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TRIGONOMETRY

FOR

INTERMEDIATE CLASSES

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ELEMENTARY TRIGONOMETRY

CHAPTER I

ANGULAR MEASURE

Art. 1. Trigonometry is derived from a Greek word meaning ‘measurement of triangles.’ Plane Trigonometry deals with triangles drawn in a plane, while Spherical Trigonometry gives the properties of triangular figures situated on the surface of a sphere. In either case, since every triangle has three angles and three sides, the definition and the measurement of angles form a necessary preliminary to a study of the subject. Trigonometry is a very useful subject ; for with its help only, the earth has been surveyed, the heights and distances of inaccessible objects have been determined, and advance in every branch of knowledge has been made possible. It has got a functional notation of its own, which plays an important part in all branches of Mathematical Analysis.

Art. 2. Measurement of Angles.—When two straight lines meet, they form an angle and it is a right angle if they are perpendicular. The right angle is taken as the fundamental unit to define angles. In Trigonometry, we define an angle as being formed by the rotation in a plane of a straight line about a fixed point on it. The fixed point is termed the origin, and the starting position of the line is termed the initial line. When the line is in any position, it is called the radius vector, and the amount of rotation is a measure of the angle made. If the rotation is clockwise, the angle described is taken as negative, while a rotation in the anti-clockwise direction gives a positive angle. Thus,

by definition, we have angles of any magnitude whatever ranging from $-\infty$ to $+\infty$. But, it is clear that whenever the radius vector revolves through a multiple of four right angles in either direction, it comes back to the initial position, since a complete revolution of a line takes it through four right angles.

To measure angles of any magnitude, the right angle is sub-divided into ninety equal parts, each part being called a degree ; a degree is again divided into sixty minutes, and a minute is further sub-divided into sixty seconds. Thus, 60 seconds make one minute ; 60 minutes make one degree and 90 degrees make one right angle. This is the Sexagesimal system of measuring angles. An angle which measures twenty degrees fifteen minutes forty-seven seconds is written as $20^{\circ} 15' 47''$, the symbols $^{\circ}$, $'$, $''$ denoting degrees, minutes and seconds respectively.

Art. 3. Centesimal System.— A second system of measuring angles is that one in which a right angle is divided into 100 grades, a grade into 100 centesimal minutes and a minute into 100 centesimal seconds, the symbol for the grade being 'g'. This system was introduced at the time of the French Revolution in France, but was dropped subsequently. The measure of an angle is specified by the number of grades, minutes and seconds which it contains.

Art. 4. Circular Measure.— Another system of measuring angles is adopted in Higher Mathematics, the unit being in this case *the radian*. **The radian is defined as the angle subtended at the centre of a circle by an arc measured along the circumference equal in length to the radius of the circle.** It is not an exact sub-multiple of a right angle as is the degree or grade ; but one can be

expressed in terms of the other by means of an incommensurable number that will be introduced presently.

Art. 5. It has been experimentally found in Practical Geometry that the circumference of a circle is a little more than three times its diameter and that the ratio of their magnitudes is the same for all circles. It is only an experimental verification of the property to be established that the ratio of the circumference of a circle to its diameter is a constant, whatever be the size of the circle. The circumference of the circle may be taken as the limit to the perimeter of a polygon inscribed in the circle, as the number of sides n of the polygon is increased in such manner that the length of each side tends to zero. For the sake of convenience, we shall take the polygon to be equilateral and therefore regular.

Art. 6. Consider two circles of different radii r, r' with their centres at O and O' respectively. Inscribe two regular polygons of the same number of sides n in them, one in each. AB is one side of the first polygon and $A'B'$ one side of the other. Join $OA, OB, O'A', O'B'$. Now, $\angle AOB = \angle A'O'B'$ each being equal to $\frac{360^\circ}{n}$. Also $OA = OB$ and $O'A' = O'B'$. Hence, the two triangles OAB

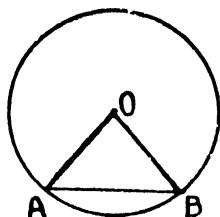


Fig. 1

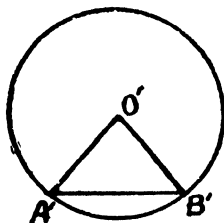


Fig. 1a

and $O'A'B'$ are similar, and their corresponding sides proportional.

$$\begin{aligned}\frac{AB}{A'B'} &= \frac{OA}{O'A'} = \frac{r}{r'} \\ \therefore \frac{n \cdot AB}{n \cdot A'B'} &= \frac{2r}{2r'}.\end{aligned}$$

But $n \cdot AB$ is the perimeter of one polygon and $n \cdot A'B'$ that of the other. The perimeters of the two polygons are in the ratio of the diameters of the circles. This result does not depend on n at all. As the number of sides of the polygon is increased, the perimeter differs from the circumference of the circle by a smaller and smaller quantity, till in the limit when n tends to infinity, the perimeter tends to coincidence with and equality in value to the circumference of the circle. Hence, it follows that in two circles, the circumferences are in the ratio of their diameters; or in other words, *the ratio of the circumference of a circle to its diameter is a number which is independent of the radius of the circle.* This constant is denoted by the Greek letter π and is an incommensurable number, *i.e.*, it cannot be expressed exactly as the ratio of one integer to another; but its value to any required degree of accuracy can be computed by methods of Higher Mathematics. Its value is roughly $\frac{22}{7} = 3\frac{1}{7}$;

a more accurate value is $\frac{355}{113} = 3\frac{16}{113}$; a still more accurate value to five decimal places is 3.14159.

Art. 7. Radian is an Angle of Constant Magnitude.—The radian is the unit of angular measure in all branches of Mathematical Analysis. The first requisite of a unit is that it must be invariable in magnitude. Hence it is necessary to prove that the radian defined as in Art. 4

in relation to the radius of the circle has the same value, whatever be the radius of the circle. Since an angle of 60° at the centre cuts off a chord equal in length to the radius, it is clear that a radian is smaller than 60° . AOB is a diameter of a circle centre O. OC is drawn perpendicular to OB so that $\angle BOC = 1$ right angle. An arc BD is measured along the circumference equal in length to the radius, so that according to the definition $\angle BOD$ is a *radian*. In the

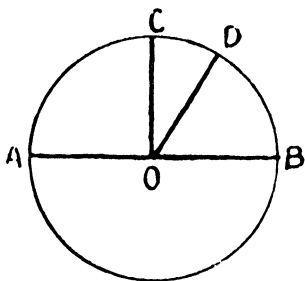


Fig. 2

same circle, angles at the centre are proportional to the arcs on which they stand. Hence

$$\frac{\angle BOD}{\angle BOC} = \frac{\text{Arc BD}}{\text{Arc BC}},$$

$$\text{i.e., } \frac{1 \text{ radian}}{1 \text{ right angle}} = \frac{\text{radius of circle}}{\frac{1}{4} \text{ of the circumference}};$$

but it has been already shown that the length of the circumference of a circle is π times the diameter or 2π times the radius.

$$\begin{aligned} \text{Hence } \frac{1 \text{ radian}}{1 \text{ right angle}} &= \frac{r}{\frac{1}{4} \times 2\pi r} = \frac{2}{\pi} \\ \therefore 1 \text{ radian} &= \frac{2}{\pi} \text{ of a right angle.} \end{aligned}$$

A right angle is known to be of constant magnitude at whatever point it is drawn and π has been proved to be a constant independent of the radius of the circle taken. Hence, a radian is of invariable size; its magnitude does not depend on the radius of the circle in which the radian is measured. From this it follows that $1 \text{ right angle} = \frac{\pi}{2}$

radians or in other words the circular measure (*i.e.*, the measure of angle when the radian is taken as the unit angle) of a right angle is $\frac{\pi}{2}$. A straight angle equals π radians.

Art. 8. The Value of a Radian in Sexagesimal Measure.

From the previous article, it follows that

$$\begin{aligned} 1 \text{ radian} &= \frac{2}{\pi} \text{ of a right angle.} \\ &= \frac{2}{\pi} \times 90^\circ \\ &= \frac{1}{\pi} \times 180^\circ \\ &= .3183 \times 180^\circ \text{ (from Clark's Tables)} \\ &= 57^\circ.294 \text{ (correct to 3rd decimal).} \\ &= \underline{57^\circ 17' 38''} \end{aligned}$$

Art. 9. Formulae of Conversion from the Sexagesimal into the Circular measure and vice versa.

$$\begin{aligned} 1 \text{ radian} &= \frac{2}{\pi} \text{ of a right angle.} \\ \therefore \pi \text{ radians} &\cong 2 \text{ right angles} = 180^\circ \\ \therefore 1 \text{ radian} &= \frac{180^\circ}{\pi} = \frac{180}{\pi} \times 60' = \frac{180}{\pi} \times 60 \times 60''. \end{aligned}$$

The equivalent of x radians is $\frac{180}{\pi} x^\circ$ in sexagesimal measure. Similarly, the circular measure of an angle y° is $\frac{\pi y}{180}$. This may be easily remembered in the following way. A straight angle is 180° ; and it is subtended at

the centre by the semi-circumference which is π times the radius ; as each arc equal to the radius subtends a radian, there are π radians in a straight angle.

Note.—The circular measure of an angle is denoted by a number only, the unit not being explicitly specified. Thus, when we speak of an angle whose magnitude is π , we mean that the angle measures π radians.

Art. 10. Length of a circular arc.—The radian being defined as in the previous articles, it is now possible to calculate the length of an arc of a circle in terms of the radius of the circle and the angle subtended by the arc at the centre.

AP is arc of a circle centre O and radius OA. Measure off an arc AQ equal in length to the radius.

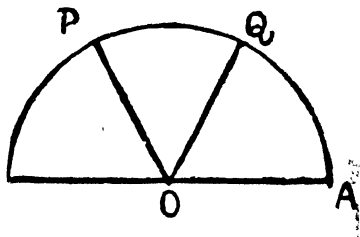


Fig. 3

It is now clear that $\frac{\text{Arc AP}}{\text{Arc AQ}} = \frac{\angle AOP}{\angle AOQ}$

$$\text{i.e., } \frac{\text{Arc AP}}{\text{radius}} = \frac{\angle AOP}{1 \text{ radian}}$$

If the circular measure of $\angle AOP$ is x ,

$$\frac{\text{Arc AP}}{\text{radius}} = \frac{x \text{ radians}}{1 \text{ radian}} = x.$$

Hence, Arc AP = radius $\times x$ = radius $\times$ **circular measure of angle AOP at the centre.**

Rule.—To get the length of a circular arc, multiply the length of the radius of the circle by the number of radians in the angle subtended by the arc at the centre.

Ex. 1. Prove that 1 radian = 206265 seconds.

$$\pi \text{ radians} = 180^\circ = 180 \times 60 \times 60 \text{ seconds}$$

$$\begin{aligned} \therefore 1 \text{ radian} &= \frac{180 \times 60 \times 60}{\pi} \text{ seconds} \\ &= \frac{180 \times 3600}{3.1416} \text{ seconds} \\ &= 206265 \text{ seconds.} \end{aligned}$$

Ex. 2. The angular diameter of the moon is 32'. If the distance of the moon from the earth is 240,000 miles, calculate the diameter of the moon in miles.

The moon being very far away, the diameter of the moon may be taken to be an arc of a circle, whose centre is at the observer's eye, and whose radius is 240,000 miles, the arc subtending 32' at the centre of the circle.

Using the formula of Art. 10, length of moon's diameter = 240,000 $\times$ circular measure of 32'

$$= 240,000 \times .0093 \text{ (from Clark's Tables)}$$

$$= \underline{2232} \text{ miles.}$$

Ex. 3. At what distance from the eyes should a rupee coin be held so as to just hide the sun, the angular diameter of the sun being 33', and the diameter of the coin being 1.5 inches.

The angular diameter being small, 1.5 inches is approximately the arc of a circle radius x inches subtending 33' at the centre, where x inches is the required distance.

$$1.5 = x \times \text{circular measure of } 33'$$

$$\text{i.e., } 1.5 = x \times .0096$$

$$\text{i.e., } x = \frac{1.5}{.0096} = 156.25'' = \underline{13 \text{ ft.}} \text{ nearly.}$$

Art. 11. Area of a sector of a circle.—In Fig. 3, AOP is a sector of a circle of radius r and angle x radians.

$$\begin{aligned} \text{Now, } \frac{\text{area of sector}}{\text{area of circle}} &= \frac{\text{Angle AOP}}{\text{Angle at centre}} \\ &= \frac{x \text{ radians}}{2\pi \text{ radians}} \\ &= \frac{x}{2\pi} \end{aligned}$$

$$\begin{aligned} \text{Hence, area of sector} &= \frac{x}{2\pi} \times \pi r^2 \\ &= \frac{1}{2} r^2 x. \end{aligned}$$

Rule.—Area of a sector of circle equals half the square of the radius multiplied the angle of sector *in radians*.

EXAMPLES I a

(1) Express in sexagesimal, circular and centesimal measures the angles of each of the regular figures having (i) three sides, (ii) five sides, (iii) six sides, (iv) eight sides, (v) ten sides, (vi) twelve sides.

(2) If the value of an angle is D degrees and also G grades, prove that $\frac{D}{90} = \frac{G}{100}$. Show that there are the same number of degrees in the angle of a dodecagon as the number of grades in the angle of an octagon.

(3) Show that the value of an angle varies directly as the arc of the circle on which it stands and inversely as the radius. When the measure of an angle is $\frac{\pi \times \text{arc}}{\text{diameter}}$ find approximately the magnitude of the unit of measurement.

(4) Two tangents are drawn to a circle radius $3''$ so as to make 82° with each other. Calculate the length of the circumference of the circle included between them.

(5) A distant telegraph post subtends at the observer on the ground an angle of $10'$. Taking the height of the post to be 15 feet, calculate approximately the distance of the post from the place of observation.

(6) A leather belt passes tightly round a circular wheel of radius 4 feet and a thin cylindrical shaft placed at a distance from the wheel. If

the two straight portions of the belt are inclined at 46° to each other, calculate the length of the belt in contact with the wheel, neglecting the thickness of the shaft.

(7) A flywheel is rotating about an axis through its centre making 200 revolutions per minute. Find its angular velocity in radians per second.

(8) A train is moving along a curve in the form of an arc of a circle radius 200 yards. If the length of the curve is 400 yards, find the angle turned through by the train in rounding this curve.⁴ How many radians will it turn through in one second taking the speed of the train to be 40 miles per hour.

(9) If the angle subtended at the centre of a circle by an arc equal in length to the diameter is 127 grades 20 minutes calculate the value of π .

(10) Find the length in inches of an object which subtends an angle of two seconds at the distance of 12 miles.

(11) The perimeter of a certain sector of a circle is equal to the length of the arc of half the circle. Show that the angle of the sector is very nearly $65\frac{1}{2}^\circ$.
[Bombay, 1892]

(12) The angles of a triangle are in arithmetic progression ; the least is a fourth part of the greatest. Find the circular measure of each angle.

(13) A person of normal sight can read printed matter when each letter subtends $7'$ at the observer's eye. Calculate the height of a letter when it is held at a distance of 20 feet. At what distance must letters 1 inch high be placed so as to be read distinctly.

(14) The greatest angle subtended by a radius of the earth at the sun is found to be $8.80''$. Calculate the distance of the sun taking the earth's radius to be 4000 miles. Also find the radius of the sun in miles given that its angular diameter is $31'$.

(15) The angular diameter of the moon is $32'$. Calculate how far from the eye a quarter anna coin of radius $\frac{1}{2}''$ must be placed so as to just hide the moon from the observer.

(16) The measure of an angle in degrees exceeds its measure in radians by two. Find the value of the angle in degrees and in radians.

(17) The sum of two angles is 60° and their difference is half a radian. Find the values of the angles in circular measure and in degrees.

(18) Two equal arcs AB, CD are measured along the circumferences of two circles of radii $3''$ and $4''$, centres O and O' . If $CO'D$ measures $24^\circ 15' 18''$, calculate the value of AOB .

(19) Find the area of a sector of circle radius $4''$ and angle 20° .

(20) Three equal circles of radii $3''$ touch one another externally two by two. Calculate the area included between the circles.

Art. 12. Latitude and Longitude.—The earth is approximately a sphere, the difference between the equatorial and polar radii being only 13.5 miles. Hence a knowledge of the geometry of the sphere is

necessary for a study of geography and geodesy. Every point on the surface of a sphere is at the same distance from a point within it. The fixed point is the *centre* and the fixed distance is the radius of the sphere. A section of the sphere by any plane is a circle ;

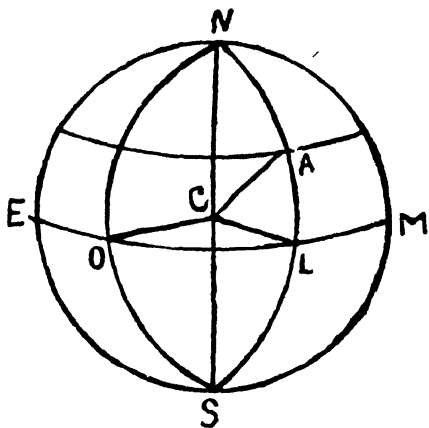


Fig. 4

i.e., all points common to the surface of the sphere and to the plane lie on the curve of intersection, which can be proved to be a circle whose centre is the foot of the perpendicular from the centre of the sphere to the plane of section. If the plane passes through C , the centre of the sphere, the circle is a *great circle*, the centre of which is at C , and the radius of which is equal to the radius of the sphere. When the plane of section does not pass through the centre, the circle got is a small circle of smaller radius than the sphere itself. Any line through the centre of the

sphere terminated by the surface of the sphere is a diameter and it is bisected at C. A section of the sphere by any plane through this line is a great circle.

Fig. 4 above represents the earth taken to be a sphere of radius r . C is its centre. NS is the diameter of the earth about which it rotates from west to east in the course of a day. It is called the earth's polar axis. N and S, the extremities of the axis, are the poles of the earth, N being the north pole and S the south pole. The great circle ELM on the earth is got by drawing a plane through C perpendicular to the polar axis, so that every point on this great circle is equidistant from either pole. It is called the *Equator*. The position of any point A on the surface of the earth is defined by two sets of co-ordinates. These are the spherical co-ordinates of A. Any great circle drawn to pass through the two poles is a *meridian of longitude*; and through each point on the surface, one and only one meridian of longitude can be drawn. Let NAS be the meridian of longitude through the place A, and let it meet the equator at L. AL, the angular distance of A from the equator measured along the meridian of longitude through A is the *latitude* of A. It is measured by the angle ACL. It is north or south according as A is situated on the same side of the equator as N or S. Madras for example is north of the equator; it is in the northern hemisphere and it has a north latitude $13^{\circ} 4'$. The latitude of the pole is 90° . Hence the latitude of a place ranges from 0° to 90° north or south. The other co-ordinate of A is the arc of the equator measured from some fixed point O of the equator to the point L, where the meridian of longitude through A meets the equator. It is measured by the angle OCL. It ranges between 0° and 180° west and an equal range east of O.

The meridian of longitude through O is the *Prime Meridian*. The Prime Meridian is taken to pass through Greenwich. Latitude and longitude determine uniquely the position of a place on the earth. The distance of A from the equator is the length of arc AL of the meridian and it is easily calculated when the radius CA of the earth and the angle ACL, the latitude of A are given. Thus, taking the radius of the earth to be 4000 miles and the latitude of a place A as θ° , A is distant from the equator $4000 \times \frac{\theta \times \pi}{180}$ miles. Similarly, the distance of

A from the north pole P is $4000 \times \frac{(90 - \theta) \times \pi}{180}$ miles.

A plane parallel to the plane of the equator meets the surface of the earth along a small circle, whose centre is the point where the plane of section meets the polar axis. All places on this circle have the same latitude : and the circle is called a *Parallel of latitude*. Through every place passes one parallel of latitude.

Ex. 1. The latitude of Madras is $13^\circ 4' \text{ N}$. Calculate its distance from N. pole of the earth.

Let A in Fig. 4 denote Madras on the globe. It is given that $\angle LCA$ is equal to $13^\circ 4'$. Hence $\angle ACN = 90^\circ - 13^\circ 4' = 76^\circ 56'$. The required distance is the length of arc AN of the great circle subtending $76^\circ 56'$ at its centre C. It is got by multiplying the radius of the earth by the radian measure of $76^\circ 56'$. Taking the earth's radius to be 4000 miles, the result is $4000 \times 1.3428 = 5371.2$ miles.

Ex. 2. A nautical mile is defined as the distance between two places on the same meridian of longitude, whose latitudes differ by $1'$. Find the equivalent in miles of a nautical mile and deduce the distance in miles between the pole and a place on the earth's equator.

$$\text{The radian measure of } 1' = \frac{60}{206265} \left(\begin{array}{l} \text{by Art. 10,} \\ \text{Ex. 1} \end{array} \right)$$

$$\begin{aligned} \text{Hence, 1 nautical mile} &= \frac{4000 \times 60}{206265} \text{ miles} \\ &= \underline{1.163 \text{ miles.}} \end{aligned}$$

The angular distance between the pole and any point of the equator is 90° . Hence the two points are distant $5400 \text{ nautical miles} = 5400 \times 1.163 \text{ miles} = 6280 \text{ miles}$.

EXAMPLES I b

(1) Greenwich has a latitude of $51^\circ 30'$ North. How far is it from the earth's equator and from the north pole, taking the earth's radius to be 4000 miles?

(2) Find the distance in miles between two places on the same meridian of longitude, the difference of their latitudes being $4^\circ 32'$.

(3) A and B are two places on the earth's equator in longitudes $20^\circ 34'$ E and $97^\circ 28'$ E. How far are they from each other?

(4) The latitude of a place in South Africa is $32^\circ 41'$ S. Calculate its distance from the equator.

(5) The Arctic Circle is the parallel of latitude corresponding to $66^\circ 33'$ N. latitude. How far is it from this circle to the pole?

(6) A globe of the earth is 6' in diameter. Find the distance in inches on the globe between two places, given that the two places have the same longitude and that their latitudes differ by $4^\circ 13'$.

(7) A place on the surface of the earth is 400 miles from the equator. What is its latitude?

(8) Find the radius of a globe, on which two places on the equator are $2''$ apart, when the difference of their longitudes is 4 hours. [1 hour of longitude = 15° .]

(9) Find the difference of latitude of two places on the surface of the earth distant 100 miles apart, taking the two places on the same meridian.

(10) A metre is defined as a ten-millionth part of the distance from the pole to the earth's equator. Find from this the value of 1 metre in feet and also the length of a nautical mile in metres. [Answer correct to 2 decimals.]

(11) Find in metres the distance of a place (Lat. $32^\circ 14'$ S) from the earth's equator, and from the north pole.

(12) Find in nautical miles the distance of two places on the same meridian (i) 20 miles apart, (ii) one kilometre apart.

CHAPTER II

TRIGONOMETRICAL FUNCTIONS AND THEIR RELATIONS TO ONE ANOTHER

Art. 13. Trigonometrical ratios defined for acute angles.—Let AOB be an acute angle of measure θ . From any point P on OB , draw PM perpendicular to OA . Then the ratios of any two of the three sides of the triangle OMP are independent of the position of the point P . Thus if P' be some other point on the line OB , and $P'M'$ drawn perpendicular to OA , it is easily seen that the two triangles OMP and $OM'P'$ are similar, and hence that the corresponding sides are proportional.

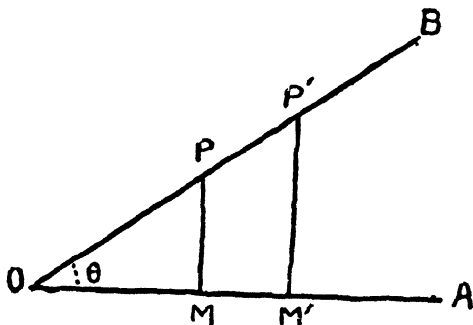


Fig. 5

Thus, $\frac{OM}{OP} = \frac{OM'}{OP'}$; $\frac{MP}{OP} = \frac{M'P'}{OP'}$; and so on for the ratios of corresponding sides. Thus, these ratios depend only on the magnitude of the angle θ . Six ratios can be formed from the three sides, as, for the measure of the ratio as a fraction we can choose the numerator in three ways and for every such way, the denominator in two ways. These six ratios have special names in Trigonometry. Thus, **with reference to angle θ** ,

$\frac{MP}{OP}$, i.e., $\frac{\text{opposite side}}{\text{hypotenuse}}$ is termed its **sine**.

OM, i.e., adjacent side is termed its **cosine**.

OP, i.e., hypotenuse

$\frac{MP}{OM}$, i.e., opposite side is termed its **tangent**.

The reciprocals of these ratios are respectively called the **cosecant**, the **secant** and **cotangent** of θ .

These trigonometric ratios are abbreviated into **sin** θ , **cos** θ , **tan** θ , **cosec** θ , **sec** θ and **cot** θ .

$$\text{Thus, Cosec } \theta = \frac{1}{\text{Sin } \theta}$$

$$\text{Sec } \theta = \frac{1}{\text{Cos } \theta}$$

$$\text{and Cot } \theta = \frac{1}{\text{Tan } \theta}$$

These six ratios are clearly positive for an acute angle.

Art. 14. Trigonometric ratios defined for angles of any magnitude.

Let $X'OX$ and YOY' be two straight lines at right angles, drawn, the one from left to right and the other from top to bottom. Let a line starting from OX , revolve about O in the counter clockwise direction and occupy the position OP . Then it is a convention to affix the positive sign to the measure of the angle XOP . The moving line may occupy the position OP , before completing a revolution, or after one or more revolutions.

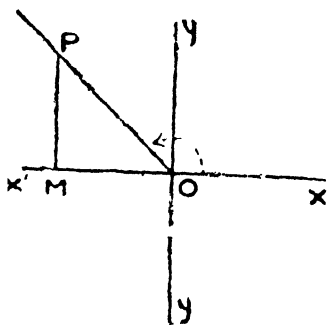


Fig. 6

Again, the same position may also be got if OP starts from OX and revolves about O in the clockwise sense. Then the measure of the angle is given the negative sign. But, so long as the initial and final positions of the revolving line are the same, the trigonometrical ratios attached to the angle described are the same, as will be seen from the following definitions.

If P be any point on OP, and a perpendicular PM to OX or XO produced be drawn, we define

the **sine** of XOP to be the ratio $\frac{MP}{OP}$,

the **cosine** of XOP to be the ratio $\frac{OM}{OP}$ and

the **tangent** of XOP to be the ratio $\frac{MP}{OM}$

The reciprocals of these are again called respectively the cosecant, the secant and the cotangent of XOP.

Care should here be taken to take the several lengths involved in the expression of the ratios with their proper signs. Thus all lengths measured along OX, that is, from O to the right carry the positive sign, while all lengths measured along OX', that is, from O to the left are given the negative sign. Again all lengths measured in the direction of OY, that is from any point in X'OX upwards take the positive sign, while all lengths measured from any point in X'OX downwards, that is, in the direction of OY' take the negative sign. Lengths measured along all other lines proceeding from the origin are always counted as positive.

Thus, it is quite material, for instance, to write the cosine of XOP as $\frac{OM}{OP}$ and *not* as $\frac{MO}{OP}$.

Art. 15. Signs of the Trigonometric Functions.

With these conventions as regards the signs, and remembering the usual convention that the four quadrants into which the lines XOX' and YOY' divide the plane are called the first, second, third and fourth quadrants respectively, in the order in which they are passed by a line starting from OX and revolving in the counter-clockwise sense, we deduce that for all angles, whose bounding lines are

- in the first quadrant, MP and OM are both positive ;
- in the second quadrant, MP is positive, but OM is negative ;
- in the third quadrant, MP and OM are both negative ;
- and in the fourth quadrant MP is negative, but OM positive.

Hence, we deduce that (i) *all* the Trigonometric ratios of angles whose bounding lines are in the first quadrant are positive ;

(ii) *only* the sine and cosecant of angles whose bounding lines are in the second quadrant are positive ;

(iii) *only* the tangent and cotangent of angles whose bounding lines are in the third quadrant are positive and

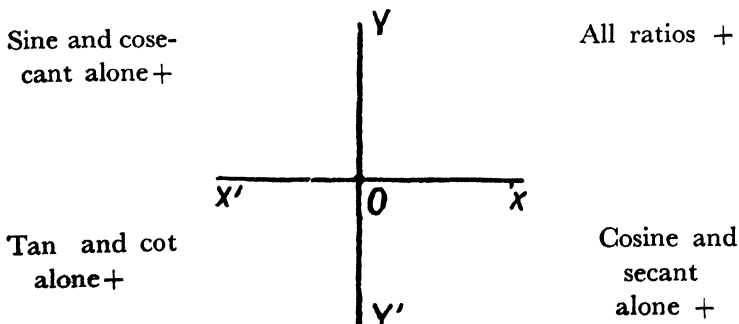


Fig. 7

(iv) *only* the secant and cosine of angles whose bounding lines are in the fourth quadrant are positive. These facts are brought out in the figure.

In particular, acute angles have all their trigonometric ratios positive, while for an obtuse angle, only its sine and cosecant are positive.

Note.—The results (i) to (iv) are easily *remembered* by the sentence ‘*All silver tea cups*’, where the four words begin with A, S, T, C corresponding to All ratios being + in I quadrant, Sine alone + in II quadrant, Tangent alone in III and Cosine alone in IV quadrant.

Art. 16. Limits to the value of the Trigonometric Functions.

As OM and MP can never exceed OP, the numerical measure of the sine and cosine of an angle can never exceed unity. Hence their reciprocals, namely, the cosecant and secant of an angle can never be below unity. As MP and OM can for different positions of OP be made as small as we please, and as they occur in the denominators of the expressions for cotangent and tangent respectively of XOP, it is clear that the cotangent and tangent can be made numerically as great as we please.

Art. 17. To construct an angle whose sine or cosecant is given.

Let the sine of an angle θ be given as k , which of course must not exceed unity. It is required to construct the angle θ .

Draw a circle centre O and radius unity. OA, OB are the radii bounding the first quadrant (Ref. Fig. 8). Along OB, measure off OL equal to k . If k is positive, L falls between O and B; if negative, on BO produced and

if zero, at O itself. Through L draw a line parallel to OA to meet the circle again at P and Q. Then AOP

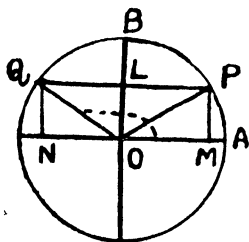


Fig. 8 (a)

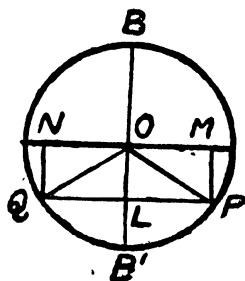


Fig. 8 (b)

and AOQ are two possible values of θ , having the given sine. For, by definition,

$$\sin \theta = \frac{MP}{OP} = \frac{k}{1} = \frac{NQ}{OQ}.$$

Note.—To construct the angle whose cosecant is given to be k , is the same as constructing one whose sine is $\frac{1}{k}$.

Art. 18. To construct the angle whose cosine or secant is given to be k , say.

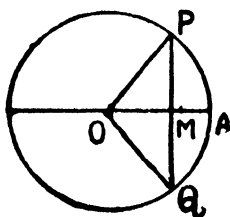


Fig. 9 (a)

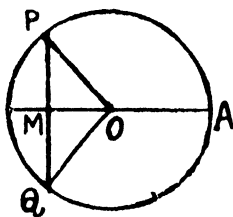


Fig. 9 (b)

Construction.—Draw as before, a circle of unit radius, and measure off along OA, or AO produced (Ref. Fig. 9) according as k is positive or negative, OM equal to the

numerical value of k . From M erect a length perpendicular to OA to cut the circle at P and Q; then AOP and AOQ are two possible values.

For by definition, in all cases,

$$\cos AOP = \frac{OM}{OP} = \frac{k}{1}$$

$$\text{and } \cos AOQ = \frac{OM}{OQ} = \frac{k}{1}.$$

Art. 19. Given the tangent and cotangent of an angle, to construct it.

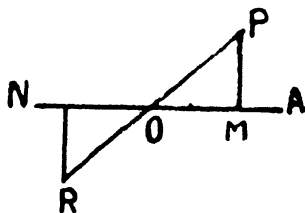


Fig. 10 (a)

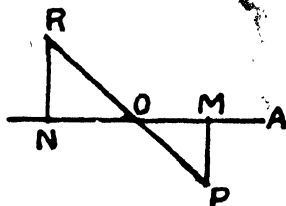


Fig. 10 (b)

Let the given tangent be equal to $\frac{q}{p}$.

Along the initial line OA or AO produced according as p is +ve or negative, measure off a length OM equal to the numerical value of p . From M, draw a line perpendicular to OA, upwards or downwards according as q is positive or negative and cut off a length MP equal to its numerical value. Then clearly,

$$\tan MOP = \frac{q}{p}.$$

If PO be produced to R so that PO = OR and RN drawn perpendicular to the initial line,

$$\tan AOR = \frac{NR}{ON} = \frac{-MP}{-OM} = \frac{q}{p}.$$

Thus, AOR is another

angle between 0° and 360° whose tangent is $\frac{q}{p}$. If the cotangent of an angle is given, its tangent is also known. Hence the angle can be constructed.

EXAMPLES II a.

- (1) Construct an angle whose cosine is $-\frac{1}{2}$.
- (2) Construct an angle whose cotangent is $\frac{1}{2}$ and show that there are two such angles between 0° and 360° .
- (3) Construct an angle whose cosecant is 3 and find its other ratios.
- (4) Are the values of θ given by the following equations real? State reasons for your answers.

(i) $2 \sin^2 \theta + \sin \theta - 1 = 0.$

(ii) $6 \cos^2 \theta + 11 \cos \theta - 6 = 0.$

(iii) $3 \tan^2 \theta + \tan \theta - 14 = 0.$

(iv) $4 \operatorname{cosec}^2 \theta + 19 \operatorname{cosec} \theta - 5 = 0.$

(v) $10 \sec^2 \theta - \sec \theta - 2 = 0.$

(vi) $2 \cot^2 \theta - 13 \cot \theta - 7 = 0.$

- (5) Construct the angle whose cotangent is 1.02. Find from the figure the other trigonometrical ratios of the angle. [Hint $1.02 = \frac{10.2}{10}$]

Art. 20. Trigonometrical ratios of certain angles.

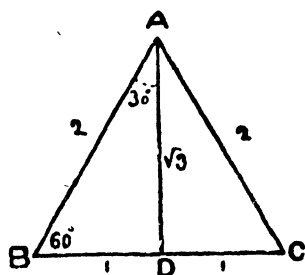


Fig. 11

(i) **Angle of magnitude 30° .**—ABC is an equilateral triangle of side 2 units. AD drawn from A perpendicular to BC bisects it, so that BD measures 1 unit. By the theorem of Pythagoras, the length of AD is $\sqrt{3}$ units. Since AD bisects the angle at A, $\angle BAD = 30^\circ$.

From the figure, $\sin 30^\circ = \frac{BD}{AB} = \frac{1}{2}$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{1}{\sqrt{3}}$$

Hence, $\operatorname{cosec} 30^\circ = 2$, $\sec 30^\circ = \frac{2}{\sqrt{3}}$ and $\cot 30^\circ = \sqrt{3}$

(ii) **Angle of magnitude 60° .**—Referring to Fig. 11, $\angle ABC = 60^\circ$ being an angle of the equilateral triangle.

Hence, from the right-angled triangle ABD,

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{BD}{BA} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \sqrt{3}$$

Also, $\operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}$, $\sec 60^\circ = 2$ and $\cot 60^\circ = \frac{1}{\sqrt{3}}$.

(iii) **Angle of magnitude 45° .**—Describe an isosceles right-angled triangle ABC in which $\angle B = 90^\circ$ and each side measures one unit.

Since $AC^2 = AB^2 + BC^2$,
 $AB/AC = 1/\sqrt{2}$.

Each of the angles A, C measures 45°

$$\text{So, } \sin 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{AB}{BC} = 1.$$

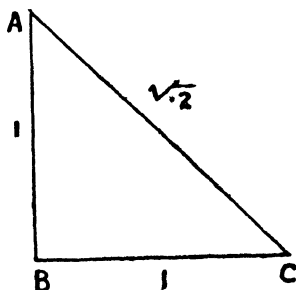


Fig. 12

$\operatorname{Cosec} 45^\circ = \sqrt{2}$, $\sec 45^\circ = \sqrt{2}$ and $\cot 45^\circ = 1$.

(iv) **Angles of magnitudes 0° , 90° , 180° , 270° .**—Draw a circle of radius *one unit*. Take A'OA and B'OB,

any two diameters of this circle perpendicular to each other. Starting from OA as the initial line, let the radius vector rotate about the centre O in the anti-clockwise direction through any angle θ . If the final position of the radius vector cuts the circle at P, draw PM perpendicular to OA meeting OA at M. Now, from the right-angled triangle MOP,

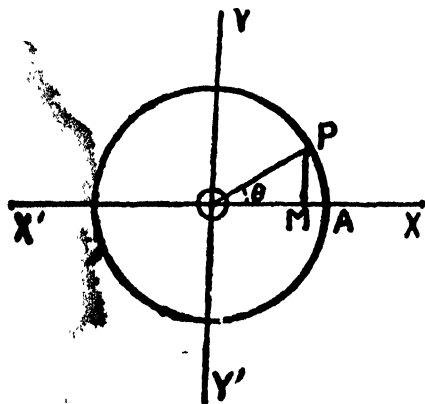


Fig. 13

$$\sin \theta = \frac{MP}{OP} = \frac{MP}{1 \text{ unit}} \\ = \text{no. of units in MP.}$$

$$\cos \theta = \frac{OM}{OP} = \text{num-} \\ \text{ber of units in OM.} \\ \text{OY, OX', OY' cut the circle in B, A', B' res-} \\ \text{pectively.}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

(a) Now, when $\theta = 0^\circ$, P and M both coincide with A. So, $MP = 0$, $OM = OA = 1$ unit. Hence, $\sin 0^\circ = 0$, $\cos 0^\circ = 1$, $\tan 0^\circ = 0$.

(b) When $\theta = 90^\circ$, P coincides with B and M with O, so that $MP = OB = 1$, $OM = 0$. So, $\sin 90^\circ = 1$, $\cos 90^\circ = 0$, $\tan 90^\circ \rightarrow \infty$.

(c) When $\theta = 180^\circ$, P coincides with A' as also M, $MP = 0$, $OM = OA' = -1$. $\sin 180^\circ = 0$, $\cos 180^\circ = -1$, $\tan 180^\circ = 0$.

(d) For $\theta = 270^\circ$, P coincides with B' so that $MP = OB' = -1$, $OM = 0$.

$$\sin 270^\circ = -1, \cos 270^\circ = 0, \tan 270^\circ \rightarrow \infty,$$

Note:— ∞ is not a number. It is the limit to which a variable $x > 1$ tends, as x increases more and more.

(v) **Angle of magnitude 360° .**—When the radius vector turns through 360° , it has made one complete revo-

lution, and comes back to the initial line wherefrom it started. Since the values of the trigonometrical functions of an angle depend only on the position of the bounding line of the angle, it follows that all ratios of an angle of 360° are identically equal to those of zero. Hence,

$$\begin{aligned}\sin 360^\circ &= 0, \\ \cos 360^\circ &= 1, \\ \tan 360^\circ &= 0, \text{ etc.}\end{aligned}$$

Art. 21. The results of the last article may be exhibited in a tabular form as follows:—

	0	30°	45°	60°	90°	180°	270°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0

The student will however do well to remember these values in relation to the geometrical figures, from which they were derived.

Art. 22. Relations between the trigonometrical functions of the same angle.

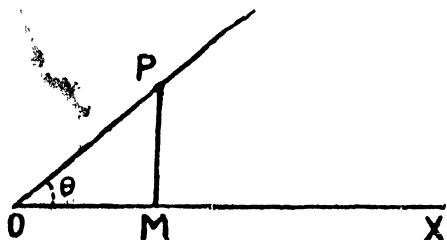


Fig. 14

XOP is any angle of magnitude θ , OX being the initial line.

By definition, cosec θ , sec θ and cot θ are reciprocals of sin θ , cos θ and tan θ respectively.

$$\text{Again, } (\sin \theta)^2 + (\cos \theta)^2 = \left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1$$

It is usual to write $(\sin \theta)^2$ and $(\cos \theta)^2$ in the form $\sin^2 \theta$ and $\cos^2 \theta$ and to read them as 'Sine square θ ' and 'Cos square θ ' respectively. Hence, we get—

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \dots \dots \dots (1)$$

$$\text{Also, } \sec^2 \theta = \frac{OP^2}{OM^2} = \frac{OM^2 + MP^2}{OM^2} = 1 + \frac{MP^2}{OM^2} \\ = 1 + \tan^2 \theta$$

$$\text{i.e., } \sec^2 \theta = 1 + \tan^2 \theta \quad \dots \dots \dots (2)$$

$$\text{Similarly, } \operatorname{cosec}^2 \theta = \frac{OP^2}{MP^2} = \frac{OM^2 + MP^2}{MP^2} \\ = 1 + \cot^2 \theta \quad \dots \dots \dots (3)$$

$$\text{Further, } \frac{\sin \theta}{\cos \theta} = \frac{MP}{OP} \times \frac{OP}{OM} = \frac{MP}{OM} = \tan \theta \quad \dots (4)$$

$$\text{and } \frac{\cos \theta}{\sin \theta} = \frac{OM}{OP} \times \frac{OP}{MP} = \cot \theta \quad \dots \dots \dots (5)$$

These are the fundamental relations that exist between the six different trigonometrical functions of the

same angle. They help us to calculate the values of the other ratios of an angle given the value of one of them.

Ex. 1. Given the sine of an acute angle, find the values of all the other ratios.

Let $\angle MOP$ be the acute angle whose sine is equal to the given number, 's' say. Take OP equal to 1 unit and draw PM perpendicular to OM .

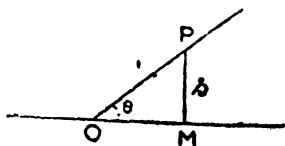


Fig. 15

$$\sin \angle MOP = \frac{MP}{OP} = \frac{MP}{1} = s \text{ (by hypothesis)}$$

$$\therefore MP \text{ measures 's' units and } OM = \sqrt{OP^2 - MP^2} = \sqrt{1 - s^2}$$

$$\text{Hence, } \cos \angle MOP = \sqrt{1 - s^2}$$

$$\tan \angle MOP = \frac{MP}{OM} = \frac{s}{\sqrt{1 - s^2}}$$

$$\operatorname{cosec} \angle MOP = \frac{1}{s}; \sec \angle MOP = \frac{1}{\sqrt{1 - s^2}}$$

$$\text{and } \cot \angle MOP = \frac{\sqrt{1 - s^2}}{s}$$

Ex. 2. Express all the trigonometrical ratios of an acute angle in terms of its cotangent.

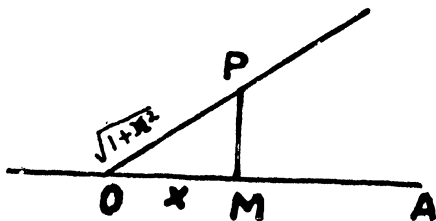


Fig. 16

Let $\angle MOP$ be the angle having the given value x for its cotangent. Choose a point P on OP , so that its distance

PM from the line OM is equal to unity. Draw PM perpendicular to OM.

From the triangle OPM, $\text{Cot MOP} = \frac{\text{OM}}{\text{MP}} = \frac{\text{OM}}{1} = x$.

Hence OM measures x units and $\therefore \text{OP} = \sqrt{1+x^2}$

$$\text{Sin MOP} = \frac{1}{\sqrt{1+x^2}}.$$

$$\text{Cos MOP} = \frac{x}{\sqrt{1+x^2}} \text{ and } \text{Tan MOP} = \frac{1}{x}.$$

The reciprocals of these ratios can easily be written down.

Art. 23. Similarly, we can express all the trigonometrical ratios of any acute angle in terms of the value of any one of them. The results are all shown in the table on the next page.

	Sine θ	Cosine θ	Tangent θ	Cosecant θ	Secant θ	Cot θ
Sin θ	$\sin \theta$	$\sqrt{1 - \cos^2 \theta}$	$\frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$	1	$\frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}$	$\frac{1}{\sqrt{1 + \cot^2 \theta}}$
Cos θ	$\sqrt{1 - \sin^2 \theta}$	$\cos \theta$	$\frac{1}{\sqrt{1 + \tan^2 \theta}}$	$\frac{\sqrt{\csc^2 \theta - 1}}{\csc \theta}$	1	$\frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}}$
Tan θ	$\frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$	$\frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta}$	$\tan \theta$	1	$\frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}$	$\frac{1}{\cot \theta}$
Cosec θ	$\frac{1}{\sin \theta}$	$\frac{1}{\sqrt{1 - \cos^2 \theta}}$	$\frac{\sqrt{1 + \tan^2 \theta}}{\tan \theta}$	$\csc \theta$	$\frac{\sec \theta}{\sqrt{\sec^2 \theta - 1}}$	$\sqrt{1 + \cot^2 \theta}$
Sec θ	$\frac{1}{\sqrt{1 - \sin^2 \theta}}$	$\frac{1}{\cos \theta}$	$\frac{\sqrt{1 + \tan^2 \theta}}{\tan \theta}$	$\frac{\csc \theta}{\sqrt{\csc^2 \theta - 1}}$	$\sec \theta$	$\frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta}$
Cot θ	$\frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta}$	$\frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}}$	$\frac{1}{\tan \theta}$	$\frac{\csc \theta}{\sqrt{\csc^2 \theta - 1}}$	$\frac{1}{\sec \theta}$	$\cot \theta$

The beginner may think it speedier to work out the question by applying the formulae of Art. 23 ; but he is advised not to commit them to memory, but to derive them from a figure. If the angle is not acute, the proper sign has to be affixed to each ratio, the sign being determined by the quadrant in which the angle is situated.

Ex. 3. The radius of a circle is 3 feet. Find the lengths of the two arcs into which the circumference is divided by a chord, which is 4 feet long.

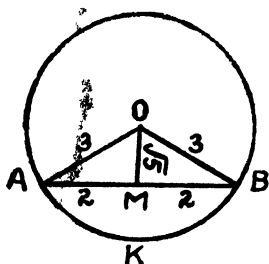


Fig. 17

AB is a chord of the circle centre O and radius $OA = 3$ units, and its length is 4 feet.

Arc AKB = radius $\times$ No. of radians in AOB.

Draw OM perpendicular to AB, and join OA, OB. It is evident that $\angle AOB = 2 \angle AOM$, and $AM = \frac{1}{2} AB$. From the right angled triangle AOM.

$$\sin AOM = \frac{AM}{OA} = \frac{2}{3} = .6667. \quad (\text{approx.})$$

Referring to the Table of sines, $\angle AOM = 41^\circ 49'$
 $= .7298$ radian

$$\therefore \text{Arc AKB} = 3 \times 2 \times .7298 \\ = 4.3788 \text{ feet.}$$

The length of the whole circumference $= 2\pi r$
 $= 18.8496 \text{ feet.}$

Hence, subtracting the length of the minor arc, the length of remaining arc $= 14.4708 \text{ feet.}$

EXAMPLES II *b*.

1. Given that the secant of an angle is 9, find the values of the other ratios of the angle.

2. If the cosine of an angle is 0.07, find its tangent and cosecant.

3. If $\tan A = \frac{3}{4}$, find the other ratios of A.

4. Express the value of $\frac{a \cos \theta + b \sin \theta + c \tan \theta}{d \cot \theta + c \operatorname{cosec} \theta}$

in terms of m and n , given that $\sec \theta = \frac{m}{n}$ and θ is acute

5. PQR is a right-angled triangle in which $P = 90^\circ$, and $Q = 30^\circ$. QS is drawn bisecting the angle at Q, to meet the opposite side in S. Calculate from the figure the values of all the trigonometric ratios of 15° .

6. PQR is a right-angled isosceles triangle, right-angled at P. The angle at Q is bisected by the line QS drawn to the opposite side. Use this figure to calculate the values of $\sin 22\frac{1}{2}^\circ$, $\cos 22\frac{1}{2}^\circ$ and $\tan 22\frac{1}{2}^\circ$.

7. AB is a chord of a circle subtending an angle of 90° at the centre of a circle of radius 10 feet. Calculate the lengths of the two arcs into which the circumference is divided by the chord.

8. If in the former question, the chord AB subtends (i) 120° or (ii) 60° at the centre, calculate the lengths of the arcs in the two cases.

9. A rod of length 8 feet lies wholly within a smooth hemispherical bowl of radius 5 feet. Find the distance of the rod from the centre and the angle subtended by the rod at the centre.

10. Given $\sin \theta = -\frac{12}{13}$, in which quadrants may θ lie? Calculate the values of $\cos \theta$ and $\cot \theta$ in each case.

11. If $\cos \theta = -\frac{7}{9}$, show that θ lies in either of the second and third quadrants. Calculate the values of the other ratios of θ , for both the values of θ .

12. For a tangent galvanometer, the current c is given by $c = K \tan \theta$, where $K = 1.769$ and θ is the angle of deflection. Find the current flowing in the circuit, when the deflection recorded is (i) 30° , (ii) 15° , (iii) 45° , (iv) 60° .

Art. 24. Trigonometrical identities :—Depending upon the results of the fundamental identities connecting the trigonometrical ratios of any angle, it is possible to build up a number of identities. The two sides of an identity contain two different expressions in the trigonometrical functions of one or more angles, and the difference lies only in their form. When they are reduced separately, they lead to the same result for *all* values of the variable angles involved. Conversely, given an identity if we are asked to prove it, we reduce one side of the identity to a simpler form by several steps applying the well known formulae, and show this to be equal identically to the other side. There are some cases when both the sides are quite complicated expressions, so much so, it is not possible to prove it by the method indicated. In such a case, reduce each side separately and show that the simplified results are equal to each other. To put the beginner in the way, some rules are given for guidance.

(1) Start with the more complicated expression of the two.

(2) The cosecant and secant being less familiar than their reciprocals, replace them by their values in terms of the sine and cosine of the angle, if it may help the simplification.

(3) Some expressions get more easily reduced by putting $\frac{\sin A}{\cos A}$ for $\tan A$ and $\frac{\cos A}{\sin A}$ for $\cot A$ and by subsequently using the formula $\cos^2 A + \sin^2 A = 1$.

(4) Avoid radicals as far as possible.

Ex. 1. Prove that $(\operatorname{cosec} A + \sin A)(\operatorname{cosec} A - \sin A) = \cot^2 A + \cos^2 A$.

$$\begin{aligned}
 \text{Left hand side} &= \left(\frac{1}{\sin A} + \sin A \right) \times \left(\frac{1}{\sin A} - \sin A \right) \\
 &= \frac{1 + \sin^2 A}{\sin A} \times \frac{1 - \sin^2 A}{\sin A} \\
 &= \frac{(1 + \sin^2 A) \cdot \cos^2 A}{\sin^2 A} \\
 &= \frac{\cos^2 A}{\sin^2 A} + \cos^2 A = \cot^2 A + \cos^2 A
 \end{aligned}$$

Ex. 2. Show that $\sin^6 A - \cos^6 A = (\sin^2 A - \cos^2 A) \times (1 - 2 \sin^2 A \cos^2 A)$.

$$\begin{aligned}
 \sin^6 A - \cos^6 A &= (\sin^4 A - \cos^4 A) (\sin^2 A + \cos^2 A) \\
 &= (\sin^2 A + \cos^2 A) (\sin^2 A - \cos^2 A) (\sin^2 A + \cos^2 A) \\
 &= (\sin^2 A - \cos^2 A) [(\sin^2 A + \cos^2 A)^2 - 2 \sin^2 A \cos^2 A], \\
 &\quad \text{since } \sin^2 A + \cos^2 A = 1. \\
 &= (\sin^2 A - \cos^2 A) \cdot (1 - 2 \sin^2 A \cos^2 A).
 \end{aligned}$$

Note.—This is an application of the algebraical identity $a^2 - b^2 = (a + b)(a - b)$, and of the trigonometrical formula $\sin^2 A + \cos^2 A = 1$.

Ex. 3. Prove that

$$\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A$$

$$\begin{aligned}
 \text{Left hand side} &= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}} \\
 &= \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A} \\
 &= \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} \\
 &= \cos A + \sin A
 \end{aligned}$$

EXAMPLES II c.

Prove the following identities :—

1. $\tan A - \cot A = (\tan A - 1) \cdot (\cot A + 1).$
2. $\tan A + \cot A = \operatorname{cosec} A \cdot \sec A.$
3. $2 + \sin^2 \theta + \operatorname{cosec}^2 \theta = (\sin \theta + \operatorname{cosec} \theta)^2.$
4. $\frac{\cot \theta - \tan \theta}{\cot \theta + \tan \theta} = 1 - 2 \sin^2 \theta.$
5. $\frac{\tan A - \tan B}{\cot B - \cot A} = \frac{\tan B}{\cot A}.$
6. $\frac{\tan B + \tan A}{\cot B + \cot A} = \tan A \cdot \tan B.$
7. $\frac{1 - \cot^2 A}{1 + \cot^2 A} = 1 - 2 \cos^2 A.$
8. $\frac{1 - \sin \theta}{1 + \sin \theta} = (\sec \theta - \tan \theta)^2$
9. $\frac{1 + \cos \theta}{1 - \cos \theta} = (\operatorname{cosec} \theta + \cot \theta)^2$
10. $\sin^4 A + \cos^4 A = 1 - 2 \sin^2 A \cos^2 A.$
11. $\sin^6 A + \cos^6 A = 1 + 3 \sin^4 A - 3 \sin^2 A.$
12. $\cot^2 A \frac{\sec A - 1}{1 + \sin A} + \sec^2 A \frac{\sin A - 1}{1 + \sec A} = 0.$
13. $(\tan A - \cot A) \sin A \cos A = 1 - 2 \cos^2 A.$
14. $\sin^2 A \cos^2 B - \sin^2 B \cos^2 A = \sin^2 A - \sin^2 B$
 $= \cos^2 B - \cos^2 A$
15. $(2 \tan A - \sec A) \cos A (2 \sec A + \tan A)$
 $= 3 \tan A - 2 \cos A.$
16. $2 (1 + \sin A) (1 + \cos A) = (1 + \sin A + \cos A)^2.$
17. $\cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A - \sin^2 B.$
18. $(\cot A - 3) (3 \cot A - 1) = (3 \operatorname{cosec}^2 A - 10 \cot A).$
19. If $a \cos \theta + b \sin \theta = c$, prove that
 $a \sin \theta - b \cos \theta = \sqrt{a^2 + b^2 - c^2}.$
20. Prove that $\sqrt{2} \cdot \sin A = \sin \theta - \cos \theta$, where
 $\tan A = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}.$

CHAPTER III

EASY PROBLEMS ON HEIGHTS AND DISTANCES

Art. 25. The Trigonometrical ratios of an angle being defined in terms of the sides of a right-angled triangle suitably drawn, it follows that we are now in a position to deal with the computation of the elements of a right-angled triangle, given one side and any trigonometrical function of one angle. The immediate application of this principle is made use of in solving easy questions on heights and distances. More difficult problems of the kind and easier methods of working them out by the use of logarithms will be the subjects of a later chapter.

Art. 26. Angles of elevation and depression.—The *projection* of a point on a line (or a plane) is the foot of the perpendicular drawn from this point on the line (or plane). The projection of a line on a plane is the locus of the projections of all the points of the line on the plane. It is easily seen that the point of intersection of the line and plane projects into itself, and the projection of a straight line is a straight line through its point of intersection with the plane. *The angle between a straight line and a plane* is defined to be the acute angle between the straight line and its projection on the plane.

When one views external objects, not in level with his eye, his line of sight has to be tilted from the horizontal either upwards or downwards according as the

object observed is above or below his eye level. The angle turned through, measured from the horizontal is termed '*angle of elevation*' in the first case and '*angle of depression*' in the second. They are mathematically defined as the angle between the horizontal plane through the observer's eye and the line joining the eye to the object (considered as a point).

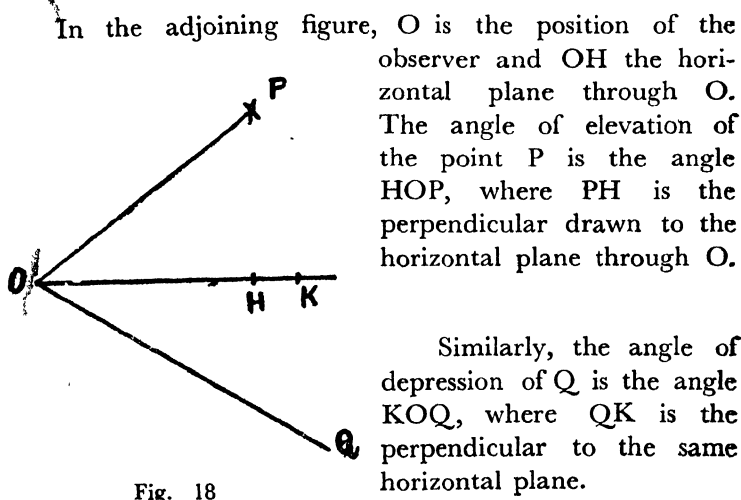


Fig. 18

Art. 27. Shadows.—Any body opaque to the transmission of light rays casts a shadow in the presence of a source of light. We cast shadows on the ground as we walk along the street ; and the telegraph post casts a shadow on the ground. The direction of the shadow is determined by the direction of the source of light relative to the object, being in the opposite direction to it and its length depends on the elevation of the source. Taking the case of a vertical rod h ft. high from the

ground, it casts a shadow of length $h \cot \theta$, where θ is the angle of elevation of the light, with respect to the top of the rod.

Referring to the adjoining figure CD is the rod, B is the source of light, and CD' is the shadow, D' being its extremity.

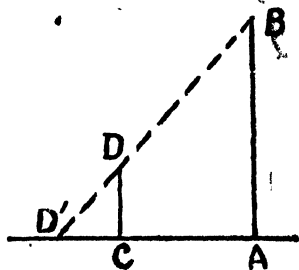


Fig. 19

| $CD'D = \theta$, being the elevation of B as seen from D. From the right-angled triangle CD'D,

$$\frac{CD'}{CD} = \cot \theta,$$

$$CD' = CD \times \cot \theta = \underline{h \cot \theta}$$

Art. 28. The following examples illustrate the methods of working out questions on heights and distances by the trigonometrical method.

Ex. 1. An observer on the ground finds the angle of elevation of the top of a temple tower to be 60° . Find the height of the tower from the ground level, given that his distance from the tower is 230 feet.

Let Q be the top of the tower, and P the point on the ground vertically below it.

A is the position of the observer, and $\angle QAP = 60^\circ$.

It is required to calculate the length of PQ.

From the right-angled triangle APQ,

$$\frac{PQ}{AP} = \tan \angle QAP = \tan 60^\circ = \sqrt{3}.$$

$$\therefore PQ = AP \times \sqrt{3} = 230 \times \sqrt{3} = 230 \times 1.732 = 398.36.$$

The height of the tower is equal to 398.36 feet.

Ex. 2. To find the height of an inaccessible mountain, an observation of the elevation of the top is taken from a place, the angle of elevation being 45° . On moving a distance of 2000 feet towards the mountain, the elevation of the top is 60° . Find the height of the mountain.

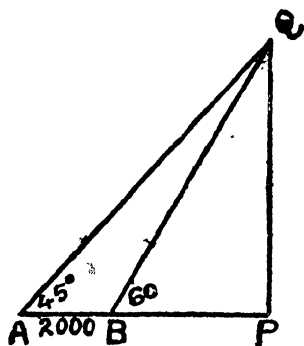


Fig. 20

and 2000 feet apart. It is required to find length of PQ.

From the right-angled triangle APQ, $\frac{AP}{PQ} = \cot 45^\circ$.

$$\therefore AP = PQ \cot 45^\circ.$$

Similarly, from the triangle BPQ, we get

$$BP = PQ \cot 60^\circ.$$

$$\text{Subtracting, } AP - BP = PQ \left\{ \cot 45^\circ - \cot 60^\circ \right\}$$

$$\text{i.e., } 2000 = PQ \left[1 - \frac{1}{\sqrt{3}} \right]$$

Simplifying, $PQ = 4732$

The height of the mountain = 4732 feet.

Ex. 3. The elevations of the top of a tower from the top and bottom of a small mound 50 feet high are

found to be 45° and 60° respectively. Find the height of the tower.

AB is the mound, and PQ is the tower. BM is the horizontal line through B in the vertical plane containing the mound and the tower, so that $\angle QBM = 45^\circ$ and $\angle QAP = 60^\circ$.

It is required to calculate the length PQ.

From the right-angled triangle BQM, $\frac{QM}{BM} = \tan 45^\circ = 1$

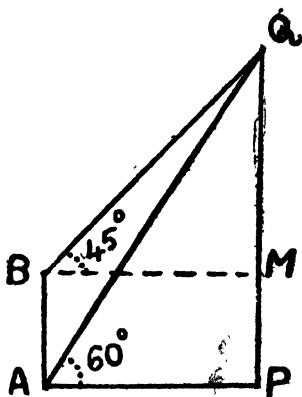


Fig. 21

$$\therefore QM = BM$$

Similarly, from the right-angled triangle AQP

$$QP = AP \times \tan 60^\circ = AP \times \sqrt{3}.$$

But $BM = AP$.

$$\therefore \frac{QP}{QM} = \frac{\sqrt{3}}{1}, \text{ i.e., } \frac{QP}{QP - MP} = \frac{\sqrt{3}}{1}$$

$$\text{i.e., } \frac{QP}{QP - 50} = \frac{\sqrt{3}}{1} \therefore \frac{QP}{50} = \frac{\sqrt{3}}{\sqrt{3} - 1}$$

$$\therefore QP = \frac{50\sqrt{3}}{\sqrt{3} - 1} = \frac{50\sqrt{3}(\sqrt{3} + 1)}{2} = 25(3 + \sqrt{3}) \\ = 25 \times 4.732 = 118.3.$$

The height of the tower is therefore 118.3 feet.

Ex. 4. From the top of a lighthouse 120 feet high the angles of depression of two ships at anchor in the harbour in line with the foot of the lighthouse are found to be 60° and 45° . Find the distance between the ships.

PQ is the lighthouse, and A, B the two ships. It is given that PQ measures 120 feet and the length AB is required.

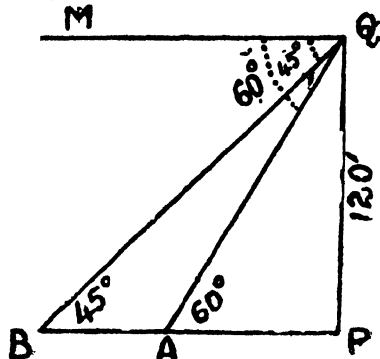


Fig. 22

$$\text{But } \frac{PB}{PQ} = \cot 45^\circ$$

$$\text{and } \frac{PA}{PQ} = \cot 60^\circ$$

$$\therefore PB = PQ \text{ and}$$

$$PA = PQ \cot 60^\circ = \frac{PQ}{\sqrt{3}}$$

$$\text{Substituting, } AB = PQ \left\{ 1 - \frac{1}{\sqrt{3}} \right\} = 50.72$$

The distance between the ships is 50.72 feet.

Ex. 5. To find the width of a river, a surveyor observes a tree on the opposite bank to be so situated that the line joining him to the tree makes an angle of 45° with the stream. On going 200 feet up the stream along the bank, he finds the angle to have changed to 30° . Calculate the width of the river.

The adjoining figure is a horizontal section through the two places of observation A and B. T is the foot of the tree on the opposite bank.

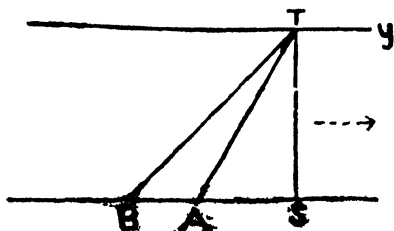


Fig. 23

$\angle TAS = 45^\circ$ and $\angle TBS = 30^\circ$. AB is given to be 200 feet.

The length of the perpendicular TS from T to AB is the width required.

From triangle BTS, $BS = TS \cot 30^\circ = \sqrt{3} \cdot TS$ and from triangle ATS, $AS = TS \cot 45^\circ = TS$.

Subtracting, $200 = BA = (\sqrt{3} - 1) TS$.

$$\therefore TS = \frac{200}{\sqrt{3} - 1} = 100 (\sqrt{3} + 1) = 273.2.$$

The width of the stream is therefore 273.2 feet.

Ex. 6. Find the length of the shadow of a gnomon of height 7 feet, standing vertically on the ground, when the sun is at an altitude of 60° ; and find the height of a tower which casts a shadow 173 feet long, then.

[*N.B.*—The sun, being at a very great distance from the earth, may for purposes of problems on shadows, be taken as a point source of light, the angle of elevation of which is the same at the same time for all points considered in such problems.]

The length of the shadow of the gnomon is evidently equal to $7 \times \cot 60^\circ$ feet.

$$\begin{aligned} &= 7 \times \frac{1}{\sqrt{3}} \text{ feet.} \\ &= \frac{7\sqrt{3}}{3} \text{ feet.} \end{aligned}$$

As to the latter part of the question, PQ' is given to be 173 feet and $QQ'P = 60^\circ$, where PQ' is the shadow of the tower PQ .

$$\frac{PQ}{Q'P} = \tan 60^\circ = \sqrt{3}.$$

$$PQ = \sqrt{3} Q'P = 173 \times \sqrt{3}.$$

The height of the tower is $173 \sqrt{3}$ feet.

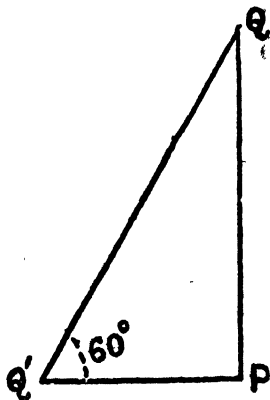


Fig. 24

Ex. 2. The altitude of a tower is observed to be 45° . After walking a distance of 200 feet up a slope inclined at 30° to the horizontal, the altitude is 60° . Find the height of the tower. T is the top and F the foot of the tower. A and B are the two places of observation and AB is the slope.

AB = 200 feet.

Let $h = FT$.

From triangle ATF,

$$AF = h \cot 45^\circ$$

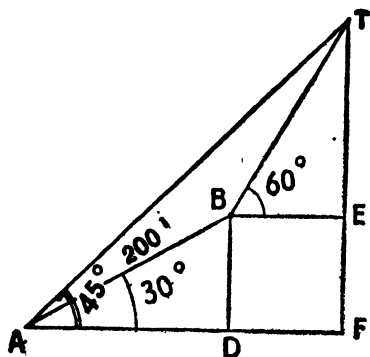


Fig. 25

$$= h = AD + DF = AB \cos 30^\circ + BE$$

$$\text{i.e., } BE = h - 200 \sqrt{3}/2 = h - 100 \sqrt{3} \dots \dots \dots (1)$$

Again in triangle BTE, $TE = FT - FE$

$$\begin{aligned} \text{i.e., } TE &= h - BD = h - AB \sin 30^\circ \\ &= h - 100 \end{aligned}$$

$$\text{and } \frac{BE}{TE} = \cot 60^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore BE = TE/\sqrt{3} = (h - 100)/\sqrt{3} \dots \dots \dots (2)$$

From (1) and (2), $h - 100 \sqrt{3} = (h - 100)/\sqrt{3}$

$$\text{i.e., } \sqrt{3} h - 300 = h - 100$$

$$\text{i.e., } h(\sqrt{3} - 1) = 200 ; h = \underline{273.2}.$$

EXAMPLES III

1. A statue 7 feet high stands on the top of a column. The angles of elevation of the top and bottom of the statue being 60° and 45° respectively, find by calculation the height of the column and the distance of the observer from the foot of the column.

2. A tower and a flagstaff placed on it subtend equal angles of 30° each at a point on the ground 62 feet off. Calculate the heights of the tower and flagstaff.

3. At a point A on the ground, the elevation of a distant mountain top is observed to be 45° . After ascending 75 feet vertically in an aeroplane, the altitude of the same point is found to be 30° . Find the height of the mountain.

4. The angles of depression of a buoy in the sea as seen from two places, A and B one vertically above the other are 60° and 45° . Find the height of A above B if it is known that the buoy is situated 500 yards off from AB.

5. Two ships A and B are sighted from the top of a lighthouse, and their angles of depression are found to be 60° and 45° . If they are in line with the foot of the lighthouse find the distance between the ships, given that the height of the lighthouse is 300 feet from sea-level.

6. A balloon is observed to have an elevation of 45° from each of two places A and B distant two miles from each other. Find the height of the balloon if the balloon is vertically above a point between the observers.

7. A man sailing due south observes two hill tops in directions 30° east of south and 60° east of south. An hour afterwards, the hill tops are found to be in line with the ship to his east. Find the distance between the hills if the speed of the ship is 15 miles an hour.

8. A lighthouse facing east sends out a fan-shaped beam extending from N.E. to S.E. A steamer sailing due north first sees the light when 7 miles away from the lighthouse and continues to see it for 40 minutes. What is the speed of the steamer?

9. A person standing on the right bank of a stream finds the altitude of a tree on the other bank just opposite to him to be 60° . On going back a distance of 100 feet in a direction perpendicular to the stream, the elevation of the tree is found to be 45° . Find the width of the stream and the height of the tree.

10. A man standing at a point A on the bank AB of a river observes that the line joining A to a point C on the other bank makes an angle of 30° with AB. He then goes a distance of 175 yards along the bank to B and finds the angle ABC to be 60° . Find the width of the river.

11. An observer facing east observes a train moving from north to south. If at two instants of time separated by 15 minutes, the train is found to be 60° and 45° north of east, calculate the speed of the train in terms of the shortest distance of the train from the observer.

12. Two towers stand on a horizontal plane. A & B are two points on the line joining their feet. From A, the angles of elevation of the towers are 30° and 60° and as seen from B are 45° and 30° . If AB is 40 feet, find the heights of the towers and the distance between them.

13. PQ is a horizontal line 350 yards long. A balloon ascends from a point R in the line PQ vertically. After a time, its elevations from P and Q are found to be 45° and 30° . Half an hour later, it is seen at an altitude of 60° from P. What is the height of the balloon at the first instant of observation and calculate the average velocity of the balloon between the two observations?

14. At the foot of a mountain, the angle of elevation of its summit is 45° . On climbing up an incline of 30° to the horizontal a distance of 600 feet, the elevation is found to be 60° . Calculate the height of the mountain from the ground and its distance from the first place of observation.

15. The shadow of a tower when the altitude of the sun is 60° measures 125 feet. What is the change in the length of the shadow when the sun has gone down by 30° ?

CHAPTER IV

ANGLES OF ANY MAGNITUDE

Art. 29. The trigonometrical functions of an angle have been defined in the preceding chapters for an angle of any magnitude whatever, not necessarily between 0° and 360° . P Being any point taken on the final position of the radius vector, PM is drawn perpendicular to the initial line or the line produced, to meet it in M. MP is then the opposite side and is taken as positive or negative according as MP is in the direction of OY or OY' ; OM is the adjacent side and is positive if it is along OX and negative if along OX'. The radius vector is always taken to be a positive length. With these conventions in regard to the signs of the lengths concerned, the trigonometrical ratios of an angle are positive for some values of the angle, and negative for others. In the following articles, the trigonometrical ratios of angles of certain forms, $-\theta$, $(90^\circ - \theta)$, $(90^\circ + \theta)$, $(180^\circ - \theta)$, $(180^\circ + \theta)$, etc., will be evaluated in terms of those of θ .

Art. 30. Trigonometrical ratios of $(-\theta)$ in terms of those of θ .

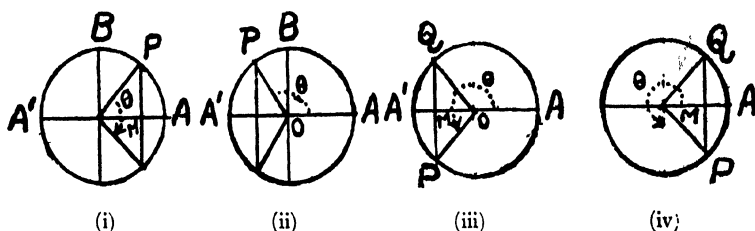


Fig. 26

OA is the initial line, and the radius vector, starting from OA revolves about O in the anti-clockwise direction

through any angle θ . OP is the final position of the radius vector. The four possible positions of OP in any one of the four quadrants are exhibited in the four adjoining figures. With O as centre and any convenient radius draw a circle to cut OP at P, and OA, OA' at A, A' respectively. PMQ is drawn perpendicular to the initial line to cut the line at M and the circle again at Q. In all positions of OP, AOP measures θ and AOQ = $-\theta$;

for $\triangle OMP \equiv \triangle OMQ$.

Also, MQ = $-MP$ and OQ = OP both in sign and magnitude.

$$\sin(-\theta) = MQ/OQ = -MP/OP = -\sin \theta.$$

$$\cos(-\theta) = OM/OQ = OM/OP = +\cos \theta.$$

$$\tan(-\theta) = MQ/OM = -MP/OM = -\tan \theta.$$

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta.$$

$$\sec(-\theta) = +\sec \theta.$$

$$\cot(-\theta) = -\cot \theta.$$

Art. 31. Trigonometrical ratios of $(180^\circ - \theta)$ in terms of those of θ .

Let OP be the position of the radius vector when it has revolved in the positive direction through an angle θ from the initial line OA. With O as centre and any convenient radius, draw a circle to cut OP at P. PQ is drawn parallel to the initial line to cut the circle again at Q. PM and QN are the perpendiculars from P and Q to A'OA. Now, AOP being equal to θ , AOQ measures $180^\circ - \theta$; for $\triangle MOP \equiv \triangle NOQ$.

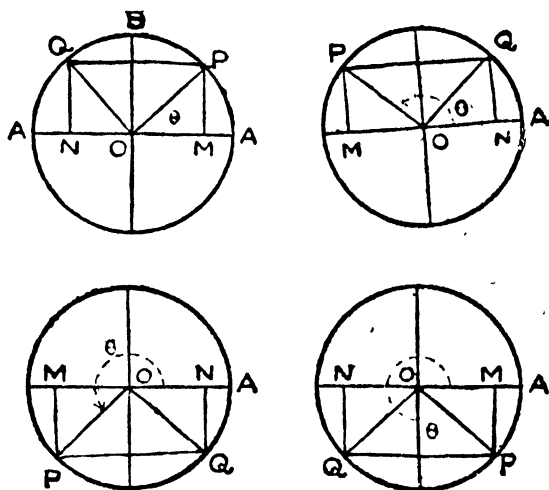


Fig. 27

Also, for all positions of OP, $NQ = + MP$
 $ON = - OM$; $OQ = OP$.

$$\sin (180^\circ - \theta) = NQ/OQ = MP/OP = + \sin \theta.$$

$$\cos (180^\circ - \theta) = ON/OQ = - OM/OP = - \cos \theta.$$

$$\tan (180^\circ - \theta) = NQ/ON = MP/- OM = - \tan \theta.$$

$$\operatorname{cosec} (180^\circ - \theta) = + \operatorname{cosec} \theta.$$

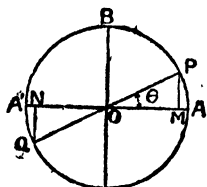
$$\sec (180^\circ - \theta) = - \sec \theta.$$

$$\cot (180^\circ - \theta) = - \cot \theta.$$

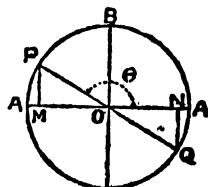
Art. 32. Trigonometrical ratios of $(180^\circ + \theta)$ in terms of those of θ .

OA is the initial line and AOP is measured in the positive direction equal to θ . If OP be produced backwards to meet the circle centre O and any convenient

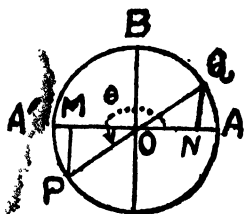
radius OQ at Q , then it is clear from the various figures that whatever may be the value of θ , the angle AOQ



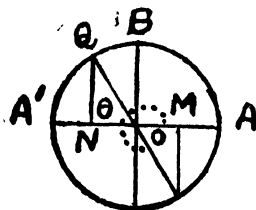
(i)



(ii)



(iii)



(iv)

Fig. 28

measures $180^\circ + \theta$. Draw MP and NQ perpendicular to the initial line. $\triangle ONQ \equiv \triangle OMP$.

$$NQ = -MP; ON = -OM; OQ = OP.$$

$$\sin(180^\circ + \theta) = NQ/OQ = -MP/OP = -\sin \theta.$$

$$\cos(180^\circ + \theta) = ON/OQ = -OM/OP = -\cos \theta.$$

$$\tan(180^\circ + \theta) = NQ/ON = -MP/-OM = MP/OM = \tan \theta.$$

$$\operatorname{cosec}(180^\circ + \theta) = -\operatorname{cosec} \theta.$$

$$\sec(180^\circ + \theta) = -\sec \theta.$$

$$\cot(180^\circ + \theta) = \cot \theta,$$

Art. 33. Trigonometrical ratios of $(90^\circ - \theta)$ in terms of those of θ .

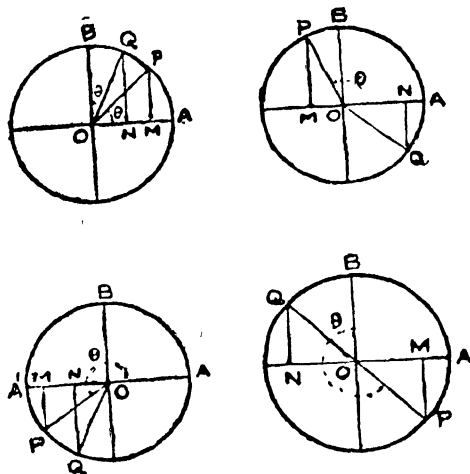


Fig. 29

The radius vector, starting from the initial line OA revolves through any angle θ and occupies the final position OP , so that $\angle AOP$ measured in the anti-clockwise direction is of magnitude θ . According to the different values of θ , OP will occupy any one of the positions marked in the figures annexed. To mark the angle $(90^\circ - \theta)$, let the radius vector start from OB and revolve through an equal angle (*i.e.*, θ) in the clockwise direction, occupying the final position OQ . Then $\angle AOQ = 90^\circ - \theta$. Draw a circle centre O cutting OP , OQ at P , Q . Draw PM and QN perpendicular to OA or AO produced.

$$\triangle NOQ \equiv \triangle MOP. \quad ON = MP; \quad NQ = OM; \quad OQ = OP.$$

$$\therefore \sin(90^\circ - \theta) = NQ/OQ = OM/OP = \cos \theta.$$

$$\cos(90^\circ - \theta) = ON/OQ = MP/OP = \sin \theta.$$

$$\tan (90^\circ - \theta) = NQ / ON = OM / MP = \cot \theta.$$

$$\operatorname{cosec} (90^\circ - \theta) = \sec \theta.$$

$$\sec (90^\circ - \theta) = \operatorname{cosec} \theta.$$

$$\cot (90^\circ - \theta) = \tan \theta.$$

Thus the **cosine** of an angle is the **sine** of its complement, and *vice versa*.

the **cotangent** of an angle is the **tangent** of its complement, and *vice versa*.

the **cosecant** of an angle is the **secant** of its complement and *vice versa*.

Or any trigonometrical ratio of an angle is the cofunction of the complement of the angle.

Art. 34. Trigonometrical ratios of $(90^\circ + \theta)$ in terms of those of θ .

Let the revolving line, starting from OA, describe any angle θ , occupying the final position OP. To get $(90^\circ + \theta)$,

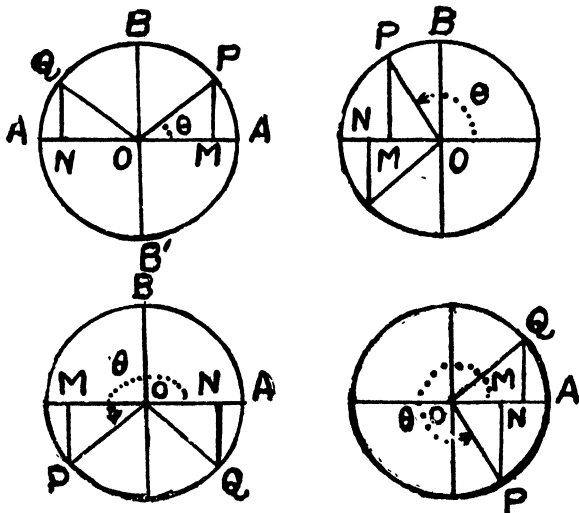


Fig. 30

the radius vector turns through an additional angle of 90° starting from OP in the same positive direction of rotation, occupying the position OQ, so that angle AOQ represents $90^\circ + \theta$, when AOP measures θ . Draw PM and QN perpendicular to the initial line to meet it in M and N respectively. Whatever be the size of the angle θ , the triangles OMP and ONQ, are equal in all respects.

$$NQ = OM ; ON = - MP ; OQ = OP.$$

$$\text{Hence, } \sin (90^\circ + \theta) = NQ/OQ = OM/OP = \cos \theta.$$

$$\cos (90^\circ + \theta) = ON/OQ = - MP/OP = - \sin \theta.$$

$$\tan (90^\circ + \theta) = NQ/ON = + OM/-MP = - OM/MP \\ = - \cot \theta.$$

$$\operatorname{cosec} (90^\circ + \theta) = \sec \theta.$$

$$\sec (90^\circ + \theta) = - \operatorname{cosec} \theta, \text{ and}$$

$$\cot (90^\circ + \theta) = - \tan \theta.$$

Art. 35. Trigonometrical ratios of ($n. 360^\circ + \theta$) in terms of those of θ , for all integral values of n .

Each revolution of the radius vector increases the angle of rotation by 360° . Hence, θ and $n. 360^\circ + \theta$ are co-terminal angles, *i.e.*, the final position of the radius vector for $n. 360^\circ + \theta$ is the same as that for θ , n being any integer positive or negative. Hence, every trigonometrical function of $n. 360^\circ + \theta$ is equal to the same function of θ ; thus, $\sin (n. 360^\circ + \theta) = \sin \theta$; $\tan (n. 360^\circ + \theta) = \tan \theta$ and so on.

Art. 36. Trigonometrical ratios of an angle expressed in terms of those of a positive acute angle less than 45° .

In trigonometrical work, we get angles of all magnitudes, but to prepare a table of trigonometrical ratios for all angles

is tedious, and also unnecessary. In Clark's Mathematical Tables, the values of sines, cosines and tangents of angles from 0° to 90° alone are given for an interval of $1'$, since it is possible to express the values of the ratios of **any** angle in terms of these. If the given angle exceeds 360° numerically, subtract from or add to it 360° as many times as will make the **result** in each case equal to a positive angle lying between 0° and 360° . As an example, $\sin 952^\circ = \sin (952^\circ - 2 \times 360^\circ) = \sin 232^\circ$; $\tan (-1543^\circ) = \tan (-1543^\circ + 5 \times 360^\circ) = \tan 257^\circ$. The angle now assumes a positive value less than 180° or a value of the form $(180^\circ + \theta)$ where θ lies between 0° and 180° . Any trigonometrical ratio of $(180^\circ + \theta)$ can be expressed in terms of those of θ by the formulae of Art. 32, $180^\circ > \theta > 0$.

As it stands now, θ may be a positive acute angle or a positive obtuse angle; but a positive obtuse angle is written in the form $(180^\circ - Q)$, $0 > Q > 90^\circ$ and the circular functions of θ can easily be written down in terms of those of Q using the results of Art 31. For example,

$$\begin{aligned}\sin 232^\circ &= \sin (180^\circ + 52^\circ) = -\sin 52^\circ \\ \tan 257^\circ &= \tan (180^\circ + 77^\circ) = +\tan 77^\circ \\ \cos 300^\circ &= \cos (180^\circ + 120^\circ) = -\cos 120^\circ \\ &= -\cos (180^\circ - 60^\circ) = +\cos 60^\circ.\end{aligned}$$

We have thus reduced the circular functions of an angle of any magnitude to those of a positive acute angle. A ready reference to Clark's tables for the ratio required, taking the reduced acute angle for argument leads to the value of the function of the given angle.

But, in some other Tables, only functions of positive acute angles less than 45° are tabulated as in Chamber's

Seven-Figure Tables. The argument here to be referred to is an angle of magnitude less than 45° . To effect this reduction, use the rule for complementary angles.

‘Any trigonometrical ratio of an angle is equal to the co-function of its complement.’ Thus, in the examples previously chosen, we have

$$-\sin 52^\circ = -\sin (90^\circ - 38^\circ) = -\cos 38^\circ$$

$$\tan 77^\circ = \tan (90^\circ - 13^\circ) = +\cot 13^\circ.$$

$$\cos 60^\circ = \cos (90^\circ - 30^\circ) = \sin 30^\circ, \text{ so that finally}$$

$$\sin 952^\circ = -\cos 38^\circ; \text{ and}$$

$$\tan (-1543^\circ) = \cot 13^\circ.$$

Art. 37. In the first Chapter, it has been noted that a given angle, of whatever magnitude it may be, has one and *only one* value for each one of its trigonometrical functions.

For example, $\sin 120^\circ = \sqrt{3}/2$, and there is no other value possible for the same. But, there are several angles, having the same number for their sine or cosine, etc. Thus, $1/\sqrt{2}$ is the sine of not only 45° but also of 135° , ($n. 360^\circ + 45^\circ$) and ($n. 360^\circ + 135^\circ$), n being any integer positive or negative. Again $\frac{1}{2}$ is the cosine of 60° and also of ($n. 360^\circ + 60^\circ$); (n being an integer). Similarly $1/\sqrt{3}$ is the tangent of 30° and also of all angles included in the formula $n. 180^\circ + 30^\circ$, (n . an integer), an unlimited number of them.

Ex. Find the two values of A between 0° and 360° such that $\sin A = \frac{1}{2}$ and hence, write down all the angles, having $\frac{1}{2}$ for their sines.

It is easily remembered that $\sin 30^\circ = \frac{1}{2}$. The other

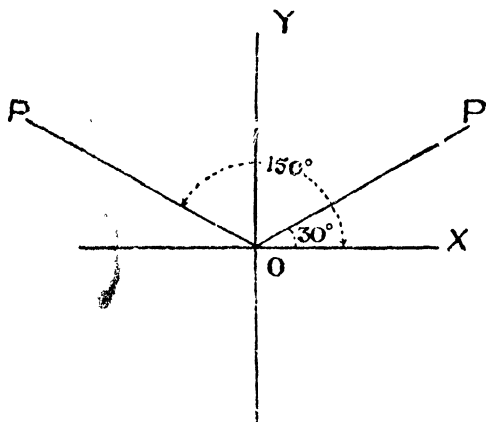


Fig. 31

angle, whose sine is $\frac{1}{2}$ cannot lie in the third or fourth quadrants, since sine is negative in them. We have therefore to seek for the angle in the second quadrant only. As the general form of an angle in the second quadrant is $180^\circ - \theta$

and because $\sin(180^\circ - \theta) = \sin \theta$, we get $\sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$. Hence 30° and 150° are the only two angles lying between 0° and 360° satisfying the given equation, the corresponding positions of the radius vector being OP and OP' respectively. Also, all angles, coterminal with these two have the same sine ($n \cdot 360^\circ + 30^\circ$) and ($n \cdot 360^\circ + 150^\circ$), where n is any integer positive or negative,

$\therefore$ the equation $\sin A = \frac{1}{2}$ is satisfied by all values of A included in either of the formulae ($n \cdot 360^\circ + 30^\circ$) and ($n \cdot 360^\circ + 150^\circ$), n being an integer or zero.

Note. Similarly, it is clear that there are only two positions of the revolving line for a given value of cosine or tangent. If, for example $\cos A = \frac{1}{2}$, 60° and 300° are the two values of A between 0° and 360° . Other values of A , such that $\cos A = \frac{1}{2}$ have their bounding radii coinciding with OP or OP' , where $XOP = 60^\circ$ and $XOP' = 300^\circ$. All the values of A are therefore contained in the formulae

($n \cdot 360^\circ + 60^\circ$) and ($n \cdot 360^\circ + 300^\circ$), n being an integer or zero. Also, $\tan A = 1$ is satisfied by values of A included in the formulae ($n \cdot 360^\circ + 45^\circ$) and ($n \cdot 360^\circ + 225^\circ$). Both are included in the expression ($p \cdot 180^\circ + 45^\circ$), p being an integer or zero.

EXAMPLES IV

(1) Find the values of the trigonometrical ratios of the following angles :—(i) 780° , (ii) 540° , (iii) 210° , (iv) 1350° , (v) -120° , (vi) -405° , (vii) -300° , (viii) 855° , (ix) $\frac{2\pi}{9}$.

(2) Express the following ratios in terms of ratios of positive acute angles less than 45° :—(i) $\sin 1234^\circ$, (ii) $\cos 2345^\circ$, (iii) $\tan 3456^\circ$, (iv) $\operatorname{cosec} 786^\circ$, (v) $\sec 546^\circ$, (vi) $\cot 640^\circ$, (vii) $\tan (-567^\circ)$, (viii) $\sin (-754^\circ)$, (ix) $\sec (-1256^\circ)$.

(3) Write down values of θ between 0° and 360° satisfying each of the following equations, and hence write down the formulae for all angles satisfying the equations :—(i) $\sin \theta = \sqrt{3}/2$, (ii) $\tan \theta = -\sqrt{3}$, (iii) $\cos \theta = -1$, (iv) $\sec \theta = \sqrt{2}$, (v) $2 \cos^2 \theta = 1$, (vi) $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$.

(4) If an angle in the second quadrant satisfies the equation $\cos^2 A = \frac{1}{4}$, find the values of its other trigonometrical ratios.

(5) If an angle in the third quadrant satisfies the equation $\sin^3 A = -3\sqrt{3}/8$ find its secant and sine.

(6) An angle in the fourth quadrant has its cosine equal to $5/13$. Calculate the values of the remaining trigonometrical functions of the angle.

(7) Find the limits to the value of A , such that $\sec A$ is greater than $\operatorname{cosec} A$, A being a positive acute angle.

(8) If A, B, C be the angles of a triangle, show that (i) $\sin A = \sin (B + C)$, (ii) $\cos \frac{A}{2} = \sin \frac{B + C}{2}$, (iii) $\cos \frac{C - A}{2} = \sin \frac{B + 2C}{2}$, (iv) $\sin C + \cos \frac{3C + A + B}{2} = 0$.

(9) Given that $\cos 65^\circ = .4226$ and $\sin 65^\circ = .9063$, find the angle whose cosine is $-.4226$ and sine $-.9063$ and deduce a general expression for such angles.

(10) Given that $\cot 194^\circ 1'$ is 4, find its sine and cosine. Write down all the angles between -360° and $+360^\circ$ having the same (i) sine, (ii) cosine.

(11) Simplify

$$\frac{\sin (540^\circ - A) \cdot \cos (-90^\circ + A) \cdot \tan (270^\circ - A)}{\operatorname{cosec} (1170^\circ + A) \cdot \sec (540^\circ + A) \cot (-90^\circ - A)}$$

CHAPTER V

FUNCTIONS OF COMPOUND ANGLES.

Art. 38. $\sin (A + B) = \sin A \cos B + \cos A \sin B$
and $\cos (A + B) = \cos A \cos B - \sin A \sin B$.

Let the radius vector, starting from the initial line OX , revolve in the positive direction through the two angles XON and NOP in succession where $XON = A$, and $NOP = B$. Then $XOP = A + B$, the total angle described. Take any point P on the final position of the radius

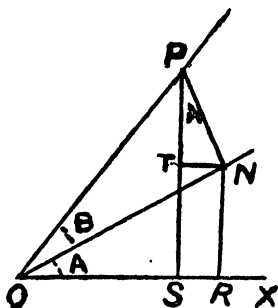


Fig. 32

vector, and draw PN , PS perpendicular to ON , OX respectively. Draw also NR , NT perpendicular to OX , PS respectively.

From the right-angled triangle SOP ,

$$\begin{aligned} \sin (A+B) &= \sin SOP = \frac{SP}{OP} = \frac{ST + TP}{OP} \\ &= \frac{RN + TP}{OP} = \frac{RN}{OP} + \frac{TP}{OP} \\ &= \frac{RN}{ON} \times \frac{ON}{OP} + \frac{TP}{PN} \times \frac{PN}{OP} \end{aligned}$$

But, since $PNO = 90^\circ = PSO$ by construction, the points P , N , S , O lie on a circle. Hence, $NPS = NOS = A$.

$$\frac{RN}{ON} = \sin A, \quad \frac{ON}{OP} = \cos B, \quad \frac{TP}{PN} = \cos A, \quad \frac{PN}{OP} = \sin B.$$

$$\therefore \sin (A+B) = \sin A \cos B + \cos A \sin B.$$

From the same figure,

$$\begin{aligned} \cos (A+B) &= \frac{OS}{OP} = \frac{OR-SR}{OP} = \frac{OR-TN}{OP} = \frac{OR}{OP} - \frac{TN}{OP} \\ &= \frac{OR}{ON} \cdot \frac{ON}{OP} - \frac{TN}{PN} \cdot \frac{PN}{OP} \\ &= \cos A \cdot \cos B - \sin A \cdot \sin B. \end{aligned}$$

Note.—In evaluating $\frac{RN}{OP}$ or $\frac{OR}{OP}$ the student will

remember that this is done only by relating each of these to a third length, this length being chosen as the side common to the two right-angled triangles of which the lengths are sides, e. g., to find the value of $\frac{RN}{OP}$,

note that RN is a side of the right-angled triangle ORN and OP of triangle OPN. ON is a side of both the triangles. $\frac{RN}{OP}$ is therefore written as $\frac{RN}{ON} \div \frac{ON}{OP}$, i.e.,

$\frac{RN}{ON} \cdot \frac{ON}{OP}$. Similarly for $\frac{OR}{OP}$ and such others.

Art. 39. $\sin (A-B) = \sin A \cos B - \cos A \sin B.$
 $\cos (A-B) = \cos A \cos B + \sin A \sin B.$

The radius vector, starting from the initial line OX revolves first through an angle XON equal to A in the positive direction and then, from ON, describe the angle NOP equal to B in the opposite direction.

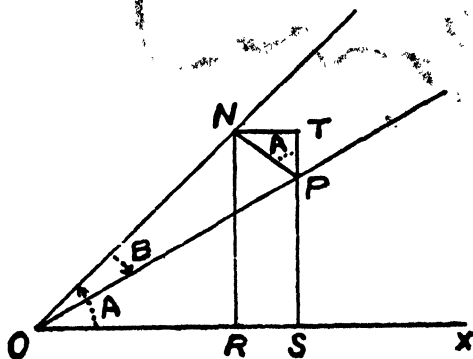


Fig. 33

XOP is of magnitude $(A - B)$.

Taking any point P on the final position of the revolving line, PS, PN, NR, NT are drawn perpendiculars to OX, ON, OX, SP produced respectively as in the previous article.

Since $PNO = 90^\circ$ and $PSO = 90^\circ$, NPSO is a cyclic quadrilateral, in which $NPT = SON$ (interior opposite angle) $= A$.

From the triangle OSP.

$$\begin{aligned}\sin (A - B) &= \frac{SP}{OP} = \frac{ST - PT}{OP} = \frac{RN - PT}{OP} = \frac{RN}{OP} - \frac{PT}{OP} \\ &= \frac{RN}{ON} \cdot \frac{ON}{OP} - \frac{PT}{PN} \cdot \frac{PN}{OP} \\ &= \sin A \cdot \cos B - \cos A \cdot \sin B.\end{aligned}$$

$$\begin{aligned}\cos (A - B) &= \frac{OS}{OP} = \frac{OR + RS}{OP} = \frac{OR}{OP} + \frac{NT}{OP} \\ &= \frac{OR}{ON} \cdot \frac{ON}{OP} + \frac{NT}{PN} \cdot \frac{PN}{OP} = \cos A \cdot \cos B + \sin A \cdot \sin B.\end{aligned}$$

Note.—The formulae for the sine and cosine of $(A + B)$, and $(A - B)$ are given only for $A, B, (A + B), (A - B)$ being all positive acute angles. The student can take them to be valid for all values of the angles, since this can be proved by projection.

$$\text{Art. 40. } \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

These formulae may be deduced assuming the values of $\sin (A \pm B)$ and $\cos (A \pm B)$ or independently from figures.

Thus, as $\sin (A + B) = \sin A \cos B + \cos A \sin B$,
and $\cos (A + B) = \cos A \cos B - \sin A \sin B$,

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing numerator and denominator by $\cos A \cos B$,

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\begin{aligned} \text{Similarly, } \tan (A - B) &= \frac{\sin (A - B)}{\cos (A - B)} \\ &= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B} \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}$$

Art. 41. Now for the geometrical proofs of these formulae :

Referring to Fig. 32, $XOP = A + B$

$$\begin{aligned} \tan (A + B) &= \tan XOP = \frac{SP}{OS} = \frac{ST + TP}{OR - SR} \\ &= \frac{RN + TP}{OR - TN} = \frac{\frac{RN}{OR} + \frac{TP}{OR}}{1 - \frac{TN}{OR}} \end{aligned}$$

The two triangles TPN and RON are equiangular because $T = 90^\circ = R$; $TPN = A = RON$.

Their corresponding sides are proportional.

$$\frac{TP}{OR} = \frac{TN}{RN} = \frac{NP}{ON} ; \text{ but } \frac{NP}{ON} = \tan B \text{ from } \triangle ONP.$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \frac{TN}{RN} \cdot \frac{RN}{OR}} = \frac{\tan A + \tan B}{1 - \tan B \tan A}$$

Similarly, from Fig. 33, $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

Caution.—The beginner is likely to commit the mistake of simplifying $\sin (A + B)$ as $(\sin A + \sin B)$, $\cos (A + B)$ as $(\cos A + \cos B)$, $\tan (A + B)$ as $(\tan A + \tan B)$. To prove the incorrectness of these statements, he has only to turn his attention to the simplified results for the functions of the compound angles. Numerical examples too may be taken. Thus, $\sin 90^\circ = 1 = \sin (60^\circ + 30^\circ)$, whereas $\sin 60^\circ + \sin 30^\circ = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3} + 1}{2}$.

Ex. 1. Evaluate $\sin 75^\circ$, $\cos 75^\circ$ and $\tan 75^\circ$, without referring to Tables.

$$75^\circ = 45^\circ + 30^\circ$$

$$\sin 75^\circ = \sin (45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\cos 75^\circ = \cos (45^\circ + 30^\circ) = \cos 45^\circ \cdot \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= (\sqrt{3} - 1) / 2\sqrt{2}$$

$$\tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}.$$

Ex. 2. Find the values of the trigonometrical functions of 15° .

$$15^\circ = 60^\circ - 45^\circ.$$

$$\sin 15^\circ = \sin (60^\circ - 45^\circ) = \sin 60^\circ \cdot \cos 45^\circ - \cos 60^\circ \cdot \sin 45^\circ$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\text{Similarly, } \cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\tan 15^\circ = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}.$$

Note. (1) The student will verify that since 75° and 15° are complementary, any function of 75° is equal to the corresponding co-function of 15° , so that knowing the values of the trigonometrical ratios of one of these two angles, the ratios of the other can be written down.

(2) It is advisable to commit the values of $\sin 15^\circ$, $\cos 15^\circ$ and $\tan 15^\circ$ to memory.

Art. 42. $\cot (A \pm B)$ in terms of $\cot A$ and $\cot B$.

$$\begin{aligned} \cot (A + B) &= \frac{\cos (A + B)}{\sin (A + B)} \\ &= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B} \\ &= \frac{\cot A \cot B - 1}{\cot B + \cot A}, \text{ dividing numerator} \end{aligned}$$

and denominator by $\sin A \cdot \sin B$.

$$\therefore \cot (A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A} \quad (4)$$

$$\text{Similarly, } \cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Art. 43. $\sin (A + B + C)$ and $\cos (A + B + C)$.

$$\begin{aligned} \sin (A + B + C) &= \sin (A + B) \cos C + \sin C \cos (A + B) \\ &= (\sin A \cos B + \cos A \sin B) \cos C \\ &\quad + (\cos A \cos B - \sin A \sin B) \sin C. \end{aligned}$$

$$\begin{aligned}
&= \sin A. \cos B. \cos C + \sin B \cos C. \cos A \\
&\quad + \sin C \cos A. \cos B - \sin A. \sin B. \sin C. \\
&= \sum \sin A. \cos B. \cos C - \sin A. \sin B. \sin C \\
&\quad \text{the result being symmetrical in A, B and C.}
\end{aligned}$$

$$\begin{aligned}
\sin (A+B+C) &= \sum \sin A. \cos B. \cos C \\
&\quad - \sin A \sin B \sin C.
\end{aligned}$$

$$\begin{aligned}
\text{Also } \cos (A+B+C) &= \cos (A+B) \cos C - \sin (A+B) \sin C. \\
&= (\cos A \cos B - \sin A \sin B) \cos C. \\
&\quad - (\sin A \cos B + \cos A \sin B) \sin C. \\
&= \cos A. \cos B. \cos C - \cos C \sin A. \sin B. \\
&\quad - \cos B. \sin A. \sin C. - \cos A. \sin B \sin C. \\
&= \cos A. \cos B. \cos C - \sum \cos A. \sin B \sin C.
\end{aligned}$$

$$\begin{aligned}
\therefore \cos (A+B+C) &= \cos A. \cos B. \cos C \\
&\quad - \sum \cos A. \sin B. \sin C.
\end{aligned}$$

Cor.—Extending this again to the sine and cosine of four angles, A, B, C, D we get working in the same manner,

$$\begin{aligned}
\sin (A+B+C+D) &= \sum \sin A. \cos B. \cos C. \cos D. \\
&\quad - \sum \cos A. \sin B. \sin C. \sin D. \\
&= \cos A. \cos B. \cos C. \cos D \times \\
&\quad (\sum \tan A - \sum \tan B. \tan C. \tan D).
\end{aligned}$$

Similarly, it can be deduced, and the result proved by mathematical induction that

$\sin (A+B+C \dots) = \cos A \cos B \cos C \dots (S_1 - S_3 + S_5 \dots)$ &
 $\cos (A+B+C + \dots) = \cos A \cos B \cos C \dots (1 - S_2 + S_4 \dots)$,
 where S_r = sum of the products of the tangents of the angles A, B, C, etc., taking r for each product.

Art. 44. Tan (A+B+C). and cot (A+B+C).

$$\begin{aligned}\tan (A+B+C) &= \frac{\sin (A+B+C)}{\cos (A+B+C)} \\ &= \frac{\cos A. \cos B. \cos C (S_1 - S_3)}{\cos A. \cos B. \cos C (1 - S_2)} \text{ from Art. 43.} \\ &= \frac{S_1 - S_3}{1 - S_2}\end{aligned}$$

$$\cot (A+B+C) = \frac{1 - S_2}{S_1 - S_3}.$$

Note.—This formula can be proved independently also by expanding $\tan [(A+B)+C]$ and substituting for $\tan (A+B)$ in terms of $\tan A$ and $\tan B$.

Cor. Similarly this formula may be extended to the tangent of the sum of any number of angles, the formula corresponding being—

$\tan (A+B+C\dots) = \frac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}$, an easy deduction from the values of $\cos (A+B+C\dots)$ and $\sin (A+B+C\dots)$, as obtained in the last article.

Ex. 1. Find the values of $\sec (A+B)$ and $\operatorname{cosec} (A+B)$ in terms of secants and cosecants of A and B .

$$\begin{aligned}\sec (A+B) &= \frac{1}{\cos (A+B)} = \frac{1}{\cos A. \cos B - \sin A \sin B} \\ &= \frac{\operatorname{cosec} A. \operatorname{cosec} B. \sec A \sec B}{\operatorname{cosec} A. \operatorname{cosec} B - \sec A. \sec B'}\end{aligned}$$

on multiplying both the numerator and denominator by $\operatorname{cosec} A. \operatorname{cosec} B. \sec B \sec A$.

$$\begin{aligned}
 \text{Similarly, } \operatorname{cosec} (A + B) &= \frac{1}{\sin (A + B)} \\
 &= \frac{1}{\sin A \cos B + \cos A \sin B} \\
 &= \frac{\operatorname{cosec} A \cdot \operatorname{cosec} B \cdot \sec A \cdot \sec B}{\sec A \operatorname{cosec} B + \operatorname{cosec} A \sec B}
 \end{aligned}$$

Ex. 2. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{40}{41}$, find the trigonometrical ratios of $(A + B)$ and $(A - B)$, supposing A and B to be acute angles.

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

$$\sin A = \frac{4}{5} \therefore \cos A = \sqrt{1 - \sin^2 A} = \frac{3}{5}, \text{ as } A \text{ is acute}$$

$$\cos B = \frac{40}{41}; \sin B = + \sqrt{1 - \frac{40^2}{41^2}} = \frac{9}{41}, \text{ as } B \text{ is acute.}$$

Substituting these values,

$$\sin (A + B) = \frac{4}{5} \times \frac{40}{41} + \frac{3}{5} \times \frac{9}{41} = \frac{160}{205} + \frac{27}{205} = \frac{187}{205}$$

$$\begin{aligned}
 \cos (A + B) &= \cos A \cos B - \sin A \sin B \\
 &= \frac{3}{5} \times \frac{40}{41} - \frac{4}{5} \times \frac{9}{41} = \frac{120 - 36}{205} = \frac{84}{205}
 \end{aligned}$$

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{187}{84}.$$

$$\text{Similarly, } \sin (A - B) = \sin A \cos B - \cos A \sin B = \frac{133}{205}$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B = \frac{156}{205}$$

$$\tan (A - B) = \frac{133}{156}.$$

Ex. 3. Prove that

$$\sin (A+B) \cdot \sin (A-B) = \sin^2 A - \sin^2 B.$$

$$\begin{aligned} \sin (A+B) \times \sin (A-B) &= (\sin A \cos B + \cos A \sin B) \\ &\quad \times (\sin A \cos B - \cos A \sin B) \\ &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ &= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B \\ &= \sin^2 A - \sin^2 B. \end{aligned}$$

It may also be written as $\cos^2 B - \cos^2 A$.

Ex. 4. Prove that

$$\begin{aligned} \cos (A+B) \cdot \cos (A-B) &= \cos^2 A - \sin^2 B \\ &= \cos^2 B - \sin^2 A \end{aligned}$$

$$\begin{aligned} \cos (A+B) \cdot \cos (A-B) &= (\cos A \cos B - \sin A \sin B) \\ &\quad \times (\cos A \cos B + \sin A \sin B) \\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ &= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\ &= \cos^2 A - \sin^2 B \end{aligned}$$

$$\begin{aligned} \text{But, } \cos^2 A - \sin^2 B &= (1 - \sin^2 A) - (1 - \cos^2 B) \\ &= \cos^2 B - \sin^2 A. \end{aligned}$$

Ex. 5. Show that

$\cos \theta + \cos (120^\circ + \theta) + \cos (120^\circ - \theta) = 0$ for all values of θ .

$$\begin{aligned} \cos (120^\circ + \theta) &= \cos 120^\circ \cos \theta - \sin 120^\circ \sin \theta \\ \cos (120^\circ - \theta) &= \cos 120^\circ \cos \theta + \sin 120^\circ \sin \theta \\ \therefore \cos \theta + \cos (120^\circ + \theta) + \cos (120^\circ - \theta) \\ &= \cos \theta + 2 \cos 120^\circ \cos \theta \\ &= \cos \theta + 2 \cos (180^\circ - 60^\circ) \cos \theta \\ &= \cos \theta - 2 \cos 60^\circ \cos \theta \\ &= \cos \theta - \cos \theta = 0 \end{aligned}$$

Ex. 6. One angle of a triangle is 45° , and the tangent of the other angle is $\frac{4}{3}$. Find the sine of the third angle.

Let ABC be the triangle in which $A = 45^\circ$, and $\tan B = \frac{4}{3}$. It is required to find $\sin C$.

The sum of the three angles of a triangle is equal to 2 right angles. $A + B + C = 180^\circ$.

$$\begin{aligned} C &= 180^\circ - (A + B) \\ \sin C &= \sin (180^\circ - A - B) = \sin (A + B) \\ &= \sin A \cos B + \cos A \sin B \\ &= \frac{1}{\sqrt{2}} \cdot \frac{3}{5} + \frac{1}{\sqrt{2}} \cdot \frac{4}{5} = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}. \end{aligned}$$

Ex. 7. Show that in a triangle ABC, the length of the altitude from A is given by the formula $\frac{a \sin B \sin C}{\sin A}$.

Draw AD perpendicular to the base, BC meeting it in D. Triangles ADB and ADC are right-angled at D.

Hence, $\frac{BD}{AD} = \cot B$ and $\frac{DC}{AD} = \cot C$.

Adding $\frac{BD + DC}{AD} = \cot B + \cot C$

i.e., $\frac{BC}{AD} = \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$

i.e., $\frac{a}{AD} = \frac{\sin C \cos B + \cos C \sin B}{\sin B \sin C}$

$$= \frac{\sin (B + C)}{\sin B \cdot \sin C} = \frac{\sin (180^\circ - A)}{\sin B \cdot \sin C} = \frac{\sin A}{\sin B \cdot \sin C}$$

$$AD = a \div \frac{\sin A}{\sin B \cdot \sin C} = \frac{a \sin B \sin C}{\sin A}.$$

Ex. 8. The sines of two angles of a triangle are $\frac{2}{3}$ and $\frac{1}{2}$. Find the third angle from its cosine.

Let $\sin A = \frac{2}{3}$ and $\sin B = \frac{1}{2}$.

Then $\cos A = \pm \sqrt{1 - \frac{4}{9}} = \pm \frac{\sqrt{5}}{3}$ and $\cos B = \pm \frac{\sqrt{3}}{2}$.

We have to consider two cases :

(i) If A and B are both acute, $\cos A$ and $\cos B$ are both positive, in which case,

$$\begin{aligned}\cos C &= \cos (180^\circ - A + B) = -\cos (A + B) \\ &= -\cos A \cos B + \sin A \sin B \\ &= -\frac{\sqrt{5}}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{\sqrt{3}}{2} = \frac{2 - \sqrt{15}}{6} = -.3122 \\ C &= 108^\circ 12' .\end{aligned}$$

(ii) If one of the angles A, B is obtuse, it must be the angle having the larger sine. Hence A is obtuse and B acute.

So, $\cos A = -\frac{\sqrt{5}}{3}$ and $\cos B = +\frac{\sqrt{3}}{2}$

$$\cos C = (\frac{\sqrt{15}}{6} + \frac{2}{6}) = \frac{5.8730}{6} = 0.9789$$

$$C = 11^\circ 48'$$

EXAMPLES V

1. Show that

$$\cos A \cdot \sin (B - C) + \cos B \cdot \sin (C - A) + \cos C \cdot \sin (A - B) = 0$$

for all values of A, B, C.

2. Prove that

$$\sin (m + 1) \theta \cdot \cos (n - 1) \theta + \cos (m + 1) \theta \cdot \sin (n - 1) \theta = \sin (m + n) \theta .$$

3. Find the value of $\tan (A + B)$ if A and B are acute angles whose sines are $\frac{3}{5}$ and $\frac{5}{13}$ respectively.

4. Show that

$$\tan (45^\circ + A) - \tan (45^\circ - A) = \frac{4 \tan A}{1 - \tan^2 A}$$

5. The sines of two angles of a triangle are $\frac{3}{5}$ and $\frac{12}{13}$. If one of the angles is acute and the other obtuse, find the third angle from its sine.

6. The cosines of two angles of a triangle are $\frac{3}{4}$ and $-\frac{5}{8}$. Find the value of the third angle from its cosine.

7. If $\sin A = \frac{4}{5}$ and $\sin B = \frac{8}{17}$ find $\sin (A + B)$

(i) when A and B are acute angles, (ii) when A is obtuse and B acute.

8. If $\tan A = \frac{1}{3}$ and $\tan B = \frac{1}{2}$, prove that $(A + B) = \frac{\pi}{4}$.

9. Show that $\sin \theta + \cos \theta$ and $\sin \theta - \cos \theta$ can be written in the forms $\sqrt{2} \sin \left(\frac{\pi}{4} + \theta \right)$ and $\sqrt{2} \sin \left(\theta - \frac{\pi}{4} \right)$ respectively. Hence deduce that the maximum value of $\sin \theta + \cos \theta$ or $\sin \theta - \cos \theta$ is $\sqrt{2}$.

10. Evaluate the trigonometrical ratios of 105° and deduce those of 285° .

11. Using the formula for $\tan (A + B + C)$, or otherwise, prove that in a triangle ABC,

$$(i) \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

$$(ii) \sum \cot \frac{A}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

12. Show that for all values of θ ,

$$\sin \left(\frac{2\pi}{3} + \theta \right) + \sin \theta - \sin \left(\frac{2\pi}{3} - \theta \right) = 0.$$

13. Prove that $3 \cos \theta - 4 \sin \theta$ can be written in the form $5 \cos (\theta + a)$, where a is an angle whose sine is 0.8 . Given that $a = 53^\circ 8'$ find the value of θ that makes the expression (i) vanish, (ii) a maximum, (iii) a minimum.

14. Find the maximum value of the expression $a \cos x + b \sin x$, and a value of x for which it is a maximum. Evaluate the results when $a = 5$ and $b = -12$.

15. Prove that

$$(1) \cos (30^\circ - A) - \cos (30^\circ + A) = \sin A.$$

$$(2) \cos (45^\circ - A) - \cos (45^\circ + A) = \sqrt{2} \sin A.$$

Prove the following identities:—

$$16. \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A.$$

$$17. \tan 4A \tan 3A \tan A + \tan 3A + \tan A - \tan 4A = 0.$$

18. $\cot (A + B + C) =$

$$= \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}$$
19. $\cot A. \cot 2 A. - \cot 2 A. \cot 3 A - \cot 3 A \cot A = 1.$
20. $\cot 2 B. \cot 3 B - \cot 3 B. \cot 5 B - \cot 5 B. \cot 2 B = 1;$
21. $\sin A. \cos 3 A \cos 4 A + \sin 3 A. \cos 4 A \cos A$
 $- \sin 4 A. \cos A. \cos 3 A + \sin A \sin 3 A. \sin 4 A = 0.$
22. $\cot A - \cot 2 A = \operatorname{cosec} 2 A.$
23. $\operatorname{Cosec} 2 A + \operatorname{cosec} 2^2 A + \operatorname{cosec} 2^3 A + \operatorname{cosec} 2^4 A$
 $= \cot A - \cot 2^4 A.$
24. $1 - \cot 2 A. \tan A = \tan A. \operatorname{cosec} 2 A.$
25. $1 - \cot A. \tan 2 A = -\sec 2 A.$
26. $1 + \tan A. \tan 2 A = \sec 2 A.$
27. $\frac{\tan 2 A + \tan A}{\tan 2 A - \tan A} = \frac{\sin 3 A}{\sin A}.$
28. If $\cos (A - B) = 3 \cos (A + B)$, prove that $\cot A \cot B = 2.$
29. Using Ex. 9, find the sign of $\sin \theta + \cos \theta$ when $\theta = 9^\circ, 82\frac{1}{2}^\circ,$
 $280^\circ, -340^\circ, 130^\circ.$
30. Find also the sign of $\sin \theta - \cos \theta$ for the same values of $\theta.$

CHAPTER VI

MULTIPLE ANGLES

Art. 45. Trigonometrical functions of 2 A in terms of those of A.

From the last chapter, we have derived expressions for the values of sine, cosine, and tangent of $(A + B)$ in terms of the ratios of A and B . Substituting in these formulae $B = A$, we get the results.

$$\begin{aligned}\sin 2A &= \sin (A + A) \equiv \sin A \cdot \cos A + \cos A \cdot \sin A \\ &= 2 \sin A \cdot \cos A \dots\dots\dots(1)\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos (A + A) = \cos A \cdot \cos A - \sin A \cdot \sin A \\ &= \cos^2 A - \sin^2 A \dots\dots\dots(2)\end{aligned}$$

As it is sometimes necessary to find $\cos 2A$ in terms of $\cos A$ or $\sin A$ only, two other expressions equivalent to result (2) have to be remembered.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) \\ &= 2 \cos^2 A - 1 \dots\dots\dots(3)\end{aligned}$$

$$\begin{aligned}\text{Also, } \cos 2A &= \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A \\ &= 1 - 2 \sin^2 A \dots\dots\dots(4)\end{aligned}$$

$$\begin{aligned}\tan 2A &= \tan (A + A) = \frac{\tan A + \tan A}{1 - \tan^2 A} \\ &= \frac{2 \tan A}{1 - \tan^2 A} \dots\dots\dots(5)\end{aligned}$$

$$\begin{aligned}\cot 2A &= \cot (A + A) = \frac{\cot^2 A - 1}{\cot A + \cot A} \\ &= \frac{\cot^2 A - 1}{2 \cot A} \dots\dots\dots(6)\end{aligned}$$

Ex. 1. Find the values of $\sin 2A$ and $\cos 2A$ in terms of $\tan A$.

$$\sin 2A = 2 \sin A \cos A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$$

Dividing both numerator and denominator by $\cos^2 A$,

$$\begin{aligned} \sin 2A &= \frac{2 \sin A \cos A}{\cos^2 A} \div \frac{\cos^2 A + \sin^2 A}{\cos^2 A} \\ &= \frac{2 \tan A}{1 + \tan^2 A} \dots\dots\dots (7) \end{aligned}$$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A} \dots\dots\dots (8)$$

Note. These are important results to be remembered by the student.

Art. 46. Owing to the importance of these results, separate geometrical proofs are given below for values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ in terms of the ratios of θ . They depend on the theorem that the angle at the centre of a circle subtended by an arc is equal to twice the angle at the circumference standing on the same arc. But, they are deducible by this method only for values of θ less than 90° , though the results are true for all values of θ .

ABP is a semicircle of any convenient radius, C is its centre and AB its bounding diameter. AP is a chord making the given angle θ ($< 90^\circ$) with the diameter AB. CP and BP are joined and from P, PM is drawn perpendicular to AB to meet it in M.

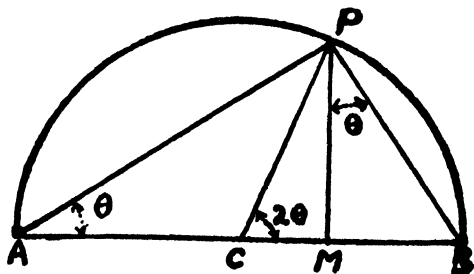


Fig. 34

Then, $BCP = 2BAP = 2\theta$.

Also, since APB is an angle in the semi-circle, its magnitude is 90° , and triangle APB is a right-angled triangle.

$MPB = \text{comp. } APM = MAP = \theta$.

From triangle MCP ,

$$\begin{aligned}\sin 2\theta = \sin MCP &= \frac{MP}{CP} = \frac{2MP}{2CP} = \frac{2MP}{AB} \\ &= \frac{2MP}{AP} \times \frac{AP}{AB} \\ &= 2 \sin \theta \cdot \cos \theta\end{aligned}$$

$$\begin{aligned}\text{Again, } \cos 2\theta &= \frac{CM}{CP} = \frac{2CM}{2CP} = \frac{2CM}{AB} \\ &= \frac{AM - MB}{AB} \\ &= \frac{AM}{AB} - \frac{MB}{AB} \\ &= \frac{AM}{AP} \cdot \frac{AP}{AB} - \frac{MB}{PB} \cdot \frac{PB}{AB} \\ &= \cos^2 \theta - \sin^2 \theta.\end{aligned}$$

Also,

$$\begin{aligned}\cos 2\theta &= \frac{2CM}{AB} = \frac{2(AM - AC)}{AB} = \frac{2AM}{AB} - \frac{2AC}{AB} \\ &= 2 \frac{AM}{AB} - 1. \text{ (since } AB = 2AC) \\ &= 2 \frac{AM}{AP} \cdot \frac{AP}{AB} - 1 \\ &= 2 \cos^2 \theta - 1\end{aligned}$$

Again,

$$\cos 2\theta = 2 \frac{CM}{AB} = \frac{2(CB - MB)}{AB} = 2 \frac{CB}{AB} - 2 \frac{MB}{AB}$$

$$\begin{aligned}
 &= 1 - 2 \frac{MB}{AB} \\
 &= 1 - 2 \frac{MB}{PB} \cdot \frac{PB}{AB} \\
 &= 1 - 2 \sin^2 \theta.
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \tan 2\theta &= \frac{MP}{CM} = \frac{2 MP}{2 CM} = \frac{2 MP}{AM - MB} \\
 &= \frac{2 MP}{AM} \div \left(1 - \frac{MB}{AM}\right) \quad \text{dividing both numerator and denominator by AM} \\
 &= \frac{2 MP}{AM} \div \left(1 - \frac{MB}{MP} \cdot \frac{MP}{AM}\right) = \frac{2 \tan \theta}{1 - \tan^2 \theta}
 \end{aligned}$$

Formulae (7) and (8) can also be proved geometrically

$$\begin{aligned}
 \sin 2\theta &= \frac{MP}{CP} = \frac{2 MP}{2 CP} = \frac{2 MP}{AB} = \frac{2 MP}{AM + MB} \\
 &= \frac{2 MP}{AM} \left(1 + \frac{MB}{AM}\right) \\
 &= \frac{2 \tan \theta}{1 + \tan^2 \theta}
 \end{aligned}$$

$$\begin{aligned}
 \cos 2\theta &= \frac{CM}{CP} = \frac{2 CM}{2 CP} = \frac{2 CM}{AB} \\
 &= \frac{2 CM}{AM + MB} = \frac{AM - MB}{AM + MB} = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}
 \end{aligned}$$

Ex. Prove geometrically :—

$$\cot \theta - \tan \theta = 2 \cot 2\theta$$

$$\text{Hint. } 2 \cot 2\theta = \frac{2 CM}{MP} = \frac{AM - MB}{MP}$$

Art. 47. Trigonometric functions of 3 A in terms of functions of A.

$$\begin{aligned}
 \sin 3A &= \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A \\
 &= 2 \sin A \cos^2 A + (1 - 2 \sin^2 A) \sin A \\
 &= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A \\
 &= 3 \sin A - 4 \sin^3 A.
 \end{aligned}$$

$$\begin{aligned}
 \cos 3A &= \cos (2A + A) = \cos 2A \cos A - \sin 2A \sin A \\
 &= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A \\
 &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A) \\
 &= 4 \cos^3 A - 3 \cos A.
 \end{aligned}$$

$$\begin{aligned}
 \tan 3A &= \tan (2A + A) = \frac{\tan 2A + \tan A}{1 - \tan A \tan 2A} \\
 &= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}
 \end{aligned}$$

Art. 48. Trigonometric ratios of A in terms of those of $\frac{A}{2}$.

Substituting $\frac{A}{2}$ for A in the preceding formulae,

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \quad \dots (1)$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \quad \dots (2)$$

$$= 2 \cos^2 \frac{A}{2} - 1 \quad \dots (3)$$

$$= 1 - 2 \sin^2 \frac{A}{2} \quad \dots (4)$$

$$\tan A = 2 \tan \frac{A}{2} \div \left(1 - \tan^2 \frac{A}{2}\right) \quad \dots (5)$$

$$\sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} \quad \dots (6)$$

$$\cos A = \left(1 - \tan^2 \frac{A}{2}\right) / \left(1 + \tan^2 \frac{A}{2}\right) \quad \dots (7)$$

$\sec A$, $\operatorname{cosec} A$, and $\cot A$ are only reciprocals of the other three ratios of A . The last three formulae

deserve special attention, since all the trigonometrical ratios of any angle A are thereby expressed *rationally* in terms of $\tan \frac{A}{2}$. Use is made of them in solving equations involving $\sin A$, $\cos A$ and $\tan A$.

Ex. 1. If $\tan B = \frac{1 - \cos A}{\sin A}$, show that

$$\tan A = \tan 2B.$$

$$\begin{aligned} \frac{1 - \cos A}{\sin A} &= \frac{1 - \left(1 - 2 \sin^2 \frac{A}{2}\right)}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{2 \sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \\ &= \tan \frac{A}{2} \end{aligned}$$

$$\tan B = \tan \frac{A}{2} = \tan \left(n\pi + \frac{A}{2}\right) \text{ where } n \text{ is any integer.}$$

$$B = n\pi + \frac{A}{2}; \quad 1 \quad A = 2B - 2n\pi$$

$$\tan A = \tan (2B - 2n\pi) = \tan 2B.$$

Ex. 2. If $\tan \theta = \frac{1}{5}$, find the trigonometrical ratios of 2θ .

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{10}{26} = \frac{5}{13}$$

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{24}{26} = \frac{12}{13}; \quad \tan 2\theta = \frac{5}{12}$$

$$\text{Ex. 3. Show that } \cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{3\pi}{9} \cos \frac{4\pi}{9} =$$

$$= \frac{1}{16} \text{ and } \sin \frac{\pi}{9} \cdot \sin \frac{2\pi}{9} \cdot \sin \frac{3\pi}{9} \cdot \sin \frac{4\pi}{9} = \frac{3}{16}$$

$$\begin{aligned}\cos 40^\circ \cdot \cos 80^\circ &= \cos (60^\circ - 20^\circ) \cos (60^\circ + 20^\circ) \\ &= \cos^2 60^\circ - \sin^2 20^\circ = \frac{1}{4} - \sin^2 20^\circ \\ &= \cos^2 20^\circ - \frac{3}{4} = \frac{4 \cos^2 20^\circ - 3}{4}\end{aligned}$$

$$\begin{aligned}\therefore \cos 20^\circ \cdot \cos 40^\circ \cos 60^\circ \cos 80^\circ &= \cos 60^\circ \cdot \cos 20^\circ \times \frac{(4 \cos^2 20^\circ - 3)}{4} \\ &= \frac{1}{2} \cdot \frac{(4 \cos^2 20^\circ - 3) \cos 20^\circ}{4} \\ &= \frac{1}{2} \cdot \frac{\cos (3 \times 20^\circ)}{4} = \frac{1}{2} \cdot \frac{\cos 60^\circ}{4} = \frac{1}{2} \times \frac{1}{2 \times 4} = \frac{1}{16}\end{aligned}$$

Similarly, using the formulae

$\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$ and

$\sin 3A = 3 \sin A - 4 \sin^3 A$, the other result is obtained.

Ex. 4. Show that

$$\sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8} = \frac{3}{2}$$

$$\frac{7\pi}{8} = \pi - \frac{\pi}{8} \therefore \sin \frac{7\pi}{8} = \sin \frac{\pi}{8}$$

$$\text{Also } \frac{5\pi}{8} = \pi - \frac{3\pi}{8} \therefore \sin \frac{5\pi}{8} = \sin \frac{3\pi}{8}$$

$$\therefore \text{L.H.S.} = 2 \sin^4 \frac{\pi}{8} + 2 \sin^4 \frac{3\pi}{8}$$

$$\text{But } \sin \frac{3\pi}{8} = \sin \left(\frac{\pi}{2} - \frac{\pi}{8} \right) = \cos \frac{\pi}{8}$$

$$\begin{aligned}\text{L.H.S.} &= 2 \left[\sin^4 \frac{\pi}{8} + \cos^4 \frac{\pi}{8} \right] \\ &= 2 \left[\left(\sin^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{8} \right)^2 - 2 \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \right] \\ &= 2 - 4 \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} = 2 - \sin^2 \frac{\pi}{4} = \frac{3}{2}\end{aligned}$$

EXAMPLES VI.

(1) If $\sin A = 0.8$, find $\sin 3 A$, $\cos 2 A$ and $\tan 2 A$.

(2) If $\cos A = 0.6$, find $\cos 3 A$, $\sin 2 A$ and $\cot 2 A$.

(3) If $\tan \theta = 1.25$, find the values of $\sin 2 \theta$, $\sec 2 \theta$ and $\tan 3 \theta$.

(4) Show that $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$

(5) If $\tan A = \frac{1}{2}$ and $\tan B = \frac{4}{3}$, find the value of $\tan 3 B$ and $\tan (3 B - A)$.

(6) Show that $\frac{\cos \theta + 1 + \sin \theta}{1 - \cos \theta + \sin \theta} = \cot \frac{\theta}{2}$

Prove that the following equalities are identically true :—

(7) $\tan A - \cot A = -2 \cot 2 A$.

(8) $1 + \sin A = \left(\cos \frac{A}{2} + \sin \frac{A}{2} \right)^2$

(9) $1 - \sin A = \left(\cos \frac{A}{2} - \sin \frac{A}{2} \right)^2$

(10) $\sin^6 A + \cos^6 A = \left(1 - \frac{3}{4} \sin^2 2 A \right)$

(11) $\cos^6 A - \sin^6 A = \cos 2 A \left(1 - \frac{1}{4} \sin^2 2 A \right)$
 $= \frac{3}{4} \cos 2 A + \frac{1}{4} \cos^3 2 A$

(12) $\tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}}$

(13) $2 \operatorname{cosec} \theta = \tan \frac{\theta}{2} + \cot \frac{\theta}{2}$.

(14) $\cot A = \frac{1 + \cos 2 A}{\sin 2 A} = \frac{\sin 2 A}{1 - \cos 2 A}$

(15) $\cot^2 A = \frac{1 + \cos 2 A}{1 - \cos 2 A}$.

(16) $\sin (A + \theta) \sin (A - \theta) = \frac{1}{2} (\cos 2 \theta - \cos 2 A)$.

$$(17) \cos (A + \theta) \cdot \cos (A - \theta) = \frac{1}{2} (\cos 2 A + \cos 2 \theta).$$

$$(18) (\cos A + \cos B) (\cos A - \cos B) = \frac{1}{2} (\cos 2 A - \cos 2 B).$$

(19) If $\sin (A + B) = \frac{1}{2}$ and $\sin (A - B) = \frac{1}{3}$ find $\sin 2 A$ and $\sin 2 B$.

$$(20) \text{ Given that } \cos v = \frac{\cos u - e}{1 - e \cos u},$$

$$\begin{aligned} \text{prove that } \tan \frac{v}{2} &= \sqrt{\frac{1 - \cos v}{1 + \cos v}} \\ &= \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2}. \end{aligned}$$

CHAPTER VII

SUM AND PRODUCT FORMULAE

Art. 49. Starting from the fundamental formulae of Chapter V, we are in a position to deduce others, whereby, several results in Trigonometry admit of being simplified. The first set enables us to express the product of two sines or cosines as a sum or difference ; the second expresses the sum or difference of sines or cosines as a product.

$$\begin{aligned}\sin (A+B) &= \sin A \cos B + \cos A \sin B \\ \text{and } \sin (A-B) &= \sin A \cos B - \cos A \sin B.\end{aligned}$$

Adding the two,

$$\sin (A+B) + \sin (A-B) = 2 \sin A \cos B \dots\dots(i)$$

Subtracting the second from the first,

$$\sin (A+B) - \sin (A-B) = 2 \cos A \sin B \dots\dots(ii)$$

Similarly, by adding and subtracting the corresponding cosines, in the formulae,

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

we get

$$\cos (A+B) + \cos (A-B) = 2 \cos A \cos B \dots\dots(iii)$$

$$\cos (A-B) - \cos (A+B) = 2 \sin A \sin B \dots\dots(iv)$$

These four results are remembered in a slightly altered form by interchanging the expressions on the two sides of each equality thus :

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B) \dots\dots(1)$$

$$2 \cos A \sin B = \sin (A+B) - \sin (A-B) \dots\dots(2)$$

$$2 \cos A \cos B = \cos (A+B) + \cos (A-B) \dots\dots(3)$$

$$2 \sin A \sin B = \cos (A-B) - \cos (A+B) \dots\dots(4)$$

Ex. 1. Transform the following products into sums :—

- (i) $2 \sin 2 A. \cos A.$
- (ii) $2 \cos 5 A. \sin 3 A.$
- (iii) $2 \cos 8 A. \cos 4 A.$
- (iv) $2 \sin A. \sin 4 A.$
- (v) $2 \sin 3 A. \sin 5 B.$
- (vi) $2 \sin (A + B). \cos (A - B).$

Solution.

(i) $2 \sin 2 A. \cos A.$

This comes under formula (1).

$$\begin{aligned} \therefore 2 \sin 2 A \cos A &= \sin (2 A + A) + \sin (2 A - A) \\ &= \sin 3 A + \sin A. \end{aligned}$$

(ii) $2 \cos 5 A \sin 3 A.$

We use formula (2).

$$\begin{aligned} \therefore 2 \cos 5 A \sin 3 A &= \sin (5A + 3A) - \sin (5A - 3A) \\ &= \sin 8 A - \sin 2 A. \end{aligned}$$

(iii) $2 \cos 8 A \cos 4 A.$

Being the product of two cosines, it is equal to the sum of two cosines, coming under formula (3).

$$\begin{aligned} \therefore 2 \cos 8 A \cos 4 A &= \cos (8A + 4A) + \cos (8A - 4A) \\ &= \cos 12 A + \cos 4A. \end{aligned}$$

(iv) $2 \sin A \sin 4 A.$

From formula (4), $2 \sin 4 A \sin A =$

$$= \cos (4 A - A) - \cos (4 A + A) = \cos 3 A - \cos 5 A.$$

(v) $2 \sin 3 A \sin 5 B.$

By formula (4),

$$2 \sin 3 A \sin 5 B = \cos (3 A - 5 B) - \cos (3 A + 5 B)$$

(vi) By formula (1),

$$\begin{aligned} 2 \sin (A + B) \cos (A - B) &= \sin (A + B + A - B) + \\ &\quad + \sin (A + B - A + B) = \sin 2 A + \sin 2 B. \end{aligned}$$

Ex. 2. Express $4 \cos A \sin B \sin C$ as a sum of cosines.

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B).$$

$$\begin{aligned} 4 \cos A \sin B \sin C &= 2 \sin C. [\sin (A + B) - \sin (A - B)] \\ &= 2 \sin C \sin (A + B) - 2 \sin C \sin (A - B) \\ &= \cos (A + B - C) - \cos (A + B + C) - \cos (A - B - C) \\ &\quad + \cos (A - B + C). \end{aligned}$$

Ex. 3. Show that if A, B, C be any angles whatever,
 $\cos A \sin (B - C) + \cos B \sin (C - A) +$
 $+ \cos C \sin (A - B) = 0.$

$$\begin{aligned} \cos A \sin (B - C) &= \frac{1}{2} \cdot 2 \cos A \sin (B - C) \\ &= \frac{1}{2} [\sin (A + B - C) - \sin (A - B + C)] \end{aligned}$$

Hence, the left hand side equals

$$\frac{1}{2} \left\{ \begin{aligned} &\sin (A + B - C) - \sin (A - B + C) \\ &+ \sin (B + C - A) - \sin (B - C + A) \\ &+ \sin (C + A - B) - \sin (C - A + B) \end{aligned} \right\} = 0$$

Note. An alternative method is to expand $\sin (B - C)$, $\sin (C - A)$, $\sin (A - B)$ and simplify.

Ex. 4. Prove that for all values of A ,

$$\begin{aligned} \cos A \cos \left(A + \frac{2\pi}{3} \right) + \cos \left(\frac{2\pi}{3} + A \right) \cos \left(\frac{2\pi}{3} - A \right) \\ + \cos A \cos \left(\frac{2\pi}{3} - A \right) + \frac{3}{4} = 0, \end{aligned}$$

$$\begin{aligned} \text{L. H. S.} &= \frac{1}{2} \left[2 \cos \left(A + \frac{2\pi}{3} \right) \cos A + \right. \\ &\quad \left. + 2 \cos \left(\frac{2\pi}{3} + A \right) \cos \left(\frac{2\pi}{3} - A \right) \right] \end{aligned}$$

$$\begin{aligned}
& + 2 \cos \left(\frac{2\pi}{3} - A \right) \cos A + \frac{3}{2} \cdot \Big] \\
& = \frac{1}{2} \left[\cos \left(2A + \frac{2\pi}{3} \right) + \cos \frac{2\pi}{3} + \cos \frac{4\pi}{3} \right. \\
& \quad \left. + \cos 2A + \cos \frac{2\pi}{3} + \cos \left(2A - \frac{2\pi}{3} \right) + \frac{3}{2} \right] \\
& = \frac{1}{2} \left[\cos \left(2A + \frac{2\pi}{3} \right) + \cos \left(2A - \frac{2\pi}{3} \right) + \right. \\
& \quad \left. + \cos 2A + 2 \cos \frac{2\pi}{3} + \cos \frac{4\pi}{3} + \frac{3}{2} \right] \\
& = \frac{1}{2} \left[2 \cos 2A \cos \frac{2\pi}{3} + \cos 2A - 1 - \frac{1}{2} + \frac{3}{2} \right] \\
& = 0 \left(\text{since } \cos \frac{2\pi}{3} = -\frac{1}{2} = \cos \frac{4\pi}{3} \right)
\end{aligned}$$

Art. 50. The other set of formulae, to be deduced now are used whenever a sum or difference of two sines or two cosines has to be expressed as a product.

From the last article,

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B.$$

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B.$$

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B.$$

Choose two new angles θ and ϕ such that $A + B = \theta$ and $A - B = \phi$,

Adding and subtracting,

$$2A = \theta + \phi; A = \frac{\theta + \phi}{2}; 2B = \theta - \phi; B = \frac{\theta - \phi}{2}$$

Substituting these values of A , B , $(A + B)$, and $(A - B)$ in the foregoing results, we have

$$\sin \theta + \sin \phi = 2 \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \dots \dots \quad (\text{I})$$

$$\sin \theta - \sin \phi = 2 \cos \frac{\theta + \phi}{2} \sin \frac{\theta - \phi}{2} \dots \dots \quad (\text{II})$$

$$\cos \theta + \cos \phi = 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \dots \dots \quad (\text{III})$$

$$\cos \theta - \cos \phi = -2 \sin \frac{\theta + \phi}{2} \sin \frac{\theta - \phi}{2} \dots \dots \quad (\text{IV})$$

Since $-\sin \frac{\theta - \phi}{2} = \sin \frac{\phi - \theta}{2}$, formula (IV) can be written as

$$\cos \theta - \cos \phi = 2 \sin \frac{\theta + \phi}{2} \sin \frac{\phi - \theta}{2} \dots \dots \quad (\text{V})$$

As the sum or difference of a number of sines or cosines cannot be simplified by logarithms, these formulae enable us to transform them into a product and thence to evaluate them using logarithms.

Ex. 1. Express the following as products :

- (i) $\sin 7 A + \sin 3 A.$
- (ii) $\cos 2 A - \cos 10 A.$
- (iii) $\sin 5 A - \sin 12 A.$
- (iv) $\cos 5 A + \cos 2 A.$
- (v) $\sin 2 A + \sin 2 B.$
- (vi) $\cos 2 B - \cos 2 C.$

(i) $\sin 7 A + \sin 3 A.$

Being the sum of two sines,

$$\begin{aligned} \sin 7 A + \sin 3 A &= 2 (\text{sine half sum} \times \text{cosine half diff.}) \\ &= 2 \sin \frac{7 A + 3 A}{2} \cos \frac{7 A - 3 A}{2} \\ &= 2 \sin 5 A. \cos 2 A. \end{aligned}$$

(ii) $\cos 2A - \cos 10A$.

This comes under the last head, the difference of two cosines.

$$\begin{aligned}\cos 2A - \cos 10A &= 2 \left(\text{sine half sum} \times \text{sine half the difference of the } \mathbf{2^{\text{nd}} \text{ angle and } 1^{\text{st}}} \right) \\ &= 2 \sin \frac{2A + 10A}{2} \sin \frac{10A - 2A}{2} \\ &= 2 \sin 6A \cdot \sin 4A.\end{aligned}$$

(iii) $\sin 5A - \sin 12A$.

This is the difference of the sines of two angles.

$$\begin{aligned}\therefore \sin 5A - \sin 12A &= 2 \cos \frac{5A + 12A}{2} \times \sin \frac{5A - 12A}{2} \\ &= -2 \cos \frac{17A}{2} \sin \frac{7A}{2}.\end{aligned}$$

(iv) $\cos 5A + \cos 2A = 2 \cos \frac{7A}{2} \cos \frac{3A}{2}$.

(v) $\sin 2A + \sin 2B = 2 \sin (A + B) \cos A - B$.

(vi) $\cos 2B - \cos 2C = 2 \sin (B + C) \sin (C - B)$.

Ex. 2. Express $\cos A + \cos B + \cos C + \cos (A + B + C)$ as a product.

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos C + \cos (A + B + C) = 2 \cos \frac{A+B+2C}{2} \cos \frac{A+B}{2}$$

$$\begin{aligned}\cos A + \cos B + \cos C + \cos (A + B + C) \\ = 2 \cos \frac{A+B}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{A+B+2C}{2} \right\}\end{aligned}$$

$$\begin{aligned}\text{But, } \cos \frac{A-B}{2} + \cos \frac{A+B+2C}{2} \\ = 2 \cos \frac{2A+2C}{4} \cos \frac{2B+2C}{4} \\ = 2 \cos \frac{C+A}{2} \cos \frac{B+C}{2}\end{aligned}$$

$$\begin{aligned} \text{Hence, } \cos A + \cos B + \cos C + \cos (A + B + C) \\ = 4 \cos \frac{A+B}{2} \cdot \cos \frac{B+C}{2} \cdot \cos \frac{C+A}{2} \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Similarly, } \sin A + \sin B + \sin C - \sin (A + B + C) \\ = 4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \cdot \sin \frac{C+A}{2}. \end{aligned} \quad (2)$$

Both these results are to be committed to memory as being important identities. Note that on the left hands of these identities, $\cos (A + B + C)$ takes the positive sign in the first, while $\sin (A + B + C)$ takes the negative sign in the second.

Ex. 3. Prove that $\cos 10^\circ + \cos 110^\circ + \cos 130^\circ = 0$.
 L.H.S. $= 2 \cos 60^\circ \cos 50^\circ + \cos (180^\circ - 50^\circ)$
 $= \cos 50^\circ - \cos 50^\circ = 0$.

Art. 51. The sum and product formulae derived in the preceding articles from the fundamental results can also be proved easily by geometry. Geometrical proofs are given here for positive values of θ , ϕ , $\theta - \phi$, and $\theta + \phi$, each less than 90° ; but the results are true for all values of the angles.

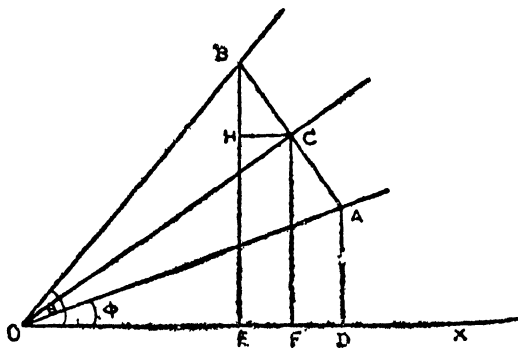


Fig. 35.

Starting from the initial line OX, let the radius vector sweep out angles XOB, XOA of magnitudes θ , ϕ respectively. Draw OC to bisect AOB; and take any point C on this line. ACB is drawn through C perpendicular to OC to meet OA, OB in A and B respectively.

AD, BE, CF are drawn perpendicular to OX meeting OX in D, E, F respectively. H is the projection of C on BE.

In the two triangles AOC and BOC, OC is common,
 $\angle OCA = \angle OCB$, each being a right angle.

$\angle AOC = \angle COB$, since OC bisects angle AOB.

$$\therefore \triangle AOC \equiv \triangle COB.$$

Hence, $OA = OB$ and $AC = CB$.

DF and FE being the projections of the equal lengths AC, CB on OX, it follows $EF = FD$.

$$OE + OD = 2 OF \text{ and } OD - OE = 2 EF = 2 HC.$$

$$\text{Also, } BE + AD = 2 CF \text{ and } BE - AD = 2 BH.$$

$$\text{Now, } \angle AOC = \angle COB = \frac{1}{2} \angle AOB = \frac{\theta + \phi}{2}.$$

Adding to both sides $\angle XOA$, we have

$$\angle XOC = \angle XOA + \angle AOC = \frac{\theta + \phi}{2}.$$

Since HC is parallel to OX,

$$\angle HCO = \angle XOC = (\theta + \phi)/2$$

$$\text{Hence, } \angle HBC = 90^\circ - \angle BCH = \angle HCO = (\theta + \phi)/2$$

$$\sin \theta + \sin \phi = \sin \angle XOB + \sin \angle XOA$$

$$= \frac{EB}{OB} + \frac{DA}{OA} = \frac{EB}{OB} + \frac{DA}{OB}$$

(Since $OB = OA$, proved)

$$= \frac{EB + DA}{OB} = \frac{2FC}{OB} = 2 \frac{FC}{OC} \cdot \frac{OC}{OB}$$

$$= 2 \sin \angle XOC \cdot \cos \angle COB.$$

$$= 2 \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2}$$

$$\begin{aligned} \sin \theta - \sin \phi &= \frac{EB}{OB} - \frac{DA}{OA} = \frac{EB - DA}{OB} = \frac{2 HB}{OB} \\ &= 2 \cdot \frac{HB}{BC} \times \frac{BC}{OB} = 2 \cos \angle HBC \cdot \sin \angle COB \\ &= 2 \cos \frac{\theta + \phi}{2} \cdot \sin \frac{\theta - \phi}{2}. \end{aligned}$$

$$\begin{aligned} \cos \theta + \cos \phi &= \frac{OE}{OB} + \frac{OD}{OA} = \frac{OE + OD}{OB} \\ &= \frac{2 OF}{OB} = \frac{2 OF}{OC} \cdot \frac{OC}{OB} \\ &= 2 \cdot \cos \angle XOC \cdot \cos \angle COB \\ &= 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \end{aligned}$$

$$\begin{aligned} \cos \theta - \cos \phi &= \frac{OE}{OB} - \frac{OD}{OA} = \frac{OE - OD}{OB} \\ &= \frac{-2 HC}{OB} = - \frac{2 HC}{BC} \times \frac{BC}{OB} \\ &= -2 \sin \angle HBC \cdot \sin \angle COB \\ &= -2 \sin \frac{\theta + \phi}{2} \sin \frac{\theta - \phi}{2} \end{aligned}$$

EXAMPLES VII a

1. Prove that $\frac{\sin 2A + \sin 2B}{\cos 2A + \cos 2B} = \tan (A + B).$

2. $\frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B} = \tan (A - B).$

3. $\frac{\sin 2B - \sin 2A}{\cos 2A - \cos 2B} = \cot (A + B).$

4. $\cos (A - B) + \sin (A + B) = 2 \sin (45^\circ + A) \cos (45^\circ - B).$

$$5. \sin A + \sin B + \sin (A + B) = 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{A+B}{2}.$$

$$6. \sin A + \sin B - \sin (A + B) = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{A+B}{2}.$$

$$7. \cos A + \cos B + \cos (A + B) = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{A+B}{2} - 1.$$

$$8. \cos A + \cos B - \cos (A + B) = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{A+B}{2} + 1.$$

$$9. \sin \frac{\pi}{18} + \sin \frac{\pi}{9} + \sin \frac{2\pi}{9} + \sin \frac{5\pi}{18} = \sin \frac{7\pi}{18} + \sin \frac{4\pi}{9}.$$

$$10. \sin 2A + \sin 2B + \sin 2(A - B) = 4 \sin A \cos B \cos (A - B)$$

$$11. \cos 70^\circ + \cos 50^\circ - \cos 10^\circ = 0.$$

$$12. \frac{\sin A + \sin B - \sin (A + B)}{\cos A + \cos B - \cos (A + B)} - 1 = \tan \frac{B + A}{2}.$$

13. If A and B be two angles, such that $\sin A - \sin B = x$, and $\cos A + \cos B = y$, then $\tan \frac{A - B}{2} = \frac{x}{y}$ and $\cos \frac{A + B}{2} = \frac{1}{2} \sqrt{x^2 + y^2}$.

14. If $\tan A \tan B = \tan^2 \frac{C}{2}$ then will $\cos (A + B) = \cos (A - B) \cos C$.

15. If $\cot A = 2x - 1$ and $\cot B = x \div (x - 1)$, prove that one value of $(A + B)$ is $\frac{\pi}{4}$.

16. Show that for all values of A ,
 $\sin A \sin 4A + \cos 2A \cos 3A = \cos A \cos 2A$.

17. $\sin B + \sin C - \sin A + \sin (A + B + C)$ is equal to
 $4 \sin \frac{B+C}{2} \cos \frac{C+A}{2} \cos \frac{A+B}{2}$.

18. Show that $\cos (p + 1)A = 2 \cos pA \cos A - \cos (p - 1)A$.
Hence, deduce that for values of x numerically < 1 ,

$$\cos \theta + x \cos 2\theta + x^2 \cos 3\theta + \dots \text{to } \infty = \frac{\cos \theta - x}{1 - 2x \cos \theta + x^2}.$$

19. If in a triangle ABC , $\cos B = \sin A / (2 \sin C)$, show that the triangle is isosceles.

20. Express $\cos 3x + \sin 2x$ as a product of (i) two cosines (ii) two sines. Hence find values of x between 0° & 360° that make $(\cos 3x + \sin 2x)$ vanish.

[Hint $\sin 2x = +\cos(90^\circ - 2x) = -\cos(90^\circ + 2x)$]

21. If A and B are two variable angles whose sum is constant, prove that $\sin A \sin B$ is maximum when $A = B$.

[Hint $2 \sin A \sin B = \cos A - B - \cos A + B$]

22. The range of a projectile on a plane of given inclination β is $\frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta}$ where u, g are constants & α is the variable angle of projection. Find the value of α for which the range is maximum.

Art 52. Trigonometrical Identities :—Using the sum and product formulae stated above, we shall establish a number of useful identities ;

Ex. 1. If $A + B + C = 180^\circ$, prove that $\sin 2A - \sin 2B + \sin 2C = 4 \cos C \cos A \sin B$.

$$\begin{aligned} \text{L.H.S} &= 2 \cos(A + B) \sin(A - B) + \sin 2C. \\ &= 2 \cos(180^\circ - C) \sin(A - B) + 2 \sin C \cos C. \\ &= -2 \cos C \sin(A - B) + 2 \sin C \cos C. \\ &= 2 \cos C [\sin C - \sin(A - B)] \\ &= 2 \cos C [\sin(A + B) - \sin(A - B)] \\ &= 4 \cos C. \cos A. \sin B. \end{aligned}$$

*Note :—*The student will note the various steps in the proof. To find the sum of three terms in the left hand side, two of them are added up and the sum expressed as a product. The third term is then split up in such a way as to admit of being added to the sum already obtained, by taking out a common factor. The quantity within the bracket which is a sum or difference is again transformed into a product and the simplification effected.

Ex. 2. If $A + B + C = 180^\circ$, prove that

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$$

$$\begin{aligned} \text{L.H.S.} &= 2 \cos (A+B) \cos (A-B) + \cos 2C \\ &= 2 \cos (180^\circ - C) \cdot \cos (A-B) + \cos 2C \\ &= -2 \cos C \cdot \cos (A-B) + 2 \cos^2 C - 1 \\ &= -2 \cos C [\cos (A-B) - \cos C] - 1 \\ &= -2 \cos C [\cos (A-B) + \cos (A+B)] - 1 \\ &= -4 \cos C \cdot \cos A \cos B - 1 \end{aligned}$$

Ex. 3. If $A + B + C = 180^\circ$, show that

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\begin{aligned} \text{L.H.S.} &= 2 \sin (A+B)/2 \cos (A-B)/2 + \sin C \\ &= 2 \cos (C/2) \cos (A-B)/2 + 2 \sin (C/2) \cos C/2 \\ &= 2 \cos (C/2) \left\{ \cos (A-B)/2 + \sin (C/2) \right\} \\ &= 2 \cos (C/2) \left\{ \cos (A-B)/2 + \cos (A+B)/2 \right\} \\ &= 4 \cos (A/2) \cdot \cos (B/2) \cdot \cos (C/2). \end{aligned}$$

Ex. 4. If $A + B + C = \pi$, prove that

$$\cos A - \cos B - \cos C = -1 - 4 \cos (A/2) \sin (B/2) \sin (C/2).$$

$$\begin{aligned} \text{L.H.S.} &= 2 \sin (A+B)/2 \cdot \sin (B-A)/2 - \cos C \\ &= 2 \sin (-C/2) \cdot \sin (B-A)/2 - \cos C \\ &= -2 \sin (C/2) \sin (B-A)/2 - (1 - 2 \sin^2 C/2) \\ &= -1 - 2 \sin (C/2) \left\{ \sin (B-A)/2 - \sin (C/2) \right\} \\ &= -1 - 2 \sin \frac{C}{2} \left\{ \sin \frac{B-A}{2} - \sin \left(-\frac{B+A}{2} \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= -1 - 2 \sin \frac{C}{2} \left\{ \sin \frac{B-A}{2} + \sin \frac{B+A}{2} \right\} \\
&= -1 - 4 \sin \frac{C}{2} \cdot \sin \frac{B}{2} \cdot \cos \frac{A}{2} \\
&= -1 - 4 \cos \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}.
\end{aligned}$$

Ex. 5. If $A + B + C = 180^\circ$, prove that

$$\sin^2 A - \sin^2 B + \sin^2 C = 2 \sin A \cdot \cos B \cdot \sin C.$$

1st Method

$$\begin{aligned}
\text{Since } \sin^2 A - \sin^2 B &= \sin(A+B) \cdot \sin(A-B) \\
&= \sin C \cdot \sin(A-B),
\end{aligned}$$

$$\begin{aligned}
\sin^2 A - \sin^2 B + \sin^2 C &= \sin C [\sin(A-B) + \sin C] \\
&= \sin C [\sin(A-B) + \sin(A+B)] \\
&= 2 \sin C \cdot \sin A \cdot \cos B.
\end{aligned}$$

2nd Method

$$\begin{aligned}
2(\sin^2 A - \sin^2 B + \sin^2 C) &= (1 - \cos 2A) - (1 - \cos 2B) + 2 \sin^2 C \\
&= \cos 2B - \cos 2A + 2 \sin^2 C \\
&= 2 \sin(A+B) \sin(A-B) + 2 \sin^2 C \\
&= 2 \sin C \cdot \sin(A-B) + 2 \sin^2 C \\
&= 2 \sin C [\sin(A-B) + \sin C] \\
&= 2 \sin C [\sin(A-B) + \sin(A+B)] \\
&= 4 \sin A \cos B \sin C.
\end{aligned}$$

Note :— The same methods may be adopted for evaluating $\cos^2 A \pm \cos^2 B \pm \cos^2 C$ by using also the corresponding identities $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \times \cos(A-B)$ and $2 \cos^2 A = 1 + \cos 2A$.

Ex. 6. If $A + B + C = 180^\circ$, prove that

$$(1) \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

$$(2) \sum \cot \frac{A}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$(3) \sum \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

1st Method

$$(1) B + C = 180^\circ - A, \text{ (by hypothesis)}$$

$$\therefore \tan (B + C) = \tan (180^\circ - A)$$

Expanding,

$$\frac{\tan B + \tan C}{1 - \tan B \tan C} = -\tan A.$$

Clearing of fractions,

$$\tan B + \tan C = -\tan A + \tan A \tan B \tan C.$$

$$\text{i.e., } \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

$$(2) \frac{B}{2} + \frac{C}{2} = 90^\circ - \frac{A}{2}.$$

$$\therefore \cot \left(\frac{B}{2} + \frac{C}{2} \right) = \cot \left(90^\circ - \frac{A}{2} \right)$$

$$\frac{\cot \frac{B}{2} \cot \frac{C}{2} - 1}{\cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{1}{\cot \frac{A}{2}}$$

$$\text{i.e., } \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} - \cot \frac{A}{2} = \cot \frac{B}{2} + \cot \frac{C}{2}$$

$$\text{i.e., } \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

(3) Multiplying both sides of the previous result by

$\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$, we have

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1$$

Otherwise, proceeding independently,

$$\tan \left(\frac{B}{2} + \frac{C}{2} \right) = \tan \left(90^\circ - \frac{A}{2} \right)$$

$$\text{i.e., } \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{1 - \tan \frac{B}{2} \tan \frac{C}{2}} = \frac{1}{\tan \frac{A}{2}}$$

$$\text{i.e., } \sum \tan \frac{B}{2} \tan \frac{C}{2} = 1;$$

2nd Method

Use the formula

$$\tan (A + B + C) = \frac{\sum \tan A - \tan A \tan B \tan C}{1 - \sum \tan A \tan B}$$

and $(A + B + C) = 180^\circ$; $(A + B + C)/2 = 90^\circ$, so that $\tan (A + B + C) = 0$ and $\tan (A + B + C)/2$ is infinite.

Note :—These relations are important and they must be remembered.

Ex. 7. If A, B, C be three angles, such that $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$, then $A + B + C$ equals some multiple of π ; and the converse.

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

Transposing and grouping,

$$\tan A (1 - \tan B \tan C) + (\tan B + \tan C) = 0.$$

$$\therefore \tan A + \frac{\tan B + \tan C}{1 - \tan B \tan C} = 0.$$

$\therefore \tan (B + C) = -\tan A = \tan (p\pi - A)$, where p is any integer.

$$B + C = p\pi - A.$$

$$\text{i.e., } A + B + C = p\pi \text{ (a multiple of } \pi).$$

The converse is got by reversing the steps.

Ex. 8. If A, B, C be any angles whatever, prove that $\sin 3A \cdot \sin (B - C) + \sin 3B \cdot \sin (C - A) + \sin 3C \cdot \sin (A - B) = 4 \sin (A + B + C) \cdot \sin (A - B) \sin (B - C) \sin (C - A)$.

$$2 \sin 3A \cdot \sin (B - C) = \cos (3A - B + C) - \cos (3A + B - C)$$

$$2 \sin 3B \cdot \sin (C - A) = \cos (3B - C + A) - \cos (3B + C - A)$$

$$2 \sin 3C \cdot \sin (A - B) = \cos (3C - A + B) - \cos (3C + A - B)$$

Adding up, and re-arranging the R.H.S. terms,

$$2 \sum \sin 3A \sin (B - C) = \sum [\cos (3A - B + C) - \cos (3B + C - A)]$$

$$= 2 \sin (A + B + C) \sum \sin 2 (B - A)$$

$$\therefore \sum \sin 3A \sin (B - C) = -\sin (A + B + C) \times$$

$$[\sin 2 (B - C) + \sin 2 (C - A) + \sin 2 (A - B)].$$

$$= -\sin (A + B + C) [2 \sin B - A \cdot \cos B + A - 2C$$

$$- 2 \sin B - A \cdot \cos B - A]$$

$$= + 2 \sin A + B + C \cdot \sin A - B [\cos A + B - 2C - \cos B - A]$$

$$= 4 \sin (A + B + C) \cdot \sin (A - B) \cdot \sin (B - C) \cdot \sin (C - A).$$

EXAMPLES VII *b*

If $A + B + C = \pi$, prove the following identities,

$$(1) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \cdot \sin B \cdot \sin C$$

$$(2) \cos 2A - \cos 2B + \cos 2C = 1 - 4 \sin A \cdot \cos B \cdot \sin C.$$

$$(3) \sin A - \sin B + \sin C = 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

$$(4) \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}.$$

$$(5) \cos A - \cos B + \cos C = 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1.$$

$$(6) \sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.$$

$$(7) \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$$

$$(8) \cos^2 A - \cos^2 B + \cos^2 C = 1 - 2 \sin A \cos B \sin C.$$

$$(9) \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$(10) \cos^2 \frac{A}{2} - \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

$$(11) \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$(12) \sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = -1 + 2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

$$(13) \cos 3A + \cos 3B + \cos 3C = 1 - 4 \sin^3 \frac{A}{2} \sin^3 \frac{B}{2} \sin^3 \frac{C}{2}.$$

$$(14) \cos 4A + \cos 4B + \cos 4C = -1 + 4 \cos 2A \cos 2B \cos 2C.$$

$$(15) \sin A \sin (B - C) + \sin B \sin (C - A) + \sin C \sin (A - B) = 0.$$

$$(16) \sum \sin 3A \sin (B - C) = 0.$$

$$[\text{Hint : } -\sin^3 A \sin (B - C) = \sin^2 A \sin (B + C) \sin (B - C)$$

$$= \sin^2 A (\sin^2 B - \sin^2 C)].$$

$$(17) \sum \sin^3 A \cos (B - C) = 3 \sin A \sin B \sin C.$$

$$(18) \tan \left(\frac{\pi}{6} + \frac{A}{2} \right) \tan \left(\frac{\pi}{6} + \frac{B}{2} \right) \tan \left(\frac{\pi}{6} + \frac{C}{2} \right)$$

$$= \tan \left(\frac{\pi}{6} + \frac{A}{2} \right) \tan \left(\frac{\pi}{6} + \frac{B}{2} \right) \tan \left(\frac{\pi}{6} + \frac{C}{2} \right)$$

$$(19) \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1 = 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4}$$

$$\sin \frac{\pi - C}{4}.$$

$$(20) \cos \frac{A}{2} - \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \left(45^\circ + \frac{A}{4} \right) \sin \left(45^\circ + \frac{B}{4} \right).$$

$$\cos \left(45^\circ + \frac{C}{4} \right) = 4 \cos \frac{\pi + A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi + C}{4}.$$

If $A + B + C = 0$, prove that

$$(21) \sin 2A + \sin 2B + \sin 2C = -4 \sin A \sin B \sin C.$$

$$(22) \cos 2A - \cos 2B + \cos 2C = 1 + 4 \sin A \cos B \sin C.$$

$$(23) 1 + 2 \sin A \cos B \sin C + \cos^2 B = \cos^2 A + \cos^2 C.$$

$$(24) \sin^2 A + \sin^2 B + \sin^2 C = 2 - 2 \cos A \cos B \cos C.$$

$$(25) \text{ If } \cos(B - C) + \cos(C - A) + \cos(A - B) = -\frac{3}{2}, \text{ show that}$$

$\cos A + \cos B + \cos C$ and $\sin A + \sin B + \sin C$ separately vanish.

(26) Given.

$$Y_1 = a \sin \alpha \cos wt.$$

$$Y_2 = a \sin \left(\alpha + \frac{2\pi}{3} \right) \cos \left(wt + \frac{2\pi}{3} \right),$$

$$Y_3 = a \sin \left(\alpha + \frac{4\pi}{3} \right) \cos \left(wt + \frac{4\pi}{3} \right),$$

express Y_1 , Y_2 and Y_3 each as the sum of two terms and prove that

$$Y_1 + Y_2 + Y_3 = \frac{3}{2} a \sin(\alpha - wt).$$

CHAPTER VIII

SOLUTION OF SIMPLE TRIGONOMETRICAL EQUATIONS

Art. 53. A trigonometric equation is a statement of equality of two trigonometrical functions of one or more variable angles, which is true only for a certain value or values of the angles involved; and it is said to be solved when *all* the values of the variables satisfying the equality are found out. As the solution of every equation ultimately depends on any one of three fundamental equations, they shall be set forth in the next three articles.

Art. 54. Solution of the equation $\sin \theta = s$, s being any proper fraction, positive or negative.

It is always possible to find the numerically smallest angle α , the sine of which is s , so that $\sin \alpha = s$. It is evident that α is positive or negative according as s is positive or negative. We have to solve $\sin \theta = \sin \alpha$.

$$\text{i.e., } \sin \theta - \sin \alpha = 0.$$

$$\text{i.e., } 2 \cos \frac{\theta + \alpha}{2} \sin \frac{\theta - \alpha}{2} = 0.$$

$$\therefore \cos \frac{\theta + \alpha}{2} = 0 \text{ or } \sin \frac{\theta - \alpha}{2} = 0.$$

All angles whose cosine is zero must have their bounding line along OY or OY', so that the general expression for all of them is $(2p - 1) \pi/2$ (p being zero or any integer). Similarly as the bounding line of angles whose sine is zero is OX or OX', they are equal to some multiple of π radians.

Hence, $\frac{\theta + \alpha}{2} = (2p - 1) \frac{\pi}{2}$ or $\frac{\theta - \alpha}{2} = p \pi$, p being zero or any integer.

i.e., $\theta = (2p - 1)\pi - \alpha$ or $\theta = 2p\pi + \alpha$.

But, both these are included in the single formula $\theta = m\pi + (-1)^m \alpha$, m being an integer, positive or negative, odd or even or zero. So, $\sin \theta = \sin \alpha$, when θ takes any value included in the formula $\theta = m\pi + (-1)^m \alpha$.

Note 1. — The expression $m\pi + (-1)^m \alpha$ is clearly the general expression for all angles having the same sine as the angle α .

Note 2. — It is also evident that $\operatorname{cosec} \theta = \operatorname{cosec} \alpha$, if θ is any one of the angles included in the formula $m\pi + (-1)^m \alpha$; for, $\operatorname{cosec} \theta = \operatorname{cosec} \alpha$, if $\sin \theta = \sin \alpha$.

Art. 55. Solution of the equation $\cos \theta = c$, ‘ c ’ being any proper fraction positive or negative. Let β be the numerically smallest angle whose cosine is c . As the cosine of a negative angle is not negative, β is a positive angle for all values for c , being acute when c is positive and obtuse when c is negative. Thus, when $c = \sqrt{3}/2$, $\beta = \pi/6$ and when $c = -\sqrt{3}/2$, $\beta = 5\pi/6$. The given equation reduces to $\cos \theta = \cos \beta$.

$$\text{i.e., } \cos \theta - \cos \beta = 0$$

$$\text{i.e., } -2 \sin (\theta + \beta)/2 \sin (\theta - \beta)/2 = 0$$

$$\therefore \sin (\theta + \beta)/2 = 0, \text{ or } \sin (\theta - \beta)/2 = 0.$$

i.e., $\theta = 2p\pi - \beta$ or $\theta = 2p\pi + \beta$ where $p = 0$ or any integer. But, both these are included in the formula $\theta = 2p\pi \pm \beta$.

Note 1. — The general expression for all angles having the same cosine as β is $2p\pi \pm \beta$, where p equals zero or any integer positive or negative.

Note 2. — The general solution of the equation $\sec \theta = \sec \beta$ is $\theta = 2p\pi \pm \beta$.

Art. 56. Solution of the equation $\tan \theta = t$, t being any number positive or negative :

Let α be the numerically smallest angle having t for its tangent. α is positive or negative according as t is positive or negative. We have now to solve the equation

$$\tan \theta = \tan \alpha.$$

$$\text{i.e., } \tan \theta - \tan \alpha = 0.$$

$$\text{i.e., } \frac{\sin \theta}{\cos \theta} - \frac{\sin \alpha}{\cos \alpha} = 0.$$

$$\text{i.e., } \sin (\theta - \alpha) = 0.$$

$$\theta - \alpha = p \pi, p \text{ being equal to zero or any integer.}$$

$$\therefore \theta = p \pi + \alpha.$$

Note 1.—The general expression for all angles having the same tangent as α is $p \pi + \alpha$, where p equals zero or any integer.

Note 2.—Similarly. the general solution of $\cot \theta = \cot \alpha$ is $\theta = p \pi + \alpha$.

Note 3.—From the formulae above, it is clear that the sine, cosine and tangent are all *periodic* functions, the period being 2π radians for the first two and π for the third.

Ex. 1. Solve the equation $\sin \theta = -(\sqrt{5} - 1)/4$ given that $\sin 18^\circ = (\sqrt{5} - 1)/4$.

Evidently (-18°) is the numerically smallest angle having $-(\sqrt{5} - 1)/4$ for its sine.

$$\sin \theta = \sin (-18^\circ).$$

$\therefore \theta = p \pi + (-1)^p (-\pi/10) = p \pi - (-1)^p \pi/10$
where p equals zero or any integer.

Ex. 2. Solve $\cos 2 \theta = (\sqrt{3} + 1)/2\sqrt{2}$.

The student will remember that $\frac{\sqrt{3} + 1}{2\sqrt{2}}$ is the cosine of 15° .

$$\cos 2 \theta = \cos 15^\circ = \cos \frac{\pi}{12}$$

$$2 \theta = 2 n \pi \pm \frac{\pi}{12}$$

$$\theta = n \pi \pm \frac{\pi}{24}, \text{ where } n \text{ is zero or any integer.}$$

Ex. 3. Solve the equation $\cos 5 A = -0.5664$, given that $\cos 55^\circ 30' = 0.5664$.

The numerically smallest angle having the given negative number—0.5664 for its cosine is $(180^\circ - 55^\circ 30')$ *i.e.*, $124^\circ 30'$. and *not*— $55^\circ 30'$.

$$\cos 5 A = \cos 124^\circ 30'$$

$$5 A = n 360^\circ \pm 124^\circ 30'$$

$$A = n 72^\circ \pm 24^\circ 54' \text{ when } n \text{ equals zero or any integer.}$$

Ex. 4. Find all the values of θ for which $\tan 3 \theta = \tan 2 \theta$.

By Art. 56. Note 1, $\tan 2 \theta = \tan (p\pi + 2\theta)$.

As $\tan 3 \theta = \tan 2 \theta$, 3θ must be one of the angles included in the formula $(p\pi + 2\theta)$

$$\text{i.e., } 3 \theta = p\pi + 2\theta. \text{ Transposing, } \theta = p\pi.$$

Art. 57. Use of Tables of Natural Sines, Cosines and Tangents. To solve Trigonometric equations by the method of the previous articles, it is necessary to read off the sine, cosine or tangent of a given angle and also to write down the positive acute angle having a given positive number for its sine or cosine or tangent. Tables are

constructed for this purpose, one for each of the ratios, giving their values to four decimal places. In each, the argument is the angle expressed in degrees and minutes. The first column contains degrees from 0 to 89° : and the topmost row gives the minutes of angle from 0 to 54, as are multiples of $6'$, *i.e.*, $0'$, $6'$, $12'$, $18'$, $24'$, $30'$, $36'$, $42'$, $48'$, $54'$. From these two, ratios of any angle whose sexagesimal measure is a multiple of $6'$ are read off. Thus $\sin 32^\circ 48'$ is read off from the Table of Natural sines in the horizontal row against 32° , and down the vertical column headed $48'$, and found to be 0.5417. The sine or cosine of a real angle is always numerically less than unity, so that the integral part of the sine or cosine of an angle is zero ; but the tangent of an angle ranges from zero to infinity as the angle varies from zero to 90° , being unity for 45° . Hence the integral part of the tangent of an angle less than 45° being zero, that of an angle exceeding 45° is greater than 1, its value for any angle being written in the first column of the corresponding row. Where there is a change of integral part (by one, of course), this is indicated by a dash,—being placed over such a number in the row. Thus $\tan 71^\circ 30' = 2.9887$ and $\tan 71^\circ 36' = 3.0061$. For all other angles, reference has to be made also to the column of “Mean Differences” given for $1'$, $2'$, $3'$, $4'$, or $5'$. The figure given in this column against the horizontal row has to be **added** to the fourth decimal place of the value found from the main column in the case of the Sine and Tangent Tables : but with the Cosine Tables, it has to be **subtracted**. The results after **addition** in the first case or after **subtraction** in the second case give the value of the desired ratio. For example $\sin 32^\circ 50'$ got by *adding* to $\sin 32^\circ 48'$ the mean difference for $2'$ and is found to be $0.5417 + 0.0005 = 0.5422$.

$$\begin{aligned}\cos 55^{\circ} 40' &= \cos 55^{\circ} 36' \text{ minus mean difference for } 4' \\ &= 0.5650 - .0010 = 0.5640.\end{aligned}$$

$$\begin{aligned}\tan 49^{\circ} 21' &= \tan 49^{\circ} 18' \text{ plus mean difference for } 3' \\ &= 1.1626 + .0021 = 1.1647.\end{aligned}$$

Since the sine, cosine, and tangent of any angle positive or negative can be expressed in terms of those of a **positive acute angle**, using the formulae of Ch. IV, it is possible by reference to the Tables to write down the three ratios for an angle of any magnitude whatever.

Art. 58. Given a number ' s ', less than unity, we proceed to indicate how to write down by reference to Table of Natural Sines, the numerically smallest angle having ' s ' for its sine. First, let s be positive. See if ' s ' occurs in any part of the main column of the Tables. If it does, the angle is an exact multiple of $6'$. The first number in the row gives the number of degrees, and the topmost number in the column gives the number of minutes in the angle.

If ' s ' is not to be found anywhere in the main column, choose the row and column, wherein the figure is *just less than* ' s '. Read along the same row to the right in the Mean Difference column, the figure, which added to the figure in the Main column makes up ' s '. The angle in minutes corresponding to the mean difference *added* to the angle given by the number in the main column of the Tables gives the angle required. If however, s is negative, read off the angle a in degrees and minutes, whose sine is the numerical value of s . Then, $(-a)$ in the required angle.

Ex. 1. Find the least angle having 0.8354 for its sine. 0.8354 is not to be found anywhere in the main column; and 0.8348 is the figure in it nearest to the given number and *less* than that. The difference 6 corresponds in the Mean

Difference column to 4'. Since 0.8348 is the sine of $56^{\circ}36'$, the required angle is $56^{\circ}40'$.

Art. 59. Given a proper fraction ' c ', it is required to read off by reference to the Table of Natural Cosines the numerically smallest angle having ' c ' for its cosine. Let ' c ' be positive. The method of procedure is quite similar to that of the previous article. But as the cosine **decreases** with **increase** of angle and as the figures in the Mean Difference column have to be subtracted from those of the main column, a slight alteration is necessary. If c is not to be found anywhere in the main column, choose the figure of the main column that *just exceeds* c . The **excess** of this figure over ' c ' is referred to in the Difference column, and the corresponding angle in minutes noted. This, *added* to the angle already obtained, is the angle required. If ' c ' is however negative, let β be the angle referred to, whose cosine is the numerical value of ' c '; $180^{\circ} - \beta$ is the required angle.

Ex. 2. Evaluate $\cos^{-1} 0.4777$.

0.4787 is the figure in the main column that is just **greater** than the given number, corresponding to $61^{\circ}24'$, and 10 is the mean difference in the same row for 4'. Hence, $61^{\circ}28'$ is the value of $\cos^{-1} 0.4777$.

It follows that $\cos^{-1} (-0.4777) = 180^{\circ} - 61^{\circ}28' = 118^{\circ}32'$.

Art. 60. To read off by reference to the Table of Natural Tangents, the numerically smallest angle having any given number t for its tangent, the rules given in Art. 58, wholly apply without any modification.

Ex. 3. Find $\tan^{-1} (1.2145)$.

1.2131 is found to be $\tan 50^{\circ}30'$

14 is the mean difference for 2'

Hence, $\tan^{-1} (1.2145) = 50^\circ 32'$

Also, $\tan^{-1} (-1.2145) = -50^\circ 32'$.

Note.—In case where the exact difference between the given number and the figure in the main column nearest to it is not found in the Mean difference column, use is made of the Principle of Proportional Parts which states—"Consequent on a small change in the 'angle, the corresponding change in the value of the Trigonometric function is approximately proportional to the change in the angle itself."

Art. 61. When angles are defined by their cosecant, secant or cotangent or when it is required to write down the cosecant, secant or cotangent of a given angle, no direct reference is possible, as we are not furnished with Tables for these functions; but their reciprocals can easily be read from the Tables, and thence their values calculated.

Ex. 4. Evaluate $\cot^{-1} 0.5$.

$$\cot^{-1} 0.5 = \tan^{-1} 2 = 63^\circ 26'$$

Ex. 5. Write down cosec $42^\circ 39'$

$$\sin 42^\circ 39' = 0.6775$$

$$\therefore \operatorname{cosec} 42^\circ 39' = \frac{1}{0.6775} = \frac{10000}{6775} = 1.476.$$

EXAMPLES VIII a

1. Write down by reference to four-figure Tables the values of—

- (i) $\sin 12^\circ 36'$. (ii) $\cos 15^\circ 44'$. (iii) $\tan 33^\circ 59'$. (iv) $\sec 44^\circ 27'$.
 (v) $\operatorname{cosec} 57^\circ 41'$. (vi) $\cot 53^\circ 53'$. (vii) $\sin 65^\circ 31' 30''$.
 (viii) $\tan 72^\circ 17' 15''$. (ix) $\cos 200^\circ 12'$. (x) $\cos 765^\circ 43'$.
 (xi) $\tan (-101^\circ 12')$. (xii) $\cot (-354^\circ 2')$.

2. Find the numerically smallest angles whose sines are (i) 0.8660.
 (ii) 0.5. (iii) -0.8888. (iv) -0.2349.

3. Evaluate (i) $\cos^{-1} 0.5$. (ii) $\cos^{-1} 0.2456$. (iii) $\cos^{-1} (-0.4567)$.
 (iv) $\tan^{-1} 0.3467$. (v) $\tan^{-1} (-2.3456)$. (vi) $\cot^{-1} 2$. (vii) $\sec^{-1} (-12)$.

4. Find the general value of θ satisfying the following equations.

(i) $\sin \theta = 0.3450$. (ii) $\cos \theta = 0.2345$. (iii) $\tan \theta + 2.45 = 0$.
 (iv) $\sin 3\theta + 0.9734 = 0$. (v) $\cos 2\theta + 0.4567 = 0$. (vi) $\tan p\theta = \tan q\theta$.
 (vii) $\sin m\theta + \sin n\theta = 0$. (viii) $\cos \theta - \cos 5\theta = 0$. (ix) $\operatorname{cosec} 4\theta = 7$.
 (x) $\cot p\theta + 4.5 = 0$.

Art. 62. Equations involving one ratio only of the unknown angle. When only one ratio of θ and its powers are present, the equation is reduced to an algebraic equation in y where y is substituted for that ratio. Solve for y by algebraical methods. If the various values of y are $a, b, c \dots$ etc., each of the equations $y = a, y = b, y = c \dots$ furnishes a general value of θ satisfying the given equation. The aggregate of all these is the complete solution.

Ex. 1. Find the general value of θ satisfying the equation. $2 \sin^3 \theta - \sin^2 \theta - 5 \sin \theta - 2 = 0$.

Put $\sin \theta = y$. Then, the given equation reduces to $2y^3 - y^2 - 5y - 2 = 0$.

$y + 1$ is a factor, and $y - 2$, and $2y + 1$ are also factors.

Hence, $y = -1, y = 2, y = -1/2$.

i.e., $\sin \theta = -1, \sin \theta = 2$ or $\sin \theta = -1/2$.

The first gives $\theta = n\pi + (-1)^n 3\pi/2$

The second value being > 1 , θ is not real.

As the numerically smallest angle whose sine is $-1/2$ is $-\pi/6$, the third value of $\sin \theta$ gives $\theta = n\pi - (-1)^n \pi/6$, n being zero or any integer.

Ex. 2. Find the general value of x satisfying the equation $\tan^4 x - 4 \tan^2 x + 3 = 0$.

Ex. 4. Solve : $-2 \cos^3 \theta - 7 \cos^2 \theta + 2 \cos \theta + 3 = 0$.

Art. 63. If the equation involves several trigonometrical functions of the unknown angle, express them all in terms of one—either the sine, or the cosine. Clear the equation of surds by rationalisation and proceed as in the previous article. It must not be forgotten that in this process, *extraneous* roots are also introduced, which must certainly be detected by actual substitution and rejected accordingly.

Ex. 1. Solve

$$\sec x - \tan x = \cos x.$$

$$\frac{1}{\cos x} - \frac{\sin x}{\cos x} = \cos x$$

$$\text{i.e., } 1 - \sin x = \cos^2 x.$$

$$\therefore 1 - \sin x = 0 \text{ or } 1 = 1 + \sin x, \text{ giving}$$

$$x = n\pi + (-1)^n \pi/2 \text{ or } x = n\pi.$$

Ex. 2. Solve $\cot \theta - \tan \theta = 2$.

Ex. 3. Solve $\cos \theta + \sin \theta - \sqrt{2} = 0$.

Ex. 4. Find the general values of A satisfying the equation $\cos 3A - \sin 2A = 0$.

Art. 64. Find the values of $\sin 18^\circ$ and $\cos 36^\circ$.

As 18° is a fifth part of a right angle, it is a root of the equation $5x = 90^\circ$ *i.e.*, of the equation $2x = 90^\circ - 3x$. Taking the sines of both sides, $\sin 2x = \cos 3x$.

$$\text{i.e., } 2 \sin x \cos x = 4 \cos^3 x - 3 \cos x.$$

$$\text{i.e., } 2 \sin x = 4 \cos^2 x - 3, \text{ since } \cos x \text{ is not equal to } 0.$$

$$\text{i.e., } 4 \sin^2 x + 2 \sin x - 1 = 0.$$

$$\text{i.e. } \sin x = \frac{-1 \pm \sqrt{5}}{4}.$$

Now, 18° being a positive acute angle, the negative value of $\sin x$ is inadmissible.

Thus, $\sin 18^\circ = (\sqrt{5} - 1)/4$.

$$\begin{aligned}\cos 36^\circ &= 1 - 2 \sin^2 18^\circ = 1 - (\sqrt{5} - 1)^2/8 \\ &= (\sqrt{5} + 1)/4.\end{aligned}$$

Note 1.—Both these results must be committed to memory.

Note 2.— $\cos 18^\circ$ and $\sin 36^\circ$ can be evaluated as radicals from $\sin 18^\circ$ and $\cos 36^\circ$.

Note 3.— 54° and 72° being respectively the complements of 36° and 18° , $\sin 54^\circ = \frac{\sqrt{5} + 1}{4}$ and $\cos 72^\circ = \frac{\sqrt{5} - 1}{4}$.

Art. 65. Geometrical method for evaluating $\sin 18^\circ$ and $\cos 36^\circ$. A B C is an isosceles triangle

having each of the base angles double the vertical angle. BAC is therefore 36° and each of the angles at the base BC is 72° . Draw AF perpendicular to BC meeting it in F. In AB mark D so that DC equals BC and draw DE perpendicular to AC. Then, $\angle BDC = 72^\circ$ so that $\angle DCA = 36^\circ$, and $\angle BCD = 36^\circ = \angle DAC$. Hence, BC touches the circle ADC at C. $BC^2 = BD \cdot BA$; But as $BC = CD = AD$, $AD^2 = BD \cdot BA$. So AB is divided at D internally in medial section. If AB measures two units, and

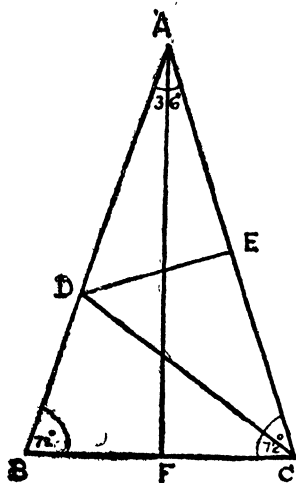


Fig. 36

$AD = x$ units, $x^2 = 2 \times (2 - x)$. Taking the positive value of x , $x = (\sqrt{5} - 1)$. In the triangle BAC, AF bisects angle A and hence $\angle BAF = 18^\circ$. From the right angled triangle

BAF, right angled at F, $\sin 18^\circ = \sin \text{BAF} = \frac{BF}{AB} = \frac{BC}{2AB}$
 $= (\sqrt{5} - 1)/4$. [Since $BC = AD = (\sqrt{5} - 1)$]

Again in the triangle ADE, $AE = \frac{1}{2} AC = 1$.

$\angle AED = 90^\circ$; $\angle DAE = 36^\circ$ and $AD = (\sqrt{5} - 1)$

Hence, $\cos 36^\circ = \frac{AE}{AD} = \frac{1}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{4}$.

Art. 66. Solution of the equation

$a \cos \theta + b \sin \theta = c$, $|c| < \sqrt{a^2 + b^2}$.

Choose two subsidiary quantities R and α such that $R \cos \alpha = a$, and $R \sin \alpha = b$. Squaring and adding $R^2 = a^2 + b^2$. We shall take R as the positive value of $\sqrt{a^2 + b^2}$. Then α is determined uniquely from the equations $\cos \alpha = a/\sqrt{a^2 + b^2}$; $\sin \alpha = b/\sqrt{a^2 + b^2}$ ($0 < \alpha < 360^\circ$).

Substituting these values for a and b , the equation becomes, $R \cos \theta \cos \alpha + R \sin \theta \sin \alpha = c$.

i.e., $R \cos (\theta - \alpha) = c$.

i.e., $\cos (\theta - \alpha) = c/\sqrt{a^2 + b^2}$.

i.e., $\cos (\theta - \alpha) = \cos \beta$, say where $\beta = \cos^{-1} (c/\sqrt{a^2 + b^2})$

$\therefore \theta = \alpha + 2n\pi \pm \beta$, where n equals zero or any integer.

Note 1.—An alternative method is suggested to the student. Substitute $(2 \tan \theta/2)/(1 + \tan^2 \theta/2)$ for $\sin \theta$ and $(1 - \tan^2 \theta/2)/(1 + \tan^2 \theta/2)$ for $\cos \theta$ in the given equation; solve for $\tan \theta/2$ and thence write down the general values of θ . The method is quite general.

Note 2.—Equations of type **$a \sec \theta + b \tan \theta = c$ or $a \operatorname{cosec} \theta + b \cot \theta = c$** can be reduced to **$a + b \sin \theta = c \cos \theta$** and **$a + b \cos \theta = c \sin \theta$** respectively.

Note 3.—Incidentally, it follows that the expression $a \cos \theta + b \sin \theta$ can be written in the form $R \cos (\theta - \alpha)$, R being equal to $\sqrt{a^2 + b^2}$ and α being defined by its sine and cosine. For various values of θ , this expression attains the maximum value R or minimum value $-R$ corresponding to $(\theta - \alpha)$ being equal to an even multiple of π or an odd multiple of π .

Ex. 1. Solve $\sqrt{3} \cos x + \sin x = 1$.

$R = 2$, and, $\cos \alpha = \sqrt{3}/2$; $\sin \alpha = 1/2$; $\alpha = \pi/6$.

The given equation becomes

$$2 \cos (x - \pi/6) = 1$$

$$\text{i.e., } \cos (x - \pi/6) = 1/2 = \cos \pi/3$$

$$\text{i.e., } x = (\pi/6) + 2n\pi \pm \pi/3$$

$$\text{i.e., } x = 2n\pi + \pi/2 \text{ or } 2n\pi - \pi/6.$$

Ex. 2. Express $\cos 3x + \sin 2x$ as a product and hence or otherwise solve the equation $\cos 3x + \sin 2x = 0$ and find all the values in the range from 0° to 360° .

$$\begin{aligned} y &= \cos 3x + \sin 2x \\ &= \sin (90^\circ + 3x) + \sin 2x \\ &= 2 \sin (45^\circ + 5x/2) \cos (45^\circ + x/2). \end{aligned}$$

$$\begin{aligned} [\text{It may also be expressed as } \cos 3x + \cos (90^\circ - 2x) \\ = 2 \cos (45^\circ + x/2) \cos (45^\circ - 5x/2)]. \end{aligned}$$

Now, when $\cos 3x + \sin 2x = 0$, $\sin (45^\circ + 5x/2) = 0$ or $\cos (45^\circ + x/2) = 0$, i.e., $5x/2 + 45^\circ = n \cdot 180^\circ$ or $45^\circ + x/2 = (2n + 1) 90^\circ$, i.e., $x = (n \times 72^\circ - 18^\circ)$ or $x = (4n + 1) 90^\circ$, where $n = 0$ or any integer.

For $n = 0$, $x = -18^\circ$ or 90° ;

$$n = 1, x = 54^\circ \text{ or } 450^\circ ;$$

$$n = 2, x = 126^\circ ;$$

$$n = 3, x = 198^\circ ;$$

$$n = 4, x = 270^\circ;$$

$$n = 5, x = 342^\circ;$$

$$n = 6, x = 414^\circ.$$

Hence the values of x between 0° and 360° , satisfying the equation are $54^\circ, 90^\circ, 126^\circ, 198^\circ, 270^\circ, 342^\circ$.

Ex. 3. If $\cos (A - B) = \frac{1}{2}$ and $\sin (A + B) = \frac{1}{2}$, find the smallest values of A and B and their most general values.

$$\cos (A - B) = \frac{1}{2} = \cos 60^\circ \text{ and } \sin (A + B) = \sin 30^\circ.$$

$$\therefore A + B = n. 180^\circ + (-1)^n. 30^\circ \text{-----} (1)$$

$$A - B = p. 360^\circ \pm 60^\circ \text{-----} (2)$$

$$2A = n. 180^\circ + p. 360^\circ \pm 60^\circ + (-1)^n 30^\circ.$$

$$2B = n. 180^\circ - p. 360^\circ + (-1)^n 30^\circ \mp 60^\circ.$$

$$\therefore A = n. 90^\circ + p. 180^\circ \pm 30^\circ + (-1)^n 15^\circ;$$

$$B = n. 90^\circ - p. 180^\circ + (-1)^n 15^\circ \mp 30^\circ.$$

where n and p are zero or any integers.

The least positive values of A and B are $A = 105^\circ$
 $B = 45^\circ$. ($n = 1, p = 0$).

Ex. 4. Explain why the same two series of angles are given by the equations $\theta + \pi/4 = n\pi + (-1)^n \pi/6$ and $\theta - \pi/4 = 2n\pi \pm \pi/3$.

Evidently, these are the general solutions of the two equations $\sin (\theta + \pi/4) = \sin \pi/6$ and $\cos (\theta - \pi/4) = \cos \pi/3$ respectively. But $\sin \pi/6 = \frac{1}{2} = \cos \pi/3$; and $\cos (\theta - \pi/4) = \cos (\pi/4 - \theta) = \sin [\pi/2 - (\pi/4 - \theta)] = \sin (\theta + \pi/4)$.

Hence the two equations are identical.

Note.—Similarly the same equation $\sin \theta = \cos \alpha$ can be written in two different ways as $\sin \theta = \sin (\pi/2 - \alpha)$ and $\cos (\theta - \pi/2) = \cos \alpha$ and the same values of θ obtained in different forms. *Vide Q. 34.*

EXAMPLES VIII *b*

Solve the following equations giving in each case the most general value of the variable angle.

- (1) $\sin x = 1 + \cos x$.
- (2) $\cos x - \cos 2x + \cos 3x = 0$.
- (3) $5 \sin^2 x + 3 \sin x - 2 = 0$.
- (4) $4 \sin^2 x - \sin x \cos x - 3 \cos^2 x = 0$.
- (5) $\tan x - \cot x = 5$.
- (6) $\tan x + \cot x = 4$.
- (7) $4 \tan^2 x - 5 \tan x + 1 = 0$.
- (8) $2 \sin x + \operatorname{cosec} x = 3$.
- (9) $\cos x + \sec x = 2$.
- (10) $\sin 9x + \sin 7x = \sin 8x$.
- (11) $\cos^2 x + 7 \sin x \cos x = 4$.
- (12) $\sin^2 x + 13 \sin x \cos x = 7$.
- (13) $5 \cos^2 x + 3 \sec^2 x = 8$.
- (14) $1 \sin \theta + m \cos \theta = \sin (\theta - \alpha)$.
- (15) $\sin (\theta + \alpha) = 3 \cos (\theta + \beta)$.
- (16) $\sin (\pi \sin x) = \cos (\pi \cos x)$.
- (17) $\tan px = \cot qx$.
- (18) $3 \sin A + 4 \cos A = 5$.
- (19) $5 \cos x + 12 \sin x = 10$.
- (20) $\cos x + \sin x = \cos 2x$.
- (21) $2 \cos x + 3 \sin x = 2$.
- (22) $4 \sin \theta - 3 \cos \theta = 3$.
- (23) $\sqrt{2} \sec \theta - 3 \tan \theta = 1$.
- (24) $\operatorname{cosec} x - \cot x + 1 = 0$.
- (25) $\operatorname{cosec} x + \cot x + \sqrt{3} = 0$.

(26) Find all the roots of the cubic $4x^3 - 3x + 1 = 0$ by putting $x = \sin A$.

(27) Solve $4x^3 - 3x - 0.2 = 0$ putting $x = \cos A$.

(28) Prove that a $h \sec \theta - bk \operatorname{cosec} \theta = (a^2 - b^2)$ is satisfied by four different values of θ lying between 0° and 360° .

(29) Use the formula for $\tan 3A$ to solve completely the equation $3x - x^3 = (1 - 3x^2) \times 1.6003$.

(30) $\tan x + \tan 2x + \tan 3x = 0$.

(31) Solve $\sin(3\theta + a) + \sin(3\theta - a) + \sin(a - \theta) - \sin(a + \theta) = \cos a$.

(32) Solve $\cos 3x - \cos 4x = \cos 5x - \cos 6x$.

(33) Show that there is always a value of x less than 90° satisfying the equation $\cot x - \tan x = a$, for all values of 'a'.

(34) Show by solving in two different ways the equation $\sin \theta = \cos a$ that the same angles are included in the two formulae $(2n + \frac{1}{2})\pi \pm a$ and $n\pi + (-1)^n(\frac{\pi}{2} - a)$.

(35) Solve $5 \tan(\theta - 30^\circ) = \tan(\theta + 30^\circ)$.

CHAPTER IX

PROPERTIES OF TRIANGLES AND THE CIRCLES CONNECTED WITH THEM

Art. 67. In this chapter, it is proposed to derive the various relations that exist between the sides and angles of a triangle and their application to establish several other identities in a triangle. They are most necessary for the solution of triangles. All the fundamental formulæ may be classified under the sine formula, the cosine formula and the tangent formula.

Art. 68. Relation between the sides and the angles opposite them.—In any triangle the lengths of the sides are proportional to the sines of the opposite angles. Denoting the sides opposite the angles A, B, C of triangle ABC by corresponding small letters a, b, c , we have to establish that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

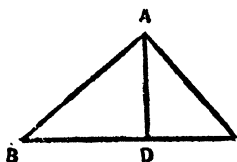


Fig. 37 a.

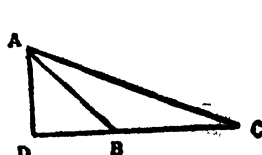


Fig. 37 b.

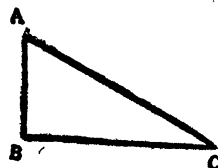


Fig. 37 c.

ABC is a triangle in which AD is drawn from A perpendicular to BC,

Since triangle ABD is right-angled at D,

$$\sin ABD = \sin B = \frac{AD}{AB} = \frac{AD}{c} \quad \text{---(1)}$$

Also, triangle ADC is right-angled at D

$$\therefore \sin ACD = \sin C = \frac{AD}{AC} = \frac{AD}{b} \quad \text{---(2)}$$

Eliminating AD between (1) and (2) by division,

$$\text{we get, } \frac{\sin B}{\sin C} = \frac{b}{c}, \text{ i.e., } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing BE perpendicular to CA, and taking the triangles BCE and BAE, we can show that

$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$

$$\text{Hence } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Note. In the obtuse angled triangle (Fig. 37-b),

$$\frac{AD}{AB} = \sin (180^\circ - B) = \sin B.$$

When $B = 90^\circ$ (Fig. 37-c), $\sin C = c/b$ and $\sin B = \sin 90^\circ = 1$, so that, $b/\sin B = c/\sin C$. Hence, in all cases, $a/\sin A = b/\sin B = c/\sin C$.

Aliter. Draw the circumcircle of the triangle. Let S be the circumcentre. Join BS and produce it to cut the circle again at D. Join DC. From triangle BDC, we deduce that $a = 2R \sin A$.

Case 1. A is acute—D and A are on the same side of BC, so that $BDC = BAC = \angle A$; $BCD = 90^\circ$, BD being a diameter, $BC = a$ and $BD = 2R$, R being the circum-radius. From triangle BDC, $BC/BD = \sin BDC$, i.e., $a/2R = \sin A$, i.e., $a = 2R \sin A$.

Case 2. A is obtuse—D and A are on opposite sides of BC and in the cyclic quadrilateral BACD, $BDC = 180^\circ - A$. As before, from triangle BDC, $BC/BD = \sin (180^\circ - A)$, i.e., $a/2R = \sin (180^\circ - A) = \sin A$.
 $\therefore a = 2R \sin A$.

Case 3. $A = 90^\circ$ — $\sin A = 1$, $BC = a = 2R$
 $= 2 R \sin A$. Hence in all cases, $a = 2 R \sin A$.
 Similarly $b = 2 R \sin B$ and $c = 2 R \sin C$.

So, $a / \sin A = b / \sin B = c / \sin C = 2 R$.

Note. This is a very useful formula since it enables a function of the sides a, b, c to be expressed as a function of the angles A, B, C .

Art. 69. Relation between the sides and one angle.

$$b^2 = c^2 + a^2 - 2 ca \cos B$$

Draw the altitude of the triangle from **either extremity of b** . In figure 37, the altitude from A is drawn to meet the opposite side BC or the side produced at D according as $\angle B$ is acute or obtuse.

Case 1. $\angle B$ acute — (Fig. 37-a).

$$\begin{aligned} \text{From triangle ADC, } b^2 &= AC^2 = AD^2 + DC^2 \\ &= AD^2 + (BC - BD)^2 = AD^2 + BD^2 + BC^2 \\ &\quad - 2 BC \cdot BD \\ &= AB^2 + BC^2 - 2 BC \cdot AB \cdot BD / AB. \\ &= c^2 + a^2 - 2 ca \cos B, \end{aligned}$$

since from triangle ABD, $BD / AB = \cos B$.

Case 2. $\angle B$ obtuse—(Fig. 37-b).

$$\begin{aligned} b^2 &= AC^2 = AD^2 + DC^2 = AD^2 + (DB + BC)^2 \\ &= AD^2 + DB^2 + BC^2 + 2 BC \cdot DB. \\ &= AB^2 + BC^2 + 2 BC \cdot AB \cdot DB / AB. \end{aligned}$$

But in triangle ADB, $\angle ABD = 180^\circ - B$, so that

$$DB / AB = \cos (180^\circ - B) = -\cos B.$$

$$\therefore b^2 = c^2 + a^2 - 2 ca \cos B.$$

Case 3. B is right. $\cos B = 0$

$$\therefore c^2 + a^2 - 2 ca \cos B = c^2 + a^2 = b^2.$$

Hence, in all cases, $b^2 = c^2 + a^2 - 2ca \cos B$

Similarly, $c^2 = a^2 + b^2 - 2ab \cos C$

and $a^2 = b^2 + c^2 - 2bc \cos A$.

Notes. (1) To prove the value of a^2 draw the altitude from B or C and **not** from A.

(2) From this formula, we deduce

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = (b^2 + c^2 - a^2) / 2bc,$$

an expression for $\cos A$ in terms of sides.

Art. 70. Another important formula that deserves attention is :— **$a = b \cos C + c \cos B$** .

$a = BC = BD + DC = AB \cos B + AC \cos C$
 $= c \cos B + b \cos C$ in Fig. 37a.

In the obtuse-angled triangle.

$a = BC = DC - BD = AC \cos C - AB \cos (180^\circ - B)$
 $= AC \cos C + AB \cos B = b \cos C + c \cos B$.

Similarly, by cyclical interchange, the other two results may be written down.

$b = c \cos A + a \cos C$; $c = a \cos B + b \cos A$.

Note. Since $c \cos A$, and $a \cos C$ are the projections of the sides c and a on the side b , the formulae are also stated thus in words.

The length of any side of a triangle is equal to the sum of the projections of the other two sides on it.

Art. 71. Tangent Formula.

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$$

By Art. 68, $\sin A / \sin B = a / b$

$$\therefore \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a - b}{a + b} \quad (\text{Componendo and dividendo}),$$

$$i.e., \frac{2 \cos [(A + B) / 2] \sin [(A - B) / 2]}{2 \sin [(A + B) / 2] \cos [(A - B) / 2]} = \frac{a - b}{a + b}$$

$$i.e., \tan \frac{A - B}{2} = \frac{a - b}{a + b} \tan \frac{A + B}{2}$$

$$i.e., \tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}.$$

$$\text{Similarly, } \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

$$\text{and } \tan \frac{C - A}{2} = \frac{c - a}{c + a} \cot \frac{B}{2}.$$

Note. If $a < b$, $A < B$, the formula is altered to

$$\tan \frac{B - A}{2} = \frac{b - a}{b + a} \cot \frac{C}{2}.$$

Second Method. The same theorem may be easily proved by Geometry.

ABC is a triangle in which A is greater than B.

With C as centre and CA as radius, draw a circle meeting BC and BC produced in L & M respectively. Join AL and AM. Through L draw LN parallel to MA to meet AB in N. Let CAL be equal to α and LAB equal to β .

$$\text{Then } A = CAB = CAL + LAB = \alpha + \beta.$$

$$B = ABC = ALC - LAB = \alpha - \beta. \quad (CA = CL).$$

Adding and subtracting, $\alpha = (A + B)/2$ and $\beta = (A - B)/2$. Also $CA = CL = CM$; $\angle LAM = 90^\circ = \angle ALN$. ($LN = MA$).

$$\therefore BL \parallel a - b; \quad BM = a + b.$$

In the $\triangle ALN$, right-angled at L, $\tan \beta = LN / AL$.

In the triangle LAM, $\tan \alpha = \tan ALM = AM/AL$.

Dividing one by the other,

$$\frac{\tan \beta}{\tan \alpha} = \frac{LN}{AM} = \frac{BL}{BM} = \frac{a-b}{a+b}.$$

Substituting for β, α their values in terms of A, B,

$$\tan \frac{A-B}{2} \bigg/ \tan \frac{A+B}{2} = \frac{a-b}{a+b},$$

$$\text{i.e., } \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

Art. 72. Sin A in terms of sides.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$\cos A = (b^2 + c^2 - a^2) / 2bc.$$

$$\sin A = [1 - \cos^2 A]^{\frac{1}{2}} = [1 - (b^2 + c^2 - a^2)^2 / 4b^2c^2]^{\frac{1}{2}}.$$

$$= [(2bc)^2 - (b^2 + c^2 - a^2)^2]^{\frac{1}{2}} / 2bc.$$

$$= [(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)]^{\frac{1}{2}} / 2bc.$$

$$= [(b+c+a)(b+c-a)(a-b+c)(a-c+b)]^{\frac{1}{2}} / 2bc.$$

Taking s to denote the semi-perimeter, $a+b+c=2s$,

$$b+c-a = (b+c+a) - 2a = 2s - 2a = 2(s-a),$$

$$c+a-b = 2(s-b); \quad a+b-c = 2(s-c),$$

$$\text{Hence, } \sin A = 2 \sqrt{s(s-a)(s-b)(s-c)} / bc.$$

$$\text{Similary, } \sin B = 2 \sqrt{s(s-a)(s-b)(s-c)} / ca.$$

$$\text{and } \sin C = 2 \sqrt{s(s-a)(s-b)(s-c)} / ab.$$

Art. 73. Ratios of Half-angles in terms of sides.

Starting from $\cos A = (b^2 + c^2 - a^2) / 2bc$, we shall deduce formulae for the sine, cosine and tangent of $A/2$.

$$2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= (2bc - b^2 - c^2 + a^2) / 2bc$$

$$\begin{aligned}
 &= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc} \\
 &= \frac{2(s - c) \cdot 2(s - b)}{2bc} = \frac{2(s - b)(s - c)}{bc}
 \end{aligned}$$

$$\sin^2 A/2 = (s - b)(s - c)/bc$$

$$\sin \frac{A}{2} = + \sqrt{\frac{(s - b)(s - c)}{bc}}, \text{ the negative sign being}$$

rejected, since $A/2$ is always less than 90° .

$$\text{Similarly, } \sin \frac{B}{2} = + \sqrt{\frac{(s - c)(s - a)}{ca}}$$

$$\sin \frac{C}{2} = + \sqrt{\frac{(s - a)(s - b)}{ab}}$$

$$\begin{aligned}
 2 \cos^2 \frac{C}{2} &= 1 + \cos C = 1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{2ab + a^2 + b^2 - c^2}{2ab} \\
 &= \frac{(a + b)^2 - c^2}{2ab} = \frac{2s(s - c)}{ab},
 \end{aligned}$$

$$\cos \frac{C}{2} = \sqrt{\frac{s(s - c)}{ab}}$$

$$\text{Similarly, } \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}, \cos \frac{B}{2} = \sqrt{\frac{s(s - b)}{ca}}$$

Dividing the value of $\sin A/2$ by that of $\cos A/2$, we have

$$\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}; \tan \frac{B}{2} = \sqrt{\frac{(s - c)(s - a)}{s(s - b)}}, \text{ etc.}$$

It may be remarked that this formula for $\tan C/2$ in terms of the sides can be directly derived from either of the formulae.—

$$(i) \tan \frac{C}{2} = \sqrt{\frac{1 - \cos C}{1 + \cos C}}, (ii) \tan \frac{C}{2} = \frac{1 - \cos C}{\sin C}.$$

Note.—(1) From these values of $\cos A/2$ and $\sin A/2$, we may deduce $\sin A = 2 \sin A/2 \cos A/2$.

$$= 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$

$$= 2 \sqrt{s(s-a)(s-b)(s-c)/bc}.$$

(2) These formulae, especially the last for the tangents of the half angles in terms of the sides are of much use in evaluating by logarithms the angles of a triangle, given the lengths of the sides.

Art. 74. Area of triangle.—We shall now proceed to deduce various expressions for the area of a triangle ABC.

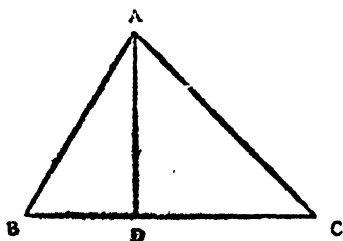


Fig. 40

(1) The first and foremost is that derived from geometry. **Area of triangle** = $\frac{1}{2}$ **base** $\times$ **altitude**. Any side may be taken as base.

(2) Draw AD perpendicular to BC, denoting the area of triangle by S.

$$S = \frac{1}{2} BC \cdot AD = \frac{1}{2} BC \cdot AB \sin ABD = \frac{1}{2} ac \sin B.$$

$$\text{Similarly, } S = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C.$$

Area of a triangle is equal to half the product of any two sides and the sine of the included angle.

(3) In the formula $S = \frac{1}{2} bc \sin A$, substitute for b, c their values in terms of the circum-radius R and the opposite angles. Since $b = 2R \sin B$ and $c = 2R \sin C$, $S = \frac{1}{2} \times 2R \sin B \times 2R \sin C \cdot \sin A$.

$$S = 2 R^2 \sin A \cdot \sin B \cdot \sin C.$$

(4) $S = (b c \sin A)/2 = \sqrt{s(s-a)(s-b)(s-c)}$, substituting for $\sin A$ in terms of sides.

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$(5) \quad b/\sin B = c/\sin C; \quad b = c \sin B/\sin C.$$

$$2S = bc \sin A = c^2 \sin B \sin A/\sin C.$$

$$\therefore S = \frac{1}{2} \frac{c^2 \sin B \sin A}{\sin (180^\circ - A - B)}$$

Note. $2S = c \times \text{altitude from } C.$

$$= c^2 \sin B \sin A/\sin C.$$

Hence, altitude from $C = c \sin B \sin A/\sin C$, a logarithmic formula for the altitude.

$$(6) \quad BC = BD + DC \quad (\text{Fig. 40})$$

$$= AD \cot B + AD \cot C.$$

$$= AD (\cot B + \cot C).$$

$$AD = a/(\cot B + \cot C).$$

$$2S = a \cdot AD = a^2/(\cot B + \cot C)$$

$$S = a^2/2 (\cot B + \cot C).$$

$$(7) \quad 2S = bc \sin A \text{ and } a = 2R \sin A. \quad \text{Eliminating } \sin A, \\ S = abc/4R.$$

$$\text{Art. 75.} \quad \frac{a+b}{c} = \frac{\cos(A-B/2)}{\sin(C/2)} \quad (1)$$

$$\frac{a-b}{c} = \frac{\sin(A-B/2)}{\cos(C/2)} \quad (2)$$

Substitute for a, b, c their values in terms of R and the angles.

$$(1) \quad \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} \\ = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} / 2 \sin \frac{C}{2} \cos \frac{C}{2}.$$

$$= \cos \frac{A-B}{2} / \sin \frac{C}{2}.$$

$$(2) \frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C}$$

$$= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} / 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= \sin \frac{A-B}{2} / \cos \frac{C}{2}.$$

Art. 76. Another important Theorem, which finds an application in questions on Statics is—

If X is a point taken in the base BC of a triangle ABC, so as to divide it in the ratio $l : m$, the angle AXC, θ is given by the formulae,

$$(l+m) \cot \theta = l \cot \beta - m \cot r$$

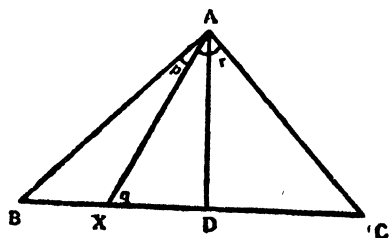


Fig. 41

$= m \cot B - l \cot C$, where
 $BAX = \beta$ and $XAC = r$.
 Draw AD perpendicular
 to BC to meet it in D.
 $BX = l$
 $XC = m$ by hypothesis.

In triangle ABX, $ABX = (\theta - \beta)$, $BAX = \beta$.

$$BX / AX = \sin \beta / \sin (\theta - \beta)$$

Similarly, in triangle AXC, $XAC = r$,

$$ACX = 180^\circ - \theta - r,$$

$$AX / XC = \sin ACX / \sin XAC = \sin (\theta + r) / \sin r$$

$$\therefore \frac{l}{m} = \frac{BX}{CX} = \frac{BX}{AX} \times \frac{AX}{XC} = \frac{\sin \beta}{\sin (\theta - \beta)} \times \frac{\sin (\theta + r)}{\sin r}$$

$$= \frac{\sin \beta (\sin \theta \cos r + \cos \theta \sin r)}{\sin r (\sin \theta \cos \beta - \cos \theta \sin \beta)}$$

$$= \frac{\cot r + \cot \theta}{\cot \beta - \cot \theta} \text{ dividing both numerator}$$

and denominator by $\sin \beta \sin r \sin \theta$.

Clearing of fractions, $(1 + m) \cot \theta = 1 \cot \beta - m \cot r$.

To the second part of the question, we have to express θ in terms of B and C. $BAX = \theta - B$,
 $XAC = 180^\circ - (\theta + C)$.

$$\frac{l}{m} = \frac{BX}{CX} = \frac{\sin (\theta - B) \sin C}{\sin B \sin (\theta + C)}$$

$$= \frac{\sin C [\sin \theta \cos B - \cos \theta \sin B]}{\sin B [\sin \theta \cos C + \cos \theta \sin C]}$$

$$= \frac{\cot B - \cot \theta}{\cot C + \cot \theta}$$

$$\text{i.e., } (l + m) \cot \theta = m \cot B - l \cot C.$$

Note. A note is also added, expressing the length AX in terms of the sides.

$$\text{From } \triangle AXC, b^2 = AX^2 + XC^2 - 2 AX \cdot XC \cdot \cos \theta. \text{---(1)}$$

$$\text{From } \triangle ABX, c^2 = AX^2 + BX^2 + 2 BX \cdot XA \cdot \cos \theta. \text{---(2)}$$

Multiplying (1) and (2) by l and m respectively and adding,

$$l b^2 + m c^2 = (l + m) AX^2 + l XC^2 + m BX^2 + 2 AX \cdot \cos \theta [m \cdot BX - l XC] \text{---(3)}$$

$$= (l + m) AX^2 + l XC^2 + m BX^2 (\because m BX = l XC)$$

$$\text{Also } \frac{BX}{l} = \frac{XC}{m} = \frac{BX + XC}{l + m} = \frac{BC}{l + m} = \frac{a}{(l + m)}$$

$$BX = la/(l + m); \quad XC = ma/(l + m).$$

Substituting these values of BX and XC in (3),

$$l b^2 + m c^2 = (l + m) AX^2 + l m a^2 / (l + m)$$

AX is given by

$$(l + m) AX^2 = l b^2 + m c^2 - l m a^2 / (l + m)$$

Art. 77. The applications of these formulae to solve examples are exemplified by typical questions worked out.

Ex. 1. Prove that in a triangle

$$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$$

$$\begin{aligned} \text{1st term} &= (b^2 - c^2) \cot A \\ &= [4 R^2 \sin^2 B - 4 R^2 \sin^2 C] \cot A \\ &= 4 R^2 [\sin^2 B - \sin^2 C] \cot A \\ &= 4 R^2 \sin B + C \times \sin B - C \cot A \\ &= 4 R^2 \sin A \times \sin B - C \times \cos A / \sin A \\ &= 4 R^2 \sin (B - C) \cos A \\ &= 4 R^2 \cos (C + B) \sin (C - B), \\ &= 2 R^2 [\sin 2 C - \sin 2 B], \end{aligned}$$

$$\text{Hence } \Sigma (b^2 - c^2) \cot A = 2 R^2 \Sigma (\sin 2 C - \sin 2 B) = 0.$$

Ex. 2. Prove that $a = (b + c) \sin \phi$, where ϕ is a subsidiary angle satisfying the relation

$$\begin{aligned} \cos \phi &= \frac{2 \sqrt{bc}}{b + c} \times \cos \frac{A}{2} \\ a^2 &= b^2 + c^2 - 2 bc \cos A \\ &= b^2 + c^2 - 2 bc (2 \cos^2 A/2 - 1), \\ &\quad \text{expressing } \cos A \text{ in terms of } \cos (A/2) \\ &= b^2 + c^2 + 2 bc - 4 bc \cos^2 A/2 \\ &= (b + c)^2 - 4 bc \cos^2 A/2 \\ &= (b + c)^2 \left[1 - 4 bc \cos^2 \frac{A}{2} / (b + c)^2 \right] \end{aligned}$$

$$\text{But, by hypothesis, } \cos^2 \phi = 4 bc \cos^2 A/2 / (b + c)^2$$

$$\therefore a^2 = (b + c)^2 \sin^2 \phi ; a = (b + c) \sin \phi.$$

Note. This a very important formula, being the adaptation of the Cosine formula $a^2 = b^2 + c^2 - 2 bc \cos A$

for logarithmic use, to find directly the third side of a triangle, given two sides and the included angle.

Similarly, there are two other formulae. In one, $\cos A$ is expressed as $\left(1 - 2 \sin^2 \frac{A}{2}\right)$, and in the other, $\cos A$ is expressed as $\left(\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}\right)$.

The corresponding results are

$$a = (b - c) \sec \phi, \text{ where } \tan \phi = \frac{2\sqrt{bc} \sin \frac{A}{2}}{b - c} \quad \text{---(1)}$$

$$a = (b - c) \cos \frac{A}{2} \cdot \sec \theta \text{ where } \tan \theta = \frac{b + c}{b - c} \tan \frac{A}{2} \quad \text{---(2)}$$

The student will work out these results for himself.

Ex. 3. Show that in a triangle ABC,

$$b^2 + c^2 - 2bc \cos (A + 60^\circ) = c^2 + a^2 - 2ca \cos (B + 60^\circ).$$

$$\begin{aligned} b^2 + c^2 - 2bc \cos (A + 60^\circ) &= b^2 + c^2 - 2bc \cos A \cdot \cos 60^\circ \\ &\quad + 2bc \sin A \sin 60^\circ \\ &= b^2 + c^2 - 2c \cos 60^\circ \times \\ &\quad (c - a \cos B) \\ &\quad + 2c \sin 60^\circ \cdot a \sin B \end{aligned}$$

$$\begin{aligned} \text{(Since } c &= b \cos A + a \cos B \\ \text{and } b \sin A &= a \sin B) \end{aligned}$$

$$\begin{aligned} &= c^2 + a^2 - 2ca \cos B \\ &+ ca \cos B + 2ca \sin 60^\circ \sin B \\ &= c^2 + a^2 - 2ca (\cos B \cos 60^\circ \\ &\quad - \sin 60^\circ \sin B) \\ &= c^2 + a^2 - 2ca \cos (B + 60^\circ) \end{aligned}$$

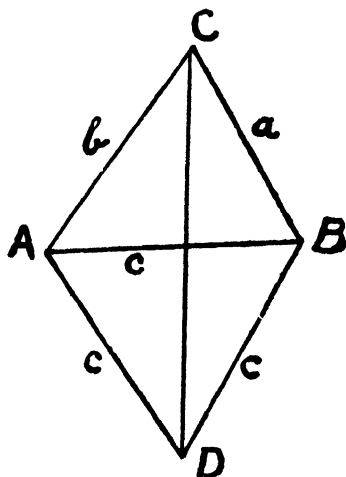


Fig. 42

Aliter.—This is solved very easily by Geometry. Describe an equilateral triangle ADB on AB, D being on the side of AB remote from C. Then $CAD = A + 60^\circ$ and $CBD = B + 60^\circ$. Join CD.

Taking triangle CAD,

$$CD^2 = b^2 + c^2 - 2bc \cos (A + 60^\circ)$$

From triangle CBD,

$$CD^2 = c^2 + a^2 - 2ca \cos (B + 60^\circ).$$

$$\therefore b^2 + c^2 - 2bc \cos (A + 60^\circ) = c^2 + a^2 - 2ca \cos (B + 60^\circ).$$

Ex. 4. If the sides a, b, c are in arithmetic progression, show that $\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

$$\cot \frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \frac{s}{S} \cdot (s-a), \text{ etc.}$$

But, a, b, c are in A.P. by hypothesis.

So, $s-a, s-b, s-c$ are in A.P.

i.e., $\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

The converse can be proved by reversing the steps.

$$\text{Ex. 5. } \frac{\cot A + \cot B + \cot C}{\cot(A/2) + \cot(B/2) + \cot(C/2)} = \frac{a^2 + b^2 + c^2}{(a+b+c)^2}$$

$$\cot(A/2) = s(s-a)/S. \quad (\text{Vide Ex. 4}).$$

$$\therefore \sum \cot(A/2) = s(s-a+s-b+s-c)/S = s^2/S.$$

$$\cot A = 2bc \cos A / 2bc \sin A = (b^2 + c^2 - a) / (4S).$$

$$\sum \cot A = (a^2 + b^2 + c^2) / 4S.$$

$$\begin{aligned} \therefore \frac{\cot A + \cot B + \cot C}{\cot(A/2) + \cot(B/2) + \cot(C/2)} &= \frac{a^2 + b^2 + c^2}{4s^2} \\ &= \frac{a^2 + b^2 + c^2}{(a+b+c)^2}. \end{aligned}$$

Ex. 6. Prove that in a triangle

$$(i) \frac{\sin 2A - \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \frac{\tan B}{\tan C}.$$

$$(ii) \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

(iii) Apply (i) and (ii) to show that if

$$\sin 2A : \sin 2B : \sin 2C = 3 : 4 : 5, \text{ then } \tan A = 3, \tan B = 2, \tan C = 1.$$

$$\begin{aligned} (i) \sin 2A - \sin 2B + \sin 2C &= 2 \cos A + B. \sin A - B + 2 \sin C \cos C. \\ &= 2 \cos C (\sin A + B - \sin A - B) \\ &= 4 \sin B \cos C \cos A. \end{aligned}$$

$$\begin{aligned} \text{Interchanging } B \text{ \& } C, \sin 2A + \sin 2B - \sin 2C &= 4 \sin C \cos A \cos B. \end{aligned}$$

Dividing one by the other, we get (i).

$$(ii) A + B = 180^\circ - C.$$

$\tan (A + B) = -\tan C$. Expanding and simplifying, we get (ii).

(iii) Let $\sin 2A = 3k$. Then $\sin 2B = 4k$, $\sin 2C = 5k$.
Substituting in (1) $\tan B / \tan C = (3 - 4 + 5) / (3 + 4 - 5)$
 $= 4/2 = 2 : 1$.

From the corresponding relation

$$\frac{-\sin 2A + \sin 2B + \sin 2C}{\sin 2A - \sin 2B + \sin 2C} = \frac{\tan A}{\tan B},$$

we have $\tan A / \tan B = 3 : 2$.

$$\text{Hence } \tan A : \tan B : \tan C = 3 : 2 : 1.$$

Let them be $3x$, $2x$, x respectively.

Substituting in (ii), $3x + 2x + x = 6x^3$, i.e., $x = \pm 1$.

But, $\tan A$, $\tan B$, $\tan C$ cannot all be negative.

Hence $\tan A = 3$, $\tan B = 2$, $\tan C = 1$ *deduced by*

Ex. 7. In a triangle ABC, $b \cos B = c \cos C$.

Show that it is isos or right-angled.

$$b \cos B - c \cos C = 0$$

$$\text{i.e., } 2R \sin B \cos B - 2R \sin C \cos C = 0$$

$$\text{i.e., } \sin 2B - \sin 2C = 0$$

$$\text{i.e., } 2 \cos (B + C) \sin (B - C) = 0$$

$$\text{i.e., } -2 \cos A \cdot \sin (B - C) = 0$$

Either $\cos A = 0$ or $\sin (B - C) = 0$

$$\text{i.e., } A = 90^\circ \text{ or } B = C.$$

Ex. 8. If $y \sin^2 A + x \sin^2 B = z \sin^2 B + y \sin^2 C$
 $= x \sin^2 C + z \sin^2 A$, where A, B, C are the angles of a
 triangle, show that

$$x : y : z = \sin 2A : \sin 2B : \sin 2C.$$

$$y \sin^2 A + x \sin^2 B = z \sin^2 B + y \sin^2 C.$$

$$\text{i.e., } y (\sin^2 A - \sin^2 C) = \sin^2 B (z - x).$$

$$y \cdot \sin (A + C) \sin A - C = \sin^2 B \cdot (z - x).$$

$$y \cdot \sin B \cdot \sin (A - C) = \sin (A + C) \cdot \sin B \cdot (z - x).$$

$$y ; (z - x) = (\tan A + \tan C) : (\tan A - \tan C).$$

$$(y + z - x) : (y - z + x) = \tan A : \tan C.$$

Then $x : y : z$

$$= (\tan B + \tan C) : (\tan C + \tan A) : (\tan A + \tan B)$$

$$= \sin A \cos A : \sin B \cos B : \sin C \cos C$$

$$= \sin 2A : \sin 2B : \sin 2C.$$

EXAMPLES IX a.

In any triangle, prove that

$$(1) (a+b) \cos C + (b+c) \cos A + (c+a) \cos B = a+b+c.$$

$$(2) a^2 = (b+c)^2 \sin^2 \frac{A}{2} + (b-c)^2 \cos^2 \frac{A}{2}.$$

$$(3) a \cos A - b \cos B + c \cos C = 2b \cos C \cos A.$$

$$(4) (a^2 - b^2) \cot \frac{C}{2} + (b^2 - c^2) \cot \frac{A}{2} + (c^2 - a^2) \cot \frac{B}{2} = 0.$$

$$(5) a^2 + b^2 + c^2 = 2ab \cos C + 2bc \cos A + 2ca \cos B.$$

$$(6) \cot A : \cot B = b^2 + c^2 - a^2 : c^2 + a^2 - b^2.$$

$$(7) \sin \frac{A-B}{2} \sin \frac{C}{2} + \sin \frac{B-C}{2} \sin \frac{A}{2} + \sin \frac{C-A}{2} \sin \frac{B}{2} = 0.$$

$$(8) c \sin \frac{A-B}{2} \sin \frac{C}{2} + a \sin \frac{B-C}{2} \sin \frac{A}{2} + b \sin \frac{C-A}{2} \sin \frac{B}{2} = 0$$

$$(9) a(b \cos c - c \cos B) = b^2 - c^2.$$

$$(10) \text{The sides of a triangle are } (n^2 + n + 1), (2n + 1), (n^2 - 1).$$

Then, the greatest angle is 120° .

$$(11) \frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0.$$

$$(12) a \cos (B - C) + b \cos (C - A) + c \cos (A - B) = 4a \sin B \sin C.$$

$$(13) a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$$

$$(14) \text{If } a \tan A + b \tan B = (a+b) \tan \frac{1}{2}(A+B), \text{ then } A = B.$$

$$(15) a^3 \sin (B - C) + b^3 \sin (C - A) + c^3 \sin (A - B) = 0.$$

$$(16) a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc.$$

$$(17) \text{If } (b+c) : (c+a) : (a+b) = 4 : 5 : 6, \text{ then,} \\ \cos A : \cos B : \cos C = -7 : 11 : 13.$$

$$(18) a^2 + b^2 + c^2 = 8R^2 [1 + \cos A \cos B \cos C].$$

$$(19) c \cos^2 \frac{A}{2} + a \cos^2 \frac{C}{2} = a \cos^2 \frac{B}{2} + b \cos^2 \frac{A}{2} \\ = b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = s.$$

$$(20) abc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{S^2}{s}$$

$$(21) \frac{a}{\cot \frac{B}{2} + \cot \frac{C}{2}} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$(22) \frac{c}{\tan \frac{A}{2} + \tan \frac{B}{2}} = 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}$$

$$(23) (a^2 + b^2 + c^2) \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right) = (a + b + c)^2 (\cot A + \cot B + \cot C).$$

$$(24) b^2 \sin 2C + c^2 \sin 2B = 4S.$$

(25) ABC is a triangle in which $\cos A = -\frac{1}{2}$, $\tan \frac{B}{2} = \frac{2}{7}$. Find the sines of the angles. If the circumradius of the triangle is 2.5 inches, calculate the lengths of its sides.

(26) The cotangents of the angles A, B, C of a triangle are in A.P; show that $a^2 + c^2 = 2b^2$.

(27) Given b, c, C of a triangle ABC, prove that the third side 'a' is a root of the equation $x^2 - 2b \cos C + (b^2 - c^2) = 0$. If a_1 and a_2 are the values of 'a' satisfying the given conditions, show that $a_1 + a_2 = 2b \cos C$; $a_1 a_2 = b^2 - c^2$ and $B_1 + B_2 = 180^\circ$.

(28) If in a triangle ABC, $\sin A : \sin B = (\cos C + 2 \cos B) : (\cos C + 2 \cos A)$, prove that $A = B$ or $C = 90^\circ$.

(29) Prove that if three lines AD, BE, CF through the vertices of a triangle are concurrent at a point, $\sin BAD \cdot \sin ACF \cdot \sin CBE = \sin DAC \cdot \sin FCB \cdot \sin EBA$. [Use Ceva's Theorem.]

(30) γ is the Brocard point of a triangle, so that $\angle B A \gamma = \angle A C \gamma = \angle C B \gamma = w$. Prove that $\cot w = \cot A + \cot B + \cot C$.

(31) If $(a^2 + b^2) \sin(A - B) = (a^2 - b^2) \sin(A + B)$, show that the triangle is isosceles or right-angled.

(32) If in a triangle, $\tan \frac{A}{2} = \frac{5}{6}$ and $\tan \frac{B}{2} = \frac{20}{37}$, find $\tan \frac{C}{2}$; and prove that $a + c = 2b$.

(33) Prove that in a triangle, the altitude from A is $(b^2 \sin C + c^2 \sin B) / (b + c)$.

(34) Show that the perimeter of a triangle is equal to $2c \cos \frac{A}{2} \cos \frac{B}{2} \sec \frac{1}{2}(A + B)$.

(35) If p_1, p_2, p_3 be the altitudes of a triangle, show that $a \sin A + b \sin B + c \sin C = 2 [p_1 \cos A + p_2 \cos B + p_3 \cos C]$.

Art. 78. Radius of circumcircle—

From Art. 68, $\frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = R$.

To find an expression for R in terms of the sides alone, express $\sin A$ in terms of sides. $\sin A = 2S/bc$

and $\sin A = a/2R$.

$$\therefore R = \frac{a b c}{4S} = \frac{a b c}{4 \sqrt{s(s-a)(s-b)(s-c)}}.$$

Art. 79. In radius.—If the internal bisectors of the angles of a triangle meet at I , a circle can be drawn with I as centre, touching all the three sides internally. The radius of this circle is clearly the distance of I from any of the sides. Several expressions can be deduced for the length of the in-radius, usually denoted by r .

(1) r in terms of sides.

Join I to the vertices, and to the points of contact of the in-circle with the sides. Since IP is perpendicular to BC , area of triangle $IBC = \frac{1}{2} BC \times IP = \frac{1}{2} BC \times r$.

Similarly, Area of triangle $CIA = \frac{1}{2} CA \times IQ = \frac{1}{2} CA \times r$.

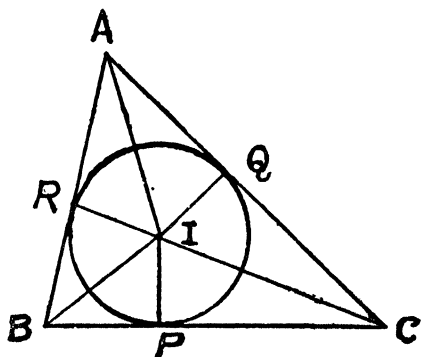


Fig. 43

Area of triangle $AIB = \frac{1}{2} AB \times IR = \frac{1}{2} AB \times r$.

Adding up, area $ABC = \triangle IBC + \triangle CIA + \triangle AIB$
 $= \frac{1}{2} (BC + CA + AB) \times r$

i.e., $S = s \times r$, where s is the semiperimeter.

$$\therefore r = \frac{S}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

(2) *r* in terms of base and base angles.

$$BP = IP \cot IBP = r \cot B/2.$$

$$\text{Similarly, } PC = r \cot C/2.$$

$$\text{Adding up, } a = r \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right).$$

$$r = a / \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right).$$

(3) *r* in terms of *R* and the angles.

$$\text{From (2), } r = a / \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right)$$

$$\text{But, } a = 2 R \sin A \doteq 4 R \sin \frac{A}{2} \cos \frac{A}{2}.$$

$$\cot \frac{B}{2} + \cot \frac{C}{2} = \cos \frac{A}{2} / \sin \frac{B}{2} \cdot \sin \frac{C}{2}.$$

$$\therefore r = 4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

(4) *r* involving one angle only and the sides.

In Fig. 41, ARI is a right-angled triangle, right-angled at R. $\therefore IR/AR = \tan RAI = \tan A/2$.

$$r = IR = AR \tan A/2 = (s - a) \tan A/2.$$

Similarly, taking the triangles IBP and IQC, we get corresponding results $r = (s - b) \tan B/2$;
 $r = (s - c) \tan C/2$.

$$\therefore r = (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}.$$

Art. 80. Corresponding formulae shall now be derived for the radius of each of the ex-circles, touching one side internally and the other two externally. The centre of the escribed circle opposite A is evidently the point of intersection of the internal bisector of angle A and the external

bisectors of angles B and C. We shall denote the point by I_1 and the radius of this circle by r_1 . Similarly I_2 and I_3 are the centres of the ex-circles opposite B and C respectively and r_2, r_3 their radii.

(1) r_1 in terms of sides :—

Let the ex-circle opposite to A touch BC, AC produced and AB produced at P_1, Q_1, R_1 , respectively.

Join I_1 to the vertices and to the points of contact.

S = area of triangle ABC

$$= I_1 AB + I_1 AC - I_1 BC$$

$$\triangle I_1 AB = \frac{1}{2} AB \cdot I_1 R_1$$

$$= \frac{1}{2} c \cdot r_1.$$

$$\triangle I_1 AC = \frac{1}{2} AC \cdot I_1 Q_1 = \frac{1}{2} b r_1.$$

$$\triangle I_1 BC = \frac{1}{2} BC \cdot I_1 P_1 = \frac{1}{2} a \cdot r_1.$$

$$\therefore S = \frac{1}{2} (c + b - a) r_1$$

$$= \frac{1}{2} (2s - 2a) \cdot r_1 = (s - a) r_1.$$

$$r_1 = \frac{S}{s - a} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s - a}$$

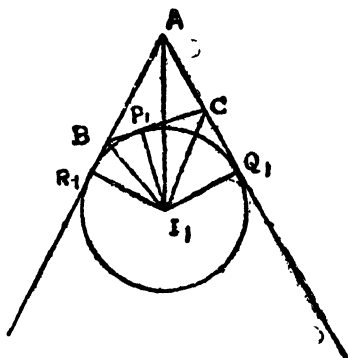


Fig. 42

Similarly, it can be proved that $r_2 = \frac{S}{s - b}$

$$\text{and } r_3 = \frac{S}{s - c}$$

(2) r_1 in terms of one side and angles.

$$a = BC = BP_1 + P_1C = I_1 P_1 [\cot I_1 BP_1 + \cot I_1 CP_1]$$

$$= r_1 \left[\tan \frac{B}{2} + \tan \frac{C}{2} \right]$$

$$\therefore r_1 = a / \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$$

Simplifying, $r_1 = a \cos \frac{B}{2} \cos \frac{C}{2} / \cos \frac{A}{2}$

Similarly, taking triangle $I_1 CA$,

$$r_1 = b \sin \frac{A}{2} \cdot \cos \frac{C}{2} / \sin \frac{B}{2}$$

and from triangle $I_1 AB$, $r_1 = c \sin \frac{A}{2} \cdot \cos \frac{B}{2} / \sin \frac{C}{2}$.

Proceeding in the same way, it can be shown that

$$r_2 = \frac{b \cos \frac{C}{2} \cdot \cos \frac{A}{2}}{\cos \frac{B}{2}} \text{ and } r_3 = \frac{c \cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{C}{2}} \text{ etc.}$$

(3) r_1 in terms of R and the angles.

$$r_1 = a \cos \frac{B}{2} \cos \frac{C}{2} / \cos \frac{A}{2}$$

Replacing 'a' by its value $4 R \sin A/2 \cos A/2$,

$$r_1 = 4 R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}.$$

$$\text{Similarly, } r_2 = 4 R \sin \frac{B}{2} \cdot \cos \frac{C}{2} \cdot \cos \frac{A}{2} \quad (1)$$

$$r_3 = 4 R \sin \frac{C}{2} \cdot \cos \frac{A}{2} \cdot \cos \frac{B}{2} \quad (1)$$

Note.—These are very important results, and the student must commit them to memory. Very many questions are easily solved by application of these formulae.

(4) r_1 in terms of one angle alone and the sides.

From triangle $A I_1 R_1$ where $A R_1 = s$, $\angle R_1 A I_1 = \frac{A}{2}$,

$I_1 R_1 = A R_1 \tan R_1 A I_1$ and we have $r_1 = s \tan \frac{A}{2}$.

$$\text{Similarly, } r_2 = s \tan \frac{B}{2}, \quad r_3 = s \tan \frac{C}{2}$$

Taking triangle $B P_1 I_1$ or $C P_1 I_1$, we get the corresponding results,

$$r_1 = (s - c) \cot \frac{B}{2} = (s - b) \cot \frac{C}{2}.$$

$$\text{Similarly, } r_2 = (s - a) \cot \frac{C}{2} = (s - c) \cot \frac{A}{2}$$

$$r_3 = (s - b) \cot \frac{A}{2} = (s - a) \cot \frac{B}{2}.$$

Art. 81. We shall in this article evaluate the lengths of some important lines in the triangle $I_1 I_2 I_3$.

Since AI_1 is the internal bisector of angle A , and $I_3 AI_2$ the external bisector of the same angle, it follows that $I_1 A$ is perpendicular to $I_2 I_3$.

Similarly, $I_2 B$ is perpendicular to $I_1 I_3$ both meeting at I .

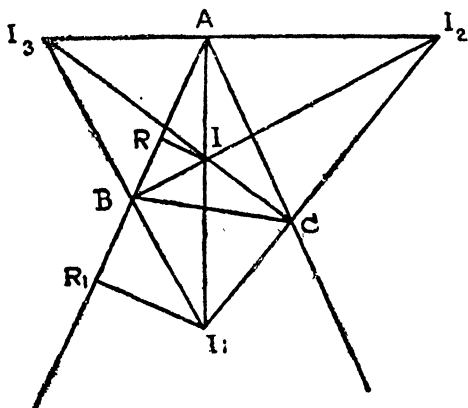


FIG. 43

Hence I is the orthocentre of triangle $I_1 I_2 I_3$ and ABC is its pedal triangle.

(1) *Lengths of AI , AI_1 etc.* :—Draw IR , $I_1 R_1$ perpendicular to AB . Then, R and R_1 are the points where the incircle and excircle of ABC opposite to A touch AB .

$$AR = (s - a) \text{ and } AR_1 = s.$$

$$\frac{AI}{AR} = \frac{1}{\cos A/2} = \frac{AI_1}{AR_1}.$$

$$\therefore AI = (s - a) \sec A/2 \text{ and}$$

$$AI_1 = s \cdot \sec A/2$$

$$\therefore II_1 = a \sec A/2 = 4 R \sin A/2$$

$$\text{Also } AI = r/\sin A/2 = 4 R \sin B/2 \sin C/2$$

$$\text{and } AI_1 = r_1/\sin A/2 = 4 R \cos B/2 \cos C/2$$

Similarly $BI_2 = s \sec B/2$ etc.

(2) *Sides and angles of triangle $I_1I_2I_3$:—*

$$\angle BI_1C = 180^\circ - \angle I_1BC - \angle I_1CB = 90^\circ - A/2.$$

Hence, the angles of the triangle $I_1I_2I_3$ are $90^\circ - A/2$, $90^\circ - B/2$ and $90^\circ - C/2$.

(2) Since ABC is the pedal triangle of $I_1I_2I_3$, the circum-radius of the latter is twice that of the former. Hence the radius R_1 of the circumcircle of $I_1I_2I_3 = 2 R$.

$$\begin{aligned} \text{From triangle } I_1I_2I_3, \quad I_2I_3 &= 2 R_1 \sin \angle I_2I_1I_3 \\ &= 4 R \sin (90^\circ - A/2) \\ &= 4 R \cos A/2 = a \operatorname{cosec} A/2 \end{aligned}$$

$$\text{Similarly, } I_3I_1 = 4 R \cos B/2 = b \operatorname{cosec} B/2$$

$$I_1I_2 = 4 R \cos C/2 = c \operatorname{cosec} C/2$$

Note.—The nine-points circle of a triangle passes through the mid-point of the line joining its orthocentre to each vertex. Hence the circumcircle of ABC bisects II_1 , II_2 , II_3 and the sides of the ex-central triangle, so that if the circumcircle of ABC meets II_1 at K , then

$$IK = I_1K = \frac{1}{2} II_1 = \frac{1}{2} \times 4 R \sin \frac{A}{2} = 2 R \sin \frac{A}{2}.$$

Ex. 1. In a triangle ABC, with the usual notation, show that $a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = abc$.

Since the right hand side is a function of the sides, express AI, BI, CI in terms of the sides.

$$\begin{aligned} \bullet \quad AI &= (s - a) \cdot \sec \frac{A}{2} = (s - a) \sqrt{\frac{bc}{s(s - a)}} \\ a \cdot AI^2 &= abc (s - a) / s. \end{aligned}$$

Similarly, $b \cdot BI^2 = abc (s - b) / s$; $c \cdot CI^2 = abc (s - c) / s$.

Hence, $a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2$
 $= abc (s - a + s - b + s - c) / s = abc.$

Ex. 2. Show that

$$(i) \triangle I_1 I_2 I_3 = 2 R s.$$

$$(ii) \triangle II_2 I_3 = 2 R (s - a)$$

$$\begin{aligned} (i) \text{ From Fig. 43, } \triangle I_1 I_2 I_3 &= I_2 I_3 \times AI_1 / 2 \\ &= \frac{1}{2} \times 4 R \cos \frac{A}{2} \times \frac{s}{\cos A/2} \\ &= 2 R \cdot s. \end{aligned}$$

$$\begin{aligned} (ii) \triangle II_2 I_3 &= (I_2 I_3 \times AI) \div 2 \\ &= \frac{1}{2} \times 4 R \cos \frac{A}{2} \times \frac{s - a}{\cos A/2} \\ &= 2 R \cdot (s - a) \end{aligned}$$

Ex. 3. In a triangle ABC, show that

$$\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$$

Transposing $-\left(\frac{1}{2R}\right)$ to L. H. S.,

$$\begin{aligned}
& \frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} + \frac{1}{2R} \\
&= \frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} + \frac{2 \triangle ABC}{abc} \\
&= \frac{ar_1 + br_2 + cr_3 + 2 \triangle ABC}{abc} \\
&= \frac{2 \triangle I_1 I_2 I_3}{abc} = \frac{4R \cdot s}{abc} \text{ Ex. 2 (i)} = \frac{s}{\Delta} = \frac{1}{r}
\end{aligned}$$

$$\begin{aligned}
\text{Aliter. } \frac{r_1}{bc} &= \frac{4R \sin A/2 \cdot \cos B/2 \cdot \cos C/2}{4R^2 \sin B \sin C} \\
&= \frac{\sin A/2}{4R \sin B/2 \cdot \sin C/2} \\
&= \frac{\sin^2 A/2}{4R \sin A/2 \sin B/2 \sin C/2} \\
\therefore \sum \frac{r_1}{bc} &= \frac{\sin^2 A/2 + \sin^2 B/2 + \sin^2 C/2}{4R \sin A/2 \sin B/2 \sin C/2} \\
&= \frac{1 - 2 \sin A/2 \sin B/2 \sin C/2}{4R \sin A/2 \sin B/2 \sin C/2} \\
&= \frac{1}{r} - \frac{1}{2R}.
\end{aligned}$$

Ex. 4. Show that the area of the ex-central triangle is $8R^2 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$.

[Use— $\triangle ABC = 2R^2 \sin A \sin B \sin C$.]

Art. 82. Distance between the circumcentre and the incentre —

S is the circumcentre and I is the incentre of triangle ABC. AI produced meets the circumcircle at K.

It is required to find the length SI .

Join SI and produce it both ways to meet the circumcircle at L, M .

By the rectangle theorem, $AI \cdot IK = LI \cdot IM$
 $= (LS + SI) (SM - SI)$
 $= (R + SI) (R - SI)$
 $= R^2 - SI^2.$

$$SI^2 = R^2 - AI \cdot IK.$$

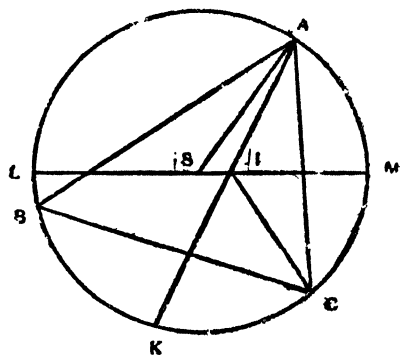


Fig. 44

$$AI = r \operatorname{cosec} \frac{A}{2} \text{ and } IK = 2R \sin \frac{A}{2}; AI \cdot IK = 2Rr.$$

$$SI^2 = R^2 - 2Rr.$$

Art. 83. Distance between circumcentre and an excentre —

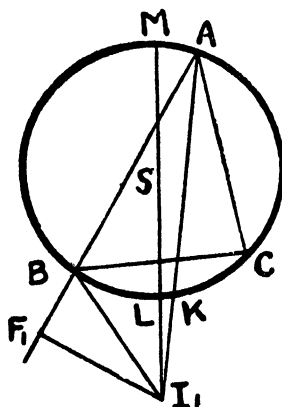


Fig. 45

I_1 is the ex-centre of triangle ABC opposite to A , and AI_1 meets the circumcircle at K .

Let the diameter of the circumcircle through I_1 meet the circle at L, M .

By the rectangle theorem, $I_1 K \cdot I_1 A = I_1 L \cdot I_1 M = I_1 S^2 - R^2$

$$SI_1^2 = R^2 + I_1 K \cdot I_1 A.$$

But, from Art. 81,

$$\begin{aligned} I_1 A &= r_1 \operatorname{cosec} \frac{A}{2}; I_1 K = \frac{1}{2} II_1 \\ &= 2R \sin \frac{A}{2} \end{aligned}$$

$$\therefore SI_1^2 = R^2 + 2Rr_1.$$

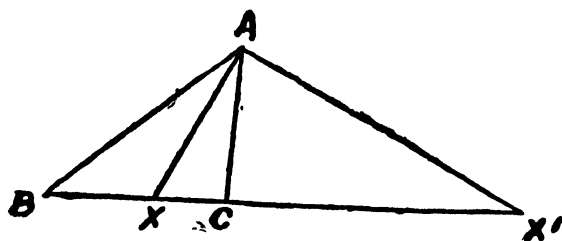
Art. 84. Lengths of the bisectors :

Fig. 46

AX and AX' bisect internally and externally the angle BAC of the triangle ABC .

It is required to find the lengths of AX and AX' .

$$\triangle ABC = \triangle BAX + \triangle XAC.$$

$$\frac{1}{2} bc \sin A = \frac{1}{2} AX \cdot c \sin \frac{A}{2} + \frac{1}{2} AX \cdot b \sin \frac{A}{2}$$

$$\text{i.e., } bc \sin A = AX \cdot \sin \frac{A}{2} (b+c)$$

$$2bc \sin \frac{A}{2} \cdot \cos \frac{A}{2} = AX \cdot \sin \frac{A}{2} (b+c)$$

$$AX = \frac{2bc \cos \frac{A}{2}}{b+c}.$$

$$\text{Also, } AX = \frac{2bc}{b+c} \sqrt{s(s-a)} = \frac{2}{b+c} \sqrt{s(s-a)bc}$$

To find AX' , we have

$$\triangle BAX' - \triangle CAX' = \triangle BAC$$

$$\begin{aligned} \text{i.e., } \frac{1}{2} AB \cdot AX' \cdot \sin \angle BAX' - \frac{1}{2} AC \cdot AX' \cdot \sin \angle CAX' \\ = \frac{1}{2} bc \sin A. \end{aligned}$$

$$\text{But, } \angle BAX' = 90^\circ + \frac{A}{2} \text{ and } \angle CAX' = 90^\circ - \frac{A}{2}$$

Hence, $\frac{1}{2} A X' \cdot \cos \frac{A}{2} \times (AB - AC) = \frac{1}{2} b c \sin A$

$$A X' = 2 b c \sin \frac{A}{2} / (c-b)$$

In terms of the sides only,

$$\begin{aligned} A X' &= \frac{2 b c}{c-b} \sqrt{\frac{(s-b)(s-c)}{b c}} \\ &= \frac{2}{(c-b)} \sqrt{b c (s-b)(s-c)} \end{aligned}$$

EXAMPLES IX b.

(1) The sides of a triangle are 23, 34, 31. Calculate the lengths of the in and ex-radii.

(2) A B C is a triangle in which $a = 12$, $b = 5$, $c = 13$. Find its circum-radius, and the area of the triangle.

(3) Given that in a triangle A B C, $a = 27$, $b = 38$, $c = 33$, find the lengths of the radii of the circle inscribed in the triangle, and of the circumcircle.

(4) The sides of a triangle are 59, 65, 48; calculate the lengths of the ex-radii. Hence calculate the values of $\tan \frac{A}{2}$, $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$.

Prove the following results :—

$$(5) \quad r_1 + r_2 + r_3 - r = 4 R.$$

$$(6) \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

$$(7) \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$$

$$(8) \quad r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 - a^2 - b^2 - c^2$$

$$(9) \quad r_1 r_2 r_3 : r^3 = 1 : \tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2}$$

$$(10) \quad (r - r_1)(r - r_2)(r - r_3) + 4 R r^2 = 0.$$

$$(11) \quad (r_2 - r_3) \cos A + (r_3 - r_1) \cos B + (r_1 - r_2) \cos C = 0.$$

$$(12) \quad R : r_1 = (\sin B + \sin C - \sin A) : 2 \sin A \sin B \sin C.$$

$$(13) \quad S = \sqrt{r_1 r_2 r_3} = s^2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$$

$$(14) \quad 2R \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right) = 4R + r.$$

$$(15) \quad r_1 \cot \frac{A}{2} = r_2 \cot \frac{B}{2} = r_3 \cot \frac{C}{2}.$$

$$(16) \quad \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}.$$

$$(17) \quad a \cdot AI_1^2 - b \cdot BI_1^2 - c \cdot CI_1^2 = abc.$$

$$(18) \quad (r_2 + r_3) \tan \frac{A}{2} = (r_1 + r) \cot \frac{A}{2} = a$$

$$(19) \quad a \cot A + b \cot B + c \cot C = 2(R + r)$$

$$(20) \quad \sum 2bc - \sum a^2 = 4r(r + 4R)$$

$$(21) \quad AI \cdot BI \cdot CI = \frac{r}{s} abc.$$

(22) Prove that the area of the triangle formed by the ex-centres of triangle ABC is equal to $8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

$$(23) \quad \text{Show that } \triangle ABC : \triangle I_1 I_2 I_3 = 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} : 1$$

(24) If D, E, F be the points of contact of the incircle with the sides, prove that triangle DEF is similar to the ex-central triangle. Also, find the lengths of the sides of triangle DEF, and of its circum radius.

(25) ABC is triangle in which $a = 7, b = 12, c = 8$. Calculate the lengths of the bisectors of $\angle B$ of the triangle.

(26) The bisectors of the angles of a triangle meet the opposite sides in L, M, N. Show that triangle LMN : triangle ABC = $2abc : (a+b)(b+c)(c+a)$.

(27) Find the angles of a triangle given the values of r_1, r_2 and r_3 .

$$\left(\text{Hint} - \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{rr_3}{r_1 r_2}} \right)$$

Prove the following results :—

$$(28) \quad \left(\frac{1}{r} - \frac{1}{r_1} \right) \left(\frac{1}{r} - \frac{1}{r_2} \right) \left(\frac{1}{r} - \frac{1}{r_3} \right) = \frac{16R}{r^2 (a+b+c)^2}$$

$$(29) \left(\frac{1}{r_2} + \frac{1}{r_3} \right) \left(\frac{1}{r_3} + \frac{1}{r_1} \right) \left(\frac{1}{r_1} + \frac{1}{r_2} \right) = \frac{64 R^3}{a^2 b^2 c^2}$$

$$(30) \sum a(s-a)(\cos B - \cos C) = 0.$$

(31) $r + r_1 + r_2 - r_3 = 4R \cos C$. Hence if in $\triangle$,
 $r + r_1 + r_2 = r_3$, the $\triangle$ is right-angled.

$$(32) \text{ Show that the area of the quadrilateral } BICI_1 \\ = \frac{1}{2} (b + c) a \tan \frac{A}{2}.$$

(33) Given s , $\triangle$, and A , find the sides of the triangle.

(34) ABC is a triangle. D is any point in BC and ADC measures θ . S' and S'' are the circumcentres of triangles ABD and ADC . Show that the distance $S' S'' = \frac{a}{2} \operatorname{cosec} \theta$.

Art. 85. Properties of the Pedal Triangle —

The three altitudes of a triangle are concurrent at the ortho-centre O . In the adjoining figure, AD , BE , CF are the altitudes. X and Y are the midpoints of BC , CA respectively. Perpendiculars to these sides at the corresponding points meet at S , the circumcentre of the triangle.

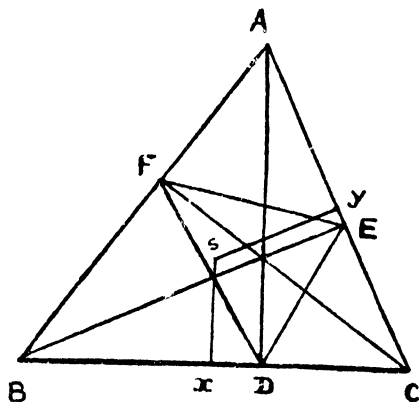


Fig. 47

Join XS , SY , DE , EF , FD .

(1) Triangle DEF is the pedal triangle of ABC ; and we know from Geometry that the circumcircle of DEF is the nine-points circle of ABC . Hence the circumcentre of DEF is the mid-point of SO , and its radius $\frac{R}{2}$.

(2) Again, since $\angle AFC = 90^\circ = \angle ADC$, A, F, D, C are concyclic.

$$\therefore \angle FDB = \angle FAC = A.$$

Similarly, A, E, D, B are concyclic and $\angle EDC = \angle BAE = A$.

Hence DE, DF, the sides of the pedal triangle are equally inclined to the side BC of the triangle ABC, the angle of inclination being A.

$$\text{So, } \angle FDE = 180^\circ - \angle BDF - \angle EDC = 180^\circ - 2A.$$

$$\text{Similarly, } \angle DEF = 180^\circ - 2B, \angle EFD = 180^\circ - 2C.$$

(3) Having found the angles of the pedal triangle and its circumradius, it is easy to calculate the lengths of its sides. The side EF subtending $180^\circ - 2A$ at the circumference of circle DEF, of which it is a chord is of length equal to $2 \times \frac{R}{2} \times \sin(180^\circ - 2A) = R \sin 2A = a \cos A$.

Similarly the other sides can be proved to be of lengths $R \sin 2B$, $R \sin 2C$ or $b \cos B$, $c \cos C$ respectively.

Note. These formulae can otherwise be deduced by comparing the triangles AFE and ACB which are similar since $\angle AFE = \angle ACB$ of the cyclic quadrilateral BFEC.

$$\frac{FE}{a} = \frac{FE}{BC} = \frac{AF}{AC} = \cos \angle FAC = \cos A. \quad FE = a \cos A.$$

(4) Again, the area of triangle DEF is equal to

$$\begin{aligned} & \frac{1}{2} DE \cdot DF \cdot \sin \angle EDF \\ &= \frac{1}{2} \times c \cos C \times b \cos B \cdot \sin(180^\circ - 2A) \\ &= bc \sin A \cos A \cos B \cos C. \end{aligned}$$

Hence triangle DEF: $\triangle ABC = 2 \cos A \cos B \cos C$:

(5) In triangle BSX, $\angle X = 90^\circ$, $BSX = \frac{1}{2} \angle BSC$

But, $\angle BSC$ at centre = 2 $\angle BAC$ at circumference.

$$\therefore \angle BSX = A. \quad SX / SB = \cos A.$$

$$SX = R \cos A ; \text{ Similarly } SY = R \cos B, \\ SZ = R \cos C.$$

$$(6) AO = 2 SX = 2 R \cos A ; OB = 2 R \cos B.$$

$$\text{Also, } OD = OB \cos BOD = 2 R \cos B \cos C.$$

$$\therefore AO \cdot OD = BO \cdot OE = CO \cdot OF ;$$

$$\text{and } AO^2 + BC^2 = 4 R^2 \cos^2 A + 4 R^2 \sin^2 A = 4 R^2.$$

Art. 86. Distance between circum-centre and ortho-centre.

Let the altitude AD of the triangle meet the circum-circle at H. It is required to find the distance SO. Join SO and produce it both ways to meet the circle at L, M.

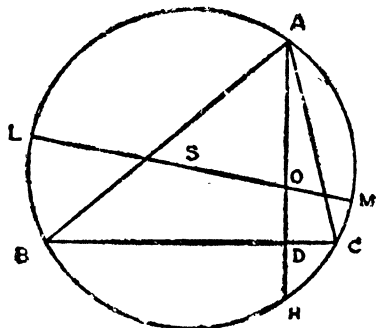


Fig. 48

By the rectangle theorem, $AO \cdot OH = LO \cdot OM$.

$$= (LS + OS) (SM - OS) \\ = R^2 - OS^2$$

$$OS^2 = R^2 - AO \times OH.$$

$$= R^2 - AO \times 2 \cdot OD. (OD = DH)$$

$$\text{i.e., } OS^2 = R^2 - 8 R^2 \cos A \cos B \cos C.$$

Cor. 1. By a previous example, $a^2 + b^2 + c^2 = 8 R^2 + 8 R^2 \cos A \cos B \cos C$.

$$\therefore SO^2 = 9 R^2 - (a^2 + b^2 + c^2).$$

Cor. 2. Since the nine-points centre N is midway between S and O, $SN = \frac{1}{2} SO = R \sqrt{1 - 8 \cos A \cos B \cos C} / 2$.

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Cor. 3. G is also on this line dividing OS internally in the ratio 2 : 1. So, $OG = 2 \times R \sqrt{1 - 8 \cos A \cos B \cos C} / 3$.

$$GS = R \sqrt{1 - 8 \cos A \cos B \cos C} / 3.$$

Art. 87. Centroid and Medians.

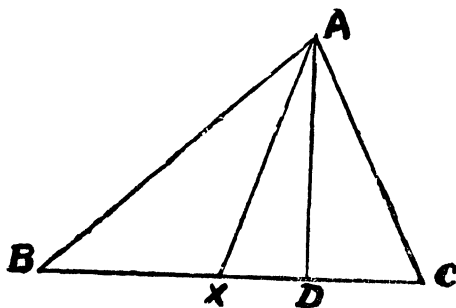


Fig. 49

ABC is triangle, in which X, Y, Z are the mid-points of the sides BC, CA, AB respectively. AX, BY, CZ are the medians of the triangle. They meet at a point G called the centroid of the triangle. G divides each of the medians in the ratio 2 : 1.

$$AG : GX = BG : GY = CG : GZ = 2 : 1.$$

Lengths of medians :—

By the Theorem of Apollonius,

$$AB^2 + AC^2 = 2BX^2 + 2AX^2$$

$$2 AX^2 = AB^2 + AC^2 - 2BX^2$$

$$= b^2 + c^2 - a^2/2$$

$$AX^2 = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4} = (2b^2 + 2c^2 - a^2)/4.$$

$$AX = \sqrt{2b^2 + 2c^2 - a^2}/2.$$

$$AG \text{ being } \frac{2}{3} AX, AG^2 = \frac{4}{9} AX^2 = \frac{2}{9} \left(b^2 + c^2 - \frac{a^2}{2} \right)$$

$$\therefore AG^2 + BG^2 + CG^2 = \frac{2}{9} \sum \left(b^2 + c^2 - \frac{a^2}{2} \right) = \frac{1}{3} (a^2 + b^2 + c^2)$$

EXAMPLES IX c.

(1) O is the ortho-centre of triangle ABC and O' is the circum-centre of BOC. Prove that the circum-radius of BO'C is $\frac{R}{2 \cos A}$ where R is the circum-radius of ABC.

(2) The perpendicular from one angle of a triangle upon the opposite side is a harmonic mean between the radii of the excircles touching the other two sides.

(3) If the r radii of the four circles touching the sides of a triangle be proportionals, the triangle is right-angled.

(4) The inradius of the pedal triangle of ABC is $2 R \cos A \cos B \cos C$ and the exradii are $2 R \cos A \sin B \sin C$, $2 R \cos B \sin C \sin A$, $2 R \cos C \sin A \sin B$.

[Hint— $r = 4 R \sin A/2 \sin B/2 \sin C/2$, etc.]

(5) I is the centre and D, E, F are the points of contact of the incircle of ABC with the sides BC, CA, AB respectively. R_1, R_2, R_3 are the radii of the circles inscribed in the quadrilaterals AFIE, BFID, CDIE respectively. Prove that $r = R_1 \left(1 + \tan \frac{A}{2}\right)$.

(6) S is the area of the triangle formed by joining the points of contact of the incircle with the sides and S_1, S_2, S_3 the corresponding areas for the escribed circles.

Prove that $\frac{S}{r} = \frac{S_1}{r_1} = \frac{S_2}{r_2} = \frac{S_3}{r_3}$.

(7) If, r, r_1, r_2, r_3 be the radii of the nine-points-circles of triangles ABC, OBC, OCA, OAB respectively, O being the circum-centre of ABC, prove that

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = \frac{2S}{r^2} \text{ where } S \text{ is the area of } ABC.$$

(8) p_1, p_2 , and p_3 are the lengths of the three altitudes of a triangle. Prove that $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$.

(9) The area of the triangle, whose sides are equal in length to the medians of triangle ABC is three-fourths of ABC.

(10) Prove that if $I I_1$ subtends a right angle at the circum-centre S, the angle A is 60° .

(11) S is the circum-centre of ABC ; if R_1, R_2, R_3 be the radii of the circles about BSC, CSA, ASB , prove that $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$.

(12) x, y, z are the lengths of the three bisectors of the angles of a triangle. Prove that $\sum (b+c)^2 \frac{x^2}{bc} = (a+b+c)^2$.

(13) Prove that in a triangle, $abc = 4R \Delta$.

If x, x_1, x_2, x_3 be the radii of the circumcircles of the triangles ABC, OBC, OCA, OAB respectively, O being the circum-centre of ABC , show that $\frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3} = \frac{4\Delta}{x^2}$.

(14) Prove that $r \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right) = s$.

(15) In a triangle, prove that $2Rr = (abc)/(a+b+c)$.

CHAPTER X

SOLUTION OF TRIANGLES

Art. 88. Every triangle has six elements, the three sides and the three angles, denoted in the usual notation by a, b, c, A, B, C . These are not all independent. Any three of these, excepting the three angles, together form a set of independent quantities; for, given two angles, the third can be written down, since the three angles are together equal to two right angles. When three elements of a triangle, of which at least one is a side are given, we can in general determine the other three elements. The determination is called the *solution of the triangle*.

Art. 89. At this stage, the student must be introduced to tables of Logarithmic sine, Logarithmic cosine and Logarithmic tangent, in each of which, the angle expressed in degrees and minutes is the argument. The figure corresponding to each argument is the logarithm of the sine (or cosine or tangent) of the angle, *increased by ten*. This avoids negative values of the integral part of the quantity sought. This figure as given in Tables is written with a capital L to distinguish it from the actual value (before it is increased by 10). The advantage of this reference is that without it we have to refer to two Tables, first to the sine (or cosine or tangent of the angle) and then to the Table of logarithms for the logarithm of this ratio. Thus for example $\sin 30^\circ = 0.5000$ and $\log \sin 30^\circ = \log 0.5000 = \bar{1}.6990$. Referring to the Table of logarithmic sines, $\text{Log } \sin 30^\circ = 9.6990$, so that $\text{L. Sin } 30^\circ = \log \sin 30^\circ + 10$. Similarly for Logarithmic cosine and Logarithmic tangent of angles.

Art. 90. Solution of Right-angled Triangles.—If ABC is a triangle right-angled at B , clearly, one of the elements is known; viz., the magnitude of B is given to be 90° . Hence, for the solution of such a triangle, it is sufficient if we are given any two of the remaining elements except the two angles. Hence the following cases arise :

- (i) the lengths of the two sides a, c .
- (ii) the lengths of the hypotenuse and one side, i.e., b, a or b, c .
- (iii) the length of the hypotenuse and one angle, i.e., b, A or b, C .
- (iv) one of the sides and one of the angles, i.e., a, C .

Art. 91. (i) Given the values of a, c , of a right-angled triangle ABC , right-angled at B , solve the triangle.—The side b is got from the formula:— $b^2 = c^2 + a^2$ (1)

$$C \text{ is got from } \tan C = \frac{c}{a} \text{ ————— (2)}$$

$$\text{and } A = 90^\circ - C \text{ ————— (3)}$$

(ii) **Given b, a , solve the triangle.**—

The third side c is got from the formula $c^2 = b^2 - a^2$ —(1)

$$\cos C = \frac{a}{b} \text{ ————— (2)}$$

$$A = 90^\circ - C \text{ ————— (3)}$$

(iii) **Given b, A , solve the triangle.**

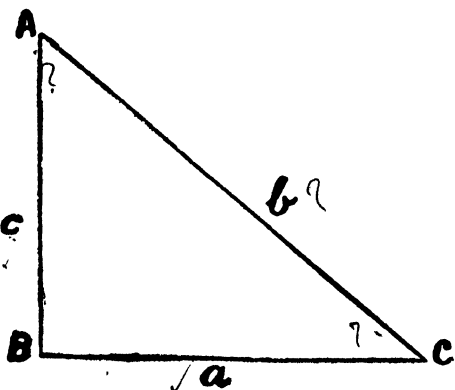


Fig. 50

$$C = 90^\circ - A \text{-----} (1)$$

$$c = b \cos A \text{-----} (2)$$

$$a = b \sin A \text{-----} (3)$$

(iv) **Given a, C solve the triangle,**

$$A = 90^\circ - C \text{-----} (1)$$

$$c = a \tan C \text{-----} (2)$$

$$b = a \sec C \text{-----} (3)$$

Ex. 1. Solve the right-angled triangle ABC in which $B = 90^\circ$, $c = 345.6$, $a = 456.7$.

$$\tan C = \frac{345.6}{456.7}; \text{Log } \tan C - 10 = \log 345.6 - \log 456.7$$

$$L \tan C = 9.8789 = L \tan 37^\circ 7'$$

$$C = 37^\circ 7', A = 52^\circ 53'$$

To find the hypotenuse, the formula $b \equiv \sqrt{c^2 + a^2}$ is not adapted for logarithmic calculation, since the values of 'c' and 'a' are given in big numbers. Hence, find it from the angle.

$$\frac{b}{a} = \frac{1}{\sin A}, \text{ giving } b = \frac{a}{\sin A}.$$

$$\begin{aligned} \log b &= \log a - \text{Log } \sin A + 10 \\ &= 12.6597 - 9.9017 = 2.7580. \end{aligned}$$

$$b = \text{Antilog } 2.7580 = 572.8.$$

$$\therefore C = 37^\circ 7'; A = 52^\circ 53'; b = 572.8.$$

Ex. 2. Solve the right-angled triangle ABC given $B = 90^\circ$, $b = 678.9$, $a = 345.7$.

$$\sin A = 345.7 / 678.9.$$

$$\begin{aligned} L \sin A &= 10 + \log 345.7 - \log 678.9 = 9.7069 \\ &= L \sin 30^\circ 36'. \end{aligned}$$

$$A = 30^\circ 36'; C = 59^\circ 24'$$

$$c = b \sin C = 678.9 \times \sin 59^\circ 24'$$

$$\log c = 2.8318 + 9.9349 - 10 = 2.7667$$

$$c = \text{Antilog } 2.7667 = 584.3.$$

Ex. 3. Solve a triangle ABC given $B = 90^\circ$,
 $b = 789.4$, $A = 62^\circ 34'$.

$$a = b \sin A = 789.4 \times \sin 62^\circ 34'.$$

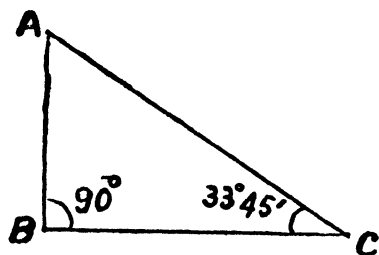
$$\log a = 2.8973 + 9.9482 - 10 = 2.8455$$

$$a = \text{Antilog } 2.8455 = 700.6 ; C = \text{comp } A = 27^\circ 26'$$

$$c = b \sin C = 759.4 \times \sin 27^\circ 26' ; \log c = 2.5607$$

$$c = \text{Antilog } 2.5607 = 363.7.$$

Ex. 4. Solve a triangle ABC right-angled at B in which $a = 235.7$, $C = 33^\circ 45'$.



$$A = 90^\circ - C = 56^\circ 15'.$$

$$c = a \tan C.$$

$$= 235.7 \times \tan 33^\circ 45'.$$

$$\log c = 2.3724 + 9.8249 - 10$$

$$= 2.1973$$

$$c = \text{Antilog } 2.1973 = 157.5.$$

Fig. 51

$$b = 157.5 / \sin 33^\circ 45'.$$

$$\log b = 2.1973 - 9.7448 + 10 = 2.4525$$

$$b = 283.4$$

EXAMPLES X a.

Solve the following right-angled triangles in which $B = 90^\circ$.

(1) $a = 98.76$, $c = 54.32$.

(5) $a = 123.4$, $c = 594.7$.

(2) $a = 87.63$, $c = 76.53$.

(6) $a = 234.5$, $b = 987.6$.

$a = 30.94$, $c = 92.07$.

(7) $a = 135.7$, $b = 246.8$.

(4) $a = 94.57$, $c = 19.86$.

(8) $c = 567.8$, $b = 579.4$.

- (9) $c = 200\cdot6$, $b = 345\cdot6$. (14) $a = 98\cdot76$, $A = 60^\circ 13'$
 (10) $b = 680\cdot2$, $A = 43^\circ 29'$ (15) $a = 86\cdot42$, $C = 69^\circ 12'$
 (11) $b = 680\cdot2$, $A = 75^\circ 38'$ (16) $c = 123\cdot4$, $A = 37^\circ 15'$
 (12) $b = 753\cdot1$, $C = 63^\circ 44'$ (17) $c = 567\cdot8$, $C = 24^\circ 13'$
 (13) $b = 975\cdot2$, $C = 21^\circ 57'$

(18) Two ships start from the same place, one to the east and the other to the north at the rate of 12·5 and 10·25 miles per hour. Find the distance between the ships after 5 hours.

(19) The shadow of a vertical post 12 ft. high is 7·3 feet. Find the altitude of the sun.

(20) When the sun is at an elevation of $23^\circ 42'$ a certain pole planted vertically on the ground casts a shadow 23·56 cm. long. Calculate the height of the pole.

Art. 92. We shall now consider the solution of any oblique-angled triangle. From geometry, it is clear that it is always possible to construct one and only one triangle in each of the following cases: (i) Given the lengths of the three sides, provided any two of them are together greater than the third; (ii) given the lengths of two sides and the magnitude of the angle **included** by them; (iii) given one side and any two angles of the triangle. Also in case the lengths of two sides and the value of the angle **opposite** to one of them are given, we get two triangles, or only one or no triangle at all. Four cases naturally arise, and we have to calculate the remaining three elements in terms of the given ones.

Art. 93. Case 1.—Given the lengths of the three sides.—The reader will remember the formula for the cosine or sine of an angle in terms of the sides.

$$\cos A = (b^2 + c^2 - a^2)/2bc \quad \dots \quad \dots \quad (1)$$

$$\sin A = 2\sqrt{s(s-a)(s-b)(s-c)}/bc \quad \dots \quad \dots \quad (2)$$

The corresponding symmetrical formula for $\cos B$ or $\sin B$ fixes B . C is the supplement of $A + B$. (It can be used

only when a, b, c are simple numbers but in other cases, the calculation becomes very tedious, as the formula is not adapted for logarithmic calculation. Formula (2) admits of logarithmic simplification, but the value of $\sin A$ does not determine A **uniquely**, except in the case when A is one of the smaller angles, since $\sin A$ is also equal to $\sin (180^\circ - A)$. This ambiguity can be got over in case ' a ' is the greatest side, by finding $|B|$ and $|C|$ (taking them acute) and then A is the supplement of $(B + C)$. This process is evidently long and pointed out only as a theoretical possibility. Another disadvantage in using formula (2) is that, for finding one angle B , say, we have to take logarithms of six quantities :— $s, (s - a), (s - b), (s - c), b, c$; and for completely solving the triangle, one more logarithm, *viz.*, that of ' a '. This formula is not used in practice for the purpose of the solution.

So, we resort to the formulae for the ratios of half the angles in terms of the sides. In case **only one** angle is to be found, any one of the three formulae

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

will be equally useful as each involves the finding of the logarithms of four quantities. But, in case **both** the angles are to be found, we use the tangent formula since in both cases we require to refer to the logarithms of only the same four quantities.

The formulae for the required solution are

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \dots \quad \dots \quad (1)$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \quad \dots \quad (2)$$

$$A + B + C = 180^\circ \quad \dots \quad (3)$$

Note.—By properties of similar triangles, it is evident that the magnitudes of the angles are the same so long as the sides are in the same ratio. Hence, to determine the angles, it is sufficient if we know the ratios of the sides.

Ex. Solve a triangle ABC given $a = 12.34$, $b = 45.67$, $c = 36.47$

$$2s = a + b + c = 94.48$$

$$s = 47.24 ; \log s = 1.6743$$

$$s - a = 34.90 ; \log (s - a) = 1.5428$$

$$s - b = 1.57 ; \log (s - b) = 0.1959$$

$$s - c = 10.77 ; \log (s - c) = 1.0323$$

Substituting these values in the formula for $\tan \frac{A}{2}$ we have,

$$\log \tan \frac{A}{2} = \frac{1}{2} [\log (s - b) + \log (s - c) - \log s - \log (s - a)]$$

$$L \tan \frac{A}{2} = \frac{1}{2} [20 + \log (s - b) + \log (s - c) - \log s - \log (s - a)]$$

$$= [20 + 0.1959 + 1.0323 - 1.6743 - 1.5428]/2$$

$$= 0.0056 = L \tan 5^\circ 47'$$

$$A/2 = 5^\circ 47' ; A = 11^\circ 34'.$$

$$L \tan \frac{B}{2} = \frac{1}{2} \times [20 + \log (s - c) + \log (s - a) - \log s - \log (s - b)]$$

$$= [20 + 1.0323 + 1.5428 - 1.6743 - 0.1959]/2$$

$$= 10.3525 = L \tan 66^\circ 3'$$

$$B/2 = 66^\circ 3', \text{ i.e., } B = 132^\circ 6'$$

$$C = 180^\circ - (11^\circ 34' + 132^\circ 6') = 36^\circ 20'.$$

Art. 94. Case 2.—Given two sides and the included angle.—Let b, c and A be given. It is required to calculate the third side ' a ' and the remaining two angles B and C .

We have

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \dots \quad \dots \quad (1)$$

$$a \sin B = b \sin A \quad \dots \quad \dots \quad (2)$$

$$\text{and } C = 180 - A - B \quad \dots \quad \dots \quad (3)$$

Formula (1) admits of easy simplification to find ' a ' only when b and c are simple numbers; otherwise the actual calculation becomes very laborious, since the right hand side of (1) is not adapted for logarithmic computation.

Hence, in such cases, the angles are calculated first by the use of the formula for $\tan \frac{B-C}{2}$ and thence the side.

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cdot \cot \frac{A}{2} \quad \dots \quad \dots \quad (1)$$

$$(B+C)/2 = 90^\circ - (A/2) \quad \dots \quad \dots \quad (2)$$

$$a \sin B = b \sin A \quad \dots \quad \dots \quad (3)$$

$$\text{Log } \tan \frac{B-C}{2} = \log(b-c) - \log(b+c) + \text{Log } \tan \left(90^\circ - \frac{A}{2}\right).$$

$(B-C)/2$ is read off from the Table of Logarithmic Tangents.

$$B = (B+C)/2 + (B-C)/2.$$

$$C = (B+C)/2 - (B-C)/2.$$

Formula (3) then gives the value of ' a ' on substituting this value of B .

Note 1.—If $b < c$, $(b-c)$ and $(B-C)$ are negative quantities and so the formula is modified as

$$\text{Log } \tan \frac{C-B}{2} = \log(c-b) - \log(c+b) + \text{Log } \tan \left(90^\circ - \frac{A}{2}\right).$$

Note 2.—In case the third side has to be found without calculating the other two angles, use the cosine rule adapted for logarithmic computation. (Vide Art. 97.)

Ex. Solve a triangle ABC given $b = 98.76$, $c = 75.31$, $A = 39^\circ 42'$.

$$\text{As } b > c, B > C; \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

$$\begin{aligned} \tan \frac{B - C}{2} &= \frac{23.45}{174.07} \cot 19^\circ 51' \\ &= \frac{2345}{17407} \tan 70^\circ 9'. \end{aligned}$$

$$\begin{aligned} \text{Log tan } \frac{B - C}{2} &= \log 2345 + L \tan 70^\circ 9' - \log 17407. \\ &= 3.3701 + 10.4425 - 4.2407 = 9.5719 \end{aligned}$$

$$\frac{B - C}{2} = 20^\circ 28'; \quad \frac{B + C}{2} = 90^\circ - \frac{A}{2} = 70^\circ 9'.$$

Adding and subtracting, $B = 90^\circ 37'$, $C = 49^\circ 41'$.

$$a = c \sin A / \sin C = 75.31 \times \sin 39^\circ 42' / \sin 49^\circ 41'.$$

$$\begin{aligned} \log a &= \log 75.31 + \text{Log sin } 39^\circ 42' - L \sin 49^\circ 41' \\ &= 1.8769 + 9.8053 - 9.8822 = 1.8000 \end{aligned}$$

$$a = \text{Antilog } 1.8000 = 63.10.$$

Hence, $B = 90^\circ 37'$, $C = 49^\circ 41'$, $a = 63.10$.

Art. 95. Case 3.—Given a side and any two angles.—This is the easiest. Two angles being given, the third can be calculated; one side and the angle opposite to it being known, the sides opposite to the other angles can be determined. So, we have, if we are given a , B , A ,

$$C = 180^\circ - B - A \quad \dots \quad \dots \quad (1)$$

$$c = a \sin C / \sin A \quad \dots \quad \dots \quad (2)$$

$$b = a \sin B / \sin A \quad \dots \quad \dots \quad (3)$$

Ex. Solve the triangle ABC given $a = 57.91$, $B = 42^\circ 34'$, $C = 54^\circ 28'$

$$B + C = 97^\circ 2' \quad \therefore \quad A = 82^\circ 58'$$

$$c = a \sin C / \sin A = 57.91 \sin 54^\circ 28' / \sin 82^\circ 58'$$

$$\log c = \log 57.91 + L \sin 54^\circ 28' - L \sin 82^\circ 58'.$$

$$= 1.7628 + 9.9105 - 9.9968 = 1.6765$$

$$c = \text{Antilog } 1.6765 = 47.47.$$

$$\text{Again, } b = a \sin B / \sin A = 57.91 \sin 42^\circ 34' / \sin 82^\circ 58'$$

$$\log b = 1.7628 + 9.8303 - 9.9968 = 1.5963$$

$$b = \text{Antilog } 1.5963 = 39.48.$$

Hence, $A = 82^\circ 58'$, $c = 47.47$ and $b = 39.48$.

Art. 96. Case 4.—Given any two sides and the angle opposite to one of them.—Suppose we are given b , c , C .

$$\text{Then, } \sin B = b \sin C / c \quad \dots \quad \dots \quad \dots \quad (1)$$

$$A = 180^\circ - B - C \quad \dots \quad \dots \quad \dots \quad (2)$$

$$a = c \sin A / \sin C \quad \dots \quad \dots \quad \dots \quad (3)$$

Formula (1) determines B , *but not uniquely*, since B can be one or other of two angles supplement to each other. When both the values of B are admissible consistent with the given data, there are two triangles satisfying the given data, and this constitutes the *ambiguous case*. We shall now investigate under what circumstances we get this ambiguous solution.

Trigonometrical discussion.—Taking formula.—

$$\sin B = b \sin C / c,$$

(i) When $c < b \sin C$, $\sin B > 1$ and there is no real value of B . There is no triangle possible in this case.

(ii) When $c = b \sin C$, $\sin B = 1$, B has only one value, viz., 90° . Hence, there is one and *only one* real triangle and it is right-angled at B .

(iii) When $b > c > b \sin C$, $\sin B$ is a positive proper fraction, and $B > C$. Hence *both* the acute and obtuse values of B are admissible *provided C is acute*; for if C is obtuse or right, B must necessarily be obtuse; but this is impossible in a triangle. In this case when C is obtuse, there is *no triangle* satisfying the given data.

(iv) When $c = b$, $B = C$. Hence both have to be acute. So, no solution is possible if C be obtuse and only one solution is possible otherwise.

(v) When $c > b$, $C > B$ and B can take only one value. There is only one triangle.

Hence, we conclude that two triangles are possible, or in other words the ambiguous case arises only when $b > c > b \sin C$ and C is acute, i.e., when two sides and the angle opposite to the smaller of them is given.

Geometrical discussion.—The same discussion can be done geometrically. Proceeding as we do to construct the triangle from the given data, the side ' b ' is first taken and an angle equal to C made at one extremity C . With the other extremity A as centre and radius equal to ' c ' a circle is drawn to meet the other arm CX of angle C in B . If AB is joined, ABC is the required triangle. Five cases arise in this construction.

(i) If $c < b \sin C$, the circle centre A does not meet CX in a real point B . There is no real triangle ABC possible (Fig. 52).

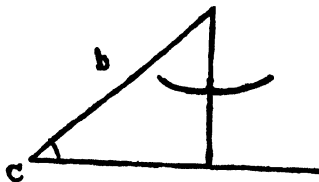


Fig. 52

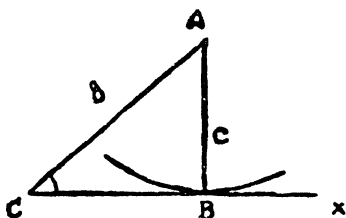


Fig. 53

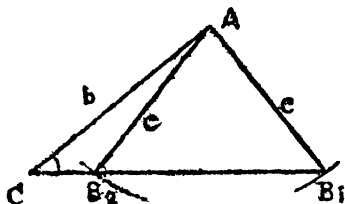


Fig. 54

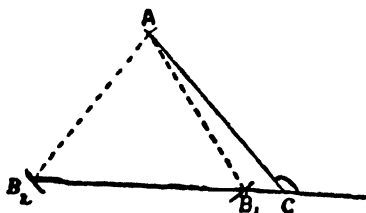


Fig. 54-a

there is only one triangle ACB_1 provided C is acute; there is no solution however if C is obtuse or right (Fig. 55).

(ii) If $c = b \sin C$, CX is touched by the circle at B . So, we get the right-angled triangle ABC , right-angled at B (Fig. 53).

(iii) If c lies intermediate in magnitude between b and $b \sin C$, the circle centre A meets CX in two real and distinct points B_1 and B_2 provided C is acute (Fig. 54). When C is obtuse, both B_1 and B_2 will lie in CX produced, giving no solution (Fig. 54-a). Hence when $C < 90^\circ$ and $b > c$ and $c > b \sin C$, there are two different triangles ACB_1 and ACB_2 (Fig. 54).

(iv) If $c = b$, B_2 coincides with C and

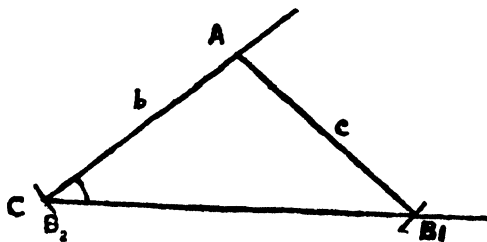


Fig. 55

(v) If $c > b$, only B_1 lies in CX, B_2 lying in XC produced. There is therefore only one triangle ACB_1 in this case. (Figs. 56 and 56-a.)

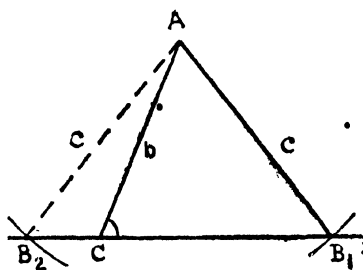


Fig. 56

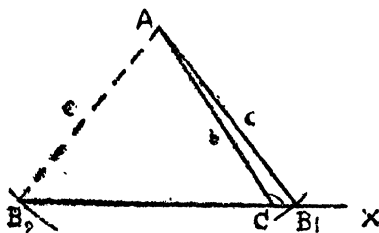


Fig. 56-a.

Algebraical discussion.—It will be useful to see how the same results follow from algebraical considerations. Given ' b ,' ' c ,' C , the third side ' a ' is connected with them by the formula $c^2 = a^2 + b^2 - 2ab \cos C$.

$$\text{i.e., } a^2 - 2ab \cos C + (b^2 - c^2) = 0$$

Hence, ' a ' is a root of the quadratic

$$x^2 - 2bx \cos C + (b^2 - c^2) = 0.$$

This equation has got two roots a_1 and a_2 which are connected by the relations,

$$a_1 + a_2 = 2b \cos C; a_1 a_2 = b^2 - c^2.$$

(i) When $c < b \sin C$, the discriminant $4(c^2 - b^2 \sin^2 C)$ is negative and there is no real value of ' a '. No solution is possible.

(ii) When $c = b \sin C$, the discriminant is zero. There are two equal and coincident values of ' a ' each equal to $b \cos C$. Since $a = b \cos C$ and $c = b \sin C$, $a^2 + c^2 = b^2$. The triangle is right-angled at B.

(iii) When $b > c > b \sin C$ and C is acute, $a_1 + a_2$ and $a_1 a_2$ are both positive and the discriminant positive. Hence

there are two real and distinct values of 'a', both of them positive and correspondingly, two triangles ACB_1 and ACB_2 satisfying the given data.

If C is obtuse, $\cos C$ is negative and so, is $a_1 + a_2$ negative, but a_1, a_2 positive. Both the values are negative and therefore inadmissible. No triangle is possible in this case.

(iv) When $c = b$, $a_1 a_2 = 0$. One of the roots is zero and there is only one solution possible.

(v) When $c > b$, $a_1 a_2$ is negative; one of the values is negative and there is no possibility of two triangles.

Ex. 1. Solve the triangle ABC given $c = 78.92$, $a = 32.34$, $A = 29^\circ 48'$.

$$\sin C = c \sin A / a = 78.92 \sin 29^\circ 48' / 32.34$$

$$\begin{aligned}\log \sin C &= \log 78.92 + \log \sin 29^\circ 48' - \log 32.34 \\ &= 1.8972 + 9.6963 - 1.5097 = 10.0838\end{aligned}$$

Since $\log \sin$ of any real angle is not greater than 10, no real value of C is possible. Hence, there is no real triangle satisfying the given data. This impossibility of solution is seen at once, since $c \sin A > a$.

Ex. 2. Solve the triangle ABC given $c = 98.76$, $a = 32.91$, $A = 19^\circ 28'$.

$$\sin C = c \sin A / a = 98.76 \sin 19^\circ 28' / 32.91$$

$$\begin{aligned}\log \sin C &= \log 98.76 + \log \sin 19^\circ 28' - \log 32.91 \\ &= 1.9946 + 9.5227 - 1.5173 = 10.\end{aligned}$$

$C = 90^\circ$. There is only one value of C and hence only one triangle ABC is possible and this is right-angled at C .

The remaining angle $B = 90^\circ - 19^\circ 28' = 70^\circ 32'$

$$b = c \cos A = 98.76 \cos 19^\circ 28'$$

$$\begin{aligned}\log b &= \log 98.76 + \log \cos 19^\circ 28' - 10 \\ &= 1.9946 + 9.9744 - 10 = 1.9690\end{aligned}$$

$$\therefore b = 93.11.$$

Ex. 3. Solve completely the triangle ABC, in which $c = 123.4$, $a = 98.76$, $A = 39^\circ 48'$.

$$\sin C = c \sin A / a = 123.4 \sin 39^\circ 48' / 98.76.$$

$$\begin{aligned}\log \sin C &= \log 123.4 + \log \sin 39^\circ 48' - \log 98.76 \\ &= 2.0913 + 9.8063 - 1.9946 = 9.9030\end{aligned}$$

$$\log \sin C = \log \sin 53^\circ 7';$$

$$\therefore C = 53^\circ 7' \text{ or } 180^\circ - 53^\circ 7'.$$

$$\text{Let } C_1 = 53^\circ 7' \text{ and } C_2 = 126^\circ 53'.$$

$$\therefore A + C_1 = 39^\circ 48' + 53^\circ 7' = 92^\circ 55'.$$

$$A + C_2 = 39^\circ 48' + 126^\circ 53' = 166^\circ 41'.$$

Since in both cases $(A + C)$ is less than 180° , there are two solutions corresponding to the two triangles BAC_1 and BAC_2 both of which satisfy the given data.

$$B_1 = 180^\circ - (A + C_1) = 87^\circ 5' \text{ and}$$

$$B_2 = 180^\circ - (A + C_2) = 13^\circ 19'.$$

$$b = a \sin B / \sin A = 98.76 \sin B / \sin 39^\circ 48'$$

$$b_1 = 98.76 \sin 87^\circ 5' / \sin 39^\circ 48'$$

$$\log b_1 = 1.9946 + 9.9994 - 9.8063 = 2.1877.$$

$$b_1 = \text{Antilog } 2.1877 = 154.$$

$$\text{Similarly, } b_2 = 98.76 \sin 13^\circ 19' / \sin 39^\circ 48'$$

$$\log b_2 = 1.9946 + 9.3623 - 9.8063$$

$$b_2 = \text{Antilog } 1.5506 = 35.53.$$

Hence there are two distinct solutions.

For one triangle BAC_1 , $C = 53^\circ 7'$; $B = 87^\circ 5'$ and $b = 154.0$ and for the other triangle BAC_2 , $C = 126^\circ 53'$; $B = 13^\circ 19'$ and $b = 35.53$.

Note.—In calculating b , the formula $a \sin B / \sin A$ for b has been used in preference to $c \sin B / \sin C$ since we have already read off $\log a$ and $L \sin A$.

Ex. 4. Solve a triangle ABC given $c = 38.95$, $a = 45.67$, $A = 41^\circ 36'$.

$$\sin C = 38.95 \sin 41^\circ 36' / 45.67$$

$$\log \sin C = 1.5905 + 9.8221 - 1.6597 = 9.7529.$$

$$\therefore C = 34^\circ 29' \text{ or } 145^\circ 31'$$

This latter value of C makes $(A + C)$ greater than 180° .

Hence, it is inadmissible.

There is only one triangle corresponding to

$$C = 34^\circ 29', B = 103^\circ 55'.$$

$$\begin{aligned} b &= \frac{a \sin B}{\sin A} = \frac{45.67 \times \sin 103^\circ 55'}{\sin 41^\circ 36'} \\ &= \frac{45.67 \times \sin 76^\circ 5'}{\sin 41^\circ 36'}. \end{aligned}$$

$$\log b = 1.6597 + 9.9871 - 9.8221 = 1.8247$$

$$b = 66.79.$$

Hence we have $C = 34^\circ 29'$; $B = 103^\circ 55'$; $b = 66.79$.

Ex. 5. Find the other elements of triangle ABC, given $a = 235.9$, $c = 357.8$, $A = 112^\circ 18'$.

It is an obtuse-angled triangle and hence there cannot be two solutions as can be shown in the actual working of the question.

$$\sin C = \frac{c \sin A}{a} = \frac{357.8 \times \sin 112^\circ 18'}{235.9} = \frac{357.8 \times \sin 67^\circ 42'}{235.9}$$

$$\log \sin C = 2.5537 + 9.9662 - 2.3728 = 10.1471$$

Hence, there is no solution at all.

Ex. 6. Solve a triangle ABC, in which

$$a = 357.8, c = 235.9, A = 112^\circ 18'.$$

$$\sin C = \frac{c \sin A}{a} = \frac{235.9 \times \sin 112^\circ 18'}{357.8} = \frac{235.9 \times \sin 67^\circ 42'}{357.8}$$

$$\text{Log sin C} = 2.3728 + 9.9662 - 2.5537 = 9.7853.$$

$$\text{Hence, C} = 37^\circ 36' \text{ or } 180^\circ - 37^\circ 36'$$

But, since the triangle is obtuse angled at A, the latter value is inadmissible.

$$\therefore C = 37^\circ 36'. \quad B = 30^\circ 6'$$

$$\text{To find } b, b = \frac{a \sin B}{\sin A} = \frac{357.8 \times \sin 30^\circ 6'}{\sin 112^\circ 18'}$$

$$\log b = 2.5537 + 9.7003 - 9.9662 = 2.2878$$

$$b = \text{Antilog } 2.2878 = 194.$$

Art. 97. Given two sides and the included angle, the third side is calculated *directly* by the formula

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

In its present form, it is not adapted for logarithmic computation ; but it can be expressed in a form suitable for logarithmic simplification, and there are three ways of doing so.

$$(i) a^2 = b^2 + c^2 - 2bc \cos A.$$

$$= (b^2 + c^2) \left(\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} \right) - 2bc \left(\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \right).$$

$$= (b^2 + c^2 - 2bc) \cos^2 \frac{A}{2} + (b^2 + c^2 + 2bc) \sin^2 \frac{A}{2}.$$

$$= (b - c)^2 \cos^2 \frac{A}{2} + (b + c)^2 \sin^2 \frac{A}{2}$$

$$= (b - c)^2 \cos^2 \frac{A}{2} \left\{ 1 + \left(\frac{b + c}{b - c} \right)^2 \tan^2 \frac{A}{2} \right\}$$

$$\equiv (b - c)^2 \cos^2 \frac{A}{2} \times \sec^2 \theta, \text{ when } \theta \text{ is a subsidiary}$$

$$\text{angle chosen to satisfy the equation } \tan \theta = \frac{b + c}{b - c} \tan \frac{A}{2}.$$

$$a = (b - c) \cos \frac{A}{2} \sec \theta$$

Note.—(1) If $b < c$, the formula becomes

$$a = (c - b) \cos \frac{A}{2} \sec \theta, \text{ where } \tan \theta = \frac{b + c}{c - b} \tan \frac{A}{2}.$$

(2) The student will not fail to note that the subsidiary angle θ is only the complement of $\frac{1}{2} (B - C)$.

$$(ii) a = (b + c) \sin \phi,$$

$$\text{where } \cos \phi = 2\sqrt{bc} \cos (A/2) / (b + c)$$

For proof, refer to Ex. 2, Art. 77.

$$(iii) a^2 = b^2 + c^2 - 2bc \cos A$$

$$= b^2 + c^2 - 2bc [1 - 2\sin^2 (A/2)]$$

$$= (b - c)^2 + 4bc \sin^2 A/2$$

$$= (b - c)^2 \left\{ 1 + \frac{4bc}{(b - c)^2} \sin^2 \frac{A}{2} \right\}$$

$$= (b - c)^2 \sec^2 \phi$$

$$\text{where } \tan^2 \phi = \frac{4bc}{(b - c)^2} \sin^2 \frac{A}{2}.$$

$$\therefore a = (b - c) \sec \phi, \quad \text{where } \tan \phi = \frac{2\sqrt{bc}}{b - c} \cdot \sin \frac{A}{2}.$$

Art. 98. Given the three altitudes, solve a triangle.—In addition to the four cases of solution of triangles cited above, many more can be cited; for, any set of three *independent* elements of a triangle fix the figure and so furnishes a question for calculating all the rest. An interesting case is to solve a triangle given the lengths of the three altitudes. Let h_1 , h_2 and h_3 denote the three altitudes of the triangle from A, B and C respectively.

Then, $ah_1 = bh_2 = ch_3 = 2 \times \text{area of triangle ABC.}$

$$\frac{a}{h_2 h_3} = \frac{b}{h_3 h_1} = \frac{c}{h_1 h_2} = k.$$

The proportionals to the lengths of the three sides being known, the angles A, B, C can be calculated (Art. 93). It remains to determine k to know the actual lengths of the sides.

$$h_1 = c \sin B = k h_1 h_2 \sin B.$$

$$\therefore k = h_1 / (h_1 h_2 \sin B) \text{ and } a = k h_2 h_3, \text{ etc.}$$

Art. 99. Given the area, the side a and $B - C$, solve the triangle ABC —

$$\begin{aligned} \text{Area } S &= \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin (B + C)} \\ &= \frac{1}{4} a^2 \frac{\cos (B - C) - \cos (B + C)}{\sin (B + C)} \end{aligned}$$

Since S , a , and $(B - C)$ are known, this equation when simplified reduces to the form—

$l \cos (B + C) + m \sin (B + C) = n$, where l , m , n , are known constants.

Solving this equation for $(B + C)$, we get the values of B and C separately; thence, the values of b and c are arrived at.

EXAMPLES X b

1. Solve the triangle ABC given

- (i) $a = 8$, $b = 12$, $c = 14$.
- (ii) $a = 33.57$, $b = 35.79$, $c = 57.92$
- (iii) $a = 40.08$, $b = 51.76$, $c = 71.32$
- (iv) $a : b : c = 4 : 5 : 6$ and $R = 2''$
- (v) $a : b : c = 12.57 : 31.92 : 28.65$ and $R = 2.5''$

2. Find the other elements of the triangle ABC given

- (i) $a = 1236$, $b = 2345$, $C = 43^\circ 28'$.
- (ii) $b = 345.6$, $c = 456.7$, $A = 13^\circ 16'$

- (iii) $c = 75.98$, $a = 96.14$, $B = 103^\circ 54'$
 (iv) $a : b = 12 : 35$, $C = 62^\circ 44'$ and $R = 1.5''$
 (v) $b : c = 31.84 : 91.7$, $A = 34^\circ 9'$ and $R = 2.3''$

3. Solve the triangle ABC given,

- (i) $a = 24.68$, $B = 72^\circ 36'$, $C = 102^\circ 53'$
 (ii) $b = 46.85$, $C = 57^\circ 38'$, $B = 86^\circ 49'$
 (iii) $c = 35.79$, $C = 58^\circ 37'$, $B = 85^\circ 48'$
 (iv) $c = 35.79$, $C = 58^\circ 37'$, $A = 45^\circ 14'$
 (v) $B = 21^\circ 36'$, $C = 113^\circ 32'$, $a = 20.93$

4. Calculate all the elements of triangle ABC stating when the solution is ambiguous.

- (i) $a = 12.56$, $b = 10.53$, $A = 63^\circ 20'$
 (ii) $a = 1322$, $b = 722$, $B = 38^\circ 45'$
 (iii) $b = 91.06$, $c = 78.05$, $C = 50^\circ 12'$
 (iv) $a = 9288$, $b = 8133$, $A = 115^\circ 20'$
 (v) $a = 2176$, $b = 1893$, $A = 129^\circ 36'$
 (vi) $a : b = 7 : 9$, $B = 38^\circ 9'$, $R = 2''$

5. The diagonals of a parallelogram are 232 and 178 and they include an angle of $54^\circ 36'$. Calculate the sides and angles of the parallelogram.

6. In a parallelogram ABCD, $AB = 38$, $AC = 72.68$, $BD = 53.6$. Find the remaining side and compute the area.

7. Given that $B = \frac{\pi}{3}$, and $c : a$ as 3 : 5 find $\tan C$ and $\tan A$.

8. To find the width of a stream, two places A and B are chosen on one bank at a distance of 1,200 yds. apart. A temple T on the opposite bank is observed from both the stations and it is found $TAB = 37^\circ 42'$ and $TBA = 78^\circ 53'$. Calculate the breadth of the river.

9. Given that the altitude from A measures 23.4 ft., $B = 48^\circ$, $C = 69^\circ 30'$, find the sides of the triangle.

$$\left[\text{Hint - Use the formula } h = \frac{a \sin B \cdot \sin C}{\sin (B + C)} \right]$$

10. Two trains start from a junction simultaneously with speeds of 30 miles per hour and 45 miles per hour. If their courses include an angle of $31^\circ 45'$, find their distance apart after 2 hrs 45 minutes.

11. ABC is a triangle, in which $AB = 86.3$, $AC = 27.4$ and $C - B = 60^\circ$. Find the angles and the remaining side.

12. Given $A = 26^\circ 39'$, $B = 31^\circ 57'$, and semiperimeter = 23, solve the triangle.

13. Solve the triangle ABC, given

- (i) $c = 96.75$, $C = 65^\circ 18'$, $(a + b) = 136.4$.
- (ii) $b = 42.9$, Altitude $BE = 31.7$, $C - A = 10^\circ 12'$.
- (iii) $C = 65^\circ 30'$, Altitude $CF = 15.6$, $a - b = 4.7$.
- (iv) $A = 47^\circ 29'$, Altitude $BE = 6.78$,

Altitude $CF = 7.54$.

14. Two circles of radii $3.5''$ and $4''$ intersect. If the distance between their centres is $6''$, calculate the angle subtended by the common chord at each of the centres. Hence, find the area enclosed by the radii to the points of intersection.

15. ABC is a triangle in which $AB = 3.7''$, $AC = 4.9''$ and $BAC = 46^\circ 30'$. Find the third side and also the length of the circum-radius.

16. PQRS is a quadrilateral in which $PQ = 3''$, $QR = 4''$, $RS = 5''$, $PQR = 107^\circ 12'$, $QRS = 56^\circ 18'$. Calculate the lengths of the diagonals and of the remaining side. Write down the expression for the area of the quadrilateral.

17. Indicate the method of solving the triangle in the following cases:—

- (i) Given two angles and the sum of the sides opposite to them.
- (ii) „ b, c, R .
- (iii) „ $B, C, s - b$.
- (iv) „ $c, C, a b$.
- (v) b, h_2, h_1 . (h_2, h_1 being the altitudes).
- (vi) $A, B, a - b$.
- (vii) a, B, h_1 .
- (viii) s, A, B .

18. The sides of a triangle are 29.54 and 14.72 and the difference of the angles opposite to them 45° . Find all the angles.

19. Solve a triangle given $s = 12.5$, $A = 70^\circ$, $B = 40^\circ$.

20. Solve a triangle given r_1, r_2, r_3 .

CHAPTER XI

HEIGHTS AND DISTANCES

Art. 100. The solution of oblique-angled and right-angled triangles takes its importance from its application to calculate the heights and distances of inaccessible objects and to work out several problems in land-surveying. Simple questions on this topic have been worked out in a previous chapter without employing logarithms; but all questions given in this chapter admit of easy simplification by logarithms. So, the reader will note that wherever possible, a logarithmic formula must be derived to compute the required length or angle.

Art. 101. To find the height of an inaccessible object on a horizontal plane by observations made at two places in the same vertical plane as the object.

- (i) Let T (Fig. 57) be an inaccessible object whose height is required. Take two places P and Q on the horizontal plane through B and in line with B, the point on the ground vertically below T. Observe with the help of a theodolite the angles of elevation of T from P and Q, and measure with a chain the

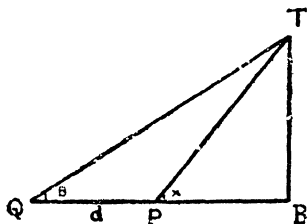


Fig. 57

distance PQ. These three data help us to calculate TB and BP.

Suppose PQ measures d units and $TPB = \alpha$, $TQB = \beta$.

Then, in triangle QTP, $QTP = \alpha - \beta$.

$$\frac{TB}{PQ} = \frac{TB}{PT} \cdot \frac{PT}{PQ} = \frac{\sin \alpha \cdot \sin \beta}{\sin (\alpha - \beta)}.$$

$$TB = PQ \cdot \frac{\sin \alpha \cdot \sin \beta}{\sin (\alpha - \beta)} = \frac{d \sin \alpha \sin \beta}{\sin (\alpha - \beta)}.$$

From right-angled triangle TPB,

$$PB = TB \cdot \cot \alpha = d \cos \alpha \sin \beta \div \sin (\alpha - \beta).$$

(ii) It may happen that the two places of observation P and Q co-planar with TB may be one vertically above the other. This contingency will arise when it is not possible to proceed horizontally from P to Q either towards or away from B. (Fig. 58.)

$$\begin{aligned} PTQ &= QTB - PTB \\ &= \alpha - \beta. \end{aligned}$$

$$\frac{TB}{PQ} = \frac{TB \cdot TP}{TP \cdot PQ}.$$

$$\frac{TP}{PQ} = \frac{\sin TQP}{\sin PTQ}.$$

$$\therefore \frac{TB}{PQ} = \frac{\sin \alpha \cdot \cos \beta}{\sin (\alpha - \beta)};$$

$$TB = \frac{d \sin \alpha \cos \beta}{\sin (\alpha - \beta)}.$$

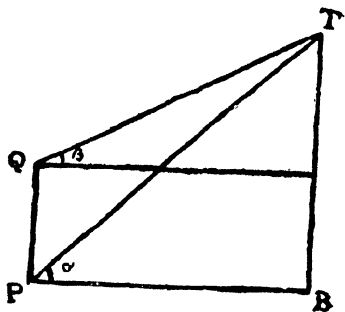


Fig. 58

(iii) A third case arises when P is at the foot of a sloping hill-side, the inclination of which to the horizon is known. Suppose the inclination of the hill is γ , and on walking up the hill from P a distance 'd' along the line of greatest slope, the observer reaches Q and observes the elevation of T. Let α and β be the angles of elevation of T from P and Q respectively. Draw QX the horizontal through Q; and produce TQ to meet PB in Y.

$$\angle TYB = \angle TQX \text{ (since QX and YB are parallel)} = \beta.$$

From triangle TPY, $PTQ = TYB - TPY = (\beta - \alpha)$ and from triangle QPY, $PQY = QYB - QPY = (\beta - \gamma)$.

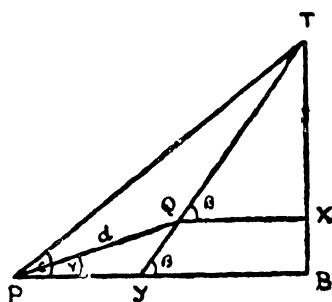


Fig. 59

$$\begin{aligned}
 \frac{TB}{PQ} &= \frac{TB}{TP} \times \frac{TP}{PQ} \\
 &= \sin \alpha \times \frac{\sin TQP}{\sin PTQ} \\
 &= \sin \alpha \times \frac{\sin (180^\circ - \beta + \gamma)}{\sin (\beta - \alpha)} \\
 &= \frac{\sin \alpha \sin (\beta - \gamma)}{\sin (\beta - \alpha)} \\
 TB &= \frac{d \sin \alpha \cdot \sin (\beta - \gamma)}{\sin (\beta - \alpha)}.
 \end{aligned}$$

Note.—The reader will note that (i) and (ii) are only particular cases of (iii), being deducible from it by putting $\gamma = 0$ and $\gamma = 90^\circ$ respectively.

Ex. 1. The angles of elevation of the top of a spire as seen from the top and bottom of a mound are $32^\circ 36'$ and $36^\circ 18'$ respectively. If the mound is 225 feet high, calculate the height of the spire above the base of the mound.

Let TB be the spire (Fig. 60) and PQ the mound. $TPX = 32^\circ 36'$ and $TQB = 36^\circ 18'$. $PQ = 225$. It is required to calculate TB.

$$\begin{aligned}
 \frac{TB}{PQ} &= \frac{TB}{TQ} \times \frac{TQ}{PQ} \\
 &= \frac{\sin 36^\circ 18' \times \cos 32^\circ 36'}{\sin 3^\circ 42'}
 \end{aligned}$$

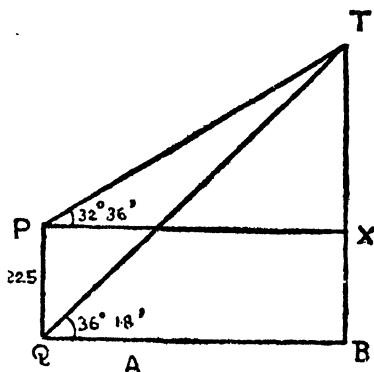


Fig. 60

$$TB = \frac{225 \sin 36^\circ 18' \times \cos 32^\circ 36'}{\sin 3^\circ 42'}$$

$$\log TB = \log 225 + L \sin 36^\circ 18' + L \cos 32^\circ 36' - L \sin 3^\circ 42' - 10.$$

$$\therefore TB = \text{Antilog } 3.2402 = 1739.$$

The spire is 1739 feet high.

Art. 102. To find the height and distance of an inaccessible object by observations from *any* two places on the ground.

In the last article the two places of observation P and Q were taken in the same vertical plane as the object. If they are not so taken, the question becomes a problem in more than one plane: TBP and TBQ being the two planes of observation. In this case, one more observation is necessary to determine the height.

With the same notation as in the previous article, let the elevation of T from P be α , and from Q be β and PQ be 'd' units distant, both the places being on the horizontal plane through B.

The angle made by PQ with PB may be determined with a sextant or theodolite. Let it be γ .

If the height TB equals 'h',

$$PB = h \cot \alpha ;$$

$$QB = h \cot \beta$$

$$PQ = d \text{ and } \angle BPQ = \gamma.$$

Hence, by the cosine rule applied to triangle PQB,

$$QB^2 = PQ^2 + PB^2 - 2 PQ \cdot PB \cos \gamma$$

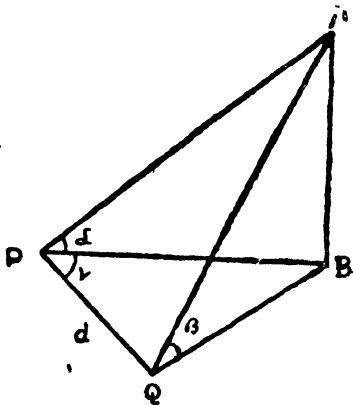


Fig. 61

i.e., $h^2 \cot^2 \beta = d^2 + h^2 \cot^2 \alpha - 2dh \cot \alpha \cos \gamma$.
 'h' is therefore a root of the equation.

$$h^2 (\cot^2 \alpha - \cot^2 \beta) - 2dh \cot \alpha \cos \gamma + d^2 = 0$$

This value of 'h' when substituted in the equation

$$PB = h \cot \alpha \text{ determines the distance PB.}$$

(ii) Another set of four observations will also lead to the required result. The angles TPQ, TQP, the length 'd' and *either* α or β suffice to solve the problem. Let $TPQ = \delta$ and $TQP = \Sigma$.

Since one side and two angles of triangle TPQ are known, solve for TP or TQ.

$$\text{Thus, } TP = \frac{d \sin \Sigma}{\sin (\delta + \Sigma)} \text{ and } TQ = \frac{d \sin \delta}{\sin (\delta + \Sigma)}$$

$$\begin{aligned} \text{The height } TB &= TP \sin \alpha \text{ or } TQ \sin \beta \\ &= \frac{d \sin \Sigma \cdot \sin \alpha}{\sin (\delta + \Sigma)} \text{ or } \frac{d \sin \delta \sin \beta}{\sin (\delta + \Sigma)} \end{aligned}$$

Note 1.—As a special case, $\gamma = 90^\circ$, PQ is at right angles to PB; $\cos \gamma = 0$. Hence 'h' is determined by the equation $h^2 (\cot^2 \alpha - \cot^2 \beta) + d^2 = 0$.

$$\begin{aligned} h^2 &= \frac{d^2}{\cot^2 \beta - \cot^2 \alpha} = \frac{d^2}{\operatorname{cosec}^2 \beta - \operatorname{cosec}^2 \alpha} \\ &= \frac{d^2 \sin^2 \alpha \cdot \sin^2 \beta}{\sin^2 \alpha - \sin^2 \beta} = \frac{d^2 \sin^2 \alpha \cdot \sin^2 \beta}{\sin (\alpha + \beta) \cdot \sin (\alpha - \beta)} \\ h &= d \sin \alpha \cdot \sin \beta / \sqrt{\sin (\alpha + \beta) \sin (\alpha - \beta)}. \end{aligned}$$

Note 2.—Instead of $QPB = 90^\circ$, Q may be chosen so that $PQB = 90^\circ$, in which case, $PB^2 = PQ^2 + QB^2$,
i.e., $h^2 \cot^2 \alpha = d^2 + h^2 \cot^2 \beta$.

Simplifying in a similar way,

$$h = d \sin \alpha \sin \beta / \sqrt{\sin (\alpha + \beta) \sin (\beta - \alpha)}$$

Note 3.—When Q is so situated that $PBQ = 90^\circ$
 $PQ^2 = PB^2 + BQ^2$, i.e., $h^2 (\cot^2 \alpha + \cot^2 \beta) = d^2$

$$h = d / \sqrt{\cot^2 \alpha + \cot^2 \beta}$$

This formula is not suitable for logarithmic simplification.

Note 4.—The reader will note that in all such questions in more than one plane, we reduce them to a problem in one plane. This means expressing the elements of the triangle PBQ in the ground plane or of the triangle TPQ in terms of the given data and 'h' and thence framing an equation to solve for 'h' using the formulae connecting the sides and angles of a triangle.

Ex. 2. The angle of elevation of the top T of a tower CT from a place P on the ground is $11^\circ 18'$. Q is another place on the ground distant 726 feet from P. The bearings of the foot C are observed from P and Q and it is found that $CPQ = 84^\circ 36'$ and $CQP = 62^\circ 24'$. Calculate the height of the tower.

Evidently, we have to solve triangle CPQ for CP.

$$\angle PCQ = 33^\circ$$

$$PC = \frac{726 \cdot \sin 62^\circ 24'}{\sin 33^\circ}$$

$$\log PC = \log 726 + L \sin 62^\circ 24' - L \sin 33^\circ$$

$$PC = \text{Antilog } 3.0723 = 1181$$

$$TC = PC \cdot \tan 11^\circ 18' = 236.$$

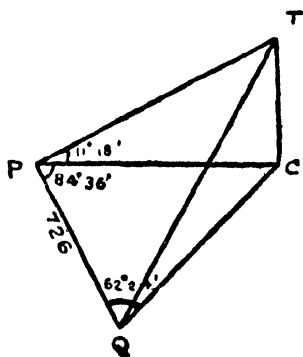


Fig. 62

Ex. 3. Standing due east of a tower, its elevation is $35^\circ 18'$; on walking 235 feet south of this position, its eleva-

tion is $22^\circ 12'$. Find the height of the tower and its distance from the first station.

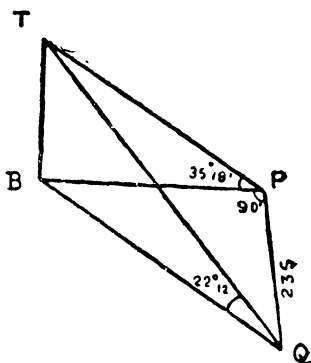


Fig. 63

P and Q are the two places of observation and TB the tower. Since P is east of B and Q south of P, $BPQ = 90^\circ$.

Let TB measure h feet. Expressing the sides of triangle BPQ in terms of the given data and h , we have,

$$BP = h \cot 35^\circ 18',$$

$$BQ = h \cot 22^\circ 12'.$$

$$BQ^2 - BP^2 = PQ^2 \text{ since } BPQ = 90^\circ, \\ \text{i.e., } h^2 [\cot^2 22^\circ 12' - \cot^2 35^\circ 18'] = 235^2.$$

$$\text{Simplifying, } h = \frac{235 \cdot \sin 22^\circ 12' \cdot \sin 35^\circ 18'}{\sqrt{\sin 57^\circ 30' \cdot \sin 13^\circ 6'}}$$

$$\log h = \log 235 + L \sin 22^\circ 12' + L \sin 35^\circ 18' \\ - \frac{1}{2} (L \sin 57^\circ 30' + L \sin 13^\circ 6') - 10 = 2.0695.$$

$$h = \text{Antilog } 2.0695 = 117.3$$

$$\text{The distance } BP = h \cot 35^\circ 18' = 117.3 \tan 54^\circ 42'.$$

$$BP = \text{Antilog } 2.2192 = 165.7 \text{ feet.}$$

Ex. 4. The angle of elevation of a tower 123.4 feet high and due north of an observer is $48^\circ 23'$. What will be its elevation from a place on the ground 750 feet west of his former position.

P and Q are two positions from which the tower TB is observed, Q being due west of P, and B north of P, so that $BPQ = 90^\circ$.

TB = 123.4 feet, QP = 750 feet, and TPB = $48^{\circ} 23'$.
It is required to calculate TQB.

From triangle TBP,

$$\begin{aligned} PB &= TB \cdot \cot TPB \\ &= 123.4 \cot 48^{\circ} 23' \\ &= 123.4 \tan 41^{\circ} 37' \end{aligned}$$

$$\begin{aligned} QB^2 &= QP^2 + PB^2 \\ &= 750^2 \\ &\quad + 123.4^2 \tan^2 41^{\circ} 37' \end{aligned}$$

$$QB = \sqrt{750^2 + 123.4^2 \tan^2 41^{\circ} 37'}$$

$$\tan TQB = \frac{123.4}{\sqrt{750^2 + 123.4^2 \tan^2 41^{\circ} 37'}}$$

The quantity within the radical in the denominator of the right hand side doesn't admit of logarithmic simplification. Let $x = 123.4^2 \tan^2 41^{\circ} 37'$.

$$\begin{aligned} \log x &= 2 [2.0913 + 1.9486] = 2 \times 2.0399 = 4.0798. \\ x &= 12010. \end{aligned}$$

$$750^2 + 123.4^2 \tan^2 41^{\circ} 37' = 574510.$$

$$\begin{aligned} L \tan TQB &= \log 123.4 + 10 - \frac{1}{2} \log 574510. \\ &= 12.0913 - 2.8797 = 9.2116 \end{aligned}$$

$$TQB = 9^{\circ} 15'.$$

Hence the elevation of the tower from Q is $9^{\circ} 15'$.

Ex. 5. From a place P, the elevation of a hill top T is $21^{\circ} 32'$. A base line PQ is measured in a convenient
V. T.—12

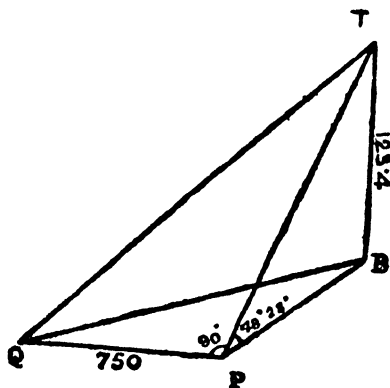


Fig. 64

direction 145 yards long, and the angles TPQ, TQP are found to be $49^{\circ} 24'$ and $78^{\circ} 48'$ respectively. Calculate the height of the hill above the ground.

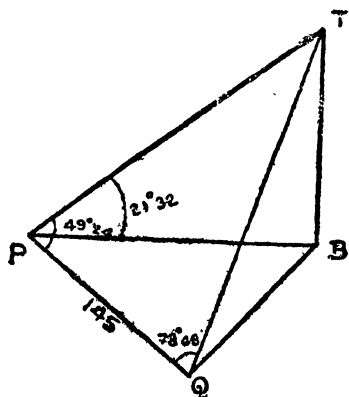


Fig. 65

Let B be the point vertically below the hill top T.

Then, from triangle TPB,

$$TB = PT \cdot \sin 21^{\circ} 32'.$$

But in triangle QTP,

$$PQ = 145.$$

$$\angle TPQ = 49^{\circ} 24',$$

$$\angle TQP = 78^{\circ} 48',$$

$$\angle PTQ = 51^{\circ} 48'$$

$$TP = \frac{145 \sin 78^{\circ} 48'}{\sin 51^{\circ} 48'}$$

$$\therefore TB = \frac{145 \sin 78^{\circ} 48' \cdot \sin 21^{\circ} 32'}{\sin 51^{\circ} 48'}$$

$$\log TB = \log 145 + L \sin 78^{\circ} 48' + L \sin 21^{\circ} 32' \\ - L \sin 51^{\circ} 48' - 10$$

$$= 2.1614 + 19.5563 - 19.8953 = 1.8224$$

$$\therefore TB = \text{Antilog } 1.8224 = 66.43.$$

The hill is 66.43 yards high.

Art. 103. To find the distance between two inaccessible objects on the ground by making observations from two points at a known distance.

Let R, S be the two points the distance between which is required. P and Q are two stations on the ground from each of which the two objects R and S are visible.

Let $PQ = d$ units. Four angles are to be determined with a theodolite, viz., $\angle RPQ$, $\angle SPQ$, $\angle SQP$, $\angle RQP$.

Let their values as observed be α , β , γ , δ respectively.

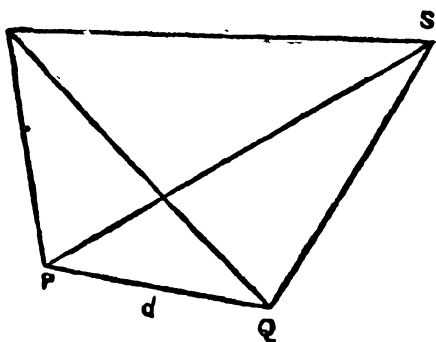


Fig. 66

Then, from triangle RPQ , where

$$PQ = d, \angle RPQ = \alpha, \angle RQP = \delta, RP = d \sin \delta / \sin (\alpha + \delta)$$

Similarly, from triangle SPQ , where $PQ = d$, $\angle SPQ = \beta$

$$\angle SQP = \gamma, PS = d \sin \gamma / \sin (\beta + \gamma)$$

$$\text{Now, in triangle } RPS, RP = \frac{d \sin \delta}{\sin (\alpha + \delta)},$$

$$PS = \frac{d \sin \gamma}{\sin (\beta + \gamma)},$$

$$\text{and } \angle RPS = (\alpha - \beta).$$

Hence, $RS^2 = PR^2 + PS^2 - 2 PR \cdot PS \cdot \cos RPS$

$$= \frac{d^2 \sin^2 \delta}{\sin^2 (\alpha + \delta)} + \frac{d^2 \sin^2 \gamma}{\sin^2 (\beta + \gamma)} - \frac{2 d^2 \sin \delta \sin \gamma}{\sin (\alpha + \delta) \sin (\beta + \gamma)} \times \cos (\alpha - \beta),$$

which can be evaluated, since d , α , β , γ , δ are known quantities.

Ex. 6. AB is a straight road. From A, two objects P, Q level with AB are observed so that $PAB = 45^\circ 31'$, $QAB = 36^\circ 12'$; and from a place C on the road AB, P and Q are found to be in line with the observer. If AC measures 100 yards, and $PCA = 62^\circ 18'$, calculate the distance between the objects. This is a simple case of the general question.

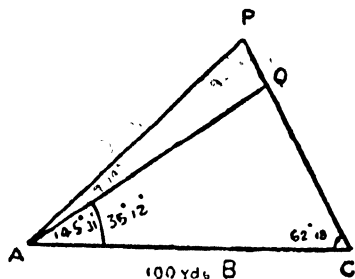


Fig. 67

From triangle PAC, since $APC = 72^\circ 11'$,

$$PA = \frac{100 \sin 62^\circ 18'}{\sin 72^\circ 11'}$$

$$\log PA = 2.0000 + 9.9471 - 9.9787 = 1.9684$$

$$PA = \text{Antilog } 1.9684 = 92.98.$$

Then, in triangle PAQ.

$$PA = 92.98, \quad PAQ = 9^\circ 19', \quad AQP = 98^\circ 30'$$

$$PQ = 92.98 \sin 9^\circ 19' / \sin 98^\circ 30'$$

$$= 92.98 \sin 9^\circ 19' / \cos 8^\circ 30'$$

$$= 15.23 \text{ yards.}$$

Art. 104. In some questions, a knowledge of geometrical Theorems is required.

Ex. 7. A spire placed on the top of a tower subtends equal angles θ at each of two places P and Q on the ground, in line with the foot of the tower on the same side of it and distant d_1 and d_2 from the foot. Prove that the height of the

spire is given by $(d_1 + d_2) \tan \theta$; if the tower subtends at P an angle ϕ , calculate the height of the tower.

BT is the tower and TS the spire on it.

P and Q are the two points, such that

$$\text{SPT} = \text{SQT} = \theta.$$

Hence, the four points S, T, P, Q are concyclic.

Let O be the centre of this circle, and M, N the mid-points of PQ, TS respectively.

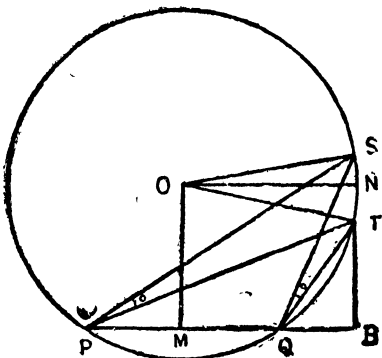


Fig. 68

Then, OM is perpendicular to PQ and bisects it ;
and ON is perpendicular to TS and bisects it.

Figure ONBM is a rectangle, in which

$$\begin{aligned} \text{ON} &= \text{MB} = \frac{1}{2} (\text{PB} + \text{QB}) \\ &= \frac{1}{2} (d_1 + d_2). \end{aligned}$$

Again, $\text{SOT} = 2 \text{SPT}$ at the circumference $= 2\theta$.

But, since $\text{OS} = \text{OT}$, $\text{SON} = \theta$

$$\text{SN} = \text{ON} \times \tan \theta = \frac{1}{2} (d_1 + d_2) \tan \theta$$

$$\text{and TS} = (d_1 + d_2) \tan \theta.$$

TPQ being given equal to ϕ , it is required to calculate the height of the tower, i.e., BT.

In the cyclic quadrilateral PQTS, the exterior angle QTB = SPQ = $\theta + \phi$ (interior opp. angle).

$$\text{Now } \text{TB}/\text{BQ} = \cot \text{QTB} = \cot (\theta + \phi)$$

$$\therefore \text{TB} = \text{BQ} \cdot \cot (\theta + \phi) = d_2 \cot (\theta + \phi).$$

Cor. In case P and Q coincide, $d_1 = d_2$ and θ is the maximum angle subtended by the spire at points on the ground. Hence the height of the spire in this case equals $2d \tan \theta$, d being the distance from the foot of the tower to the point on the ground at which this maximum angle is subtended by the spire.

Ex. 8. From three consecutive furlong stones on a straight road, which does not pass through the foot of a spire, the angles of elevation of the top of the spire are 15° , 45° and 75° . Find the height of the spire.

Let F be the foot of the spire and A, B, C, the three places of observation. Suppose the height of the spire to be h furlongs.

Then, in triangle FAC,

$FA = h \cot 15^\circ$, $FB = h \cot 45^\circ$, $FC = h \cot 75^\circ$ and $AB = BC = 1$.

By the theorem of Apollonius,

$$FA^2 + FC^2 = 2FB^2 + 2AB^2,$$

$$\text{i.e., } h^2 [\cot^2 15^\circ + \cot^2 75^\circ - 2] = 2,$$

$$\text{i.e., } h^2 (\cot^2 15^\circ + \tan^2 15^\circ - 2) = 2,$$

$$\text{i.e., } h^2 (\cot 15^\circ - \tan 15^\circ)^2 = 2,$$

$$\text{i.e., } h = \sqrt{2}/(\cot 15^\circ - \tan 15^\circ) = 1/\sqrt{6}.$$

Art. 105. A certain type of questions coming in this chapter relates to the sizes of the shadows cast by objects such as a rod, or a wall, or a target or a cloud.

Ex. 8. A vertical wall faces due south. When the sun is θ° south of east at an altitude α , calculate the area

of the shadow cast by the wall in terms of its length and height.

Let l and h be the length and height of the wall $ABCD$, so that $AB = l$ and $BC = h$.

The sun being θ° south of east, the shadow of AD falls in a direction θ° north of west and its length $AD' = AD \cot \alpha$.

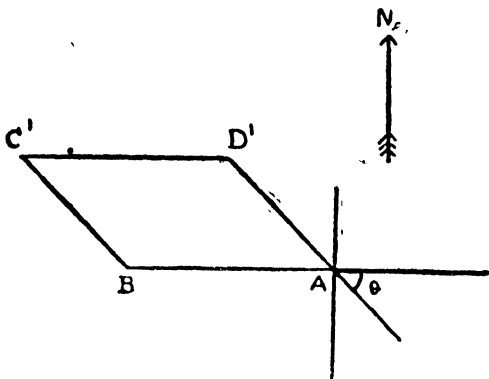


Fig. 69

BC being equal and parallel to AD , its shadow BC' is a parallel line of equal length to AD' .

Hence, the shadow cast by the wall is a parallelogram, $ABC'D'$, in which $AB = l$, $AD' = h \times \cot \alpha$,

$$\angle D'AB = \theta.$$

$$\begin{aligned} \text{Hence, Area } ABC'D' &= AB \times AD' \cdot \sin \angle D'AB \\ &= l h \cot \alpha \cdot \sin \theta. \end{aligned}$$

Art. 106. The projection of a line AB on any plane not parallel to it is got by joining A' and B' , the feet of the perpendiculars from A and B to the plane. The angle made by AB with its projection $A'B'$ measures the angle of inclination of the line to the plane. If it is a right angle, the line is perpendicular to the plane. Every horizontal line points to a certain direction in space, which can be defined with respect to the cardinal points, *e.g.*, $N. 25^\circ. W.$, etc. A verti-

cal line has no such direction. If a line is oblique, its direction is taken to be that of its projection on the horizontal plane.

Again, two planes, unless parallel, always meet along a straight line, *viz.*, a section of two planes is a straight line. If at *any* point P on this common section '*l*' of two planes, two lines PL, PM are drawn perpendicular to *l*, one in each of the two planes, the angle included by them is a measure of the inclination between the planes. Of all lines in one plane, only those that are perpendicular to the common section make the greatest angle with the other plane; such lines are therefore called lines of greatest slope on the plane.

Ex. 9. A line inclined to the horizon at any angle θ runs up an inclined plane of inclination i in a direction making α with the line of greatest slope. If the projection of the line on the horizontal plane make an angle ϕ with the common section of the planes, prove that $\tan i = \tan \theta \times \operatorname{cosec} \phi$ and $\cos \alpha = \sin \theta \cdot \operatorname{cosec} i$.

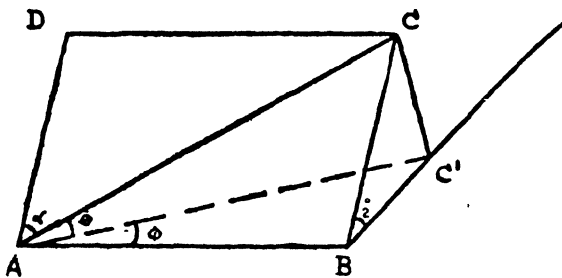


Fig. 70

ABCD is the plane inclined to the horizontal ABC' at an angle i , AB being the common section of the planes; and AC is the straight line taken on it. Draw CB perpendicular to AB, and mark C' the projection of C on the horizontal.

Then, BC' and AC' are respectively the projections of BC and AC . BC being the line of greatest slope, $CBC' = i$ and $CAC' = \theta$. Also AB and AC' are inclined at ϕ . There are evidently three right-angled triangles—

ACC' right angled at C' ,

ABC' right angled at B ,

BCC' right angled at C' .

From triangle ACC' ,

$$\tan \theta = \frac{CC'}{AC'}$$

But, $CC' = BC' \tan i$

and $AC' = BC' \operatorname{cosec} \phi$.

$$\therefore \tan \theta = \frac{\tan i}{\operatorname{cosec} \phi}$$

i.e., $\tan i = \tan \theta \cdot \operatorname{cosec} \phi$.

For the latter part, ACB is given to be α .

From triangle ACC' , $\sin \theta = \frac{CC'}{AC}$.

But, $CC' = BC \sin i$

and $AC = BC \sec \alpha$.

$$\therefore \sin \theta = \frac{\sin i}{\sec \alpha}, \text{ i.e., } \cos \alpha = \sin \theta \cdot \operatorname{cosec} i.$$

EXAMPLES XI

(1) From the top of a cliff 135 feet high, two ships are seen at anchor in line with the foot of the cliff. Their angles of depression being $20^\circ 31'$ and $17^\circ 18'$, find the distance between the ships.

(2) To find the breadth of a river, the elevation of the top of a telegraph post on one bank is observed from the point on the other bank directly opposite to be $70^\circ 42'$. On receding perpendicular to the bank a distance of 250 yards, the angle of elevation of the top of the post is found to be $62^\circ 39'$. Find the width of the river.

(3) A balloon is observed at the instant when it is vertically above a point of the road. The observers on the road on opposite sides of the point note the elevations of the balloon to be $66^{\circ} 48'$ and $43^{\circ} 29'$. Find the height of the balloon given that the observers are 800 feet apart.

(4) The elevations of a spire from a point on the floor of a house and from a place 60 feet above it on the terrace are found to be $34^{\circ} 15'$ and $28^{\circ} 17'$. Calculate the height of the spire from the level of the floor.

(5) A is the foot of a hillock, rising up at an inclination of $30^{\circ} 17'$ to the horizon. B is a point on its side 100 yards up. The elevations of a distant steeple as seen from A and B are $38^{\circ} 23'$ and $41^{\circ} 12'$. Find the height of the steeple and its distance from A.

(6) A statue 15 feet high standing on the top of a column subtends an angle of $40^{\circ} 12'$ at a point on the horizontal plane through the foot of a column, the column subtending $20^{\circ} 15'$ at the same point. Find the height of the column.

(7) The elevation of a certain rock is 41° and after walking towards it 720 feet up an inclined at $27^{\circ} 41'$ to the horizon, it is found to be $69^{\circ} 35'$. Calculate the height of the rock.

(8) The angles of elevation of a distant spire from two points A and B on the ground are found to be $30^{\circ} 24'$ and $27^{\circ} 12'$. A and B subtend 90° at the foot of the spire and are distant 200 feet from each other. Find the height of the spire from the ground and its distance from A.

(9) From a place due north of a temple tower, the elevation of the tower top is observed to be $23^{\circ} 12'$. On moving 120 yards east, it is found to be $20^{\circ} 18'$. Find the height of the tower.

(10) To find the breadth of a river, the elevation of the top of a tree on one bank is observed from the point of the other bank directly opposite to be $15^{\circ} 10'$. After walking 220 yards along the bank, it is found to be $12^{\circ} 17'$. Find the breadth required.

(11) A and B are two points on the ground subtending a right angle at the foot of a column and 250 yards apart. If the elevation of the top from A be $10^{\circ} 7'$, find the elevation of the same from B, given that the height of the column is 30 feet.

(12) From two places P and Q on level ground 120 feet apart, it is found that the angles APQ and AQP are $54^{\circ} 15'$ and $72^{\circ} 19'$ respectively. If the elevation of A from P is $12^{\circ} 13'$, calculate the height of A above the ground.

(13) A balloon is observed by two persons on a road 200 yards apart to have elevations of 62° and 43° when it is directly over the road and between them. Half an hour later, the elevations are 41° and 28° . Find the speed of the balloon in miles per hour, assuming the balloon to fly horizontally.

(14) A and B are two points on one bank of a stream and C a point on the other, so that AB measures 200 feet, $\angle CAB = 47^{\circ} 24'$, $\angle CBA = 62^{\circ} 18'$. Find the width of the stream.

(15) PQR is an isosceles triangle on the ground in which the base QR = 120 feet, and $\angle QPR = 124^{\circ}$. If P is the point vertically below an aeroplane, and the elevation of the aeroplane from Q be $72^{\circ} 18'$, calculate its height above the ground.

(16) An observer standing on the bank of a tank 10 feet above the water level notes the angle of elevation of a tree top and the angle of depression of its reflection in the water to be $30^{\circ} 41'$ and $33^{\circ} 47'$. How high is the tree above the water level?

(17) A wall 15 feet high and 100 feet long points 10° east of north. Calculate the area of the shadow cast by it when the sun is 22° north of east at an altitude 45° .

(18) A rectangular frame 5 feet by 3 feet is placed vertically on the ground facing east. Calculate the area of the shadow cast when the sun is $20^{\circ} 31'$ south of west at an altitude $47^{\circ} 12'$.

(19) A target in the form of an isosceles triangle is placed vertically, its base resting on the ground, facing east. When the sun is E. 12° N. at an elevation 60° , find the area of the shadow it casts, given that the base and altitude of the triangle measure 4 feet and 5 feet respectively.

(20) From a point on a straight road, the angular elevation of the top of a spire is found to be 37° . Fifty yards further up, I am at the point of the road nearest the tree, and the elevation of the spire from this point is 42° . Calculate the height of the spire and its distance from the road.

(21) A spire is observed to be due south at an elevation $15^{\circ} 12'$ from a place A on the ground. After walking 100 feet due west,

the bearing of the spire is S. 72° E. Calculate the height of the spire and its distance from A.

(22) From two places A and B on the ground, a distant flagstaff on a building bears 20° north of east and 10° east of north. If the elevation of the top be 35° from A, find the height of the flag from the ground given that AB measures 200 feet and points E. 60° S.

(23) The elevation of a mountain summit as seen at a place P due east of it is $28^\circ 51'$, and at a place due south of P and two miles from P it is $19^\circ 12'$. Find the height of the summit.

(24) A, B, are two points on opposite banks of a river, one directly opposite the other. From the top of a tower 200 feet high and in line with AB and at some distance from the bank, the angles of depression of A and B are found to be 18° and 27° respectively. Find the breadth of the river.

(25) To find the distance between two inaccessible objects P and Q, a base line AB is taken coplanar with PQ. Given that $AB = 500$ yards, $PAB = 75^\circ 12'$, $QAB = 51^\circ 13'$, $PBA = 43^\circ 17'$, $QBA = 65^\circ 12'$, find the distance PQ by calculation if P and Q are on the same side of AB.

(26) PQ is a level base line 2442 feet long measured along a straight road running north east and south west. The bearing of a distant spire S from P is N. 60° W. and from Q is N. 10° E. L is a lake on the other side of the road, the bearings of which from P and Q are respectively S. 20° E. and N. 65° E. Calculate how far the spire is from the lake.

(27) From two points B, A on a level road 1000 feet apart, the bearings of P are 30° east of north and 25° east of north; and those of another object Q on the same side of the road are 70° east of north and 61° east of north respectively. If B bears 80° north of west from A, calculate the distance PQ.

(28) A man travels from A to B 5 miles in a direction N. 56° E, from B to C 7 miles in a direction N. 30° E, and from C to D 5 miles in a direction N. 25° W. Find by calculation the distance and bearing of D from A.

(29) A flagstaff 15 feet high placed on the top of a cliff subtends an angle whose tangent is 0.15 at a point on the ground distant 50 feet from the foot of the cliff. Find the height of the cliff.

(30) The upper one-third of a ship's mast subtends an angle whose sine is .06 at a point on the deck 10 yards from it. Calculate the height of the mast from the level of the deck.

(31) A statue 15 feet high standing on the top of a column 50 feet high subtends the same angle at a point on the ground as a man 6 feet high standing at the foot of the column. Find the distance of the point from the foot of the column.

(32) AB is a vertical pole 37 feet high on one bank of a river. C and D are two points of the pole distant 6 and 18 feet respectively from A. P is the point on the other bank directly opposite to A. It is found that AC and BD subtend equal angles at P. Calculate the breadth of the river.

(33) A tower h feet high is surmounted by a spire of height h' feet. Both the tower and spire subtend equal angles to an observer standing on the ground x feet from the base of the tower. Taking the height of the observer's eyes above the ground level to be ' a ' feet find a formula connecting h , h' , a and x .

(34) A flagstaff on the top of a column subtends the same angle θ at two places P and Q on the ground in level with the foot of the column and in line with it. The elevation of the top of the column from P is ϕ . PQ measures d units. Prove that the heights of the column and flagstaff are respectively $d \sin \phi \cos (\theta + \phi) \times \sec (2\phi + \theta)$ and $d \sin \theta \times \sec (2\phi + \theta)$.

(35) P is a point of a straight road at which two trees A and B on the same side of the road subtend the greatest angle θ . Walking d yards further up, I am in line with AB. Prove that the distance between the trees is $d \sin \theta \sin \phi \sec \frac{\theta + \phi}{2} \sec \frac{\theta - \phi}{2}$ where ϕ is the angle between AB and the road.

(36) Up a hill sloping at 30° to the horizon runs a road, the inclination of which to the horizontal plane is 23° . If the hill slopes south, find the direction of the road.

(37) P and Q are two places on the ground 1345 feet apart. From P, a tower is observed to be $22^\circ 18'$ east of north, and the angle of elevation of its top is $16^\circ 10'$. From Q, the tower is $12^\circ 15'$ west of north, and P $16^\circ 48'$ south of west. Calculate the height of the tower.

(38) From each of three consecutive telegraph posts A, B, C on a straight road, the angular elevations of a distant tree top are $30^{\circ} 22'$, $27^{\circ} 19'$ and $24^{\circ} 33'$. Calculate the height of the tree given that $AB = BC = 72$ yards.

(39) The plane side of a hill makes an angle of 30° with the horizon, and has its lines of greatest slope running due north from the base. Towards what point of the compass does a man walk up the hill if in walking to the top, he travels a distance equal to double the distance along the line of greatest slope? [M.U., 1921]

(40) A balloon moving in a straight line passes vertically above two points A and B on a horizontal plane 1000 feet apart. When above A, it has an altitude of 50° as seen from B, and when above B, it has an altitude of 40° as seen from A. Find the distance from A of the point at which it will reach the plane. [M.U., 1915]

CHAPTER XII

SUB-MULTIPLE ANGLES

Art. 107. Given $\cos A$, to find the trigonometrical ratios of $\frac{A}{2}$.

$$2 \sin^2 \frac{A}{2} = 1 - \cos A ; \sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}$$

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}} \quad \dots \quad \dots \quad \dots \quad (1)$$

Similarly $2 \cos^2 \frac{A}{2} = 1 + \cos A$

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \quad \dots \quad \dots \quad \dots \quad (2)$$

Dividing one by the other,

$$\tan \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \quad \dots \quad (3)$$

It is clear from the above that each value of $\cos A$ gives rise to two different values of each of the ratios of $\frac{A}{2}$.

Art. 108. Given the value of $\cos A$, A is not uniquely fixed. $A = 2\pi n \pm a$, where a is one value of A having the given cosine, n being 0 or any integer positive or negative. Hence, the value of $\sin \frac{A}{2}$ derived from the value of $\cos A$ must give the value of $\sin [\frac{1}{2}(2n\pi \pm a)]$, i.e., of $\sin(n\pi \pm \frac{a}{2})$. But $\sin(n\pi \pm \frac{a}{2}) = \pm \sin \frac{a}{2}$.

$$\text{Similarly } \cos \left(n\pi \pm \frac{a}{2} \right) = \pm \cos \frac{a}{2}.$$

$$\tan \left(n\pi \pm \frac{a}{2} \right) = \pm \tan \frac{a}{2}.$$

Hence for a value of $\cos A$ there are two equal and opposite values of each of the ratios of $\frac{A}{2}$. This ambiguity can be cleared only if we know the value of A or suitable limits to the value of A .

Ex. 1. Evaluate the ratios of $22\frac{1}{2}^\circ$.

As $\cos 45^\circ$ is known to be equal to $\frac{1}{\sqrt{2}}$, putting $A = 45^\circ$, $\frac{A}{2} = 22\frac{1}{2}^\circ$. All the ratios of $\frac{A}{2}$ are therefore positive since $\frac{A}{2}$ is positive and acute.

$$\text{Hence, } \cos 22\frac{1}{2}^\circ = + \sqrt{\frac{1 + \cos 45^\circ}{2}} = + \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$$

$$\sin 22\frac{1}{2}^\circ = + \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}$$

$$\tan 22\frac{1}{2}^\circ = + \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1}} = (\sqrt{2} - 1).$$

Ex. 2. Find the sine and cosine of $202\frac{1}{2}^\circ$

$$\text{Evidently, } 202\frac{1}{2}^\circ = \frac{1}{2} \times 405^\circ$$

$$\cos 405^\circ = \cos (360^\circ + 45^\circ) = \frac{1}{\sqrt{2}}$$

Also, $202\frac{1}{2}^\circ$ being an angle having its bounding line in the third quadrant, its tangent and cotangent alone are positive.

$$\text{Hence, } \sin 202\frac{1}{2}^\circ = -\sqrt{\frac{1 - \cos 405^\circ}{2}} = -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$\cos 202\frac{1}{2}^\circ = -\sqrt{\frac{1 + \cos 405^\circ}{2}} = -\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$\tan 202\frac{1}{2}^\circ = +\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

Ex. 3. Given the value of $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$, evaluate the ratios of 162° .

$$162^\circ = \frac{1}{2} \times 324^\circ \text{ and } \cos 324^\circ = \cos 36^\circ.$$

Art. 109. Given $\sin A$, find $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$

$$\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1$$

$$2 \sin \frac{A}{2} \cdot \cos \frac{A}{2} = \sin A$$

Adding and subtracting, we have

$$\left(\cos \frac{A}{2} + \sin \frac{A}{2} \right)^2 = 1 + \sin A$$

$$\text{and } \left(\cos \frac{A}{2} - \sin \frac{A}{2} \right)^2 = 1 - \sin A$$

$$\cos \frac{A}{2} + \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \quad \dots \quad \dots \quad (1)$$

$$\cos \frac{A}{2} - \sin \frac{A}{2} = \pm \sqrt{1 - \sin A} \quad \dots \quad \dots \quad (2)$$

Adding and subtracting equations (1) and (2),

$$\cos \frac{A}{2} = \frac{1}{2} \left\{ \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A} \right\} \dots \dots (3)$$

$$\sin \frac{A}{2} = \frac{1}{2} \left\{ \pm \sqrt{1 + \sin A} \mp \sqrt{1 - \sin A} \right\} \dots \dots (4)$$

A single value of $\sin A$ gives four values of $\sin \frac{A}{2}$ and four values of $\cos \frac{A}{2}$, since in the equation (3) or (4), the positive or negative sign of the second term in the bracket can be associated with either of the signs in the first term.

This ambiguity is cleared only when we know the value of $\frac{A}{2}$ or the limits to the value of $\frac{A}{2}$ defining the position of the bounding line of the angle in any one of the four quadrants.

Art. 110. The general expression for all angles having the same sine as A is $n\pi + (-1)^n A$, where n is zero or any integer. So, the value of $\sin \frac{A}{2}$ derived from a given value of $\sin A$ must give the values of $\sin \frac{1}{2} \left\{ n\pi + (-1)^n A \right\}$.

When n is an even integer $2m$, this expression becomes $\sin \left(m\pi + \frac{A}{2} \right)$ which equals $\sin \frac{A}{2}$ or $-\sin \frac{A}{2}$ according as m is even or odd. Again if n is an odd integer $(2m+1)$, the expression becomes

$$\sin \left(m\pi + \frac{\pi}{2} - \frac{A}{2} \right) \text{ which simplifies to } \pm \sin \left(\frac{\pi}{2} - \frac{A}{2} \right)$$

$= \pm \cos \frac{A}{2}$, according as m is even or odd. Thus, for a given value of $\sin A$, there are four values of $\sin \frac{A}{2}$. A similar working shows the possibility of four values of $\cos \frac{A}{2}$ also.

Art. 111. The signs of the expressions $\cos \frac{A}{2} + \sin \frac{A}{2}$ and $\cos \frac{A}{2} - \sin \frac{A}{2}$ for different values of A .

The determination of the values of $\cos \frac{A}{2}$ and $\sin \frac{A}{2}$ from a given value of $\sin A$ depends on the signs of $\cos \frac{A}{2} + \sin \frac{A}{2}$ and of $\cos \frac{A}{2} - \sin \frac{A}{2}$ for a given value of $\frac{A}{2}$.

$$\begin{aligned} \cos \frac{A}{2} + \sin \frac{A}{2} &= \sqrt{2} \left\{ \frac{1}{\sqrt{2}} \cos \frac{A}{2} + \frac{1}{\sqrt{2}} \sin \frac{A}{2} \right\} \\ &= \sqrt{2} \left\{ \sin \frac{\pi}{4} \cos \frac{A}{2} + \cos \frac{\pi}{4} \sin \frac{A}{2} \right\} \\ &= \sqrt{2} \sin \left\{ \frac{\pi}{4} + \frac{A}{2} \right\} \end{aligned}$$

Similarly,

$$\begin{aligned} \cos \frac{A}{2} - \sin \frac{A}{2} &= \sqrt{2} \left\{ \frac{1}{\sqrt{2}} \cos \frac{A}{2} - \frac{1}{\sqrt{2}} \sin \frac{A}{2} \right\} \\ &= \sqrt{2} \left\{ \cos \frac{\pi}{4} \cos \frac{A}{2} - \sin \frac{\pi}{4} \sin \frac{A}{2} \right\} \\ &= \sqrt{2} \cos \left\{ \frac{\pi}{4} + \frac{A}{2} \right\} \end{aligned}$$

Hence, the expressions $\cos \frac{A}{2} + \sin \frac{A}{2}$ and $\cos \frac{A}{2} - \sin \frac{A}{2}$ have the same signs as $\sin \left(\frac{A}{2} + \frac{\pi}{4} \right)$ and $\cos \left(\frac{A}{2} + \frac{\pi}{4} \right)$ respectively.

$\sin \theta$ is positive only when θ lies between $2n\pi$ and $(2n+1)\pi$; also $\cos \theta$ is positive only for values of θ between $\left(2n\pi - \frac{\pi}{2}\right)$ and $\left(2n\pi + \frac{\pi}{2}\right)$.

Hence, $\cos \frac{A}{2} + \sin \frac{A}{2}$ is positive for values of A as make $\frac{\pi}{4} + \frac{A}{2}$ lie between $2n\pi$ and $(2n+1)\pi$, i.e., for values of A as make $\frac{A}{2}$ lie between $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$. Also $\cos \frac{A}{2} - \sin \frac{A}{2}$ is positive for values of $\frac{A}{2}$ lying between $2n\pi - \frac{3\pi}{4}$ and $2n\pi + \frac{\pi}{4}$.

Ex. 1. If $270^\circ < A < 315^\circ$, find the signs of $\sqrt{1 + \sin A}$ and $\sqrt{1 - \sin A}$ in the formulae for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in terms of $\sin A$. Hence, write down the values of $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$, given $\sin A$.

Since A lies between 270° and 315° , $\frac{A}{2}$ lies between 135° and $157\frac{1}{2}^\circ$. So, $\left(\frac{A}{2} + 45^\circ\right)$ lies between 180° and $202\frac{1}{2}^\circ$.

$\sin \left(\frac{A}{2} + 45^\circ \right)$ and $\cos \left(\frac{A}{2} + 45^\circ \right)$ are both negative.

$$\cos \frac{A}{2} + \sin \frac{A}{2} = -\sqrt{1 + \sin A}$$

$$\cos \frac{A}{2} - \sin \frac{A}{2} = -\sqrt{1 - \sin A}$$

$$\cos \frac{A}{2} = -\frac{1}{2} \left\{ \sqrt{1 + \sin A} + \sqrt{1 - \sin A} \right\}$$

$$\sin \frac{A}{2} = -\frac{1}{2} \left\{ \sqrt{1 + \sin A} - \sqrt{1 - \sin A} \right\}$$

Ex. 2. If A is an angle lying between 180° and 270° find the value of $\sin \frac{A}{2}$ in terms of $\sin A$.

$\frac{A}{2}$ lies between 90° and 135° ; and $\left(\frac{A}{2} + 45^\circ \right)$ lies between 135° and 180° . Its sine is positive and cosine negative. Hence $\cos \frac{A}{2} + \sin \frac{A}{2}$ is positive, while $\cos \frac{A}{2} - \sin \frac{A}{2}$ is negative.

$$\cos \frac{A}{2} + \sin \frac{A}{2} = +\sqrt{1 + \sin A}$$

$$\cos \frac{A}{2} - \sin \frac{A}{2} = -\sqrt{1 - \sin A}$$

$$\sin \frac{A}{2} = \frac{1}{2} \left\{ \sqrt{1 + \sin A} + \sqrt{1 - \sin A} \right\}$$

Ex. 3. Find the limits to the value of A , such that $\sin \frac{A}{2}$ may have each of the four values in terms of $\sin A$ in turn.

$$\cos \frac{A}{2} + \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \dots\dots\dots (1)$$

$$\cos \frac{A}{2} - \sin \frac{A}{2} = \pm \sqrt{1 - \sin A} \dots\dots\dots (2)$$

$$2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \mp \sqrt{1 - \sin A} \dots\dots\dots (3)$$

(i) If the positive sign be taken for both the terms on the right hand side of the last equality, evidently

$$\cos \frac{A}{2} + \sin \frac{A}{2} = + \sqrt{1 + \sin A}$$

$$\text{and } \cos \frac{A}{2} - \sin \frac{A}{2} = - \sqrt{1 - \sin A}.$$

$\sin \left(\frac{A}{2} + 45^\circ \right)$ is positive, while $\cos \left(\frac{A}{2} + 45^\circ \right)$ is negative.

$\therefore \frac{A}{2} + 45^\circ$ lies between $n \cdot 360^\circ + 90^\circ$ and $n \cdot 360^\circ + 180^\circ$,

i.e., $\frac{A}{2}$ lies between $n \cdot 360^\circ + 45^\circ$ and $n \cdot 360^\circ + 135^\circ$.

So, A lies between $n \cdot 720^\circ + 90^\circ$ and $n \cdot 720^\circ + 270^\circ$.

(ii) Taking the positive sign for the first term and the negative sign for the second term in the right hand of equality (3), we get

$$\cos \frac{A}{2} + \sin \frac{A}{2} = + \sqrt{1 + \sin A}$$

$$\text{and } \cos \frac{A}{2} - \sin \frac{A}{2} = + \sqrt{1 - \sin A}.$$

$\left(\frac{A}{2} + 45^\circ \right)$ has its sine and cosine both positive. Hence,

$\left(\frac{A}{2} + 45^\circ\right)$ lies between $n \cdot 360^\circ$ and $n \cdot 360^\circ + 90^\circ$. So, A lies between $n \cdot 720^\circ - 90^\circ$ and $n \cdot 720^\circ + 90^\circ$.

(iii) If $2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$,

$$\cos \frac{A}{2} + \sin \frac{A}{2} = -\sqrt{1 + \sin A}$$

$$\text{and } \cos \frac{A}{2} - \sin \frac{A}{2} = -\sqrt{1 - \sin A}$$

A lies between $4n\pi + \frac{3\pi}{2}$ and $4n\pi + \frac{5\pi}{2}$.

(iv) Lastly, if $2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$

$$\cos \frac{A}{2} + \sin \frac{A}{2} = -\sqrt{1 + \sin A}, \text{ and}$$

$$\cos \frac{A}{2} - \sin \frac{A}{2} = +\sqrt{1 - \sin A}$$

A has some value between $4n\pi + \frac{5\pi}{2}$

and $4n\pi + \frac{7\pi}{2}$.

Art. 112. $\sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$. Hence $\tan \frac{A}{2}$ is a

root of the equation $\sin A \cdot x^2 - 2x + \sin A = 0$.

$$\tan \frac{A}{2} = \frac{1 \pm \sqrt{1 - \sin^2 A}}{\sin A}$$

There are two values of $\tan \frac{A}{2}$ for each value of $\sin A$.

Art. 113. Given $\tan A$, find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$ and $\tan \frac{A}{2}$.

$$\text{Since } \tan A = 2 \tan \frac{A}{2} \left(1 - \tan^2 \frac{A}{2} \right)$$

$$\tan A \cdot \tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} - \tan A = 0$$

This being a quadratic equation in $\tan \frac{A}{2}$,

$$\tan \frac{A}{2} = -1 \pm \sqrt{1 + \frac{\tan^2 A}{\tan A}}$$

For a given value of $\tan A$, A has an infinite number of values, all included in the formula, $n\pi + A$. So, the value of $\tan \frac{A}{2}$ in terms of $\tan A$ must give the values of $\tan \left(\frac{n\pi}{2} + \frac{A}{2} \right)$ for all integral values of n . If n is an even integer equal to $2m$, this becomes $\tan \left(m\pi + \frac{A}{2} \right) = \tan \frac{A}{2}$. When n is an odd integer $(2m+1)$, this becomes $\tan \left(m\pi + \frac{\pi}{2} + \frac{A}{2} \right) = -\cot \frac{A}{2}$. Hence, for a given value of $\tan A$, there are two values of $\tan \frac{A}{2}$. This ambiguity is cleared if we know the limits to the value of A .

To find the value of $\cos \frac{A}{2}$ from that of $\tan A$, we have $2 \cos^2 \frac{A}{2} = 1 + \cos A = 1 + \frac{1}{\sec A} = 1 \pm \frac{1}{\sqrt{1 + \tan^2 A}}$

$$\cos \frac{A}{2} = \pm \left\{ \frac{1}{2} \pm \frac{1}{2\sqrt{1 + \tan^2 A}} \right\}^{1/2} \dots\dots\dots (1)$$

$$\text{Similarly, } 2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{1}{\sec A}$$

$$\sin \frac{A}{2} = \pm \left\{ \frac{1}{2} \pm \frac{1}{2\sqrt{1 + \tan^2 A}} \right\}^{1/2} \dots\dots\dots (2)$$

Thus, we see that for one value of $\tan A$, there are four different values for $\sin \frac{A}{2}$, and four different values for $\cos \frac{A}{2}$, while there are only two different values for $\tan \frac{A}{2}$. The proof is on the same lines.

Ex. 1. Prove that (i) $\tan 7\frac{1}{2}^\circ = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$
(ii) $\cot 142\frac{1}{2}^\circ = \sqrt{2} + \sqrt{3} - \sqrt{4} - \sqrt{6}$.

$$\begin{aligned} \text{(i) } \tan A &= \frac{\sin A}{\cos A} \\ &= \frac{2 \sin^2 A}{2 \sin A \cos A} \quad \begin{array}{l} \text{(multiply numerator} \\ \text{and denominator by} \\ 2 \sin A) \end{array} \\ &= \frac{1 - \cos 2A}{\sin 2A} \\ \tan 7\frac{1}{2}^\circ &= \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \frac{1 - \cos (60^\circ - 45^\circ)}{\sin (60^\circ - 45^\circ)} \\ &= \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1} \\ &= \frac{(2\sqrt{2} - \sqrt{3} - 1)(\sqrt{3} + 1)}{2} \\ &= \sqrt{6} - \sqrt{3} + \sqrt{2} - 2. \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } \cot A &= \frac{\cos A}{\sin A} = \frac{2 \cos^2 A}{2 \sin A \cos A} \quad \begin{array}{l} \text{(multiplying nume-} \\ \text{rator and denomina-} \\ \text{tor by } 2 \cos A) \end{array} \\
 &= \frac{1 + \cos 2A}{\sin 2A} \\
 \cot 142\frac{1}{2}^\circ &= \frac{1 + \cos 285^\circ}{\sin 285^\circ} \\
 &= \frac{1 + \cos 75^\circ}{-\sin 75^\circ} \\
 &= \frac{2\sqrt{2} + \sqrt{3} - 1}{-(\sqrt{3} + 1)} \\
 &= \frac{(2\sqrt{2} + \sqrt{3} - 1)(\sqrt{3} - 1)}{-2} \\
 &= \sqrt{2} + \sqrt{3} - \sqrt{4} - \sqrt{6}.
 \end{aligned}$$

Aliter.

(iii) $\cot 142\frac{1}{2}^\circ = \cot (90^\circ + 52\frac{1}{2}^\circ) = -\tan 52\frac{1}{2}^\circ$.
 Calculating the value of $\tan 52\frac{1}{2}^\circ$, we can write down that of $\cot 142\frac{1}{2}^\circ$.

Ex. 2. Given that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, find the sine and cosine of 9° .

$\cos 9^\circ + \sin 9^\circ = \sqrt{1 + \sin 18^\circ}$, since $\sin (9^\circ + 45^\circ)$ is positive.

$\cos 9^\circ - \sin 9^\circ = +\sqrt{1 - \sin 18^\circ}$, since $\cos (9^\circ + 45^\circ)$ is also positive.

$$\begin{aligned}
 \text{Adding, } \cos 9^\circ &= \frac{1}{2} \left\{ \sqrt{1 + \sin 18^\circ} + \sqrt{1 - \sin 18^\circ} \right\} \\
 &= \frac{1}{2} \left\{ \frac{\sqrt{3} + \sqrt{5}}{2} + \frac{\sqrt{5} - \sqrt{3}}{2} \right\} \\
 &= \frac{1}{4} \left\{ \sqrt{3} + \sqrt{5} + \sqrt{5} - \sqrt{3} \right\};
 \end{aligned}$$

subtracting, $\sin 9^\circ = \frac{1}{4} \left\{ \sqrt{3} + \sqrt{5} - \sqrt{5 - \sqrt{5}} \right\}.$

Substituting the value of $\sqrt{5}$ to as many decimal places as are necessary to secure the required degree of accuracy and evaluating the expression within the bracket, we get the values of $\sin 9^\circ$ and $\cos 9^\circ$.

Art. 114. Ratios of $\frac{A}{3}$ in terms of $\sin A$, $\cos A$ or $\tan A$:—

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}.$$

This can be rewritten in the form

$$4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3} - \cos A = 0.$$

Hence $\cos \frac{A}{3}$ is a root of the cubic equation $4x^3 - 3x - \cos A = 0$, from which we get **three** values of $\cos \frac{A}{3}$ for a given value of $\cos A$.

$$\text{Again } \sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}.$$

Hence $\sin \frac{A}{3}$ is a root of the cubic equation

$4x^3 - 3x + \sin A = 0$, giving **three** values of $\sin \frac{A}{3}$ for one value of $\sin A$.

Also, since $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$, $\tan A$ is a root of the cubic equation.

$$3 \tan A - \tan^3 A - \tan 3A + 3 \tan 3A \tan^2 A = 0$$

$$\text{i.e., } \tan^3 A - 3 \tan A = 0. \quad \tan^2 A - 3 \tan A + \tan 3A = 0.$$

$$\text{i.e., } x^3 - 3 \tan 3A \cdot x^2 - 3x + \tan 3A = 0, \quad \text{putting } x \text{ for } \tan A.$$

Using this result, we can solve the cubic equation $x^3 - 3t x^2 - 3x + t = 0$, where t is any number whatever.

Ex. 1. Given $\sin A$, find all the values of $\sin \frac{A}{3}$ and explain all the three solutions.

Ex. 2. Solve the equation $x^3 - 3.75x^2 - 3x + 1.25 = 0$. From the form of the equation, we infer that the question can be solved by putting $\tan A$ for x . The equation then becomes

$$\tan^3 A - 3.75 \tan^2 A - 3 \tan A + 1.25 = 0.$$

$$\text{i.e., } 3 \tan A - \tan^3 A = 1.25 - 3.75 \tan^2 A.$$

$$\text{i.e., } \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} = 1.25$$

$$\text{i.e., } \tan 3A = 1.25.$$

Referring to Tables of Natural tangents, $\tan 51^\circ 20' = 1.25$. Hence, $51^\circ 25'$ is *one* value of $3A$ satisfying the equation. The general value of $3A$ is given by the formula, $3A = n \cdot 180^\circ + 51^\circ 20'$. n being an integer.

$$\therefore A = n \cdot 60^\circ + 17^\circ 7' \quad (\text{approximately})$$

Giving successive integral values to n , and writing down the corresponding values of A , and of $\tan A$, we have

$$n = 0, \quad A = 17^\circ 7'; \quad x = \tan A = \tan 17^\circ 7' = 0.3079$$

$$n = 1, \quad A = 77^\circ 7'; \quad x = \tan 77^\circ 7' = 4.372$$

$$n = 2, \quad A = 137^\circ 7'; \quad x = \tan 137^\circ 7' = -\tan 42^\circ 53' \\ = -0.9287.$$

For the other values of n , the same three values of x repeat in the same order. There is no necessity to give to n other integral values and get the corresponding values of x .

EXAMPLES XII

(1) If $\cos A = \frac{4}{5}$ find all the ratios of $\frac{A}{2}$, if $\frac{A}{2}$ is an obtuse angle.

(2) Given that $\cos \theta = \frac{1}{2}$, calculate the value of $\cot \frac{\theta}{2}$, if θ lies between 750° and 800° .

(3) If A is angle between 360° and 270° , and $\sin A$ is given to be equal to $-\frac{1}{25}$, find the values of $\sin 3A$ and $\tan \frac{A}{2}$.

(4) A lies between 90° and 180° , and $\sin A = .7$. Calculate the ratios of $\frac{A}{2}$.

(5) Using the values of $\sin A$ and $\cos A$ in terms of $\tan \frac{A}{2}$, solve for $\tan \frac{A}{2}$ from the equation, $3 \cos A + 4 \sin A = 2$. Hence write down the general value of A .

(7) If cosecant $A = 2.25$, find the value of $\tan \frac{A}{2}$ and explain the ambiguity. Find the limits to the values of A for positive values of $\tan \frac{A}{2}$.

(8) Deducing the value of $\tan 22\frac{1}{2}^\circ$ from $\tan 45^\circ$ either by the formula or by a figure suitably drawn, calculate the value of $\tan 11\frac{1}{4}^\circ$ correct to two places of decimals.

Show that,

$$(9) \sin^2 A + \sin^2 \left(\frac{\pi}{3} + A \right) + \sin^2 \left(\frac{\pi}{3} - A \right) = \frac{3}{2}.$$

$$(10) \cos^2 A + \cos^2 \left(\frac{\pi}{3} + A \right) + \cos^2 \left(\frac{\pi}{3} - A \right) = \frac{3}{2}.$$

(11) Evaluate $\sin 9^\circ$ and $\cos 9^\circ$ to three places of decimals.

(12) If $\cos C = \frac{\cos A + \cos B}{1 + \cos A \cos B}$ show that one value of $\tan \frac{C}{2}$ is equal to $\tan \frac{A}{2} \times \tan \frac{B}{2}$.

(13) Write down the values of $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in terms of $\sin A$ in each of the following cases :

- (i) $\frac{A}{2} = 71^\circ$. (ii) $\frac{A}{2} = 117^\circ$. (iii) $\frac{A}{2} = 205^\circ$. (iv) $\frac{A}{2} = 285^\circ$.
 (v) $A = -40^\circ$. (vi) $A = -51^\circ$. (vii) $A = -417^\circ$.
 (viii) $A = -1332^\circ$.

(14) Find the limits to the values of A so that

$$2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}.$$

(15) Show that $\cos \frac{\pi}{4} + \cos \frac{3\pi}{4} + \cos \frac{5\pi}{4} + \cos \frac{7\pi}{4} = 0$; Hence or otherwise evaluate $\cos^4 22\frac{1}{2}^\circ + \cos^4 67\frac{1}{2}^\circ + \cos^4 112\frac{1}{2}^\circ + \cos^4 157\frac{1}{2}^\circ$ and also $\sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8}$.

(16) If $13a = \pi$, prove that

$$[\cos a. \cos 2a. \cos 3a. \cos 4a. \cos 5a. \cos 6a = \frac{1}{64}]$$

$[\cos a. \cos 2a. \cos 3a. \cos 4a. \cos 5a. \cos 6a \times 2^6 \sin a. \sin 2a. \sin 3a. \sin 4a. \sin 5a. \sin 6a = \sin 2a. \sin 4a. \sin 6a. \sin 8a. \sin 10a. \sin 12a = \sin 2a. \sin 4a. \sin 6a. \sin 5a. \sin 3a. \sin a$ (since $13a = \pi$).

Removing the common factor $\sin a. \sin 2a. \sin 3a. \sin 4a. \sin 5a. \sin 6a$, we get the required result.]

CHAPTER XIII

GRAPHS OF TRIGONOMETRIC FUNCTIONS

Art. 115. The graph of the function $\sin x$ is the curve passing through all those points, whose x and y co-ordinates satisfy the equation $y = \sin x$, so that the ordinate of any point of the curve is the sine of the angle equal to the corresponding abscissa. The graph of any other trigonometrical function— $\cos x$, $\tan x$, $\operatorname{cosec} x$, or $\cot x$, etc., is similarly drawn. The abscissa of any point of the curve is the measure of the angle x either in radians or in degrees, and the corresponding ordinate the value of the trigonometrical function of that angle.

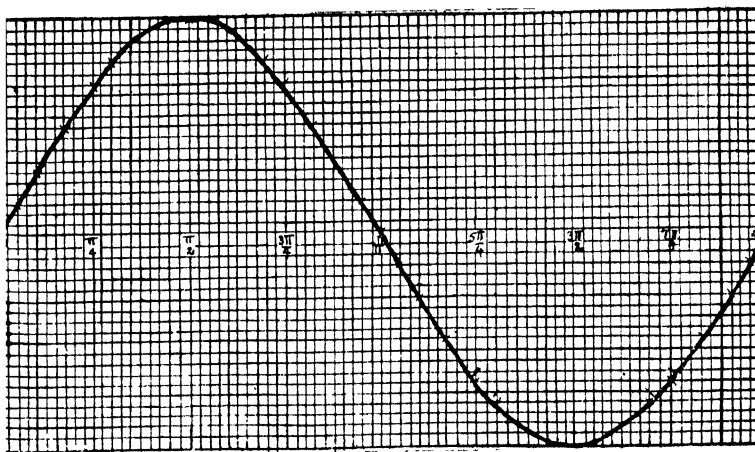


Fig. 71.

Values of $\sin x$, $\cos x$, $\tan x$ are given in the tables of Natural sines, Natural cosines and Natural tangents, for values of x from 0° to 90° ; and those of the same ratios for any other value of x can be deduced by using the formulae

of the earlier chapters so that it is easy to set down a number of values of x at sufficiently close intervals. The corresponding values of y are noted down against these. One set of values of x and y thus noted gives one point of the curve. A good number of such points smoothly joined by a free-hand curve is the graph of the function $\sin x$, $\cos x$, etc., or the graph of the equation $y = \sin x$, $y = \cos x$, etc.

Art. 116. Graph of $\sin x$.—Fig. 71 is the graph of $\sin x$ drawn to a scale of $1'' = \frac{\pi}{4}$ radian along the x axis, and $2'' = 1$ unit along the y axis, from points plotted from the following Table of values for values of x between 0 and 2π .

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$y = \sin x$	0	0.5	.707	.866	.966	1	.966	.866	.707	.50

x	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	0	-0.50	-.707	-.866	-.966	-1	-.966	-.866	-.707	-.50	0

It is clear from the graph that $\sin x$ is a *continuous* function of x and that there is no abrupt change in the value of the function. For every value of x there is a finite value of $\sin x$, and there is no value of y outside the limit -1 to $+1$, these being the minimum and maximum values, which correspond to $x = \frac{3\pi}{2}$ and $x = \frac{\pi}{2}$ respectively.

That the graph can be extended indefinitely both to the left and to the right and that the same wave from 0 to 2π repeats itself is also clear from the fact that $\sin x$ is a periodic function of x , its period being 2π radians. It will be noted that the function is positive and increasing for values of x between 0 and $\frac{\pi}{2}$, positive and decreasing for values of x between $\frac{\pi}{2}$ and π , negative and increasing (numerically) from $x = \pi$ to $x = \frac{3\pi}{2}$ and negative and decreasing (numerically) from $x = \frac{3\pi}{2}$ to $x = 2\pi$. The graph may be used to read off the sine of any angle between 0 and 2π , and to write down the angles between 0 and 2π having a given number (less than unity numerically) for its sine, viz., to solve the equation $\sin x = a$. ($|a| < 1$).

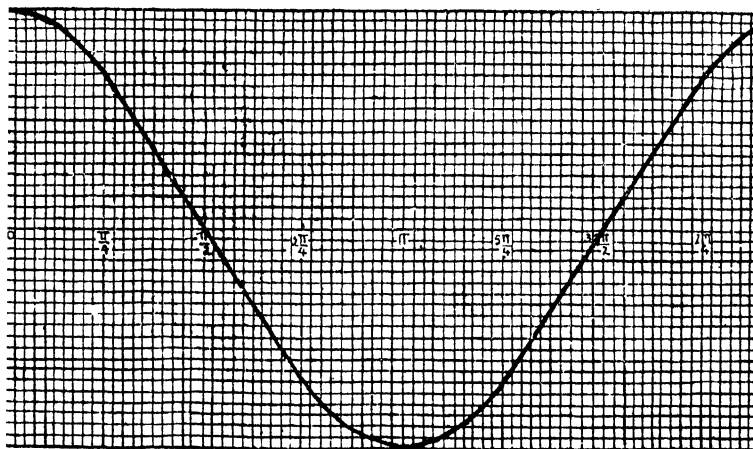


Fig. 72

Art. 117. Fig. 72 is the graph of $y = \cos x$ drawn similarly. It is also found to be a *continuous* wavy curve, the wave length being 2π , and the height being unity. But, while the sine graph passes through the origin, the cosine graph passes through the point $(0, 1)$ and meets the x axis at the points $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$ etc. Again it is seen from the graph that the cosine is positive and decreasing from $x = 0$ to $x = \frac{\pi}{2}$, negative and increasing (numerically) from $x = \frac{\pi}{2}$ to $x = \pi$, reaching the maximum negative value—1, for $x = \pi$, diminishes (numerically) from $x = \pi$ to $x = \frac{3\pi}{2}$, becomes positive and increases from $x = \frac{3\pi}{2}$ to $x = 2\pi$, reaching the maximum positive value at $x = 2\pi$.

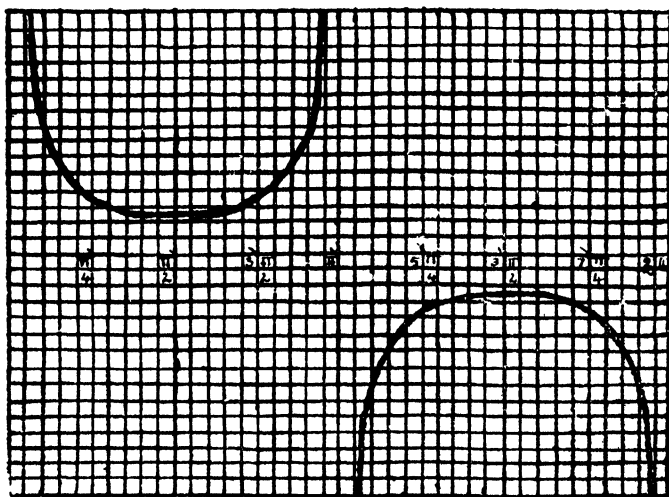


Fig. 73

Art. 118. Figure 73 is the graph of $y = \operatorname{cosec} x$. The student will draw the graph of $\sec x$. It may be observed that the graphs are not continuous, the discontinuities in the graphs occurring at $x = 0, \pi, 2\pi$, etc., in the former and at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$...etc., in the latter. These graphs also repeat at intervals of 2π radians. There is no part of the graph between $y = -1$ and $y = +1$ and the graph extends to infinity in both directions. The lines $x = 0, x = \pi$, etc., are the asymptotes of the cosecant graph and $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ etc., are the asymptotes of the secant curve.

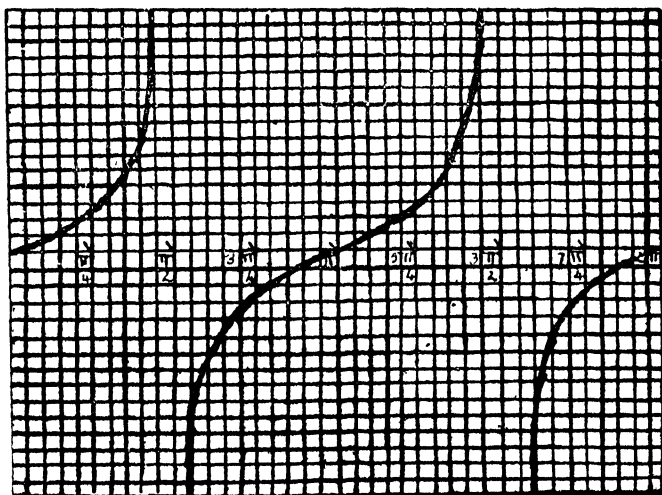


Fig. 74

Figures 74 and 75 are the graphs of $y = \tan x$ and $y = \cot x$ respectively. They are not continuous. When $x = 0$, $\tan x = 0$, and it increases till as x tends to $\frac{\pi}{2}$,

$\tan x$ becomes infinitely large. As x passes the value $\frac{\pi}{2}$, $\tan x$ changes sign becoming negative and decreases numerically till when $x = \pi$, $\tan x = 0$. Now, $\tan x$ becomes positive and increasing till as x tends to $\frac{3\pi}{2}$, $\tan x$ becomes infinitely great. Again it changes sign and

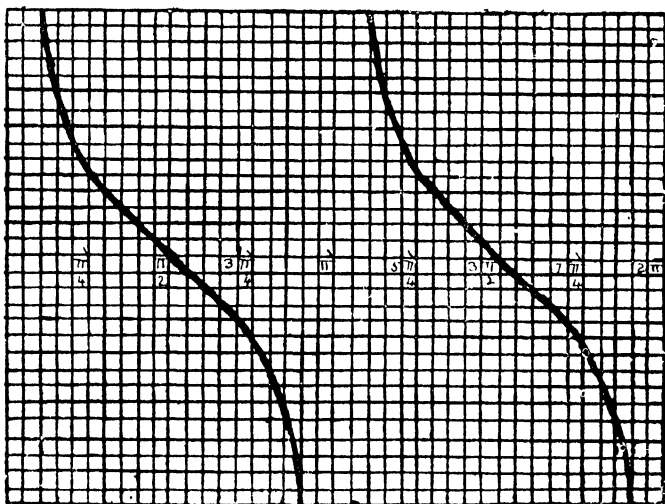


Fig. 75

decreases numerically till when $x = 2\pi$, $\tan x = 0$. Thus the graph of $\tan x$ repeats itself at intervals of $x = \pi$, and has for its asymptotes $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. The graph of $\cot x$ has a period π radians and has $x = 0$, $x = \pi$, $x = 2\pi$, etc., for asymptotes. The changes in the sign and value of the function may be studied from the graph.

Ex. 1. Draw the graph of $y = \sin 4x$ for values of x between 0 and 2π . As usual, tabulate values of x and corresponding values of y and draw the graphs.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$y = \sin 4x$	0	1	.866	0	-.866	-1	0

Ex. 2. Draw the graph of $\frac{1}{4} \sin x$.

Tabulate and plot points as usual and draw the graph through these points.

Note.—It is to be noted that the graphs of the two examples are exactly similar to the graph of $\sin x$; only the length of each complete wave in the graph of $\sin 4x$ is not 2π but $\frac{1}{4}(2\pi)$, i.e., $\frac{\pi}{2}$; the height of the wave equals 1; whereas, the height of the wave in the graph of $\frac{1}{4} \sin x$ is only $\pm \frac{1}{4}$, while the length of each wave is 2π . From this we may deduce that the graph of $\sin bx$ will be a wavy sine curve, of height unity and wave-length $\frac{2\pi}{b}$; whereas that of $a \sin x$ is a wavy sine curve of wave length 2π and height ‘ a ’ units. In general, the graph of $(a \sin bx)$ is the sine curve, wherein the height and length of each complete wave are $\pm a$ and $\frac{2\pi}{b}$ respectively. This affords an easy way of constructing the curve from the graph of $\sin x$ by merely changing the scales along the axes. Thus the graph of $y = \sin x$ drawn to a certain scale represents the function $\sin bx$, provided one unit of the old scale

along the x axis is made to represent $\frac{1}{b}$ of a unit. But, the easier way for the student to construct the graph of the function is to tabulate corresponding values of x and y , plot the points and draw the curve.

EXAMPLES

Draw the graphs of the following functions for one full period : —

- | | |
|---------------------------------|---|
| (1) $\sin 2x$ | (8) $\sec 3x$ |
| (2) $2 \sin x$ | (9) $\cot \frac{x}{3}$ |
| (3) $2 \sin 2x$ | (10) $4 \cos \frac{5x}{6}$ |
| (4) $2 \tan 3x$ | (11) $7 \sin \frac{2x}{3}$ |
| (5) $4 \cos 5x$ | (12) $6 \tan \frac{x}{4}$ |
| (6) $2 \operatorname{cosec} 3x$ | (13) $\sin \left(2x + \frac{\pi}{3} \right)$ |
| (7) $\cot 2x$ | (14) $\cos \left(3x - \frac{\pi}{4} \right)$ |

Art. 119. Graph of the function

$a \cos px + b \sin qx$ where a, b, p, q are numbers.

The usual method is that of giving x a set of values at sufficiently close intervals, writing down the corresponding values of $\cos px$, $a \cos px$, $\sin qx$, $b \sin qx$ and of $(a \cos px + b \sin qx)$ in tabular columns, plotting points corresponding to each pair of values of x and y and joining the points plotted by a smooth curve. This is the routine method to be followed. Another method is also suggested. Draw the graphs of the functions ' $a \cos px$ ' and ' $b \sin qx$ ' separately with respect to the same axes in one figure. At close intervals, draw the ordinates of both the graphs. Let the ordinates of the two graphs corres-

ponding to any value of x be y_1 and y_2 so that the graphs drawn are representations of the equations $y_1 = a \cos px$ and $y_2 = b \sin qx$. At the points on the x -axis where the ordinates are drawn, mark off with a pair of dividers ordinates equal to the algebraic sum of the two corresponding ordinates, so that the ordinate y of each new point marked is equal to $y_1 + y_2$. Join these points by a smooth graph. The x and y co-ordinates of any point on the graph last drawn satisfy the equation $y = a \cos px + b \sin qx$ and hence this is the required graph.

In case $p = q$, $a \cos px + b \sin qx$ can be written as $\sqrt{a^2 + b^2} \cos (px - \theta)$, where $\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$ and $\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}$. This reduces to a form already dealt with. The graph is the cosine graph, *i.e.*, the graph of $y = \cos x$, the wave length being $\frac{2\pi}{p}$, and height $\sqrt{a^2 + b^2}$ the graph passing through the point $\left(\frac{\theta}{p}, \sqrt{a^2 + b^2}\right)$.

Exercises.—Draw the graphs of the following functions of x .

- | | |
|----------------------------|--------------------------------------|
| (1) $3 \sin x + 4 \cos x$ | (5) $7 \cos (x - 10^\circ)$ |
| (2) $5 \cos x - 12 \sin x$ | (6) $2 \sin x - \frac{1}{3} \sin 3x$ |
| (3) $2 \sin x - 3 \cos x$ | (7) $4 \cos x + 3 \cos 2x$ |
| (4) $4 \cos 2x + 5 \sin x$ | |

Find in each case the maximum and minimum values of the functions and the values of x between 0 and 2π radians for which each function attains the value (i) zero, (ii) -1.5 .

CHAPTER XIV

REGULAR POLYGONS

Art. 120. A regular polygon is a rectilinear figure in which all the sides are equal and all the angles are also equal. A circle can be drawn passing through the vertices and it is the circumcircle of the polygon.

There is also another circle which is inscribed in the figure and which touches all the sides. This circle is the incircle of the polygon, and the polygon is said to be circumscribed to the circle. These two circles have the same centre. Let O be the centre of a regular polygon ABC ... of 'n' sides inscribed in a circle of radius R and circumscribed to a circle of radius r, so that if OX be drawn perpendicular to side AB, OX measures r and OA measures R.

It is evident that $\angle AOB = \frac{2\pi}{n}$ radians, and that
 $OA = OB$. $\angle AOX = \frac{\pi}{n}$; $\frac{AX}{OA} = \sin AOX = \sin \frac{\pi}{n}$
 $\therefore AB = 2 AX = 2 OA \cdot \sin \frac{\pi}{n}$

i. e., $a = 2 R \sin \frac{\pi}{n}$, where 'a' is the length of a side of the polygon.

Again, $\frac{OX}{OA} = \cos \frac{\pi}{n}$; $OX = OA \cos \frac{\pi}{n}$

i. e., $r = R \cos \frac{\pi}{n}$.

Also, area of triangle $AOB = \frac{1}{2} OA \cdot OB \sin AOB$
 $= \frac{1}{2} R^2 \cdot \sin \frac{2\pi}{n}$.

Directed line

$$\therefore \text{Area of polygon} = n \cdot \triangle AOB = \frac{n}{2} R^2 \sin \frac{2\pi}{n}.$$

$$\text{Similarly, } R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}; r = \frac{a}{2} \cot \frac{\pi}{n};$$

$$\text{Area of polygon} = \frac{na^2}{4} \cot \frac{\pi}{n}.$$

In the same way,

$$R = r \sec \frac{\pi}{n}, a = 2r \tan \frac{\pi}{n}.$$

$$S = n r^2 \tan \frac{\pi}{n}.$$

Art. 121. Area of sector.

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{angle of sector}}{\text{angle at the centre}}$$

If θ radian be the angle of a sector of radius r , we have, substituting in the formula—

$$\frac{\text{Area of sector}}{\pi r^2} = \frac{\theta \text{ radians}}{\text{angle at the centre}}$$

$$\text{i.e., Area of sector} = \frac{\pi r^2 \times \theta}{2\pi} = \frac{1}{2} r^2 \theta$$

It must be noted that θ is the *circular* measure of the angle of the sector and not the sexagesimal measure.

Art. 122. Area of segment.—AB is a chord of a circle centre O and radius r . The circle is divided into two segments—the minor segment AEB and the major segment AFB.

The minor segment is evidently the difference between the minor sector AOB and the triangle AOB. If AOB measures θ radians,

$$\text{Area of minor sector AOB} = \frac{1}{2} r^2 \theta.$$

$$\text{Area of triangle AOB} = \frac{1}{2} r^2 \sin \theta.$$

$$\text{Area of minor segment} = \frac{1}{2} r^2 \times (\theta - \sin \theta)$$

$$\begin{aligned} \text{Area of major segment} &= \pi r^2 - \frac{1}{2} r^2 (\theta - \sin \theta) \\ &= \frac{1}{2} r^2 (2\pi - \theta + \sin \theta). \end{aligned}$$

Ex. 1. Find the area of the minor segment cut off from a circle of radius 3" by a chord 2" long.

Taking figure for Art. 122, AB = 2", OA = 3". Hence AOD is determined by the equation (D being the mid-point of AB), $\sin \text{AOD} = \frac{\text{AD}}{\text{AO}} = \frac{1}{3} = 0.3333$. From the Table of Natural Sines, $\text{AOD} = 19^\circ 28'$. So, $\text{AOB} = 38^\circ 56'$. Its circular measure as read off from Clark's Mathematical Tables is 0.6795.

$$\begin{aligned} \text{Hence, area of minor sector} &= \frac{1}{2} \times 3^2 \times 0.6795 \\ &= 3.0578 \text{ sq. in.} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle AOB} &= \frac{1}{2} \times 3^2 \times \sin 38^\circ 56' \\ &= 2.8283 \text{ sq. in.} \end{aligned}$$

Subtracting, area of segment = 0.2295 sq. in.

Ex. 2. Two circles of radii 3" and 4" have their centres 5" apart. Calculate the area common to the two circles.

Hint.—If PQ be the common chord, the required area is the sum of the areas of two segments, each bounded by PQ, one being a segment of the first circle and the other of the second circle.

EXAMPLES XIV

1. Calculate the areas of the two segments into which a circle of radius $4''$ is divided by a chord $3''$ long.

2. Three equal circular discs of radius $1''$ are placed on the table, so that each touches the other two. Calculate the area included by them.

3. A tight belt passes round two wheels of radii $3''$ and $2''$, the distance between the centres being $6''$. Calculate the length of the belt (i) if the belt does not cross itself, (ii) if it crosses itself.

4. The span of an arch of a bridge is 4 feet and its height 1 foot. If water flows through the arch up to its highest point at three miles per hour, find the volume in cubic feet of water that passes through the arch in one minute.

5. In the same circle, regular figures of five, six and ten sides are inscribed, their sides being p , h , and d units respectively. Prove that $p^2 = h^2 + d^2$.

6. Compare the areas of two regular hexagons, one inscribed in and the other circumscribed to the same circle of radius r .

7. The radius of a circle is 3 feet. Find the lengths of the two areas into which the circumference is divided by a chord 4 feet long.

8. The corners of a square ' a ' feet long are cut off to form a regular octagon. Show that a side of the octagon is a $(\sqrt{2} - 1)$ feet.

APPENDIX

MADRAS UNIVERSITY QUESTIONS

March 1941

I. Prove that in a triangle ABC.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

The perimeter of a triangle is 12" and two of its angles are 47° and 53° . Find the sides of the triangle.

II. With the usual notation prove that

$$(i) \ r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$(ii) \ r_1 r_2 r_3 = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}$$

$$(iii) \ \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

p_1, p_2, p_3 being the altitudes of the triangle.

III. From the top of a cliff 150 feet high, the angle of depression of a boat due North of the observer is 20° , and that of a boat in a direction $22^\circ 30'$ North of East is 43° . Find to the nearest foot the distance between the boats.

IV. (a) Prove geometrically that

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

(b) If θ be an acute angle and $\sin \theta = 0.6$, calculate the values of $\cos 3\theta$ and $\tan \frac{1}{2} \theta$ without the use of the tables.

V. (a) Prove that the angle subtended at the centre of a circle by an arc equal in length to the radius is the same for all circles.

(b) Calculate, correct to a hundredth of a square inch, the area of the smaller of the two segments into which a circle of radius 4 inches is divided by a chord 5 inches long.

(c) Solve the equation $3 \sin x + 4 \cos x = 1\frac{1}{2}$.

September 1941

I. (a) By means of a graph, trace the changes in the value of the function $y = 3 \sin \theta + 4 \cos \theta$ as θ increases from 0° to 360° .

(b) Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

II. In any triangle ABC, prove that

$$(a) \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$(b) \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} \\ = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

$$(c) a + b + c = abc \div 2 Rr.$$

III. Obtain a general expression for all angles which have a given cosine.

Solve the equation—

$$\sin \theta + \sin 2\theta = \sin 3\theta.$$

IV. In a triangle ABC, prove that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cdot \cot \frac{A}{2}.$$

Two sides of a triangle are in the ratio 7 : 9, and the included angle is $64^\circ 12'$. Find the other angles.

V. A ship steaming at a speed of 15 miles an hour towards a harbour A was observed from a station B, 10 miles due west of A, to lie 42° North of East. If the ship reached the harbour after three quarters of an hour, find its distance from B and its bearing from A when first observed.

March 1942

I. (a) Prove geometrically that

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

Express $\cot 3A$ in terms of $\cot A$:

(b) Prove that

$$\cot a + \cot (60^\circ + a) - \cot (60^\circ - a) = 3 \cot 3a.$$

II. Trace the changes in the value of $\tan x$ as x increases from 0° to 360° .

Prove that

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13.$$

III. (a) Find an expression for all angles with a given cosine.

$$\text{Solve } 2 \cos 3\theta + 6 \cos \theta + 1 = 0$$

(b) If $A + B + C = 180^\circ$, prove that

$$a \cos^2 \frac{A}{2} + b \cos^2 \frac{B}{2} + c \cos^2 \frac{C}{2} = s + \frac{\Delta}{R}.$$

IV. In a triangle ABC, prove that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

Solve the triangle ABC given $a = 36.21$, $c = 30.14$, $B = 78^\circ 10'$.

V. From a point O due south of a tower, the angle of elevation of its top is found to be 60° . From A, due west of O, the elevation is 45° and from B due west A, it is 30° . If $AB = 200$ feet and B, A, O and the foot of the tower are in a horizontal plane, find the height of the tower.

September 1942

I. Trace the changes in sign and value of $4 \sin x + 3 \cos x$. Illustrate by drawing the graph of $y = 4 \sin x + 3 \cos x$ for values of x between 0° and 360° . Solve graphically or otherwise the equation $4 \sin x + 3 \cos x = 2$.

II. Show that II_1 is a diameter of circle BIC. Hence prove that $II_1 = a \sec \frac{A}{2}$;

$$II_1 \cdot II_2 \cdot II_3 = 16 R^2 r.$$

III. Prove geometrically

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}.$$

$$\text{Show that } \cos A + \cos B + \cos C + \cos (A+B+C) = 4 \cos \frac{A+B}{2} \cos \frac{B+C}{2} \cos \frac{C+A}{2}$$

IV. Prove that

$$(i) \sum a \cot A = 2 (R + r)$$

$$(ii) (r_1 - r) (r_2 - r) (r_3 - r) = 4 R r^2.$$

$$V. \text{ Prove that } \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Solve ABC given

$$a = 68.47, b = 92.68, c = 71.42.$$

VI. A church tower AB standing on a level plain is surmounted by a spire BC of the same height as the tower. D is a point in AB such that $AD = \frac{1}{3} AB$. At a point on the plain, 100 feet from the foot of the tower, the angles subtended by AD and BC are the same. Find the height of the tower.

March 1943

I. (a) Prove geometrically that $\cos 2A = \cos^2 A - \sin^2 A$.

(b) If $A + B + C = \frac{\pi}{2}$ prove that

$$\sin^2 A + \sin^2 B + \sin^2 C + 2 \sin A \sin B \sin C = 1.$$

II. In a triangle ABC, prove that

$$(i) SI^2 = R^2 - 2Rr$$

$$(ii) rr_1r_2r_3 = \Delta^2.$$

$$(iii) (b+c) \sin \frac{A}{2} = a \cos \frac{B-C}{2}$$

III. (a) Prove geometrically that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

A and B being acute angles.

(b) If $A + B + C = n\pi$, prove that $\sum \cot B \cot C = 1$.

IV. Find an expression for all angles having a given tangent.

Solve $1 + \sin^2 \theta = 3 \sin \theta \cos \theta$, completely.

V. In a triangle ABC prove

$$\frac{r_2 + r_3}{1 + \cos A} = \frac{r_3 + r_1}{1 + \cos B} = \frac{r_1 + r_2}{1 + \cos C}.$$

Solve a triangle ABC, given $A = 35^\circ 17'$, $C = 45^\circ 13'$, $b = 42.1$.

VI. The angles of elevation of the top of a tower as seen from the top and bottom of a building 60 feet high are 50° and 75° respectively. Find the height of the tower to the nearest foot.

September 1943

I. Prove geometrically that

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

Prove that $4(\sin 24^\circ + \cos 6^\circ) = \sqrt{3} + \sqrt{15}$.

II. (a) Solve $\sqrt{3} (\tan \theta + \sin \theta) = 4$.

(b) In a triangle ABC, prove

$$(i) \Delta = \frac{a^2}{4} \sin 2B + \frac{b^2}{4} \sin 2A$$

$$(ii) r_2 r_3 + r_3 r_1 + r_1 r_2 \equiv s^2$$

III. Solve a triangle ABC, given $a = 182.5$, $b \equiv 82.5$, $A = 72^\circ 15'$.

IV. A man in a balloon observes the angle of depression of an object on the ground due north is 33° . The balloon drifts 3 miles due west and the angle of depression of the object is then found to be 21° . Find the height of balloon in miles correct to two decimal places.

March 1944

I. Prove the following in a Δ

$$(i) (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2}.$$

$$(ii) abc = 4 R. r. s.$$

$$(iii) \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

II. Find an expression for the area of a triangle with angles A, B, C, inscribed in a circle of radius R. If A and B are such that $A > 60^\circ > B$, show that the area of a triangle with $A' = 60^\circ$, $B' = A + B - 60^\circ$, $C' = C$ inscribed in the same circle is greater than that of the triangle with angles A, B, C. Hence, find the maximum of the areas of triangles inscribed in a given circle.

III. Give a geometrical proof of the formula $\cos 2A = \cos^2 A - \sin^2 A$

$$\text{Prove that } \cos^2 76^\circ + \cos^2 16^\circ - \cos 76^\circ \cos 16^\circ = \frac{3}{4}.$$

IV. (a) Solve *completely* the equation $\sin 6\theta = \sin 4\theta - \sin 2\theta$.

(b) In a triangle ABC , prove that

$$a^2 = (b + c)^2 \sin^2 \frac{A}{2} + (b - c)^2 \cos^2 \frac{A}{2} \text{ and}$$

$$(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2.$$

V. Solve a triangle ABC given $c = 10.9$, $a = 6$, $B = 56^\circ 36'$.

VI. Two points A and B are on a straight horizontal road, at a distance 400 feet apart. A vertical pole in the same horizontal plane as the road is 100 feet high and is at equal distances from A and B . The angle subtended by AB at the foot of the pole being 80° , find the distance from the road to the foot of the pole and the angle subtended by AB at the top of the pole.

September 1944

I. Prove geometrically that $\cos(B - A) = \cos A \cos B + \sin A \sin B$ when A is acute and B lies between 90° and 180° .

Prove that $\cos 35^\circ + \cos 85^\circ + \cos 155^\circ = 0$.

II. (a) Solve giving the general value of θ .

$$\cos 2\theta + 5 \cos \theta = 2.$$

(b) In any triangle ABC , prove that $\cot A + \cot B +$

$$\cot C = \frac{a^2 + b^2 + c^2}{4\Delta} \text{ and } r_1 + r_2 + r_3 = 4R + r.$$

III. Solve the triangle ABC given $b = 6.5$, $c = 3.3$, $C = 30^\circ 31'$.

IV. An observer sees an aeroplane bearing $N. 40^\circ. W.$ at an elevation of $20^\circ 33'$. Twenty-five seconds later, its bearing is found to be $N. 80^\circ. E$ and its elevation $23^\circ 30'$. Find the direction in which the aeroplane is flying and its speed assuming it to be travelling horizontally in a straight line at a height of 1000 feet.

March 1945

I. (a) If the sum of the diameters of two escribed circles is the same as the difference of the diameters of the third escribed circle and the incircle, show that the triangle is right-angled.

(b) In a triangle, show that

$$1 - \sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = \frac{r}{2R}.$$

II. Prove that in a triangle, $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$.

If $a = 13$, $b = 22$, $C = 80^\circ$, calculate c , A , B .

III. (a) Draw the graph of $\cos x$ in the interval $(0-2\pi)$.

(b) Solve completely

$$\operatorname{cosec}^4 \theta + 9 \operatorname{cosec}^2 \theta - 6 \operatorname{cosec}^2 \theta \cot \theta - 10 = 0.$$

IV. A balloon is observed simultaneously from three places A, B, C, lying due west of it on a horizontal straight line, passing directly underneath it. $AB = 220'$, $BC = 100'$, and the angular elevations at B and C are respectively twice and thrice that at A. Calculate the height of the balloon.

Hint.—If O is the balloon, OB is the bisector of $\angle AOC$ in triangle AOC. Hence, $OA : OC = AB : BC$.

September 1945

I. (a) Give a geometrical proof of the formula $\cos 2\theta = 1 - 2 \sin^2 \theta$.

(b) Derive the value of $\sin 18^\circ$, and show that $4 \sin 18^\circ \cos 36^\circ = 1$.

II. (a) Show that the area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$. Deduce that $(r_1 - r)(r_2 + r_3) = a^2$.

(b) Show that $4Rrs = abc$.

III. Discuss the case of solution of a triangle when two sides and the angle opposite one of them is given.

Solve the triangle given $a = 4.7$, $c = 1.3$, $\angle C = 15^\circ$.

IV. A pole leaning east is inclined to the horizontal at an angle θ . A person walks due west a distance b from the foot of the pole and observes the elevation of the top of the pole to be α . On walking a further distance ' d ' in the same direction, the elevation of the top is observed to be further diminished by β .

Show that $\tan \theta = \frac{d}{(b+d)} \cdot \cot \alpha - b \cot (\alpha - \beta)$.

March 1946

I. (a) Prove geometrically that

$$\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

(b) If $A + B + C = 180^\circ$, prove that
 $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$.

II. (a) Draw the graph of $2 \cos x + 5 \sin x$ for values of x from 0° to 360° .

(b) Hence or otherwise solve the equation
 $2 \cos x + 5 \sin x = 4$.

III. (a) In a triangle ABC show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

(b) Prove that in a triangle,

$$(b+c) \cos A + (c+a) \cos B + (a+b) \cos C = a + b + c$$

$$(ii) \sum \frac{b^2 - c^2}{a} \cos A = 0$$

IV. From a point 10 feet above the level of water in a lake, the angle of elevation of the top of a tree is found to be 10° ; the angle of depression of its reflection in the lake is 15° . Find the height of the tree-top above the water level.

September 1946

I. (a) Prove geometrically :

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

(b) If $A + B + C = 180^\circ$, prove that

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

II. (a) Find the general expression for all angles having a given sine.

(b) Solve $\tan \theta = 1 - \sec 2\theta$.

III. (a) In a triangle ABC prove that

$$\tan \frac{A}{2} = \sqrt{\frac{s-b}{s} \frac{s-c}{s-a}}$$

(b) Solve the triangle ABC given that $a = 345.6$, $b = 456.6$, $c = 567.8$.

Give the angles to the nearest minute.

IV. From three consecutive furlong stones on a straight road, which does not pass through the foot of a tower, the angles of elevation of the top of a tower are 15° , 45° , 75° . Find the height of the tower.

March 1947

I. (a) Prove geometrically that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$.

(b) If $2 \sin \frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$, find the limits within which A must lie.

II. (a) Solve: $\tan \theta + 4 \cot 2\theta + 1 = 0$.

(b) If $A + B + C = 130^\circ$, prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C)$.

III. (a) Given two sides of a triangle and an angle opposite to one of them, show how two solutions may arise.

If $b = 16$, $c = 25$, $B = 33^\circ 15'$, find C.

(b) With the usual notation, prove that

$$r_1 + r_2 + r_3 - r = 4R$$

IV. A statue AT stands at a point A on one side of a straight road. B and C are points on the opposite side such that B is directly opposite to A and M is mid-way between B and C. If the angles of elevation of the top T of the statue from B, C and M are α , β and θ respectively, show that

$$4 \cot^2 \theta = 3 \cot^2 \alpha + \cot^2 \beta$$

September 1947

I. (a) Prove geometrically :—

$$\cot A - \tan A = 2 \cot 2A.$$

(b) If $A + B + C = 180^\circ$, show that

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

II. Draw the graph of $\operatorname{cosec} 2x$ for values of x from 0° to 180° . Hence or otherwise, find the general solution of $\operatorname{cosec} 2x = 5$.

III. With the usual notation, prove that

$$(i) \quad r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$(ii) \quad (r_2 + r_3)(r_1 - r) = a^2$$

IV. (a) The sides of a triangle are in the ratio of 4 : 5 : 6. Show that one angle of the triangle is twice another.

(b) A pilot flying horizontally in a moth plane above a straight road observes in front of him a jeep moving along the road in the same direction. The pilot observes the angle of depression of the jeep to be 32° and two minutes later to be 64° . If the speeds of the moth plane and the jeep be respectively 60 and 30 miles per hour, find the height at which the plane is flying and the time at which it is vertically above the jeep.

March 1948

I. (a) Prove geometrically that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Express $\sin 3\theta$ in terms of $\sin \theta$.

(b) Prove that $\sin \theta = \frac{\sin 3\theta}{1 + 2 \cos 2\theta}$; and hence find the value of $\sin 15^\circ$.

II. Show that, in a triangle ABC

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Hence find the value of $\tan \frac{1}{2} A$ in terms of the sides.

If s is the semi-perimeter and Δ the area of the triangle prove that

$$\Delta [\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C] = s^2.$$

III. Draw the graph of $\sin x + \cos 2x$ for values of x lying between 0° and 360° ; and use it to find, roughly, the values of x in the given range, for which

$$\sin x + \cos 2x = 0.2.$$

Verify by solving the equation as a quadratic in $\sin x$.

IV. At a point A on the same level as P, the foot of a vertical tower PQ, the angle of elevation of the top of the tower is α . At a point B distant a from A and such that angle PAB is β , AP subtends angle γ . Prove that the height of the tower is,

$$a \sin \gamma \cdot \operatorname{cosec} (\beta + \gamma) \tan \alpha.$$

From the data $\alpha = 9^\circ 18'$, $\beta = 60^\circ$, $a = 750$ feet and $\gamma = 79^\circ 23'$, calculate the height of the tower.

September 1948

I. Prove geometrically that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}.$$

If $A + B = 45^\circ$, prove that $(1 + \tan A)(1 + \tan B) = 2$; deduce the value of $\tan 22\frac{1}{2}^\circ$.

Without using tables, prove that
 $\cos 40^\circ \cos 80^\circ \cos 160^\circ = -\frac{1}{8}$.

II. In a triangle, with the usual notation, prove that

$$(i) \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

$$(ii) abc = 4R \Delta.$$

If I_1, I_2, I_3 , are the excentres of the triangle ABC, show that $I_1 I_2 = 4R \cos \frac{C}{2}$. Hence prove that the area of triangle $I_1 I_2 I_3 = \frac{abc}{2r}$.

III. Find the general expression for all angles which have a given tangent.

Solve the equation $\tan A = -\cot 2A$.

Express the function $5 \sin x - 12 \cos x$ in the form $R \sin(x - a)$; hence find the maximum and minimum values of the function.

4. A lightship is at a distance 5 miles due south of a point A; a launch starts from the lightship in a direction 10° East of South and at the end of half an hour alters its course to due East. At the end of another half an hour, the bearing of the launch from A is 31° East of South. Find the speed of the launch in miles per hour.

March 1949

I. (i) Deduce geometrically the expansion of $\cos(A - B)$ in terms of the sines and cosines of the angles A and B.

(ii) Find the value of $\cos^2 36^\circ + \sin^2 18^\circ$.

II. (i) Show that in a triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(ii) Hence prove that $(b-c) \cos \frac{1}{2} A = a \sin \frac{1}{2} (B-C)$.

III. (i) If $\sin \theta + \sin \phi = a$, and $\cos \theta + \cos \phi = b$, find the value of $\tan \frac{1}{2} (\theta - \phi)$.

(ii) Prove that $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \frac{1}{2} (\alpha - \beta)$.

IV. A and B are two points on the banks of a river opposite to each other. Between them is the mast CD of a ship. The breadth of the river is 1000 feet, and the angular elevation of C at A is $14^\circ 20'$ and at B is $8^\circ 10'$. What is the height of C above AB?

September 1949

I. (i) Determine geometrically the expansion of $\sin (A + B)$.

(ii) If $\tan \alpha = \sqrt{3}$ and $\tan \beta = 1$, find $\sin (\alpha + \beta)$.

II. (i) Prove that

$$\cos \alpha + \cos \left(\alpha + \frac{2\pi}{3} \right) + \cos \left(\alpha + \frac{4\pi}{3} \right) = 0.$$

(ii) In any triangle ABC, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

III. (i) If θ be the angle which the median AD of a triangle ABC makes with the side BC, show that $2 \cot \theta = \cot B - \cot C$.

(ii) A triangle ABC is such that $a \cos A = b \cos B$. Show that the triangle is either isosceles or right-angled.

IV. A person observes that two objects respectively bear N.E. and N.W. of him. He walks due east for one mile and then observes that both the objects appear in a direction N. 60° . W. How far were the objects from him at first and how far were they apart?

March 1950

I. (a) Obtain geometrically the expansion of $\cos (A + B)$ in terms of the sines and cosines of A and B .

(b) Find the value of $\sec 72^\circ - \sec 36^\circ$.

II. (a) Solve the equation $\cos 3\theta + \cos 2\theta + \cos \theta = 0$.

(b) If $A + B + C = 180^\circ$, prove that

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

III. (a) Show that in any triangle ABC ,

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}.$$

(b) In a triangle ABC if $AB = 1400$, $AC = 1300$ and $A = 60^\circ$, find the angles B and C .

IV. (a) Prove that $r_1 + r_2 + r_3 = r + 4R$, where the letters have the usual meaning.

(b) A person walking along a straight road observes that at two consecutive milestones the angles of elevation of a hill in front of him are 30° and 75° . Find the height of the hill.

September 1950

I. Prove the relation—

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$

for arbitrary values of A and B .

Use this relation to obtain the value of $\cos 40^\circ$ by choosing $A = 80^\circ$, $B = 40^\circ$.

Prove the formula

$$\cos^2 A - \sin^2 A = \cos 2A = \frac{1}{2} (\cos 2A + \cos 4A).$$

II. (a) Prove that in any triangle,

$$(1) \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

$$(2) \sum \tan \frac{B}{2} \cdot \tan \frac{C}{2} = 1.$$

(b) Prove that in a triangle ABC,

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

Explain how to solve the triangle when a, b, C are known.

III. Find correct to a minute the values of x between 0° and 360° , which satisfy the equation

$$5 \sin x - 7 \cos x = 6.$$

Give a rough sketch of the graph of $y = 5 \sin x - 7 \cos x$, and find its maximum and minimum values.

IV. The top P of a mountain is observed from two points A and B at sea-level. If N is the point at sea-level vertically below P, and $\angle NAB = \alpha$, $\angle NBA = \beta$, $\angle NAP = \theta$, $\angle NBP = \phi$, show that $\cot \phi \sin \beta = \cot \theta \sin \alpha$.

If the angles α, β, θ are observed and found to be 60° , 45° , and $8^\circ 54'$ respectively, and $AB = 7000$ metres, find the height of the mountain.

If ϕ is also observed and found to be $7^\circ 24'$, how would you use this fact?

March 1951

I. Show that in any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

In a triangle ABC, it is given that $b = 2.5$ in, $c = 3$ in. and the altitude $AD = 1.5$ in. Show that two solutions are possible.

II. If θ is any angle in the third quadrant, draw the angles denoted by the general expression $n\pi + (-1)^n \theta$ where $n = 0, \pm 1, \pm 2$, etc., and show that

$$\sin [n\pi + (-1)^n \theta] = \sin \theta.$$

Explain why the same two series of angles are given by the equations

$$\theta + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6} \text{ and}$$

$$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

III. (a) Prove that,

$$\cot(A + 15^\circ) - \tan(A - 15^\circ) = \frac{4 \cos 2A}{2 \sin 2A + 1}.$$

(b) If $A + B + C = \pi$, show that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$$

IV. (a) Find the values of $\sin 18^\circ$ and $\cos 36^\circ$, either geometrically or theoretically.

(b) Prove that the square of the side of a regular pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

September 1951

I. (a) Find a general expression for all angles having a given cosine.

(b) Express $\sin 10x + \cos 20x$ as a product and hence or otherwise solve the equation $\sin 10x + \cos 20x = 0$ giving the general values of x . Find the values of x between 0° and 90° .

II. (a) If x is any angle in the fourth quadrant, draw the angle $270^\circ - x$ and express the trigonometrical ratios of $(270^\circ - x)$ in terms of those of x .

(b) If $\tan \frac{A}{2} = t$, prove that

$$\sin A + \tan A = 4t / \left(\frac{5}{t^4} \right).$$

III. In any triangle, prove that

$$c^2 = a^2 + b^2 - 2ab \cos C \text{ and that}$$

$$c = (a - b) \cos \frac{C}{2} \sec \phi \text{ if}$$

$$\tan \phi = \frac{(a + b)}{a - b} \tan \frac{C}{2}$$

If $a = 27.3$, $b = 16.8$, $C = 45^\circ 12'$ find ϕ and thence, find c .

IV. Show that

$\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B)$ and prove that in a triangle ABC,

$$\frac{a^2 - b^2}{2} \cdot \frac{\sin A \cdot \sin B}{\sin(A - B)} = \Delta.$$

March 1952

I. (a) Express $\cos 3x + \sin 2x$ as a product and hence or otherwise solve the equation $\cos 3x + \sin 2x = 0$ and find all the values in the range from 0° to 360° .

(b) If $\cos(A - B) = \frac{1}{2}$ and $\sin(A + B) = \frac{1}{2}$, find the smallest positive values of A and B and also their most general values.

II. If the sides of a triangle be in arithmetical progression, prove that so also are the cotangents of half the angles.

In a triangle ABC, if $\tan \frac{A}{2} = \frac{5}{6}$ and $\tan \frac{B}{2} = \frac{20}{37}$, find $\tan \frac{C}{2}$ and prove that in this triangle $a + c = 2b$.

III. A flagstaff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane which lie on a line passing through the centre of the base of the tower, and whose distance from one another is $2x$, and an angle β at a point half-way between them. Prove that the height of the flagstaff is

$$x \sin \alpha \sqrt{\frac{2 \sin \beta}{\cos \alpha \cdot \sin(\beta - \alpha)}}$$

Calculate the height if $x = 125$ feet, $\alpha = 22^\circ 48'$, $\beta = 36^\circ 24'$.

IV. DEF is the pedal triangle of a triangle ABC. Prove that

(a) its area is $2 S \cos A \cos B \cos C$,

(b) the radius of its circumcircle is $\frac{R}{2}$,

(c) the radius of its incircle is $2 R \cos A \cos B \cos C$, where the symbols have the usual meanings with reference to the triangle ABC.

September 1952

I. Express $4 \cos x - 3 \sin x$ in the form $k \cos (x + \theta)$. If the angles are measured in degrees, draw an accurate graph of $y = 4 \cos x - 3 \sin x$ for the range $x = -360^\circ$ to $x = 360^\circ$. If $y = -2$, find from the graph the corresponding values of x for this range and verify your result by solving the equation $4 \cos x - 3 \sin x + 2 = 0$.

How many roots has the equation
 $4 \cos x - 3 \sin x = 4$.

II. Prove that in a triangle ABC,

(a) $\frac{\sin 2A - \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \frac{\tan B}{\tan C}$,

(b) $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$.

Apply (a) and (b) to show that if $\frac{1}{3} \sin 2A = \frac{1}{4} \sin 2B = \frac{1}{5} \sin 2C$, then $\tan A = 3$, $\tan B = 2$, $\tan C = 1$.

III. From three collinear points A, B, C on a straight level road, the angles of elevation of a hill top are θ , θ and ϕ respectively. If $AB = BC = 800$ feet, prove that the height of the hill is

$$\frac{800 \sqrt{2} \sin \theta \cdot \sin \phi}{\sqrt{\sin (\theta + \phi) \cdot \sin (\theta - \phi)}} \text{ feet.}$$

Calculate the height when $\theta = 16^\circ 36'$, $\phi = 13^\circ 24'$.

IV. O_1, O_2, O_3 are the ex-centres of triangle ABC. Prove that

(1) its sides are $4 R \cos \frac{A}{2}, 4 R \cos \frac{B}{2}, 4 R \cos \frac{C}{2}$,

(2) its angles are $\left(\frac{\pi}{2} - \frac{A}{2}\right), \left(\frac{\pi}{2} - \frac{B}{2}\right), \left(\frac{\pi}{2} - \frac{C}{2}\right)$,

(3) its area is $2 R \cdot s$.

March 1953

1. (a) Prove geometrically that

$$\cos 2A = \cos^2 A - \sin^2 A.$$

(b) Give the general values of x satisfying the equation $\sin x = \cos 2x$, and state the particular solutions lying between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Verify the particular solutions graphically.

2. In any triangle, prove that

(a) $\sin^2 A - \sin^2 B - \cos^2 C = 1 + 2 \cos A \cos B \cos C$.

(b) $r = 4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

(c) $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$.

3. Prove that in a triangle ABC,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

Given that $c = 148.3$, $b = 175.8$ and angle $A = 47^\circ 20'$, determine the values of the angles B and C.

4. A person observes the angle of elevation of a peak from a point A on the ground to be α . He goes up an incline, of inclination β to the horizontal, towards the top of the peak and observes that the angle of elevation of the peak now is γ . If the second place of observation is B and if AB is l yards, show that the height of the peak above the ground is $\frac{l \sin \alpha \cdot \sin (\gamma - \beta)}{\sin (\gamma - \alpha)}$ yards. Calculate this height if $l = 200$,

$$\alpha = 42^\circ 30', \beta = 27^\circ 12' \text{ and } \gamma = 55^\circ 42'.$$

MISCELLANEOUS EXAMPLES

1. If $\cos (A - B) = 3 \cos (A + B)$, show that $\cot A \cdot \cot B = 2$.
2. If $\tan^2 A = 2 + 3 \tan^2 B$, show that,

$$\cos 2B = 2 + 3 \cos 2A.$$
3. Show that $\cos^2 (45^\circ - a) + \cos^2 (15^\circ + a) - \cos^2 (15^\circ - a) = \frac{1}{2}$.
4. Find the value of

$$\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8}.$$
5. Show that in a $\triangle ABC$,

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{B+C}{4} \times \cos \frac{C+A}{4} \cos \frac{A+B}{4}.$$
6. Show that in a $\triangle ABC$,

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 1 + 4 \sin \frac{B+C}{4} \cdot \sin \frac{C+A}{4} \sin \frac{A+B}{4}.$$
7. Show that $\tan 75^\circ = 2 + \sqrt{3}$.
8. Show that $4 \cos 18^\circ \cdot \cos 42^\circ \cdot \cos 78^\circ = \cos 54^\circ$.
9. If in a $\triangle ABC$, $\tan A \tan C = 3$, show that $\tan A$, $\tan B$, and $\tan C$ are in A.P.
10. If $\frac{\tan^2 \theta - \tan^2 a}{1 - \tan^2 \theta \times \tan^2 a} = n$, show that

$$\cos 2 \theta = \frac{1-n}{1+n} \cdot \cos 2 a.$$
11. If O is the orthocentre of a $\triangle ABC$, show that
 $a \cdot OB \cdot OC + b \cdot OC \cdot OA + c \cdot OA \cdot OB = abc.$

12. Show that in a
- $\triangle ABC$
- ,

$$\tan \frac{A}{2} \tan \frac{C-B}{2} = \tan^2 \theta \text{ where } \cos 2\theta = \frac{b}{c}.$$

13. Show that
- $S = \frac{a^2 - b^2}{2} \cdot \frac{\sin A \cdot \sin B}{\sin(A-B)}$
- .

14. Show that in a
- $\triangle ABC$
- ,
- $\sum AI^2 \left(\frac{1}{c} - \frac{1}{b} \right) = 0$
- .

15. Show that in a
- $\triangle ABC$
- ,

$$\frac{a+b}{c} = \frac{1 + \tan \frac{A}{2} \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}}.$$

16. In a
- $\triangle ABC$
- , prove that

$$\sum (b \cos B - c \cos C) \tan A = 0.$$

17. Show that in a
- $\triangle ABC$
- ,

$$\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0.$$

18. If in a
- $\triangle ABC$
- ,
- $r_1 = 2R + r$
- , show that
- $\angle A$
- is a right angle.

19. If in a
- $\triangle ABC$
- ,
- $r_1 + r_2 + r = r_3$
- , then show that the
- $\triangle$
- must be right-angled at C.

20. In a
- $\triangle ABC$
- , the product of two of the radii of the 4 circles touching the sides is equal to the product of the two remaining radii. Show that the
- $\triangle$
- must be right-angled.

21. AD, BE, CF are the altitudes of a
- $\triangle ABC$
- whose lengths are
- p_1
- ,
- p_2
- and
- p_3
- ; show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

22. Show also that

$$2R(p_1 + p_2 + p_3) = bc + ca + ab.$$

23. Show also that

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta}.$$

24. Show that in a $\triangle ABC$

$$r^2 + s^2 + 4Rr = bc + ca + ab$$

25. In a $\triangle ABC$, prove that

$$b^2r - 2b \triangle + r(r^2 + 4Rr + s^2 - ac) = 0.$$

26. O is the orthocentre of a $\triangle ABC$. Show that

$$a^2 + AO^2 = b^2 + BO^2 = c^2 + CO^2.$$

27. AD, BE and CF are the altitudes of a $\triangle ABC$ in which O is the orthocentre. Show that

$$(i) \frac{OD}{DA} + \frac{OE}{EB} + \frac{OF}{FC} = 1.$$

$$(ii) \frac{OA}{AD} + \frac{OB}{EB} + \frac{OC}{FC} = 2.$$

28. Show that in a $\triangle ABC$

$$\frac{\sin^2 \frac{A}{2}}{a} + \frac{\sin^2 \frac{B}{2}}{b} + \frac{\sin^2 \frac{C}{2}}{c} = \frac{r^2 + 4Rr}{abc}$$

29. Show also that

$$\frac{\cos^2 \frac{A}{2}}{a} + \frac{\cos^2 \frac{B}{2}}{b} + \frac{\cos^2 \frac{C}{2}}{c} = \frac{s^2}{abc}$$

30. If the in-circle of a $\triangle ABC$ passes through the circumcentre, then show that $\cos A + \cos B + \cos C = \sqrt{2}$.

31. If the circumcircle of a $\triangle ABC$ touches the nine-points-circle, show that the $\triangle$ must be right-angled.

32. Given a , b , and $\angle A$, find when two $\triangle$ s can be constructed. Show further that the difference in areas of the two $\triangle$ s so constructed is $ab \sin A \cos B$.

33. In the preceding problem, if x and y are the radii of the in-circles of the two $\triangle$ s, show that

$$xy = b(b-a) \sin^2 \frac{A}{2}.$$

34. It is known that the centre of gravity of a circular wire lies at a distance of $a \frac{\sin B}{B}$ from the centre on the radius which bisects the arc, where a is the radius of the circle and $2B$ is the radian measure of the angle subtended at the centre by the arc. If a wire 14 inches long be bent in the form of an arc of a circle of radius 8 inches, determine the position of its centre of gravity.

35. Show that the values of θ that satisfy the equation $a = b \cos \theta$ form one or other of two arithmetic progressions.

36. (a) Show that all the angles having the same sine as a are included in $(2n + \frac{1}{2})\pi \pm \left[\frac{\pi}{2} - a \right]$

(b) Solve $4 \sin x - 3 \operatorname{cosec} x = 2 \cot x$.

37. Show that the maximum value of $\sin x \sin y$ occurs when $x = y = k$ where $2k = x + y$ and k is a constant.

Solve the equations—

38. $\tan^2 x = \sec x - \cos x$.

39. $4 \sin \theta \sin (\theta - B) = 2 \cos B - 1$.

40. $\tan m\theta = \cot n\theta$.

41. $4 \sin^2 \theta - \sqrt{3} = 2(1 - \sqrt{3}) \sin \theta$.

42. Find all the values of θ satisfying $\cos \theta = 2 \sin (\theta + 10^\circ)$

43. Find the least +ve value of θ satisfying the equation $\cos \theta = \tan \theta$ by drawing the graphs of $\cos \theta$ and $\tan \theta$.

44. Trace the changes in the value of $3 \cos \theta + 4 \sin \theta$ as θ increases from 0° to 180° , and illustrate graphically. Find the value of θ for which the expression is a maximum.

45. Plot the curve $y = 2 \sin x + \frac{1}{2} \sin 3x$ for values of x from 0° to 180° . Read the values of x for which y has maximum values and the value of x for which y is least between the maxima.

46. Draw the graph of $2 \cos \theta + 3 \sin \theta$ for all values of θ between 0° and 360° . Use the graph to solve the equation $2 \cos \theta + 3 \sin \theta = 1.5$. Verify your result by calculation.

47. The elevation of a certain rock is found to be $47^\circ 13'$ and after walking 1153 feet towards the rock, up a slope inclined at an angle of $32^\circ 30'$ to the horizon, the observer finds that the elevation is $77^\circ 42'$. Find the vertical height of the top of the rock above the first point of observation.

48. A, B are two points exactly opposite each other on the banks of a river and in line with the foot of a tower at some distance from the river. If the height of the tower is 210 feet and the angles of depression of A, B as seen from the top of the tower are 25° and 28° , calculate the breadth AB of the river.

49. An object $2h$ feet high placed on the top of a tower subtends an angle a at a place whose horizontal distance from the foot of the tower is h feet. Show that the height of the tower is $h(\sqrt{2} \cot a - 1)$.

50. A person standing on a cliff observes that the angles of depression of a light on the mast of a ship 500 yds. away and its reflection in water are α and β . Show that the height of the cliff is $250(\tan \alpha + \tan \beta)$.

51. A boat leaves a light-house at the end of a harbour and sails 20° east of north for 1000 yds., then 16° west of north for 550 yards and finally due west for 400 yards. Find the distance and bearing of the final position of the boat with respect to the light-house. Check your result by a figure.

52. The angle of elevation of an aeroplane observed from a station due south of it was $47^{\circ} 18'$. From a second station 220 yards due west of the first, the elevation was found to be $41^{\circ} 14'$. Find the height of the aeroplane above the horizontal plane through the two stations.

53. The elevation of a tower was observed at a station to be 25° and its bearing N.E. At a second station 1000 yds. due south of the former and on the same level with it, the bearing of the tower was N. $11^{\circ} 15'$. E. Find the height of the tower above the horizontal plane through the two stations.

54. A and B are two points in the same horizontal plane, the distance between them being 876 yds. From A, a hill is observed to be N. $20^{\circ} 5'$. E. and its angle of elevation is $13^{\circ} 35'$. From B, the hill is N. 11° . W. and A is W. $14^{\circ} 37'$. S. Find the height of the hill.

55. Up a hill sloping at 26° to the horizontal plane runs a path which makes 60° with the line of greatest slope. What is the length of the path to the top of the hill which is 1200 feet high.

56. A ship is east of a light-house. It sails in a direction S. $27^{\circ} 23'$. W. for 5 miles till it is due south of the light-house and then sails for 8 miles in a direction W. $37^{\circ} 49'$. S. What are now its bearing and distance from the lighthouse?

57. If the circumference of a circle be divided into 4 parts in A. P., the greatest being 4 times the least, express in radians the angle each subtends at the centre.

58. The bisector of the angle A meets the side BC in D and the circumscribed circle in E. Show that

$$DE = \frac{a^2 \sec \frac{A}{2}}{2(b+c)}$$

59. If $\tan A = \frac{x \sin B}{1 - x \cos B}$ and $\tan B = \frac{y \sin A}{1 - y \cos A}$,

prove that $\frac{\sin A}{\sin B} = \frac{x}{y}$.

60. Prove that $(a + b + c) d_1 d_2 d_3 = 8 R a b c$, where d_1, d_2, d_3 are the distances of the incentre from the excentres.

61. In a triangle, $\tan \frac{A}{2} = \frac{5}{6}$,

$\tan \frac{B}{2} = \frac{20}{37}$. Find $\tan \frac{C}{2}$ and

show that a, b, c are in A. P.

62. Solve the equation $\sin \theta = \cos a$ and show that the two expressions $(2n + \frac{1}{2})\pi \pm a$ and

$n\pi + (-1)^n \left\{ \frac{\pi}{2} - a \right\}$ denote the same set of angles.

63. Solve the equations

(i) $\tan \theta + \tan 2\theta + \tan 3\theta = 0$.

(ii) $\tan 3\theta - 4 \tan \theta = 0$.

64. A wireless signal from an aeroplane is intercepted at two direction-finding stations A and B which are 5 miles apart in a north and south line. From A, the direction of the aeroplane is N. 66° . W. and from B it is found to be S. 26° . W. At the same time, the altitude of the aeroplane is observed from A to be 40° . Find the height of the aeroplane above A.

65. If $\sin \theta = n \sin (\theta + 2a)$, prove

$\tan (\theta + a) = \frac{1+n}{1-n} \tan a$. Hence, solve the equation

$\sin \theta = 5 \times \sin (\theta + 60^\circ)$.

66. Solve the equations

$\sin x + \sin y = \frac{1}{2}$; $\cos x + \cos y = \frac{1}{2}$.

67. The internal bisectors of the angles of a triangle meet the circumcircle in D, E, F. Show that

$$\triangle DEF : \triangle ABC = R : 2r.$$

68. (a) Find the real values of θ and ϕ which lie between 0° and 360° and satisfy the equations

$$\tan \theta = \cot 2\phi : \tan 2\theta = \cot \phi.$$

(b) If $\frac{\cos(\theta - \phi)}{\cos(\theta + \phi)} = \frac{a}{b}$, prove that

$$(a^2 + b^2 - 2ab \cos 2\theta) = (a^2 + b^2 - 2ab \cos 2\phi) = (a^2 - b^2)^2.$$

69. Calculate the value of $\tan 11\frac{1}{4}^\circ$ correct to three decimal places.

70. In a semicircle standing on the diameter AB, a chord AP is drawn bisecting the area of the semicircle. If BAP equals θ radians, prove that $2\theta + \sin 2\theta = \frac{\pi}{2}$.

71. A flagstaff on the top of a tower of height h subtends a small angle θ at the eye of an observer of negligible height at a distance ' d ' from the foot of a tower; show that the length of the flagstaff is $\frac{h^2 + d^2}{d} \theta$ nearly.

72. If $\sin \theta + \sin \phi = a$ and $\cos \theta + \cos \phi = b$, prove that

$$\tan \frac{\theta - \phi}{2} = \pm \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}.$$

73. Solve the equation

$$3 \tan \theta + \cot \theta = 5 \operatorname{cosec} \theta.$$

74. Two circles of radii ' a ' and ' b ' touch each other externally; θ is the angle contained by the common tangents to these circles. Show that

$$\sin \theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

75. If A , A_1 , A_2 , A_3 be the areas of the incircle and the escribed circles, show that

$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$$

76. (a) In a triangle,

$\cos^2 \frac{A}{2}$, $\cos^2 \frac{B}{2}$, $\cos^2 \frac{C}{2}$ are in A.P. Prove that $\cos A$, $\cos B$, $\cos C$ are in A.P.

(b) D is the mid-point of the side BC of triangle ABC . Show that $\cot BAD - \cot B = 2 \cot A$.

77. Show by using $\cos 36^\circ$ and $\sin 18^\circ$ or otherwise

$$2^7 \cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{3\pi}{15} \cos \frac{4\pi}{15} \cos \frac{5\pi}{15} \cos \frac{6\pi}{15} \cos \frac{7\pi}{15} = 1.$$

78. The angles of elevation of the top of a tower from two points distant p and q from its base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{pq}$, and if a is the angle subtended at the top of the tower by the line joining these points, then

$$\sin a = \frac{p - q}{p + q}.$$

79. Prove that in a triangle,

$$(i) \sum \frac{bc}{r_1} = 2R \left(\sum \frac{a}{b} + \sum \frac{b}{a} - 3 \right)$$

$$(ii) \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} = 4 \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \right)$$

where p_1 , p_2 , p_3 are the altitudes.

80. If $xy + yz + zx = 1$, prove trigonometrically that

$$(i) \quad \sum \frac{x}{1+x^2} = \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

$$(ii) \quad \sum 4yz(1-x^2) = (1-x^2)(1-y^2)(1-z^2) \\ + (1+x^2)(1+y^2)(1+z^2).$$

[Hint—Use the identity $\sum \cot B \cot C = 1$ in a triangle.]

81. Prove that $r + r_1 + r_2 - r_3 = 4R \cos C$. Deduce that the triangle is right-angled if $r + r_1 + r_2 = r_3$.

82. Show that

$$\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}.$$

ANSWERS

I a, Page 9

(1) (i) 60° , $\frac{\pi}{3}$, 66.6s. (ii) 108° , $\frac{3\pi}{5}$, 120s. (iii) 120° , $\frac{2\pi}{3}$, 133.3s. (iv) 135° , $\frac{3\pi}{4}$, 150s. (v) 144° , $\frac{4\pi}{5}$, 160s. (vi) 150° , $\frac{5\pi}{6}$, 166.6s. (3) $2/\pi$ radians. (4) 5."1312. (5) 5172 feet. (6) 15.78 feet. (7) $20\pi/3$. (8) $114^\circ 35' 30''$; .0978. (9) 3.145. (10) 7."37. (12) 0.4189, 1.0472, 1.6765. (13) 0."48; 41 feet 8 inches. (14) 93,780,000 miles. (15) 8 feet 10 inches. (16) $2^\circ 2' 1''$; 0.0355. (17) 0.7736; 0.2736; $44^\circ 19' 30''$; $15^\circ 40' 30''$. (18) $32^\circ 20' 24''$. (19) 2.79 sq. in. (20) $9\pi/2$.

I b, Page 14

(1) 3595.2 miles; 2688 miles. (2) 316.4 miles. (3) 5368.8 miles. (4) 2282 miles. (5) 4646 miles. (6) 0.2208". (7) $5^\circ 44'$. (8) $1."91$. (9) $1^\circ 26'$. (10) 3.32 feet; 1851.85 metres. (11) 3581481.48 metres; 13581481.48 metres. (12) 17 nautical miles; 0.54.

II a, Page 22

(4) (i) Real. (ii) One real and the other imaginary. (iii) Real. (iv) One real and the other imaginary. (v) Both imaginary. (vi) Real.

II b, Page 31

(1) $\cos \theta = .111$; $\sin \theta = 0.994$; $\tan \theta = 8.944$. (2) $\tan \theta = 14.25$; $\operatorname{cosec} \theta = 1.0025$. (3) $\sin A = \frac{3}{5}$; $\cos A = \frac{4}{5}$. (5) $\tan 15^\circ = (2 - \sqrt{3})$ etc. (6) $\tan 22\frac{1}{2}^\circ = (\sqrt{2} - 1)$ etc. (7) 15.708; 47.124. (8) (i) 20.944; 41.888. (ii) 10.472; 52.360. (9) 3 feet; $53^\circ 8'$. (10) 3rd and 4th; $-\frac{5}{13}, \frac{5}{12}, \frac{5}{13}, -\frac{5}{12}$. (12) (i) 1.021. (ii) 0.4741. (iii) 1.769. (iv) 3.064.

III, Page 42

(1) 9.562 ; 9.562. (2) $\frac{62}{\sqrt{3}}$; $\frac{124}{\sqrt{3}}$. (3) 177.45. (4) 366 yards. (5) 126.8 feet. (6) 1 mile. (7) 17.32 miles. (8) 14.847 miles per hour. (9) 136.7 ; 236.7. (10) 75.78 yards. (11) $4(\sqrt{3}-1)$ times the shortest distance, per hour. (12) $20(\sqrt{3}+1)$; $20\sqrt{3}$ feet. (13) 128.1 yards ; 187.6 yards per hour. (14) $300(\sqrt{3}+1)$. (15) 250 feet.

IV, Page 56

(1) The sine, cosine and tangent of each angle are given in order. (i) $\frac{\sqrt{3}}{2}$; $\frac{1}{2}$; $\frac{\sqrt{3}}{1}$. (ii) 0 ; -1 ; 0. (iii) $-\frac{1}{2}$; $-\frac{\sqrt{3}}{2}$; $+\frac{1}{\sqrt{3}}$. (iv) -1 ; 0 ; ∞ . (v) $-\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$; $\sqrt{3}$. (vi) $-\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{2}}$; -1. (vii) $\frac{\sqrt{3}}{2}$; $\frac{1}{2}$; $\sqrt{3}$; (viii) $\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; -1. (ix) .6428 ; .7660 ; .8391. (2) (i) $\sin 26^\circ$. (ii) $-\cos 5^\circ$. (iii) $\tan 36^\circ$. (iv) $\sec 24^\circ$. (v) $-\sec 6^\circ$. (vi) $-\tan 10^\circ$. (vii) $-\tan 27^\circ$. (viii) $-\sin 34^\circ$. (ix) $-\sec 4^\circ$. (3) (i) $n. 360^\circ+60^\circ$ and $n. 360^\circ+120^\circ$. (ii) $n. 360^\circ+120^\circ$ and $n. 360^\circ-60^\circ$. (iii) $n. 360^\circ+180^\circ$. (iv) $n. 360^\circ\pm 45^\circ$. (v) $n. 360^\circ\pm 45^\circ$; $n. 360^\circ\pm 135^\circ$. (vi) $n. 360^\circ+210^\circ$ or $n. 360^\circ-30^\circ$. (4) $\sin A = \frac{\sqrt{3}}{2}$; $\tan A = -\sqrt{3}$. (5) -2 ; $-\frac{\sqrt{3}}{2}$. (6) $\sin A = -\frac{12}{13}$; $\tan A = -\frac{12}{5}$. (7) $90^\circ > A > 45^\circ$. (9) 245° ; $n. 360^\circ+245^\circ$. (10) $\cot 43^\circ 37' = 1.0495$ etc. (11) $-\cos^4 A$.

V, Page 67

(3) 1.28. (5) $\frac{33}{65}$. (6) 0.985. (7) $\frac{84}{85}$; $-\frac{36}{85}$. (10) $105^\circ = (180^\circ - 75^\circ)$ and $285^\circ = (360^\circ - 75^\circ)$. (13) $36^\circ 52'$; $-53^\circ 8'$; $126^\circ 52'$. (14) $\sqrt{a^2+b^2}$; 13. (29) + ; + ; - ; + ; +. (30) - ; + ; - ; - ; +.

VI, Page 77

- (1) .352 ; - .28 ; $-\frac{24}{7}$. (2) - 0.936 ; 0.96 ; $-\frac{7}{24}$.
 (3) 0.932 ; 0.9620 ; 0.2836. (4) $\frac{40}{41}$; $-\frac{41}{9}$; $-\frac{115}{236}$.
 (5) $-\frac{44}{117}$; $-\frac{41}{38}$. (19) 0.7601 ; 0.1827.

VII-a, Page 87

- (20) 90° ; 54° ; 126° . (22) $\alpha = \frac{90^\circ + \beta}{2}$.

VIII-a, Page 104

- (1) (i) 0.2181. (ii) 0.9625. (iii) 0.6699. (iv) 1.401.
 (v) 1.352. (vi) 0.7297. (vii) 0.9102. (viii) 3.1314. (ix)
 -0.9385 . (x) 0.6986. (xi) 5.0504. (xii) 9.5792. (2)
 (i) 60° . (ii) 30° . (iii) $-62^\circ 43' 30''$. (iv) $-13^\circ 34'$. (3)
 (i) 60° . (ii) $69^\circ 47'$. (iii) $117^\circ 10' 30''$. (iv) $19^\circ 7'$.
 (v) $-66^\circ 55'$. (vi) $26^\circ 34'$. (4) (i) $n\pi + (-1)^n 20^\circ 13'$.
 (ii) $2n\pi \pm 76^\circ 26'$. (iii) $p\pi - 67^\circ 48'$. (iv)
 $\frac{n\pi}{3} - (-1)^n 25^\circ 35'$. (v) $n\pi \pm 56^\circ 35'$. (vi) $\theta = \frac{n\pi}{pq}$.
 (vii) $\theta = \frac{p\pi}{m+n}$ or $\frac{(2p-1)\pi}{m-n}$. (viii) $\theta = \frac{p\pi}{3}$ or $\frac{p\pi}{2}$. (ix)
 $\frac{p\pi}{4} + (-1)^p 2^\circ 3' 15''$. (x) $\frac{n\pi - 12^\circ 32'}{p}$.

Page 105

- (2) $n\pi \pm 45^\circ$ or $n\pi \pm 60^\circ$. (4) $2n\pi$; $2n\pi \pm \frac{2\pi}{3}$.

Page 106

- (2) $\frac{n\pi}{2} + \frac{\pi}{8}$. (3) $n\pi + (-1)^n \frac{\pi}{2} - \frac{\pi}{4}$. (4)
 $A = \frac{2n\pi}{5} + \frac{\pi}{10}$ or $A = 2n\pi - \frac{\pi}{2}$.

VIII-b, Page 111

- (1) $x = (2n + 1)\pi$ or $2n\pi + \frac{\pi}{2}$. (2) $x = (2n + 1)\frac{\pi}{4}$ or $2n\pi \pm \frac{\pi}{3}$. (3) $x = n\pi - (-1)^n \frac{\pi}{2}$ or $n \cdot 180^\circ + (-1)^n 23^\circ 35'$. (4) $n\pi + \frac{\pi}{4}$; $n \cdot 180^\circ - 36^\circ 52'$. (5) $n \cdot 90^\circ - 10^\circ 54'$. (6) $n \cdot 90^\circ + (-1)^n 15^\circ$. (7) $n \cdot 180^\circ + 45^\circ$ or $n \cdot 180^\circ + 14^\circ 2'$. (8) $x = n \cdot 180^\circ + (-1)^n 22^\circ 27'$. (9) $x = 2n\pi$. (10) $\frac{n\pi}{8}$ or $2n\pi \pm \frac{\pi}{3}$. (11) $n\pi + \frac{\pi}{4}$ or $n \cdot 180^\circ + 36^\circ 52'$. (12) $n\pi + \frac{\pi}{4}$ or $n \cdot 180^\circ + 49^\circ 24'$. (13) $n\pi$ or $n \cdot 360^\circ \pm 39^\circ 14'$; $n \cdot 360^\circ \pm 140^\circ 46'$. (14) $n\pi + \tan^{-1}[(\sin \alpha + m) / (\cos \alpha - l)]$. (15) $n\pi + \tan^{-1}[(3 \cos \beta - \sin \alpha) / (3 \sin \beta + \cos \alpha)]$. (16) $\sin \left(x \pm \frac{\pi}{4} \right) = \frac{4n + 1}{2\sqrt{2}}$. (17) $\frac{(2n + 1)\pi}{2(p + q)}$. (18) $A = n \cdot 360^\circ + 36^\circ 52'$. (19) $x = n \cdot 360^\circ + 67^\circ 23' \pm 39^\circ 43'$. (20) $n\pi - \frac{\pi}{4}$ or $2n\pi$ or $2n\pi - \frac{\pi}{2}$. (21) $2n\pi$ or $n \cdot 360^\circ + 112^\circ 36'$. (22) $(2n + 1)\pi$ or $n \cdot 360^\circ + 73^\circ 44'$. (23) $n \cdot 360^\circ + 71^\circ 34' \pm 63^\circ 26'$. (24) $2n\pi$ or $2n\pi - \frac{\pi}{2}$. (25) $(2n + 1)\pi$ or $2n\pi - \frac{\pi}{3}$. (26) $\frac{1}{2}, \frac{1}{2}, -1$. (27) No real value of x . (28) 0.3509 ; 5.3091 ; -0.859 . (29) $\frac{n\pi}{3}$ or $n \cdot 180^\circ \pm 35^\circ 16'$. (30) $n\pi$ or $(2n + 1)\pi / 9$. (31) No real value of θ .

IX-a, Page 129

- (25) $C = 28^\circ 6'$, $a = 2.165$; $b = 1.321$; $c = 1.1775$. (32) 0.4 .

IX-b, Page 141

- (1) $r = 7.88$; $r_1 = 16.50$; $r_2 = 34.66$; $r_3 = 26.66$.
 (2) $R = 6.5$; $S = 30$. (3) $R = 19.43$; $r = 8.89$. (4)
 50.42 ; 64.83 ; 35.83 ; 0.586 ; 0.754 ; 0.417 . (24) $2r \cos \frac{A}{2}$
 etc.; r . (25) 4.49 ; 89.59 .

X-a, Page 152

- (1) $C = 28^\circ 49'$; $A = 61^\circ 11'$; $b = 112.7$. (2)
 $C = 41^\circ 8'$; $A = 48^\circ 52'$; $b = 116.3$. (3) $C = 71^\circ 25'$;
 $A = 18^\circ 35'$; $b = 97.14$. (4) $C = 11^\circ 52'$; $A = 78^\circ 8'$;
 $b = 96.56$. (5) $C = 78^\circ 17'$; $A = 11^\circ 43'$; $b = 907.4$.
 (6) $A = 13^\circ 44'$; $C = 76^\circ 16'$; $c = 959.4$. (7) $A = 33^\circ 22'$;
 $C = 56^\circ 38'$; $c = 206.1$. (8) $C = 78^\circ 30'$; $A = 11^\circ 30'$;
 $a = 115.5$. (9) $C = 35^\circ 29'$; $A = 54^\circ 31'$; $a = 281.5$.
 (10) $C = 46^\circ 31'$; $a = 468$; $c = 493.5$. (11) $C = 14^\circ 22'$;
 $a = 658.9'$; $c = 168.8$. (12) $A = 26^\circ 16'$; $a = 333.3$;
 $c = 675.3'$. (13) $A = 68^\circ 3'$; $c = 364.6$; $a = 904.4$. (14)
 $C = 29^\circ 57'$; $c = 56.93$; $b = 114$. (15) $A = 20^\circ 48'$;
 $c = 227.5$; $b = 243.3$. (16) $C = 52^\circ 45'$; $a = 93.85$;
 $b = 155$. (17) $A = 65^\circ 41'$; $a = 1257$; $b = 1379$. (18)
 80.81 miles. (19) $58^\circ 41'$. (20) 10.34 .

X-b, Page 167

- (1) (i) $A = 34^\circ 46'$; $B = 58^\circ 48'$; $C = 86^\circ 26'$. (ii)
 $A = 32^\circ 12'$; $B = 34^\circ 36'$; $C = 113^\circ 12'$. (iii) $A = 33^\circ 23'$;
 $B = 45^\circ 24'$; $C = 101^\circ 8'$. (iv) $A = 41^\circ 25'$; $B = 55^\circ 47'$;
 $C = 82^\circ 4'$; $a = 2.6460$; $b = 3.3076$; $c = 3.9690$. (v)
 $A = 23^\circ 8' 30''$; $B = 86^\circ 39'$; $c = 70^\circ 12' 30''$; $a = 1.965$;
 $b = 4.992$; $c = 4.705$. (2) (i) $A = 30^\circ 26'$; $B = 106^\circ 6'$;
 $c = 1679$. (ii) $C = 133^\circ 21'$; $B = 33^\circ 23'$; $a = 144.1$.
 (iii) $A = 43^\circ 17' 30''$; $C = 32^\circ 48' 30''$; $b = 136.1$. (iv)
 $A = 19^\circ 52' 30''$; $B = 97^\circ 23' 30''$; $a = 102$; $b = 2.98$;
 $c = 2.67$. (v) $C = 130^\circ 32' 30''$; $B = 15^\circ 18' 30''$;
 $c = 3.50$; $b = 1.21$; $a = 2.58$. (3) (i) $A = 4^\circ 31'$;
 $b = 299$; $c = 305.5$. (ii) $A = 35^\circ 33'$; $c = 39.64$;
 $a = 27.29$. (iii) $A = 35^\circ 35'$; $a = 24.4$; $b = 41.81$. (iv)
 $B = 76^\circ 9'$; $a = 29.77$; $b = 40.70$. (v) $A = 44^\circ 52'$;
 $b = 10.91$; $c = 27.19$. (4) (i) $B = 48^\circ 30'$; $C = 68^\circ 10'$;

$c = 13.05$. (ii) No solution. (iii) $A = 66^\circ 9'$; $B = 63^\circ 39'$; $a = 92.94$; another solution. $A = 13^\circ 27'$; $B = 116^\circ 21'$; $a = 23.68$. (iv) $B = 52^\circ 19'$; $C = 12^\circ 21'$; $c = 2198$. (v) $B = 48^\circ 29'$; $C = 10^\circ 55'$; $c = 478.8$. (vi) $A = 28^\circ 42'$; $C = 113^\circ 9'$; $a = 1.92$; $c = 2.47$; $c = 3.66$. (5) 97.03 and 182.5 units; $71^\circ 48'$. (6) 51.33; Area = 1855. (7) $\tan C = 0.7423$; $\tan A = 8.6651$. (8) 805 yards. (9) $a = 29.82$; $b = 24.98$; $c = 31.49$. (10) 68.96 miles. (11) $a = 87.70$; $B = 18^\circ 6'$; $C = 78^\circ 6'$; $A = 83^\circ 48'$. (12) $a = 11.27$; $b = 13.29$; $c = 21.44$. (13) (i) $A = 97^\circ 50'$; $B = 16^\circ 52'$; $b = 30.9$; $a = 105.5$. (ii) $A = 51^\circ 8'$; $B = 67^\circ 32'$; $C = 61^\circ 20'$; $a = 36.15$; $c = 40.74$. (iii) $A = 68^\circ 2'$; $B = 46^\circ 28'$; $a = 21.54$; $b = 16.84$; $c = 21.13$. (iv) $B = 73^\circ 8'$; $C = 59^\circ 23'$; $a = 7.879$; $b = 10.23$; $c = 9.19$. (14) $79^\circ 40'$; $61^\circ 11'$; 13.45 sq. in. (15) $a = 3.571$; $R = 2.461$. (16) $PR = 5.67$; $QS = 4.337$; $SP = 2.479$. Area = 11.906. (18) $A = 77^\circ 55'$; $B = 73^\circ 32' 30''$; $C = 28^\circ 32' 30''$.

XI, Page 185

(1) 72.68 feet. (2) 524.2 yards. (3) 608.9. (4) 292.8. (5) 239.5; 302.4. (6) 3.967 feet. (7) 659.2. (8) 77.32. (9) 87.86. (10) 80.39 yards. (11) $2^\circ 24'$. (12) 30.82. (13) 303.4 yards per hr. (14) 138.5 feet. (15) 212.9 feet. (16) 166.8 feet. (17) 1272 sq. feet. (18) 13.01. (19) 16.94. (20) 23.22; 25.79. (21) 8.828 feet. (22) 104 feet. (23) 0.8986 mile. (24) 223. (25) 218.7. (26) 2201. (27) 4348. (28) 14.49 miles; E. $67^\circ 33'$ N. (29) 43.1 feet. (30) 81.5 or 1.9 yards. (31) 30 feet. (32) 17.54. (33) E. $47^\circ 20'$ N. (37) 685.3. (38) 222.7. (39) E. $26^\circ 34'$ N. (40) 3379.

XII, Page 205

(1) $+0.3162$; -0.9487 ; -0.3333 . (2) $\sqrt{3}$. (3) -0.7522 ; -7 or $-\frac{1}{7}$. (4) $\sin \frac{A}{2} = 0.9258$; $\cos \frac{A}{2} = 0.3781$; $\tan \frac{A}{2} = 2.4482$. (5) 1.72 or -0.12 . (7) 4.265 or 0.265. (8) 0.2004. (11) 0.157; 0.988.

- (13) (i) $2 \cos \frac{A}{2} = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}$
 (ii) $2 \cos \frac{A}{2} = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}$
 (iii) $2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$
 (iv) $2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$
 (v) $2 \cos \frac{A}{2} = +\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$
 (vi) $2 \cos \frac{A}{2} = +\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$
 (vii) $2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$
 (viii) $2 \cos \frac{A}{2} = +\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$

- (14) A lies between $2\pi(2r+1) \pm \frac{\pi}{2}$. (15) $\frac{3}{2}$; $\frac{3}{2}$.

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- (1) 0.588; 49.6776. (2) $\pi/2$. (3) 27.875. (4) 2.796.
 (6) 3 : 4. (7) 2.0961.

MADRAS UNIVERSITY QUESTIONS

March 1941

- I. 3.489; 3.819; 4.699. II. 270.5. IV. $-\frac{44}{125}$; $\frac{1}{3}$.
 V. (b) 300. (c) $n \cdot 180^\circ + (-1)^n 17^\circ 28' - 53^\circ 8'$.

September 1941

- III. $n\pi$ or $\frac{2n\pi}{3}$. IV. B = $69^\circ 10'$; C = $46^\circ 38'$.
 V. 16.47; $78^\circ 30'$ N of E.
 V. T.—17

March 1942

III. $n \cdot 360^\circ \pm 120^\circ$. IV. $A = 57^\circ 21'$; $C = 43^\circ 29'$; $b = 42.39$. V. 100 $\sqrt{6}$ feet.

September 1942

V. $A = 47^\circ 10'$; $B = 82^\circ 58'$; $C = 49^\circ 52'$. VI. 100 feet.

March 1943

IV. $n \cdot 180^\circ + 45^\circ$; $n \cdot 180^\circ + 26^\circ 34'$. V. $a = 24.06$; $c = 29.57$, $B = 99^\circ 30'$. VI. 88.16 feet.

September 1943

II. $n \cdot 180^\circ + 43^\circ 10'$. III. $B = 25^\circ 30'$; $C = 82^\circ 10'$; $c = 190$. IV. 1.428 miles.

March 1944

IV. $\frac{n\pi}{4}$ or $n \cdot 180^\circ \pm 30^\circ$. V. $b = 9.101$; $C = 90^\circ$, $A = 33^\circ 24'$. VI. 238.36; $75^\circ 29'$.

September 1944

II. $2n\pi \pm \frac{\pi}{3}$. III. No solution. IV. 117.4 miles per hr.; S. $67^\circ 33'$ E.

March 1945

II. $B = 67^\circ 2'$; $A = 32^\circ 58'$; $c = 25^\circ 53'$. III. $n\pi + \frac{\pi}{2}$ or $n \cdot 180^\circ + 26^\circ 34'$ or $n \cdot 180^\circ + 18^\circ 26'$ or $n \cdot 180^\circ + 45^\circ$. IV. 176.

September 1945

III. $A = 69^\circ 23'$; $B = 95^\circ 37'$; $b = 4.998$; $A_1 = 110^\circ 37'$; $B_1 = 54^\circ 23'$; $b_1 = 4.083$.

March 1946

II. $n \cdot 360^\circ + 68^\circ 12' \pm 42^\circ 1'$. IV. 48.49 feet.

September 1946

II. $\theta = n\pi$ or $n \cdot 180^\circ + 67^\circ 30'$ or $n \cdot 180^\circ - 22^\circ 30'$.
 III. $B = 53^\circ 32'$; $A = 37^\circ 30'$; $C = 88^\circ 58'$. IV. 4082 furlongs.

March 1947

I. A lies between $(4n + 2)\pi \pm \frac{\pi}{2}$. II. (a) $\theta = n \cdot 180^\circ + 63^\circ 26'$ or $n \cdot 180^\circ - 45^\circ$. III. $C = 58^\circ 56'$ or $121^\circ 4'$.

September 1947

IV. (b) 0.8986 mile; 52.6 seconds still later.

March 1948

IV. 185.5.

September 1948

III. $A = (2n + 1)\frac{\pi}{2}$; $13 \sin(x - 67^\circ 23')$. IV. 10.33 miles per hr.

March 1949

I. 0.75. IV. 91.87 feet.

September 1949

IV. 0.5176; 1.9318; 2 miles.

March 1950

I. 2. II. $(2n + 1)\frac{\pi}{4}$ or $2n\pi \pm \frac{2\pi}{3}$. III. $C = 63^\circ 40'$; $B = 55^\circ 20'$. IV. 0.6289 mile.

September 1950

I. The positive root of the equation $2(2x^2 + x - 1)^2 - (1 + x) = 0$. III. $98^\circ 42'$; $190^\circ 14'$. IV. 815.1 metres.

March 1951

I. $B = 30^\circ$; $C = 36^\circ 52'$; or $B_1 = 30^\circ$, $C_1 = 143^\circ 8'$.

September 1951

I. $x = n \cdot 12^\circ - 3^\circ$ or $(2n + 1) 18^\circ - 9^\circ$. $\pi = 9^\circ$, 21° , 33° , 45° , 57° , 69° , 81° . III. $\phi = 60^\circ 14'$; $c = 19.53$.

March 1952

I. $x = n \cdot 72^\circ - 18^\circ$ or $(4n + 1) 90^\circ$. $x = 54^\circ$, 90° , 126° , 198° , 270° , 342° . I. (b) $A = n \cdot 90^\circ + p 180^\circ \pm 30^\circ + (-1)^n 15^\circ$. $B = n \cdot 90^\circ + (-1)^n 15^\circ - p 180^\circ \mp 30^\circ$. $A = 105^\circ$; $B = 45^\circ$ ($n = 1$, $p = 0$.) III. 113.3.

September 1952

I. $x = -102^\circ 17'$; $257^\circ 43'$; $-330^\circ 27'$, $29^\circ 33'$. III. 448.2.

March 1953

I. $x = n\pi + (-1)^n \frac{\pi}{6}$; $n\pi + (-1)^n \frac{\pi}{2}$; 30° , 90° , 270° . III. $B = 77^\circ 17'$; $C = 55^\circ 23'$. IV. 282.3 yards.

MISCELLANEOUS EXAMPLES

(4) 1.5. (34) 7.018. (36) $n \cdot 360^\circ \pm 72^\circ$; $n \cdot 360^\circ \pm 144^\circ$.
 (38) $n\pi$. (39) $n\pi \pm \frac{\pi}{6} + \frac{B}{2}$. (40) $\frac{(2p+1)\pi}{m+n \cdot 2}$. (41)
 $n\pi + (-1)^n \frac{\pi}{2}$. (42) $n \cdot 180^\circ + 18^\circ 20'$. (47) 1183 feet.
 (48) 55.40. (51) 1485 yards. N. $8^\circ 7'$ W. (52) 328 yards.
 (53) 163.7. (54) 400.3. (55) 5475 feet. (56) 11.28 miles.
 S. $34^\circ 4'$ W. (57) $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{3\pi}{5}$, $\frac{4\pi}{5}$. (61) (i) 0.4. (63)
 $n\pi$; $n\pi \pm \frac{\pi}{3}$; $n \cdot 180^\circ \pm 35^\circ 16'$. (ii) $n\pi$; $n \cdot 180^\circ \pm 16^\circ 47'$.
 (64) 1.841 miles. (65) $n \cdot 180^\circ - 70^\circ 54'$. (66) $x = (n + 2m) 180^\circ + 61^\circ 56' \pm 64^\circ 51'$. $y = (n - 2m) 180^\circ + 61^\circ 56'$
 $\mp 64^\circ 51'$. (68) $\frac{\pi}{6}$, $\frac{\pi}{6}$; $\frac{\pi}{2}$, $\frac{\pi}{2}$. (73) $\theta = 2n\pi \pm \frac{\pi}{3}$.

