *Berechnung turbulenter Ausbreitungsvorgänge, Zeitschr. f. angew. Math. u. Mech.*, Vol. 6, p. 468 (1926).

as proportional to $2\tau'_m/b$. Thus for a volume element of the central flow the equation Mass × Acceleration = Force gives

$$\rho u \frac{\partial u}{\partial x} = \frac{\partial \tau'}{\partial y} \sim -8\rho \frac{\alpha^2 u_1^2}{b},$$

and we may put $u = u_1$ on the left-hand side, so that

$$\frac{du_1}{dx} \sim -8\frac{\alpha^2 u_1}{b}.$$

As $u_1 = C/b$ (by the momentum theorem) and $\frac{du_1}{dx} = -\frac{C}{b^2}\frac{db}{dx}$, we have

$$\frac{db}{dx} \sim 8\alpha^2,$$

i.e. $b \sim 8\alpha^2 x$, in accordance with the experimental result $b = x/8$.

Another case not without importance is that of the spread of the edge of a jet (fig. 3.11). Here $u_1 = \text{const.}$; if we put $l = \alpha b$ we have, as before, $\tau'_m \sim \alpha^2 \rho u_1^2$, i.e. τ'_m is also constant. The loss of momentum * of the part of the flow coming from the pipe is proportional to $\rho u_1^2 b$, and the corresponding resistance is proportional to

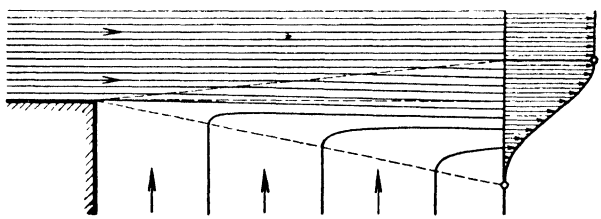


Fig. 3.11.—Spread of the edge of a jet

$\tau_m \,.\, x$, so that $b \sim \alpha^2 x$, as in the previous example. The fluid sucked in from the surrounding region at rest shows an equal gain of momentum. The slope of the boundary between the undisturbed portion of the jet and the turbulent zone is of practical importance; it may be taken †[9] as 1 : 10.

Yet another case is that of the wake in the rear of a moving body (cf. § 14(*c*), p. 184).

* The loss of momentum and the resistance are calculated for a cross-section of unit depth in the direction perpendicular to the plane of the paper.

†This case and others are discussed by W. Tollmien, loc. cit.; see also L. Prandtl, "Über die ausgebildete Turbulenz", *Proc. 2nd Internat. Congr. Appl. Mach.*, 1926 (Zürich, 1927), p. 62. In *VDI-Forschungsheft* 414 (1942), H. Reichardt sets himself the task of calculating all the quantities of importance in connexion with turbulence from experimental results, using as a basis his own experiments on the phenomena of free turbulence. For a theory of free turbulence developed from this, see *Zeitschr. f. angew. Math. u. Mech.*, Vol. 21, p. 257 (1941). Another hypothesis was put forward by L. Prandtl and tested in some particular cases by H. Görtler; see *Zeitschr. f. angew. Math. u. Mech.*, Vol. 22, pp. 241, 244 (1942).

(*b*) In cases of flow along boundaries the " mixture length " must, as is obvious from its definition, tend to zero as the boundaries are approached. It follows from this that $\partial u_m/\partial y$ becomes very small in the interior of the flow but reaches large values in the neighbourhood of the boundaries. Figs. 3.12 and 3.13 show the difference between the distributions of velocity for turbulent flow and laminar flow in a tube.

The layer of fluid next the boundary adheres to the latter even in the case of turbulent flow, and in the immediate neighbourhood of the boundary a thin layer is formed in which the flow is laminar, so that in fact $\partial u_m/\partial y = \tau_{\text{wall}}/\mu$, provided that the boundary is smooth.

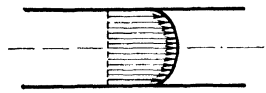


Fig. 3.12.—Distribution of velocity in turbulent flow in a pipe

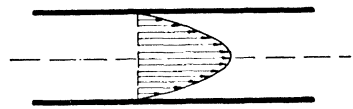


Fig. 3.13.—Distribution of velocity in laminar flow in a pipe

For large values of Reynolds' number, the value of τ_{wall} is quite considerable owing to the vigorous mixing in the interior of the fluid, so that the rate of increase $\partial u_m/\partial y$ is extremely rapid and the laminar boundary layer is accordingly very thin. On superficial observation, then, it seems as if in turbulent motion the velocity had a finite value even at the boundary itself.

From the theoretical point of view a general idea of the state of affairs may be got very simply if we assume that the shearing stress is constant throughout the region.* If this is positive, $\partial u_m/\partial y$ is also positive, so that the modulus sign can be dropped from equation (19), p. 118. The total shearing stress (the mean value of the viscous stress plus the apparent shearing stress due to turbulence) is then given by

$$\tau = \mu \frac{\partial u_m}{\partial y} + \rho l^2 \left(\frac{\partial u_m}{\partial y}\right)^2. \quad . \quad . \quad . \quad . \quad (20)$$

The first term is important only for very small distances from the boundary; if Reynolds' number is fairly large the second term is so much greater than the first until close to the boundary that the first may be neglected in comparison with it. Taking the square root of the resulting simplified form of equation (20), we have

$$\sqrt{\frac{\tau}{\rho}} = l \frac{\partial u_m}{\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

* In actual cases of flow τ usually decreases in a continuous way as the distance from the boundary increases (for the pipe it becomes zero on the axis). Nevertheless, the formulæ obtained by putting $\tau = \text{const.} = \tau_{\text{wall}}$ give very useful approximations even in this case, for by far the greater part of the fall in velocity occurs at the boundary or very close to it. For pipes the formulæ (24) and (28) hold good nearly to the axis, since here l lags behind κy as the distance from the wall increases (see p. 126).

From the right-hand side we readily see that $\sqrt{(\tau/\rho)}$ has the dimensions of a velocity. For simplicity we introduce the symbol v_* for it and call v_* the shearing-stress velocity. From § 4, p. 117, we see that v_* is of the same order of magnitude as the velocities u', v' due to turbulence (more accurately, $v_* = \sqrt{|\,(u'v')_m\,|}$). With the assumptions we have made here, however, v_* may be constant.

We shall now suppose that $y = 0$ represents a smooth wall, and for simplicity we shall regard it as extending to an infinite distance in both directions. We shall suppose that the other wall is at such a distance that it may be regarded as being at an infinite distance so far as the state of affairs in the neighbourhood of the first wall is concerned. Then u_m depends on y only; in what follows, therefore, we shall write du_m/dy for $\partial u_m/\partial y$, and as we shall no longer be concerned with fluctuations, we shall also drop the suffix m.

We have now to find a reasonable law for the mixture length l, i.e. one which gives the correct dimensions. If we make the further assumption (suggested by observation) that l is unaffected by the viscosity, the only length we have at our disposal is the distance from the wall, y. The only dimensionally correct formula for l is then

$$l = y \times \text{constant}.$$

The numerical factor, which in future we shall denote by κ, is then an essential (universal) constant of this problem in turbulent flow. From equation (21) we then have

$$v_* = \kappa y \frac{du}{dy},$$

and since v_* is constant, this can be immediately solved, giving

$$u = v_* \left(\frac{1}{\kappa} \log y + C \right). \quad . \quad . \quad . \quad . \quad . \quad (22)$$

For large values of Reynolds' number this expression is in good agreement with observation; the latter gives 0·417 as the value of κ. (It is true that for $y = 0$ the formula gives the value $-\infty$ instead of the value 0, but we know already that our simplified calculation would not apply here; instead, we should have to use the more accurate equation (20) and set up a modified formula for l involving the second length ν/v_*.)

We can also obtain an expression for C, the constant of integration, from the fact that the viscosity becomes important in the immediate neighbourhood of the wall. The expression in the bracket in equation (22) must be a pure number and must not depend on the units

employed. This is achieved if we subtract from $\log y$ the logarithm of the length ν/v_* mentioned above, i.e. if we put

$$C = C_1 - \frac{1}{\kappa} \log \frac{\nu}{v_*}.$$

C_1 is then a second universal number, and we have

$$u = v_* \left(\frac{1}{\kappa} \log \frac{y v_*}{\nu} + C_1 \right). \quad . \quad . \quad . \quad . \quad (23)$$

Since the greatest velocity differences occur in the immediate neighbourhood of the wall, formula (23) may also be used as a good approximation in cases in which the shearing stress τ depends on y. We have merely to put $v_* = \sqrt{(\tau_{\text{wall}}/\rho)}$, and obtain values of the velocity which are found to lie very close to the observed values. For these cases of deviation from theory, e.g. for flow in pipes, the observed values of u/v_* can be plotted against $\log_{10} \frac{y v_*}{\nu}$; the curve obtained is almost a straight line, for which an equation may be found. If equation (23) is used in this way as an approximation to the distribution of velocity in smooth-walled straight pipes, Nikuradse's experiments † give $\kappa = 0{\cdot}40$ and $C_1 = 5{\cdot}5$. Passing from natural logarithms to ordinary logarithms ($\log_e x = 2{\cdot}3026 \log_{10} x$), we obtain the practical formula

$$u = v_* \left(5{\cdot}75 \log_{10} \frac{y v_*}{\nu} + 5{\cdot}5 \right). \quad . \quad . \quad . \quad (24)$$

Using experimental values only for points close to the wall, we obtain an approximation for the ideal case $\tau = \text{const.}$ Here we obtain $\kappa = 0{\cdot}417$, as was mentioned above, and $C_1 = 5{\cdot}84$. The practical formula ‡ for this case is then

$$u = v_* \left(5{\cdot}52 \log_{10} \frac{y v_*}{\nu} + 5{\cdot}84 \right). \quad . \quad . \quad . \quad (25)$$

(*c*) For small values of the ratio of the two lengths y and ν/v_* viscosity plays an important part in two ways: directly by the contribution $\mu \, du_m/dy$ to the shearing stress in equation (20) and indirectly by its effect on the mixture length l. The two effects are taken account of by proving that if we assume that $\tau = \text{const.}$, the ratios u/v_* and $y v_*/\nu$ are the only independent dimensionless quantities involved, and

† "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren", *VDI-Forschungsheft* 356 (1932).

‡ J. Nikuradse, loc. cit., p. 17; also L. Prandtl, *Ergebn. d. Aerodynamischen Versuchsanst. zu Göttingen*, Vol. 4, p. 21 (1932).

hence that if $\tau = \text{const.}$, u/v_* must be a universal function of yv_*/ν. This is in fact the case. For points close to the wall we have the laminar boundary layer already mentioned, in which $\tau = \mu u/y$; since $\tau = \rho v_*^2$, this gives $u/v_* = yv_*/\nu$. As an approximation for the lower region of turbulence (yv_*/ν equal to about 40 to 600) we obtain a convenient power formula

$$\frac{u}{v_*} = 8{\cdot}7\left(\frac{yv_*}{\nu}\right)^{1/7}; \qquad \ldots\ldots \quad (26)$$

according to this, then, u is proportional to the seventh root of y, which has been well established for pipes up to about $\mathscr{R} = 100{,}000$. For greater values of $\mathscr{R}$, u is approximately proportional to the eighth, ninth and tenth roots of yv_*/ν; these formulæ, however, are merely approximations to the more accurate formula (25).

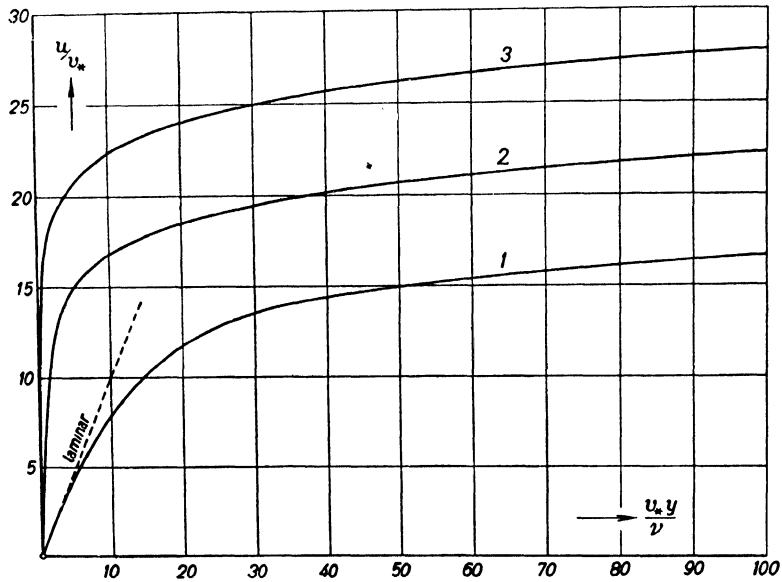


Fig. 3.14.—Universal velocity function
("law of flow past a wall")

The behaviour of the universal velocity distribution obtained by experiment for $\tau = \text{const.}$ is shown in fig. 3.14. For curve 1 the values shown for the abscissæ apply; for curve 2 these are to be multiplied by 10 and for curve 3 by 100. The three curves together give a clear picture of the velocity distributions observed for the same shearing stress at the wall if the viscosity is reduced to a tenth or a hundredth

as compared with curve 1. In fig. 3.15 the same function is represented, so far as it is backed by experiment, by the heavy continuous curve, the abscissæ being plotted logarithmically. We see that from about $yv_*/\nu = 50$ onwards it is practically a straight line, which enables us to extrapolate for values of yv_*/ν as large as we please. The dotted

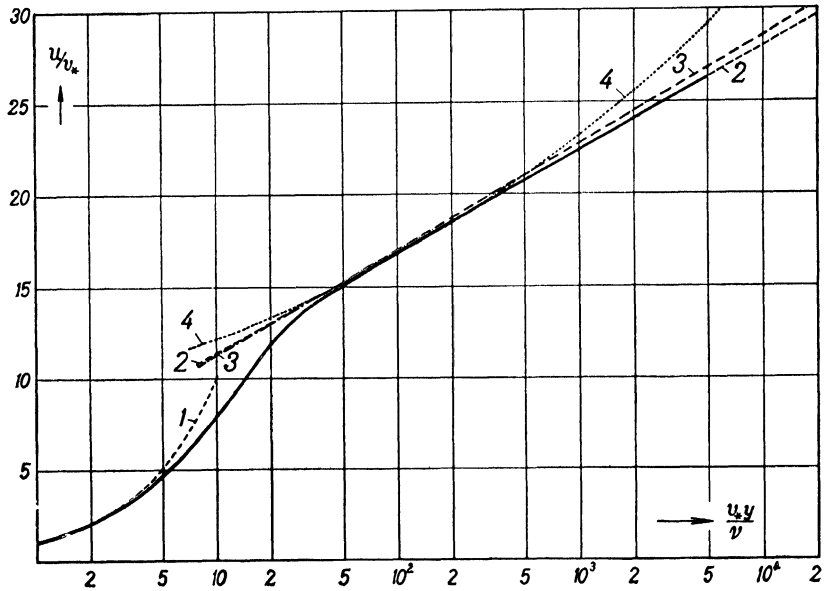


Fig. 3.15.—Universal velocity function plotted on semi-logarithmic paper

curves represent the formulæ given above: 1—the law of laminar flow $u/v_* = yv_*/\nu$; 2—the "law of flow past a wall" [equation (25)]; 3—the "pipe formula" [equation (24)]; and 4—the power formula [equation (26)].

It may be mentioned here that equations (23) to (26) can of course also be solved for v_*, so that we can obtain the shearing stress at the wall for given values of u and y. From equation (26), for example, we have

$$\tau_{\text{wall}} = \rho v_*^2 = (8{\cdot}7)^{-7/4}\rho u^{7/4}\left(\frac{\nu}{y}\right)^{1/4} = 0{\cdot}0225\,\rho u^2\left(\frac{\nu}{uy}\right)^{1/4}, \quad (27)$$

a relationship which can also be deduced from an empirical law, discovered by H. Blasius, for the fall of pressure in a smooth pipe [see § 11, equation (46), p. 163].

Historically the investigation took a different line. It started from the law of Blasius, from which equation (27) above was deduced. This gave the relation-

ship† $u \sim y^{1/7}$ for the case $\tau_{\text{wall}} = \text{const.}$ The advance which led to our equation (23) started with an argument based on similarity due to Th. von Kármán,‡ in which he postulated that in turbulent flow where Reynolds' number was so large that the effect of viscosity could be neglected, the turbulent mixing took a similar course everywhere; thus it could be distinguished only by its unit of length and its unit of time (the unit of length is then identical, apart from a numerical factor, with our mixture length l, and the unit of time is proportional to $1\Big/\frac{du}{dy}\Big)$. From Euler's equation von Kármán§ deduced (by a not very convincing argument, it is true) that $l = \kappa \left| \frac{du}{dy} \Big/ \frac{d^2u}{dy^2} \right|$, and was thus in a position to solve several problems numerically. In the case $\tau = \text{const.}$ discussed in (*b*), we had $\frac{du}{dy} = \frac{\text{const.}}{y}$. In this case von Kármán gets $\left| \frac{du}{dy} \Big/ \frac{d^2u}{dy^2} \right| = y$, so that his assumption becomes identical with ours ($l = \kappa y$).

(*d*) The line of argument which led to equation (22) still holds good for turbulent flow past a *rough* boundary, except for the determination of the constant of integration C. A second length is available here, namely the height (k) of the rough projections, and the question is now whether the length ν/v_* is also involved or not. The criterion is the ratio of these two lengths, kv_*/ν, which may be regarded as a Reynolds' number characteristic of the single projection. If kv_*/ν is sufficiently large, ν/v_* becomes small relative to k, and an argument similar to that which led to equation (23) gives

$$C = C_2 - \frac{1}{\kappa} \log k,$$

i.e.

$$u = v_* \left(\frac{1}{\kappa} \log \frac{y}{k} + C_2 \right), \quad . \quad . \quad . \quad . \quad . \quad (28)$$

or, changing to ordinary logarithms,

$$u = v_* \left(5{\cdot}75 \log_{10} \frac{y}{k} + C_2 \right). \quad . \quad . \quad . \quad . \quad (29)$$

For smaller values of kv_*/ν, C_2 is replaced by a function of kv_*/ν, which for very small values of kv_*/ν takes the form

$$C_1 + \frac{1}{\kappa} \log \frac{kv_*}{\nu},$$

so that equation (28) is replaced by equation (23), i.e. the slightly rough pipe behaves as if "hydraulically smooth".

† See v. Kármán, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 233 *et seq.* (1921)., and Prandtl and Betz, *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 3, p. 1 *et seq.* (1927).

‡ *Göttinger Nachrichten* (1930), p. 58, and a report given at the third International Congress on Applied Mechanics, Stockholm, 1930 (see the *Proceedings* of this congress, Vol. 1, p. 85).

§ See *Aerodynamic Theory*, Vol. III, p. 132.

J. Nikuradse † tested these conclusions with pipes made rough by adhering sand grains of various sizes, and obtained good confirmation of them. For $kv_*/\nu < 4$ the pipes behaved as if they were practically smooth, while for $kv_*/\nu > 80$, ν had no effect at all, i.e. equation (28) held good. C_2 was then found to be 8·5 when the size of grain k was defined as the size of mesh of the coarser of the two sieves through which the sand was sifted. When equation (28) was solved for v_*, v_* was proportional to u for given values of y/k, i.e. τ_{wall} was proportional to u^2 (cf. also § 11, p. 161).

With rough surfaces as they occur in practice there is the difficulty that individual irregularities vary very much in size and, moreover, their shape is ill-defined, so that usually it is not possible to deduce a trustworthy value of k from the form of the surface. The constant C of equation (22) can, however, be found by experiment and the "equivalent sand-grain roughness" k_s calculated from $C = 8{\cdot}5 - 5{\cdot}75 \log_{10} k_s$. For natural wind passing over ground covered with vegetation, experiments by W. Paeschke ‡ gave $k_s = 4k$ approximately, where k was the height of the vegetation.§ It comes to this, that C_2 in formula (28) must be lowered to about 5·0 if k is to be the height of the vegetation.(10)

As we have said, in rough surfaces occurring in practice the individual irregularities usually vary very much in size; hence as compared with the case of uniform medium roughness the limit of hydraulic smoothness is lowered and that where viscosity begins to have an effect is raised.‖

Related to roughness is "waviness" or "corrugation", a form of surface where there is no separation of the flow (see next section, p. 136) at the individual smooth irregularities, but yet the resistance to flow is noticeably raised.¶ Here the relationships are similar to those for smooth walls, i.e. for moderate values of Reynolds' number τ is given approximately by formula (27), but with a higher numerical coefficient.

These relationships have also been applied to the study of water waves and wind friction on water surfaces (cf. Chapter II, § 15, p. 91). H. U. Roll** gives a formula for the wind profile which resembles that for a smooth boundary and which, expressed in the notation of equation (24), p. 126, is

$$u(z) = v_*\{5{\cdot}75 \log_{10}(v_* z/\nu) + 1{\cdot}85\},$$

where $v_* = \sqrt{(\tau'/\rho_{\text{air}})}$, ρ_{air} is the density of the air at the height of the anemometer, and z is the height above the mean water level.†† For values of v_* below about 10 cm./sec. the flow in the lowest 2 metres seems to be laminar (cf. (c) and

† "Strömungsgesetze in rauhen Rohren", *VDI-Forschungsheft* 361 (1933).

‡ *Diss. Göttingen*, 1937; part published in *Beiträge zur Physik der freien Atmosphäre*, Vol. 24, p. 163 (1937).

§ For dense vegetation (e.g. grain crops) the roughness height k is to be taken as the difference between the heights of the tallest plant and the shortest plant. In the case of the "sand-grain roughness" k_s it was not this difference that was used, but the size of grain, or more accurately the size of mesh of the sieve, for convenience of measurement. This explains why the "equivalent sand grain" is about four times the size of the variation in height of the vegetation.

‖ For details, see e.g. F. Schultz-Grunow, *Jahrb. d. Schiffbautechn. Ges.*, Vol. 39, p. 176 (1938).

¶ See e.g. Hopf and Fromm, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 3, pp. 329, 339 (1923), and Hopf, *Handbuch der Physik*, Vol. VII, p. 146 *et seq.*

** *Ann. d. Meteorologie*, Vol. 1, p. 139 (1948).

†† The change in the additive term as compared with that in the formula for the smooth boundary (5·5) means that for a given value of v_* the velocity of the air above the surface of the water is less by $3{\cdot}65\, v_*$ than for a smooth surface.

fig. 3.14). For τ' the apparent shearing stress of the turbulent wind, G. Neumann * gives an empirical formula, namely,

$$\tau' = 0{\cdot}009\rho_{\text{air}} q_{\text{air}}^{3/2},$$

where q_{air} is the wind velocity in metres/sec. at the height of the anemometer and $\rho_{\text{air}} = 0{\cdot}125$ Kg. sec.^{2}/m.^{4} These formulae hold good for fully developed flow. In the initial stages, however, frictional effects are much greater.

(*e*) For experiments with *plates* in fluid moving as in fig. 3.4 (p. 107), the momentum argument developed in connexion with fig. 3.4 may be repeated for the law of friction with turbulent flow [equation (27)]. We thereby obtain a frictional layer increasing proportionally to $x^{4/5}$ (x being the distance from the front edge) and a resistance proportional to $\frac{\rho q^2}{2}\left(\frac{ql}{\nu}\right)^{-1/5}$. Like equation (27) on which it is based, this relationship holds only for a limited region and must be replaced by other formulæ when Reynolds' number has higher values (cf. § 15, p. 188). If the flow remains laminar at the front of the plate, as is the case with bevelled plates, the resistance falls somewhat below that given by the formula,† and the more so the greater the region of laminar flow (see also p. 194).(11)

The phenomena which occur when a stream of fluid enters a pipe (Reynolds' number being large) bear a certain resemblance to those associated with a plate subject to parallel flow. If the entering fluid is comparatively free from disturbance, the length of the region of laminar flow up to the region of turbulence is given approximately by $ql/\nu = 300{,}000$, or

$$\frac{l}{d} = \frac{ql/\nu}{qd/\nu} = \frac{300{,}000}{\mathscr{R}}.$$

(*f*) If in a flow which is predominantly horizontal the density of the medium diminishes rapidly upwards (as e.g. in a mass of air with the temperature increasing upwards, or where there is a layer of fresh water superimposed on salt), the process of turbulent mixing must cause heavier layers to be moved above lighter, and lighter layers to be pushed down below heavier. That is, part of the store of work available for the maintenance of turbulence (derived from the main flow) is used up against gravity.‡ Hence the turbulent motion is diminished and may die out altogether. This is the explanation of the cessation of turbulence and " dying-down " of the wind at night in the lower layers of the atmosphere (the wind still continuing unabated at a higher level). Conversely, turbulence is increased by irradiation

* *Zeitschr. f. Meteorologie*, Vol. 2, p. 193 (1948).

† See e.g. L. Prandtl, *Ergebnisse der Aerodynamischen Versuchsanstalt*, Vol. 3, pp. 4, 87 (Munich and Berlin, 1927); Vol. 4, p. 27 (1931).

‡ For a quantitative discussion, see Chapter V, § 12(*d*), p. 381.

of the ground, which causes a reversal of the stratification in order of density.*

In the case of flow round curves, turbulence is also diminished or increased as a result of the centrifugal forces, according as the velocity increases or decreases from the centre of curvature outwards.† Here the variation in magnitude of the centrifugal forces plays the same part as the variation of the force of gravity does in the flow of layers differing in density.

According to Couette,‡ if a fluid occupies the space between two cylinders, the *outer* rotating and the inner at rest, and if the distance $d = r_2 - r_1$ between the cylinders is small compared with r_1 and r_2, there is a critical peripheral velocity v such that Reynolds' number $vd/\nu = 1900$. If the distance between the cylinders is greater, the stabilizing effect mentioned above comes into play and the critical value of Reynolds' number rises sharply. On the other hand, if the *inner* cylinder is rotating and the outer at rest, even the stream-line motion is unstable, regular eddies being formed with their axes parallel to the direction in which the periphery is moving and with alternate right-handed and left-handed directions of rotation. The condition for the occurrence of this instability has been obtained theoretically by G. I. Taylor § and confirmed by experiment. It may be expressed as follows:

$$\frac{vd}{\nu} > 41{\cdot}3\sqrt{\left(\frac{r}{d}\right)},$$

where r is the mean of the two radii.

The stability or instability just mentioned is noticeable in turbulent boundary layers adjacent to surfaces of comparatively slight curvature. The turbulent mixing is enfeebled at convex surfaces and strengthened at concave surfaces.|| Eddies similar to those of Taylor described above may even occur in laminar flow past concave surfaces. According to H. Görtler,¶ if δ is the thickness of the boundary layer, $U\delta/\nu = 16\sqrt{(r/\delta)}$ may be regarded as the limit of stability.

Examples of the turbulent velocity profile between a rotating cylinder and a non-rotating cylinder are given in fig. 3.16, based on experiments by F. Wendt.** The exchange of momentum is small if

* See e.g. the article by L. Prandtl in *Vorträge aus dem Gebiete der Aerodynamik und verwandten Gebieten*, Aachen, 1929 (Berlin, 1930), p. 1.

† L. Prandtl, loc. cit.; A. Betz, ibid., p. 10; H. Wilcken, *Ing.-Arch.*, Vol. 1, p. 357 (1930).

‡ M. Couette, *Ann. de chim. et phys.* (6), Vol. 21, p. 433 (1890).

§ G. I. Taylor, *Phil. Trans.* (A), Vol. 223, p. 317 (1923).

|| See e.g. H. Wilcken, *Ing.-Arch.*, Vol. 1, p. 351 (1930), and H. Schmidbauer, *Diss.* Technische Hochschule, Munich (1934).

¶ *Göttinger Nachrichten, Math.-Phys. Kl., Neue Folge*, Vol. 2, p. 1 (1940).

** *Ing.-Arch.*, Vol. 4, p. 577 (1933); see also G. I. Taylor, *Proc. Roy. Soc.* (A), Vol. 151, p. 494 (1935), and Vol. 157, pp. 546, 565 (1936).

the outer cylinder rotates, but very great if the inner cylinder rotates.

(g) Much attention has recently been devoted to *turbulence in wind tunnels* [cf. § 22(a), p. 246].[12] Here turbulence is undesirable, as the object of experiments in wind tunnels is to simulate the state of affairs when a body is moving at a uniform speed through air at rest. Turbulence, however, cannot be entirely avoided; a small amount still remains after the air has passed through the " honeycomb ". It particularly affects the occurrence of turbulence in the boundary layers adjacent to the bodies under investigation, and hence also the separation of the flow from the bodies (see the two following sections). An older way of measuring turbulence in a wind tunnel was by the fall in the resistance of a sphere (cf. § 15, p. 189) due to the onset of turbulence in the boundary layers.

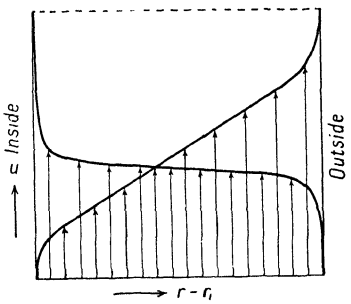


Fig. 3.16

More recently Dryden * has worked out very reliable methods using hot-wire anemometers [cf. § 22(b), p. 252], by which numerical values for the small fluctuations of velocity can be obtained. It was found that wind-tunnel turbulence (or, more generally, any turbulence arising from flow through a wire grid or the like) has extremely simple properties at a sufficient distance from the object giving rise to the turbulence. It is found to be " isotropic ", that is, the fluctuations of velocity are of the same magnitude in all directions. Isotropic turbulence is therefore the simplest form of turbulence and can be dealt with by statistical theory and by experiments suggested by theoretical work. Special reference may be made to the papers by G. I. Taylor and Th. von Kármán.†[13]

The simplest statistical quantity is the mean energy of fluctuation

$$E = \tfrac{1}{2}\rho\{(u'^2)_m + (v'^2)_m + (w'^2)_m\},$$

written as $\frac{1}{2}\rho q^2$ for brevity. Dryden established that for a grid of meshes of width m through which fluid moves with mean velocity U,

$$q = \frac{m}{T+t} \times \text{const.},$$

where t is the time elapsed since the particle passed through the grid, and T is a time-constant such that $T \sim (m/U)$. The constant of proportionality in the formula for q depends on the ratio d/m, where d is the diameter of the wire.

An idea of the distribution in space of the fluctuations in velocity may be got

* H. L. Dryden and others, *Reports of the National Advisory Committee for Aeronautics*, Washington, Nos. 320, 342, 448 and 581 (1929–36).

† See e.g. the comprehensive review by G. I. Taylor, *Proc. Fifth Internat. Congr. for Applied Mech.*, Cambridge, Mass., 1938, p. 294.

by studying the correlation * between the velocities at neighbouring points A and B. For isotropic turbulence there are only two non-zero correlations, both functions of the distance $r = \overline{AB}$, R_1 the correlation between the components of velocity at A and at B parallel to the line AB, and R_2 the correlation between two parallel components of velocity at A and at B at right angles to the line AB. In virtue of continuity it can easily be proved,† as was done by von Kármán, that R_1, R_2 are connected by the relation

$$r\frac{dR_1}{dr} = 2(R_2 - R_1).$$

The way in which R_1 and R_2 vary with r/m is shown in fig. 3.17 (based on the average of various series of measurements).

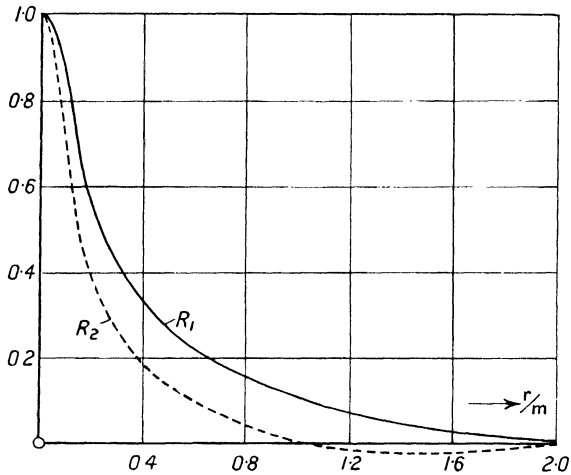


Fig. 3.17

From the graph of R_1 (and R_2) two characteristic lengths can be derived, of which one, $\int_0^\infty R_1(r)\,dr = L$, is a measure of the size of the masses moving as a whole and therefore is closely related to the mixture length l, and the other, by § 4(*f*), p. 120, is a measure of the small eddies in the turbulent motion, in which the energy of turbulence is transformed (dissipated) in heat. The latter Taylor denotes by λ; it has the same meaning as on p. 120. According to Taylor the statistical mean value of the dissipation is proportional to

$$\mu q^2 \left(\frac{d^2R_1}{dr^2}\right)_{r=0},$$

where

$$\left(\frac{d^2R_1}{dr^2}\right)_{r=0} = \tfrac{1}{2}\left(\frac{d^2R_2}{dr^2}\right)_{r=0} = \frac{1}{\lambda^2}.$$

* By the correlation of two varying quantities $x(t)$ and $y(t)$ is meant the expression $R = \dfrac{(xy)_m}{\sqrt{\{(x^2)_m(y^2)_m\}}}$; it is equal to 1 or -1 if the two quantities are directly or inversely proportional, and equal to zero when there is no connexion between them at all.

† This can be done by considering a hemispherical shell of thickness dr with A as centre, the mean velocity being along the axis of symmetry.

Further, λ is proportional to $\sqrt{(\nu L/q)}$, or, using the above formula for q and putting $L \sim m$, λ is proportional to $\sqrt{[\nu(T+t)]}$, as might be expected on dimensional grounds or from formula (12) of § 3 (p. 107).

(*h*) Isotropic turbulence is not covered by the system of formulæ in § 4, since here $\partial u_m/\partial y = 0$ and yet u', v' do not vanish. Prandtl * has recently succeeded in constructing a system of formulæ which includes this case also. Starting from the mean energy of fluctuations $\frac{1}{2}\rho q^2$ of p. 133, we consider the change of this quantity with time for the case of an individual particle of fluid, arising from the supply of energy from the main flow, the destruction of energy in the element of fluid itself, and the loss of energy to regions poorer in energy, or gain of energy from regions richer in energy. We are above all interested in the rate of destruction of energy in unit volume in unit time, dE/dt (in the steady case, this is equal to D (p. 120)). A formula for dE/dt can be found by the following approximate argument.

Let R be the resistance of a ball of fluid moving with velocity u' relative to its surroundings, and let $V = al^3$ be the volume of the ball, where a is a number; then $dE/dt = -Ru'/V$. The resistance is proportional (§ 13, p. 174) to $\rho u'^2 . l^2$ provided $\mathscr{R} = u'l/\nu$ is not too small. Since $u' \sim q$, this gives † $dE/dt = -a_1\rho q^3/l$, where a_1 is another number. If the other portions of dE/dt are non-existent the relation $dE/dt = \rho q\, dq/dt$ then gives the differential equation

$$\frac{1}{q^2}\frac{dq}{dt} = -\frac{a_1}{l},$$

i.e.

$$\frac{1}{q} = a_1 . \frac{t+T}{l},$$

which is in agreement with the experimental result of § 5(*g*), p. 133.

By what has been said before [§ 4(*f*), p. 120)], the supply of energy per unit volume is $\tau'\partial u_m/\partial y$, where, as before, $\tau' = A_\tau \partial u_m/\partial y$, but here, in contrast to p. 119, $A_\tau = a_2 . lq$. As before, l must be found empirically. The transfer of energy from regions richer in energy to regions poorer in energy is governed by an exchange formula in such a way that the flow of energy is $A_E\partial(q^2)/\partial y$, where $A_E = a_3 . \rho lq$. The resulting system of formulæ is capable of removing certain discrepancies between the profiles calculated from formula (19), p. 118, and those observed.

6. **Eddy Formation.**

The retarded boundary layers formed at the surface of a body as a result of slight viscosity have another very important property, no matter whether the motion in them is smooth or turbulents under certain conditions they give rise to the formation of free layer: of discontinuity and eddies or vortices (cf. §§ 6 and 11 of last chapter,

* *Göttinger Nachr., Math.-Phys. Kl.* (1945) p. 6.

† This formula can also be applied to turbulence of the second and higher orders (§ 4(*f*), p. 120) if l_i is a length associated with the ball of the ith order of eddies and q_i is their velocity relative to their surroundings. Since in the lower orders of turbulence not much energy is lost as a result of viscosity, q_i^3/l_i is approximately constant. Further information will be found in the author's report on turbulence in *F.I.A.T. Review*, Vol. 3, *Aerodynamics* (1947); see especially abstracts, pp. 63, 65, 66, of papers by Prandtl, von Weizsäcker, and Heisenberg.

pp. 50, 76). If any accelerating or retarding pressure differences exist in the outer fluid which adjoins the boundary layer, these differences of pressure affect the fluid in the boundary layer also. If the external flow is accelerated by a fall of pressure in the direction of motion, the fluid particles which are travelling more slowly in the boundary layer also receive an impulse in the direction of motion. Hence all the particles will continue on their way past the surface of the body. If, on the other hand, there is a fall of pressure in the direction opposite to that of the motion, retarding the external flow, the slower-moving particles in the boundary layer are still more strongly retarded and finally, when all their kinetic energy has been consumed, are made to turn back. Thus while the external flow with its greater supply of energy continues to flow on, close to the boundary the fluid comes to a standstill or has its flow reversed. Now as fresh fluid is continually suffering the same fate all along the boundary, more and more retarded fluid is piled up in a short time between the boundary and the external flow; the backward flow rapidly broadens out and the external flow is pushed farther and farther away from the boundary (becomes "separated" from the boundary). Fig. 3.18 shows the details of a process of separation (the vertical scale considerably increased). The layer of discontinuity arising in this way then rapidly coils itself up into an eddy or eddies as described in § 6 of last chapter (p. 50). As a result of these phenomena the distribution of pressure over the body is markedly altered, not only downstream but also to some extent upstream. The phenomena which have been described here are well illustrated * in figs. 3.19 and 3.20, in which successive stages of the flow round a circular cylinder are reproduced.† Thus initially the motion is irrotational, in agreement with theory, closing up behind the cylinder just as it opened to make way for it. But while this condition persists in front of the cylinder, where the flow is accelerated, backward flow very soon sets in at the rear, where the fluid is retarded along the boundary, and well-marked eddies arise in its train. The layer of discontinuity, which consists of particles which originally were at the cylinder boundary, is clearly shown in several of the pictures by the

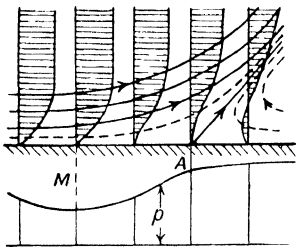


Fig. 3.18.—The process of separation

M = Point of maximum velocity
A = Point of separation

* Figs. 3.19, 3.20, 3.24 and 3.25 are taken from *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 3, by Prandtl and Betz, with the kind permission of the publisher (R. Oldenbourg, Munich).

† From special cinema photographs with long exposures and rapid jumps from one picture to the next; see Tietjens, *Naturwissenschaften*, Vol. 13, p. 1050 (1925).

crowding together of aluminium filings scattered on the water.* The last pictures of the series show how the eddies keep on taking in new parts of the layer of discontinuity and growing until they finally become unstable and break down, making way for a more or less regular alternation of eddies (see also fig. 3.61, p. 183).

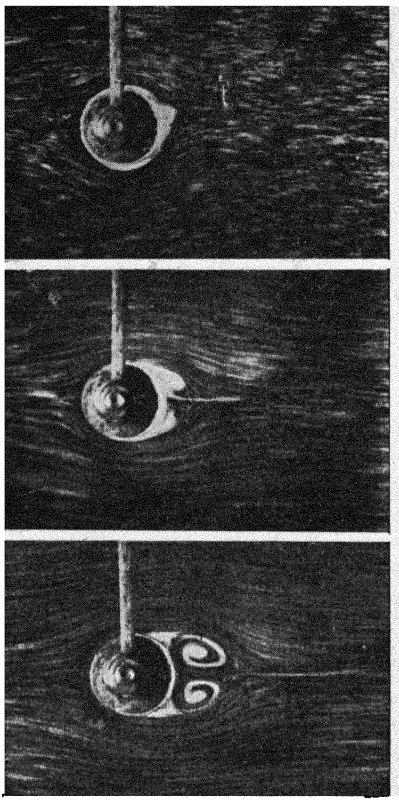


Fig. 3.19

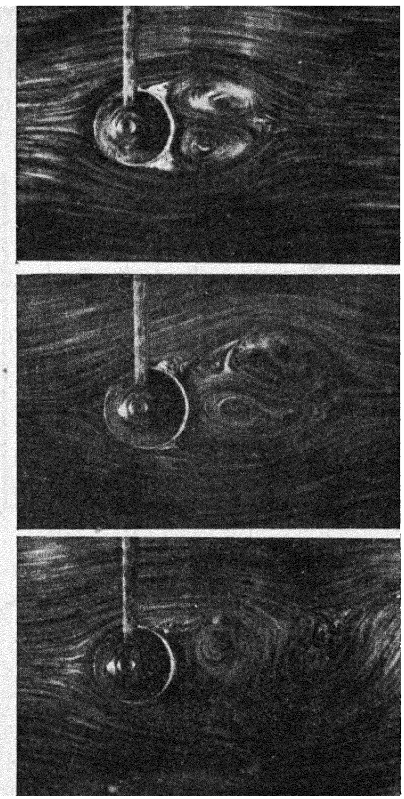


Fig. 3.20

Development of the system of vortices behind a non-rotating cylinder

The essential conclusion from the foregoing, therefore, is that in a motion where the velocity increases in the direction of flow the fluid moves forward always adhering to the boundary, whereas *in a motion where the velocity diminishes in the direction of motion the external flow*

* This is due to the experimental body being coated with a substance which repels water (in this case paraffin wax). Owing to capillarity the water slopes downwards towards the body, the surface of which is a little below the mean level of the liquid; the particles fall down this slope and thus accumulate at the boundary of the experimental body.

may become separated from the wall and eddies of considerable size may be formed owing to the backward flow of the inner parts of the boundary layer. Eddies will certainly arise if the decrease in velocity is sufficiently great, but they may not if the velocity falls slowly (cf. § 7).

These relationships of course apply not only to flow about a body, but also to flow in pipes, water-courses, &c. Where the cross-section diminishes in the downstream direction, i.e. the velocity increases, the flow will always be distributed very uniformly over the whole cross-section. Where the cross-section increases, however, a backward flow will readily be set up, and this may give rise to all manner of irregularities. If the cross-section widens suddenly the flow will become separated from the boundary and a free jet similar to those issuing from sharp-edged orifices will be formed (see fig. 3.26, p. 143). If the widening is slighter or more gradual, the jet breaks down into eddies or attaches itself irregularly to some part of the walls of the pipe.

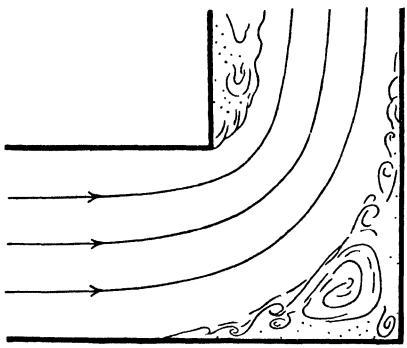


Fig. 3.21.—Flow round a sharp corner

If the flow is deviated to one side (fig. 3.21), there is a transverse fall of pressure in the curved portion of the flow (cf. Chapter II, § 5(*b*), p. 46). By Bernoulli's equation, the velocity is therefore increased on the inner side of the curve and decreased on the outer; that is, the conditions associated with separation of the flow are satisfied at the outer boundary. The flow in fig. 3.21 clearly shows the separation of the flow from the plane wall. This disposes of the view which was sometimes held formerly, that separation of the flow cannot occur unless the wall is convex. There is a difference from the ordinary case of separation in that here the flow becomes reattached to the wall downstream. Similar cases of separation of flow occur at the entrance to a curved pipe, or ahead of a sudden narrowing of a pipe, where the stream-lines also turn their convex sides to the wall of the pipe. Again, on the windward side of a house or on the upstream side of a bridge pier or a post in a river the flow becomes separated from the ground and the corresponding eddies occur [cf. fig. 3.22; also fig. 3.33 (p. 147)].

The " separation " of the flow past a sharp edge, with the formation of a layer of discontinuity, which we described in § 6 of last chapter (p. 50), may be explained from our new point of view by considering a rounded-off edge in the first instance. The velocity of irrotational motion is greatest at the edge and rapidly decreases

as we pass downstream from it. Thus here again the conditions for backward flow and the formation of the layer of discontinuity are determined by frictional phenomena; the layer of discontinuity will start from the edge. Now if we imagine the radius of curvature of the edge to become smaller and smaller, the only change is that owing to the more and more abrupt difference of velocity the time required for the setting-up of a backward flow and the subsequent phenomena will become smaller and smaller. Thus we come to the conclusion that at sharp edges a layer of discontinuity is formed with great rapidity, i.e. that the assumption on p. 50 of the occurrence of layers of discontinuity at sharp edges is fully justified.

For the sake of simplicity we omitted in our discussions hitherto to mention a circumstance which may prevent eddy formation taking place when the retardation of flow is inconsiderable. It is as follows. The boundary layer is not only retarded by its contact with the boundary, but is forced on by its contact with the external current. Hence if retardation sets in very gradually this forward urge may suffice to prevent backward flow, and then the layer will not become separated from the wall. In laminar boundary layers the accelerating effect of the external current on the material of the boundary layer is of course fairly small; owing to the extensive exchange of momentum by mixing, it is much greater in the case of turbulent flow. This fact also has the result that the point of separation in the case of bodies which are blunt at the rear (e.g. spheres) lies distinctly farther back when the boundary layer is turbulent, so that the region occupied by eddies is smaller (cf. § 15, p. 189, on the resistance of a sphere).

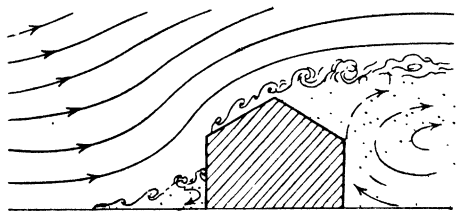


Fig. 3.22.—Flow of air over a house

In engineering there arises the practical problem how separation of the flow may be prevented from occurring when there is a rise of pressure, in order that the losses may be kept small. The problem is solved by making the pipe widen very gradually or giving the body a shape sufficiently slender to make the forward urge of the external current predominate over the rise of pressure. In general, however, the result is only satisfactory in cases where the flow in the boundary layer in the retarded region is turbulent.

In this connexion the precise circumstances under which transition from laminar to turbulent flow in the boundary layer takes place when there is a rise in pressure are of interest. Turbulence of any sort in the oncoming fluid is an important factor which strongly favours the onset

of turbulence. If, however, the oncoming fluid is very free from turbulence (as is the case e.g. in flight at a considerable height above the ground), the flow may remain laminar almost until the point of separation of laminar flow is reached. The occurrence of turbulence just behind (or just in front of) this point, at which the boundary-layer profile has a point of inflexion (p. 115), may then cause the flow which has just separated itself to re-attach itself again, or may prevent separation of the flow occurring at all. Which of these alternatives occurs

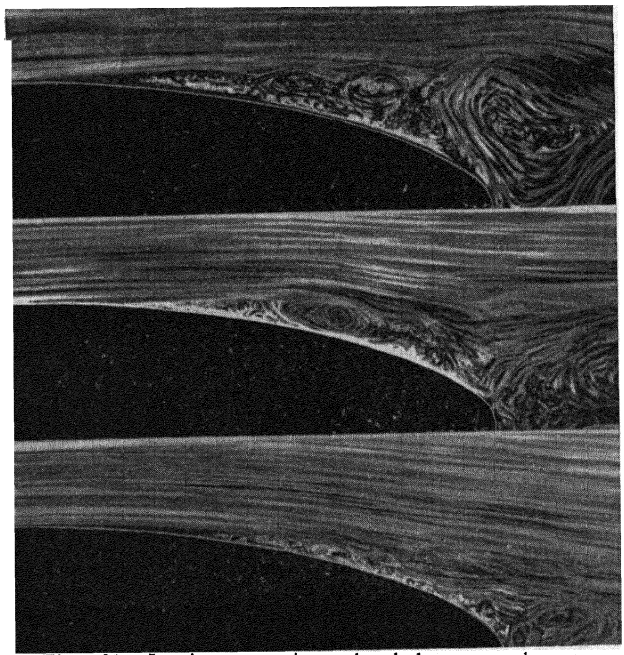


Fig. 3.23.—Laminar separation and turbulent re-attachment

in a particular case depends on Reynolds' number and also very much on the shape of the boundary behind the point of separation. Hitherto no clear-cut rule has been discovered. In addition to the Reynolds' number formed from the radius of curvature at the point of separation, the decrease or increase of curvature in the direction of flow seems to be important. Fig. 3.23 shows the transition from the flow separating itself for low values of Reynolds' number to that re-attaching itself for higher values of Reynolds' number. The photographs correspond to values of qr/ν of approximately 2×10^4, 5×10^4, and 6×10^4.

A remarkable relationship between the turbulence pre-existing in the fluid and the position of transition from laminar to turbulent flow has been suggested

on theoretical grounds by G. I. Taylor* and is in good agreement with experiment. If u' is the mean velocity fluctuation of the fluid approaching with mean velocity U, and s the distance from the stagnation point to the point of transition, Reynolds' number Us/ν is a function of $\frac{u'}{U}\left(\frac{s}{L}\right)^{1/5}$, and similarly the critical value of Reynolds' number for geometrically similar bodies (e.g. spheres of diameter d; cf. § 15, p. 189), Ud/ν, is a function of $\frac{u'}{U}\left(\frac{d}{L}\right)^{1/5}$, where L is the length mentioned on p. 134, which is defined by the correlation coefficient R_1, and which, in the case of turbulence produced by a wire grid or the like, is nearly equal to the size of mesh of the grid. Taylor's argument is based on the assumption that the occurrence of turbulence is determined by the magnitude of the mean *pressure* fluctuation in the pre-existing turbulence in the fluid. A short method of deducing Taylor's formula has been given by K. Wieghardt.†

Mathematical Extension.—The process of laminar separation is amenable to calculation when the pressure distribution is known, by means of the methods indicated in § 3(*c*), p. 110. The first calculation of this kind was made by H. Blasius.‡ The most accurate mathematical calculation hitherto made is that of H. Görtler,§ based on measurements of pressure distribution made by K. Hiemenz on circular cylinders.‖ It gave the position of the point of separation at an angle of 80° from the stagnation point, compared with the mean value of 81° experimentally obtained by Hiemenz.

E. Gruschwitz ¶ has shown how to make the calculations for turbulent boundary layers at a plane boundary. His method is based on Kármán's integral condition (footnote 3, p. 110), which also applies to turbulent boundary layers, together with an assumption, which has some experimental backing, about the distribution of total pressure for $y = \theta$, where θ is the momentum thickness (p. 110). He puts

$$\frac{\theta}{\rho U^2}\frac{d}{dx}(p + \tfrac{1}{2}\rho u_\theta^2) = A\eta - B,$$

where $\eta = 1 - u_\theta^2/U^2$ is his profile parameter and A and B are two numbers. The calculation leads to two simultaneous differential equations of the first order for θ and η, which can be solved by iterative methods. The method has been extended to higher values of Reynolds' number and to converging or diverging flow by A. Kehl;** the constant B is then replaced by a function of Reynolds' number $U\theta/\nu$. For convex boundaries H. Schmidbauer †† found that B is diminished by a quantity depending on θ/r.

According to Gruschwitz, separation of the turbulent boundary layer occurs when the value of the parameter η is approximately 0·8; the matter, however, has not yet been completely cleared up.‡‡[14]

* *Proc. Roy. Soc.* (A), Vol. 156, p. 307 (1936).

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 58 (1940).

‡ *Diss.*, Göttingen (1907); *Zeitschr. f. Math. u. Phys.*, Vol. 56, p. 1 (1908). See also the references given on p. 109 *et seq.*

§ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 19, p. 129 (1939).

‖ *Diss.*, Göttingen (1911); *Dinglers Polytechn. Journ.*, Vol. 326, p. 321 (1911).

¶ *Diss.*, Göttingen; *Ing.-Arch.*, Vol. 2, p. 321 (1931).

** *Diss.*, Göttingen; *Ing.-Arch.*, Vol. 13, p. 293 (1943).

†† *Diss.*, Techn. Hochsch., Munich (1934).

‡‡ The experiments of Wieghardt, *UM.* 6603 (1943); Wieghardt and Tillman, *UM.* 6617 (1944); also the author's *F.I.A.T. Review*, were all made by the momentum loss method of § 22(*d*), p. 256, and, as has been shown by investigations by H. Ludwieg and W. Tillmann with a skin friction meter (see the end of § 22, p. 257), are falsified in the vicinity of the point of separation by secondary flow.

7. Prevention of Eddy Formation.

Boundary layers may also be influenced artificially so as to prevent separation of the flow taking place. If, for example, a cylinder past which fluid is moving is made to rotate so that its peripheral velocity is equal to or greater than the greatest velocity of flow occurring at the surface of the cylinder, the boundary layer on the side where the

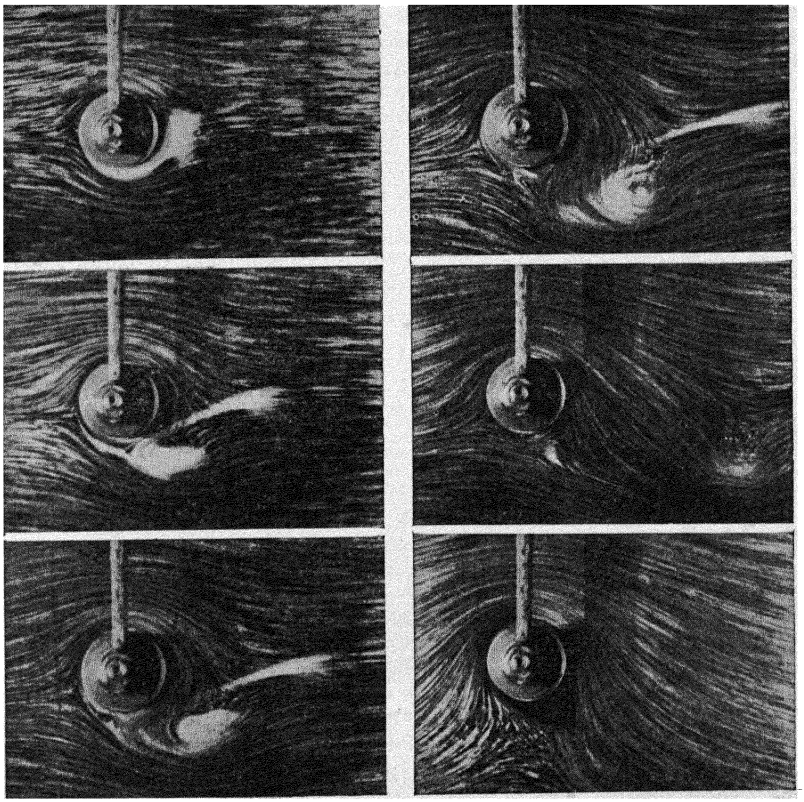


Fig. 3.24 Fig. 3.25

Development of the flow round a rotating cylinder

fluid and the surface of the cylinder are moving in the same direction will not be retarded at all, but will on the contrary be accelerated by the motion of the cylinder. This layer will then have a more powerful effect than the external current in overcoming the rises of pressure causing retardation, so that on this side backward flow does not set in at all, and the flow therefore does not become separated

from the boundary. On the other side the surface of the cylinder is moving in the direction opposite to that of the fluid, retarding the boundary layer, so that first backward flow and subsequently the separation of a strong eddy takes place there. Together with the eddy there arises a circulation about the cylinder in the direction opposed to that of the eddy, as described in § 10 of last chapter

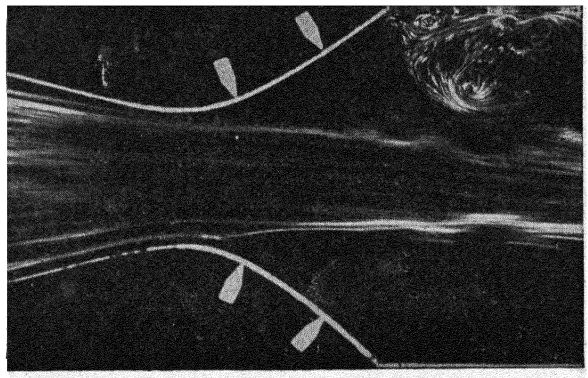


Fig. 3.26.—Ordinary flow in a suddenly enlarged channel

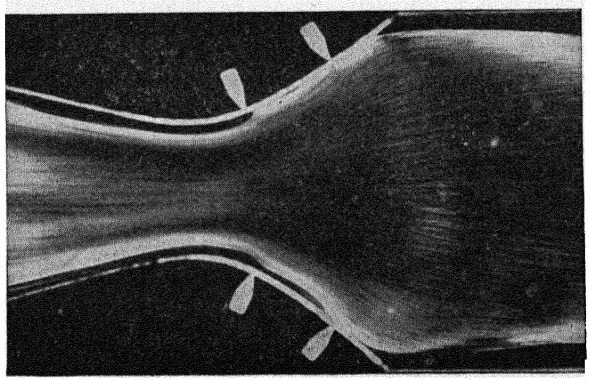


Fig. 3.27.—Flow in a suddenly enlarged channel with suction at the walls

(p. 68); this gives rise to the transverse force known as the " Magnus effect ". In ordinary irrotational motion the maximum velocity at the surface of the cylinder is $2v$; if there is in addition a circulatory motion $2v$, the velocity will be zero on one side and $4v$ on the other. Experiments with rotating cylinders show that this state of flow occurs when the peripheral velocity u is taken about equal to $4v$ and gives approximately the maximum Magnus effect. Stages in the

formation of eddies for $u = 4v$ are shown by photographs of the flow in figs. 3.24 and 3.25.

Another method of influencing the boundary layer consists in sucking fluid into the interior of the body at places where backward-flowing fluid would accumulate, through slits or holes in the surface of the body. If the accumulation of retarded fluid is thereby prevented, separation of the flow does not occur. The suction has the additional effect of lowering the pressure in the region immediately in front of a slit and thus preventing separation there. This does not enable us to cut down the resistance if it is already small, but we can obtain a flow without appreciable eddy formation with decidedly shorter or blunter bodies than we otherwise could.(15)

Figs. 3.26 and 3.27 * illustrate the flow in a widening channel, in the one case without and in the other with suction at the side walls. Here, owing to the increase in cross-section, the fluid must be retarded rapidly in order that the flow may remain smooth. It is only in the latter case that this is so; in the former case a jet is formed.†

The sucking-out of part of the boundary layer by means of a permeable area, i.e. an area with small holes or slits very close together, has been investigated theoretically, and it has again been established that the tendency to separation of the flow is greatly diminished. Since the boundary-layer profile is much more convex with suction than without, the tendency to turbulence is also less.‡

A third way of influencing the boundary layer is to increase the accelerating effect of the external flow. This method is especially effective when used as in the slotted wing of Handley-Page and Lachmann. The boundary layer of the first part of the wing in fig. 3.28 is extended into the external flow and thereby rendered harmless, so to speak. A new boundary layer is formed at the second part of the wing; this has to overcome part only of the rise in pressure on the suction side. With this kind of wing the flow is made to remain attached for much larger angles of incidence than is otherwise possible, and the lift is therefore considerably increased, but of course at the expense of a great increase in the resistance.



Fig. 3.28.—Slotted wing

Occasional attempts have been made to achieve a similar result by blowing air out from inside the wing, but probably it would be difficult to provide sufficiently large quantities of air. In the case of suction considerably smaller quantities suffice.

* From *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt* (1927), by kind permission of the publisher, R. Oldenbourg, Munich.

† For other examples of suction of the boundary layer, see J. Ackeret, *VDI-Zeitschr.*, Vol. 70, p. 1153 (1926).

‡ See § 4(*c*), p. 116; also J. Pretsch, *Jahrb. d. Deutsch. Luftfahrtforschung* (1942), Vol. I, p. 1, and A. Bussmann and H. Münz, ibid. (1942), Part I, p. 36.

A somewhat similar process is the use of auxiliary vanes to improve a flow in which disturbances from separation would otherwise arise. The vanes used to direct the flow in wind tunnels (cf. fig. 3.130, p. 249) are an older example of this. Vanes are also used in other cases of flow round sharp corners, to prevent too great losses * (see fig. 3.29). Here the non-occurrence of separation is to be explained by the fact that owing to the pressure field of the vanes the total pressure at the boundary wall, towards

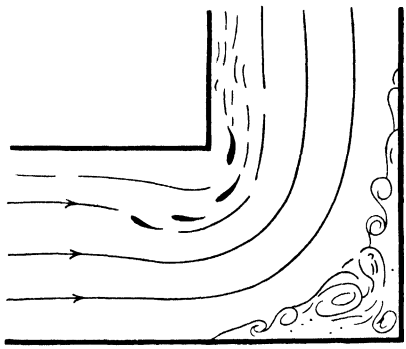


Fig. 3.29.—Direction of the flow by auxiliary vanes

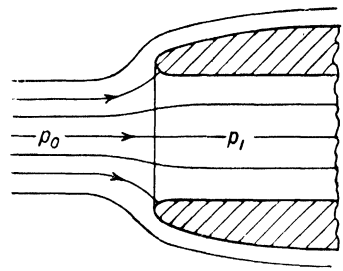


Fig. 3.30.—Küchemann's diffuser

which the vanes turn their pressure sides, is higher than it would be in the absence of the vanes, and hence the rise of pressure which the boundary layer has to withstand either does not exist at all or is very much reduced. This explanation also applies of course to the slotted wing just as the explanation given above of the way in which the slotted wing makes the boundary layer harmless can be applied here.

In this connexion we may also mention a case in which considerable increases in pressure may be achieved without noticeable losses, merely as a result of geometrical arrangement. The problem to be solved is that of sucking air from outside into a rapidly moving aeroplane and thereby gaining as much pressure as possible. These diffusers are used in cooling installations, jet engines and the like. The raising of the pressure in the absence of boundaries entails practically no losses. If matters are so arranged that the contact with the wall which begins at a stagnation point takes place with falling pressure only, the problem is solved. It is a question of finding bodies of the right shape, which can now be done by calculation.† A typical form of diffuser is shown in fig. 3.30.[16]

8. **Secondary Flow.**

If a fluid flowing along a boundary is subjected to a lateral displacement parallel to the boundary, the layers near the boundary are

* Cf. G. Flügel, *Jahrb. d. Schiffbautechn. Ges.*, Vol. 19, p. 385 (1918); also K. Frey, *Forschung*, Vol. 4, p. 67 (1933), Vol. 5, p. 105 (1934) where the results of numerous comparative experiments with and without auxiliary vanes are given.

† D. Küchemann and J. Weber, *Mitt. der D.A.L.*, No. 5004 (1943), for the case symmetrical about an axis; P. Ruden, *Jahrb. d. Deutsch. Luftfahrtforschung* (1941), Part I, pp. 377, 388, for the two-dimensional case.

displaced to a greater extent than those farther away, owing to their smaller velocity. From Chapter II, § 5(*b*), p. 47, we see that the ratio of the radii of curvature is given by $r_1 : r_0 = q_1^2 : q_0^2$. In reality, however, the flow is not frictionless; friction at the boundary, and the dragging effect of the flow farther out, in conjunction with the acceleration effect just mentioned, deflect the boundary layer towards the side where the pressure is less, the deflection amounting to about 45° in the laminar case, but not more than about 25° to 30° in the turbulent case. The phenomenon may be regarded as a combination of the main flow with a " secondary flow " at right angles to it, which for reasons of continuity is not usually confined to the boundary layer, but affects the " core " of the flow and may considerably influence it. If, for example, fluid flows through a curved pipe, the core of the flow tends to go straight ahead owing to its greater velocity, whereas the slower layers along the boundary are markedly deflected, and hence tend towards the inner side of the curve. Thus in the curved pipe the main flow parallel to the central line of the pipe has superposed on it a secondary flow at right angles to it, outwards at the centre of the pipe, and inwards in the neighbourhood of the walls. This secondary flow is shown in fig. 3.31. It has the effect of displacing the region of maximum velocity close to the outer wall; see also the passage in small type at the end of § 11 (p. 167).

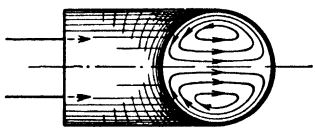


Fig. 3.31

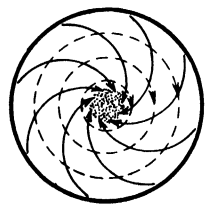


Fig. 3.32

Cases of secondary flow

Another striking example of secondary flow is the flow at the bottom of a flat-bottomed circular vessel when the fluid in the vessel is given a circular motion. The flow in the layer next the bottom is directed inwards owing to its smaller " centrifugal force " (see fig. 3.32). It is a matter of everyday observation that small particles on the bottom of the vessel are carried to the centre and piled up there. (Try the experiment in a wash-basin.) The bottom flow which we have just mentioned is the explanation of this phenomenon.

Again, in natural river-courses the secondary flow at bends results in the sand, gravel, &c., carried along by the current at the bottom of the river being removed from the outer side of the bend and piled up at the inner, so that the river is deepened at the outer side and made shallower at the inner. The combination of this effect with the erosion on the outer side by the particularly rapid flow there causes the bend to become more and more pronounced. Hence, where

they can, rivers always follow a winding course ("meandering").

The direction of flow in the layer of fluid immediately adjacent to the boundary can be made visible by a suitable surface coating. In the case of water a coating of oil paint will serve; in air, a mixture of lampblack or other colouring matter with petrol. When the moving

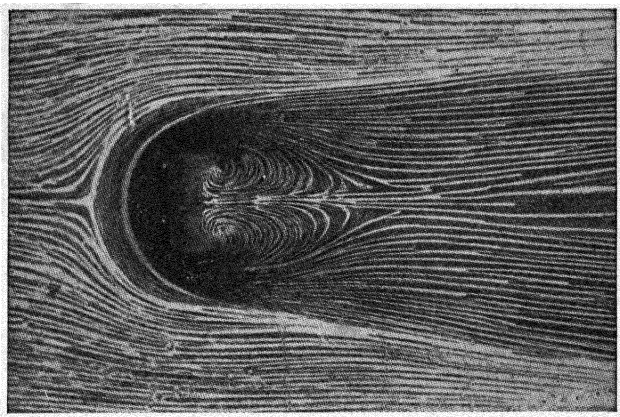


Fig. 3.33.—"Paint-picture" of the disturbance of the flow past a boundary due to a plate at right angles to it, after A. Hinderks

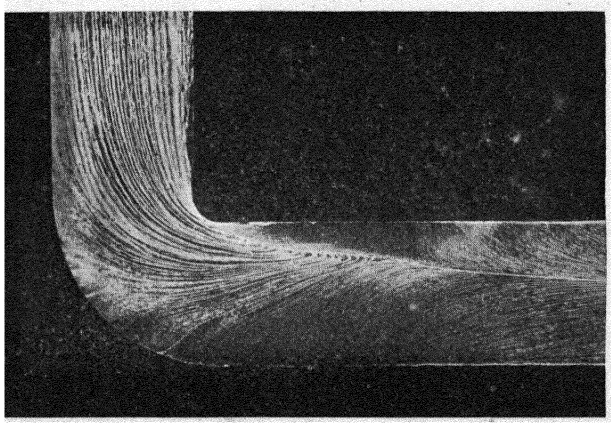


Fig. 3.34.—"Paint-picture" of the flow round a bend, after A. Hinderks

fluid is allowed to act on the coating for a suitable length of time (about 5 minutes in the case of water), a design is formed, which is particularly clear in the case of turbulent flow, and shows up the direction of the mean velocity in the layer next the boundary; it thereby yields valuable information about the course of flow, points of separation, &c. In order to interpret these pictures correctly, however, it is neces-

sary to bear in mind that they show only the paths in the layers next the boundary, not the main body of the flow. Two pictures of water flow by A. Hinderks * are reproduced in figs. 3.33 and 3.34. The first shows the flow at the bottom of a channel in which there is a flat plate at right angles to the flow. From the broad white strip surrounding the front of the plate we see that the flow along the boundary leaves the latter at a considerable distance in front of the plate, avoiding the region of excess pressure there. In the two eddies behind the plate the spiral inward flow of fig. 3.32, which might be expected to occur here, is clearly shown. It is noteworthy that in this region where the turbulence is particularly great the picture of the flow is particularly clear. Hitherto no explanation of the way in which the picture is formed has been given. Fig. 3.34 shows the flow in a bent channel of rectangular cross-section. Here the deflection of the layer next the boundary towards the inner side of the bend is very clear, and also the separation of the flow behind the bend, where there is a rise of pressure on the inner side.

In turbulent flow in straight channels of non-circular section another type of secondary flow is observed (" secondary flow of the second kind "). Its distinguishing features are peculiarities in the distribution of velocity, which can only be explained if we assume transverse flow reaching into the corners of the profile of the

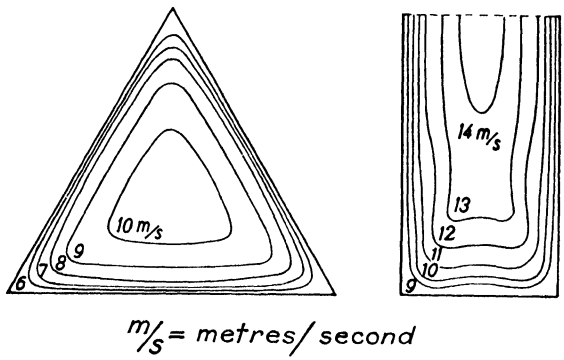


Fig. 3.35.—Curves of equal velocity

channel and then returning towards the centre. The curves of equal velocity for channels of rectangular and triangular cross-section (fig. 3.35) † are readily explainable if we assume that the secondary flow is as in fig. 3.36. It is also possible to show this secondary flow by using dye,‡ or directly by the surface coating method. Here we may also mention the fact that small objects floating in rivers tend to move to the middle, which is explained by the existence of a surface current from the banks

* Made in the Franzius-Institut, Technische Hochschule, Hanover.

† Taken from experiments by J. Nikuradse, *Ing.-Arch.*, Vol. 1, p. 306 (1930); see also L. Prandtl, *Proc. 2nd Internat. Congr. Appl. Mech.* (1926), p. 71 *et seq.* (Zürich, 1927).

‡ J. Nikuradse, loc. cit., p. 328.

to the middle. The ultimate cause of this type of secondary flow is not yet established. The observed facts, however, justify the idea that at places where the shearing stress at the boundary is greater, fluid is impelled towards the centre of the channel as a result of turbulence, and on the other hand at places where the shearing stress is less (e.g. at the corners), fluid flows from the inside towards the wall. This has the effect of lowering the velocity at the former places and raising it at the latter, so that the shearing stress at the boundary is largely evened out. This fact serves as a foundation for the engineer's formulæ for resistance (cf. § 11, equation (43), p. 161) based on the idea of the hydraulic mean depth.

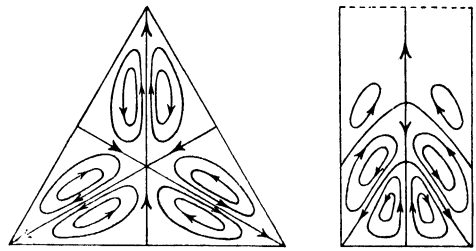


Fig. 3.36.—Secondary flow

We may also mention here another phenomenon, which leads to "secondary flow of the third kind", namely, the peculiar flow set up by small oscillations of solid bodies in fluids, which is particularly striking in the case of experiments with ultrasonic waves. It is also to be observed near a boundary in the case of standing waves in a fluid. These cases of flow have been explained by H. Schlichting * by phenomena in the boundary layer at the surface of the vibrating body or at the boundary of the fluid. It is a case of a second-order effect, related to the interaction of the mean pressures, momenta and frictional forces. If $U(x)\cos\omega t$ is the velocity outside the boundary layer, there is, according to Schlichting, an additional velocity, which has the value $u' = \frac{3}{4}\frac{U}{\omega}\frac{\partial U}{\partial x}$ in the neighbourhood of the boundary but outside the boundary layer, and is directed from points of greater velocity to points of smaller velocity. Fig. 3.37 shows a photograph of the flow of water in a tank round a circular cylinder swinging to and fro, taken by a camera moving with the cylinder. The metal particles used for making the motion visible give rise to broad bands as they move to and fro during the very long exposure. The flow approaches the cylinder from above and below, and leaves it on either side along the line of vibration. (The asymmetry of the photograph is due to a slight motion of the water.)

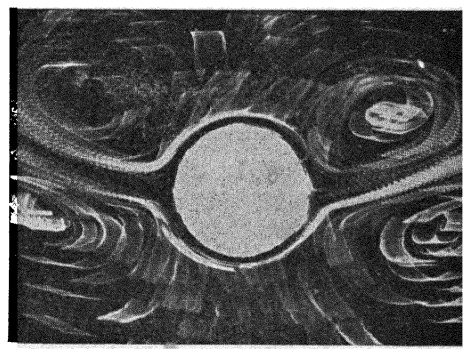


Fig. 3.37.—Secondary flow round a body moving to and fro, after Schlichting

The phenomenon described here gives a good explanation of flow round bodies which are placed in a field of standing acoustic waves † and also of Kundt's well-known dust figures.

* "Berechnung ebener periodischer Grenzschichtströmungen", *Phys. Zeitschr.*, Vol. 33, p. 327 (1932).

† See e.g. the smoke photographs by Andrade, *Proc. Roy. Soc.* (A), Vol. 134, p. 445 (1931).

9. Cases of Flow where Viscosity predominates.

As we have already explained in § 3 (p. 105), when viscosity strongly predominates the forces of inertia may be neglected in comparison with frictional forces. In all cases this type of flow has the property that the resistance to motion is proportional to the first power of the velocity. In the case of steady flow the predominance of viscosity is outwardly shown by Reynolds' number being very small, as a small value of Reynolds' number may arise either from very large viscosity, very small velocity, or very small space dimensions (we may also mention the case of very small density, which occurs in the flow of a gas through highly evacuated tubes). We shall discuss three examples of this type of motion in greater detail in this section and the following one.

(*a*) *The Flow of Water and Air through Sand, Gravel, &c.*—This is an important practical case; for example, the laws obtained here are used in numerical calculations on the flow of subsoil water. The main features of the underground flow, which really takes place through the interstices between the separate grains of sand, may be traced out by finding the quantity of fluid flowing per second across unit area of a cross-section of the mass of sand. This quantity is a velocity and may be represented by the components u, v, w (the quantities flowing across areas normal to the x-, y-, and z-axes). By what we have said above, or in analogy with the Hagen-Poiseuille law for flow through tubes, we may take this velocity as being proportional to the fall of pressure in the case of slow motion,* i.e. the components of velocity are:

$$u = -k\frac{\partial p}{\partial x};\quad v = -k\frac{\partial p}{\partial y};\quad w = -k\frac{\partial p}{\partial z}. \quad . \quad . \quad (30)$$

From the equation of continuity (p. 37)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

which may be used as it stands, we get

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

by using (30). Thus p satisfies the same equations as the velocity potential Φ does [see §§ 8, 9 of last chapter (p. 56)], and u, v, w are here derived from p in the same way as they were from Φ in the case of irrotational motion without friction. That is, subsoil flow is an irrotational motion of the type described in Chapter II, § 9 (p. 61). There

* Law of filters, also known as Darcy's law: H. Darcy, *Les fontaines publiques de la ville de Dijon*, p. 590 (1856).

is, however, the essential difference that owing to its physical nature p must everywhere be one-valued and must vary continuously, whereas Φ may be discontinuous on surfaces of discontinuity and is actually many-valued when the flow involves circulation. Such solutions therefore do not apply here.

The foregoing equations enable us to trace out the flow of subsoil water in various typical cases, e.g. to investigate the relationships which hold in the neighbourhood of a well from which water is being pumped, that is, to find the velocity distribution and also the lowering of the water table in the neighbourhood of the well. We cannot, however, go into further details here.*

The linear relation between velocity and fall of pressure assumed above holds good only so long as the Reynolds' number formed from the grain diameter is sufficiently small. G. Kling,† working with observations by various workers, found that for flow through beds of spherical particles the critical value of Reynolds' number ud/ν was 10. Below this, the equation

$$-\frac{dp}{dx} = 1000\,\frac{\mu u}{d^2}$$

held, where u has the same meaning as above. From $\mathscr{R} = 200$ to $\mathscr{R} = 5000$, the highest figure investigated, the equation

$$-\frac{dp}{dx} = \frac{\zeta}{d}\rho\frac{u^2}{2}$$

held, where $\zeta = 94\mathscr{R}^{-0\,16}$. In the intermediate region there was a gradual transition from the first to the second.

(*b*) *The Flow between two Parallel Plates.*—This bears a certain relation to the flow of subsoil water. In detail the motion resembles the Hagen-Poiseuille flow with a parabolic distribution of velocity (cf. fig. 3.13, p. 124). When the velocities are sufficiently small, however, the mean value of the velocity over the gap between the plates may again be taken as proportional to the fall of pressure, i.e. we have

$$u = -k\frac{\partial p}{\partial x}, \quad v = -k\frac{\partial p}{\partial y} \quad . \quad . \quad . \quad . \quad (30a)$$

for the velocity components. Hence the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

gives

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31a)$$

* The older problems are discussed in detail by Forchheimer, *Hydraulik*, 3rd ed., pp. 51–110. For more recent references see the first two footnotes on p. 153.

† *VDI-Zeitschr.*, Vol. 84, p. 85 (1940); further data in W. Barth and W. Esser, *Forschung*, Vol. 4, p. 82 (1933), and H. G. Kayser, *Forschung*, Vol. 6, p. 81 (1935).

again in analogy with the velocity potential for the flow of frictionless fluid in two dimensions. Hence the stream-lines for frictionless irrotational motion where the velocity potential is one-valued and continuous may be imitated by an experiment in which the friction is very large. This was first done by Hele-Shaw,* who introduced jets of coloured liquid into the liquid (water, glycerine), which he made to flow between glass plates, thus obtaining a very fine representation of the stream-lines of many different cases of irrotational flow. A more modern form of apparatus on the same principle, working with water and ink, is due to R. W. Pohl: fig. 3.38 was obtained by means of this apparatus.†

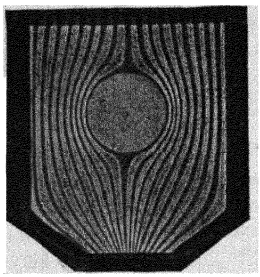


Fig. 3.38.—Flow round a cylinder, after Pohl

F. Prašil has obtained similar effects by filling a large shallow basin a few millimetres deep with water and sprinkling the latter with small crystals of potassium permanganate; cf. fig. 3.39,‡ which represents the flow between a source and a sink (inflow through the left-hand hole, outflow through the right). As the distribution of pressure is quite different from what it is when the friction is slight, the fall of pressure being in fact always in the direction of flow,§ no backward flow is observable at any point here.

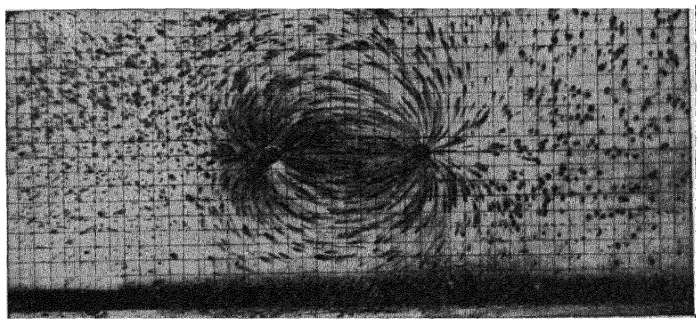


Fig. 3.39.—Flow between a source and a sink, after Prašil

Hence we would expressly emphasize the fact that the cases of flow obtained in the way described are on exactly the same footing as cases

* Hele-Shaw, *Nature*, Vol. 58, p. 34 (1898); *Roy. Inst. Proc.*, Vol. 16, p. 49 (1899).

† See Pohl, *Physical Principles of Mechanics and Acoustics*, Chapter X. Blackie & Son, Ltd. (1932).

‡ Taken from *Technische Hydrodynamik*, second edition, by kind permission of the publisher, Julius Springer, Berlin (1926).

§ In frictionless irrotational motion, on the other hand, the fall of pressure is in the direction of the gradient of $\frac{1}{2}q^2$.

of irrotational flow calculated from the formulæ. They give no real explanation of the actual flow when Reynolds' number is large. Cases of flow with surfaces of discontinuity may, if necessary, be obtained with this apparatus by making an actual surface of discontinuity by means of a partition and suitably regulating the quantities of fluid flowing on either side (but no separation of the flow actually occurs).

We may also remark that in the cases of flow described here irregularities in the paths of the inner and outer layers of fluid occur when the fluid layer is too thick or the velocity too great. The paths of layers near the boundary are then more strongly curved, while those of the inner layers are opened out [secondary flow, cf. § 8 (p. 146)]. During the experiments care must be taken to keep this source of disturbance within the limits of observational error.*

E. Günther † has imitated two-dimensional subsoil flow (e.g. seepage of water through dams) by means of the flow of very viscous fluid through comparatively wide slits. Wide slits are necessary in order to avoid disturbances due to capillary forces.

10. **Hydrodynamical Theory of Bearing Lubrication.**‡

The phenomena occurring here form a further example (and one of great practical importance) of fluid motion where viscosity is predominant. It has been found that between parts of a machine in relative motion (journal and bearing or slipper and guide), there exists —especially if the motion is rapid—a thin layer of moving oil which prevents the two solid bodies from touching at all. The capacity of a bearing or guide to take up heavy loads, the friction remaining small, must therefore be explained as being caused by motion in the layer of oil. As a particularly simple case we may discuss the case of a slipper moving over a plane guide, and for further simplification we shall assume that the surfaces sliding past one another extend a long way in the direction at right angles to the motion, so that we may assume that the motion, in the central region at least, is in one plane.§ To obtain a steady motion we shall choose a system of co-ordinates in which the slipper is at rest and the (considerably longer) guide moves to the right with velocity v. As a preliminary we shall investigate the flow through an interspace of height h under the assumption that its upper surface (slipper) is at rest and its lower surface (guide)

* A theory of these disturbances due to the inertia of the fluid is given by F. Riegels, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 18, p. 95 (1938).

† *Forschung*, Vol. 11, pp. 76, 147 (1940).

‡ For recent German literature, see G. Vogelpohl, *F.I.A.T. Review of Hydro- and Aerodynamics*, Chapter VIII.

§ This two-dimensional theory was first developed by O. Reynolds (not only for slippers but also for journals and bearings); cf. *Phil. Trans. Roy. Soc.*, 1886, Part I (*Collected Papers*, Vol. II, p. 228; *Ostwalds Klassiker*, No. 218, p. 39).

moves parallel to the upper one with velocity v, and that a rise or fall of pressure simultaneously exists in this direction. Let the x-axis be in the direction of motion and the y-axis at right angles to the surfaces. The rise of pressure is then dp/dx, for which we shall write p' for short. Here p' may be regarded as independent of y, on account of the small thickness of the oil film (h). Let the velocity of flow, which lies along the x-axis, be u (even in the more general case where the boundaries are not absolutely parallel the velocity is sufficiently determined by the x-component, the only importance of the y-component being in connexion with continuity). By the remarks at the end of § 1 (p. 103), if we neglect inertia and assume that conditions only change slowly in the x-direction (slowly compared with the marked changes with change of y, which means that we neglect $\partial^2 u/\partial x^2$ in comparison with $\partial^2 u/\partial y^2$), we have

$$\mu \frac{\partial^2 u}{\partial y^2} = p'.$$

By integration it follows that

$$\mu \frac{\partial u}{\partial y} = p'y + C_1$$

and

$$\mu u = \tfrac{1}{2}p'y^2 + C_1 y + C_2.$$

The stipulation that u should equal v, the velocity of the guide relative to the slipper, for $y = 0$ is satisfied if $C_2 = \mu v$, and $u = 0$ when $y = h$ provided

$$C_1 = -\left(\frac{\mu v}{h} + \frac{p'h}{2}\right).$$

Hence

$$u = \frac{p'}{2\mu}(y^2 - hy) + \frac{v}{h}(h - y). \quad . \ . \ . \quad (32)$$

The force of friction, reckoned positive, on the upper surface per unit area is

$$\tau_1 = -\mu \left(\frac{\partial u}{\partial y}\right)_h = \mu \frac{v}{h} - p' \frac{h}{2}. \quad . \ . \quad (33a)$$

Similarly, the force of friction on the lower surface is

$$\tau_0 = -\mu \left(\frac{\partial u}{\partial y}\right)_0 = -C_1 = \mu \frac{v}{h} + p' \frac{h}{2}. \quad . \ . \quad (33b)$$

In discussing these results it must be borne in mind that a rise of pressure in the positive direction of the x-axis corresponds to a positive value of p' and a fall of pressure to a negative value of p'. We

have still to find the quantity of moving fluid necessary to ensure continuity. The quantity for height h and unit breadth is

$$Q = \int_0^h u\,dy.$$

Carrying out the very simple calculation, we have

$$Q = \frac{vh}{2} - \frac{p'h^3}{12\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Having considered these preliminaries, we can now turn to the actual problem. We have to find a solution such that the pressure p, starting from the atmospheric pressure p_0, increases rapidly with x and subsequently diminishes to p_0, for the slipper must of course be able to bear a load, which is possible only if the pressure behaves in the above way. This does not happen if h, the height of the interspace, is constant, for we must have $Q = \text{const.}$ for reasons of continuity. v is the velocity of the slipper, i.e. is constant, so that the rise of pressure p' is also constant. We must therefore make h depend on x. Equation (34) then gives

$$p' = \frac{dp}{dx} = 12\mu\left(\frac{v}{2h^2} - \frac{Q}{h^3}\right), \quad . \quad . \quad . \quad . \quad (35)$$

which gives $p(x)$ by integration. The condition that p must be equal to p_0 at either end of the slipper then gives us the hitherto unknown value of Q. This determines the behaviour of p, and we can now evaluate the resultant pressure on the slipper by further integration $\left(\int_0^l p\,dx \text{ per unit breadth}\right)$ and also the moment $\int_0^l px\,dx$. The distance of the point of application of the force from the point $x = 0$ is then obtained by dividing the moment by the force. The friction is equal to $\int_0^l \tau_0\,dx$, where τ_0 is given by equation (33*b*), so that we can now find the resultant force on the slipper in magnitude and direction for any given variation in h, the height of the interspace. As a rule we are given the resultant pressure, from which we may deduce an expression for the height of the interspace.

The friction may also be calculated from τ_h, but it must be noted that here the pressure p on the surface inclined at the angle $\delta = \tan^{-1}(dh/dx)$ to the direction of motion also contributes a component of force along the line of motion. As the pressure at the back of the slipper is p_0, this component of the force is $-\int_0^l (p - p_0)\frac{dh}{dx}\,dx$. Since $p = p_0$ at $x = 0$ and $x = l$, integration by parts gives $+\int_0^l p'h\,dx$. If we take account of equation (33*a*), we obtain the same expression for the friction as was obtained from τ_0.

The simplest case of a variable interspace occurs when the slipper and guide are plane but inclined to one another at a small angle δ. Let the slipper extend from $x = 0$ to $x = l$ and let the depth be

$$h = (a - x)\delta; \qquad \ldots\ldots\ldots \quad (36)$$

this means that the line of intersection of the two planes is at a distance a from the front edge of the slipper ($x = 0$) (see fig. 3.40). Then

$$\int_0^x \frac{dx}{h^3} = \frac{1}{2\delta^3}\left(\frac{1}{(a-x)^2} - \frac{1}{a^2}\right) = \frac{2ax - x^2}{2\delta^3 a^2(a-x)^2}$$

and

$$\int_0^x \frac{dx}{h^2} = \frac{1}{\delta^2}\left(\frac{1}{a-x} - \frac{1}{a}\right) = \frac{x}{a\delta^2(a-x)};$$

hence

$$p = p_0 + \frac{6\mu x}{a\delta^2(a-x)}\left(v - \frac{Q(2a-x)}{a\delta(a-x)}\right). \qquad \ldots \quad (37)$$

By equation (37), $p = p_0$ when $x = 0$: in order that p may also be equal to p_0 when $x = l$, the term inside the bracket in (37) must vanish,

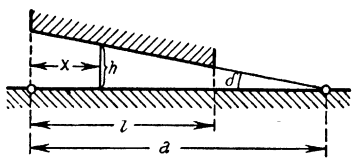


Fig. 3.40

i.e.

$$Q = \frac{va\delta(a-l)}{2a-l}. \qquad (38)$$

Putting h for $\delta(a - x)$ and simplifying the result we readily obtain

$$p = p_0 + \frac{6\mu v x(l-x)}{h^2(2a-l)} \qquad (39)$$

To estimate the mean pressure we first evaluate p_1, the pressure at the centre of the slipper ($x = l/2$). This, of course, is not the maximum pressure, as h varies with x in accordance with equation (36), but if this variability is not too great the pressure p_1 is of the same order as the maximum pressure. By equation (39), putting $h = \delta(a - l/2) = h_m$, we have

$$p_1 - p_0 = \frac{3}{2}\frac{\mu v l^2}{h_m^2(2a-l)}.$$

If we assume that the distribution of pressure is approximately parabolic, the mean excess pressure p_m is equal to $\frac{2}{3}(p_1 - p_0)$, i.e.

$$p_m = \frac{\mu v l^2}{h_m^2(2a-l)}. \qquad \ldots\ldots \quad (40)$$

This formula makes it clear that even when μ is comparatively small very high pressures may be reached if h_m, the mean thickness of the oil film, is very small. As h decreases in the direction of flow, the

maximum pressure lies behind the middle, by equation (39), so that the resultant force also lies behind the middle. A distribution in accordance with formula (39) is shown in fig. 3.41. The distributions of velocity are shown below; the variations of curvature clearly show how the pressure varies. The particular nature of the pressure distribution also depends on the ratio l/a, while the position of the resultant force depends on this ratio only. In the Michell thrust block the slippers are pivoted somewhat behind the middle (fig. 3.42), uniformly satisfactory working at all loads being thereby obtained. The slipper automatically takes up a definite oblique position (more accurately. a definite value of a), for if the inclination is increased, the point of application of the pressure moves farther back and, if it is decreased, farther forward, so that the correct position is very stable.

In actual thrust blocks some of the oil enclosed by the front edge escapes at the sides, so that the pressure in the oil is considerably lowered;* qualitatively, however, the phenomena are as described above.

Owing to the way in which the pressure is distributed, the shearing stresses on the thrust block are smaller at the front and larger at the rear than those due to the ordinary friction in a narrow interspace; the reverse is true for those on the guides. Formulæ for them can readily be calculated from equations (33a), (33b), (35) and (38). Instead of stating these formulæ we shall content

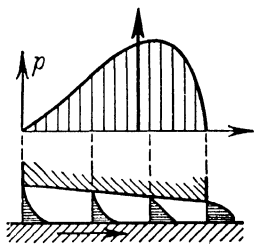


Fig. 3.41

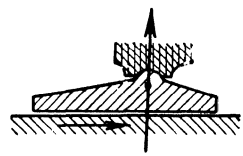


Fig. 3.42.—Thrust block

ourselves with an approximate expression for the friction, which is more accurate the greater the value chosen for the ratio a/l. In this case the distribution of the shearing stresses is approximately trapeziform, so that the mean force of friction per unit area can be taken as approximately equal to the friction at the middle. As the value of p' there is trifling, we may put

$$\tau_m \approx \mu v/h_m$$

by equation (33b).

* Cf. A. G. M. Michell, *Zeitschr. f. Math. u. Phys.*, Vol. 52, p. 123 (1905) (*Ostwalds Klassiker*, No. 218, p. 202); the case of a slipper of finite width with the oil moving out at either side is also discussed there.

The thickness of the layer of lubricant h_m can be eliminated from this expression by means of equation (40). We have

$$h_m = \sqrt{\frac{\mu v l^2}{p_m(2a - l)}}, \quad \ldots \ldots \quad (41)$$

and hence

$$\tau_m = \sqrt{\left(\frac{\mu v p_m}{l}\right)} \sqrt{\left(\frac{2a - l}{l}\right)}. \quad \ldots \quad (42)$$

The expression $\mu v/l$ represents the (very small) frictional stress which would arise from a layer of oil of thickness l. As regards order of magnitude, therefore, the actual frictional stress represents the geometric mean of this small frictional stress and the mean load on the slipper. For fixed values of l and a the resistance accordingly varies as the square root of the viscosity and also as the square root of the load and as the square root of the velocity. This relationship, in fact, is not merely associated with the mean values we have considered here but could have been equally well derived from the exact formula (in the latter case the function of a/l would of course be different).

The "coefficient of friction" (friction/normal force) considered in Coulomb's theory of the friction of solid bodies would here be represented by the dimensionless quantity

$$f = \tau_m/p_m.$$

Thus for fixed values of l and a, i.e. for given dimensions of the slipper in fig. 3.42, it would be proportional to the pure number $\sqrt{(\mu v/p_m l)}$.

Numerical Example.—Let $\mu = 1$ poise (p. 105), $v = 100$ cm./sec., $p_m = 30$ Kg./cm.$^2 \approx 3 \times 10^7$ dynes/cm.^{2}, $l = 10$ cm. Then the pure number

$$\sqrt{\left(\frac{\mu v}{p_m l}\right)} = 0{\cdot}577 \times 10^{-3}.$$

If, further, $a = 2l$, the numerical factor in equation (42) is $\sqrt{3} = 1{\cdot}732$, so that $f = 0{\cdot}001$. By equation (41), which may also be written

$$h_m = l\sqrt{\left(\frac{\mu v}{p_m l} \cdot \frac{l}{2a - l}\right)},$$

the mean thickness of the layer of oil is 10 cm. $\times$ 0·577 $\times$ $10^{-3}/\sqrt{3} = 0{\cdot}0333$ mm.

In the case of a journal and bearing the relationships are not so simple, as here we have one new fixed quantity, namely s, the interval between the journal and bearing when the former is in a central position (the difference between $r + s$, the radius of the bearing, and r, the radius of the journal), which is called the clearance, and two additional unknown quantities when the centre of the bearing is displaced vertically and horizontally. The essential point in this case also is that a wedge-shaped cushion of oil is formed, through which the oil is

dragged from the wide side to the narrow as the journal rotates (fig. 3.43). The calculations are very intricate, but are simplified if we may assume that e, the eccentricity of the journal, is small compared with s, the clearance. This is the case for journals dipping into oil and rotating rapidly under a moderate load. In this case we may put $h = s + e\cos(\phi + \alpha)$, where ϕ is an angular co-ordinate, and expand the denominators in the formulæ by the binomial theorem. As a result of this calculation, which is carried out in a way similar to that for the slipper, e/s turns out proportional to the dimensionless quantity $L = p_m s^2/\mu v r$, where p_m is the mean pressure on the bearing, r the radius of the journal, and v the peripheral velocity. The nature of this coefficient also follows from formula (40) for the slipper, which may be written

$$\frac{l}{2a - l} = \frac{p_m h_m^2}{\mu v l}.$$

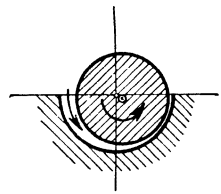


Fig. 3.43.—Journal and bearing

The left-hand side of this equation is of a purely geometrical nature and its meaning is closely related to that of e/s, h_m and l taking the places of s and r on the right-hand side respectively.

In the more general case * where e/s is no longer small this quantity turns out to be a function of the coefficient L, and so does α, the angle between the direction of the resultant force and the line joining the centre of the journal to the centre of the bearing. This angle is usually nearly 90°, the point where the distance between the journal and the bearing is least lying ahead of the direction of the pressure on the journal.

The effects of different loads on the bearing, different amounts of clearance, different degrees of viscosity of the oil, and different peripheral velocities are all represented in the coefficient L, so that experimental results may justifiably be classified by means of this number.†

The frictional coefficient f_1 of a bearing (tangential force/load on bearing) can be expressed in a way analogous to that for the slipper:

$$f_1 = \sqrt{\frac{\mu v}{p_m r}} \times \text{constant}.$$

From experiments on a hemispherical bearing of length $1{\cdot}23d$, Walger ‡ found the value 2·4 for the constant.§

* Cf. the paper by O. Reynolds referred to in the footnote on p. 153, and also A. Sommerfeld, *Zeitschr. f. Math. u. Phys.*, Vol. 50, p. 97 (1904) (*Ostwalds Klassiker*, No. 218, p. 108).

† For details, see e.g. Gümbel and Everling, *Reibung und Schmierung im Maschinenbau* (Berlin, 1925).

‡ " Schmiertechnische Untersuchungen ", *VDI-Zeitschr.*, p. 205 (1932).

§ It may be noted that the formula can also be written in the form $f_1 = 2{\cdot}4\, s/r\sqrt{L}$.

In the above account we have of course tacitly assumed that oil is liberally supplied and the velocity is not too low or the load on the bearing too great, so that there exists an oil film covering the entire surface of the bearing and preventing any metallic contact. In view of the fact that there are limits to the accuracy with which journals and bearings or slippers and guides can be constructed, metallic contact cannot be avoided if the interspace h is too small, and phenomena then arise which are better explained by the ordinary friction of solid bodies. The case where fairly large negative pressures occur in the oil film must also be excluded when we use the above formulæ. In this case we may expect the oil film to be torn apart. Careful measurements by W. Frössel * of the pressure in a bearing running in oil showed that the pressure where the oil film tore apart was nearly equal to the atmospheric pressure, owing to vigorous evolution of gas from the oil, whereas from observations on cavitation in water (Chapter V, § 1, p. 312) a vacuum might have been expected to occur. Tearing of the oil film regularly occurs with heavy loads, and the situation is then the same as for a bearing which only partly encloses the journal; but we cannot go further into the theory of heavily loaded bearings here.

The relationships described above are in good agreement with experiment in the case of moderately loaded slippers and bearings.* For heavy loads considerable discrepancies arise from the heating of the oil, which brings about a marked lowering of its viscosity. G. Vogelpohl,† however, has shown that this very complicated state of affairs can be discussed in greater detail. Of his results we shall mention only that the more satisfactory oils are those in which the viscosity does not decrease so much as the temperature rises. He also proved that even in the case of the so-called semi-fluid or mixed friction the greater part of the load on the bearing is borne "hydrodynamically", i.e. by the oil contained between the irregularities of the two opposing surfaces, and only a very small part by the tops of the irregularities which meet one another.

There is a not inconsiderable rise in viscosity with pressure, so that the state of affairs is more favourable with heavily loaded bearings and small tangential velocities than if the viscosity remained constant; cf. S. Kiesskalt, *VDI-Forschungsheft* 291 (1927). According to his experiments it is two to four times as great at 600 atmospheres as at atmospheric pressure.

* *Forschung*, Vol. 9, p. 261 (1938).

† G. Vogelpohl, "Beiträge zur Kenntnis der Gleitlagerreibung", *VDI-Forschungsheft* 386 (1938).

11. Flow through Straight Pipes and Water-courses of Constant Cross-section.

(*a*) The Hagen-Poiseuille law given in § 1 (p. 99), according to which the fall of pressure increases in proportion to the velocity, is valid only for velocities below the critical velocity (cf. p. 111). For velocities above the critical velocity, i.e. for turbulent flow, the fall of pressure is, as we mentioned on p. 111, more or less accurately proportional to the square of the velocity. The shearing stress on the boundary, τ_w, or in the case of non-circular cross-sections its mean value, may then be put equal to $\frac{1}{2}\lambda'\rho q^2$, where λ' is a number depending on the particular conditions and especially on the roughness of the boundary, and q is the mean velocity. The fall of pressure in a section of a pipe or water-course of length l must counterbalance the shearing stresses on the boundary surface (cf. the discussion of fig. 3.2 on p. 99). Hence if a is the area of the cross-section and P the " wetted perimeter ",*

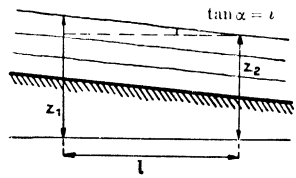


Fig. 3.44

$$(p_1 - p_2)a = \tau_w l P = \tfrac{1}{2}\lambda'\rho q^2 l P,$$

so that

$$\frac{p_1 - p_2}{l} = \tfrac{1}{2}\lambda'\frac{P}{a}\rho q^2. \qquad \ldots\ldots \quad (43)$$

$a/P = m$ is called the hydraulic mean depth. In a liquid flowing under the influence of gravity, e.g. in a river, the fall of level $i = (z_1 - z_2)/l$ is usually given (fig. 3.44); this is connected with the fall of pressure along a horizontal line by the relationship $p_1 - p_2 = g\rho(z_1 - z_2) = g\rho l i$ (Chapter I, § 6, p. 12), so that

$$\tau_w = g\rho m i \text{ and } i = \frac{1}{g\rho}\frac{p_1 - p_2}{l} = \tfrac{1}{2}\frac{\lambda'}{m}\frac{q^2}{g}.$$

Reversing this formula we obtain

$$q = \sqrt{\left(\frac{2g}{\lambda'} m i\right)}.$$

When applied to rivers and canals this formula is usually written

$$q = C\sqrt{(mi)}, \qquad \ldots\ldots \quad (44)$$

and is then known as Chézy's formula. The value of C, which must still be regarded as a function of the hydraulic mean depth and of

* In an open water-course (river or canal) the free surface does not form part of the " wetted perimeter ".

the roughness of the boundary surfaces, varies for depths from 0·5 to 3 metres from 80 m.$^{\frac{1}{2}}$sec.$^{-1}$ for channels of smooth wood or smoothly plastered masonry to 30–50 for earth walls and 24–40 for gravel. Many formulæ have been proposed in attempts to reproduce the results of observation more accurately,* e.g. C may be put equal to say the sixth to eighth root of the hydraulic mean depth, the nature of the boundary being constant, but there still remains the difficulty of correctly estimating the exact degree of roughness of the boundary. According to Strickler,† for wide natural rivers with a stony bed, C may be put equal to $21{\cdot}1\ (t/d_g)^{1/6}$ m.$^{\frac{1}{2}}$ sec.$^{-1}$, where t is the depth of the water and d_g the mean diameter of the stones. Here d_g, the roughness of the bed, depends to some extent on the fall and the depth of water [cf. Chapter V, § 5(*a*), p. 334], so that every type of stream has its own empirical formula not involving d_g. Since, however, natural and artificial irregularities in the bed (banks of gravel, groynes, &c.) contribute to the "roughness", many of the formulæ are very like one another.

The effects of variously shaped cross-sections are satisfactorily reproduced by means of the hydraulic mean depth, which indicates that, as we have tacitly assumed in the above argument, the shearing stress is nearly the same at all points of the boundary of a cross-section.

In § 16 of the last chapter (p. 94) we mentioned that the velocity $\sqrt{(gh)}$ with which a slight elevation is propagated forms the boundary between "tranquil flow" and "shooting". If we put $h = m$, which is approximately true for broad river-courses, we have

$$q = C\sqrt{(mi_1)} = \sqrt{(gm)},$$

i.e.

$$i_1 = g/C^2,$$

so that we have e.g. $i_1 = 1/250$ for $C = 50\text{m.}^{\frac{1}{2}}\,\text{sec}^{-1}$. For $i < i_1$ there is "tranquil flow", for $i > i_1$ "shooting".

(*b*) For *pipes of circular cross-section* (radius r, diameter d) we have

$$m = \frac{a}{P} = \frac{\pi r^2}{2\pi r} = \frac{r}{2} = \frac{d}{4}.$$

If we write λ_1 for $4\lambda'$,

$$\frac{p_1 - p_2}{l} = \tfrac{1}{2}\frac{\lambda_1}{d}\rho q^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

by equation (43). λ_1 may be called the coefficient of resistance. Here we have followed engineering usage; in the physics literature r is used instead of d and the corresponding coefficient of resistance is $\frac{1}{2}\lambda_1$. The meaning of λ_1 is illustrated by the fact that $l = d/\lambda_1$ is the length of

* See e.g. the numerous references in Forchheimer's *Hydraulik*, 3rd ed., pp. 139–71.

† *Mitteilung* 16 *des Eidgen. Amtes für Wasserwirtschaft*, Berne (1923).

pipe in which the pressure decreases by an amount equal to the dynamic pressure corresponding to the mean velocity q.

Considerations of similarity in the case of *rough pipes* lead us to expect the same numerical values of λ_1 for two pipes of differing diameter, provided the dimensions of the irregularities are proportional to the breadths of the tubes. Hopf and Fromm * found that if k is a linear measure of the roughness (e.g. its height), λ_1 is proportional to $(k/m)^{0\cdot314}$ for geometrically similar roughnesses, which is in good agreement with the above rule for the relation between C and m. By comparison with equation (44) we have $\lambda_1 = 8g/C^2$, which for the above values of C gives values of λ_1 ranging from 0·012 to 0·136.

Extensive series of experiments, some of great accuracy, have been made on the behaviour of turbulent flow in *smooth pipes*. From the theoretical point of view all that can be said is that in such cases the coefficient of resistance must have the same numerical value when Reynolds' number qd/ν (see § 2, p. 104) has the same value. Thus λ_1 may be a function of Reynolds' number also. In actual fact λ_1 decreases as Reynolds' number increases. Up to $\mathscr{R} = qd/\nu = 80{,}000$ we may write †

$$\lambda_1 = \frac{0\cdot3164}{\mathscr{R}^{1/4}}. \qquad \ldots\ldots\ldots \quad (46)$$

(given by Blasius,‡ following experiments by Saph and Schoder). Later experiments extending to values of Reynolds' number up to 400,000 showed that the formula

$$\lambda_1 = 0\cdot0072 + \frac{0\cdot611}{\mathscr{R}^{0\cdot35}} \qquad \ldots\ldots\ldots \quad (47)$$

represents the whole range better (Lees, following experiments by Stanton and Pannell,§ also Jakob and Erk from their own experiments).‖

The formula

$$\lambda_1 = 0\cdot0054 + \frac{0\cdot396}{\mathscr{R}^{0\cdot3}} \qquad \ldots\ldots\ldots \quad (47a)$$

was subsequently proposed by R. Hermann¶ on the basis of his own experiments, in which the values of Reynolds' number ranged up to nearly 2×10^6.

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 3, pp. 329, 339 (1923).

† In the literature the length used for Reynolds' number for flow in pipes is sometimes the radius, sometimes the diameter, or more rarely the hydraulic mean depth m. Here, following engineering custom, we have used d.

‡ *Forschungsarbeiten des Ver. Deutsch. Ing.*, No. 131 (1913).

§ *Phil. Trans. Roy. Soc.* (A), Vol. 214, p. 199 (1914); *Proc. Roy. Soc.* (A), Vol. 91, p. 46 (1915).

‖ *Forschungsarbeiten des Ver. Deutsch. Ing.*, No. 267 (1924).

¶ *Dissertation*, Leipzig, 1930.

(*c*) Later, an attack on the problem of the resistance of pipes from the theoretical side also met with success. Th. von Kármán,† using the considerations on velocity distribution given in § 5(*c*), p. 126, has established a second important relationship which, to distinguish it from the "law of flow past a wall" discussed in § 5, we shall call the "middle law". This holds for both smooth and rough pipes, but strictly speaking only for very large values of Reynolds' number, since the viscosity is regarded as negligible. According to it, the difference between the maximum velocity at the centre of the pipe and the velocity at distance y from the boundary is v_* times a universal function of the ratio y/r, where $v_* = \sqrt{(\tau_{\text{wall}}/\rho)}$, that is,

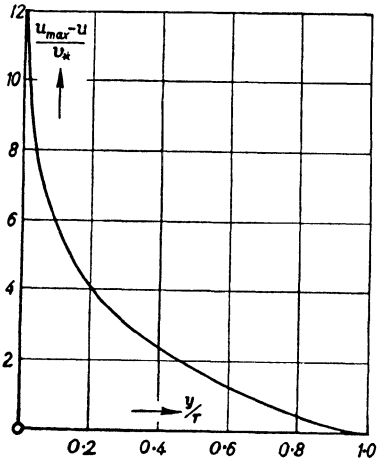


Fig. 3.45.—Middle law

$$u_{\max} - u(y) = v_* f\left(\frac{y}{r}\right). \quad (48)$$

From his assumption of similarity von Kármán deduced a formula for the function f which gives good agreement with experiment and in which only one numerical value, namely the quantity κ in § 5, had to be established experimentally.

Fig. 3.45 shows the behaviour of $f(y/r)$ as found by experiment. An expression for the mean velocity q can readily be deduced from equation (48). From the empirical behaviour of $f(y/r)$, we have

$$q = u_{\max} - 4{\cdot}07 v_*. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Since the fall of pressure $-dp/dx$ can be expressed by $\tau = \rho {v_*}^2$ and $u_{\max}$ in terms of q by (49), a relation between $\lambda_1 = 8{v_*}^2/q^2$ and $\mathscr{R} = qd/\nu$ can be deduced by combining (48) with the laws of flow past a wall for smooth and rough pipes. We shall not go into the details of the calculations here.‡ For smooth pipes, using equation (24) of § 5 (p. 126), we obtain the relation

$$1/\sqrt{\lambda_1} = 2{\cdot}0 \log_{10}(\mathscr{R}\sqrt{\lambda_1}) - 0{\cdot}8, \quad . \quad . \quad . \quad (50)$$

which can be used in practice for all values of Reynolds' number in the turbulent region.

† See first footnote on p. 129.

‡ For these, see Prandtl-Betz, *Ergeb. d. AVA*, Vol. 4 (1932), or *Aerodynamic Theory*, Vol. 3, p. 142 *et seq.*; also Prandtl, *VDI-Zeitschr.*, Vol. 77, p. 109 (1933).

Using equation (25) of § 5, we have the equation

$$1/\sqrt{\lambda_1} = 1{\cdot}95 \log_{10}(\mathscr{R}\sqrt{\lambda_1}) - 0{\cdot}55, \quad . \quad . \quad (50a)$$

which is more accurate for large values of Reynolds' number.

No serious difficulty arises from the fact that λ_1 occurs on both sides of equations (50) and (50*a*); all that is required is to assume a suitable approximate value for λ_1 on the right-hand side and repeat the calculation if necessary.

(*d*) For rough pipes, once the " roughness flow " has set in, equation (29) of § 5 (p. 129) with $C_2 = 8{\cdot}5$ (equivalent to a coating of sand) gives

$$1/\sqrt{\lambda_1} = 2{\cdot}0 \log_{10}(r/k) + 1{\cdot}74. \quad . \quad . \quad . \quad (51)$$

Fig. 3.46 shows the relation between λ_1 and Reynolds' number plotted on logarithmic paper. All but the lowest of the curves are based on J. Nikuradse's *[16] experiments with pipes having an internal coating of

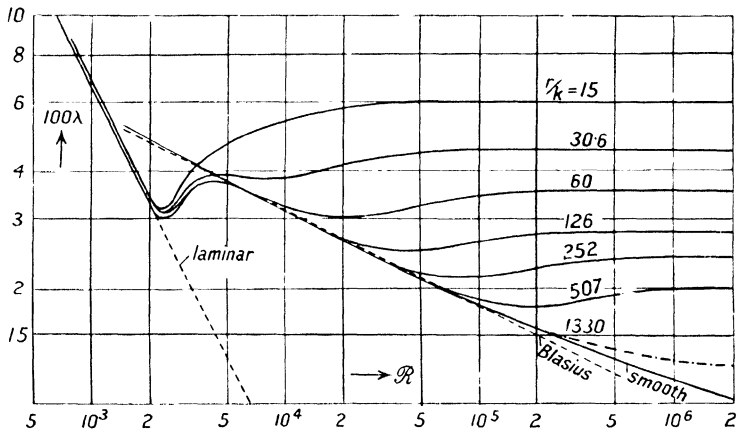


Fig. 3.46.—The relation between λ_1 and Reynolds' number for rough pipes according to Nikuradse (continuous), and Bauer and Galavics (dots and dashes)

sifted sand with grains of various sizes. The rapid transition from the "hydraulically smooth" state to the definitely "rough" state which the curves show for the smaller values of Reynolds' number is characteristic of surfaces of this type, consisting of irregularities close together and of about the same size.†

The roughness which occurs in practice usually consists of a background of small humps with a few larger humps here and there. In

* " Strömungsgesetze in rauhen Rohren ", *VDI-Forschungsheft* 361 (1933).

† The condition of similarity required for this passage to the case of § 5(*d*) (p. 129), that the quantity C_2 should be a function of kv_*/ν, is well confirmed by Nikuradse's experiments; see his paper, or Prandtl, *VDI-Zeitschr.*, Vol. 77, p. 108 *et seq.* (1933).

this case the transition is much more gradual; this is exemplified by the curve of dots and dashes, from experiments by B. Bauer and F. Galavics * with hot water in an iron pipe which was " smooth " in the practical sense.

A general idea of the behaviour of λ_1 for smooth pipes may be obtained from the following values, given in round numbers:

	$\lambda_1 = 0{\cdot}040$	0·030	0·020	0·015	0·010
for	$\mathscr{R} = 4000$	12,000	60,000	240,000	2,500,000.

The Hagen-Poiseuille law (equation (4), p. 99) is also included in fig. 3.46. Using $q = Q/\pi r^2$, we have

$$\frac{p_1 - p_2}{l} = \frac{8\mu q}{r^2} = \frac{32\mu q}{d^2}. \quad . \quad . \quad . \quad . \quad (52)$$

Comparison with equation (45), p. 162, shows that if we put $\lambda_1 = 64\mu/\rho qd = 64/\mathscr{R}$ the two equations become formally the same.

(*e*) The fall of pressure in a pipe given by equations (46)–(47*a*) and (50)–(52) and illustrated in fig. 3.46 is only obtained if the section at the beginning and end of which the pressures are measured (by means of perforations; see § 7 of last chapter, p. 53) is at a sufficient distance, say 60 diameters, from the inflow, and even then certain irregularities may occur when the flow is laminar. If measurements are made nearer the inflow, or if we measure e.g. the fall of pressure between the reservoir and the far end of the pipe, the *phenomena at the inflow* must be taken into account. Assuming that the mouth of the pipe is rounded off, we first have the fall of pressure $\frac{1}{2}\rho q^2$ required to produce the velocity q at the entrance of the pipe. At first the distribution of velocity is practically uniform, as each particle has been subject to the same fall of pressure. Frictional retardation sets in from the walls inwards and an increasing layer of retarded fluid is formed (fig. 3.47). The velocity of flow at the centre of the pipe must increase in such a way that the same quantity flows across every cross-section. This acceleration of the central flow corresponds to a fall of pressure along the axis of the pipe, which is obtained by applying Bernoulli's theorem to the central flow. This fall of pressure, which exerts a forward urge even on the marginal regions, is greater than that associated with the Hagen-Poiseuille flow. As the flow advances the normal state of flow according to the Hagen-Poiseuille law is gradually established owing to the extension of the

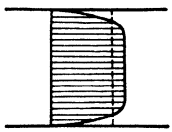


Fig. 3.47.—Profile of the velocity at inflow

* Report of *Fernheiz-Kraftwerks der Eidgenössischen Technischen Hochschule*, Zürich, 1936.

frictional region. According to L. Schiller's calculations and experiments * the transition is practically complete after a distance

$$l_1 = 0{\cdot}03\,d^2q/\nu = 0{\cdot}03\,d\mathscr{R}.$$

Thus if Reynolds' number runs from 200 to 20,000, l_1 runs from 6 to 600 times the diameter of the pipe. Hence for short lengths of pipe rounded off at the inflow the velocity is certainly not as given by equation (3), p. 99, i.e. distributed paraboloidally, unless Reynolds' number is quite small. In fact, if the fluid is coming from a large vessel the velocity is in the main distributed uniformly, the boundary layer being alone retarded by friction. The corresponding two-dimensional case (flow into a broad rectangular channel) has been worked out analytically by H. Schlichting,† and a similar calculation has been made for the circular pipe by B. Punnis.‡ H. Hahnemann and L. Ehret § have made calculations and comprehensive experiments, extending into the turbulent region, on the initial flow in channels. Their calculations followed the lines of Schiller's.

Turbulent flow occurs in considerably shorter lengths of pipe if the necessary eddies are set up by sharp edges, bends, &c., at the entrance to the pipe. When the entrance is smooth there is at first a region of laminar flow, at the end of which turbulence sets in. If Reynolds' number is large and the inflowing fluid is undisturbed the length of this region may be taken as about $500{,}000\,\nu/q$ or $500{,}000\,d/\mathscr{R}$ (cf. resistance of plates, § 15, p. 188).

(*f*) It must be expressly emphasized that the above statements apply only to straight pipes. In curved pipes the resistance is always greater. In laminar motion the effects even of a very slight curvature are particularly striking when Reynolds' number is not too small. The more rapidly flowing central parts of the flow are forced outwards by centrifugal action, while the slower parts along the walls are forced inwards, i.e. towards the centre of curvature, as has already been mentioned in § 8 (p. 146). A theory of this phenomenon has been developed by W. R. Dean.||

The phenomena may be accounted for qualitatively as follows. Let R, the radius of curvature, be large compared with the radius of the pipe. If the distribution of velocity is paraboloidal, the velocity at the centre of the pipe is $2q$; the corresponding rise of pressure due to centrifugal forces is $\rho(2q)^2/R$ per unit length. In the marginal regions the centrifugal forces are small. Hence the difference of the pressures at the inner wall and the outer is approximately

$$\tfrac{1}{2} \,.\, 2r \,.\, \frac{4\rho q^2}{R} = 4\rho q^2 \frac{r}{R}.$$

* *Forschungsarbeiten des Ver. Deutsch. Ing.*, No. 248; *Zeitschr. f. angew. Math. u. Mech.*, Vol. 2, p. 96 (1922).

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 14, p. 368 (1934).

‡ *Diss.*, Göttingen (1946).

§ *Jahrb. d. Deutsch. Luftfahrtforschung* (1941), Part 1, p. 21, and (1942), Part 1, p. 186.

|| W. R. Dean, *Phil. Mag.* (7), Vol. 4, p. 208 (1927); Vol. 5, p. 673 (1928).

As in the marginal regions there are no forces to oppose this pressure difference, it gives rise to secondary flow of the first type mentioned above. In the case of very slight curvature the velocity of the secondary flow may be estimated as follows. Suppose that a central strip has the velocity v outwards and two outer strips the same velocity inwards (with transitional regions interposed, of course). If the shearing stress between the two strips is τ_1, the force on the central strip per unit length in the direction of the axis of the pipe is approximately $2\tau_1 . 2r$; this has to balance the resultant pressure $b\Delta p$, where b is the breadth of the central strip. For simplicity we put $b = r$, and we may also take τ_1 as approximately equal to $\frac{2\mu v}{b/2}$ (cf. the calculations in § 1, p. 98).

Then we have $$16\mu v = 4\rho q^2 r^2/R$$

or $$\frac{v}{q} = \tfrac{1}{4} . \frac{qr}{\nu} . \frac{r}{R} = \tfrac{1}{8}\mathscr{R}\frac{r}{R}.$$

This is a dimensionless quantity associated with our problem, which we could of course have written down from simple dimensional considerations.

If the curvature is more marked, the distribution of velocity is entirely altered by the secondary flow, the greatest velocity now being near the outside wall; the secondary flow takes place chiefly in a sort of boundary layer in the regions near the walls. C. M. White found by experiment * that the resistance of laminar flow in a curved pipe is equal to that of the straight pipe as given by (52) multiplied by a function $f(D)$, where D is half the geometric mean between Reynolds' number and the dimensionless quantity above, Reynolds' number $\times r/R$, that is,

$$D = \frac{qr}{\nu}\sqrt{\frac{r}{R}}.$$

The function $f(D)$ does not differ greatly from unity for $D < 20$; for $20 < D < 1000$ we may use† the approximation formula $f(D) = 0{\cdot}37D^{0\,36}$.

In the case of turbulent flow the effect of a slight curvature is not so marked, but sharp curvatures give rise to considerable increases in resistance. If the pipe is bent through a definite angle and is thereafter straight, the resistance even in the straight portion is found to be increased; this is due to the change in velocity distribution on entering the straight piece. The total additional resistance due e.g. to a right-angled bend does not vary much with the curvature, as a short sharply-bent piece has much the same effect as a longer slenderer one.

For rough estimates of the losses in smooth bends with radii of curvature ranging from $R = 4d$ to $R = 10d$ it may suffice to add that the resistance coefficient ζ is approximately 0·045, 0·075, 0·09, and 0·10 for bends of $22\frac{1}{2}°$, 45°, 60°, and 90° respectively, the values being somewhat greater for sharper curves and rough walls ‡ (the formula is $p_2 - p_1 = \frac{1}{2}\zeta\rho q^2$ + ordinary pipe friction).

12. **Flow through Channels of Varying Cross-section.**

(*a*) The simplest case of this kind is the flow from orifices. In so far as no hydraulic losses occur the problem has already been discussed in § 4 of the last chapter (p. 42): in particular we emphasized the fact that owing to the contraction of the jet its cross-section is not in general the same as that of the orifice (*a*), the cross-section of the jet being

* C. M. White, *Proc. Roy. Soc.* (A), Vol. 123, p. 645 (1929).

† See M. Adler, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 5, p. 257 (1934).

‡ Cf. D. Thoma, *Transactions Tokyo Sectional Meeting, World Power Conference*, 1929, Vol. II, p. 446.

αa, where α is the coefficient of contraction (α is about 0·61 for sharp-edged orifices in plane walls). When fluid flows out of a vessel with a cross-section which is large compared with that of the orifice, the velocity at the centre of the jet is usually very nearly $\sqrt{(2gh)}$, but in the marginal regions the friction between the fluid emerging from the hole and the wall gives rise to a diminution of velocity, which turns out to be larger for orifices like that in fig. 2.8, p. 43, than for those like fig. 2.7, p. 43. Thus the mean velocity is somewhat less than the theoretical and may be written as $\phi\sqrt{(2gh)}$, where ϕ is called the coefficient of velocity. In the main the coefficient of velocity differs considerably from unity for small orifices and low velocities (i.e. for small values of Reynolds' number); for large openings and high velocities it is almost always close to unity (we are still assuming that the cross-section of the vessel is large compared with that of the orifice).

Quantities of fluid may readily be measured by means of the flow from an orifice. We have only to measure a, the cross-section of the orifice, and h, the depth of the liquid; then the quantity flowing out per second is $\alpha\phi a\sqrt{(2gh)}$. The product $\alpha\phi$ is usually replaced by the single symbol μ, which is called the coefficient of discharge. To find μ we can weigh the amount of liquid flowing out in a definite time.

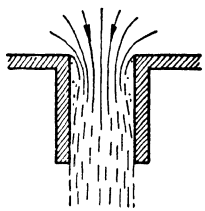


Fig. 3.48.—Sharp-edged side tube

(*b*) On the other hand, when side tubes join the wall of a vessel in sharp edges, as e.g. in fig. 3.48, considerable losses occur. We first have a contraction as in the case of an opening in a thin-walled vessel; the jet then broadens out again, mixing with the surrounding dead water (dotted in fig. 3.48). The theory in § 12(*c*) of the last chapter (p. 80) gives the following results. If q is the mean velocity of efflux at the end of the side tube, the velocity at the narrowest part is q/α. Thus a loss of head equal to $\frac{1}{2}q^2(1/\alpha - 1)^2/g$ must be added to the pressure head $\frac{1}{2}q^2/g$. The sum of the two heights must be equated to the height of the water in the vessel. Solving for the velocity, we obtain $\phi = \alpha(1 - 2\alpha + 2\alpha^2)^{-1/2}$ for the coefficient of velocity. For $\alpha = 0{\cdot}61$, $\phi = 0{\cdot}84$.

(*c*) Sudden contractions in a pipe (fig. 3.49) cause a loss of pressure (in addition to the fall of pressure given by Bernoulli's equation), analogous to that discussed above, as contraction of the jet again takes place. According to Weisbach, the coefficient of contraction α may be taken as $0{\cdot}63 + 0{\cdot}37(a_1/a_0)^3$.

If the contraction is followed by a sudden widening (fig. 3.50) the loss of pressure is similarly

$$p_0 - p_2 = \tfrac{1}{2}\rho q_0^2(a_0/\alpha a_1 - 1)^2.$$

The difference of pressure $p_0 - p_1$, which is greater, is frequently used in making measurements of discharges; by Bernoulli's theorem (apart from small losses; see discussion of fig. 3.51 below)

$$p_0 - p_1 = \tfrac{1}{2}\rho q_0^2[(a_0/\alpha a_1)^2 - 1].$$

If $p_0 - p_1$ is determined by measuring the pressure by an orifice in front of the contraction (orifice plate) and another behind, and α is known from other experiments, q_0 can be found from this equation and hence $q_0 a_0$, the quantity discharged per second. According to E. Buckingham * the formula $a = 0{\cdot}598 + 0{\cdot}395(a_1/a_0)^2$ can be used for α between $a_1/a_0 = 0{\cdot}05$ and $a_1/a_0 = 0{\cdot}55$.

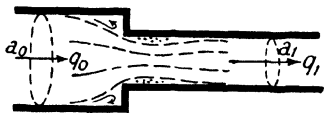


Fig. 3.49.—Sharp-edged contraction of a pipe

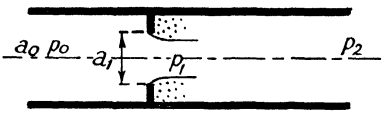


Fig. 3.50.—Orifice plate

Elbows and right-angle bends may be discussed on the same principles as the pipe in fig. 3.50; here again contraction phenomena occur at the sharp bends, the velocity subsequently becoming uniform again. The loss of head may be written as $\frac{1}{2}\zeta q^2/g$, where ζ is a resistance coefficient to be found empirically in each particular case. Numerical values for typical cases are to be found in engineering textbooks.†[17]

(*d*) Gradual contractions result in very small losses, whereas gradual widenings cause larger losses, as here there is a tendency for the flow to become separated from the walls. Nevertheless, the recovery of pressure is considerably greater than in the case of a sudden widening. The loss of head $p_0 - p_2$ in fig. 3.51 may be written as $\frac{1}{2}\zeta\rho(q_1^2 - q_2^2)$, where $\zeta = 0{\cdot}15$ to $0{\cdot}2$. A pipe like that shown in fig. 3.51 is called a Venturi tube and is likewise used for measuring quantities of fluid (Venturi meter); p_0 and p_1 are observed and the calculation is similar to the above, except that if the shape of the pipe is suitably chosen α may be taken equal to unity. The coefficient of velocity, on the other hand, is not exactly unity, and lack of uniformity in the entering fluid also has a disturbing effect, so that it

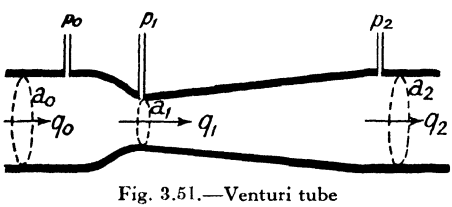


Fig. 3.51.—Venturi tube

* Personal communication.

† See e.g. Hütte, Vol. 1, 27th ed., pp. 482 *et seq.*; *Mitteilungen des Hydraulischen Instituts der Technischen Hochschule, München*, Heft 2 (1928).

is advisable to have the meter calibrated empirically if a high degree of accuracy is required. This is also true of the orifice plate of fig. 3.50.*[18]

The rise of pressure $p_2 - p_1$ in a pipe widening either suddenly or gradually is used in the jet apparatus shown in fig. 3.52 to suck in and thus remove other fluid.[19] Among the practical applications we may mention the water-jet aspirator, with which a fair vacuum can be obtained (q_1 must be about 20 metres per second in order that $p_2 - p_1$ may be equal to an atmosphere), and the bunsen burner, in which the jet of gas issuing from a small orifice sucks in air and becomes mixed with it. Another application is the blast of a locomotive, in which the steam issuing from the cylinder is used to suck the gases of combustion out of the smoke-box and thus maintains combustion. Another very remarkable application is the injector which by means

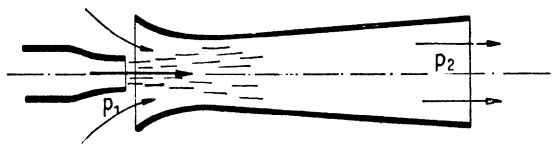


Fig. 3.52.—Jet pump

of steam from a boiler sucks up cold water, say from a well, and pumps it into the same boiler (the effect is explained by the increase in density when the steam is condensed to water; a large volume is removed from the boiler and a much smaller volume returned to it). The theory of jet instruments, which depends on the same relationships as in the apparatus described above, cannot be discussed in greater detail here.†

(*e*) A very useful way of measuring the discharge of an open water-course is by means of the flow over a weir (fig. 3.53). We have already obtained an approximate formula for the quantity of liquid passing over the weir (see the discussion at the end of last chapter, p. 95). This formula may, however, also be deduced from quite a different and apparently only remotely related problem, namely that of the flow through a narrow vertical notch in the boundary of an open water-course (fig. 3.54). The cross-section of the orifice may at once be divided up into elements bdh; the quantity flowing through one

* Further details on the measurement of quantities of fluid will be found in the article *Geschwindigkeits- und Mengenmessung von Flüssigkeiten* by H. Mueller and H. Peters in *Handbuch der Experimentalphysik*, Vol. IV, 1, pp. 553 *et seq.*, especially on pp. 575–88; also in *VDI-Durchflussmessregeln* (*DIN*, 1952, 5th ed., 1943), where particulars of standard nozzles and diaphragms are given. In viscous fluids the effect of Reynolds' number is very noticeable. Forms of nozzle suitable for values of Reynolds' number up to 1000 are also given in the *Durchflussregeln*; see also R. Witte, *VDI-Zeitschr.*, Vol. 87, p. 289 (1943), and H. G. Giese, *Forschung*, Vol. 4, p. 12 (1933), for small values of Reynolds' number.

† For the theory, see G. Flügel, *VDI-Forschungsheft* 395 (1938).

of these elements is $\alpha b dh\sqrt{(2gh)}$. Integrating from 0 to h_1, we obtain the total discharge

$$Q = \tfrac{2}{3}\alpha b h_1^{3/2}\sqrt{(2g)},$$

which agrees with the formula on p. 95 if α is put equal to $1/\sqrt{3}$ (h_1 taking the place of z there). The formal agreement is due to the fact that, owing to dimensional reasons, the quantities defining the fall cannot enter into the formulæ in any other way. Experiment shows, however, that even the numerical

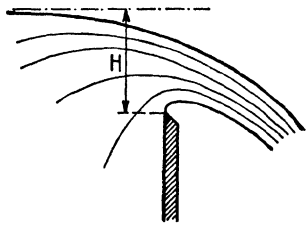


Fig. 3.53.—Flow over a sharp-edged weir

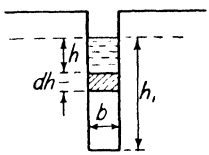


Fig. 3.54

values are almost exactly the same for the vertical fall in fig. 3.53 and for the notch in fig. 3.54. Particularly careful measurements have been made for weirs bounded by parallel walls (with no side contractions) with a sharp-edged vertical crest, air having free access to the underside of the falling sheet of water. According to Rehbock these observations give *[20]

$$\alpha = 0{\cdot}605 + \frac{h'}{3(h_1 - h')} + \frac{0{\cdot}08 h_1}{a},$$

where h' is a small height of 2·85 mm., probably depending on capillary effects, and a is the distance of the sharp edge from the bottom of the channel conveying liquid to it; h_1 is the head, i.e. the height of the surface of the water above the edge of the weir, measured at some distance from the latter. This may be measured with a good degree of accuracy by means of e.g. an apparatus like that shown in fig. 3.55 (hook gauge), as the position where the metal point emerges from the water can be determined very accurately (better than the position in which a point directed downwards dips into the water).

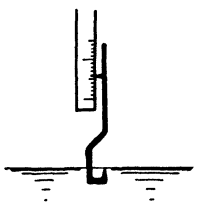


Fig. 3.55.—Hook gauge for measuring the level of a water surface

For weirs with a flat crest the theoretical value of α, $1/\sqrt{3} = 0{\cdot}577$ (see above), is in good agreement with experimental results.

* See e.g. Rehbock's paper in De Thierry and Matschoss, *Die Wasserkraftlaboratorien Europas*, p. 104 (Berlin, 1926); translated in *Hydraulic Laboratory Practice*, edited by J. R. Freeman (New York, 1929).

(*f*) Hydraulic engineers have also made very detailed investigations of the phenomena in an open channel where uniform flow is disturbed by artificial structures. In the case of disturbances affecting only a short length of the channel the frictional effects may be ignored

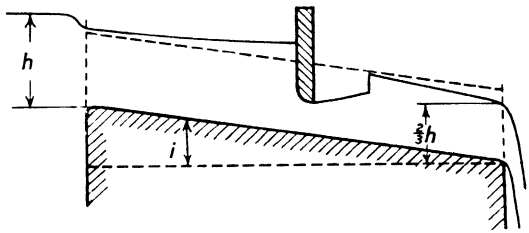


Fig. 3.56.—Backwater curves for $i < i_1$

as a first approximation, and we then obtain the relationships described in § 16 of the last chapter (p. 95). In many cases, however, taking frictional resistance into account makes all the difference. If the vertical acceleration can be neglected everywhere (that is, if the curvature of the surface is not too great), it is sufficient to consider the mean longitudinal acceleration $q\,dq/dx$. The fall of level

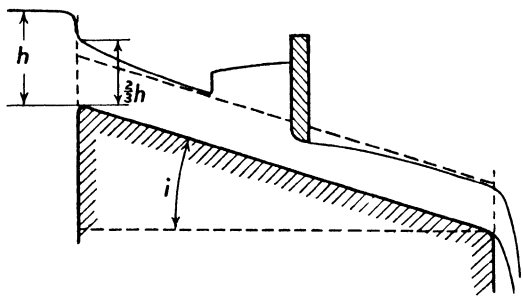


Fig. 3.57.—Backwater curves for $i > i_1$

$-dz/dx$ is then the combination of the frictional fall corresponding to q and the depth of water h, and the acceleration fall $\frac{q}{g}\frac{dq}{dx}$. If i is the fall of the bed, obviously

$$-\frac{dz}{dx} = -\frac{dh}{dx} + i.$$

In the simplest case, that of a very wide channel of the same depth all the way across, continuity is expressed by the equation $qh = \text{const.}$ Eliminating q, we have a differential equation of the first order for h for a given function $i(x)$. The solution is of course simplest when

$i = \text{const.}$ It differs in character according as the equilibrium velocity q_1 corresponding to the given fall is less than or greater than the velocity of a slight elevation $\sqrt{(gh_1)}$, i.e. according as the motion is "tranquil" or "shooting" (cf. Chapter II, § 16, p. 94, and § 11 of the present chapter, p. 162). For the details of the calculations we would refer the reader to Forchheimer's *Hydraulik*.*[21] Here we shall merely mention the most important result, namely, that in "tranquil" flow any disturbances of the state of equilibrium are propagated upstream, gradually dying out, while in "shooting" they are propagated downstream only. If in the latter case they are forced to spread upwards by a powerful obstacle, there is a jump in level, and broken water between this and the obstacle. Figs. 3.56 and 3.57 give an exaggerated representation of the surface of a portion of a water-course starting from a lake, containing a sluice in the middle, which dams up the water, and ends in a waterfall. (The problems under discussion arose from the problem of the damming up of the flow of a stream above an obstacle and are frequently referred to as the "theory of the backwater curve".) The two figures clearly show the differing characters of the backwater curves for "tranquil" flow ($i < i_1$) and "shooting" ($i > i_1$).

Note.—To the hydraulic engineers of the nineteenth century we owe detailed investigations of the flow of water in rivers and artificial channels, and e.g. a theory of "tranquil" and "shooting" flow in which vertical acceleration is taken into account (see the last paragraph on p. 97). They also discussed non-steady motion, including advancing waves in "shooting" flow, the ebb and flow of the tide in estuaries, floods in rivers, and the bursting of dams.†

13. Resistance of Bodies Immersed in Fluid.

(*a*) *General Remarks on the Resistance Formula.*—The conclusion that the resistance with which a fluid opposes the motion of a body immersed in it in virtue of its inertia must be proportional to the area of the section of the body at right angles to the direction of motion (*a*), and also proportional to the density of the fluid (ρ) and to the square of the velocity (v), dates back to Newton. This result may be explained by the following very simple argument. Each second the body must move a mass of fluid $M = \rho a v$ out of its way, and in doing so imparts a velocity, which may be taken as proportional to its own velocity, to each element of the fluid. The resistance is equal to the momentum imparted to the fluid per second and is therefore proportional to $Mv = \rho a v^2$.

In Newton's theory the laws of collision of elastic bodies are applied to the resistance of a fluid (Newton regarded the medium as consisting of particles free to move but *at rest*, which are regularly reflected

* Ph. Forchheimer, *Hydraulik*, 3rd ed. (1930), p. 217 *et seq.*

† The literature is given in Forchheimer's *Hydraulik*, 3rd ed., pp. 246–303.

by the moving body): the detailed results, however, have proved untenable. The Newtonian conception of fluid resistance has been replaced by the hydrodynamical conception, according to which the resistance consists of the pressure differences and frictional stresses arising from the fluid flowing round the body. A fundamental difference between the conclusions drawn from the old and from the new conception is that in the former only the shape of the front portion of the body is considered, whereas we now know that the phenomena giving rise to resistance are really to be sought for in the rear of the body, so that the shape of the rear portion may be of very great importance. We must also emphasize that in the older theory the resistance of any body was obtained by simple summation over all the elements of surface (using a law obtained for flat plates), whereas from the hydrodynamical point of view it is immediately obvious that this is inadmissible. This may be illustrated by the following example. The flow round a pair of plates inclined at an angle (fig. 3.58) must be quite different from that round two separate plates inclined at the same angle, seeing that in the latter case the fluid can flow between the plates and in the former case it cannot. According to experiments by Eiffel,* in fact, the resistance of a body consisting of two square plates inclined at 150° to the direction of motion is about 60 per cent of that of the plates when separated, whereas according to the older theory the resistance should be the same in the two cases. Another very striking example is as follows. A circular disc, a circular cylinder with a length equal to its diameter and one with a length equal to one and a half times its diameter (all moving in a direction at right angles to the circular section) have resistance coefficients of 1·12, 1·10 and 0·80 respectively, according to Eiffel's experiments (in which the bodies were dropped from the Eiffel Tower).† These numbers have been tested and confirmed in the *Modellversuchsanstalt* at Göttingen (in an artificial current of air). The fact that the longer cylinder has a lower resistance than the shorter can only be explained by the fact that in its case the system of eddies is reduced in extent owing to the flow re-attaching itself to the surface of the cylinder, so that the suction on the rear of the body is less than in the other two cases.

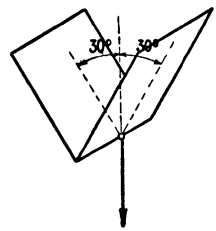


Fig. 3.58.—A pair of plates inclined at an angle

As regards the general form of the law of resistance which hydrodynamical principles may lead us to expect for a definite type of body, we may begin with the following considerations. The resistance is

* *Recherches expérimentales sur la résistance de l'air exécutées à la tour Eiffel*, p. 78 *et seq.* (Paris, 1907).

† Ibid., p. 54.

produced by pressure differences and frictional stresses. In general the former predominate and may be taken as proportional to the dynamic pressure corresponding to the velocity v, i.e. as proportional to $\frac{1}{2}\rho v^2$ (see § 4(*c*) of the last chapter, p. 45), i.e. the resistance, being the product of the pressure difference and the area exposed to it, is proportional to $\frac{1}{2}\rho a v^2$. The dynamic pressure, which we shall denote by p_d, may be measured by means of the Pitot and static tube described on p. 55. As the velocity is usually obtained from measurements of the dynamic pressure, the latter, and not the velocity, is the original experimental quantity. In addition, the representation of a resistance as a pressure acting on a surface is very suggestive. Hence it has become customary, not only in Germany but also in most other countries, to write

$$R = \tfrac{1}{2}\rho a v^2 \times \text{constant}$$

instead of

$$R = \rho a v^2 \times \text{constant}.$$

The numbers in the first case are of course twice those in the second: in accordance with aerodynamic usage we shall denote the former by c, possibly with a suffix to indicate components.

Making use of our discussion of dynamical similarity we may proceed to make the following statements about the behaviour of this "resistance coefficient". So long as we are comparing cases which are geometrically and dynamically similar, i.e. which have the same value for Reynolds' number vl/ν (l being any comparison length), the pressure differences and frictional stresses will be in the same ratio: in the cases under comparison the frictional stresses alter in proportion to the pressure differences, which, again, may be taken proportional to the dynamic pressure, so that here the above form of the law of resistance, i.e.

$$R = cap_d = c\rho v^2/2, \qquad \ldots\ldots \quad (53)$$

represents a true law. The resistance coefficient c of course only remains the same as long as Reynolds' number remains the same: in general it alters with Reynolds' number and may therefore be written as a function of Reynolds' number:

$$c = f(\mathscr{R}) = f(vl/\nu).$$

If in any particular case there are no marked frictional effects to be looked for, the resistance (by the above) is fairly accurately proportional to $a\rho v^2/2$, i.e. the function becomes a constant. This is true with fair accuracy for plates moved in a direction at right angles to their own plane and for similar sharp-edged objects. For circular plates the factor c is about 1·12.

If, on the other hand, frictional effects predominate, as e.g. in the case of *plates moved in their own plane*, large deviations from Newton's law may be expected (cf. § 15, p. 188). For extremely low velocities (Reynolds' number small compared to unity) viscosity alone is involved. In this case we apply Stokes' law, which we have already mentioned on p. 106 (resistance proportional to v). This law may also be expressed by an equation of the form of (53) by making c inversely proportional to Reynolds' number.

(*b*) *Analysis of the Resistance.*—The resistance of a body moving in fluid may always be divided into two parts, the resistance due to pressures and that due to frictional forces.

The force exerted by the fluid on the body can be resolved into a normal component and a tangential component for every element of surface, i.e. into a pressure and a frictional force. The resultant of all the pressures gives the pressure resistance * and the resultant of all the frictional forces the frictional resistance or skin friction. (For rough surfaces we shall for practical reasons take the resolution as being with respect to an ideal smooth surface following the average course of the actual surface. That part of the pressure resistance, which, strictly speaking, is borne by the individual irregularities, is then added to the frictional resistance.) The separation of the total resistance into these two parts may be carried out experimentally by observing the distribution of pressure over the surface of the body by means of perforations and thereby calculating the pressure resistance as the resultant of the pressures on the elements of the surface. If the total resistance is then obtained by balance measurements, the difference gives the resistance due to skin friction.† Starting with the assumption that the pressure resistance depends largely on the shape of the body, but that the resistance due to skin friction depends essentially on the extent of the surface and is independent of the shape of the body, it has been proposed to divide the resistance into a resistance due to *form* and a resistance due to *surface area*. Meanwhile more recent experiments indicate that the frictional resistance also depends to a very considerable extent on the shape, so that this subdivision cannot be strictly maintained.

In the case of bodies moving on the free surface of a liquid there is a further resistance, the *wave resistance* caused by the system of waves produced by the body in motion. As the wave motion takes place under the influence of gravity (we ignore capillary forces) the law of dynamical similarity involved differs from that associated with frictional phenomena. From the velocity (v), the length (l), and the acceleration of gravity (g) we may form the dimensionless quantity

* [That part of the total resistance which is not due to skin friction is sometimes called the *residuary resistance*.]

† For a direct measurement of the skin friction, see § 22(*d*), p. 256.

v^2/gl. The wave system which we may expect to be set up will be geometrically similar for two different sizes of a ship's form (e.g. model and ship) if this number has the same value, i.e. if the velocities are proportional to the square roots of the lengths (Froude's law of corresponding speeds).

The wave resistance is markedly affected by small changes in the velocity or shape of the ship: lengthening the hull may raise or lower it according as the stern wave, which interferes with the system of waves coming from the bow, strengthens or weakens these.[22] The resistance is large if the stern lies in a trough of the bow wave system, small if it coincides with a peak of the bow wave system.* In shallow water the wave system may be considerably modified. The wave resistance rises very noticeably if the ship is moving with exactly the same velocity as a slight elevation (§ 15 of the last chapter, p. 92). For velocities above this it diminishes again.

As the system of waves spreads out behind the ship, the work done by the ship against the wave resistance is carried away in the form of the kinetic energy of the wave motion. The other part of the pressure resistance, which corresponds to the ordinary resistance of a body surrounded on all sides by fluid, reappears in the momentum of the eddies in the wake, together with the frictional resistance, and may therefore also be called the wake resistance. The corresponding work reappears partly as heat and partly in the kinetic energy of the eddies, which subsequently is also gradually transformed into heat.

As, apart from disturbances due to wave formation, the frictional resistance and the wake resistance obey Reynolds' law of similarity, whereas the wave resistance obeys Froude's law of similarity, it is impossible to apply the results of experiments on models on the full scale without some modification. As in ships it is the wave resistance that is most important, experiments on the towing of models in experimental tanks are based on Froude's law of similarity, the variability of the other parts of the resistance with the size of the body being taken account of by means of empirical rules.

There are other cases in which resistance phenomena give rise to kinetic energy in a form which persists for a long time. The wings of an aeroplane form an example; as was described in § 6 of the last chapter (p. 53), they give rise to a strong and very uniform eddying motion (cf. also § 17, p. 206). Here again the pressure resistance can be subdivided into two parts, one of which provides the work necessary to produce the kinetic energy of vortex motion all through the fluid, the other, together with the frictional resistance, being called the wake resistance. The latter may be found by an application of the momentum theorem (cf. § 14(*c*), p. 184).

* For the theoretical calculation of the wave resistance, see e.g. G. Weinblum, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 10, p. 453 (1930); also *Hydromech. Probl. d. Schiffsantriebs* (Hamburg, 1932), pp. 99–159.

(*c*) *Moving Bodies and Moving Fluid.*—There still remains the important question how the resistance of a body moving in fluid at rest is related to the force exerted by a moving fluid on a body at rest. If the fluid is moving perfectly uniformly there can by the laws of mechanics be no difference between the two cases, as the superposition of a common uniform motion (equal and opposite to the velocity of the body, so that the latter is brought to rest) makes no difference to mechanical phenomena. It does make a difference, however, whether the fluid moves perfectly uniformly with respect to the body or whether it flows *turbulently*. The resistances are usually greater in the second case, but to this there are exceptions (a sphere in the critical region: cf. § 15, p. 188). As ordinary fluid motions (flow of water or air through pipes, &c.) are generally turbulent when of large dimensions, such differences always occur. If for experimental purposes we wish to imitate the behaviour of a body moving in fluid at rest by means of a model at rest in moving fluid, it is necessary to make the stream of fluid as uniform as possible by suitable means (cf. § 22(*a*), p. 248). For turbulence in wind tunnels, see also § 5(*g*).

Here we may also refer to the striking behaviour of a boat drifting downstream on a river, although strictly speaking it does not belong here. The boat, in fact, hurries ahead of the water, and gets enough way on her to be steered. The force acting is obviously the fall of the water level; the boat is, so to speak, on an oblique plane and is thus subject to a force Wi downwards, where W, the weight of the boat, is equal to the weight of water displaced, and i is the fall. The fact that the boat moves more rapidly than the water is easily seen in the following way. If it were a case of an ordinary mass of water of weight W, the force acting would be the same; but owing to turbulent mixing with the surrounding water, the mass of water would meet with a very great resistance. The solid form of the boat, however, prevents mixing taking place and instead gives rise only to a turbulent boundary layer with a very much smaller resistance. Thus the boat may acquire considerable way.

14. **Fluid Resistance in Hydrodynamical Theory.**

(*a*) *Fundamental Ideas.*—When the theory of the ordinary irrotational motion of frictionless fluid is applied to a body moving uniformly in a fluid extending indefinitely in all directions there never results any resistance in the direction of motion or any transverse force at right angles to it, no matter how the body is shaped. This result, which may at first seem remarkable, may readily be understood by applying the momentum theorem to a control surface (p. 78) surrounding the body at some little distance. Closer investigation shows that the disturbing velocities caused by the body decrease very rapidly in all directions, according to the inverse cube of the distance at least, and the pressure differences likewise; if the control surface is allowed to extend indefinitely, its surface area increasing in proportion to the

square of the radius, the contributions to the momentum, &c., tend to zero. As the momentum theorem must give the same result for every control surface, the result cannot be different from zero.

If we form the moments of momentum, the factors by which the momenta are multiplied are powers of the distance; hence it is clear that the moments of momentum need not vanish, and couples are in fact exerted on bodies oblique to the flow (in a direction tending to place the body at right angles to the flow, as readily follows from the discussion on the distribution of pressure).

The above argument does not contradict the fact that bodies moving in the neighbourhood of a boundary or of another body are acted on by forces due to the flow (here it is impossible to move the control surface to infinity). For example, a sphere moving parallel to a boundary is attracted with a force proportional to the square of the velocity and to the inverse fourth power of its distance from the boundary.

It is worth mentioning that closer examination of the case of a very elongated body (fig. 2.34, p. 64) shows that not only does the resistance vanish as a whole, but the resultant pressures on the front and rear parts are each equal to zero.

In the case of accelerated bodies forces exist between the fluid and the body even when the fluid is frictionless; these, however, merely have the same effect as if the mass of the body were increased by the amount of fluid carried along with it, and we speak of the "apparent mass" of the fluid carried along. In the case of a sphere the apparent mass is half as great as the mass of fluid displaced by the sphere. As the motion is always approximately irrotational when motion is first set up from rest, this result is of importance for the case of actual fluid also.

The fact that the resistance to a body moving uniformly in frictionless fluid is equal to zero may also be deduced from energy considerations. When friction is absent the work which would have been required to overcome the resistance can only be stored up in the fluid in the form of kinetic energy. If, however, as in irrotational motion, the fluid closes in behind the body just as it gave way in front of it, so that no disturbance is left in the fluid, it is impossible that any resistance should exist. A case in which the fluid is left with kinetic energy, so that resistance is developed, has already been briefly referred to in § 13(*b*), p. 178, and will be discussed in detail in connexion with the aeroplane wing in §§ 17, 18 (p. 206). Energy considerations do not exclude the possibility of a lifting force, or lift, occurring when no kinetic energy is left in the fluid, as the lift acts in a direction at right angles to the path of the body through the fluid and therefore involves no expenditure of work. The occurrence of a lift or transverse force of this kind has already been discussed in § 10 of last chapter (p. 70).

Another case is that of the ship travelling on the surface of water, in which, as we have already mentioned on p. 178, energy is dissipated

in the formation of a system of waves which extend more and more with the distance traversed. Here the energy dissipated is equal to the work done against the wave resistance.

The forces on the fluid corresponding to the apparent mass when a body is accelerated may readily be explained from the energy standpoint if we bear in mind that the work corresponding to these forces is necessary to produce the kinetic energy present in the motion of the fluid or—in the case of retardation—to abstract it from the fluid.

The fact that most bodies experience a resistance far greater than the unavoidable resistance due to friction (cf. §§ 3, 5, pp. 105, 121) is due to the formation of layers of discontinuity and vortices in their neighbourhood, as described in § 6 (p. 135). These are to be regarded as the real seat of the resistance. They prevent the fluid from closing up behind the body as it opens in front and thus cause the distribution of pressure to be unsymmetrical (the full value of the dynamic pressure existing in front and not behind); they also cause the continual removal of kinetic energy from the body.

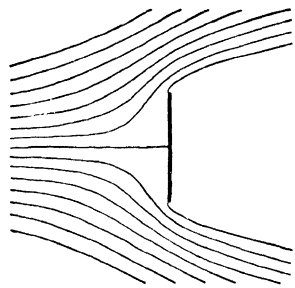


Fig. 3.59.—Kirchhoff's flow past a flat plate

(*b*) *Particular Cases.*—Of the various attempts to discuss the problem of resistance without going beyond the confines of the theory of the ideal fluid we shall discuss two specially typical ones, Kirchhoff's flow round a plate when surfaces of discontinuity are formed and Kármán's vortex street.

Kirchhoff * discovered a very beautiful example of the application of Helmholtz's surfaces of discontinuity (see § 7 of last chapter, p. 53) in the flow round a flat plate shown in fig. 3.59. The current divides in front of the plate and flows laterally past the edges, the space behind the plate being occupied by fluid at rest. As apart from gravity the pressure is the same throughout this space, we obtain for the surface of discontinuity the condition that the pressure must be constant on it also, so that the velocity must be constant, by Bernoulli's theorem. If these conditions are observed, the theory gives only solutions such that the surfaces of discontinuity extend to infinity and the velocity in the surface of discontinuity is equal to the velocity of the undisturbed fluid (i.e. equal to the velocity at infinity). It is easy to get a general idea of the distribution of pressure: at the centre of the front of the plate it is the dynamic pressure, falling away towards the edges to the pressure of the undisturbed fluid; on the rear side of the plate the pressure is constant

* *Zur Theorie freier Flüssigkeitsstrahlen, Crelles Journ.*, Vol. 70 (1869).

and equal to the pressure of the undisturbed fluid. In all, this gives a pressure proportional to the area of the plate and to the dynamic pressure, i.e. c is constant (equal to $2\pi/(4+\pi) = 0{\cdot}880$).

In reality, as we have previously emphasized, these surfaces of discontinuity are extremely unstable and break up to form large and small eddies. Hence the dead water does not actually extend to infinity and the flow soon closes in again behind the plate. This is connected with the fact that the pressure behind the plate is considerably less than the undisturbed pressure, and owing to this "suction effect" the resistance is considerably greater than that obtained from Kirchhoff's calculations. For the case of the "plate of infinite width" (a plate between parallel walls at either side) we obtain c equal to about 2·0; in the case of a rectangular plate where the lengths of the sides are in a finite ratio fluid flows past the narrow sides into the suction region and thus diminishes the reduction of pressure considerably. For sides in the ratio of

	1 : 20	1 : 10	1 : 4	1 : 1
we have	$c =$ 1·45	1·29	1·19	1·10.

Kirchhoff's calculations refer to a plate of infinite length, and are therefore very far from agreeing with the results of experiment; on the other hand, good agreement with theory is obtained in the case where a current of water flowing against a plate divides and the vacant space formed is filled with air (or with the vapour of the liquid, as

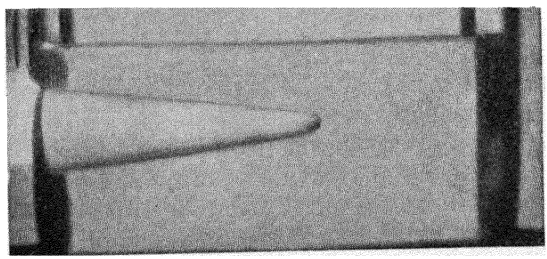


Fig. 3.60.—Shot through water, after C. Ramsauer

occurs when the velocity is very high; this is known as *cavitation* (p. 312)). As in this case the surface of discontinuity does not break down at all, or not to any noticeable extent, the conditions of the theory are satisfactorily fulfilled, so that one would expect the resistance to agree with the theoretical value. A symmetrical phenomenon of the same kind, the result of shooting through the side of a glass trough full of water, is shown in fig. 3.60. (Here, according to Ramsauer,* $c = 0{\cdot}288$.)

* *Ann. d. Physik*, Vol. 84, p. 697 (1927).

In cases of flow round an elongated plate or similar obstacle where the fluid does not split up as described above, a regular oscillatory motion arises behind the obstacle in certain circumstances, resulting in the formation of a strong eddy on the right and left sides alternately. The eddies form a more or less regular series as in fig. 3.61. Observations

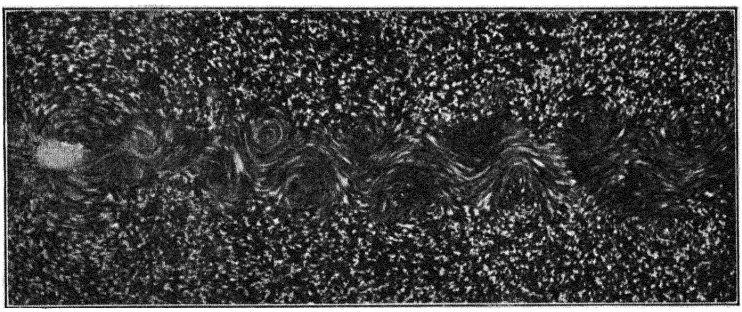


Fig. 3.61.—Vortex street behind a narrow plate (camera at rest)

of this kind led von Kármán * to investigate the stability of straight parallel vortex filaments arranged in a similar way. Most configurations turn out to be more or less unstable: only in one case, resembling that shown in fig. 3.62, in which the ratio of h, the distance between the two rows of vortices, to l, the distance between successive vortices on the same side, is 0·283, does stability exist.† (23)

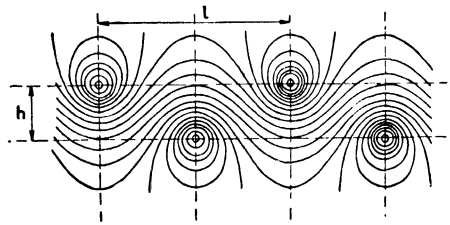


Fig. 3.62.—Stream-lines of a vortex street, after Kármán

Rows of vortices actually observed reproduce the above value of the ratio of h/l very accurately provided the vortices have fairly de-

* Kármán and Rubach, *Phys. Zeitschr.*, Vol. 13, p. 49 (1912); see also R. Fuchs, p. 239 *et seq.* A. W. Maue, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 129 (1940), has demonstrated stable oblique vortex streets, which give a transverse force in addition to the resistance.

† More accurately speaking, the configuration is stable with respect to all small deviations from the initial position of the vortex filaments, with the exception of a special perturbation in which the vortices at a distance l apart are always displaced to an exactly equal extent in opposite directions. With respect to this perturbation the equilibrium is neutral, i.e. such perturbations may continue uniformly in time.

finite cores. (When the shape of the vortices becomes indistinct as a result of friction they move farther apart, as may be seen from fig. 3.61.)

By the principle of energy the continual formation of fresh eddies corresponds to a resistance, which may also be shown to exist by means of the momentum theorem * (between the eddies the fluid continues to follow the body). This problem has been worked out by von Kármán. The theory has yielded the fine result that the resistance coefficient of a body giving rise to vortices can be calculated from a photographic measurement of the system of vortices and observation of the velocity of the vortices as compared with the velocity of the body, in good agreement with the value found by experiment. So far, however, it has not been found possible to give a theoretical explanation of the relationship between the dimensions of the system of vortices and the dimensions of the body.

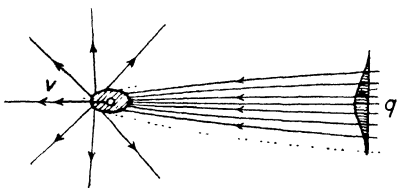


Fig. 3.63.—Flow from a source, and wake

(*c*) *Connexion between the Resistance and Conditions at some Distance.*—We must first clear up the problem of what the velocity field is like at a considerable distance from the body which is moving through the fluid. In the rear of the body we find the so-called *wake*, consisting of the fluid set in motion by the resistance of the body. In front of the body the fluid moves outwards in all directions as if from a "source" (cf. Chapter II, § 9(*b*), p. 62). Thus with a system of co-ordinates for which the undisturbed fluid is at rest, we have the state of affairs shown diagrammatically in fig. 3.63. The strength of the source (Q) is equivalent to the strength of the wake and is closely related to the resistance of the body. If the velocity of the fluid in the wake relative to the medium at rest is denoted by q, we have

$$Q = \iint_W q\,da, \qquad \ldots\ldots\ldots \quad (54)$$

where W means that the surface integral is taken over the area of the wake only, provided that the distance from the moving body is so great that the flow from the source is negligible. Applying the momentum theorem to the flow from the source and the wake at a great distance,† we have

$$R = \rho Q v. \qquad \ldots\ldots\ldots \quad (55)$$

* The argument from the principle of energy would require data about the diameter of the cores of the vortices, whereas that based on the momentum does not, so that the latter only is applicable here.

† See e.g. Prandtl-Tietjens, *Aero- und Hydromechanik*, Vol. II, p. 137 *et seq.*; also the note at the end of § 20(*b*), p. 234.

From equations (54) and (55) we see that the resistance can be obtained by measurements on the wake. As the pressure disturbance due to the moving body dies away rapidly as the distance from it increases, but the velocity in the wake much less rapidly (see below), a very convenient formula can be given for the resistance if the distance from the body is considerable. Within the wake the velocity relative to the body is $v - q$. The " total pressure " $g = p + \frac{1}{2}\rho(v-q)^2$ is measured by a Pitot tube (see Chapter II, § 4(*c*), p. 45) at rest relative to the body. If the undisturbed total pressure $p_0 + \frac{1}{2}\rho v^2$ is denoted by g_0, then, by (54) and (55), the integral $\iint_W (g_0 - g)\,da$ taken over an area at a considerable distance behind the body gives the resistance R apart from the term $\frac{1}{2}\rho q^2$, which is unimportant there. A. Betz * was the first to point out the possibility of measuring the resistance in this way; he has also extended the method so that it can be applied to points near the moving body (cf. § 22(*c*), p. 255).

From the properties of the flow round a moving body described above we can also make an important deduction about the pressure field associated with the flow. This is dominated by the source. The radial velocity for a point source in laminar flow is $q_r = Q/4\pi r^2$, or $Q_1/2\pi r$ for a line source of strength Q_1 per unit length. In forming the square of the resultant velocity at a considerable distance from the source, only the x-component $u = q_r \cos\phi$ has to be taken into account, if we confine ourselves to quantities of the first order. Putting u_0 equal to v, the velocity of the body, in the expression $\frac{1}{2}\rho\{(u_0+u)^2 - u_0^2\} = \frac{1}{2}\rho(2u_0u + u^2)$ in Bernoulli's equation and neglecting quantities of the second order, we have

$$p - p_0 = -\rho v u = -\rho \frac{Qv}{4\pi r^2}\cos\phi \quad \text{or} \quad -\rho\frac{Q_1 v}{2\pi r}\cos\phi.$$

By equation (55), this gives

$$p - p_0 = -\frac{R\cos\phi}{4\pi r^2} \quad \text{or} \quad -\frac{R_1\cos\phi}{2\pi r}.$$

These quantities are by no means negligible, especially in the latter case, even at a considerable distance from the moving body, a fact which must be taken into account in experiments if the flow is disturbed by a rod, say the stem of a measuring instrument, lying across the line of flow. In front of the body there is an excess pressure, behind it a deficit of pressure. The flow in the wake, which is a flow with

* A. Betz, " Ein Verfahren zur direkten Ermittlung des Profilwiderstandes ", *Zeitschr. f. Flugtechnik u. Motorluftschiffahrt*, Vol. 16, p. 42 (1925); see also Prandtl-Tietjens, Vol. II, p. 141 *et seq.*

friction and so does not obey Bernoulli's equation, does not give rise to any appreciable pressure field.

As regards the distribution of velocity in the wake, we may mention that for small values of Reynolds' number, so small that no periodic transverse motions arise (up to above $vd/\nu = 50$ for a circular cylinder lying across the flow), the breadth of the wake b increases proportionally to $\sqrt{(\nu t)} = \sqrt{(\nu x/v)}$ at a considerable distance from the moving body [the same formula as for boundary layers; cf. equations (11), (12), p. 107]. From formula (55) it follows that the velocity in the wake in the two-dimensional case decreases proportionally to $1/\sqrt{x}$, and in the case of a solid of revolution to $1/x$ (here the cross-section of the wake is proportional to b^2, where b satisfies the same relation as above). As in the case of heat conduction, the distribution of q over the cross-section of the wake is represented by the function $q = q_1 e^{-(y/b)^2}$. For $\mathscr{R} < 1$, in fact, there is a mathematical theory of the velocity and pressure fields for a sphere, by C. W. Oseen,* and for the circular cylinder, by Lamb.† The results are in good agreement with fig. 3.63. The only feature peculiar to the case where Reynolds' number is small compared with unity is that the moving body is surrounded on every side by a ball of the viscous fluid which moves along with it;‡ that is, the flow from the source and the flow in the wake start, so to speak, not immediately from the body but from the surrounding ball. This is related to the fact that, in the region where Stokes' law holds, the resistance is proportional not to the area of cross-section but to the diameter of the body, i.e. increases as the size of the body diminishes.

The transition from this case of "laminar" flow to the vortex street is clearly shown in figs. 3.64–68, obtained by F. Homann with circular cylinders in moving oil.§

As Reynolds' number increases the vortex streets become irregular, that is, the wake becomes turbulent. The powers of x which occur here can be found in the following way. The mixture length clearly bears a fixed relation to b, the breadth of the wake, and hence, by § 4, p. 118, the velocity fluctuations u', v' are proportional to q_1, the mean velocity in the axis of the wake. The rate of increase in the breadth of the wake can be taken as proportional to v', i.e. to q_1. Hence

$$\frac{db}{dt} = v\frac{db}{dx} \sim q_1;$$

but by equation (55) $q_1 \sim Q_1/b$ or Q/b^2, according as it is the two-dimensional case or the case of symmetry about an axis. Integrating db/dx, we easily find that b is proportional to $\sqrt{x}$ in the two-dimensional case, but to $\sqrt[3]{x}$ in the case of a solid of revolution. This gives q_1 proportional to $1/\sqrt{x}$ or to $1/\sqrt[3]{x^2}$. A stricter treatment of both cases is to be found in papers by H. Schlichting || and Miss L. M. Swain.¶ The result for the two-dimensional case has also been confirmed very satisfactorily by experiment.||

* *Arkiv för mat., astr. och fis.*, Vol. 6, No. 29 (1910); cf. also the account in H. Lamb's *Hydrodynamics*, 2nd Ger. ed., § 340. A short account is also given in *Aerodynamic Theory*, Vol. III, p. 75 *et seq.*

† H. Lamb, *Hydrodynamics*, § 343, 6th ed. (1932); *Phil. Mag.*, 5th ser., Vol. 21, p. 120 (1911).

‡ This is not to be taken too literally, seeing that the velocities in the ball are not constant, but diminish steadily as the distance from the body increases. As Reynolds' number increases the ball is gradually transformed into the boundary layer.

§ F. Homann, *Forschung auf d. Geb. d. Ing.-Wes.*, Vol. 7, p. 1 (1936).

|| H. Schlichting, *Ing.-Arch.*, Vol. 1, p. 533 (1930).

¶ L. M. Swain, *Proc. Roy. Soc.* (A), Vol. 125 (1929), p. 647.



Fig. 3.64.—$\mathscr{R} = 32$

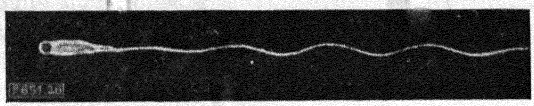


Fig. 3.65.—$\mathscr{R} = 55$

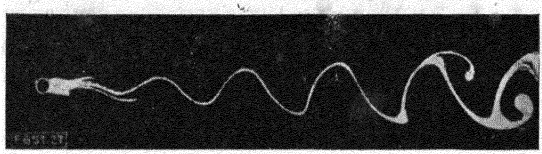


Fig. 3.66.—$\mathscr{R} = 65$



Fig. 3.67.—$\mathscr{R} = 71$



Fig. 3.68.—$\mathscr{R} = 101$

Flow of oil behind circular cylinders, after F. Homann

15. Experimental Results on Fluid Resistance.*

(*a*) In the case of bodies with a fairly large fluid resistance, the occurrence of resistance is, as we have already mentioned, bound up with the formation of layers of discontinuity. If, as is usually the case with sharp-edged bodies, the points at which the layers of discontinuity begin, i.e. the points at which the flow becomes separated from the body, are definitely fixed, the coefficient of resistance is found by experiment to be constant for quite a wide range of conditions. Thus, for example, experiments at Göttingen † on smooth circular discs set normal to the stream show no recognizable alteration in the resistance coefficient for values of Reynolds' number (vd/ν) ranging from about 4000 to 1,000,000, and it may be taken as certain that no change of behaviour would take place even for much larger values of Reynolds' number; the value of c is 1·10 to 1·12. As for smaller values of Reynolds' number, for which data on freely falling bodies only are available, the region between $\mathscr{R} = 3000$ and $\mathscr{R} = 80$ has to be left out, since the discs oscillate vigorously to and fro and hence give values of the resistance up to 50 per cent greater than if they fell steadily. When Reynolds' number is below 80, the discs fall steadily, so that the measurements can again be used to find the coefficient of resistance; the law of resistance gradually passes into Stokes' law of resistance, which gives

$$c = \frac{20{\cdot}4}{\mathscr{R}}$$

and is sufficiently accurate for circular plates when Reynolds' number is less than 0·5. According to experiments by Schmiedel,‡ for values of Reynolds' number equal to

80 20 5 2

the resistance coefficient c is

1·5 2·4 5·6 11·5

What is true of the resistance coefficient is also true of the distribution of pressure. Eiffel § made measurements of the distribution of pressure on three geometrically similar models of a house, with lengths of 0·8, 5, and 40 cm. In spite of the great difference in size

* See also *Handb. d. Exp.-Phys.*, Vol. IV, 2, p. 235 *et seq.* (article by H. Muttray).

† Cf. *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 2, p. 28 *et seq.* (Munich, 1923).

‡ J. Schmiedel, *Experimentelle Untersuchungen über die Fallbewegung von Kugeln und Scheiben* (*Diss.*, Leipzig (1928); *Phys. Zeitschr.*, Vol. 29, p. 593 (1928)).

§ G. Eiffel, *Nouvelles recherches sur la résistance de l'air et l'aviation*, p. 286 *et seq.* (Paris, 1914).

the results agreed with each other very well: they are illustrated in fig. 3.69. It may be pointed out that varying configurations of the ground in front of a house may affect the distribution of pressure on the house; see e.g. a paper by R. Gran Olsson.* It is the separation of the flow on the windward side of the house that is chiefly affected (cf. fig. 3.22, p. 139).

Rounded bodies for which the flow does not separate from the surface at a point determined by the shape of the body behave in an essentially different way. Here the position of the point of separation is determined by the phenomena taking place in the boundary layer at the surface of the body, which is usually thin (cf. §§ 6, 7, p. 135), with the result that apparently quite unimportant circumstances, such as slight roughness of the surface, more or less vorticity present in the approaching fluid, or the like, frequently have a very marked effect on the separation point and hence also on the size and position of the vortex system and therefore on the resistance. Whether the flow in the boundary layer remains laminar up to the separation point or becomes turbulent before reaching it is, however, decisive. As we have already mentioned on p. 139, in the latter case the separation point moves a long way to the rear and the resistance is lowered very considerably.

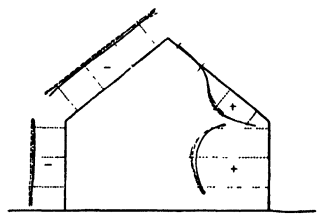


Fig. 3.69.—Wind pressures on a house

This phenomenon was first established in investigating the resistance of spheres,† which in the region of $vd/\nu = 3 \times 10^5$ falls from $c = 0{\cdot}45-0{\cdot}48$ to values which under particularly good experimental conditions actually lie below 0·10; for larger values of Reynolds' number (about 2×10^6) c again rises to about 0·18 (distinctly higher for rough surfaces). The critical range of Reynolds' number is displaced towards lower values by the presence of eddies in the current of air used in the experiment.‡ The fact that the fall in the resistance is actually caused by the boundary layer becoming turbulent is clearly demonstrated by the following experiment. If a hoop of thin wire (with a diameter e.g. 1/300 that of the diameter of the sphere) is fixed to a large sphere somewhat ahead of the point where the flow would separate if it were

* "Versuche über Winddruck auf Bauwerke", *Der Bauingenieur*, Vol. 15, No. 49/50 (1934).

† G. Eiffel, *Compt. rend.*, Vol. 155, p. 1597 (1912); L. Prandtl, *Göttinger Nachr., Math.-Phys. Kl.*, p. 177 (1914), where further references are given; see also C. Wieselsberger, *Zeitschr. f. Flugtechn. u. Motorluftsch.*, Vol. 5, p. 140 (1914).

‡ This has led to the practice of taking as an empirical measure of the turbulence of a wind tunnel the value of Reynolds' number for which $c = 0{\cdot}30$. The highest value is found to be about $\mathscr{R} = 3{\cdot}9 \times 10^5$, and for bad tunnels about $1{\cdot}5 \times 10^5$. A refinement of this criterion is mentioned at the end of § 6 (p. 141).

laminar, so as to set up eddies in the boundary layer, the resistance is diminished even for values of Reynolds' number below 3×10^5. The separation point, which would otherwise be situated at an angle of about 80° from the foremost point of the body, is simultaneously displaced backwards to about 110°–120°.

The resistance of spheres has been studied from very small values of Reynolds' number up to $3{\cdot}6 \times 10^6$.* For values less than 0·4 Stokes' law is sufficiently accurate; in our notation it is given by

$$c = \frac{24}{\mathscr{R}}.$$

For

$\frac{vd}{\nu} =$	0·1	1	10	10^2	10^3	10^4	10^5	10^6
$c =$	245	28	4·4	1·10	0·46	0·42	0·49	0·14

approximately. Circular cylinders behave in a similar way. For very long cylinders ($l \geqslant 100$ diameters) or for cylinders bounded by parallel walls at either end we have the following set of values:†

$\frac{vd}{\nu} =$	0·1	1	10	10^2	10^3	10^4	10^5	10^6
$c =$	58	10	2·6	1·45	0·98	1·12	1·23	0·35.

The transition from large values of the resistance to small ones takes place between $\frac{vd}{\nu} = 1{\cdot}8 \times 10^5$ and 5×10^5, the resistance falling from $c = 1{\cdot}2$ to $c = 0{\cdot}3$. For small values of Reynolds' number Stokes' formula is here replaced by a more complicated one due to Lamb.‡

$$c = \frac{8\pi}{\mathscr{R}(2{\cdot}002 - \log_e \mathscr{R})}.$$

Values of c for circular plates, spheres, and cylinders are shown in fig. 3.70.

For cylinders of finite length we again obtain much smaller resistances than for those of infinite length. As with finite plates, the reason is to be sought for in the fact that the region of eddies behind the cylinder becomes aerated owing to air passing round the ends of the

* From $0{\cdot}2$ to 8×10^3, Allen, *Phil. Mag.* (5), Vol. 50, p. 323; from 8×10^2 to 8×10^5, *Ergebnisse der Aerodyn. Vers.-Anstalt, Göttingen*, Vol. 2, p. 28; from 2×10^5 to $3{\cdot}6 \times 10^6$, Bacon and Reid, *Report of the National Advisory Committee for Aeronautics*, No. 185 (1923) (Washington, 1924).

† *Ergebnisse der AVA*, Vol. 2, p. 23, and Relf, *Technical Report of the Advisory Committee for Aeronautics*, 1913–14, p. 47 (London, 1914).

‡ *Phil. Mag.* (5), Vol. 21, p. 120 (1911); H. Lamb, *Hydrodynamics*, § 343, 6th ed., (p. 614).

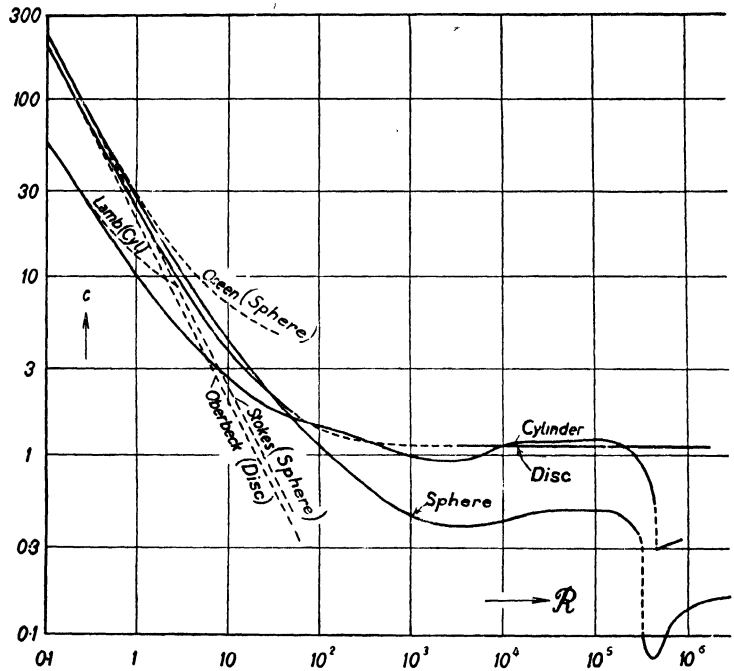


Fig. 3.70.—Relationship between Reynolds' number and the resistance coefficient of a sphere, a cylinder, and a disc

cylinder, so that the reduction of pressure there is less than if the air were not present. According to experiments carried out at Göttingen we have

$\frac{vd}{\nu} =$	10^3	10^4	10^5	10^6
$c =$	0·67	0·73	0·75	0·37 *

when $l/d = 5$.

Note.—If Reynolds' number is large, freely falling bodies frequently move more or less irregularly. This has the effect of raising the resistance. Details are given by L. Schiller, *Handbuch der Experimentalphysik*, Vol. IV, 2, p. 368 *et seq.*

(*b*) The problem of finding bodies with as small an air-resistance as possible has become of special importance in connexion with aviation. We know that if in frictionless fluid the flow is such that the fluid closes in behind the body just as it opens in front of it the resistance is zero, no matter how the body is shaped, as no disturbance is left in the fluid. Thus the above problem may be stated as follows. How must a body be shaped in order that the flow does not become separated

* By extrapolation.

from the body when it is moving in actual fluid? If this is attained, we may expect from the foregoing that the resistance of the body will practically consist of frictional resistance only. Experiments show that this is actually the case. The shapes of body for which the flow remains attached are all well rounded and rather slender, ending in a point or edge behind, which, however, may be somewhat rounded off. The front, where there is no danger of eddies separating themselves, need not be pointed, a shape like that of an elongated

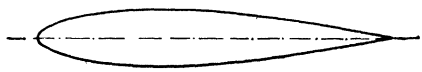


Fig. 3.71.—Good shape for an airship

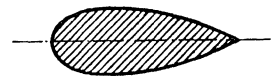


Fig. 3.72.—Good strut profile

ellipsoid being perfectly satisfactory. (It is otherwise with ships travelling on the surface of water, which must have a sharp bow in order to prevent a high wave piling up in front of the ship.) Examples are shown in figs. 3.71 (hull of airship) and 3.72 (strut).

Here, as a rule, the observed distribution of pressure agrees very well with that calculated for irrotational motion, except for the rear end, where owing to the losses of energy in the boundary layer enclosing the body the pressure fails to rise, as it theoretically should, to the dynamic pressure. Two characteristic cases of pressure distribution are shown in figs. 3.73, 3.74.*

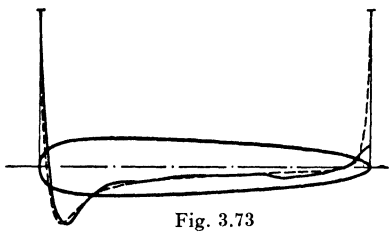


Fig. 3.73

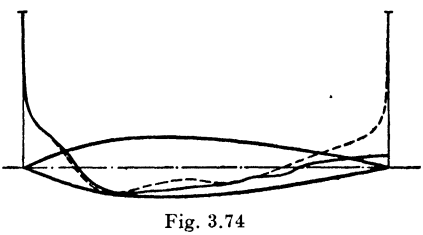


Fig. 3.74

Distribution of pressure on airship models: the dotted line represents the calculated distribution and the continuous line the observed distribution

For solids of revolution like that in fig. 3.71 experiments give $c = 0{\cdot}04$, i.e. the resistance is only a twenty-eighth of that of a circular plate with the same diameter. For a strut with the section shown in fig. 3.72, experiments give $c = 0{\cdot}06$ to $0{\cdot}08$ when Reynolds' number exceeds about 50,000.

The following data on the *frictional resistance* † will suffice here

* Taken from calculations and experiments by G. Fuhrmann, *Diss.*, Göttingen (1910); *Jahrb. d. Motorluftschiff-Studienges* (1911–12), p. 63.

† See also *Hydromech. Probl. d. Schiffsantriebs* (Hamburg, 1932), pp. 1–98. For the measurement of frictional resistance, see § 22(*d*), p. 256.

(cf. also § 5(*e*), p. 131). This is usually referred to the whole surface exposed to friction, i.e. for a rectangular plate of breadth b and length l with fluid flowing past both sides the total surface is $2bl$. The resistance may in general be taken as directly proportional to the breadth at right angles to the direction of motion, so that the breadth is properly taken account of already in the above formula and therefore does not occur in the coefficient of frictional resistance (c_f). The length l (in the direction of motion) is involved in a different way, as the rear parts are situated in the current which has already been partly retarded

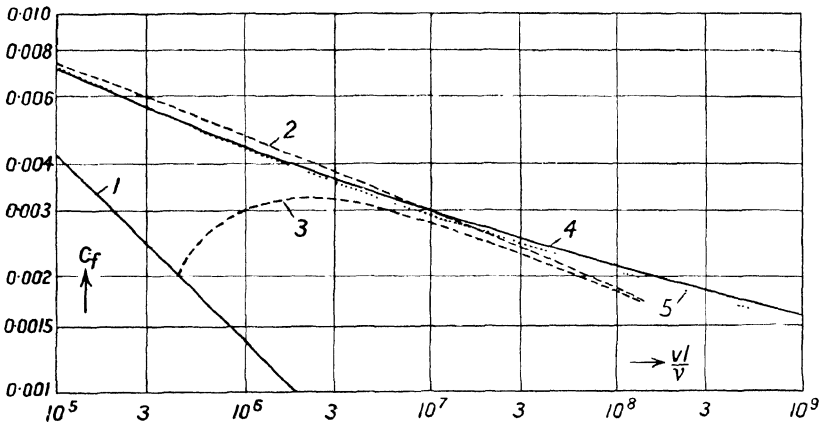


Fig. 3.75.—Relation between Reynolds' number and the frictional resistance of smooth plates

by the front parts, so that doubling the length does not exactly double the resistance. When the flow in the boundary layer is laminar, the resistance is proportional to $\sqrt{l}$: when it is turbulent and Reynolds' number is sufficiently great, the resistance varies as $l^{0\cdot8}$ to $l^{0\cdot85}$ for a smooth surface and as $l^{0\cdot65}$ to $l^{0\cdot75}$ for a rough surface. If we insert Reynolds' number vl/ν, the results of experiment for smooth surfaces are as shown in fig. 3.75, where c_f and vl/ν are plotted logarithmically. The continuous and dotted lines represent different formulæ. For laminar flow (cf. formula (13), p. 108)

$$c_f = 1{\cdot}328/\sqrt{(vl/\nu)} = 1{\cdot}328/\sqrt{\mathscr{R}} \quad . \ . \ . \quad (56)$$

(curve 1 in fig. 3.75): for turbulent flow, where turbulence sets in at the very front of the plate (cf. § 5(*e*), p. 131),

$$c_f = 0{\cdot}074/(\mathscr{R})^{0\cdot2}. \quad . \ . \ . \ . \ . \ . \quad (57)$$

This approximate formula (shown by curve 2 in fig. 3.75) is very closely related to Blasius' formula for the resistance of pipes * (equation (46), p. 163). If the flow is laminar at the front and turbulence sets in only towards the rear we may put

$$c_f = 0{\cdot}074/(\mathscr{R})^{0{\cdot}2} - 1700/\mathscr{R}, \quad . \quad . \quad . \quad . \quad (58)$$

corresponding to a critical value of Reynolds' number equal to 500,000 (curve 3 in fig. 3.75). These formulæ hold for values of Reynolds' number up to 5×10^6. The last of them is in good agreement with the results of towing experiments in a water tank by Gebers. Corresponding to the deviations of the resistance of pipes from Blasius' relationship for large values of Reynolds' number, the resistances of plates are somewhat greater here also.

The logarithmic laws given in formulæ (24) and (25) in § 5 (p. 126) for the distribution of velocity in turbulent flow lead to theoretical formulæ for the resistance of a plate which hold good for the whole range of values of Reynolds' number,† and for large values of Reynolds' number (up to 5×10^8) are in agreement with towing experiments by G. Kempf.‡ As these formulæ are inconvenient, however, H. Schlichting has given the interpolation formula

$$c_f = \frac{0{\cdot}455}{\log_{10}\mathscr{R}^{2{\cdot}58}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

which represents the relationships pretty accurately. On the basis of similar calculations by Th. von Kármán, K. Schönherr § has put forward the following formula, which resembles equation (50), § 11 (p. 164), in some of its features and is in good agreement with the results of old and new series of experiments:

$$\sqrt{c_f} = \frac{0{\cdot}242}{\log_{10}(\mathscr{R} \,.\, c_f)}. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Formulæ (59) and (60) correspond to curves (4) and (5) in fig. 3.75. If the flow is laminar at the front of the plate the term $1700/\mathscr{R}$ can again be subtracted from formula (59) or (60) (but for other critical values of Reynolds' number 1700 is replaced by a different number).‖ [24]

* Cf. also Prandtl, *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 3, p. 3 (Munich, 1927).

† Cf. e.g. Prandtl, *ibid.*, Vol. 4 (1932), p. 24.

‡ *Werft, Reederei, Hafen* (1929), pp. 234, 247.

§ *Trans. Soc. Nav. Arch. and Marine Eng.*, Vol. 40 (1932).

‖ For more recent experiments by F. Schultz-Grunow, see *Lufo.*, Vol. 17, p. 239 (1940). His formulæ are also to be found in *F.I.A.T. Review of Aerodynamics*, Chapter III, §§ 1, 2, where further references are also given.

Note.—In very fast aeroplanes it is necessary to keep down the frictional resistance or frictional drag, above all that of the wings, as much as possible. A glance at fig. 3.75 shows that this resistance can be made very much smaller if the flow can be kept laminar over a large part of the wing surface. Now it is easier to keep accelerated fluid laminar than retarded fluid, and it must therefore be arranged that the maximum velocity is moved as far as possible towards the rear of the aeroplane. This is done by making the point where the profile is thickest as far to the rear as possible. Of course the surface must also be devoid of any projections, and of corrugations, even very slight ones. According to American reports, critical values of Reynolds' number of 2·5 to 3 × 10^6 have been attained with these laminar-flow profiles.

The theory of turbulent flow past rough surfaces sketched in § 5(*d*) (p. 129; see also the relevant remarks on the resistance of pipes in § 11, p. 163) enables us to calculate the frictional resistance of *rough plates*. We might expect that for a fully developed flow past a rough surface the resistance for a plate of given length with irregularities of height k would be proportional to the square of the velocity, the factor of

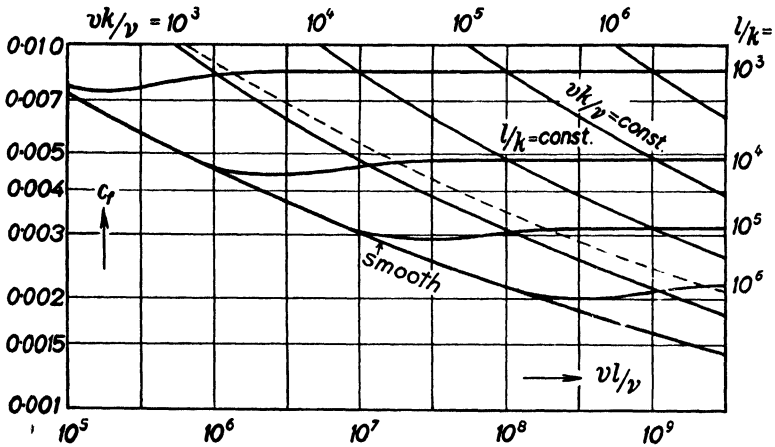


Fig. 3.76.—Frictional resistance of rough plates

proportionality being higher, the greater the ratio k/l was. Since, if k is constant, this ratio falls as the length l increases, if v is constant, c_f diminishes as the length increases, i.e. as Reynolds' number vl/ν increases. The results of H. Schlichting's * calculations based on Nikuradse's experiments with rough pipes are reproduced in fig. 3.76. For the smaller irregularities we see the transition, shown in fig. 3.46 (p. 165), from the hydraulically smooth state to the definitely rough state. The dotted line shows the lengths of plate beyond which the fully developed rough flow occurs. F. Schultz-Grunow † has obtained

* *Werft, Reederei, Hafen*, Vol. 15, p. 1 (1934).

† *Jahrb. d. Schiffbautechn. Gesellsch.*, Vol. 39, p. 176 (1938).

data from a water channel for calculating the resistance of moderately rough plates, such as are used in shipbuilding, and was able to bring his results into agreement with those on pipes by Bauer and Galavics (cf. p. 166).

A problem related to the resistance of a plate is the calculation of the resistance to a circular disc rotating in its own plane. This is discussed in Chapter V, § 10 (p. 364).

16. Aerofoils.* (25)

(*a*) In discussing resistance problems we have hitherto dealt only with cases where, owing to the existence of symmetry, the resistance lies in the direction of motion. In general, however, the resistance may consist of a force in any direction, and possibly a couple in addition, likewise orientated in any direction. An aeroplane wing and a windmill sail are cases where a lateral force and a couple respectively are wanted. In both cases there is a resistance in the direction of motion in addition to the desired effect.

In the case of aeroplane wings or *aerofoils* † the problem is obviously to obtain maximum "lift" with minimum resistance or "drag". Flat plates inclined at a small angle (*angle of incidence* or *angle of attack*) to the direction of motion are not wholly unsuitable

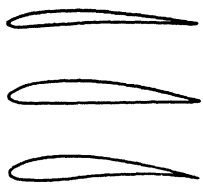


Fig. 3.77.—Older aerofoil profiles

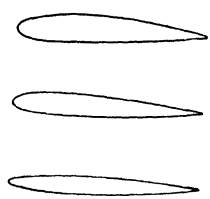


Fig. 3.78.—More recent aerofoil profiles

for this purpose; but as was shown by Lilienthal ‡ in 1873, slightly cambered plates are better. Thin plates may with advantage be replaced by fairly thick "profiles" such as may be formed by describing a fish-like outline, as in fig. 3.71, round a slightly arched centre line ("camber line"). The fact that even comparatively thick profiles may be very useful was first demonstrated by Junkers about 1917 by the construc-

* In connexion with §§ 16–21, see the comprehensive survey by A. Betz in *Handb. d. Phys.*, Vol. 7, p. 215 *et seq.*, and, for the most recent advances, his contribution to *F.I.A.T. Review, Aerodynamics*, Chapter I, § 1.

† [The wing when considered apart from the rest of the machine is referred to as an *aerofoil* or *airfoil*.]

‡ See O. Lilienthal, *Der Vogelflug als Grundlage der Fliegekunst*, 3rd ed., p. 76 *et seq.* (Munich, 1938).

tion of his all-metal aeroplanes, the wings of which were entirely supported by stiffening members in their interior. Some older wing profiles are shown in fig. 3.77; they closely resemble the profiles of the wings of birds which have great powers of flight. While at that time special emphasis was laid on producing large lift, more recent development has been more and more in the direction of the greatest possible speed, i.e. of cutting down drag. At the same time greater ease of manœuvrability was demanded; this was attained by profiles with a fixed centre of pressure (cf. p. 201). Some of these modern profiles are shown in fig. 3.78, the lowest being an extremely good profile for a fast aeroplane.

Experiments in a wind tunnel particularly free from turbulence have shown, according to G. W. Lewis,* that it is possible to keep the boundary layer at an aerofoil laminar over a considerable part of it, and thus reduce the frictional drag, by making the place where the profile is thickest far to the rear (cf. e.g. the third profile in fig. 3.78 and the profile *b* in fig. 3.79). For in the absence of external disturbances, and if the surface is very smooth, the transition from laminar flow to turbulence in the boundary layer can be postponed to very high values of Reynolds' number, so long as the pressure falls in the direction of flow; if the pressure rises, it sets in markedly earlier (p. 116).

Fig. 3.79 shows the distribution of pressure, calculated by Tani and Mituisi † and satisfactorily confirmed by experiment, for two profiles, an older form *a* and a more modern form *b* designed according to the above rule. For Reynolds' number vb/ν equal to 3×10^6 at zero angle of incidence, experiments gave the drag coefficients (see below) $C_D = 0{\cdot}0063$ for profile *a* and 0·0030 for profile *b*.

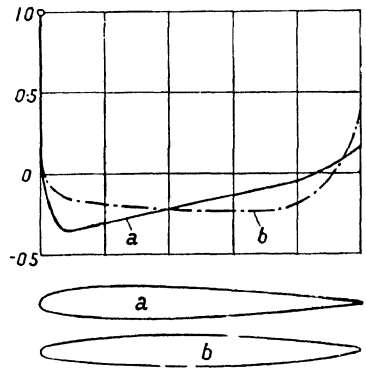


Fig. 3.79.—Theoretical distribution of pressure over two symmetrical profiles for zero angle of incidence, after Tani and Mituisi

(*b*) In order to get numerical results expressing the properties of aerofoils, we may split up the resultant force exerted by the air into two components, either in directions fixed in the aerofoil (normal force N and tangential force S perpendicular and parallel to the " chord " of the aerofoil or to some other suitably chosen direction), or in the direction of motion and at right angles to it (lift L and drag D). Here it is convenient to choose (dimensionless) resistance coefficients in such a way that they refer not to the variable projection of the surface in the direction of motion but to the " projection of maxi-

* *Journ. Roy. Aeronautical Soc.*, May, 1939 (27th Wilbur Wright Memorial Lecture).

† *Report Aeronaut. Research Institute, Tokio Imp. Univ.*, Vol. 15, p. 399 (1940): see also a valuable paper by W. Hausammann, *Flugwehr und Technik*, Vol. 4 (Zürich, 1942), p. 179 *et seq.*, giving the results of Swiss experiments.

mum area ", which e.g. in the case of a rectangular wing is simply $a = lb$, where l is the length (span *) and b the breadth (in the direction of flight). Again writing p_d for the dynamic pressure $\frac{1}{2}\rho v^2$, we put

$$\left.\begin{aligned} L &= C_L a p_d, \; D = C_D a p_d \\ \text{or} \quad N &= C_N a p_d, \; S = C_S a p_d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

(C_L, C_D are known as the *lift coefficient* and *drag coefficient* respectively.) If α is the angle between the chord of the profile and the direction of motion (fig. 3.80), L and D and N and S are connected by the well-known formula for transforming from one set of axes to another set making an angle α with the first. Dividing by $\rho a v^2$, we obtain the corresponding formulæ

$$\left.\begin{aligned} C_N &= C_L \cos\alpha + C_D \sin\alpha \\ C_S &= -C_L \sin\alpha + C_D \cos\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (62)$$

The air flow varies according to the magnitude of α, the angle of incidence, so that C_L and C_D (and also C_N and C_S, by equation (62)) are to be regarded as functions of α. Typical graphs of these functions for the first of the profiles in fig. 3.78 (for a rectangular wing with *aspect ratio* 5, i.e. with the lengths of the sides in the ratio 5 : 1) are given in fig. 3.81. The range of angle between A and B corresponds to unstalled flow, i.e. a flow which lies close to the wing at every point. In the regions beyond A and B the flow breaks away from the under or upper side of the wing respectively, and the intervening space is filled with eddies, giving rise to a considerable increase in drag and a simultaneous decrease in lift; see the photographs in figs. 3.82 (unstalled flow) and 3.83 (stalled flow on upper side of aerofoil).

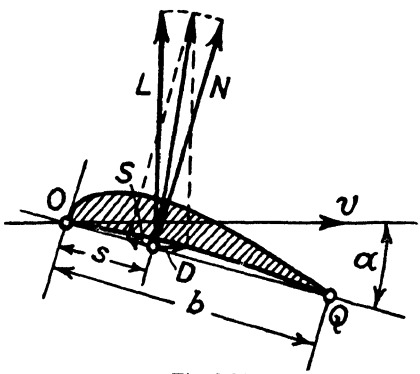


Fig. 3.80

Another common way of representing the data contained in fig. 3.81 is to plot L against corresponding values of D, or C_L against C_D; as L and D are rectangular components of the resultant force, the radius vector from the origin O to a point of the L/D curve represents the resultant force in magnitude and direction. The same is true of the C_L/C_D curve; these curves are called polar curves. This mode of

* [In an aeroplane the *span* is the distance from wing-tip to wing-tip.]

representation, which apparently was first used by Lilienthal, was improved by Eiffel,* who plotted the drag (which is usually very small or drag coefficient on a scale five times that of the lift or lift coefficient). This method of plotting has been very largely adopted, as it is very convenient for aeronautical practice. If each point of the lift/drag curve has the corresponding value of the angle of incidence inserted beside it, the graph contains all the information given by the two curves in fig. 3.81 (see the continuous line in fig. 3.84).

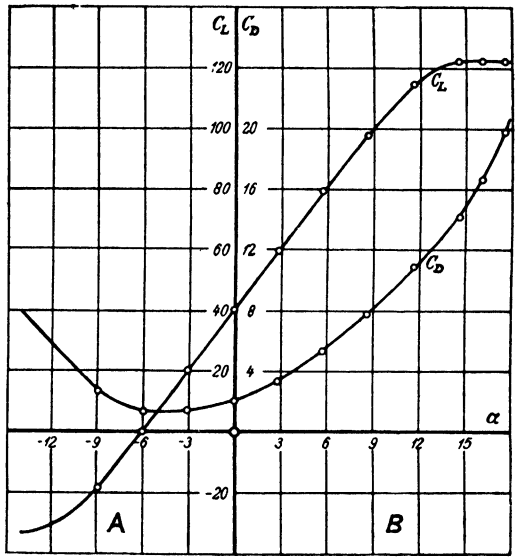


Fig. 3.81.—Variation of the lift and drag of an aerofoil with the angle of incidence

The numbers are C_L, C_D multiplied by 100

(*c*) Another important problem in connexion with the practical properties of an aerofoil is the *position* of the resultant force on the aerofoil for varying angles of incidence; this may be obtained from the distance between the origin O and the point of intersection of the resultant force with the chord OQ (the *centre of pressure*), i.e. s in fig. 3.80. As, however, s usually becomes infinite when $N = 0$, the graph of s as a function of α (the shift of the centre of pressure) is inconvenient for drawing and also for interpolation purposes: it is better to plot M, the moment of the resultant force about O, which gives a practically continuous curve. As we easily see, $M = Ns$, from which s may be found when M and N are given. In order that M may be

* *Nouvelles recherches sur la résistance de l'air et l'aviation* (Paris, 1914).

expressed by a dimensionless quantity another length must be introduced: a suitable one is OQ, the length of the chord of the profile (b). Then

$$M = C_M a p_d b.$$

C_M is called the *moment coefficient*. The dotted curve in fig. 3.84 shows the result of plotting C_M (abscissa) against C_L (ordinate). This method

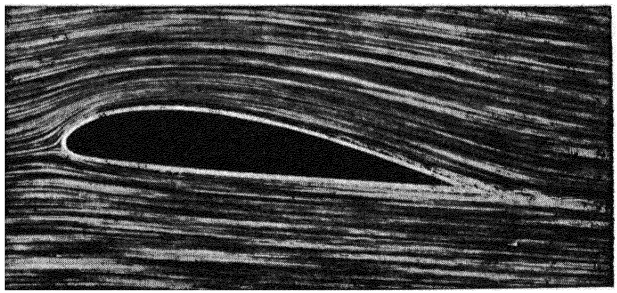


Fig. 3.82.—Unstalled flow round an aerofoil

of plotting is useful, as in the region of unstalled flow which is chiefly involved in aeronautical problems C_L increases fairly uniformly as the angle of incidence increases. Another reason why it is advisable to choose C_L as independent variable will appear from what follows (see next section).

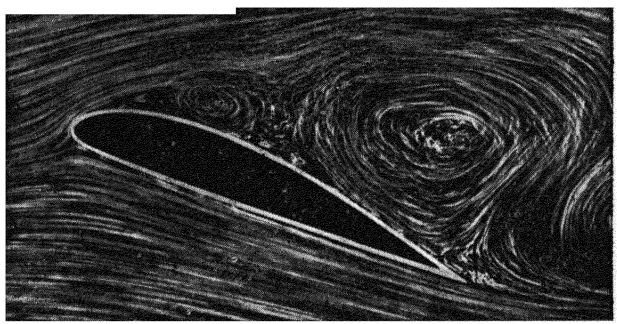


Fig. 3.83.—Stalled flow above an aerofoil

As regards the position of the centre of pressure, it may be stated that for symmetrical profiles (i.e. those with a straight centre line) it lies fairly accurately at a distance behind the front edge equal to one-fourth of the chord. For a centre or camber line forming an arc of a circle and zero angle of incidence (relative to the chord) it lies halfway back. At other angles of incidence, in addition to the lift for

$\alpha = 0°$ there is an additional lift at one-fourth of the chord equal to that for a straight centre line with the same value of α. Hence the resultant lift found by combining these two moves to and fro as the ratio of the components varies. Certain curved (slightly S-shaped) centre lines resemble the straight centre line in that the centre of pressure is fixed, lying at one-fourth of the chord.

Fig. 3.85 shows the distributions of pressure over a profile similar to the middle one in fig. 3.77 for four different angles of incidence ($\alpha = 0°$, 6°, 12°, and 18°). In the last, the flow is stalled. We see that near the front edge on the suction side there is a pressure deficit amounting to as much as three times the dynamic pressure; on the pressure side the highest excess pressure is merely equal to the dynamic pressure, as follows from Bernoulli's equation.

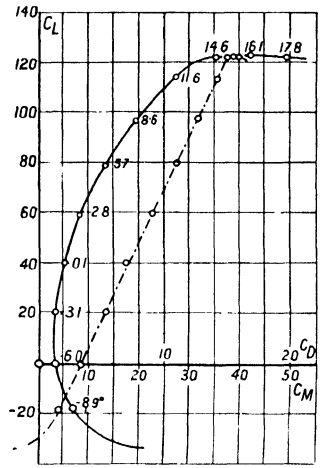


Fig. 3.84.—Lift and drag coefficients (continuous line) and lift and moment coefficients (dotted line)

The numbers are C_L, C_D, C_M multiplied by 100

(*d*) As has already been mentioned, the maximum lift of an aerofoil depends on the separation of the flow on the suction side. In general it rises to a moderate extent as Reynolds' number increases (and in wind tunnels it also increases with the turbulence of the stream of air). For values of Reynolds' number below 100,000 it is easy to reach the " subcritical " region,* in which the maximum lift is very much less. This is a point to be taken into account in the

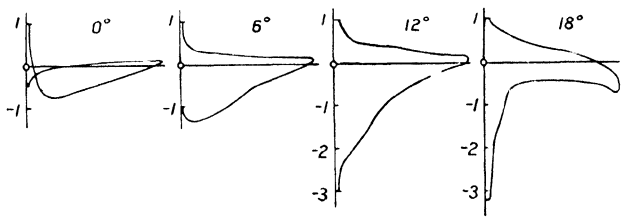


Fig. 3.85.—Pressure distributions on an aerofoil

construction of models. For this reason the best model profiles have shapes different from those of the best aeroplane profiles; see e.g. the experiments of F. W. Schmitz,† who has among other points recom-

* Cf. pp. 139–40.

† " Aerodynamik des Flugmodells ", *Tragflügelmessungen*, Part I (Berlin, 1941).

mended the use of profiles with a sharp front edge as advantageous. As he has shown, profiles of the type usual in aeroplanes can be used for models if a thin wire or thread is stretched ahead of the front edge to make the onflowing air turbulent.

To obtain a large lift at moderate speeds for the take-off and landing of very fast aeroplanes, various types of " landing device " are employed. Of these the most satisfactory are " flaps ", which represent the rear fourth or fifth of the breadth of the wing, and at landing are made to hang down steeply so that a markedly curved wing results. With split flaps the upper side of the wing remains as it was, and in the space between the wing and the flap a considerable pressure deficit is produced, which spreads to the suction side. In the slotted flap with suction we have the lift-increasing effect described on p. 138 and sometimes Handley-Page leading-edge slots are used as well (fig. 3.28, p. 144). Two practical forms are shown in fig. 3.86. By these methods the maximum value of C_L, which is about 1·2 to 1·5 for the profiles in fig. 3.77, and about 1·0 to 1·1 for those in fig. 3.78, can be raised to 2·3 to 2·5 with split flaps and 2·5 to 2·7 with slotted flaps, and, with Handley-Page leading-edge slots in addition, to nearly 3·0.

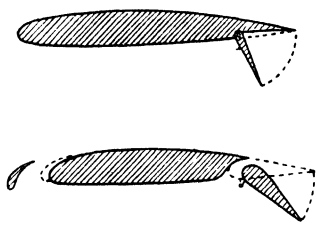


Fig. 3.86.—*Above*, wing with split flap; *below*, wing with slotted flap and leading edge slot

According to the experiments of O. Schrenk,* even greater values of $(C_L)_{max}$, up to about 4·2, can be obtained by sucking away the boundary layer as in § 7, p. 144, preferably at the hinge of a displaced flap. See also two reports on flight experiments by J. Stüper † and a retrospective survey by A. Betz.‡

By suction immediately above the trailing edge, as in fig. 3.87, B. Regenscheit § has achieved surprisingly large lifts without increasing the angle of incidence, even up to $C_L = 6$ and more with very powerful suction. The explanation is that eddies are produced at the trailing edge by the suction flow about that edge, and are carried away by the stream of air, until the flow round the edge ceases owing to the increase in circulation.



Fig. 3.87. — Suction above the trailing edge

Another property of flow at the stall that is worth mentioning is that the process of stalling takes a certain time. Hence, as was first shown by M. Kramer || by wind-tunnel experiments, very great lifts are obtained at first if the angle of incidence is suddenly raised above the critical value, i.e. as long as the flow remains attached.

* *Zeitschr. f. Flugtech. u. Motorluftschiffahrt*, Vol. 22, p. 259 (1931); *Lufo*, Vol. 12, p. 10 (1935).

† *LGL-Bericht*, 006 (1938); *FB* 1821 (1942).

‡ *Schrift. d. Deutsch. Akad. d. Luftfahrtforschung*, No. 49, p. 51 (1939).

§ *FB*, 1474 (1941) and 1594 (1942); see also *UM*, 3104 (1944).

|| *Zeitschr. f. Flugtech. u. Motorluftschiffahrt*, Vol. 23, p. 185 (1932).

It is only after some time that the lift falls to the steady value corresponding to the separated flow (the effect made noticeable by unpleasantly large vertical accelerations when an aeroplane enters a vertical gust). The details of these and related phenomena have been studied very thoroughly by W. S. Farren.*

(*e*) The properties of the aerofoil, like those of flat plates, are found to depend very much on the aspect ratio l/b: the greater the aspect ratio, i.e. the more elongated the wing, the smaller is the value of

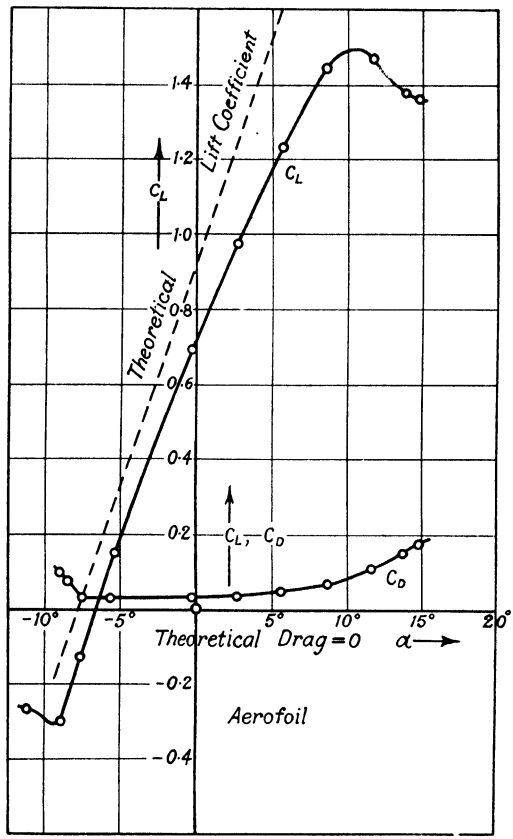


Fig. 3.88.—Lift and drag for two-dimensional flow round a Joukowski aerofoil, according to A. Betz

C_D corresponding to a definite value of C_L. The same is true of the corresponding values of the angle of incidence. These facts may be explained by theoretical considerations of the flow of frictionless fluid

* *Rep. and Mem.*, No. 1561 (1933), and No. 1648 (1935); cf. also de Haller, *Mitt. a. d. Inst. f. Aerodynamik Eidg. Techn. Hochsch. Zürich*, No. 4/5 (1936), which shows the effect of such phenomena on wing flutter.

round aerofoils for angles of incidence below the stalling angle: this of course takes no account of the resistance due to skin friction or any pressure resistance ("form resistance") due to the flow becoming separated from the wing, a fact which must always be borne in mind when one compares the results given by theory with those given by experiment.

The simplest case is that of a wing of infinite length, or what comes essentially to the same thing, a wing enclosed between parallel walls closely fitting to it. The steady motion of frictionless fluid in such a case is irrotational motion with circulation (cf. § 10 of the last chapter, p. 68). How well this theory (developed by W. M. Kutta * and N. Joukowski †) agrees with experiment, may be gathered from fig. 3.88,‡ in which the lift and drag of an aerofoil between parallel walls are plotted against the angle of incidence from theoretical figures and experimental data due to A. Betz.§ The profile of the aerofoil is the lowest shown in fig. 3.77, p. 196 ("Joukowski profile").

Theoretically the drag is zero: in practice it is chiefly due to skin friction; the lift obtained in practice is somewhat less than that indicated by theory, a fact which is likewise due to friction (the circulation is diminished by the boundary layer being somewhat broader on the suction side, which has the same effect as a slight diminution of the angle of incidence). If this diminution of the angle of incidence is taken into account, the theoretical and experimental distributions of pressure on an aerofoil of this type are in very good agreement below the stall.

The theoretical discussions of Kutta and Joukowski, which started from the fact that a circle in a ζ-plane round which fluid was flowing could be conformally represented on a z-plane in such a way that another circle cutting (or touching) it in the ζ-plane became a straight line in the z-plane, gave only a certain limited number of types of aerofoil profiles. The possibilities were considerably extended by von Kármán and Trefftz,|| by making use of the transformation of a circle into a pair of intersecting circular arcs.

R. von Mises¶ brought forward a number of important general relationships and added other modes of representation, by means of which e.g. profiles with the centre of pressure fixed were obtained. In a large number of other papers,**

* *Sitzungsber. d. Bayr. Akad. d. Wiss., M.-Ph. Kl.*, 1910, 1911.

† *Zeitschr. f. Flugt. u. Motorluftsch.*, Vol. 1, p. 281 (1910); see also Joukowski, *Aérodynamique* (Paris, 1916), or R. Fuchs in Vol. 2 of Fuchs, Hopf, and Seewald, *Aerodynamik*, 2nd ed., p. 61 *et seq.*

‡ The figures 3.88, 94, 95, 96, 97, 98 are taken from Kármán and Levi-Civita, *Vorträge aus dem Gebiete der Hydro- und Aerodynamik*, by kind permission of the publisher (Julius Springer, Berlin).

§ *Zeitschr. f. Flugt. u. Motorluftschiffahrt*, Vol. 6, p. 173 (1915).

|| *Ibid.*, Vol. 9, p. 111 (1918).

¶ *Ibid.*, Vol. 8, p. 157 (1917); Vol. 11, pp. 68, 87 (1920).

** For the oldest of these, see R. Fuchs, loc. cit., pp. 67–82.

among which those by Th. Theodorsen and J. E. Garrick * are of special importance, methods were developed for calculating the irrotational flow with circulation, and hence the corresponding pressure distribution for any arbitrary profile. The problem of finding the profile corresponding to a prescribed pressure distribution could also be solved by a process of approximation.†[26]

Potential theory methods, which depend on doubly-connected spaces, have also been developed for the two-dimensional theory of the biplane. Here we specially mention the fundamental work of J. E. Garrick,‡ whose methods are also applicable to double wings (wing with flaps, slotted wing).§

Apart from these theories depending on conformal representation, approximation theories have also been developed, starting from distributions of vortices taking the place of the aerofoil and regarded for simplicity as being in the horizontal plane (instead of on the centre line of the wing). Three-dimensional problems can also be attacked by this method. In the two-dimensional case it gives very neat relationships, e.g. for the way in which the lift and moment coefficients depend on the angle of incidence, on the camber of the wing,|| etc., e.g. $C_L = 2\pi(\alpha + 2f/b)$, $C_M = (\pi/2)(\alpha + 4f/b)$, where b is the chord of the wing, f the camber of the centre line of the wing, assumed to be an arc of a circle.¶ It was also possible, by suitable expansions in series, to calculate the distribution of lift for bent profiles such as arise from the displacement of a hinged flap.**

A camber line in the form of an arc of a circle, as is easily proved by means of the formula for C_L given above, has the property that for arbitrary (slight) camber the lift is as great as for a plane surface of the same chord touching the camber line at 3/4 of the breadth. This rule is useful in practical cases where the flow of air is not only altered in direction by neighbouring objects but has its streamlines curved. The lift can be calculated from the formula for a plate, $C_L = 2\pi\alpha$, where α is the angle between the tangent to the centre line and that of the disturbed flow of air, both at three-quarters of the breadth.

(*f*) A curious paradox arises from the hydrodynamical theory of the lift on an aerofoil as applied to a flat plate inclined at an angle α to the direction of flow, when it is based on the assumption of ideal frictionless fluid. Here the resultant force is composed of the pressure differences at the individual points of the plate. But all the pressures are inclined at the same angle α to the vertical, as they are all at right angles to the plane of the plate; the resultant R must therefore be in this direction also, so that a drag $D = R\sin\alpha$ will be present as well as the lift $L = R\cos\alpha$.

On the other hand, the circulation theory of Chapter II, § 10 (p. 69), and the momentum considerations of § 13(*a*) of the same chapter (p. 83), give a lift alone, and on this theory it is not possible to introduce a drag so long as we are dealing with two-dimensional flow. Kutta has also discussed a type of irrotational flow about a flat inclined plate; he obtained a lift $L = \pi\rho v^2 . b\sin\alpha$ and a zero drag.

* *Report of National Advisory Council for Aeronautics*, Washington, No. 411 (1931) (Theodorsen), and No. 452 (1933) (Theodorsen and Garrick).

† A. Betz, *Luftfahrtforschung*, Vol. 11 (1934), No. 6; also W. Mangler, *Jahrb. d. D. Luftfahrtforschung*, Part I, p. 46 (1938).

‡ *Report of National Advisory Council for Aeronautics*, Washington, No. 542 (1936).

§ See I. Flügge-Lotz and I. Ginzel, *Ing.-Arch.*, Vol. 11, p. 268 (1940).

|| Birnbaum, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 3, p. 290 (1923); Glauert, *Rep. and Mem.*, No. 910 (1924); see also Glauert-Holl, p. 76, or R. Fuchs, loc. cit., pp. 82–100.

¶ The remarks about the position of the centre of pressure on pp. 199–201 are deductions from these formulæ.

** Glauert, *Rep. u. Mem.*, No. 1095 (1927).

More detailed analysis of this flow yields the explanation * of the paradox. At the rear edge of the plate the flow moves off smoothly as in fig. 2.41, p. 70. At the front edge, however, the flow is of the type shown in fig. 2.37*e*, p. 67, with an infinite velocity at the actual edge. To the infinite velocity corresponds an infinite negative pressure. We can get a clearer idea of the relationships by assuming that the plate, instead of being infinitely thin, has a finite but small thickness, and rounding off the region of the front edge like one of the streamlines in fig. 2.37*e*, p. 67. Then we have only to do with finite velocities and finite negative pressures. The negative pressures on the front end of finite breadth now combine to give a " suction " which opposes the drag component of the resultant pressures on the main part of the surface of the plate. More detailed calculation shows that the resultant suction is practically independent of the thickness of the plate, the negative pressures increasing in magnitude as the radius of curvature at the front edge diminishes. It is natural to suppose that the suction effect will be of the same magnitude in the case of an infinitely thin plate. With the above value of L the magnitude of the suction effect is found to be $S = L \sin\alpha$, its direction lying in the plane of the plate; the resultant of the other pressures, which is at right angles to the plate, is found to be $R = L \cos\alpha$. The horizontal component of R is $R \sin\alpha = L \sin\alpha \cos\alpha$ pointing backwards, and the horizontal component of the suction is $S \cos\alpha = L \sin\alpha \cos\alpha$ pointing forwards, so that the resultant horizontal force is zero.

This result, however, applies only to the irrotational flow of ideal fluid with circulation. In the actual flow about a plate bevelled at the front edge no infinite negative pressures arise, of course, but the flow becomes separated from the plate; as a result of a turbulent process, however, it returns to the plate on the suction side, provided the angle of incidence is low, so that the resulting flow is broadly speaking not dissimilar to the theoretical one and gives rise to a lifting force of about the same magnitude. In the absence of suction we of course get a drag † which exceeds the frictional drag by $L \tan\alpha$. The equivalent of this additional drag is to be sought for in the velocity losses which arise on the suction side as a result of the turbulence there.

The value of the theoretical argument about the suction at the front edge consists on the one hand in the clearing-up of the paradox mentioned above, and on the other in the insight it gives into the significance of the rounding-off of the aerofoil profile in front. If the profile is well rounded off the suction effect becomes quite important in practice. If the resultant force on an aerofoil with an angle of incidence of 6° makes an angle of only 2° with the vertical (as is often observed), it is clear indication of the presence of this suction.

17. **Theory of the Aerofoil.**‡

(*a*) With an aerofoil of finite span a surface of discontinuity is formed at the rear edge, as described in connexion with fig. 2.22 on p. 53. The rolling-up of the surface of discontinuity illustrated there gives rise to two eddies extending backwards all along the line of flight. As part of these eddies must continually be in fresh formation as time goes on, it is obvious from the principle of energy that work must

* Pointed out by Kutta himself (*Sitzungsber. d. Bayr. Akad. d. Wiss.*, 1910, Part II, pp. 25, 42).

† Here, in contrast to the state of affairs above, $L = R \cos\alpha$, $D = R \sin\alpha$.

‡ Cf. Prandtl, *Götting. Nachr.*, 1918, p. 451, and 1919, p. 107 (reprinted in *Vier Abhandlungen zur Hydrodynamik* (Göttingen, 1927)); also the lectures by Trefftz and Wieselsberger in *Hydrodyn. Vorträge* (Innsbruck, 1922), edited by von Kármán and Levi-Civita (Berlin, 1924).

be continually expended, and this work can only consist in the overcoming of a resistance. From this follows the fact that *a finite aerofoil is subject to a drag even in frictionless fluid.* This drag may be approximately evaluated as explained below.

We know from § 10 of the last chapter (p. 70) that in frictionless fluid the lift is always at right angles to the direction of the velocity v. In the case of the finite aerofoil the fluid on the pressure side moves outwards and flows past the wing-tips to the suction side. This motion may be regarded as meaning that the fluid gives way under the pressure exerted by the aerofoil, giving rise to components of velocity in the directions of the vertical and lateral components of the pressure gradient. Hence the resultant flow round the aerofoil is compounded from the undisturbed velocity v and the velocity caused by the aerofoil itself. The component of this flow that we are most interested in is the mean downward velocity q at the surface of the wing (weighted with respect to the lift distribution). Hence the mean flow at the wing is inclined at an angle given by $\tan\phi = q/v$ (cf. fig. 3.89). The resultant force in the air, by what has been said above, is at right angles to this direction and hence has two components, the lift L and the drag $D_i = L\tan\phi = Lq/v$. Now the problem of aerofoil theory is to find out q, the velocity "induced" by the aerofoil.

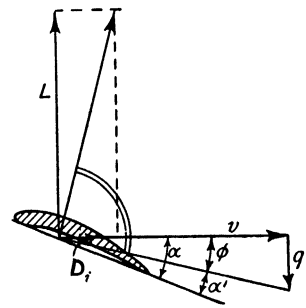


Fig. 3.89

In order to ascertain the induced velocity q we make the simplifying assumption that the lift L is so small that all changes in the direction of the flow are small, so that it is sufficient to take account of the first order of each quantity which occurs, neglecting all amounts of a higher order of small quantities. This assumption includes e.g. the assumption that the sine, tangent, and circular measure of the angles made by the relative stream-lines with the direction of motion may be interchanged and the cosine may be equated to unity.

The velocity field induced by an aerofoil may now be calculated from the system of vortices by which the aerofoil and its surfaces of discontinuity can be replaced. The lift is always associated with circulation; in fact, the Kutta-Joukowski theorem, which is still true here, states that the lift per unit length is $\rho\Gamma v$, where Γ is the circulation; see §§ 10, 13(*a*) (pp. 68, 83) of the last chapter. (Another proof will be given on p. 211.) Thus the effect of the aerofoil on the parts of the fluid at some little distance from it is the same as that of a portion of vortex filament of strength (circulation) Γ. As, however, a vortex of this kind, which really consists of an aerofoil, does not satisfy Helmholtz's theorem that the vortex always consists of the same particles

of fluid, we speak of a "bound vortex" in contradistinction to the "free vortices" or "trailing vortices" for which Helmholtz's theorem is true.

The simplest arrangement of vortices for a finite aerofoil consists, then, of a bound vortex of strength Γ together with two parallel free vortices of equal strength starting from the wing-tips and stretching backwards to an infinite distance (fig. 3.90). (The theorem that a vortex filament can never come to an end and must have the same strength at every point is of a purely kinematic nature, and therefore holds for a combination of free and bound vortices just as it does for free vortices.) In actual fact the lift usually diminishes in magnitude towards the wing-tips. In the vortex method of representation this may be attained by superposing a large number of vortex combinations of infinitesimal strength, like that in fig. 3.90, each having a different

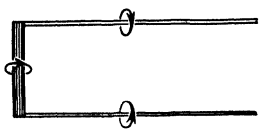


Fig. 3.90.—Simplified vortex system of an aerofoil

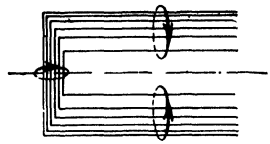


Fig. 3.91.—More accurate vortex system

span from the others (fig. 3.91). Proceeding in this way we obtain an approximate representation of the surface of discontinuity described above, in which, however, we have deliberately omitted to take account of the change of shape in the surface of discontinuity as its distance from the aerofoil increases. The smaller the lift is, the more slowly does this change of shape take place, so that in the limiting case where the lift is very small we can in actual fact neglect the change of shape entirely if we have to calculate the velocity field in the immediate neighbourhood of the aerofoil.

For the problem meanwhile under discussion we need only calculate the velocity in the direction of the lifting force at points on the aerofoil. Taking the vortex system as shown in fig. 3.90, we have the following argument for the centre of the aerofoil. A straight vortex filament of strength Γ extending to infinity in both directions gives a velocity $q = \Gamma/2\pi d$ at a distance d (see § 11 of the last chapter, p. 74). A vortex extending in a backward direction only from the transverse plane through the aerofoil gives just half this value at points in the transverse plane, from reasons of symmetry, i.e. $\Gamma/4\pi d$. If l is the span of the aerofoil, $d = l/2$ for the centre of the aerofoil, and the contributions of the right-hand and left-hand vortices have to be combined, so that $q_0 = 2\dfrac{\Gamma}{4\pi l/2} = \dfrac{\Gamma}{\pi l}$. By the Kutta-Joukowski theorem

$\Gamma = L/\rho l v$, so that $q_0 = L/\pi\rho v l^2$. Near the centre the velocities are larger, as is readily verified, and near the ends of the aerofoil they tend towards infinity. This merely means that it is not permissible to assume that the lift is constant all the way to the ends of the aerofoil. The more accurate theory based on fig. 3.91 shows that we obtain a constant value for q over the length of the aerofoil if the lift is distributed elliptically as in fig. 3.92. As here the circulation in the centre is $4/\pi$ times the mean value and, moreover, the single vortex filaments are on the average nearer the middle, q is greater than q_0; by integration over all the vortex filaments,* in fact, we have

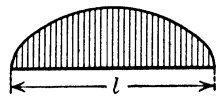


Fig. 3.92.—Elliptic distribution of lift

$$q = 2q_0 = \frac{2L}{\pi\rho v l^2}, \quad \ldots\ldots\ (63)$$

so that

$$\tan\phi = \frac{q}{v} = \frac{2L}{\pi\rho v^2 l^2} = \frac{L}{\pi p_d l^2}. \quad \ldots\ldots\ (64)$$

Since, as we have just said, q is constant all over the span when the lift is distributed elliptically, $\tan\phi$ is so as well. Thus the equation

$$\tan\phi = \frac{D_i}{L}$$

may immediately be applied, so that we have

$$D_i = \frac{L^2}{\pi p_d l^2}. \quad \ldots\ldots\ (65)$$

This drag on an aerofoil moving in frictionless fluid is called the *induced drag* (the name is due to the similarity of the relationship between vortex filaments and velocities to that between electrical conductors and their magnetic field, cf. p. 75). Close investigation has shown that the drag given by equation (65) represents the minimum induced drag attainable † by any distribution of the lift L over the span l. This minimum property is related to the constancy of the induced velocity over the span. As, moreover, a quantity does not usually alter much in the neighbourhood of a minimum, formula (65) may also be used as an approximation formula for other distributions of lift, provided they do not differ too widely from the elliptic distribution. This is the case e.g. for rectangular aerofoils where the aspect ratio is not too large.

* Cf. p. 212.

† It must be expressly emphasized that the elliptical distribution of lift is certainly not that giving the least induced drag, since D_i can always be decreased by increasing the span. The latter is of course limited by practical considerations such as the strength of the spars; see e.g. L. Prandtl, *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, Vol. 24, p. 305 (1933).

Formula (65) shows that the induced drag associated with the production of a definite lift is the smaller, the greater the length over which the lift is distributed. This is the reason why aeroplanes have wings of span large compared to their chord. The chord of the wing in the direction of flight does not occur in formula (65), which means that the essential point is merely the nature of the flow left behind by the wing, and that it does not matter whether this flow was produced by large pressure differences over a narrow region or by slight pressure differences over a rather broader region.

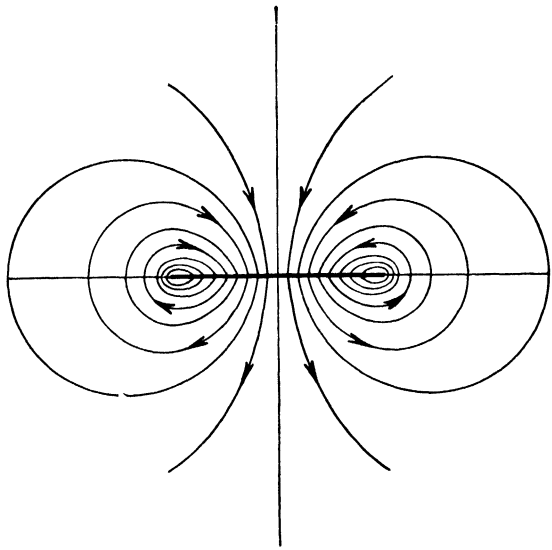


Fig. 3.93.—Flow in the rear of an aerofoil

(*b*) Consideration of the disturbance left behind by the aerofoil leads to another vivid way of representing the relationships involved. Instead of *one* point after another along the path of the aerofoil in rapid succession experiencing pressures due to the aerofoil, we may imagine that all the points on the path *simultaneously* experience these pressures for a brief interval of time, say as a result of a stiff board of this length and of span l being suddenly pushed down. The resulting motion is a two-dimensional irrotational motion (cf. § 9(*c*) of the last chapter, p. 64); the surface on which the pressures are exerted becomes a surface of discontinuity. A motion of this kind is represented in fig. 3.93.* The pressure impulses $\int_0^{t_1} p\,dt$ may be brought

* Taken from *Jahrbuch der wiss. Gesellschaft für Luftschiffahrt* (1920) by kind permission of the publisher (Julius Springer, Berlin).

into relation with the velocity potential on the one hand and with the distribution of the lift on the other, thus giving rise to formulæ which are equivalent to those mentioned above.

The course of the argument is briefly as follows. By equation (18*a*), p. 61, neglecting the squares of the velocity fluctuations, we have

$$\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} = \text{const.} = \frac{p_\infty}{\rho},$$

where p_∞ is the undisturbed pressure at an infinite distance. If Φ was initially zero everywhere, integration over the time during which the pressures exist gives

$$\rho\Phi + \int_0^{t_1} (p - p_\infty)\, dt = 0.$$

If p_u is the pressure below the surface of discontinuity and p_o that over it, we accordingly have

$$\int_0^{t_1} (p_u - p_o)\, dt = \rho(\Phi_o - \Phi_u) = \rho\Gamma.$$

The impulsive force on a portion of the surface of discontinuity of length b (or, in other words, the momentum in an element of the path of flight of length b) is thus

$$\int_{y_1}^{y_2} b\, dy \left(\int_0^{t_1} (p_u - p_o)\, dt \right) = \rho b \int_{y_1}^{y_2} \Gamma dy.$$

The lift L is the increase in this momentum with time, i.e. putting $db/dt = v$, we have

$$L = \rho v \int_{y_1}^{y_2} \Gamma\, dy, \qquad \ldots\ldots\ldots \quad (66)$$

which is none other than the Kutta-Joukowski theorem applied to the finite aerofoil.

For simplicity the velocity q_1 with which the surface of discontinuity moves downwards after the impulse may be taken as constant [this corresponds to the minimum problem mentioned in connexion with equation (65)]. It is double the induced velocity q (cf. the approximate argument on p. 207 with reference to fig. 3.89). The relation between Γ, the circulation on the surface of discontinuity, and q_1 is uniquely found from the second boundary-value theorem of potential theory. Γ is proportional to q_1; for dimensional reasons we may put

$$\int_{y_1}^{y_2} \Gamma\, dy = a_1 q_1, \qquad \ldots\ldots\ldots \quad (67)$$

where a_1 is a surface defined by (67). Equation (66) then becomes

$$L = \rho a_1 v q_1. \qquad \ldots\ldots\ldots \quad (68)$$

The drag is now found by applying the work theorem. The work done against the drag in unit time is $D_i v$; the work supplied by the lift in the same time is $Lq = \frac{1}{2} L q_1$. Equating the two work expressions and expressing q_1 in terms of L by means of equation (68), we have

$$D_i = \frac{L^2}{2\rho v^2 a_1} = \frac{L^2}{4 p_d a_1}. \qquad \ldots\ldots\ldots \quad (69)$$

It may also be pointed out here that in the straight monoplane, where the y-distribution of Γ is semi-elliptical, a_1 is equal to the area of the circle on the span l as diameter, i.e. to $\pi l^2/4$. With this value of a_1, (69) again gives the value found in (65). The formula (69), moreover, holds quite generally, provided $q_1 =$ constant, for any aerofoil system moving in a straight line (biplane, wing with discs at the ends, and also an aerofoil in a wind tunnel). In every case, of course, the area a_1 has to be specially found by solving the appropriate potential problem.

The following simplified discussion based on momentum and energy considerations may serve to throw more light on equations (68) and (69). Here we shall assume that after the aerofoil has passed, a quantity of fluid of cross-section a_1 is moving with constant velocity q_1 in the direction of the force exerted on the fluid, while the rest of the fluid remains at rest. With these assumptions, m, the mass of fluid freshly set in motion per second, is $\rho a_1 v$. The change of momentum in the direction of q_1, which is equivalent to the effect of the lift L per second, is thefore $mq_1 = \rho a_1 v q_1$, and the kinetic energy produced is $\frac{1}{2}mq_1^2 = \frac{1}{2}\rho a_1 v q_1^2$. This must be equal to the work done against the induced drag on the aerofoil per second, i.e.

$$\tfrac{1}{2}\rho a_1 v q_1^2 = D_i v.$$

Using the relation

$$\rho a_1 v q_1 = L,$$

we can eliminate q_1, and we again obtain equation (69).

(*c*) The aerofoil problem can also be solved for any distribution of circulation $\Gamma(y)$ not giving a constant downward velocity q. Taking account of the fact that $q(y)$ may be regarded as due to the superposition of the velocity fields of separate infinitesimal vortices of strength $-(d\Gamma/dy')dy'$, where y' is the variable of integration, we readily obtain *

$$q(y) = \frac{1}{4\pi}\int_{y_1}^{y_2} \frac{d\Gamma}{dy'}\frac{dy'}{y - y'}, \qquad \ldots\ldots (70)$$

by what has been said on p. 209. Further, Γ can be expressed in terms of the "effective angle of incidence" α' of fig. 3.89. If α' is suitably defined (i.e. $\alpha' = 0$ for the direction of zero lift), Γ is proportional to α' and

$$\Gamma(y) = Cvb\alpha'. \qquad \ldots\ldots (71)$$

Comparing this with equation (66), we easily see that $C = \frac{1}{2}dC_L/d\alpha'$, where α' is the angle of incidence for two-dimensional flow. (The theoretical approximate value (cf. p. 205) is $C = \pi$.)

Eliminating α' by means of the relation

$$\alpha' = \alpha_g - \phi = \alpha_g - \frac{q(y)}{v}, \qquad \ldots\ldots (72)$$

readily deduced from fig. 3.89, α_g being the geometrical angle of incidence, we obtain a relation (an integro-differential equation) for $\Gamma(y)$ in terms of α_g and

* In evaluating this integral the infinite value at $y' = y$ requires special consideration. The so-called principal value is to be taken.

the chord of wing $b(y)$, i.e. for the straight monoplane of any shape. As it is linear in $\Gamma(y)$, it can be solved by replacing $\Gamma(y)$ by a suitable series. In the course of time a great many methods of carrying out the calculations have been put forward.* In the first, due to A. Betz,† the problem was solved for the simple rectangular aerofoil by means of the series $\Gamma = \sqrt{(1-\eta^2)} \cdot \sum_0^\infty A_n \eta^n$, where $\eta = y/\frac{1}{2}l$. Here the contributions of the individual terms of the series to $q(\eta)$ are integral functions of the nth degree in η. The calculation of the coefficients A_n was found to be fairly laborious.

A series which is closely related to the preceding was obtained by E. Trefftz ‡ by introducing the angle θ by means of the relation $\eta = \cos\theta\ (0 < \theta < \pi)$. With the series

$$\Gamma = \sum_1^\infty B_n \sin n\theta$$

equation (70) gives

$$q = \sum_1^\infty \frac{nB_n}{2l} \frac{\sin n\theta}{\sin\theta}.$$

In the case of the flat rectangular aerofoil Trefftz was able to reduce the solution to a boundary-value problem in potential theory.[27] In the general case, however, the calculation of the coefficients B_n for a series terminating at a finite value of n is also very laborious; the position was so much improved by the following idea of Irmgard Lotz § that relatively convenient and rapidly convergent methods of calculating the coefficients B_n are now available, with the result that more difficult problems, e.g. a wing with displaced aileron, can be attacked with success. The new idea consisted in replacing $\alpha_g(\theta)\sin\theta$ and $\sin(\theta)/b(\theta)$ by Fourier series (which, moreover, converge more rapidly than those for α_g and $1/b$); thus all the terms in equation (72) have the denominator $\sin\theta$, which can then be cancelled. The solution for the unknowns B_n can be obtained by iteration, beginning with B_1.

An important advance was brought about by Multhopp's method,‖ which saved the trouble of the two Fourier series. It is based on a mechanical integration process, using certain selected discrete values of η, as in the Gaussian quadrature, for which the coefficients can be calculated once and for all; any desired degree of accuracy can be attained by iteration, so that the method has now been generally adopted.

Note.—The calculations in this section, in which the aerofoil is replaced by a bound vortex line, fail when the line is not at right angles to the direction of flow of the approaching air, since the expression taking the place of the integral in equation (70) then becomes infinite. In the theory of the wing in side slip and the swept-back wing, as well as in that of the flapping wing, a "bound vortex surface" must be considered instead. The first approximate calculation for the wing in side slip is due to H. Blenk;¶ a solution for the wing in side slip with elliptical

* See e.g. R. Fuchs in Vol. 2 of Fuchs, Hopf and Seewald, *Aerodynamik*, 2nd ed., pp. 139–80 (Berlin, 1935).

† *Diss.*, Göttingen (1919); *Ber. u. Abhandl. d. Wiss. Ges. f. Luftfahrt* (Supplement to *Zeitschr. f. Flugtechnik u. Motorluftschiffahrt*), No. 2, p. 1 (1920).

‡ *Math. Ann.*, Vol. 82 (1921), Part 3–4.

§ *Zeitschr. f. Flugtechnik u. Motorluftschiffahrt*, Vol. 22, p. 189 (1931).

‖ *Luftf.-Forsch.*, Vol. 15, p. 153 (1938); *Jahrb. d. Deutsch. Luftf.-Forschung* (1938), Part I, p. 101.

¶ *Diss.*, Göttingen (1923); a summary is given in *Zeitschr. f. angew. Math. u. Mech.*, Vol. 5, p. 36 (1925).

boundary has been given by Kl. Krienes,* and for the flapping wing by W. Birnbaum.† In more recent papers successful use has been made of the approximation rule deduced on p. 205, to calculate the vertical component of velocity at three-quarters of the chord for a bound vortex assumed to be at one-quarter of the chord (the fact that this velocity becomes infinite at the position of the bound vortex is then immaterial). Further references ‡ are given in the *Bericht über das Preisausschreiben der LGL*, 1939, S 2, and in the *F.I.A.T. Review of Aerodynamics*, Chapter I, section 1.5.[28]

(*d*) In a biplane each wing is subject not only to its own effects but to those of the other wing (" self-induction " and " mutual induction "); with the assumption that the lift is small, the separate effects may be combined additively, and the induced drag consists of four terms:

$$D_i = D_{11} + D_{12} + D_{21} + D_{22}$$

(D_{12} being the drag on the first plane due to the second, and so on). The total induced drag of a biplane is smaller for the same total lift than that of a monoplane with the same span (since a_1, the cross-section of air affected by it, is greater than that of the monoplane). An approximate formula will be found at the end of the next section. More exact study of the mutual effects of the two wings § shows that the velocity, and hence the lift, on the upper wing is strengthened and that on the lower wing weakened. In addition, the flow past each wing is curved by the flow past the other, so that the wing profile behaves as if the curvature of its centre line were correspondingly diminished. Good confirmation of the theoretical results has been obtained from experiments.‖

Detailed methods of calculation have been developed for the biplane of prescribed form. A reference to the accounts by R. Fuchs ¶ and D. Küchemann ** may suffice.[29]

Related to the biplane problem is the problem of the behaviour of an aerofoil in the neighbourhood of the ground, which can be transformed into a problem in free space by taking into account the image of the aerofoil in the surface of the ground. (Owing to the symmetry of the flow thus obtained, all the velocities in the plane of symmetry

* *Diss.*, Göttingen; *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 65 (1940).

† *Diss.*, Göttingen; summary in *Zeitschr. f. angew. Math. u. Mech.*, Vol. 4, p. 177 (1924).

‡ Among these may be mentioned J. Weissinger (wing in side slip); H. Multhopp (wing in side slip and swept-back wing), both in *LGL-Ber.*, S 2, pp. 5, 53. See also H. Weissinger, *FB.* 1553 (1942), and *UM.* 1392 (1944).

§ See e.g. Prandtl, *Gött. Nachr.* (1919), pp. 107–23; reprinted in *Vier Abhandlungen zur Hydrodyn. u. Aerodynamik*, pp. 36–52; also *Ergebn. d. Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 2, p. 9 (1923); Vol. 3, p. 9 (1927).

‖ *Ergebn. d. Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 2, p. 35 *et seq.* (1923).

¶ R. Fuchs, loc. cit., p. 188 *et seq.*

** *Luftf.-Forschung*, Vol. 15, p. 543 (1938); *Jahrb. d. Deutsch. Luftf.-Forschung* (1938), Part 1, p. 136.

representing the ground are parallel to this plane.) As is easily seen, the effect of the image is to reduce the induced velocity q and hence also the induced drag, and also to reduce the velocity of approach of the flow by a " following wind ". This theory is also confirmed by experiment.*

A modified form of the image principle can also be applied to the case of an aerofoil in a closed tunnel or in a free jet of air; it thus provides certain corrections which have to be applied when values obtained in wind-tunnel experiments are applied to free space. The theory † of these corrections for streams of air of circular cross-section is in good agreement with experiment.‡ The rectangular wind tunnel and rectangular jet of air have been discussed by H. Glauert.§

Note.—In contrast to this theory of a three-dimensional aerofoil based on the bound vortex line, an aerofoil theory depending on potential theory has been developed, of which the two-dimensional analogue is the linearized profile theory referred to in footnote 5, p. 205. The more complicated three-dimensional theory, however, permits the accurate solution of the problem only for surfaces of circular and elliptical outline, but for these gives very full information; see the papers by W. Kinner || (circle) and Kl. Krienes ¶ (ellipse and ellipse with side slip; see note on p. 213). The rectangle has been dealt with by methods of approximation; see the papers by K. Wieghardt ** and I. Ginzel.††

18. Practical Application of the Theory of Aerofoils: Experimental Verifications.

In applying the results of aerofoil theory in practice, it must be borne in mind that in real fluids both frictional resistance and resistance due to separation of the flow from the body are present. The latter part of the drag can be observed in isolation by experimenting with aerofoils extending close to the walls of a closed tunnel [cf. the experiments by Betz ‡‡ mentioned in § 16 (p. 204)]; in such cases the induced drag is zero.§§

In the case of a finite aerofoil in free space there is in addition to this drag D_o, called the *profile drag*, the induced drag D_i:

$$D = D_o + D_i.$$

* See A. Betz, *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, Vol. 13, p. 86 (1912); C. Wieselsberger, ibid., Vol. 22, p. 145 (1921); *Ergebn. d. Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 2, p. 41 (1923).

† Cf. e.g. Prandtl, *Gött. Nachr.* (1919), pp. 123–8; *Vier Abh.*, pp. 52–7.

‡ *Ergeb. d. Aerodynamischen Versuchsanstalt zu Göttingen*, Vol. 2, p. 17 (1923).

§ Cf. Glauert-Holl, p. 168 *et seq.*; also Glauert, *Rep. and Mem.*, No. 867 (1923), and No. 1566 (1933), for a general account and experiments. See also A. Toussaint, *Aerodynamic Theory*, Vol. 3, p. 280 *et seq.*

|| *Diss.*, Göttingen; *Ing.-Arch.*, Vol. 8, p. 47 (1937).

¶ *Diss.*, Göttingen; *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 65 (1940).

** *Diss.*, Göttingen; *Zeitschr. f. angew. Math. u. Mech.*, Vol. 19, p. 257 (1939).

†† *Jahrb. d. Deutsch. Luftfahrtforschung* (1940), Part I, p. 238.

‡‡ Another method, in which the momentum theorem is applied to the region of diminished velocity behind the aerofoil, is described in more detail in § 22(*c*), p. 255.

§§ When there is a free jet between parallel side walls there is an induced drag as given by formula (69) in last section, a_1 being the cross-section of the current.

In order that this subdivision may be permissible, however, we may not compare results for the same angle of incidence, but the nature of the flow at the aerofoil element must be the same: if this is the case the lift on unit area of the aerofoil is the same (neglecting small quantities). We see then that if we are to apply the relationships obtained from the theory of an aerofoil validly, it is advisable to start from the lift per unit area, i.e. from the lift coefficient C_L. (Of course the lift coefficient C_L already represents the formation of an average of the varying intensities of lift as we pass from the centre of the aerofoil to its ends; but as C_L is approximately a linear function of α in the region of unstalled flow and the profile drag coefficient C_{Do} does not alter greatly there, the fact that C_L represents an average may be left out of account.) Introducing C_L and C_{Di} into formula (65) of last section (p. 209) instead of L and D_i ($L = C_L a p_d$ and $D_i = C_{Di} a p_d$), we have

$$C_{Di} = \frac{a C_L^2}{\pi l^2},$$

and hence for a given value of C_L

$$C_D = C_{Do} + C_{Di} = C_{Do} + \frac{a C_L^2}{\pi l^2} \qquad . \; . \; . \quad (73)$$

by the above. This formula may be used to calculate the drag coefficient of a finite aerofoil from that of an infinitely long aerofoil. The drag coefficient of a finite aerofoil may, however, be more accurately evaluated by experimenting with another finite aerofoil with a different value of a/l^2, for here the distribution of the lift over the span is approximately the same in the two cases and the average mentioned in the sentence in brackets above is formed in a similar way both for the given aerofoil and that under investigation. Using equation (73) twice, first for aerofoil 1 (span l_1, area a_1) and then for aerofoil 2 (span l_2, area a_2), and putting $C_{L2} = C_{L1}$ in accordance with what we have said above, we may verify that the following relationship holds:

$$C_{D2} = C_{D1} + \frac{C_L^2}{\pi}\left(\frac{a_2}{l_2^2} - \frac{a_1}{l_1^2}\right). \qquad . \; . \; . \quad (74)$$

For rectangular aerofoils $a = lb$, i.e. in this case, as is worth noting, $l^2/a = l/b$, i.e. is equal to the aspect ratio. The conversion formula (74), although deduced in the first instance for infinitesimal values of the lift, has been found to be substantially true for all values of lift occurring for the range of aspect ratio (3 to 7) common in practice. In this range, with unstalled flow, the deviation of the stream-lines caused by such aerofoils is fairly small, except in the immediate neighbourhood of the aerofoil. When the aspect ratio is 2 or actually

1, it is pushing matters rather too far to replace the aerofoil by a "bound vortex filament", and we cannot reasonably expect agreement with the formulæ to be still maintained; nevertheless, even

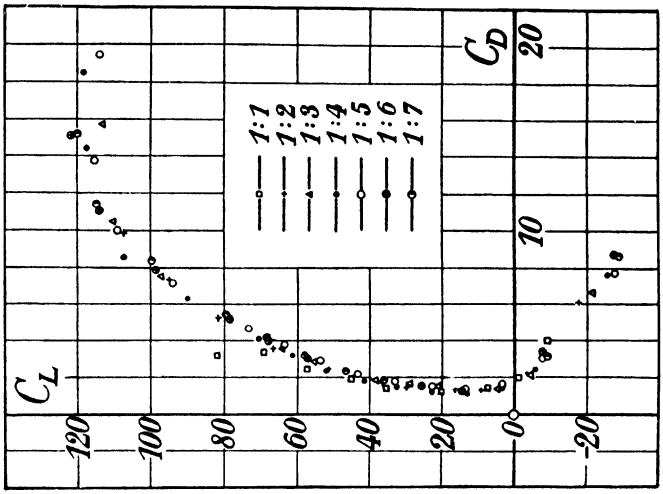


Fig. 3.95.—Recalculation of the experimental results shown in fig. 3.94 for an aspect ratio of 5

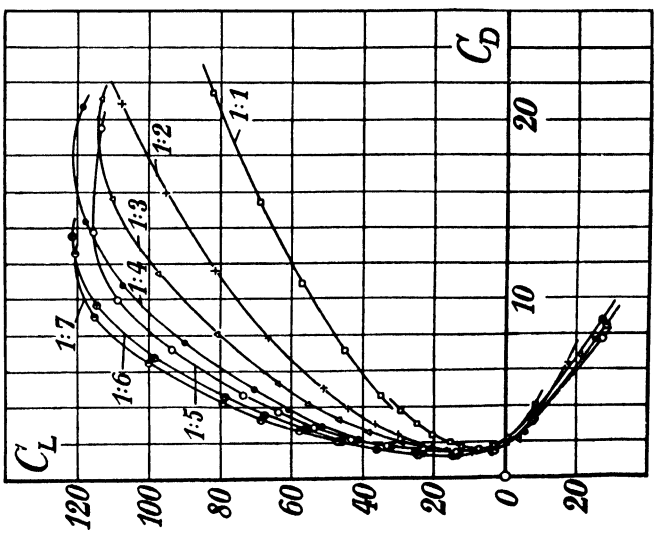


Fig. 3.94.—Lift/drag curves for seven rectangular aerofoils with various aspect ratios

The numerical values are the values of C_L and C_D multiplied by 100: the ratios represent chord of aerofoil/span of aerofoil, i.e. the reciprocal of the aspect ratio

in these extreme cases the discrepancies are not so very extensive. The degree of agreement with formula (74) may be illustrated by the following results of Göttingen experiments.* Fig. 3.94 shows the

* *Ergebnisse der Aerodyn. Vers.-Anst.*, Vol. 1, p. 50 (1921).

lift/drag curves for a set of rectangular aerofoils with the same profile and aspect ratios varying from 1 to 7. In fig. 3.95 all these results have been recalculated for an aspect ratio of 5 by means of formula (74), and we see that most of the points plotted are close to the experimental curve for aspect ratio 5. It is only in the case of a square aerofoil that the deviations are somewhat greater.

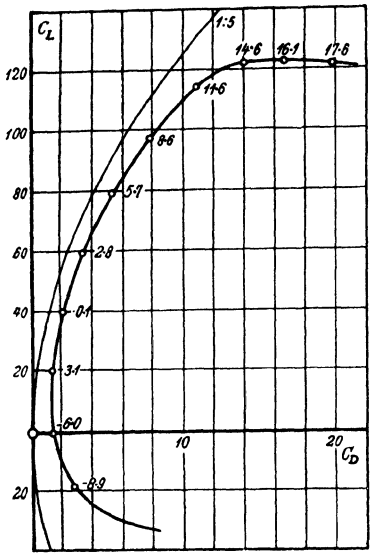


Fig. 3.96.—Parabola of the induced drag

Another useful application of the above relationships consists in drawing the "parabola" of the induced drag beside the lift/drag curves obtained by experiment; by equation (73), p. 216, the profile drag may then be read off from the horizontal intercept between the parabola and the experimental curve. Fig. 3.96 shows this parabola in conjunction with the lift/drag curve ("polar curve") of fig. 3.84, p. 201.

The recalculation of the angles of incidence for a given value of C_L is carried out in the same way as for the drag coefficient. Here the following considerations enable us to attain our object. To the angle of incidence of the infinitely long aerofoil, the flow being horizontal, there corresponds in the case of the finite aerofoil the angle α' between the chord and the direction of flow, the latter being inclined at an angle ϕ to the horizontal (see fig. 3.89, p. 207). We are thus led to expect that the relationship between C_L and the "effective angle of incidence" $\alpha' = \alpha - \phi$ will be the same for all values of the aspect ratio considered. By equation (64), p. 209,

$$\tan\phi = \frac{L}{\pi p_d l^2} = \frac{aC_L}{\pi l^2},$$

i.e. if α' is written as $f(C_L)$,

$$\alpha = \alpha' + \phi = f(C_L) + \frac{aC_L}{\pi l^2}. \quad . \quad . \quad . \quad . \quad (75)$$

If, again, we wish to calculate α_2, the angle of incidence of a second aerofoil (2), from experiments on an aerofoil (1) for the same value of C_L, we have

$$\alpha_2 = \alpha_1 + \frac{C_L}{\pi}\left(\frac{a_2}{l_2^2} - \frac{a_1}{l_1^2}\right). \quad . \quad . \quad . \quad . \quad (76)$$

This equation also agrees well with experiment. Fig. 3.97 shows how C_L varies with the angle of incidence in the experiments with seven different aerofoils mentioned above, while fig. 3.98 gives the angles of

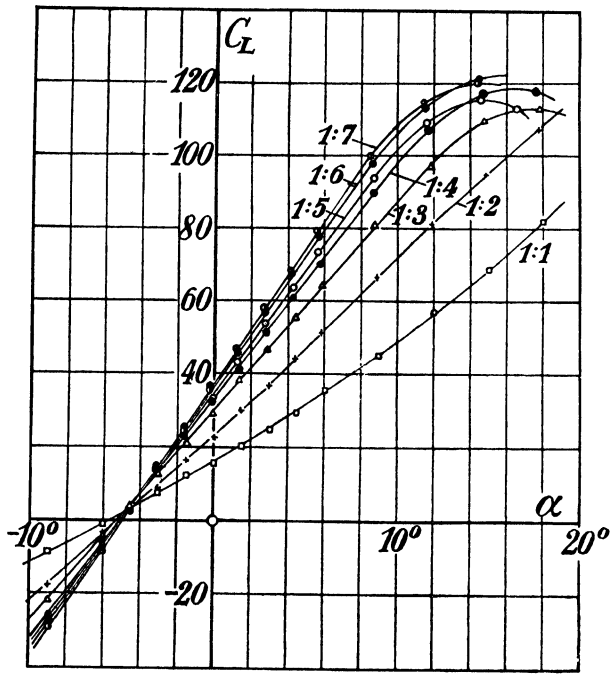


Fig. 3.97.—Relationship between the angle of incidence and the lift coefficient for rectangular aerofoils with various aspect ratios

incidence recalculated for aspect ratio 5. Here too the agreement within the limits previously referred to is very satisfactory.

The relation between α', the effective angle of incidence, and C_L, expressed above as $\alpha' = f(C_L)$, is given by

$$f(C_L) = \alpha - \frac{aC_L}{\pi l^2},$$

by equation (75); here α and C_L are taken from wind-tunnel experiments. It was pointed out by F. Weinig * that this formula is accurate only in the case where the wind-tunnel experiment is made on an elliptical aerofoil [$b = b_0\sqrt{\{1 - (y/\frac{1}{2}l)^2\}}$]; it was of course for this case, where the lift per unit area and the downward velocity $q(y)$ each have the same values at every point of the wing, that the basic

* *VDI-Zeitschr.*, Vol. 80, p. 299 (1936). I am indebted to F. Riegels for the information that H. Glauert had put forward ideas similar to those of Weinig as early as 1926 (see e.g. Glauert-Holl (1929), p. 131). J. Hueber (*Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, Vol. 24, p. 269 (1933)) made corresponding calculations for tapered wings.

formula was worked out. If, however, the measurements were made on a rectangular aerofoil, neither the lift per unit area nor the downward velocity is constant over the span; the distribution of lift is more nearly rectangular, and the downward velocity smaller in the middle, but greater at either side, than in the elliptical aerofoil. What we want to calculate is the behaviour of an infinitely

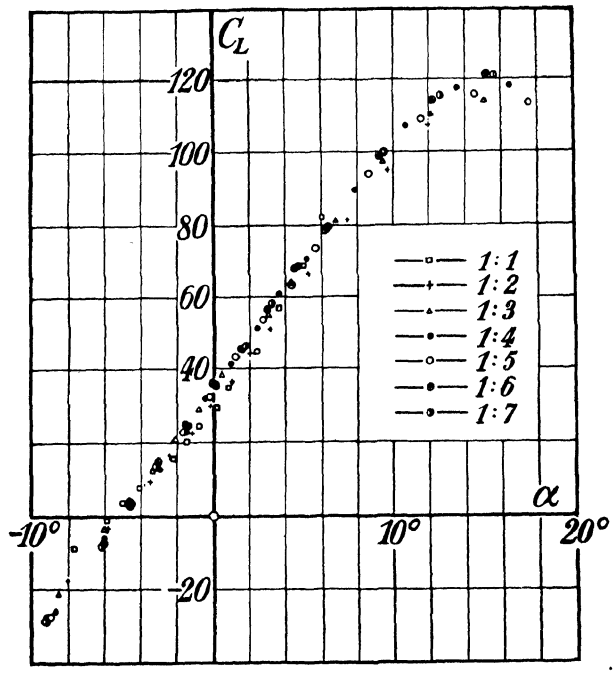


Fig. 3.98.—Recalculation of the experimental results shown in fig. 3.97 for an aspect ratio of 5

long aerofoil, or in other words, that of the aerofoil profile for two-dimensional flow. If we consider the middle of the experimental aerofoil, we must find both the local lift per unit area, possibly expressed in the local lift cofficient C_{L_0}, and also the local correction to the angle of incidence, q_0/v. For the aspect ratio of 5 frequently employed in experiments Weinig obtains $C_{L_0} = 1{\cdot}17 C_L$ and

$$\frac{q_0}{v} = 0{\cdot}855 \frac{C_L a}{\pi l^2}, \quad \text{i.e. for } a/l^2 = 1/5,\ \alpha_0 = \alpha - 0{\cdot}0545\, C_L.$$

Using values of α and C_L from the experiments, $\alpha_0(C_{L_0})$ then gives the required behaviour of the "infinitely long aerofoil".

Finding the profile drag C_{D_0} from equation (73) when C_L and C_D are found by experiment is a simpler matter. Here we have merely to take account of the fact that owing to the slight departure of the lift from the elliptical distribution, C_{Di} is about 4 per cent greater than the value $C_L{}^2 a/\pi l^2$ for the elliptical distribution. Hence for $l/b = 5$ we have

$$C_{D_0} = C_D - 1{\cdot}04 \frac{C_L{}^2}{5\pi} = C_D - 0{\cdot}6662\, C_L{}^2.$$

Very similar relationships may be obtained for a number of aerofoils combined in any way, as e.g. in the biplane. It is found that formulæ (73) and (74) (p. 216) may also be used for these combinations of aerofoils provided a/l^2 is replaced by $\kappa a/l^2$, where κ is a number depending only on the particular arrangement; in good combinations κ is generally less than unity. For biplanes with both wings of the same length

$$\kappa = \frac{0{\cdot}975 + 1{\cdot}44h/l}{1 + 3{\cdot}5h/l}$$

approximately, where l is the span and h the distance between the two wings. This formula can be applied between $l = 2h$ and $l = 15h$. Formulæ (75) and (76) may likewise be replaced by similar approximate formulæ, but with a number κ' instead of κ, κ' being greater than κ (owing to the effect of the curvature of the stream-lines, due to the other wing).*

Note.—The original papers by W. Birnbaum † and H. Wagner ‡ on the aerodynamics of flapping wings and of unsteady motion have been followed by an extensive literature. For a good comprehensive account, see H. G. Küssner, L. Schwarz, Th. Schade and H. Söhngen, *Lufo.*, Vol. 17, No. 11/12 (1940). The latest developments are discussed in detail by H. G. Küssner and H. Billing in the *F.I.A.T. Review of Aerodynamics*, Chap. VI; there is a shorter account in the section on *Mechanics*, also by H. G. Küssner, in the *F.I.A.T. Review of Applied Mathematics*.

19. **Propellers.**§ (30)

In contradistinction to aerofoils, which are meant to give rise to a force at right angles to the direction of motion, propellers are for producing a force in the direction of motion and thereby doing work against the medium. If the force is T ("propeller thrust") and the velocity of the ship or aircraft relative to the medium is v, the work usefully applied per second is $E_0 = Tv$. According to the momentum theorem the propeller works by continually setting fresh masses of fluid in motion. If the mass moved per second is M and the velocity imparted to it q, the propeller thrust T is equal to the momentum Mq. The setting of the masses of fluid in motion requires that kinetic energy should be transferred to them: this involves an additional expenditure of energy E' equal to $\frac{1}{2}Mq^2$ per second. If the propeller thrust, i.e. the product Mq, is given, E' is obviously smaller the smaller q is, i.e. the larger M is. Thus in order to make the work required $(E_0 + E')$ as

* See *Ergebnisse der Aerodyn. Versuchsanstalt*, Vol. 3, p. 9 (1924).

† *Diss.*, Göttingen; summary in *Zeitschr. f. angew. Math. u. Mech.*, Vol. 4, p. 177 (1924).

‡ *Diss.*, Charlottenburg; *Zeitschr. f. angew. Math. u. Mech.*, Vol. 5, p. 17 (1925).

§ For the most recent German references on propellers, see A. Betz, *F.I.A.T. Review of Aerodynamics*, Chapter I, 2.

small as possible it is important to act on as large a mass as possible and impart a small velocity to it. This is not the total amount of work required; another item E'' must be added, representing the work used up in friction and eddy formation. The efficiency η means the ratio of the work usefully applied to the total amount of work supplied, i.e.

$$\eta = \frac{E_0}{E_0 + E' + E''}. \quad . \quad . \quad . \quad . \quad . \quad (77)$$

From this "total efficiency" we distinguish the theoretical efficiency

$$\eta_{th} = \frac{E_0}{E_0 + E'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

and the "hydraulic efficiency"

$$\eta_h = \frac{E_0 + E'}{E_0 + E' + E''}. \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Obviously $\eta = \eta_{th} \cdot \eta_h$. As $E' = Mq^2/2 = Tq/2$ (see above), we may also write

$$\eta_{th} = \frac{Tv}{Tv + Tq/2} = \frac{v}{v + q/2} \quad . \quad . \quad . \quad . \quad (80)$$

for the theoretical efficiency. We see from this that η_{th} can always be made nearly equal to unity by making q small compared with v, which, by what we have said above, is attained by acting on a large mass of fluid, i.e. by making the cross-section of the stream of fluid affected by the propeller large. In actual cases, of course, practical considerations often require that this should be kept within fairly close limits: if the propeller is too large the hydraulic efficiency is reduced, so that the gain may in certain circumstances be cancelled. Further, the velocity of the tips must remain below the velocity of sound (cf. Chapter IV, § 8, at end, p. 295).

The most important kinds of propeller* are the *paddle wheel* and the *screw propeller*. The first is used in paddle steamers: the paddles on the wheel dip in succession into the water and impart a backward velocity to the water between the paddles. The edges of the paddles then give rise to surfaces of discontinuity (cf. § 6 of the last chapter, p. 50); the water between the paddles attains approximately the velocity of the paddles and retains this velocity when the paddles are first withdrawn (subsequently, of course, mixing with the rest of the

* Recently, jet propellers have come into use. In these an outflowing jet is produced by means of combustion, and the reaction of the jet (see Chapter II, § 12(*b*), p. 79) propels the aeroplane. The rocket propulsion engine, as well as the actual jet propeller, is of this type. In the former the oxygen necessary to maintain combustion must be carried in addition to the fuel; in the latter, atmospheric air, brought to a higher pressure by means of a compressor driven by a turbine, is used.

water takes place as described in § 5, p. 121). If the steamer is travelling with velocity v and the backward velocity imparted to the water is q (the *slip*), the peripheral velocity of the wheel is approximately $v + q$; if the force acting at the circumference is denoted by T (strictly speaking, T is merely the horizontal component of the pressures on the paddles, the directions of which in general deviate from the horizontal), the work supplied is $T(v + q)$, i.e. the efficiency η is $Tv/T(v + q) = v/(v + q)$. Comparison with the foregoing discussion shows that here work amounting to $E_0 + 2E'$ is used up, the second E' being due to the eddies caused by the paddles. Further losses occur as a result of all kinds of subsidiary effects. The efficiency may nevertheless be raised to quite a high figure by making q small, i.e. by making the effective paddle area large (e.g. by arranging the paddle wheels right across the stern of the ship, as is done in the Danube, Mississippi and Yukon steamers). From the point of view of engineering construction, however, these large wheels revolving at a slow rate are undesirable, so that the use of paddle wheels is confined to rivers and lakes, where screw propellers cannot be used owing to the small draught of the steamers.

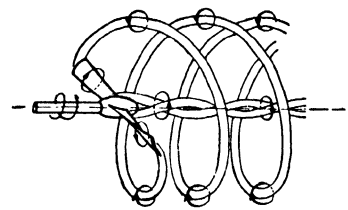


Fig. 3.99.—The vortex system of an airscrew *

Where sufficient depth is available screw propellers are superior to paddle wheels, owing to their smaller diameter and greater speed of revolution (involving lighter engines). For driving aircraft paddle wheels are not used.

The thrust of a screw propeller arises in the same way as the lifting force of an aerofoil, the only difference being that, owing to the rotation and simultaneous translation of the propeller, the blades advance in spirals instead of in straight lines. Here again vortices start from the ends of the blades (fig. 3.99): the vortices arising from either blade near the boss rotate in the same direction and combine to form a "boss vortex", while the outer vortices form a spiral enclosing the whole mass of air traversed by the propeller.† If we meanwhile assume

* [The term *airscrew* is often taken to include fans, windmills, &c., as well as air propellers.]

† Cf. H. Föttinger, *Verhandl. der Versammlung von Vertretern der Flugwissenschaft*, Göttingen, 1911 (Munich, 1912), p. 40; *Jahrb. d. Schiffbautechn. Ges.* (1918), p. 385. The system of vortices shown in fig. 3.99 was first recorded photographically in the water-propeller experiments of O. Flamm, *Jahrb. d. Schiffbautechn. Gesellsch.* (1908), Vol. 9, p. 427; see also his book, *Die Schiffsschraube und ihre Wirkung auf das Wasser* (Munich, 1909). This was done by leading air into the vortices; it collected at their cores and so made them visible. In air these vortices and others have been made visible in great detail by means of beautiful smoke experiments by J. Valensi (*Science Aérienne*, Vol. 4, p. 97 (1935); *Proc. 5th Internat. Congr. Appl. Mech.*, 1938, p. 520).

that the propeller has a large number of blades, the outer vortices form the boundary of an almost completely closed stream of fluid (the rapid succession of vortices is almost equivalent to a surface of discontinuity; cf. § 11(*c*) of the last chapter (p. 76)). As compared with the rest of the fluid this stream of fluid (*propeller race* or *slipstream*) is in rotational and translational motion. The translational motion is directly related to the task for which the propeller is intended, i.e. the production of a thrust, whereas the rotational motion is a necessary evil; in favourable cases the kinetic energy associated with it is comparatively small.

A completely satisfactory theory of the complicated phenomena of screw propellers is not yet available. In so far as the subsidiary velocities set up by the propeller may be regarded as very small compared with the rate of advance (propeller with a light load) we may use arguments analogous to those of aerofoil theory:* as a rule, however, this is not true for actual propellers. Here we shall content ourselves with a somewhat closer consideration of two methods of dealing with the problem, which to some extent give us a general idea of the main phenomena.

The first of these consists in the application of momentum and energy theorems to an idealized screw in which it is assumed only that every element of fluid which passes through the screw disc has its pressure raised. That a rise of pressure should result from a system of aerofoils is intelligible in view of the discussion in § 13(*a*) of the last chapter (p. 81). For greater simplicity we shall now assume that all the elements of fluid passing through the screw disc have their pressures raised by exactly the same amount. If the chord and angle of incidence of the blades are suitably chosen, this can be attained over the greater part of the screw disc but is impossible in the neighbourhood of the boss, where the velocity of points on the blades is insufficient, and at the tips of the blades, where the fluid may leave the blades. These deviations may, however, be taken account of by considering the area remaining when the non-effective regions are deducted from the actual area of the screw disc as the area of an ideal screw disc.

Further, we shall neglect the rotational motion actually imparted to the fluid by the screw.† Thus we now have an ideal propeller which raises the pressure of every element of fluid passing through a given circular area a by Δp, i.e. raises the constant of Bernoulli's equation, whereas the elements of fluid flowing along outside the range of the propeller of course do not experience this change. As the pressure is completely equalized at some little distance from the propeller, we

* See e.g. A. Betz, *Schraubenpropeller mit geringstem Energieverlust, Göttinger Nachr.*, 1919, p. 193 (= *Vier Abhandlungen zur Hydrodynamik und Aerodynamik*, p. 68).

† For a more refined form of this screw jet theory in which the rotational motion of the fluid in the slipstream is taken into account, see A. Betz, *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, Vol. 11, p. 105 (1920).

thus obtain a stream of fluid with an increased velocity behind the propeller. The stream-lines passing through the rim of the disc form the boundary of the slipstream (fig. 3.100). In detail, then, we have the following state of affairs.

In a system of co-ordinates with respect to which the centre of gravity of the screw is at rest, the fluid approaches the screw with a velocity v equal to the actual rate of advance of the screw. The velocity of the fluid moving away from the screw is then $v + q$, both velocities being measured at distances from the screw at which the pressure field of the screw may be regarded as equalized, i.e. at which the pressure has its undisturbed value p_0. Owing to the effects of the pressure field the velocity of passage through the screw differs from v: it lies between v and $v + q$ (see fig. 3.100). We shall further idealize the screw by neglecting its length in the direction of flow. Then, by continuity, the velocity immediately behind the screw is identical with that immediately in front of the screw; let this velocity be denoted by v'. The difference of pressure Δp now arises from the fact that p', the pressure immediately in front of the plane of the screw, is less than the undisturbed pressure, while p'', the pressure immediately behind the screw, is greater than the undisturbed pressure. Applying Bernoulli's theorem to the conditions in front of and behind the screw, we have

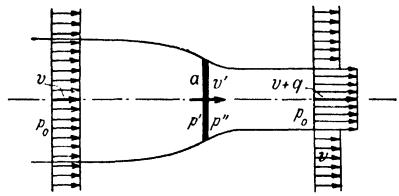


Fig. 3.100

$$p_0 + \tfrac{1}{2}\rho v^2 = p' + \tfrac{1}{2}\rho v'^2,$$

$$p_0 + \tfrac{1}{2}\rho(v+q)^2 = p'' + \tfrac{1}{2}\rho v'^2.$$

By subtraction we get for the rise of pressure ($\Delta p = p'' - p'$)

$$\Delta p = \tfrac{1}{2}\rho(v+q)^2 - \tfrac{1}{2}\rho v^2 = \rho q(v + \tfrac{1}{2}q). \quad . \quad . \quad (81)$$

We now apply the momentum theorem in two different ways:

(1) to a control surface enclosing the plane of the screw disc as closely as possible. In the limit the velocity is the same behind and in front of the plane of the screw and is equal to v', i.e. contributes nothing to the momentum, but the rise of pressure Δp causes a pressure equal to T, the thrust of the screw, on the area a;

$$T = a\Delta p = \rho a q(v + \tfrac{1}{2}q); \quad . \quad . \quad . \quad . \quad (82)$$

(2) to a control surface bounded by the stream-lines passing through the rim of the disc, an area a long way in front of the screw, and an

area a long way behind the screw. Here the quantity of fluid crossing the control surface is $\rho av'$; previously it has the velocity v, subsequently the velocity $v + q$. As closer investigation shows, the pressure integral contributes nothing,* so that the thrust is also given by

$$T = \rho av'q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Comparison of (82) and (83) yields the remarkable relationship

$$v' = v + \tfrac{1}{2}q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Using either of the above control surfaces we can calculate the amount of work required by our ideal propeller. The first gives

$$E = av'.\Delta p = Tv' \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

(volume moved per unit time $\times$ difference of pressure); the second gives the mass moved per unit time multiplied by half the difference of the squares of the velocities, namely

$$E = \rho av'[\tfrac{1}{2}(v + q)^2 - \tfrac{1}{2}v^2],$$

which is also equal to Tv'. As before, the work usefully applied is $E_0 = Tv$, so that the efficiency is given by

$$\eta = \frac{Tv}{Tv'} = \frac{v}{v'} = \frac{v}{v + q/2}, \quad . \quad . \quad . \quad . \quad (86)$$

which agrees with the expression for η_{th} in equation (80). The ideal propeller is therefore an ideal corresponding to any type of propeller, and accordingly forms a suitable standard of comparison for actual propellers.

Another important relationship is obtained by finding q from equation (82) and substituting this value in the formula for the efficiency (80). We immediately obtain

$$q = -v + \sqrt{\left(v^2 + \frac{2T}{\rho a}\right)}, \quad . \quad . \quad . \quad . \quad (87)$$

which if we introduce the thrust coefficient †

$$T_c = \frac{T}{\frac{4}{\pi}\rho av^2}$$

* The so-called half-body theorem; cf. e.g. Prandtl-Tietjens, Vol. 2, p. 135, or *Aerodynamic Theory*, Vol. 3, p. 192 *et seq.*

† See footnote, p. 232.

may also be written

$$\frac{q}{v} = -1 + \sqrt{\left(1 + \frac{T_c}{\pi}\right)} \quad . \quad . \quad . \quad . \quad (87a)$$

(the other sign of the square root having no physical meaning). This gives

$$\eta_{th} = \frac{2}{1 + \sqrt{(1 + 8T_c/\pi)}}. \quad . \quad . \quad . \quad . \quad (88)$$

We see that as the thrust coefficient tends to zero, η_{th} tends to unity; with increasing values of T_c, on the other hand, η_{th} tends to zero. Formula (88) gives an upper limit for the efficiency of any screw which is to produce a given thrust T and has a given area a and a given rate of advance v. The formula also provides a very welcome basis of comparison for experiments. The ratio of the measured efficiency η to η_{th} is the hydraulic efficiency mentioned above.

The above discussion obviously does not apply to propellers which have to sustain a load in the air (as in the helicopter). Here indeed energy must continually be used up, but if no rise takes place no effective work is done. Thus we can only inquire as to the amount of work theoretically required to sustain the same weight and compare it with the work actually required.

We may apply our system of formulæ to the case where $v = 0$. Then $v' = q/2$, and

$$E_{th} = \tfrac{1}{2}Tq$$

by equation (85), where T here means the weight sustained. Equation (82),

$$T = \tfrac{1}{2}\rho a q^2,$$

again enables us to eliminate q. We have $q = \sqrt{(2T/\rho a)}$, and hence

$$E_{th} = \sqrt{\frac{T^3}{2\rho a}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

from which we can again form * the hydraulic efficiency E_{th}/E.

Thus if the hydraulic efficiency remains the same, the work which must be done by the propeller is actually smaller, the greater a is. On the other hand, in actual cases the weight of the propeller itself and hence T, the weight to be sustained, increases with increase in size.

* Equation (89) can also be applied to the thrust of an airscrew in a test stand, and when solved for T gives a convenient approximate formula for the static thrust, if the hydraulic efficiency is known.

If a propeller is moving horizontally with a velocity v which is large compared with q, the vertical velocity of flow through its plane, the whole flow becomes similar to that round a plane circular disc inclined at an angle $\alpha = \tan^{-1}q/v$ and acting as an aerofoil. Here q is no longer found from the formula just given but from formula (63), p. 209: E_{th} is approximately equal to $D_i v$, where D_i is given by equation (65), p. 209. To maintain the motion the propeller shaft must be tilted forward through the angle $\beta = \tan^{-1}D/T$, where D is the total drag, unless it is driven in some other way.

For the autogyro and some other details of the helicopter see note 2, p. 235.

20. Further Discussion of Screw Propellers; Windmills; Other Types of Propeller.

(*a*) To give the reader greater insight into the way in which a propeller works, we shall now study the behaviour of a single propeller blade more closely.

We shall treat a section of a propeller blade lying between the radii r and $r + dr$ as a part of an aerofoil, or more accurately as a part of an infinitely long aerofoil, except that in our calculations we take as the velocity the velocity of the blade element relative to the fluid at the point where the blade element happens to be. Now this velocity is affected by the propeller as a whole, so that the discussion on which we are entering is based on the same principles as that of aerofoil theory. For the sake of greater simplicity we shall again neglect the rotation of the fluid in the screw. The effect of the screw on the axial motion may be taken account of by equating the velocity of this motion to the value v' given by equation (84) of the last section (p. 226), where in virtue of equation (87*a*) for q we may put

$$v' = \tfrac{1}{2}v\{1 + \sqrt{(1 + 8T_c/\pi)}\}.$$

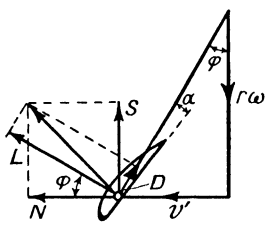


Fig. 3.101

The peripheral velocity of the blade is $r\omega$ (where ω is the angular velocity). Thus the velocity of the blade relative to the fluid is $\sqrt{(v'^2 + r^2\omega^2)}$, the direction being given by $\tan\phi = v'/r\omega$ (see fig. 3.101). If the blade profile is inclined at a small angle α to this direction, there results a lift L corresponding to the dynamic pressure due to the relative velocity and to this angle of incidence, in a direction normal to the relative velocity, and a drag D in the direction opposite to the relative velocity, where L and D are taken with reference to unit length of the propeller blade (i.e. for the element dr we should have $L\,dr$ and $D\,dr$).

For further calculation we shall find it convenient to form the components normal to the plane of the screw (N) and tangential to the line of motion (S); i.e. the former act in the direction of v

(fig. 3.101), and the latter in the direction opposite to $r\omega$. It is easy to see that

$$\left.\begin{aligned} N &= L\cos\phi - D\sin\phi \\ S &= L\sin\phi + D\cos\phi \end{aligned}\right\}. \qquad (90)$$

Integration of the normal components gives the thrust, that of the moments of the tangential components the torque Q, i.e. if there are n blades

$$T = n\int_0^r N dr; \quad Q = n\int_0^r S r dr. \qquad (91)$$

We could instead have integrated the work done by the element, $Ndr\,.\,v'$ and $Sdr\,.\,r\omega$, obtaining Tv', the thrust exerted against the velocity v', and $Q\omega$, the work to be supplied by the engine. We shall also mention another simple idea, that of the efficiency of a single blade element:

$$\eta_b = \frac{Ndr\,.\,v'}{Sdr\,.\,r\omega}. \qquad (92)$$

With the above values of N and S, bearing in mind the fact that $v' = r\omega\tan\phi$, and inserting the lift/drag ratio $L/D = 1/\epsilon$, we have

$$\eta_b = \frac{1 - \epsilon\tan\phi}{1 + \epsilon\cot\phi}. \qquad (93)$$

This expression has a maximum for an angle of pitch ϕ equal (approximately) to $\pi/4 - \epsilon/2$. As, however, the ratio ϵ is very small ($\epsilon = 1/20$ to $1/50$), the quantity varies so slightly in the neighbourhood of the maximum that from the point of view of blade efficiency angles of pitch very different from the value corresponding to the maximum are almost as useful. It is only for very small pitch that the blade efficiency becomes too small. In practice the decisive factor is that large values of ϕ mean that the rotational velocity imparted to the fluid is large, which is disadvantageous, owing to the loss of kinetic energy.* For this reason the outer parts of the blade, which, owing to their large velocities, are most important in this connexion, are inclined at an angle considerably smaller than the above optimum value, e.g. $\tan\phi = 1/3$ to $1/5$. In the propellers of fast aeroplanes, however, the attempt to keep the relative velocity of the blades below

* The investigation of these relationships may be based on the theory of systems of blades which was developed (though for frictionless fluid) in § 13(a) of the last chapter (p. 81). The velocity denoted there by u_1 is balanced by the equally great tangential velocity $r\omega$ in the system of co-ordinates used here; for the tangential velocity u of the fluid behind the propeller we have to introduce the quantity $u_1 - u_2$ from the former argument. There the effective relative tangential velocity was $\frac{1}{2}(u_1 + u_2)$, i.e. in this case $r\omega - \frac{1}{2}u$. We thus obtain a more accurate form of the argument based on fig. 3.101 by substituting this expression for $r\omega$.

that of sound leads to the use of much larger angles of pitch ($\tan\phi$ may reach nearly 1). To obtain satisfactory thrusts for starting and also good climbing efficiency, the setting of the blades is made adjustable (variable-pitch propellers).* (31)

The connexion between L, the lift on a blade element, and the corresponding dynamic pressure now enables us to find the breadth which the blade should have at various distances from the centre, and the corresponding angle of incidence. For rapidly rotating marine propellers, however, it is also necessary to take account of the fact that if the absolute pressure in the water approaches zero the flow is interrupted by the separation of air from the water and the formation of water vapour. This phenomenon is known as *cavitation* (see Chapter V, § 1, p. 312). For high-speed marine propellers it is therefore necessary to

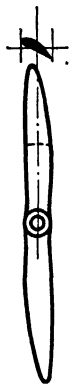


Fig. 3.102.—Air propeller

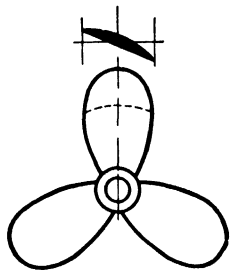


Fig. 3.103.—Low-speed marine propeller

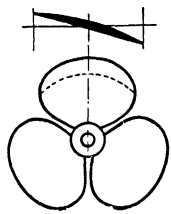


Fig. 3.104.—High-speed marine propeller

use very flat profiles and very small angles of incidence, in order to keep the reduction of pressure on the suction side of the profile small, so that the blades must be made very broad. In figs. 3.102–104 the front view and a section of an air propeller, a low-speed marine propeller for a cargo steamer, and a high-speed marine propeller for a turbine passenger steamer are given in juxtaposition. Very flat profiles should be used for air propellers if the velocity of the tips reaches the velocity of sound (see § 8 of the next chapter, p. 293). These, however, are usually very narrow owing to the high dynamic pressure. Occasionally propellers with tip speeds exceeding the speed of sound are used. These, however, give rise to a very loud noise, which spreads mainly in a direction at right angles to the direction of flight; the sound waves arise as in fig. 4.47, p. 301.

* For the aerodynamical properties of variable-pitch propellers, see e.g. the report of experiments by B. Dirksen in Vol. 16 of *Abhandl. aus. d. Aerodyn. Inst. d. T. H. Aachen.*

A more accurate theory of the screw propeller can be developed on the lines of aerofoil theory, which more closely represents the actual behaviour of propellers and is now very frequently used in the design of aeroplane propellers. It is based on theorems by A. Betz.* A stricter calculation of the distribution of the circulation along the radius of the propeller blade was made by S. Goldstein.†[32] Fig. 3.105 shows the distribution of circulation for a propeller with many blades; it is proportional to the distribution of thrust per unit area over the various zones of the screw disc. The distribution of thrust on the individual blades is found from this by multiplying by $2\pi r/n$, where n is the number of blades. As an example of the working out of the theory in a particularly easy case (blades narrow and the coefficient λ (see below) small), and its confirmation by experiments, we may mention a paper by F. Walchner.‡

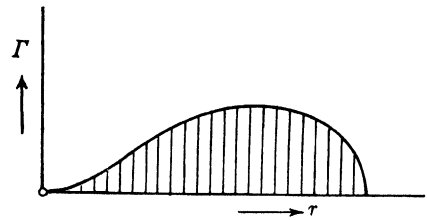


Fig. 3.105.—Distribution of circulation along the radius of a propeller

The actual behaviour of marine and air propellers is usually determined by experiment. Here we shall discuss in detail only experiments in which the propeller is not tested in conjunction with the ship or aircraft, but is running by itself in undisturbed fluid. Otherwise there arise disturbances due to the ship or aircraft, and mutual actions between the vessel and the propeller, making the whole matter very complicated.§[33] A basis of comparison for experimental results is obviously furnished by the ratio of the velocity of advance to the peripheral velocity. We write $v/u = v/r\omega = \lambda$ (v being the velocity of advance, u the peripheral velocity, r the radius of the propeller disc, and ω the angular velocity).|| For a screw propeller consisting of helicoid surfaces of pitch h, λ obviously has a special value λ_1, for which the screw moves through the fluid in such a way as to produce frictional resistance only. Obviously $\lambda_1 = h/2\pi r$. It is only for values of λ less than λ_1 that the angle of incidence between the blades and the flow relative to them is such as to give rise to a positive thrust. If λ were greater than λ_1 we should have a windmill and not a propeller. Similar relationships hold for the forms of propeller in actual use, except that λ_1 no longer has a simple geometrical meaning.

The forces measurable by experiment are T, the thrust, and Q, the torque exerted by the motor driving the screw. The work supplied

* "Schraubenpropeller mit geringstem Energieverlust", mit einem Zusatz von L. Prandtl, *Gött. Nachr.* (1919), p. 193, or *Vier Abhandl. z. Hydrodyn. u. Aerodyn.*, p. 68.

† *Proc. Roy. Soc.* (A), Vol. 123, p. 440 (1929).

‡ *Luftf.-Forsch.*, Vol. 13, p. 103 (1936).

§ See e.g. A. Betz, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 7, p. 431 (1927).

|| [In British textbooks the ratio v/nD (where D is the diameter of the propeller and n the number of revolutions per second) is generally used: $v/nD = \pi\lambda$.]

is $E = Q\omega$, the work utilized $E_0 = Tv$ as before, so that the efficiency is given by

$$\eta = \frac{Tv}{Q\omega}. \qquad \ldots\ldots\ldots \quad (94)$$

In order to exhibit the results of experiment in a form independent of the particular dimensions of the screw experimented on we use dimensionless quantities (*thrust coefficient*, *torque coefficient*) formed in analogy to those for the aerofoil. In British textbooks these are generally defined as follows:*

$$k_t = \frac{T}{\rho n^2 D^4}, \qquad \ldots\ldots\ldots \quad (95a)$$

$$k_q = \frac{Q}{\rho n^2 D^5}, \qquad \ldots\ldots\ldots \quad (95b)$$

where D is the diameter of the propeller and n the number of revolutions per second. As $v = \lambda u$, it readily follows from equation (94) that the efficiency is

$$\eta = \tfrac{1}{2}\lambda \frac{k_t}{k_q} = \frac{1}{2\pi}\left(\frac{v}{nD}\right)\frac{k_t}{k_q}.$$

Fig. 3.106 shows the behaviour of k_t, k_q, and η for an airscrew like that in fig. 3.102. We see that k_t, k_q pass through zero in the neighbourhood of $\lambda = 0{\cdot}35$ ($v/nD = 1{\cdot}1$ approximately). In view of its meaning η must be zero for $\lambda = 0$ and for $k_t = 0$, so that it has a maximum somewhere in between. In order to compare it with η_{th} (given by formula (88), p. 227), we may first find the thrust coefficient T_c, which, as is easily shown, is equal to $k_t/(v/nD)^2$. We can then calculate η_{th} for various points and plot it against corresponding values of λ. The result is shown by the dotted line in fig. 3.106. We see that the values of η approach those of η_{th} fairly closely, except in the region of λ_1 where η_{th} tends towards 1 and η towards 0.

(*b*) Similar considerations apply in the case of windmills, which we shall briefly touch on here.† [34] When the wind impinges on the disc of blades and passes through them it suffers a *fall* of pressure Δp, which supplies work to the amount $\pi r^2 v' \Delta p$ (v' again being the axial velocity of passage through the plane of the wheel), which is transformed into useful work at the shaft of the wheel according to the degree of efficiency of the windmill. The formulæ of last section may at once

* [$T_c = T/\rho v^2 D^2$ and $Q_c = Q/\rho v^2 D^3$ are sometimes used.]

† See e.g. A. Betz, *Windenergie und ihre Ausnützung durch Windmühlen* (Göttingen, 1926); *Naturwiss.*, Vol. 15, p. 905 (1927); also U. Noetzlin, *Die strömungstechn. Grundlagen der Windkraftmaschinen*, Denkschrift 2 der Reichsarbeitsgemeinschaft "Windkraft" (Berlin-Steglitz, 1941).

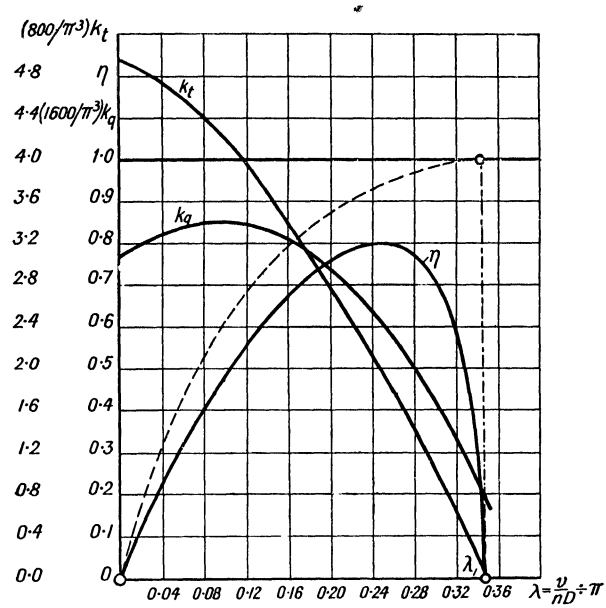


Fig. 3.106.—Performance curves for an airscrew: values of $(800/\pi^3)k_t$ in left-hand column, values of $(1600/\pi^3)k_q$ in right-hand column

be applied, except that here the velocity behind the wheel is less than in front. We must therefore introduce a negative value for the difference of velocity q. If we write $-q$ for q in order to keep to a positive quantity, we have

$$\Delta p = \rho q(v - \tfrac{1}{2}q)$$

by equation (81), p. 225, and

$$v' = v - \tfrac{1}{2}q$$

by equation (84), so that the available work is

$$E_1 = \pi r^2 . v' \Delta p$$
$$= \pi r^2 \rho q (v - \tfrac{1}{2}q)^2.$$

It may easily be shown that this is a maximum for given values of v and r when $v - q$, the velocity behind the wheel, is taken equal to $v/3$, i.e. $v' = 2v/3$. Then * $(E_1)_{\max} = 8\pi r^2 \rho v^3/27$. With this we may compare the energy E_0 which would be transferred across the

* A. Betz, *Zeitschr. f. d. gesamte Turbinenwesen*, Vol. 17, p. 307 (1920).

wheel if the latter offered no resistance and hence find the " efficiency " $c_l = E/E_0$. As the mass passing through the wheel per second is $\rho\pi r^2 v$ and the kinetic energy per unit mass is $\frac{1}{2}v^2$, $E_0 = \frac{1}{2}\rho\pi r^2 v^3$, so that the maximum theoretical " efficiency " $(c_l)_{\max}$ is $(E_1)_{\max}/E_0 = 16/27 = 0{\cdot}5926$.

In ordinary windmills the idea of efficiency has no meaning, as the energy of the wind is available to an unlimited amount. Their commercial efficiency can only be judged from the ratio of the work supplied to the cost of construction, so that in certain circumstances a cheaply constructed windmill may prove more satisfactory than a more expensive one which is superior from the aerodynamic point of view. The same is not true for windmills attached to aircraft for driving auxiliary machinery by the wind due to the motion. Here T, the thrust of the windmill, must be met by the main propeller of the aircraft. Thus work Tv is supplied and the efficiency is given by $\eta = E/Tv$.

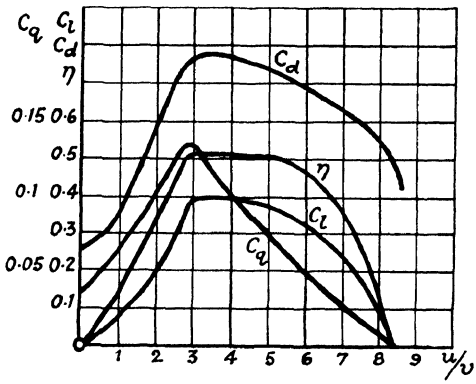


Fig. 3.107.—Performance curves for a windmill

$$c_d = \frac{\text{Drag}}{\frac{1}{2}\rho v^2 \pi r^2}, \; c_q = \frac{\text{Torque}}{\frac{1}{2}\rho v^2 \pi r^3} \left(= \frac{16}{\pi} k'_q\right)$$

In experiments on model windmills retarding torques of varying strength may be applied for a given strength of wind, the peripheral velocity decreasing as the torque increases. As in the case of screws, we may form torque coefficients and plot these against the ratio u/v (peripheral velocity/wind velocity). The " efficiency " may also be plotted. Windmills exhibit different " speeds " according to the pitch of their blades. Curves for a high-speed windmill * are given in fig. 3.107.

Notes.—1. By equation (82), p. 225, the thrust of the wind on the shaft of the windmill is

$$T = a\Delta p = \rho a q(v - \tfrac{1}{2}q),$$

where a is the area of the disc. Thus the windmill is a very simple case of resistance to flow (at least so long as we are content with the simple jet theory). The connexion with the relation between the resistance and the wake, described in § 14(*c*), p. 184, can be brought out in the following way. If a_s is the cross-section of the jet at such a distance from the windmill that the pressure field vanishes and the velocity in the wake is equal to q, the strength of the wake, Q, of § 14(*c*),

* For further examples see e.g. the papers by Seiferth in *Ergebn. d. AVA*, part 3, p. 139 and part 4, p. 118.

p. 184 (the intensity of the flow due to displacement of the fluid near the windmill), is obviously equal to $a_s q$. By continuity, we have

$$a(v - \tfrac{1}{2}q) = a_s(v - q).$$

Hence
$$T = D = \rho a q(v - \tfrac{1}{2}q)$$
$$= \rho a_s q(v - q) = \rho Q(v - q)$$

is a more exact expression for D than ρQv given in § 14(*c*). There the simpler formula was justifiably used, since we were considering distances from the body exposed to the flow at which q could be neglected in comparison with v.

2. The autogyro (in which the rotors revolve freely as compared with the helicopter, in which they are power-driven) exhibits a certain resemblance to the windmill, in so far as the work necessary to keep the propeller running is supplied by the air passing through the propeller with a reduction of pressure. As its lifting capacity is completely developed only when it is moving forward, the behaviour of the flow is similar in other respects to that of a helicopter moving forward (cf. the small type on p. 228), except that the propeller shaft is tilted backwards instead of forwards. As in the ordinary aeroplane, the resistance to motion is overcome by an ordinary propeller.

In both the autogyro and the helicopter the torques about the horizontal axes are balanced by having the individual blades fixed near the hub in such a way that the advancing blades, which are working under a higher dynamic pressure than the retiring, can move upwards and so have a smaller relative angle of incidence than the latter, which move downwards. Hence the plane in which the tips of the blades move is inclined further back than the shaft is.

(*c*) Of other propellers the *Voith-Schneider propeller* is the most notable.[35] It is being used more and more in ships where specially high manœuvrability is called for (river and lake steamers, speed-boats, &c.). In the Voith-Schneider propeller a number of blades are carried round in a horizontal circle, their angle of incidence being altered from zero to positive values, and then back through zero to negative values, in the course of each revolution. The direction of the position in which the angle of incidence is zero, and the value of this angle, can be altered at will, so that a propeller thrust of any strength and in any direction can be produced without altering the number of revolutions per minute (hence the Voith-Schneider propeller is particularly well suited for use with internal-combustion engines). With two propellers the ship can even be made to rotate on its own axis or be moved to one side or the other while remaining parallel to its original position!

Fig. 3.108 is a diagrammatic representation of the Voith-Schneider propeller. As the plan shows, the normals to the individual blades are directed towards a fixed central point M throughout the revolution. The point M can be made to move anywhere in the neighbourhood of the point A on the shaft by remote control from the bridge of the ship. The thrust produced is approximately proportional to the length AM and at right angles to it. For reasons of space the adjustment of the propeller blades is in practice carried out by a more complicated

mechanism. The elevation in fig. 3.108 gives an idea of the actual construction.

(*d*) It is not without interest to inquire how Nature has solved the problem of forward drive in living creatures in air and water. They have available only the to-and-fro motion which is also used by man in primitive means of water transport (the rowing-boat). The state of affairs resembles that in a paddle wheel or a screw propeller according as the motion of the driving organ is parallel to the direction of advance or perpendicular to it. The ordinary flight of birds * represents a special case, inasmuch as the same organ provides both the lift and the forward drive. The forward drive arises from the fact that the downward stroke of the wings is made with greater force, but the upward more passively, with the minimum drag. The greater part of the forward drive comes from the outer parts of the wings, which move through the greatest vertical distance. The forces acting are shown in fig. 3.109. In favourable circumstances the efficiency of this mechanism is very high. Part of the drag consists of the induced drag of § 17 (p. 207); there is also an additional drag due to the vortices with axes at right angles to the direction of flight which are produced by the strokes of the wings. These are not very intense if the beat of the wings is slow and steady. The type of flight † in which smaller birds rise almost vertically, or even hover in one place, is comparable to the action of a helicopter. In it the " down beat " is associated with a powerful forward movement of the bird, the " up beat " with a corresponding backward movement, in which the wings are made to approach the body as closely as possible. The helicopter principle is manifested in an even more pronounced form in the hovering flight ‡ of humming birds and many insects; in this the back stroke is also used to produce a lift,

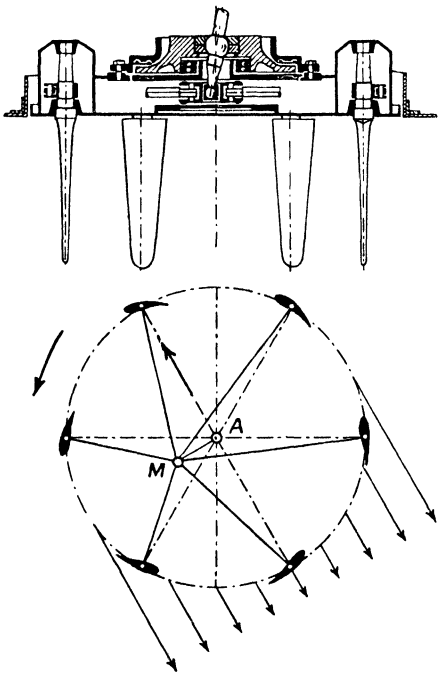


Fig. 3.108.—Voith-Schneider propeller

* [Ger. *Ruderflug.*] † [Ger. *Rüttelflug.*] ‡ [Ger. *Schwirrflug.*]

the wing being turned over about its longitudinal axis (see fig. 3.110). This enables the bird or insect to hover where it will and also fly forwards or backwards as it will, or even turn round in the same spot, like the helicopter.*

In aquatic animals, in addition to a driving apparatus resembling that of the bird's wing (e.g. in penguins and in certain turtles), we

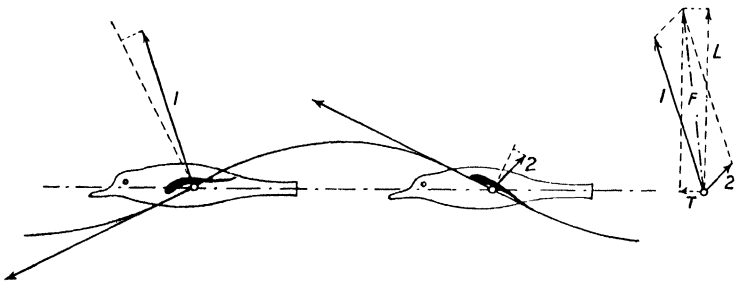


Fig. 3.109.—Diagram of a bird in ordinary flight

above all find the " fish-tail propeller ", which depends for its working on the flexibility of the thin surface of the tail. If the tail is moved to and fro by the moving of the body of the fish from side to side, the tail always sets itself obliquely in such a way that it gives a thrust. The bending of the body itself in a wriggling motion, as in the eel and the ray, is another means of producing a thrust by means of pressures on oblique surfaces.

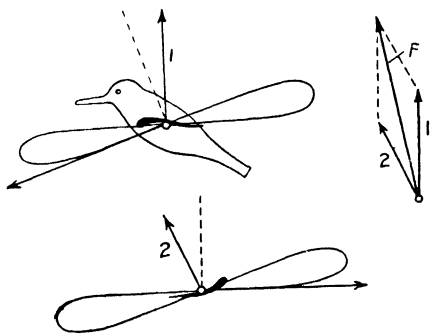


Fig. 3.110.—Diagram of a hovering bird

A very curious means of providing a thrust is found in the cuttle-fish. It fills the hollow of its sack-like body with water and ejects this forcibly to the rear by contracting the sack. Considerable velocity is

* Experiments to measure the forces on wings of this type have been made by E. von Holst (see *Luftwissen*, Vol. 10, p. 146 (1943)).

thereby produced by the principle of the reaction of a jet.* The "jet propeller" is sometimes used in ships, water being sucked in at the bows or at the side of the ship by means of a pump and ejected at the stern. If a large quantity of water is ejected at a low velocity (see § 19, first paragraph, p. 221) the efficiency is quite good, but the same effect is more easily achieved with the screw propeller.†

Finally, we may mention the motion of micro-organisms by means of lashes, or, as the zoologists call them, *cilia*. Here Reynolds' number is extremely small compared with unity, so that all inertia effects disappear and the phenomena are not in any way comparable with those just described, which depend on inertia. A much more accurate idea

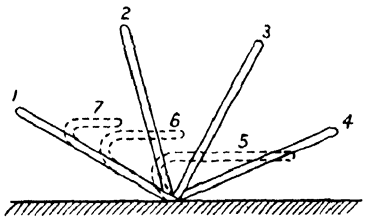


Fig. 3.111

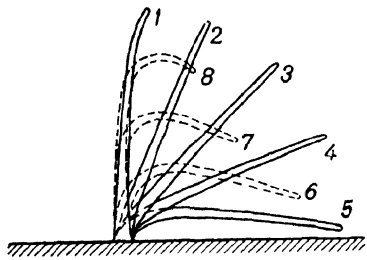


Fig. 3.112

Motion of the cilia of infusoria, after Gray

of them is got by thinking of very slow motions in a very viscous fluid. Film microscopy ‡ has shown that the cilia move forward with a peculiar sharp bend near the body of the creature and backward in a more extended position (cf. fig. 3.111). In this way rather more of the outer layer of the surrounding fluid is pushed backwards than is carried forwards on the forward stroke. This peculiar motion may be explained by assuming that the cilia have the capacity to become stiff and then relax. Another form of ciliary motion is shown in fig. 3.112.

21. **Water Turbines, Pumps, and Compressors.**

Water wheels and water turbines transform work supplied by water falling from a height (H) into work done by a rotating shaft:

* The motion of a jellyfish depends on a similar principle (ejection of a vortex ring).

† Mention may be made here of Kort's ducted fan, in which a screw turns in the narrowest part of a widening nozzle. We thus obtain a thrust equal to that of a screw propeller of diameter equal to that of the mouth of the nozzle, but with an engine running considerably faster and a better angle of pitch for the blade (which is important for tugs, particularly for use in canals; an additional advantage here is that the bed of the canal is not damaged by being stirred up by eddies).

‡ J. Gray, *Ciliary Movement* (Cambridge, 1928).

pumps use work supplied by a rotating shaft to raise water to a height, while compressors or blowers supply energy to moving air. Where the pressure differences are small, compressors are called fans.

If the quantity of water moving per second (volume/sec.) is Q and its weight per unit volume w, the work supplied to the wheel or turbine is $E_0 = wQH$.

The distinguishing feature of water wheels is that the water between the blades is practically brought to rest, whereas in turbines it remains in motion relative to the blades. In water wheels (at least in the better types) the effect is chiefly due to the weight of the water but partly also to the impulse given by the water entering the buckets. Here we shall not discuss water wheels in more detail, as they must now be regarded as quite out of date and, moreover, are not of very great interest from the hydrodynamical point of view.

Turbines work on an entirely different principle; they transform the kinetic energy of water flowing out under a difference of pressure into useful work. The rapidly flowing water is led in without shock between the blades of the turbine runner by means of guide vanes, and has its course deviated by the blades in such a way that it leaves the wheel with as small an absolute velocity as possible. As large losses do not occur at any point in this process, it follows from the principle of energy that by far the greater part of what is left after the kinetic energy of the emerging water has been deducted from the total head supplied appears at the rotating shaft of the turbine.

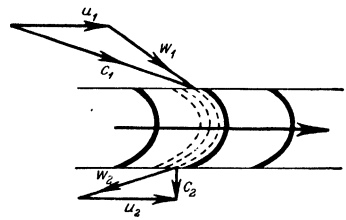


Fig. 3.113.—Velocity diagram of an impulse turbine

As regards the method by which the transference of energy is carried out we distinguish two types of turbines, " impulse turbines " and " reaction turbines ". In the former the whole head available is transformed into kinetic energy before it reaches the wheel, by letting the water flow freely through suitable openings or pipes. The water, which impinges with an absolute velocity * c_1 on the wheel moving with peripheral velocity u_1, has a velocity w_1 relative to the wheel (fig. 3.113). The inflow edges of the blades are practically in the direction of w_1. The water is then deviated by the blades into the direction w_2, the relative velocity, apart from moderate losses owing to friction, remaining the same. The absolute velocity c_2 is obtained from w_2 and u_2 by the triangle of velocities. Under favourable circumstances it is nearly at right angles to the system of blades and very small. In

* We use the notation c for absolute velocities, u for peripheral velocities and w for relative velocities.

the case of reaction turbines only a moderate amount of the head is transformed into velocity at the outset, so that the water enters the wheel with a considerable excess pressure. The details are shown in fig. 3.114. The water then has its relative velocity considerably raised in the narrowing passages between the blades, the rest of the head being thereby used up. In favourable circumstances the absolute velocity of the water after passing through the wheel is again very small and nearly at right angles to the system of blades.

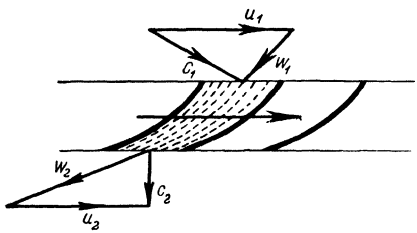


Fig. 3.114.—Velocity diagram of a reaction turbine

Reaction turbines necessarily work with the passages between the blades full of water. In impulse turbines, on the other hand, it is usual to make the blades of such a form that the water retains a free air surface during its passage. As the water then comes into contact with the blades on one side only, the friction is less than when both sides of the blades are wetted.

If we neglect friction and centrifugal effects in the wheel, we have

$$H = \frac{1}{2g}(c_1^2 + w_2^2 - w_1^2). \quad . \quad . \quad . \quad . \quad (96)$$

The first term gives the part of the head used up in the guide vanes, and the remaining terms the part of the head utilized in the runner (if $w_2 = w_1$, as in the impulse turbine, the latter part is zero).

The theory of turbines (in which all the particles of water are assumed to be deflected by the same amount) goes back to Euler. The main equation of turbine theory, which is given below and which is based on the theorem about the moment of momentum (§ 13(*b*) of the last chapter, p. 84), is also due to him. The peripheral components of the absolute velocity of the water at inflow and outflow, which were previously denoted by $q_1 \cos\beta_1$ and $q_2 \cos\beta_2$, will now be called c_{u_1} and c_{u_2}. Then the torque exerted on the turbine shaft is

$$M = \rho Q(r_1 c_{u_1} - r_2 c_{u_2}). \quad . \quad . \quad . \quad . \quad . \quad (97)$$

The useful work done is $M\omega$, so that if we put $u_1 = r_1\omega$, $u_2 = r_2\omega$, we obtain

$$E = \rho Q(u_1 c_{u_1} - u_2 c_{u_2}). \quad . \quad . \quad . \quad . \quad . \quad (98)$$

We also have $E = \eta E_0 = \eta wQH$, where η is the hydraulic efficiency and $w = g\rho$, so that

$$\eta = \frac{u_1 c_{u_1} - u_2 c_{u_2}}{gH}. \qquad \ldots\ldots \quad (99)$$

If the water emerges at right angles to the blades c_{u_2} vanishes, so that $u_1 c_{u_1} = \eta gH$. From this equation and consideration of figs. 3.113 and 3.114 it is clear that for the same head a reaction turbine will have a much higher peripheral velocity than an impulse turbine. For this reason reaction turbines are generally used for low or medium heads and impulse turbines for large heads.

In the latest hydrodynamical theory of the turbine problem attempts are made to do without the assumption that all the particles of water are deviated to the same extent. The space between two blades of a reaction turbine is no longer regarded as a "channel", but the blades are regarded as separate aerofoils situated in the flow and having mutual effects on one another.* (36) For modern high-speed turbines in which the distance between the blades is very large this point of view is the only one which in any degree represents the actual phenomena taking place. But even in cases where we may still speak of channels between the blades it turns out that we may not equate the mean of the directions of inflow and outflow to the corresponding value of the angle of the blades, as was done in the older theory. On the contrary, the angle of the blades represents a maximum deflection, and all the water filaments not immediately adjacent to the blades are subject to smaller deviations. These relationships have been discussed very recently in various papers for two-dimensional flow.† An example of the flow past blades is shown in fig. 3.115. We see that the mean deviation of the stream-lines is considerably less than the difference between the direction of the blades at inflow and outflow.

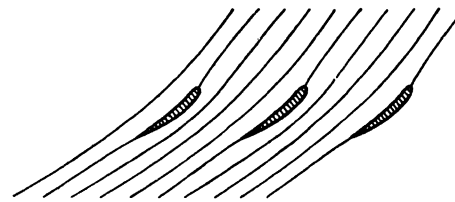


Fig. 3.115.—Flow past blades

Illustrations of some typical forms of turbine runners now in use are reproduced here. Fig. 3.116 shows a modern form of impulse turbine, as developed from the original form of Pelton wheel. One

* See Bauersfeld, *Zeitschr. d. Ver. Deutsch. Ing.*, Vol. 66, pp. 461, 514 (1922), or Prandtl, ibid., Vol. 65, p. 959 (1921), and F. Weinig, *Die Strömung um die Schaufel von Turbomaschinen* (Leipzig, 1935); see also the calculation for a row of blades in § 13(*a*) of last chapter (p. 81).

† See Busemann, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 8, p. 372 (1928), where further references are given; also A. Betz and I. Flügge-Lotz, *Ing-Arch*, Vol. 9, p. 499 (1938).

or more water-jets of circular cross-section are directed on to the sharp edges visible on the lower left-hand side of the figure and are deviated through angles of nearly 180° by the buckets on either side. The peripheral velocity of the wheel for maximum efficiency is about half that of the water in the jet. Fig. 3.117 is a somewhat simplified sketch of the general construction of a Pelton wheel. Fig. 3.118 shows a common form for the runner of a reaction turbine (the so-called Francis turbine). The water, approaching the runner from a set of guide vanes surrounding it, enters by the openings on the left and flows out by the openings on the right. Though the path of the water through the wheel is tortuous, the essential features of the diagram in fig. 3.114 still apply. Forms like that

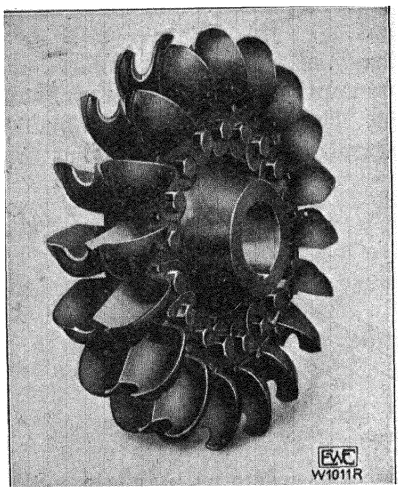


Fig. 3.116.—Pelton wheel

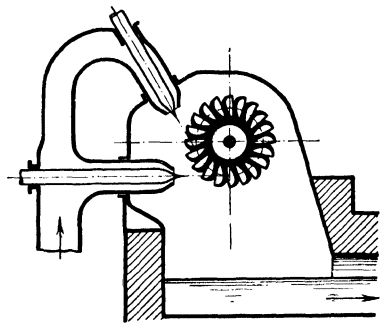


Fig. 3.117.—Pelton wheel with regulating needles

in fig. 3.119 (Kaplan turbine) have been developed in order to attain high speeds of rotation with comparatively low heads. The arrangement of guide vanes is similar to that of the Francis turbine, giving rise to a spiral inward motion of the water. The water then passes axially through the wheel. This type may be regarded as arising from the Francis turbine by retaining only the outermost parts of the blades on the right-hand side of the figure, leaving out the rest of the runner. A sketch of the general construction and guide vanes of a Kaplan turbine is given in fig. 3.120.

In the reaction turbine the energy at outflow is generally diminished by means of a " draft-tube " (*c* in fig. 3.120) which is usually somewhat tapering and conical. The total head is utilized as far as possible by making the draft-tube open under water.

In order to adapt the turbine to suit the load at any time, the quantity of water must be regulated. This is usually done automatically in such a way that the rate of rotation of the turbine is

kept constant. In the Pelton wheel this is effected by means of the regulating needles or spears in the jets, which can be moved backwards and forwards and thereby decrease or increase the jet cross-section (fig. 3.117). In reaction turbines the guide vanes (*a* in fig. 3.120) are rotated so that the water-channels are widened or narrowed, while in the Kaplan turbine the blades of the runner (*b*) can also be rotated, giving better adjustment to large variations of load.

In well-constructed turbines of various types the efficiency is in the region of 85 to 90 per cent. It may be noted that steam turbines may also be classified as impulse turbines and reaction turbines. The compressibility of steam is of course responsible for various modifications of their design as compared with water turbines.

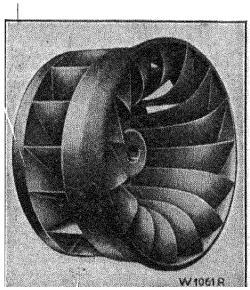


Fig. 3.118.—Runner of Francis turbine

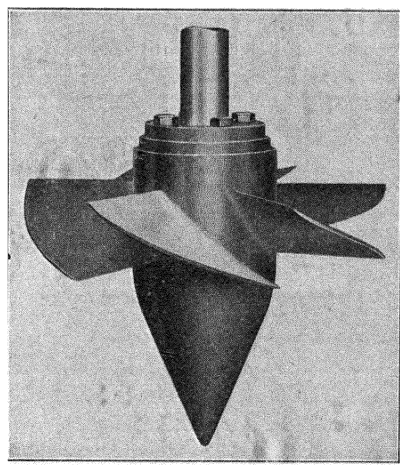


Fig. 3.119.—Runner of Kaplan turbine

The pumps depending on the motion of a bladed wheel, which to a certain extent represent reversed turbines (and in particular reversed reaction turbines), may be divided into those where the water passes through in an axial direction (rotary pumps) and those where it passes through in a radial direction (centrifugal pumps). Similar machines (screw fans, centrifugal fans) are used for pumping air. Rotary pumps and screw fans resemble screw propellers in their construction; the simplest ones differ from propellers only in that they work inside a pipe or in the wall separating two enclosures. By means of guide vanes the greater part of the energy of rotation produced in the rotating part of the pump can be transformed into pressure (see the diagram in fig. 3.121).

In centrifugal pumps and blowers, in addition to the effect of the blades described above, there is the centrifugal effect in the rotating passages between the blades (in a system of passages rotating with

angular velocity ω the centrifugal force appears as a body-force and causes a rise of pressure $\frac{1}{2}\rho\omega^2(r_2^2 - r_1^2)$ for the state of relative rest and also—along a stream-line—for steady motion (cf. Chapter I, § 10, p. 26; Chapter V, § 7, p. 346)). The blades of centrifugal pumps may be radial or bent backwards or even forwards. The highest efficiencies are obtained if the blades are bent backwards, but somewhat higher pressures are reached if the blades are bent forwards. Instead of a system of guide vanes the simplest types of centrifugal pumps and blowers have a suitably shaped spiral volute chamber (fig. 3.122). In more powerful machinery of this kind, especially in high-pressure pumps and turbo-compressors consisting of several pumps arranged in series, on the other hand, there is a special system of guide vanes (fig. 3.123).

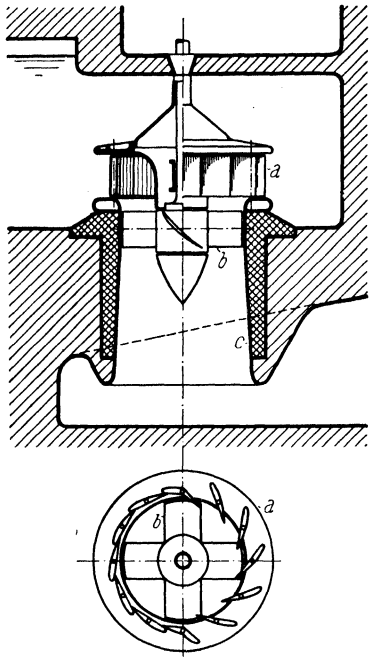


Fig. 3.120.—Kaplan turbine

a, Guide vanes; *b*, runner; *c*, draft-tube. On the right-hand side of the plan the guide vanes are shown open, on the left closed.

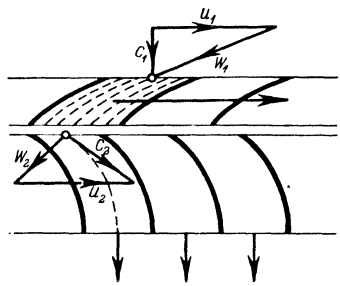


Fig. 3.121.—Diagram of the blades of a rotary pump

If Q is the volume of water passing per second and h the height through which it is raised, E_0, the work done by the pump, is given by

$$E_0 = wQh. \quad . \quad . \quad . \quad . \quad . \quad . \quad (100a)$$

In the case of fans, where the work done in compression may be disregarded as being comparatively small, wh is replaced by $p_2 - p_1$, the difference of the static pressures in front of and behind the fan, so that

$$E_0 = Q(p_2 - p_1). \quad . \quad . \quad . \quad . \quad . \quad (100b)$$

If air is sucked out of a space where the pressure is p_0 and the connecting tube is short, $p_1 = p_0 - \frac{1}{2}\rho q^2$, where q is the mean velocity over the area of the inlet. The kinetic energy leaving the fan or other apparatus may be partly recovered by means of a diffuser (a widening piece of piping) which reduces the pressure p_2 behind the fan, &c., while the opposing pressure remains the same. In the case of compressors, where the difference of pressure is large, i.e. where the work done in compression is not negligible, the formulæ for piston engines are immediately applicable.

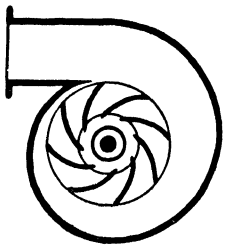
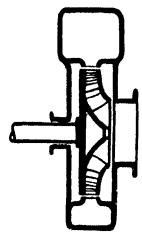


Fig. 3.122.—Centrifugal pump with spiral volute chamber



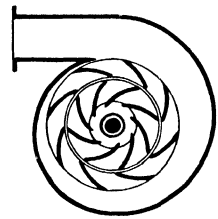


Fig. 3.123.—High-pressure centrifugal pump

Owing to flow losses and friction on bearings, E, the work supplied, is greater than E_0. The ratio $E_0/E = \eta$ is the "efficiency" of the pump or compressor. If we neglect the friction on the bearings, the torque M in $E = M\omega$ may be calculated from Euler's equation (97) (p. 240) for both rotary and centrifugal pumps; c_{u_1} is usually equal to zero, so that

$$M = \rho Q r_2 c_{u_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Hence, since $r_2\omega = u_2$,

$$E = \rho Q u_2 c_{u_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Using $E_0 = \eta E$ and bearing equation ($100a$) in mind, we have

$$h = \frac{\eta u_2 c_{u_2}}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

where c_{u_2} may be found from the appropriate triangles of velocities and the value of η must be estimated.

The hydraulic transmitter first introduced by H. Föttinger * is related to turbines and pumps.[37] It consists of a pump and a turbine in the same casing; by means of it power can be transferred from one shaft to another. This is done by filling or emptying the casing; a very

* *Jahrb. d. Schiffbautechn. Ges.*, Vol. 11, p. 157 (1910).

fluid oil is generally used as the working fluid. Two types of hydraulic transmitter are shown in figs. 3.124 and 3.125—a " coupling " which transfers angular momentum without change, and a " transformer "

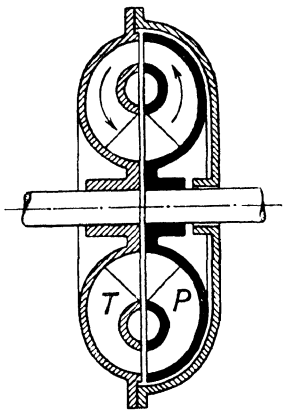


Fig. 3.124.—Hydraulic coupling

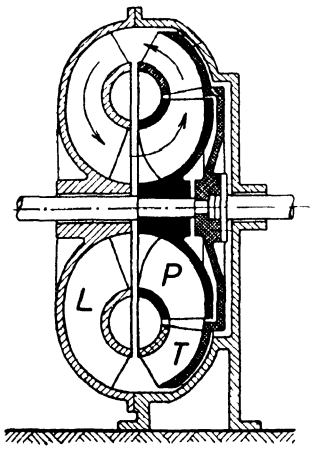


Fig. 3.125.—Angular momentum transformer

P = pump, T = turbine,
L = fixed guide ring

in which a fast-running pump is coupled to a slower-running turbine; in the latter a fixed system of guide blades takes up the difference in angular momentum. The efficiency of the hydraulic transmitter is fairly high, since there are not the usual exit losses.

22. Hydrodynamical and Aerodynamical Methods of Experiment.

(*a*) *The Production of Satisfactory Experimental Conditions.*—We frequently meet with the problem of investigating the motion of a body relative to an otherwise undisturbed mass of fluid. Here two methods are at our disposal. We may either cause the body to move in the fluid, the latter having previously been brought to a sufficient degree of rest, or we may cause a uniform stream of fluid to flow past the body. The first method is well adapted to experiments on water and is used in experimental tanks for testing shipbuilding models. A model ship, usually made of paraffin wax, is towed through the water by means of a travelling carriage; the velocity of the carriage and the resistance of the model are measured, the forms of the waves caused are observed, and so on. The tank must be so large and deep that any disturbances due to the walls are inappreciable. In very refined experiments the models are driven by their own propellers.

Dynamical similarity as regards wave formation, which is specially important here, is attained if the ratio of the velocity of the model to that of the ship is equal to the square root of the ratio of the length of the model to that of the ship (Froude's similarity: by Chapter III, § 13(*b*) (p. 178), the lengths of the waves are then proportional to the lengths of the bodies causing them). Similarity with respect to friction is unattainable, as this would require the values of Reynolds' number to be the same in both cases, which is incompatible with the condition that the waves should be similar. No attempt, therefore, is made to attain similarity as regards friction. To keep the resulting error small the size of the models is made as large as possible.[38]

In the case of air the towing method has not proved suitable. As it is necessary to make the model relatively heavy as compared with the mass of air displaced, the body-forces resulting from unintentional small accelerations of the model predominate over the forces under measurement to such an extent that the results are largely vitiated. Besides this, the disturbances due to the vehicle carrying the recording instruments make themselves very noticeable, as here the latter is moving in the same medium as the model. In the case of air, therefore, the second method is invariably used: that is, the body is suspended at rest and the air made to flow past it by powerful fans. The current of air must be very constant both from point to point and in time, and must be of a cross-section sufficiently large not only to give a good margin round the body but to allow the disturbances caused by the body to die away before they reach the boundaries of the current of air. Otherwise the behaviour of the body in the bounded current of air may be expected to differ from its behaviour in a space of indefinite extent.

The experimental air current may either be conducted along a pipe (wind tunnel, wind channel), or traverse a space in which the air is at rest in the form of a free jet. In the first case there is a fall of pressure in the direction of flow if the walls of the wind tunnel are parallel, owing to friction at the walls; the effect of this on bulky bodies (such as model airships) is like that of the upward thrust of the air where the pressure decreases upwards, i.e. causes a horizontal force $V\partial p/\partial x$ which increases the drag (V is the volume of the model). This source of disturbance may be avoided by making the tunnel widen slightly, so that the pressure is exactly constant along its axis (strictly speaking, this can be attained only for one velocity). In the case of the free jet of air the pressure at its boundary is constant, so that the difficulty just mentioned does not occur; on the other hand, however, the effective region of the jet is limited by the fact that along its boundary the jet gradually mixes with the surrounding air at rest, whereas the boundary layer along the walls of the wind tunnel penetrates at a much slower rate into the interior of the current of air. Owing to greater accessibility, it is naturally more convenient to carry out experiments using the jet of air.

The best way of obtaining a current of air which is as uniform as possible and is free from eddies is to raise its velocity very rapidly from a very small value to that required in the experiment by making the air traverse a pipe of very wide cross-section passing into a narrow one (with the cross-section to be used in experiments) by way of a suitably rounded-off mouthpiece (fig. 3.126). The fall of pressure $p_1 - p_2$ experienced by each particle has the effect of increasing the kinetic energy of each particle by the same amount. All that is necessary, therefore, is that the trifling kinetic energy with which the air arrives in the wide pipe should not be distributed too irregularly among the individual particles. If, for example, the ratio of the velocities is 1 : 5, that of the dynamic pressures (of the kinetic energies) is 1 : 25, so that the fall of pressure gives each particle 24/25 of its subsequent energy. Thus variations of one in four in the approaching energy give rise to variations of one per cent in the final energy or of one-half per cent in the final velocity. Another important point to be attended to is that any rotational motions in the approaching current of air must be removed by making the air pass through a system of parallel channels ("honeycomb") consisting of sets of metal plates crossing at right angles (fig. 3.127); for a mass of air rotating about an axis parallel to the direction of flow would have its angular velocity multiplied by n if its cross-section were divided by n.* As the diameter at right angles to the stream-lines is decreased in the ratio $1/\sqrt{n}$, the transverse velocity ($r\omega$) is multiplied by $\sqrt{n}$, whereas the longitudinal velocity is multiplied by n. In contradistinction to the above, rotation about an axis *at right angles* to the stream-lines gives a *decrease* of the angular velocity proportional to the decrease in linear dimensions, i.e. a decrease in the ratio $1/\sqrt{n}$: here, then, the disturbing velocity $r\omega$ is divided by n as above, in agreement with the above discussion based on Bernoulli's theorem. Wire screens, varying if necessary in size of mesh in different places, may also be placed in front of the honeycomb in order to equalize differences in longitudinal velocity.[39]

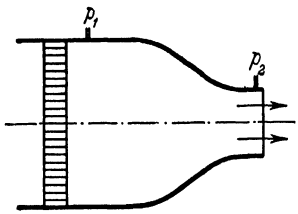
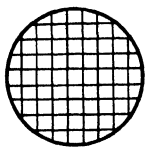


Fig. 3.126.—Mouthpiece


Fig. 3.127.—Honeycomb

Several types † [40] of wind tunnel which have proved satisfactory are shown in figs. 3.128–130; × marks the place where experiments are carried out. It must be remembered that in the open types of wind tunnel the emerging air current must eventually come back to the

* By Helmholtz's theorem for vortices: see Chapter II, § 11, p. 72.

† For more details, see e.g. *Handb. d. Exp.-Phys.*, Vol. IV, 2, p. 65 *et seq.*

inflow, carrying along irregular vortices with it. In fig. 3.128 a honeycomb is also fixed in front of the fan, as otherwise the fan would give a variable pressure when it sucked in these vortices rotating now in the same direction as the fan, now in the opposite direction. In fig. 3.129 the space behind the fan is closed in (forming a " diffuser " or " distributor "), so as to reduce the velocity of the outflowing air.

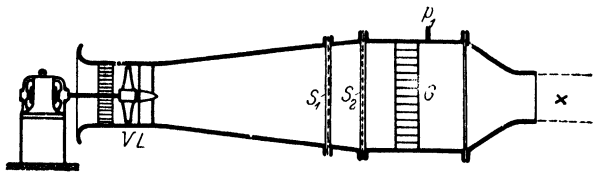


Fig. 3.128

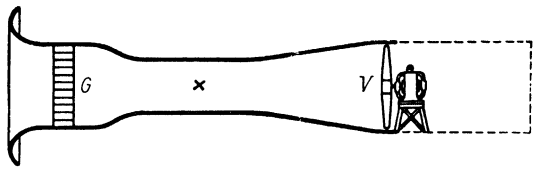


Fig. 3.129

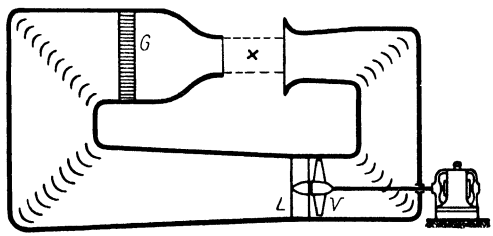


Fig. 3.130

Types of wind tunnel

V = fan; L = vanes; G = honeycomb; S = screen

In the closed type of wind tunnel shown in fig. 3.130 the current of air leaving the place of experiment is again caught up by a funnel, and inconvenience to the observers from draughts, which would otherwise be considerable, is thereby avoided. In the types in figs. 3.129 and 3.130 a considerable amount of the kinetic energy produced is recovered by means of " diffusers " (gradually widening tunnels; cf. § 12(*d*), p. 170).

For experiments on the behaviour of models in a current of air there does not exist any rule (like that for ship models) determining the relationship between the velocity of the model and its other properties. Strictly speaking, of course, Reynolds' law of similarity should be observed. This, however, is nearly always

out of the question, as in order to maintain the same value of Reynolds' number with a model on the scale of 1 : 10 a velocity ten times that of the full-scale case would be required. Apart from the fact that a velocity of this magnitude is in most cases absolutely unattainable, we should usually land in regions where the compressibility of the air comes in (see Chapter IV, p. 258), and this would prevent similarity being realized. Thus in this case also exact similarity as regards friction must be dispensed with.

(*b*) *Measurement of Velocity, Pressure, and Direction of Flow.*—*Velocities* may be measured by means of pressure differences, using Bernoulli's theorem. This is the usual method for wind tunnels. The pressure difference $p_1 - p_2$ in fig. 3.126 may be measured, or a Pitot and static tube like that in fig. 2.28, p. 55, may be employed.

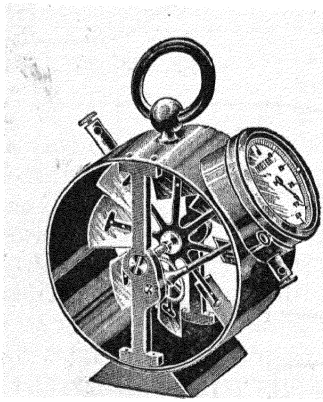


Fig. 3.131.—Fan anemometer

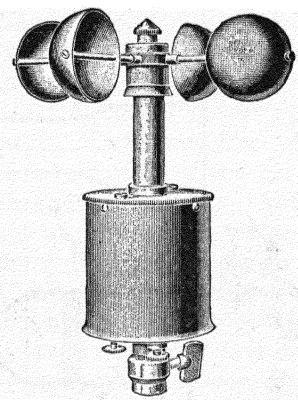


Fig. 3.132.—Cup anemometer

Another instrument for measuring velocity, which is much used in practice but which is generally inferior in accuracy to the above methods, is the anemometer, two types of which, the fan anemometer or airmeter and the cup anemometer, are shown in figs. 3.131, 3.132. The corresponding instrument for measuring the velocity of water is called a current meter; a form used in practice is shown in fig. 3.133. We observe either the number of revolutions (indicated by a mechanical counter) in a definite time, e.g. a minute, or else the time required for a definite number of revolutions, e.g. 500, which may be indicated by some signal. All these instruments require to be calibrated. In the case of anemometers this used generally to be done by means of a whirling arm, i.e. an arm rotating about a vertical axis which carries the instrument round with the desired velocity in a large circle. Here, however, apart from undesirable centrifugal forces acting on the mechanism of the instrument, considerable inaccuracy arose from the "draught" or circulatory motion of the surrounding air

caused by the motion of the arm, which led to the distance actually traversed by the instrument relative to the air being diminished by an amount which was difficult to measure accurately. For this reason it is now considered preferable to calibrate instruments in a wind tunnel. Current meters for water are calibrated by towing them by means of a carriage along a trough of water resembling an experimental tank. This method of calibration is much more reliable than the whirling-arm method.

The cooling of electrically heated wires has also been applied (e.g. in the Tucker hot-wire microphone) to measure the velocity of air currents.[41] As the electrical resistance of metals varies with temperature

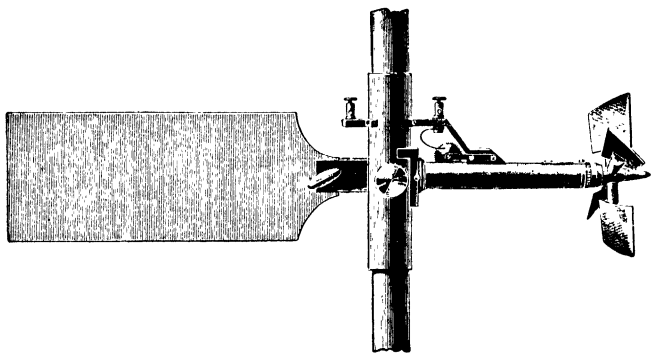


Fig. 3.133.—Current meter for water

a suitable form of bridge circuit is generally used to determine the temperature. We either keep the wire at a constant temperature by means of variable resistances and proceed to find the quantity of heat lost (equivalent to the amount of energy supplied), which in the case of thin wires is very nearly proportional to the cube root of the velocity,* or we work with a constant-energy supply and a falling temperature, when the relation between the resistance of the wire and the velocity of the air is found empirically by calibration. The method is particularly suitable for low air velocities, for which the other methods fail, and also for measurements at points very close to the body.† By using very fine wires, very rapid variations of velocity in time may be traced by means of the hot wire.‡ The properties of turbulent

* For more details, see p. 409.

† J. M. Burgers and v. d. Hegge-Zijnen have measured the boundary layers for flat plates by this method: see *Koningl. Akad. van Wetenschappen*, Proc. XIII, No. 3 (Amsterdam, 1924).

‡ See J. M. Burgers, *Koningl. Akad. van Wetenschappen*, Proc. XXIX, No. 4 = Communication 8, *Labor. f. Aerodyn. und Hydrodyn.*, Delft: see also Fage and Johansen, *Proc. Roy. Soc.* (A), Vol. 116 (1927).

fluctuations have recently been studied very extensively in this way; see § 5(*g*), p. 133. The papers by H. L. Dryden and his collaborators * are of special importance for wind-tunnel work. The technique of testing in wind tunnels which are particularly free from turbulence has been developed further by H. Schuh.† [42]

In order to observe the turbulent fluctuation u' in the direction of the main flow and that (v') at right angles at the same time, H. Reichardt ‡ has developed a "three-wire meter" (see fig. 3.134); the two potential differences (the first hot wire by itself and the difference between the other two) are connected to the crossed pair of plates of a cathode-ray oscillograph in order to make the correlation between u' and v' visible. For time exposures made in this way in a shallow rectangular tunnel, see fig. 3.135 (right, in the median plane; left, to one side of it; correlations 0 and 0·45, respectively).

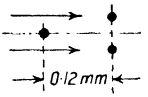
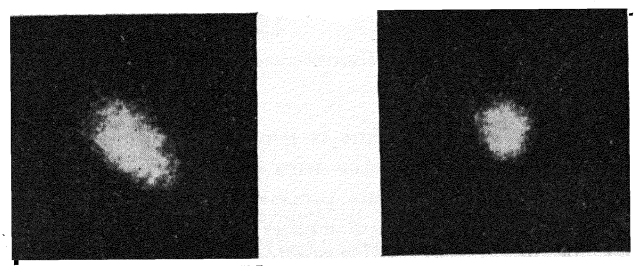


Fig. 3.134

Pressures are measured by means of manometers and micromanometers § (see Chapter I, § 8, p. 18). In certain circumstances there may be difficulty in transferring the pressure to the recording instrument. The method of transferring pressures by means of perforations in a smooth wall is fairly satisfactory (see Chapter II, § 7, p. 54.) The rear opening of the ordinary Pitot static tube may be used to measure the pressure at points in the interior of the fluid, but care must be taken that the instrument points along the direction of flow


Fig. 3.135.—Figures obtained with the three-wire meter

with an accuracy of within one or two degrees. Such instruments are best calibrated by bringing them into a suitable parallel jet of air not too small in size. They should then indicate the pressure of the air at rest in the neighbourhood of the jet, or more accurately an excess pressure of about one-half per cent of the dynamic pressure

* See footnote 1, p. 133.

† *UM*, 6607 (1944) and 6623 (1945). For a new method of calibrating hot wires, see *UM*, 6624 (1945).

‡ *Naturwiss.*, Vol. 26, p. 404 (1938); *Zeitschr. f. angew. Math. u. Mech.*, Vol. 18, p. 358 (1938).

§ See also *Handb. d. Exp.-Phys.*, Vol. IV, 1, p. 487 *et seq.*, p. 511 *et seq.*; also B. Eck, *Technische Strömungslehre* (Berlin, 1941), p. 228 *et seq.*

of the jet, as owing to mixing at the surface of the jet there exists an excess of pressure in its interior * (this excess pressure must likewise be taken into account in the calibration of very accurate anemometers).

Direction of flow can be detected by so-called tuft probes, which consist of thin stems with a light thread fastened to one end, which sets itself fairly accurately in the line of flow provided the velocity is sufficient. By exploring the whole space with the tuft probe it is easy to get a good idea of the distribution of flow. Sometimes tufts of wool fixed at regular intervals to the surface of the body under investigation (e.g. aeroplane models or even actual aeroplanes) are used; these set themselves in the line of flow and thereby show the regions where the flow becomes separated particularly clearly. The surface coating mentioned on p. 147 has also been used in this connexion. In wind tunnels, smoke apparatus (thin tubes from the end of which dense smoke is allowed to emerge slowly) is sometimes of use.

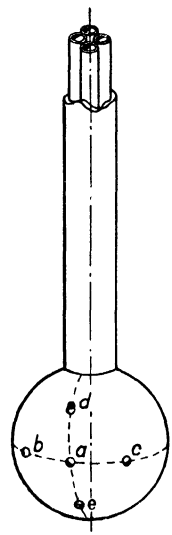


Fig. 3.136.—Five-hole sphere

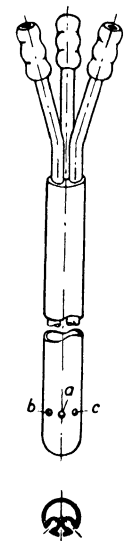


Fig. 3.137.—Three-hole apparatus

Various types of static tube which give indications of the direction of flow from pressure differences measured by manometers have also been developed. The most complete apparatus of this kind is the five-hole sphere proposed by B. G. van der Hegge Zijnen.† The sphere has five holes drilled in it as shown in fig. 3.136 and can be rotated about the stem. Separate tubes lead outwards from each hole. In making measurements the sphere is first turned so that the pressure difference $b - c$ vanishes, and its orientation is read off. The pressure differences $a - d$ and $a - e$ are then read off. Previously obtained calibration curves ‡ then enable the dynamic pressure and the angle which the direction of the flow makes with the stem to be found. If one of the above pressures, or the pressure $b = c$, is measured in absolute value, the static pressure can be obtained

* W. Tollmien, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 6, p. 468 (1926).

† *Verh. d. Kgl. Akad. van Wetenschappen*, Amsterdam, Vol. 32, 1, p. 552 (1929).

‡ A detailed account of the method of calibration and of calibration curves is to be found in a paper by F. Krisam, *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, Vol. 23, p. 369 (1932).

from it in conjunction with the dynamic pressure by means of another calibration curve.

If directions in one plane only are required, a cylinder with three holes *a*, *b*, *c* drilled to correspond with the similarly lettered holes in the sphere may be used (fig. 3.137).* A similar apparatus with only a single hole can be used for measurements in very turbulent flow by making three readings at definite angles, and the intensity of the turbulence may thereby also be obtained.†

(*c*) *Model Experiments: Direct Measurement of the Forces by means of Balances.*—The model to be towed or held in the wind is usually secured by suspensions in such a way that it is held fast in position. The forces acting on the model are made to act on balance beams and are thus measured. Many arrangements are possible: one which has proved useful for measurements in air is as follows.

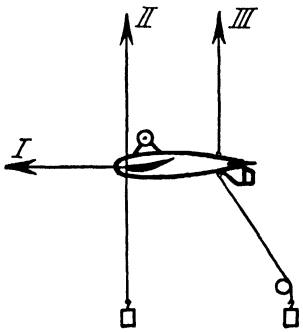


Fig. 3.138.—Three-wire suspension

The model (fig. 3.138) is held in position by three systems of wires, of which one runs horizontally forward and two run vertically upwards. The first serves to transfer the drag, and the second and third the lift, to the measuring instrument. The other end of each set of wires is fastened to a balance beam by means of which the forces just mentioned may be observed. The position of the resultant lifting force may also be found from the ratio of the readings on the second and third balances. The weights shown at the bottom of the figure—the wire to the right-hand one passing over a pulley which can readily rotate—serve to give the wires a suitable initial tension. The model is usually suspended upside down, as shown in fig. 3.138, in order that the lifting force may stretch the wires instead of reducing their tension. Observations are made by first bringing all the balances into equilibrium with the wind shut off and then counterbalancing the deflections which occur when the wind is on. While observations are being made on one balance it is convenient to keep the other balances fixed in their zero positions, so that the angle of incidence does not alter during the experiment.

Other Methods of Obtaining the Drag.—As has already been explained in § 13(*b*), p. 177, the *distribution of pressure* all over the body under investigation can be found by measuring the pressure at a sufficiently large number of holes in the surface, and the resultant pressure can

* E. Gruschwitz used this type of apparatus to measure the secondary flow at the side wall of a curved tunnel; see *Ing.-Arch.*, Vol. 6, p. 355 (1935).

† J. G. von Bohl, *Ing.-Arch.*, Vol. 11, p. 295 (1940).

then be obtained by integration, numerical or otherwise. The measurements may be made by connecting the interior of the hollow body with a manometer and then stopping up all the holes but one with wax or similar material. As this method takes up a good deal of time, it is now common to attach a small piece of tubing to each orifice and to lead these through the interior of the model to as many tubes as there are holes. In the simplest type of apparatus there are a large number of vertical tubes half-full of liquid, arranged close to one another and communicating below. The two end ones are open at the top and accordingly give the undisturbed pressure, the others being connected with the holes. The recording is conveniently done by photographing the multi-tube manometer. In this way the pressure distribution can be obtained not only for models but also for actual aeroplanes, airships, &c.

The " wake resistance " mentioned in § 13(*b*), p. 178, may be found by a method first proposed by A. Betz † in 1925, and mentioned in § 14(*c*) (p. 185), in which the disturbances or wake left behind by the body is measured. Strict proof of the validity of the method is by no means easy, as the pressure disturbances set up by the body spread into the fluid in all directions, and it must first be established whether they contribute anything to the drag, and if so, how much, or, in the case of wave resistance or induced drag, that the part of the drag due to them and that detected in the wake add up to give exactly the total drag. Here, therefore, we shall give only the final result of Betz's argument. In a plane at right angles to the direction of motion of the body and somewhere in its rear, let the distribution of the total pressure g be determined by means of a Pitot tube at rest relative to the body, and that of the static pressure p by means of a static tube. If the velocity of the body is v and the component parallel to v of the velocity of flow relative to the undisturbed fluid is q, the undisturbed total pressure is

$$g_0 = p_0 + \tfrac{1}{2}\rho v^2,$$

and, if, as is assumed here, the squares of the components of velocity at right angles to q can be neglected, the total pressure in the wake is

$$g = p + \tfrac{1}{2}\rho(v - q)^2.$$

Outside the wake Bernoulli's equation holds good, that is, $g = g_0$ everywhere. A fictitious velocity q_* is now introduced which also satisfies Bernoulli's equation in the region of the wake; that is,

$$p + \tfrac{1}{2}\rho(v - q_*)^2 = g_0.$$

(Within the wake q_* represents the velocity which would be produced

† A. Betz, *Ein Verfahren zur direkten Ermittlung des Profilwiderstandes, Zeitschr. f. Flugtechnik u. Motorluftschiffahrt*, Vol. 15, p. 42 (1925).

by the pressure field alone, without friction or apparent friction; outside it, q_* is obviously equal to q.)

With this notation † Betz's formula is

$$D = \iint \{g_0 - g - \tfrac{1}{2}\rho(q^2 - q_*^2)\} da. \quad . \quad . \quad (104)$$

In the case where $p = p_0$, i.e. $q_* = 0$, this result, as a short calculation shows, is in complete agreement with the relation obtained in Note 1 on p. 234.

The method just described is successfully used in wind-tunnel experiments to obtain the profile drag of aerofoils (separate from the induced drag, of course). It has also been used to obtain the profile drag on actual aeroplanes in flight.‡[43]

(*d*) The *frictional drag* of any body can, according to § 13(*b*) (p. 177), be obtained as the difference between the total drag found by the balance and the pressure drag obtained from the distribution of pressure (see above).

For plane flow the frictional stress τ can be evaluated by the "Kármán integral condition" (p. 110, footnote 3). In the case of constant pressure (e.g. in flow with constant velocity along a smooth plate) we can trace out the variation in loss of momentum per unit depth,

$$I = \int_0^x \tau dx = \rho \int_0^\delta u(U - u) dy, \quad . \quad . \quad . \quad (105)$$

where $u(x, y)$ is measured by a Pitot tube. It is convenient to have a number of Pitot tubes joined to form a sort of comb and record the dynamic pressures photographically by a multi-tube manometer.§ (The integral is then replaced by a sum of terms uU and $-u^2$ with fixed numerical coefficients.) Then $\tau = dI/dx$. If the pressure varies with x, the integral condition can be used in the simple form given by the author,‖

$$\tau = \frac{dI}{dx} - \delta^* \frac{dp}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (106)$$

where $\delta^* = \int_0^\delta (1 - u/U) dy$ is the displacement thickness (see the note on p. 109).

† In Betz's paper and in Prandtl-Tietjens, Vol. II, p. 141 *et seq.*, u_0 is used instead of v, $u_0 - u_2$ instead of q, $u_0 - u_2'$ instead of q_*, and g_2 instead of g, so that the expression in brackets above is there

$$g_0 - g_2 + \tfrac{1}{2}\rho(u_2' - u_2)(u_2' + u_2 - 2u_0).$$

‡ M. Schrenk, *Luftfahrtforschung*, Vol. II, Part 1 (Munich and Berlin, 1928); see also B. M. Jones, *Rep. and Mem.* 1688 (1936).

§ K. Wieghardt, *Techn. Ber.*, Vol. 11, p. 207 (1944).

‖ Durand, *Aerodynamic Theory*, Vol. III, p. 108, formula (17.10).

The method can be applied where the roughness is discontinuous and the resistances are concentrated.*

In attempts to find the laws of turbulent separation, however, this method, elsewhere very useful, has led to disappointments.† These were found to be due to secondary flow, as a result of which lateral losses of momentum came into the cross-section being investigated. The matter has been fully cleared up by recent measurements by H. Ludwieg and W. Tillmann with a skin friction meter (see below).

Of direct methods for finding frictional stresses we may mention that of Fage and Falkner.‡ The essentials of this method consist in finding the velocity quite close to the surface of the body, where the flow is laminar and the shearing stress τ may accordingly, by § 1, p. 98, be put equal to $\mu q/y$, where q is the measured velocity and y the distance from the surface. In practice the distances are of the order of $\frac{1}{20}$ mm. The measuring apparatus consists of a perforation in the surface, covered at this minute distance by a sharply ground knife-edge (fig. 3.139). The apparatus must of course be calibrated and is therefore made so that it can be unscrewed. The calibration is carried out by inserting the apparatus in a narrow pipe in which the flow is known to be laminar, and for which the shearing stress can be calculated from the observed fall of pressure. Another method for finding the shearing stress, due to H. Ludwieg,§ is by the device mentioned above. The method depends on the cooling effect of the stream of air or fluid on a small block of copper firmly inserted into the surface of the body and supplied electrically with a definite amount of heat. By Chapter V, § 15(c), p. 403, the laminar sub-layer at the wall plays an important part here. The relation between the shearing stress τ and the excess temperature ΔT predicted by theory is confirmed by experiment. If the supply of heat to the copper is kept constant, an approximation formula gives



Fig. 3.139.—Apparatus for measuring the frictional drag

$$\tau^{1/3} \sim \frac{1}{\Delta T} - \frac{1}{\Delta T_0}, \qquad \ldots\ldots \quad (107)$$

where ΔT_0 is the excess temperature in the fluid at rest. As the laws of plate resistance are sufficiently well known, the numerical values of the constants can be found by wind-tunnel experiments.

* W. Jacobs, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 19, p. 87 (1939); K. Wieghardt, *FB*, 1563 (1942); W. Tillmann, *UM*, 6619 (1944), and *UM*, 6627 (1945); see also the author's *F.I.A.T. Review* (Vol. V, 3.3).

† See footnote 9, p. 141.

‡ A. Fage and V. M. Falkner, *Proc. Roy. Soc.* (A), Vol. 129, p. 378 (1930).

§ H. Ludwieg and W. Tillmann, *Ing.-Arch.*, Vol. 17, pp. 207, 288 (1949).

CHAPTER IV

Flow with Appreciable Volume Changes (Dynamics of Gases)

Preliminary Remarks.

Appreciable volume changes chiefly occur in the flow of gases and vapours where there are large differences of pressure. In the introduction to Chapter II (p. 31) we emphasized the fact that for moderate differences of pressure (small compared with the absolute pressure of the gas) changes of volume are of no importance, so that in such cases the flow of the gas may to a first approximation be discussed on the basis of the laws which hold for incompressible fluids. If, however, the volume changes are appreciable, the relationship between the velocities and pressures differs from that for flow where volume changes are negligible, owing to the variability of the density, and in addition continuity requires different distances between the particles and hence different paths for the particles for given velocities—or different velocities for given cross-sections.

The cases where appreciable volume changes and considerable pressure changes occur may broadly speaking be classified as follows:

(1) *Masses of gas subject to gravity and extending over a great range of height.*—The chief application of this case is to flow phenomena in the free atmosphere. To consider the matter apart from the special meteorological problems would be of little purpose, so that here we shall not discuss the subject, which in any case has been developed but little as yet.* A few isolated problems are mentioned in Chapter V.

(2) *Gases moving with high velocities.*—This case occurs (*a*) when two regions between which there is a pressure difference † comparable

* A variety of literature on the subject is to be found in the meteorological journals. Two useful books in German are V. Bjerknes and others, *Physikalische Hydrodynamik* (Berlin, 1933), and H. Koschmieder, *Dynamische Meteorologie* (Leipzig, 1941). See also the list of books in the appendix.

† This difference of pressure need not always be large absolutely; what is essential is that it should not be negligible compared with the absolute pressures in the two regions. If for example the difference of pressure between the vaporizer and condenser of a mercury-vapour pump such as is used to produce high vacua is 1/10 mm. of mercury, and the absolute pressure in the latter space (which is connected with the fore-vacuum) is also 1/10 mm. of mercury, the difference of pressure is "appreciable" in our sense and the flow must be discussed in accordance with the laws of the "dynamics of gases".

with the absolute pressures are connected by an opening or tube, (*b*) when bodies are moving in a gas with very high velocities.

These two cases are closely related theoretically; it is only necessary to place the body in a stream of air flowing with a certain velocity in order to imitate the behaviour (referred to a different system of reference) of a body moving with that velocity in air at rest. The relationships in question find practical application in the flow in steam turbines and similar machines, and on the other hand in the motion of projectiles, fast aeroplanes, and aeroplane propellers, where very high velocities are likewise reached. The theory of flow at high velocities * may be briefly referred to as the " dynamics of gases ".

(3) *Large accelerations.*—These arise in a mass of gas either at rest or in motion when portions of the boundaries or bodies moving in the gas move with large accelerations or when the mass of gas is subject to other sudden changes. To this class belong the phenomena of the propagation of rapid vibrations (which are dealt with in acoustics), the propagation of explosions, &c., the phenomena resulting from the rapid opening and shutting of valves of various types, and so on.

The velocity of propagation of any pressure change in a mass of gas plays a decisive part in determining the special nature of these phenomena, as also in the case of high-velocity flow. We shall therefore begin by discussing how propagation takes place.

1. Propagation of Pressure: the Velocity of Sound.

We shall base our treatment of the propagation of pressure on the following simple case, which is related to the motion of a slight elevation on the surface of water (see § 15 of Chapter II (p. 92)). We suppose that by suitable motion of a piston an increase of pressure is produced in a mass of gas at rest, enclosed say in a wide tube, and is propagated through the mass of gas as indicated in fig. 4.1.

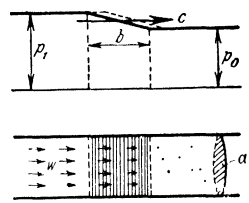


Fig. 4.1.—Pressure wave in a pipe

We assume that the pressure distribution moves towards the right with the constant velocity c, without any change in form. As the gas is thereby compressed, the part over which the rise of pressure has already passed must have a velocity q towards the right. For purposes of calculation we shall assume that the increase of pressure, $p_1 - p_0$, is small

* Detailed surveys of the subject are to be found in Ackeret, *Handb. d. Phys.*, Vol. 7, p. 289 *et seq.*; Busemann, *Handb. d. Exp.-Phys.*, Vol. IV, 1, p. 341 *et seq.*; Taylor and Maccoll, *Aerodyn. Theory*, Vol. III, p. 209 *et seq.*; Prandtl, Taylor, Kármán, Pistolesi, Busemann, *Convegno Volta*, Vol. 5, p. 169 *et seq.* The theory is dealt with in Sauer's book (see p. 428). For the most recent German literature, see A. Betz, *F.I.A.T. Review*, Aerodynamics, Chapter IV.

compared with the pressure p_0; the change of density, $\rho_1 - \rho_0$, will then be small also, and q will likewise be small, as will appear from what follows.

We can now make two statements about the phenomena in the region of transition of breadth b (cf. fig. 4.1):

(1) About continuity. While the region b passes over a particular point, the density there rises from ρ_0 to ρ_1. The time taken for this is $\tau = b/c$. The resulting increase in mass in the region of transition per unit time * is accordingly equal to ab, the volume of this region, multiplied by the increase in density in unit time $(\rho_1 - \rho_0)/\tau$, i.e. equal to $a(\rho_1 - \rho_0)c$. This increase in mass is produced by the mass flowing into the region of compression in unit time, namely $\rho_1 aq$, so that

$$\rho_1 q = (\rho_1 - \rho_0)c. \quad \ldots\ldots\ldots \quad (1)$$

(2) The theorem Mass $\times$ Acceleration = Resultant Force leads to the following statement. Within the region of transition the velocity increases from 0 to q in time τ, so that the mean acceleration is $q/\tau = qc/b$. The mass to be accelerated is $\rho_m ab$, where ρ_m is the mean density. The resultant force is $a(p_1 - p_0)$, i.e. we have the dynamical relationship

$$\rho_m qc = p_1 - p_0. \quad \ldots\ldots\ldots \quad (2)$$

In equation (1) we may subsequently replace ρ_1 by ρ_m without giving rise to any error more serious than that already involved in the assumptions above. If we now divide (2) by (1) and cancel out $\rho_m q$, we have

$$c^2 = \frac{p_1 - p_0}{\rho_1 - \rho_0}.$$

Now the expression on the right-hand side depends only on the law of compressibility of the substance; replacing it by the differential coefficient $dp/d\rho$, we obtain

$$c^2 = \frac{dp}{d\rho}. \quad \ldots\ldots\ldots \quad (3)$$

Thus if we assume that the changes of pressure are small, the velocity of propagation of the variations of pressure is independent of their magnitude and of the breadth of the region of transition and depends only on the law of compressibility of the material in question. Hence positive and negative pressure variations may follow one another in any way without affecting one another. As sound consists of such

* For this discussion it is convenient to choose a unit of time which is small compared with the time τ. As the time τ does not occur in the final result, this does not limit the validity of the proof in any way.

successions of positive and negative changes of pressure, advancing with the velocity c (by the above), we call c the velocity of sound.

For gases we have the adiabatic law (cf. Chapter I, § 5, p. 10), $p = \text{const. } \rho^\gamma$, i.e.

$$\frac{dp}{d\rho} = \gamma \cdot \text{const. } \rho^{\gamma-1} = \gamma \frac{p}{\rho},$$

where $\rho = \rho_0 \frac{p}{p_0} \cdot \frac{1}{1+\alpha\theta}$ (ρ_0 being the normal density at the normal pressure p_0 and temperature 0° C.). Hence

$$c = \sqrt{\gamma \frac{p_0}{\rho_0}(1+\alpha\theta)}. \quad \ldots\ldots (4)$$

That is, in any particular gas the velocity of sound depends on the temperature alone. For air of moderate humidity, we have $\sqrt{(\gamma p_0/\rho_0)} = 334$ m./sec. (1096 ft./sec.), which is in good agreement with the observed velocity of sound.

The case (which is important for us) of the propagation of pressure in a *moving* gas may be reduced to the case considered here by using a system of co-ordinates moving with the gas. It follows that the pressure is propagated with the velocity c *relative* to the gas. Its velocity of propagation relative to a space in which gas is flowing with velocity q is $c+q$ downstream and $c-q$ upstream. We see at once that in the case where q is greater than c there is no propagation of pressure upstream at all. Hence for velocities of flow exceeding that of sound, the behaviour of gases and vapours is essentially different from the ordinary.

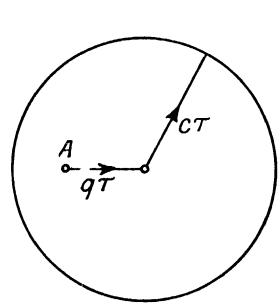


Fig. 4.2.—A pressure wave: $q < c$

If we consider the propagation in space of pressure disturbances originating in a limited region, we obtain a further characteristic property of flow with velocity either smaller (*subsonic*) or greater (*supersonic*) than the velocity of sound. A momentary point disturbance at A (fig. 4.2) * is propagated through a gas in uniform motion in the form of a spherical wave, the centre of which advances with the velocity of flow of the gas. A continuous disturbance at A, such as may be caused e.g. by a small obstacle situated there, may be regarded as a succession of momentary disturbances. If q, the velocity

* Figs. 4.2, 3, 5–14, 16–19, 21–31, 43–46 are taken from the author's article "Gasbewegung" in *Handwörterbuch der Naturwissenschaften*, Vol. 4, by kind permission of the publisher (Gustav Fischer, Jena).

of flow, is smaller than c, the velocity of sound, the effect of the obstacle is propagated outwards in all directions, but possibly with differing intensities in different directions. If, however, the velocity of flow exceeds the velocity of sound, all the spherical waves are contained in a cone behind the point A (cf. fig. 4.3). The space in front of the cone is completely unaffected by the obstacle. Similar relationships hold in the case of a body (e.g. a bullet) passing through a mass of gas at rest. If the velocity of the projectile exceeds the velocity of sound, its effects are confined to a cone like that in fig. 4.3 (E. Mach *).

The angle of the cone may be found as follows. During a small interval of time τ, a point source of disturbance expands to a sphere of radius $c\tau$, the centre of which has moved through a distance $q\tau$. The cone touches this sphere, so that

$$\sin\alpha = \frac{c\tau}{q\tau} = \frac{c}{q}; \qquad \ldots\ldots (5)$$

α is called the Mach angle.

If the gas flows past solid boundaries which exhibit irregularities, the differences of pressure caused by these are rapidly equalized towards the interior of the region of flow when the velocity is subsonic; if, on the other hand, the velocity is supersonic any irregularity gives rise to a wave starting off at the Mach angle, which spreads throughout the region of flow and may even be reflected at an opposite wall. Fig. 4.16 (p. 276) shows a flow past artificially roughened walls; the decrease of the Mach angle from left to right indicates the increase in the velocity from left to right.

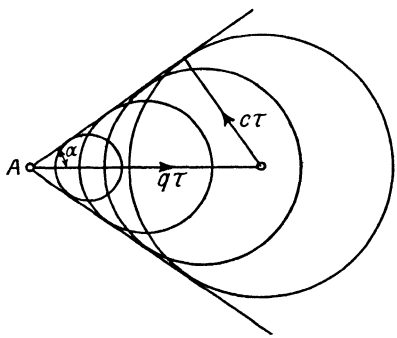


Fig. 4.3.—Pressure waves: $q > c$

The same relationships apply to the motion of bodies in a mass of air at rest. If the velocity of motion exceeds the velocity of sound, the disturbance caused by the body merely spreads out in a cone as in fig. 4.3; in advance of this cone the medium remains wholly at rest. An example of this is offered by the motion of a bullet illustrated in fig. 4.4.† The velocity of the bullet may be calculated fairly accurately from the angle made by the

* E. Mach, *Sitzungsber. d. Wien. Akad.*, IIa, Vol. 95, p. 164 (1887); Vol. 98, p. 1310 (1889); Vol. 105, p. 605 (1896).

† Taken from C. Cranz, *Lehrb. d. Ballistik*, Vol. 2, p. 451, by kind permission of the publisher (Julius Springer, Berlin (1926)).

nose-wave, provided we use the parts of the wave at some little distance from the bullet. In the immediate neighbourhood of the latter the differences of pressure are so great that in these regions the nose-wave is propagated with a velocity exceeding that of sound, so that the angle α is increased. The characteristic note emitted by propellers with tip speeds greater than the speed of sound (p. 230) arises from compression waves of this type.

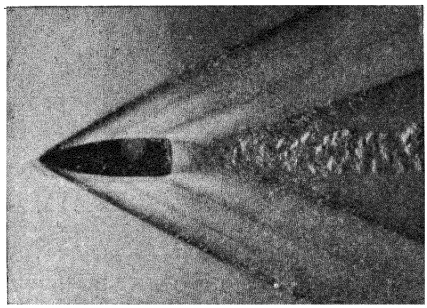


Fig. 4.4.—Photograph of a bullet in flight, after Cranz

Note.—A finite change of pressure may be thought of as a succession of many small pressure changes. At the points of high pressure there is (by equation (2), p. 260) a velocity in the direction of advance approximately proportional to the difference of pressure (and conversely a velocity in the reverse direction at the points of low pressure). According to this, the velocity of propagation of a compression following upon a previous compression is $c + q$, i.e. is larger; similarly, that of a second rarefaction is smaller, so that we must conclude that strong compressions are propagated at a rate exceeding that of sound and strong rarefactions at a rate less than that of sound. These differences are further increased by the fact that compression increases the temperature of the gas, and hence, by (4), the speed of sound, whereas rarefaction decreases it. The exact theory of these phenomena is due to B. Riemann.* It

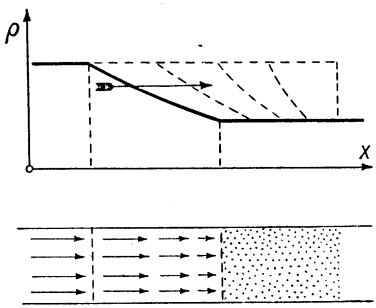


Fig. 4.5.—Wave of compression

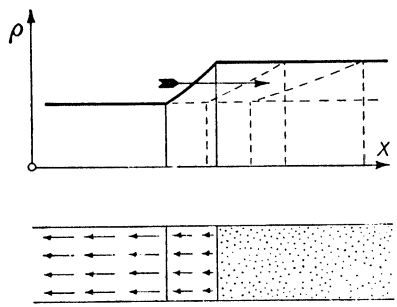


Fig. 4.6.—Wave of rarefaction

yields the result, consistent with the above, that in a pressure wave where the compression is finite the velocity of advance of the partial compressions is greater the higher the value to which the pressure has already risen, so that the crest of the wave advances more rapidly than the trough and after some time overtakes

* *Abh. d. Gött. Ges. d. Wiss.*, Vol. 8, p. 43 (1858–9), and *Gesammelte Werke*, 2nd ed., p. 156 (Leipzig, 1892); also Lamb, *Hydrodynamics*, 6th ed., § 278. For further examples, see K. Bechert, *Ann. d. Phys.*, Vol. 37, p. 89 (1940); Vol. 38, p. 1 (1940).

the latter. Thereafter the process of compression is discontinuous: more will be said on this subject later. In waves of rarefaction the rear lags behind the front, so that the wave becomes flatter and flatter. These phenomena are illustrated by figs. 4.5 and 4.6.

2. One-dimensional Treatment of Steady Flow with Appreciable Volume Changes.

Here the general form of Bernoulli's theorem (equation (7*b*), p. 41) is true for a stream-tube. Neglecting gravity and assuming (as we do in most parts of this chapter) that there is no friction, we have

$$\mathbf{P} + \tfrac{1}{2}q^2 = \text{const.} = \mathbf{P}_1, \qquad \ldots\ldots \quad (6)$$

where the pressure function $\mathbf{P} = \int dp/\rho$ is to be taken for an adiabatic change of state (the only kind of change involved here). If we put $1/\rho = v$ the integral * may be written as $\int v dp$. For an ideal gas $\rho = \rho_1(p/p_1)^{1/\gamma}$; evaluating the integral, we obtain

$$\mathbf{P} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1}\left(\frac{p}{p_1}\right)^{(\gamma-1)/\gamma}. \qquad \ldots\ldots \quad (7)$$

If p_1 is the pressure for which $q = 0$, e.g. the pressure in a tank from which gas is allowed to flow out, we have

$$q = \sqrt{2(\mathbf{P}_1 - \mathbf{P})} = \sqrt{\left[\frac{2\gamma}{\gamma - 1}\frac{p_1}{\rho_1}\left\{1 - \left(\frac{p}{p_1}\right)^{(\gamma-1)/\gamma}\right\}\right]}. \quad (8)$$

It is easy to see that the velocity produced when the gas expands to the last degree of rarefaction is finite. It is found by putting $p = 0$ in equation (8) and is therefore given by

$$q_{\max} = \sqrt{\frac{2\gamma}{\gamma - 1}\frac{p_1}{\rho_1}}. \qquad \ldots\ldots \quad (9)$$

For air at ordinary temperatures (about 15° C.) with a pressure at the outlet equal to that of the atmosphere, this gives in Kilograms and metres

$$\sqrt{\left(\frac{2 \times 1{\cdot}405}{0{\cdot}405} \cdot \frac{10{,}330}{0{\cdot}125}\,\frac{\text{Kg./m.}^2}{\text{Kg. sec.}^2/\text{m.}^4}\right)} = 757 \text{ metres per second.}$$

The relationship between q and p is graphed in fig. 4.7, which also includes the graph of the specific volume v $(= 1/\rho)$ as a function of p according to the adiabatic equation. The shaded area $\int_p^{p_1} v dp$ is equal to $\mathbf{P}_1 - \mathbf{P}$. For the graph of v/q as a function of p, see below.

* Here v is the volume per unit *mass*, or specific volume; in engineering literature, however, v is usually defined as the volume of unit *weight*, i.e. $v = 1/g\rho = 1/w$.

In the case of the steady flow of a compressible fluid, continuity (see p. 35) requires that the same mass should flow across every section of a stream-tube in unit time: that is,

$$a\rho q = \text{const.} \qquad \ldots\ldots\ldots \quad (10)$$

along a stream-tube. The way in which a, the cross-section of the stream-tube, varies with the pressure p is given by the behaviour of the function $1/\rho q = v/q$ (see the graph in fig. 4.7); it may also be explained as follows, using equations (8) and (10). If we begin with $p = p_1$, we have $q = 0$, so that a is infinite. If p now decreases, q

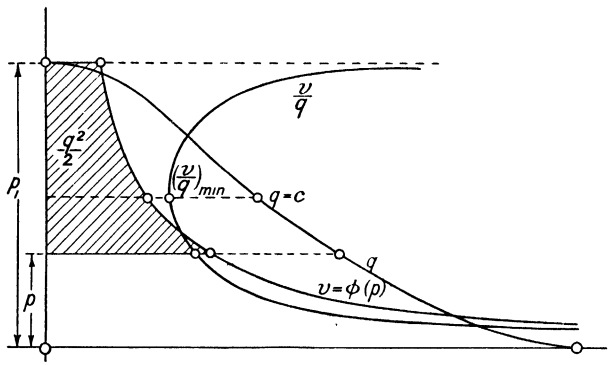


Fig. 4.7.—The graphs of v, q, and v/q as functions of p

gradually increases without there being much change in ρ, i.e. a must decrease. If, however, p becomes very small, q approaches the value q_{max}, i.e. no longer varies to any great extent, but ρ decreases indefinitely as p does, so that in this region a must increase and tend to infinity.

Between the region where a is decreasing and that where it is increasing there must obviously lie a minimum of a; it is situated where the relative increase in velocity, dq/q, is numerically equal to the relative decrease in density, $-d\rho/\rho$. Now calculation shows that this minimum is situated where the velocity of flow is equal to the velocity of sound under the conditions in question. On account of the adiabatic cooling which takes place, the velocity of sound is not that corresponding to the initial conditions but is reduced to correspond with the lower temperature (for air issuing at ordinary temperature it is about 315 m./sec.). The relationship just mentioned may be deduced without any calculation from the considerations of last section. For if we consider the phenomenon represented in fig. 4.1 (p. 259) from the point of view of a system of co-ordinates moving to the right with velocity c, the gas at places of rest *appears* to be flowing from right to left with the velocity c, and the pressure wave remains fixed in

position. Thus from this point of view we have a case of steady flow with the peculiarity that a difference of pressure exists without any change in the cross-section of the stream-tube. But this is just the characteristic feature of the point in our stream-tube where the cross-section is a minimum, i.e. is neither increasing nor decreasing.

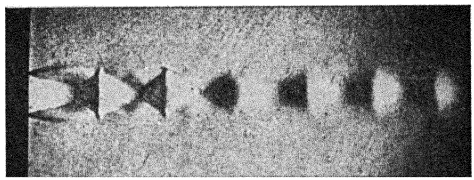


Fig. 4.8.—Waves produced by the release of compressed air (L. Mach)

After we pass the minimum the velocity exceeds that of sound. It follows from what we have said above that in a flow of this kind the cross-section increases (instead of decreasing, as would be the case in an incompressible fluid) as the pressure decreases, i.e. as the velocity increases; as the pressure increases, i.e. as the velocity decreases, the cross-section decreases. Thus the behaviour in a case of this kind differs markedly from the ordinary. A regular increase in velocity up to the terminal value given by Bernoulli's theorem is not possible in the case where this terminal value exceeds the velocity of sound unless the pipe in which the flow is taking place is first narrowed and then widens by a definitely fixed amount (as in de Laval's steam nozzle, discussed below). In the case of a simple unwidened opening, although the external pressure may be so small as to give rise to a velocity of flow exceeding that of sound, the pressure at the opening itself retains the value corresponding to a velocity equal to that of sound (about 0·53 of the pressure of the gas at rest in the case of air and other diatomic gases *). The quantity flowing out is then quite independent of the external pressure. Once outside the

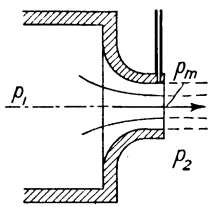


Fig. 4.9.—Measurement of the pressure at the opening

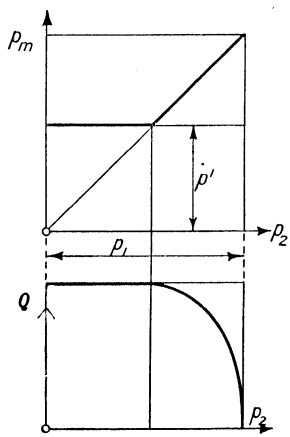


Fig. 4.10

* It is given by the formula $p' = p\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$; the corresponding velocity is $q' = c' = \sqrt{\{2\gamma p_1/(\gamma+1)\rho_1\}}$.

opening, the cross-section of the jet of gas increases, and owing to inertia does so to such an extent that a region of reduced pressure is left in its interior. Owing to this reduction of pressure the flow again becomes convergent, condensing to a pressure about equal to that at the opening, and the process begins afresh.

Fig. 4.8 shows a photograph of a jet of air of this kind taken by L. Mach,* using Töpler's "Schlieren" method (for the latter, see p. 275).

The pressure p_m at the opening can be measured by boring a hole in the nozzle close to the free edge (fig. 4.9). For external pressures p_2 which are less than the "critical pressure" p', p_m is to a good approximation constant and equal to the critical pressure. For higher external pressures it is practically equal to the external pressure. If p_2 is allowed to decrease gradually from the value p_1, the quantity of gas flowing out,

$$Q = a\rho_2 q_2 = a\left(\frac{p_2}{p_1}\right)^{1/\gamma}\sqrt{\left[\frac{2\gamma}{\gamma-1}p_1\rho_1\left\{1-\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right\}\right]}, \quad (11)$$

gradually increases to a maximum at the critical pressure, given by

$$Q_{\max} = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} a\sqrt{\left(\frac{2\gamma}{\gamma+1}p_1\rho_1\right)}. \quad . \quad . \quad (12)$$

If p_2 decreases further Q then remains constant and equal to $Q_{\max}$. The variations of p_m and Q with p_2 are illustrated in fig. 4.10.

The graph of Q for $p_1 > p_2 > p'$ is nearly a quadrant of an ellipse; this fact may be used, according to Fliegner, to establish approximation formulæ for practical use. For atmospheric air these are as follows:

(1) for $p_2 > \frac{1}{2}p_1$,
$$Q = 0{\cdot}76a\sqrt{\{(p_1 - p_2)p_2/T_1\}};$$

(2) for $p_2 < \frac{1}{2}p_1$,
$$Q = 0{\cdot}38ap_1/\sqrt{T_1}.$$

Here p is measured in Kg./cm.^{2} and a in cm.^{2}, or p in Kg./m.^{2} and a in m.^{2}, and T_1 is the absolute temperature inside the tank; the formula then gives the discharge per second in kilograms.

The behaviour described above may also be explained very well from the point of view of last section. We shall assume that at the exit end of the nozzle there is a chamber in which the pressure can be regulated by means of an outlet valve or otherwise (fig. 4.11). Let p_2, the pressure in the chamber, be greater for the time being than the critical pressure. Then if p_2 is lowered, e.g. by opening the outlet valve

* L. Mach, *Sitzungsber. d. Wien. Akad.*, IIa, Vol. 106, p. 1025 (1897); see also R. Emden, *Ann. d. Phys. u. Chem.* (3), Vol. 69, pp. 264, 426 (1899), and L. Prandtl, *Physik. Zeitschr.*, Vol. 8, p. 23 (1907).

further, a wave of rarefaction runs into the nozzle and sets up a new type of flow. (There is an additional velocity in a direction opposite to that in which the waves are propagated, i.e. tending to increase the velocity of flow.) If p_2 decreases further, however, this only goes on until the velocity at the opening is equal to that of sound. Subsequent lowerings of the pressure can no longer advance into the nozzle, as they are propagated through the gas with only the velocity of sound, so that the conditions within the nozzle henceforth remain unaltered. The result accordingly agrees with that given above.

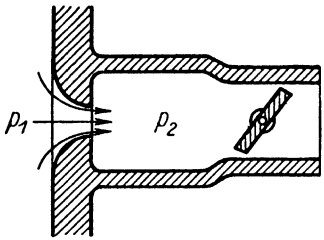


Fig. 4.11

In order to obtain regulated expansion when the pressure ratio is far above the critical value, the Swedish engineer de Laval used out-flow nozzles of the shape shown in fig. 4.12 in the construction of his steam turbine.* The phenomena in such nozzles, being of very great practical interest, have been investigated in great detail both theoretically and experimentally. To these studies we owe the solution of a large number of fundamental problems in the flow of gases and vapours.

Here we shall only consider frictionless flow through a Laval nozzle. Let the pressure p_1 in front of the nozzle be given; then the values

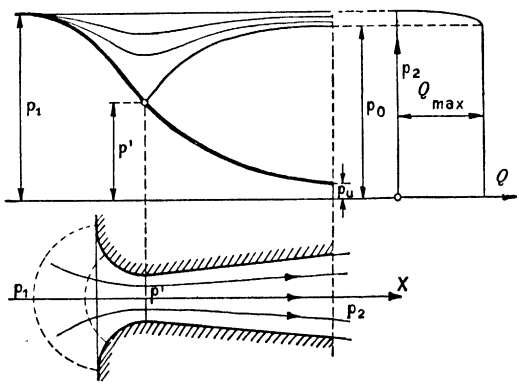


Fig. 4.12.—Flow through a Laval nozzle

of q and v/q corresponding to every lower pressure p may be obtained from the formulæ or from fig. 4.7, p. 265. As the mass of gas discharged per second is $Q = a\rho q = aq/v$, the value of q/v corresponding to any

* I was informed by Herr Johann Körting that his great-uncle Ernst Körting, the well-known inventor of many forms of jet apparatus, used conical nozzles as early as 1878. De Laval's steam turbine became known in 1883.

cross-section a may be determined for every given value of Q and the corresponding pressure may be found from fig. 4.7 (p. 265). Under normal circumstances the minimum cross-section of the stream-tube, i.e. the minimum of v/q, must coincide with the minimum cross-section (*throat*) of the nozzle; here the quantity discharged is a maximum and is determined, as in the case of a simple opening, by equation (12). The pressure obtained in the way described above gives the heavy line in fig. 4.12 (from p_1 to the lowest pressure p_u). But as there are always two pressures corresponding to a single value of v/q, by fig. 4.7, from the narrowest point onwards there is a second possible pressure graph, which leads to the upper final pressure p_o.

If we plot the pressure curve for smaller discharges in the same way we obtain the lines ending above p_o in fig. 4.12. The discharge accordingly varies with the pressure p_2 at the end of the nozzle (cf. the figure at the right of fig. 4.12) in such a way that the discharge increases from zero when $p_2 = p_1$ to $Q_{\max}$ when $p_2 = p_o$. Thereafter the velocity of sound is attained at the throat of the nozzle, and hence, according to our previous discussion of the part played by the velocity of sound, we may in the absence of exact knowledge of the phenomena occurring expect that as p_2 decreases still further the phenomena above the throat will remain unchanged, so that the discharge remains constant. The behaviour of the discharge which we have described is well confirmed by a large number of experiments.

The above discussion shows that no flow without losses can lead to values of the pressure p_2 between p_o and p_u. Observations of the pressure variations led A. Stodola to recognize the fact that here discontinuous compressions arise, which are related to the discontinuities predicted theoretically by Riemann (cf. p. 263). In such phenomena losses do actually occur, so that they cannot be discussed by means of the formulæ for flow without losses. On the other hand, a satisfactory theory of the phenomena has been obtained by using the momentum theorem deduced in Chapter II, § 12 (p. 77), together with an energy theorem which is important in all cases where flow with appreciable volume changes is associated with resistance. We accordingly proceed to give a general proof of this energy theorem.

3. Energy Theorem for Flow where Resistance occurs.

Resistances to the flow have a double effect on gases: in addition to the mechanical hindrance to the flow, the gas is supplied with an amount of heat energy corresponding to the mechanical energy which is destroyed. Thus, in contradistinction to the flow of incompressible fluids, there is the possibility of re-utilizing part of the resistance energy in a further expansion, seeing that it is converted into heat.

To obtain a general idea of the relationships prevailing here, it is

convenient to establish an energy theorem which is obtained by considerations analogous to those for the momentum theorems in Chapter II, § 12 (p. 77). We consider the change of energy in a limited portion of a mass of gas in steady motion. For this it is most convenient to take a part of a stream-tube (cf. fig. 2.51, p. 77). As the flow is to be steady, the change in the condition of the portion of gas in time dt simply consists in a particle $dm = \rho_A a_A q_A dt$ vanishing at A and a particle $dm' = \rho_B a_B q_B dt$ appearing at B; in view of continuity $dm = dm'$.

We have next to express the fact that the change in energy content arising from this displacement in the mass of gas must be equal to the energy supplied from outside during time dt. Now the energy content of a particle consists of its kinetic energy, its potential energy, and its heat energy, the so-called " internal energy ". The value of this for unit mass we shall call u; u is to be measured not in heat units but in work units like a mechanical energy. Hence, if the potential energy arises from gravity only, the energy content of the mass dm is $dm(\frac{1}{2}q^2 + gz + u)$. The supply of energy to the mass in the stream-tube consists of work done by the pressures on the end surfaces and any heat which may pass through the side walls. Frictional work does not appear in the statement, as it only means a change of work into heat, not a change in energy content. The work done by the pressure on the surface a_A is given* by the product of force and distance, $a_A p_A . q_A dt$, or, if we introduce dm, $p_A dm/\rho_A = p_A v_A dm$: similarly, for B, we have $-p_B v_B dm$. Any gain in heat between A and B we shall denote by $h_{AB} dm$ (h_{AB} then means the heat gained by every unit mass in passing from A to B). The above statement about the change in energy content is accordingly as follows:

$$dm(\tfrac{1}{2}q_B^2 + gz_B + u_B) - dm(\tfrac{1}{2}q_A^2 + gz_A + u_A) = dm(p_A v_A - p_B v_B + h_{AB}).$$

Hence

$$\tfrac{1}{2}q_B^2 + gz_B + u_B + p_B v_B = \tfrac{1}{2}q_A^2 + gz_A + u_A + p_A v_A + h_{AB},$$

or, as the end section B may be altered at will,

$$\tfrac{1}{2}q^2 + gz + u + pv = \text{const.} + h. \quad . \quad . \quad . \quad (13)$$

The equation is also frequently used in its differential form

$$q\,dq + g\,dz + du + d(pv) = dh. \quad . \quad . \quad . \quad (13a)$$

The quantity $u + pv$ is much used in thermodynamics. It is called the *total heat* or *enthalpy*, and is usually denoted by i.

* For v, see footnote on p. 264.

For permanent gases we have the formulæ

$$u = \frac{1}{\gamma - 1} pv = c_v T,$$

$$i = u + pv = \frac{\gamma}{\gamma - 1} pv = c_p T,$$

where c_v, c_p are measured in work units, i.e. their ordinary values have to be multiplied by the mechanical equivalent of heat (see p. 299).

To the energy equation we may add a further statement, corresponding to the so-called first law of thermodynamics. Every particle of the gas must satisfy the condition that the heat received by conduction and the frictional work transformed into heat go to increase its internal energy and give rise to work done in expansion. Let dF be the frictional work per unit mass on an element of stream-tube; then

$$dh + dF = du + p\,dv. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Adding this to the differential equation (13a) and replacing $d(pv)$ by $p\,dv + v\,dp$, we have

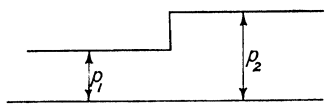
$$q\,dq + g\,dz + v\,dp + dF = 0. \quad . \quad . \quad . \quad (15)$$

Integrating this we obtain the extension of Bernoulli's theorem to motion where there is resistance:

$$\tfrac{1}{2}q^2 + gz + \int v\,dp + F = \text{const.} \quad . \quad . \quad . \quad (15a)$$

Here F denotes the frictional work done on unit mass from the initial cross-section up to the cross-section under consideration.

The quantity F/g may be called the " frictional head ", so that here the sum of velocity head, potential head, pressure head (cf. Chapter II, § 3, p. 40), and frictional head is constant.



4. Theory of the Normal Shock Wave.

(*a*) The simplest case of a discontinuous compression is the normal steady shock wave * which was first discussed by Stodola.† The gas is initially flowing in parallel lines with velocity q_1, pressure p_1, and volume v_1, and is compressed at a plane AA (fig. 4.13) into the small volume v_2, the velocity being reduced to q_2 and the pressure

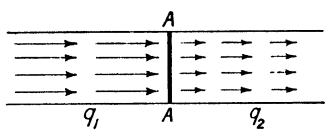


Fig. 4.13.—Normal shock wave

* German, *Verdichtungsstoss.*

† Cf. e.g. Stodola, *Dampf- und Gasturbinen*, 5th ed., p. 68 *et seq.* (Berlin, 1922).

raised to p_2. The phenomenon, which bears some resemblance to the slight elevation on the surface of water described in § 15 of Chapter II (p. 92), is governed by the following equations:

(1) the equation of continuity (in which the mass m flowing per second refers to unit area),

$$m = \frac{q_1}{v_1} = \frac{q_2}{v_2},$$

(2) the momentum equation,

$$m(q_1 - q_2) = p_2 - p_1;$$

(3) the energy equation (no gain or loss of heat),

$$\tfrac{1}{2}q_1^2 + i_1 = \tfrac{1}{2}q_2^2 + i_2.$$

Here the total heat i is a function of the corresponding p and v (see p. 270).

From the first two equations we can eliminate q_1 and q_2, and thereby obtain

$$p_2 - p_1 = (v_1 - v_2)m^2,$$

which with $m^2 = \text{const.}$ also holds for the transitional conditions within the shock wave (that is, in the pv diagram we have a straight line with slope m^2). Using this relation to simplify the third equation, we have

$$(p_2 - p_1) \cdot \tfrac{1}{2}(v_1 + v_2) = i_2 - i_1.$$

This is known as the Rankine–Hugoniot relationship. The p_2v_2 curve which it gives for given p_1, v_1 is sometimes referred to as the Hugoniot curve * or dynamic adiabatic.

Thus if three quantities, e.g. p_1, v_1 and p_2, are given, v_2 can be found from this curve, and hence m and therefore also q_1 and q_2. If, however, p_1, v_1, and q_1 are given, the calculation is more troublesome.

Of the results obtained by such calculations, assuming the simple gas laws, we may mention the following. We obtain the relation

$$q_1 q_2 = c'^2$$

for the velocities, where c' is the critical velocity of sound given on p. 266. Thus of the velocities q_1, q_2, the one is always greater and the other always less than the velocity of sound c'. According to the formulæ, which are all completely symmetrical, discontinuous compression and discontinuous rarefaction are equally possible. But by

* Given by Rankine, *Phil. Trans. Roy. Soc.*, Vol. 160, p. 277 (1870), and by H. Hugoniot, *Journ. de l'école polytechnique* (Paris, 1887, 1889); fuller details are given e.g. by R. Becker, *Zeitschr. f. Physik*, Vol. 8, p. 321 (1922). In the previous editions Hugonoit only was mentioned as the discoverer of this relationship. Rankine's paper is referred to by G. I. Taylor and J. W. MacColl (Durand, *Aerodynamic Theory*, Vol. III, p. 217).

considering the entropy, which in closed systems can only increase and can never decrease, we see that the shock wave is alone physically possible. This is also in excellent agreement with the discussion at the end of § 1 (p. 263), according to which it is only the discontinuous compression that persists, the discontinuous rarefaction being immediately transformed into a continuous wave of rarefaction. Thus q_1 is always greater than the velocity of sound c'.

By changing the system of reference we may at once apply the relationships for the steady shock wave to a wave of compression advancing over a mass of air at rest. For if we give the system in fig. 4.13 a velocity of magnitude q_1 to the left, the velocity in front of the shock is reduced to zero, the shock itself moves with velocity $c = q_1$ to the left, and the mass of gas behind the shock wave follows it with the velocity $q = q_1 - q_2$.

The momentum equation here takes the form $p_2 - p_1 = \rho_1 c q$, which gives the relationship between the increase of pressure and q, the velocity of the following gas, for small values of q, in the case where c is nearly equal to the velocity of sound. Here, therefore, the velocity of propagation c is always greater than the velocity of sound, and may, in fact, increase indefinitely as the difference of pressure does. These abnormally large (supersonic) velocities of propagation are observed in the case of explosions.

If we take into account the fact that gases have a definite, though small, thermal conductivity, we obtain a gradual transition from p_1 to p_2, which usually takes place over an interval of less than 1/1000 mm. (Prandtl,* Becker †),[44] instead of a mathematically sharp discontinuity. The increase of entropy mentioned above is chiefly brought about by transference of heat from the mass of gas which has already been compressed, and therefore heated, to that which has not yet been compressed.

In the shock wave the heat content is increased by the amount $\frac{1}{2}(q_1^2 - q_2^2)$; only part of this can be won back in the form of kinetic energy by expansion from p_2 to p_1. In the non-homogeneous shock wave (e.g. the nose-wave of a bullet (fig. 4.43, p. 298) or the discontinuity in fig. 4.24, p. 283), the different stream-tubes are heated to different extents, so that the homogeneity of the mass of gas is destroyed and hence the flow is no longer irrotational.

Note.—These discontinuous compressions are related to the "explosion-waves" which may occur in the combustion of inflammable gas mixtures. The three equations set up at the beginning of this section can be applied to these phenomena in exactly the same way, provided the amount of chemical energy transformed until the particular state under consideration is reached is inserted on the left-hand side of the energy equation. More detailed investigation, due to

* L. Prandtl, *Zeitschr. f. d. gesamte Turbinenwesen*, Vol. 3, p. 241 (1906).

† R. Becker, *Zeitschr. f. Physik*, Vol. 8, p. 321 (1922); see especially sections 6 and 7.

R. Becker,* shows that two regions are to be distinguished, that of slow or moderately fast "combustion" and that of "detonation". In the first, ignition advances by heat conduction; in the second, it is brought about by the heating which results from a shock wave, and advances along with the latter. In combustion the rate of advance depends on physical and chemical data (conduction of heat, velocity of reaction, &c.). On the other hand, a detonation such as can be produced by a mechanical shock acting on a small quantity of an explosive has a definite velocity of propagation, which can be calculated from heat data in advance and which is characterized by the fact that the combustion front is never caught up again by the wave of rarefaction following it in the "fumes", i.e. the burnt-up gases.

If we look more closely into this, we find that there are two distinct phenomena, firstly the shock wave which is necessary for detonation, and secondly the considerably slower chemical reaction. In order to investigate the steady advance of the process which sets in after a preliminary non-steady stage depending on the intensity of the detonating impulse, we have to consider a Hugoniot curve which rises gradually, corresponding to gradual combustion; the momentum line, whose slope m^2 has already been established by the rule given above,† gives the point (p, v) representing the state at a particular time, by its intersection with the particular Hugoniot curve. As the Hugoniot curves are strongly concave, we see that the pressure at the point of detonation is markedly higher than at the end of combustion. It is only comparatively recently that these phenomena were explained theoretically by J. B. Seldovitsch ‡ in 1940 and independently by W. Döring § in 1941.

If the shock wave which originally started the detonation gave rise to a higher pressure than that of the stationary detonation, the wave of rarefaction catches up on the shock wave and lowers it to the stationary value. Whether a shock of lower pressure is raised automatically to the detonation pressure or not is again a physico-chemical question.

What we have said above applies also to liquid and solid explosives. In the explosive mixture of hydrogen and oxygen obtained by the electrolysis of water the velocity of propagation of detonation at room temperature and atmospheric pressure is about 2800 metres per second, and with nitroglycerine about 7400 metres per second. The respective detonation pressures are about 18 and 100,000 Kg./cm.^{2}

(*b*) Now that we have discussed the theory of shock waves, we are in a position to answer the question left open in § 2 (p. 269) as to what may be expected to occur in a Laval nozzle when the external pressure p_2 behind the nozzle lies between p_o and p_u. We shall assume that in such cases a normal shock wave is set up somewhere behind the throat, in which the transition is made from a supersonic velocity to a subsonic velocity. As in the shock wave the total energy remains

* *Zeitschr. f. Elektrochemie*, Vol. 23, pp. 40, 304 (1917); *Zeitschr. f. Physik*, Vol. 8, p. 348 (1922); see also W. Jost, *Explosions- und Verbrennungsvorgänge in Gasen*, p. 161 *et seq.* (Berlin, 1939). For further references, see W. Döring, *F.I.A.T. Review. Hydrodynamics and Aerodynamics*, Chapter V.

† The rule means that the momentum line must be drawn from the initial state (p_1, v_1) to touch the Hugoniot curve of the final state of combustion. The velocity of propagation found in this way for gases is in very good agreement with experiment. (The calculation cannot be made for solid explosives for want of an equation of state.)

‡ *Journ. f. exp. u. theor. Physik* (Russian), Vol. 10, p. 542 (1940).

§ *Ber. d. Ballist. Inst. d. Luftwaffe*, 5a (1941); *Ann. d. Physik*, Vol. 43, p. 421 (1943).

unaltered, we may obtain information about the phenomena which then occur by supplementing fig. 4.12, p. 268, with pressure curves corresponding to the same rate of discharge and the same total energy but to smaller initial pressures p_1. This is done in fig. 4.14. The transition from the line representing the normal pressure distribution $p_1 - p_u$ to the new curves takes place by way of the shock wave, the position of which is uniquely determined by the condition that the momentum equation is satisfied. In actuality, however, the phenomena are by no means so definitely determined, as oblique shock waves (cf. § 6, p. 283) or other resistance phenomena may occur instead of the normal shock wave. In actual cases the process is also somewhat altered by the fact that, owing to friction at the boundary, separation of the flow (cf. Chapter III, § 6, p. 135) arises from any sudden increase of pressure. As a result, in two-dimensional flow the normal shock wave is replaced by two oblique shock waves crossing one another as in fig. 4.19, often followed by further waves if the flow remains separated. Owing to the turbulence arising from separation of the flow, the rise in pressure remains considerably below the theoretical value.

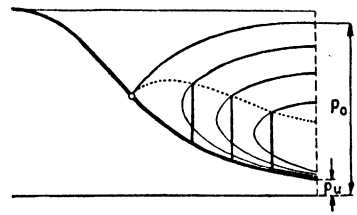


Fig. 4.14.—Theoretical distribution of pressure in the Laval nozzle with shock waves

These phenomena are illustrated by figs. 4.16–19, which are reproduced from photographs by the author * of the flow through rectangular nozzles with glass walls, taken by the " Schlieren " method. This method (due to Töpler †) may be briefly explained as follows. Its purpose is to make small differences in density visible by means of

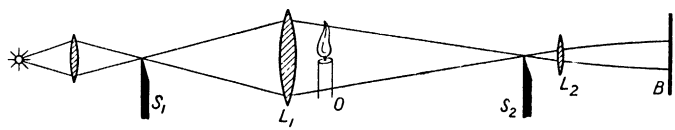


Fig. 4.15.—The principle of the " Schlieren " method

differences of illumination. In order to do this the light from a suitable source is concentrated in a small region by means of a convex lens (fig. 4.15); at this point a screen S_1 is set up so that its sharp edge cuts off part of the light. A sharp image of the spot of light and the edge of the screen is then thrown on a second screen S_2 with a sharp edge

* *Physikal. Zeitschr.*, Vol. 8, p. 23 (1907), from which figs. 4.26–4.31 are also taken.

† A. Töpler, *Beobachtungen nach einer neuen optischen Methode* (Bonn, 1864: *Ostwalds Klassiker*, No. 157).

parallel to the edge of the first screen, by means of an accurately corrected lens L_1. The second sharp edge is placed so that a narrow pencil of light passes over the edge of S_2, the rest of the light being cut off. If we then place in the path of the rays an object O, e.g. a candle flame, which causes the rays of light to be slightly deviated, more of

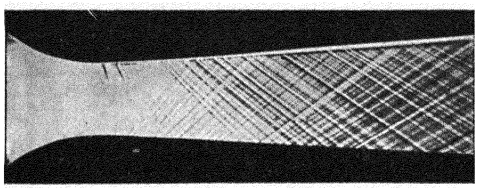


Fig. 4.16

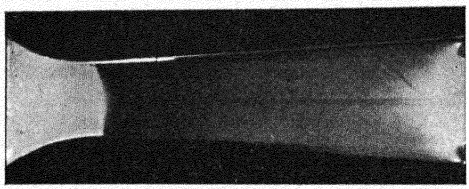


Fig. 4.17

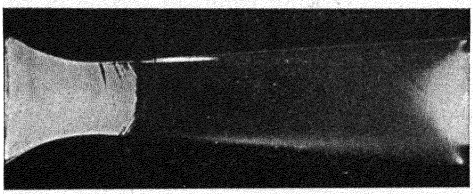


Fig. 4.18

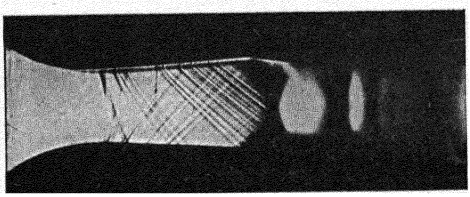


Fig. 4.19

the light passing through a part of the object which gives rise to a deviation upward passes over the edge of S_2 than if there were no deviation, whereas if the deviation is downward less light, or possibly none at all, gets through. Hence if a sharp image of the object O is thrown by a second lens L_2 on a ground-glass screen or photographic

plate, we obtain an image B of the object Θ in which the places giving an upward deviation appear bright and the places giving a downward deviation appear dark. In a non-homogeneous mass of air the deviations are proportional to the component of the density gradient in a direction at right angles to the sharp edge. Thus in a particular case a decrease in density from left to right may give brightness and an increase of density darkness; here, then, brightness means expansion, darkness means compression, provided the flow is from left to right.

The results of a series of experiments are shown in figs. 4.16–19. Fig. 4.16 illustrates the undisturbed expansion of compressed air from an initial pressure of about 7 atmospheres to atmospheric pressure. In the region where the velocity exceeds that of sound we see fine stationary waves of sound crossing one another; these are particularly well marked here because the walls of the nozzle were deliberately roughened with a file. Measurement of the angle at which the waves cross one another gives a value of q/c in good agreement with the calculated value.* Fig. 4.17 shows how the density is distributed when the velocity of sound is not attained ($p_2 > p_o$); the density diminishes until the throat is reached and then increases again. There are no stationary sound waves anywhere. Fig. 4.18 shows a shock wave ($p_2 < p_o$); the first few stationary sound waves are clearly to be seen in front of the shock wave, but behind it only the uniform area where the velocity is less than that of sound. Fig. 4.19 shows a case where the external pressure is still less; there is distinct separation of the jet from the boundary and transverse waves occur. Provided the velocity exceeds that of sound, every fine sound wavelet is identical with the corresponding one in fig. 4.16, which again shows that for supersonic velocities the phenomena downstream have no effect whatever.†

Note.—The similarity between the flow through a nozzle and the flow of water over a weir (Chapter II, § 16, p. 94) is unmistakable. In fact the velocity of low elevations plays the same part in the flow over a weir as the velocity of sound does in the present case. This analogy is frequently referred to in the literature. E. Preiswerk ‡ has worked it out mathematically and shown that in the flow of water over a smooth bed where the horizontal dimensions are large compared with the depth of water, there is quantitative agreement with the laminar flow of a hypothetical gas for which $\gamma = 2$. The depth of water at any point, h, corresponds to the density of the gas, and, since $\gamma = 2$, to the temperature of the gas also. The pressure of the gas is replaced [neglecting small vertical accelerations (p. 97)] by the shearing force per unit length $\frac{1}{2}wh^2$ (hence the assumption $\gamma = 2$).

* L. Magin, *Dissertation*, Göttingen (*Forschungsheft* 62 *des Vereins Deutscher Ingenieure*, 1908).

† Slight differences between fig. 4.16 and the supersonic part of fig. 4.19 are due to slightly different setting of the screen S_2.

‡ *Dissertation*, Zürich; *Mitteilungen aus dem Institut für Aerodynamik an der Eidgenössischen Technischen Hochschule*, Zürich, No. 7 (1938).

5. Flow with Resistance.

(*a*) *Steady Flow.*—In steady flow where there is resistance but where no heat is transmitted through the walls the total energy remains constant, the frictional energy being entirely transformed into heat.* As differences of height usually play no part, the energy equation [equation (13), p. 270] applies, so that we have

$$\tfrac{1}{2}q^2 + u + pv = \text{const.};$$

for permanent gases this equation takes the simple form

$$\tfrac{1}{2}q^2 + \frac{\gamma}{\gamma - 1}pv = \text{const.} \quad . \quad . \quad . \quad . \quad (16)$$

or

$$\tfrac{1}{2}q^2 + c_p T = \text{const.}, \quad . \quad . \quad . \quad . \quad (16a)$$

where c_p is measured in mechanical units (cf. p. 271). From this we at once deduce the important fact that—if we assume the simple gas laws—the fall of temperature as compared with the initial state is independent of the magnitude of the resistance and depends only on the velocities at the point in question. If the velocity is everywhere negligible, as in cases where the resistance to the flow is very great, the gas expands at constant temperature during the flow. The work done in expansion is completely transformed into frictional work, and the cooling due to the expansion is balanced by the heat due to friction. The same is true when a velocity already existing is annihilated by resistances.

Actual gases exhibit a slight deviation from this behaviour, which was first discovered by W. Thomson and Joule. They found that when air is made to flow through a plug of cotton-wool, the difference in pressure on the two sides of the plug being considerable, there is a slight cooling, which is explained as due to molecular attractions. This cooling amounts to 0·25° C. per atmosphere for air at ordinary temperature, but is considerably greater at low temperatures; it is on this effect that Linde's apparatus for making liquid air depends.

The flow through tubes, &c., with friction may be calculated by taking into consideration the equation of continuity, and Bernoulli's theorem [equation (15*a*), p. 271], in which a special assumption is to be made regarding the frictional work F, in addition to the energy equation [equation (13), p. 270, or equation (16)]. From this it follows that in the case of the Laval nozzle (fig. 4.12) certain corrections have to be made in the values given earlier for the velocity and pressure. These are not very large as a rule, but e.g. the velocity of sound is reached at a point somewhat behind the narrowest part of the tube and at a rather lower pressure. We shall give a more detailed account

* If there is any energy due to turbulence, it must be included in the total energy in a cross-section.

of the case of a tube of constant cross-section, for which the calculations were made by Grashof * in the way indicated above, taking friction into account.

As a preliminary we shall consider a flow with friction in which the velocity q remains constant; then $pv = \text{const.}$, by equation (16). As the resistances must be met by a fall of pressure in the direction of flow, the volume increases in the direction of flow and hence the cross-section does likewise, by the equation of continuity. In order to return to the tube of constant cross-section, we must assume that the tube subsequently narrows again; for velocities below that of sound this gives an increase in velocity and a further fall of pressure, which owing to the renewed increase of volume resulting therefrom becomes greater the closer the velocity approaches the velocity of sound (cf. the v/q curve in fig. 4.7, p. 265). The velocity of sound c' (cf. the footnote on p. 266) here represents the limiting velocity, which can only be attained at the end of the tube, if at all.

If the velocity exceeds that of sound (which of course can be achieved only by putting a Laval nozzle in front of the tube) the narrowing, as compared with the widening tube for which $q = \text{const.}$, gives rise to a decrease in velocity corresponding to the rise in pressure. Here again the velocity of sound (c') represents the last state compatible with continuous flow. This final state, however, corresponds to a length of tube which is fixed by the initial state and the amount of frictional resistance and is not very large. If the tube is actually longer than this, somewhere within it there will be a transition to a velocity less than sound, by way of a shock wave, and the rest of the flow will follow the laws described above for velocities less than that of sound.

According to experiments by W. Frössel,† the resistance coefficient λ_1 for velocities of flow below and above that of sound in smooth pipes is not noticeably different from that for non-compressible fluids. Fig. 4.20 shows pressure distribution curves based on Frössel's experiments. The curves coming from the top of the figure relate to flow at velocities below that of sound; those coming from below start with velocities above that of sound and, if the pipe is long enough, jump to velocites below that of sound by way of a shock wave. The numbers on the curves are the quantities flowing expressed as fractions of the maximum flow for a short nozzle of the same diameter and for the same reservoir pressure.(45)

M. Koppe,‡ following a suggestion of Kl. Oswatitsch and using equations (14) and (15), p. 271, with the equation of continuity and the equation of state, has

* Grashof, *Theoretische Maschinenlehre*, Vol. I, p. 592 (Leipzig, 1875); cf. also Zeuner, *Technische Thermodynamik*, 2nd ed., Vol. I, p. 261 (Leipzig, 1900).

† *Forschung*, Vol. 7, p. 75 (1936).

‡ *Diss.*, Göttingen (1946).

established a set of formulæ for flow with friction in straight pipes of constant or variable cross-section. He introduced as independent variable Mach number $\mathcal{M} = q/c$ and thereby was able to get rid of all other quantities depending on state, and set up a differential equation of the first order for $\mathcal{M}(x)$ which can be solved numerically if the cross-section $a(x)$ and the coefficient of resistance λ_1 are given (x is the distance of the cross-section $a(x)$ from a fixed point; for λ_1, see pp. 163–6). If a is a constant $\pi d^2/4$, $G = \int \lambda_1 d(x/d)$ is a second dimensionless variable. By integration $G = G(\mathcal{M})$ is obtained as a universal function of $\mathcal{M}$. Comparison with Frössel's experiments confirms the above-mentioned agreement of λ_1 with the value for incompressible flow, except in the neighbourhood of the velocity of sound, where the value is smaller. The formulæ also enable the position of any shock wave to be found when the velocity of flow is initially above that of sound.

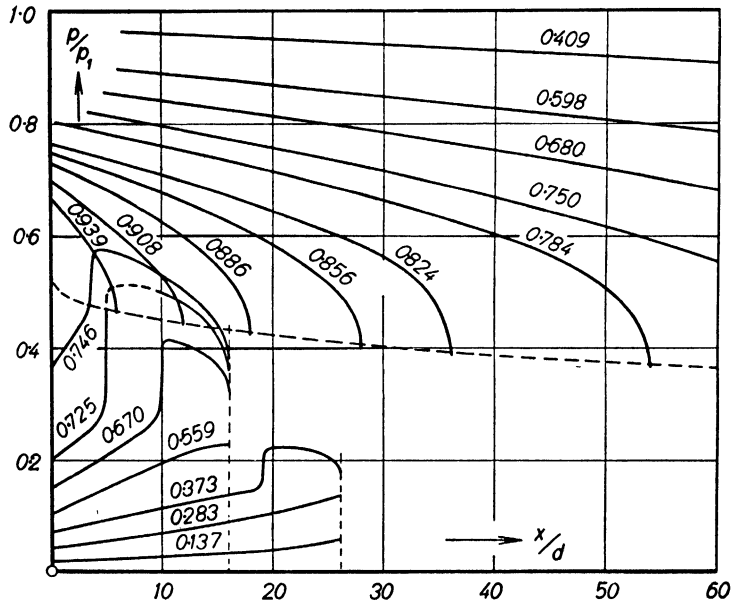


Fig. 4.20.—Pressure distribution in a gas flowing through a pipe, after Frössel

(*b*) *The Flow into and out of Vessels.*—The phenomena of flow into and out of vessels are also of interest. If we neglect any exchange of heat between the gas in the vessel and the walls, flow out of a vessel gives rise to adiabatic expansion and hence to cooling. The ejected gas, if brought to rest without mixing with the surrounding air, would have the temperature of the vessel at the instant when it flowed out, by equation (16*a*), p. 278. As the temperature of the vessel gradually falls from T_1 to T_2, the mixture of gas, which could be collected e.g. in a bell-jar, has an intermediate temperature T_m. The cooling of the gas in the vessel to T_2 and of the gas in the bell-jar to T_m is equivalent

to the work done against the external pressure in raising the bell-jar. When gas at atmospheric pressure flows into a partly evacuated vessel, the gas flowing in at any moment would revert to the external temperature if it were brought to rest and did not mix with the rest of the gas; the gas already in the vessel would be adiabatically compressed and therefore heated. In the actual state of affairs, where the gas is mixed, the heating will therefore be less than that corresponding to adiabatic compression. The heating is equivalent to the work done by the external atmospheric pressure as the gas enters the vessel. In the case of a gas at constant pressure flowing into a completely evacuated vessel we have the remarkable result, which may readily be proved by the energy relationship just mentioned, that T, the temperature in the vessel, remains constant throughout the flow and equal to γT_0, where T_0 is the external temperature.

If gas flows from one vessel into another, the gas in the first is cooled and that in the second heated. As no external work is done here, the total heat of the two vessels remains constant during the process.

6. Supersonic Flow in more than One Dimension. Flow round a Corner. Jets of Gas.

(*a*) *Flow round a Corner with a Velocity exceeding that of Sound.*— By way of an introduction to the present section we shall first consider a mass of gas moving with a velocity greater than that of sound, in which a small lowering of pressure occurs abruptly at the point A of the boundary (fig. 4.21). This diminution of pressure will be propagated in the direction given by the Mach angle α (see § 1, p. 262), and will give rise to an acceleration of the flow in the direction at right angles to the pressure jump. The velocity of flow is thereby somewhat increased and at the same time the flow is slightly deviated. If another diminution of pressure now occurs abruptly at the point A, it is propagated through the modified flow at a different angle α' (smaller than α), giving rise to a further increase in the velocity of flow and change of its direction, and so on.

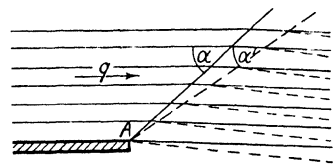


Fig. 4.21.—Flow with a velocity exceeding that of sound, the pressure being suddenly diminished

This process, which in actual fact goes on continuously, has been theoretically discussed as an irrotational motion by L. Prandtl * and Th. Meyer.† The flow has the following properties. Along any radius

* L. Prandtl, *Physikal. Zeitschr.*, Vol. 8, p. 23 (1907).

† Th. Meyer, *Dissertation*, Göttingen (*Forschungsheft* 62 *des Vereins Deutsch. Ing.*, 1908).

vector drawn from the point A the pressure is constant and so are the magnitude and direction of the velocity. Every radius vector makes an angle equal to the Mach angle with the corresponding direction of flow, so that the component of velocity at right angles to the radius vector is always equal to the velocity of sound corresponding to the conditions there.

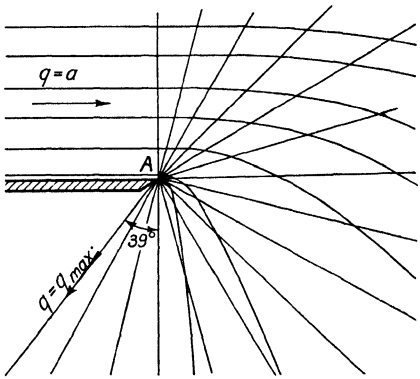


Fig. 4.22.—Flow into a vacuum

The stages of progress of the complete flow from the velocity of sound up to the maximum velocity (cf. § 2, p. 264), i.e. to that corresponding to expansion into a perfect vacuum, are shown in fig. 4.22 for $\gamma = 1{\cdot}405$. Here the direction of the jet is turned through an angle of 129°.

The principal value of this solution of the theoretical equations of motion lies in the fact that, owing to the pressure being propagated along the radius vector, any regions bounded by two radii vectores may be combined with flow in straight lines. If, for example, a stream of gas is flowing parallel to a fixed boundary with a velocity q_1 exceeding that of sound, and beyond the end of the boundary (A in fig. 4.23) there is a pressure (p_2) which falls short of that in the jet of gas (p_1) by a finite amount, the flow will continue unaltered up to the radius vector 1, which makes the Mach angle α_1 with the direction of flow ($\sin\alpha_1 = c_1/q_1$); thereafter the gas expands from pressure p_1 to p_2 in the wedge-shaped region between the radii vectores 1 and 2; once it has reached the pressure p_2 at the radius vector 2 the flow advances uniformly and in a straight line in the new direction. The direction of flow makes the angle α_2, corresponding to q_2, with the radius vector 2.

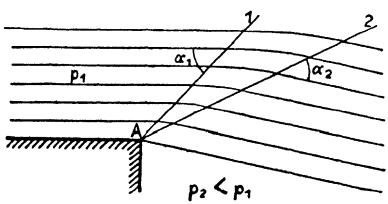


Fig. 4.23.—Wave of rarefaction

If the boundary has one or more *convex angles*, the flow is again a combination of rectilinear flow and wedge-shaped regions of expansion. The flow along a continuously curved boundary may also be discussed by combining elements where the flow is of the first type described. The boundary may even be concave; in this case, however, the solution is only accurate so long as two of the radii vectores drawn at the Mach angles do not intersect (fig. 4.24). If they do intersect, the flow at that point is discontinuous.

In the case of a *concave angle*, which corresponds to a rise of pressure, and of the flow of a jet of gas into a region of higher pressure, the flow is always discontinuous; we have an oblique shock wave (fig. 4.25; for the normal shock wave, see § 4, p. 271). Here, in fact, the radius vector 2 in fig. 4.23 would come to lie in front of the radius vector 1, which is impossible; instead there is discontinuous compression, the plane of compression lying between the regions 1 and 2. The equations for the components of velocity at right angles to the plane of compression are the same as for the normal shock wave; the transverse components of velocity which remain unaltered throughout the shock wave are merely

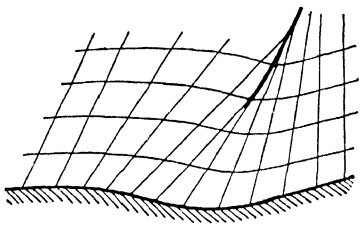


Fig. 4.24.—Flow along a curved boundary

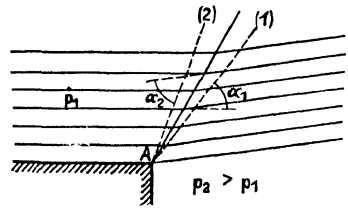


Fig. 4.25.—Oblique shock wave

superposed on the motion represented in fig. 4.13, p. 271. The figures 4.28–30, obtained by Töpler's "Schlieren" method, form good evidence for these theoretical results. A graphic method for tracing the course of oblique shock waves by means of the "shock polar" has been developed by A. Busemann.*

(*b*) *Phenomena in Free Jets of Gas.*—In the jets formed by gas flowing out of an orifice in a reservoir at high pressure, E. Mach and P. Salcher† discovered the existence of regular waves by means of "Schlieren" methods. These were subsequently demonstrated by other workers by means of pressure observations. These waves, which always occur when the velocity in the jet is greater than that of sound, may be readily explained when we know that the oblique waves of rarefaction and compression which we have described above can penetrate one another without appreciably disturbing one another, and further that they are totally reflected at the boundaries of the free jet in such a way that a wave of rarefaction is reflected as a wave of compression and conversely.

In the case of a jet of gas flowing in parallel lines from an orifice into a free space with a velocity exceeding that of sound, the following statements may be made if we assume that the motion is two-dimen-

* *Aachener Vorträge*, p. 162 (1929).

† E. Mach and P. Salcher, *Sitzungsber. d. Wiener Akad., Math.-Naturw. Klasse*, Vol. 98, 1*Ia*, p. 1303 (1889).

sional, i.e. the mouthpiece is a fairly long rectangle. If the pressure in the outside region is less than that in the jet (fig. 4.26), wedge-shaped waves of rarefaction like that in fig. 4.23 start from each edge of the orifice; these cross one another and are reflected from the opposite boundaries of the jet as waves of compression. As these advance they taper away and after traversing the jet are reflected as waves of rarefaction; the whole process then begins afresh. The pressure in the middle of the wave, p_3, bears the same ratio (less than unity) to the external pressure p_2 as the latter does to p_1. If the external pressure is greater than the pressure in the jet, oblique shock waves are first

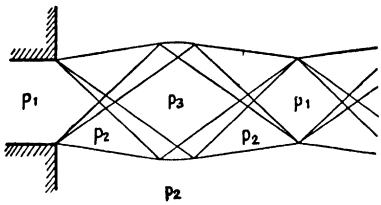


Fig. 4.26.—Wave diagram; initial velocity greater than that of sound

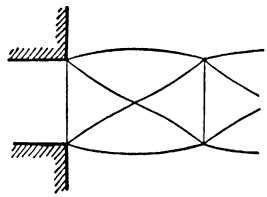


Fig. 4.27.—Wave diagram; initial velocity equal to that of sound

formed as in fig. 4.25; these are reflected as wedge-shaped waves of rarefaction, which thenceforward behave as in fig. 4.26. If the initial velocity is equal to that of sound (as is always the case with openings which do not widen like the Laval nozzle (§ 2, p. 266)), the initial value of the Mach angle is 90°, and owing to the wedge-shaped region spreading over the whole region fig. 4.26 is replaced by fig. 4.27, the double cross becoming a single one.

Photographs of such waves by Prandtl are shown in figs. 4.28–31. In fig. 4.28 the pressure in the jet exceeds that of the surrounding region, in fig. 4.29 it is equal to it, and in fig. 4.30 it falls short of it; in all three photographs the velocity of efflux is the same. Fig. 4.31 exemplifies the case where the velocity at the opening is equal to the velocity of sound. (In all the photographs brightness indicates rarefaction and darkness compression.)

If, as is usually the case, the motion in the jet is not wholly along parallel lines on leaving the opening, the wave diagrams become considerably more complicated; the wave-length, however, remains fairly constant. As what we have to deal with is essentially a wave traversing the jet twice at the Mach angle, the wave-length for two-dimensional motion, taking equation (5), p. 262, into consideration, is

$$\lambda = 2d_m \cot\alpha_m = 2d_m \sqrt{\{(q/c)_m{}^2 - 1\}},$$

where d_m is the mean diameter of the jet and α_m, $(q/c)_m$ are mean values of α and q/c.

In jets emerging from circular orifices the relationships are not

nearly so simple, owing to the conical intersections of the waves causing considerable disturbance. A photograph of the jet from a narrowed mouthpiece, by L. Mach, was given in fig. 4.8 (p. 266). The wave-length in such jets (the velocity of efflux being equal to the velocity of sound) was experimentally found to be

$$\lambda = 0{\cdot}89d\sqrt{\{(p_1 - 1{\cdot}9p_2)/p_2\}}$$

by R. Emden; d is the diameter of the orifice, p_1 the pressure in the reservoir, and p_2 that in the external space.

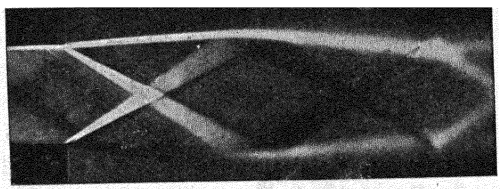


Fig. 4.28.—Flow with a velocity exceeding that of sound; pressure at the opening greater than that of the surrounding region

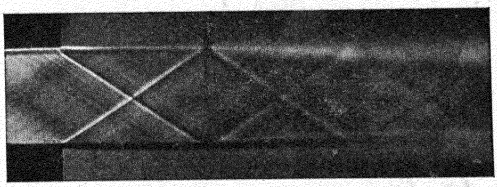


Fig. 4.29.—Flow with a velocity exceeding that of sound; pressure at the opening equal to that of the surrounding region

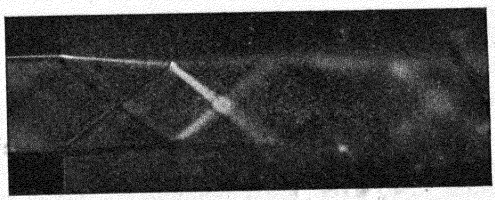


Fig. 4.30.—Flow with a velocity exceeding that of sound; pressure at the opening less than that of the surrounding region

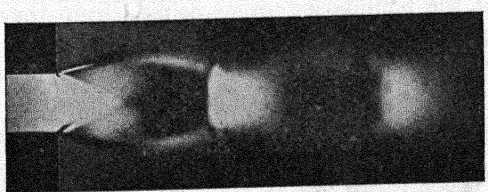


Fig. 4.31.—Flow with the velocity at the opening equal to that of sound

7. General Approximation Method for Two-dimensional Supersonic Flow.

The considerations of the previous section may be generalized as follows by introducing a method of approximation.* Here the changes of state, which are actually continuous, as shown in fig. 4.22 or in the separate portions of figs. 4.23, 4.26, and 4.27, must be broken down into elementary discontinuous stages, each of which is of the type shown in figs. 4.21 and 4.25 (just as a continuously curving line is replaced for purposes of integration by a series of steps or polygons). The individual waves of rarefaction and compression are now supposed to be all of such a type that the stream-line on meeting them is always deflected through a definite angle chosen once for all, e.g. 2°, so that the velocities between the waves can only have certain definite directions, all of which differ from one another by exact multiples of 2°. In virtue of the relationship between the magnitudes of the velocity before and after each wave and the given angle of deviation, only a discrete series of absolute values of the velocity can occur. If we draw a velocity diagram or hodograph, i.e. plot all the velocity vectors from a single origin O, we therefore obtain a regular network with circular symmetry (cf. fig. 4.32). The points of the network are the permissible velocities; each line joining two neighbouring points represents one of the transitions taking place in a 2° wave. The conditions of the flow in fig. 4.22, p. 282, where the velocity is constant along each radius vector, are represented by the black curve in fig. 4.32. (So long as we have to do with waves from one side only, as in the cases of figs. 4.22 and 4.24, we move backwards or forwards along *one* curve only in the velocity diagram; in general, however, there are waves from both sides.) The problem of drawing the stream-lines for a given case is not completely determinate unless the velocities and directions of flow are given for a cross-section and the pressure or the direction of flow is given along the two lateral boundaries of the flow. Having the pressure given as constant (and hence, by Bernoulli's theorem, the corresponding velocity) corresponds to the boundary of a free jet; having the direction of flow given corresponds to a given fixed boundary. These boundary conditions are satisfied by making compression waves or rarefaction waves start afresh from the boundaries in such a way as to fulfil the conditions. At the boundaries waves coming from the interior of the jet are reflected, and in such a way, in fact, that at the boundary of a free jet, where the velocity is to remain constant, a compression wave becomes a rarefaction wave and conversely, whereas at a fixed boundary, where the direction is to remain constant, a compression is

* The method is due to L. Prandtl and A. Busemann (*Festschrift zum 70. Geburtstag von Prof. A. Stodola*, p. 499 (Zürich, 1929); *Handb. d. Exp.-Phys.*, Vol. IV, 1, p. 421 *et seq.*).

reflected as a compression and a rarefaction as a rarefaction. The two relationships may easily be deduced from the velocity diagram.

It is also important to note that the method is applicable only to cases where the constant of Bernoulli's equation is the same for all

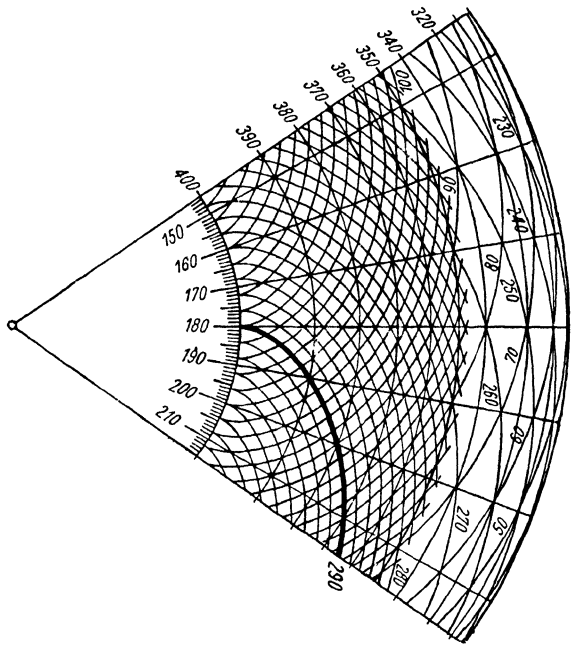


Fig. 4.32.—Velocity diagram. The difference of the numbers attached to two intersecting curves at once gives the angle in degrees of the radius vector drawn to the point of intersection; the sum gives a number by means of which the velocity and the pressure can be read off from a table

the stream-lines, i.e. to irrotational motion (cf. Chapter II, § 8, p. 58). It is only under these conditions that the simple velocity diagram suffices.

When several compression waves of the same type run together they form a shock wave. So long as the pressure jump in the shock wave is not very large the shock may as an approximation be taken as reversible, i.e. the velocities and directions may again be derived from the velocity diagram. For more intense shock waves, however, the deviations are considerable. A graphical process applicable to these shock waves has been given by A. Busemann.*

There are a number of points which are worth noting as useful in the carrying out of the constructions. In the first place, the line repre-

* A. Busemann, " Verdichtungsstösse in ebenen Gasströmungen ", *Aachener Vorträge*, p. 162 (1929), from which figs. 4.47–4.49 are taken; see also *Handb. d. Exp.-Phys.*, Vol. IV, 1, p. 431 *et seq.*

senting the wave of rarefaction or of compression is in each case at right angles to the line joining the two corresponding points in the velocity diagram. This may be seen as follows. As we mentioned above, the component parallel to the wave-front is unaltered by its passage through the wave, and only the component at right angles to the wave-front increases or decreases, according as it is a wave of rarefaction or a wave of compression. This merely means that the vector joining the points in the velocity diagram which correspond to the conditions in front of and behind the wave is at right angles to the wave-front.

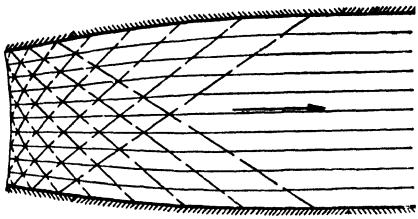


Fig. 4.33.—Production of a parallel jet

If we are dealing with a perfect gas, i.e. with the simple adiabatic formula $p = \text{const.}\ \rho^{\gamma}$, the heavy line in fig. 4.32 (and similarly the other lines of the same type) will become an epicycloid, and then special simple methods for finding the directions of the tangents to the curves are available. Here we cannot go any further into these details, and must refer the reader to the publications cited.

As a practical example we shall give the solution of the problem of the shape which must be given to a widening nozzle in order that it may give rise to a parallel jet. Here the condition is that the waves arriving at the wall shall no longer be reflected, so that the jet may be free from deflecting waves. This is attained by giving the wall a suitable change of direction at every point where a wave impinges on it. (If there were no change of direction a wave of rarefaction would be reflected as a wave of rarefaction; if the wall is given a bend which would by itself produce an equally intense wave of compression, the rarefaction and the compression cancel each other.) The result of the construction with changes of direction of 2° is shown in fig. 4.33.

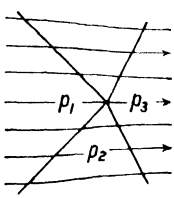


Fig. 4.34

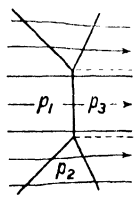


Fig. 4.35

The flows shown in figs. 4.26 and 4.27 (p. 284) can of course also be constructed throughout by the above method. Two other cases will be discussed * in § 10, p. 300.

* Certain axially symmetric cases of supersonic flow have also been discussed theoretically; see G. I. Taylor and J. W. MacColl, *Proc. Roy. Soc.* (A), Vol. 139, p. 278 (1933), (conical-nosed bullets); A. Busemann, *Luftfahrtforschung*, Vol. 19, p. 137 (1942), (general conical flow); R. Sauer, ibid., p. 148 (flow about the nose of a shell of any shape for small angles of incidence).

Note.—Two shock waves of moderate intensity as in fig. 4.30, p. 285, will meet and pass each other without much mutual disturbance. If the intensity is fairly great, however, it must be noted that propagation beyond the point of meeting occurs in regions of the gas which not only are subject to a higher pressure but also are heated to more than an adiabatic extent by the non-reversible phenomenon of the previous shock wave. For shock waves beyond a certain intensity, in addition to the intersection of the two shock waves (fig. 4.34), there is also the situation shown in fig. 4.35, in which the rise of pressure $p_3 - p_1$ in one shock wave is exactly the sum of the two original rises of pressure $p_2 - p_1$ and $p_3 - p_2$. Since the total loss of pressure in a shock wave is approximately proportional to the third power of the jump in pressure, we find that the loss behind the one shock wave is greater than behind the two, and the temperature is likewise higher. The theory of the "forked shock wave" has been put forward several times independently. A comprehensive account has been given by W. Wüst.* The phenomena which occur when two spherical explosion waves meet, discovered by E. Mach in 1875 and often observed since, are related to the forked shock wave; an explanation of them has recently been given by F. Schultz-Grunow.†

8. **Flow with Subsonic Velocities.**‡

To this section we shall preface a general discussion of steady flow in which the velocity differs only very little either in magnitude or in direction from a given velocity u_0. This velocity u_0 may itself be taken large, and we shall at first leave open the question whether it lies below or above the velocity of sound. For the sake of simplicity we shall assume that the flow is two-dimensional. We shall denote the small deviations of the velocity from u_0 by u and v, and shall carry out our calculations only for quantities of the first order throughout. Let the flow be irrotational.

To begin with, we have the generalized form of Bernoulli's theorem [equation (6), p. 264],

$$\int \frac{dp}{\rho} + \frac{(u_0 + u)^2 + v^2}{2} = \text{const.},$$

or in its differential form

$$\frac{dp}{\rho} + u_0 du + \text{quantities of 2nd order} = 0.$$

We may, however, replace $\frac{dp}{\rho}$ by $\frac{dp}{d\rho} \cdot \frac{d\rho}{\rho} = c^2 \frac{d\rho}{\rho}$ [cf. equation (3), p. 260]. Hence

$$d\rho = -\rho \frac{u_0}{c^2} du. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 28, p. 73 (1948).

† *Ibid.*, p. 30.

‡ For a review of the literature, see F. Eser, *Luftfahrtforschung*, Vol. 20, p. 220 (1943).

Continuity is expressed by the equation

$$\frac{\partial}{\partial x}[\rho(u_0+u)]+\frac{\partial}{\partial y}(\rho v)=0$$

(cf. the corresponding formula for flow without volume changes in Chapter II, § 2, p. 37). If ρ_0 is the mean value of ρ the first-order terms give

$$u_0\frac{\partial \rho}{\partial x}+\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0,$$

or, taking account of (17), where ρ may again be put equal to ρ_0, and cancelling the factor ρ_0, we have

$$\frac{\partial u}{\partial x}\left(1-\frac{{u_0}^2}{c^2}\right)+\frac{\partial v}{\partial y}=0. \qquad \ldots \quad (18)$$

If we introduce the potential ϕ (cf. Chapter II, § 9, p. 61) by putting

$$u=\frac{\partial \phi}{\partial x} \quad \text{and} \quad v=\frac{\partial \phi}{\partial y},$$

(18) becomes

$$\frac{\partial^2 \phi}{\partial x^2}\left(1-\frac{{u_0}^2}{c^2}\right)+\frac{\partial^2 \phi}{\partial y^2}=0. \qquad \ldots \quad (19)$$

This differential equation again brings out very clearly the difference between flow with a velocity below that of sound and flow with a velocity above that of sound. The coefficient of $\partial^2\phi/\partial x^2$ passes through zero when u_0 passes through the velocity of sound. Below this limit the differential equation is of the (elliptic) type corresponding to irrotational flow without volume changes, and above it of the (hyperbolic) type corresponding to the vibrations of a string. For $u_0=c$, $\partial^2\phi/\partial y^2=\partial v/\partial y=0$ and $\partial^2\phi/\partial x^2=\partial u/\partial x$ is arbitrary, i.e. stationary waves with a front parallel to the y-axis can exist. For $u_0>c$ any function F of $y\pm x\tan\alpha$ which is continuous and twice-differentiable but otherwise arbitrary is a solution, provided α is suitably determined. For

$$\frac{\partial^2 \phi}{\partial x^2}=F''\tan^2\alpha \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2}=F'';$$

in order to satisfy equation (19), we must have

$$\tan^2\alpha\left(\frac{{u_0}^2}{c^2}-1\right)=1,$$

i.e.

$$\tan\alpha=\pm\frac{c}{\sqrt{({u_0}^2-c^2)}}.$$

Hence $$\sin\alpha = \frac{\tan\alpha}{\sqrt{(1+\tan^2\alpha)}} = \pm\frac{c}{u_0}.$$

The solution therefore represents waves of arbitrary form, whose straight fronts ($y = \pm x\tan\alpha + \text{const.}$) are inclined to the x-axis (the mean direction of the stream-lines) on either side at an angle α equal to the Mach angle; that is, we have been led to the same conclusion as that resulting from the simplified discussion in § 1, p. 262.

In the case where the velocity is less than that of sound, to which we shall now return, characteristic solutions are obtained in the following form. We shall make a comparison between our flow and a corresponding flow without volume changes in which the density and the undisturbed velocity u_0 are the same. We shall denote the co-ordinates of the flow without volume changes by X and Y, the small additional velocities by U and V, and the corresponding velocity potential by Φ. By equation (20), p. 61, the latter must satisfy the equation

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2} = 0. \quad \ldots\ldots \quad (20)$$

The two flows can accordingly be correlated by making the potentials ϕ, Φ at corresponding points proportional, i.e. by putting

$$\phi(x, y) = \alpha\Phi(X, Y), \quad \ldots\ldots \quad (21)$$

where α is a numerical factor by means of which the magnitude of the velocity potential ϕ relative to Φ, that of flow without volume changes, can be altered in certain cases. The velocities u, v, which are assumed to be small, enter into it, subject to another change resulting from equation (22); so do the small transverse dimensions of the disturbing object, which to a first order depend only on v/u_0, and possibly also the angle of incidence of the disturbing object.

In order that ϕ shall satisfy equation (19) and Φ equation (20) simultaneously, the relationships between corresponding values of x and X on the one hand and of y and Y on the other must hold for *different* scales. If we put $Y/y = \beta X/x$, the potentials may be made to satisfy (21) by suitable choice of β. For simplicity we arbitrarily put $X = x$ and thereby obtain

$$Y = \beta y. \quad \ldots\ldots \quad (22)$$

Using (21) and (22) we may write equation (19) in the following form:

$$\alpha\frac{\partial^2\Phi}{\partial X^2}\left(1 - \frac{{u_0}^2}{c^2}\right) + \alpha\beta^2\frac{\partial^2\Phi}{\partial Y^2} = 0. \quad \ldots \quad (23)$$

This equation is identical with (20) if $\beta^2 = 1 - {u_0}^2/c^2$; α remains arbitrary in the first instance. The ratio u_0/c, which is very important

in connexion with the type of flow, is now usually known as the *Mach number*, and denoted by $\mathscr{M}$. That is,

$$\beta = \sqrt{(1 - \mathscr{M}^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

As β is always less than unity, it follows from (22) that the upward extension (y) of a velocity field or pressure field is greater in the compressible flow than in the incompressible flow. As the velocity of sound is approached β tends to zero, the height of the disturbed flow then increasing without limit.

The angle δ at which a stream-line is inclined to the x-axis is given by

$$\tan\delta = \frac{v}{u_0 + u},$$

which in our approximate calculation may be replaced by $\dfrac{v}{u_0} = \dfrac{1}{u_0}\dfrac{\partial\phi}{\partial y}$. Similarly we have

$$\tan\Delta = \frac{V}{u_0} = \frac{1}{u_0}\frac{\partial\Phi}{\partial Y}.$$

If the two flows are set up by one and the same body, say one like that in fig. 4.36, we must have $\tan\delta = \tan\Delta$ for corresponding points on the boundary stream-line, i.e. $\partial\phi/\partial y = \partial\Phi/\partial Y$ when $X = x$, or, by (21) and (22), $\alpha\beta = 1$; that is,

$$\alpha = \frac{1}{\beta} = \frac{1}{\sqrt{(1 - u_0^2/c^2)}} = \frac{1}{\sqrt{(1 - \mathscr{M}^2)}}. \quad . \quad . \quad (25)$$

In order to compare the distribution of pressure in the two flows we have merely to compare the pressure gradients in the x-directions. As the scale is unaltered in this direction, the finite pressure differences in the two flows are in the same ratio as these gradients. We form the principal term of Euler's equation (p. 41), which in this case is $\rho(u_0+u)\partial u/\partial x$ and may again be simplified to $\rho u_0 \partial u/\partial x = \rho u_0 \partial^2\phi/\partial x^2$, and compare it with the expression for the incompressible flow, $\rho u_0 \partial^2\Phi/\partial X^2$. The ratio of the two expressions is obviously equal to α; from this we conclude that to a first approximation the pressure differences in the compressible flow are $(1 - \mathscr{M}^2)^{-\frac{1}{2}}$ times as great as in the incompressible flow about the same (sharp-ended) body. This relation, as has been confirmed by experiment,* is also approximately true for slender aerofoils and aerofoils at small angles of incidence, but only

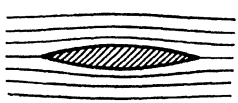


Fig. 4.36

* Examples of measurements on aerofoils near the velocity of sound are given in the report of Stack, Lindsay and Littel, Report No. 646, *National Adv. Comm. for Aeronautics*, Washington, 1938.

if the velocity of sound is not reached (see below) at any part of the aerofoil. The lift then increases as a result of compressibility in the ratio given by equation (25) (Prandtl's rule).*(46)

E. Krahn † has shown that the above rule may also be applied to fairly thick non-tapering bodies by starting from the pressure distribution $P - P_\infty$ calculated or found by experiment for the incompressible flow for a body broadened by the factor $1/\sqrt{(1 - \mathcal{M}^2)}$. From this the pressure distribution $p - p_\infty$ for the compressible flow is deduced by the formula (slightly altered from the form in which it is given by Krahn)

$$p - p_\infty = (p_0 - p_\infty)\frac{P - P_\infty}{P_0 - P_\infty},$$

where P_∞, p_∞ are the pressures in the undisturbed flow and P_0, p_0 the pressures at the stagnation point; $p_0 - p_\infty$ is to be calculated from the adiabatic compression for the velocity v_∞. For small values of the Mach number we may use the rough-and-ready formula

$$p_0 - p_\infty = \tfrac{1}{2}\rho_\infty v_\infty^2(1 + 0{\cdot}27\,\mathcal{M}^2);$$

cf. the value β on p. 299.

Conversely, we may raise the question how a body must be shaped in order that the pressure differences in the compressible flow may be the same as in the incompressible flow. This question is obviously of importance in the case where the pressure in the corresponding incompressible flow approaches the limit beyond which the flow becomes separated from the body as a result of friction. Here α must obviously be taken equal to unity. Then, however, $\tan\delta = \beta\tan\Delta$, i.e. the closer u_0 approaches the velocity of sound, the slenderer must the body in the compressible flow be made, if separation of the flow is to be avoided. This is in good agreement with experimental fact.

An easy example of the calculations based on this theory is provided by the following. Fluid is moving with mean velocity u_0 past a slightly undulating boundary, $y_1 = a\sin\mu x$, where $\mu = 2\pi/\lambda$, λ being the wave-length. Since $v_0/u_0 = dy_1/dx$, we have $v_0 = u_0 a\mu\cos\mu x$ in the neighbourhood of $y = 0$. In the comparison flow $V_0 = v_0$ for $Y = 0$, and hence the corresponding potential, which must decrease without limit as Y increases (cf. the case of water waves on p. 87), is

$$\Phi = -u_0 a\cos\mu X\, e^{-\mu Y}.$$

The potential corresponding to this in the compressible flow is

$$\phi = -A\cos\mu x\, e^{-\mu y\sqrt{(1-\mathcal{M}^2)}}, \qquad \ldots\ldots \quad (26)$$

which for $y = 0$ gives $v_0 = \partial\phi/\partial y = A\mu\sqrt{(1 - \mathcal{M}^2)}\cos\mu x$. Comparison with the above gives $A = u_0 a/\sqrt{(1 - \mathcal{M}^2)}$ in agreement with equation (25). Fig. 4.37

* Given in the author's lectures on aerodynamics (1922). It was rediscovered by H. Glauert, *Proc. Roy. Soc.* (A), Vol. 118, p. 113 (1928).

† *AVA-Bericht*, 43/A/20.

shows the incompressible flow; fig. 4.38 that for $u_0 = 0{\cdot}9c$. The corresponding flow for $u_0 = 1{\cdot}25c$ ($\sin\alpha = 0{\cdot}8$) is shown in fig. 4.39. The corresponding formula for the potential (p. 290) is

$$\phi = \frac{u_0 a}{\sqrt{(\mathscr{M}^2 - 1)}} \sin\mu\{x - y\sqrt{(\mathscr{M}^2 - 1)}\}. \quad . \ . \ . \quad (27)$$

Here, for the sake of simplicity, the approximate theory given above was stated for two-dimensional flow only. It can be written down for the three-dimensional case in the same way; each extension of the flow region in a direction other than that of the main flow u_0 must be dealt with as was done for the y-direction above, to make the

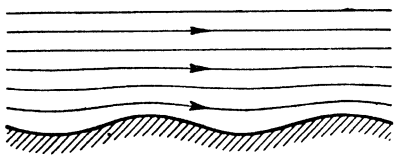


Fig. 4.37. $u_0 \ll c$

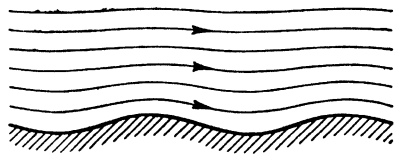


Fig. 4.38. $u_0 = 0{\cdot}9c$

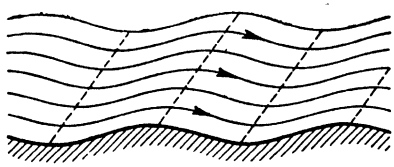


Fig. 4.39. $u_0 = 1{\cdot}25c$

transition from incompressible fluid to compressible. Instead of this, we may leave the transverse dimensions unaltered and shorten the length * in the ratio $\sqrt{(1 - \mathscr{M}^2)}$. (This method corresponds to that usual in relativity theory, which is the natural one when it is a question of real shortening of the same object.) Wing thicknesses and angles of incidence may be treated in the same way as the span (so that the whole form is an affine transformation of the original form). This was suggested by B. Göthert.† [47] Here we have $\tan\delta = \tan\Delta/\sqrt{(1 - \mathscr{M}^2)}$ and hence $\alpha = 1/(1 - \mathscr{M}^2)$. In this linearized theory it seems equally feasible, however, to deal with the span, &c., by Göthert's method and determine the wing thicknesses and angles of incidence independently by the rules discussed above.

If for fairly large values of the Mach number the velocity at any point on the slender body approaches the velocity of sound, the approximation in which the deviation of the velocity from u_0 was taken

* A. Busemann, *Jahrb. d. Wiss. Ges. f. Luftfahrt*, 1928, p. 96.

† *FB* 1275 (1940); *Jahrb. d. Deutsch. Luftfahrtforschung* (1941), part I, p. 156.

as linear no longer suffices. The accuracy of the pressure field is considerably increased by making the Mach number refer not to the mean velocity of flow but to the velocity at the point in question. This means that the velocity of sound is reached locally for a smaller value of u_0 than would be expected from the approximate theory.

The hints given here have been extended on the theoretical side to higher values of the Mach number by means of expansions in series: see especially papers by A. Busemann * and by W. Hantzsche and H. Wendt.† E. Krahn ‡ has developed a different method of calculation on an empirical basis; in collaboration with A. Betz § he carried it to a second approximation which gives very accurate results even for thick profiles.

The behaviour of the flow in the immediate neighbourhood of the velocity of sound has been qualitatively explained. True, the methods developed for subsonic velocities are not of the same type as those for supersonic velocities, and so fail on the verge of the velocity of sound, so that cases of flow in which the velocity of sound is surpassed cannot be treated by means of them.(48) Isolated examples of such flow, however, have been discussed || and serve to show that a continuous transition without singularities is possible in regions of finite extent,¶ but certainly not for every prescribed form of boundary (even if of continuous curvature).

This is obviously connected with the phenomena shown in fig. 4.24, p. 283. It is found by observation that the transition from subsonic to supersonic velocities is always continuous, but that the reverse transition readily leads to the appearance of shock waves (which in their turn lead to separation of the flow, and that in turn increases the shock strength). This phenomenon, known as the "compressibility burble", is the reason for the extraordinary falling off in the profile qualities of aerofoils in which the velocity of sound is locally surpassed at points on the suction side. Figs. 4.40 and 4.41 give "Schlieren" photographs of a local transition to supersonic velocities. The supersonic region is made visible by light scratches on the surface of the model. It extends wherever the waves starting from the scratches run

* *Schriften d. Deutsch. Akad. d. Luftfahrtforschung*, No. 18 (1940).

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 22, p. 72 (1942); Vol. 23, p. 185 (1943); Vol. 24, p. 234 (1944).

‡ *AVA-Bericht* 42/A/01.

§ *AVA-Bericht* 45/A/18.

|| See e.g. the reviews by W. Tollmien and H. Görtler in *Zeitschr. f. angew. Math. u. Mech.*, Vol. 17, p. 117 (1937); Vol. 19, p. 325 (1939); Vol. 20, p. 254 (1940).

¶ Here belong the analytical solutions which cannot be extended beyond so-called "limit lines", see e.g. W. Tollmien, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 21, pp. 140, 308 (1941). In actual cases of flow, shock waves occur before the limit line is reached, and then the flow usually continues at a speed below that of sound. For another more recent account of the limit line theory see W. Tollmien and M. Schäfer, *AVA-Bericht* 114 (1946). This paper also gives a method for calculating the shock wave that starts off with zero strength at the beginning of the limit line.

obliquely to the stream-lines. The adjacent parts of the waves which are perpendicular to the stream-lines belong to the subsonic region. They die out upwards as calculated in equation (26), p. 293. In both figures we see the shock wave which cuts off the subsonic region.

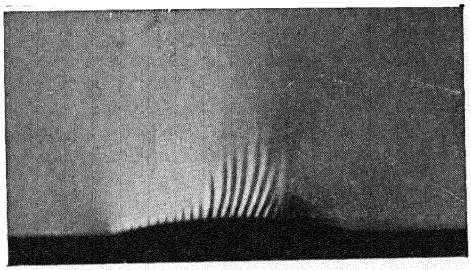


Fig. 4.40.—Reservoir pressure 1·60 atmosphere

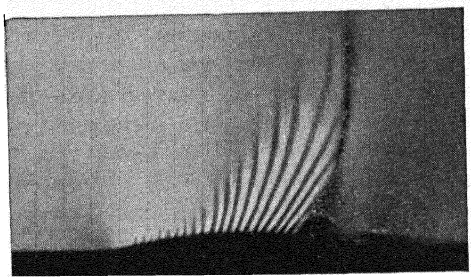


Fig. 4.41.—Reservoir pressure 1·89 atmosphere

"Schlieren" photographs of a local transition to supersonic velocities

The distribution of pressure in such a transition is given in fig. 4.42, based on American experiments.* W. Frössel † has made extensive measurements of the pressure distribution on and in the neighbourhood of similar curved bodies in the small supersonic wind tunnel at Göttingen, where figs. 4.40 and 4.41 were obtained.

The cause of the deterioration of profiles mentioned above is understandable from fig. 4.42. Owing to the occurrence of fairly intense shock waves the drag increases so markedly that flight becomes impossible when the Mach number is close to unity. The velocity of sound can be approached more closely without larger losses the smaller the curvature of the profile is. The profile *b* of fig. 3.79 (p. 197) is therefore faster than the profile *a* from this point of view also.

The peculiar features of flow past convex surfaces at velocities above that of sound also affect the behaviour of spheres and cylinders described on pp. 190–1. As Mach's number rises the critical value of Reynolds' number rises to about double. For $\mathcal{M} = 0{\cdot}7$ the effect of Reynolds' number vanishes almost entirely; for $\mathcal{M} = 0{\cdot}8$, c is practically constant (with a value of about 0·65). New detailed

* See footnote, p. 292. † *UM*, 6608, 6611, 6622 (1944–1945).

experiments are described by A. Ferri, *Atti di Guidonia*, Nos. 67–69, p. 49 (1942), briefly reported in *Luftwissen*, Vol. 10, p. 90 (1943).

When the velocity of sound is exceeded and the flow is separated, "forked" shock waves are sometimes observed.* Continuity requires that the two jumps

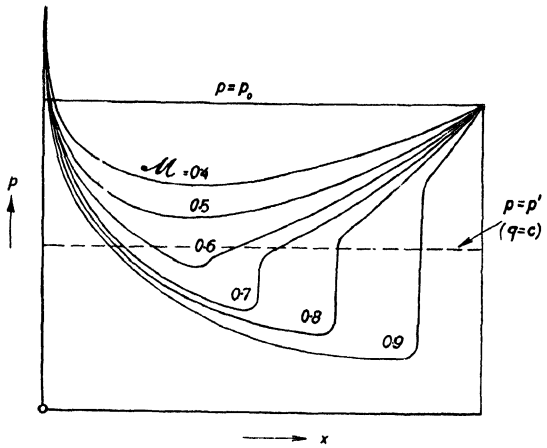


Fig. 4.42.—Change in the pressure distribution as the velocity of sound is exceeded

from p_1 to p_2 and from p_2 to p_3 should give the same final pressure as the single jump from p_1 to p_3. From the point of forking there runs a stream-line along which the velocity and the temperature are discontinuous. See also the remarks on p. 289.

9. Motion of Bodies at Supersonic Velocities. Resistance of Projectiles.

It has been already explained in § 1 (p. 262) that when a small body is moving through a mass of gas—or when a mass of gas is moving uniformly past a small obstacle at rest, which comes to the same thing—the pressure effects due to the body can be propagated only in a conical region behind the body when the velocity exceeds the velocity of sound for the gas in question. This result, however, merely represents the main features of the phenomena. Whenever the body can no longer be regarded as small and its extension in space must be taken into consideration, further details have to be considered. If the body is assumed blunt in front, it will push some of the gas in front of it and a stagnation point (Chapter II, § 4(*c*), p. 44) will be formed at the centre of its front (A in fig. 4.43). As this mass of gas is moving relative to the body with a velocity less than that of sound, the pressure in it is also propagated in a forward direction; it is bounded in that direction by a shock wave, the so-called bow-wave or nose-wave, in which

* See e.g. J. Ackeret, F. Feldmann and N. Rott, *Mitteilungen aus dem Inst. f. Aerodyn. an der Eidgenössischen Technischen Hochsch. Zürich*, No. 10 (1946).

the velocity (relative to the body) diminishes from a value greater than that of sound (on the outside) to a value less than that of sound (on the inside). The intensity of the compression is such that the shock wave advances with the same velocity as the body. The pressure jump at the nose-wave extends sideways as an oblique shock wave. As the distance from the body increases the rise of pressure steadily

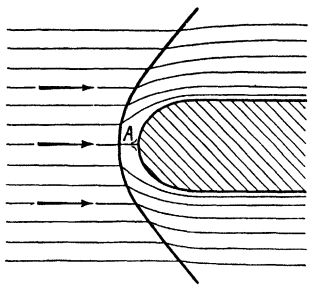


Fig. 4.43.—Shock wave formed at the front of a body

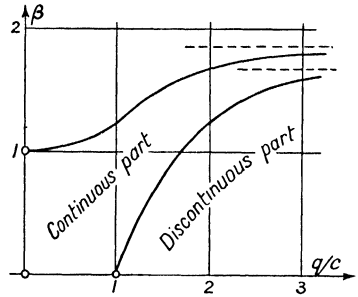


Fig. 4.44.—The relationship between β and q/c

diminishes, the phenomenon approaching an ordinary conical wave more and more. For large velocities the shock wave lies close to the body, for smaller velocities it lies farther away. Fig. 4.4, p. 263 (from a photograph obtained by C. Cranz * using Töpler's "Schlieren" method) and fig. 4.43 show the motion of the air about a bullet.

The conical waves which start from a body moving faster than sound and advance like other sound waves in the direction normal to the surface of the cone are audible as a sharp report. It may be mentioned that the crack of a whip is of a similar nature, being due to the extremity of the lash moving through the air with a velocity exceeding that of sound. If such reports follow in regular rapid succession, as is the case for a propeller with tip speeds above the speed of sound, a penetrating musical note results.

The rise in the pressure of the air at the stagnation point at the front of the body is of some interest. When the velocity exceeds that of sound it consists of two parts, the discontinuous rise of pressure in the shock wave and the continuous rise of pressure from the nose-wave to the stagnation point. Calculation shows that this rise of pressure is proportional to the square of the velocity, not merely for small velocities [Chapter II, § 4(*c*), p. 45] but also for very high velocities; for intermediate velocities it increases somewhat more rapidly. Its behaviour may be represented by the formula

$$p_1 - p_0 = \tfrac{1}{2}\rho q^2 \beta,$$

* Cf. footnote 2 on p. 262.

where the factor β is a function of the Mach number; the values of β for $\gamma = 1{\cdot}405$ (air) may be obtained from the table below, and the general behaviour of β is shown in fig. 4.44.

The details given here may be similarly applied to the determination of velocity * by a Pitot tube (measurement of dynamic or velocity pressure, cf. Chap. II, § 4(*c*), p. 45, and § 7, p. 54). A shock wave is again formed in front of the Pitot tube when $q > c$. Here the Pitot pressure may well be compared with the excess pressure which would be required in a tank if the velocity of efflux were q. If we put this excess pressure equal to $\frac{1}{2}\rho q^2 \cdot \beta_0$, $\beta_0 = \beta$ for $q \leqslant c$, but $\beta_0 > \beta$ for $q > c$, which agrees with the loss in the nose-wave.

$\mathcal{M} = q/c =$	0	0·5	1	1·5	2	3	∞
$\beta =$	1	1·065	1·275	1·53	1·655	1·75	1·85
Discontinuous part =	—	—	0	0·92	1·25	1·48	1·65
$\beta_0 =$	1	1·065	1·275	1·69	2·48	4·85	∞

The rise of temperature resulting from the damming-up of the gas in front of the body may be obtained from equation (16*a*), p. 278; it is

$$T_1 - T_0 = \tfrac{1}{2}q^2/c_p;$$

if q is measured in metres per second,

$$c_p \text{ for air} = 0{\cdot}2388 \frac{\text{kilocal.}}{\text{kg. deg.}} \times 427{\cdot}2 \frac{\text{m. Kg.}}{\text{kilocal.}} = 102{\cdot}0 \frac{\text{m. Kg.}}{\text{kg. deg.}},$$

where 1 Kg. = 1 kg. × 9·81 m./sec.^{2} (footnote, p. 6), so that c_p is approximately 1000 m.^{2}/sec.^{2} deg.

For 1000 m./sec. (shells) this gives about 500° C., for 20,000 m./sec. (meteors) 200,000° C. This latter temperature is presumably not reached, owing to the strong radiation from the compressed air.

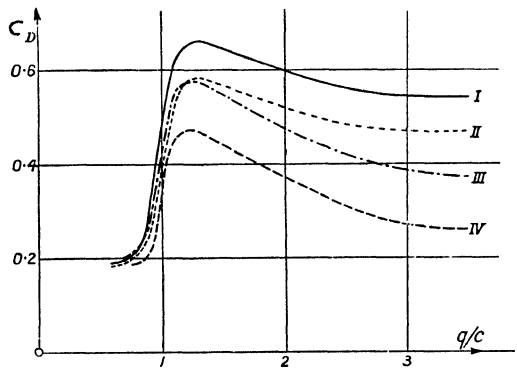


Fig. 4.45.—Resistance of projectiles

By analogy with the behaviour of the dynamic pressure we may conclude that the *resistance of a projectile* again becomes proportional

* See also O. Walchner, *Jahrb. d. Deutsch. Luftfahrtforschung*, 1938, part I, p. 578.

to the square of the velocity for very high velocities, although the constant of proportionality is different from that for small velocities.

Moreover, even where this relationship does not hold, the resistance or drag may on dimensional grounds be represented by the formula $D = \frac{1}{2}C_D \rho a q^2$, just as in the case of flow without volume changes, a being the cross-section of the projectile. Here, however, C_D may not be regarded as a constant as before, but as a function of the Mach number q/c.

The experimental values obtained by measuring the retardation of projectiles with increasing velocity first show a slight decrease below the velocity of sound, but beyond the velocity of sound a marked increase. This is to be explained by the fact that, in addition to the previously existing resistance mainly due to the formation of eddies behind the body, there is now a wave resistance corresponding to the acoustic energy produced. For greater velocities the resistance of the air again decreases somewhat and then appears to approximate to a constant value. The decrease is partly due to the change in the wave form and partly due to the fact that the suction effect at the rear of the projectile cannot increase further quadratically, but reaches its limit at the absolute vacuum.

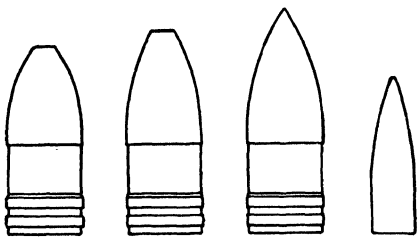


Fig. 4.46.—Shapes of projectiles

In fig. 4.45 values of C_D are plotted for some types of shell after O. von Eberhard * and for a bullet after K. Becker and C. Cranz;† the corresponding forms of projectile are shown in fig. 4.46.

10. Two-dimensional Supersonic Flow past Tapering Bodies and Aerofoils.

If the bodies are sufficiently tapering, the flow at a velocity greater than that of sound may be traced out more closely by the methods given in § 7, p. 286.‡ The pressure on each element of surface is then given by the velocity of the approaching fluid and the inclination of the element, apart from the small loss in the shock wave in front of the body. If the cross-section is lenticular, as in fig. 4.47, the flow begins at the front with an oblique shock wave with an excess of pressure behind it; the convex curvature of the surface gives rise to

* *Artilleristische Monatshefte*, No. 69, p. 196 (1912).

† Ibid., No. 69, p. 189 (1912), and No. 71, p. 833 (1912).

‡ Applied for the first time by J. Ackeret, *Zeitschr. f. Flugtechnik u. Motorluftsch.*, Vol. 16, p. 72 (1925); see also A. Busemann, *Luftfahrtforschung*, Vol. 19, p. 137 (1942).

waves of rarefaction, as a result of which the excess pressure gradually decreases and passes into a region of reduced pressure at the back of the body. At the rear the two flows, one from either side of the body, meet at a finite angle, giving rise to another shock wave. Behind the second shock wave the pressure is again approximately equal to the undisturbed pressure. The waves of rarefaction diverge from one another as they advance; those starting from the front of the body meet the front shock wave impulse and those starting from the rear

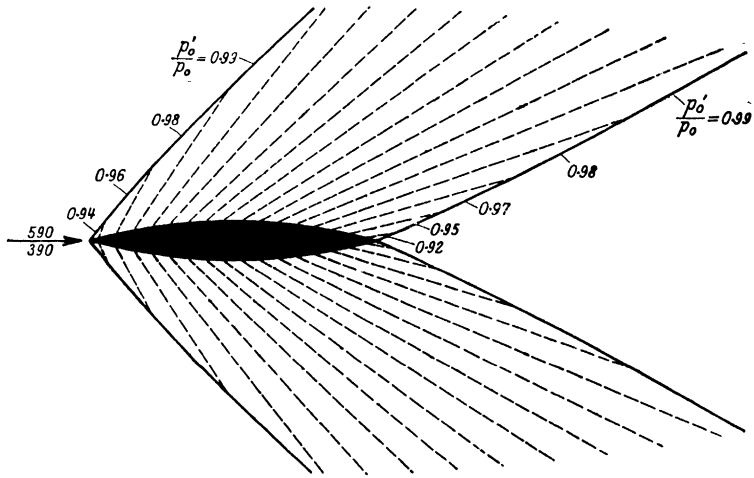


Fig. 4.47.—Flow past a tapering body at a velocity greater than that of sound *

of the body meet the rear shock wave, gradually diminishing the strength of these shock waves. This theoretical sketch of the configuration has been satisfactorily confirmed by "Schlieren" photographs (cf. fig. 4.48).

The pressure distribution illustrated yields a resistance or drag (without friction being taken into account). According to A. Busemann † the increase in entropy of the mass of air traversed by the two shock waves in unit time corresponds exactly to the work done per second by this pressure drag. From the momentum theorem it also follows that as a further consequence of the shock waves there is a subsequent motion of the air on either side of the body, whose momentum (the integral of the product of the mass struck per second

* In the figure p_0 denotes the "reservoir pressure" which would be required to produce the flow; p_0' is the reservoir pressure minus the losses in the shock wave.

† *Handbuch d. Experimentalphysik*, Vol. IV, Part I, p. 443 (Akademische Verlagsgesellschaft, Leipzig, 1931), from which figs. 4.47–49 are reproduced by kind permission of the publishers.

and the velocity of its subsequent motion) is equal to the above pressure drag. In addition to the pressure drag there is in actuality a frictional drag also, which corresponds to a special frictional wake in the boundary layer.

The two theorems are true quite generally. Kl. Oswatitsch has given the proof * for the theorem relating to increase of entropy. According to Oswatitsch the theorem can be expressed in such a way that the work which has to be done in order to tow a body with constant velocity in a medium completely surrounding it is equal to the temperature of the approaching air multiplied by the resultant flow of entropy through a closed surface which encloses all the changes in entropy caused by the body.

As the type of an " aerofoil ", that is, of a body which is subject to a lifting force owing to its being inclined at a small angle to the direction of motion or direction of flow (angle of incidence), we shall here consider the thin smooth plate (fig. 4.49). At the front edge the

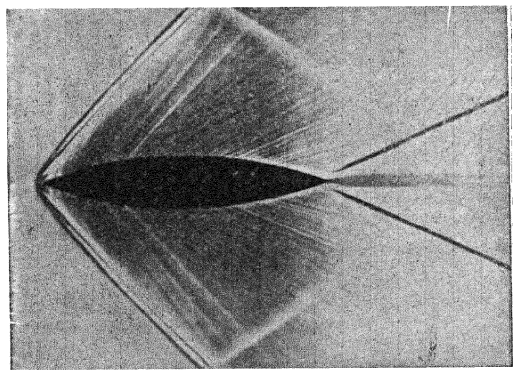


Fig. 4.48.—" Schlieren " photograph of flow of the type shown in fig. 4.47 †

flow divides (the flow with " infinite " velocity at the front edge being absent here, and likewise the corresponding suction; cf. p. 206); on the pressure side a shock wave is set up, on the suction side a wave of rarefaction, both of such intensity that the flow has its direction deflected through the angle ϵ, which is the angle at which the plate

* *Gött. Nachr., Math. u. Phys. Kl.* (1945), p. 88.

† Correspondence with a critical reader has caused the author to give the following explanations. The worker responsible for this figure has set the screen (S_2 in fig. 4.15, p. 275) in the way opposite to that in the other pictures in this book, so that here brightness means increase in density in a horizontal direction and darkness decrease in density. (As is clearly to be seen, the warming of the boundary layers as a result of friction gives darkness in front, brightness behind.) The photograph of the flow, which was from a reservoir of compressed air into free air, was taken at rather too low a pressure; hence the oblique white stripes due to the edges of the nozzle. Behind these the picture differs from the theoretical diagram.

is inclined to the approaching flow. So long as the direction remains constant, the pressure is also constant, so that here the resultant force has its point of application exactly at the middle of the plate. At the rear edge there is an equalization of pressure which gives rise to a shock wave on the suction side and a wave of rarefaction on the pressure side. Here again the flow may be represented by a shock wave on either side of the plate, giving rise to a wake at the rear of the body. For small angles of incidence the resultant force is nearly proportional to the angle of incidence, and as there is no suction effect it is exactly at right angles to the plate if there is no friction. The

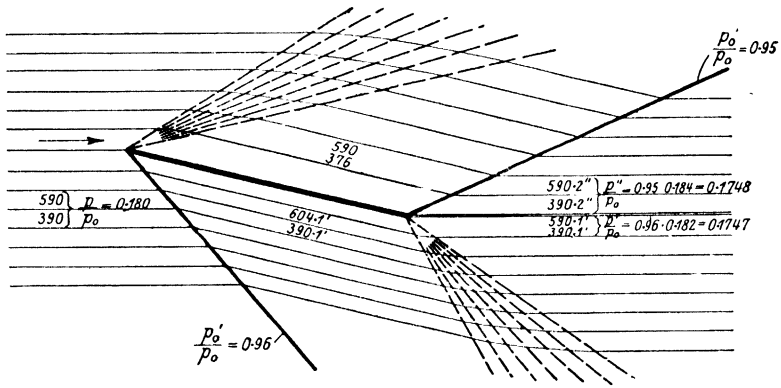


Fig. 4.49.—Flow about an oblique plate at a velocity exceeding that of sound. The numbers are experimental values after Busemann

equivalent of the lifting force is contained in the lateral flow in the two waves. In these, to be sure, the transverse velocity diminishes after a certain distance, owing to waves of rarefaction meeting the shock waves, but the breadth increases in the same proportion, so that behind the plate the lift is carried on as momentum in any cross-section at right angles to the direction of flow.

The thin plate tapering to both ends, possibly slightly arched on the suction side, is found to be the most suitable profile for supersonic speeds; the usual aerofoil profiles thickened at the front behave rather poorly under these conditions, owing to their large drag. In contradistinction to the case of moderate velocities, the best ratio for D/L is never less than $\tan \epsilon$.

The relationships involving the induced drag of aerofoils of finite span (Chapter III, § 17, p. 206) no longer hold when the velocity exceeds that of sound, since drag work is removed in the waves and shock waves.* Here the variation with aspect ratio is much smaller.

* If the velocity is below that of sound, on the other hand, the relationships still hold good if we assume that the lift is as given in § 8 (p. 289), as what remains in the field is a flow in which the velocities are small.

To obtain approximate formulæ for the pressure distribution in cases of flow like those in figs. 4.47 and 4.49 we may apply the differential equation (19), p. 290. As was explained there, every potential $\phi = F(y - x\tan\alpha)$ gives a possible disturbance of the main flow u_0. We put $\phi = 0$ for $y - x\tan\alpha > 0$ and $\phi = C(y - x\tan\alpha)$ for $y - x\tan\alpha < 0$, where α is the Mach angle, i.e. $\tan\alpha = 1/\sqrt{(\mathscr{M}^2 - 1)}$ (fig. 4.50). The disturbances of velocity in front of this plane then become zero; behind the plane $u = -C\tan\alpha$ and $v = +C$. The angle ε through which the streamlines are bent is given by

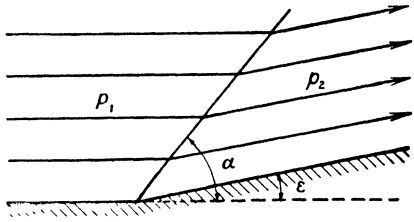


Fig. 4.50

$$\tan\varepsilon = \frac{v}{u_0 + u} \approx \frac{v}{u_0} = \frac{C}{u_0}.$$

By Bernoulli's equation the rise of pressure $p_2 - p_1$ is found as a first approximation to be $-\rho u_0 u = \rho u_0 C\tan\alpha$, for which we may also put $\rho u_0^2 \tan\varepsilon \,.\, \tan\alpha$, and since ε is small, we can replace $\tan\varepsilon$ by ε. Thus for flow at supersonic velocities the pressure at the surface of the slender body depends only on the angle ε, as could already be seen from what was said in §§ 6, 7, provided waves from a neighbouring body do not happen to impinge on the surface. The body in fig. 4.47, as has already been mentioned, gives an excess pressure on the front and a deficit of pressure on the rear, proportional on each element of area to the local value of ε. The pressures are everywhere at right angles to the surface, i.e. have a component $p\sin\varepsilon$ in the direction of the flow. Integrated over the whole body, these give rise to a pressure drag

$$D_p = \iint p\sin\varepsilon\, da \approx \rho u_0^2 \tan\alpha \iint \varepsilon^2 da.$$

Since the oblique plate of fig. 4.49 is subject above to an excess pressure, below to a deficit of pressure, each of magnitude $\rho u_0^2 \tan\alpha \,.\, \varepsilon$, it is subject to a normal force $N = 2a\rho u_0^2 \tan\alpha \,.\, \varepsilon$, which to a first approximation gives the lift L. Thus

$$C_L = 4\tan\alpha \,.\, \varepsilon = \frac{4\varepsilon}{\sqrt{(\mathscr{M}^2 - 1)}}.$$

By the above, if friction is neglected, $C_D = C_L \sin\varepsilon$, that is,

$$C_D \text{ is approximately } \frac{4\varepsilon^2}{\sqrt{(\mathscr{M}^2 - 1)}}.$$

The formulæ above were first given by J. Ackeret * in a somewhat different form. Measurements of lift and drag, as well as a more accurate form of the theory, will be found in a paper by A. Busemann and O. Walchner.†

Much light was thrown on the three-dimensional aerofoil problem for supersonic velocities in a paper by H. Schlichting.‡ The effect of the ends of the wing on either side extends only to the region which is included in the Mach cone starting from the ends of the wing, as might be expected from the properties of

* *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, Vol. 16, p. 72 (1925).

† *Forschung*, Vol. 4, p. 87 (1933).

‡ "Tragflügeltheorie bei Überschallgeschwindigkeit", *Luftfahrtforschung*, Vol. 13, p. 320 (1936).

flow at velocities above that of sound. Elsewhere, provided the front edge is straight and the angle of incidence is constant, the flow is two-dimensional. For the plane rectangular plate, according to A. Busemann (see below), the pressure distribution is as in fig. 4.51.[49]

A. Betz * has extended the theory of a row of blades [Chapter II, § 13(*a*), p. 81] to the case of a compressible fluid, expressing the results by means of a nomogram and a number of curves.

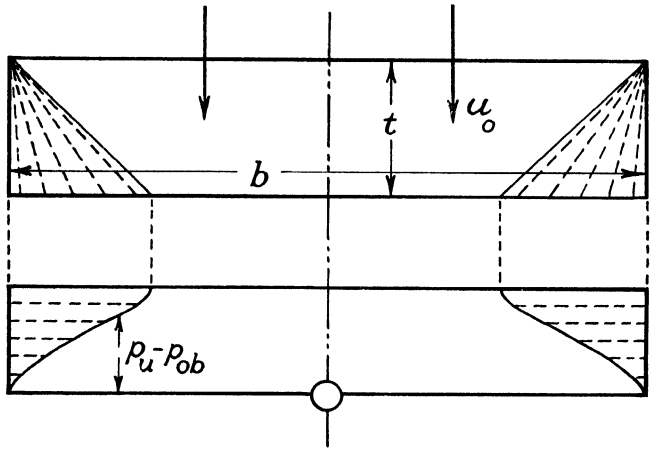


Fig. 4.51.—Distribution of pressure over a plane rectangular wing for supersonic velocities

Various Theoretical Extensions.

1. There is an interesting type of pure supersonic flow in which a point P is marked out and the pressure and velocity vector is constant along every radius vector from P. The surfaces of constant pressure are therefore cones with P as their common vertex. Flows of this type occur at the nose of a bullet † and also at the sharp edges of thin flat plates.

For the case where all the pressure differences are small A. Busemann ‡ has developed a very useful theory in which the calculation is reduced to the solution of problems of incompressible irrotational flow. The flow behind the edge of a rectangular plate in fig. 4.51 is an example of this theory.[50]

2. Cases of flow which are symmetrical about an axis can be dealt with by methods of approximation similar to that for two-dimensional

* *Ing.-Arch.*, Vol. 16, p. 249 (1948).

† G. I. Taylor and J. W. Maccoll, *Proc. Roy. Soc.* (A), Vol. 139, p. 278 (1933); Maccoll, ibid., Vol. 159, p. 459 (1937); Th. von Kármán and N. B. Moore, *Trans. Soc. Mech. Eng. Appl. Mech.*, p. 303 (1932); H. S. Tsien, *J. Aeron. Soc.*, Vol. 5, p. 480 (1938).

‡ " Infinitesimale kegelige Überschallströmungen ", *Schriften der DAL*, 7B, Part 3, p. 105 (1943).

flow on p. 287 (fig. 4.32). Since, however, there is no hodograph plane in which the characteristic curves would be integrable, it is necessary to make calculations or auxiliary constructions between each two steps in the meridional plane, so that the work is much more troublesome than in the two-dimensional case. These methods have been developed by various workers; Kl. Oswatitsch has * given a comprehensive account of them.(51)

3. The propagation of pressure waves with moderate degrees of compression in pipes † and in free air forms part of acoustics and we shall not deal with it here beyond what we have said on pp. 259–64. On the other hand, various cases of the propagation of pressure when the differences in density are large are of interest to engineers. Thus, following the classical work of B. Riemann,‡ K. Kobes § has discussed the phenomena in Continental railway braking systems in a very valuable paper. Graphical methods of approximation like those of paragraph (2) above have recently been applied in the theoretical discussion of non-steady phenomena in engines, &c.; see e.g. a paper by F. Schultz-Grunow || which is also based on the work of Riemann.

11. Experimental Technique for High Velocities.¶

Continuing the account of hydrodynamical and aerodynamical methods of experiment given in § 22 of the last chapter (p. 246), we shall give some brief notes on the special points that have to be borne in mind in experiments with air velocities of the order of the velocity of sound. For a very long time there was a hesitation to install the high-power machinery necessary for prolonged experiments with velocities approaching those of sound, unless with very minute cross-sections, and hence large tanks were used as a source of energy. These could be filled with compressed air or pumped free of air by an engine of modest dimensions in a few minutes, and were then capable of maintaining a stream of air of fairly respectable cross-section for 5 to 10 seconds. (As an example, we may mention the 1929 plant of the Aachen *Aerodynamisches Institut*, with a tank of volume 90 cubic metres and a wind tunnel 10 cm. by 10 cm.) Most of these plants used a vacuum, since the flow could then be held perfectly constant by inserting a throat, downstream of which the flow was supersonic,

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 25–27, pp. 195, 264 (1947).

† A. Wood, *Acoustics*, p. 250 (Blackie, 1940).

‡ See footnote, p. 263.

§ *Zeitschr. d. Österr. Ing.-u. Arch.-Vereins*, Vol. 62, p. 553 (1910).

|| *Forschung*, Vol. 13, p. 125 (1942); *Zeitschr. f. angew. Math. u. Mech.*, Vol. 24, p. 284 (1944).

¶ Cf. also *Convegno Volta*, Vol. 5, p. 369 *et seq.* (papers by Jacobs, Panetti, Ackeret and others), and A. Busemann, *Schriften d. Deutsch. Akad. d. Luftfahrtforschung*, No. 30, p. 17 *et seq.*

unless the supersonic velocity was destroyed by a shock wave moving upstream from the interior of the tank. By using compressed air it is possible to store up more energy in the same space, but there is the disadvantage that owing to the adiabatic expansion of the air in the boiler, the density and temperature are continually altering while it is flowing out. Both the velocity of flow and the velocity of sound are proportional to the square of the temperature, so that "Schlieren" diagrams for velocities above that of sound are independent of the particular degree of expansion, as they depend only on the ratio of the two velocities. (Nevertheless, it is desirable to work quickly, as otherwise fog formation and icing occur as a result of the adiabatic expansion of the air in the tank.) Compressed air is made use of indirectly in the Langley Field method at some research stations with compressed-air wind tunnels. When the pressure in a tunnel is let down after an experiment with compressed air (as is necessary before the tunnel can be entered again), the compressed air is made use of to generate a vacuum by means of a jet blower (fig. 4.52). By regulating the compressed air it is easy to adjust the velocity of the experimental air sucked in from the atmosphere (the maximum velocity is about that of sound).

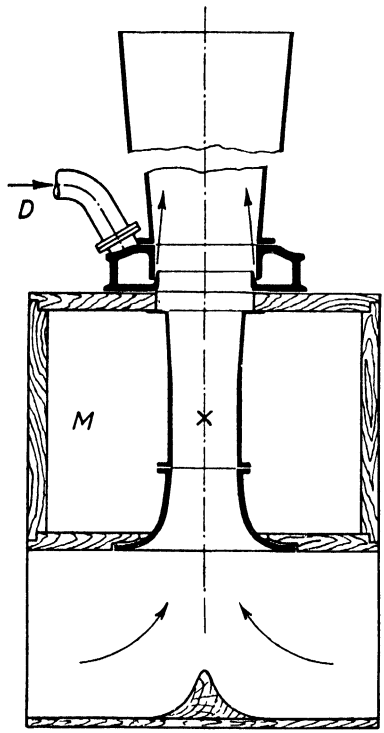


Fig. 4.52.—American high-velocity wind tunnel. × marks the spot where experiments are carried out. M is a space for instruments. D is the pipe introducing compressed air

Working with the vacuum tank, it is possible to produce velocities both below and above the velocity of sound, but different types of tunnel are required. For subsonic velocities the space where experiments are made is immediately behind the entry where the air is sucked in, and behind this there is an adjustable throat in which the velocity is that of sound, so that the quantity of air flowing through can be determined (fig. 4.53). For supersonic velocities the narrow part of the tunnel must come first and then the space where experiments are carried out, which is wider (fig. 4.54). The greater the value of the Mach number which is to be attained, the more marked

must the constrictions be. To avoid troublesome transverse waves in the stream of air, the shapes of the nozzles must be carefully designed, more or less as in fig. 4.33, p. 288, with a separate nozzle for each value of the Mach number (whereas for subsonic velocities a single nozzle will suffice). If stress is laid on regaining pressure, it is advisable to place

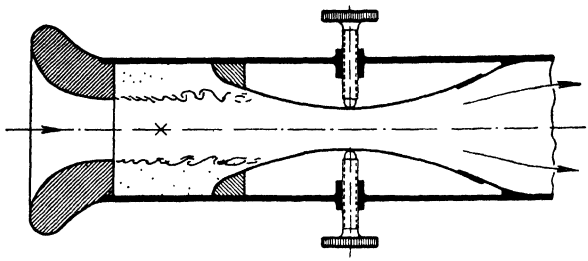


Fig. 4.53.—Tunnel for subsonic velocities

an adjustable diffuser behind the supersonic region as shown in fig. 4.54. The second narrowing, however, must be much wider than the first, otherwise the supersonic region will not " start up ". In both figures the position of the model under investigation is marked by $\times$.

To make as good use as possible of the short space of time available for experiment with this method, a motor-driven valve with a wide opening is installed between the point where experiments are made and the tank. Usually it can be controlled electrically from the point

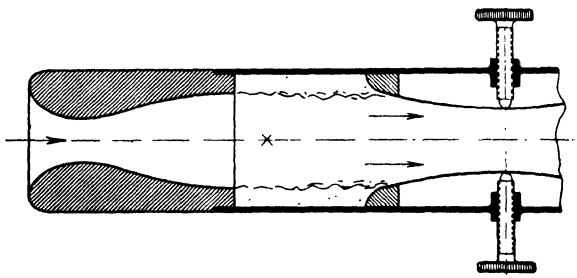


Fig. 4.54.—Tunnel for supersonic velocities

where observations are made, and opens or shuts * in a fraction of a second. In this method of experiment, especially for values of the Mach number above unity, serious difficulty arises from the humidity of the air. Owing to the adiabatic cooling, which here is more than 50° C., the greater part of the humidity would have to be liberated in the

* See the description of the first plant of this kind by A. Busemann, *Jahrbuch d. Wissensch. Gesellsch. f. Luftfahrt*, 1928, p. 95.

form of liquid water; but the brief space of time (of the order of 10^{-4} sec.) in which the particles of air move through the nozzle is not sufficient for the formation of water droplets of the usual sort. According to experiments by K. Oswatitsch,* in fact, spontaneous condensations occur in the absence of particles of foreign matter in a way which has been investigated theoretically by R. Becker and W. Döring,† once the supersaturation has reached a high value.(52) As this phenomenon sets in very suddenly, there is a rise of pressure in the middle of the expanding mass of gas, owing to the latent heat which is set free. This rise of pressure frequently manifests itself by a shock wave (which is usually oblique), and gives rise to noticeable deviations from the calculated distribution of velocities. As the position where the rise of pressure occurs varies according to the humidity of the air, this phenomenon is very awkward if accurate measurements are to be made. The difficulty can be got over by warming or artificially drying the air before entry. With the latter good experimental results have been obtained; a convenient method is to use a silica-gel filter.‡

The difficulties arising from the short time of observation are, however, still very real. They can be avoided by setting up a steady circulation of the air by powerful blowers—usually turbo-compressors with several rotors in series. The first installation of this kind was constructed at the *Technische Hochschule* at Zürich by J. Ackeret.§ It takes nearly 1000 h.p. and for the higher velocities it is necessary to use rarefied air, otherwise the power requirement would be even greater. A second installation of the same type taking about 4000 h.p. was constructed at Guidonia, near Rome.‖(53) As in plants of this type the energy supplied remains in the form of heat in the current of air, a special cooling system must be included to keep the temperature constant. The experimental region is arranged as already described. Fig. 4.55 is a diagram of the Zürich installation.

Note.—The above was written in 1942. Many larger plants of this kind have been built since, especially in America.

The velocity of air flow in the experimental region is best calculated from the measured fall of pressure, by formula (8), p. 264; in addition to the pressure at rest p_1, the density at rest ρ_1 must also

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 22, p. 1 (1942), where a quantitative theory of the phenomenon is developed, which gives results in good agreement with those of observation. R. Hermann, in *Luftfahrtforschung*, Vol. 19, p. 201 (1942), gave an account of detailed experiments carried out in 1934–36 in the Aachen Aerodynamic Institute on the way in which the position and intensity of the " condensation shock " depend on the humidity of the air and the Mach number for the nozzle.

† *Ann. d. Phys.* (5), Vol. 24, p. 719 (1935).

‡ Made by the *Silikagel-Gesellschaft*, Berlin.

§ *Convegno Volta*, Vol. 5, p. 487 (1935).

‖ For a description, see M. Gasperi, loc. cit., p. 537; also *Gesammelte Vorträge der Hauptvers. der Lilienthalges. f. Luftfahrtforsch.*, p. 187 (1937).

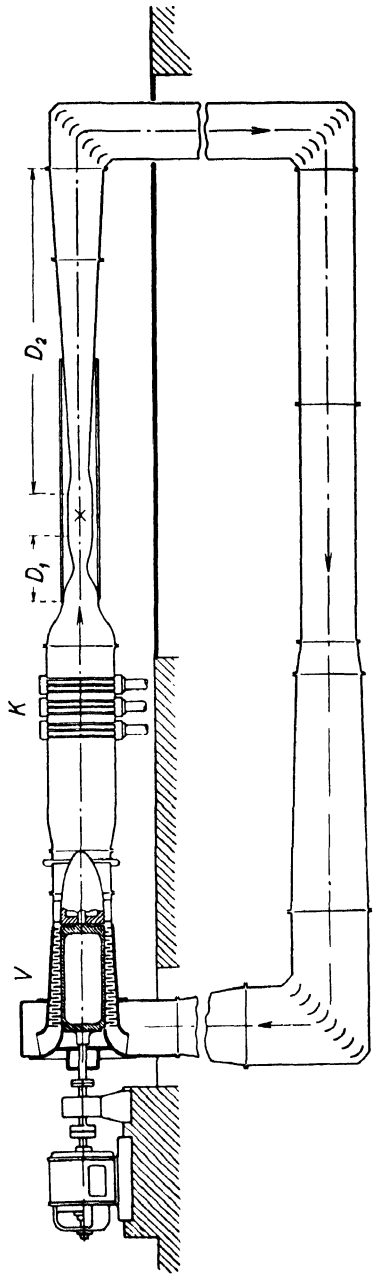


Fig. 4.55.—Supersonic wind tunnel at Zürich

V = compressor, K = cooler, D_1 = nozzle, D_2 = diffuser

be known (e.g. from temperature measurements). If, however, condensation of moisture takes place ahead of the experimental region, this method is not very accurate.

To determine the nature of the flow at a given point it is necessary to know three quantities, e.g. the pressure p, the density ρ and the velocity q. The pressure can be measured directly by a hole in the wall or by a static tube. The density can be measured by interferometry.* Instead of the density, the temperature may be measured, as it is connected with the pressure and the density by the equation of state. Measurement of the temperature of a moving gas, however, is difficult (see below).

For the third quantity we can take either the Pitot pressure p_p or the Mach number, which is given by the Mach angle, determined by "Schlieren" photographs. As $\rho c^2 = \gamma p$ and $\rho q^2 = 2(p_p - p)/\beta$, where β, by § 9, p. 299, is a function of the Mach number q/c, we see that p_p/p is a new function of the Mach number, $1 + \frac{1}{2}\gamma \mathcal{M}^2 \beta$, which can be given in the form of a curve or table from which $\mathcal{M}$ can be found when p_p and p are measured. If ρ is also known, c and q follow from the above relationships. If the cross-section of the flow is known, the measurement of the density can be replaced by the equation of continuity $a\rho q = a_0\rho_0 q_0$, provided the product $\rho_0 q_0$ is known for the cross-section at entry a_0; ρ and q are then found separately from ρq and ρq^2.

For a given temperature ρ can be found from p by means of the equation of state. A thermometer held in the current of gas, however, does not record the temperature of the gas; at the stagnation point we obtain the "stagnation-point temperature", which, as we saw in § 9, p. 299, is greater. Moreover, there is also a heating effect at the side walls owing to friction by the current of air; this varies according to the interaction of viscosity and heat conduction,† and in air, for example, is not much below the temperature at the stagnation point. A relatively reliable measurement can be made by means of a thermocouple surrounded by air at rest in the interior of a hollow body resembling a Pitot tube. To obtain the true temperature of the moving gas, the quantity $q^2/2c_p$ must be subtracted from the temperature recorded. According to the theory of E. Pohlhausen,‡ which takes into account only the friction and heat conduction arising from laminar flow, 0·85 of this quantity must be subtracted for air, and 1·04 of it for water vapour, when a "plate thermometer" is used. (For c_p, see p. 299.)(54)

Notes.—1. Light has been thrown on the temperature relationships for laminar and turbulent boundary layers around unheated bodies in high-velocity currents of gas by the experiments of E. Eckert and W. Weise in two papers, § and also on the local distribution over the body in the current of gas. The calculation of such temperature fields in laminar boundary layers has been worked out in a paper by E. Eckert and O. Drewitz.‖(55)

2. Recently the Mach-Zehnder interference method has been used in addition to Töpler's "Schlieren" method for observations on compressible flow; this was first done by Th. Zobel.¶

* Kl. Oswatitsch, *FB*, 1285 (1940); he used a double-slit interferometer.

† It is a question of the dimensionless quantity ν/a; cf. p. 407.

‡ *Zeits. f. angew. Math. u. Mech.*, Vol. 1, p. 115 (1921). For the practical details of apparatus for temperature measurement by these two methods and experiments with it, see the report by E. Eckert in *VDI-Zeitschr.*, Vol. 84, p. 813 (1940).

§ *Forschung*, Vol. 12, p. 40 (1941); Vol. 13, p. 246 (1942).

‖ *Luftfahrtforschung*, Vol. 19, p. 189 (1942).

¶ *FB* 1008 (1938); see also *FB* 1167 (1940) and 1934 (1944).

CHAPTER V

Miscellaneous Topics

Preliminary Note.—This chapter includes a number of topics which did not fit conveniently into the previous chapters but cannot be omitted from a comprehensive account of hydrodynamics. They fall under the following heads:

A. *Combined effects of more than one state of matter*: fluid-gas (including cavitation and water hammer,* mixtures of water and air, atomization); fluid-solid and gas-solid (transport of gravel and suspended material in fluids, sandstorms, snowstorms); also hydrodynamical action at a distance.

B. *Laws of flow in rotating enclosures* (application to pumps and turbines, and to the motion of wind and water over the rotating surface of the earth; resistance of rotating discs).

C. *Motion of stratified media under the influence of gravity* (including internal waves in stratified media; cyclones and anticyclones arising from vertical displacements of masses of air, in conjunction with stratification and the rotation of the earth; the general circulation of the atmosphere).

D. *Convection and transfer of heat in moving media* (forced and natural heat transfer, and so on); interaction of flow phenomena and diffusion.

A. Combined Effects of More than One State of Matter

1. Cavitation.

(*a*) The name of *cavitation* is given to a group of phenomena of great importance in engineering, in which cavities filled with vapour or gas are formed in the interior of a moving fluid. These phenomena arise in the following way. In a fluid whose undisturbed pressure p_0 exceeds the vapour pressure p', the motion gives rise to reduced pressures p which are less than p'. Vapour is then suddenly formed, the pressure rising to p'. The flow is naturally caused to deviate from its ordinary course as a result of the increase of volume in the change from the liquid state to the vapour state. When the pressure rises again at a subsequent stage of the flow the vapour condenses and the

* Loosely connected with these are planing surfaces on the surface of water.

cavities fill up, their collapse being usually accompanied by a loud rushing sound.*(56)

Cavitation is markedly favoured by air bubbles carried along by the flow and by gases dissolved in the water. Water from which most of the air has been removed, however, will for a short time stand not only reductions of pressure below the vapour pressure but even very considerable tensile stresses without cavities being formed. This is shown experimentally e.g. by the fact that cavitation frequently begins at the point of separation of the flow (cf. Chapter III, § 6, p. 136), i.e. at the point where the particles in the boundary layer come to rest and remain for some time in this condition. No pressure lower than the vapour pressure can be measured by holes in the surface, as the liquid at rest in the hole invariably begins to boil when the pressure outside it falls below the vapour pressure. In what follows we shall assume that air or bubbles of gas are always present in the moving fluid, and that the cavitation begins at these when the vapour pressure p' is reached. This is almost always the case in practice.

A particularly clear and simple case of cavitation arises when liquid flows through a pipe which first narrows and then widens. If the velocity of flow is high there is a large rise in pressure in the widening part, particularly when the widening is gradual, so that it is possible for efflux to take place at atmospheric pressure even although the pressure is very low at the throat upstream. If the vapour pressure is reached, the main body of the liquid after passing the throat flows as a free (but usually very turbulent) jet through the pipe, accompanied on either side by a foaming mixture of liquid and bubbles. Farther on in the direction of flow the jet impinges as a sort of shock wave against more slowly moving liquid filling the whole breadth of the pipe; owing to the rise in pressure the bubbles of vapour suddenly collapse, leaving behind only small air-bubbles (the air previously dissolved in the water).† Phenomena of this kind often occur in water-taps, especially when the mains pressure is high, and are betrayed by the well-known loud noises (" water hammer ").

F. Numachi ‡ (57) studied cavitation in water containing a varying proportion of air and found that in a glass tube which narrows and then widens again a steady jet of water as clear as glass is formed when the amount of air is less than half that required to saturate the water completely, but that there is more and more foaming when the amount of air is greater than this. The length of the free jet depends

* These phenomena are discussed in detail in a memoir by J. Ackeret, *Techn. Mech. u. Thermodyn.* = *Forschung*, Vol. 1, pp. 1, 63 (1930); see also the article on cavitation by the same author in *Handb. d. Exper.-Physik*, Vol. IV, 1, pp. 461–86. Many additional references are given in *Hydromech. Probl. d. Schiffsantr.*, pp. 227–342 (Hamburg, 1932).

† The theory of these shock waves is given by J. Ackeret, loc. cit.

‡ *Ing.-Arch.*, Vol. 7, p. 397 (1936); see also Vol. 9, p. 86 (1938).

on the opposing pressure. It is fairly nearly equal to that calculated for a suddenly widening tube from the momentum equation (Chapter II, § 5(*b*), p. 46). Thus the pressure regained is smaller, the smaller the ratio of the cross-section of the jet and that of the tube. Hence for

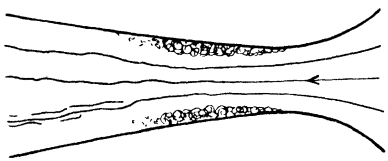


Fig. 5.1

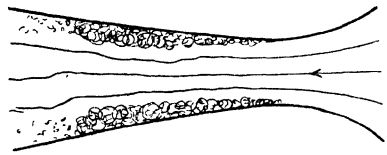


Fig. 5.2

fairly high opposing pressures * the hollow spaces are short (fig. 5.1), for fairly low opposing pressures they are long (fig. 5.2). If they are of medium length the jet often wobbles violently.

To exemplify the variation of pressure in a pipe which first narrows and then widens again, we shall reproduce the results obtained by H. Hochschild † on measuring the pressures at orifices in the wall of the pipe.[58] The same quantity of water was always pumped through pipes with rectangular cross-sections of equal heights and varying breadths, the pressure at the outlet being altered by a throttle-valve. If the opposing pressure is sufficiently high the pressure curves differ only by an additive constant (curves *a* and *b* in fig. 5.3). If, however, the pressure at the narrowest point falls so low that there is an appreciable separation of air, there are large losses (curves *e* to *i*); the pressure recovery, which is otherwise considerable, is greatly diminished or entirely absent. The point at which the rise of pressure sets in marks the end of the region where cavitation occurs.

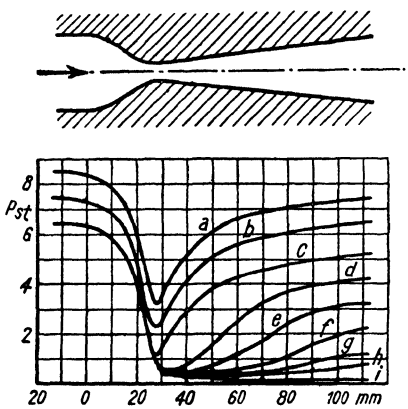


Fig. 5.3.—Distribution of pressure in cavitation

(*b*) The greatest practical significance of cavitation is in connexion with high-speed water turbines and ships' propellers. In both these types of machinery it is advantageous to use as high velocities as pos-

* For the same value of the opposing pressure, fig. 5.1 would represent a smaller pressure and fig. 5.2 a greater pressure upstream in the tube.

† H. Hochschild, *Versuche über die Strömungsvorgänge in erweiterten und verengten Kanälen* (*Forschungsheft* 114 *des Vereins Deutsch. Ing.*, 1912).

sible; on the other hand, there are very weighty reasons for avoiding cavitation, not only on account of the losses resulting therefrom, but also on account of the peculiar type of corrosive damage associated with it (see below). The simple rule for avoiding cavitation is to avoid low pressures. This is done by using flat profiles and very small angles of incidence, i.e. by making the load on the surface small. The fraction

$$\lambda = \frac{p_0 - p'}{\frac{1}{2}\rho v^2}$$

gives a measure of the risk of cavitation. In the case of propellers it has been found possible to obtain satisfactory working down to about $\lambda = 0{\cdot}2$.

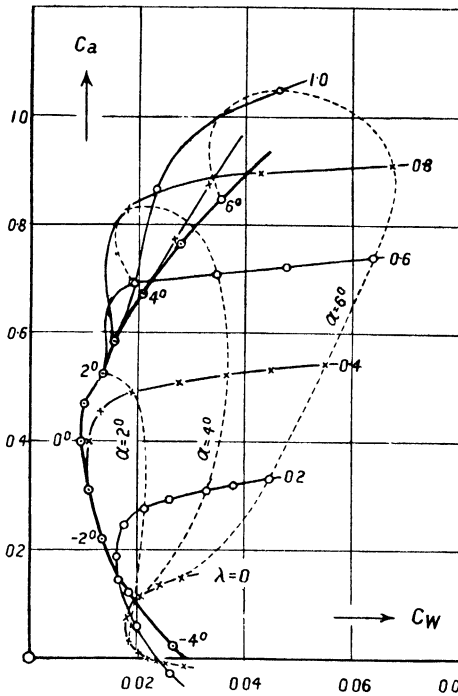


Fig. 5.4.—Cavitation polars

A good idea of the phenomena which occur at a propeller when it is running can be obtained from experiments on models in a cavitation tunnel, i.e. a water circulation in which the absolute pressure can be altered at will.*[59] A simpler and clearer picture, however, is furnished by experiments on an isolated blade profile. A profile polar is obtained for each value of λ and at the same time the cavitation is observed through the glass wall of the apparatus. The results are plotted along with the group of polar curves (fig. 5.4) in a cavitation diagram (fig. 5.5). As an example, we give the two figures from experiments by O. Walchner †[60] for a profile in the form of the arc of a circle, such as is very commonly used for marine propellers (thickness = 0·075*b*). The thicker line is the polar for a high absolute pressure. As λ falls, strange to say, cavitation gives better results for large angles of incidence (more thrust and less resistance). The reason

* A description of the Hamburg installation is given by H. Lerbs in *Werft, Reederei, Hafen*, Vol. 12, p. 191 *et seq.* (1931); see also *Hydromech. Probl. d. Schiffsantr.*, p. 287.

† Personal communication; see also Walchner's report, *Hydromech. Probl. d. Schiffsantr.*, Vol. I, p. 256, and the lecture by A. Betz at the *3rd International Congress on Applied Mechanics*, Stockholm, Vol. 1, p. 411 (1930).

is that the cavity above the front part of the suction side gives a markedly better flow round the profile.*(61) As λ diminishes further and the cavity extends, however, the profile soon becomes worse, and finally only a very small lift is produced. The state of cavitation can be seen from the cavitation diagram, in which the cavitation number λ is plotted as abscissa against the angle of incidence as ordinate. The region to the right of a, b, c is free of cavitation and hence the lift and drag are the same as for high values of λ. The points in the regions to the left of a, b, and c, respectively, give cavitation on the suction

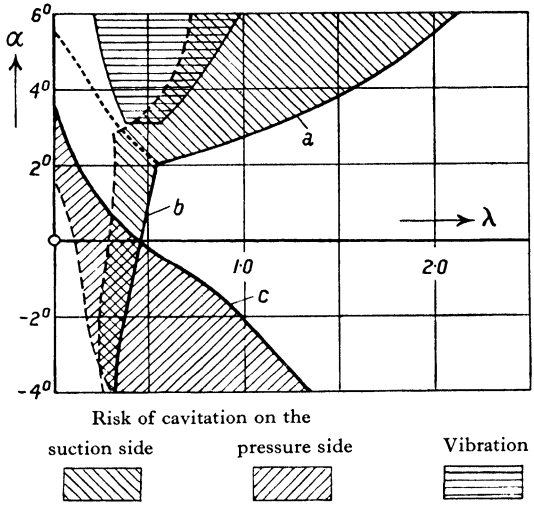


Fig. 5.5.—Cavitation diagram

side in front, or on the suction side in the middle, or on the pressure side, as shown in the sketches in fig. 5.6. (In the region to the left of b and c these two forms occur simultaneously.) In the regions shaded there is a danger of corrosion (the ends of the cavities being still on the surface of the blade; see below). In the region characterized by "vibration" the flow swings rhythmically to and fro, as already mentioned, and strong shaking forces result. In the unshaded regions to the left of a, b, or c no difficulty arises. H. B. Helmbold † has pointed out that the boundaries b for profiles of similar shape but different thickness almost coincide if both the cavitation number λ and the angle of incidence α are divided by the thickness ratio $\delta = d/b$, i.e. if λ/δ is plotted against α/δ. This is connected with the fact that the

* This would not happen if the profile were of a good shape to start with; see e.g. F. Numachi, *Forschung*, Vol. 11, p. 303 (1940).

† *Hydromech. Probl. d. Schiffsantr.*, p. 338.

differences of pressure at the profile for the same values of α/δ are approximately proportional to the thickness ratio δ. From this it may also be concluded that if it is necessary to work with very small values of λ, very slender profiles must be used. (We had the same rule for air when the velocity approached that of sound, but for other reasons.) Experiments show that the boundaries a and c also follow the same rule to some extent.

(*c*) When cavities of the kind shown in fig. 3.60 (shot passing through water, p. 182) are formed, the cavitation number λ is extremely small, owing to the very high dynamic pressure. Model experiments can be made only by achieving similar small values of λ in the experimental arrangements. This is possible only for a very small pressure p_0, since the velocity of the water is limited. The weight of the stream of water would itself give too high a pressure, but its effect can be got rid of by making the water flow as a free jet through the evacuated space (the slight curvature of the jet does not make much difference *).

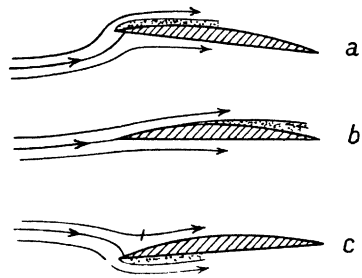


Fig. 5.6

H. Reichardt has investigated resistance phenomena of various types in this way.[62] Among other things he found † that for small values of the cavitation number λ the coefficient of resistance based on the maximum cross-section of the bubbles is equal to λ and that the ratio of bubble length to bubble diameter is a universal function of λ.

Since under these conditions the cavities are as clear as glass, it is also possible to deal with the problem of ascertaining the geometrical form which shows the least surplus velocity for given conditions; for, since the cavitation bubble is subject to a constant pressure, which here can be varied at will, a photograph of the phenomenon shows the desired form. The problem is of importance for establishing the best propeller shapes for high speeds in water, where cavitation must be avoided, and also in air, where the neighbourhood of the velocity of sound must be avoided.‡

(*d*) In connexion with the corrosion phenomena§[63] mentioned above, we may add that they usually occur at the point where the shock wave is situated, i.e. at the point where the bubbles of vapour collapse.

* For the apparatus, see *UM.* 6620 (1945).

† See *UM.* 6628 (1945).

‡ See H. Reichardt, *Betz-Festschrift* (1945); also *UM.* 6606 (1944) and 6618 (1944).

§ See Ackeret's article in the *Handb. d. Exper.-Physik*, also P. d. Haller, *Handb. d. Werkstoffprüfung*, Vol. 2, p. 471 (Berlin, 1939).

Particles of water previously separated by the bubbles suddenly impinge on one another with widely differing velocities, so that very high pressures arise locally (cf. also § 2(*a*), p. 319). In addition to this hammering effect, chemical influences are frequently at work too. The air



Fig. 5.7.—Corrosion due to cavitation *

liberated from the water is richer in oxygen than ordinary air, and this oxygenated air is forced into the pores of the metal at high pressure and then sucked out again as soon as the pressure falls, so that there is a rapid alternation of stresses reaching into the finest pores. The result of these processes, which are by no means exactly understood as yet, is that the surface of the metal becomes roughened, and if

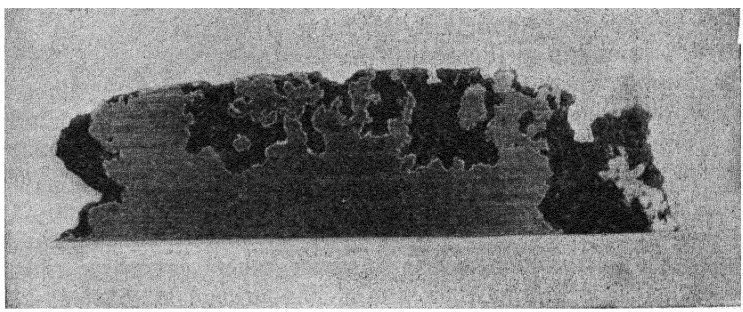


Fig. 5.8.—Section through the corroded part *

the cavitation process persists for a long time quite large hollows are eventually formed. As an example, fig. 5.7 shows the corroded surface of a destroyer propeller of high-grade bronze, fig. 5.8 a section of the same piece of metal.

* Figs. 7, 8 are reproduced from *Handbuch d. Experimentalphysik*, Vol. IV, Part 1, p. 443 (Akademische Verlagsgesellschaft, Leipzig), by kind permission of the publishers.

2. Water Hammer. Planing Surfaces.

(*a*) *Impulsive Phenomena.*—If the continuity of the fluid in a pipe is broken by the occurrence of a cavity (full of the vapour of the fluid) arising from a transient fall of pressure, and the pressure is then restored, the effect of the collapse of the cavity is that of a hard, almost metallic blow.* The two separated columns of fluid approach one another with relative velocity q and are suddenly stopped as the cavity disappears. How great is the pressure produced, and what are its effects? Although the walls of the pipe may be assumed to be rigid (in reality, of course, they are not), the fluid at least must be regarded as compressible. Thus the velocity of sound in it is finite and given by $c = \sqrt{(K/\rho)}$, where K is the modulus of elasticity, and changes of pressure are propagated with this velocity. Accordingly, the increase in pressure (p) arising from the impulse is propagated in both directions with the velocity c, and the particles of fluid affected by the pressure wave acquire velocities which are the arithmetic mean of their previous velocities. The change of velocity in each column of fluid as the impulse passes is therefore $\frac{1}{2}q$, and by the argument in Chapter IV, § 1 [equation (2), p. 260], the increase in pressure p is given by $\frac{1}{2}\rho cq$.

Numerical Example.—For water $q = 10$ metres/sec., $c = 1400$ metres/sec. In engineering units with the metre as unit of length

$\rho = 1$ g./cm.^{3} $= 1000$ Kg./m.$^2 \div 9{\cdot}81$ m./sec.$^2 = 102$ Kg. sec.^{2}/m.^{4} (approx.).

Hence $p = 102 \times 1400 \times 5$ Kg./m.^{2} $= 71{\cdot}4$ Kg./cm.$^2 = 71{\cdot}4$ atmospheres.

If the elastic deformation of the pipe is taken into account, the relationships become much more complicated. Here, not considering periodic bending of the wall of the pipe, there is a second velocity of propagation c' in addition to the velocity of sound in water; for thin-walled pipes c' may be considerably less than c. For propagation to great distances c' predominates. If d is the diameter of the pipe, s the thickness of the wall and E the modulus of elasticity of the wall,† (64)

$$c' = c\sqrt{\{Es/(Kd + Es)\}}.$$

* This is observed e.g. if a tap that has a fairly long piece of piping attached to it in the direction in which the water is moving is quickly turned off. The water in the pipe goes on moving forward by its inertia, forming a cavity, but is then forced back by the atmospheric pressure and thus acquires a velocity which if there were no friction would be equal to the velocity it had at first.

† For details, see Alliévi, *Revue de mécanique* (1904), or the German version by Dubs and Bataillard, *Allg. Theorie über die veränderliche Bewegung des Wassers in Leitungen* (Berlin, 1909). See also C. Camichel, *Proc. II Internat. Congr. for Applied Mechanics*, p. 75 (Zürich, 1926); H. Kreitner, *Die Wasserwirtschaft*, p. 258 (1926); Ch. Jäger, *Wasserkraft u. Wasserwirtsch.*, Vol. 32, p. 269 (1937), which has a large bibliography; and G. Ried, *VDI-Zeitschr.*, Vol. 85, p. 639 (1941).

The collapse of cavitation bubbles is a related phenomenon. If the bubble is squashed flat, as is sometimes seen in spark photographs, the phenomena resemble those just described. If, however, a spherical bubble shrinks to a point, very large velocities are produced in the last instant of its existence, as in the contraction of the bubble the accumulated kinetic energy is contracted into a smaller and smaller space (of the same order as the diameter of the sphere at the instant in question). Thus very high pressures may be produced over an area which is little more than a point. The same applies to hemispherical bubbles at the solid boundary and to bubbles in conical depressions, which helps to explain the destructive effects described at the end of the previous section.*

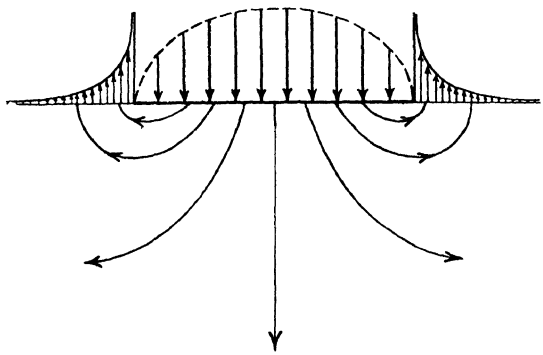


Fig. 5.9

A similar account can be given of the impact of a solid body on the surface of water, if the thin layer of air separating them is ignored. But as the " acoustic hardness " ρc of the solid body is different, the distribution of the relative velocity q between the two substances is different, though the same rise in pressure occurs in both. If the second material is heavy metal, we can reckon that practically the whole of the relative velocity will be imparted to the water.†

The pressure so calculated, however, does not long retain its full strength, for actually a low pressure is propagated inwards from the boundaries of the solid striking the water with the velocity of sound, and also the solid body loses its velocity more or less rapidly according to its mass, owing to the reaction. With respect to time the

* Cf. J. Ackeret, footnote, p. 313.

† This has a certain bearing on the attack of stormy seas on breakwaters and quay walls. As the bed of the sea rises, the crests of the waves bend forward and break. Before they break, there is an intermediate form, a wave with a perpendicular front. If this is formed just in front of the wall, the blows of the waves may be so powerful, owing to the large forward velocity of the wave front, as to loosen the largest blocks of masonry.

phenomenon behaves much as the distribution of pressure over a rectangular plate in fluid moving with a velocity above that of sound does with the breadth (cf. fig. 4.51, p. 305). As the process goes on the pressure of the water is soon completely reduced, and finally only the ordinary hydrodynamic pressures corresponding to the residual motion are left.

(*b*) *Continuous Phenomena.*—If a rigid body is placed without velocity with its plane under-surface on the surface of water, and then suddenly pushed downwards, there is a flow of the kind discussed on p. 210. There it was produced by a board moving with acceleration, as a substitute for a moving aerofoil. The medium completely surrounded the board, and below the plane of the board there was an excess of pressure and above it a deficit. In the plane of the board the pressure near the board was equal to the undisturbed pressure. It is this last property that we require here, in order to fulfil the condition at the free surface of the water. If we use only the lower half of the solution given previously, we have exactly what is needed here, at least initially. The distribution of pressure is again found from $p - p_\infty = -\rho\,\partial\Phi/\partial t$ and can be obtained by the method used previously, with suitable modifications, at least for the two-dimensional case (for a long board). The pressure distribution over the span of the board is semi-elliptical as in the case of the aerofoil (fig. 5.9).

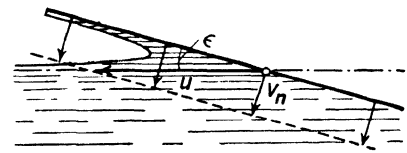


Fig. 5.10

Another phenomenon arises when a plane inclined to the surface of the water moves into the water with normal velocity v_n. In the previous example the water splashes up vertically close to the sides of the board if the acceleration is short and intense. Here, in the same way, as was shown by Herbert Wagner in an important paper,* the water splashes up along the oblique board (as shown by an arrow in fig. 5.13). The contact between the surface of the water and the board obviously advances with a horizontal velocity $u = v_n/\sin\epsilon$, as is clear from fig. 5.10. In the simplest case the flow which gives rise to the splash is

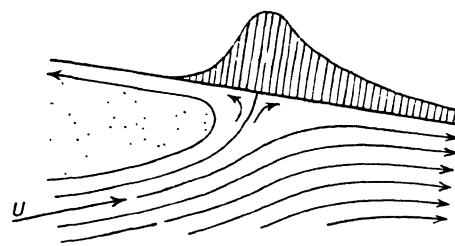


Fig. 5.11

* *Über Stoss- und Gleitvorgänge an der Oberfläche von Flüssigkeiten, Zeitschr. f. angew. Math. u. Mech.*, Vol. 12, p. 193 (1932).

steady in a system of co-ordinates moving along the surface with velocity u. In this co-ordinate system the fluid at rest is moving with velocity u to the right. Fig. 5.11 shows the stream-lines for this steady flow and the distribution of pressure over the surface. For the stagnation point which can be seen in this figure, Bernoulli's equation gives a rise of pressure equal to $\frac{1}{2}\rho u^2 = \frac{1}{2}\rho v_n^2/\sin^2\epsilon$. Thus if the angle ϵ is small very high pressures may occur locally (a fact which may easily give rise to damage if a seaplane with poorly designed floats makes a bad landing).

(*c*) *Planing Surfaces on the Surface of Water.*—Owing to the fundamental close relation between the phenomena we shall add a discussion of the processes which give rise to very considerable lifting force in speedboats and seaplane floats.* The relationship to the lift on an aerofoil is immediately obvious; as in that case, a high velocity is necessary for the production of a high lifting force. The relationship, however, goes much deeper than a superficial analogy. Here, as before, a stationary distribution of pressure moves horizontally over a fluid medium and gives rise to velocities in it. The difference between the present case and that of the aerofoil, namely, that here the moving medium is under the planing surface only instead of being all around it, merely has the effect of cutting out the pressure distribution on the "suction side", as at the beginning of subsection (*b*) above. The analogy can be regarded as complete if the planing velocity v is very large, so that the influence of gravity on the motion of the disturbed surface of the water may be neglected in the region where the planing surface extends in the direction in which it is moving. The planing surface must also be curved in such a way and set at such an angle that the front edge enters the water smoothly. In these circumstances the formulæ for the two-dimensional aerofoil and even for the three-dimensional aerofoil hold good, except that both the lift and the induced drag have to be halved. Also, as only one side is wetted, the frictional drag is approximately halved.

If the front edge does not enter the water smoothly, however, there is a fundamental difference between the two cases, inasmuch as in the aerofoil there is a flow round the front edge with "suction" (p. 206), but in the planing surface there is a splash of the kind described in subsection (*b*). The impulse associated with the splash implies a resistance; it corresponds almost exactly to cutting out the suction (which of course is again halved). For a plane surface inclined at an angle ϵ the pressures in the moving fluid give a resultant at right angles to the surface, i.e. inclined backwards at an angle ϵ from the perpendicular. If the suction is taken into account the resultant is more steeply inclined (in the two-dimensional case it is vertical).

* The game of playing ducks and drakes with a flat stone is another case of this; the surf-board and water-ski may also be mentioned.

The downward velocities behind the planing surface and the upward velocities beside it actually give rise to a wave motion which is very complicated in the three-dimensional case. For the two-dimensional case (planing surface of infinite span or enclosed by parallel walls at either side) the calculation is feasible; it gives a system of waves which follow after the planing surface with velocity $c = u$. By

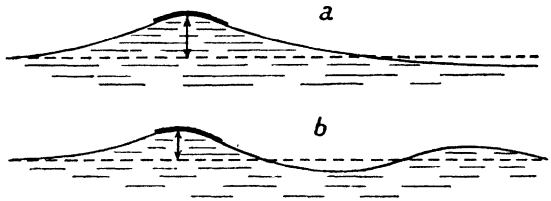


Fig. 5.12

formula (34), p. 87, $\lambda = 2\pi u^2/g$. If the velocity u is moderate and b, the breadth of the surface wetted, is fairly large, u^2/g is no longer large compared with b, and then the effect of the wave motion on the upward force and resistance must be taken into account. The fundamental explanation of the motion of planing surfaces and many details are due to Herbert Wagner;*[65] the two-dimensional problem had been discussed earlier by Lord Rayleigh and H. Lamb.†[66] Fig. 5.12 shows the variations in level for the two-dimensional case and non-impulsive

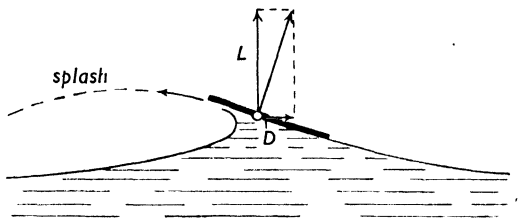


Fig. 5.13

start, (*a*) without and (*b*) with gravity. In the former case there is no pressure resistance, for reasons of symmetry; in the latter there is, balanced by the energy moving to the rear in the train of waves with velocity $c - U = u/2$ (see the discussion on p. 89). Fig. 5.13 shows the flow with splash for the case of the two-dimensional planing surface. The rise in level even ahead of the surface is a result of the pressure

* H. Wagner, footnote, p. 321, and *Proceedings of the Fourth Intern. Congr. for Applied Mech.*, 1934, p. 126 (Cambridge, 1935).

† See H. Lamb, *Hydrodynamics*, § 242 *et seq.*, 6th ed.; also E. Hogner, *Proc. First Internat. Congr. f. Applied Mech.*, 1924, p. 146 (Delft, 1925).

field, which gives an upward acceleration to particles ahead of the planing surface as well as to those in the rear; particles under the planing surface are given a downward acceleration (these accelerations are clear from fig. 5.9, which may be regarded as the acceleration diagram for the flow in fig. 5.12*a*). Here, as has already been mentioned, a resistance arises from the splash, even in the absence of gravity.

3. **Mixtures of Water and Air.**

(*a*) *Drops of Water in the Air.*—Very small freely falling drops of water are practically spherical as a result of capillary forces; since the viscosity of water is from 50 to 90 times that of air, varying according to the temperature, they may be regarded as rigid spheres, obeying Stokes' law (cf. Chapter III, § 3, equations (10) and (10*a*), p. 106). If d is the diameter of the sphere and v the velocity with which it falls, the resistance in the steady state is equal to the weight, i.e.

$$3\pi\mu_a v d = \tfrac{1}{6} g \rho_w \pi d^3,$$

whence

$$v = \frac{\rho_w g d^2}{18\mu_a} = \frac{g}{18}\frac{\rho_w}{\rho_a}\frac{d^2}{\nu_a}, \qquad \ldots\ldots (1)$$

where μ_a is the viscosity of air and ρ_w the density of water. This formula may be regarded as approximately true up to about $\mathscr{R} = 2$, corresponding to d about 0·1 mm. (droplets of mist). If for greater convenience we put the diameter in millimetres and keep the velocity in metres per second, we obtain

$$v = 31 d^2,$$

since $\rho_w/\rho_a = 800$ and $\nu_a = 0{\cdot}14$ cm.^{2}/sec., which gives 0·31 metres per second at the upper limit.[67]

For larger drops, say from $d = 1$ mm. onwards, Newton's law may be used. Here (cf. Chapter III, § 13, equation (53), p. 176), assuming that the drops are spherical, we must put

$$g\rho_w \cdot \tfrac{1}{6}\pi d^3 = c\rho_a \cdot \tfrac{1}{2}v^2 \cdot \tfrac{1}{4}\pi d^2,$$

which gives

$$v = \sqrt{\left(\frac{4}{3c}\frac{\rho_w}{\rho_a} g d\right)}. \qquad \ldots\ldots (2)$$

Here the resistance coefficient c is not a pure constant. For $c = 0{\cdot}5$, with the units used above (d in mm., v in metres/sec.) we obtain the rule-of-thumb formula *

$$v = 4{\cdot}6\sqrt{d}. \qquad \ldots\ldots (2a)$$

* For the intermediate region $d = 0{\cdot}1$ to 1 mm. and with the same units, $v = 4d$ may be used as a rough approximation.

For diameters above 4 mm. this formula soon becomes unusable, since the drop becomes more and more deformed as the pressure differences increase. In particular, it becomes flattened in the region of the stagnation point and may be compressed, so that c becomes considerably larger and d is also increased. E. Hochschwender * photographed drops falling into a vertically ascending stream of air; he found that drops only 6 mm. in diameter were markedly flattened and rather larger ones swelled out like a chef's cap. For diameters over 6·5 mm., measured while the drops are still spherical, the drops are completely broken up † by the pressure of the atmosphere, so that, as has long been known by observation, *very* large drops cannot occur even in heavy rain. The course of the phenomenon is as follows: starting from the cap-like form the middle is drawn out into a thin skin and bursts, leaving an anchor ring which then breaks up into a number of small drops. Fig. 5.14 shows some stages of the phenomenon, drawn from Hochschwender's photographs.

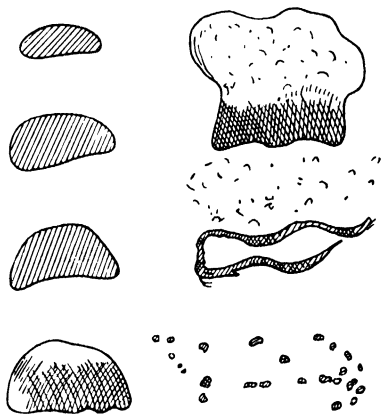


Fig. 5.14.—Break-up of large raindrops, after Hochschwender

(*b*) *Break-up of Fluid Jets in Air.*—Even at low velocities a slender jet breaks up into drops; according to Lord Rayleigh, this is to be ascribed to the effects of surface tension. As he showed,‡ surface tension renders a thin cylindrical jet statically unstable, since the jet can reduce its surface area by breaking up into not-too-small drops.§ This breaking-up || is very well seen if a thin stream of water is allowed to flow slowly from a tap. Figs. 5.15 and 5.16 are two typical spark photographs by Lord Rayleigh of the break-up of jets of water. In fig. 5.16 a very regular break-up of the jet is achieved, in spite of the high jet velocity, by bringing a vibrating tuning-fork into contact with the mouth of the tube from which the jet is issuing. The break-up is hindered if the fluid is very viscous, so that it can be drawn out into

* *Diss.*, Heidelberg (1919). † Ger. *zerblasen.*

‡ Lord Rayleigh, *Proc. Lond. Math. Soc.*, Vol. 10, p. 4 (1879) = *Scientific Papers*, Vol. I, p. 361.

§ The transition from jet to drops begins because when the diameter of the cylinder varies from its average value a higher pressure is produced inside the thin places than inside the thick as a result of surface tension. This has the effect of driving the fluid inside towards the thicker places. The thin places then draw together lengthwise like a thin rod and finally detach themselves from the large drops, themselves forming a small drop. || Ger. *zertropfen.*

threads in a remarkable way, as in the case of syrup and the like.*

With high-velocity jets of a less viscous fluid, the break-up is accelerated by currents of air. Another type of break-up † is also observed,‡ which is caused by surface waves. A further rise in jet velocity leads to complete break-up § of the jet, the wavy bulges in the jet obviously being torn to pieces by the marked turbulence of the air carried along. These phenomena are shown in the first three lines of fig. 5.17, taken from photographs by A. Haenlein; the fourth and fifth lines show corresponding phenomena in a very viscous fluid. The degree of dispersion arising from the break-up of a jet, the final product of which, in the less viscous fluids, is a mixture of fine droplets and the air carried along with them, may obviously be greatly enhanced by deliberately making the jet turbulent or by giving it a rotating motion before it leaves the nozzle.



Fig. 5.15

For thick jets, e.g. from a fireman's hose, break-up is unavoidable even when an attempt is made to cut down turbulence at the exit from the nozzle to a minimum by suitably shaping it, since the surface is covered by small waves like the surface of a pond lashed by a storm, and drops are torn off from the crests of these waves.|| Thus



Fig. 5.16

Break-up of jets of water, after Lord Rayleigh

eventually the whole jet breaks up into a rain of drops and its range is much less than it would be in a vacuum.¶ The range of the jet is increased if the jet is very free of turbulence when it emerges from the nozzle.(68)

The above considerations also throw light on the phenomena in sprays and atomizers. In most of these a rapid stream of air is made

* The increase in viscosity owing to evaporation of the solvent, or possibly by cooling, which is more marked at the thin places, may make the process quite stable; cf. the spinning of glass threads and the like.

† Ger. *zerwellen.*

‡ Cf. the observations of A. Haenlein, reported in *Forschung*, Vol. 2, p. 139 (1931), and a theoretical discussion of the phenomena by C. Weber, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 11, p. 136 (1931).

§ Ger. *zerschmettern.*

|| The instability of the surface of a fluid exposed to rapid air currents follows from the discussion of fig. 2.16, p. 51.

¶ For details, see Hütte, 26th ed., Vol. I, p. 397; also Th. Oehler, *Techn. Mech. u Thermodyn.*, p. 329 = *Forschung*, Vol. 1 (1930).

to surround a stream of liquid emerging more slowly from a nozzle; it breaks it up * into little particles which become mixed with more air as the current of air spreads out and becomes turbulent, thus giving rise to a sort of mist. This is what happens in the carburettor of an internal-combustion engine; the air sucked into the engine is led through a narrowing tube into which the fuel is sprayed and broken up into fine particles by the rapidly moving air; it partly evaporates and

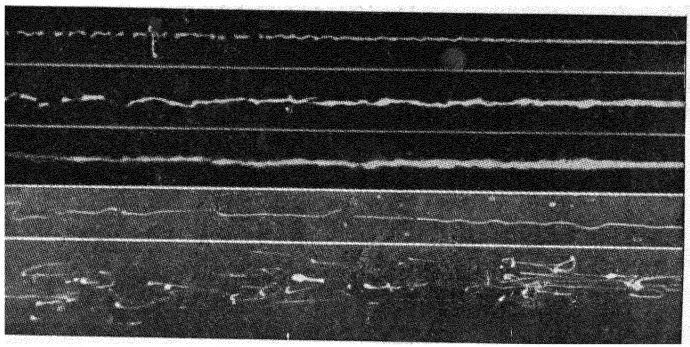


Fig. 5.17.—Break-up of thin jets of fluid, after Haenlein
(The flow is from right to left)

forms an inflammable mixture. In the Diesel engine the fuel is atomized in the cylinder itself. It is forced at very high pressure through a nozzle so shaped as to give rise to a considerable degree of turbulence in the emerging jet, so that the latter breaks up to a considerable extent itself. Still finer subdivision is then brought about by the break-up † of the rapidly moving shreds of liquid, just as in the case of the raindrops.

On this basis the maximum diameter of drop which can subsequently exist may be estimated by considering the equilibrium between the stagnation pressure of the air flowing past the drops, $\frac{1}{2}\rho_{\text{air}}v^2$, and the capillary force which may arise, say, in the cap-shaped drop described previously. The radii of curvature of a definite "cap" form are proportional to d, the diameter of the drop, and hence, by Chapter I, § 11 (p. 27), the differences of pressure at the deformed surface of the drop are proportional to T/d, where T is the surface tension. Equating the two pressures at the stagnation point, we have

$$\tfrac{1}{2}\rho_{\text{air}}v^2 = \text{constant} \times T/d.$$

The constant can be obtained from the results of Hochschwender's experiments. Here $d_1 = 0{\cdot}65$ cm., and hence, by equation (2a), p. 324, $v = 1170$ cm./sec. Taking $\rho = 0{\cdot}00125$ gm./cm.^{3} and $T = 75{\cdot}2$ dynes/cm. (= gm./sec.^{2}), we have

* Ger. *zerwellen, zerblasen.* † Ger. *zerblasen.*

the value 7·7 for the constant. Hence the maximum diameter of drop after atomization is

$$d_1 \approx 7{\cdot}7 \frac{T}{\frac{1}{2}\rho_{\text{air}} v^2}. \quad \ldots\ldots\ldots \quad (3)$$

For v we have to insert a suitable average value for the velocity of the drop relative to the air, which diminishes as the process goes on. The formula brings out the effect of this relative velocity and also that of the density of the air. For the same value of v the degree of atomization is much greater when the density of the air is high than at atmospheric pressure.

Increased viscosity increases the time required for atomization, but hitherto it has not been possible to account quantitatively for this on a theoretical basis.

Many experiments on atomization have been made in order to throw light on the phenomena in the Diesel engine. Fig. 5.18 is taken from a photograph *(69) by courtesy of M. Popoff; the nozzle with which it was obtained is shown enlarged in fig. 5.19.



Fig. 5.18.—From a spark photograph of a jet of atomized fuel, after M. Popoff

The atomization of water is also made use of in apparatus for ground spraying. A fine distribution of the water is obtained, e.g. by letting the water impinge on a plate inclined at a small angle; this spreads the jet out fanwise and it rapidly disintegrates. Still more effective are Körting's swirl atomizers, in which the water is given a strong circulation by means of vanes before it reaches the narrow part of the nozzle. The tangential component of the velocity is considerably increased as the distance from the axis is diminished (cf. the remarks on p. 47 about the volute chamber of a pump); as a result the jet spreads out conically when it emerges from the nozzle and the atomization is very effective. The nozzle used by Popoff in the

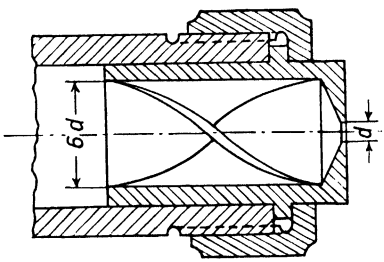


Fig. 5.19.—Popoff's atomizer nozzle

* Other photographs of the atomization of a jet are to be found in W. Oschatz, *Deutsche Kraftfahrtforschung*, No. 57 (1941); see also *Photographie und Forschung*, Vol. 3, p. 170 (1941). Slow-motion pictures taken under the conditions occurring in the Diesel engine, which also show self-ignition and the phenomena of combustion, are given by O. Holfelder, *VDI-Forschungsheft* No. 374 (1935); see also K. Blume, *Forschung*, Vol. 11, p. 284 (1940), and *Deutsche Kraftfahrtforschung*, No. 53 (1941).

experiments mentioned above is a particular form of swirl atomizer. Other swirl atomizers with a higher twist give a much wider angle of scatter.[70]

(*c*) *Bubbles of Air in Water.*—The upward thrust on a bubble of air in water is practically equal in magnitude to the weight of a drop of water of equal diameter in air. Hence, provided the bubble is spherical, the same formulæ as were given in (*a*), p. 324, for drops of water falling in air can in general be used, provided we note that at average temperatures the viscosity of water is about 60 to 80 times that of air and its density about 800 times that of air. We thus obtain the rule-of-thumb formulæ:

(*a*) for $d \leqq 0{\cdot}16$ mm., $v = 0{\cdot}4d^2 \sim 0{\cdot}5d^2$ (Stokes' law *),

(*b*) for $d \geqq 1$ mm., $v =$ constant $\times \sqrt{d}$ (Newton's law).

For bubbles ascending vertically upwards and $c = 0{\cdot}4$ to $0{\cdot}5$ the constant in the second formula would be about $0{\cdot}16$ to $0{\cdot}18$. Actually, the smaller bubbles move more or less from side to side as they ascend, or in corkscrew fashion, while larger ones become flattened like a cap and fluctuate irregularly in shape.†[71] According to hitherto unpublished experiments by W. Müller, if d in the formula denotes the diameter of the sphere equal in volume to the bubble, and d is between 1 mm. and 2·5 mm., the constant in the formula can be taken as about $0{\cdot}127$. If d is between 3 mm. and 8 mm. the shape of the bubble alters so rapidly that the velocity remains almost constant at about $0{\cdot}21$ to $0{\cdot}22$ metres per second. Above $d = 12$ mm. the cap form remains practically constant and is then very flat; and here the constant can be taken ‡ as $0{\cdot}068$ (the corresponding resistance coefficient c is $1{\cdot}7$). Fig. 5.20 shows the shapes of air bubbles from photographs by W. Müller.§

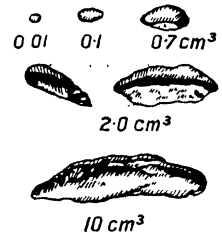


Fig. 5.20. — Shapes of ascending air bubbles, after W. Müller (half natural size)

If a bubble, or a group of bubbles, enters a vertical pipe, and the

* The surface of small bubbles seems to behave in essentials like a rigid surface, so that Stokes' law can be applied in its ordinary form. The case of a flexible surface has also been investigated theoretically. If the viscosity of the air inside is neglected, this gives a velocity 1·5 times as great (cf. e.g. *Handb. d. Exp.-Phys.*, Vol. IV, 2, p. 346, formulæ of Rybczynski, Hadamard and Boussinesq).

† Note added in December 1943: the author has only now come across the detailed series of experiments by K. Höfer *VDI-Forschungsheft* No. 138, pp. 3–12 (1913). For further references, see the short report by F. Pickert in *Forschung*, Vol. 3, p. 308 (1932).

‡ For bubbles of diameter above 1 mm. or above 12 mm., theory gives, respectively, $v = 1{\cdot}28\sqrt{(gd)}$ or $0{\cdot}69\ (gd)$, where v, d and g are in cm. (or metres).

§ The diameters of the spheres with these volumes are 2·7, 5·7, 11·0, 15·6, 26·7 mm.

total volume of the bubbles is V, the weight of water in the pipe is less by wV than if the pipe were filled to the same height without bubbles ($w = \rho g$ = weight of unit volume). The pressure in the pipe below the group of bubbles is therefore less by $p' = wV/a$ than if there were no bubbles, assuming that the pressure is given at a higher level (a is the cross-section of the pipe). This fact can be utilized to lift water by means of compressed air (air lift pump). If the water is to be lifted a height h above its original level, a volume of bubbles $V > ah$ must be kept up in the pipe by means of a constant flow of compressed air, in order to make the water flow out at the top. Owing to the bubbles overshooting the water by the velocity v (see above) and to the expansion of the air as it rises, the more accurate theory of the air lift pump is fairly complicated and we shall not go into details of it here.* For the pump to have a high efficiency it is necessary to keep the velocity v as small as possible by making the bubbles extremely small. The bubbles, however, have a tendency to coalesce and become larger and larger, eventually becoming large bubbles of the shape shown in fig. 5.21, taking up the whole cross-section of the pipe. This form is very stable and, as there are no longer any disturbances due to the detachment of eddies, it can be investigated theoretically. According to the theory and experiments of Th. Dumitrescu,† the velocity of ascent of these large bubbles is given by $v = 0{\cdot}35\sqrt{(gd)}$, i.e. it is less than that of bubbles of diameter greater than $0{\cdot}26d$, reckoned as spheres. The latter therefore catch up on the large bubble and make it even larger.

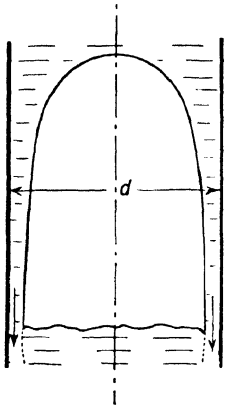


Fig. 5.21.—Air bubble in pipe, after Dumitrescu

The circulation of water in water-tube boilers also depends on the differences of pressure which arise from the presence of bubbles of steam.‡

* Cf. H. Behringer, *Diss.*, Karlsruhe (1930); a summary is given in *Forschung*, Vol. 3, p. 310 (1932).

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 23, p. 139 (1943).

‡ 1. The results of a series of experiments at Danzig by E. Schmidt and his colleagues on the velocity of ascent of bubbles of steam in boiler tubes [cf. *VDI-Forschungsheft* No. 365 (1934)] have been reduced to order according to the rules of dynamical similarity by F. Kaissling, *Forschung*, Vol. 14, p. 30 (1943).

2. *Flow of Mixtures of Water and Air.*—Let the mass of air involved, ϵ, be small and let the flow be rapid, so that the water and air remain mixed. Take the temperature of the air as equal to the practically constant temperature of the water. Then the volume of unit mass of the mixture is given by

$$v = \frac{1}{\rho} = (1 - \epsilon)v_w + \epsilon v_a,$$

4. Solid Bodies in Moving Air.

Here there are two distinct problems of note: the pneumatic transport of granular substances in pipes, and the motion of sand and snow in natural winds. Both problems are closely akin to that of the transport of sand and gravel in rivers (see § 5, p. 334). In practice, however, there is a considerable difference between them, in that in water the ratio of the specific gravity of the body being moved and that of the medium scarcely exceeds 3 : 1, whereas in air it is about 700 : 1 for snow and as much as 2400 : 1 for desert sand. Here, especially for fairly large grains, the path of an individual grain does not follow the flow exactly but is more like the path of a projectile. The theory of these motions is as yet largely undeveloped, so that for the most part we can only give an account of experimental results.

(*a*) *Practical Applications.*—So far as the transport of grain through pipes is concerned the experiments of J. Gasterstädt * are of special importance. For wheat, which we may take as an example, the equilibrium rate of fall in air at rest (v_0) is about 10 metres per second. For velocities above $2v_0$ in a horizontal pipe the resistance coefficient (cf. Chapter III, § 11, p. 163) is given by

$$\lambda_1 = \lambda_0(1 + 0{\cdot}3W/Q),$$

where W is the weight of grain moved per unit time, and Q the weight of air moved per unit time. For smaller velocities the number is greater than 0·3; for $1{\cdot}2v_0$, the grain remains at rest; λ_0 is the value for the empty pipe.

Gasterstädt also studied the motion of individual grains. They bound erratically from one wall to the other in paths more or less inclined to the direction of flow, provided the air velocity is sufficiently great, and thus on the average fill the whole cross-section of the pipe fairly evenly.

where $v_a = RT/p = K/p$. If v_w is constant, Bernoulli's theorem gives

$$\tfrac{1}{2}q^2 = \text{const.} - \int v\,dp$$
$$= \text{const.} - \{(1-\epsilon)pv_w + \epsilon K \log p\};$$

and the velocity of sound

$$c = \sqrt{\left(1\Big/\frac{d\rho}{dp}\right)}$$
$$= \sqrt{(\epsilon K)} + (1-\epsilon)pv_w/\sqrt{(\epsilon K)}.$$

The velocity of sound in the mixture is often very small. A theory which is more accurate from the thermodynamical point of view has been published by G. Heinrich, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 23, p. 117 (1943). Shock waves were discussed by J. Ackeret (cf. footnote, p. 313).

* *VDI-Forschungsheft* No. 265, p. 617 (1924), and *VDI-Zeitschr.*, Vol. 68, p. 617 (1924).

Another important practical application of pneumatic transport is the sucking away of shavings and dust in the wood and leather trades, and industrial dust of all kinds. Another important problem is that of the separation of material to be transported. This is usually done by so-called " cyclones ", which are vessels symmetrical about a vertical axis. Air bearing the material in question is brought in tangentially and owing to its weight the material reaches the outer wall, where its velocity is reduced by friction and it falls down. Owing to the increased turbulence arising from the instability of boundary layers inside curved boundaries (cf. Chapter III, § 5(f), p. 132), the finer particles are again mixed with the air and are thus separated from the coarser. Experience has shown that the separation is more effective in small cyclones than in large, so that a number of small cyclones are preferable to one large one. A simple form of cyclone * is shown in fig. 5.22.

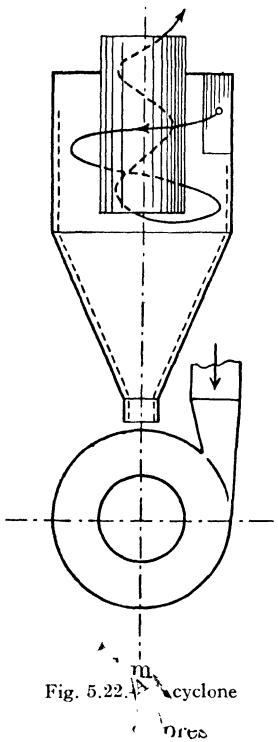


Fig. 5.22. Cyclone

Two other important practical applications of the interaction of grains and a current of air may be mentioned here: the winnower and the sand blast. In the first a mixture of heavy and light grains is led from above into a current of air of suitable velocity directed obliquely upwards. The heavier grains fall through the current of air, while the lighter ones are carried up with it. In the sand blast, a stream of air carrying grains of sand with it is forced through a very narrow nozzle and thereby acquires a high velocity. The grains of sand impinge very soon after their exit from the nozzle on the object which is to be sand-blasted. The theory of these appliances is usually based on the consideration of a single grain which is acted on by the usual air resistance. Since these theories, though useful, do not as a rule take into account the reaction of these resistance forces on the flow, they contribute but little to hydrodynamics, and we shall therefore pass over them here. W. Barth † was the first to make a more accurate calculation in connexion with apparatus for measuring the mean diameter of grains of industrial dusts.

* The opposing pressures at the entrance [see Chapter II, § 5(b), at end, p. 48] can be avoided by devices at the exit side for doing away with rotations and high velocities.

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 25/27, p. 157 (1947); *Ing.-Arch.*, Vol. 16, p. 147 (1948).

(*b*) *Effects of Natural Wind.*—The transport of sand by the wind, which is very similar to that of dry snow, has been studied very carefully by R. A. Bagnold,* both in a wind tunnel built specially for this purpose and also in the desert. He noted that there is a limiting velocity below which no motion of the sand is possible at all. Between this velocity and a certain higher velocity the sand may remain at rest, but motion occurs if a grain of sand from anywhere else impinges on the grains of sand at rest; the impact throws up other grains into the air, where they are caught in the current of air, acquire additional velocity, and set other grains in motion, so that the number of grains in motion rapidly increases. The paths of the particles are as shown in fig. 5.23. For still greater wind velocities, for which the vertical component of turbulent fluctuation is greater than v_0, the rate of fall of the grains of sand, a large number of grains are whirled up into the air in dense clouds. The density diminishes upwards; cf. the account of the motion of suspended material in a river in § 5(*b*), p. 336.



Fig. 5.23.—Paths of grains of sand, from photographs by Bagnold

The balance between grains falling and grains caught up is disturbed by obstacles of any kind. Where the velocity is less, more grains remain at rest; where it is greater, more are carried away; and thus the configuration of the surface is altered by losses here and gains there until a state of equilibrium is established. Specially noticeable, for example, is the shape of the surface of snow in the neighbourhood of a tree-trunk or post. Close to the tree-trunk a downward flow sets in from the region where the full dynamic pressure takes effect on the tree-trunk above the frictional boundary layer of the ground. In this way the full velocity is brought down to ground level and causes deep hollows on either side and on the front of the tree-trunk. Ahead of this, and also at some distance behind the tree-trunk, the level is raised owing to the velocity being smaller there. The same phenomenon is seen in the undermining of the piers of bridges. The flow corresponds fairly closely to that shown in fig. 3.33, p. 147.

If a wind of sufficient velocity impinges on a sand dune, it lifts the sand on the windward side and lets it fall on the lee side, where the wind velocity is less (if not non-existent owing to separation of

* *Proc. Roy. Soc.* (A), Vol. 157, p. 594 (1936), "Wind Tunnel Experiments"; Vol. 167, p. 282 (1938), "Observations in a Sandstorm in the Desert".

the flow); the whole dune is thus gradually moved bodily in the direction of the wind (travelling dunes).

A striking phenomenon in dunes is the occurrence of ripples, often very regular, on the windward side when the wind is steady.† This phenomenon has not yet been explained. Possibly the wave-length of the ripples is related to the mean trajectory of the grains of sand shown in fig. 5.23.

Of practical interest are measures for preventing the drifting of snow in places where public transport would be affected, e.g. sunken roads and railway cuttings, where, by what we have said above, snow would pile up to great depths in high winds.(72) This is prevented by erecting high fences with gaps between the posts at a suitable distance to windward of the cutting. On their lee there is a relatively calm zone, with a steady but slight wind, in which all the snow is deposited.‡ (A solid wall is less effective, as it gives rise to strong eddies, so that the snow is whirled up into the air; this is of course to be avoided.)

Air mixed with snow or sand is of course markedly heavier than ordinary air. Hence for example, in avalanches of dry snow in the mountains, high velocities are produced and great damage may be done. More will be said about this on p. 368.

5. Solid Bodies in Moving Water.§

(*a*) *Motion of Sand, Gravel, &c, in Rivers.*—In contrast to sand or ice in air, the specific gravity of sand is of the same order as that of water, so that here the free paths of individual grains are generally quite short. This makes the discussion simpler than in the case of air. Owing to the importance of the subject in hydraulic engineering and river control we shall go into rather more detail.

The force exerted by the turbulent flow of the water on the sand, gravel, stones, and the like which form the bed of the river can be estimated in the following way.

A particle of gravel at a height y_1 above the mean level of the bed is impinged on by the current with a mean velocity u_1, which is equal, by Chapter III, § 5(*d*) (p. 129), to $v_* (5{\cdot}75 \log y_1/k + C_2)$, where k is the "roughness" of the bed. Here d, the mean diameter of the coarser particles of gravel, serves as a measure of k. If u_1 is the limiting velocity for which the coarser particles just fail to move, the finer particles cannot remain at rest except in the sheltered corners between the

† See e.g. the beautiful photographs by A. Ehrhardt in *Die Kurische Nehrung* (Hamburg, 1938). See also Vaughan Cornish, *Ocean Waves* (Cambridge University Press, 1934).

‡ Such snow fences are to be seen in the Scottish Highlands.

§ Professor R. Seifert, formerly Director of the *Preussische Versuchsanstalt für Wasser-, Erd- und Schiffbau* in Berlin, was kind enough to read this section, and the author is indebted to him for several valuable suggestions.

larger particles.) For particles of geometrically similar shape y_1 can also be taken as proportional to d. Thus with the assumptions that all particles are of geometrically similar form and the flow is fully developed, i.e. C_2 has a fixed numerical value, the expression in brackets has a fixed numerical value. The mean force exerted by the flow on a particle of gravel, $F_m = ca \,.\, \frac{1}{2}\rho u_1^2$, can be related to τ_b, the mean apparent shearing stress on unit area of the bed, for by definition $\rho v_*^2 = \tau_b$ and hence, since $u_1 = v_* \times$ const., $\frac{1}{2}\rho u_1^2 = \tau_b \times$ const.

The question whether the particle of gravel remains at rest or is swept away is not settled by the mean force but by the highest force F_1 occurring in turbulent fluctuations, which are particularly strong near the boundary. This can be taken as proportional to F_m, and a may be taken as proportional to d^2.

F_2, the resistance which has to be overcome in pushing the grain over the rough surface of the other grains, is proportional to the "weight of the grain in water", i.e. weight less upthrust; if w is the weight of unit volume of water and w_g that of the grain of gravel, the weight is $w_g V_g$ and the upthrust wV_g. The volume V_g is proportional to d^3. Hence $F_2 = (w_g - w)d^3 \times$ const. The condition that the particle of gravel moves is obviously $F_1 > F_2$. By what has been said above, this is the same as

$$\tau_b > (w_g - w)d \times \text{const.} \qquad . \quad . \quad . \quad . \quad (4)$$

For a broad river with an even fall i, a very simple calculation (see Chapter III, § 11, p. 161, putting $m = t$, the depth of the river) gives

$$\tau_b = wti,$$

so that equation (4) becomes

$$i > \frac{w_g - w}{w} \cdot \frac{d}{t} \times \text{const.} \qquad . \quad . \quad . \quad . \quad (4a)$$

The constant depends to some extent on the shape of the particles of gravel. For fine grains, where roughness flow is not yet completely developed, there is a definite subsidiary effect due to Reynolds' number $v_* d/\nu$; see e.g. the detailed series of experiments of the *Preussische Versuchsanstalt für Wasser-, Erd- und Schiffbau* in Berlin, published in the *Mitteilungen* of this Institute.† According to A. Shields,† for fairly large values of $v_* d/\nu$ and rounded grains, an average value of the constant in equation (4) is 0·06. This value is remarkably low. When a grain begins to move there seems also to be a lifting effect due to eddies, similar to that in a waterspout, which first raises

† Especially Nos. 9, 19, 26, and 43 (published by the Institute in 1932, 1935, 1936, and 1942). No. 26, by A. Shields, contains an instructive summary of the theory of the motion of sand, gravel, &c., based on dynamical similarity and turbulence.

the grain a little so that its resistance to movement is decreased. The relationship to w_g was tested with artificial particles with values of w_g ranging from 1·27 to 4·2, and satisfactorily confirmed. The effect of $v_* d/\nu$ was also very clear. For $v_* d/\nu = 10$ the minimum value of the constant was about 0·033.

(*b*) *Behaviour of Suspended Material.*—If sand and gravel over a greater area are set in motion they are caught up from time to time and whirled along by the current. Finer material may even be carried up to the surface of the water, becoming "flotsam". Then each partical is continually falling downwards with its own appropriate velocity q_0 relative to the portion of water in which it is. The continual upward movement in spite of this fall arises from the fact that the number of particles per unit volume is greater in the lower layers than in the higher, so that a portion of water flowing upwards carries up more particles with it than are carried down by the corresponding portion of equal volume moving downwards. Thus an equilibrium is set up between the average upward transport by turbulent transfer and the uniform fall of all the particles relative to the surrounding portion of water. More rapid turbulent transfer tends to make the distribution more even; more rapid rate of fall gives rise to a more marked falling-off of the number of particles in the higher levels.

The mathematical discussion of the behaviour of suspended material starts from the theory of exchange in turbulent flow. By Chapter III, § 4 (p. 120), the coefficient of exchange of momentum $A_\tau = \rho l^2 \mid du/dy \mid$ and the coefficient of exchange of mass A_M is about 1·4 to 2 times as great. The volume exchange A_V, which is what we are concerned with here, is equal to A_M/ρ. For the flow along a plane boundary $l = \varkappa y$, and by Chapter III, § 5 (p. 125), $du/dy = v_*/\varkappa y$, so that $A_\tau = \varkappa \rho y v_*$, and hence A_V can be put equal to $\beta \,.\, y v_*$; if $\varkappa$ is about 0·4, β is equal to from 0·55 to 0·65. If n is the number of suspended particles with the same rate of fall v_0 per unit volume, the convective flow of particles is $-A_V\, dn/dy = -\beta y v_* dn/dy$. Now this current—an upward one, since dn/dy is negative—is balanced by the falling of the particles. In unit time as many particles fall through unit area as were previously contained in the volume $1 \times 1 \times v_0$, i.e. $v_0 n$. Hence

$$v_0 n = -\beta y v_* \frac{dn}{dy},$$

or

$$\frac{dn}{n} = -\frac{v_0}{\beta v_*}\frac{dy}{y},$$

which on integration gives

$$n = n_1 \left(\frac{y}{y_1}\right)^{-v_0/\beta v_*}, \qquad \ldots\ldots\ldots \quad (5)$$

where y_1 is, say, the layer close to the bed in which the lowest suspended particles are, and n_1 is the density of the particles there; details of these must be obtained from the mechanics of the flow immediately adjacent to the bed. (Provided i is considerably above the values given in (4) or (4*a*), we may assume that the layer next the bed is thoroughly stirred up, and hence that n_1 is as great as is compatible with the maintenance of the state of flow of the water.)

Formula (5) clearly shows that very fine suspended particles ($v_0 \ll \beta v_*$) distribute themselves practically uniformly over the whole depth, whereas coarser materials are found only in a layer close to the bed (cf. fig. 5.24; case *a* is for large values of v_0/v_*, case *b* for small values of v_0/v_*).

The total number of particles suspended over unit area is given by

$$N = \int_{y_1}^{t} n\,dy,$$

which, when $v_0 > \beta v_*$, can be replaced approximately by

$$\int_{y_1}^{\infty} n\,dy = \frac{\beta v_* n_1}{v_0 - \beta v_*}.$$

If we multiply N by the weight of a single grain and by an average velocity u_m, we get an approximate formula for the total amount of suspended material which is transported. (For u_m we may take, say, the velocity at y_a, the height of the centre of gravity of the suspended material.)

The calculation above holds good only for particles of one size, or, more accurately, all with the same rate of fall v_0. If particles of different sizes are in suspension simultaneously, the distribution must be found for each kind separately, the appropriate values of n_1 being found from the constitution of the mixture at the bottom of the river and from the fact that the inequality (4) or (4*a*) must hold good. The solution of this portion of the problem (the calculation of n_1) is still outstanding.[73]

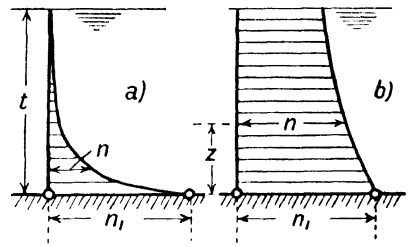


Fig. 5.24.—Distribution of suspended particles

Note.—Other relations hold good when the number of suspended particles is so great that they are in perpetual contact. These mud-like or paste-like mixtures behave like a plastic solid, with a yield point, i.e. a finite critical shearing stress, below which no displacement of the particles relative to one another takes place. The phenomena which may be expected to occur when a mass of this kind flows through a pipe have been discussed by E. C. Bingham.† D. H. Caldwell and H. E. Babbitt have reported experiments in confirmation of this.‡

(*c*) *Weight of Material Transported.*—Considerable irregularities in the bed of a stream arise from the movement of sand or gravel, even in straight watercourses which originally had a flat bed. These irregularities may take a great variety of forms; short ripples across the line of flow, or elongated banks usually occurring alternately on opposite sides of the river, as well as various intermediate forms. §[74] A. Shields

† *Fluidity and Plasticity*, p. 224 (New York, 1922); see also the second footnote on p. 101 of the present book.

‡ *Amer. Inst. Chem. Engrs.*, Vol. 37, p. 237 (1941); summarized by L. Schiller in *Forschung*, Vol. 14, p. 85 (1943).

§ For illustrations of ripples, banks, &c., see e.g. a report by H. J. Casey, *Mitteilungen d. Preuss. Vers.-Anstalt*, No. 19.

has shown † that the particular form depends ‡ on the value of v_*d/ν (ripples for values from about 2 to 6, elongated banks for values from about 20 to 70). The reason for this difference in behaviour is still not clear; it is worth noting that the values for ripples lie below the minimum of the constant in equation (4*a*), but those for banks above it.

It will be seen from the above that it is not easy to give a theoretical expression for W, the weight of material transported per metre breadth of bed per unit time, especially as the average shape of the particles and the particular mixture of differing sizes of particle are also involved. Shields § has given a graphical representation of results of experiments, in which

$$y = \frac{W}{Q}\frac{w_g - w}{wi}$$

is plotted against

$$x = \frac{\tau_b - \tau_0}{(w_g - w)d}.$$

Here W is the amount of material transported per second (i.e. its underwater weight) and Q the weight of water per second per metre breadth of the stream, d an average value for the diameter of the particle, $\tau_b = wti$ as before is the apparent shearing stress on the bed, and τ_0 its critical value, which by equation (4), p. 335, is equal to a numerical multiple of $(w_g - w)d$, the number being 0·03 to 0·06 provided the ripple stage is excluded (i.e. for values of v_*d/ν greater than about 10). The graph may be represented by the formula $y = 4x$, but shows a considerable amount of scatter.||

An empirical formula has been developed, which refers to the material in Swiss rivers (Archæan rock for which w_g is approximately 2·6 w) based on very careful experiments in an artificial channel, by E. Meyer-Peter, H. Favre and H. A. Einstein.¶ With the constants slightly modified by more recent experiments of H. Favre, it is

$$\frac{iQ^{2/3}}{d} = 16{\cdot}1 + 0{\cdot}54\frac{W^{2/3}}{d}, \qquad \ldots\ldots \quad (6)$$

taking 1 metre as the unit of length and measuring Q and W as before both in Kg./m. sec.** W. Liebs †† introduced the quantity of water Q_0 for which the motion of material just begins (i.e. $W_0 = 0$ and hence

† See footnote, p. 335.

‡ As δ, the thickness of the laminar boundary layer of the turbulent flow is proportional to ν/v_*, the ratio d/δ may be considered instead of v_*d/ν (Shields).

§ Loc. cit., p. 19.

|| In Shields' paper the number is 10 instead of 4, owing to his different definition of w_g (based on the volume including the interstices between the particles).

¶ *Schweizer Bauzeitung*, Vol. 103, No. 13 (1934).

** For Kg., see p. 6. †† See footnote 3, p. 340.

$iQ_0^{2/3}/d = 16{\cdot}1$), and thereby obtained the form of the expression which with present-day values becomes

$$W^{2/3} = 1{\cdot}85\,i(Q^{2/3} - Q_0^{2/3}). \quad . \quad . \quad . \quad . \quad (6a)$$

The latest modification of the formula also takes into account material of other densities. With $(w_g - w)/w = z$, H. Favre † obtained

$$\frac{iQ^{2/3}}{d} = 9{\cdot}57z^{10/9} + 0{\cdot}462\,z^{1/3}\frac{W^{2/3}}{d}. \quad . \quad . \quad . \quad (7)$$

With $z = 1{\cdot}6$ we again obtain equation (6). On the other hand, $Q = qtw$ Kg./m. sec., where $w = 1000$ Kg./m.³ Introducing Chézy's formula for q (equation 44, p. 161) and Strickler's approximation of p. 162 with $m = t$, we have

$$Q = 21{,}100\,\frac{i^{1/2}t^{5/3}}{d^{1/6}},$$

and hence ‡

$$\frac{iQ^{2/3}}{d} = 763\,i^{4/3}\left(\frac{t}{d}\right)^{10/9}.$$

It follows from (4a) that the index of z in the first term on the right-hand side of (7) must agree with that of t/d. Formulæ (6) and (7) are supported by experiment up to $t/d = 50$.

The rules which it is necessary to observe in order to obtain motions of sand, gravel, &c., in model experiments which correspond to those in nature form a very important problem for hydraulic research institutes. Above all, the inequality (4a) must be fulfilled, which is equivalent to Froude's law of similarity (p. 178). But the extension of Froude's law of similarity to the grains of sand must be renounced, otherwise the sand would be much too fine. (The model of a stretch of river 4 km. long can at most be made 40 metres long; then 10-mm. pebbles in the river would be represented by a very fine sand of 0·1 mm.) A way out of this difficulty while keeping inequality (4a) approximately true, is by choosing coarse material and a smaller value of w_g. Reynolds' number $\mathscr{R} = v_* d/\nu$ must be brought down at least to 20 if long banks are to be produced. Brown-coal slack with w_g/w about 1·25 has been found very suitable. Corresponding to the greater relative roughness, the inclination of the bed i must be somewhat increased.§ The velocity with which the material moves is markedly raised on account of its greater coarseness and the simultaneous diminishing of the specific gravity, and this greatly shortens the time required for the movement of banks in the model. If no wave motion is involved it is also permissible to exaggerate the

† *Recherches expérimentales sur les débits solides dans les cours d'eau* (Lyons, 1935); *Annales des ponts et chaussées* (1935), No. 8, and (1936), No. 7.

‡ Putting $Q = Q_0$ and using (7), we find that the constant in (4a) is equal to $0{\cdot}0375(t/zd)^{1/6}$. For $t/zd = 5$; 50; 500 the constant is 0·049; 0·072; 0·105. Shields' value 0·06 corresponds to $t/zd = 22$.

§ For irregular flow it is convenient to take the fall of the energy lines as the fall i; the " energy line " lies at a height above the surface of the water equal to the mean velocity head of each transverse section, $u_m^2/2g$. If the depth and breadth of the stream vary, it has a more uniform fall, depending only on the frictional losses, than the surface of the water, which in wider parts may even rise.

vertical dimensions of the model; then the fall is necessarily again increased. In this way it is even possible to work with ordinary sand. Fundamental principles are discussed in the papers by H. A. Einstein and Rob. Müller * and by R. Seifert,† as well as in the very full dissertation by W. Liebs.‡

(*d*) *Effect of the Transport of Material on the Bed of the River.*—In the higher reaches of a river the material reaches it from the mountains in the form of more or less large blocks which are mainly moved during the yearly floods, becoming smaller and smaller as a result of colliding and rubbing. The material rubbed off is carried along in suspension. If the river valley itself consists of material brought down by the river, a state of equilibrium is established between it and the river in the course of thousands of years; on the average as much solid matter enters each section of the river as leaves it.§ A "wild" river, i.e. one whose flow has not been interfered with by human measures, leaves material stranded on the banks every time it overflows them in flood, so that the banks are gradually raised and the river with them. The consequence is that now and again it breaks through to lower-lying ground and makes a new bed for itself. In this way in the course of a long period of time it works over the whole width of its valley.‖ As the material becomes finer and finer towards the mouth of the river, formula (4*a*) shows that the fall of the river valley will decrease more and more.

Human interference, like natural catastrophe, may seriously disturb the long-term equilibrium. Thus, for example, it has been observed that when mountain rivers with a large amount of transported material are dammed for power, the river digs deeper below the dam (and lays bare bridge piers and bank piles, because the material which it previously carried along is now deposited in the dam and is absent from the river lower down). A river may also scour deeper by being confined by artificial banks to too narrow a bed, so that the velocity is increased, and on the other hand, flood banks on either side of the river have the effect of raising the land between them, since there the flood water flows less rapidly than in the river-bed. Large rivers with much suspended matter, like the Mississippi and the Hwang-ho, have in the course of time become raised several metres above the level of

* *Schweizer Archiv. f. angew. Wissensch. u. Technik*, 1939), No. 8.

† *Die Bautechnik*, Vol. 20, p. 327 (1942).

‡ *Mitt. d. Preuss. Versuchsanstalt f. Wasser-, Erd- u. Schiffbau* (1942), No. 43.

§ Strictly speaking it is not a perfect equilibrium, as the river gradually moves the materials which sink to its bed and its mouth extends farther and farther into the sea (or into a lake) and the river valley must be raised by a corresponding amount.

‖ The wandering of the course of a river over the surrounding land is favoured by the phenomenon of *meandering*, described in Chapter III, § 8, p. 147. Not only does the secondary flow direct suspended material towards the inner side of the curved portion of the river, where it is deposited in banks, but the position of maximum velocity is also deflected towards the outer side, so that the outer margin is attacked and the curvature of the stretch of river thereby increased.

the land outside their retaining banks* and eventually have broken through these and formed entirely new beds for themselves.

The behaviour of the river-bed in the neighbourhood of any artificial structure is also of great importance in hydraulic engineering. The attack of the water on the bed increases with the velocity. As it is chiefly a matter of the velocity in the vicinity of the bed, danger arises particularly at places where more rapidly flowing water from the interior of the mass of water impinges directly on the bed. As has already been mentioned in § 4 (p. 333), characteristic hollows are formed in front of the piers of bridges (and also in front of isolated posts). If

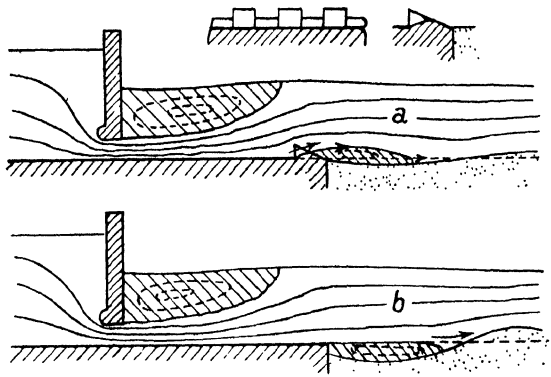


Fig. 5.25.—Hollow formed behind a structure (*a*) with, (*b*) without a toothed apron. The shaded regions are characterized by self-contained flow, so-called *Walzen*

(Above, on the left, *Deckwalze*; below, on the right, *Grundwalze*)

the piers have only shallow foundations, the latter may be undermined and the piers fall over in the upstream direction. A flatter depression is also formed downstream owing to the increased turbulence behind an obstacle, which likewise causes rapidly moving water to approach the river-bed. Fig. 5.25*b* shows the hollowing-out behind a sluice. The explanation follows readily from what has already been said. By constructing suitable obstacles, e.g. Rehbock's dentated sill,† it is possible to reduce the velocity along the bed considerably and thereby prevent the formation of deep hollows which otherwise might endanger the stability of the structure (cf. fig. 5.25*a*).

* This also explains the occurrence of deltas at the mouths of rivers.

† See e.g. G. de Thiery and C. Matchoss, *Die Wasserbaulaboratorien Europas*, p. 170 and Plate VI (Berlin, 1926), in which many other examples of banks and hollows are given; also *Hydraulic Laboratory Practice* (ed. J. R. Freeman), p. 185 and figs. 183–186 on p. 186 (Amer. Soc. Mech. Eng., New York, 1929).

6. Bodies in Accelerated Fluid. Hydrodynamical Action at a Distance.

Here we shall discuss a number of matters (some of which are theoretically instructive and others of importance in practice) relating to the forces acting on bodies which are in non-uniform motion. We shall confine ourselves to irrotational motion in fluid which completely surrounds the body; that is, we shall not consider cases in which there is a surface of discontinuity, e.g. the important phenomena which occur with aerofoils in gusts.*

One of the first problems of this kind arose when it was found, in making drag experiments on airship models in closed wind tunnels, that a noticeable additional drag, sufficient to cause a considerable error in the drag, which in itself was very small, was produced by the fall of pressure in the current of air in the wind tunnel.†

The first primitive solution was to deduct the quantity $V \mid \partial p/\partial x \mid$, where V is the volume of the airship model, from the drag recorded by the balance, that is, the idea on which Archimedes' principle is based was formally transferred to this case. More accurate hydrodynamical arguments, developed first by M. Munk ‡ and later in more and more general form by H. Glauert § and G. I. Taylor,|| show that in this case the above calculation becomes correct if to V we add V', the volume of the apparent mass (see p. 180). For an airship model, in fact, this does not make much difference, as the apparent mass, e.g. of an ellipsoid (of revolution) with axes in the ratio 1 : 6, is only 4·5 per cent of the mass of fluid displaced. For the ellipsoid with axes in the ratio 1 : 2, however, it is as much as 20 per cent and for the sphere 50 per cent. An extensive paper by W. Tollmien¶ is recommended for consultation because of the rigour of its proofs, and for other examples. (It gives, for example, the forces and torques acting on a body in slightly curved irrotational flow.)

Another problem is that of the behaviour of a body in a fluid subject to some form of acceleration which varies with time. For example, we may ask what is the acceleration f_1 of a body which arises from an acceleration f of the fluid surrounding it, say in the direction of the x-axis. Let ρ be the density of the fluid, ρ_1 that of the body. If to begin with we ignore the disturbance due to the body itself, we may assume that all parts of the fluid are subject to the same acceleration. This

* See the note on p. 221.

† Instead of making a correction for the fall of pressure it is of course better to prevent the occurrence of the fall of pressure altogether by modifying the wind tunnel, e.g. by making it widen slightly in the direction of flow.

‡ *National Advisory Committee for Aeronautics*, Rep. No. 114 (1921).

§ *Aeronautical Research Committee Rep. and Mem.* No. 1158 (1928).

|| *Proc. Roy. Soc.* (A), Vol. 120, p. 260 (1928); *Aeronautical Research Committee Rep. and Mem.* No. 1160.

¶ *Ing.-Arch.*, Vol. 9, p. 308 (1938).

can be achieved, e.g. if the fluid is incompressible and completely fills the interior of a vessel, and the vessel is then subjected to some accelerated translational motion. Then every particle of fluid partakes in this translational motion.* The acceleration f, which is the same everywhere, then corresponds to a uniform fall of pressure in the direction of f, amounting to $-\partial p/\partial x = \rho f$. The body exposed to this fall of pressure is first subjected to a " lift " $V\rho f$ in the direction of f. As we assume that ρ_1, the density of the body, is not the same as ρ, the density of the fluid, their accelerations f_1, f will also be unequal. Hence from the acceleration of the body relative to the fluid $(f_1 - f)$, there will arise an acceleration drag proportional to the apparent mass $\rho V'$ (see above), i.e. equal to $\rho(f_1 - f)V'$, acting so as to oppose the lift. The resultant force, the difference between lift and drag, is equal to the mass of the body multiplied by its acceleration, that is,

$$\rho\{fV - (f_1 - f)V'\} = \rho_1 V f_1,$$

or

$$f_1 = f\frac{V + V'}{V\rho_1/\rho + V'}. \qquad \ldots\ldots (8)$$

Thus, if $\rho_1 > \rho$, $f_1 < f$, and conversely. If $\rho_1 = \rho$, $f_1 = f$, as might be expected. This of course applies to any limited portion of the fluid, which is immediately constrained by its surroundings to take up the acceleration f.

The following experiment has been described by V. Bjerknes:† Three horizontal glass tubes are filled completely with liquid, no air bubbles being left. The first contains a body of specific gravity less than that of the fluid, the second a body of the same specific gravity as the fluid, and the third a body of specific gravity greater than that of the fluid. If the tubes are lying loose on the table and are struck with a light wooden hammer in a longitudinal direction, the tubes are accelerated by the blow and then immediately retarded by friction at the table-top. Then the body in the first tube will move farther than the tube and will thus move relative to the tube in the direction of the blow. The second body will move exactly as the tube does, while the third will hang back relative to the tube and, relative to it, will move in the direction opposite to that of the blow. As a further example Bjerknes points out that the flame of the candle in a stable-lantern, which is of course lighter than the surrounding air, moves forward when the lantern begins to move horizontally forward, and backward when the motion of the lantern ceases $(f_1 > f)$.

* The flow starts from rest and is therefore irrotational. As the above solution satisfies the boundary conditions, it must be the true solution, by the theorem of uniqueness in potential theory.

† *Zeitschr. f. d. phys. u. chem. Unterricht*, Vol. 43, p. 1 (1930).

According to Bjerknes, equation (8) suggests very peculiar behaviour in the case of a body which increases and diminishes in volume in a regular rhythm, if it is surrounded by a fluid which vibrates to and fro with the same rhythm. Suppose the volume of the body is at its greatest exactly at the right-hand point of reversal of the vibration of the fluid, where the acceleration f is directed to the left, and least at the left-hand point of reversal, where the acceleration of the fluid is directed to the right. If we assume for simplicity that the mean specific gravity of the body is *equal* to that of the fluid,* it is less than that of the fluid when the body is in the expanded state, so that the body moves to the left ahead of the fluid. When the body is in the contracted state, its specific gravity is greater than that of the fluid, so that the body is behind the fluid in its acceleration to the right, that is, the result is two relative accelerations to the left. If the body is held in position by means e.g. of a very extensible spring, the spring is therefore stretched to the left by the combined effect of these accelerations, i.e. this peculiar motion of the fluid gives rise to a force acting on the body towards the left. Bjerknes now goes farther and assumes that the periodic motion of the fluid is caused by another "pulsating" body. If the two bodies pulsate in the same rhythm, i.e. attain their greatest and least volumes simultaneously, it follows readily from what has been said above that the two bodies will attract one another; if they pulsate in opposite rhythms, they will repel one another. In an unlimited fluid it is found that the velocities in the neighbourhood of a pulsating body are proportional to the inverse square of the distance and that the forces of attraction or repulsion are inversely proportional to the square of the distance, that is, the laws are those of electrostatic or magnetic "action at a distance". Hence we may speak of "hydrodynamical action at a distance". The rule of signs, however, is the opposite to that for electric and magnetic forces at a distance, for in these the opposite charges or poles attract one another while the similar charges or poles repel one another. The theory of hydrodynamical forces at a distance originated in a paper published by C. A. Bjerknes in 1871 in the *Verhandlungen der Ges. d. Wissenschaften* in Christiania, and expanded and published in book form by his son, V. Bjerknes.† The latter also designed a number of pieces of apparatus for demonstrating the effects.‡ In addition to "pulsators", this includes "oscillators" by means of which dipoles may be represented, and which behave like elementary magnets.

* This assumption does not in itself affect the final result.

† *Vorlesungen über hydrodynamische Fernkräfte*, Vols. 1 and 2 (Leipzig, 1900 and 1902).

‡ V. Bjerknes, *Die Kraftfelder* (Brunswick, 1909), No. 28 of the series *Die Wissenschaft*; see also Vol. 2 of the *Vorlesungen*.

The force of attraction between two *spherical* pulsators, the first of which has the volume V given by $V_1 + A_1 \cos\omega t$ and the second the volume $V_2 + A_2 \cos\omega t$, may be calculated as follows. The mass of the two pulsators does not vary with time. If that of the first pulsator ($V_1\rho_1$) is equated to $V_1\rho$ for simplicity, and $V' = \frac{1}{2}V$, equation (8) gives

$$f_1 = f\frac{\frac{3}{2}(V_1 + A_1\cos\omega t)}{\frac{3}{2}V_1 + \frac{1}{2}A_1\cos\omega t},$$

that is, for small amplitudes A,

$$f_1 = f\left(1 + \tfrac{2}{3}\frac{A_1}{V_1}\cos\omega t\right)$$

to the first order; the momentary force is

$$\rho V_1 f_1 = \rho V_1 f + \tfrac{2}{3}\rho A_1 \cos\omega t \,.\, f. \qquad \ldots\ldots \quad (9)$$

Owing to f the first term is purely periodic and so disappears when the mean value is formed, but not the second.

The acceleration f at the first pulsator is produced by the second pulsator, which is at a distance r from the first. Here we have to deal with a non-steady flow from a source. By the theory of Chapter II, § 9(*b*), p. 63, with $Q = dV/dt$, we have

$$q = \frac{1}{4\pi r^2}\frac{dV}{dt} = -\frac{A_2\omega}{4\pi r^2}\sin\omega t;$$

hence to the first order the acceleration

$$f = \partial q/\partial t = -\frac{A_2\omega^2}{4\pi r^2}\cos\omega t.$$

As the mean value of $\cos^2\omega t$ is $\frac{1}{2}$, by equation (9) the mean value of the force is

$$F = -\frac{\rho A_1 A_2}{12\pi r^2}. \qquad \ldots\ldots \quad (10)$$

The minus sign shows that the force is in the direction opposite to that of r, that is, it is an attraction if A_1 and A_2 have the same sign. The formula is symmetric in A_1 and A_2, i.e. the principle of action and reaction being equal is fulfilled, as must of course be the case.

A practical application of these "forces at a distance" arises in the case of a fluid full of air bubbles which is set into ultrasonic vibration. Owing to the variations of pressure, neighbouring air bubbles are subject to changes of volume in the same phase, and therefore attract one another. Thus they unite to form bigger and bigger bubbles which are quickly got rid of. In this way molten metal containing gases may readily be de-gassed by means of ultrasonic vibrations.

Note.—This process is probably helped by the de-gassing effect of shearing movements in the boundary layers of the pulsating bubbles. This effect may be observed in lubricating oil, which gives off dissolved gases when it is subjected to shearing movements.*

* W. Frössel, *Öl und Kohle*, Vol. 39, p. 257 (1943); see also *Forschung*, Vol. 9, p. 273 (1938).

B. ROTATING BODY AND ROTATING SYSTEM OF REFERENCE

7. Fundamental Principles. Bernoulli's Equation and some Particular Cases.

(*a*) In studying the flow about *a body which is rotating steadily* or the flow *in a hollow space which is rotating steadily* (here we shall consider only the most important case, that of uniform rotation about an axis), it is convenient to consider the phenomena from the point of view of an observer rotating with the body or with the hollow space. For such an observer the body or hollow space is at rest, so that for him the flow in certain cases appears steady. As is well known, the laws of mechanics may be applied immediately to a rotating system of reference by adding to the forces present in the absolute system of reference two body-forces,* of which one depends only on the particular point, but the other involves the velocity also.† The "first additional force" is equal to the mass in question multiplied by the absolute acceleration (taken negatively) of the point of the system which is in the position of the mass at the time in question (i.e. here that of a point in the position of the mass which is at rest relative to the rotating system of reference). In this case the acceleration is the centripetal acceleration $\omega^2 r$, that is, the additional force is in the direction opposite to it, so that we have the so-called centrifugal force $m\omega^2 r$, where ω is the angular velocity. The "second additional force" arises from the fact that in the case of a mass which is moving uniformly in a straight line in the rotating system of reference, the component v_1 of the velocity vector, which is at right angles to the axis of rotation, is rotating in absolute space with angular velocity ω, and that the mass moving with velocity v_1 moves on into regions where the "system velocity" is different and is directed in different directions. The two parts of the absolute acceleration which are not apparent in the system are of the same magnitude ωv_1 and are perpendicular both to the axis of rotation and to the direction of v_1. Hence the apparent force which makes these parts of the acceleration vanish for the moving system of reference is $2m\omega v_1$. It is known as the Coriolis force, after its discoverer. Its direction is opposite to that of the absolute velocity of the vector end point of the relative velocity vector.(75)

Illustrative Examples.

(1) A particle of mass m, at rest in absolute space, describes a circle with angular velocity $-\omega$ in the rotating system of reference. It is acted on by (1) the

* Body-forces are forces which act on a mass and give a prescribed acceleration to every particle of it irrespective of its size.

† The laws of motion relative to an accelerated system will be found in any good-sized textbook of mechanics.

centrifugal force $mr\omega^2$, which would occur even in the case of an equal particle at rest relative to the moving system, and (2) by the Coriolis force $2mr\omega^2$ corresponding to the velocity $v = -r\omega$. According to the rule above, the latter force is in the centripetal direction. Hence the resultant force is $mr\omega^2$ in the centripetal direction, which is just what is required, by the laws of mechanics, to produce the circular motion in the moving system of reference.

(2) A particle is situated in a straight tube which is rotating with constant angular velocity about an axis at right angles to its middle line. If we consider the motion relative to a system of reference rotating with the tube, the particle is subject to centrifugal force and is therefore accelerated outwards in the tube. If its velocity in the tube at a particular instant is v, it is forced against the side of the tube by the Coriolis force $2m\omega v$, and the side of the tube reacts with an equal and opposite force. The kinetic energy of the particle, considered in the rotating system of reference, comes from the work done by the centrifugal force. The Coriolis force is at right angles to the path of the particle and therefore does no work. In the absolute system of reference the particle is absolutely free in the radial direction (so that no work is done by the centrifugal force, which in this case does not exist). The work is done by the applied force which generates the ever-increasing tangential velocity $r\omega$.

(*b*) In deducing Bernoulli's theorem for a steady relative flow in a uniformly rotating system, all we have to do in applying the argument of p. 38 is to add to the local components of force in the direction of the flow the component of the centrifugal acceleration $\omega^2 r$ in that direction. The Coriolis acceleration is always at right angles to the direction of the relative flow and hence yields no contribution. By what was said on p. 26, however, the centrifugal acceleration has a potential given by const. $-\frac{1}{2}\omega^2 r^2$. Hence Bernoulli's equation is immediately integrable and, by analogy with equation (7), p. 40, takes the form

$$\frac{p}{\rho} + gz + \tfrac{1}{2}q^2 = \text{const.} + \tfrac{1}{2}\omega^2 r^2, \qquad . \; . \; . \quad (11)$$

if ρ is constant.

Since in general the constant in the equation will not be the same for different stream-lines, this equation holds only for points on one and the same stream-line. Comparison with the equation for p on p. 26, however, shows that in the case of rest relative to the rotating system ($q = 0$ everywhere) the equation holds good throughout the region without limitation to the stream-lines (which in this case of course do not exist).

There is another case of greater practical importance where the stream-line limitation does not apply, namely, where the flow is free from rotation in the system of reference at rest (i.e. where the flow is irrotational but in general not steady). This case occurs e.g. when fluid (assumed to be frictionless) moves from a system of pipes at rest to a rotating system as in turbines, centrifugal pumps, &c. Since an individual particle remains without rotation as before, in the rotating system a flow of this kind will show a constant rotation with angular

velocity $-\omega$ in the rotating system, about an axis parallel to the axis about which the system is rotating. Instead of a general proof * that here the constant of equation (11) has the same value throughout the region, we shall give a simple but instructive example. Let a straight pipe (fig. 5.26) rotate uniformly about an axis (assumed vertical for simplicity) at right angles to the axis of the pipe, and let it be full of fluid free from rotation in the system of reference at rest. Let x, y, z be the co-ordinates parallel to the axis of the pipe, at right angles to this axis in the plane of rotation and parallel to the axis of rotation, and let u, v, w be the corresponding components of the velocity. We put $u = u_1 + 2\omega y$, $v = w = 0$, so that $\omega_x = \omega_y = 0$ and $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\omega$. Since every particle of fluid moves uniformly in a straight line, all the relative accelerations are zero. The Coriolis acceleration is of magnitude $2\omega u$ in the direction of the negative y-axis. The centrifugal acceleration $\omega^2 r$ has an x-component $\omega^2 x$ and a y-component $\omega^2 y$; in the direction of the negative z-axis there is the acceleration of gravity g. Hence

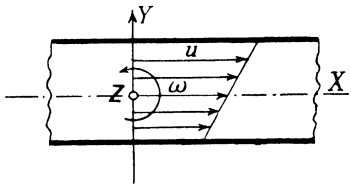


Fig. 5.26

$$\frac{\partial p}{\partial x} = \rho\omega^2 x,$$

$$\frac{\partial p}{\partial y} = -2\rho\omega u_1 - 4\rho\omega^2 y + \rho\omega^2 y,$$

$$\frac{\partial p}{\partial z} = -\rho g.$$

Therefore
$$\frac{p}{\rho} = p_0 + \tfrac{1}{2}\omega^2(x^2 - 3y^2) - 2\omega u_1 y - gz. \quad . \quad . \quad (12)$$

* This proof is not difficult in itself. We have only to apply equation (18a), p. 61, to the absolute motion, taking note that the relative motion must be steady, i.e. the velocity potential must remain constant for every point rotating with velocity ωr. This leads to the relation $\frac{\partial\Phi}{\partial t} + \omega r \frac{\partial\Phi}{\partial s'} = 0$, where ds' denotes an element in the direction of rotation. $\partial\Phi/\partial s'$ is then obviously the tangential component of the absolute velocity of flow, called c_u by turbine engineers. Thus $\partial\Phi/\partial t = -\omega r c_u$. The transformation of this absolute relationship into the velocities of the relative flow then leads to our theorem. The relation

$$\frac{p}{w} + z + \frac{c^2}{2g} - \frac{\omega r c_u}{g} = \text{const.}$$

was given first by F. Prášil, *Schweizer Bauzeitung*, Vol. 48, p. 300 (1906), and the theorem in the text by R. von Mises, *Theorie der Wasserräder* (Leipzig, 1908), and *Zeitschr. f. Math. u. Physik*, Vol. 57, p. 1 (1909). For these two references I am indebted to W. Spannhake.

From this and $\frac{1}{2}u^2 = \frac{1}{2}u_1^2 + 2\omega u_1 y + 2\omega^2 y^2$, where u corresponds to q in equation (11), we see that the left-hand side of equation (11) gives

$$p_0 + \tfrac{1}{2}u_1^2 + \tfrac{1}{2}\omega^2(x^2 + y^2),$$

that is, the " constant " in equation (11) is here actually independent of y and z, in agreement with the statement made at the beginning of this section.†

(*c*) The device used in Chapter II, § 5(*a*) (p. 46), of splitting up the pressure p into the " equilibrium pressure " p' and a " dynamic pressure " p^* can also be applied with advantage in the case of flow in a rotating system of reference. Here, by the formula on p. 26, the equilibrium pressure is given by

$$p' = \text{const.} - g\rho z + \tfrac{1}{2}\rho\omega^2 r^2.$$

Hence, by equation (11), the dynamic pressure in steady relative flow is

$$p^* = \text{const.} - \tfrac{1}{2}\rho q^2, \qquad \ldots\ldots \quad (13)$$

i.e. it is formally the same as in the non-rotating system of reference, but the stream-lines are different from those in a region at rest. As before, equation (13) applies in the first instance only to stream-lines, but this restriction again disappears in the " irrotational " motions we have just discussed.

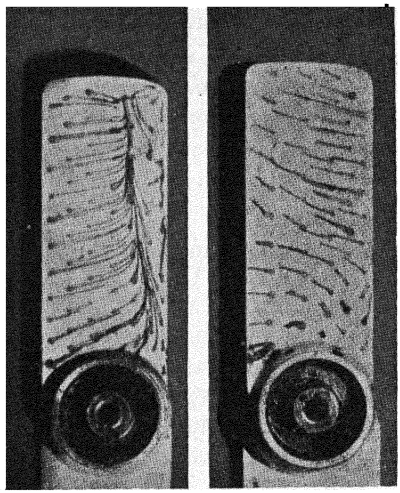
Any portions of fluid which become separated from a rotating body partake more or less of its rotation and are therefore thrown outwards (" centrifuged "). The portions of the boundary layers close to the body are likewise subject to this centrifugal effect, so that the flow takes the forms already discussed under secondary flow [cf. Chapter III, § 8 (p. 145)]. Owing to the moving outwards of portions of fluid torn away from the body, the portions nearer the centre of rotation are subject to a sort of " boundary-layer suction " (Chapter III, § 7, p. 144). In fig. 5.27 are reproduced photographs by F. Gutsche ‡ showing the boundary-layer flow on the suction and pres-

† Numerous examples of two-dimensional irrotational motion in rotating pipes will be found in the book *Strömungen einer reibungsfreien Flüssigkeit bei Rotation fester Körper*, by W. Kucharski (Munich and Berlin, 1918), from which the above example is taken. Of related experimental work we may mention the dissertation by H. Fette, *Zeitschr. f. techn. Physik*, Vol. 14, p. 257 (1933). Theoretical calculations relating to flow in turbines are to be found in W. Spannhake, *Hydraulische Probleme* (Berlin, 1926), p. 180, W. Spannhake and W. Barth, *Zeitschr. f. angew. Math. u. Mech.*, Vol. 9, p. 466 (1929), and W. Spannhake, *Mitteilungen des Instituts für Strömungsmaschinen der Techn. Hochsch. Karlsruhe*, No. 1 (Munich and Berlin, 1930), p. 4.

A new hydrodynamical theorem for systems of rotating blades is given by A. Busemann in Vol. II of Föppl's *Vorlesungen über techn. Mechanik*, 9th ed., p. 417 (1942).

‡ *Jahrb. d. Schiffbautechn. Ges.*, Vol. 41, p. 188 (1940), where further illustrations are to be found.

sure sides of an experimental propeller (profile a segment of a circle). Before the experiment (which was made in water) the propeller was dotted over with spots of oil paint, which were drawn out by the boundary-layer flow. The separation of the flow on the suction side is clearly seen, and also the fact that it is less clearly marked in the inner regions owing to the tendency of the point of separation to move outwards as a result of centrifugal forces.



Suction side Pressure side

Fig. 5.27.—Boundary-layer flow at a rotating propeller, after Gutsche

The question whether there is a risk of the flow becoming separated depends merely on the behaviour of p^* (if p^* increases in the direction of flow such a risk exists). The centrifugal acceleration $\omega^2 r$, like the acceleration of gravity g, acts on fluid at rest and moving fluid alike, and therefore does not affect the question of separation of the flow at all.†

In connexion with the example of fig. 5.26 it may be mentioned that the velocity is diminished on the pressure side and increased on the suction side as a result of the rotation $-\omega$; the corresponding differences in p^* exactly balance the resultant of the Coriolis forces. The effect of these pressure differences on the walls of the pipe is to produce an angular momentum resisting the rotation of the pipe; this must be overcome by the machine which is causing it to move. The work required is the product of the angular momentum and the angular velocity ω. The work done by a radial pipe extending from radius r_1 to a greater radius r_2, and revolving with angular velocity ω, is given by the rise in pressure due to the centrifugal force $\frac{1}{2}\rho\omega^2(r_2^2 - r_1^2)$, provided that the exit velocity is not subsequently transformed into pressure in a pipe or spiral volute chamber at rest. The relationships discussed here find an application in the design of centrifugal pumps and compressors, and also, of course, in that of radial turbines.

8. The Effect of the Earth's Rotation on Frictionless Currents in the Atmosphere and in the Ocean.

(*a*) The behaviour of the earth's atmosphere and of the oceans is quite markedly affected by the earth's rotation. As the resulting centrifugal acceleration on masses at rest and in motion is the same

† It may also be pointed out that if the axis of rotation is replaced by another line parallel to it, as is often convenient in experiments, it makes no difference to the behaviour of p^*, i.e. it does not affect the separation of the flow.

and simply forms a constituent of the sensible gravitational acceleration, we need pay special attention only to the Coriolis forces. Of these the horizontal component $2\omega v \sin\phi$ is particularly important, where ϕ is the geographical latitude, v the velocity (taken to be horizontal) relative to the surface of the earth, and ω for one revolution per sidereal day (86,164 seconds) is $2\pi/86{,}164 = 7{\cdot}29 \times 10^{-5}$ sec.$^{-1}$. Here the velocity v may be in any direction in the horizontal plane. To prove the formula it is most convenient to resolve the angular velocity ω which is a vector in the direction of the earth's axis into a component in the vertical direction and one in the horizontal south-north line. The latter, which is equal to $\omega\cos\phi$, gives vertical Coriolis forces for horizontal velocities. These Coriolis forces may usually be regarded as negligible in comparison with the forces of gravitational stratification (cf. §§ 11–13, p. 368). The former component of the angular velocity is equal to $\omega\sin\phi$ and is at right angles to v, so that it gives $2\omega v\sin\phi$ in the horizontal direction at right angles to v. In the northern hemisphere it is to the right, looking along the line of v, and in the southern to the left; at the equator it is zero, and it is a maximum at the poles.

It is instructive to consider the path described by any particle moving without friction in the horizontal plane under the influence of this acceleration. As the acceleration is at right angles to the velocity, and the latter therefore remains constant in magnitude, the particle will describe a circle.* Since $v^2/r = 2\omega v\sin\phi$, the radius of this "circle of inertia" is given by $r = v/(2\omega\sin\phi)$, so that the time of revolution $T = 2\pi r/v = \pi/(\omega\sin\phi)$. The time $2\pi/(\omega\sin\phi)$ which a Foucault pendulum takes to make one revolution is called a "pendulum-day", that is, the circle of inertia is always described in half a pendulum-day, no matter how great the velocity v is. Such inertial variations have been observed as after-effects of rapidly passing disturbances in the atmosphere and in the ocean.†

Of the various types of atmospheric movements two particularly characteristic types deserve closer consideration, namely, those of *short duration* and those which *persist for a long time.*(76) In motions which are completed in fractions of a pendulum-hour (and usually in fairly small space) the deviations due to the earth's rotation are only trifling. Here, therefore, the ordinary laws of flow are obeyed with sufficient accuracy. On the other hand, in extensive movements which persist in the same form for more than half a pendulum-day,

* In the deduction of this relationship $\sin\phi$ is treated as constant. This is no longer permissible when the radius of the circle of inertia is no longer small compared with the distance of its centre from the equator. Here the paths are no longer closed, but are loops similar to those in fig. 2.59, p. 87; these are more curved the greater the distance from the equator.

† See F. Defant, *Meteorol. Zeitschr.*, Vol. 58, p. 53 (1941), where further references are given.

and are subject to only moderate accelerations relative to the earth, the Coriolis force predominates so markedly that quite different types of motion arise, which are unfamiliar to many students of flow. Here a rough approximation is obtained by neglecting all accelerations other than the Coriolis acceleration [for details, see (*b*), below]. Velocities of a magnitude and direction such that their horizontal Coriolis forces exactly balance the existing field of force of the horizontal pressure gradient are called *geostrophic velocities*.

In the intermediate types of motion, i.e. those which last for several pendulum-hours, the ordinary accelerations relative to the earth are in general of the same order of magnitude as the Coriolis accelerations. As a rule such problems are very difficult to handle. If necessary the calculation may be made stepwise according to ordinary hydrodynamics for time intervals of 10–30 pendulum-minutes; the appropriate deviations due to the earth's rotation (of the order of 5° to 15°) can then be found subsequently for each step.* The author, however, has not come across any calculations of this kind. For a simpler type of problem relating to phenomena of similar duration, see § 9(*d*), p. 362.

Note.—As the equator is approached the length of a "pendulum-hour" $\pi/(12\omega \sin \phi)$ steadily increases. At the equator it becomes infinite, so that here the difference between the short- and long-period types of atmospheric movement completely disappears, and—apart from the small deviations of vertical air currents to west or east—everything is as if there were no rotation of the earth.

In extensive phenomena the *vertical* accelerations are so small compared with the horizontal that they can be neglected in the calculations, since weather phenomena extend much farther horizontally than they do vertically (about a hundred times as far). The relation between pressure and density along a vertical line is then given by the formulæ for the equilibrium of a heavy gas (see Chapter I, § 7, p. 14).

(*b*) In order to get rid of the complications arising from the curvature of the earth's surface, we may inquire into the behaviour of a mass of fluid on a plane disc rotating with velocity $\omega' = \omega \sin \phi$. Since on the rotating earth the centrifugal force is a part of the earth's gravitation, it is convenient to assume, in passing from the curved surface of the earth to the plane disc, a centripetal field of force in such a way as to get rid of the centrifugal force in this case also. This enables us among other things to have a mass of fluid at rest relative to the disc, with a flat surface. Let us assume that the density of the fluid is constant and, for still greater simplicity, that the fluid is non-viscous.

If now a portion of this fluid is given a velocity q which is the same in magnitude and direction relative to the rotating disc at all depths, the Coriolis acceleration will cause the surface of this portion of fluid to be inclined at an angle $2\omega' q/g$ to the direction of the velocity, so that the contour lines of the surface run in the direction of q (and the sur-

* Thus the effect of a momentum applied at zero time is exactly reversed in 6 pendulum-hours. Hence the whole accelerating effect of a horizontal force in a direction fixed in the earth exactly cancels out after 12 pendulum-hours.

face rises towards the right, provided the direction of rotation, like that of the northern hemisphere of the earth, is counter-clockwise). If we now consider extensive regions, in which the magnitude and direction of q gradually change in space at a given time, and if the horizontal velocity q is sufficiently small, an approximate solution is obtained by neglecting the relative accelerations proportional to q^2 in comparison with the Coriolis acceleration proportional to q [" geostrophic " calculation; cf. under (*a*)]. Then above horizontal ground it is possible to have two-dimensional flow with stream-lines of any form, in which the contour lines of the surface are also the stream-lines. [The right-hand and left-hand edges of a stream-tube then have the constant difference of level dh. From $dh = 2\omega' q\,ds/g$ (see above) we then have $q\,ds = \text{const.}$, i.e. continuity is satisfied.] In the portions of the surface which are lowered (" low-pressure regions ") the direction in which the closed stream-lines are described is that of the rotating ground beneath; in the regions which are raised (" high-pressure regions ") it is in the opposite direction [see also (*c*), p. 355].

It is possible to take the relative accelerations into account subsequently (cf. p. 352), say by stepwise refinement of the velocity field which corresponds to a given pressure field.* We would expressly point out that the non-viscous nature of the fluid is an essential condition for the state of affairs we have described, and so too is the evenness and horizontality of the ground. If the ground is uneven or sloping, the relationships become more complicated (see the following paragraph). For the peculiar role played by friction at the ground in flow of this type, see § 9(*c*), p. 360.

If H, the depth of the layer of fluid, is not constant as assumed above but varies owing to the ground being uneven or sloping, we may state (by Helmholtz's theorem) that the vertical component, measured in a system at rest, of the angular velocity of an element of fluid alters along a stream-line in proportion to H. Here it is assumed that the irregularities of the ground are not too sudden, so that—provided the fluid is non-viscous—we can assume that the horizontal velocity is constant for all points on each vertical line. For example, consider a sufficiently gentle ridge of height h in an otherwise flat region, and let the height of the layer of the medium of constant density above the flat region be H_0. If now the initial velocity in the rotating system of reference is constant in magnitude and direction, the angular velocity of the medium relative to the ground is $(\omega_{\text{rel}})_0 = 0$ and thus the absolute angular velocity $(\omega_{\text{abs}})_0 = \omega'$, i.e. is equal to the angular velocity of the ground. Then by the foregoing, neglecting slight differences of level, we have

$$(\omega_{\text{abs}})_1 = (\omega_{\text{abs}})_0 \frac{H_0 - h}{H_0}$$

above the ridge. Thus the rotation of the relative flow is

$$(\omega_{\text{rel}})_1 = (\omega_{\text{abs}})_1 - \omega' = -\omega' h/H_0.$$

* See e.g. L. Prandtl, *Zur Berechnung des Wetterablaufs, Gött. Nachr. Math.-Phys. Kl.* (1946), p. 102.

This, in conjunction with our previous explanation, implies an additional flow above the ridge like that in a region of high pressure (i.e. in the northern hemisphere the direction of rotation is clockwise, in the southern hemisphere counter-clockwise). In the atmosphere, where instead of a free surface we have a density *gradually* diminishing upwards, the relationships are numerically different, but qualitatively the state of affairs is similar (cf. § 13(*b*), p. 386). Hence the rise

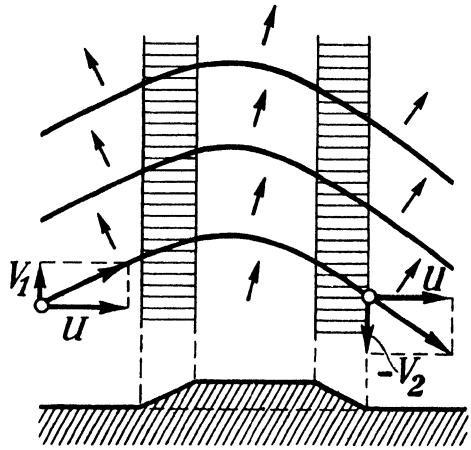


Fig. 5.28.—Flow over a mountain ridge

of pressure above elevated regions of the earth is actually observed (e.g. the "high" over the Alps). A wind blowing across a long mountain range is deviated to the right in the northern hemisphere. If the range runs in the y-direction (fig. 5.28), and in front of the mountain ($x = x_1$) u = a constant U, and v = a constant V_1; then, since

$$\omega_{\text{rel}} = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

and $\partial u/\partial y = 0$ above the mountain, $\partial v/\partial x = -2\omega' h/H_0$. From this, by integration, we have, at $x = x_2$,

$$V_2 = V_1 - \frac{2\omega'}{H_0}\int_{x_1}^{x_2} h\,dx = V_1 - \frac{2\omega'}{H_0}\,a,$$

where a is the cross-sectional area of the mountain between x_1 and x_2. Thus the deviation of the flow $(V_2 - V_1)/U$ is greater the smaller U is. Hence the phenomenon is much more marked in the case of ocean currents, which are much slower than winds.

For the long mountain range the calculation can be made in another way depending on the pressure field. By what has been said earlier, $\partial p/\partial y = -2\rho\omega' U$ over the flat region; over the ridge, by the principle of continuity, we have $u = UH_0/(H_0 - h)$. But here $\partial p/\partial y$ must have the same value as it has in front of the mountain, for, owing to the initial-flow conditions being the same for all values of y, the pressure differences in the x-direction must be the same for all values of y. The excess Coriolis force due to the greater value of u over the

ridge must therefore be compensated by an acceleration $u\partial v/\partial x$ in the y-direction. Completion of the calculation gives $\partial v/\partial x = -2\omega' h/H_0$ as before. Thus important relative accelerations are already taken into account by the application of Helmholtz's theorem.

(*c*) A peculiar case is that of the behaviour of a fluid originally at rest relative to a rotating horizontal bed when displacements produced by some outside agency occur. The best means of attacking this problem is by applying Thomson's theorem for a closed horizontal fluid line * to the absolute motion. For a mass of fluid in relative rest this is a rigid rotation with angular velocity ω', hence by what has already been said † the circulation is $\Gamma_0 = 2\omega' a_0$, where a_0 is the area enclosed by the fluid line. In the case of relative rest the circulation Γ_0' relative to the rotating disc is of course zero. Now suppose that the fluid line is deformed in such a way by external agency that it encloses the area a_1. By Thomson's theorem the absolute circulation remains equal to Γ_0, and $2\omega' a_1$ must now be subtracted for the transition to the relative circulation, so that $\Gamma_1' = 2\omega'(a_0 - a_1)$. The cross-section a is altered, e.g. when the height of the column of fluid is altered, and also if fluid is removed from or added to the region inside the fluid line. If the cross-section decreases, there is a relative circulation in the same sense as ω'; if it increases the circulation is in the opposite sense (the former is called *cyclonic*, the latter *anticyclonic*). To the velocities of circulation correspond Coriolis forces which, as it is easy to see, act outwards in the cyclonic case, inwards in the anticyclonic, i.e. produce a deficit of pressure in the former case and an excess in the latter [cf. also § 13(*b*), p. 386], that is, they always tend to prevent the change in cross-section. Thus the Coriolis forces bring about a dynamical stability of the mass of fluid which acts like a (very soft) elastic stability.

Note.—For motion in a *plane*, which starts from rest relative to the rotating disc, every a_1 is permanently equal to a_0; that is, $\Gamma' = 0$, so that here the relative flow is always irrotational. If fluid is added or removed, there is irrotational motion with circulation outside the region disturbed. If the disturbed area is assumed to be a circular region of radius R, and R is altered by ΔR, then Γ' is approximately equal to $-2\omega' \cdot 2\pi R \cdot \Delta R$. If q is the tangential velocity in the sense of ω' (i.e. counter-clockwise in the northern hemisphere), $\Gamma' = 2\pi R q$, so that $q = -2\omega' \Delta R$. Hence the radial change in pressure due to the Coriolis force is

$$\frac{dp}{dR} = 2\omega' \rho q = -4\rho\omega'^2 \cdot \Delta R,$$

i.e. a fall or rise in the direction of R according as ΔR is positive or negative. If the disturbed area is displaced laterally, a_0 remaining the same, an ordinary irrotational motion is produced during the displacement and no velocity field or pressure field is left behind when it comes to an end.

* See p. 56. † See the small type on p. 58.

9. Frictional Wind and Similar Phenomena.*

(*a*) Hitherto we have deliberately neglected friction. If we take into account the viscosity of the fluid and the fact that it sticks to the rotating surface, the state of affairs in a boundary layer (or, in the case of the earth's atmosphere, a layer near the ground) resembles what we have already described in Chapter III, § 8 (p. 145), under the title " secondary flow ". The pressure field which in free flow is balanced by the Coriolis forces is also present in the layers near the boundary; here the Coriolis forces are diminished by the lowered velocity and equilibrium is no longer maintained. On the contrary, owing to the preponderance of the fall of pressure, part of the flow takes place in the direction in which the pressure falls, to such an extent that the fall in Coriolis forces is compensated by frictional forces. Owing to the pull of the flow above, the resultant flow near the boundary is deviated by about 45° for laminar flow, and by about 20° to 30° for turbulent flow (the angle is smaller because here the pull is greater).(77)

The simplest example is as follows: Suppose that relative to the rotating disc of the previous section there is a parallel flow outside the frictional region with constant velocity $u = U$ in the x-direction. For the sake of simplicity we assume that the density ρ is constant. Let the pressure be $p = p_0 - 2\rho\omega' U \,.\, y$, so that the Coriolis forces called forth by U are in equilibrium with the fall of pressure $-\partial p/\partial y$ (here we may leave the decrease in pressure upwards, due to gravity, out of account). In the frictional layer u differs from U; it is a function of z, the distance from the boundary. Here there is also a velocity component v in the y-direction, which is likewise a function of z. For $z = 0$, we must have $u = v = 0$; for z infinite, $u = U$ and $v = 0$. In the laminar case the Navier-Stokes equations give viscous forces $\mu\partial^2 u/\partial z^2$ in the x-direction and $\mu\partial^2 v/\partial z^2$ in the y-direction per unit volume. As every layer moves uniformly, the accelerations, other than the Coriolis ones, are zero; further, $\partial p/\partial x = 0$, $\partial p/\partial y = -2\rho\omega' U$. Thus we have the equations

$$\mu \frac{\partial^2 u}{\partial z^2} = -2\rho\omega' v,$$

$$\mu \frac{\partial^2 v}{\partial z^2} = 2\rho\omega'(u - U).$$

If we put $u = U - Ae^{\lambda z}$ and $v = Be^{\lambda z}$, we have $\lambda^4 = -(2\rho\omega'/\mu)^2$. Of the four complex roots, there are two for which the real part is positive. They diminish from above downwards and hence are suitable

* For the historical development of these problems, see e.g. the comprehensive account by Th. Hesselberg and H. U. Sverdrup in *Veröffentlichungen des Geophysikal. Instituts d. Univ. Leipzig*, Ser. 2, Vol. 1, pp. 241–309 (1913–16).

solutions for disturbances at an upper boundary layer.* There are also two for which the real part is negative and which diminish upwards, and hence are the only ones relevant to our problem. With the above boundary conditions, the solutions in real form are

$$u = U(1 - e^{-\beta z}\cos\beta z),$$

$$v = Ue^{-\beta z}\sin\beta z,$$

where $\beta = +\sqrt{(\omega'/\nu)}$. The velocity vector varies as is indicated by the plan in fig. 5.29 (Ekman spiral). We see that here the wind at

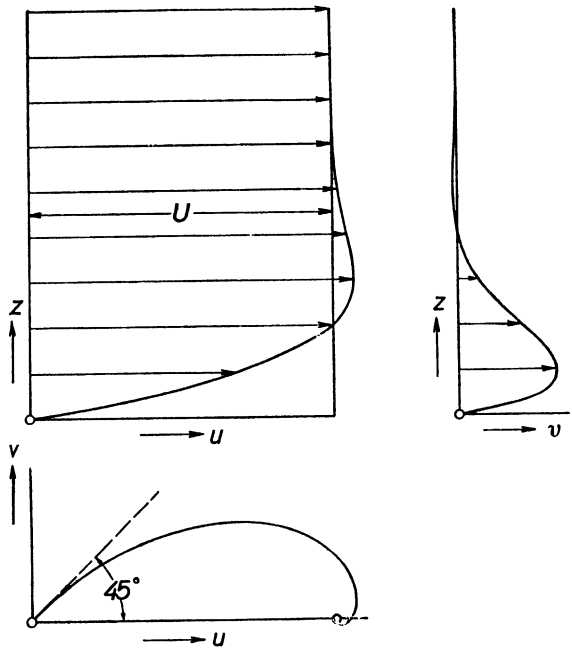


Fig. 5.29.—Laminar flow along the ground, after Ekman

ground level makes an angle of 45° with the wind at a considerable height.

In the case of turbulent flow the viscosity μ must be replaced by the exchange coefficient A [cf. Chapter III, § 4(*e*), p. 119]. As A is much greater than μ and is roughly proportional to the wind velocity, the frictional layer is much deeper and its depth increases with the wind velocity. Since, however, A is not constant over the height z, the distribution of velocities in space does not agree with Ekman's formulæ. As in other cases of turbulent flow, the greatest velocity gradients are near

* These solutions serve e.g. for the calculation of the ocean currents produced by wind; cf. W. Ekman, *Dynamische Gesetze der Meeresströmungen, Innsbrucker Vorträge* (1922), p. 97.

the ground. The mean velocity in the frictional layer is thereby increased, and hence the mean difference of the Coriolis forces from those of the wind at a height becomes smaller, and so does the angle of deviation, so that spirals like that in fig. 5.30 are obtained. (With regard to the observations that can be made both from pilot balloons and from photographs of the course of free balloons, it may be pointed

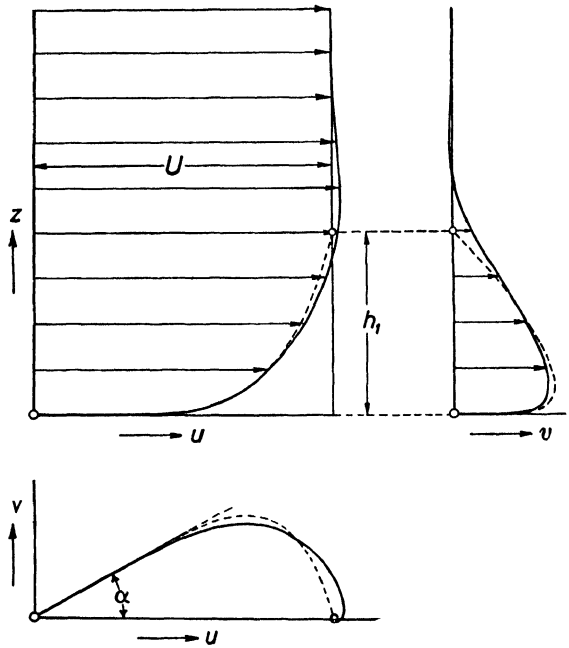


Fig. 5.30.—Turbulent flow along the ground, after Prandtl

out that, strictly speaking, the calculation given here applies only to straight steady isobars, but that the curvature of the isobars and the changes in the pressure field as time goes on *do* actually affect the direction of the wind as well.)

(*b*) To obtain numerical relationships we start from two formulæ for τ_x, τ_y, the components of the shearing stress on the ground, which are independent of the particular law of friction and are readily deduced by applying the momentum theorem to the Coriolis forces of the flow along the ground. They are

$$\tau_x = 2\omega' \int_0^h \rho v dz, \quad \ldots\ldots\ldots \quad (14)$$

$$\tau_y = 2\omega' \int_0^h \rho (U - u) dz, \quad \ldots\ldots \quad (15)$$

where h denotes a height at which the deviations $(U - u)$ and v from the wind at a height have vanished. The resultant shearing stress must be in the direction of the wind at the ground; that is, the angle α between this direction and that of the wind at a height must satisfy the relation

$$\tan\alpha = \frac{\tau_y}{\tau_x} = \lim_{z\to 0}\left(\frac{v}{u}\right). \quad . \quad . \quad . \quad . \quad . \quad (16)$$

According to the theory of turbulent flow past rough surfaces given in Chapter III, § 5(*d*), p. 129, the frictional stresses τ are approximately proportional to the second power of the velocity. Hence, using equation (14), we conclude that as the wind velocity increases the height of the frictional layer increases nearly in proportion to it.

It is impossible to carry the strict calculation further meanwhile, owing to our ignorance of the more exact laws of accelerated turbulent flow. We must accordingly content ourselves with an approximation:*

Here again, for simplicity, we may take the density as constant. For u and v let us make the following rough assumptions. Near the ground u is to be proportional to the nth root of z, i.e. we put $u = U(z/h_1)^{1/n}$, and v is to be equal to $u\tan\alpha$. To make $v = 0$ also for $z = h_1$, we may put $v = u(1 - z/h_1)\tan\alpha$. The height h_1 is less than the h used above; it may be chosen in such a way that the behaviour of u near the ground conforms on the average to the more exact law.† From formulæ (14) and (15) we readily obtain

$$\tan\alpha = \frac{1}{n}\sqrt{(2n+1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Hence τ_x, the shearing stress on the ground, is given, by equation (14), by

$$\tau_x = \frac{2n}{(n+1)\sqrt{(2n+1)}}\rho\omega' U h_1, \quad . \quad . \quad . \quad . \quad (18)$$

where $2\rho\omega' U = -dp/dy$ denotes the barometric gradient (expressed, of course, in the same units as τ_x).

To obtain h_1 and the index $1/n$, which depends on h_1 and the roughness of the ground, we may use the formulæ for the distribution of velocity above rough surfaces which were given in Chapter III, § 5(*d*), p. 129, provided a stable temperature stratification does not prevent the development of the normal degree of turbulence.(78) If k is the height of the irregularities (the height of the vegetation or of houses and the like), we may, by p. 129 *et seq.*, put

$$u = \left(5{\cdot}75\log_{10}\frac{z}{k} + C_2\right)\sqrt{\left(\frac{\tau_x}{\rho}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where C_2 varies between 5 and 8·5. From this, since $u = U$ for $z = h_1$, we have

$$\tau_x = \rho U^2 \Big/ \left(5{\cdot}75\log_{10}\frac{h_1}{k} + C_2\right)^2. \quad . \quad . \quad . \quad (20)$$

* An older approximation is given in the paper by Hesselberg and Sverdrup (loc. cit.), who also give a comparison with results obtained from pilot balloons (p. 267).

† The behaviour of u and v corresponding to these formulæ is shown by the dotted curve in fig. 5.30.

That here, in contrast to the foregoing, the logarithmic formula is used does not greatly matter so far as the approximation is concerned, as the two functions follow much the same course provided the value of n is suitably chosen. A convenient assumption to make is that the two functions

$$u = U\left(\frac{z}{h_1}\right)^{1/n} \text{ and } u = U\,\frac{5{\cdot}75\,\log_{10} z/k + C_2}{5{\cdot}75\,\log_{10} h_1/k + C_2}$$

must give the same value of $\int_0^{h_1}(U-u)\,dz$. Calculation gives

$$n = -1 + (\log_{10} h_1/k + C_2/5{\cdot}75)/\log_{10} e,$$

which leads to the relation

$$\log_{10} h_1/k + C_2/5{\cdot}75 = (n+1)\log_{10} e.$$

Eliminating τ_x between equations (18) and (20), we now obtain

$$h_1\left(5{\cdot}75\,\log_{10}\frac{h_1}{k} + C_2\right)^2 = \frac{(n+1)\sqrt{(2n+1)}}{2n}\cdot\frac{U}{\omega'}, \quad \ldots \quad (21)$$

from which it is easy to obtain the value of h_1 corresponding to a given value of U by trial and error. (It is, however, more convenient to assume a value for h_1 and calculate the corresponding value of U.) Formula (21) confirms the statement made above that h_1 increases nearly in proportion to U.

Numerical Data.—Take $C_2 = 5{\cdot}0$. For $5{\cdot}75\,\log_{10}(h_1/k) + C_2 = 20$; 25; 30, which corresponds to h_1/k = about 400; 3000; 22,000 and $\tau_x/\rho U^2 = \zeta = 0{\cdot}0025$; 0·0016; 0·0011, we obtain $n = 7$; 9; 11. The angle of deviation $\alpha = 29°$; 26°; 23·5°. By equation (21), $h_1\omega'/U = 0{\cdot}00553$; 0·00388; 0·002905.

Example.—Lat. 50°: $\omega' = \omega\sin\varphi = 5{\cdot}58 \times 10^{-5}$ sec. For U about 12 metres/sec. and k = about 3·0; 0·3; 0·03 metre, corresponding to, say, low trees; a crop of cereals or beet; a flat surface of water or snow, we have h_1/U = about 100; 70; 50 sec., giving a convenient rule-of-thumb formula * for h_1.

Note.—The calculation above was made with ρ = const. Actually ρ decreases as z increases. If the pressure gradient, which is often brought about by differences of density in higher layers, is regarded as constant within the frictional layer, there is a geostrophic velocity U which for increasing values of z increases in inverse proportion to ρ. The relationships of pp. 358–9 can also be used in this case; the curves of fig. 5.30 remain unchanged if ρU, ρu and ρv are plotted instead of U, u and v.† As the mechanism of friction at the ground is predominantly determined by phenomena near the ground, it is sufficiently accurate to insert the ground values $\rho = \rho_0$ and $U = U_0$ in formulæ (19) to (21).

(*c*) An important consequence of the secondary flow v at right angles to the given "gradient wind" U, which we have described above, is the gradual breakdown of the pressure field which it brings about. The flow of mass in the direction in which the pressure falls, which is given by

$$M = \int_0^h \rho v\,dz$$

* For wind friction on extensive sheets of water see also pp. 91, 130.

† With the assumptions of p. 359, this statement is confirmed by the results of observations by Hesselberg and Sverdrup (see footnote, p. 356).

per unit length in the U-direction, or, by equation (14), $\tau_x/2\omega'$, continually transfers fluid from regions of higher pressure to regions of lower pressure. In return, there must be balancing currents in the upper layers from low-pressure regions to high-pressure regions. The resulting displacements, however, give rise to changes in the original velocity field and pressure field, by what has been said in § 8(*c*) (p. 355). These, it is easy to see, are always such that both fields are thereby weakened. Thus we have the peculiar state of affairs that although it is only in a layer along the ground, which may in certain circumstances be comparatively shallow, that real frictional effects occur, the kinetic energy even of the upper layers is also dissipated, owing to the peculiar coupling between them resulting from the Coriolis forces. The energy destroyed by frictional work per unit area per unit time, $-dE/dt$, is supplied by the work done by the fall of pressure, $-\frac{\partial p}{\partial y}\int_0^h v\,dz$. For this, using $-\partial p/\partial y = 2\rho\omega' U$ and the expression for τ in equation (14), we obtain the very illuminating expression $\tau_x U$. Putting $\tau_x = \zeta\rho U^2$ (for numerical values of ζ, see p. 360), we therefore have

$$-\frac{dE}{dt} = \zeta\rho U^3. \qquad \ldots\ldots\ldots \quad (22)$$

The kinetic energy of the column of air above unit area can be expressed as $\frac{1}{2}H\rho U^2$, where H is the height of the moving mass of air recalculated to constant density ρ. From this, in so far as there is no supply or removal of energy by thermal influences, we obtain $-dE/dt = -\rho HU\,dU/dt$. From (22) we then have $dU/dt = -\zeta U^2/H$, which, on integrating and putting $U = U_0$ for $t = 0$, gives

$$U = \frac{U_0}{(1 + \zeta U_0 t/H)}. \qquad \ldots\ldots\ldots \quad (23)$$

For $\zeta = 0{\cdot}0025$, $H = 5000$ metres and $U_0 = 20$ metres/sec. we get a 20 per cent diminution in velocity for $t_1 = \frac{1}{4}H/\zeta U_0 = 25{,}000$ sec., i.e. about 7 hours.

Another case of the destruction of a pressure field by the secondary flow in the frictional layer is that occurring in the flow over a mountain which was discussed for the frictionless case in the passage in small type in § 8(*b*), p. 353. From the shape of the stream-lines in fig. 5.28 it can be seen that here the secondary flow, which is at right angles to the direction of the main flow, diverges over the mountain. Thus, as in an ordinary region of high pressure, air is carried away from that region and fresh air flows in from above to replace it. Calculations for this case by H. Görtler,* however, show that it is only if the flow is very slow or the surface very rough that the upper flow is markedly altered.

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 21, p. 279 (1941); a brief account is also to be found, along with some related problems, in *Naturwissenschaften*, Vol. 29, p. 473 (1941).

The change is such as to diminish the deflection of the flow over the mountain as in fig. 5.28.

In the case of wind fields of any kind above a plain which vary from place to place, there are vertical velocities above the frictional layer given by $\rho w = -\text{div}\,\mathbf{M}$, where "div" is the two-dimensional divergence and $\mathbf{M}$ the two-dimensional vector of the mass flow M (see p. 359, where we put $M = \tau_x/2\omega'$). In view of what follows, the vector $\mathbf{M}$ may also have a second component of magnitude $\tau_y/2\omega'$, corresponding to equation (15), p. 358, and directed at right angles to the gradient wind. If we combine the two components τ_x and τ_y into a vector $\mathbf{F}_g$ representing friction at the ground, $\mathbf{M}$ is at right angles to $\mathbf{F}$ and, since

$$\text{div}\,\mathbf{M} = -\text{curl}\,\mathbf{F}_g/2\omega',$$

we have

$$\rho w = \text{curl}\,\mathbf{F}_g/2\omega', \qquad \ldots\ldots\ldots \quad (24)$$

where "curl" is the two-dimensional curl.

If we take as an example a wind field which has on either side of it regions where the wind is slight, then on the left side there are upward currents and on the right side downward currents, both spreading out upwards. In accordance with footnote 1, p. 370, we may expect cloud formation over the left-hand border, but disappearance of any clouds over the right-hand border.

If the temperature is variable in the region, the temperature field in the frictional layer moves towards the left-hand border with the secondary flow and here is carried away from the ground. Here, since there is practically no change in the region of slight winds, a marked jump in temperature is formed, a so-called "front". Above the frictional layer, air which has flowed out from it is found over an extensive region, and near by the old temperature distribution unchanged (except possibly for moderate changes arising from exchange with the frictional layer). To this corresponds the observation that marked temperature jumps are in general found only in layers near the ground. For the development of these wind fields with a front, see also p. 384 and fig. 5.43.

(*d*) What has been said in (*b*) and (*c*) refers to wind flow with a pressure gradient constant in both space and time, when the roughness of the ground is constant. As was indicated at the end of § 9(*a*), p. 358, if there is much change in the pressure gradient from place to place or from time to time, or a change in the roughness of the ground, we may expect changes in the velocity profile and in the deviation of the wind. Hitherto satisfactory results have been obtained by calculation only in particularly simple isolated cases; see e.g. the investigation of L. Roux.* Among other things, he studied the change in the wind profile on passing from a water surface, assumed perfectly smooth, to a flat coast of given roughness. He also studied the changes with time in a wind flow separated from the ground by night by a layer of cold air, which first moves without friction, but after sunrise comes into frictional contact with the ground as a result of dissipation of the cold layer owing to irradiation of the ground by the sun. In both cases the production of the steady state takes several pendulum-hours and is achieved only after marked fluctuations.

Roux made his calculations with the help of momentum theorems similar to equations (14) and (15) (p. 358), and accordingly used the logarithmic approximate profile of fig. 5.30; but they were very troublesome. It is much more convenient to carry out the work if we use the power formulæ of section (*b*) with n and ζ constant. (We have two simultaneous differential equations of the first order for h_1 and $h_1 \tan\alpha$). H. Merbt† has repeated the calculations for Roux's second

* *Diss.*, Göttingen; *Geophysikal. Zeitschr.*, Vol. 11, p. 167 (1935).

† *Veröffentl. d. Geophys. Inst. d. Univ. Leipzig*, Ser. 2, Vol. 15, p. 138 (1949).

problem in this way and also taken into account variation of ζ due to variation in h_1. Fig. 5.31, in which the numbers give the number of pendulum-hours since friction first set in, gives a good idea of the succession of profiles in time. At first, when the effect of the rotation of the earth is scarcely noticeable, h_1 is much greater than its subsequent equilibrium value. In this layer, which has become too thick, inertia (see § 8(*a*)) gives rise to too marked a rotation. Hence the frictional component τ_y is too large, and h_1 is therefore decreased too much, and so on.

These considerations show how much winds blowing for only a short time at ground level may deviate from the state of equilibrium. The phenomena associated with passage from a smooth surface to a rough are qualitatively of much the same kind. Here the pendulum-hour ($\pi/12\omega'$) is replaced by the distance covered ($x_1 = U\pi/12\omega'$).

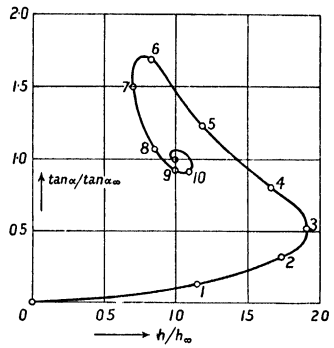


Fig. 5.31.—Development of a frictional layer, after Merbt

(*e*) In the case of ocean currents it is necessary to distinguish between the "drift" caused by the wind and the current proper, which depends on differences of level as in § 8(*b*), pp. 352–3, and if the density is uniform,* has a constant velocity in a constant direction at all depths, except for the frictional layer at the ocean bed. The arguments already put forward for the atmosphere are immediately transferable to the frictional layer. The drift is produced by the "friction" of the wind on the surface of the sea (more accurately, mainly by the momentum of the wind, which is transferred by pressure effects to the waves on the surface). Owing to their inertia the layers of water set in motion are deviated to the right by the rotation of the earth; as a result there is a current in a direction to the right of that of the wind. (This is for the northern hemisphere; in the southern hemisphere it is to the left.) The distribution at different depths in the laminar case is again an "Ekman spiral".† In general, however, the motion will be turbulent, if only on account of the irregularity of the wind, and we get a spiral of the turbulent type. Fig. 5.32 shows a plan of a flow of this type. If we consider the change in direction of the wind with height, we have the rule that the direction of the drift on the surface of the water is almost the same as that of the wind above the frictional layer, and hence is practically at right

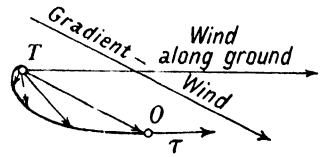


Fig. 5.32.—Drift
O surface, T down below

* A horizontal fall of density arising from differences in temperature or salinity at different places causes the current to alter with the depth [cf. § 14(*b*), p. 390].

† Cf. e.g. W. Ekman, *Innsbrucker Vorträge* (1922), p. 97.

angles to the barometric gradient. Slight deviations from this rule may occur as a result of stratification effects increasing or decreasing the turbulence.

If the drift is hemmed in by a coastline, a current must likewise set in to bring back the water in the drift impinging on the coast. Hence in the steady state the surface of the sea becomes inclined. Taking into consideration the rules given above we find that there is a horizontal pressure gradient in the sea, whose component parallel to the coastline has the sign opposite to that of the barometric gradient. These effects, however, are generally too small to be noticeable unless the sea is shallow.

10. Flow past Rotating Discs and Formulæ for the Resistance.

Suppose that a circular disc of diameter D is rotating, about an axis through its centre and at right angles to its plane, in a fluid which is otherwise at rest. The layer of fluid close to the disc, which is set in circular motion by friction at the disc, tends to flow outwards because of its inertia and leaves the disc at its outer edge, while fresh layers of fluid approach the disc and are thrown outwards in turn. Thus after a short initial period a steady flow sets in which involves a considerable resistance to the rotation of the disc. If the disc is moving, not in a mass of fluid extending for a long way in every direction, but in a closed casing concentric with the disc, the mass of fluid which has been set in motion by the disc again comes into contact with the disc, but it has already lost some of its circulation as a result of friction against the walls of the casing. To get a clearer idea of what happens we must study the phenomena at the disc and those at the casing more closely. So long as these motions are laminar they are capable of analytical discussion.

We shall begin with a simple approximate investigation. Let the direction of flow across the disc, which is parallel to the shearing stress at the wall, τ_w, make an angle ϕ with the tangential direction. The radial component of the shearing stress, $\tau_w \sin\phi$, along with the radial increase in momentum, must balance the centrifugal force of the fluid carried round with the disc, and is therefore proportional to $\rho r\omega^2\delta$, where δ is a measure of the thickness of the layer of fluid carried round. Again, the tangential component of the shearing stress, $\tau_w \cos\phi$, is proportional to $\mu r\omega/\delta$. Hence, eliminating τ_w and assuming that ϕ is independent of the radius, as is in accordance with observation, we obtain a statement about δ. In fact, $\delta \sim \sqrt{(\nu/\omega)}$, i.e. is the same no matter what the radius! We could also have predicted that there would be a frictional layer whose thickness would be of the order $\delta = \sqrt{(\nu/\omega)}$ from dimensional considerations, as in Chapter III, § 3 (p. 107). (The formula given there was $\delta = \sqrt{(\nu t)}$, but here the time characteristic of the phenomenon is $1/\omega$. If we start from this formula

for δ, the two relationships for τ_w given above lead to the statement that $\phi = \text{const.}$) With this value for δ we have $\tau_w \sim \rho r\omega\sqrt{(\nu\omega)}$. The angular momentum M is proportional to $\tau \times$ area $\times$ arm of couple, i.e.

$$M \sim \rho r^4 \omega \sqrt{(\nu\omega)}.$$

The case of a disc rotating in a fluid at rest has been worked out by W. G. Cochran.* If u is the tangential component of the velocity, v the radial component, and w the component at right angles to the

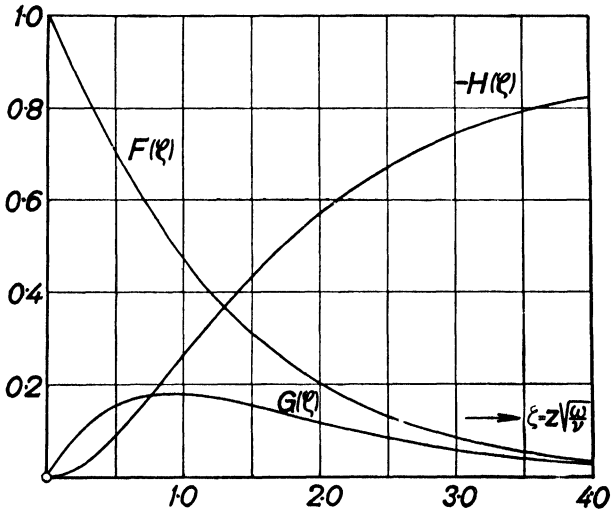


Fig. 5.33.—To illustrate the case of a disc rotating in fluid at rest

plane of the disc, u and v are proportional to $r\omega$. The factor of proportionality is a function of $z/\delta = z\sqrt{(\omega/\nu)}$, where z is the distance from the plate; for reasons of continuity, w is then proportional to $\delta \,.\, \omega = \sqrt{(\nu\omega)}$. We accordingly put †

$$u = r\omega F\{z\sqrt{(\omega/\nu)}\},\ v = r\omega G\{z\sqrt{(\omega/\nu)}\},\ w = \sqrt{(\nu\omega)}\,.\,H\{z\sqrt{(\omega/\nu)}\}.$$

The behaviour of the three functions F, G, H is shown in fig. 5.33. As we can immediately see from the figure, the distance from the wall at which the tangential velocity is half that of the disc is $\delta_{0\cdot5} \approx \sqrt{(\nu/\omega)}$. The inclination of the relative stream-lines close to the wall is $\phi_1 = 39{\cdot}6°$, corresponding to

$$\tan\phi_1 = -G'(0)/F'(0) = 0{\cdot}510/0{\cdot}616 = 0{\cdot}828.$$

* *Proc. Cambridge Phil. Soc.*, Vol. 30, Part 3 (1934).

† Here F and G have been interchanged, as compared with the papers by Cochran and Bödewadt (see below).

The frictional momentum * for a disc of diameter D and tangential velocity u_1 is

$$M = 0{\cdot}684 D^3 \,.\, \tfrac{1}{2}\rho u_1^2 \sqrt{(\nu/u_1 D)}. \quad . \quad . \quad . \quad (25)$$

If we remember that $u_1 = \omega r_1 = \frac{1}{2}\omega D$, the factors in formula (25) agree exactly with those of the previous expression.

A similar calculation was made by U. T. Bödewadt † for the case of a disc at rest in fluid rotating with angular velocity ω. The corresponding functions F, G and H are plotted in fig. 5.34. Here the secondary flow (the same as was described in Chapter III, § 8 (p. 146)) is

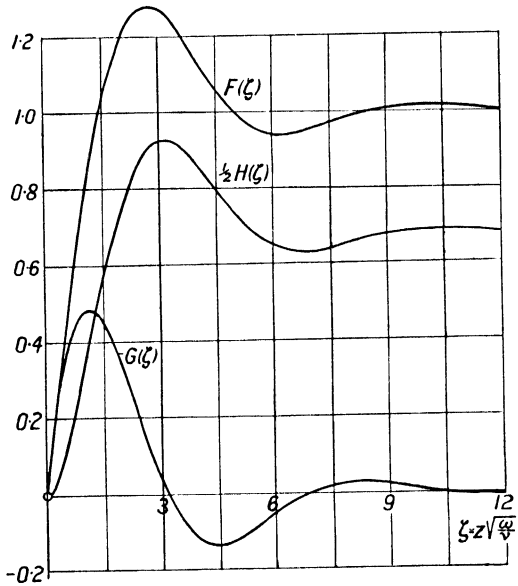


Fig. 5.34.—To illustrate the case of rotating fluid on a bed at rest

directed towards the centre. Hence the component w parallel to the axis here is in the direction away from the disc, with the limiting value $w_\infty = 1{\cdot}349\sqrt{(\omega\nu)}$. (It may be pointed out that the two flows show the typical forms of anticyclonic or cyclonic flow.) Here the angle of the stream-lines close to the disc is $\phi_2 = 50{\cdot}6°$, corresponding to $\tan\phi_2 = 1{\cdot}218$. As we see, the angles ϕ_1, ϕ_2 both deviate from the Ekman angle of 45°, one being below and one above, a fact which is

* The same formula with a different number was obtained by an approximate argument by Th. von Kármán (see *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 247 (1921)). In Cochran's paper an arithmetical error in von Kármán's result is corrected.

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 241 (1940).

related to the curvature of the stream-lines. For the state of affairs in the initial stages of these motions, see a paper by K. H. Thiriot.*

In the case of turbulent flow of the boundary layer at the rotating disc we obtain formulæ which correspond exactly to those of Chapter III, § 5 (p. 121), provided the turbulence begins close enough to the centre. Using the power formula (27) (p. 128), we may put

$$\tau \cos\phi \sim \rho(\omega r)^{7/4}\left(\frac{\nu}{\delta}\right)^{1/4};$$

also, $\tau \sin\phi \sim \rho\omega^2 r\delta$ as before, owing to the centrifugal force. Combining these two relationships, we have $\delta \sim r^{3/5}(\nu/\omega)^{1/5}$. If we again put $M \sim D^3\tau$, some manipulation and introduction of the peripheral velocity u_1 give

$$M = D^3 \cdot \tfrac{1}{2}\rho u_1^2(\nu/u_1 D)^{1/5} \times \text{const.} \quad . \quad . \quad . \quad (26)$$

According to an approximate calculation by Th. von Kármán,† the constant is 0·021. According to experiments by W. Schmidt ‡ and G. Kempf § the value 0·025 would be more correct once the turbulent state has fully developed. The passage from the laminar state to the turbulent takes place in the same way as for plates; cf. curve 3 of fig. 3.75 (p. 193).

If a disc rotates in a cylindrical casing, the fluid rotates along with it to a considerable extent, as we mentioned above, and the relative velocity of the disc and the fluid is less, and hence the resistance is also less. According to experiments by F. Schultz-Grunow ‖ the constant in formula (26) is 0·0089 for $u_1 D/\nu > 6 \times 10^5$, and is practically independent of the distance of the plane walls of the casing from the disc, provided this distance is not too small. In fact, a frictional boundary layer is formed both at the disc and at the flat ends of the casing, the first with a flow outwards, the others with a flow inwards. Between them is a passive layer rotating fairly uniformly, which gradually moves from the ends of the casing to the disc. Between $u_1 D/\nu = 3 \times 10^4$ and 6×10^5 Schultz-Grunow found that the flow was laminar and that

$$M = 0{\cdot}47 D^3 \cdot \tfrac{1}{2}\rho u_1^2 \sqrt{(\nu/u_1 D)}. \quad . \quad . \quad . \quad (27)$$

For narrow casings, where the distance between the disc and the end

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 1 (1940).

† *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 249 (1921).

‡ *VDI-Zeitschr.*, Vol. 65, p. 441 (1921).

§ *Innsbrucker Vorträge* (1922), p. 168.

‖ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 15, p. 191 (1935), where a theoretical discussion is also given. See also the calculations by T. Okaya and M. Hasegawa, *Japan Journ. of Physics*, Vol. 13, p. 29 (1939).

of the casing is s, and small values of $u_1 D/\nu$, it was found by calculation, and confirmed by experiment, that

$$M = \frac{\pi}{8}\mu u_1 \frac{D^3}{s}. \qquad \ldots\ldots \quad (27a)$$

C. Flow in Heavy Stratified Fluids

11. **Two Fluids Differing in Density.**

(*a*) If two fluids differing in density are in two layers one above the other, the argument in Chapter II, § 5 (p. 46), about the subdivision of the pressure into gravity pressure and dynamic pressure may be applied by subtracting from the actual pressure the gravity pressure corresponding to the less dense fluid. Then the latter fluid is left with only the dynamic pressure, but in the denser fluid there is not only the dynamic pressure but a gravity pressure corresponding to a fluid of density $(\rho_2 - \rho_1)$. This line of thought makes the treatment considerably easier. The less dense fluid may then be regarded as if it were weightless and the more dense as if it were subjected to a gravitational acceleration $g(\rho_2 - \rho_1)/\rho_2$. From this it follows immediately that the denser fluid always tends to occupy the lowest possible position.

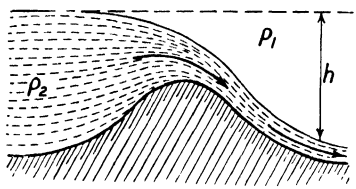


Fig. 5.35.—Downflow of cold air in a mountainous region

In mountainous regions it is sometimes found that in a valley shut in by high mountains on three sides there is a strong wind blowing down the valley, although the mountains would seem to make a wind in such a direction impossible. The explanation is that there has been a " cold front " on the other side of the mountains and the cold air flowing over the passes rushes into the valley like a waterfall (fig. 5.35). If, for example, the difference in height h is 1000 metres and the difference in temperature 10° C., we have $(\rho_2 - \rho_1)/\rho_2 \approx 1/30$. Then the velocity, assuming the motion to be frictionless, is

$$q_0 = \sqrt{\{2gh \,.\, (\rho_2 - \rho_1)/\rho_2\}} \approx \sqrt{(20{,}000/30)} = 25{\cdot}8 \text{ metres/sec.},$$

which is a pretty strong gale.*

If an avalanche of powdery snow falls over a precipice and the snow becomes mixed with air, a mixture may result which has a density

* In this calculation it is assumed that the mass of warm air above the mass of cold air in the valley is stratified more or less adiabatically (cf. p. 17). If the stratification were stable the temperature differences in the lower layers would be smaller, as the cold air flowing down behaves adiabatically. Then $(\rho_2 - \rho_1)/\rho_2$ would be smaller there too, and hence also the final velocity.

several times that of air and yet still retains the properties of a fluid.* If, for example, the density is five times that of air, the acceleration of the mass would be 4/5 of the gravitational acceleration, that is, after a fall of 500 metres the mass would have a velocity of about 90 metres per second, which would give a dynamic pressure of about 2500 Kg. per sq. metre. In actual fact, of course, the motion is turbulent. In the heart of the flow, however, similar destructive pressures may very well occur, so that it is easy to understand how avalanches of this type can carry whole houses away.

The form in which the front of a mass of cold air breaks into a mass of warm air, which may be assumed at rest, may be briefly described here.(79) When the cold air begins to move, it first forms a roll like that shown in fig. 2.21, p. 52, if we imagine the lower half of the two figures to be removed. The vortex thus formed first grows continually in size and gradually becomes somewhat flattened as a result of gravity, owing to its greater density. In this way is produced a type of flow which is shown in idealized form in fig. 5.36 for a system of reference moving with the front. A formula for v, the rate of advance of the front, may be deduced from the condition that the pressure must everywhere vary continuously, and hence must be the same to the right and to the left of the stagnation point. If the velocity of the advancing cold air is q and the previous velocity of the warm air zero, the common dynamic pressure, with the system of reference of fig. 5.36, is $\frac{1}{2}\rho_2(q-v)^2 = \frac{1}{2}\rho_1 v^2$, which gives $v = q/\{1+\sqrt{(\rho_1/\rho_2)}\}$. If, as usually happens, ρ_1 and ρ_2 do not differ very much, the cold front advances with velocity about $\frac{1}{2}q$; this is confirmed by observation.† If, however, ρ_2 is considerably greater than ρ_1, as is the case in an avalanche, v is nearly equal to q.

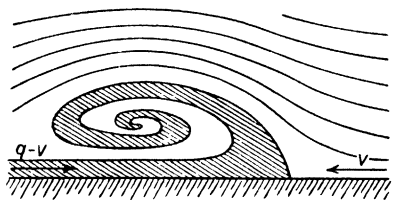


Fig. 5.36.—Advancing mass of cold air; flow relative to the front (idealized)

In the discussion just given friction was left out of account, as was also the fact that boundaries where heavier material lies above lighter are unstable. Hence the actual phenomenon is more of the form shown in fig. 5.37. This is confirmed by observations on sandstorms.‡ Fig. 5.37 shows the stream-lines for a system of reference at rest relative to the ground, from which the behaviour of the velocity for an observer on the ground is clear.

* Particularly loose dry snow often has so much air enclosed in it that it can form a fluid emulsion of this type without the further admixture of air, and it may remain "fluid" even for turbulent flow, provided the conditions of equilibrium on p. 334 are satisfied.

† See e.g. H. Koschmieder, *Danziger Seewinduntersuchungen, Forsch.-Arb. des Observ. Danzig*, Vol. 8 (1936).

‡ For illustrations of sandstorms, see e.g. Koschmieder, *Kleinräumige Luftbewegungen*, pp. 119, 125 (Leipzig, 1943).

Similar phenomena occur when a rising mass of air (often visible as a cumulus cloud *) reaches an " inversion " where the colder air surrounding it impinges on warmer air into which the rising mass cannot penetrate. The latter is then deflected horizontally, still retaining its kinetic energy, and then spreads out between the two layers, its front being rolled up as in fig. 5.36, but simultaneously above and below. This double vortex may in certain circumstances persist for a very long time and fresh condensation may occur in its upper half. If such motion goes on for some time it results in long isolated clouds which seem to reach from one side of the horizon to the other (line squall).

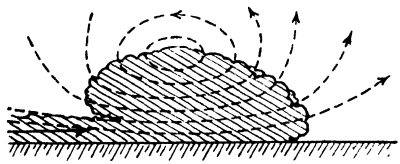


Fig. 5.37.—Advancing mass of cold air. Flow relative to the ground (actual phenomenon)

Repetition of this sort of process frequently leads to the occurrence of a laminated structure in the upper layers of the atmosphere. The individual layers have different wind directions and degrees of humidity which are shown by the zigzag course of snow clouds slowly falling through them (cirrus clouds) or, if the whole mass of air is rising, by the formation of thin layers of mist.*

(*b*) The horizontal boundary between two fluids of differing density is capable of wave motion under gravity, just like the free surface of a liquid. If the two fluids are relatively at rest, and both layers are sufficiently thick, the waves advance, as was mentioned on p. 90, with the velocity

$$c = \sqrt{\left(\frac{g\lambda}{2\pi} \cdot \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\right)}. \qquad \ldots \quad (28)$$

Here $g(\rho_2 - \rho_1)$ is the gravity force on the wave motion and $(\rho_1 + \rho_2)$ a measure of the mass inertia of the two layers. The velocity fluctuations are of the same magnitude in both layers and decrease in both directions away from the surface of separation with $e^{-2\pi z/\lambda}$, where λ is the wave-length. The horizontal velocities are in opposite directions.†

Stationary waves are also possible, as in the case of a free surface. They are obtained e.g. by the superposition of two waves advancing in opposite directions. Hence for waves of length λ the period of vibration is $T = \lambda/c$, where c is given by formula (28).

Formula (28) is based on the assumption that the depth of both layers is large compared with $\lambda/2\pi$. For shallower layers the formulæ are more complicated. A simple result, however, is obtained for the opposing waves when one layer is of depth h and the other layer is of very great depth, and $(\rho_2 - \rho_1)$ is small compared with ρ_m. Here the greatest rate of propagation, which is for very long waves, is

* Rising masses of air reach regions of lower pressure and are thereby adiabatically cooled; if their moisture content is sufficiently great, part of the water vapour condenses in the form of a cloud.

† For further details ,see Lamb, *Hydrodynamics*, § 231 *et seq.*

$c = \sqrt{\{gh(\rho_2 - \rho_1)/\rho_i\}}$, where ρ_i is the density of the layer of depth h. If there is a free surface a wave can also exist in the same direction; if the wave-length is large, the rate of advance of the wave is approximately $c = \sqrt{(g\lambda/2\pi)}$.

By formula (28) the rates of propagation of internal waves are fairly small compared to those for surface waves if the difference of density $(\rho_2 - \rho_1)$ is small; this is the case e.g. when fresh water lies above salt water, as when drift ice in the Arctic Ocean melts. It is reported that in this case, if the fresh-water layer is of a certain depth, the speed of the ship cannot be raised above that of the internal waves, because any extra power is used up in producing internal waves of considerable amplitude. The ship is held fast above the crest of the wave by the pressure field of the upper flow. Only if the speed of the ship can be raised very quickly above the critical velocity before the wave has formed can the normal speed be reached and maintained.*

Waves at the boundary of two fluids differing in density are also of some interest in the case where one fluid has the velocity U relative to the other. Just as in the case of fluids at rest relative to one another, two solutions exist for a given wave-length, giving two waves moving in opposite directions with the same velocity c; here, too, there are two solutions. For wave-lengths so great that the velocity of propagation of formula (28) is large compared to U, the two velocities of propagation are still of opposite sign. As λ diminishes, however, the difference between them becomes smaller and smaller, and for a critical value of λ,

$$\lambda_{\text{crit}} = \frac{2\pi U^2}{g} \frac{\rho_1\rho_2}{\rho_2^2 - \rho_1^2}, \qquad \ldots \quad (29)$$

they coincide. This value of c lies between the two rates of flow and is given by the formula

$$c_1 = c_2 = \frac{U_1\rho_1 + U_2\rho_2}{\rho_1 + \rho_2}. \qquad \ldots \quad (30)$$

For wave-lengths below λ_{crit} an ordinary wave motion is no longer possible. Here there are waves which increase with time (and also those which decrease with time, but these are of no importance). All the longer waves are stable. Their mode of propagation may be qualitatively deduced from what has been said above. Waves shorter than λ_{crit} all have the same phase-velocity as in equation (30); they grow exponentially, i.e. with increasing velocity, and finally topple over. This causes a mixing which leads to the formation of a transition layer

* Cf. V. Bjerknes *et al.*, *Phys. Hydrodynamik*, p. 387 *et seq.*, and Lamb, *Hydrodynamics*, §§ 248, 249.

of finite depth instead of the original discontinuity of density. Once this layer has formed all imaginable wave disturbances become stable.

The theory indicated above finds application to waves in air at the boundary separating a colder mass of air from a warmer, first studied by H. von Helmholtz. These are often made visible by regular cloud formations * corresponding to the crests of the waves. Often, however, they remain invisible but can still always be detected by small regular variations in atmospheric pressure.

12. Continuous Variation of Density.

(*a*) *Theorem of V. Bjerknes.*—Thomson's theorem, that the circulation along a closed fluid line is constant in time in a frictionless fluid (cf. Chapter II, § 8, p. 56), involves the assumption that the density is constant, or, as is the case in homogeneous gases, depends on the pressure only. For any density distributions of a more general kind, however, the theorem requires modification. The term

$$-\int \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy + \frac{\partial p}{\partial z}\,dz\right)$$

on p. 60 must obviously be recalculated, and it is convenient to do so again for the closed line. As the argument would take up a good deal of space in ordinary co-ordinates, we shall give a brief account in the vector notation.† By what we have said, it is a case of evaluating the line integral

$$-\int \frac{1}{\rho}\,\mathrm{grad}\,p \,.\, d\boldsymbol{s}$$

for the closed line. By Stokes' theorem this can be transformed into the surface integral

$$-\iint \mathrm{curl}\left(\frac{1}{\rho}\,\mathrm{grad}\,p\right) .\, d\mathbf{A}$$

taken over an area bounded by the closed line. But

$$\mathrm{curl}\left(\frac{1}{\rho}\,\mathrm{grad}\,p\right) = \nabla \times \left(\frac{1}{\rho}\,\mathrm{grad}\,p\right) = \left(\nabla\,\frac{1}{\rho}\right) \times \mathrm{grad}\,p + \frac{1}{\rho}\,\nabla \times \mathrm{grad}\,p.$$

Since the curl of a gradient vanishes, the last term disappears and we have

$$-\int \frac{1}{\rho}\,\mathrm{grad}\,p \,.\, d\boldsymbol{s} = -\iint \mathrm{grad}\,\frac{1}{\rho} \times \mathrm{grad}\,p \,.\, d\mathbf{A}.$$

* Cf. footnote 1, p. 370.

† Gibbs' notation: · for scalar product, × for vector product.

Hence if Γ is the circulation the new formula is

$$\frac{d\Gamma}{dt} = -\iint \operatorname{grad}\frac{1}{\rho} \times \operatorname{grad} p \,.\, d\mathbf{A}. \quad . \quad . \quad . \quad (31)$$

If grad $1/\rho$ is parallel to grad p everywhere the integrand vanishes and, as before, $d\Gamma/dt = 0$. If, however, the surfaces $\rho = \text{const.}$ and the surfaces $p = \text{const.}$ (each of them at right angles to the corresponding gradient) intersect, formula (31) gives non-zero values.

There is a very illuminating geometrical interpretation of the integral on the right-hand side of (31), which is also due to V. Bjerknes. Suppose all the surfaces $p = \text{const.}$ are drawn which correspond to an arithmetic series of p-values with increment Δp, and all the surfaces $v = 1/\rho = \text{const.}$ (where v is the mass per unit volume) which correspond to an arithmetic series of v-values with increment Δv. Then each pair of p-surfaces and a pair of v-surfaces together form a " tube " with a parallelogram as cross-section (cf. fig. 5.38). Now Bjerknes found that *the integral in equation* (31) *is proportional to the number of tubes enclosed by the fluid line.*

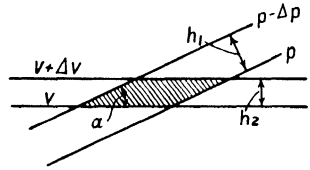


Fig. 5.38

This may be proved as follows. We begin by calculating the area of a parallelogram which is at right angles to the lines in which the two sets of surfaces intersect. If h_1 is the distance between two neighbouring p-surfaces and h_2 the distance between two neighbouring v-surfaces, then the area of the parallelogram is obviously $a' = h_1 h_2/\sin\alpha$. In general the area of the parallelogram is inclined to the area a' at the angle β; then $a = a'/\cos\beta$. Now it is clear that $\Delta p = h_1 \,.\, |\operatorname{grad} p|$ and $\Delta v = h_2 \,.\, |\operatorname{grad} v|$, so that

$$a = \frac{\Delta p \,.\, \Delta v}{|\operatorname{grad} p| \,.\, |\operatorname{grad} v| \,.\, \sin\alpha \,.\, \cos\beta}.$$

The agreement between the denominator and the integrand of equation (31) is immediately obvious (the sine comes from the vector product, the cosine from the scalar product). Now the number of parallelograms of area a in unit area is $n = 1/a$. Hence N, the total number of " tubes " in the area A which has the " fluid line " as its boundary, is

$$N = \int n\,dA = \int \frac{|\operatorname{grad} p| \,.\, |\operatorname{grad} v| \,.\, \sin\alpha \cos\beta \, dA}{\Delta p \,.\, \Delta v}.$$

By comparison with (31) we therefore have

$$\left|\frac{d\Gamma}{dt}\right| = \Delta p \,.\, \Delta v \,.\, N. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In a particular case the sign of $d\Gamma/dt$ is easily found, since the denser

portions tend to move downwards and the less dense upwards. Bjerknes' theorem is particularly suited to qualitative considerations. Quantitative applications are held up by the fact that as a rule the pressure field for the accelerated flow is not known exactly. An exception is formed by the flow in regions where the horizontal extension is several times the vertical, as holds in most meteorological applications. In this case the vertical accelerations are small compared with the horizontal ones, for reasons of continuity, so that the pressures at different points on a vertical line can be found by the rules of hydrostatics.* Here, however, it is usually better to employ the initial form of the integral, $-\int \frac{1}{\rho} \operatorname{grad} p \,.\, d\boldsymbol{s}$. (Here, since $|\operatorname{grad} p| = \rho g$, we have merely $g(h_1 - h_2)$ for the vertical parts of the fluid line; portions of the line on isobaric surfaces give contributions which are zero.)

(*b*) *Internal Waves.*—These may occur in any medium which is stratified in stable layers. Hitherto most researches have been confined to the simple forms. Among these are "cellular waves", in which the whole space is divided up into cells in which vibrations take place, as it were, within solid walls, the frequency being the same for all cells. The simplest case is that of *two-dimensional stationary waves in stratified incompressible fluid above a plane.* If we assume that the density at rest is distributed according to the law $\rho_0 = \rho_g e^{-z/H}$, where ρ_g is the density at the ground, all the cells become identical and the amplitude increases upwards in such a way that the energy in each cell is the same. Then the velocities must obviously increase upwards in proportion to $e^{+z/2H}$. Thus if this type of vibration occurs in infinite space, an infinite amount of energy is stored up, so that it is a question of a vibration which has worked up in the course of infinite time. The conditions under which it can arise in finite time have not yet been investigated.

If the density varies from point to point it becomes necessary in the calculation to distinguish the individual particles of fluid from one another, so that the Eulerian formulæ no longer suffice.† Since, in order to avoid mathematical difficulties, we are forced to confine ourselves to small amplitudes, it is, however, not necessary to have recourse to the more troublesome Lagrangian formulæ.‡ Instead, we can represent (ξ, η, ζ), the displacements of the particles, as functions of the mean-position co-ordinates (x, y, z) fixed in space and of the time t.

In the two-dimensional case (ξ and ζ functions of x, z and t) we obtain for incompressible fluid the following linearized equations:

1. As a result of the displacements, $\rho(x, z, t)$, the density at any instant at a particular point differs from $\rho_0(z)$, the density at rest, by a quantity σ: that is,

* See also p. 352. † Equations (2), p. 33. ‡ Equations (1), p. 32.

$\rho = \rho_0 + \sigma$. To the first order, then, $\sigma = -\zeta \partial \rho_0 / \partial z$, for at the point (x, z) there is a particle which originally belonged to the level $(z - \zeta)$. For the above density distribution, therefore, we have $\sigma = \rho_0 \zeta / H$.

2. The equation of continuity is

$$\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial z} = 0.$$

3. The linearized forms of the dynamical equations for the x- and z-directions, with ρ_0 on the left instead of $\rho_0 + \sigma$ (a permissible simplification), are

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x}, \quad \rho_0 \frac{\partial^2 \zeta}{\partial t^2} = -\frac{\partial p}{\partial z} - g(\rho_0 + \sigma). \quad . \quad . \quad . \quad (33)$$

In the case mentioned above a possible solution for ξ and ζ ($\eta = 0$) is

$$\left.\begin{aligned} \zeta &= A e^{z/2H} \sin \alpha x \cos \beta z \cos \omega t, \\ \xi &= A e^{z/2H} \cos \alpha x \left(\frac{1}{2H\alpha} \cos \beta z - \frac{\beta}{\alpha} \sin \beta z \right) \cos \omega t. \end{aligned}\right\} \quad (34)$$

These formulæ represent a cellular standing wave (fig. 5.39); the horizontal wave-length is $\lambda_x = 2\pi/\alpha$; the vertical, $\lambda_z = 2\pi/\beta$; the

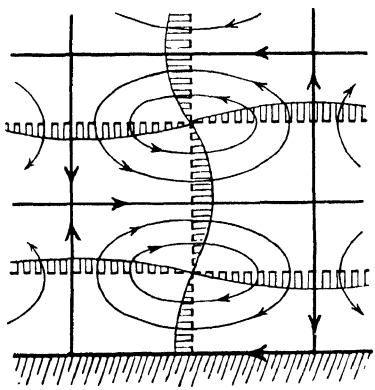


Fig. 5.39.—Cellular standing wave of gravity type

breadth or height of a cell is half a wave-length; the frequency ω (if T is the period of vibration $T = 2\pi/\omega$) is found to be

$$\omega = \sqrt{\left\{ \frac{g}{H} \frac{\alpha^2}{\alpha^2 + \beta^2 + 1/4H^2} \right\}}. \quad . \quad . \quad . \quad (35)$$

Note.—Fluctuations in a heated room or hall can be calculated in the same way, for the individual possible vibrations α, β can each be put equal to π/l or π/h, or any multiple of these, where l and h are the length (or breadth) and height

of the room. H has to be chosen so that the ratio of the density of the air at the ceiling and that at the ground is $e^{-h/H}$. Such fluctuations are easily observed, being made visible by stable layers of tobacco smoke.

By superposing on the wave (34) a similar wave differing in phase by a quarter-wave in x and a quarter-period in t, we obtain waves advancing horizontally with the phase-velocity

$$c = \frac{\omega}{\alpha} = \pm\sqrt{\left\{\frac{g}{(\alpha^2+\beta^2)H + 1/4H}\right\}}. \quad . \quad . \quad (36)$$

If we superpose on this a velocity of flow $U = -c$, the phenomenon becomes a steady flow with wave-like stream-lines (fig. 5.40). For a definite value of the ratio β/α, say $\beta = 0$, which gives $\lambda_z = \infty$, or in other words, the same phase for all values of z, the wave-length λ_x of the stationary waves can be calculated for a given value of U from (36). If $\beta = 0$, a brief calculation gives

$$\lambda_x = \frac{2\pi}{\alpha} = 2\pi U\sqrt{\left\{\frac{H}{g(1 - U^2/4gH)}\right\}}. \quad . \quad . \quad (37)$$

For velocities U which are small compared with $\sqrt{(2gH)}$, the rate of fall corresponding to the height H, the term in U^2 in the denominator is negligible compared with unity. We then have the remarkable result that the stationary wave-length is proportional to the velocity of flow U, whereas in waves on the free surface of water of great depth the stationary wave-length is $\lambda = 2\pi U^2/g$. These relationships are brought out more clearly by equation (35), especially in the case $\beta = 0$. Here for wave-lengths λ_x which are small compared with $8\pi H$ we have a period of vibration $T = 2\pi/\omega = 2\pi\sqrt{(H/g)}$ which is practically constant and independent of λ_x; for water waves, on the other hand, the corresponding expression (p. 87) for T is $\sqrt{(2\pi\lambda_x/g)}$, that is, it depends on λ_x.

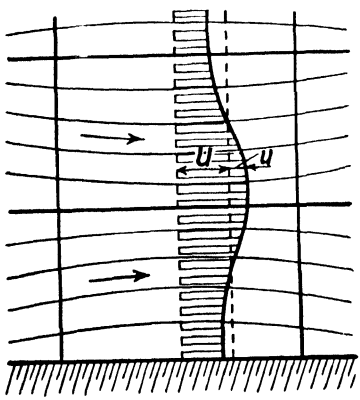


Fig. 5.40.—Stationary wave system

The general forms of cellular waves are three-dimensional. By superposing several cellular waves each with its own wave-length λ_x, λ_y, λ_z we obtain any number of more general types of waves, which as a rule are no longer cellular; hitherto these have been little studied. A type more interesting from the practical point of view is discussed in more detail below (p. 378). The many and varied

possible types of vibration of stratified heavy fluids with the density decreasing continuously upwards mark them off sharply from uniform fluids. Here the solutions are no longer unique, unless the existence of frictional forces which damp the vibrations is assumed, or unless the development of motion from a state of complete rest is followed step by step. Both cases generally involve very difficult calculations.

(*c*) *Internal Waves in Compressible Media.*—In stratified compressible media, e.g. in *an isothermally stratified heavy gas*, the calculations for cellular waves become much less simple. Here we have to express the fact that the individual particle of gas behaves adiabatically in its vibrations. By comparison with Chapter IV, § 1, particularly equation (3), p. 260, we see that here the change of pressure Δp is connected with the change of density $\Delta\rho$ by the equation $\Delta p = c_s^2 \Delta\rho$, where $c_s = \sqrt{(\gamma p/\rho)}$, the velocity of sound. Here, therefore,

$$\sigma = -\zeta \frac{\partial \rho_0}{\partial z} + \frac{\psi}{c_s^2},$$

where $\psi(x, z)$ is the pressure change. Here two types of cellular wave occur, one the type described above, with fairly slow vibrations, caused mainly by displacements of the centre of gravity, and another type, markedly faster, caused mainly by compression and expansion. The latter type of motion is shown in fig. 5.41, where the stream-lines of several cells are given.

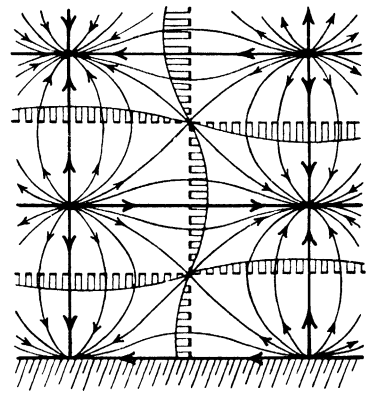


Fig. 5.41.—Cellular standing wave of acoustic type

For the calculations reference may be made to Bjerknes' book, *Physikalische Hydrodynamik* (p. 341 *et seq.*), where the Lagrangian formulæ are consistently used (a, b, c are the co-ordinates and x, z are used instead of ξ, ζ). Instead of giving the exact theory we shall content ourselves here with a few approximate formulæ for cases where $(\alpha^2 + \beta^2)$ is large compared to $1/2H^2$, i.e. λ_x or λ_z or both are small compared with $4\pi H$. H is, as before, the height of the homogeneous atmosphere (about 8000 metres; cf. p. 16). Here for both types of wave we have the approximate formulæ for the frequency ω:

$$\omega_1 \sim \sqrt{\left\{\frac{\gamma - 1}{\gamma} \frac{g}{H} \frac{\alpha^2}{\alpha^2 + \beta^2 + \ldots}\right\}}. \quad \ldots \quad (38)$$

$$\omega_2 \sim c_s \sqrt{\{\alpha^2 + \beta^2 + \ldots\}}, \quad \ldots \quad (39)$$

where α and β have their previous meanings; for γ, see p. 10. Apart

from the term dropped here from the denominator and other modifications, formula (38) differs from (35) only by the factor $\sqrt{\{(\gamma-1)/\gamma\}}$.

From the standing waves we may pass to the progressive waves as in (*b*). For the second type we obtain the rate of propagation $c \sim c_s\sqrt{(1+\beta^2/\alpha^2+\ldots)}$, i.e. a velocity exceeding that of sound. (These waves may in fact be regarded as arising from the superposition of two sound waves with wave fronts inclined to the vertical at angles given by $\tan\delta = \pm\beta/\alpha$. In terms of the angle δ the formula for c becomes $c \sim c_s/\cos\delta$.)

For the simpler case of the purely vertical oscillations of an isothermal atmosphere, i.e. for the case where $\alpha = 0$ and hence $\lambda_x = \infty$, we shall give the exact formulæ for all values of ω or λ_z. According to *Physikalische Hydrodynamik*, p. 329, for the vertical amplitude $\zeta(z,t) = Ae^{\delta z}\cos\omega t$ of the stationary oscillation the relation $\delta^2 - \delta/H + \omega^2/{c_s}^2 = 0$ holds; hence for real values of ω

$$\delta = \frac{1}{2H} \pm \sqrt{\left\{\frac{1}{4H^2} - \frac{\omega^2}{{c_s}^2}\right\}}. \qquad \ldots\ldots\ (40)$$

For $|\omega| < c_s/2H$ there are two real values δ_1 and δ_2. The condition for free oscillations above an earth at rest, $\zeta(0) = 0$, is satisfied by $\zeta = A(e^{\delta_1 z} - e^{\delta_2 z})\cos\omega t$.

For $|\omega| > c_s/2H$,

$$\delta = \frac{1}{2H} \pm i\sqrt{\left\{\frac{\omega^2}{{c_s}^2} - \frac{1}{4H^2}\right\}} = \frac{1}{2H} \pm i\beta; \qquad \ldots\ (41)$$

here, with the same boundary condition, the amplitude is $\zeta = Ae^{z/2H}\sin\beta z\cos\omega t$. Here, therefore, there are horizontal nodal planes (places where ζ is zero) at distances $\pi/\beta = \lambda_z/2$ apart. By superposition of a second solution

$$\zeta' = -Ae^{z/2H}\cos\beta z\sin\omega t$$

displaced in phase relative to the first, we again have a moving wave

$$\zeta_1 = \zeta + \zeta' = Ae^{z/2H}\sin(\beta z - \omega t)$$

with phase-velocity $c = \omega/\beta$. By (41), $\omega = c_s\sqrt{\{\beta^2 + 1/(4H^2)\}}$, so that

$$c = c_s\sqrt{\{1 + 1/(4H^2\beta^2)\}}.$$

Thus c is greater than the velocity of sound; for sufficiently short waves (i.e. if β is sufficiently great), however, it is practically equal to the velocity of sound, in accordance with the approximate formula (39). As we approach the aperiodic limiting case β tends to 0, and hence c tends to infinity. This behaviour seems very peculiar. The phase-velocity, however, is not a criterion so far as mechanical considerations are concerned; it is the group-velocity (see p. 88 *et seq.*) that we have to consider. Here this is given by $c^* = d\omega/d\beta$. A brief calculation gives

$$c^* = c_s/\sqrt{\{1 + 1/(4H^2\beta^2)\}},$$

i.e. c^* is less than the velocity of sound. For the aperiodic limiting case ($\beta = 0$) we have $c^* = 0$. As the group-velocity is also the velocity with which energy is transferred (p. 89), the flow of energy due to the spread of the waves is given

by $\varepsilon = Ec^*$, where E is the energy density; the flow of energy therefore vanishes in the limiting case $\omega = c_s/2H$.

In the sun turbulent motion in the so-called hydrogen convection layer gives rise to advancing waves which move up into the photosphere lying above. Their amplitude, moderate at the outset, increases with height in proportion to $e^{z/2H}$, the energy of the waves thus remaining constant. According to L. Biermann † this gives rise to considerable heating as a result of shock waves (p. 271) in the still higher chromosphere.

Note.—The vibrations of an isothermal atmosphere above the spherical earth have also been investigated in detail; the first to do this was M. Margules ‡ (1890–93).

A wave motion of practical interest is that calculated by G. Lyra §[80] for an isothermal atmosphere flowing with velocity U whose horizontal motion is disturbed by a range of mountains of moderate height running across the direction of the wind. Here the perturbations are expressed as sums of Bessel functions of argument $2\pi r/\lambda$ multiplied by suitable trigonometrical functions of ϕ, and, like all the solutions in this section, by $e^{z/2H}$, where $r = \sqrt{(x^2 + z^2)}$ is the radius vector (the obstacle being at $x = 0$), ϕ the corresponding angle ($\tan\phi = z/x$), and λ the wave-length of the disturbance. Here λ is given approximately by

$$\lambda = 2\pi U \sqrt{\left(\frac{\gamma}{\gamma - 1}\frac{H}{g}\right)}, \qquad . \quad . \quad . \quad . \quad (42)$$

which corresponds to formula (37) for $\beta = 0$ when we take into account the remarks made in connexion with equation (38).

Here again the solution is not unique in the first instance, but is made so by the condition that the resultant flow to windward of the mountain must be aperiodic, or else by the introduction of frictional forces which are subsequently made to tend to zero in the limit. (Both methods were used first in the discussion of waves on the free surface of water.‖) The results of Lyra's calculations may be illustrated by an example. Fig. 5.42 shows the distribution of vertical disturbances of velocity above an escarpment. The figure shows the regions where there are upward currents (of interest to those practising gliding); the vertical scale is magnified three times. In addition to the region immediately above the escarpment there is a second one at a distance

† *Naturwiss.*, Vol. 33, p. 118 (1946); for a more detailed account, see *Zeitschr. f. Astrophysik*, Vol. 25, p. 161 (1948).

‡ *Sitz.-Ber. d. Wiener Akad.*, II*a*, Vol. 99, p. 204 (1890); Vol. 101, p. 597 (1892); Vol. 102, pp. 11, 1369 (1893). See also H. Koschmieder, *Dynamische Meteorologie*, 2nd ed., p. 297 *et seq.* (Leipzig, 1941), where further references are given.

§ *Beitr. z. Physik d. freien Atmosphäre*, Vol. 26, p. 197 (1940); *Zeitschr. f. angew. Math. u. Mech.*, Vol. 23, p. 1 (1943). Fig. 5.42 is taken from the latter paper, which contains other examples; in it the calculations have been considerably refined.

‖ See fig. 5.12*b* and footnote 2, p. 323. For the conditions for unique solutions see the critical article by W. Magnus, *Abhandl. Math. Seminar Univ. Hamburg*, Vol. 16, p. 77 (1947).

of about a wave-length from the edge, and further feebler ones at distances 2λ, 3λ, and so on. Between them are regions with downward currents, the first being at a distance about $\frac{1}{2}\lambda$ from the edge. (In the example a polytrope (p proportional to ρ^n (p. 18), where $n = 1{\cdot}2$, is used instead of an isothermal. According to Lyra, equation (42) then

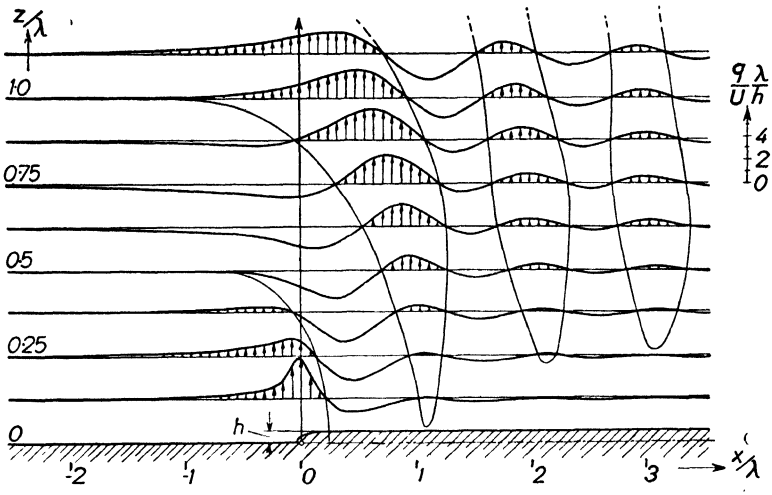


Fig. 5.42.—System of waves above an escarpment, after Lyra

remains approximately valid if $\gamma/(\gamma - 1)$ is replaced by $n\gamma/(\gamma - n)$, and H is the height of the homogeneous atmosphere (p. 16) corresponding to the state at the ground. Here $e^{z/2nH}$ takes the place of the factor $e^{z/2H}$.)

The upward currents are then proportional to $h\sqrt{\left(\frac{\gamma - n}{n\gamma}\frac{g}{H}\right)}$, where h is the height of the plateau; the numerical factor can be obtained from fig. 5.42. It is remarkable that the strength of the upward currents is independent of the strength of the wind, U. The latter affects only λ, that is, the extension of the region of upward currents; also, the calculation holds only for the case where the upward currents are small compared with U, i.e. only for limited heights h.

According to calculations by H. Stümke,* u, the horizontal component of the perturbation, exhibits very peculiar behaviour, which again goes to show how different the flow of a stratified fluid is from that of a homogeneous fluid. The velocity u does not completely disappear at a distance from the steep edge of the plateau, but a considerable part of it remains all over the plateau. As the height z increases it changes its sign from positive to negative and back again, and so on; the amplitude increases upwards, and again depends not on

* *Beitr. z. Phys. d. freien Atm.*, Vol. 26, p. 207 (1940).

the strength of the wind, but only on h, the height of the plateau. The wave-length λ is again given by equation (42).

In our notation the asymptotic formula given by Stümke is

$$u_{as}(z) = h\sqrt{\left(\frac{\gamma-1}{\gamma}\,\frac{g}{H}\right)}\cdot e^{z/2H}\left(\sin\frac{2\pi z}{\lambda} + \frac{2-\gamma}{2\gamma}\,\frac{\lambda}{2\pi H}\cos\frac{2\pi z}{\lambda}\right). \quad (43)$$

Again this formula of course only holds good so long as these velocities can be regarded as small compared with U. As z increases they increase without limit (double themselves every 10 or 11 kilometres). What happens if u_{as} is no longer small compared with U is not known. Observations of the condensation trail of a meteor falling over Scotland, which long remained visible in the western sky from Holland and Norway, even after sundown, because it was still lit up by the sun's rays, showed * that such distributions of velocity do actually occur at great heights. The trail was at first a fairly straight line, then changed into a fairly regular sine curve with several bends, and gradually broadened out horizontally and disappeared.

(*d*) *Limit of Stability of Shearing Flow in a Stably Stratified Fluid.*— As has already been explained in Chapter III, § 5(*f*), p. 131, the stability of a heavy stratified fluid, if the velocity varies with the height, tends to hinder turbulence, since the vertical motions of individual " balls " of fluid shift heavier material upwards and lighter material downwards, and in both cases work is done. A striking phenomenon of this type is the dying away of the wind on bright evenings, when very markedly stable stratification occurs near the ground owing to radiation from the ground to the clear sky. (At a height the wind is in no wise diminished, but only the shearing stress and the turbulence.) There now arises the question of the position of the boundary at which an existing turbulent motion with vertical mixing must be destroyed by the aforesaid expenditure of work (horizontal mixing is, of course, not affected). Shearing motion, i.e. the increase of velocity upwards, measured by dU/dz, obviously plays a decisive part here. According to the statements in Chapter III, § 4(*d*), especially those on p. 118, the order of magnitude of the vertical velocity fluctuations in turbulent motion is given by $w' \sim l\,dU/dz$, where U is the mean velocity and l the mixing length. The kinetic energy of the vertical motion of a ball of fluid of volume V is accordingly $\frac{1}{2}\rho V w'^2 \sim \frac{1}{2}\rho V l^2 (dU/dz)^2$. For the sake of simplicity we assume that the fluid is incompressible. Then the work done in raising a ball of fluid which is in equilibrium in a layer $z = z_1$ is found from the downward force at $z = z_2$ (weight minus upward thrust), namely,

$$F = g(\rho_1 - \rho_2)V = g\,\frac{d\rho}{dz}\,(z_1 - z_2)V,$$

to be

$$\int_{z_1}^{z_1+l} F\,dz_2 = -\tfrac{1}{2}gVl^2\,\frac{d\rho}{dz}.$$

* See the report by C. Størmer in *Astr. Norv.*, Vol. 3, No. 5 (1939), summarized in *Nature*, Vol. 145, 195 (1940).

Equating kinetic energy and work done, we have

$$-g\,d\rho/dz = \rho(dU/dz)^2.$$

This relationship obviously gives only an upper limit for $-d\rho/dz$ (decrease of density with height), for which every turbulent motion must immediately die away. In reality, if any particular disturbance is so enfeebled that the next one is weaker still, this is sufficient to ensure the destruction of any existing turbulence.[81] The ratio of the two expressions above, i.e.

$$R_i = -\frac{g}{\rho}\frac{d\rho}{dz}\Big/\left(\frac{dU}{dz}\right)^2,$$

is called *Richardson's number* (R_i) in honour of L. F. Richardson,* who was the first to put forward these ideas. G. I. Taylor,† by investigating the small oscillations of a stratified flow with velocity $U = U_0 + az$, showed that it is probable that the limit of stability lies at $-\frac{g}{\rho}\frac{d\rho}{dz}\Big/a^2 = \frac{1}{4}$. Calculations by H. Schlichting,‡ based on Tollmien's theory of the stability of laminar boundary layers (cf. p. 114), showed that for values of Richardson's number above 1/20 the boundary layers remain perfectly stable. The theoretical results are in good agreement with the experiments made a few years earlier by H. Reichardt § on the onset of turbulence in the flow of air above a cooled, or below a heated, horizontal plate.

If the guiding effect of a plate or the ground is absent, the limit may lie higher. Until further information is available, therefore, the limit of stability for free flow may be taken as $R_i = 1/4$.

With a view to applying the argument given above to the case of a compressible medium, the following simple considerations may be put forward here. Let a volume V_1 of the medium of the same density ρ_1 as its surroundings be raised through a height dz, and let the pressure be thereby altered from p_1 to $p_1 + dp$; then by equation (5*a*), p. 14, $dp = -g\rho_1\,dz$. Let the volume raised behave adiabatically; then, using the adiabatic law $pV^\gamma = \text{const.}$, we have

$$\frac{dV_{\text{ad}}}{V_1} = -\frac{1}{\gamma}\frac{dp}{p_1}.$$

We now inquire what is the difference between the weight W and the upward thrust L of this volume, resulting from the displacement dz. The weight is still $W = g\rho_1 V_1$; the upward thrust (p. 13) in the state of rest is equal to the weight,

* *Proc. Roy. Soc.* (A), Vol. 97, p. 354 (1920), and *Phil. Mag.* (6), Vol. 49, p. 81 (1925); see also L. Prandtl, *Aachener Vorträge*, p. 1 (1929), and J. M. Burgers, ibid., p. 7.

† *Proc. Roy. Soc.* (A), Vol. 132, p. 499 (1931); see also S. Goldstein, ibid., p. 524.

‡ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 15, p. 313 (1935), where numerous references are given.

§ Details are given in Schlichting's paper (p. 333 *et seq.*).

but in an upward displacement dz it is altered in the first place by the increase in volume (dV) of the volume displaced, due to the fall of pressure, and in the second place by the vertical change in density of the surrounding layers of air or gas. The former gives $dL_1 = g\rho_1 dV_{ad}$, the latter $dL_2 = gV_1 d\rho_{la}$. Hence, by what we have said above, $dL_1 = -g\rho_1 V_1 dp/\gamma p_1$; from this, since $dp = -g\rho_1 dz$, $g\rho_1 V_1 = W$, and $p_1 = g\rho_1 H_1$, we have $dL_1 = +W\,dz/\gamma H_1$, where H_1 is the height of the homogeneous atmosphere (p. 16) for the state $p_1\rho_1$. Further, if $(d\rho/dz)_{la}$ means the vertical density gradient of the surrounding medium,

$$dL_2 = gV_1(d\rho/dz)_{la}\,dz.$$

If the stratification is polytropic with index n ($p\rho^n = \text{const.}$; see p. 18), $d\rho/\rho_1 = dp/np_1$ and by the same calculation as above $(d\rho/dz)_{la} = -\rho_1/nH_1$ and hence $dL_2 = -W\,dz/nH_1$. Hence the force F (see above)

$$= W - L = W\frac{z_2 - z_1}{H}\left(\frac{1}{n} - \frac{1}{\gamma}\right).$$

Comparison shows that for a polytropically stratified mass of air, $\frac{1}{\rho}\frac{d\rho}{dz}$ in Richardson's formula must be replaced by $\frac{1}{H}\left(\frac{1}{n} - \frac{1}{\gamma}\right)$.

The argument above also explains the occurrence of the factor $(\gamma - 1)/\gamma$ in the formula for the vibrations of masses of air [see under (c), p. 377]. There an isothermal atmosphere was assumed, for which $n = 1$, so that the bracket becomes $1 - 1/\gamma = (\gamma - 1)/\gamma$.

Note.—A more general expression for Richardson's number is, by the above,

$$R_i = g\left\{\left(\frac{d\rho}{dz}\right)_{ad} - \left(\frac{d\rho}{dz}\right)_{la}\right\}\Big/\rho\left(\frac{dU}{dz}\right)^2.$$

Using the equation of state $\rho = p/RT$, where R is the gas constant and T the absolute temperature, we may also write

$$R_i = g\left\{\left(\frac{dT}{dz}\right)_{la} - \left(\frac{dT}{dz}\right)_{ad}\right\}\Big/T\left(\frac{dU}{dz}\right)^2.$$

Again, we may replace

$$\left(\frac{dT}{dz}\right)_{la} - \left(\frac{dT}{dz}\right)_{ad} \quad \text{by} \quad \left(\frac{d\theta}{dz}\right)_{la},$$

where θ is the "potential temperature", that is, the temperature which a particle of air would reach if it were compressed *adiabatically* to normal pressure, e.g. 1000 millibars.

In stably stratified air the mixing which results from the turbulence produced by a strong wind gives rise to a gradual approach to the adiabatic (neutral) state of stratification. According to W. Schmidt,*[82] the turbulent heat flow which brings about this change is

$$Q = c_p A_\varrho\left\{\left(\frac{dT}{dz}\right)_{ad} - \left(\frac{dT}{dz}\right)_{la}\right\}.$$

If the stratification is stable this has a negative value, i.e. the flow of heat is downwards, which tends to equalize the potential temperature. If $(dT/dz)_{la} < 0$, as is the case under average conditions, heat is transferred from cold places to warmer. This is thermodynamically possible owing to the work done by the

* See footnote 1, p. 119.

turbulent flow against the stratification as described on pp. 381–2. This process therefore represents a " refrigerator "; cf. H. Ertel's account in *Meteorol. Zeitschr.*, Vol. 56, p. 109 (1939), the main theme of which is Richardson's theorem.

13. Combined Effects of Stratification and the Earth's Rotation.

If we combine the theory of flow on the rotating earth given in §§ 8–10 (pp. 350–68) with that of stratified media we obtain a number of results which yield valuable information when applied to currents in the atmosphere.

(*a*) *Two Densities.*—If two incompressible fluids of constant but differing density are stratified one above the other, and both have constant but differing velocities, the question arises, what position is taken up by the boundary surface of the two fluids.

For the sake of simplicity we shall assume that the two velocities are horizontal and parallel to one another. Let their direction be that of the x-axis. Let the velocities be U_1, U_2 and the corresponding densities ρ_1, ρ_2. Let the z-axis be vertical and the y-axis in the horizontal plane at right angles to the direction of the velocities. Let the positive direction of z be upwards, and the positive direction of y be northwards, if the positive direction of U is eastwards. The relevant vertical component ω' of the angular velocity with which the earth rotates, equal to $\omega \sin\phi$, gives rise to Coriolis accelerations in the southward direction for positive values of U, of amount $2\omega' U$. The combined effects of gravity and Coriolis force give a pressure distribution of the same type in each fluid, namely,

$$p_1 = p_0 - g\rho_1 z - 2\omega' U_1 \rho_1 y,$$
$$p_2 = p_0 - g\rho_2 z - 2\omega' U_2 \rho_2 y.$$

Now at the required boundary surface the pressure must be continuous. We accordingly obtain the equation of the boundary surface by putting $p_1 = p_2$. For the sake of simplicity we choose a point of the boundary surface at which the pressure is $p_1 = p_2 = p_0$ for the origin of co-ordinates. The equation of the boundary surface is then

$$\frac{z}{y} = \frac{2\omega'(U_2\rho_2 - U_1\rho_1)}{g(\rho_1 - \rho_2)}, \qquad . \quad . \quad . \quad . \quad (44)$$

which represents an inclined plane.* This equation was first given by M. Margules.† If we now introduce the mean density $\rho_m = \frac{1}{2}(\rho_1 + \rho_2)$, it is easily seen that we may also write

$$\frac{z}{y} = \frac{2\omega'}{g}\left\{\frac{\rho_m}{\rho_1 - \rho_2}(U_2 - U_1) - \frac{U_1 + U_2}{2}\right\}. \quad . \quad (44a)$$

* The results of this calculation are likewise applicable to two continuously stratified masses of air if we replace z/y by dz/dy.

† *Meteorol. Zeitschr.*, Hann volume, p. 243 (1906).

If, as is usually the case, $(\rho_1 - \rho_2)$ is small compared with ρ_m, and $(U_2 - U_1)$ is of considerable magnitude, the second term in the bracket is trifling compared with the first, and may therefore be left out of account.

For stability it is necessary that the lower fluid (1) should have the greater density. By formula (44*a*) the upper fluid must then have the greater velocity U_2, in order that z/y shall be positive for the chosen direction of U_2 towards the east (a *west* wind), i.e. the boundary surface rises towards the north. This is the situation most commonly met with in the northern hemisphere. A warm flow with a strong west wind lies over a colder one with a small velocity (west wind or east wind). The inclination of the boundary surface is generally of the order of 1 in 100. Fig. 5.43 shows a cross-section perpendicular to U_2 with the isobars drawn for the case $U_1 = 0$.

Numerical Example.—Let $\omega' = 0{\cdot}5 \times 10^{-4}$ sec.$^{-1}$, corresponding to $\phi = 43{\cdot}5°$, $g = 10$ metres/sec.^{2}, $U_2 - U_1 = 10$ metres/sec., $\rho_1 - \rho_2 = 0{\cdot}01\rho_m$, corresponding to a temperature difference of about 3° C. Then

$$\frac{z}{y} = \frac{10^{-4}}{10} \cdot \frac{10}{0{\cdot}01} = 10^{-2}.$$

As the boundary surface falls towards the south, it must meet the earth's surface somewhere. Then here the warm flow with the strong wind, and the cold flow with the slight wind, are close together at ground level. As a result of additional velocities of the whole formation at right angles to the boundary surface,* the individual observer experiences a sudden change in the direction and strength of the wind as well as in the temperature—a so-called " front ". The phenomena at ground level are modified in several respects as a result of friction, but we cannot go into details here. In referring to the discussion on p. 362 we may point out, however, that in general the changes in density and velocity are not nearly so abrupt as in fig. 5.43. It is only near the ground that comparatively sharp fronts are produced, as a result of frictional phenomena. Thanks to Bjerknes and his school of meteorologists, the movements and transformations of fronts have been studied in all their details and their great value in interpreting local weather conditions has been recognized. Typical diagrams of fronts

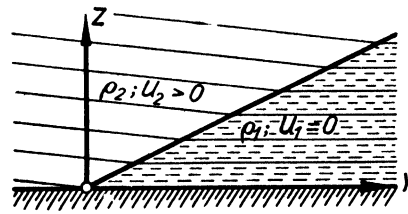


Fig. 5.43.—Boundary surface between warm flow and cold flow, after Margules

* It is now know that such velocities involve deviations from the geostrophic state; cf. p. 352, and footnote, p. 353.

are to be found in a book by Chromow,* which also contains much other valuable illustrative material.

If the boundary surface is at a considerable height above the ground and the upper flow is not rectilinear, but e.g. contains a cyclonic vortex, so that the pressure at the centre of the vortex is less than in the surrounding region, the lower flow is sucked up as shown in the cross-sectional view in fig. 5.44. Thereby, according to the argument in § 8 (p. 355), it is itself set in rotation, in the same sense as that of the upper vortex. This induced vortex dies out after the upper vortex has moved on, but the latter causes a fresh part of the heavier medium to be set revolving. Thus the upper vortex is accompanied by a lower vortex, in apparent contradiction of Helmholtz's theorem. This phenomenon is very common in the atmosphere, since the upper layers frequently contain vortices which may have arisen from the ascent of a mass of air † and which may persist for a very long time at the boundary of a colder mass of air, owing to the very slight friction. If only observations made on the ground are available, it is difficult to predict the future motion of one of these "stratospherically governed lows", as this depends on the velocity of the "stratospheric low".

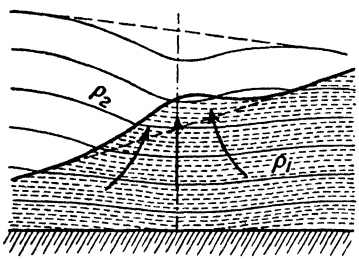


Fig. 5.44.—Stratospheric "low"

(*b*) *Continuous Stratification.*—Here again, owing to the effects of the earth's rotation discussed in § 8 (p. 350), new forms of motion arise. Of particular interest is the case where a mass of air leaves its original position (say owing to local heating or cooling) and finds a new position of equilibrium at a different level. For theoretical discussion the process can be split up into two stages: the vanishing of the mass of air from one place, so that the surrounding masses of air rush together from all sides to fill the vacant space, and the appearance of the same mass of air at the second place, so that the masses of air already there must be pushed apart in all directions.

In practice the change-of-place phenomenon may be regarded as occurring in stages, in local ascents in the form of cumulus clouds, or in downward currents which, though invisible, are of similar nature, each stage lasting for so short a time that the rotation of the earth need not be taken into account. The total effect, however, is to bring about a new state of equilibrium (see below) involving considerable horizontal transport of air; this gives rise to noticeable velocity fields and pressure fields, by what was said in § 8(*c*), p. 355. In typical cases these fields can be calculated correct to the first order, if the displacements are assumed to be moderate. The place where the mass of air is removed represents a sink; that where it arrives, a source. At the

* *Einführung in die synoptische Wetteranalyse* (Vienna, 1940); see especially pp. 313–402 and 484–502.

† See next page.

first place there remains behind a diminution of pressure which, owing to the rushing together of the remaining portions of air, is associated with a cyclonic rotation; at the second, there is a corresponding increase in pressure and an anticyclonic rotation.

In the simplest solution * both sink and source are regarded as points. The singularities may be avoided by integrating over continuous distributions of infinitesimal sources and sinks.†

The combined effects of the forces of stratification and the Coriolis forces, as in § 14(*a*), p. 389, bring about a general state of equilibrium like that of an elastic body, except that the "softness" of this body is markedly greater in the horizontal direction than in the vertical. In the neighbourhood of the disturbance the surfaces of equal pressure disturbance, which for an isotropic elastic body would be spherical, are flattened ellipsoids of revolution with their heights in the ratio of $1 : \frac{1}{2\omega'}\sqrt{\left\{\frac{g}{H}\left(\frac{1}{n}-\frac{1}{\gamma}\right)\right\}}$ to their breadths. The corresponding final form of the rising mass is that of a shallow stratus cloud. (If $H = 8000$ metres, $\gamma = 1{\cdot}4$, $n = 1{\cdot}2$ and $2\omega' = 10^{-4}$, the ratio is approximately 1 : 120.) If masses of air rise in this way from a lower level to a higher over a considerable area, the pressure fields and the rotations overlap and thus very powerful effects may arise (a cyclone at the lower level, an anticyclone at the higher level). The energy increases as the square of the disturbance of pressure, and this is proportional to the square of the tangential velocity. The greater and greater amounts of work required to raise the masses of air arise from the fact that the later masses to ascend have to fight against the pressure field which is already in existence, and are thereby forced to do additional work.

The work done originates in important thermodynamical changes of state of the following kind. When air saturated with moisture is adiabatically cooled, as is the case when it rises into a region of lower pressure, it loses part of its moisture in liquid form as a mist. This sets free the latent heat of evaporation, which imparts itself to the mass of air. Hence the mass of air is no longer cooled at the rate of 1° C. per 100 metres, but only by 0·5° to 0·7° per 100 metres, according to the initial temperature (this change of state is called a "wet adiabatic"). If the stratified air which does not share in the rising motion is closer to the dry adiabatic, i.e. 0·8° to 0·9° per 100 metres, the mass which is rising as a wet adiabatic may gain superfluous energy as it rises, and hence may do work.‡[83]

* H. Stümke, *Zeitschr. f. Geophysik*, Vol. 16, p. 127 (1940); *Diss.*, Göttingen (1940) —the symmetrical problem; cf. also L. Prandtl, *Report to the Meteorological Congress*, Edinburgh, 1936, p. 186 *et seq.* (Paris, 1939), where the two-dimensional problem is discussed.

† Unpublished calculations by W. Rothstein (1944).

‡ More details about these "wet-labile" states and the part they play in supplying energy in atmospheric phenomena will be found in any modern textbook of meteorology. A particularly clear account is given by H. von Ficker, *Wetter und Wetterentwicklung*, 2nd ed. (Berlin, 1940), especially on p. 70. For a quantitative discussion, see e.g. H. Koschmieder, *Dynamische Meteorologie*, 2nd ed. (Leipzig, 1941), especially p. 94 *et seq.*, or P. Raethjen, *Einführung i. d. Physik d. Atmosphäre*, Vol. I, p. 65 *et seq.*

So long as the " high " which we have described remains immediately above the " low ", the two weaken one another. If, however, as in § 14(*b*), p. 391, there is a flow all through the mass of air, and the velocity increases with increasing height, the " sources " and hence the high will gradually move ahead of the " sinks " and the low, and it is only then that the effects of the two pressure fields on the regions around the disturbance are fully displayed.

Note.—The process we have described represents only *one* possible way in which a low may arise, i.e. that which explains how a low can arise from a mass of air originally at rest. Other possibilities occur when two flows of air with differing velocities impinge on one another, and as a result of some disturbance the boundary layer assumes first a wavy form and eventually a vortex form as in figs. 2.16, 17 (p. 51). Here we shall not go into further details of this possible explanation of the development of depressions in high latitudes, which was postulated by V. Bjerknes on theoretical grounds and investigated in detail on an observational basis by J. Bjerknes and others.* On the other hand, the account given here may serve as a good explanation of the initial stages of tropical cyclones. The stratospheric cyclones and anticyclones mentioned in (*a*), p. 386, may be regarded as residual phenomena from such processes in tropical regions.

14. Combined Effects of Horizontal Density and Velocity Gradients and the Earth's Rotation. General Circulation.

(*a*) *Problems of Stability*—The stability or instability of a stratified atmosphere has already been discussed in Chapter I, § 7 (p. 17), for the case of rest. It was shown there that adiabatic stratification represents the neutral case corresponding to the state of equilibrium of an incompressible fluid of the same density everywhere (in which every particle of fluid can be displaced in any way without showing any tendency to return to its original level or to move beyond the new level in the direction of the displacement). The " potential temperature " defined in the note at the end of § 12 (p. 383) is a quantity which is suited to play a part for the stratified gas subject to gravity corresponding to that of the density of a stratified incompressible liquid (i.e. which is constant in the neutral case); for by definition adiabatic stratification, which, by what we have said above, represents the neutral case, gives a constant potential temperature. The way in which the potential temperature increases with height is a measure of the stability of a stratification. In the case of rest the surfaces of constant potential temperature are horizontal. If, however, the temperature also varies in the horizontal direction (in moving air equilibrium is possible in this case also, owing to the Coriolis effect (see under (*b*) below), these surfaces are slightly inclined, but only slightly, provided the vertical stability is sufficient.

A particle of a stably stratified atmosphere can be displaced in any

* A detailed account is given by V. Bjerknes and his collaborators in *Physikalische Hydrodynamik* and in most modern textbooks of meteorology.

way within a surface of constant potential temperature, unless prevented by the Coriolis effects of § 8(*c*), p. 355. If, however, these are taken into account, we find that a displacement $y - y_0$ gives rise * to an x-velocity $u = 2\omega'(y - y_0)$, and this in turn gives rise to a Coriolis acceleration in the direction towards the original position of the particle. The particle will move according to this acceleration provided it is not prevented by a pressure field arising from the general velocity distribution. Here in general, then, there is " dynamical stability " even within the surface of equal potential temperature.

This stability, however, may be endangered by velocity fields. The simplest case occurs when all the particles have velocities parallel to the x-axis, but varying according to y, and the density is constant in every horizontal plane. Here we may inquire into the neutral state. This occurs when, as in the discussion of stability of p. 17, every particle displaced in the y direction automatically takes up, as a result of the Coriolis effect, the same velocity as the particle which was there had originally. A particle which originally belonged to the plane $y = y_1$ obviously fits into its new environment after a displacement $(y_2 - y_1)$, by what we have said above, if $u(y) = \text{const.} + 2\omega' y$. By definition, the rotation relative to the " disc " is

$$\omega_1 = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\omega'$$

(the direction of rotation being anticyclonic); the absolute rotation is accordingly zero. Thus the neutral state turns out to be identical with the motion absolutely free of rotation which was studied in § 7 (p. 346). In this form the statement is true also for steady relative flow in curves.† Such flow is unstable if the anticyclonic rotation exceeds the value $-\omega'$, i.e. if, for example, $du/dy > 2\omega'$ for the rectilinear relative flow considered. For a fluid which previously was at rest relative to the rotating disc, i.e. had the absolute rotation ω', this unstable condition normally cannot arise as a result of pressure effects from outside.‡

In cyclonic and anticyclonic velocity fields the inertial rotations caused by local fluctuations in velocity no longer follow the rule given on p. 351, but are more rapid in cyclonic fields and less rapid in anticyclonic fields. In the motion described above, $u = \text{const.} - \alpha y$, $v = 0$, the time of revolution, according to calculations by P. Raethjen,§ is $T = \pi/\omega'\sqrt{(1 + \alpha/2\omega')}$. The paths relative to an undisturbed particle are ellipses with semi-axes a in the direction of u and b

* In the note in § 8(*c*) we considered a displacement ΔR which gave rise to a velocity $q = -2\omega'\Delta R$.

† See e.g. H. Ertel, *Meteorol. Zeitschr.*, Vol. 57, p. 397 (1940), where the proof is extended to zonal flow on a sphere.

‡ An exception may occur if the uppermost layer becomes undermost. This, however, will not in general occur with pressure effects of this kind.

§ See his *Einführung in die Physik der Atmosphäre*, Vol. II, p. 204.

at right angles to it, so that for an initial fluctuation v_0 in the direction at right angles to u, $a = v_0/2\omega'(1 + \alpha/2\omega')$ and $b = v_0/2\omega'\sqrt{(1 + \alpha/2\omega')}$. For $\alpha = 0$ the formulæ lead back to the circle of inertia; as α approaches $-2\omega'$, T tends to ∞ and the ellipse becomes a parabola; beyond this critical value it becomes a hyperbola. The changes as compared with the circle of inertia are to be explained by the fact that the pressure gradient in the y-direction is not constant but alters in the y-direction in proportion to α, and hence there result additional positive or negative accelerations proportional to the y-displacement from the original position.

So far as the applications of these relationships are concerned, it must be pointed out that in the interior of a mass of air either at rest or moving, there can be no hour-long motion of an enclosed mass of air of the same density but with a different velocity, owing to turbulent mixing. It is, however, possible to have a foreign mass, e.g. like that described in § 13(*b*), p. 386, in a stably stratified mass of air, which, owing to its previous history, brings in a horizontal velocity of its own and spreads out in a shallow layer. If Richardson's number is sufficiently great the subsequent motion is laminar and hence may persist for a long time. As the pressure field of the main mass is more or less impressed on the foreign mass, the conditions assumed in Raethjen's calculation hold good. The condition that the motion shall take place in a plane of constant potential temperature is satisfied *a priori* by the way in which the motion is brought about.

(*b*) *The Connexion between Density Field and Velocity Field.*— Here we shall consider states in which the density is also variable in a horizontal direction. For the sake of simplicity we shall assume that both u and ρ are constant in *one* horizontal direction. In actual cases this direction, which here we have taken to be the x-direction, is usually the line from west to east; y is then in the line from south to north. Then $u = u(y, z)$, $\rho = \rho(y, z)$. Let the rotation be with angular velocity ω' on the horizontal disc. Then the Coriolis acceleration $-2\omega' u$ is in the direction of the negative y-axis (south) when u is in the positive direction (towards the east). The pressure field is then defined by the two following statements:

$$\frac{\partial p}{\partial z} = -g\rho, \qquad \ldots\ldots\ldots \quad (45a)$$

$$\frac{\partial p}{\partial y} = -2\omega'\rho u. \qquad \ldots\ldots\ldots \quad (45b)$$

The requirement that $\partial^2 p/\partial z\,\partial y$ must be equal to $\partial^2 p/\partial y\,\partial z$, in order that it may be possible to have the forces of gravity and the Coriolis forces in equilibrium with a pressure field $p = p(y, z)$, then leads to a notable relationship (also due to M. Margules). Obviously we must have

$$g\frac{\partial \rho}{\partial y} = 2\omega'\frac{\partial}{\partial z}(\rho u). \qquad \ldots\ldots\ldots \quad (46)$$

Thus the density gradient in the south-north direction is coupled with the gradient of ρu with respect to height. If, then, the density in-

creases in a horizontal line from the equator to the pole owing to the decreasing temperature, the product of the west-east velocity and the density must increase with height. As the density decreases regularly upwards, the increase in velocity is all the more considerable.

If we assume a polytropic law for the variation of density with height, $p/p_g = (\rho/\rho_g)^n$, where the suffix g refers to the ground, an easy manipulation of the statements $dp/\rho = -g\,dz$ and $p/g\rho = H$ leads to

$$\frac{\partial \rho}{\partial z} = -\frac{\rho}{nH}.$$

Then

$$\frac{\partial u}{\partial z} = \frac{u}{nH} + \frac{g}{2\omega'\rho}\frac{\partial \rho}{\partial y}. \qquad \ldots\ldots\ (46a)$$

The increase in the west wind (or decrease in the east wind) with height is actually observed regularly.

For the case of uniform polytropic behaviour with respect to x and y, equation (45*b*) gives

$$\frac{\partial \rho}{\partial y} = \frac{\rho}{np}\frac{\partial p}{\partial y} = -\frac{2\omega'\rho u}{ngH}.$$

Equation (46*a*) then gives $\partial u/\partial z = 0$, i.e. $u = \text{const.}$ for $y = \text{const.}$; the relation to y is undetermined. This relationship means that particles vertically above one another remain perpetually above one another if the density depends only on the pressure. It also holds in the case where u and v are arbitrary, provided the relative accelerations are negligible compared with the Coriolis acceleration (cf. p. 351).

Note.—In weather phenomena in the temperate regions so-called *frontal zones* occur, regions in which the temperatures over a space of several hundred kilometres exhibit very considerable differences.* By equation (46) or (46*a*), there is then a very great rise in velocity with the height. From the temperature distribution it is found that the surfaces of equal potential temperature fall markedly from the cold side to the warm. For the velocity distribution in a surface of constant potential temperature it follows from the two relationships that there is a very marked anticyclonic shearing stress du/dy; a still more marked one is found in the air of clouds in the surfaces of constant " wet potential temperature ".† In the latter case, in particular, it seems that shearing stresses of such magnitude occur as to give rise to marked instability—in spite of vertical stability—which leads to motions along these surfaces.‡[84] It is supposed that the study of such phenomena may throw light on the genesis of new cyclones, which is still obscure.[85]

(*c*) *General Circulation of the Earth's Atmosphere.*—The following statements on this fundamental problem in meteorology may be made from the point of view of hydrodynamics. From the fact that the air in the neighbourhood of the equator is warm, and that at the poles cold (that is, at the same height z above the ground the density is less

* See e.g. Raethjen, *Einführung in die Physik der Atmosphäre*, Vol. II, p. 177 *et* seq.

† Definition to correspond with that of potential temperature on p. 383, applied to the wet adiabatic change of state (see the small type on p. 387).

‡ E. Kleinschmidt, *Diss.* (Hamburg, 1941); *Ann. d. Hydrogr.* (1941), p. 305. See also Raethjen, *Einführung in die Physik der Atmosphäre*, Vol. II, p. 194 *et seq.*

at the equator and greater at the poles), it follows that there is a tendency to rise at the equator and fall at the poles. For reasons of continuity this is associated with a tendency to flow from the equator towards the poles in the upper regions, and the reverse tendency in the lower regions of the atmosphere.[86] In the earlier theories a flow like this, regulated by apparent frictional forces over a non-rotating earth, was taken as the main flow, and an additional flow was superposed on it as a sort of secondary flow; this was produced by the Coriolis forces of the fundamental flow and of course also gave rise to Coriolis forces.* This decomposition into a fundamental flow and an additional flow started from the fact that the angular velocity of the earth ω is, after all, small (about 7×10^{-5} sec.$^{-1}$), and hence the effect of the earth's rotation on the flow might be regarded as a secondary effect of the order of ω, which is small. This, however, was a false conclusion in that 1 second is an unsuitable unit of time for a problem in which the average time for the masses of air to traverse paths relative to the earth of the order of the earth's circumference amounts, perhaps, on the average to 10 days. This refers to the west-east motions at a considerable height; the south-north motions are much slower, of the order of 100 days from equator to pole. In reality, therefore, ω must be regarded as large and the " secondary flow " is perhaps 40 times as strong as the " main flow ",† for the tendency to the latter flow is inhibited by the Coriolis forces of the west-east motion, apart from a small residue related to frictional phenomena (see below), and in general gives rise to a pressure field only.

On the other hand, if the density distribution is taken as given, the equation (46) above gives the correct approach to an approximate theory of the large-scale circulation in the atmosphere. On integration with respect to z we immediately have the distribution of east and west winds for every latitude ϕ, except for a constant of integration (u_0, the wind at ground level),‡ namely,§

$$\rho u = \rho_0 u_0 + \frac{g}{2\omega \sin\phi} \frac{1}{R} \int_0^z \frac{\partial \rho}{\partial \phi}\, dz. \quad . \quad . \quad . \quad (47)$$

If there were no friction the Coriolis force $-2\omega \sin\phi \,.\, \rho u$ would be completely balanced by the pressure field given by equations (45a) and

* See e.g. A. Oberbeck, *Sitzungsber. d. Preuss. Akad. d. Wiss.*, p. 383 (1888).

† Nevertheless, Oberbeck's calculation gives some qualitative statements correctly, e.g. the occurrence of trade winds in the lower latitudes and of westerly winds at higher latitudes.

‡ More accurately, the gradient wind calculated for the region near the ground.

§ In this calculation as in later ones it is assumed that H, the height of the region of the atmosphere where vertical motions occur (the so-called *troposphere*), is small compared with the earth's radius R (10 to 15 km. compared with 6380 km.), so that we may always write R for $(R + z)$. Here dy in equation (46) is duly replaced by $R\,d\phi$, ω' by $\omega \sin\phi$.

(45*b*). This ideal state, however, is not realized owing to the apparent friction τ' of the layers flowing one over the other, which can be taken as $A\partial u/\partial z$, where A is a suitably chosen "exchange coefficient" (see p. 119). The difference between the values of τ' in two layers unit length apart, $\partial\tau'/\partial z$, is a force per unit volume in the west-east direction, which is balanced by the Coriolis force of a flow v from south to north. Hence, ignoring trifling acceleration terms and taking account of sign, we have the equation

$$2\rho\omega \sin\phi \,.\, v = -\frac{\partial}{\partial z}\left(A\frac{\partial u}{\partial z}\right) \quad . \quad . \quad . \quad . \quad (48)$$

for v.

By continuity, the field of the south-north velocities is related to that of the vertical velocities w. We have

$$\frac{\partial(\rho w)}{\partial z}\cos\phi = -\frac{1}{R}\frac{\partial}{\partial\phi}(\rho v \cos\phi). \quad . \quad . \quad . \quad (49)$$

In the integration we have to take account of the "mass flow" M in the frictional layer in the neighbourhood of the ground by adding a term to ρw at the ground. By § 9(*c*), p. 360, we have $M = \tau_0/2\omega\sin\phi$, where τ_0 is the component of the shearing stress of the ground wind in the direction of the gradient wind (formerly denoted by τ_x). The gradient wind at the ground, which is the integration constant u_0 mentioned above, may also be expressed in terms of τ_0, by what was said in small type in § 9(*b*) (p. 359). As a rough estimate we may put $\tau_0 = 0{\cdot}0025\,\rho_0 u_0^2$, which gives $u_0 = 20\sqrt{(\tau_0/\rho_0)}$. We now have to find τ_0. This is done by the moment of momentum theorem (p. 84), as follows.

Across a parallel of latitude of radius $R\cos\phi$ there flows per area $2\pi R\cos\phi \,.\, H$, momentum $2\pi R\cos\phi\int_0^H \rho uv\,dz \,.\, R\cos\phi$, where H is the height of the layer of the atmosphere where the masses of air are moving up or down (the so-called troposphere). At the equator the above momentum integral is zero, for there $v = 0$ for reasons of symmetry. At the poles it is likewise zero, for there $\cos\phi = 0$. It follows among other things from this that the total angular momentum of all wind frictional forces must vanish between the equator and the poles, i.e. that as much angular momentum must be produced by west winds as is produced with the opposite sign by east winds. Between the equator and latitude ϕ the angular momentum of wind friction is $\int_0^\phi \{\tau_0(\phi) \,.\, 2\pi R\cos\phi R\,d\phi \,.\, R\cos\phi\}$. This must be numerically equal to the flow of momentum for the parallel of latitude ϕ given above. If positive u means a west wind and positive v a south

wind, and the shearing stress of a west wind is reckoned positive, we have to insert the minus sign, by the laws of signs, on p. 77. Differentiating the resulting relation with respect to ϕ, we then have

$$\tau_0(\phi) = -\frac{1}{R\cos^2\phi}\frac{\partial}{\partial\phi}(\cos^2\phi\int_0^H \rho uv\,dz). \quad . \quad . \quad (50)$$

This forms the basis for calculation, but the actual computation is difficult. A helpful fact, however, is that the integral of equation (50) is obtained with fair accuracy if we first put $u_0 = 0$ and thus obtain u and v from equations (47) and (48). From the approximate value of τ_0 we obtain an approximate value for u_0 as above and repeat the calculation of equation (50) if necessary.

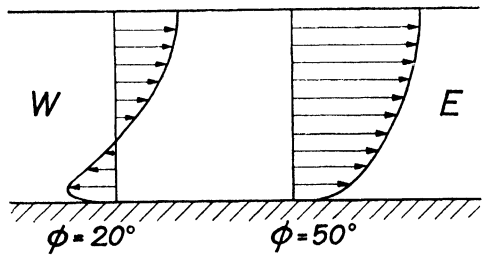


Fig. 5.45.—Wind profiles in the trade-wind zone (left) and the zone of the westerlies (right)

Hitherto only a somewhat simpler problem has been discussed as a preliminary, that in which it is assumed that a fluid of constant density surrounding a rotating celestial body in a flat layer is subject to body-forces upwards in the neighbourhood of the equator, and downwards in the neighbourhood of the poles. With suitable assumptions about the magnitude and distribution of the body-forces and the magnitude of the exchange coefficient A, it has been found possible to solve the problem completely.* Of the results we shall state the following:

1. The calculation agrees with observation in giving east winds in the lower layers on both sides of the equator, with a frictional wind component from the north in the northern hemisphere, and a frictional wind component from the south in the southern hemisphere. This trade-wind zone extends to the point where the momentum integral has its maximum value. Beyond the trade winds there are west winds,

* F. Kropatscheck, *Diss.*, Göttingen (1935); *Beitr. z. Phys. d. freien Atm.*, Vol. 22, p. 272 (1935); see also L. Prandtl, *Report to the Meteorological Congress*, Edinburgh, 1936, p. 171 (Paris, 1939).

more and more marked as the latitude increases—the "westerlies". At the equator there is practically a calm. North of the maximum momentum the west wind descends to ground level; in the northern hemisphere it has a frictional component from the south. Fig. 5.45 shows two profiles of the u-components, in the trade-wind zone and in the zone of the westerlies.*

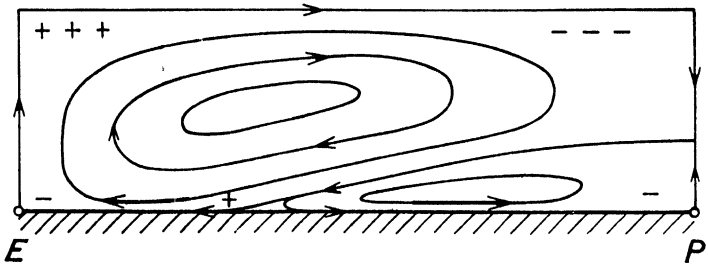


Fig. 5.46.—Meridional section of the flow

2. As might be expected, the two meridional components v and w give rise to a circulation in the main mass of the atmosphere, the warm air of hot regions rising and the polar air, which has lost its heat by radiation at great heights, falling. Lower down there is a counter-circulation in the zone of the westerlies caused by friction at the ground; if the simple type of force distribution is assumed, the vertical extent is only trifling. Fig. 5.46 shows the approximate course of these two circulations in exaggerated diagrammatic form for the northern hemisphere (E equator, P pole; + high, — low).

In a more accurate theoretical discussion it would be necessary to take into account the change in density distribution brought about by the two circulations. The trade winds carry masses of air which have been heated by the ground to the equator from both sides, so that there there is a special concentration of warm air. This rises particularly high as a result of wet-adiabatic expansion, so that high up particularly low temperatures are reached. The low-level circulation in the zone of the westerlies carries air from sub-tropical regions northwards. There it has colder masses of air superposed on it, so that marked instability results, and to this are due the depressions which regularly occur in this region. The warm current becomes separated more and more from the ground in the more or less regular succession of depressions. Still farther north there is a third circulation of polar winds, in which the cooling of the air near the ground as a result of radiation plays the chief part. This cold air flows from the north towards the depressions region, where it is carried up aloft again by admixture with the winds coming from the south in the frictional layer. Such phenomena are taken into account in the calculation by assuming an increased value for the exchange coefficient A in equation (48) in the depressions region.

* This is for ρ = const. If ρ diminishes with height the increase in the wind with height is more marked [cf. end of (*b*), p. 360].

3. The pressure field in the simple case is obtained as follows. The pressure gradients at the ground correspond to u_0, the gradient wind in the neighbourhood of the ground, and, like it, are small. There is a high-pressure region at the ground at the boundary between the trade-wind and westerly zones, and low-pressure regions at the equator and in the region of the poles. At higher levels the pressure differences are much greater and find their counterpart in the Coriolis forces of the west wind at a high level. Above the equator there is a very strong high owing to the hot masses of air pushing upwards, and in the polar region there is a corresponding low.

If, as is actually the case, the troposphere is surmounted by a stratosphere which is practically free of up and down movements ($v = w = 0$) and in which the temperature is constant (i.e. a polytrope with $n = 1$), $u = u_1(y)$ by the discussion on p. 391 and equation (46a).

D. HEAT TRANSFER IN MOVING FLUIDS; CASES OF FLOW DUE TO HEAT.

Preliminary Note.—It is difficult to give an exact theoretical account of this branch of the subject, as all the properties of the fluid depend on the temperature, and in addition to those hitherto considered (density and viscosity) we also have thermal conductivity and specific heat. The relationships are more readily discussed theoretically if we assume that the differences of temperature are very small, so that the properties may still be regarded as constant. Even here we have to select particularly simple problems in order to get satisfactory results. One of our groups of problems is concerned with flow produced by external causes, and here it is assumed that the velocities are exactly as they would be if there were no temperature differences, that is, the parts of the motion which arise from differences of density produced by thermal expansion are neglected. In the first instance we are seeking to find only the temperature field which arises as a result of thermal conduction and convection, and from this follows the amount of heat transferred. Problems of this type may be referred to under the name of " forced flow ".

The other part of the subject is concerned with the flow brought about by gravity as a result of thermal expansion, with the assumption that no other causes of motion exist at all. Here the velocity field has to be determined simultaneously with the temperature field. Problems of this type are referred to as " natural flow ".

The literature of this whole branch of the subject, especially that relating to experimental work, is fairly extensive. A detailed discussion would require more space than is available in this book, and we must instead content ourselves by mentioning the main features

of the fundamentals. More detailed information is to be found in specialized textbooks, of which we may especially mention the following: H. Gröber and S. Erk, *Die Grundgesetze der Wärmeübertragung* (Berlin, 1933), which especially stresses the theoretical relationships; M. ten Bosch, *Die Wärmeübertragung; Ein Lehr- und Nachschlagebuch für den praktischen Gebrauch*, 3rd ed. (Berlin, 1936), which contains many numerical data, tables and nomograms as well as an extensive bibliography.

We may also mention the accounts in *Handbuch der Experimentalphysik*, edited by W. Wien and F. Harms, Vol. 9, Part 1, pp. 268–317 (Leipzig, 1929); *Wärmeübergang durch Mitführung*, by H. Reiher (see also pp. 331–44); *Der Chemie-Ingenieur*, by A. Eucken and M. Jakob, Vol. 1, Part 1 (Leipzig, 1933), pp. 230–85 (M. Jakob, general principles) and pp. 380–517 (S. Erk, special cases).

For the latest state of research on heat transfer in different types of flow, and on evaporation and condensation, which are not discussed here, see a review article with numerous references by H. Kraussold in *Die Technik*, Vol. 3, pp. 205, 257 (1948), and *Modern Developments in Fluid Dynamics*, by S. Goldstein, Vol. 2, pp. 601–75.[87]

15. General Principles of Heat Transfer. Special Cases of Forced Flow.

(*a*) The problem is usually as follows: a liquid or a gas at temperature θ_1 flows along a boundary formed by a good conducting body at temperature θ_0. The question is, what is the amount of heat transferred per second (or per hour) from the fluid to the body, or from the body to the fluid, according as $\theta_1 > \theta_0$ or $\theta_1 < \theta_0$. Frequently it is a case of the heat transfer between two moving media at temperatures θ_1 and θ_2 separated by a metal boundary. In this case, if $\theta_1 > \theta_2$ heat passes from the first medium to the boundary on one side, and on the other side it is transferred from the boundary to the second medium.

Hot-water radiators and aeroplane-engine cooling systems are two examples, the first of natural flow, the second of forced flow. In both cases heat is transferred from hot water to air, but the purpose is different in the two cases (in the first case it is to heat the air, in the second to cool the water). When equilibrium is reached on both sides the temperature of the metal boundary is such that it takes up as much heat on one side as it gives up on the other.

The transfer of heat comes about partly by transport with the moving fluid (convection), partly by conduction, and partly by radiation. As a rule the last plays a very subordinate part if the temperatures are moderate, and here we shall accordingly neglect it altogether.* In convection we have to distinguish between convection

* All the books mentioned above have a section on heat radiation.

by laminar flow and convection by turbulent exchange. The first gives rise to transfer of heat in the direction of flow, the amount of heat transferred per second across unit cross-section at right angles to the line of flow being $q_1 = c_p \rho w \,.\, \theta$, where ρ is the density, c_p the specific heat of unit mass at constant pressure, and w the rate of flow. The second gives rise to transfer of heat in the direction of maximum temperature gradient, arising from the fact that in turbulent mixing the masses of air coming from the warmer side bring more heat with them than those on the colder side carry away. The result of this mixing is measured by the quantity introduced in Chapter III, § 4 (p. 119), and here termed the " exchange coefficient ". The transfer of heat per unit area as a result of exchange is

$$q_2 = -c_p A_q \frac{\partial \bar{\theta}}{\partial n},$$

where A_q is the coefficient of heat exchange, $\bar{\theta}$ the time-average of the temperature, and dn the line element in the direction normal to the surface $\bar{\theta} = \text{const}$. Similarly, the heat transfer due to conduction is

$$q_3 = -\lambda \, \partial\theta/\partial n,$$

where λ is the thermal conductivity. We see that just as A_τ measures the " apparent viscosity " of the turbulent flow, the quantity $c_p A_q$ represents an " apparent heat conduction " due to turbulence.*

Numerically the apparent heat conduction preponderates over the molecular conduction λ to the same extent as the apparent viscosity does over the molecular viscosity μ. The exchange disappears as we approach the boundary, so that here only the thermal conduction remains. (If the flow is laminar, of course we have thermal conduction only.) Hence, as the thermal conductivity of fluids is generally small, the boundary layer forms an obstacle to the transfer of heat, so that here there is an abrupt fall in temperature, whereas in the interior of the liquid the temperatures are largely equalized as a result of mixing. Hence formerly a finite jump in temperature was often assumed and calculations were based on the assumption that a quantity of heat proportional to the temperature difference $(\theta_m - \theta_0)$ was exchanged with the boundary, where θ_m was the mean temperature † of the liquid and θ_0 the temperature of the boundary. If q was the quantity of heat transferred per unit area, q was put equal to $\alpha(\theta_m - \theta_0)$, in the hope that the " heat transfer coefficient " α would depend only on the nature of the liquid and perhaps on that of the metal used for the boundary.

* For the ratio A_q/A_τ, see the discussion on p. 120; cf. also p. 406.

† For the definition of θ_m, see p. 402 below.

For flow through a pipe with mean velocity u_m this gave a transfer of heat in the direction of flow amounting to

$$Q = \pi r^2 q_{1m} = \pi r^2 \cdot \rho u_m c_p \theta_m.$$

By continuity,

$$\frac{dQ}{dx} + 2\pi r q = 0,$$

which with the above assumption leads to

$$\frac{d\theta_m}{dx} = \frac{-2\alpha(\theta_m - \theta_0)}{\rho c_p r u_m}.$$

The solution of this simple differential equation for $\theta_0 = \text{const.}$ is

$$\theta_m = \theta_0 + Ce^{-2\alpha x/\rho c_p r u_m}, \qquad \ldots \quad (51)$$

where C is an arbitrary constant which enables the formula to be adjusted to the initial temperature of the fluid. It is only over a certain limited interval, about $x_1 = \rho c_p r u_m/\alpha$ or a little more, that there is an effective transfer of heat; beyond that the temperature of the fluid is practically equal to that of the pipe (see fig. 5.47).

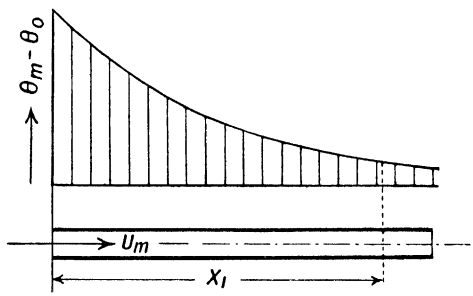


Fig. 5.47

(*b*) Actually it is found that the heat-transfer coefficient α depends very greatly on the state of flow, though the solution given above retains its practical value so long as the state of flow remains the same at different cross-sections of the pipe. A theory of the heat-transfer coefficient which gives a better account of the actual relationships is obtained as follows. If the velocity of flow is sufficiently great the surfaces of constant temperature are also nearly parallel to the boundary, hence the heat flow $q_2 + q_3$ may with sufficient accuracy be regarded as perpendicular to the boundary,* whereas the flow q_1

* The small components parallel to the boundary are negligible compared with q_1.

is parallel to the boundary. Hence in so far as we may ignore any curving of the boundary, the condition of continuity for the flow of heat may be written in the form

$$\frac{\partial q_1}{\partial x} + \frac{\partial (q_2 + q_3)}{\partial y} = 0, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

x being in the direction of flow and y at right angles to the boundary. For the sake of simplicity let us now assume that the flow is entirely parallel to the boundary, and hence that the velocity u depends only on y. Then by what we have said above, equation (52) gives

$$\rho c_p u \frac{\partial \theta}{\partial x} - \frac{\partial}{\partial y}\left\{(c_p A_q + \lambda)\frac{\partial \theta}{\partial y}\right\} = 0. \quad . \quad . \quad . \quad (53)$$

This is the differential equation for the temperature distribution in the two-dimensional problem; in three dimensions there is the additional term

$$\frac{\partial}{\partial z}\left\{(c_p A_q + \lambda)\frac{\partial \theta}{\partial z}\right\}.$$

In theory the solution of this differential equation is completely determined provided the temperature distribution $\theta(y, z)$ is known for an initial cross-section $x = 0$, and also the temperature of the boundary, or the temperature of the layer of fluid adjacent to the boundary, which is equal to it, is known for all positive values of x. In practice, however, the calculation of the solution is very difficult. It is simplest for laminar flow, where $A_q = 0$. This last problem for the pipe was solved independently by L. Grätz * in 1883 and W. Nusselt † in 1910, with the assumptions of a paraboloidal distribution of velocity, the same for all values of x, and a constant boundary temperature. Let this be θ_0 and let the fluid impinge on the boundary at $x = 0$ with a different temperature θ_1. In view of the fact that a certain initial interval is required ‡ for the parabolic velocity profile to develop (p. 166), it may be assumed that a suitable length of the pipe ahead of the cross-section $x = 0$ contains an insulating material in which the parabolic profile is developed, but in which the temperature remains equal to θ_1.

From $x = 0$ onwards, a " thermal boundary layer " is in the first instance formed as a result of contact with the highly conducting wall of the pipe, and the change in temperature gradually advances as far as the middle of the pipe. After a certain transition region the tem-

* L. Grätz, *Ann. d. Phys. u. Chem.*, New series, Vol. 18, p. 79 (1883); Vol. 25, p. 337 (1885).

† W. Nusselt, *VDI-Zeitschr.*, Vol. 54, p. 1155 (1910).

‡ This fact was not known at the time the papers referred to above were written.

perature differences begin to fall away exponentially as in equation (51) and fig. 5.47. Grätz and Nusselt made their calculations with series in which the terms were functions of y multiplied by exponential functions of x. For the middle of the pipe the largest term gives the value *

$$\theta = \theta_0 + 1{\cdot}477(\theta_1 - \theta_0)e^{-14\,63\,\lambda x/\rho c_p \bar{u} d^2}. \quad . \quad . \quad (54)$$

Here the quantity $\rho c_p \bar{u} d/\lambda = \bar{u} d/a$ is a dimensionless number of structure similar to that of Reynolds' number (the " temperature conductivity " $a = \lambda/\rho c_p$, like the kinematic viscosity ν, has the dimensions L^2/T). This number was introduced by H. Gröber † and called Péclet's number. Péclet was the first to apply Fourier's work on heat conduction in engineering; he wrote a book which was very important in its day.‡ Fig. 5.48 illustrates the temperature distribution in the above example; it is based on one of Gröber's figures, but more recent calculations of greater accuracy by K. Yamagata § have been taken into account in it.

It is of considerable practical interest to obtain the amount of heat transferred across a boundary in unit time. This is obtained in the first place from the continuity of heat flow, with the help of the relation $q_1 = c_p \rho u \theta$; Q, the amount of heat transferred across the portion of the boundary between the sections $x = x_1$ and $x = x_2$, is thus found as the difference between the total heat flow across the section $x = x_1$ and that across the section $x = x_2$.

If we ignore the variations of c_p and ρ with temperature (as we did in finding the temperature distribution), we have $Q = (Q_1 - Q_2) = c_p \rho \iint u(\theta_1 - \theta_2) da$. In the second place the same quantity of heat can also be calculated directly from the flow of heat past the boundary, $q_3 = \lambda(\partial\theta/\partial n)_{\text{wall}}$; that is, $Q = \iint \lambda(\partial\theta/\partial n)_{\text{wall}} da'$, where a' is the surface element of the portion of the boundary. This relation holds even if the flow is turbulent, since the exchange coefficient A_q is zero in the immediate neighbourhood of the boundary.

To obtain convenient quantities for a rough estimate it is usual to give the quantity of heat transferred across unit area of the boundary per unit time (second or hour) when the temperature difference is 1° C. This quantity is none other than the heat-transfer coefficient α

* Nusselt's formula contains three such terms with increasing numbers in the index; cf. e.g. Gröber-Erk, equation (79), p. 179.

† In the first edition of his book (1921) he gives the reciprocal of this quantity. Subsequently, however, it was standardized in the form given above, corresponding to Reynolds' number.

‡ There is a German translation of the 3rd edition by C. Hartmann in 3 volumes (Leipzig, 1860).

§ *Mem. Fac. Engng. Kyushu Imp. Univ.*, Vol. 8, p. 365 (1936–40).

mentioned above. The temperature difference of 1° C. still requires definition. We could take the difference between the centre of the pipe and the wall, or that between the mean temperature of the fluid over the cross-section, $\bar{\theta} = \frac{1}{a}\int \theta \, da$, and that of the wall. In view of the calculation of the transfer of heat by means of the heat flow across

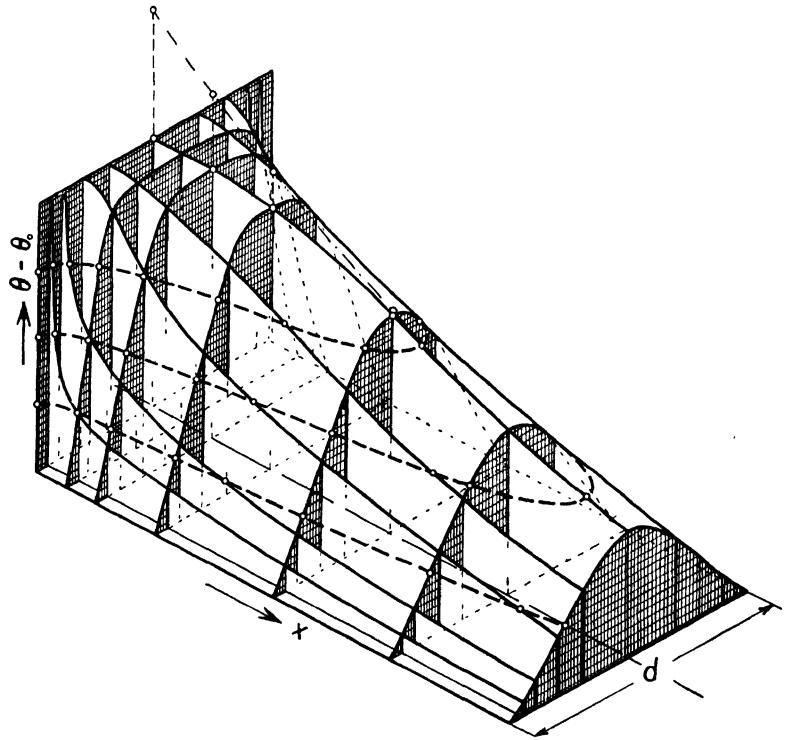


Fig. 5.48.—Temperature distribution near the entrance to a pipe, after Grätz and Nusselt. The temperature profiles drawn correspond to the values $ax/\bar{u}d^2 = 0{\cdot}005$, 0·01, 0·02, 0·04, 0·06, 0·08. The dotted line represents formula (54).

sections 1 and 2 (see above), however, it is convenient to take that mean value which in conjunction with $c_p \rho V \theta_m$, where $V = a\bar{u}$ is the volume transferred per unit time, gives the heat flow, i.e. to put $\theta_m = \frac{1}{a}\iint u\theta \, da$. This mean temperature can also be measured by the thermometer if the fluid is sent through some form of mixing apparatus, any further heat exchange beyond the cross-section in question being prevented. Then $\alpha = q_3/(\theta_m - \theta_0)$ is the heat-transfer coefficient.

For a given temperature profile we can put

$$(\partial\theta/\partial n)_{\text{wall}} = \text{const.} \times (\theta_m - \theta_0)/d,$$

whence, with the above formula for q_3, we have $\alpha = \text{const.} \times \lambda/d$. Gröber and Erk propose that this constant should be called Nusselt's number, denoted by $\mathcal{N}_u$, in honour of W. Nusselt, who was also responsible for fundamental work in the experimental investigation of heat transfer, that is,

$$\mathcal{N}_u = \alpha d/\lambda. \qquad \ldots\ldots\ldots \quad (55)$$

In the above example $\mathcal{N}_u = \text{const.} = 3{\cdot}66$, apart from an initial interval up to about $x = 0{\cdot}05 d^2\bar{u}/a$, in which $\mathcal{N}_u$ is greater.*

(*c*) *Heat Transference when the Flow is Turbulent.*—This case is more important than that of laminar flow in engineering applications. Since A_q now comes in as well as λ, the relationships are considerably more complicated, since A_q varies with the distance from the wall of the pipe; there it is zero and, the higher Reynolds' number is, the more rapidly does it increase towards the centre of the pipe.† In most liquids other than molten metals the thermal conductivity λ is fairly small, so that even quite close to the wall A_q greatly preponderates over λ. Hence, as has already been mentioned, in the "core" or main flow there is a marked exchange by mixing, that is, there are comparatively small temperature differences, whereas in a thin border zone in which there is no exchange there are very marked temperature differences, except in such cases as that of mercury, for which the thermal conductivity is large. It is clear, therefore, that the phenomena in the border zone have a decisive influence on the amount of heat passing to and fro between the wall of the pipe and the liquid.

The exact theory of the phenomena based on equation (53), p. 400, leads to very complicated calculations, which, moreover, necessitate very accurate knowledge of the velocity distribution in the immediate neighbourhood of the wall of the pipe. Yet a good general view of the main relationships may be obtained by assuming that there are two regions: a border zone, in which the flow is laminar, so that conduction alone is involved, and a core, in which the thermal conductivity λ can be neglected in comparison with A_q. Let the velocity at the boundary between the two regions be u' and the temperature θ'. For the sake of simplicity we shall take the temperature of the wall as zero; this merely implies a shift of the origin of the temperature scale.

The subsequent argument can be simplified still further by assuming a hypothetical transfer of heat to the core of the liquid, say by an

* Gröber-Erk, p. 183.

† The formulæ on p. 125 give the approximation $A_\tau = \kappa\rho v_* y$.

electric current, such that the fall of the temperature difference $(\theta_1 - \theta_0)$ in the x-direction in fig. 5.47 is exactly prevented. If this transfer of heat is assumed to be the same for every volume element of the " core ", the flow of heat q_2, at right angles to the axis, will everywhere be proportional to r, the distance from the axis,* that is, $q_2 = (q_2)_0 r/r_0$, which corresponds to the distribution of the apparent shearing stress $\tau = \tau_0 r/r_0$, where $r = r_0 - y$, and r_0, $(q_2)_0$, τ_0 are values at the wall of the pipe, which owing to the thinness of the boundary layer are sufficiently close to the values at the boundary of the core to be substituted for them. Now $q_2 = c_p A_q d\theta/dy$ and $\tau = A_\tau du/dy$, and let us make the assumption that A_q and A_τ are in a fixed ratio. Then the above relationships for q_2 and τ lead to similarity between the temperature profile of the core and the velocity profile there, which makes the subsequent calculations very much easier.† The similarity condition, which holds for the mean values also,‡ may be written in the form

$$c_p A_q(\bar{\theta} - \theta')/(q_2)_0 = A_\tau(\bar{u} - u')/\tau_0,$$

from which, since $A_q = mA_\tau$, for $(q_2)_0$ at the boundary of the core (q_3 in the boundary layer) we have

$$(q_2)_0 = mc_p\tau_0 \frac{\bar{\theta} - \theta'}{\bar{u} - u'}. \qquad \ldots\ldots \quad (a)$$

For the laminar boundary layer of thickness δ we have, on the one hand, $q_3 = \lambda\theta'/\delta$, and on the other hand, $\tau_0 = \mu u'/\delta$, whence, by eliminating δ, we have

$$q_3 = \frac{\lambda}{\mu}\frac{\tau_0\theta'}{u'}. \qquad \ldots\ldots\ldots \quad (b)$$

Equating the right-hand sides of (a) and (b), we have an equation for θ',

$$\theta' = \bar{\theta}\, u'/\{\sigma\bar{u} + (1 - \sigma)u'\}, \qquad \ldots\ldots \quad (c)$$

where $\lambda/m\mu c_p = a/m\nu$ has been put equal to σ. Substituting from (c) in (b), we now have

$$q_3 = \frac{\lambda\bar{\theta}\tau_0}{\mu\{\sigma\bar{u} + (1 - \sigma)u'\}}. \qquad \ldots\ldots \quad (56)$$

* If q' is the heat transferred to unit volume in unit time, we have $\pi r^2 l q' = 2\pi r l q_2$.

† Without the use of this device the two profiles would not be similar, but the discrepancies in the actual problem (with the temperature difference falling in the x-direction) are not large. If high accuracy is required, of course, the difference must be taken into account (see below).

‡ Of course the mean values must be formed in the same way, i.e. corresponding to $\bar{u} = \frac{1}{a}\int u\, da$ we have the mean value already mentioned, $\frac{1}{a}\int \theta\, da = \bar{\theta}$, which in fact is more suitable in the problem in question here, in which $dq_1/dx = 0$.

The problem is thus reduced to a problem in hydrodynamics. In Chapter III, § 11, the resistance coefficient λ_1 corresponding to τ_0 is given as a function of Reynolds' number $\mathscr{R} = qd/\nu$ by the approximations (46), (47), (47*a*) and (50), pp. 390–4. As we have already used λ in this section with another meaning we shall denote the resistance coefficient by ζ; and the former velocity (q) is identical with the mean value $\bar{u}$ used here. By comparison with p. 164 we now have $\tau_0 = \frac{1}{8}\zeta\rho\bar{u}^2$. If we introduce Nusselt's number $\mathscr{N}_u = q_3 d/\lambda\bar{\theta}$, we immediately obtain

$$\mathscr{N}_u = \tfrac{1}{8}\zeta \frac{\mathscr{R}}{\sigma + (1-\sigma)u'/\bar{u}}. \quad . \quad . \quad . \quad . \quad (57)$$

Péclet's number $\bar{u}d/a = \mathscr{P}_e$ may also be introduced instead of Reynolds' number $\bar{u}d/\nu$, using the relation

$$\mathscr{R} = \frac{a}{\nu}\mathscr{P}_e = m\sigma\mathscr{P}_e;$$

then

$$\mathscr{N}_u = \tfrac{1}{8}\zeta \frac{m\mathscr{P}_e}{1 + (1/\sigma - 1)u'/\bar{u}}. \quad . \quad . \quad . \quad (57a)$$

In this equation, however, the true value of $u'/\bar{u}$ is still unknown; it is a function of Reynolds' number, which is related to the universal law of velocity distribution.* Since, however, there are difficulties in determining the exact position of the discontinuity with which we have to replace the continuous transition from laminar flow to turbulent flow, the better way is to determine $u'/\bar{u}$, from equation (54), p. 401, by heat measurements when σ is small.†[88]

Formula (57) becomes particularly simple when $\sigma = 1$, which is approximately true for gases. Then $\mathscr{N}_u = \frac{1}{8}\zeta\mathscr{R}$, and using the power formula of equation (46), p. 163, we have $\mathscr{N}_u = 0{\cdot}04\mathscr{R}^{3/4}$ approximately. Experiments by Nusselt ‡ with air at pressures of 1 to 16 atmospheres, and also with carbon dioxide and coal gas, gave similar power formulæ, with powers about $\mathscr{R}^{0{\cdot}786}$. According to ten Bosch,§ working with Nusselt's results, the numerical coefficient is 0·0255. For Reynolds' number equal to 10^4 or 10^5, this gives $\mathscr{N}_u = 3{\cdot}55$ or 21·7, whereas the formula above gives 4·0 or 22·5. When we consider that owing to the assumption of heat sources our argument gives a too full-shaped temperature profile, and hence too great a transfer of heat in the neighbourhood of the wall, the agreement is closer than we

* Cf. fig. 3.15, p. 128; u' is proportional to v_*, hence, using the power formula (26), p. 127, we have $u'/\bar{u}$ proportional to $\mathscr{R}^{-1/8}$.

† L. Prandtl, *Phys. Zeitschr.*, Vol. 11, p. 1072 (1910).

‡ *Habilitationsschrift*, 1909 (*VDI-Forschungsheft*, No. 89 (1910)); a summary is given in *VDI-Zeitschr.*, Vol. 53, pp. 1750, 1808 (1909).

§ *Wärmeübertragung*, equation (63), p. 127.

could reasonably expect. For more viscous media it is markedly less good.

(*d*) Since in the more viscous liquids the viscosity varies very greatly with the temperature (in the case of lubricating oil at 20° C. it is actually reduced to half by a rise in temperature of 8° to 10°), it is a serious matter, even when the temperature differences are very moderate, to decide which of the various viscosities is to be used in finding the value of Reynolds' number to be inserted in the formula. As it mainly turns on the fall of temperature in the boundary layer, we must choose a mean temperature in the latter, say $\theta = \frac{1}{2}(\theta_0 + \theta')$. From this it is easy to see that with the same quantity of fluid flowing through the pipe, and the same initial temperature, the heat-transfer coefficient may differ very widely according as the fluid is to be heated or cooled. (It is greater in the former case owing to the thinner boundary layer; the two velocity profiles also differ, the former being the fuller.)

For numerical data for the practical calculations from experiment the reader is referred to ten Bosch's book with its numerous formulæ, tables and nomograms. Much work has been directed towards refinement of the theory by ten Bosch and others; see the bibliography on p. 273 of his book.

A very complete theoretical discussion of the problem of heat transfer in a pipe has recently been given by H. Reichardt.* It is based on the universal velocity profile of turbulent flow, which Reichardt himself has measured right up to the immediate neighbourhood of the wall. The region of flow is divided into three zones instead of the two considered hitherto: a purely laminar boundary zone; an intermediate zone in which molecular viscosity and thermal conductivity have an effect as well as exchange phenomena; and a purely turbulent core, in which the molecular viscosity and thermal conductivity can be neglected in comparison with exchange phenomena. The values for the particular material, thermal conductivity and specific heat as well as viscosity, are determined in each zone separately. This theory is of course very complicated, but it has thereby been found possible to include all known experimental results from the least viscosity to the greatest. A subsidiary result of some importance refers to the number $m = A_q/A_\tau$ introduced above. The calculations gave the best agreement with experiment for the value $m = 1{\cdot}1$. How this result is to be brought into agreement with the value $m = 1{\cdot}4$ or $1{\cdot}5$ obtained from simultaneous measurements of velocity and temperature profiles for the case of a plate (cf. p. 120) meanwhile remains an open question.

(*e*) *Some Historical Remarks.*—Osborne Reynolds seems to have been the first who clearly recognized the part played in heat transfer by the velocity of

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 20, p. 297 (1940).

flow. In a short paper on the effective heating area of locomotive boilers * he refers to the considerable confusion which had hitherto prevailed in papers on heat transfer, in which conductivity had alone been taken into account, and hence the view had arisen that the velocity of flow had no effect on the amount of heat transferred in unit time. He points out that nevertheless the cooling effect of wind as compared with air at rest is not to be overlooked, and that there must therefore be something wrong with the theories as hitherto developed. He goes on to say that the main phenomenon in heat transfer is that particles move from the interior of the fluid up to the boundary and bring their heat with them. The resistance to flow arises in the same way as a result of the particles bringing their velocities from the interior of the fluid up to the boundary, where they give rise to frictional forces. The latter are experimentally known to be proportional to ρw^2 (the momentum of an individual particle being ρw and the other w-factor denoting the quantity of fluid carried up to the boundary in unit time). The heat transfer is accordingly to be taken as proportional to $w\theta$, where θ is the temperature difference. Reynolds takes the simultaneous effect of conduction into account by means of a constant term added on, and that of viscosity by a term in the resistance proportional to w, i.e. $q = (a + bw)\theta$ and $\tau = (a' + b'w)\rho w$, where it is assumed that $a/a' = b/b'$.

Subsequent measurements showed that w^2 had to be replaced by about $w^{1.75}$ in the resistance, and w in the heat transfer by about $w^{0.75}$ to $w^{0.8}$. W. Nusselt, who obtained the last result by very careful experiments, found from dimensional considerations that the law for the transfer of heat from a gas (or a liquid) to the wall of the pipe enclosing it, if capable of being represented by a power formula at all, must have the following form:

$$q = \text{const.} \times \frac{\lambda\theta}{d}\left(\frac{wd}{\nu}\right)^{n_1}\left(\frac{\nu}{a}\right)^{n_2},$$

where $a = \lambda/c_p\rho$ and $\nu = \mu/\rho$. Experiments with air gave $n_1 = n_2$ approximately; this index was less by almost exactly unity than that in Blasius' resistance formula $(p_1 - p_2)/l \sim w^{1.75}$.

In the attempt to clear up the connexion the author of this book, who was unaware of Reynolds' little paper of long ago, rediscovered the same train of thought, but in a more accurate mathematical form.† It turned out that the simple relationship put forward by Reynolds was valid only if ν/a was equal to unity [according to present knowledge it would have been more accurate to put $(A_q/A_\tau)(\nu/a) = m\nu/a = 1/\sigma = 1$; cf. equation (57), p. 405]. The difference between the thermal conductivity and the transfer of momentum in the laminar layer, compared with the "apparent conduction" in the turbulent region for $m\nu/a \neq 1$, then led to the formulation of equation (57), with the value $m = 1$, of course.

For laminar flow where the difference of pressure vanishes, the temperature field is exactly proportional‡ to that of the velocity component u when $\nu/a = 1$, and only then. Reynolds' argument and that of the author are referred to in the literature as the "momentum theory of heat transfer".§

* *Proc. Lit. and Phil. Soc. Manchester*, Vol. 14 (1874–75); *Papers on Mechanical and Physical Subjects*, Vol. 1, p. 81 (1900).

† *Phys. Zeitschr.*, Vol. 11, p. 1072 (1910). The account in subsection (c) is partly based on this account.

‡ The number $\nu/a = \mu c_p/\lambda$, which, as we have mentioned, occurs in Nusselt's work of 1909, was later called Prandtl's number. The author was unwilling to encourage this piece of historical inaccuracy and therefore preferred the equally brief expression ν/a.

§ Gröber and Erk, *Die Grundgesetze der Wärmeübertragung*, p. 195 (Berlin, 1933).

(*f*) *Heat Transfer where the Body is surrounded by Moving Fluid.*— The simplest case, that of a thin plate in two-dimensional laminar flow in its own plane, has been worked out numerically by E. Pohlhausen.* In the case $\nu/a = 1$, as we have already mentioned, the temperature field is exactly the same as the field of the longitudinal component of the velocity; that is, the lines $\theta = \text{const.}$ are identical with the lines $u = \text{const.}$ and $\theta = \theta_0 + (\theta_1 - \theta_0)u/u_1$, where θ_0 is the temperature of the plate, θ_1 and u_1 the temperature and velocity of the undisturbed fluid, and θ and u the values anywhere in the boundary layer (cf. pp. 106–8). By the "momentum theory" the amount of heat Q transferred on *one* side of the plate is $\dfrac{\lambda R}{\mu u_1}(\theta_1 - \theta_0)$, where $R = 0{\cdot}664b\sqrt{(\mu\rho u_1{}^3 l)}$ is the frictional resistance † on one side of the plate; R is the integral of the shearing stresses τ_0; see also equation (*b*), p. 404. This relation, as is easily seen, can also be written in the dimensionless form

$$\mathcal{N}_u = \frac{\alpha l}{\lambda} = 0{\cdot}664\sqrt{\mathcal{R}}. \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The transfer of heat is greatest at the front edge and diminishes towards the rear in proportion to $x^{-1/2}$, corresponding to the increase in the thickness of the boundary layer. If $\nu/a \neq 1$, the temperature distribution is no longer similar to the velocity distribution. If $\nu/a > 1$ the "temperature boundary layer" is thinner than the frictional boundary layer; if $\nu/a < 1$ it is thicker. The profiles were calculated by Pohlhausen for some values which occur in practice, and also the amounts of heat transferred; Pohlhausen remarks that the latter are obtained to a good degree of accuracy by inserting the factor $\sqrt[3]{(\nu/a)}$ in the formula above.‡

The difficulties, mentioned at the beginning of (*d*), p. 406, in the theoretical calculation of heat transfer in oils, arising from the natural variation in the viscosity of oils with temperature, have been solved in a numerically satisfactory way for the flow round a plate in a new paper by H. Schuh.§ The calculation is based on an iterative method for solving an integral equation, developed by N. A. V. Piercy and J. H. Preston.‖ Application can be made to problems of temperature boundary layers arising at high velocities as a result of frictional heating, and also to diffusion phenomena at high concentrations (see under (*g*), p. 409).

* *Zeitschr. f. angew. Math. u. Mech.*, Vol. 1, p. 115 (1921).

† The number in equation (13), p. 108, is 1·328; see also formula (56), p. 193. Here it is halved.

‡ This is true for other forms also; see Nils Frössling, *Lunds Universitets Arsskrift N.F. Avd.* 2, Vol. 36, No. 4 (Lund and Leipzig, 1940).

§ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 25/27, p. 54 (1947).

‖ *Phil. Mag.* (7), Vol. 21, p. 995 (1936).

For values of Reynolds' number $\mathscr{R} = u_1 l/\nu$ above 10^5, turbulent flow is possible and this increases the amount of heat transferred. Calculations would have to be based on equation (56). Factors of about $\mathscr{R}^{0.8}$ may be expected to occur; for details, see e.g. ten Bosch, *Wärmeübertragung*, p. 139 *et seq.*

The case of the body surrounded by fluid which is of greatest practical importance is that of the circular cylinder about which fluid is moving at right angles to the axis of the cylinder; the latter occurs in practice in the form of an electrically heated wire, or a pipe through which fluid of a different temperature is flowing. For a wide range of Reynolds' number ud/ν from 30 to 5000 the transfer of heat is predominantly in the boundary layer, and accordingly obeys a law corresponding to equation (58). According to experiments by R. Hilpert * in air, $\mathscr{N}_u = \alpha d/\lambda$ is approximately equal to $0{\cdot}63\mathscr{R}^{0\,46}$ in the region referred to above. For lower values of Reynolds' number, down to about 4 (e.g. hot wires), the index is about 0·38; below that, about 0·33. For values of Reynolds' number above 5000 (heating systems and the like) the index rises gradually to 0·8 (for $\mathscr{R}$ between 4×10^4 and 4×10^5), which indicates that here the ever-increasing eddies and the turbulent re-attachment of the flow cause even the back of the pipe to affect the heat transfer. For moderate values of Reynolds' number turbulence in the oncoming fluid markedly increases the heat transfer. In a pipe which had two rows of similar pipes in front of it, H. Reiher † observed that the heat transfer was $1\frac{1}{2}$ times as great. In this connexion the experiments of E. Schmidt and K. Wenner ‡ on the distribution of heat transfer over the circumference of cylinders §(89) are also very instructive.

Note.—The "momentum theory" which has been successfully applied to the problem of heat transfer in the case of the plate is associated with the relation between heat transfer and frictional resistance. Unless the body is very slender, however, a very important part of the resistance is pressure resistance (cf. p. 177). The latter is obviously not directly related in any way to the heat transfer. Indirectly it may lead to an increase in the heat transfer in that it increases the eddying near the parts of the surface behind the point of separation of the flow. The same is true for rough surfaces, which also give rise to pressure resistance. Here the heat transfer is noticeably higher, as compared with smooth surfaces, when the resistance is higher.|| A systematic investigation of the matter, however, has not yet been made.

(*g*) *Diffusion.*—Laws very similar to those for the propagation of heat hold for the diffusion of a chemical substance into a solvent or

* *Forschung*, Vol. 4, p. 215 (1933). † *VDI-Forschungsheft* No. 269 (1925).

‡ *Forschung*, Vol. 12, p. 65 (1941).

§ See also the theoretical account by E. Eckert, *VDI-Forschungsheft* No. 416 (1942).

|| That is, if the rough surface does not happen to be hydraulically smooth; cf. p. 129.

of one kind of gas into another.* In the limiting case of very great dilution, where the volume of the diffusing substance itself is trifling compared to that of the solvent, in fact, the form of the differential equation for the spread of the substance is exactly the same as for the propagation of heat. Instead of the temperature θ we have the concentration C, instead of the thermal conductivity a we have the diffusion coefficient D. If C is expressed as a pure number, e.g. as the percentage of foreign substance, D has the dimensions L^2/T, i.e. the same dimensions as a and ν. For gases D is approximately equal to a (for the diffusion of water vapour in air D is equal to about $0{\cdot}86\,a_{\text{air}}$). The analogy of diffusion into a moving solvent, which of course is accurate only if $D = a$, serves on the one hand to provide formulæ relating to the chemical reaction in the moving fluid which have been derived from laws of heat transfer already obtained otherwise; or, on the other hand, enables conclusions to be drawn on the heat transfer problem from observations of the corresponding diffusion phenomena.

An example of the former type is the investigation of combustion of layers of coal on the bars of a firebox by W. Nusselt.† As a result of the draught in the chimney, or that produced by a special blower, the air for combustion is forced through the interstices between the pieces of coal. The oxygen of the air diffuses on to the glowing surface of the coal, which absorbs it and forms carbon dioxide; this then diffuses out into the nitrogen which is left. The flow carries the carbon dioxide away along with the heat and draws in more oxygen. These three parts of the process (diminution of the oxygen content of the air, increase in its CO_2 content, and increase in the temperature of the gas) to a first approximation obey the same law, at least so long as the diffusion constants of the two phenomena (diffusion of O_2 into N_2 and of CO_2 into N_2) are taken as equal to the thermal conductivity of N_2, which is approximately true.‡ In his calculations Nusselt simplified matters by assuming that smooth straight tubes passed through the hot mass of coal and applying his formula for heat transfer in pipes to these.§ Nusselt also applied the same considerations to the gasification of coal. If there is a lack of oxygen, CO_2 unites with a further atom of carbon at the glowing surface of the coal to form $2\,CO$, and similarly water vapour (H_2O) unites with carbon to form $H_2 + CO$. Here again there are processes of diffusion to and from the surface of the coal, associated in this case with a cooling effect.

Similar relationships hold for the evaporation of water by a stream of air. If no heat is transferred to the water the latent heat of evaporation must be abstracted from the air; the surface of the water or the surface of a porous body saturated with water then assumes a temperature lower than that of the air (cf. the wet-and-dry-bulb thermometer), heat is transferred from the air to the wet

* See e.g. the discussion by M. Jakob in *Der Chemie-Ingenieur*, Vol. I, Part 1, pp. 251–60.

† *VDI-Zeitschr.*, Vol. 60, p. 102 (1916).

‡ As a rough approximation the ratio a/D for a gas of molecular weight M_1 diffusing in small quantities into another gas of molecular weight M_0 is actually $\sqrt{(M_1/M_0)}$. The above condition is very well satisfied for the diffusion of O_2 and CO into N_2, rather less so for the diffusion of H_2O and CO_2 into air.

§ A paper by H. G. Kayser, *Forschung*, Vol. 6, p. 89 (1935), where further references are given, deals with the aerodynamics of actual layers of fuel in furnaces, but does not discuss the chemical phenomena.

surface, and from the latter a corresponding amount of water vapour diffuses into the air and is carried away in the same way as heat would be. For the details the reader may consult e.g. a paper of basic importance by E. Schmidt.*

The first work in which diffusion experiments were applied to the study of heat-transfer phenomena was done by H. Thoma,† who wished to find out the most effective way of arranging the rows of tubes in a water-tube boiler. He led a stream of air mixed with a little ammonia gas through a model in which the system of water-tubes was imitated by tubes of blotting-paper soaked in phosphoric acid, which eagerly absorbs ammonia. After an experiment in which a known amount of ammonia was used, the amount of ammonium phosphate could be found by titration, so that the "heat-transfer coefficient" at any point could be obtained. By soaking the blotting-paper tubes with hydrochloric acid, whose vapour along with the ammonia gave dense white clouds, Thoma obtained striking illustrations of the flow. In these the regions where diffusion and mixing took place stood out clearly from the clear zones where no mixing took place, i.e. where no contribution to the heat transfer would be made either.‡

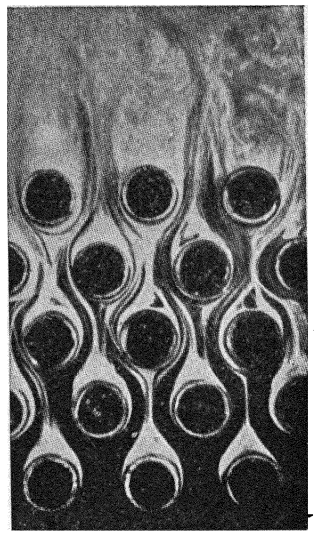


Fig. 5.49.—Regions of mixing made visible by Thoma's method

Fig. 5.49, taken from a paper by W. Lohrisch,§ gives a good idea of the results to be obtained from this method of making flow visible. This paper also gives the results of very accurate measurements by the phosphoric-acid method, which gave very satisfactory agreement in cases where actual heat-transfer measurements had also been made.

E. Schmidt pointed out in the paper of 1929 mentioned above that good model experiments on heat transfer could also be made with evaporation experiments. See also the detailed account by L. Kettenacker.‖

These methods have been critically discussed by W. Nusselt,¶ as well as by E. Schmidt, in the paper mentioned above.

The similarity between two different diffusion phenomena has been utilized by P. Rosin and H. G. Kayser, who used the dissolving of cakes of salt in a stream of water as a model experiment for the gradual burning away of similar pieces of coal in a stream of air.**

* *Gesundh.-Ingenieur*, Vol. 52, p. 525 (1929).

† See his book *Hochleistungskessel*, p. 38 *et seq.* (Berlin, 1921).

‡ Loc. cit., p. 19 *et seq.*

§ *VDI-Forschungsheft* No. 322, p. 46 (1929); reproduced by permission of the *VDI-Verlag*, Berlin.

‖ *Techn. Mech. u. Thermodyn.*, Vol. 1, p. 439 (1930).

¶ *Zeitschr. f. angew. Math. u. Mech.*, Vol. 10, p. 105 (1930).

** *VDI-Zeitschr.*, Vol. 75, p. 849 (1931).

16. Natural Flow produced by Differences in Density resulting from Heat Transfer.

(*a*) *Introduction.*—When the density differences associated with temperature differences as a result of thermal expansion are the only causes of motion, we speak of *natural flow*, in contrast to *forced flow* brought about by outside causes. If the natural flow is not subject to constraining boundaries, we refer to it as *free flow*. In the latter the pressures are generally almost exactly the same as would arise in the fluid at rest as a result of gravity.* Hence it is again convenient to make use of the device, explained in § 11(*a*) (p. 368), of removing the gravity pressure of the undisturbed medium at the very start. By what we have just said, the residual pressure due to the flow is then practically zero. As a cause of the motion we then have only the difference between weight and Archimedean upthrust arising from gravity, $g(\rho - \rho_1)$ per unit volume. The corresponding acceleration will then be $g(\rho - \rho_1)/\rho$ vertically downwards. For this we may substitute $g(\rho - \rho_1)/\rho_1$, since as usual we are working to a first order only; ρ_1 is the undisturbed density, which may usually be regarded as constant. For a *gas* where the absolute temperature is T_1 at undisturbed points and $(T_1 + \theta)$ at a disturbed point, we have $\rho = \rho_1 T_1/(T_1 + \theta)$, which we may write approximately † as $\rho_1(T_1 - \theta)/T_1$. Then the resultant acceleration of the gravity field is $+g\theta/T_1$ if we take the positive direction vertically upwards. Let the expansion coefficient for a *liquid* be β; then the acceleration is $g\beta\theta$ (upwards if θ is positive, by what we have said above).

As an introduction we shall consider a particularly simple example. Suppose that within a tube of diameter d, open above and below, there is a temperature higher by θ than outside. By the above, the force per unit volume will be $g\rho\beta\theta$. In the laminar case (cf. p. 99) this is balanced by a viscous resistance ‡ proportional to $\mu\bar{w}/d^2$. Since $\mu = \rho\nu$, this leads to $\bar{w} \sim g\beta\theta d^2/\nu$, so that the corresponding expression for Reynolds' number is

$$\bar{w}d/\nu = \text{const.} \times g\beta\theta d^3/\nu^2.$$

If instead we assume that there is no friction, at the entrance to the tube there is a fall of pressure $\frac{1}{2}\rho w^2 = g\rho\beta\theta \,.\, l$, whence $w = \sqrt{(2g\beta\theta l)}$. If we work out a formal expression for Reynolds' number in this case, we have

$$\frac{wd}{\nu} = \sqrt{\left(\frac{2g\beta\theta d^2 l}{\nu^2}\right)} = \sqrt{\left(\frac{2g\beta\theta d^3}{\nu^2}\,\frac{l}{d}\right)}.$$

* This is particularly true if the flow is of boundary-layer type; see R. Hermann, *Phys. Zeitschr.*, Vol. 34, p. 211 (1933).

† Divide out and break off the series after the second term.

‡ The former quantity $(p_1 - p_2)/l$; if $\bar{w}$ is the mean velocity, this becomes $8\mu\bar{w}/r^2 = 32\mu\bar{w}/d^2$.

From the two expressions we see that $g\beta\theta d^3/\nu^2$ is a dimensionless number which plays an important part in this type of flow. It has been called the Grashof number ($\mathscr{G}$) in honour of F. Grashof, who developed the theory of the draught in a chimney and other valuable though now somewhat antiquated work on heat transfer,* that is,

$$\mathscr{G} = \frac{g\beta\theta d^3}{\nu^2}; \qquad \ldots\ldots\ldots \quad (59)$$

where in the case of gases β can be put equal to $1/T_m$ where T_m is the mean absolute temperature at which the phenomenon takes place.

(*b*) The Grashof number keeps turning up in all the laws of natural flow in which a *temperature difference* is given. An exception is formed by the flow in which the *quantity of heat given up per unit time* is given instead, for example, in the stream of air rising from a candle flame or other source of heat. These phenomena are amenable to calculation for distances from the heat source such that the difference in temperature between the stream of air and its surroundings is small compared with the absolute temperature at the point in question. The calculation for the turbulent case is given in a paper by W. Schmidt.† His calculations are based on Tollmien's calculations for the spread of a jet,‡ and like these give a breadth of the turbulent flow increasing proportionally to the distance, i.e. here to the height z above the source of heat. In the case symmetrical about an axis the temperature and velocity distributions are almost the same as the velocity distribution of fig. 3.10, p. 122. The velocity which results from the upward thrust of the heated air is greatest at the flame, and owing to the high degree of mixing decreases in proportion to $z^{-1/3}$; the temperature, on the other hand, decreases in proportion to $z^{-5/3}$. Good confirmation of this has been obtained in experiments by W. Schmidt.

The laminar case does not seem to have been investigated hitherto. Approximate estimates by the author gave a velocity independent of z and a temperature proportional to $1/z$ at the central line of the jet. Here the breadth of the jet increases in proportion to $\sqrt{z}$.

These estimates, which can be deduced as indicated on p. 122 for the spreading out of a jet, are based on the assumption that the quantity of heat Q transported by the jet per second is proportional to $c_p\rho\theta_1 w_1 b^2$ and must have the same value for all heights z (w_1 is the velocity at the centre of the jet, θ_1 the excess

* *Theoretische Maschinenlehre*, Vol. I, pp. 935–70 (Leipzig, 1875). Grashof, however, did not use the number which bears his name.

† *Diss.*, Göttingen; summary in *Zeitschr. f. angew. Math. u. Mech.*, Vol. 21, pp. 265, 351 (1941).

‡ Cf. footnote 2, p. 122.

temperature there, and b the breadth of the jet). For the central stream-tube we have

$$\rho w_1 \frac{dw_1}{dz} = g\rho\beta\theta_1 + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}.$$

If a power law holds, we may expect the relation between the terms to be the same for all values of z, so that we have

$$\rho {w_1}^2/z \sim g\rho\beta\theta_1 \sim \tau/b.$$

If we take τ as proportional to $\rho {w_1}^2$ in the turbulent case, and to $\mu w_1/b$ in the laminar case, we have all the necessary relations for the estimate in question.

As a supplement to the above we may mention that for a given z, $w_1 \sim Q^{1/3}$ and $\theta_1 \sim Q^{2/3}$ in the turbulent case; in the laminar case $w_1 \sim Q^{1/2}$ and $\theta_1 \sim Q$.

The corresponding two-dimensional problem (a flow of air rising from a long horizontal source of heat) differs from the symmetrical three-dimensional case just discussed in that in the calculation we have to take Q_1, the quantity of heat given up per unit length in unit time, proportional to $c_p \rho \theta_1 w_1 b$, which gives somewhat different powers. The turbulent case has been discussed by W. Schmidt (loc. cit.). Here we again have $b \sim z$, but $w_1 = \text{const.}$ and $\theta_1 \sim z^{-1}$.

(*c*) *Natural Flow near a Vertical Heated Wall.*—Here a layer of heated air is formed at the wall and moves upwards. In general the flow is laminar, since the velocities are not very large. Provided the temperature differences θ are small, the powers to be expected are found by an approximate estimate very similar to that in subsection (*b*). Let w_1 be the maximum velocity in a cross-section $z = \text{const.}$, θ_1 the excess temperature of the wall, assumed to be constant, δ the thickness of the layer which is flowing upwards. In the equation

$$w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = g\beta\theta + \nu \frac{\partial^2 w}{\partial y^2},$$

we have $\partial w/\partial y = 0$ and $\partial w/\partial z = dw_1/dz$ for $w = w_1$, and θ there is to be taken as proportional to θ_1. Hence if we are to obtain a power formula, we must have

$${w_1}^2/z \sim g\beta\theta_1 \sim \nu w_1/\delta^2,$$

from which $\delta \sim \sqrt{(\nu z/w_1)}$ and $w_1 \sim \sqrt{(g\beta\theta_1 z)}$, i.e. $\delta \sim (\nu^2 z/g\beta\theta_1)^{1/4}$. According to this approximation Q_1, the quantity of heat carried away from unit breadth of a plate of height $z = h$, is *

$$Q_1 \sim \rho c_p \theta_1 \,.\, w_1 \delta \sim (\nu^2 g\beta {\theta_1}^5 h^3)^{1/4}.$$

In this argument the differential equation for the temperature distribution (in the boundary-layer form $w\,\partial\theta/\partial z + v\,\partial\theta/\partial y = a\,\partial^2\theta/\partial y^2$)

* It is a remarkable fact that these formulæ, with h for z, actually appear implicitly in an old paper by the physicist L. Lorenz of Copenhagen [*Wied. Ann. d. Phys. u. Chem.*, Vol. 13, p. 582 (1881)]. This paper is thus at once the first paper on free heat convection and the first paper on boundary layers. The relation between w_1, δ and z, however, is not given in it.

has not been taken into account at all. In the exact solution of the problem, of course, the temperature distribution has an important effect on the velocity distribution, as can be seen immediately from the structure of the equations; the ratio ν/a is accordingly important for *both* distributions. We conclude that in our approximation the coefficients entering into the equations for w_1, δ and Q_1 are still unknown functions of ν/a.

If we use the above formulæ to construct the dimensionless numbers δ/z, $w_1\delta/\nu$ and $Q_1/\lambda\theta_1 = (\mathscr{N}_u)_h$, we find that $\delta/z \sim \mathscr{G}_z^{-1/4}$, $w_1 z/\nu \sim \mathscr{G}_z^{1/2}$ and $(\mathscr{N}_u)_h \sim (\nu/a)\mathscr{G}_h^{1/4}$. If we start, not as above from heat transfer through the flow, but from heat conduction across the flow, i.e. if we put $Q_1 \sim h\lambda\theta_1/\delta$, we obtain $(\mathscr{N}_u)_h \sim h/\delta \sim \mathscr{G}_h^{1/4}$. This makes it particularly clear that from primitive arguments like that above no safe conclusions can be drawn about the way in which the dimensionless number ν/a enters into the formulæ.

For $\nu/a = 0{\cdot}733$ (the case of air) E. Pohlhausen * has given an exact solution of the differential equations which result from the introduction of the powers of the fundamental quantities given above. These gave $(\mathscr{N}_u)_h = 0{\cdot}478\,\mathscr{G}_h^{1/4}$, $w_1 z/\nu = 0{\cdot}550\,\mathscr{G}_z^{1/2}$ and δ/z about $5{\cdot}4\,\mathscr{G}_z^{-1/4}$, if the point was chosen where $\theta = 0{\cdot}02\theta_1$ and $w = 0{\cdot}1\,w_1$. The way in which θ/θ_1 and w/w_1 vary with $\mathscr{G}_z^{1/4}(y/z)$ is shown in fig. 5.50. These theoretical results have been confirmed very accurately by careful experiments by E. Schmidt and W. Beckmann.†[90]

The natural flow about a circular cylinder with its axis horizontal has been discussed theoretically by R. Hermann ‡ subject to the assumption that δ is small compared with the diameter d, so that the usual methods of calculation with boundary layers can be applied. He found that the functions of fig. 5.50 are again applicable and even that the powers obtained for the vertical plate apply here if h is replaced by d. The variations of the velocity and the thickness of the boundary layer with the angular co-ordinate naturally follow special laws in this case (see fig. 5.52, p. 417). These calculations have also received good confirmation from experiments by K. Jodlbauer.§ For heat transfer the calculation gives $\mathscr{N}_u = 0{\cdot}372\,\mathscr{G}_d^{1/4}$. For values of $\mathscr{G}$ above 10^5 experiments gave values only slightly higher, 0·372 being replaced by 0·395 approximately.

The flow remains laminar up to very high values of the Grashof number. For a vertical plate 1 metre high and for a cylinder of diameter 58·5 cm. Hermann found that the change to turbulent motion took place for $\mathscr{G} = 10^9$ or $3{\cdot}5 \times 10^8$ respectively. In the two cases the

* Quoted by E. Schmidt and W. Beckmann in *Forschung*, Vol. 1, pp. 341, 391 [= *Techn. Mech. u. Thermodyn.* (1930)].

† *Forschung*, Vol. 1, pp. 341, 391 (1930).

‡ *Physik. Zeitschr.*, Vol. 34, p. 211 (1933).

§ *Diss.*, Danzig [= *Forschung*, Vol. 4, p. 157 (1933)]. See also the valuable review by E. Schmidt, *VDI-Zeitschr.*, Vol. 76, p. 1025 (1932).

critical value of Reynolds' number was in the neighbourhood of 500 or 400. (The lower values for Reynolds' number arise from the special type of velocity profile.) For values of the Grashof number below 10^5 the " thickness of the boundary layer " is no longer small compared with the height of the plate or diameter of the cylinder, so that the formulæ given above become less and less useful the smaller $\mathscr{G}$ becomes. In such cases, however, the problem is still of practical interest

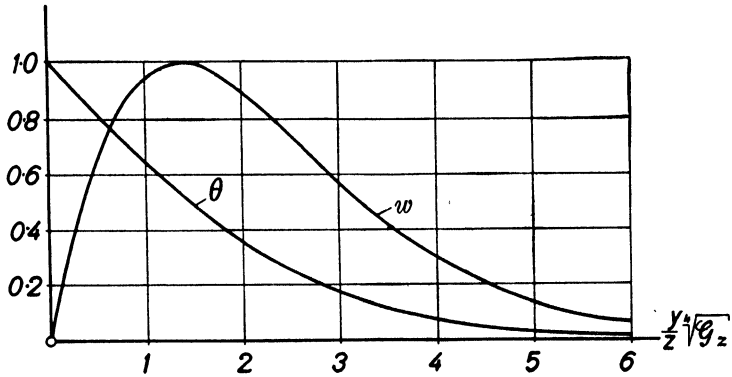


Fig. 5.50.—Temperature and velocity distributions in the neighbourhood of a heated vertical wall, after E. Pohlhausen

in connexion with the heat loss of horizontal electrically heated wires in still air. On the basis of experiments by others, R. Hermann * gave a numerical table from which we reproduce the following extract for small temperature differences:

$\mathscr{G}^{1/4} =$	0·1	0·316	1	3·16	10	31·6	100
$\mathscr{N}_u =$	0·484	0·612	1·10	2·18	4·47	12·4	39·3.

Slight movement of the air markedly increases the heat transfer.

A great advance in the making visible of the flow about hot bodies is provided by the " Schlieren " method discovered by E. Schmidt † and extensively used by him. It is particularly well suited to two-dimensional flow and in such cases yields quantitative results. It consists in shadow images thrown on a distant screen by practically parallel light from a point source. If the light rays run parallel to a surface which is giving out heat, within the layer which is heated, they are deviated outwards owing to the density gradient of the air, the deviation being greatest where the gradient is steepest, i.e. in the immediate neighbourhood of the hot body. This gives a definite bright border to the

* *VDI-Forschungsheft* No. 379 (1936).

† *Forschung*, Vol. 3, p. 181 (1932); *Proc. IV Internat. Congr. Appl. Mech.* (Cambridge, 1934), p. 92.

" Schlieren " image. As the transfer of heat at any point is proportional to the same gradient, the distance of the bright border from the shadow of the body when it is cold gives a numerical measure of the transfer of heat at the particular point. Outside the boundary layer the light remains parallel; inside the boundary layer it is deviated outwards, but not to such an extent as in the layer closest to the body. It follows that if the screen is sufficiently distant the whole space occupied by the heated layer remains dark. Fig. 5.51 shows a photograph for the case of a heated vertical plate. The shadow of the plate when cold is shown by the dotted lines. We can clearly see from the shadow that the thickness of the boundary layer δ varies as $z^{1/4}$, and from the outer bright border that the heat transfer at a particular point varies as $1/\delta$. Fig. 5.52 shows the corresponding phenomena for the circular cylinder with its axis horizontal. We see that the heat loss diminishes from the maximum at the lowest point, at first gradually, but in the upper half very rapidly.

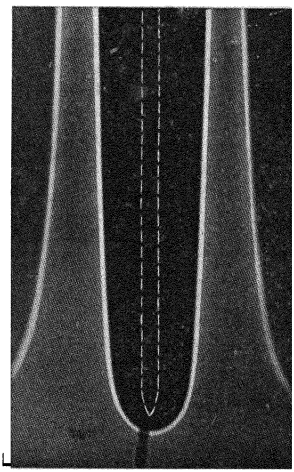


Fig. 5.51.—" Schlieren " image of the heated vertical plate

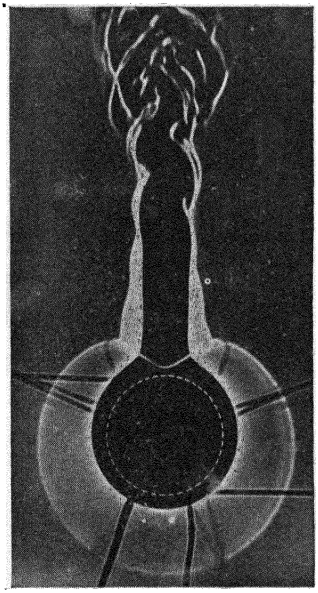


Fig. 5.52.—" Schlieren " image of the horizontal heated circular cylinder

(*d*) *Heat Flow about a Horizontal Plate.*—Here there is a fundamental difference between the state of affairs at the lower surface and that at the upper surface. Below the plate the stratification is stable. The heated air moves away only gradually at the side edges. Above the plate, on the contrary, there is instability, and there are irregular currents of air up and down. This is clearly shown in the " Schlieren "

image of fig. 5.53. The experimental results obtained hitherto are not consistent, but there seems to be no great difference between this case and that of the vertical plate in the total amount of heat transferred.

The instability which we have mentioned is of particular interest because of the types of flow which occur. These are exhibited in an especially clear form when a shallow layer of a fluid with a not too small coefficient of thermal expansion is heated from below. The striking phenomena which occur have been studied in great detail by H. Benard * and his pupils as well as by numerous other workers.† Under the simultanous effects of thermal conduction and viscosity it is possible to have a stable state with the density increasing from

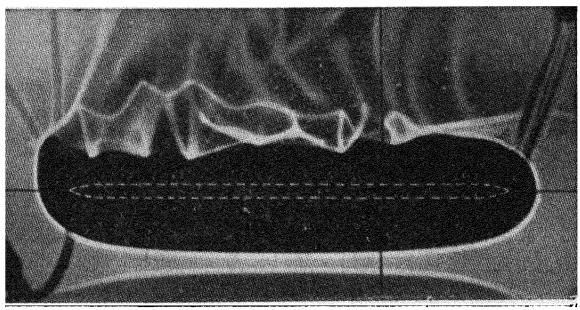


Fig. 5.53.—" Schlieren " image of the horizontal heated plate

below upwards when the temperature differences are moderate, in contradiction to the elementary discussion of stability on pp. 14 and 17. If, however, the temperature difference (see below) exceeds a certain value depending on the depth of the layer, there are upward and downward currents in the interior of the fluid which after an irregular transition stage lead to a more or less regular subdivision of the whole layer of fluid into cells. If the critical temperature difference is only slightly exceeded, the cells are hexagonal and form a regular honeycomb arrangement; the fluid rises in the centre of the cell and falls along the boundaries. For larger temperature differences or deeper layers the subdivisions become more or less irregular (fig. 5.54) but are still stable. If the temperature differences become still greater the cellular flow is replaced by unsteady and irregular up and down currents; the motion becomes " turbulent " (fig. 5.55).‡

* *Thesis*, Paris (1901); see also *Revue gén. des sciences pures et appl.*, Vol. 11, pp. 1261 and 1309 (1900).

† For a detailed bibliography, see D. Avsec, *Publ. scientif. et techn. du ministère de l'air*, No. 155 (Paris, 1939).

‡ For figs. 5.54, 5.55 the author is indebted to H. Siedentopf, Jena. The flow is made visible by admixed aluminium filings. The scale is 2/5 natural size. For other illustrations, see e.g. S. Mal, *Beitr. z. Phys. d. fr. Atm.*, Vol. 17, p. 40 (1931); D. Avsec (loc. cit.); V. Volkovisky, *Publ. scientif. et techn. du ministère de l'air*, No. 151 (1939).

If there is a horizontal flow in addition, the cellular vortices in the stable cells are replaced by parallel vortex tubes with right-hand and left-hand rotations alternately.*

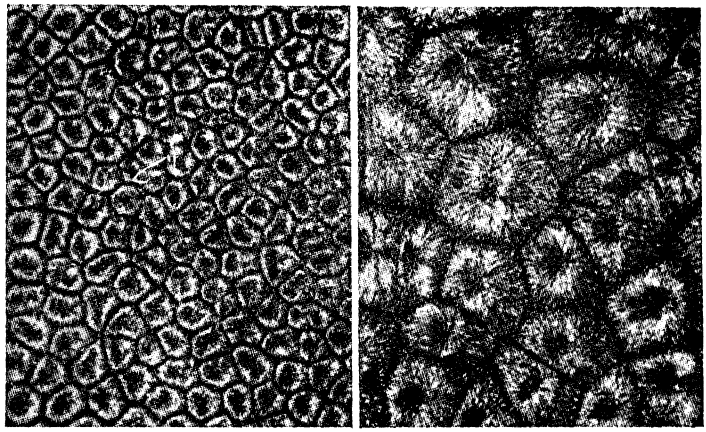


Fig. 5.54.—Stable thermal convection, after Siedentopf; depth of layer on the left 4 mm.; on the right 10 mm.



Fig.5.55.—Turbulent thermal convection, after Siedentopf; depth of layer 20 mm.

The theory of this group of phenomena was first discussed by Lord Rayleigh,† although in a deliberately simplified form. Nevertheless, he obtained the correct result that for the limit of stability it is a

* For illustrations, see footnote 3 on p. 418.

† *Phil. Mag.* (6), Vol. 32, p. 529 (1916) = *Collected Papers*, Vol. VI, p. 432.

matter of the quantity $\Lambda = g\beta(\theta_1 - \theta_2)h^3/a\nu$, which to-day may be written in the form $\mathscr{G}\nu/a$. The critical value was first calculated by H. Jeffreys * and confirmed by A. R. Low † and D. Avsec.‡ For rigid, highly-conducting walls above and below, Λ is about 1705. When $(\theta_1 - \theta_2)$ was plotted against the amount of heat transferred, from the experimental values of R. J. Schmidt and O. A. Saunders § for water of mean temperature 18° to 20° C., there was a definite break in the curve at $\Lambda = 1700$ to 1800,|| and another at about 47,000 (transition to turbulence). According to their experiments, which, however, extend only to 150,000, the heat transfer above 47,000 can be represented by the formula $\mathscr{N}_u = 0{\cdot}113\sqrt[3]{(\mathscr{G}\nu/a)}$.

The following brief account of the fundamental theory may suffice. For the sake of simplicity we shall consider a two-dimensional flow. Let (x, z) be the co-ordinates, and $u(x, z)$, $w(x, z)$ the velocities. Let the temperature θ be given by $\theta = \theta_1 - (\theta_1 - \theta_2)z/h + \theta'(x, z)$; see fig. 5.56.¶ Let the density be

$$\rho = \rho_0(1 - \beta\theta).$$

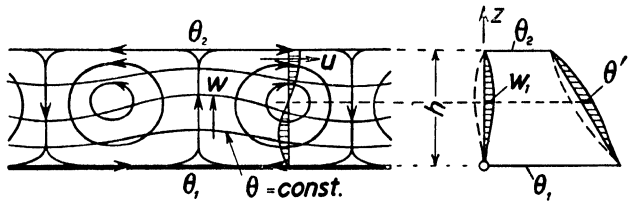


Fig. 5.56.—Heat convection

If we assume slow motion, i.e. if we neglect inertia effects, we have

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \qquad \ldots\ldots (60)$$

$$\frac{\partial p}{\partial z} = -g\rho + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \qquad \ldots (61)$$

$$u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = a\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right). \qquad \ldots\ldots (62)$$

* *Proc. Roy. Soc.* (A), Vol. 118, p. 195 (1928).

† Ibid., Vol. 125, p. 180 (1929).

‡ See footnote 2, p. 418.

§ *Proc. Roy. Soc.* (A), Vol. 165, p. 216 (1938).

|| These values indicate that the surface of the fluid behaves as if it were rigid relative to the small frictional forces; this is often the case for surfaces which have not been specially cleaned. For the case of a non-rigid surface and a rigid bottom A. R. Low obtained $\Lambda_{crit} = 1107$ (see footnote † and *Proc. 3rd Internat. Congr. Mech.*, Stockholm, 1930, Vol. 1, p. 109, where accurate illustrations of stream-lines are given).

¶ The figure corresponds not to two-dimensional flow but to a vortex ring; at $z = 0$ there is solid ground, at $z = h$ a free surface.

In addition, we have the equation of continuity $\partial(\rho u)/\partial x + \partial(\rho w)/\partial z = 0$, in which we put $\rho = \text{const.}$ as an approximation. If we are investigating the critical condition, we may regard u, w and θ' as very small. Then, neglecting small quantities of the second order, we are left with only $w(\theta_1 - \theta_2)/h$ on the left-hand side of (62).

Instead of giving a complete treatment of the linearized system of equations, in which three of the four quantities u, w, p, θ' can be eliminated, giving a proper-value problem, we shall content ourselves here with a rough approximation. If the extent of *one* vortex in the x-direction is of the same order of magnitude as that in the z-direction, we need only consider the variation with z in further detail. p is a dependent quantity which can be obtained from the others, and hence may be left out of account in the approximation. Of $-g\rho$ there remains, after we have removed the effect of this quantity on the pressure at rest, $-g\rho_0\{1 - \beta\theta_1 + \beta(\theta_1 - \theta_2)z/h\}$, the portion $g\rho_0\beta\theta'$ which is important for the flow. $\partial^2 w/\partial z^2$ and $\partial^2\theta/\partial z^2$ may be represented as regards order of magnitude by w_1/h^2 and θ_1'/h^2. Hence, using $\nu = \mu/\rho_0$, equations (61) and (62) give the relationships

$$g\beta\theta_1' \sim \nu w_1/h^2$$

and

$$(\theta_1 - \theta_2)w_1/h \sim a\theta_1'/h^2.$$

The product of the left-hand sides is then also proportional to the product of the right-hand sides; hence

$$g\beta(\theta_1 - \theta_2)/h \sim \nu a/h^4,$$

that is,

$$g\beta(\theta_1 - \theta_2)h^3/\nu a = \mathscr{G}_h \nu/a = \text{a fixed number.}*$$

The similarity between the phenomena considered here and many cloud formations is striking, and consequently meteorological literature contains many references to the theory of these phenomena.† Here stable stratification replaces horizontal boundaries formed by the ground and a free surface. For the phenomena to occur it is only necessary for a particular layer to gain or lose heat by absorbing or emitting radiation and thus disturb the equilibrium. Genuine phenomena of this kind are, for example, the characteristic structures in thin stratus clouds at night, which can be observed by moonlight and in which the driving force is the cooling of the upper side of the layer by radiation to the night sky. Here, however, we may mention that apparently very similar phenomena of short duration, e.g. a mackerel sky, can also be produced by mere instability of the stratification arising from the lifting of a mass of moist air with a layer of dry air above it, so that it expands according to the wet adiabatic with the formation of a fog, while the dry mass expands according to the dry adiabatic and thus becomes colder than the moist mass.‡

* It may be pointed out here that the expression $\mathscr{G}\nu/a$ always comes in when it is a case of slow motion.

† See e.g. the paper by S. Mal (*Beitr. z. Phys. d. fr. Atm.*, 1931), mentioned in footnote 3, p. 418, also the report by A. R. Low to the Stockholm Congress of Mechanics, also quoted above.

‡ The reverse phenomenon, which leads to the formation of mammato-cumulus, occurs when a dense layer of cloud lying above a dry layer descends.

According to H. Siedentopf,* the structure known as granulation, which is clearly exhibited in photographs of the sun, is a genuine flow of heat brought out by ionization of hydrogen in a comparatively thin layer near the sun's surface. Here the motion is definitely of the turbulent type; the individual elements persist for only a few minutes.†

A coefficient Λ' has often been considered for comparison in which the kinematic viscosity ν and the thermal conductivity a are replaced by the corresponding quantities for turbulent exchange, i.e. $lu' = A/\rho$ is introduced instead of both. In many cases, however, no information is available about the mixing length l and the velocity u' of the turbulent motion, unless we can relate the former to the height of the active layer, and the latter to the rate of ascent.

(*e*) *Mountain and Valley Winds in Stratified Air.*—Provided there is no stratification, the phenomena in the neighbourhood of an inclined plane surface at a temperature different from that of its surroundings can be deduced theoretically from the formulæ for the vertical plate by putting $g \sin\alpha$ for g. In practice it is necessary to note that when the inclined plate is hotter than its surroundings the disturbance of the temperature field gives rise to statical instability or an increased tendency to become turbulent. This is for the upper side of the plate. On the other hand, on the under side of the hot plate, or on the upper side of a plate cooler than its surroundings, there is statical stability and hence a decreased tendency to become turbulent.

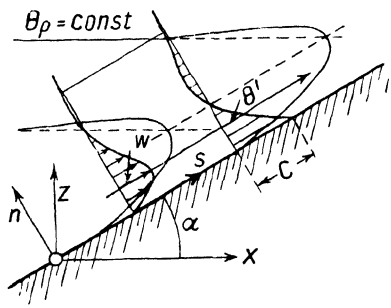


Fig. 5.57.—Wind at a slope

If, however, the whole air mass is already in stable equilibrium as a result of stratification, that is, for example, by increase of the potential temperature ‡ upwards, the phenomena are different. Here a heated particle of air will be able to rise only until it is surrounded by particles at the same temperature. If, however, more heat is meanwhile supplied to it from the surface, it will go on rising in a way which depends on the supply of heat to it.

The simplest case to follow is the laminar one (fig. 5.57). It is found that for a surface inclined at an angle α which everywhere has a definite excess of temperature over the stratified mass of air, every individual particle has an approximately uniform velocity § so that the inertia of the mass has no effect.

* *Vierteljahrsschrift der Astronom. Gesellschaft*, Vol. 76, p. 185 (1941).

† See also the associated vibration phenomena, p. 378. ‡ Cf. note, p. 383.

§ Apart from initial phenomena at the lower edge of the heated surface.

For the calculation it is convenient to transform the co-ordinates x and z (fig. 5.57) into the co-ordinates s and n parallel and perpendicular to the inclined plane. Taking account of θ', the disturbance of temperature by conduction from the inclined plane, the potential temperature is

$$\theta_p = A + Bz + \theta'(n). \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

By what has been said above, the velocity w parallel to s may be expected to be a function of n only. Hence, leaving out the pressure at rest, the Navier-Stokes equation reduces to

$$0 = g \sin\alpha \,.\, \beta\theta' + \nu \frac{\partial^2 w}{\partial n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

The combined effects of heat transfer and heat conduction give

$$w \frac{\partial \theta_p}{\partial s} = a\left(\frac{\partial^2 \theta_p}{\partial n^2} + \frac{\partial^2 \theta_p}{\partial s^2}\right),$$

which in conjunction with (63) and the relation $z = s \sin\alpha + n \cos\alpha$ gives

$$wB \sin\alpha = a \frac{\partial^2 \theta'}{\partial n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Differentiating (65) twice with respect to n and substituting for $\partial^2 w/\partial n^2$ in (64), we have

$$0 = g \sin\alpha \,.\, \beta\theta' + \frac{\nu a}{B \sin\alpha} \frac{\partial^4 \theta}{\partial n^4}. \quad . \quad . \quad . \quad . \quad (66)$$

A suitable solution of this well-known differential equation is

$$\theta' = Ce^{-n/l} \cos(n/l). \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where

$$l = \left(\frac{4\nu a}{g\beta B \sin^2\alpha}\right)^{1/4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

From (64) we now readily obtain

$$w = C\left(\frac{g\beta a}{\nu B}\right)^{1/2} e^{-n/l} \sin\frac{n}{l}. \quad . \quad . \quad . \quad . \quad . \quad (69)$$

It is remarkable that w is independent of the angle of inclination α. This is related to the fact that although for small values of α the upward thrust resulting from the density differences due to heating is less, the opposition to motion in the direction s arising from the stratification is likewise diminished. The latter effect is expressed in the increased value of l [by the boundary-layer calculation on p. 107, a doubling of the "free path" s_1 means multiplying l (the thickness of the boundary layer) by $\sqrt{2}$, in agreement with equation (68) above].

The behaviour of the functions in equations (67) and (69) is shown in fig. 5.57. It is noteworthy that negative values can occur in the distribution of the temperature disturbance θ'. We see the necessity of this, however, if we bear in mind that as a result of friction due to the thermal upward thrust layers are set in motion which have not themselves been heated, and that after they have risen to a new position they are colder than the particles which were there when the stratification of the air was undisturbed. The two curves $\theta_p = \text{const.}$ in fig. 5.57 show the deformed potential isotherms in the case of strong heating, where turbulence will probably occur.

The above formulæ become easier to understand if as before we put $\beta = 1/T$ and introduce a height H such that $B = T/H$. C is the disturbance of temperature at the boundary. Then e.g. the factor in w in equation (69) becomes $\frac{C}{T}\sqrt{\left(gH \cdot \frac{a}{\nu}\right)}$.

For the turbulent case a brief approximate discussion may suffice. Instead of equation (64) we may express the fact that τ, the shearing stress at the rough inclined plane, which may be put equal to $\zeta\rho w_1^2$ as an approximation, is equal to the thermal upward thrust on the whole heated layer of thickness l (w_1 is the maximum velocity), that is,

$$\zeta\rho w_1^2 \sim g\rho \sin\alpha \cdot \beta C \cdot l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64a)$$

In addition, we have the relation corresponding to equation (65), in which the thermal conductivity a must be replaced by some quantity connected with turbulence. This is A/ρ, where A is the exchange coefficient. If we put $\tau = A\,\partial w/\partial n$, since $\partial w/\partial n \sim w_1/l$ the above relation for τ gives $A \sim \zeta\rho w_1 l$, that is $a \sim \zeta w_1 l$. Hence equation (65) leads to

$$w_1 B \sin\alpha \sim \zeta w_1 l \cdot C/l^2 = \zeta w_1 C/l, \quad . \quad . \quad . \quad . \quad (65a)$$

from which we have

$$l \sim \zeta C/B \sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68a)$$

Then (64a) gives

$$w_1 \sim C\sqrt{\left(\frac{\zeta g\beta}{B}\right)} = \frac{C}{T}\sqrt{(\zeta gH)}, \quad . \quad . \quad . \quad . \quad (69a)$$

which is again independent of α. Here, however, we have to note that, particularly if the inclination is slight, the resistance coefficient ζ varies not only with l/k (p. 129) but also with the degree of turbulence, which is measured by Richardson's number (p. 382). Owing to this and to the increase in l, we have a gradual transition as α decreases to the phenomena above horizontal surfaces described in subsection (*d*).

To the above discussion we may relate the important remark that the flow of air in the neighbourhood of mountain slopes, upwards when warmed, downwards when cooled, gives rise to "sources" and "sinks" at the foot of the mountains and along their ridges, which affect the main flow. If, for example, a long valley between high mountains is full of stably stratified air, the air carried away into the sinks at the foot of the warm slopes can only be replaced if air flows into the valley at its lower end and, owing to the stratification, this flow will be almost exclusively horizontal. Hence it may be expected that this wind, the so-called "valley wind", which is a very striking phenomenon in all long valleys among the mountains, will be strongest in the lowest layers of the atmosphere, in which the sinks lie. When the air is cooled by radiation from the slopes at night, the cold currents of air flow together to the foot of the valley and flow down the valley like a river, forming the so-called "mountain wind". The currents in the upper air, by which the upper sources and sinks are equalized, are less noticeable, because here there is no funnel effect, but should still

exist to some extent in the layers in which the sources or sinks are found (of course in so far as the whole system is not destroyed by large-scale meteorological phenomena).

So far as the author knows, no quantitative investigation has been made of winds at a slope, e.g. into their intensity in the direction n (fig. 5.57) or their turbulent character.

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* A full account is given in the article by A. Betz on aerofoils and hydraulic machines in *Handbuch der Physik* (p. 429), Vol. VII, p. 215; see also Durand, *Aerodynamic Theory* (p. 429), Vol. II, article by Th. von Kármán and I. M. Burgers, p. 367: Vol. IV, articles by A. Betz, pp. 2–129, and H. Glauert, pp. 169–360.

† For the literature on the dynamics of gases, see also footnote, p. 259; for that on cavitation, see footnote 1, p. 313; for that on heat transfer, see the text, p. 397.

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APPENDIX

BOOKS

I. GENERAL

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NOTES AND REFERENCES

The numbers correspond to the parenthetical indices used in the main text

1. The Chattock manometer is commonly used in England for the measurement of small pressure differences.
   See OWER, p. 215. (Details of books are given on p. 431.)

2. See also books by GLAUERT and MILNE-THOMSON.

3. See also DURAND (Vol. II) and books by MILNE-THOMSON.

4. Some recent papers are:

   Boundary Layers with Suction

   B. THWAITES: " An Exact Solution of the Boundary Layer Equations under Particular Conditions of Porous Surface Suction ", Aeronautical Research Council, R. & M. 2241 (1946).

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   Boundary Layers at High Speeds

   BRAINERD and EMMONS: *Jour. App. Mech.*, 1941, Vol. 8: *Trans. Amer. Soc. Mech. Eng.*, 1942, Vol. 64A.

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5. Surveys are also given in the following papers:

   BATCHELOR: Proceedings 7th International Congress Applied Mechanics, London, 1948 (General lecture).

   DRYDEN: *Quarterly of Applied Mathematics*, 1943, Vol. I, p. 7.

   VON KÁRMÁN: " Progress in the Statistical Theory of Turbulence ", *Proc. Nat. Acad. Sc.*, 1948, Vol. 34, pp. 530–539.

6. See also:

   LIN: *Quarterly of Applied Mathematics*, 1945–46, Vol. 3, pp. 117–142, pp. 218–234, pp. 277–301.

7. The term " Austausch " is sometimes used in English papers, but it is more usual to make use of Boussinesq's " eddy viscosity ", which is equal to A/ρ.

8. See also:

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9. Experiments on a round jet and on the mixing at the edge of a jet are described by KUETHE: *Jour. App. Mech.*, 1935, Vol. 2, pp. 87–95.

10. Experiments on the shear stress of natural wind over ground are described by P. A. SHEPPARD: *Proc. Roy. Soc.* A, 1947, Vol. 188, p. 208.

11. See also:
VON KÁRMÁN: "Turbulence and Skin Friction", *Jour. Aero. Soc.*, 1934, Vol. 1, pp. 1–18.

12. Görtler's theory has been confirmed by experiments: see LIEPMANN, N.A.C.A. A.C.R. No. 3H30 (Wartime Report No. W107) (1943), N.A.C.A. A.C.R. No. 4J28 (Wartime Report No. W87) (1945).

13. See particularly:
DRYDEN and SCHUBAUER: "The use of Damping Screens for the Reduction of Wind Tunnel Turbulence", *Jour. Aero. Sc.*, 1947, Vol. 14.

14. Some work in this field is described by DRYDEN in N.A.C.A. Tech. Note No. 1188 (1947).

15. Much study had been given to the application of boundary layer suction to the improvement of aerofoil characteristics; see for example:
E. F. RELF: *Jour. Roy. Aero. Soc.*, 1946, Vol. 50, p. 241.
S. GOLDSTEIN: *Jour. Aero. Sc.*, 1948, Vol. 15, p. 189.
A. VON DOENHOFF and L. K. LOFTIN: *ibid.*, 1949, Vol 16, p. 729.

16. See also:
WHITE and COLEBROOK: "Experiments with Fluid Friction in Roughened Pipes", *Proc. Roy. Soc.*, 1937, Vol. A46.

17. See also:
PATTERSON: "Note on the Design of Corners in Duct Systems", Aeronautical Research Council R. & M. No. 1773;
and
HILDING BEIJ: "Pressure Losses for Fluid Flow in 90° Bends", *Journal of Research*, National Bureau of Standards, Washington, July, 1938, Vol. 21.

18. See also:
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19. See also:

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20. See also: "Test Code for Hydraulic Prime Movers", A.S.M.E., New York, 1938.

21. See also:

BAKHMETEFF: "Hydraulics of Open Channels" (McGraw–Hill), A.S.M.E. Monograph, 1932;

and

HUNTER ROUSE (see p. 431).

22. See also:

WIGLEY: "Calculated and Measured Wave Resistance of a Series of Forms defined Algebraically", *Trans. I.N.A.*, 1942, p. 52.

23. See also:

GOLDSTEIN: pp. 553–570 (see p. 431).

24. A useful review is given by VON KÁRMÁN, see note 11.

25. See papers by RELF and GOLDSTEIN referred to in note 15, where many references are given. Also, "Summary of Airfoil Data" by I. H. ABBOTT, A. E. VON DOENHOFF and L. S. STIVER, N.A.C.A. Report 824. These papers give information on the design and characteristics of modern aerofoils.

26. See reports by S. GOLDSTEIN on "Approximate Two-Dimensional Aerofoil Theory" (to be published by the Aeronautical Research Council). Also M. J. LIGHTHILL, Aeronautical Research Council R. & M. No. 2112 (1945).

27. See also:

GLAUERT (see p. 431).

28. See also:

FALKNER: Aeronautical Research Council R. & M. 1895 (1941).

29. See also:

C. B. MILLIKAN: N.A.C.A. Report No. 362 (1930).

30. See GLAUERT'S article on propellers in DURAND (see p. 431).

31. See also:

J. LOCKWOOD TAYLOR: "The Variable Pitch Marine Propeller", *Proc. Inst. Mech. Eng.*, War Emergency Paper No. 19 (1946).

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38. See also: "Fifth International Conference of Ship Tank Superintendents, London, Sept. 1948", H.M.S.O., 1949.

39. DRYDEN and SCHUBAUER: "The Use of Damping Screens for the Reduction of Wind Tunnel Turbulence", *Jour. Aero. Sci.*, 1947, Vol. 14.

40. DURAND (see p. 431);
and

A. POPE: "Wind Tunnel Testing" (Wiley, 1947).

41. See OWER (see p. 431);
also

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42. Further references to the hot-wire technique of measurement of turbulence are:

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S. CORRSIN: "Extended Applications of the Hot-Wire Anemometer", *Review Sci. Inst.*, 1947, Vol. 18, pp. 469–471.

43. A different method of determining the wake drag is given by B. M. JONES (*loc. cit.*, p. 256) which in most practical cases leads to values of the drag which are little different from the values given by Betz's method. Jones assumes that no mixing takes place in the wake downstream of the plane of measurement, and this leads to the formula

$$W = 2\iint \sqrt{g - p}\,(\sqrt{g_0 - p_0} - \sqrt{g - p_0})\,dF,$$

where the symbols have the same significance as on p. 256.

44. L. H. THOMAS: "Note on Becker's Theory of the Shock Front", *Journal of Chemical Physics*, 1944, Vol. 12, pp. 449–453.

45. Further experiments are described by:
J. H. KEENAN and E. P. NEUMANN: "Friction in Pipes at Supersonic and Subsonic Velocities", N.A.C.A. Tech. Note No. 963 (1945).
A theoretical investigation has been given by
YOUNG and WINTERBOTTOM: "High Speed Flow in Smooth Cylindrical Pipes of Circular Section", Aeronautical Research Council R. & M. 2068 (1942).

46. For later developments see list of books on p. 431. Also VON KÁRMÁN: "Compressibility Effects in Aerodynamics", *Jour. Aero. Sciences*, 1941, Vol. 8, and "Supersonic Aerodynamics—Principles and Applications", *ibid.*, 1947, Vol. 14, p. 373.

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S. GOLDSTEIN and A. D. YOUNG: "The Linear Perturbation Theory of Compressible Flow, with Application to Wind Tunnel Interference", Aeronautical Research Council R. & M. 1909 (1943).

48. A step forward in the analysis of transonic flow had been made by VON KÁRMÁN in a paper entitled "The Similarity Law of Transonic Flow", *Jour. Math. and Phys.*, 1947, pp. 182–190.

49. See J. C. GUNN: *Phil. Trans. Roy. Soc.*, 1947, Vol. 240, pp. 327–373.

50. Extensive applications of the Busemann cone field theory have been made; see book list on p. 431 and VON KÁRMÁN *loc. cit.* in note 46; also
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51. See also FERRI (book list on p. 431).

52. See also:
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53. See also:
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54. See also:
W. J. KING: "Measurement of Temperature in High Velocity Gas Streams", *Trans. A.S.M.E.*, 1943, Vol. 65, p. 421.

55. See also:
H. B. SQUIRE: "Heat Transfer Calculation for Aerofoils", Aeronautical Research Council R. & M. 1986 (1942).

56. ACKERET's memoir is translated in N.A.C.A. Tech. Memo No. 1078. See also R. T. KNAPP and A. HOLLANDER, "Laboratory Investigations of the Mechanism of Cavitation", *Trans. A.S.M.E.*, 1948, Vol. 70; and M. S. PLESSET, "The Dynamics of Cavitation Bubbles", *Jour. of App. Mech.* (Sept. 1949).

57. NUMACHI's paper has been translated by the School of Engineers, Pennsylvania State College (1946).

58. See also:
J. G. BURNELL: "Flow of Boiling Water through Nozzles, Orifices and Pipes", *Engineering*, Dec. 1947, Vol. 164.

59. F. B. BROWN, R. T. KNAPP, J. LEVY: "The Hydrodynamics Laboratory of the California Institute of Technology", *Trans. A.S.M.E.*, July, 1948.
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60. See also:
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