

CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS

VOLUME III

G. SEIFERT	F. HAAS
L. MARKUS	G. HUFFORD
E. PINNEY	W. T. KYNER
V. B. HAAS	S. P. DILBERTO
R. E. GOMORY	M. D. MARCUS
S. BAROCIO	P. KOOSIS

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PREFACE

The papers in this third volume of Contributions fall under two distinct categories. The second beginning with the paper by Hufford, to the end of the collection, consists of contributions by Hufford, Diliberto and students of the latter, essentially on biperiodic systems. This part of the volume offers a certain degree of homogeneity and may be left to speak for itself.

The first category is more heterogeneous. In the paper by Seifert there is discussed an equation arising in the study of the oscillations of a synchronous motor about its average velocity.

L. Markus considers the equation

$$\dot{x} = f(x; t)$$

(x, f are n -vectors) where $f(x; t) \longrightarrow f(x)$ as $t \longrightarrow +\infty$ and compares initial and limiting systems.

Pinney attacks the problem of "practical" determination of oscillations in n -th order systems.

The paper by Violet Haas deals with the general problem of approach of a solution of

$$\epsilon \ddot{x} + f(t, x, \dot{x}, \epsilon) = 0$$

as $\epsilon \longrightarrow 0$ to a solution of

$$f(t, x, \dot{x}, 0) = 0.$$

Gomory's paper is a contribution (the first so far) to the relation between the critical points at infinity of a system

$$\dot{x} = X(x), \quad (x, X \text{ are } 2\text{-vectors})$$

and the periodic solutions of

$$\dot{x} = X(x) + E(t)$$

where E is a periodic vector. He also analyzes the behavior in the large

of the solutions of

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

where f and g are polynomials with degree $f \geq$ degree g , and applies this to periodic solutions of the same system with a periodic right handside.

Barocio fully analyzes the critical point at the origin of a system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y)$$

where X, Y are analytic and have first degree terms whose coefficient matrix, although not zero, does have both characteristic roots zero.

Felix Haas studies equations on a two-dimensional manifold with vector not necessarily a gradient, and obtains relations extending those of Morse for the gradient case. These relations involve not only the numbers of critical points but also the numbers of limit-cycles of various types.

A number of the papers of this collection have been written under sponsorship by certain governmental agencies. The nature of the agency and the contract number are indicated in each case.

Solomon Lefschetz

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I. A ROTATED VECTOR APPROACH TO THE PROBLEM OF STABILITY OF SOLUTIONS OF PENDULUM-TYPE EQUATIONS

George Seifert

We consider the differential equation

$$(1) \quad \frac{d^2\theta}{dt^2} + f(\theta, \alpha) \frac{d\theta}{dt} = g(\theta)$$

where

- (i) $f(\theta, \alpha)$ is defined and has continuous second partial derivatives for $\alpha > 0$, $-\infty < \theta < \infty$, and in this domain

$$f(\theta, \alpha) > 0, \quad \frac{\partial f}{\partial \alpha} > 0,$$

and $f(\theta + 2\pi, \alpha) = f(\theta, \alpha)$.

- (ii) $g(\theta)$ has continuous second derivatives everywhere, $g(\theta) = g(\theta + 2\pi)$, $g(\theta)$ has zeros θ_1 for each of which $g'(\theta_1) \neq 0$, and

$$\int_0^{2\pi} g(\theta) d\theta > 0.$$

This equation, of the so-called pendulum-type, arises in the study of the oscillations of a synchronous motor about its average angular velocity [1]. If it has a solution $\theta(t, \alpha)$ such that if $\frac{d\theta}{dt}(t, \alpha) = y = y(\theta, \alpha)$ is periodic in θ with period 2π , then $\theta(t, \alpha)$ will be called a periodic solution of the second kind of Eq. (1). With respect to the performance of the synchronous motor such a solution would correspond to a continuous slipping of the rotor; that is, asynchronous performance.

It is known [3] that for fixed α if such a solution exists, it is unique and its time derivative is non-negative, while if no such solution

exists, then each solution of Eq. (1) tends to a finite limit as $t \rightarrow \infty$, i.e., is stable.

Define the set L^* to be the set of α for which periodic solutions of the second kind of Eq. (1) exist. Define the class P_2^* of functions $y(\theta, \alpha) = \frac{d\theta(t, \alpha)}{dt}$ where $\theta(t, \alpha)$ is a periodic solution of Eq. (1) of the second kind and where $\alpha \in L^*$; i.e.,

$$P_2^* = \bigcup_{\alpha \in L^*} y(\theta, \alpha);$$

denote by P_2 the subset of P_2^* consisting of the positive functions of P_2^* and by L the subset of L^* for which $y(\theta, \alpha) \in P_2$ exist.

In this paper a method analogous to that of rotated vector fields used by G. F. D. Duff [4] is used to obtain information on the structure of the set L^* , primarily with respect to upper bounds. In Part 1 we show that L^* is an interval, that the functions $y(\theta, \alpha) \in P_2$ move upward analytically as $\alpha \rightarrow 0$, $\alpha \in L^*$, and obtain an equation for $\frac{\partial y(\theta, \alpha)}{\partial \alpha}$ in terms of $y(\theta, \alpha)$ for $y(\theta, \alpha) \in P_2$. In Part 2 the special case $f(\theta, \alpha) = \alpha f(\theta)$ is considered, and using the results of Part 1, a method is given such that corresponding to a known $\alpha \in L$, an upper bound α^* on the set L can be determined; i.e., a value α^* can be determined such that for $\alpha > \alpha^*$, all solutions of Eq. (1) will be stable.

Our point of departure is the equation of the phase trajectories of Eq. (1):

$$(2) \quad y'(\theta, \alpha) = \frac{g(\theta) - f(\theta, \alpha)y(\theta, \alpha)}{y(\theta, \alpha)}$$

obtained by putting $\frac{d\theta}{dt} = y$, noting that $\frac{dy}{dt} = g(\theta) - f(\theta, \alpha)y$ from Eq. (1), and dividing the latter equation by the former. Here, as in what follows, the prime denotes differentiation with respect to the first indicated variable.

The singularities (in the sense of Poincaré) of Eq. (2) consist of the points $(\theta_1, 0)$ of the phase plane (θ, y) , the θ_1 being the zeros of $g(\theta)$. We note that as a consequence of condition $g'(\theta_1) \neq 0$ for each θ_1 , the number of zeros of $g(\theta)$ in $(0, 2\pi)$ is finite. An analysis of these singularities shows that the points $(\theta_{21}, 0)$ for which $g'(\theta_{21}) > 0$ are saddle points, each other singularity being either a focus or a node.

Since the functions $f(\theta, \alpha)$ and $g(\theta)$ have period 2π in θ , a cylindrical phase space (see Minorsky [5] pp. 116-117) may also be used. On it, the phase trajectories of P_2 solutions are closed curves encircling the phase cylinder and the Bendixson criteria for existence of

closed phase trajectories become applicable. In view of the fact that Amerio's work [3] to which we will frequently refer, employs the usual cartesian phase plane, we will suggest the use of cylindrical phase space in an auxiliary capacity only.

Part 1: The General Case

We shall assume in this case that the set P_2 is non-void and begin by proving two non-intersection theorems for integral curves of Eq. (2) corresponding to distinct values of α .

THEOREM 1.1. Let $\alpha_1 < \alpha_2$, and $y(\theta, \alpha_1)$ and $y(\theta, \alpha_2)$ be solutions of Eq. (2) positive on $a \leq \theta \leq b$ such that $y(a, \alpha_1) > y(a, \alpha_2)$. Then $y(\theta, \alpha_1) > y(\theta, \alpha_2)$ for $a \leq \theta \leq b$.

PROOF. Suppose c to be the smallest value of θ in (a, b) such that $y(c, \alpha_1) = y(c, \alpha_2)$. From Eq. (2), $y'(c, \alpha_1) - y'(c, \alpha_2) = f(c, \alpha_2) - f(c, \alpha_1) > 0$; i.e., $y'(c, \alpha_1) > y'(c, \alpha_2)$. Since $y(\theta, \alpha_1)$ and $y(\theta, \alpha_2)$ have continuous derivatives for $a \leq \theta \leq b$, this is impossible.

THEOREM 1.2. Let $\alpha_1 < \alpha_2$ and $y(\theta, \alpha_1)$ and $y(\theta, \alpha_2)$ be solutions of Eq. (2) positive on $a \leq \theta \leq b$. Then the integral curves of $y(\theta, \alpha_1)$ and $y(\theta, \alpha_2)$ cannot be tangent nor intersect more than once in this interval.

PROOF. If at $\theta = c$, $a \leq c \leq b$, we have $y'(c, \alpha_1) = y'(c, \alpha_2)$ and $y(c, \alpha_1) = y(c, \alpha_2)$, an impossibility. Hence the integral curves cannot be tangent. If there are two points of intersection, then at one of these two points, say $\theta = c_1$, we must have $y'(c_1, \alpha_1) < y'(c_1, \alpha_2)$, which implies $f(c_1, \alpha_1) > f(c_1, \alpha_2)$, a contradiction. This completes the proof.

The next theorem shows that each integral curve of a P_2 function has a neighborhood, each point of which contains an integral curve corresponding to another P_2 function, and that in this neighborhood the P_2 curves vary analytically with α .

We shall need the following

LEMMA. Each P_2 solution of Eq. (2) is stable; i.e., given θ_0 and $y_0(\theta, \alpha) \in P_2$, there exists an $\epsilon > 0$ such that if $|y(\theta_0, \alpha) - y_0(\theta_0, \alpha)| \leq \epsilon$ where $y(\theta, \alpha)$ is a solution of Eq. (2), then

$$\lim_{\theta \rightarrow \infty} |y(\theta, \alpha) - y_0(\theta, \alpha)| = 0.$$

PROOF. Let $\xi(\theta, \alpha) = y(\theta, \alpha) - y_0(\theta, \alpha)$. Then for $|\xi|$ small, the well-known equation for this first variation ξ is

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$$\xi'(\theta, \alpha) = - \frac{g(\theta)}{y_0^2(\theta, \alpha)} \xi(\theta, \alpha)$$

from which we obtain

$$\xi(\theta, \alpha) = c \exp \left[- \int_{\theta_0}^{\theta} \frac{g(t)}{y_0^2(t, \alpha)} dt \right].$$

But since $g(\theta) = y_0'(\theta, \alpha)y_0(\theta, \alpha) + f(\theta, \alpha)y_0(\theta, \alpha)$, we have that

$$\xi(\theta, \alpha) = c \frac{y_0(\theta, \alpha)}{y_0(\theta_0, \alpha)} \exp \left[- \int_{\theta_0}^{\theta} \frac{f(t, \alpha)}{y_0(t, \alpha)} dt \right].$$

Hence $\xi(\theta, \alpha) \rightarrow 0$ for $\theta \rightarrow \infty$.

THEOREM 1.3. Let $y_0(\theta, \alpha_0) \in P_2$ and θ_0 be fixed; then there exists an $\epsilon_0 > 0$ such that to each ϵ , $|\epsilon| < \epsilon_0$, there corresponds an $\alpha = \alpha(\epsilon)$ and a $y(\theta, \alpha(\epsilon)) \in P_2$ for which $y(\theta_0, \alpha(\epsilon)) = y(\theta_0, \alpha_0) + \epsilon$, the $\alpha(\epsilon)$ being differentiable for $|\epsilon| < \epsilon_0$.

PROOF. We shall prove the theorem only for lower neighborhoods. The corresponding proof for upper neighborhoods follows along the same lines and will be omitted.

Let $\alpha > \alpha_0$ and denote by $y(\theta, \alpha, \epsilon)$ the solution of Eq. (2) for which $y(\theta, \alpha, \epsilon) = y_0(\theta_0, \alpha_0) - \epsilon$, $\epsilon > 0$. Define $h(\theta, \alpha, \epsilon) = y_0(\theta, \alpha_0) - y(\theta, \alpha, \epsilon)$ and note that for $\alpha - \alpha_0$ and ϵ sufficiently small, $h(\theta, \alpha, \epsilon)$ is defined and hence differentiable with respect to each variable for $\theta_0 \leq \theta \leq \theta_2 = \theta_0 + 2\pi$. In what follows, these restrictions on α and ϵ are presumed.

We show first that $\frac{\partial}{\partial \alpha} h(\theta_2, \alpha, \epsilon) > 0$. By Eq. (2)

$$h'(\theta, \alpha, \epsilon) = \frac{g(\theta)}{y_0(\theta, \alpha_0)} - f(\theta, \alpha_0) - \frac{g(\theta)}{y(\theta, \alpha, \epsilon)} + f(\theta, \alpha).$$

Differentiating with respect to α , we have, after changing the order of differentiation on the left,

$$(3) \quad \eta'(\theta, \alpha, \epsilon) = \frac{g(\theta)}{y^2(\theta, \alpha, \epsilon)} \eta(\theta, \alpha, \epsilon) + \frac{\partial f}{\partial \alpha}(\theta, \alpha)$$

where $\eta(\theta, \alpha, \epsilon) = \frac{\partial}{\partial \alpha} h(\theta, \alpha, \epsilon)$. Multiplying Eq. (3) by the usual integrating factor, and integrating from θ to θ_2 we obtain

$$\begin{aligned} \eta(\theta_2, \alpha, \epsilon) \exp \left[- \int_{\theta_0}^{\theta_2} G(\theta, \alpha, \epsilon) d\theta \right] - \eta(\theta_0, \alpha, \epsilon) \\ (4) \quad = \int_{\theta_0}^{\theta_2} \frac{\partial f}{\partial \alpha} \exp \left[- \int_{\theta_0}^{\theta} G(t, \alpha, \epsilon) dt \right] d\theta \end{aligned}$$

where $G(\theta, \alpha, \epsilon) = g(\theta)/y^2(\theta, \alpha, \epsilon)$. Since by Theorem 1.1 $y(\theta, \alpha, \epsilon) < y_0(\theta, \alpha_0)$ for $\theta_0 \leq \theta \leq \theta_2$, and since $\eta(\theta_0, \alpha, \epsilon) = 0$, we have that $\eta(\theta_2, \alpha, \epsilon) = \frac{\partial}{\partial \alpha} h(\theta_2, \alpha, \epsilon) > 0$. Now for fixed $\alpha_1 > \alpha_0$ we have $h(\theta_2, \alpha_1, 0) - h(\theta_2, \alpha_0, 0) = h(\theta_2, \alpha_1, 0) = [\frac{\partial}{\partial \alpha} h(\theta_2, \xi, 0)](\alpha_1 - \alpha_0)$, where $\xi = \xi(\alpha_1, \alpha_0)$ is such that $\alpha_0 < \xi < \alpha_1$. Put $\frac{\partial}{\partial \alpha} h(\theta_2, \xi, 0) = M_0(\xi)$ and note that $M_0(\xi) > 0$. Since $h(\theta_2, \alpha_1, \epsilon)$ is continuous in ϵ , there exists an $\epsilon_0 > 0$ such that $\epsilon_0 \leq M_0(\xi)(\alpha_1 - \alpha_0)/2$ and for $\epsilon < \epsilon_0$,

$$(5) \quad h(\theta_2, \alpha_1, \epsilon) > \frac{M_0(\xi)}{2} (\alpha_1 - \alpha_0) \geq \epsilon_0 > \epsilon.$$

If, now, ϵ_0 is sufficiently small, we also have for $\epsilon < \epsilon_0$, $h(\theta_2, \alpha_0, \epsilon) < y_0(\theta_2, \alpha_0) - y(\theta, \alpha_0, \epsilon) < \epsilon$; for if not, $h(\theta, \alpha_0, \epsilon)$ would not tend to 0 for $\theta \rightarrow \infty$ as is required by the lemma. (On the phase cylinder, the curve of $y(\theta, \alpha_0, \epsilon)$ must move up after one circuit in the positive θ direction in order that it approach the closed curve of $y_0(\theta, \alpha_0)$ for $\theta \rightarrow \infty$.) Hence for ϵ_0 sufficiently small corresponding to each ϵ , $0 \leq \epsilon < \epsilon_0$, there exists an $\alpha = \alpha(\epsilon) \geq \alpha_0$ such that $h(\theta_2, \alpha(\epsilon), \epsilon) = \epsilon$. This shows that $y(\theta, \alpha(\epsilon), \epsilon) \in P_2$.

That this $\alpha(\epsilon)$ is differentiable for $0 \leq \epsilon < \epsilon_0$ is a consequence of the implicit function theorem. This completes the proof.

From this last theorem we see that L is open; the next theorem shows that the set L is an interval, not necessarily finite, and that for fixed θ , $y(\theta, \alpha) \in P_2$ is a strictly decreasing function of α for $\alpha \in L$.

THEOREM 1.4. If $y(\theta, \alpha_0) \in P_2$, then for any α , $0 < \alpha \leq \alpha_0$, there exists a $y(\theta, \alpha) > y(\theta, \alpha_0)$ such that $y(\theta, \alpha) \in P_2$; i.e., if $\alpha_0 \in L$, then the interval $0 < \alpha \leq \alpha_0$ is contained in L .

PROOF. Fix $\theta = \theta_0$ and $\alpha < \alpha_0$, and consider the solution $y_0(\theta, \alpha)$ of Eq. (2) such that $y_0(\theta_0, \alpha) = y(\theta_0, \alpha_0)$. Since $f(\theta_0, \alpha) < f(\theta_0, \alpha_0)$ we have $y'_0(\theta_0, \alpha) > y'(\theta_0, \alpha_0)$. Hence, by Theorem 1.2, we must have $y_0(\theta_0 + 2\pi, \alpha) > y(\theta_0 + 2\pi, \alpha_0) = y(\theta_0, \alpha_0) = y_0(\theta, \alpha)$. From this we conclude that there exists a $y(\theta, \alpha) \in P_2$ (Cf. [2], [3]); and since the curves of P_2 functions cannot intersect by Theorem 1.2, $y(\theta, \alpha) > y(\theta, \alpha_0)$. This proves the theorem.

This theorem shows that the function $\alpha(\epsilon)$ determined in the proof of Theorem 1.3 is strictly decreasing; hence has an inverse $\epsilon(\alpha)$ which is also strictly increasing and differentiable and may be defined by the equation: $\epsilon(\alpha) = y(\theta_0, \alpha) - y(\theta_0, \alpha_0)$, for $y(\theta, \alpha)$ and $y(\theta, \alpha_0)$ each of class P_2 and $|\alpha - \alpha_0| < \delta_0$, δ_0 sufficiently small.

Consider now the function $\epsilon^*(\alpha) = y(\theta_0, \alpha) - y(\theta_0, \alpha_1)$ for $y(\theta, \alpha)$ and $y(\theta, \alpha_1)$ of class P_2 ; $\alpha_1 \in L$ is fixed and $\alpha \in L$ is the variable. Suppose at $\alpha = \alpha_0 \in L$, $\epsilon^*(\alpha)$ is not differentiable. By Theorem 1.3 $\epsilon(\alpha) = y(\theta_0, \alpha) - y(\theta_0, \alpha_0)$ is differentiable with respect to α at $\alpha = \alpha_0$; but $\epsilon^*(\alpha) = [y(\theta_0, \alpha) - y(\theta_0, \alpha_0)] + [y(\theta_0, \alpha_0) - y(\theta_0, \alpha_1)] = \epsilon(\alpha) + C$ where C is independent of α , and we conclude that $\epsilon^*(\alpha)$ is differentiable on L .

We now consider the set L^* . Clearly, if L is the interval $0 < \alpha < \infty$, then since $L \subset L^*$, we have $L = L^*$, and everything we have proved for the sets L and P_2 holds also for the sets L^* and P_2^* .

Suppose now L is bounded, let α_c be its least upper bound. If $y \in P_2$ then for each θ , $\lim_{\alpha \rightarrow \alpha_c^-} y(\theta, \alpha) = y^*(\theta)$ exists.

Suppose first $y^*(\theta) > 0$ for all θ . Consider the solution $y_0(\theta, \alpha_c)$ of Eq. (2) such that $y_0(\theta_0, \alpha_c) = y^*(\theta_0)$. If $y^*(\theta_0 + a) - y_0(\theta_0 + a, \alpha_c) = \epsilon > 0$ for some positive $a \leq 2\pi$, then there exists an $\alpha \in L$ so close to α_c such that if $y(\theta, \alpha) \in P_2$ and $y(\theta_0, \alpha) = y_0$, the solution $y_1(\theta, \alpha_c)$ for which $y_1(\theta_0, \alpha_c) = y_0$ will be such that $y_1(\theta_0 + a, \alpha_c) - y_0(\theta_0 + a, \alpha_c) < \frac{\epsilon}{2}$, and that $y(\theta_0 + a, \alpha) - y_1(\theta_0 + a, \alpha_c) < \frac{\epsilon}{2}$. Hence $y^*(\theta_0 + a) - y_0(\theta_0 + a, \alpha_c) < y(\theta_0 + a, \alpha) - y_0(\theta_0 + a, \alpha_c) < \epsilon$, a contradiction. Suppose now that for some positive $b \leq 2\pi$, $y_0(\theta_0 + b, \alpha_c) - y^*(\theta_0 + b) > \epsilon_1$. This is also impossible since we can find a $y(\theta, \alpha) \in P_2$ with $\alpha \in L$ so close to α_c that $y(\theta, \alpha)$ must intersect $y_0(\theta, \alpha_c)$ at some point $\theta = \theta_0 + c$, $0 < c \leq b$; this is impossible by Theorem 1.1.

Hence we can only conclude that $y_0(\theta, \alpha_c) = y^*(\theta)$; i.e., $\alpha_c \in L$, which is a contradiction, since L is open.

Thus we see that $y^*(\theta) > 0$ is impossible; i.e., $y^*(\theta)$ must

have zeros. Since the slope of solutions of Eq. (2) on the line $y = 0$ is ∞ or undefined except at the saddle points, the zeros of $y^*(\theta)$ must clearly be found among these saddle points. If we denote one of the zeros of $y^*(\theta)$ by θ_1 , then the solution $y_0(\theta, \alpha_c)$ for which $y_0(\theta_1, \alpha_c) = 0$ can be shown to be coincident with $y^*(\theta)$ in a manner similar to that of the previous case. Hence $\alpha_c \in L^*$, and $y^*(\theta) \in P_2^*$.

Suppose now that for some $\alpha_1 > \alpha_c$, $\alpha_1 \in L^*$. Then the corresponding $y(\theta, \alpha_1) \in P_2^*$ must also have a zero at the saddle point $\theta = \theta_1$. The slope of the integral curve of $y(\theta, \alpha_1)$ approaches the value

$$m_1 = \frac{-f(\theta_1, \alpha_1) - \sqrt{f^2(\theta_1, \alpha_1) + 4g'(\theta_1)}}{2}$$

as $\theta \rightarrow \theta_1$ from the left. Since this is less than the slope of $y^*(\theta) = y_0(\theta, \alpha_c)$ defined above, it follows that for values of $\theta < \theta_1$, $y(\theta, \alpha_1)$ must pass above $y_0(\theta, \alpha_c)$. Hence there exists a solution $y(\theta, \alpha) \in P_2$ whose curve is intersected by $y(\theta, \alpha_1)$ twice. This is impossible by Theorem 1.2. Hence an $\alpha_1 > \alpha_c$ such that $\alpha_1 \in L^*$ is impossible.

We can now assert that if L is bounded, then L^* simply consists of L plus its least upper bound α_c ; also that in this case P_2^* consists of P_2 plus a periodic solution of Eq. (2) which has zeros.

The next theorem gives an expression for $\frac{\partial}{\partial \alpha} y^2(\theta, \alpha)$ for $y(\theta, \alpha) \in P_2$.

THEOREM 1.5. Let $y(\theta, \alpha) \in P_2$. Then for $\theta = \theta_0$,

$$(6) \quad \frac{1}{2} \frac{\partial}{\partial \alpha} y^2(\theta_0, \alpha) = - \int_{\theta_0}^{\theta_2} \frac{y(\theta, \alpha) E(\theta, \alpha) \frac{\partial}{\partial \alpha} f(\theta, \alpha) d\theta}{E(\theta_2, \alpha)^{-1}}$$

where $\theta_2 = \theta_0 + 2\pi$, and

$$E(\theta, \alpha) = \exp \left[\int_{\theta_0}^{\theta} \frac{f(t, \alpha)}{y(t, \alpha)} dt \right].$$

PROOF. If we put $\eta(\theta, \alpha) = \frac{\partial}{\partial \alpha} y(\theta, \alpha)$, differentiate Eq. (2) with respect to α , and interchange the order of differentiation on the left, we obtain

$$\eta'(\theta, \alpha) = - \frac{g(\theta)}{y^2(\theta, \alpha)} \eta(\theta, \alpha) - \frac{\partial f}{\partial \alpha}(\theta, \alpha).$$

Multiplying this equation by the usual integrating factor, and integrating from θ_0 to θ_2 , we obtain

$$\begin{aligned} (7) \quad \eta(\theta_2, \alpha) \exp \left[\int_{\theta_0}^{\theta_2} \frac{g(\theta)}{y^2(\theta, \alpha)} d\theta \right] - \eta(\theta_0, \alpha) \\ = - \int_{\theta_0}^{\theta_2} \frac{\partial f}{\partial \alpha} \exp \left[\int_{\theta_0}^{\theta} \frac{g(t)}{y^2(t, \alpha)} dt \right] d\theta. \end{aligned}$$

Since $\eta(\theta_2, \alpha) = \eta(\theta_0, \alpha)$, and since, as before,

$$\int_{\theta_0}^{\theta} \frac{g(t)}{y^2(t, \alpha)} dt = \log \frac{y(\theta, \alpha)}{y(\theta_0, \alpha)} + \int_{\theta_0}^{\theta} \frac{f(t, \alpha)}{y(t, \alpha)} dt;$$

we obtain, solving Eq. (7) for $\eta(\theta_0, \alpha) = \frac{\partial}{\partial \alpha} y(\theta_0, \alpha)$;

$$\frac{\partial}{\partial \alpha} y(\theta_0, \alpha) = - \frac{1}{y(\theta_0, \alpha)} \int_{\theta_0}^{\theta_2} \frac{y(\theta, \alpha) E(\theta, \alpha) \frac{\partial}{\partial \alpha} f(\theta, \alpha)}{E(\theta_2, \alpha)^{-1}} d\theta$$

from which Eq. (6) follows easily.

COROLLARY. Let $y(\theta, \alpha) \in P_2$, $M_1(\alpha) = \max_{\theta} [g(\theta)/f(\theta, \alpha)]$,

$$M_2(\alpha) = \max_{\theta} \left[\frac{\partial}{\partial \alpha} f(\theta, \alpha) \right],$$

and

$$E_1 = \exp \left[\frac{1}{M_1(\alpha)} \int_{\theta_0}^{\theta_2} f(\theta, \alpha) d\theta \right];$$

then

$$(8) \quad \frac{\partial}{\partial \alpha} y^2(\theta, \alpha) > - \frac{4\pi M_1(\alpha) M_2(\alpha) E_1(\alpha)}{E_1(\alpha)^{-1}}$$

PROOF. We show first that if $y(\theta, \alpha) \in P_2$, then $y(\theta, \alpha) < M_1(\alpha)$. Consider the graph of $y = g(\theta)/f(\theta, \alpha)$ in the phase plane. In the region R_1 of the phase plane bounded by this graph and in the θ axis, the phase

trajectories have positive slope; on the graph the slopes are zero, and elsewhere the slopes are negative. Let θ_m be a value of θ at which $g(\theta_m)/f(\theta_m, \alpha) = M_1(\alpha)$ and suppose $y(\theta, \alpha) \in P_2$ is such that $y(\theta_m, \alpha) \geq M_1(\alpha)$. Then clearly $y(\theta_m - 2\pi, \alpha) > y(\theta_m, \alpha)$, which is a contradiction. This shows that $y(\theta, \alpha) < M_1(\alpha)$.

To complete the proof we note that

$$\int_{\theta_0}^{\theta_2} y(\theta, \alpha) E(\theta, \alpha) \frac{\partial}{\partial \alpha} f(\theta, \alpha) d\theta < 2\pi E(\theta_2, \alpha) M_1(\alpha) M_2(\alpha)$$

and that since $E(\theta_2, \alpha) > E_1(\alpha)$, we have

$$\frac{E(\theta_2, \alpha)}{E(\theta_2, \alpha) - 1} < \frac{E_1(\alpha)}{E_1(\alpha) - 1}.$$

Using the above inequalities in Eq. (6) we obtain (8) which completes the proof.

The usefulness of inequality (8) in obtaining information on values of α for which $\alpha \in L$ clearly depends on the availability of a lower bound on $y^2(\theta, \alpha_0) - y^2(\theta, \alpha_1)$ for some fixed θ , $\alpha_0 < \alpha_1$, α_0 and α_1 being members of L . Since this lower bound seems rather difficult to obtain, the special case $f(\theta, \alpha) = \alpha f(\theta)$ in which an inequality of type (8) from which dependence on $y(\theta, \alpha) \in P_2$ can be eliminated is of interest. We also note that the equation of the synchronous motor given by Edgerton and Fourmarier [1] contains a damping function of the type of this special case.

Part 2: The Special Case $f(\theta, \alpha) = \alpha f(\theta)$

We first note that the conditions previously imposed on $f(\theta, \alpha)$ imply that if $f(\theta, \alpha) = \alpha f(\theta)$, then $f(\theta)$ has a continuous first derivative everywhere, is of period 2π , and has a positive lower bound. The equation of the phase trajectories is now

$$(2a) \quad y'(\theta, \alpha) = \frac{g(\theta) - \alpha f(\theta) y(\theta, \alpha)}{y(\theta, \alpha)}.$$

We also note that for α sufficiently small, a $y(\theta, \alpha) \in P_2$ may exist [2]; i.e., the class P_2 is in this case non-empty.

Let us replace $f(\theta, \alpha)$ by $\alpha f(\theta)$ in Eq. (6) and consider the

numerator of the fraction on the right. Since

$$E'(\theta, \alpha) = \frac{\alpha f(\theta)}{y(\theta, \alpha)} E(\theta, \alpha)$$

where $E(\theta, \alpha)$ is defined as in Theorem 1.5, we have

$$\begin{aligned} \int_{\theta_0}^{\theta_2} y(\theta, \alpha) f(\theta) E(\theta, \alpha) d\theta &= \frac{1}{\alpha} \int_{\theta_0}^{\theta_2} y^2(\theta, \alpha) E'(\theta, \alpha) d\theta \\ (9) \quad &= \frac{[E(\theta_2, \alpha) - 1] y^2(\theta_2, \alpha)}{\alpha} - \frac{1}{\alpha} \int_{\theta_0}^{\theta_2} y^{2'}(\theta, \alpha) E(\theta, \alpha) d\theta. \end{aligned}$$

But by Eq. (2a), $y^{2'}(\theta, \alpha) = 2[g(\theta) - \alpha f(\theta)y(\theta, \alpha)]$; hence

$$\begin{aligned} \frac{1}{\alpha} \int_{\theta_0}^{\theta_2} y^{2'}(\theta, \alpha) E(\theta, \alpha) d\theta &= \frac{2}{\alpha} \int_{\theta_0}^{\theta_2} g(\theta) E(\theta, \alpha) d\theta \\ &\quad - 2 \int_{\theta_0}^{\theta_2} y(\theta, \alpha) f(\theta) E(\theta, \alpha) d\theta. \end{aligned}$$

Substituting this in Eq. (9) and solving for

$$\int_{\theta_0}^{\theta_2} y(\theta, \alpha) f(\theta) E(\theta, \alpha) d\theta$$

we have

$$\begin{aligned} \int_{\theta_0}^{\theta_2} y(\theta, \alpha) f(\theta) E(\theta, \alpha) d\theta &= - \frac{E(\theta_2, \alpha) - 1}{\alpha} y^2(\theta_2, \alpha) \\ (10) \quad &\quad + \frac{2}{\alpha} \int_{\theta_0}^{\theta_2} g(\theta) E(\theta, \alpha) d\theta. \end{aligned}$$

Using Eq. (10) in Eq. (6) we obtain

$$(11) \quad \frac{1}{2} \frac{\partial}{\partial \alpha} y^2(\theta_0, \alpha) = \frac{y^2(\theta_2, \alpha)}{\alpha} - \frac{2}{\alpha} \int_{\theta_0}^{\theta_2} \frac{g(\theta) E(\theta, \alpha)}{E(\theta_2, \alpha) - 1} d\theta$$

We now consider the integral

$$\begin{aligned}
 (12) \quad & \int_{\theta_0}^{\theta_2} g(\theta) E(\theta, \alpha) d\theta = \int_{\theta_0}^{\theta_2} E(\theta, \alpha) d \left[\int_{\bar{\theta}}^{\theta} g(t) dt \right] \\
 & = E(\theta_2, \alpha) \int_{\bar{\theta}}^{\theta_2} g(\theta) d\theta - \int_{\bar{\theta}}^{\theta_0} g(\theta) d\theta - \int_{\theta_0}^{\theta_2} E'(\theta, \alpha) \int_{\bar{\theta}}^{\theta} g(t) dt d\theta.
 \end{aligned}$$

The critical points of the function

$$\Phi(\theta) = \int_{\bar{\theta}}^{\theta} g(t) dt$$

are clearly the zeros $\theta_1, \theta_1 < \theta_{1+1}$, of $g(\theta)$ and are independent of the choice of $\bar{\theta}$. Denote by $\theta_{2k+1}, k = 0, \pm 1, \pm 2, \dots$, the values of θ at which $\Phi(\theta)$ has a relative minimum. We now prove the

LEMMA. There exists an odd integer m such that for all $n = m + 2k, k = 1, 2, 3, \dots$, for which $\theta_n < \theta_m + 2\pi$ we have

$$\int_{\theta_m}^{\theta_n} g(\theta) d\theta = \Phi_m(\theta_n) \geq 0.$$

PROOF. Assume the lemma false; that is, suppose for each odd integer m there exists an odd integer $n_m > m$ for which $\theta_{n_m} < \theta_m + 2\pi$ and $\Phi_m(\theta_{n_m}) < 0$. Starting with $m = 1 = n_0$, we determine an n_1 for which $\Phi_{n_0}(\theta_{n_1}) < 0$ where $\theta_{n_0} < \theta_{n_1} < \theta_n + 2\pi$. Consider the interval $(\theta_{n_1}, \theta_{n_1} + 2\pi)$. Clearly, $\theta_{n_0} + 2\pi$ is contained in it. Now corresponding to n_1 there exists an n_2 in this interval for which $\Phi_{n_1}(\theta_{n_2}) < 0$. If $\theta_{n_2} = \theta_n + 2\pi$, we have $\Phi_{n_0}(\theta_{n_1}) + \Phi_{n_1}(\theta_{n_2}) = \Phi_n(\theta_{n_2}) = \Phi_{n_0}(\theta_{n_0} + 2\pi) < 0$, a contradiction. Hence suppose $\theta_{n_2} \neq \theta_{n_0} + 2\pi$. Next, consider the interval $(\theta_{n_2}, \theta_{n_2} + 2\pi)$. It must clearly contain $\theta_{n_1} + 2\pi$ and a point of the form $\theta_{n_0} + 2\pi p, p = 1$ or 2 . Now corresponding to n_2 there is an n_3 such that θ_{n_3} is in this interval and $\Phi_{n_2}(\theta_{n_3}) < 0$. If $\theta_{n_3} = \theta_0 + 2\pi p, p = 1$ or 2 , we would have $\Phi_{n_0}(\theta_{n_1}) + \Phi_{n_1}(\theta_{n_2}) + \Phi_{n_2}(\theta_{n_3}) = \Phi_n(\theta_{n_3}) = \Phi_{n_0}(\theta_{n_0} + 2\pi p) < 0$, while if $\theta_{n_3} = \theta_{n_1} + 2\pi$, then $\Phi_{n_1}(\theta_{n_2}) + \Phi_{n_2}(\theta_{n_3}) = \Phi_{n_1}(\theta_{n_3}) = \Phi_{n_1}(\theta_{n_1} + 2\pi) < 0$; each of these being impossible, we assume

$$\theta_{n_3} \neq \theta_{n_s} + 2\pi p, \quad s = 0, 1, \quad p = 1, 2.$$

By continuing in this way we must, since the number of zeros of $g(\theta)$ in any interval $(\theta, \theta + 2\pi)$ is finite, eventually arrive at integers s, r , and p , with $s < r - 1$, such that $\theta_{n_r} = \theta_{n_s} + 2\pi p$, and for which $\Phi_{n_{r-1}}(\theta_{n_1}) < 0$, $1 = 1, 2, \dots, r$. Since we have again,

$$\sum_{i=s+1}^r \Phi_{n_{i-1}}(\theta_{n_1}) = \Phi_{n_s}(\theta_{n_r}) = \Phi_{n_s}(\theta_{n_s} + 2\pi p) < 0,$$

a contradiction, the proof is complete.

It may be observed that if we put

$$c_1 = \int_{\theta_{21-1}}^{\theta_{21+1}} g(\theta) d\theta, \quad 1 = 1, 2, \dots, n$$

where θ_1 is a zero of $g(\theta)$ such that $g'(\theta_1) > 0$, and $2n$ is the number of zeros of $g(\theta)$ in $\theta_1 \leq \theta < \theta_1 + 2\pi$, then the lemma may be formulated as follows: there exists a cyclic permutation of the numbers $(c_1, c_2, \dots, c_n) \longrightarrow (c_1', c_2', \dots, c_n')$ such that

$$\sum_{i=1}^k c_i' > 0 \quad \text{for} \quad k = 1, 2, \dots, n.$$

(This was pointed out to the author and proved by Professors L. K. Jackson and H. Ribeiro.)

As in the proof of the lemma we may for convenience denote by θ_1 , the θ_m assured us, and since $\Phi(\theta)$ has a relative maximum at the zero immediately preceding θ_1 , we have, setting $\bar{\theta} = \theta_1$, that

$$\int_{\bar{\theta}}^{\theta} g(t) dt = \int_{\theta_1}^{\theta} g(t) dt \geq 0$$

for $\theta_0 \leq \theta \leq \theta_0 + 2\pi = \theta_2$ where θ_0 is the greatest zero of $g(\theta)$ less than θ_1 .

This shows that if θ_0 and $\bar{\theta}$ are chosen in the manner indicated above, the last integral in Eq. (12) is positive and we have

$$\begin{aligned}
 (13) \quad & \int_{\theta_0}^{\theta_2} g(\theta) E(\theta, \alpha) d\theta < E(\theta_2, \alpha) \int_{\theta_1}^{\theta_2} g(\theta) d\theta - \int_{\theta_1}^{\theta_2} g(\theta) d\theta \\
 & = E(\theta_2, \alpha) \int_{\theta_0}^{\theta_2} g(\theta) d\theta - (E(\theta_2, \alpha) - 1) \int_{\theta_0}^{\theta_1} g(\theta) d\theta.
 \end{aligned}$$

Since $g(\theta) < 0$ for $\theta_0 < \theta < \theta_1$, we have, using Eq. (2a), $y'(\theta, \alpha) < -\alpha f(\theta)$, and integrating from θ_0 to θ_1 , we obtain

$$y(\theta_1, \alpha) - y(\theta_0, \alpha) < -\alpha \int_{\theta_0}^{\theta_1} f(\theta) d\theta,$$

or

$$(14) \quad y(\theta_0, \alpha) > \alpha \int_{\theta_0}^{\theta_1} f(\theta) d\theta + y(\theta_1, \alpha) > \alpha \int_{\theta_0}^{\theta_1} f(\theta) d\theta$$

for $y(\theta, \alpha) \in P_2$.

Using inequalities (13) and (14) in Eq. (11) we obtain

$$\begin{aligned}
 (15) \quad & \frac{1}{2} \frac{\partial}{\partial \alpha} y^2(\theta, \alpha) > \alpha F_1^2 + \frac{2}{\alpha} \left[\int_{\theta_0}^{\theta_1} g(\theta) d\theta \right. \\
 & \quad \left. - \frac{E(\theta_2, \alpha)}{E(\theta_2, \alpha) - 1} \int_{\theta_0}^{\theta_2} g(\theta) d\theta \right]
 \end{aligned}$$

where

$$F_1 = \int_{\theta_0}^{\theta_1} f(\theta) d\theta.$$

Define

$$E_1(\alpha) = \exp \left[\frac{\alpha^2}{M} \int_{\theta_0}^{\theta_1} f(\theta) d\theta \right]$$

where $M = \max[g(\theta)/f(\theta)]$, and note that as in the proof of the corollary of Theorem 1.5, $y(\theta, \alpha) < M/\alpha$ provided $y(\theta, \alpha) \in P_2$.

Hence $E_1(\alpha) > E(\theta_2, \alpha)$, and we have

$$(16) \quad \frac{E(\theta_2, \alpha)}{E(\theta_2, \alpha) - 1} < \frac{E_1(\alpha)}{E_1(\alpha) - 1} = H(\alpha)$$

Thus (15) becomes

$$(17) \quad \frac{1}{2} \frac{\partial}{\partial \alpha} y^2(\theta_0, \alpha) > \alpha F_1^2 - \frac{2}{\alpha} [G_1 + H(\alpha)G_2]$$

where $H(\alpha)$ is defined in (16), and

$$G_1 = - \int_{\theta_0}^{\theta_1} g(\theta) d\theta, \quad G_2 = \int_{\theta_0}^{\theta_2} g(\theta) d\theta.$$

Let $\alpha_1 \in L$; then by Theorem 1.4 if $\alpha < \alpha_1$, $y(\theta, \alpha) \in P_2$ exists also. Integrating (17) from $\alpha_0 < \alpha_1$ to α_1 we obtain

$$\frac{y^2(\theta_0, \alpha_1) - y^2(\theta_0, \alpha_0)}{2} > \frac{\alpha_1^2 - \alpha_0^2}{2} F_1^2 - 2 \int_{\alpha_0}^{\alpha_1} \frac{G_1 + H(\alpha)G_2}{\alpha} d\alpha,$$

but since $\frac{\partial}{\partial \alpha} y^2(\theta_0, \alpha) < 0$ for $y(\theta, \alpha) \in P_2$ we have

$$(18) \quad (\alpha_1^2 - \alpha_0^2) F_1^2 < 4 \int_{\alpha_0}^{\alpha_1} \frac{G_1 + H(\alpha)G_2}{\alpha} d\alpha.$$

Since this last integral is in general non-elementary, we make the observation that $H(\alpha)$ is a strictly decreasing function of α ; and that therefore we may replace (18) by

$$(\alpha_1^2 - \alpha_0^2) F_1^2 < 4 \left[G_1 + H(\alpha_0) G_2 \right] \log \frac{\alpha_1}{\alpha_0}$$

which we can also write in the form

$$(19) \quad \alpha_0 \exp \left[K(\alpha_0) (\alpha_1^2 - \alpha_0^2) \right] < \alpha_1$$

where

$$K(\alpha) = \frac{F_1^2}{4[G_1 + H(\alpha)G_2]}$$

We now state and prove the following

THEOREM 2.1. Let $K(\alpha)$ be defined as above and let $\alpha \in L$. Then the equation

$$(20) \quad \alpha \exp [K(\alpha)(x^2 - \alpha^2)] = x$$

has a solution $x = \alpha^*$ such that $\alpha^* > \alpha$, and such that $\alpha^* \notin L$.

PROOF. Consider the function $\psi(x) = \alpha \exp [K(\alpha)(x^2 - \alpha^2)]$. Clearly $\psi(\alpha) = \alpha$. To show that $\psi'(\alpha) < 1$ we note that $\psi'(\alpha) = 2\alpha^2 K(\alpha) < 1$ by (17) since $\frac{\partial}{\partial \alpha} y^2(\theta, \alpha) < 0$. Hence Eq. (20) has a solution $\alpha^* > \alpha$.

If $\alpha^* \in L$, then (19) must hold, which is a contradiction.

COROLLARY 1. There exists a finite positive number α_c such that for each $\alpha < \alpha_c$ a $y(\theta, \alpha) \in P_2$ exists, while if $\alpha \geq \alpha_c$, no such solutions exist; i.e., the set L is bounded.

PROOF. We have already noted that the set P_2 is non-void in the case under consideration. Hence, if $\alpha_0 \in L$, we have a finite $\alpha^* \notin L$. Take α_c as the least upper bound of this set L which is an open interval (Theorem 1.4), and the proof is complete.

COROLLARY 2. The solution $\alpha^* = \alpha^*(\alpha)$ of Eq. (20) is a strictly decreasing function of α .

PROOF. Define $F(x, \alpha) = \alpha \exp [K(\alpha)(x^2 - \alpha^2)] - x$; then

$$\begin{aligned} \frac{dx}{d\alpha} &= - \frac{\partial F}{\partial \alpha} / \frac{\partial F}{\partial x} \\ &= - \frac{\exp[K(\alpha)(x^2 - \alpha^2)](1 + K'(\alpha)(x^2 - \alpha^2) - 2\alpha^2 K(\alpha))}{2\alpha x K(\alpha) \exp[K(\alpha)(x^2 - \alpha^2)] - 1} \end{aligned}$$

Putting $x = \alpha^*(\alpha)$ we have

$$(21) \quad \frac{d}{d\alpha} \alpha^*(\alpha) = - \frac{\exp[K(\alpha)(\alpha^{*2} - \alpha^2)](1 + K'(\alpha)(\alpha^{*2} - \alpha^2) - 2\alpha^2 K(\alpha))}{2\alpha \alpha^* [K(\alpha)(\alpha^{*2} - \alpha^2)] - 1}$$

But if $\psi(x)$ is defined as in the proof of the theorem, we must have $2\alpha \alpha^* \exp[K(\alpha)(\alpha^{*2} - \alpha^2)] = \psi'(\alpha^*) > 1$ which shows that the denominator of the fraction in Eq. (21) is positive. Since $K'(\alpha) > 0$, and since, as in

the proof of the theorem $2\alpha^2 K(\alpha) = \psi'(\alpha) < 1$ the numerator is also positive. This completes the proof.

To summarize, we have in Theorem 2.1 a method whereby in the special case $f(\theta, \alpha) = \alpha f(\theta)$ corresponding to an $\alpha \in L$, we can determine an α^* such that for $\alpha > \alpha^*$ no periodic solutions of the second kind for Eq. (1) exist; i.e., for which all solutions will be stable. Furthermore, the use of the largest available $\alpha \in L$ obtains the smallest possible α^* .

We also note, in conclusion, that better results may be obtained at the expense of the greater complexity of a better approximation to the integral

$$\int_{\alpha_0}^{\alpha_1} \frac{H(\alpha)}{\alpha} d\alpha$$

in (18), and that one could have carried through most of the procedure of Part 2 in the more general case $f(\theta, \alpha) = f_1(\alpha)f_2(\theta)$.

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II. ASYMPTOTICALLY AUTONOMOUS DIFFERENTIAL SYSTEMS

L. Markus¹

1. Introduction

In this paper we analyse topologically the solution curves of real, first order, ordinary differential systems, such as,

$$(1) \quad \mathcal{S}: \dot{x}_1 = f_1(x_1, x_2, \dots, x_n; t) = f_1(x; t) \quad (i = 1, 2, \dots, n).$$

The vector field $\{f_1(x; t)\}$ is assumed (precise hypotheses later) to approach an autonomous (time-independent) vector field $\{f_1(x)\}$ for large values of the time t . The solutions of $\mathcal{S}$ are compared with those of the limiting autonomous differential system

$$(2) \quad \mathcal{S}_\infty: \dot{x}_1 = f_1(x) \quad (i = 1, 2, \dots, n).$$

Our purpose is to extend the global qualitative theory of non-linear autonomous differential equations [see Reference 5] to certain time-dependent equations. We apply our results to generalize the Poincaré-Bendixson theory of two-dimensional systems. As a particular example, we analyse a Liénard-type equation which contains a varying damping coefficient.

DEFINITION. Let $\mathcal{S}: \dot{x}_1 = f_1(x; t)$ and $\mathcal{S}_\infty: \dot{x}_1 = f_1(x)$ ($i = 1, 2, \dots, n$). The real-valued functions $f_1(x; t)$ and $f_1(x)$ are continuous in (x, t) for $x \in \theta$ an open subset of real Euclidean n -space E^n , and for all $t > t_0$, a right half-line, and there satisfy a local Lipschitz condition in x . We say $\mathcal{S} \longrightarrow \mathcal{S}_\infty$ ($\mathcal{S}$ is asymptotic to $\mathcal{S}_\infty$) in case for each compact set $K \subset \theta$ and for each $\epsilon > 0$, there is a $T = T(K, \epsilon) > t_0$ such that $|f_1(x; t) - f_1(x)| < \epsilon$ for all $i = 1, 2, \dots, n$, all $x \in K$, and all $t > T$.

¹ Presented to the society September 1953.

EXAMPLE. Bessel's equation $t^2 \ddot{x} + t\dot{x} + (t^2 - \alpha^2)x = 0$, for each real α , becomes, in the phase plane $\theta = E^2$ and for $t > t_0 = 0$, the system

$$\mathcal{S} : \dot{x} = y, \quad \dot{y} = -\frac{1}{t}y + \left(\frac{\alpha^2}{t^2} - 1\right)x.$$

Here $\mathcal{S}$ is asymptotic to the equation of the harmonic oscillator

$$\mathcal{S}_\infty : \dot{x} = y, \quad \dot{y} = -x.$$

The technique of comparing a time-dependent differential equation with its limiting autonomous equation was originally developed by Sturm for certain linear equations [see 4]. Also, in the case where $\mathcal{S}_\infty$ is linear, there are many known results concerning the asymptotic rate of growth of the solutions of $\mathcal{S}$ [1]. However, we shall consider a qualitative theory which includes non-linear $\mathcal{S}_\infty$ and which relates such global properties as boundedness and recursiveness of the solutions of $\mathcal{S}$ to certain aspects of the solutions of $\mathcal{S}_\infty$, such as the distribution of the critical points and the closed (periodic) orbits.

2. Boundedness and Recursiveness of Solutions of $\mathcal{S}$

Through each point $x_1 \in \theta$ and initial time $t_1 > t_0$ there is a unique solution $S(t)$ of $\mathcal{S}$ which is defined for a maximal interval $t_0 \leq \tau_- < t, t < \tau_+ \leq +\infty$. The escape times, τ_- and τ_+ [7], of $\mathcal{S}$ are not determined by those of $\mathcal{S}_\infty$. In fact, it can happen that every solution of $\mathcal{S}_\infty$ has a positive escape time of $+\infty$ and yet $\tau_+ < \infty$ for each solution of $\mathcal{S}$; and vice versa.

For each solution $S(t)$ of $\mathcal{S}$ one defines the usual positive and negative limit sets

$$S_+ = \bigcap_{m < \tau_+} \overline{\bigcup_{t > m} S(t)}$$

and

$$S_- = \bigcap_{m > \tau_-} \overline{\bigcup_{t < m} S(t)}$$

Then S_+ and S_- are closed, connected subsets of θ . If $S(t)$ is bounded in the positive sense, that is, $S(t)$ lies in a compact set $K \subset \theta$ for all large $t \rightarrow \tau_+$, then S_+ is compact and non-empty.

THEOREM 1. Let $S(t)$ be a solution of $\mathcal{S} \rightarrow \mathcal{S}_\infty$ in θ . If S_+ is not empty, then $\tau_+ = +\infty$.

Furthermore, S_+ is the union of solutions of $\mathcal{S}_\infty$.

PROOF. Let $P \in S_+$ and let $B_1 \subset B_2$ be closed balls, centered at P ,

with radii $r_1 < r_2$ and lying in θ . If $S_+ \subset B_2$, then $\tau_+ = +\infty$, since $S(t)$ does not approach the boundary of θ . Otherwise, $S(t)$ must traverse the set $B_2 - B_1$ an infinite number of times. Since the "speed"

$$\left[\sum_{i=1}^n f_i(x)^2 \right]^{\frac{1}{2}}$$

of $\mathcal{S}_\infty$ is bounded above in B_2 , so is the speed of $\mathcal{S}$ bounded above in B_2 for $t \geq t_0 + \min[1, \frac{1}{2}\tau_+]$. Therefore, $S(t)$ is defined in θ for arbitrarily large t and $\tau_+ = +\infty$.

From the uniform convergence of $\mathcal{S}$ to $\mathcal{S}_\infty$ on compact subsets of θ , it is clear that $S_\infty \subset S_+$ where S_∞ is the solution of $\mathcal{S}_\infty$ through P . Thus S_+ is the union of solutions of $\mathcal{S}_\infty$.

We now relate the asymptotic behavior of the solutions of $\mathcal{S}$ to certain distinguished solutions of $\mathcal{S}_\infty$ such as critical points and periodic solutions.

THEOREM 2. Let $\mathcal{S} \rightarrow \mathcal{S}_\infty$ in θ and let P be a critical point of $\mathcal{S}_\infty$. Assume that the variational equations of $\mathcal{S}_\infty$ based on P have characteristic values with negative real parts. Then there is a neighborhood N of P and a time T such that $S_+ = P$ for every solution $S(t)$ of $\mathcal{S}$ intersecting N at a time later than T .

PROOF. There is a neighborhood (with compact closure in θ) N of P such that each solution $S_\infty(t)$ of $\mathcal{S}_\infty$ in N has (after an affine coordinate change) a radial velocity toward $P : (x_0)$, that is, $\frac{dr}{dt} < -Kr < 0$ where

$$r^2 = \|S_\infty(t) - P\|^2 = \sum_{i=1}^n (x_{\infty i}(t) - x_{0i})^2$$

and K is a positive constant. Then for sufficiently large $t > T$, $\mathcal{S}$ approximates $\mathcal{S}_\infty$ so closely in N that each solution $S(t)$ of $\mathcal{S}$ on N has radial velocity towards P . Therefore $S_+ = P$.

The example $\mathcal{S} : \dot{x} = \frac{1}{t}, \dot{y} = 0, t > 0$ which is asymptotic to $\mathcal{S}_\infty : \dot{x} = 0, \dot{y} = 0$ in E^2 shows that every solution of $\mathcal{S}$ can be unbounded even though every solution of $\mathcal{S}_\infty$ is bounded. The next theorem shows that the converse situation is not possible.

THEOREM 3. Let $\mathcal{S} \longrightarrow \mathcal{S}_\infty$ in θ and assume that every solution of $\mathcal{S}_\infty$ is unbounded in the positive sense. Then every solution of $\mathcal{S}$ is unbounded in the positive sense.

PROOF. Suppose a solution $S(t)$ of $\mathcal{S}$ is bounded, that is, when $t \longrightarrow \tau_+$ $S(t) \subset K$, a compact subset of θ . Then S_+ is compact and non-empty. But then S_+ is the union of solutions of $\mathcal{S}_\infty$. However, every solution of $\mathcal{S}_\infty$ is unbounded and this is a contradiction.

We next determine a criterion that a solution $S(t)$ of $\mathcal{S}$ be eventually periodic, that is, there is a $T > t_0$, such that, for $t > T$, $S(t)$ is periodic.

THEOREM 4. Let $S(t)$ be a solution of $\mathcal{S} \longrightarrow \mathcal{S}_\infty$ in θ . Assume $S(t)$ is eventually periodic for $t > T > t_0$. Then S_+ is a closed orbit of $\mathcal{S}_\infty$.

PROOF. Since S_+ is bounded, S_+ is a compact non-empty set. Let $P \in S_+$ and say $S(t_1 + mp) = P$ where $t_1 > T$ and $m = 0, 1, 2, \dots$. Let $S_\infty(t)$ be the solution of $\mathcal{S}_\infty$ initiating (at $t = 0$) at P .

Since $\|S(t + t_1 + mp) - S_\infty(t)\|$ on $0 \leq t \leq p$ can be made arbitrarily small when m is large, $S_\infty(t)$ has period p . Moreover, $S(t + t_1) = S_\infty(t)$ for $t > 0$. Therefore, $S_+ = S_{\infty+}$ which is the simple closed curve traced out by $S_\infty(t)$.

We recall that a solution $S(t)$ of $\mathcal{S}$, with $\tau_+ = +\infty$, is Poisson stable at a point $P \in S(t)$ in case $P \in S_+$. G. D. Birkhoff [2] brought attention to the important case, called recursive, where the return times t_{α} , for each given neighborhood N of P , are not separated by arbitrarily large gaps.

DEFINITION. Let $S(t)$ be a solution of $\mathcal{S}$ which is Poisson stable at P . Suppose for each neighborhood N of P there is a constant T such that $S(t)$ intersects N in every time interval $t_1 \leq t \leq t_1 + T$, for all large t_1 . Then $S(t)$ is recursive at P .

THEOREM 5. Let $S(t)$ be a solution of $\mathcal{S} \longrightarrow \mathcal{S}_\infty$ in θ . Assume $S(t)$ is recursive at $P \in S(t)$ and that the solution $S_\infty(t)$ of $\mathcal{S}_\infty$ initiating at P has a positive escape time of $+\infty$. Then $S_\infty(t)$ is recursive at P .

PROOF. Let B be a closed ball centered at P with radius ϵ . We shall show that $S_\infty(t)$ returns to B in every interval $0 \leq t' \leq t' + T + 1$ where T is the maximum duration between return times of $S(t)$ to the concentric ball $B_1 \subset B$ with radius $\epsilon/2$.

On $0 \leq t \leq t' + T + 1$, $S_\infty(t)$ lies in an open set K with compact closure $\bar{K} \subset \theta$. Since $P \in S_+$, there is a large t_1 such that $S(t_1) \in B_1$ and is so near to P that $S(t)$ remains in K for $t_1 \leq t \leq t_1 + t' + T + 1$ and therein $\|S(t) - S_\infty(t - t_1)\| < \epsilon/2$. But in the interval $t_1 + t' \leq t \leq t_1 + t' + T + 1$ there is a return of $S(t)$ to B_1 . Thus in the interval $t' \leq t \leq t' + T + 1$ there is a return of $S_\infty(t)$ to B .

We note that if a solution $S(t)$ of $\mathcal{S}$ is eventually almost periodic for $t > T$, then, a fortiori, $S(t)$ is recursive at each of its points $P \in S(t)$ for $t > T$. If also $\theta = E^n$, $S(t)$ is bounded in the positive sense and S_+ contains the closure of the positive ends of all solutions $S_\infty(t; P)$ of $\mathcal{S}_\infty$ which intersect $S(t)$ for $t > T$. Clearly, each $S_\infty(t; P)$ is also bounded in the positive sense and thus has infinite positive escape time. Thus each $S_\infty(t; P)$ is recursive.

3. Applications to Two-Dimensional Systems

Certain results in the plane are immediate consequences of the general theorems of the preceding section.

THEOREM 6. Let $\mathcal{S} \longrightarrow \mathcal{S}_\infty$ in E^2 . If $\mathcal{S}_\infty$ has no critical points, then

- 1) each solution of $\mathcal{S}$ is unbounded in the positive sense,
- 2) no solution of $\mathcal{S}$ is recursive at any point, provided $\tau_+ = +\infty$ for every solution of $\mathcal{S}_\infty$.

PROOF. If $\mathcal{S}_\infty$ has no critical points in E^2 , then each solution approaches infinity along both ends. Thus no solution of $\mathcal{S}_\infty$ is periodic or even Poisson stable.

For two-dimensional autonomous systems we have precise knowledge on the nature of the limit sets of solutions. The following theorem is a generalization of the famous result of Poincaré and Bendixson [5].

THEOREM 7. Let $\mathcal{S} \longrightarrow \mathcal{S}_\infty$ in $\theta \subset E^2$. Let a solution $S(t)$ of $\mathcal{S}$ lie in a compact set $K \subset \theta$ for large t and suppose S_+ contains no critical points of $\mathcal{S}_\infty$. Then S_+ is the union of closed orbits of $\mathcal{S}_\infty$.

PROOF. Let $P \in S_+$ and $S_\infty(t)$ be the solution of $\mathcal{S}_\infty$ through P . Clearly, $S_\infty(t) \subset S_+$ and thus $S_\infty(t) \subset K$ and also $S_\infty(t)$ does not approach (as $t \rightarrow +\infty$) any critical points of $\mathcal{S}_\infty$. Therefore, by the classical Poincaré-Bendixson theorem, $S_\infty(t)$ is a closed orbit of $\mathcal{S}_\infty$.

COROLLARY. If $S_\infty(t) \subset S_+$ is an isolated closed orbit of $\mathcal{S}_\infty$, then $S_\infty(t) = S_+$.

COROLLARY. If $S_\infty(t)$ is an isolated closed orbit of $\mathcal{S}_\infty$ and is stable on both sides, then there is a solution $S(t)$ of $\mathcal{S}$ with $S_+ = S_\infty(t)$.

PROOF. Both corollaries follow immediately from the known structure of the solution curves of a plane system near an isolated closed orbit.

EXAMPLE. Using polar coordinates in the plane with the origin deleted, consider $\mathcal{S} : \dot{r} = \frac{1}{t} \sin \log t$, $\dot{\theta} = 1$ for $t > 0$. Then $\mathcal{S} \rightarrow \mathcal{S}_\infty : \dot{r} = 0$, $\dot{\theta} = 1$ which has solutions of concentric circles about the origin. The solution $S(t)$, $r = 2 - \cos \log t$, $\theta = t$ of $\mathcal{S}$ through $r = 1$, $\theta = 1$, $t = 1$ intersects itself infinitely many times and also S_+ is the closed annulus $1 \leq r \leq 3$.

We complete our studies of two-dimensional systems by an analysis of Liénard equations with varying damping coefficients. This includes the generalized van der Pol equation $\ddot{x} - \mu(t)(1 - x^2)\dot{x} + x = 0$ where $\mu(t) > 0$ is continuous for $t > t_0$ and $\lim_{t \rightarrow \infty} \mu(t) = \mu_\infty > 0$ (cf. [3, 5]).

THEOREM 8. Consider $\ddot{x} + \mu(t)f(x)\dot{x} + g(x) = 0$ where

- 1) $f(x), g(x)$ in $C^{(1)}$ for all x ; $\mu(t)$ is continuous for $t \geq t_0$
- 2) $xg(x) > 0$ for $x \neq 0$; $f(0) < 0$ and $F(x) \equiv \int_0^x f(\xi) d\xi$ approaches $+\infty$ strictly monotonely for $x > x_0$ and $-\infty$ strictly monotonely for $x < -x_0$.
- 3) $M > \mu(t) > 2m > 0$ for all $t \geq t_0$.

Then each solution $S(t)$ has infinite escape time $\tau_+ = +\infty$ and $S(t)$ is bounded in the phase plane.

PROOF. In the phase plane the system is $\mathcal{S} : \dot{x} = y$, $\dot{y} = -\mu(t)f(x)y - g(x)$. If a solution $S_0(t)$ passes through the origin, then by the uniqueness theorem $S_0(t) \equiv 0$.

We introduce the Liénard coordinates $\bar{x} = x$, $\bar{y} = y + mF(x)$ which effect

a $C^{(2)}$ -homeomorphism of the phase plane onto itself. In the Liénard coordinates the system $\mathcal{S}$ becomes $\tilde{\mathcal{S}}: \dot{x} = y - mF(x), \dot{y} = -(\mu(t) - m)f(x)\dot{x} - g(x) = -(\mu(t) - m)f(x)(y - mF(x)) - g(x)$, where we have simplified notation by dropping the super bars. If a solution of $\tilde{\mathcal{S}}$ is bounded, then it is defined for all large t and the theorem is valid.

Let $S(t) \neq 0$ be a solution of $\tilde{\mathcal{S}}$ and we shall show $S(t)$ spirals clockwise around the origin and is, in fact, bounded. Consider the auxiliary curve $\mathcal{C}: y - mF(x) = 0$ in the Liénard plane. On $\mathcal{C}$ $\dot{x} = 0$ so the solutions of $\tilde{\mathcal{S}}$ cross vertically from above to below in $x > 0$ (where $\dot{y} = -g(x) < 0$) and from below to above in $x < 0$. Above $\mathcal{C}$ the x -velocity is positive and below $\mathcal{C}$ it is negative.

In order to show that $S(t)$ is a spiral define the following eight subregions of the Liénard plane: (See Figure 1.)

- | | |
|------------------------------------|---------------------------------------|
| I. $0 < x < x_0, \quad y > mF(x)$ | V. $-x_0 < x < 0, \quad y < mF(x)$ |
| II. $x_0 < x, \quad y > mF(x)$ | VI. $x < -x_0, \quad y < mF(x)$ |
| III. $x_0 < x, \quad y < mF(x)$ | VII. $x < -x_0, \quad y > mF(x)$ |
| IV. $0 < x < x_0, \quad y < mF(x)$ | VIII. $-x_0 < x < 0, \quad y > mF(x)$ |

Suppose $S(t)$ lies in region I at some instant. If $S(t)$ remains in I, then it either becomes unbounded or approaches $\mathcal{C}$ since $\dot{x} = y - mF(x) > 0$. The first alternative is impossible since

$$\left| \frac{dy}{dx} \right| < (M - m)|f(x)| + \frac{g(x)}{y - mF(x)}$$

which is bounded on $0 \leq x \leq x_0$ for $y > 1 + mF(x)$. Also $S(t)$ can not approach $\mathcal{C}$ and remain in region I since then $\dot{y}$ is negative and bounded away from zero. Thus $S(t)$ must leave I and enter either II or IV.

If $S(t)$ intersects region II, then, since $\dot{x} > 0, \dot{y} < 0$ and $\mathcal{C}$ increases monotonely to $+\infty$, $S(t)$ must enter III.

If $S(t)$ intersects III, then, since $\dot{x} < 0$, $S(t)$ either enters IV, approaches $\mathcal{C}$ or becomes unbounded in III. But $S(t)$ does not approach $\mathcal{C}$ since near $\mathcal{C}$ $-\frac{dy}{dx}$ is arbitrarily great and, furthermore, the last possibility is excluded since $|\frac{dy}{dx}|$ is bounded on $S(t)$ in III if $y < -1$.

If $S(t)$ intersects IV, then, since $\dot{x} < 0$, either $S(t)$ enters V or remains in IV. This last alternative is excluded since $S(t)$ remains bounded in IV because $|\frac{dy}{dx}|$ is bounded on $S(t)$ if $y < mF(x) - 1$. Yet $S(t)$ does not approach $\mathcal{C}$ in IV since $\frac{dy}{dx} \rightarrow -\infty$ at points of $\mathcal{C}$, not the origin, and $\dot{y} < 0$ for small $|x|$ near the origin.

From V, using similar arguments, we see that $S(t)$ continues clockwise around the origin until it again intersects the positive y -axis. Thus each

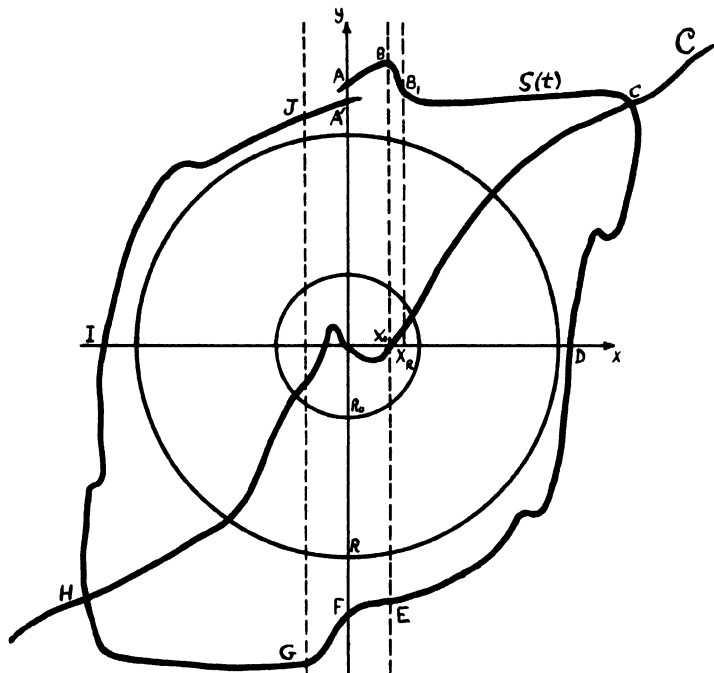


FIGURE 1

solution $S(t)$ spirals clockwise around the origin, alternately intersecting the positive and negative y -axes infinitely often.

We shall show that solutions having a positive y -intercept of y_A and remaining outside the circle $x^2 + y^2 = R^2$ (defined below) spiral inwards so that $y_{A_1} < y_A$. Furthermore, such solutions are shown to lie entirely within the circle $x^2 + y^2 = \phi(1.3y_A)$ (defined below) for the entire revolution.

Consider the Liénard radial distance ρ defined by $\rho^2 = 2G(x) + y^2$, where

$$G(x) \equiv \int_0^x g(\xi) d\xi.$$

Then $\frac{d}{dt}(\rho^2/2) = -\delta_1 - \delta_2$ where $\delta_1 = mF(x)g(x)$ and $\delta_2 = (\mu(t) - m)f(x)y\dot{x}$ along a solution $S(t)$ lying outside the circle of radius R . We shall show that $\oint_{S(t)} (\delta_1 + \delta_2) dt > 0$ for a single revolution of $S(t)$ and thus

that $y_{A'} < y_A$.

Now on $AB + EF + FG + JA'$ we have

$$\left| \int \delta_1 dt \right| \leq \left| \int \frac{mF(x)g(x)}{y-mF(x)} dx \right| < 1$$

provided $R^2 > R_1^2 = m^2 F_M^2 (4g_M x_0 + 1)^2 + x_0^2$ where

$$g_M = \max_{-x_0 \leq x \leq x_0} |g(x)|$$

and

$$F_M = \max_{-x_0 \leq x \leq x_0} |F(x)|$$

and $R^2 > R_0^2 \geq m^2 F_M^2 + x_0^2$. On each of BC, DE, GH, IJ we have $\int \delta_1 dt > 0$.

From C to D and also from H to I both $|x|$ and $|y|$ decrease and so $\int (\delta_1 + \delta_2) dt = - \int \frac{d}{dt} (\rho^2/2) dt > 0$ on CD and HI . On BC, DE, GH, IJ we have $\int \delta_2 dt = \int (\mu(t) - m)f(x)y dx > 0$.

On AB, EFG , and JA' , $|y| > (R^2 - x_0^2)^{\frac{1}{2}} > g_M + mF_M$, if $R^2 > R_2^2 = (g_M + mF_M)^2 + x_0^2$, and

$$\left| \frac{dy}{dx} \right| < (M - m) \max_{-x_0 \leq x \leq x_0} |f(x)| + 1 = K.$$

On AB , $.9y_A < y < 1.1y_A$ if $y_A > R > R_3 = 10Kx_0$. Then

$$\left| \int_A^B \delta_2 dt \right| < (M - m)(1.1y_A) \int_A^B |f(x)| dx < K_1 y_A$$

where the constant

$$K_1 = 1.1(M - m)x_0 \max_{-x_0 \leq x \leq x_0} |f(x)|$$

does not depend on R , provided $R > R_2 + R_3$.

Now $\rho(B)^2 < 2G(x_0) + (1.1y_A)^2$ and $\rho(E) < \rho(B)$ so that $|y_E| < \rho(E) < \rho(B) < 1.2y_A$ provided $y_A > R > R_4 = [10G(x_0)]^{\frac{1}{2}}$. Therefore, on EFG , $|y| < 1.2y_A + 2Kx_0 < 1.3y_A$ provided $y_A > R > R_5 = 20Kx_0$. Since $\rho(J) < \rho(G)$, we have $y_J < |y_G| < 1.3y_A$. Thus $y < 1.3y_A + Kx_0 < 1.4y_A$ on JA' .

Now it is easy to complete the estimate of $\int \delta_2 dt$ on $EF + FG + JA'$. Here

$$\left| \int \delta_2 dt \right| < 3(M - m) \int_0^{x_0} |f(x)| dx \quad (1.4y_A) < 4K_1 y_A.$$

Thus, provided $R > R_0 + R_1 + R_2 + R_3 + R_4 + R_5$ we have $\int (\delta_1 + \delta_2) dt > -1 - 5K_1 y_A$ on the complete revolution AA' omitting BC . We shall show that $\int (\delta_1 + \delta_2) dt > 0$ on BC , is very large.

If we require that $R > 3R_0$, then we need only consider

$$\int_B^{B_1} \delta_2 dt.$$

Now on BB_1 ,

$$\left| \frac{dy}{dx} \right| < (M - m) \max_{x_0 \leq x \leq R_0} |f(x)| + \left(\max_{x_0 \leq x \leq R_0} g(x) \right) / 2R_0 = K_2.$$

Thus on BB_1 we have $y > .9y_A - K_2(R_0 - x_0) > .8y_A$ provided $y_A > R > R_6 = 10K_2(R_0 - x_0)$. Then

$$\int_B^C (\delta_1 + \delta_2) dt > \int_B^{B_1} \delta_2 dt > .8y_A m \int_{x_0}^{x_R} f(x) dx > 1 + 5K_1 y_A$$

provided x_R is such that

$$F(x_R) > 1 + \frac{5K_1}{.8m}$$

and

$$y_A > R > R_7 = \frac{1}{.8m}.$$

Thus, take

$$R^2 > R_0^2 > \left[F^{-1} \left(1 + \frac{5K_1}{.8m} + m^2 F_M^2 + x_0^2 \right) \right]^2 + \left[1 + \frac{5K_1}{.8m} \right]^2.$$

Therefore, if $R > R_0 + R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7$ we have the desired result that $\oint (\delta_1 + \delta_2) dt > 0$ for a complete revolution. Thus $y_{A'}^2 = \rho(A') < \rho(A) = y_A^2$ and $y_{A'} < y_A$.

On AB , $y < 1.1y_A$ so $r^2 = x^2 + y^2 < x_0^2 + (1.1y_A)^2$. On BC , $y < 1.1y_A$ so

$$r^2 < \left[F^{-1}(1.1y_A) \right]^2 + (1.1y_A)^2.$$

On CD,

$$r^2 < \left[F^{-1}(1.1y_A) \right]^2 + (1.1y_A)^2.$$

On DE,

$$\begin{aligned} r^2 &< r_D^2 + \rho^2 < r_D^2 + \rho_D^2 < r_D^2 + \left[2G(x_D) \right]^2 \\ &< r_D^2 + \left[2G(r_D) \right]^2 < \left[F^{-1}(1.1y_A) \right]^2 + (1.1y_A)^2 \\ &+ 4G \left(\left[F^{-1}(1.1y_A)^2 + (1.1y_A)^2 \right]^{\frac{1}{2}} \right)^2 = \phi(1.1y_A). \end{aligned}$$

On EFG, $|y| < 1.3y_A$ and so $r^2 < x_O^2 + (1.3y_A)^2$. On GH, $r^2 < F^{-1}(1.3y_A)^2 + (1.3y_A)^2$. On HI, $r^2 < F^{-1}(1.3y_A)^2 + (1.3y_A)^2$. On IJ, $r^2 < r_I^2 + \rho_I^2 < \phi(1.3y_A)$. On JA', $r^2 < x_O^2 + (1.1y_A)^2$. Thus on the complete revolution $R^2 < r^2 < \phi(1.3y_A)$ where ϕ is a monotone increasing function.

If a solution $S(t)$ remains outside the circle about the origin of radius R , then it is bounded. Therefore S_+ is not empty and $\tau_+ = +\infty$ for $S(t)$.

To complete the proof of the theorem we need only show that if a solution $S(t)$ intersects the disc $x^2 + y^2 \leq R^2$ on a certain revolution, then $r < K_2$, a constant, on that entire revolution. But by the same procedure as used above we obtain $r^2 < \phi(1.3R) = K_2^2$.

We have shown that if $S(t)$ has an intercept y_A on the positive y -axis, then, on $S(t)$, $r^2 < \phi(1.3y_A) + \phi(1.3K_2)$ for all times after that intercept. Thus each solution $S(t)$ is bounded in the phase plane for $t \rightarrow \tau_+$ and has an escape time $\tau_+ = +\infty$.

COROLLARY. The positive limit set S_+ of $S(t) \neq 0$ is non-empty but does not contain the origin.

PROOF. In the phase plane the contours of $\rho^2 = 2G(x) + y^2$ are simple closed curves encircling the origin for ρ^2 near zero; because $\rho^2 = \epsilon > 0$ is compact and $\text{grad } \rho^2 \neq 0$ except at the origin. But $\frac{d}{dt}(\rho^2/2) = g(x)\dot{x} + y\dot{y} = -\mu(t)f(x)y^2 > 0$ for small $|x|$ and all $t \geq t_0$. Thus, near the origin, the solutions of $\mathcal{S}$ are all directed outward across the contours of ρ^2 and thus they do not approach the origin.

THEOREM 9. Consider $\mathcal{S}: \ddot{x} + \mu(t)f(x)\dot{x} + g(x) = 0$

in the phase plane $(x, y = \dot{x})$, where

- 1) $f(x), g(x)$ in $C^{(A)}$ for all x , $\mu(t)$ is continuous on $t \geq t_0$.
- 2) $g(x) = -g(-x)$ and $xg(x) > 0$ in $C^{(\omega)}$ for $x \neq 0$; $f(x) = f(-x)$ and

$$F(x) = \int_0^x f(\xi) d\xi$$

is negative for $0 < x < x_0$ but is positive and approaches $+\infty$ monotonely for $x > x_0$.

- 3) $\mu(t) > m > 0$ and $\lim_{t \rightarrow \infty} \mu(t) = \mu_\infty$.

Then each solution $S(t)$ of $\mathcal{S}$ is bounded in the phase plane and has an infinite escape time τ_+ . Moreover, if $S(t) \neq 0$, $S(t)$ approaches the unique closed orbit Σ of $\mathcal{S}_\infty$ and hence $S_+ = \Sigma$. There is an eventually periodic solution of $\mathcal{S}$ if and only if $\mu(t) = \mu_\infty$, for all sufficiently large t .

PROOF. By the preceding theorem each solution $S(t)$ of $\mathcal{S}$ is bounded, has $\tau_+ = +\infty$, and has a positive limit set which is the union of closed orbits of $\mathcal{S}_\infty$. Thus $S_+ = \Sigma$ and $S(t)$ lies in any prescribed neighborhood of Σ for all sufficiently large t .

If $\mu(t) = \mu_\infty$ for $t \geq T \geq t_0$, then there is an eventually periodic solution of $\mathcal{S}$, namely, any solution of $\mathcal{S}_\infty$ which traces out Σ and initiates at $t = T$. Conversely, if there exists an eventually periodic solution $S^*(t)$ of $\mathcal{S}$, then $S_+^* = \Sigma$ and, for large t , $S^*(t)$ is a solution of $\mathcal{S}_\infty$ tracing Σ .

In the phase plane the equal velocities of $S^*(t)$ and $\Sigma(t)$ at $(x, y) \in \Sigma$ are $\dot{x} = \dot{y} = y$, $\dot{y} = -\mu(t)f(x)y - g(x) = -\mu_\infty f(x)y - g(x)$. Thus, on Σ and where $yf(x) \neq 0$, $\mu(t) = \mu_\infty$. But $yf(x)$ is analytic and vanishes at only a finite number of points of Σ . Thus $\mu(t) = \mu_\infty$ at all large t excepting a discrete set of t . But by continuity $\mu(t) = \mu_\infty$ for all large t .

The hypotheses of the above theorem could be weakened so that

- 1) $f(x)$, $g(x)$, $\mu(t)$ satisfy the hypotheses of the preceding theorem and, in addition,
- 2) there exists just one periodic solution of $\mathcal{S}_\infty$ [6] and, moreover,
- 3) $f(x)$ has no accumulation point for its zeros.

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III. NONLINEAR DIFFERENTIAL EQUATIONS SYSTEMS¹

Edmund Pinney

1. Introduction

The literature dealing with nonlinear ordinary differential equations systems is largely concerned with existence, uniqueness, and boundedness theorems. These theorems are of theoretical importance and occasionally have practical applications. However, the rather more difficult theory of oscillations in nonlinear systems has received very little attention, and that of a type not well adapted to application.

As a result, although the theory of oscillations in mechanical and electrical nonlinear systems of a single degree of freedom is fairly well understood, we know very little about multiple degree of freedom systems. It is most probable that matters of considerable practical importance are yet to be discovered in this field.

The general theorems mentioned above are useful in the sense that an airplane is useful in prospecting for oil. They cover a wide territory but don't penetrate very deeply. An intimate knowledge of the field can be obtained only by detailed investigations of particular differential equations systems.

The purpose of this paper is to develop a practical method for carrying through such detailed investigations. Although the systems considered are somewhat less general than those of some other writers, the present theory has a wider scope within this limitation. Thus we have considered only polynomial nonlinearities, but we have been able to avoid restricting our attention to periodic solutions, a restriction that has seriously curtailed the usefulness of most previous theories. We may, for instance, treat some problems in which several oscillations of incommensurable periods occur. A novel feature of the present theory is that cases

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of multiple characteristic roots can be treated.

An approximate solution to the nonlinear differential equation system (3.1) below will be developed, based on the linear system gotten by setting $\epsilon = 0$ in (3.1). Such a theory cannot in itself settle the question of unboundedness of the solution, for as the dependent variables increase, the nonlinear terms must ultimately dominate, thus invalidating the base of approximation. It must therefore be emphasized that a prediction that one or more of the dependent variables tends to infinity merely means that the theory fails. Indeed, the theory may fail for a prediction of large finite values of these quantities (when conditions (4.3) and (10.7) fail to be satisfied). This will be called "large solution failure."

Sections 2 through 9 are concerned with development of the theory. In Section 10, the procedure for solving a system of equations is summarized for reference purposes. It can be applied without detailed familiarity with the theory. Section 11 is an Appendix in which some results used in the theory are proved.

The following notation is employed. References are to the numbers of the equations in which or near which the notation is defined.

A	(7.3)	$D_{1s; \alpha}(z)$	(2.3)
a	(5.1)	$E(z)$	(9.4)
$A_{1\alpha}$	(2.1), (3.1)	$f_1(y, t)$	(3.1)
$A_{1\alpha}^*$	(9.2)	$f_1^*(y, t)$	(9.3)
$a_{jq}(t)$	(2.8), (3.4)	$f_{jq}(t)$	(7.4), (8.2)
$\bar{a}_{jq}(t)$	(6.13)	$F(z, y, t)$	(3.5)
$A_{jq}(t)$	(7.1)	$F^*(z, y, t)$	(9.10)
b_1	(2.6)	$F^{(q)}(z, y, t)$	(3.6)
$B_{1\alpha}$	(9.2)	$G_1(z)$	(2.9)
$B(z)$	(2.10)	$g_{jq}(t)$	(7.4), (8.2)
C_j	(9.7)	$G_j(z, a)$	(5.7)
$C(z)$	(9.6)	$G_j^*(z, a)$	(9.11)
$D(z)$	(2.2)	$G_j^{(q)}(z, a)$	(3.6)
$\bar{D}(z)$	(2.3)	$G_j(z, a, T)$	(5.3)
$D_s(z)$	(2.3)	$g_j(z, a, t)$	(5.7)
$\bar{D}_r(z)$	(2.3)	$g_j^*(z, a, t)$	(9.11)
$D_{j1}(z)$	(2.2)	$g_j^{(p)}(z, a, t)$	(3.6)

$g_j(z, a, t, T)$	(5.3)	$y_1(t)$	(2.7), (3.3)
$G_{jq}(R, \theta, t)$	(7.2)	$Y(a, t, T)$	(5.3)
$H_{jq}(R, \theta, t)$	(7.2)	$Y_1(a, t, T)$	(5.1)
K	(4.1), (10.6)	z_j	(2.1)
K_j	(9.7)	z_j'	(5.3)
K_{jr}	(9.14)	z_{jp}^*	(9.8)
m	(2.1), (3.1)	α_{jp}	(9.9)
n	(3.1)	ϵ	(3.1)
n_j	(9.7)	θ	(7.3)
R	(7.3)	$\theta_{jq}(t)$	(7.1)
$R_{jq}(t)$	(7.1)	κ	(7.4), (8.2)
$\bar{R}_{jq}(t)$	(7.4)	λ_s	(2.3)
$\Delta R_{jq}(t)$	(7.4)	μ_j	(2.1)
S	(2.1)	v_r	(2.3)
S'	(5.4)	$\rho_{jq}(t)$	(6.6)
$S_{1\alpha}$	(2.13)	$\Delta \rho_{jq}(t) $	(7.4)
t	(3.1)	$\Delta \arg[\rho_{jq}(t)]$	(7.4)
$u_1(t)$	(2.1)	$\sum_{S^k}^j$	(6.1)
$U(z, t)$	(2.10)		
x_j	(7.1)	$\emptyset$	(7.3)
y	(3.1)	$\emptyset_{jq}(t)$	(7.1)
y_1	(2.1), (3.1)	$\bar{\emptyset}_{jq}(t)$	(7.4)
$y(t)$	(3.1)	$\Delta \emptyset_{jq}(t)$	(7.4)

2. A Nonhomogeneous Differential Equation System

As a preliminary to our discussion of the nonlinear system (3.1) below, we will consider the nonhomogeneous differential equation system

$$(2.1) \quad \frac{dy_1}{dt} = \sum_{\alpha=1}^m A_{1\alpha} y_\alpha + u_1(t), \quad 1 = 1, 2, \dots, m.$$

The quantities $A_{1\alpha}$ are constants. To discuss this system, let S denote the set of characteristic roots z_j (of multiplicity μ_j) of the characteristic equation

$$(2.2) \quad D(z) \equiv \left| A_{1j} - z\delta_{1j} \right| = 0,$$

and let $D_{j1}(z)$ be the cofactor of $A_{1j} - z\delta_{1j}$ in the determinant of (2.2). Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be an arbitrarily chosen set of constants not all of which are zero, and let $v_1, v_2, \dots, v_m$ be another such set. Define the quantities

$$(2.3) \quad \begin{aligned} D_r(z) &= \sum_{s=1}^m \lambda_s D_{rs}(z), \quad \bar{D}_s(z) = \sum_{r=1}^m v_r D_{rs}(z), \\ \bar{D}(z) &= \sum_{r=1}^m v_r D_r(z) = \sum_{s=1}^m \lambda_s \bar{D}_s(z) = \sum_{r=1}^m \sum_{s=1}^m \lambda_s v_r D_{rs}(z), \\ \bar{D}_{1\alpha}(z) &= \sum_{r=1}^m \sum_{s=1}^m \lambda_s v_r D_{1s;r\alpha}(z), \end{aligned}$$

where $D_{1s;r\alpha}(z)$ is the cofactor of $A_{\alpha r} - z\delta_{\alpha r}$ in the determinant $*D_{1s}(z)$. Then

$$(2.4) \quad \sum_{\beta=1}^m (A_{1\beta} - z\delta_{1\beta}) D_{\beta j}(z) = \delta_{1j} D(z),$$

$$(2.5) \quad D_{1\alpha}(z) \bar{D}(z) - D_1(z) \bar{D}_{\alpha}(z) + D(z) \bar{D}_{1\alpha}(z) = 0.$$

(2.5) is derived by multiplying (11.1) in the Appendix by $\lambda_s v_r$, summing over r and s from 1 to m , and using (2.3).

THEOREM 1. The differential equation system (2.1) has a unique solution set $y_1 = y_1(t)$ for which

$$(2.6) \quad y_1(0) = b_1, \quad 1 = 1, 2, \dots, m,$$

where $b_1, b_2, \dots, b_m$ are arbitrary constants. If the λ_s and v_r quantities in (2.3) are so chosen that $D(z)$ and $\bar{D}(z)$ have no common zeros, then for $1 = 1, 2, \dots, m$,

$$(2.7) \quad y_1(t) = \sum_j e^{z_j t} \frac{\mu_j!}{D^{(\mu_j)}(z_j)} \sum_{q=0}^{\mu_j-1} \frac{1}{q!} G_1^{(q)}(z_j) a_{j, \mu_j-1-q}(t),$$

where

$$a_{jq}(t) = \left[\left(t + \frac{\partial}{\partial z_j} \right)^q B(z_j) + \int_0^t e^{-z_j \tau} \left(t - \tau + \frac{\partial}{\partial z_j} \right)^q U(z_j, \tau) d\tau \right] / q!, \quad (2.8)$$

where

$$G_1(z) = -D_1(z) / \bar{D}(z), \quad (2.9)$$

$$B(z) = \sum_{\alpha=1}^m b_{\alpha} \bar{D}_{\alpha}(z), \quad U(z, t) = \sum_{\alpha=1}^m \bar{D}_{\alpha}(z) u_{\alpha}(t). \quad (2.10)$$

PROOF. The Picard theory of successive approximations may be used to show that the system (2.1) subject to the boundary conditions (2.6) has a unique solution set $y_1(t)$, which we will now show can be represented in the form

$$y_1(t) = - \sum_j \frac{\mu_j}{D(\mu_j)(z_j)} \frac{\partial^{\mu_j-1}}{\partial z_j^{\mu_j-1}} \sum_{\alpha=1}^m D_{1\alpha}(z_j) \left[e^{z_j t} b_{\alpha} + \int_0^t e^{z_j(t-\tau)} \mu_{\alpha}(\tau) d\tau \right]. \quad (2.11)$$

In fact,

$$\begin{aligned} \frac{dy_1}{dt} - \sum_{\beta=1}^m A_{1\beta} y_{\beta} &= - \sum_j \frac{\mu_j}{D(\mu_j)(z_j)} \frac{\partial^{\mu_j-1}}{\partial z_j^{\mu_j-1}} \sum_{\alpha=1}^m \left\{ \sum_{\beta=1}^m (z_j \delta_{1\beta} \right. \\ &\quad \left. - A_{1\beta}) D_{\beta\alpha}(z_j) \left[e^{z_j t} b_{\alpha} + \int_0^t e^{z_j(t-\tau)} \mu_{\alpha}(\tau) d\tau \right] \right. \\ &\quad \left. + D_{1\alpha}(z_j) \mu_{\alpha}(t) \right\}, \end{aligned}$$

so by (2.4),

$$\frac{dy_1}{dt} = \sum_{\alpha=1}^m A_{1\alpha} y_{\alpha} + \sum_{\alpha=1}^m S_{1\alpha} u_{\alpha}(t), \quad (2.12)$$

where

$$(2.13) \quad S_{1\alpha} = - \sum_j \mu_j D_{1\alpha}^{(\mu_j^{-1})} (z_j)/D^{(\mu_j)} (z_j).$$

By (2.11),

$$(2.14) \quad y_1(0) = \sum_{\alpha=1}^m S_{1\alpha} b_\alpha.$$

To evaluate $S_{1\alpha}$, let C be a circle centered at the origin enclosing all the characteristic roots of (2.2). Then by (2.9) and the theory of residues

$$(2.15) \quad S_{1\alpha} = \frac{1}{2\pi i} \int_C \left[D_{1\alpha}(z)/D(z) \right] dz.$$

Now as $z \rightarrow \infty$,

$$(2.16) \quad D_{1\alpha}(z)/D(z) = -\delta_{1\alpha}/z + o(z^{-2}).$$

This may be derived readily from (2.4) and the fact that $D_{1\alpha}(z)$ is of degree $n-1$ or $n-2$ according as $\alpha=1$ or $\alpha \neq 1$, respectively. Now, allowing C to tend to infinity and inserting (2.16) into (2.15) we get $S_{1\alpha} = \delta_{1\alpha}$. Substituting this into (2.12) and (2.14), it follows that both (2.1) and (2.6) are satisfied by (2.11), which is therefore a solution.

Now let C be a contour surrounding all the roots of $D(z)$ but none of the roots of $\bar{D}(z)$. Then by (2.11),

$$\begin{aligned} y_1(t) = & -\frac{1}{2\pi i} \int_C \sum_{\alpha=1}^m \left[D_{1\alpha}(z)/D(z) \right] \left[e^{zt} b_\alpha \right. \\ & \left. + \int_0^t e^{z(t-\tau)} u_\alpha(\tau) d\tau \right] dz. \end{aligned}$$

Solving (2.5) for $D_{1\alpha}(z)$ and inserting in this,

$$\begin{aligned} y_1(t) = & -\frac{1}{2\pi i} \int_C \sum_{\alpha=1}^m \left\{ \frac{D_1(z) \bar{D}_\alpha(z)}{D(z) \bar{D}(z)} - \frac{\bar{D}_{1\alpha}(z)}{\bar{D}(z)} \right\} \left[e^{zt} b_\alpha \right. \\ & \left. + \int_0^t e^{z(t-\tau)} u_\alpha(\tau) d\tau \right] dz. \end{aligned}$$

The second term inside the braces has no poles inside C and therefore does not contribute to the integral. The first term has poles at the characteristic roots only. By the theory of residues,

$$y_1(t) = - \sum_j \frac{\mu_j}{D^{(\mu_j)}(z_j)} \frac{\partial^{\mu_j-1}}{\partial z_j^{\mu_j-1}} G_1(z_j) \left[e^{z_j t} B(z_j) + \int_0^t e^{z_j(t-\tau)} U(z_j, \tau) d\tau \right].$$

Applying Leibnitz's rule to this expression we get (2.7) with

$$a_{jq}(t) = e^{-z_j t} \frac{\partial^q}{\partial z_j^q} \left[e^{z_j t} B(z_j) + \int_0^t e^{z_j(t-\tau)} U(z_j, \tau) d\tau \right] / q!.$$

(2.8) follows from this because $B(z)$ and $U(z, \tau)$ are polynomials in z .

3. The Nonlinear Differential Equation System

We will consider a nonlinear differential equation system

$$(3.1) \quad \frac{dy_1}{dt} = \sum_{\alpha=1}^m A_{1\alpha} y_\alpha + \epsilon f_1(y, t), \quad 1 = 1, 2, \dots, m,$$

the quantities in which are defined as follows:

- i) the $A_{1\alpha}$ are constants and ϵ is a parameter
- ii) $y = y(t)$ is the matrix $(y_1(t), y_2(t), \dots, y_m(t))$;
- iii) $f_1(y, t)$ is a polynomial of degree n in the elements of y ; of the terms of degree n , some have coefficients of the order of magnitude of unity in ϵ , and the remainder have coefficients of an order less than unity in ϵ ;
- iv) $f_1(y, t)$ depends on t explicitly only as a polynomial in circular functions of t ;
- v) for $b_1, b_2, \dots, b_m$ arbitrarily preassigned quantities,

$$(3.2) \quad y_1(0) = b_1, \quad 1 = 1, 2, \dots, m$$

Picard theory may be used to show that the system (3.1) has a unique integrable solution for $t \geq 0$ such that the initial conditions (3.2) are satisfied. Accordingly $f_1(y, t)$ in (3.1) is integrable and

Theorem 1 may be applied, using $\mu_1(t) = f_1(y(t), t)$. We get

$$(3.3) \quad y_1(t) = \sum_S j e^{z_j t} \sum_{q=0}^{\mu_j-1} \frac{\mu_j! g_1^{(q)}(z_j)}{q! D^{(\mu_j)}(z_j)} a_{j, \mu_j-1-q}(t),$$

for $1 = 1, 2, \dots, m$, where by (2.8),

$$(3.4) \quad a_{jq}(t) = \left[\epsilon \int_0^t e^{-z_j \tau} \left(t - \tau + \frac{\partial}{\partial z_j} \right)^q F(z_j, y(\tau), \tau) d\tau \right. \\ \left. + \left(t + \frac{\partial}{\partial z_j} \right)^q B(z_j) \right] / q!,$$

where

$$(3.5) \quad F(z, y, t) = \sum_{\alpha=1}^m \bar{D}_\alpha(z) f_\alpha(y, t).$$

By (3.4), if $F^{(q)}(z, y, t)$ denotes $(\partial^q / \partial z^q) F(z, y, t)$,

$$(3.6) \quad a_{jq}(T) = \sum_{r=0}^q a_{jr}(t) (T-t)^{q-r} / (q-r)! \\ + (\epsilon/q!) \int_t^T e^{-z_j \tau} \left(T - \tau + \frac{\partial}{\partial z_j} \right)^q F(z_j, y(\tau), \tau) d\tau,$$

$$(3.7) \quad a'_{jq}(t) = a_{j, q-1}(t) + (\epsilon/q!) e^{-z_j t} F^{(q)}(z_j, y(t), t),$$

$$(3.8) \quad \frac{\partial}{\partial t} \sum_{r=0}^q a_{jr}(t) (T-t)^{q-r} / (q-r)! \\ = (\epsilon/q!) e^{-z_j t} \left(T - t + \frac{\partial}{\partial z_j} \right)^q F(z_j, y(t), t),$$

for z_j in S , $q = 0, 1, \dots, \mu_j - 1$, and for $a_{j, -1}(t) = 0$. (3.6) is derived from (3.4) by means of the binomial theorem. Differentiating (3.6) with respect to T and letting $T \rightarrow t$ we get (3.7). Differentiating

(3.6) with respect to t we get (3.8).

So far the theory is exact, but it provides no method to compute the quantities $a_{jq}(t)$. Our main problem is to obtain approximate expressions for these quantities.

4. Bounded Solutions

As noted in the Introduction, we will use an approximation procedure which depends upon the dominance of the linear terms in (3.1). In particular, it is essential that the solution be bounded if this procedure is to succeed.

The boundedness of the unknown quantities $b_{jq}(t) = e^{z_j t} a_{jq}(t)$ may be determined in terms of the approximation $\sigma_{jq}(t) = e^{z_j t} \rho_{jq}(t)$ to these quantities derived below. $|\sigma_{jq}(t)|$ will be known, so an upper bound K' for it may be computed. However the difference between $b_{jq}(t)$ and $\sigma_{jq}(t)$ is estimated in terms of the $b_{jq}(t)$ quantities, whose bounds are unknown, and ϵ .

Suppose $|b_{jq}(0)|$ is less than some quantity K . Moreover, suppose we know that so long as

$$(4.1) \quad |b_{jq}(t)| = \left| e^{z_j t} a_{jq}(t) \right| < K, \quad z_j \text{ in } S,$$

$$q = 0, 1, \dots, j-1,$$

there is a quantity $E(\epsilon, K)$ such that

$$(4.2) \quad |b_{jq}(t) - \sigma_{jq}(t)| = \left| e^{z_j t} \left[a_{jq}(t) - \rho_{jq}(t) \right] \right| < E(\epsilon, K),$$

and that

$$(4.3) \quad E(\epsilon, K) < K - K', \quad K' < K,$$

where K' is an upper bound for $|\sigma_{jq}(t)|$:

$$(4.4) \quad |\sigma_{jq}(t)| = \left| e^{z_j t} \rho_{jq}(t) \right| \leq K'.$$

Then

$$|b_{jq}(t)| \leq |b_{jq}(t) - \sigma_{jq}(t)| + |\sigma_{jq}(t)| < E(\epsilon, K) + K' \\ < K - K' + K' = K.$$

That is, if a K may be found which satisfies (4.3) and which is greater than $|b_{jq}(0)|$, then $|b_{jq}(t)|$ always remains less than this K .

To illustrate this idea, think of b_{jq} as a boy who enters a haunted house (condition (4.1) at $t = 0$) with his father σ_{jq} . The boy is afraid to get farther away from his father than $E(\epsilon, K)$ (condition (4.2)). We don't know where the boy is, but we do know where the father is. However, so long as the father remains a distance at least $E(\epsilon, K)$ away from the door (condition (4.3)), we know that the boy remains inside the house (condition (4.1) for $t > 0$).

This process can be carried through if we are able to select a K greater than K' (which may depend on ϵ) for which (4.3) holds. When this cannot be done the theory suffers large solution failure. It should be emphasized that K is not necessarily a least upper bound to the quantities $|b_{jq}(t)|$. Indeed, the present theory will not enable us to calculate this bound. K may depend on ϵ . Without loss in generality we can, and will, for convenience, assume that K does not tend to 0 as $\epsilon \rightarrow 0$, i.e., $1/K = O(1)$. In addition, we will bound K above by the assumption that as $\epsilon \rightarrow 0$,

$$(4.5) \quad K = O(\epsilon^{-1/(n-1)}).$$

This eliminates from consideration bounds on the $y_1(t)$ quantities and their derivatives so large that the linear terms in (3.1) do not dominate the nonlinear terms. Our method of approximation depends on this dominance; although we would not be forced to assume (4.5) until later, some simplification results from using this assumption from the outset.

The "trend functions" $\rho_{jq}(t)$ will be defined below as the solutions to the differential equations (6.6). To complete the theory we must calculate the form of the function $E(\epsilon, K)$. This is done beginning with the assumption (4.1), where K is not specified beyond conditions (4.5) and $1/K = O(1)$ as $\epsilon \rightarrow 0$. With $E(\epsilon, K)$ and K' obtained from (4.4) we seek to specify K more particularly so that (4.3) holds. Usually there are infinitely many possible values of K . When K' satisfies (8.7) below, a value $K = O(K')$ is particularly convenient. In general we wish to render $E(\epsilon, K)$ as small as possible.

By (3.3), (3.5), and iii) of Section 3,

$$(4.6) \quad \begin{aligned} y_1(t) &= O(K), \quad 1 = 1, 2, \dots, m, \\ F^{(q)}(z_j, y(t), g) &= O(K^n), \quad z_j \text{ in } S, \quad q = 0, 1, \dots, \mu_j - 1. \end{aligned}$$

By (3.6) - (3.8), (4.6), for $T - t = O(1)$ as $\epsilon \rightarrow 0$, z_j in

$S, q = 0, 1, \dots, \mu_j^{-1},$

$$(4.7) \quad a_{jq}(T) = \sum_{r=0}^q a_{jr}(t)(T-t)^{q-r}/(q-r)! + e^{-z_j t} o(\epsilon K^n),$$

$$(4.8) \quad a'_{jq}(t) = a_{j,q-1}(t) + e^{-z_j t} o(\epsilon K^n),$$

$$(4.9) \quad \frac{\partial}{\partial t} \sum_{r=0}^q a_{jr}(t)(T-t)^{q-r}/(q-r)! = e^{-z_j t} o(\epsilon K^n).$$

5. Expansion of the Function $F(z, y, t)$

For $i = 1, 2, \dots, m$, define

$$(5.1) \quad \begin{aligned} Y_1(a, t, T) \\ = \sum_j \frac{\mu_j! e^{z_j t}}{D(\mu_j)} \sum_{(z_j)} \frac{\mu_j^{-1}}{q=0} \frac{G^{(\mu_j^{-1}-q)}(z_j)}{(\mu_j^{-1}-q)!} \sum_{r=0}^q a_{j,q-r} T^r / r!, \end{aligned}$$

where a denotes the matrix of the quantities a_{jq} for z_j in S , $q = 0, 1, \dots, \mu_j - 1$. Then by (3.3),

$$(5.2) \quad y_1(t) = Y_1(a(t), t, 0)$$

Let $Y(a, t, T)$ denote the one-row matrix of the quantities $Y_1(a, t, T)$. Then

$$(5.3) \quad \begin{aligned} F(z, Y(a, t, T), t) &= \sum_j e^{z_j t} g_j(z, a, t, T) \\ &+ \sum_{j'} e^{z_{j'} t} G_{j'}(z, a, T), \end{aligned}$$

where the following conditions hold:

1) the terms in the S summation must be such that

$$(5.4) \quad \frac{\partial}{\partial t} g_j(z, a, t, T) = e^{-z_j t} o(\epsilon K^{2n-1});$$

ii) the S' summation is over all terms in the expansion of the left-hand side of (5.3) not included in the S summation. However, we will impose the further restriction that

$$(5.5) \quad 1/|z_j - z'_k| = o(1)$$

for all z_j in S and z'_k in S' .

Condition i) simply defines the terms which are to be included in the S summation. However, condition (5.5) in ii) implies a new restriction on F and on the characteristic roots.

By (5.2), (5.3),

$$(5.6) \quad \begin{aligned} F(z, y(t), t) &= \sum_j e^{z_j t} g_j(z, a(t), t) \\ &+ \sum_{j'} e^{z_j' t} G_j(z, a(t)), \end{aligned}$$

where

$$(5.7) \quad g_j(z, a, t) = g_j(z, a, t, 0), \quad G_j(z, a) = G_j(z, a, 0).$$

When $T = o(1)$,

$$(5.8) \quad \begin{aligned} g_j(z, a(t), t, T) &= e^{-z_j T} o(K^n), \\ G_j(z, a(t), T) &= e^{-z_j T} o(K^n), \end{aligned}$$

and when $T - t = o(1)$ as $\epsilon \rightarrow 0$,

$$(5.9) \quad \begin{aligned} \frac{\partial}{\partial T} g_j(z, a(t), t, T - t) &= e^{-z_j T} o(\epsilon K^{2n-1}), \\ \frac{\partial}{\partial T} G_j(z, a(t), T - t) &= e^{-z_j T} o(\epsilon K^{2n-1}). \end{aligned}$$

(5.8) follows from (4.6) and (5.3). By (4.9) and (5.1) as $\epsilon \rightarrow 0$,

$$\frac{\partial}{\partial s} Y_1(a(s), t, T - s) = o(\epsilon K^n).$$

By (5.6),

$$F(z, Y(a(s), t, T-s), t) = \sum_j e^{z_j t} g_j(z, a(s), t, T-s) \\ + \sum_{S, j} e^{z_j^1 t} G_j(a, a(s), t-s).$$

Therefore, since the derivative of $F(z, Y, t)$ with respect to any element of Y is a polynomial of degree $n-1$ at most in the elements of Y , by (4.6), the derivative with respect to s of the left-hand side of this expression is $O(\epsilon K^{2n-1})$ as $\epsilon \rightarrow 0$. Therefore,

$$\frac{\partial}{\partial s} g_j(z, a(s), t, T-s) = O(\epsilon K^{2n-1}),$$

$$\frac{\partial}{\partial s} G_j(z, a(s), T-s) = O(\epsilon K^{2n-1}).$$

These, together with (5.4), give (5.9).

6. The Trend Equations

For z_j in S , $q = 0, 1, \dots, \mu_j - 1$, define

$$\bar{a}_{jq}(t) = a_{jq}(t)$$

$$- \epsilon e^{-z_j t} \sum_k^j \sum_{r=0}^q e^{z_k t} g_k^{(r)} \left(z_j, a(t), t, \frac{\partial}{\partial z_k} \right) (z_k - z_j)^{r-1-q/r!}$$

(6.1)

$$- \epsilon e^{-z_j t} \sum_{S, k}^j \sum_{r=0}^q e^{z_k^1 t} G_k^{(r)} \left(z_j, a(t), \frac{\partial}{\partial z_k} \right) (z_k^1 - z_j)^{r-1-q/r!},$$

where $\sum_{S, k}^j$ denotes the sum over all roots z_k in S except the root z_j ,

and where the superscript (r) denotes the r -th derivative with respect to the first place. Since $g_k^{(r)}$ and $G_k^{(r)}$ are polynomials in T ,

$$\bar{a}_{jq}(t) = a_{jq}(t)$$

$$- \epsilon \sum_S^j \sum_{k, r=0}^q g_k^{(r)} \left(z_j, a(t), t, -t + \frac{\partial}{\partial z_k} \right) \left[e^{(z_k - z_j)t} (z_k - z_j)^{r-1-q} \right] / r!$$

(6.2)

$$- \epsilon \sum_{S, k} \sum_{r=0}^q G_k^{(r)} \left(z_j, a(t), -t + \frac{\partial}{\partial z_k} \right) \left[e^{(z_k' - z_j)t} (z_k' - z_j)^{r-1-q} \right] / r!.$$

By (5.8), (6.1),

$$(6.3) \quad \bar{a}_{jq}(t) = a_{jq}(t) + e^{-z_j t} o(\epsilon K^n).$$

Therefore when $T = 0(1)$, by (5.1), (5.3),

$$g_j^{(r)}(z, a, t, T) = g_j^{(r)}(z, \bar{a}, t, T) + e^{-z_j t} o(\epsilon K^{2n-1}),$$

(6.4)

$$G_j^{(r)}(z, a, T) = G_j^{(r)}(z, \bar{a}, T) + e^{-z_j t} o(\epsilon K^{2n-1}).$$

Differentiating (6.2), by (5.7), (5.9) for $\bar{a}_{j,-1}(t) \equiv 0$,

$$\begin{aligned} \bar{a}_{jq}^{(1)}(t) - \bar{a}_{j,q-1}(t) &= a_{jq}^{(1)}(t) - a_{j,q-1}(t) \\ &= \epsilon \sum_S^j g_k^{(q)} \left(z_j, a(t), t, -t + \frac{\partial}{\partial z_k} \right) e^{(z_k - z_j)t} / q! \\ &= \epsilon \sum_{S, k} G_k^{(q)} \left(z_j, a(t), -t + \frac{\partial}{\partial z_k} \right) e^{(z_k' - z_j)t} / q! + e^{-z_j t} o(\epsilon K^{2n-1}) \\ &= a_{jq}^{(1)}(t) - a_{j,q-1}(t) \\ &= \epsilon e^{-z_j t} \sum_S^j e^{z_k t} g_k^{(q)} \left(z_j, a(t), t \right) / q! \\ &= \epsilon e^{-z_j t} \sum_{S, k} e^{z_k t} G_k^{(q)} \left(z_j, a(t) \right) / q! + e^{-z_j t} o(\epsilon K^{2n-1}). \end{aligned}$$

Therefore, by (3.7), (5.6), (6.4),

$$(6.5) \quad \begin{aligned} \bar{a}'_{jq}(t) - \bar{a}_{j,q-1}(t) &= \epsilon g_j^{(q)}(z_j, \bar{a}(t), t)/q! \\ &+ e^{-z_j t} o(\epsilon^2 K^{2n-1}) \end{aligned}$$

for z_j in S , $q = 0, 1, \dots, \mu_j - 1$.

We are interested in the degree to which the functions $\bar{a}_{jq}(t)$ may be approximated by the solutions $\rho_{jq}(t)$ of the system

$$(6.6) \quad \frac{d\rho_{jq}(t)}{dt} = \rho_{j,q-1}(t) + \epsilon g_j^{(q)}(z_j, \rho(t), t)/q!$$

for z_j in S , $q = 0, 1, \dots, \mu_j - 1$, for $\rho_{j,-1}(t) \equiv 0$. This matter will be treated in another section.

The equations (6.6) are called trend equations and the functions $\rho_{jq}(t)$ are called trend functions because $\rho_{jq}(t)$ represents the trend of $a_{jq}(t)$ after high frequency oscillations have been filtered out.

7. The Trend Functions

Some sufficient conditions that the functions $\rho_{jq}(t)$ approximate the functions $\bar{a}_{jq}(t)$ will be given in this section.

Write $x_j = \text{Re}(z_j)$, and define real quantities $R_{jq}(t)$, $\theta_{jq}(t)$, $A_{jq}(t)$, $\phi_{jq}(t)$ by

$$(7.1) \quad \begin{aligned} R_{jq}(t) e^{i\theta_{jq}(t)} &= e^{x_j t} \rho_{jq}(t), \\ A_{jq}(t) e^{i\phi_{jq}(t)} &= e^{x_1 t} \bar{a}_{jq}(t). \end{aligned}$$

Substituting these into (6.5) and (6.6), we get equations of the form

$$(7.2) \quad \begin{aligned} R'_{jq}(t) &= G_{jq}(R, \theta, t), \\ \theta'_{jq}(t) &= H_{jq}(R, \theta, t), \\ A'_{jq}(t) &= G_{jq}(A, \phi, t) + o(\epsilon^2 K^{2n-1}), \\ \phi'_{jq}(t) &= H_{jq}(A, \phi, t) + o(\epsilon^2 K^{2n-1}), \end{aligned}$$

where R , θ , A , and θ stand for matrices of the quantities R_{jq} , θ_{jq} , A_{jq} , and θ_{jq} , respectively.

The solutions to the system (7.2) may be represented as integral curves in a hyperspace H in which t , R_{jq} , θ_{jq} are coordinates for each pair jq . R_{jq} and θ_{jq} may be taken as polar coordinates in a two-dimensional plane. The individual integral curves of (7.2) will be called R-trajectories in H . The individual integral curves of the system (7.3) may also be plotted in H and will be called A-trajectories. We will be concerned with projections of these trajectories on R_{jq} -planes and θ_{jq} -cylinders. The R_{jq} -plane is the two-dimensional plane having t as abscissa and R_{jq} as ordinate. The θ_{jq} -cylinder is the two-dimensional circular cylindrical surface having t as axial distance and θ_{jq} as azimuthal angle. Slope will be defined in the usual way in the R_{jq} -plane. On the θ_{jq} -cylinder, slope will be defined as the ratio of θ_{jq} displacement to t displacement.

The A-trajectories will cut across the R-trajectories. Consider a point P on an A-trajectory. An R-trajectory will also pass through this point. Suppose these two trajectories are projected onto the various R_{jq} -planes and θ_{jq} -cylinders by straight lines drawn orthogonal to these surfaces. Then by (7.2), (7.3), the slopes of the two projections on each plane or cylinder will differ from one another by $o(\epsilon^2 K^{2n-1})$. Therefore limits for the A-trajectory may be traced by starting with the projection P'_0 of the initial point P_0 of the A-trajectory on each R_{jq} -plane and θ_{jq} -cylinder, and tracing the two curves which cut the R-trajectory projections with slope Δm greater than and Δm less than the slope of the R-trajectory projection at the point of intersection, where $\Delta m = o(\epsilon^2 K^{2n-1})$. These curves are shown dotted in Figure 1, where a typical R_{jq} -plane projection is shown. The solid lines are R-trajectory projections. The A-trajectory projections will lie between these boundary lines. If the R-trajectories are converging together with increasing t , these boundary curves may themselves remain near together, and the position of the A-trajectories will then be determined within this error. A common situation is dealt with in Lemma 1 below.

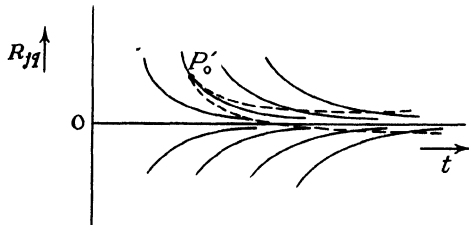


FIGURE 1

The R-trajectory for which $P_{jq}(0) = a_{jq}(0)$ is called the "principal R-trajectory". Its projections on the R_{jq} -plane and on the θ_{jq} -cylinder will be denoted by $\bar{R}_{jq}(t)$ and $\bar{\theta}_{jq}(t)$, respectively.

LEMMA 1. On the R_{jq} -plane and the θ_{jq} -cylinder let $\Delta R_{jq}(t)$ and $\Delta \theta_{jq}(t)$ denote the differences in ordinate and azimuthal angle, respectively, between the principal R-trajectory projection and neighboring R-trajectory projections. Suppose that

$$(7.4) \quad \begin{aligned} \Delta R_{jq}(t) &= f_{jq}(t)e^{-\kappa t} \\ \Delta \theta_{jq}(t) &= g_{jq}(t)e^{-\kappa t} \end{aligned}$$

where $\kappa > 0$, and

$$(7.5) \quad f'_{jq}(t)/f_{jq}(t) \leq 0, \quad g'_{jq}(t)/g_{jq}(t) \leq 0.$$

Then for $t \geq 0$,

$$(7.6) \quad \begin{aligned} A_{jq}(t) &= \bar{R}_{jq}(t) + O(\epsilon^2 K^{2n-1}/\kappa), \\ \theta_{jq}(t) &= \bar{\theta}_{jq}(t) + O(\epsilon^2 K^{2n-1}/\kappa). \end{aligned}$$

PROOF. The proof for the first equation in (7.6) only will be given; that for the second equation is similar.

In Figure 2, P'_0Q represents the principal R-trajectory projection, and the other solid lines are neighboring R-trajectory projections. The dotted lines are the A-trajectory limit curves beginning at P'_0 , as defined above.

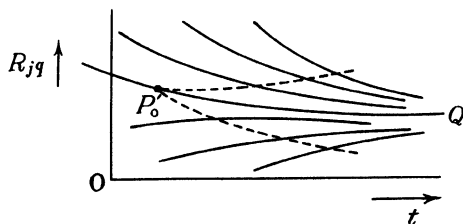


FIGURE 2

Consider the upper dotted limit curve in Figure 2. We wish to obtain a limit for the height of this curve above P'_0Q , and are therefore interested only in the case in which it is not converging toward P'_0Q , i.e., when $A'_{jq}(t) \geq R'_{jq}(t)$. By (7.4)

$$\begin{aligned}\kappa \Delta R_{jq}(t) &= e^{-\kappa t} \frac{d}{dt} \left[e^{\kappa t} \Delta R_{jq}(t) \right] - \Delta R'_{jq}(t) \\ &= f'_{jq}(t) e^{-\kappa t} + A'_{jq}(t) - R'_{jq}(t) \\ &\quad - \left[A'_{jq}(t) - \bar{R}'_{jq}(t) \right],\end{aligned}$$

where $R'_{jq}(t)$ denotes the slope of the R-trajectory projection at the point crossed by the limit curve. By (7.4), $f_{jq}(t) > 0$, so by (7.5), $f'_{jq}(t) \leq 0$. By (7.2) and (7.3), $A'_{jq}(t) - R'_{jq}(t) = O(\epsilon^2 K^{2n-1})$. Finally $A'_{jq}(t) - \bar{R}'_{jq}(t) \geq 0$ as noted above. Therefore $\kappa \Delta R_{jq}(t)$, which is positive, is bounded above by $O(\epsilon^2 K^{2n-1})$. However, $\Delta R_{jq}(t) = A_{jq}(t) - \bar{R}_{jq}(t)$, so (7.6) follows. The proof for the lower limit curve is similar.

8. The Solution

The solutions to equations (6.6) are integral curves containing parameters. The solutions having the same initial values as $a_{jq}(t)$, i.e., such that

$$(8.1) \quad \rho_{jq}(0) = a_{jq}(0)$$

will be called "principal trend functions". The function $|\rho_{jq}(t)|$ may be plotted against t on an jq-trend modulus plane, and the function $\arg[\rho_{jq}(t)]$ may be plotted against t on an jq-trend argument cylinder.

THEOREM 2. On each jq-trend modulus plane let $\Delta|\rho_{jq}(t)|$ denote the difference in ordinate between the principal jq-trend modulus and neighboring trend moduli; on each jq-trend argument cylinder let $\Delta\arg[\rho_{jq}(t)]$ denote the difference in azimuthal angle between the principal jq-trend argument and neighboring trend arguments. Suppose

$$(8.2) \quad \begin{aligned}\Delta |\rho_{jq}(t)| &= f_{jq}(t) e^{-(x_{jq} + \kappa)t}, \\ \Delta \arg [\rho_{jq}(t)] &= g_{jq}(t) e^{-\kappa t},\end{aligned}$$

where $\kappa > 0$ and

$$(8.3) \quad f'_{jq}(t)/f_{jq}(t) \leq 0, \quad g'_{jq}(t)/g_{jq}(t) \leq 0.$$

Then for $t \geq 0$, $i = 1, 2, \dots, m$,

$$(8.4) \quad \begin{aligned} y_1(t) = & \sum_S \sum_{j=0}^{\mu_j^{-1}} \frac{\mu_j! G_1^{(q)}(z_j)}{q! D^{(\mu_j)}(z_j)} e^{z_j t} \rho_{j, \mu_j - q}(t) \\ & + O(\epsilon K^n) + O(\epsilon^2 K^{2n-1}/\kappa). \\ = & Y_1(\rho, t, 0) + O(\epsilon K^n) + O(\epsilon^2 K^{2n-1}/\kappa). \end{aligned}$$

PROOF. By (6.3) and Lemma 1,

$$(8.5) \quad a_{jq}(t) = \rho_{jq}(t) + e^{-z_j t} O(\epsilon K^n) + e^{-z_j t} O(\epsilon^2 K^{2n-1}/\kappa),$$

so (8.4) follows from (3.3).

When the quantities $\rho_{jq}(t)$ are known, κ may be calculated from (8.2), and the bound K' may be calculated from (4.4). (If $K' = \infty$, large solution failure of the theory occurs.) By (8.5),

$$(8.6) \quad E(\epsilon, K) = O(\epsilon K^n) + O(\epsilon^2 K^{2n-1}/\kappa).$$

Now suppose

$$(8.7) \quad K' = o\left(\epsilon^{-1/(n-1)}\right), \quad K' = o\left(1/[2(n-1)] \epsilon^{-1/(n-1)}\right).$$

as $\epsilon \rightarrow 0$. Then (4.3) may be satisfied for $K = O(K')$, and K in (8.4) may be replaced by K' . This is done in 4^o of Section 10 in equations (10.7) and (10.8).

9. Small Changes in the Characteristic Equation

When the characteristic equation has closely clustered roots condition (5.5) will generally fail to be satisfied. To avoid this, small changes may be made in the characteristic equation that will bring the clustered roots into coincidence as a multiple root, which may be treated by the methods already given. On the other hand, simple roots can sometimes be given more convenient values by slightly changing the characteristic equation.

A small change in the left-hand side of equation (3.1) is compensated by the same change in the right-hand side, so (3.1) is equivalent to

$$(9.1) \quad \frac{dy_1}{dt} = \sum_{\alpha=1}^m A_{1\alpha}^* y_\alpha + \epsilon f_1^*(y, t), \quad 1 = 1, 2, \dots, m,$$

where

$$(9.2) \quad A_{1\alpha}^* = A_{1\alpha} + \epsilon B_{1\alpha},$$

$$(9.3) \quad f_1^*(y, t) = f_1(y, t) - \sum_{\alpha=1}^m B_{1\alpha} y_\alpha.$$

The characteristic equation corresponding to (9.1) is

$$(9.4) \quad E(z) \equiv \left| A_{1j}^* - z\delta_{1j} \right| = \left| A_{1j} + \epsilon B_{1j} - z\delta_{1j} \right| = 0.$$

Then

$$(9.5) \quad E(z) = D(z) + \epsilon C(z) + O(\epsilon^2) = 0$$

where

$$(9.6) \quad C(z) = \sum_{\alpha=1}^m \sum_{\beta=1}^m B_{\alpha\beta} D_{\beta\alpha}(z).$$

In this section we will consider (9.1) to be the given equation and (3.1) to be the modified equation. We will assume that whatever else is done the modification of the equation will be such that near characteristic roots are brought together. Then, about the characteristic root z_j of equation (2.4) a circle of radius $O(1)$, as $\epsilon \rightarrow 0$, may be drawn which contains no other roots of (2.4). Let C_j be a circle of this type centered at z_j . Since $C(z)$ and its derivatives are bounded over C_j , there exists a constant K_j and an integer n_j , $0 \leq n_j < \mu_j/2$ such that

$$(9.7) \quad C^{(v)}(z_j) \leq K_j \frac{(\mu_j - v)}{\epsilon^{(\mu_j - n_j)}} \frac{(n_j - v)}{\epsilon^{(\mu_j - n_j)}}, \quad v = 0, 1, \dots, n_j,$$

$$C^{(n_j+1)}(z) \leq K_j \quad \text{for } z \text{ in } C_j.$$

This is always possible for $n_j = 0$. In certain cases, when the first few

derivatives of $C(z)$ are small at z_j , condition (9.7) may be satisfied for $n_j > 0$.

In the Lemma in the Appendix, take $f(z) = D(z)$, $g(z) = C(z) + O(\epsilon)$. The equation (9.5) has μ_j zeros, z_{j1}^* , z_{j2}^* , ..., $z_{j\mu_j}^*$ within a circle of radius $O[(K_j \epsilon)^{1/(\mu_j - n_j)}]$ about z_j . By Taylor's theorem, for $p = 1, 2, \dots, \mu_j$,

$$\begin{aligned} & \frac{(z_{jp}^* - z_j)^{\mu_j}}{\mu_j!} D^{(\mu_j)}(z_j) + \epsilon \sum_{v=0}^{n_j} \frac{(z_{jp}^* - z_j)^v}{v!} C^{(v)}(z_j) \\ &= O(\epsilon^2) - \frac{(z_{jp}^* - z_j)^{\mu_j+1}}{(\mu_j+1)!} D^{(\mu_j+1)}(\xi_p) - \epsilon \frac{(z_{jp}^* - z_j)^{n_j+1}}{(n_j+1)!} C^{(n_j+1)}(\xi_p) \end{aligned}$$

where ξ_p is a point on the line between z_j and z_{jp}^* . This gives

$$(9.8) \quad z_{jp}^* = z_j + \alpha_{jp} + O\left[(K_j \epsilon)^{2/(\mu_j - n_j)}\right],$$

where α_{jp} is the p -th root of

$$(9.9) \quad (\alpha^{\mu_j} / \mu_j!) D^{(\mu_j)}(z_j) + \epsilon \sum_{v=0}^{n_j} (\alpha^v / v!) C^{(v)}(z_j) = 0.$$

This gives

$$\alpha_{jp} = O\left[(K_j \epsilon)^{1/(\mu_j - n_j)}\right].$$

We may now solve (9.1) by solving (3.1). Write

$$(9.10) \quad F^*(z, y, t) = \sum_{\alpha=1}^m \bar{D}_\alpha(z) f_\alpha^*(y, t),$$

$$\begin{aligned} (9.11) \quad F^*(z, Y(a, t, 0), t) &= \sum_j e^{z_j t} g_j^*(z, a, t) \\ &+ \sum_{j'} e^{z_{j'} t} g_{j'}^*(z, a), \end{aligned}$$

where for all z_j in S and z'_k in S' , as $\epsilon \rightarrow 0$

$$\frac{\partial}{\partial t} g_j^*(z, a, t) = e^{-z_j t} O(\epsilon K^{2n-1}), \quad (9.12)$$

$$1/|z_j - z'_k| = O(1).$$

Then by (3.3), (3.5), (5.6), (5.7), (9.3), (9.10),

$$g_j^*(z, a, t) = g_j(z, a, t) - \sum_{\alpha=1}^m \sum_{\beta=1}^m B_{\alpha\beta} \bar{D}_{\alpha}(z) \sum_{q=0}^{\mu_j-1} \frac{\mu_j! G_{\beta}^{(q)}(z_j)}{q! D_{\beta}^{(\mu_j)}(z_j)} a_{j, \mu_j-1-q}, \quad (9.13)$$

$$G_j^*(z, a) = G_j(z, a).$$

(3.1) may now be solved using (6.6) and (8.4).

In the particular case in which $g_j^*(z, a, t) = 0$ for a given z_j in S , by (9.13), the system (6.6) becomes linear, and has a solution

$$\rho_{jq}(t) = \sum_{r=1}^{\mu_j} K_{jr} \alpha_{jr}^{-q} e^{\alpha_{jr} t}, \quad (9.14)$$

where the quantities α_{jq} are the roots of (9.9) within an error

$$O \left[(K_j \epsilon)^{1/(\mu_j - n_j)} \right]$$

and the K_{jr} are constants. This is shown by inserting (9.13) into (6.6). The resulting linear system has a solution of the form (9.14), where the quantities α_{jr} satisfy an equation in the form of a determinant of order m involving ϵ . Expanding this and neglecting powers of ϵ higher than the first, one gets equation (9.9).

10. How to Solve Equations

For the benefit of those who have specific equation systems to solve, we will now outline the steps to be followed in the use of the theory of this paper.

It is assumed that the equation system is in the form (3.1) and

that the conditions (i) - (v) of Section 3 are satisfied. Having verified this, the following steps may be taken.

1^o Calculate the characteristic equation from (2.2). Locate its multiple roots by equating it and its derivatives to zero simultaneously. If there are parameters in the equation, the ranges of interest of these parameters must be selected. For one fixed set of values of the parameters the characteristic roots must be computed.

2^o Near roots may be brought into coincidence, and other roots may be moved small distances to more convenient locations (for instance, roots near the imaginary axis may be moved to the imaginary axis). To do this let z_{jp}^* be the old root and z_j the new root in (9.8). Then for each such root compute α_{jp} from (9.8) and substitute into (9.9). One obtains a set of simultaneous linear equations for the quantities $C^{(v)}(z_j)$, $v = 0, 1, \dots$. Having solved these, a set of simultaneous linear equations for the quantities $B_{\alpha\beta}$ are then given by (9.6). In general the quantities $B_{\alpha\beta}$ will not all be determined and some of them may be set arbitrarily.

The effect of small changes in parameters in the differential equation (9.1) can be readily investigated by giving the quantities $A_{1\alpha}$ in (3.1) appropriate fixed values and varying the quantities $B_{1\alpha}$ in (9.2).

3^o Calculate $D_r(z)$, $\bar{D}_s(z)$, and $\bar{D}(z)$ from (2.3), selecting the constants λ_s and ν_r to simplify these functions (it is particularly desirable to make $\bar{D}(z) \equiv 1$). Calculate $G_1(z)$ from (2.9). Then calculate the functions $Y_1(a, t, 0)$ from (5.1). In the particular case in which all the characteristic roots are simple, drop the second subscript on a_{jq} , and write in place of (5.1),

$$(10.1) \quad Y_1(a, t, 0) = \sum_j \left[G_1(z_j) / D'(z_j) \right] e^{z_j t} a_j.$$

The function $F(z, y, t)$ is calculated from (3.5). Then the Y_1 functions are substituted into the left-hand side of

$$(10.2) \quad F(z, Y(a, t, 0), t) = \sum_j e^{z_j t} g_j(z, a, t) + \sum_{j'} e^{z_{j'} t} G_j(z, a)$$

and expanded. The terms in the expansion for which, as $\epsilon \rightarrow 0$,

$$(10.3) \quad \frac{\partial}{\partial t} g_j(z, a, t) = e^{-z_j t} O(\epsilon K^{2n-1})$$

are put in the first sum on the right of (10.2). Other terms are put in the second sum, but in any case the z_j and z'_k quantities must not be near, i.e., as $\epsilon \rightarrow 0$,

$$(10.4) \quad 1 / |z_j - z'_k| = O(1)$$

for all z_j in S and z'_k in S' .

Since K is not yet known, (10.3) cannot be completely verified at this point. The third argument of $g_j(z, a, t)$ is restricted by it, but the extent of the restriction cannot be known until K is determined. In the usual case, when there are parameters in the equation system, the calculation may proceed anyway, with (10.3) and also (10.4) constituting restrictions on the ranges of the parameters for which the theory is valid. These ranges may be determined when K is known.

4° The trend functions $\rho_{jq}(t)$ may now be calculated by solving the differential equation system (6.6). In the case in which all the characteristic roots are simple, we drop the second subscript on $\rho_{jq}(t)$, and (6.6) becomes

$$(10.5) \quad \frac{d\rho_j(t)}{dt} = \epsilon g_j(z_j, \rho(t), t),$$

for z_j in S .

The quantities $f_{jq}(t)$, $g_{jq}(t)$, κ may be determined from (8.2) after the equations in (6.6) (or (10.5)) have been solved. Unless $\kappa > 0$ and the conditions (8.3) are satisfied, the theory fails.

The bound K' is now computed from (4.4). If $K' = \infty$ the theory fails. If K' satisfies conditions (8.7) we may take

$$(10.6) \quad K = K' + o(K')$$

as $\epsilon \rightarrow 0$. Then by (8.4) the solution is

$$\begin{aligned}
 y_1(t) = & \sum_j \sum_{q=0}^{\mu_j-1} \frac{\mu_j! G_1^{(q)}(z_j)}{q! D^{(\mu_j)}(z_j)} e^{z_j t} \rho_{j, \mu_j-1-q}(t) \\
 (10.7) \quad & + O(\epsilon K^{1n}) + O(\epsilon^2 K^{12n-1/\kappa}),
 \end{aligned}$$

for $i = 1, 2, \dots, m$, as $\epsilon \rightarrow 0$. In particular when all the characteristic roots are simple this reduces to

$$\begin{aligned}
 y_1(t) = & \sum_j \left[G_1(z_j)/D'(z_j) \right] e^{z_j t} \rho_j(t) + O(\epsilon K^{1n}) \\
 (10.8) \quad & + O(\epsilon^2 K^{12n-1/\kappa}),
 \end{aligned}$$

for $i = 1, 2, \dots, m$, as $\epsilon \rightarrow 0$.

If K' does not satisfy (8.7), although it may still be possible to satisfy (4.3), the expansion on the right of (8.4) does not dominate the order terms, so that the solution (8.4) is of little value.

⁵⁰ With the determination of K in (10.6), the restrictions on the parameters that are due to (10.3) may be determined in accordance with the remarks in the last paragraph of ³⁰.

11. Appendix

To prove (2.5), we first show that

$$(11.1) \quad D_{1\alpha}(z)D_{rs}(z) - D_{1s}(z)D_{r\alpha}(z) + D(z)D_{1s;r\alpha}(z) = 0.$$

To do this, consider the quantities $A_{1\alpha}$ in (2.4) as independent variables and differentiate, holding z fixed. Then

$$\begin{aligned}
 \sum_{\beta=1}^m \left[dA_{1\beta}D_{\beta s}(z) + (A_{1\beta} - z\delta_{1\beta}) \sum_{r=1}^m \sum_{\alpha=1}^m dA_{\alpha r}D_{\beta s;r\alpha}(z) \right] \\
 = \delta_{1s} \sum_{r=1}^m \sum_{\alpha=1}^m dA_{\alpha r}D_{r\alpha}(z).
 \end{aligned}$$

Equating the coefficients of $dA_{\alpha r}$,

$$(11.2) \quad \delta_{1\alpha} D_{rs}(z) - \delta_{1s} D_{r\alpha}(z) + \sum_{\beta=1}^m (A_{1\beta} - z\delta_{1\beta}) D_{\beta s; r\alpha}(z) = 0.$$

Now multiply by $D_{k1}(z)$ and sum over 1 from 1 to m . Replacing k by 1 in the result, we get (11.1). (2.5) now follows from (2.3) and (11.1).

LEMMA 2. Let $f(z)$ be analytic over a region S having an isolated zero of multiplicity m at z_0 . Let R be the radius of any circle centered at z_0 lying wholly within S and containing no other zeros of $f(z)$. Let $g(z)$ be analytic over S and satisfying the condition $|g(z)| \leq M$ for z in S and some constant M . Then the equation

$$(11.1) \quad f(z) + \epsilon g(z) = 0$$

has m roots inside any circle of radius r centered at z_0 which lies wholly within S and satisfies

$$(11.2) \quad r > R[\epsilon M/N(R)]^{1/m},$$

where

$$(11.3) \quad N(R) = \min \left[1, \min_{0 \leq \theta \leq 2\pi} |f(z_0 + R e^{i\theta})| \right].$$

PROOF. By the Poisson-Jensen formula (Titchmarsh, Section 3.62),

$$\ln|f(z_0 + r e^{i\theta})| = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \vartheta) + r^2} \ln|f(z_0 + R e^{i\vartheta})| d\vartheta \\ - m \ln(R/r).$$

Therefore, by (11.1), (11.2),

$$\ln|f(z_0 + r e^{i\theta})| \geq \ln N(R) - m \ln(R/r) > \ln(\epsilon M).$$

Therefore $|f(z)| > \epsilon |g(z)|$ on the circle of radius r centered at z_0 , so by Rouché's theorem (Titchmarsh, Section 3.42), equation (11.1) has m zeros inside this circle.

Reference

- [1] TITCHMARSH, E. C., "The Theory of Functions", Oxford University Press, 1932.

IV. ON A NON-LINEAR DIFFERENTIAL EQUATION CONTAINING A SMALL PARAMETER¹

Violet B. Haas

Ever since Liouville [3] derived series solutions for the linear homogeneous differential equation

$$(1) \quad \frac{d^2 z}{dx^2} + \left[\rho^2 + g(x) \right] z = 0$$

where ρ is a large real parameter, for a given set of linear boundary conditions, more than one hundred years ago, mathematicians have concerned themselves with the behavior of asymptotic solutions of linear differential equations containing large parameters. Recently it has been discovered that when the differential equation is non-linear and contains a large parameter the asymptotic solutions may behave in a rather singular way. There may exist solutions of a differential equation such as

$$(2) \quad \frac{d^2 z}{dx^2} + \rho f \left(x, z, \frac{dz}{dx}; \rho \right) = 0,$$

for large ρ , or

$$(3) \quad \epsilon \frac{d^2 z}{dx^2} + F \left(x, z, \frac{dz}{dx}; \epsilon \right) = 0,$$

for small ϵ , which converge in some fixed closed interval $I: a \leq x \leq b$, for $\epsilon \rightarrow 0$ to a solution of the degenerate equation,

$$(4) \quad F \left(x, z, \frac{dz}{dx}; 0 \right) = 0,$$

¹ This paper is based on the author's M.I.T. dissertation. The author wishes to express her appreciation to Professor Norman Levinson, under whose direction this thesis was written, for his help and guidance.

which is continuous in I but which has a point of discontinuity to the right of I . When such is the case these solutions of (3) may jump or oscillate rapidly in a neighborhood of this point. The Van der Pol equation has some solutions which offer a concrete example. For these see [1].

In this paper the author shows that under certain conditions imposed upon F for small ϵ there exist solutions of (3) which are stable in I , which are completely unstable in some open interval to the right of I , these two intervals being separated by a single boundary point.

1. We are here concerned with the behavior of the solutions of the real differential equation,

$$(1.1) \quad \epsilon \frac{d^2 x}{dt^2} + f\left(t, x, \frac{dx}{dt}; \epsilon\right) = 0,$$

where ϵ is a small parameter. We suppose that $f(t, x, y; \epsilon)$ belongs to class $C^2(t, x, y)$ in the domain $D: [T_0 \leq t < \infty, |x| \leq X, |y| < \infty]$, where T_0 and $X > 0$ are constants and where $f_{yy} \neq 0$ in D . We suppose that f is continuous in ϵ for $0 \leq \epsilon \leq \epsilon_0$, where ϵ_0 is a constant. Levinson [1] has studied equation (1.1) for the case when $f_{yy} = 0$ and also [2] for the case when $f = x - 4\dot{x}/(3 + \dot{x}^4)$.

When $\epsilon = 0$ equation (1.1) becomes

$$(1.2) \quad f\left(t, z, \frac{dz}{dt}; 0\right) = 0.$$

Let $x = x(t)$ and $z = z(t)$ be solutions in D of (1.1) and (1.2) respectively, and let

$$(1.3) \quad \left| x(t_0) - z(t_0) \right| \leq \delta_1, \quad \left| \frac{dx}{dt}(t_0) - \frac{dz}{dt}(t_0) \right| \leq \frac{\delta_2}{\epsilon}, \quad T_0 < t_0.$$

We shall show that under certain rather general conditions there exist t_1 and T such that $|x(t) - z(t)| \rightarrow 0$ on $t_0 \leq t < t_1$, and $|dx/dt - dz/dt| \rightarrow 0$ on $t_0 < t < t_1$ as $\epsilon, \delta_1, \delta_2 \rightarrow 0$, and $x = x(t)$ has rapid oscillatory behavior for $T_0 < t_1 < t \leq T$ with an "amplitude" approaching $z(t_1)$ and a "period" approaching $O(\epsilon^{1/2})$ as $\epsilon \rightarrow 0$. The words, "amplitude" and "period" are in quotation marks because, as we shall see, the magnitudes of successive maxima and minima are not constant, nor are the distances between them.

2. We suppose that t_0 satisfies $T_0 < t_0$, that $z = z(t)$ is that integral curve of (1.2) in D satisfying $z(t_0) = z_0$, and that

integral curve of (1.2) in D satisfying $z(t_0) = z_0$, and that
 (2.1) $f(t, x, y; \epsilon) = f_1(x) - f_2(t, x, y) + \epsilon f_3(t, x, y; \epsilon)$,

where

- i. $f(t, x, y; \epsilon)$ and its first and second order partial derivatives with respect to t, x, y are bounded and continuous in t, x, y, ϵ in D for $0 \leq \epsilon \leq \epsilon_0$,
- ii. $xf_1(x) > 0$ for $x \neq 0$; $f_1'(x) = df_1/dx \geq 0$, $f_1''(x) \geq 0$ for $|x| \leq X$,
- iii. there exists a positive constant D such that $f_1^2(x) \geq D^4 x^2$ for $|x| \leq X$,
- iv. there exist $T > t_0$ and $p > 0$ such that

$$\int_p^\infty |yf_2(t, x, y)| dy < \infty, \quad \int_p^\infty |yf_3(t, x, y; \epsilon)| dy < \infty,$$

- for $|x| \leq X$, $t_0 \leq t \leq T$, $0 \leq \epsilon \leq \epsilon_0$,
- v. $F(t, x, y; \epsilon) = y[f_2(t, x, y) - \epsilon f_3(t, x, y; \epsilon)] > 0$ for $y \neq 0$, $|x| \leq X$ and for ϵ sufficiently small, and the zeros of F are isolated in D ,
- vi. there exist constants $z_0 > 0$ and $z_1 > z_0$ such that $F(t, z_0, p; \epsilon) < \frac{1}{2}D^2 z_1$ for $t_0 \leq t$,
- vii. $F_y(t, x, y; 0) < 0$ and $F_x(t, x, y; 0) < 0$ for $|x| \geq z_0$, $|dz/dt| \geq p$, $t_0 \leq t \leq T$,
- viii. f_3 is a continuous function of ϵ for $0 \leq \epsilon \leq \epsilon_0$,
- ix. f_1 is an odd function of x , f_2 and f_3 are odd functions of x and of y in D ,
- x. there exist constants $a > 0$ and $q_1 > 0$ such that $f_{yy} > 0$ for $t_0 \leq t \leq T$, $|x| \leq X$, $0 < |y| \leq q_1 + a$,
- xi. the equation $f(t, x, y; 0) = 0$ implicitly defines a unique surface, H_0 in (t, x, y) -space, and contained in the strip $|x| \leq z_1$; and in any region where $f_x(t, x, y; 0) \neq 0$ this surface is defined explicitly by the equation $x = g_0(t, y)$. We suppose that there exists $k > 0$ such that $f_x(t, x, y; 0) > k$ for $|x| < X$, $t \geq t_0$, $|y| < \infty$. This implies, in particular, that $f(t, x, y; 0) \geq 0$ according as $x \geq g_0(t, y)$,
- xii. let $C_0 : z = z(t)$ be that integral curve of (1.2) satisfying $z(t_0) = z_0$. Let $dz(t_0)/dt = p$. We suppose that

$$\dot{z} = - \frac{f_t(t, x, y; 0) + y f_x(t, x, y; 0)}{f_y(t, x, y; 0)} < 0$$

on C_0 for $t_0 \leq t < t_1$. On C_0 let $\lim_{t \rightarrow t_1^-} z(t) = z_1$

and we suppose that there exist k numbers,
 $q_1, \dots, q_k, k \geq 1$, such that $f(t, x, q_1; 0) = 0$,
 $f_y(t, x, q_1; 0) = 0, t = t_1, x = z_1$ for each
 $i = 1, 2, \dots, k$, and that

$$\lim_{t \rightarrow t_1^-} dz/dt = q_1$$

for some i , say $i = 1$,

$$\text{xiii. } f_{yy}/[f_t + yf_x] > 0 \text{ at } t = t_1, x = z_1, y = q_1, \epsilon = 0,$$

$$\text{xiv. } (t - t_1) \frac{\partial f_2}{\partial t}(t, z_1, q_1) \geq 0,$$

$$(y - y_1)(t - t_1) \frac{\partial^2 f_2}{\partial t \partial y}(t, z_1, q_1) \geq 0.$$

This is the entire hypothesis needed to prove the theorem as stated in Section 4. However, much less is needed to prove the lemmas in Section 3, and we shall indicate this in the proofs.

Remark 2.1. Part xiii of the hypothesis is not really an extra assumption; it is a direct consequence of x, xi and xiv .

In the following we let K denote an upper bound in D for f and its first and second order partial derivatives with respect to t, x, y , and all finite sums, products and combinations of these which are used in the proofs, and let $m > 0$ be a lower bound for $f_{yy}(t, x, y; 0)$ in the domain $t_0 \leq t \leq T, |x| \leq X, \frac{1}{2}p < |y| \leq q_1 + a$. Throughout this paper ϵ will be made to assume only non-negative values, the theorems stated not being necessarily true otherwise.

3. Here we state some preliminary lemmas and derive some general qualitative results.

LEMMA 3.1. Let $w(t)$ satisfy the differential equation

$$(3.1) \quad \epsilon \frac{d^2 w}{dt^2} + H(t, w; \epsilon) \frac{dw}{dt} = J(t, w; \epsilon),$$

for $t_0 \leq t \leq \tau$, where H and J are continuous in the region given by $t_0 \leq t \leq \tau$, and where $|w| \leq \lambda$ for some $\lambda > 0$. In this region let $H \geq h > 0, |J| \leq K |w| + \epsilon_1$. Moreover, let $|w(t_0)| \leq \delta_1$ and $|dw(t_0)/dt| \leq \delta_2/\epsilon$. Let $\Delta = \frac{\epsilon_1}{K} + \delta_1 + \frac{\delta_2}{h}$. Then for $t_0 \leq t \leq \tau$, and if

$\epsilon_1, \delta_1, \delta_2$ are small enough,

$$(3.2) \quad \left\{ \begin{array}{l} |w(t)| \leq \Delta e^{\frac{K}{h}} (t-t_0) \\ \left| \frac{dw}{dt} \right| \leq \frac{\delta_2}{\epsilon} e^{-\frac{h}{\epsilon} (t-t_0)} + \frac{K}{h} \Delta e^{\frac{K}{h}} (t-t_0) + \frac{\epsilon_1 + K\delta_1}{h} \end{array} \right.$$

where

$$\Delta = \frac{(K+1)(h+1)^2}{h^2 K} (\delta_1 + \delta_2 + \epsilon_1).$$

For a proof of this lemma see [1].

THEOREM 3.1. Let $x = x(t)$ and $z = z(t)$ be integrals in D of (1.1) and (1.2) respectively, and let $f(t, x, y; \epsilon)$ have the continuity and differentiability properties described in Section 1. Let $w = x - z$. If $f_y(t, x, y; 0) \geq h > 0$ for $t_0 \leq t \leq \tau \leq t_1$, and if $|w(t_0)| \leq \delta_1$, $|dw(t_0)/dt| \leq \delta_2/\epsilon$, then $w(t)$ satisfies the hypothesis of Lemma 3.1, with $\epsilon_1 = 0(\epsilon)$. Thus, for $t_0 \leq t < t_1$, $x(t) \rightarrow z(t)$, and for $t_0 < t < t_1$, $dx/dt \rightarrow dz/dt$ as $\epsilon, \delta_1, \delta_2 \rightarrow 0$ provided $|x(t_0) - z(t_0)| \leq \delta_1$, $|dx(t_0)/dt - dz(t_0)/dt| \leq \delta_2/\epsilon$. Moreover, if τ_1, τ_2 are any numbers satisfying $t_0 < \tau_1 < \tau_2 < t_1$, the convergence of $w(t)$ is uniform on $t_0 \leq t \leq \tau_2$, and the convergence of dw/dt is uniform on $\tau_1 \leq t \leq \tau_2$.

PROOF. We differentiate (1.2) with respect to t and multiply the result by ϵ to obtain

$$(3.3) \quad \epsilon \frac{d^2 z}{dt^2} + \epsilon \frac{f_t(t, z(t), \dot{z}(t); 0) + \frac{dz}{dt} f_x(t, z(t), \dot{z}(t); 0)}{f_y(t, z(t), \dot{z}(t); 0)} = 0,$$

where the dot refers to differentiation with respect to t . Subtracting (3.3) from (1.1), setting $w(t) = x(t) - z(t)$ and applying the Theorem of the Mean to the term $f(t, x, dx/dt; \epsilon) - f(t, z, dz/dt; 0)$, we obtain

$$\begin{aligned} \epsilon \frac{d^2 w}{dt^2} + \frac{dw}{dt} f_y(t, z, \dot{z} + \theta_1 \dot{w}; \epsilon) &= -w f_x(t, z, \dot{z} + \theta_2 \dot{w}, x; \epsilon) \\ &+ \epsilon \frac{\dot{z} f_x(t, z, \dot{z}; 0) + f_t(t, z, \dot{z}; 0)}{f_y(t, z, \dot{z}; 0)} - \psi(\epsilon), \end{aligned}$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and where

$$\psi(\epsilon) = f(t, z, \dot{z}; \epsilon) - f(t, z, \dot{z}; 0).$$

$\psi(\epsilon)$ can be made as small as desired by choosing ϵ sufficiently small. Setting

$$H = f_y(t, z, \dot{z} + \theta_1 \dot{w}; \epsilon), \quad J = -wf_x(t, z, \dot{z} + \theta_2 \dot{w}, \dot{x}; \epsilon) - \psi(\epsilon) \\ + \epsilon \frac{\dot{z}f_x(t, z, \dot{z}; 0) + f_t(t, z, \dot{z}; 0)}{f_y(t, z, \dot{z}; 0)},$$

and letting ϵ_1 be an upper bound for

$$\psi(\epsilon) + \epsilon \left| \frac{\dot{z}f_x(t, z, \dot{z}; 0) + f_t(t, z, \dot{z}; 0)}{f_y(t, z, \dot{z}; 0)} \right|,$$

we obtain the required result.

LEMMA 3.2. Given the differential equation (1.2), where f has the continuity and differentiability properties described in Section 1; let H_0 be the surface in (t, x, y) -space defined in Section 2, and suppose that H_0 lies in the interior of D . Let S_0 be an integral curve of (1.2) with initial point (t_0, z_0) . On S_0 let $f_y > 0$ for $t_0 \leq t \leq t_1$ and let $f_y = 0$ at $t = t_1$. Then S_0 cannot be continued as a curve with continuously turning tangent for $t \geq t_1$ if

$$(3.4) \quad \frac{f_{yy}(t, x, y; 0)}{f_t(t, x, y; 0) + yf_x(t, x, y; 0)} > 0, \quad t = t_1, \quad x = z_1, \quad y = q_1,$$

where $z_1 = z(t_1)$ and q_1 is a root of $f(t_1, z_1, q; 0) = 0$.

PROOF. We suppose that S_0 can be continued for increasing $t \geq t_1$ with continuously turning tangent. Then by Taylor's theorem we have on S_0 in a neighborhood of the point (t_1, z_1) and for $t > t_1$,

$$f(t, z(t), \dot{z}(t); 0) = (t - t_1)f_t(t_1, z_1, q_1; 0) + (z - z_1)f_x(t_1, z_1, q_1; 0) \\ + \frac{1}{2}(t - t_1)^2 f_{tt}(t', z', \dot{z}'; 0) + \frac{1}{2}(z - z_1)^2 f_{xx}(t', z', \dot{z}'; 0) \\ (3.5) \quad + (t - t_1)(z - z_1)f_{tx}(t', z', \dot{z}'; 0) + (z - z_1)(\dot{z} - q_1)f_{xy}(t', z', \dot{z}'; 0) \\ + \frac{1}{2}(\dot{z} - q_1)^2 f_{yy}(t', z', \dot{z}'; 0) =$$

where $t_1 < t' < t$, z' lies between z and z_1 and $\dot{z}'$ lies between $\dot{z}$ and q_1 and where $f_t(t_1, z_1, q_1; 0)$ is

$$\frac{\partial f(t, x, y; 0)}{\partial t}$$

evaluated at (t_1, z_1, q_1) , etc. Regrouping terms and setting $t - t_1 = \Delta t$, $z - z_1 = \Delta z$, $z - q_1 = q - q_1 = \Delta q$, we obtain,

$$\begin{aligned} \Delta t \left(f_t + \frac{\Delta z}{\Delta t} f_x + \frac{1}{2} \Delta t f_{tt} + \frac{1}{2} \frac{\Delta z}{\Delta t} \Delta z f_{xx} + \frac{1}{2} \frac{\Delta q}{\Delta t} \Delta q f_{yy} + \Delta z f_{tx} \right. \\ \left. + \Delta q f_{ty} + \frac{\Delta z}{\Delta t} \Delta q f_{xy} \right) = 0, \end{aligned}$$

where all partial derivatives of $f(t, x, y; 0)$ are taken at the same values of the variables as they were taken respectively in equation (3.5). Since we have supposed z to be continuous on S_0 we may use the Theorem of the Mean on the quantity $\Delta z/\Delta t$ and obtain

$$\Delta z/\Delta t = dz(t_1 + \theta \Delta t)/dt = \bar{q},$$

where $0 < \theta < 1$. We therefore write

$$\begin{aligned} \Delta t \left(f_t + \bar{q} f_x + \frac{1}{2} \Delta t f_{tt} + \frac{1}{2} \bar{q} \Delta z f_{xx} + \Delta z f_{tx} + \Delta q f_{ty} \right. \\ \left. + \bar{q} \Delta q f_{xy} \right) = - \frac{1}{2} (\Delta q)^2 f_{yy}. \end{aligned}$$

Thus, we see that if Δt is sufficiently small then

$$\Delta t = - \frac{\frac{1}{2} (\Delta q)^2 f_{yy}}{f_t + q f_x + 0(\Delta q) + 0(\Delta t)}.$$

Thus, if $f_{yy}/(f_t + q f_x) > 0$ at (t_1, z_1, q_1) then Δt cannot be positive, and this proves the lemma.

In the rest of this section we shall obtain a general qualitative description of the behavior of the solutions $x = x(t)$ of (1.1) for $t > t_1$.

We shall suppose that $f(t, x, y; \epsilon)$ satisfies the continuity and differentiability hypothesis of Section 1, and that $C: [x = x(t), x_0 = x(t_0), \dot{x}_0 = \dot{x}(t_0)]$ is the integral curve under discussion and that $x_0 = z_0 = z(t_0)$, $\dot{x}_0 = \dot{z}_0 = \dot{z}(t_0)$, where $z = z(t)$ is an integral curve of (1.2). We suppose that $f(t, x, y; \epsilon)$ is an odd function of x and

of y , and for definiteness, we shall assume throughout this paper that $z_0 > 0$, $\dot{z}_0 > 0$. The point $(t_0, z_0, \dot{z}_0)$ lies on the surface H_0 defined in Assumption xi of Section 2. Similar results hold if $z_0 < 0$, $\dot{z}_0 < 0$. We shall further suppose that at $t = t_1 > t_0$, $z(t)$ exists, $z(t_1) = z_1$, and that

$$\lim_{t \rightarrow t_1 - 0} \dot{z}(t) = q_1,$$

where q_1 is a root of $f(t_1, z_1, q; 0) = 0$. The following theorem then holds.

THEOREM 3.2. Let $f_y(t, x, y; 0) > 0$ for $t_0 \leq t < t_1$, let x lie between z_0 and z_1 , let y lie between $\dot{z}_0$ and q_1 . If for any fixed t , $T_0 < t$, there exists a sequence of values $\{y_n\}$ with

$$\lim_{n \rightarrow \infty} |y_n| = \infty$$

and such that

$$\lim_{n \rightarrow \infty} g_0(t, y_n) = 0,$$

where $g_0(t, y)$ is defined as in Assumption xi of Section 2, then the integral curve C , in (t, x, y) -space, of the second order system

$$(3.6) \quad \begin{cases} dx/dt = y, \\ \epsilon dy/dt = -f(t, x, \frac{dx}{dt}; \epsilon), \quad x(t_0) = z_0, \quad \dot{x}(t_0) = \dot{z}_0, \end{cases}$$

spirals in the following fashion.

The curve C leaves its initial point with increasing x -coordinate equal to $g_0(t, z)$; and after an interval in (t, x, y) -space all of whose dimensions approach zero as $\epsilon \rightarrow 0$, with decreasing z -coordinate as t increases. It meets the plane $y = 0$ at right angles and then turns with positive slope, dy/dx ; with y still decreasing and now x also decreasing. C meets the surface $x = g_0(t, y)$ in the lower half-space $y < 0$, and somewhere in a neighborhood of this surface it turns once more with ascending y -coordinate and still decreasing

x-coordinate until it again meets the plane $y = 0$ at right angles. C now continues to rise with increasing x-coordinate in the upper half-space $y > 0$ until it again appears in the vicinity of the surface $x = g_0(t, y)$, and so on. In Figure 1 is portrayed a projection of the motion in the (x, y) -plane with t a parameter along the curve.

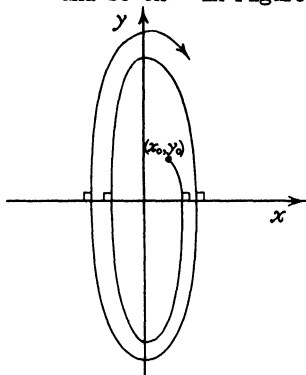


FIGURE 1

Remark 3.1. We note that we have used only the continuity of f and its first and second order partial derivatives and their boundedness in D , the oddness of f in x and y , Assumption xi of Section 2 and a modified Assumption iv of Section 2.

In order to prove Theorem 3.2 we first need the following two lemmas.

LEMMA 3.3. Integral curves of (3.6) cross the surface H_0 from left to right (in the sense of increasing x-coordinate) in the upper half space $y > 0$, and from right to left in the lower half-space $y < 0$.

LEMMA 3.4. The integral curve C of (3.6) descends (decreasing y-coordinate) for $t > t_0 + \delta$, $\delta \geq 0$, $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$, and meets the plane $y = 0$ at right angles. It then turns with $\dot{x} < 0$ and $\dot{y} < 0$ and eventually arrives in the neighborhood of H_0 . It again turns in this neighborhood now with $\dot{y} > 0$ and $\dot{x} < 0$ until it once more meets the plane $y = 0$, etc.

PROOF OF LEMMA 3.3. By the continuity in ϵ of f and its first order partial derivatives the equation

$$(3.7) \quad f(t, x, y; \epsilon) = 0$$

also defines a unique surface in (t, x, y) -space if ϵ is sufficiently small. Let us denote this surface by H_ϵ and, when $f_x(t, x, y; \epsilon) \neq 0$, express it explicitly by $x = g_\epsilon(t, y)$. Again by the continuity of f and its first order partial derivatives, we can choose ϵ so small that the two surfaces H_0 and H_ϵ lie close together and have almost parallel tangent

planes at neighboring points. For $t_0 \leq t \leq T_1$ we can choose $\delta > 0$ so small that an integral curve of (3.6) passing through a point P of H_0 crosses H_0 from left to right (right to left) at P whenever an integral curve of (3.6) passing through a point Q of H_ϵ crosses H_ϵ from left to right (right to left) at Q whenever the distance between P and Q is less than δ provided ϵ is sufficiently small. This follows from the fundamental theorem of ordinary differential equations.

Now, in the upper half-space, $y > 0$, on an integral curve of (3.6), if ϵ is sufficiently small, then $dy/dx > 0$ to the left of H_ϵ and $dy/dx < 0$ to the right of H_ϵ . On H_ϵ $dy/dx = 0$. Moreover, $dx/dt = y > 0$. Hence, integral curves of (3.6) cross H_ϵ from left to right in the upper half-space, and therefore also cross H_0 from left to right there. A similar argument holds in the lower half-space.

PROOF OF LEMMA 3.4. Suppose that for $t = t_0$, $y = y_0$, $g_\epsilon(t, y) < g_0(t, y)$. Then at (t_0, x_0, y_0) we have by Assumption xi of Section 2 that

$$dy/dt = (-1/\epsilon)f(t, x, y; \epsilon) < 0,$$

and that y decreases. If $g_\epsilon = x_0$ at $t = t_0$, $y = y_0$, then again y decreases as t increases beyond t_0 . Suppose, therefore, that $g_\epsilon > x_0$ for $t = t_0$, $y = y_0$. Then since $dx/dt = y$, C leaves H_0 with increasing x -coordinate, and we again invoke the fundamental theorem to show that it must meet H_ϵ , say for $t = t_0 + \delta$, $\delta > 0$. Once C meets H_ϵ it must cross this surface and must travel along with decreasing y -coordinate and increasing x -coordinate.

Once C has crossed H_ϵ (from left to right) we must have $dy/dt < 0$ so long as C remains to the right of H_ϵ , and so y decreases. Furthermore, since $dx/dt = y > 0$, x increases. So C moves to the right and downward. Since $f(t, x, y; \epsilon)$ is never zero so long as $x > g_\epsilon(t, y)$, then neither is $dy/dx = -(1/\epsilon y)f(t, x, y; \epsilon)$. Thus, since y is positive and decreasing, and $dy/dx < 0$, y must become zero and

$$\lim_{y \rightarrow 0} dy/dx = -\infty.$$

Thus, C crosses the plane $y = 0$ at right angles.

As C crosses into the lower half-space y becomes negative, dy/dx becomes positive and dy/dt remains negative. Then y continues to decrease and now x begins to decrease too.

Suppose that C does not meet H_0 in the octant $y \leq 0$, $x \geq 0$, $t \geq t_0$. We shall show that then C enters the octant $y < 0$, $x < 0$, $t \geq t_0$ and crosses H_0 there.

Since C does not meet H_0 for $x \geq 0, y \leq 0, t \geq t_0$, we may choose ϵ so small that C does not meet H_ϵ either in this octant for say $t_0 \leq t \leq t^*$, where $t^* < T_1, |x| \leq X$, and dy/dx does not vanish here unless $y \rightarrow -\infty$. Now y cannot approach $-\infty$ unless x also does, for $y = dx/dt$ and dx/dt is continuous on C . Thus, if $y \rightarrow -\infty$ C must cross the plane $x = 0$. If y does not approach $-\infty$ then dy/dx remains positive in this octant. If C does not cross the plane $x = 0$ then there exists a number $d > 0$ such that

$$\lim_{t \rightarrow \infty} x(t) = d,$$

and since y does not approach $-\infty$ there exists a positive number h such that

$$\lim_{t \rightarrow \infty} y(t) = -h.$$

Then given any $\delta > 0$ there exists a positive number $\bar{t}$ such that $y + h \leq \delta$ for $t \geq \bar{t}$, or $dx/dt \leq \delta - h$ for $t \geq \bar{t}$, or $x(t) - x(\bar{t}) \leq (\delta - h)(t - \bar{t})$ for $t \geq \bar{t}$. Letting $t \rightarrow t^*, t^* > \bar{t}$, we find that $x(t)$ becomes negative if t^* is sufficiently large. And so, in any case, C crosses the plane $x = 0$.

If now C does not meet H_0 for $x \leq 0, y \leq 0, t \geq t_0$, then we can again choose ϵ so small that $f(t, x, y; \epsilon)$ does not vanish on C in this octant for say, $t_0 \leq t \leq t^*$, and so dy/dx does not vanish unless again $y \rightarrow -\infty$. In any case, we can show as above, that if t^* is sufficiently large then dy/dx does not vanish unless $|x(t)|$ becomes large. But by hypothesis, for any $t \geq t_0$ the surface H_0 comes arbitrarily close to the plane $x = 0$ provided $|y|$ is sufficiently large, and so it is impossible that C and H_0 do not meet.

This proves the lemma and also Theorem 3.2.

Remark 3.2. Lemma 3.3 now tells us that C crosses H_0 exactly once in the lower half-space and exactly once in the upper half-space on each circuit.

Remark 3.3. The proof of Theorem 3.2 would have been very much the same had we supposed the existence of not one but several isolated roots, $y_1 = g_{1,0}(t, x), i = 1, \dots, n$, of $f(t, x, y; 0) = 0$ placed sufficiently far apart so that each surface lies in a strip, $a_1 \leq x \leq b_1$, where $b_1 \leq a_1 + 1$.

4. We shall now describe in detail the behavior of C under the more

restricted hypothesis of Section 2. This description is divided into two parts. The first part is devoted to that part of C between the point (t_0, x_0, y_0) and the first intersection point of C with the plane $y = 0$. The second part is concerned with the behavior of C between a point of C where C intersects the half-plane $x = 0, y > 0$, and the next intersection point of C with the plane $y = 0$. It is then shown that the behavior of C in all other octants can be easily deduced from this part of the discussion.

Remark 4.1. Since $f_1(x)$ and $-F(t, x, y; \epsilon)$ are both non-negative in the octant $t \geq t_0, x \geq 0, y \leq 0$, and since $f_1(x)$ and $-F(t, x, y; \epsilon)$ are both non-positive in the octant $t \geq t_0, x \leq 0, y \geq 0$, then C must meet H_0 only in the octants $t \geq t_0, x \leq 0, y \leq 0$ and $t \geq t_0, x \geq 0, y \geq 0$.

In the following text γ will be used to denote a constant independent of ϵ . For convenience we shall replace ϵ by ϵ^2 .

Let now $\bar{y}$ be the y -coordinate of a point P on C in the lower half-space $y < 0$, whose x -coordinate is zero, and let $\bar{x}$ be the x -coordinate of the next previous intersection point Q of C with the plane $y = 0$. Then multiplying the second of equations (3.6) by $y dt$ and integrating along C between P and Q we obtain

$$\bar{y}^2 = 2\epsilon^{-2} \int_{t_P}^{t_Q} f(t, x(t), dx/dt; \epsilon^2) y(t) dt,$$

where t_P and t_Q are the t -coordinates of P and Q , respectively. Since $f_1(x) \geq 0$ and since $F(t, x, y; \epsilon^2) \leq 0$ in this range we have

$$\bar{y}^2 \geq 2\epsilon^{-2} \int_{t_P}^{t_Q} f_1(x(t)) y(t) dt = 2\epsilon^{-2} \int_0^{\bar{x}} f_1(s) ds.$$

We must have $\bar{x} > z_1$, and employing this fact we find that

$$\bar{y}^2 > 2\epsilon^{-2} \int_0^{z_1} f_1(s) ds \geq \epsilon^{-2} D^2 z_1^2,$$

or

$$(4.1) \quad |\bar{y}| > \epsilon^{-1} z_1 D.$$

In an identical manner we can show that an inequality of this type holds in the octant $t \geq t_0$, $x \leq 0$, $y \geq 0$.

We now project the curve C for $t_0 \leq t \leq T$ onto the (x, y) -plane by eliminating the variable t from the parametric equations of C . This can be done for all of C in this range of t since $\dot{x}$ and $\dot{y}$ do not vanish simultaneously on C in this range of t if only ϵ is sufficiently small. To see this we note that when $\dot{x} = \dot{y} = 0$ then $|x| > z_1$ and we can make ϵ so small that $f(t, x, y; \epsilon^2) \neq 0$ at $y = 0$ for $t_0 \leq t \leq T$.

We now define the function $r(x, y; \epsilon^2)$. Let

$$(4.2) \quad r^2 = \epsilon^2 y^2 + 2 \int_0^x f_1(s) ds.$$

Differentiating (4.2) with respect to t we find that on C ,

$$(4.3) \quad dr^2/dt = 2y \left[f_2(t, x, y) - \epsilon^2 f_3(t, x, y; \epsilon^2) \right] \geq 0, \quad t_0 \leq t \leq T,$$

and so r^2 is a non-decreasing function of t on C in this range.

Now let t' and t'' be two different values of t satisfying $t_0 < t' < T$, $t_0 < t'' < T$, and let (x', y') and (x'', y'') be the corresponding values of (x, y) on C . We have

$$(x' - x'')^2 + (y' - y'')^2 \geq 0.$$

If $x' = x''$ and $y' = y''$ then if r' and r'' are the corresponding values of r we must have $r' = r''$, and this is not possible in view of Assumption v of Section 2. Thus, only the inequality holds and there is a one-to-one correspondence on C between the independent variable t and the number pairs (x, y) , or $(x(t), y(t))$. Hence, the points of C are in one-to-one correspondence with the points of the projection C' of C on the (x, y) -plane, and so we shall henceforth discuss only the properties of C' and denote C' by C .

We project the curve C_0 (see xii of Section 2) onto the (x, y) -plane for $t_0 \leq t < t_1$. This can be done since, by hypothesis, $\dot{z} \neq 0$ in this range. Let this projection of C_0 be denoted by C'_0 . Then the points of C_0 are in one-to-one correspondence with the points of C'_0 . This is true because $\dot{z} > 0$ and $\dot{y} < 0$ in the specified range of t .

In the (x, y) -plane let P_0 be the point (x_0, y_0) and let Q_0 be the point (z_1, q_1) defined in Assumption xii of Section 2. Let the

equation $y = \theta_\epsilon(x)$ describe that portion of the curve C which lies between P_0 and the first intersection point of C with the line $y = 0$. Construct the polygonal line ABD , where AB is a horizontal line segment whose y -coordinate is $\theta_\epsilon(z_1 - \epsilon)$. The point A is the intersection of this line with C'_0 and the x -coordinate of B is z_1 . BD is a straight line with slope $-\epsilon^{-\frac{1}{2}}$, and D is its intersection with the x -axis. We shall obtain an estimate on x_D , the x -coordinate of D , and then show that the point E , which is the intersection of C with the x -axis, has x -coordinate not greater than x_D .

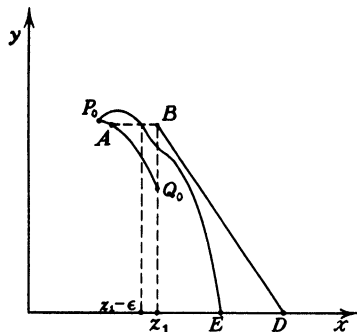


FIGURE 2

If the subscripts A, B, D, E on the coordinates refer to the corresponding point (see Figure 2) then on BD we have

$$\frac{y_D - y_B}{x_D - x_B} = \frac{-\theta_\epsilon(z_1 - \epsilon)}{x_D - z_1} = -\epsilon^{-\frac{1}{2}},$$

or

$$\epsilon^{\frac{1}{2}} \theta_\epsilon(z_1 - \epsilon) = x_D - z_1.$$

And if ϵ is sufficiently small then

$$(4.4) \quad x_D - z_1 < x_0 \epsilon^{\frac{1}{2}}.$$

Let the point on C having coordinates $(z_1, \theta_\epsilon(z_1))$ be called P_1 . We wish to show that as the point $P: (x, \theta_\epsilon(x))$ traverses C from P_1 to E the increase in x is $O(\epsilon^{\frac{1}{2}})$. For convenience we replace z_1 by x_1 and q_1 by y_1 in the following text.

The slopes of integral curves of (3.6) satisfy

$$dy/dx = -\epsilon^{-2} y^{-1} f(t, x, y; \epsilon^2),$$

and on this part of C we have

$$|dy/dx| > \epsilon^{-2} p^{-1} f(t, x, y; \epsilon^2).$$

Expanding $f(t, x, y; \epsilon^2)$ about (t, x_1, y_1) by Taylor's formula we get

$$\begin{aligned} f(t, x, y; \epsilon^2) &= f(t, x_1, y_1; 0) + (x - x_1)f_x(t, x_1, y_1; 0) \\ &\quad + (y - y_1)f_y(t, x_1, y_1; 0) + \frac{1}{2}(x - x_1)^2 f_{xx}(t, x_1, y_1; 0) \\ &\quad + \frac{1}{2}(y - y_1)^2 f_{yy}(t, x_1, y_1; 0) + (x - x_1)(y - y_1)f_{xy}(t, x_1, y_1; 0) \\ &\quad + \epsilon^2 f_3(t, x, y; \epsilon^2). \end{aligned}$$

Replacing

$$f(t, x_1, y_1; 0) + (y - y_1)f_y(t, x_1, y_1; 0)$$

by

$$\begin{aligned} f(t_1, x_1, y_1; 0) + (t - t_1)f_t(t_1, x_1, y_1; 0) + (y - y_1)f_y(t_1, x_1, y_1; 0) \\ + (t - t_1)(y - y_1)f_{ty}(t_1, x_1, y_1; 0) \end{aligned}$$

and recalling that $f(t_1, x_1, y_1; 0) = f_y(t_1, x_1, y_1; 0) = 0$, we obtain

$$\begin{aligned} f(t, x, y; \epsilon^2) &= (x - x_1)f_x(t, x_1, y_1; 0) + \frac{1}{2}(x - x_1)^2 f_{xx}(t, x_1, y_1; 0) \\ &\quad + \frac{1}{2}(y - y_1)^2 f_{yy}(t, x_1, y_1; 0) + (x - x_1)(y - y_1) \\ (4.5) \quad &\quad f_{xy}(t, x_1, y_1; 0) + (t - t_1)[f_t(t_1, x_1, y_1; 0) \\ &\quad + (y - y_1)f_{ty}(t_1, x_1, y_1; 0)] + o(\epsilon^2), \end{aligned}$$

where t' lies between t_1 and t , x' lies between x_1 and x and y' lies between y_1 and y .

Suppose that $y_A = y_B \geq y_1 + \epsilon^{1/5}$. Then for $y - y_1 \geq \epsilon^{1/5}$ and if ϵ is sufficiently small we have on BD

$$(4.6) \quad f(t, x, y; \epsilon^2) \geq \frac{1}{2}(y - y_1)^2 m,$$

and so

$$|dy/dx| \geq \epsilon^{-2} m \epsilon^{2/5} / 4p.$$

Now let $\epsilon^{1/5} > y - y_1 \geq \epsilon^{3/5}$. Then on this part of BD we again have (4.6) and $|dy/dx| > \epsilon^{-2}$.

Now let $|y - y_1| < \epsilon^{3/5}$ on C. On this part of C we have

$$f(t, x, y; \epsilon^2) \geq \frac{1}{2}(x - x_1)k,$$

and

$$(4.7) \quad -dy \geq \frac{1}{2}\epsilon^{-2}p^{-1}k(x - x_1)dx.$$

Let P_2 and P_3 be the points where the descending y -coordinate first becomes $y_1 + \epsilon^{3/5}$ and $y_1 - \epsilon^{3/5}$, respectively. Integrating equation (4.7) along C between P_2 and P_3 , we get

$$2\epsilon^{3/5} \geq \frac{1}{2}\epsilon^{-2}p^{-1}k \left[(x_3 - x_1)^2 - (x_2 - x_1)^2 \right],$$

or

$$(x_3 - x_2)^2 < \frac{8p\epsilon^{13/5}}{k},$$

and

$$(4.8) \quad (x_3 - x_2) < \sqrt{(8p/k)} \epsilon^{13/10},$$

where x_2 and x_3 are the x -coordinates of P_2 and P_3 , respectively.

Consider an integral curve of (3.6), say $\bar{C}$, starting at B . On this curve we have

$$f(t, x, y; \epsilon^2) \geq \frac{1}{4}(y - y_1)^2m,$$

and

$$(4.9) \quad -dy/dx \geq m\epsilon^{-2}(y - y_1)^2/4p.$$

Since $\bar{C}$ cannot cross BD between B and that point of BD where $y = y_1 + \epsilon^{3/5}$, this last inequality also holds at every point of $\bar{C}$ between B and $\bar{P}_2$, where $\bar{P}_2$ is the first point after B on $\bar{C}$ where $y = y_1 + \epsilon^{3/5}$. Integrating (4.9) along $\bar{C}$ between B and P_2 , letting Δx be the increase in x in this range we get

$$\int_p^{y_1 + \epsilon^{3/5}} \frac{-dy}{(y - y_1)^2} \geq \frac{m}{4p} \epsilon^{-2} \Delta x,$$

and

$$(4.10) \quad \Delta x \leq 4p\epsilon^{7/5}/m.$$

Now, when $y = y_B$ C lies to the left of $\bar{C}$. Then by the fundamental uniqueness theorem we must have C to the left of $\bar{C}$ when $y = y_1 + \epsilon^{3/5}$, and so (4.10) holds on C between P_1 and P_2 . Between P_2 and P_3 on C equation (4.8) holds. Thus, between P_1 and P_2 , C lies to the left of BD if ϵ is sufficiently small.

Now consider the slopes of integral curves of (3.6) on BD between the points where $y = y_1 - \epsilon^{3/5}$ and $y = y_1 - \epsilon^{1/5}$. On this part of BD we have $f(t, x, y; \epsilon^2) > \frac{1}{4}m(y - y_1)^2$, and the slopes of integral curves of (3.6) satisfy

$$-dy/dx > \frac{m\epsilon^{-2}\epsilon^{6/5}}{4p} = \frac{m}{4p}\epsilon^{-4/5} > \epsilon^{-1/2}$$

here. Thus, integral curves of (3.6) cross BD from left to right for $y_1 - \epsilon^{3/5} \geq y \geq y_1 - \epsilon^{1/5}$, and so C lies to the left of BD in this range.

Now let $y < y_1 - \epsilon^{1/5}$ on BD . Then $f(t, x, y; \epsilon^2) \geq (x - x_1)k$ and the slopes of integral curves of (3.6) crossing BD in this range satisfy

$$(4.11) \quad -dy/dx \geq (x - x_1)k\epsilon^{-2}/p.$$

It is now necessary to find a lower bound for $x - x_1$ on that part of BD for which $y < y_1 - \epsilon^{1/5}$. The equation of BD is

$$x - x_1 = \epsilon^{\frac{1}{2}} \left[\theta_\epsilon(x_1 - \epsilon) - y \right] = \epsilon^{\frac{1}{2}} \left[y_B - y \right].$$

Now, $y_B > y_1$, and so $(x - x_1) > \epsilon^{\frac{1}{2}}(y_1 - y)$. Thus,

$$-dy/dx \geq k\epsilon^{7/10}\epsilon^{-2}/p > \epsilon^{-1/2}.$$

Hence, integral curves of (3.6) cross BD from left to right between those points on BD where $y = y_1 - \epsilon^{3/5}$ and $y = 0$. And so C lies to the left of BD for $\theta_\epsilon(x_1 - \epsilon) \geq y \geq 0$, and the increment Δx of x on C between P_0 and E satisfies

$$(4.12) \quad \Delta x < p\epsilon^{1/2}.$$

If $y_B < y_1 + \epsilon^{1/5}$, or $y_B < y_1 + \epsilon^{3/5}$ then the first step or first two steps may be omitted from our argument and equation (4.12) still holds.

5. On the curve C from P_0 to E let $P_1': (x_1, y_1')$ be the point where the x -coordinate of C first becomes x_1 . We shall first suppose that $y_1' > y_1 + \epsilon^{2/3}$. Let P_2, P_3, P_4 be the points where the descending y -coordinate first becomes $y_1 + \epsilon^{2/3}, y_1 - \epsilon^{2/3}$ and zero, respectively, and let t_1, t_2, t_3, t_4 be the corresponding values of t . We have $x_1 < x_2 < x_3 < x_4, y_1' > y_2 > y_3 > 0$. On this arc of C let $x = u + x_1, y = w + y_1$. Using u and w as new variables we obtain from equation (4.5)

$$\begin{aligned}
 f(t, x, y; \epsilon^2) = & uf_x(t, x_1, y_1; 0) + \frac{1}{2}u^2f_{xx}(t, x', y'; 0) \\
 & + \frac{1}{2}w^2f_{yy}(t, x', y'; 0) + uwf_{xy}(t, x', y'; 0) \\
 (5.1) \quad & + (t - t_1) \left[f_t(t', x_1, y_1; 0) \right. \\
 & \left. + wf_{ty}(t', x_1, y_1; 0) \right] + o(\epsilon^2),
 \end{aligned}$$

and from equation (3.6) we get

$$(5.2) \quad \begin{cases} du/dt = w + y_1. \\ -\epsilon^2 dw/dt = uf_x(t, x_1, y_1; 0) + \frac{1}{2}u^2f_{xx}(t, x', y'; 0) \\ + \frac{1}{2}w^2f_{yy}(t, x', y'; 0) + uwf_{xy}(t, x', y'; 0) \\ + (t - t_1) \left[f_t(t', x_1, y_1; 0) + wf_{ty}(t', x_1, y_1; 0) \right] + o(\epsilon^2). \end{cases}$$

In the new system the coordinates of P_1' may be denoted by $(x_1, y_1 + w_1)$. We then have $y_1 + w_1 < \theta_\epsilon(x_1 - \epsilon)$. As $\epsilon \rightarrow 0$ we have $\theta_\epsilon(x_1 - \epsilon) \rightarrow y_1$, and so given any $\eta, 1 > \eta > 0$, we can choose ϵ so small that $\theta_\epsilon(x_1 - \epsilon) - y_1 < \eta$, and $\epsilon^{2/3} < \eta$.

We now proceed from P_1' to P_2 . On this part of C we have $0 \leq u \leq \rho\epsilon^{1/2}, 1 > \eta > w \geq w_2 = \epsilon^{2/3}$. Hence, from the second of equations (5.2) if η and ϵ are sufficiently small,

$$-\epsilon^2 dw/dt \geq \frac{1}{2}w^2f_{yy}(t, x', y'; 0).$$

Multiplying this equation through by $(2/w^2)dt$ and integrating along C from P_1 to P_2 we get

$$\int_{t_1}^{t_2} f_{yy}(t, x', y'; 0) dt < 2\epsilon^2 \int_{\epsilon^{2/3}}^1 \frac{dw}{w^2} < 2\epsilon^{4/3}.$$

Since $f_{yy} \geq m$ in this range we have

$$(5.3) \quad (t_2 - t_1') < 2\epsilon^{4/3}/m.$$

From P_2 to P_3 we have $|w| \leq \epsilon^{2/3}$, and so if ϵ is sufficiently small, from the second of equations (5.2) we get

$$(5.4) \quad -\epsilon^2 dw/dt \geq \frac{1}{2} u f_x(t, x_1, y_1; 0).$$

But between P_2 and P_3 we have

$$u = u_2 + \int_{t_2}^t (y_1 + w) dt > u_2 + y_1(t - t_2) - \epsilon^{2/3}(t - t_2),$$

and if ϵ is sufficiently small,

$$(5.5) \quad u > u_2 + \frac{1}{2} y_1(t - t_2).$$

Using equation (5.5) in equation (5.4) and multiplying (5.4) by dt and integrating on C from P_2 to P_3 we obtain

$$2\epsilon^2 \epsilon^{2/3} > y_1 k(t_3 - t_2)^2/8,$$

or

$$(5.6) \quad t_3 - t_2 < 4\epsilon^{4/3}/(ky_1)^{\frac{1}{2}}.$$

We now proceed from P_3 to P_4 . We break this arc into two parts. Let P_3^1 be the point on this arc whose y -coordinate is $y_1 - \epsilon^{1/5}$. Then from P_3 to P_3^1 we have

$$-\epsilon^2 dw/dt \geq \frac{1}{2} w^2 f_{yy} \geq \frac{1}{2} w^2 m.$$

Multiplying this inequality by dt/w^2 and integrating along C from P_3 to P_3^1 we obtain

$$(5.7) \quad (t_3^1 - t_3) \leq 2\epsilon^{8/3}/m.$$

From P_3^1 to P_4 we have

$$(5.8) \quad -\epsilon^2 dw/dt \geq \frac{1}{2} w^2 f_{yy}.$$

Since f_{yy} may vanish on this arc of C we cannot use the lower bound m

here. Instead we employ the first mean value theorem of the integral calculus to obtain

$$(5.9) \quad \int_{t_3^1}^{t_4} f_{yy} dt = \mu(t_4 - t_3^1),$$

where μ is a positive constant which does not vanish as $\epsilon \rightarrow 0$. Then multiplying (5.8) by dt/w^2 and integrating along C between P_3^1 and P_4 we obtain

$$(5.10) \quad (t_4 - t_3^1) < 4\epsilon^{9/5}/\mu.$$

Combining equations (5.3), (5.6), (5.7) and (5.10) we find that

$$(5.11) \quad (t_4 - t_1^1) < \gamma\epsilon^{4/3}.$$

If $y_1^1 < y_1 + \epsilon^{2/3}$ the first step of our argument is eliminated and (5.11) still holds.

6. We shall now describe the behavior of C for $t \geq t_4$.

Let $P_5: (x_5, y_5)$ be one of the points on C for $t_0 \leq t \leq T$ where C intersects the positive y -axis, and let us follow along C in a clockwise direction. Let $P_6: (x_6, y_6)$ be the point where the descending y -coordinate first becomes $\epsilon^{-2/3}$; let $P_7: (x_7, y_7)$ be the point where y first becomes y_0 , and let $P_8: (x_8, y_8)$ be the first point after P_7 where C meets the positive x -axis. Then between P_5 and P_6 $y \geq \epsilon^{12/3}$. If r_5, r_6, r_7, r_8 are the corresponding values of the positive quantity r , defined in Section 4, at the points P_5, P_6, P_7, P_8 , respectively, then from equation (4.3) we get

$$r_6^2 - r_5^2 = 2 \int_{t_5}^{t_6} y \left[f_2(t, x, y) - \epsilon^2 f_3(t, x, y; \epsilon^2) \right] dt,$$

where the integrand is evaluated along C . Replacing y by dx/dt we get

$$r_6^2 - r_5^2 = 2 \int_0^{x_6} \left[f_2(t, x, y) - \epsilon^2 f_3(t, x, y; \epsilon^2) \right] dx.$$

From part iv of the hypothesis we conclude that if $|y|$ is sufficiently large there exists a positive number F_1 such that $|f_2(t, x, y)| \leq F_1/y^2$,

in the range $t_0 \leq t \leq T$, $|x| \leq X$. Likewise, we conclude from part iv of the hypothesis that if ϵ is sufficiently small and if $|y|$ is sufficiently large then there exists a positive number F_2 such that $|f_3(t, x, y; \epsilon^2)| \leq F_2/y^2$ in the same range of t and x . We set $M = \max(F_1, F_2)$. Thus, if ϵ is sufficiently small, we have

$$r_6^2 - r_5^2 < 2Mx_6\epsilon^{4/3}.$$

Then,

$$r_6 - r_5 < Mx_6\epsilon^{4/3}/r_5.$$

But by equation (4.1) $r_5 > x_1 D$, and since $x_6 < x_1 + p\epsilon^{\frac{1}{2}}$ we have for ϵ sufficiently small,

$$(6.1) \quad r_6 - r_5 < \frac{M\epsilon^{4/3}}{D}.$$

From P_6 to P_7 $y \leq \epsilon^{-2/3}$. By part iii of the hypothesis and equation (4.2) since r is a non-decreasing function of t and if ϵ is sufficiently small, we have,

$$2 \int_0^x f_1(s) ds = r^2 - \epsilon^2 y^2 \geq 2 \int_0^{x_1} f_1(s) ds - \epsilon^{2/3} \geq D^2 x_1^2 - \epsilon^{2/3} > \frac{1}{2} D^2 x_1^2.$$

But,

$$\int_0^x f_1(s) ds \leq x f_1(x) \leq f_1^2(x)/D^2,$$

and so, $f_1^2(x) \geq \frac{1}{4} D^4 x_1^2$, or between P_6 and P_7 on C ,

$$(6.2) \quad f_1(x) > \frac{1}{2} D^2 x_1.$$

From (4.3) and the second of equations (3.6) we get

$$(6.3) \quad dr^2 = - \frac{2y\epsilon^2[f_2(t, x, y) - \epsilon^2 f_3(t, x, y; \epsilon^2)] dy}{f_1(x) - f_2(t, x, y) + \epsilon^2 f_3(t, x, y; \epsilon^2)}.$$

Now, at P_5 we have

$$r_5^2 > 2 \int_0^{x_1} f_1(s) ds.$$

At P_6 we have

$$r_6^2 = \epsilon^{2/3} + 2 \int_0^{x_6} f_1(s) ds,$$

and $r_6 \geq r_5$. Hence,

$$(6.4) \quad \int_0^{x_6} f_1(s) ds > \int_0^{x_1} f_1(s) ds - \frac{1}{2}\epsilon^{2/3}.$$

If $x_6 < x_1$ we obtain from (6.4) that

$$\int_{x_6}^{x_1} f_1(s) ds < \frac{1}{2}\epsilon^{2/3}.$$

If $x_6 \leq x_0$ then

$$\int_{x_0}^{x_1} f_1(s) ds \leq \int_{x_6}^{x_1} f_1(s) ds,$$

and

$$(x_1 - x_0)f_1(x_0) < \int_{x_0}^{x_1} f_1(s) ds < \frac{1}{2}\epsilon^{2/3}.$$

This is a contradiction since x_0 , x_1 and $f_1(x_0)$ do not depend on ϵ , while $(x_1 - x_0)f_1(x_0) \neq 0$. Hence, if $x_6 < x_1$ then $x_6 > x_0$. Also, since $x_1 > x_0$ then $x_6 > x_0$ in either case. Now,

$$\begin{aligned} F(t, x, y; \epsilon^2) &= F(t, x_0, y_0; \epsilon^2) \\ &\quad + (x - x_0)F_x(t, x', y'; \epsilon^2) \\ &\quad + (y - y_0)F_y(t, x', y'; \epsilon^2), \end{aligned}$$

where x' is intermediate between x and x_0 , y' is intermediate between y and y_0 , and

$$F(t, x, y; \epsilon^2) = f_2(t, x, y) - \epsilon^2 f_3(t, x, y; \epsilon^2).$$

Since $y \geq y_0$ and $x \geq x_0$ on the arc $P_6 P_7$ we have by parts vi and vii of the hypothesis,

$$F(t, x, y; \epsilon^2) \leq F(t, x_0, y_0; \epsilon^2) \leq \frac{1}{2} D^2 x_1.$$

Comparing this result with (6.2) we see that for ϵ sufficiently small there exists a positive constant d independent of ϵ such that $f(t, x, y; \epsilon^2) \geq d$ on this arc. Therefore, by (6.3),

$$dr^2 = - \frac{2\epsilon^2 y f_2(t, x, y) dy + 2\epsilon^4 y f_3(t, x, y; \epsilon^2) dy}{f(t, x, y; \epsilon^2)},$$

or,

$$dr^2 < - 2\epsilon^2 \left[|y f_2(t, x, y)| dy + \epsilon^2 |y f_3(t, x, y; \epsilon^2)| dy \right] / d.$$

Integrating this last equation along C from P_6 to P_7 we get

$$\begin{aligned} r_7^2 - r_6^2 &< 2\epsilon^2 \left[\int_{y_0}^{\infty} |y f_2(t, x, y)| dy + \epsilon^2 \int_{y_0}^{\infty} |y f_3(t, x, y; \epsilon^2)| dy \right] / d \\ &= 2\epsilon^2 \left[\int_{y_0}^{\infty} |y f_2(t, x, y)| dy + o(\epsilon^2) \right] / d, \end{aligned}$$

or

$$(6.5) \quad r_7 - r_6 < \gamma \epsilon^2.$$

At P_7 we have

$$(6.6) \quad r_7^2 = \epsilon^2 y_0^2 + 2 \int_0^{x_7} f_1(s) ds > 2 \int_0^{x_4} f_1(s) ds.$$

We wish to show that $x_7 \geq x_1$ so that the procedure used on the arc $P_7 P_8$ can be the same as that used on the arc $P_1 P_4$. Let P_2' be that point on the arc $P_2 P_3$ where $w = 0$. From equation (5.11) and the second of equations (5.2), we have on the arc $P_2' P_3$

$$- \epsilon^2 \frac{dw}{dt} \leq \gamma \epsilon^{4/3} + o(\epsilon^2) + \left[u_2^1 + \int_{t_2^1}^t (w + y_1) dt \right] \left[f_x + \frac{1}{2} u f_{xx} + w f_{xy} \right].$$

On $P_2^1 P_3$ we have $0 \geq w \geq -\epsilon^{2/3}$. If $u_2^1 \leq \epsilon^{4/3}$, if K is an upper bound for $(|f_x| + |f_{xx}| + |f_{xy}|)$ in this range, and if ϵ is sufficiently small then

$$- \epsilon^2 \frac{dw}{dt} < \gamma \epsilon^{4/3} + \left[\epsilon^{4/3} + y_1(x - x_2^1) \right] \left[K \right] \left[1 + \frac{1}{2} p \epsilon^{\frac{1}{2}} + \epsilon^{2/3} \right],$$

or

$$- \epsilon^2 dw/dt < \gamma \epsilon^{4/3} + 2Ky_1(x - x_2^1).$$

Multiplying this equation by dx and integrating over the arc $P_2^1 P_3$, we obtain

$$\epsilon^{8/3} \leq \gamma \epsilon^{4/3}(t_3 - t_2^1) + Ky_1(t_3 - t_2^1)^2.$$

Therefore, there exists a positive number γ such that $(t_3 - t_2^1) \geq \gamma \epsilon^{4/3}$. Now, $du/dt = y_1 + w$, and so $u_3 - u_1 \geq y_1 \gamma \epsilon^{4/3} - \epsilon^{2/3} \gamma \epsilon^{4/3}$, or $u_3 - u_1 \geq \gamma \epsilon^{4/3}$, provided ϵ is sufficiently small. Since $u_1 = 0$, we have

$$(6.7) \quad u_3 \geq \gamma \epsilon^{4/3}.$$

If $u_2^1 > \epsilon^{4/3}$ then certainly $u_3 > \epsilon^{4/3}$ whence we may choose γ as unity. Thus, in any case, equation (6.7) holds.

We now use this result in equation (6.6). Since $u_4 > u_3$ we have

$$\epsilon^2 y_0^2 + 2 \int_0^{x_7} f_1(s) ds > 2 \int_0^{x_1} f_1(s) ds + 2 \int_{x_1}^{x_1 + \gamma \epsilon^{4/3}} f_1(s) ds,$$

or

$$\int_0^{x_7} f_1(s) ds > \int_0^{x_1} f_1(s) ds + f_1(x_1) \gamma \epsilon^{4/3} - \frac{1}{2} \epsilon^2 y_0^2,$$

and therefore,

$$\int_0^{x_7} f_1(s) ds > \int_0^{x_1} f_1(s) ds,$$

and so $x_7 > x_1$. Thus, on the arc P_7P_8 we may proceed exactly as we did on the arc P_1P_4 . We then have

$$(6.8) \quad t_8 - t_7 < \gamma \epsilon^{4/3}.$$

If L is an upper bound for $|F(t, x, y; \epsilon^2)|$ in the range $t_0 \leq t \leq T$, $|x| \leq x_1 + a$, $|y| \leq y_0$, then since $dr^2 = 2yf_2(t, x, y)dt - 2y\epsilon^2 f_3(t, x, y; \epsilon^2)dt$, we have $dr^2 \leq 2y_0 L dt$, or

$$(6.9) \quad r_8 - r_5 < \gamma \epsilon^{4/3}.$$

The result is the same in the fourth and second quadrants and is, in fact, easier to get, since there $F(t, x, y; \epsilon^2) \leq 0$ and $f(t, x, y; \epsilon^2) \geq f_1(x)$, and so we may proceed directly from $y = 0$ to $y = -\epsilon^{-2/3}$ in one step.

7. We now consider the change in t when the point $P: (x, y)$ traverses the fourth (or second) quadrant from $P_8: (x_8, 0)$ to $P_9: (0, y_9)$. In this quadrant, using (4.2) we get

$$dt = dx/y = -\epsilon dx / \sqrt{\left[r^2 - 2 \int_0^x f_1(s) ds \right]}.$$

Integrating this equation between P_8 and P_9 we find that

$$\begin{aligned} t_9 - t_8 &= -\epsilon \int_{x_8}^0 dx / \sqrt{\left[r^2 - 2 \int_0^x f_1(s) ds \right]} \\ &\leq \epsilon \int_0^{x_8} dx / \sqrt{\left[2 \int_0^{x_8} f_1(s) ds - 2 \int_0^x f_1(s) ds \right]} \\ &= \epsilon \int_0^{x_8} dx / \sqrt{\left[2 \int_x^{x_8} f_1(s) ds \right]}. \end{aligned}$$

Using part iii of the hypothesis we obtain,

$$\Delta t \leq \epsilon \int_0^{x_8} dx / \sqrt{\left[2D^2 \int_x^{x_8} s ds \right]} = \frac{\epsilon}{D} \int_0^{x_8} dx / \sqrt{(x_8^2 - x^2)} = \frac{\pi \epsilon}{2D}.$$

Thus,

$$(7.1) \quad t_9 - t_8 \leq \pi\epsilon/2D.$$

We shall now find the change in t as the curve C traverses the first (or third) quadrant. We shall use $P_5: (0, y_5)$ and $P_8: (x_8, 0)$ to denote the intersection points of an arc of C with the positive y -axis and the positive x -axis, respectively. $P_7: (x_7, y_0)$ is the point where the descending y -coordinate of C becomes y_0 . We shall denote the point where

$$2 \int_0^x f_1(s) ds = \epsilon^2 y_5^2 = r_5^2$$

by $P_{10}: (x_{10}, y_{10})$. Since r is non-decreasing we have

$$\begin{aligned} t_{10} - t_5 &= \epsilon \int_0^{x_{10}} dx / \sqrt{\left[r^2 - 2 \int_0^x f_1(s) ds \right]} \\ &\leq \epsilon \int_0^{x_{10}} dx / \sqrt{\left[2 \int_0^{x_{10}} f_1(s) ds - 2 \int_0^x f_1(s) ds \right]} \\ &\leq \epsilon \int_0^{x_{10}} dx / D^2 \left(x_{10}^2 - x^2 \right) = \epsilon \pi / D^2 2, \end{aligned}$$

or

$$t_{10} - t_5 \leq \pi\epsilon/2D^2.$$

Now, $r_{10}^2 = r_5^2 + \epsilon^2 y_{10}^2$, and by Section 6, $r_{10} - r_5 < \gamma\epsilon^{4/3}$. Thus, if ϵ is sufficiently small,

$$x_{10} = \epsilon^{-2}(r_{10}^2 - r_5^2) \leq 3r_5\epsilon^{-2}\gamma\epsilon^{4/3}.$$

or $x_{10} < \gamma\epsilon^{-1/3}$. We remark that $x_{10} > x_0$. Then from P_{10} to P_7 on C we have, by the material immediately preceding equation (6.5) that

$$dt = -\epsilon^2 dy / f(t, x, y; \epsilon^2) \leq -\epsilon^2 dy / d.$$

Hence, in this range we have,

$$(7.2) \quad t_9 - t_8 \leq \epsilon^2 \gamma \epsilon^{-1/3} / d \leq \gamma \epsilon^5 / 3.$$

We have already seen that $\Delta t \leq \gamma\epsilon^{4/3}$ in the range $0 \leq y \leq p$. Hence,

we have

$$(7.3) \quad \Delta t = t_8 - t_5 \leq \pi\epsilon/2D + \gamma\epsilon^{4/3}.$$

A lower bound for Δt in all quadrants is easily obtained, but is not so precise as the upper bounds obtained here. In the fourth quadrant we get

$$(7.4) \quad \Delta t \geq \frac{\epsilon}{D} \text{Arc sin}(Du_8/r_9).$$

and in the third quadrant,

$$(7.5) \quad \Delta t \geq \frac{\epsilon}{D} \text{Arc sin}(Du_8/r_8).$$

We have thus proved the following theorem.

THEOREM: Under the hypothesis of Section 2, and for ϵ sufficiently small, if $C: x = x(t)$ is the integral curve of (1.1) in (t, x) -space described in Section 3, then for $t > t_1$, $x(t)$ has oscillatory behavior, the "period" of the oscillations being $O(\epsilon^{\frac{1}{2}})$ and the "amplitude" being $z_1 + O(\epsilon^{\frac{1}{4}})$. The results stated here can be easily extended to a system of the form

$$\begin{aligned} \frac{dx}{dt} &= f_1(t, x_1, \dots, x_n, u; \epsilon) \frac{du}{dt} \\ &+ f_i(t, x_1, \dots, x_n, u; \epsilon), \quad i = 1, \dots, n, \\ \frac{d^2u}{dt^2} &+ g(t, x_1, \dots, x_n, u; \epsilon) = 0. \end{aligned}$$

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V. CRITICAL POINTS AT INFINITY AND FORCED OSCILLATION

Ralph E. Gomory

INTRODUCTION

The problem of oscillations, both forced and free, is central in non-linear differential equations. Poincaré [1] and Bendixson [1] in their fundamental papers discussed the existence of free oscillations in a pair of first order equations

$$(a) \quad \frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),$$

while the interest aroused by van der Pol's equation has caused more modern writers to turn their attention especially to the system arising from the second order equation

$$(b) \quad \frac{d^2x}{dt^2} + f(x) \frac{dx}{dt} + g(x) = 0.$$

The free oscillations of this equation and the forced oscillations of

$$(c) \quad \frac{d^2x}{dt^2} + f(x) \frac{dx}{dt} + g(x) = E(t)$$

have been investigated in numerous papers, especially Levinson and Smith [1], and Levinson [1].

A central condition in these investigations is that the equations be of the type called by Levinson [2] "dissipative for large displacements." This condition ensures that with increasing time all solutions tend to the interior of some circle in the phase plane. This makes it possible to apply the Brouwer fixed point theorem to (c), or, provided there is an unstable critical point at the phase plane origin, to apply the Poincaré-Bendixson theorem to (b).

The main contribution of this paper will be to drop this condition,

and to prove the existence of forced periodic solutions to a variety of differential equations not dissipative for large displacements. In fact many of these equations will have, in addition to their periodic solutions, other solutions which become unbounded with increasing t , and still others which become unbounded with decreasing t .

The notion of critical point at infinity due to Poincaré will be used throughout. Poincaré extended the planar autonomous system

$$\frac{dx}{X(x,y)} = \frac{dy}{Y(x,y)}$$

to a sphere, (the doubly covering surface of the projective plane), and discussed his equations on the entire sphere including the equator and its critical points, (the doubly covered line at infinity and its critical points). Later Bendixson briefly discussed a single point at infinity, but since that time, with the exception of a recent paper by Lefschetz [1], authors have dealt mainly with the finite plane. This may be due to the fact that while Poincaré restricted himself to equations with simple critical points at infinity, the systems arising from (b) rarely have this property, and so are not as easily treated.

In Part I, systems of the Poincaré type will be treated, and their forced oscillations related to the nature of their critical points at infinity. In Part II the nature of the critical points at infinity arising from a wide class of equations (b) will be completely analyzed. This analysis gives the asymptotic behavior of trajectories that become unbounded. Geometric information, such as the existence of limit-cycles to equations of the van der Pol type, will be deduced. In Part III the results on critical points from II will be applied to obtain new criteria for the existence of oscillations in (c).

The general condition which in one form or another will be used to replace dissipativity may be stated roughly as follows. Let the plane be completed by the addition of a single point at infinity. This point is generally a critical point, and in a dissipative system it must be an unstable node or focus. Our conditions are roughly equivalent to requiring that the index of this point be $\neq 2$, and that it have no saddle sectors.

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PART I: FORCED OSCILLATIONS IN A CLASS OF FIRST ORDER SYSTEMS

§1. In this section we will consider the system

$$(1) \quad \frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y)$$

where $X(x, y)$, $Y(x, y)$ are both polynomials of degree $n > 0$. The theorem stated below will relate the nature of the critical points at infinity of (1) to the question of the existence of periodic solutions to

$$(2) \quad \frac{dx}{dt} = X(x, y) + E_1(t), \quad \frac{dy}{dt} = Y(x, y) + E_2(t)$$

where $E_1(t)$ and $E_2(t)$ have continuous derivatives and are periodic with period T .

We will start by extending equation (1) to the projective plane.

Adopting homogeneous coordinates (x, y, z) , with $(x, y, 1)$ a designation for the point with affine coordinates (x, y) , we see that (1) and

$$(3) \quad \frac{d}{dt} \left(\frac{x}{z} \right) = X \left(\frac{x}{z}, \frac{y}{z} \right), \quad \frac{d}{dt} \left(\frac{y}{z} \right) = Y \left(\frac{x}{z}, \frac{y}{z} \right),$$

have the same solutions. However we can easily extend (3) to cover points for which $z = 0$.

Using (3) and multiplying by z^{n+1} gives

$$(4) \quad \begin{aligned} z^{n-1} \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) &= zX^H(x, y, z) \\ z^{n-1} \left(z \frac{dy}{dt} - y \frac{dz}{dt} \right) &= zY^H(x, y, z), \end{aligned}$$

where X^H and Y^H are the original X and Y made homogeneous of degree n with z . Adopting a new parameter τ such that $\frac{dt}{d\tau} = z^{n-1}$ we obtain from (4) in the region covered by coordinates $(x, 1, z)$

$$(5) \quad \begin{aligned} \frac{dz}{d\tau} &= -zY^H(x, 1, z) \\ \frac{dx}{d\tau} &= X^H(x, 1, z) - xY^H(x, 1, z) \end{aligned}$$

and in the region covered by coordinates $(1, y, z)$ using the parameter σ such that $\frac{dt}{d\sigma} = z^{n-1}$:

$$(6) \quad \begin{aligned} \frac{dz}{d\sigma} &= -zX^H(1, y, z) \\ \frac{dy}{d\sigma} &= Y^H(1, y, z) - yX^H(1, y, z). \end{aligned}$$

It is easily seen that solutions to (5) and (6) with $z \neq 0$ trace out the same trajectories as those of (1), the parametrizations only being different. As (5) and (6) are defined for $z = 0$, we can consider them as

extending (1), which was defined only for $(x, y, 1)$, to $(x, 1, z)$, and $(1, y, z)$.

A critical point on the line at infinity is simply a critical point of (5) or (6) on $z = 0$. An elementary or simple critical point is one for which the linear terms in the Taylor expansion about the point have a non-zero determinant.

§2. We are now in a position to state the theorem.

THEOREM: If the critical points at infinity of the extended equation (1) satisfy the following conditions:

- (a) there are isolated critical points and they are simple;
- (b) their index sum is $\neq 1$;
- (c) there are no consecutive saddle points;

then equation (2) has a periodic solution of period T .

It is probably worth while to clarify exactly what is meant by "consecutive saddle points." As the line at infinity is topologically a circle we may trace it through in some fixed sense, encountering in succession all critical points. Condition (c) is that in this cyclic arrangement saddle points should not follow saddle points. In particular, if there is only one critical point and it is a saddle, condition (c) is not satisfied.

To facilitate application to specific equations, the conditions for the theorem will be restated entirely in terms of the original polynomials X and Y . To do this, designate by X_n, Y_n the terms of degree n of X, Y . Condition (a) is readily seen to be equivalent to

- (a1) $xY_n - yX_n$ has real roots, no multiple roots, and does not vanish identically.
- (a2) $xY_n - yX_n$ and X_n have no roots in common, and the same holds for $xY_n - yX_n$ and Y_n .

As the index sum of a vector distribution in the projective plane is 1, condition (b) is that the sum of the indices of the finite critical points of (1) be $\neq 0$. Condition (c) may be tested for directly by using the following observation from Poincaré ([1], p. 25).

The coordinates of critical points at infinity have ratios $\frac{y}{x} = \alpha$ which satisfy $xY_n - yX_n = 0$. If, when α increases from $\alpha - \epsilon$ to $\alpha + \epsilon$ the expression

$$\frac{Y_n}{X_n} - \frac{y}{x}$$

changes from negative to positive, the critical point is a saddle point;

if the expression changes from positive to negative, it is a node.

In terms of a single point at infinity the conditions ensure an index $\neq 2$, and the absence of saddle sectors.

§3. PROOF OF THE THEOREM. The first step in the proof will be the construction of a large closed contour J , containing all the finite critical points of (1) and with the following special property. A solution $x(t)$, $y(t)$ of (2) which at time t_0 is at a point p on J , never is at p again for $t > t_0$.

We will start by extending (2). Equation (2) is merely (1) with time dependent additions to the constant terms in X and Y . Hence from (5) the extension to $(x, 1, z)$ is

$$\begin{aligned} \frac{dz}{d\tau} &= -z \left\{ Y^H(x, 1, z) + z^n E_2(t) \right\} \\ (7) \quad \frac{dx}{d\tau} &= \left\{ X^H(x, 1, z) + z^n E_1(t) \right\} - x \left\{ Y^H(x, 1, z) + z^n E_2(t) \right\} \end{aligned}$$

or, arranging in powers of z ,

$$\begin{aligned} \frac{dz}{d\tau} &= -z \left\{ P_0 + P_1 z + \dots + (P_n + E_2) z^n \right\} \\ (8) \quad \frac{dx}{d\tau} &= \left\{ Q_0 + Q_1 z + \dots + (Q_n + E_1 - x E_2) z^n \right\} \end{aligned}$$

where P_1 and Q_1 are polynomials in x . From condition (a) we have that P_0 and Q_0 do not vanish identically and have no common real zeros. Also Q_0 has real roots, all of which are simple.

Let x^0 be a zero of Q_0 and consider (8) near x^0 on $z = 0$. Such a point is a critical point at infinity of (1). As $P_0(x^0) \neq 0$ and E_2 is periodic, $\frac{dz}{d\tau}$ is $\neq 0$ for all t throughout some sufficiently small box B , $0 < z \leq \delta_1$, $x^0 - \delta_1 \leq x \leq x^0 + \delta_1$ (Fig. 1)

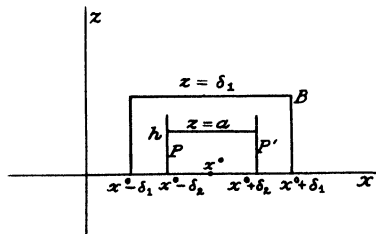


FIGURE 1

so $\frac{dz}{d\tau}$ has constant sign in B. As x^0 is a simple zero of Q_0 , there is a δ_2 such that $Q_0(x^0 + \delta_2)$, $Q_0(x - \delta_2)$ are $\neq 0$ and have opposite signs. Hence at $x = x^0 \pm \delta_2$ on $z = 0$ we may erect perpendiculars P and P' of some height $h < \delta_1$, and $\frac{dx}{d\tau}$ will have the sign of $Q_0(x^0 \pm \delta_2)$ on these lines (Fig. 1). If we then connect P and P' by any line $z = a$, $0 < a < h$ we will have formed a three sided box whose sides are segments without contact, that is they are never tangent to the vector distribution $(\frac{dx}{d\tau}, \frac{dy}{d\tau})$. With increasing τ these sides may be crossed in various directions by solutions of (7). As $\frac{dx}{d\tau}$ has opposite signs on P and on P' there are exactly four possible combinations of crossing directions as is indicated by the velocity vectors in Fig. 2.

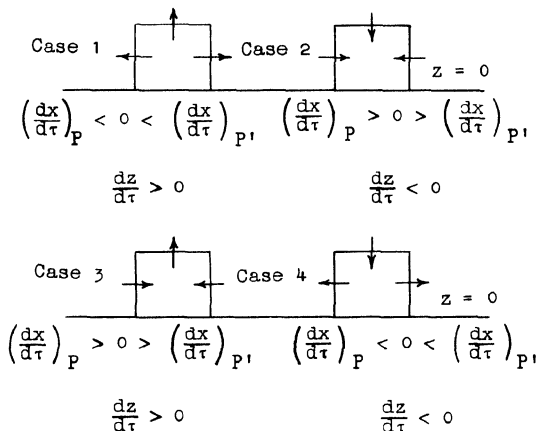


FIGURE 2

If x^0 is a node of (1), then either case 1 or case 2 occurs, if x^0 is a saddle then case 3 or case 4 occurs. We will still apply these names to x^0 even though the vector field arising from (2) is time dependent. Thus if case 1 holds we call x^0 a repulsive node with respect to the parameter τ , designating it by NR_τ , if case 2 holds, an attractive node NA_τ , if case 3, a repulsive saddle SR_τ , if case 4, an attractive saddle SA_τ .

These boxes of segments without contact, formed near critical points x^0 , will make up part of the final curve J. The boxes cannot follow each other in any random order along $z = 0$, for by considering the sign of Q_0 on the sides of successive boxes it appears that the only possible successive pairs are NR_τ, NA_τ ; NR_τ, SR_τ ; NA_τ, SA_τ ; SR_τ, SA_τ ; and

SR_τ , SA_τ is excluded by condition (c) of the theorem.

These boxes lie in the domain covered by the original x, y coordinates where they appear as flat ended wedges open toward infinity (Fig. 3).

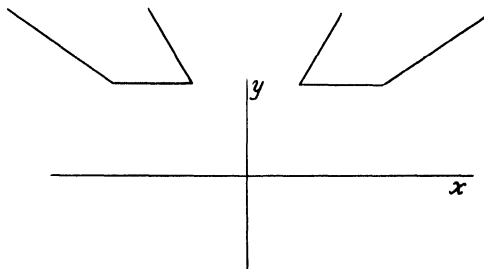


FIGURE 3

The ends are lines $y = \text{const.}$, the sides are lines $\frac{x}{y} = \text{const.}$ If $(1, 0, 0)$ is a critical point we simply adopt $(1, y, z)$ coordinates and proceed in exactly the same way to form a suitable box.

The next step is to join these boxes together using more segments without contact.

In the region covered by $(x, 1, z)$ consider a closed interval $[ab]$ on $z = 0$ which is free of critical points. Then $Q(x^0) \neq 0$ on $[ab]$. So if δ' is chosen sufficiently small $(P_0 + P_1 z + \dots + (P_n + E_2)z^n)$ is bounded, and $(Q_0 + Q_1 z + \dots + (Q_n + E_1 - xE_2)z^n)$ is bounded and bounded away from zero in the box B' , $0 \leq z \leq \delta'$, $a \leq x \leq b$, and for all t . From (8) then for points in B'

$$(9) \quad -Mz < \frac{dz}{dx} < -M'z$$

for some constants M and M' . Therefore the portions of the curves $z = z(a)e^{-M(x-a)}$, $z = z(a)e^{-M'(x-a)}$ lying in B' are arcs without contact, one having slope $\frac{dz}{dx}$ greater than the slope of solutions to (8), the other having smaller slope (Fig. 4). Since $z(b) \rightarrow 0$ as $z(a) \rightarrow 0$, it follows immediately that one may connect any arbitrarily small perpendicular at a to an arbitrarily small perpendicular at b using arcs without contact. One arc will be crossed only by parametrized solutions which, as

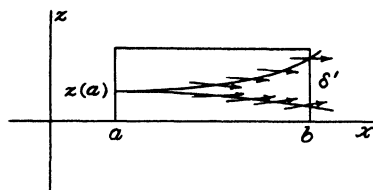


FIGURE 4

τ increases, move into the sector under it, while the other will be crossed only by solutions moving outward (Fig. 4).

Although in this construction we have assumed that a and b were in the region covered by coordinates $(x, 1, z)$, the same construction applies to any a and b on $z = 0$ whose perpendiculars form a box not containing a critical point. It is only necessary to divide $[ab]$ into subintervals lying entirely in one coordinate system or the other, and connect the separate arcs obtained for each.

We now return to x, y coordinates and the parameter t and put together boxes and arcs to form J .

For each singularity at infinity we have a flat ended wedge. We will take the solution curves as parametrized with t , thus when we refer to an attractive node, we will mean with respect to t (Fig. 5).

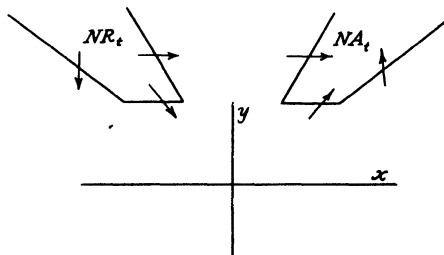


FIGURE 5

Successive boxes at infinity give rise to wedges which succeed each other angularly. It is easily seen that the only possible successive wedges are NR_t , NA_t ; NR_t , SR_t ; NA_t , SA_t ; SR_t , SA_t . The last is excluded by condition (c).

We construct J in two steps. First, we connect any pair of neighboring wedges by an arc without contact. As the sides of the wedges are perpendicular to $z = 0$, and the sides of successive wedges form the ends of a box without critical points, the previous construction applies.

If an NR_t is being connected to an NA_t , choose the arc without contact so that solutions cross it outward with increasing t , i.e., toward the line at infinity (Fig. 6).

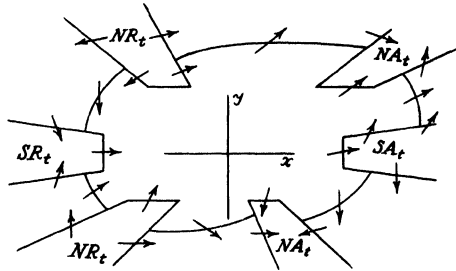


FIGURE 6

If an NR_t is being connected to an SR_t , let the connecting arc be chosen so that the solutions cross it moving away from the line at infinity. If an NA_t is connected to an SA_t let the arc be crossed toward the line at infinity (Fig. 6).

Now the wedges with their connecting arcs bound a 2-cell. With one modification the boundary of this 2-cell will be the curve J .

We remember that in constructing the boxes at critical points on $z = 0$, the top of the box was a segment of $z = a$, where a could be taken arbitrarily small. We now choose a new a' and a new segment $z = a'$ so near $z = 0$ that the segment cuts the sides of the box below the intersections with the connecting arcs (Fig. 7). This change will be made only on saddle boxes. The new boundary of the central 2-cell is the curve J .

We will now show that a solution through a point p on J never returns to p .

First, suppose p is an attractive nodal wedge (Fig. 8). Then

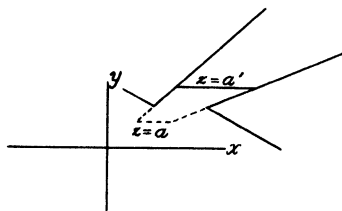


FIGURE 7

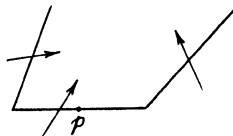


FIGURE 8

the path through p enters the wedge and can never leave it, hence cannot return to p . Similarly, if the node is repulsive, the path leaving the wedge can never re-enter it, so cannot return to p .

If now p is on an arc connecting two nodal wedges as in Fig. 9,

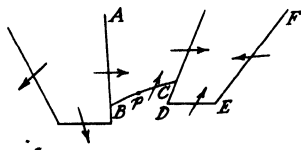


FIGURE 9

then one wedge combined with the box between the nodal wedges forms a region which is only entered and never left by solutions. See Fig. 9. — A B C D E F Thus the solution through p cannot return to p .

Finally, suppose p is on a saddle wedge, or on an arc

connecting a saddle wedge with a nodal wedge. Since there are no two consecutive saddle wedges, every saddle is flanked with nodal wedges (Fig. 10).

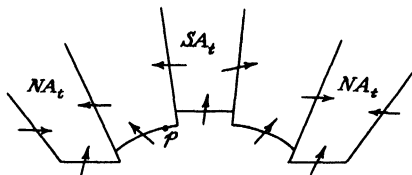


FIGURE 10

Combining all three wedges and the two enclosed boxes we form another region only entered by solutions. Thus p again has the desired property, and this last argument clearly holds whether the wedges involved are attractive or repulsive.

As we have now covered all possible locations of p we conclude that J has the desired property.

It is also clear from the mode of construction that one may select J as far out in the plane as desired. In particular, for reasons which will appear later, we will take J so large that it contains all the finite critical points of (1) and of

$$\frac{dx}{dt} = X(x, y) + E_1(0), \quad \frac{dy}{dt} = Y(x, y) + E_2(0).$$

We now consider

$$(10) \quad \begin{aligned} \frac{dx}{dt} &= \left\{ X(x, y) + E_1(t) \right\} F(x, y) \\ \frac{dy}{dt} &= \left\{ Y(x, y) + E_2(t) \right\} F(x, y), \end{aligned}$$

where $F(x, y)$ has continuous derivatives, is always positive, and is identically 1 in some circle containing J , while as $R \rightarrow \infty$, $F(x, y) \rightarrow 0$ so rapidly that the right hand sides in (10) are bounded in the entire plane. An appropriate F can easily be found.

Now the curve J has its special property for the solutions of (10) as well as for those of (2). For the velocity vectors of (10) have the same direction as those of (2) and so point into or out of the same wedges. Thus, if a solution to (10) is at p on J , it never returns there at a later time.

Also equation (10) gives us a mapping of the plane into itself. For, let $u(t, u^0)$ be the position at time t of that solution to (10) which is at u^0 for $t = 0$. Then define the mapping φ_t by $\varphi_t u^0 = u(t, u^0)$. Because of the boundedness of (10) φ_t is defined for all t .

If we assign to the point u^0 the vector $\varphi_t u^0 - u^0$ we obtain a continuous vector distribution in the plane which also varies continuously with t . We will now compute the index of this distribution on J .

First of all, the index is the same for all $t > 0$, for the index is an integer and the vector distribution varies continuously with t . In this situation the index can only change with t if for some point u^0 on J and some t , the vector $\varphi_t u^0 - u^0$ vanishes. But this means $u^0 = u(t, u^0)$ which cannot occur on J as on J no solution may return to its starting point. Therefore, the index is the same for all t .

As we have seen before, J may be taken so large that the velocity vectors $v_1(x, y) = (X(x, y), Y(x, y))$, and $v_2(x, y) = (X(x, y) + E_1(0), Y(x, y) + E_2(0))$ vanish only inside J . Therefore, both of these are vector distributions with a well defined index on J . We will see that the two indices computed with respect to these vector distributions on J are the same, and in fact that they are both equal to the index computed with respect to the vectors $\varphi_t u^0 - u^0$.

The first equality results from the fact that addition of $E_1(0)$ and $E_2(0)$ to X and Y does not affect the nature of the critical points at infinity; they remain nodes or saddles as before as is evident from (8). As the sum of all critical point indices must be 1, and in both cases all the finite critical points are in J , we have for both vector distributions

$$(11) \quad \text{index}(J) = 1 - \sum_1 \text{index}(p_1),$$

the p_1 being the critical points on $z = 0$. So the two indices are equal.

Now at any point $u^0 = (x^0, y^0)$ on J we may write

$$x(t, x^0, y^0) = x(0, x^0, y^0) + \frac{dx}{dt}(0, x^0, y^0)t + \frac{d^2x}{dt^2}(t', x^0, y^0)\frac{t^2}{2}$$

$$y(t, x^0, y^0) = y(0, x^0, y^0) + \frac{dy}{dt}(0, x^0, y^0)t + \frac{d^2y}{dt^2}(t'', x^0, y^0)\frac{t^2}{2}$$

where $x(t, x^0, y^0)$, $y(t, x^0, y^0)$ are the components of $u(t, u^0)$ and t' and t'' are less than t . Because the right hand sides in (10) are bounded in the entire plane, $u(t, u^0)$ lies inside some finite region R , for all u^0 on J and all t less than any fixed t_1 . But in

$R \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ are bounded, so as $t \rightarrow 0$,

$$\frac{u(t, u^0) - u^0}{t} \rightarrow v_2(x^0, y^0)$$

uniformly on J .

Therefore, for some t_0 sufficiently small, the angle between $v_2(x^0, y^0)$ and $\varphi_{t_0} u^0 - u^0$ is less than ϵ throughout J , and the index of the vector distribution $\varphi_{t_0} u^0 - u^0$ is the same as the index of v_2 , namely $1 - \sum_1 \text{index}(p_1)$. By condition (b) of the theorem, $1 - \sum_1 \text{index}(p_1) \neq 0$.

Since the index is the same for all φ_t , it is $\neq 0$ for φ_T , T being the period of the E_1 . Hence there is a fixed point under the map φ_T . That is, there exists at least one point u' inside J for which $\varphi_T u' - u' = 0$. The solution through u' must return to u' after T seconds, and so is evidently a periodic solution to (10).

However, because of its periodicity, this solution must lie inside J not only for $t = 0, t = T$, and so forth, but for all t . For if at any time it cut the curve J , say at p , then it could never return to p for later t , and this contradicts its periodicity. Therefore, it must be inside J for all t .

As $F(x, y)$ is identically 1 inside J , the solution through u' is in fact a periodic solution to (2). This establishes the theorem.

§4. It is not really necessary for the $E_1(t)$ to be purely time dependent. If the E_1 are replaced by $E_1'(x, y, t), E_2'(x, y, t)$, where the E_1' are polynomials of degree less than n in x and y with coefficients periodic in t , the proof of the theorem goes through unchanged.

PART II: CRITICAL POINTS AT INFINITY OF A CLASS OF SECOND ORDER EQUATIONS

§1. The second order equation

$$(12) \quad \frac{d^2x}{dt^2} + f(x) \frac{dx}{dt} + g(x) = 0$$

gives rise to the phase plane system

$$(13) \quad \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -f(x)y - g(x).$$

In this section the critical points at infinity of (13) will be completely analyzed in the case where $f(x)$ is a polynomial

$$\sum_{s=0}^m a_s x^s \quad \text{of degree } m,$$

and $g(x)$ a polynomial

$$\sum_{s=0}^n b_s x^s \quad \text{of degree } n,$$

and $m \geq n > 0$. The nature of these critical points will be shown to be completely determined by the leading coefficients a_m and b_n of the polynomials, and by the parity of m and n . From these facts geometric results will be deduced about the behavior in the large of trajectories.

As in Part I, consider in place of (13)

$$(14) \quad \frac{d\left(\frac{x}{z}\right)}{dt} = \frac{y}{z}, \quad \frac{d\left(\frac{y}{z}\right)}{dt} = -f\left(\frac{x}{z}\right)\frac{y}{z} - g\left(\frac{x}{z}\right)$$

and proceeding as before, with $\frac{dt}{d\tau} = z^m$ for the region covered by $(x, 1, z)$ we obtain

$$(15) \quad \begin{cases} \frac{dz}{d\tau} = z \left[f^H(x, z) + z^{m-n+1} g^H(x, z) \right] \\ \frac{dx}{d\tau} = x \left[f^H(x, z) + z^{m-n+1} g^H(x, z) \right] + z^m \end{cases}$$

and for the region with coordinates $(1, y, z)$, using $\frac{dt}{d\sigma} = z^m$,

$$(16) \quad \begin{cases} \frac{dz}{d\sigma} = -z^{m+1}y \\ \frac{dy}{d\sigma} = -f^H(1, z)y - z^{m-n+1}g^H(1, z) - z^m y^2 \end{cases}$$

where $f^H(x, z)$, $g^H(x, z)$ denote f and g made homogeneous with z .

It is evident from (15) and (16) that $z = 0$ is a trajectory connecting critical points, and that the only critical points on $z = 0$ are $(0, 1, 0)$ and $(1, 0, 0)$. Before proceeding with an analysis of these critical points, we will mention some lemmas on trajectories near a singularity.

§2. Let C be a trajectory of an analytic system approaching an isolated critical point P . At a regular point q on C any sufficiently

small circular neighborhood $N(q)$ is divided by C into cells lying on opposite sides of C . These cells will be called half-neighborhoods, $N_{1/2}(q)$. We will often refer to a curve C together with a side A of C along which all the half-neighborhoods are to be chosen. The choice of side will be indicated by a subscript A or B .

In terms of half-neighborhoods we will define what is meant by a prolongation.

Let C_A approach P with a definite limiting direction. Then any sufficiently small circle S around P will be cut only once by C , and the intersection point q will not be a tangency.

We will say that C_A has a prolongation with respect to S if there is another curve C'_A , cutting S at q' and approaching P such that for every $N_{1/2}(q')$ there is an $N_{1/2}(q)$ such that any trajectory through $N_{1/2}(q)$ in S cuts $N_{1/2}(q')$, and lies entirely inside S between the two neighborhoods. (Fig. 11)

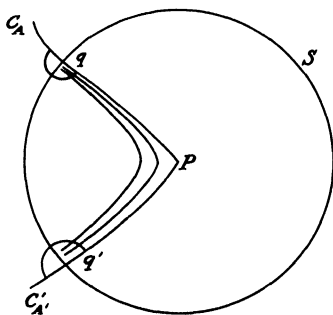


FIGURE 11

It is further stipulated that no trajectory shall tend to P between C and C' . Here between means in the cell qPq' that intersects the half-neighborhoods.

We are now in a position to state a lemma which is an immediate consequence of theorems of Bendixson [1].

LEMMA 1. Let C_A tend toward P with a definite limiting direction. Then for any sufficiently small

circle S around P cut by C at q either

- (a) C_A has a prolongation with respect to S , or
- (b) there is a half-neighborhood $N_{1/2}(q)$, such that every trajectory through $N_{1/2}(q)$ approaches P lying always inside S . (Fig. 12)

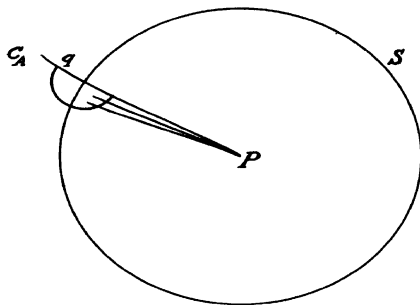


FIGURE 12

Provided we have taken S sufficiently small, it is clear that if case (a) holds, it holds also when C is referred to even smaller circles S' . Therefore, the same remark must apply when case (b) holds. So, although this is not always the case for non-analytic systems, we can use here the phrase " C_A has a prolongation" meaning (a) holds for all sufficiently small S , or " C_A finishes" meaning (b) holds, without specifying a particular circle S .

Another useful lemma is

LEMMA 2. Let S be a circle containing the isolated critical point P , and which contains no closed trajectory, no critical point other than P , and is not itself a trajectory. Then either

- (a) there is a curve tending to P which has a prolongation, or

- (b) there is a neighborhood N of P such that if $q \in N$, then the trajectory through q tends to P in one direction or the other, and without touching S .

PROOF: In this proof we will use half-trajectory or half-characteristic to indicate a trajectory pursued from a starting point in one direction or the other. By positive (negative) half-trajectory is meant the path traced out by the starting point as parameter values increase (decrease).

Now suppose (b) does not hold. Then there exists a sequence of points $p_n \rightarrow P$ such that both the half-trajectories through p_n either

- (1) fail to tend to P , or
- (2) cut S .

In fact we may assume (2) holds, for if a trajectory C fails to tend to P it must tend to another singular point, to a closed trajectory, or to a curve with prolongations, Bendixson [1]. Under our hypotheses this implies it cuts S , or else tends in S to a curve with a prolongation, which already proves (a), therefore we may assume (2) holds.

With the trajectory C_n through P_n associate the points a_n and b_n , the first intersections with S of the positive and negative half-trajectories through p_n . Then $a_n p_n b_n$ divides the interior of S into two 2-cells. We will call the one containing P the exterior, and the other the interior. The two arcs $a_n b_n$ of S will be called exterior and interior arcs.

Because of analyticity, S has only a finite number of tangencies with the vector field, these occur at points T_m . Consider all the C_n having T_m on their interior arc. For some m there is an infinity C_n of these. As the C_n cannot cross, they may be arranged in order, with each containing its predecessor in its interior 2-cell (Fig. 13).

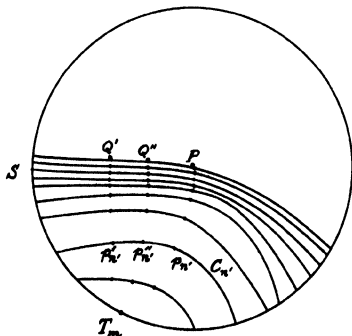


FIGURE 13

Denoting by $d(C_{n'}, P)$ the distance from $C_{n'}$ to P , it is clear that every $C_{n'}$ has points at any distance d' from P , $d(C_{n'}, P) < d' < R$, where R is the radius of S . Since $d(C_{n'}, P) \rightarrow 0$ as $n' \rightarrow \infty$, we may always find a sequence of points $p_{n'}^1, p_{n'}^2 \in C_{n'}$, such that the $p_{n'}^1$ have a limit point Q' at a distance d' from P , $0 < d' < R$.

Choose such a sequence with a limit point Q' at distance d' from P . It will appear that the characteristic C' through Q' must tend to P in one direction or the other, for if it merely connects two points of S we have the following consequences. All $C_{n'}$ lie in the exterior of C' for n' greater than some N'_0 , for these $C_{n'}$ will have points at a distance from $P < d(C', P)$, and therefore will have one point in the exterior of C' , and so lie entirely in the exterior. Now choose a new sequence $p_{n'}^{11}$ on the $C_{n'}$ with limit Q'' at distance d'' from P , $0 < d'' < d'$. The characteristic C'' through Q'' lies in the exterior of C' , and the $C_{n'}$, $n' > N'_0$ lie in the exterior of C'' , both by the same argument as before. Thus the $p_{n'}^1$ lie in the exterior of C'' , $n' > N'_0$, and so Q' cannot be their limit point, a contradiction. Hence C' tends to P in one direction or the other.

We will show C' to have a prolongation. C' approaches P with a definite limiting direction. (A characteristic either approaches a critical point with a definite limiting direction or else its angle increases without limit, this latter is impossible as C' would cross some $C_{n'}$.) Thus there is a sufficiently small circle S^* such that C' cuts S^* at Q^* and then always lies in the interior of S^* .

As C' is the characteristic through Q' , by continuity there are characteristics $C_{n'}$ arbitrarily close to Q^* on one side of C' , these $C_{n'}$ of course, do not tend to P . Therefore, the conditions for case (b), Lemma 1, are not fulfillable, and case (a), Lemma 1, must hold. There is a prolongation. This establishes Lemma 2.

§3. We will now analyze the critical points at infinity, starting with the point $(0, 1, 0)$, or the $x = 0, z = 0$ of equation (15).

Let $\theta = \arctan(-\frac{z}{x})$ and $R^2 = x^2 + z^2$, then from (15)

$$(17) \quad \therefore \frac{d\theta}{d\tau} = \frac{z^{m+1}}{R^2}$$

so that if the axes are taken as in Fig. 14, trajectories below the line $z = 0$ move counterclockwise, those above counterclockwise or clockwise accordingly as m is odd or even. It is also clear that $z = 0, x > 0$, and $z = 0, x < 0$ are trajectories tending to $(0, 0)$ with directions of

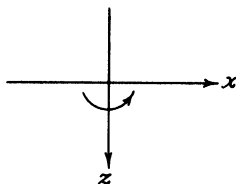


FIGURE 14

motion given by

$$(18) \quad \left(\frac{dx}{d\tau} \right)_{z=0} = a_m x^{m+1}.$$

Since these are trajectories tending to $P = (0, 0)$ with a definite limiting direction, it follows by Bendixson [1] that any characteristic approaching P will have a limiting direction for its tangent. The possible limiting directions are not arbitrary but are those that satisfy

$$(19) \quad 0 = z \left(\frac{dx}{d\tau} \right)_L - x \left(\frac{dz}{d\tau} \right)_L = z^{m+1}$$

where the subscript L indicates that only the terms of degree

$$L = \min \left(\text{degree } \frac{dx}{d\tau}, \text{degree } \frac{dz}{d\tau} \right)$$

are taken. So any trajectory approaching P must approach tangency with $z = 0$, either from above or below.

It will also be useful to distinguish the two sides of a half-trajectory $z = 0$ near $(0, 0)$. A side will be called positive if it borders on a quadrant in which the sign of $\frac{d\theta}{d\tau}$ is such that the acute angle φ between $z = 0$ and a radius vector to a point p in the quadrant constantly decreases in absolute value. It will be called negative if $|\varphi|$ increases. As $\frac{d\theta}{d\tau}$ has constant sign throughout each quadrant, there is no ambiguity.

We have at once this simple lemma.

LEMMA 3. Let C be a positive (negative) half-trajectory tending to $(0, 0)$. Then C approaches tangency with one of the half-trajectories $z = 0$ on a positive (negative) side.

PROOF: As C , a positive half-trajectory, approaches P , it must approach tangency with one of the half-characteristics $z = 0$. Since C cannot reach or cross $z = 0$ it will always lie in the same quadrant for $\tau > \tau_0$. At time τ_0 , $|\varphi| = b > 0$, while for τ , sufficiently large $|\varphi| < b$. Hence for some τ' , $\tau_0 < \tau' < \tau_2$, $\frac{d|\varphi|}{dt} < 0$, so the quadrant is such that the side approached is positive. A similar proof applies in the negative case.

We will now proceed with a case by case analysis of the singularity at $P = (0, 0)$.

Case (1a). m even, $a_m > 0$.

Equation (18) shows that $z = 0$ consists of two negative half-trajectories approaching P : C' , $z = 0$ $x > 0$, and C'' , $z = 0$ $x < 0$. If we adopt A and B for their lower and upper edges, we see from equation (17) that C'_A and C'_B are positive sides, and C''_A and C''_B negative (Fig. 15).

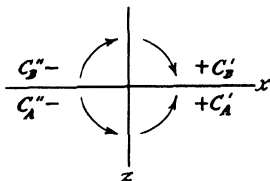


FIGURE 15

We next notice that there are in fact no positive half-trajectories approaching P . For if we consider the curve α

$$(20) \quad f^H(x, z) + z^{m+1-n} g^H(x, z) = 0$$

the tangents to the various branches of the curve at $(0, 0)$ are the directions satisfying

$$(21) \quad f^H(x, z) = 0$$

for $f^H(x, z)$ is homogeneous of degree m , while $z^{m+1-n} g^H(x, z)$ is of degree $m + 1$. Since z does not divide $f^H(x, z)$, $z = 0$ is not a tangent to any of the curves, so there will be an ϵ_1 , and an ϵ_2 such that the region D , $0 < |x| < \epsilon_1$, $|\frac{z}{x}| < \epsilon_2$ is not entered by any of the branches of α (Fig. 16).

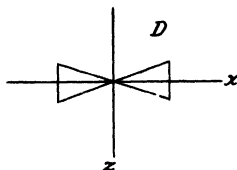


FIGURE 16

In each of the two open triangles of D $f^H(x, z) + z^{m+1-n}g^H(x, z)$ cannot change sign, and therefore has the same sign as for $z = 0$, that is sign $(a_m x^m)$. Hence from (15) in these triangles

$$(22) \quad \text{sign } \frac{dz}{d\tau} = \text{sign } (za_m x^m) = \text{sign } (z).$$

Since any positive half-trajectories must approach tangency to $z = 0$, they must lie inside the triangles for all τ greater than some τ_0 . If they are not the trajectories C' or C'' themselves, then $|z(\tau_0)| > 0$, and $|z(\tau)| \rightarrow 0$ as $\tau \rightarrow \infty$. But (22) states that for these trajectories in D , $\frac{d|z|}{d\tau} > 0$, a contradiction. As C' and C'' themselves are negative half-trajectories approaching P , no positive half-trajectories tend to P .

Since any prolongation obviously involves both a positive and a negative trajectory, there are no curves through P with a prolongation. Hence, using Lemma 2, all curves inside a sufficiently small circle S' tend to P as $\tau \rightarrow \infty$ or as $\tau \rightarrow -\infty$ without leaving a second small circle S . By the above we know that in fact curves can tend to P only as $\tau \rightarrow -\infty$, therefore they are all negative trajectories and by Lemma 3 can tend only to the negative sides of C' and C'' . Since $z = 0$ cannot be crossed, we see that for $z > 0$ the trajectories tend with decreasing τ toward tangency with C_A'' , and for $z < 0$ toward tangency with C_B'' (Fig. 17).

Case (1a)

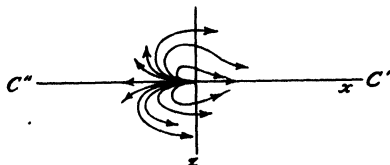


FIGURE 17

So for m even, $a_m > 0$ P is a (non-simple) unstable node.

Case (1b). m even, $a_m < 0$.

From equation (18) we see that C' and C'' are now positive half-trajectories approaching P . As (17) shows, $\frac{d\theta}{d\tau}$ is unaffected by the sign of a_m so that the sides C_A^I , C_B^I , etc. are positive or negative as before. The argument immediately preceding equation (22) applies again only (22) now becomes

$$(23) \quad \text{sign } \frac{dz}{d\tau} = \text{sign} (za_m x^m) = - \text{sign} (z)$$

and the argument which previously showed that no positive half-trajectory approaches P , now shows that no negative ones do. Again there are no prolongations, and now inside some small circle all paths are positive half-trajectories and tend to P . Only now they must tend to positive sides, and so tend to C_A^I and C_B^I (Fig. 18).

Case (1b)

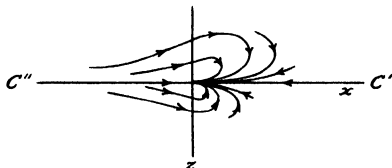


FIGURE 18

So P is a (non-simple) stable node.

Case (2a). m odd, $a_m > 0$.

Equation (18) shows that C' is a negative, C'' a positive half-trajectory. Equation (17) shows $\frac{d\theta}{d\tau}$ to be always non-negative, so C_A^I and C_B^I are positive sides, and C_B^I and C_A^I negative. Also using once more the argument based on $\frac{dz}{d\tau}$ we find

$$(24) \quad \text{sign } \frac{dz}{d\tau} = \text{sign} (za_m x^m) = \text{sign} (xz)$$

for the sign of $\frac{dz}{d\tau}$ inside the small triangles. This implies that negative trajectories cannot approach tangency to C_A^I and C_B^I , and that positive ones cannot approach C_A^I and C_B^I . As C_A^I is a negative side, this means that no characteristics at all approach tangency to it, and as C_A^I is positive, no characteristics approach it. So no characteristics other than C' and C'' enter P from the lower half plane. Therefore C_A^I

must have a prolongation, for if it finished at P , by Lemma 1 there would be a half-neighborhood on the A side, through which passed a characteristic tending to P , and clearly tending to P from the lower half plane. As this is impossible, case (a) of the lemma must hold, so C_A'' has a prolongation. The prolongation cannot lie in the upper half plane, for then characteristics passing near a point on C'' and a point on its prolongation, would cut C' . As there are no characteristics other than C' and C'' tending to P from the lower half plane, the prolongation can only be C' .

Thus the lower half plane near P is a saddle sector as shown in Fig. 19.

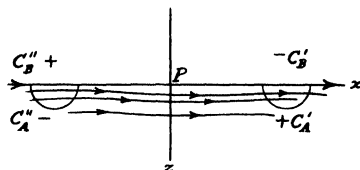


FIGURE 19

We will next show that there are no curves with prolongations tending to P from the upper half plane. For suppose there existed a C_1 , a positive semi-characteristic with a prolongation on some side A . C_1 must tend to tangency with C_B'' , and its prolongation C_2 , being negative, must tend to C_B' . By Lemma 1 there exists a trajectory T passing arbitrarily close to points of C_1 and then proceeding with increasing τ arbitrarily close to points of C_2 while lying always inside a small circle around P . Now when near C_1 , T is near C_B'' and hence has a θ coordinate greater than say $\frac{3}{4}\pi$. At a later time, near C_2 , it is near C_B' and therefore has a θ coordinate less than $\frac{1}{4}\pi$. So θ has decreased with time. However, in the region in which (17) applies, and in which T always lies during the time under discussion, $\frac{d\theta}{d\tau}$ is non-negative, a contradiction. Hence such a prolongation is impossible.

Denoting points $z \leq 0$ by U , and repeating the arguments of Lemma 2 but restricting ourselves to points in U , we conclude that as there is no prolongation, given any circle S , there is another circle S' such that if p belongs to $S' \cap U$, then the positive or the negative half-trajectory through p tends to P inside $S \cap U$. From Lemma 1 it follows that as there are no prolongations, the set of points inside S' whose positive half-trajectories tend to P in S is open, and of course the same applies to the set of points whose negative trajectories tend to

P. As the union of these two sets is the interior of the connected set $S' \cap U$, their intersection is not void, so there is a point p whose positive and negative half-trajectories tend to P . Taking a new semi-circle S_1 through p , we may find a smaller semi-circle S_2 which is divided into three sectors by the positive and negative half-trajectories $C^+(p)$ and $C^-(p)$ through p (Fig. 20). S_2 may also be taken to have the property that if q is in S_2 , either $C^+(q)$ or $C^-(q)$ tends to P inside S_1 .

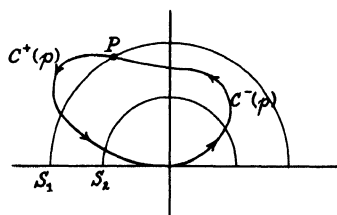


FIGURE 20

If we consider q in the central sector, $C^+(q)$ and $C^-(q)$ lie always inside $C(p)$, so both must tend to P . If we consider q in either of the other sectors, then either $C^+(q)$ or $C^-(q)$ tends to P inside S_1 and outside $C(p)$. In fact as C_B'' is a positive, and C_B' a negative side, if q is in a sector bordering C_B'' , then $C^+(q)$ tends to P in S_1 and $C^-(q)$ does not, and if q is in a sector bordering C_B' , $C^-(q)$ tends to P in S_1 , and $C^+(q)$ does not.

Thus the configuration at P is as in Fig. 21.

Case (2a)

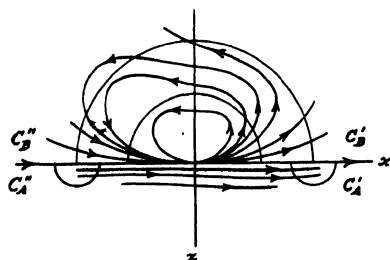


FIGURE 21

Case (2b). m odd, $a_m < 0$.

Equation (18) shows C' to be a positive, C'' to be a negative half-characteristic. The sides of the characteristics are positive and negative as before, as $\frac{d\theta}{d\tau}$ is unchanged. With obvious modifications all arguments go through as in case (2a) and the singular point again has one prolongation and one closed nodal region, although with positions reversed as indicated in Fig. 22.

Case (2b)

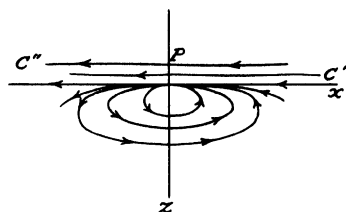


FIGURE 22

We have thus determined the nature of the singularity $x = 0$, $z = 0$, which is the only singularity on the line at infinity for $y \neq 0$. The nature of this singularity has turned out to be completely determined by the parity of the degree m of $f(x)$, and the sign of its leading coefficient a_m .

We next turn to the critical point $Q = (1, 0, 0)$. Using y, z coordinates and equation (16) our differential equations are of the form

$$(25) \quad \frac{dz}{d\sigma} = Z(y, z), \quad \frac{dy}{d\sigma} = cy + dz + Y(y, z),$$

where Y and Z consist of terms of degree ≥ 2 . In (16) $c = -a_m \neq 0$, while $d = -b_n$ if $m = n$, and is zero otherwise.

The possible directions of approach are given by

$$(26) \quad 0 = z \left(\frac{dy}{d\sigma} \right)_L - y \left(\frac{dz}{d\sigma} \right)_L = z(cy + dz),$$

and are $z = 0$ and the line $L: cy + dz = 0$. Of course $z = 0, y > 0$, and $z = 0, y < 0$, are actually trajectories of (16). Now for equations of the form (25) we may use results due to Bendixson. First there are two and only two characteristics tending to Q tangent to $z = 0$. In (16) then these are the two trajectories $z = 0$. Secondly, considering the

curves tending to Q tangent to L , we may divide them into two groups, those near Q with $z > 0$, and those near Q with $z < 0$.

If the index of Q is 1, both of these will form nodes, i.e., there will be an infinity of trajectories tending to Q in each group, and if C tends to Q , so do all trajectories through a neighborhood of C .

Index 1

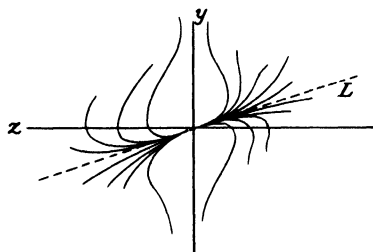


FIGURE 23

If the index is -1 , there will be a unique curve in each group tending to Q , and it will have as prolongation one of the trajectories tending to $z = 0$. That is, we will have a saddle.

Index -1

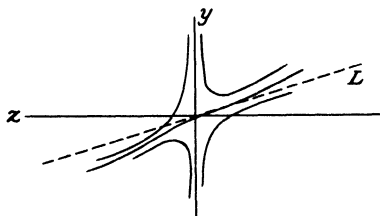


FIGURE 24

If the index is zero we will have one nodal and one saddle side. In this last case there are two possible arrangements depending on which side is nodal and which saddle. However, aside from this we see, following Bendixson, that the nature of the singularity Q is largely determined by the index. Now the index of Q is given by

$$(27) \quad \text{index } (Q) + \text{index } (P) + \sum_1 \text{index } (p_1) = 1$$

Index 0

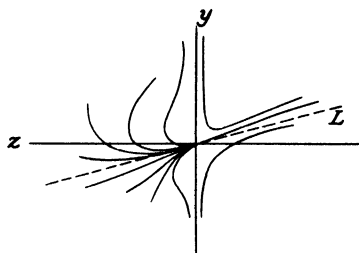


FIGURE 25

where the p_1 are the critical points not on the line at infinity. The index of P may be computed from a formula of Bendixson

$$2(-\text{index}(P) + 1) = c - n_P,$$

c being the number of curves having prolongations, n_P the number of closed nodal regions. Referring to cases (1a), (1b), (2a), (2b) we find always

$$(28) \quad \text{index}(P) = 1.$$

The p_1 are, of course, the points where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vanish simultaneously. Hence, referring to (13), all the p_1 must lie on the x -axis and have x -coordinates a_1 such that $g(a_1) = 0$. If we assume for the moment that $g(x)$ has only simple roots, a routine calculation shows that the points $(a_1, 0)$ are simple critical points, and that they have index $+1$ if $g'(a_1) > 0$, and index -1 if $g'(a_1) < 0$.

At successive intersections of the curve $g(x) = 0$ with the x -axis, $g'(x)$ will change its sign, hence at successive critical points the index alternates from $+1$ to -1 . If $g(x)$ is of even degree, i.e., n even, there will be equal numbers of both types so (27) gives

$$(29) \quad n \text{ even, } \text{ind}(Q) = 0,$$

while if $g(x)$ is of odd degree there are two cases

$$(30) \quad \begin{array}{ll} n \text{ odd, } b_m < 0 & \text{ind}(Q) = 1 \\ n \text{ odd, } b_m > 0 & \text{ind}(Q) = -1. \end{array}$$

Hence the index of Q is determined by $g(x)$ alone.

Equations (29) and (30) hold even when g has multiple roots, for these multiplicities can be removed by small changes in g which, as they change the vector field only slightly, cannot affect the index sum.

In the cases covered by equation (30) the index completely determines the nature of Q except for the direction in which the trajectories are to be pursued. This last is directly determined from (16) and depends on the sign of a_m .

In the case n even, it remains to determine which side of $z = 0$ at Q is a saddle and which side a node. Remembering that $z = 0$ is a trajectory, we observe that if the side $z > 0$ is a saddle side the vector field should rotate through $-\pi$ along any curve connecting a point on the positive y -axis with a point on the negative y -axis through the half-plane $z > 0$. Similarly, if the side is to be nodal, the rotation should be $+\pi$. We will actually obtain this rotation along a simple path.

Let the path consist of the sides formed in the half-plane $z > 0$ by the lines $y = \pm a$, $z = \epsilon > 0$ (Fig. 26).

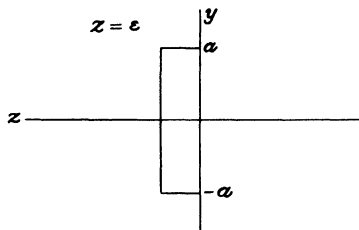


FIGURE 26

From (16) it appears that at $(a, 0)$ and $(-a, 0)$ the vectors $(\frac{dy}{d\sigma}, \frac{dz}{d\sigma})$ are vertical and oppositely directed. On $y = a$ $z > 0$, and on $z = \epsilon$ $y > 0$, we have $\frac{dz}{d\sigma} < 0$, so the vectors point toward the right half-plane $z < 0$. At $y = 0$, $\frac{dy}{d\sigma} = -\epsilon^{m-n+1} g^H(1, \epsilon)$ which for ϵ sufficiently small has the sign of $-b_n$. So at $z = \epsilon$, $y = 0$, the vector points toward the half-plane $-yb_n > 0$. On $z = \epsilon$ $y < 0$, and on $y = -a$ $z > 0$, $\frac{dz}{d\sigma} > 0$ and the vector points into the left half-plane, $z > 0$. This gives a rotation of $-\pi$ if $b_n > 0$, and $+\pi$ if $b_n < 0$.

So if n is even and $b_n > 0$, $z > 0$ is a saddle side (Fig. 27), and if $b_n < 0$, $z > 0$ is a nodal side (Fig. 28). We find that that the form of the trajectories near Q , that is, whether they are saddle or nodal sectors, is determined by the parity of n and the sign of b_n . So

$f(x)$ controls P and $g(x)$ controls Q .

n even, $b_n > 0$

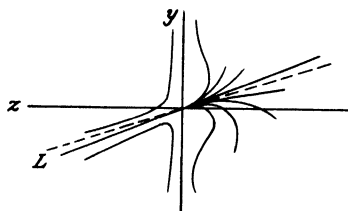


FIGURE 27

n even, $b_n < 0$

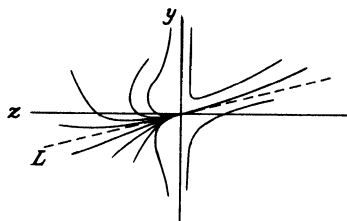
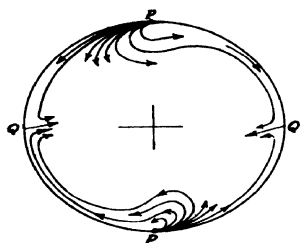
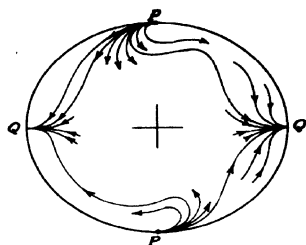
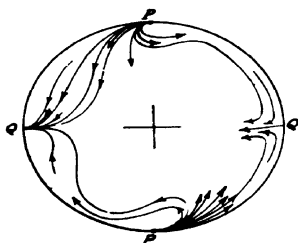
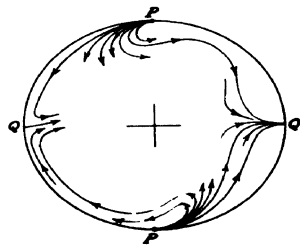
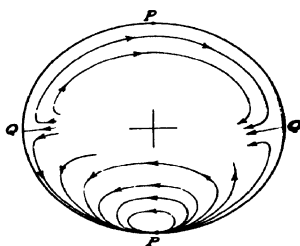
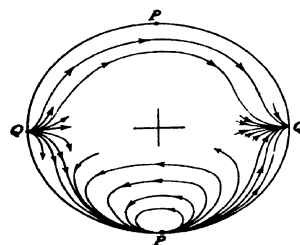
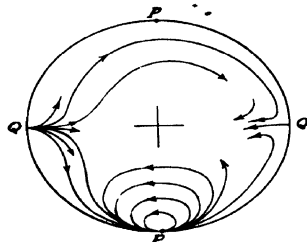
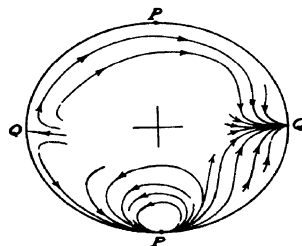


FIGURE 28

We can now give the various possible modes of behavior in the projective plane near the line at infinity, that is, the asymptotic behavior of solutions to (13) that become unbounded either for increasing or decreasing t (Fig. 29). In these figures the trajectories in the finite plane are given parametrized by t . The projective plane is represented as a circle and its interior, with opposite points on the circumference identified. The circumference itself consists of the trajectories $z = 0$ and the critical points P and Q .

Next consider $a_m < 0$. For m even the following changes should be made. First the direction of the parametrization is reversed, and secondly, the curves approaching P are now tangent at the other side. For example, from 1 comes 1A (Fig. 30). Similarly from 2, 3, 4 come 2A, 3A, 4A. For m odd the role of the upper and lower sides of P is interchanged. Thus from 5, 6, 7, 8 come 5A, 6A, 7A, 8A (Fig. 31).

The information obtained about behavior at infinity can be used to deduce results about the behavior in the large of the trajectories. For

Case 1. n odd, $b_n > 0$ Case 2. n odd, $b_n < 0$ Case 3. n even, $b_n > 0$ Case 4. n even, $b_n < 0$ FIGURE 29a. m even, $a_m > 0$ Case 5. n odd, $b_n > 0$ Case 6. n odd, $b_n < 0$ Case 7. n even, $b_n > 0$ Case 8. n even, $b_n < 0$ FIGURE 29b m odd, $a_m > 0$

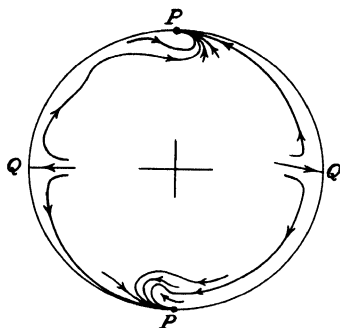


FIGURE 30

example, a glance at Fig. 29 shows that the only equations dissipative for large displacements are those of case (1), n even, m odd, a_m and b_n positive. Furthermore, if we take one of this class which, like van der Pol's equation, has a single unstable singularity in the finite plane, we can readily deduce the existence of limit cycles. For the trajectories emanating from the finite singularity must have as their limit sets either a limit-cycle, a critical point, or a graph made up of separatrices and critical points. As no trajectories tend to P or Q with increasing time, the last two possibilities are ruled out and limit cycles exist, in fact every point outside the outermost limit-cycle must tend to it with increasing time.

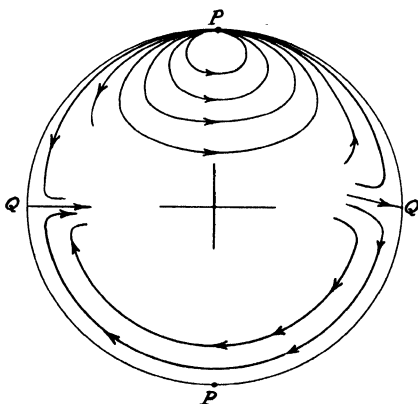


FIGURE 31

A similar remark applies to 1A with a single stable singularity.

If we take case (6) with a single simple singularity R in the finite plane, R will necessarily be a saddle. Consider the limit sets of the two positive separatrices C_1 and C_2 emerging from R . There are no limit-cycles since a limit-cycle must contain singularities with an index sum of 1. For the same reason neither separatrix can return to R , so a loop of linked separatrices is impossible. Hence, each must tend either to P or to Q along a possible direction of approach for positive trajectories. They cannot both tend to P (or Q) for if they did, RC_1PC_2 would enclose a certain region A free of singularities and every positive and negative trajectory in A would have to tend to P between C_1 and C_2 . But this is a direction of approach for positive trajectories only. Hence, C_1 tends to Q and C_2 to P (Fig. 32). The paths of the other trajectories are then completely determined. C_2 and another separatrix bound the closed nodal region of P .

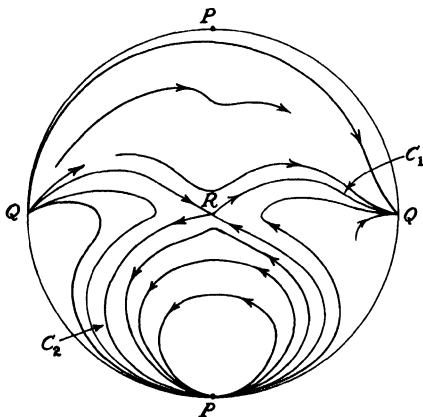


FIGURE 32

A single singularity in case (2) again gives an easily determined configuration, while in (5) there are many possibilities.

In (3), (4), (7), and (8) the simplest case is $g(x)$ never zero. With no finite singularities the destination of the unique separatrix issuing from Q is uniquely determined, and in turn it determines the destinations of the other paths. In case (3) it tends from Q to Q , in case (4) from Q to Q , in (7) from Q to P , in (8) from P to Q , so

that the paths in the finite plane are as shown in Fig. 33.

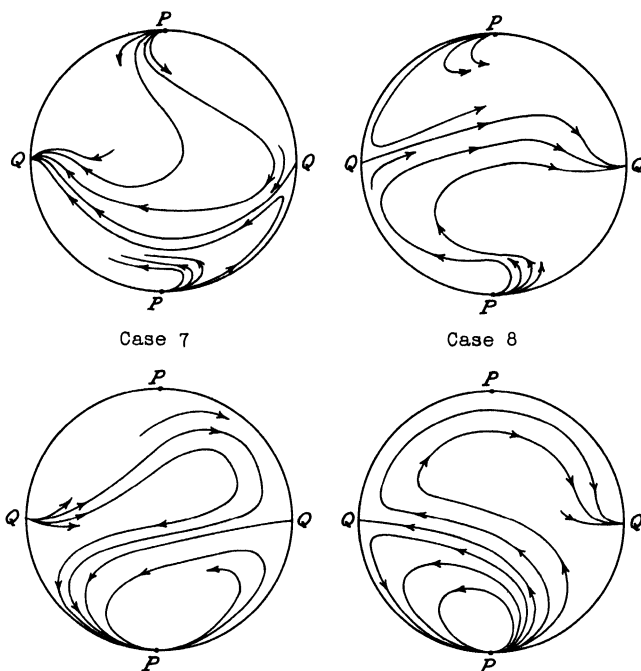


FIGURE 33

We see that in all cases (1) to (8) and (1A) to (8A) trajectories tend to infinity with a definite asymptotic direction. They are asymptotic to the line at infinity at P, and to a finite line at Q. If $m > n$, they are in fact asymptotic to the x-axis as they tend to Q. In no case is there a spiralling toward infinity.

PART III: FORCED OSCILLATIONS IN A CLASS OF SECOND ORDER EQUATIONS

§1. In this section the method of Part I and the knowledge of critical points obtained in Part II will be used to prove a forced

oscillation theorem about

$$(31) \quad \frac{d^2x}{dt^2} + f(x) \frac{dx}{dt} + g(x) = E(t).$$

Here, as before,

$$f(x) = \sum_{s=0}^m a_s x^s, \quad g(x) = \sum_{s=0}^n b_s x^s,$$

and $m \geq n > 0$, while the forcing term $E(t)$ has a continuous derivative and is periodic of period T .

THEOREM. If n is odd, and if we do not have both $b_n > 0$ and m odd, then equation (31) has a periodic solution of period T .

§2. As a first step toward establishing the theorem we will introduce a pair of equations closely related to the phase plane form of (31).

Choose any E' greater than $\sup_t |E(t)|$, and let

$$h_1(y) = \begin{cases} -E', & y \geq 0 \\ +E', & y < 0 \end{cases}$$

and

$$h_2(y) = \begin{cases} +E', & y \geq 0 \\ -E', & y < 0 \end{cases}$$

then introduce

$$(32)_1 \quad \frac{dx}{y} = \frac{dy}{-f(x)y - g(x) + h_1(y)}$$

and

$$(32)_2 \quad \frac{dx}{y} = \frac{dy}{-f(x)y - g(x) + h_2(y)}.$$

In all the following we will only consider these equations in the exterior of a circle S which surrounds all the zeros of all the polynomials $-g(x) + c$, $|c| \leq E'$.

At every point of this region the equations $(32)_1$ satisfy Lipschitz

conditions either in x or in y , so that we have unique trajectories, and in fact, in a sense to be made more precise below, these trajectories will have the same behavior at infinity as the trajectories of the corresponding equation (13).

Consider first the singularity $P = (0, 1, 0)$. Near P and on one side of the trajectory $z = 0$ the trajectories of $(32)_1$ coincide with those of

$$(33)_1 \quad \frac{dx}{dt} = y \quad \frac{dy}{dt} = -f(x)y - g(x) - E'$$

while on the other side they coincide with the trajectories of

$$(33)_2 \quad \frac{dx}{dt} = y \quad \frac{dy}{dt} = -f(x)y - g(x) + E'.$$

The arguments used in Part II to determine the behavior of the trajectories of (13) at P can be applied separately to each of the equations $(33)_j$ to find the behavior of the trajectories of $(32)_1$ on each side of $z = 0$. As the $(33)_j$ differ from (13) only by a constant, and the type of singularity obtained at P is determined by m and a_m , we will have only the same types of sides and the same combinations of sides as in the corresponding equation (13).

The same remark applies to $(32)_2$.

Near the critical point $Q = (1, 0, 0)$ consider the side $z > 0$, ($x > 0$ in x, y coordinates), and notice that $y = 0$ is a segment without contact. Suppose $z > 0$ is an attractive node or saddle side for (13) and hence for $(33)_1$ and $(33)_2$. Equation (16) shows that we have $\frac{dz}{dt} = -z^{m+1}y$ for all three equations. Therefore all the trajectories, (or the unique trajectory), tending to Q do so in the half-plane $y > 0$. This behavior then is not affected when the trajectories of the $(33)_j$ are pieced together along $y = 0$ to obtain the trajectories of the $(32)_1$. Similar remarks apply to repulsive nodes and saddles and to $z < 0$. The following more precise statements may be verified at once.

Let the trajectories of the $(32)_1$ be given the parametrization obtained naturally from the appropriate $(33)_j$. Suppose first that the side $z > 0$ of Q is an attractive nodal side for (13). Then, given any semi-circle S_1 around Q , there will exist an S_2 , such that if x^0 is in S_2 , the trajectory of $(32)_1$ through x^0 tends to Q inside S_1 . Now suppose that $z > 0$ is a saddle side of Q for (13). Then for each equation $(32)_1$ there is a unique trajectory tending to Q , and its prolongations are the parts $y > 0$ and $y < 0$ of the line $z = 0$, exactly as is the case for trajectories of (13).

These statements may be repeated for a repulsive node or saddle, and for the side $z < 0$.

§3. We will now introduce some notations needed for the main lemma of this section. By C_1^+ , C_1^- will be meant positive and negative half-trajectories of the $(32)_1$, the parametrization being obtained from the appropriate equation (33). The region $x^2 + y^2 \geq R^2$ will be denoted by $K(R)$. Also, given some $K(R)$, the set of points x^0 in $K(R)$ such that $C_1^+(x^0)$ tends to infinity in $K(R)$ will be denoted by $A_1^+(R)$. Similarly from the C_1^- we obtain the sets $A_1^-(R)$.

A large part of the proof of the theorem consists of establishing the following lemma.

LEMMA. If the $f(x)$ and $g(x)$ in (32) satisfy the conditions for the theorem, then, given any sufficiently large R there is an R_0 depending on R such that

$$A_1^+(R) \cup A_2^-(R) \supset K(R_0).$$

PROOF. The proof will proceed by a case by case analysis. The conditions for the theorem mean that we deal only with the cases 1, 2, and 6, 1A, 2A, and 6A of Fig. 29. We will give the proof of the Lemma only for cases 1, 2, and 6, the modifications for the other cases being obvious.

We take a fixed R , and consider all the following constructions as being carried out in $K(R)$.

Case 1, m even, n odd, $a_m > 0$, $b_n > 0$. P is an unstable node for both of the equations $(32)_1$. Considering especially $(32)_2$, we see that given a circle S_1 around P , there is an S_2 such that if x^0 is inside S_2 , $C_2^-(x^0)$ tends to P inside S_1 . So the interior of $S_2 \subset A_2^-(R)$. Also for $x > 0$ and near Q choose another x^0 so that $C_2^-(x^0)$ is the unique trajectory tending toward Q . As the line at infinity forms the two prolongations of this trajectory, it follows immediately from the definition of prolongation that we may find a short segment without contact, ax^0b , orthogonal to $C_2^-(x^0)$ and such that $C_2^-(a)$ and $C_2^-(b)$ cut S_2 (Fig. 34). Repeating the construction for $x < 0$ with an x^0 and a segment $a'x^0b'$ and trajectories $C_2^-(a')$ and $C_2^-(b')$ we see that S_2 , $C_2^-(a)$, $C_2^-(b)$, $C_2^-(a')$, and $C_2^-(b')$ bound a region A which clearly has the property that if $x^0 \in A$, $C_2^-(x^0)$ lies entirely in the region bounded by the same four trajectories and S_2 . Thus $A \subset A_2^-(R)$, and any $K(R_0) \subset A$ is a $K(R_0)$ fulfilling the condition of the Lemma.

Case 2, m even, n odd, $a_m > 0$, $b_n < 0$. As P is an unstable, Q a stable node for the $(32)_1$, just as in case 1 we may find

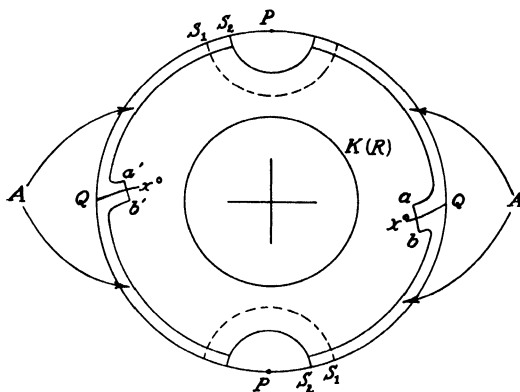


FIGURE 34

circles, S_2 around P , S_2' around Q , such that

$$S_2 \subset A_2^-(R)$$

$$S_2' \subset A_1^+(R).$$

Also, since $z = 0$ is a trajectory, using continuity with respect to initial conditions, we may find an x^0 on S_2 with z coordinate so small that $C_1^+(x^0)$ cuts S_2' , and an $x^{0'}$ on S_2' with z so small that $C_2^-(x^{0'})$ cuts S_2 (see Fig. 35). So if p lies under both

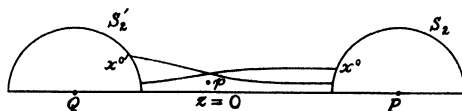


FIGURE 35

these trajectories $C_2^-(p)$ will enter S_2 , and $C_1^+(p)$ will enter S_1 . Thus p lies in $A_1^+(R)$ and in $A_2^-(R)$. In this way the circles may be connected by long pipes along $z = 0$, the pipes lying in $A_1^+(R)$ and $A_2^-(R)$. Thus combining pipes and circles we obtain a region A lying in $A_1^+(R) \cup A_2^-(R)$, and any R_0 such that $K(R_0) \subset A$ is a $K(R_0)$ as required

in the Lemma (Fig. 36).

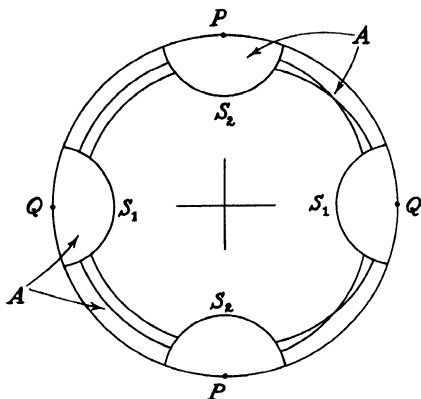


FIGURE 36

Case 6, m odd, n odd, $a_m > 0$, $b_n < 0$. In dealing with the critical point Q we construct a circle S as before. The part of the circle near Q with $x > 0$ will belong to $A_1^+(R)$, the part with $x < 0$ to $A_2^-(R)$. But in dealing with P on the side $y < 0$ we have a more complicated singularity and must slightly modify the procedure.

In Part II in discussing this critical point we saw that the interior of any sufficiently small circle S_2 around P , $y < 0$, would be divided into three sectors by a trajectory of (13) there referred to as $C(p)$ (Fig. 20). Turning to equations $(32)_1$ and $(32)_2$ we have different circles and different divisions, but choosing a circle sufficiently small to serve for both equations, we have it divided into the three closed sectors E_1, E_2, E_3 , by a suitable trajectory $C_1(a)$ of $(32)_1$, and into sectors E_1', E_2', E_3' , by a suitable $C_2(a')$ of $(32)_2$. The properties ascribed to the sectors in Part II show that $E_1 \cup E_2 \subset A_1^+(R)$, and $E_2' \cup E_3' \subset A_2^-(R)$. Also, as $C_1(A)$ and $C_2(a')$ both approach P tangent to $z = 0$, we have for a sufficiently small circle S_2 , $E_1 \cap E_3' = E_1' \cap E_3 = \emptyset$ (the null set).

Using this circle then we may write

$$(E_1 \cup E_2 \cup E_3) \cap (E_1' \cup E_2' \cup E_3') = S_2 \text{ and interior}$$

$= (E_1 \cup E_2) \cup (E_3 \cap E_2^!) \cup (E_3 \cap E_3^!) \subset (E_1 \cup E_2) \cup (E_2^! \cup E_3^!) \subset A_1^+(R) \cup A_2^-(R)$.
 So we have a circle around P , $y < 0$, of the desired type.

We may now construct pipes to the circle around Q by linking S (of Q , $x > 0$) with S_2 by a C_1^+ which starts on S_2 and cuts S with $x > 0$, and a C_2^- starting on S_2 and cutting S , $x < 0$ (Fig. 37). The points under this C_1^+ will belong to $A_1^+(R)$, those under C_2^- to $A_2^-(R)$.

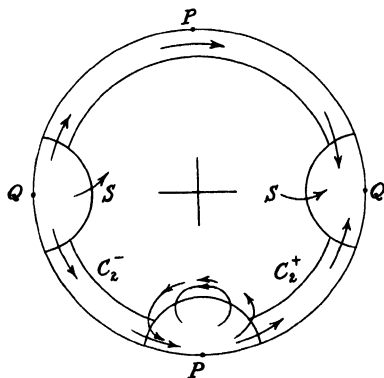


FIGURE 37

Since the line $z = 0$ is its own prolongation at $P(y > 0)$, we may construct a tube linking $Q(x < 0)$ with $Q(x > 0)$ past $P(y > 0)$. The points in the tube will have the property of belonging to $A_1^+(R)$ or $A_2^-(R)$ or to both. Together the tubes and circles form a region $A \subset A_1^+(R) \cup A_2^-(R)$. Thus any $K(R_0)$ in A is a suitable $K(R_0)$.

As we have now dealt with all the cases, the Lemma is established.

§4. Now take any specific $f(x)$ and $g(x)$ satisfying the conditions for the theorem, and consider the corresponding equations $(32)_1$ and $(32)_2$ together with an R and R_0 such that

$$A_1^+(R) \cup A_2^-(R) \supset K(R_0).$$

Choose any point $p = (a, b)$ lying in $K(R_0)$. We will associate with p a half-trajectory and a ray in the following manner. If $p \in A_1^+(R)$, the trajectory is $C_1^+(p)$, if $p \in A_2^-(R)$ the trajectory is $C_2^-(p)$, if b is positive, the ray is the ray $x = a, y \geq b$, if b is ≤ 0 , the ray is

$$x = a, y \leq b.$$

The chosen trajectory through p may behave in one of two possible ways (Fig. 38 is an example), either it cuts the chosen ray again after leaving p , or it does not. If it does cut again at p' , then the trajectory and the segment pp' divide the plane into two regions, and a path

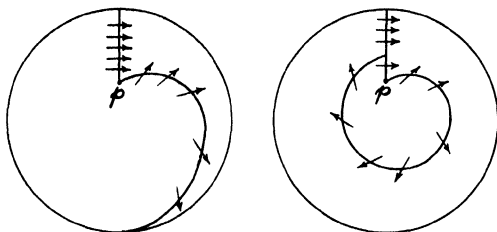


FIGURE 38

proceeding from one region to the other must cut pp' or the trajectory. If the trajectory does not cut the ray again, then by the definition of $A_1^+(R)$ or $A_2^-(R)$ the trajectory must tend to infinity inside $K(R)$. Thus the ray and trajectory combined again separate the plane into two regions. We will always call this separating curve through p , $J(p)$.

Now consider the phase plane form of (31).

$$(34) \quad \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -f(x)y - g(x) + E(t).$$

Comparing the slopes of trajectories of $(32)_1$, (34) , and $(32)_2$, we have for any point (x, y) , $y \neq 0$,

$$\frac{-f(x)-g(x)+h_1(y)}{y} < \frac{-f(x)y-g(x)+E(t)}{y} < \frac{-f(x)y-g(x)+h_2(y)}{y}.$$

This shows that if $C_1^+(p)$ is pursued in the direction of increasing time, the motions of (34) will cross it from right to left except possibly at points where $y = 0$. Similarly if $C_2^-(p)$ is pursued in the direction of decreasing time, the motions of (34) cross it from right to left. Also, as $\frac{dx}{dt} = y$, the same statement holds for the ray through p , provided it is traversed in the direction of decreasing $|y|$.

We may run through $J(p)$ by first traversing the ray with $|y|$ decreasing, and then running through the assigned trajectory after reaching

p . All along this path, with the possible exception of points $y = 0$, the trajectories of (34) cross from right to left. Since the points $y = 0$ on $J(p)$ are isolated, it follows from continuity that even at these points the motions of (34) cross $J(p)$ from right to left. Hence, all solutions starting on $J(p)$ and moving with increasing t will enter only one of the two regions into which $J(p)$ divides the plane, while with decreasing t they all enter the other.

We conclude that any trajectory of (34) cuts $J(p)$ at most once. In particular, the solution through p itself never returns to p , where p is any point in $K(R_0)$. We are now in a position to duplicate the argument of Part I.

§5. Consider in place of (34)

$$(35) \quad \begin{aligned} \frac{dx}{dt} &= y F(x, y) \\ \frac{dy}{dt} &= [-f(x)y - g(x) + E(t)]F(x, y) \end{aligned}$$

where $F(x, y)$ has continuous derivatives, is always positive, is identically 1 inside some circle of radius $R' > R_0$, and tends to zero so rapidly that the right hand sides of (35) are bounded in the entire plane.

As the special property of the separating curves $J(p)$ depended only on the direction, not the length of the velocity vectors of (34), the curves $J(p)$ have this property again with respect to solutions of (35). Thus a solution to (35) passing through a point p in $K(R_0)$ never returns to p .

Just as in Part I, equation (35) gives us a mapping $\varphi_t : u^0 \rightarrow u(t, u^0)$ of the plane into itself. If we consider the index of this mapping on any circle S lying in $K(R_0)$ with radius R'' , $R_0 < R'' < R'$, we find, just as before, that since solutions cannot return to their starting point the index of φ_t on S is independent of t . In fact, by the argument of I, it is equal to the index on S of the velocity vectors taken at $t = 0$.

But, as the theorem assumes that $g(x)$ is of odd degree, this last index, obtained from

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -f(x)y - g(x) + E(0) \end{aligned}$$

must be $\neq 0$. Hence even the index of φ_T is $\neq 0$, and there is a point

u^0 inside S with $\varphi_T u^0 = u^0$. The solution through u^0 is a periodic solution to (35). However, this solution passes through no point of $K(R_0)$, for a trajectory through p in $K(R_0)$ cannot return to p . So the solution lies always inside S . Here however, $F(x, y)$ is identically 1, so we have a periodic solution to (34) and hence to (31).

This establishes the theorem.

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VI. ON CERTAIN CRITICAL POINTS OF A DIFFERENTIAL SYSTEM IN THE PLANE

Samuel Barocio *

Suppose we have a system of the form:

$$\begin{aligned} \dot{x} &= [x, y]_p \\ \dot{y} &= [x, y]_q \end{aligned} \quad (1)$$

where the expressions in brackets are real convergent power series, whose first terms are respectively of degrees at least p and q . If $p = q =$ the system takes the general form:

$$\begin{aligned} \dot{x} &= ax + by + [x, y]_2 \\ \dot{y} &= cx + dy + [x, y]_2. \end{aligned} \quad (2)$$

It is a well known result that at the origin, the local phase-portrait is that of the linear approximation of (2); i.e., to the system obtained by neglecting the terms of two and higher degrees. (The focus-center case is a well known exception.)

In the linear case, when both of the characteristic roots of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of (2) are different from zero, their signs or reality completely define the type of singularity [1], [2].

The case of one characteristic root zero has been discussed by Bendixson [3] and in a different and with a more direct procedure by S. Lefschetz in Chapter X of his forthcoming book On Differential Equations, (referred to as [1]).

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We believe that the case where both characteristic roots are zero and nevertheless the matrix of the system is not identically the null matrix, has not been treated (as a whole phase-portrait) in the literature [4]. It is our present purpose to discuss the possible types of critical points of (2) in this case. At all events such a system may be reduced in a suitable co-ordinate system to the form

$$(3) \quad \begin{aligned} \dot{x} &= \lambda y + [x, y]_2^x, \quad \lambda \neq 0 \\ \dot{y} &= [x, y]_2^y = ax^2 + 2bxy + cy^2 + [x, y]_3. \end{aligned}$$

Here the upper indices x, y are merely to distinguish the square brackets.

There are now two possibilities. First, $c \neq 0$. Then, by replacing y by $-y/c$ we reduce the system to the same form but with $c = \pm 1$. Suppose now that $c = 0$. Then, making the transformation of variables, regular at the origin:

$$x_1 = x, \quad y_1 = y + \frac{1}{\lambda} xy,$$

the system is reduced to the form

$$(4) \quad \begin{aligned} \dot{x}_1 &= \lambda y_1 + \left[x_1, y_1 \right]_2^{x_1} \\ \dot{y}_1 &= -y_1^2 + 2bx_1y_1 + ax_1^2 + [x_1, y_1]_3. \end{aligned}$$

Applying now the Weierstrass preparation theorem together with a change of time variable, we may write

$$\begin{aligned} \dot{x}_1 &= \lambda y_1 + \left[x_1, y_1 \right]_2^{x_1} = \lambda y_1 + C(x_1) \\ \dot{y}_1 &= - \left[y_1^2 - 2A(x_1)y_1 + B(x_1) \right] E(x_1, y_1) \end{aligned}$$

where $E(0, 0) = 1$.

By changing the time units we may reduce this last system to one of the two forms:

$$\begin{aligned} \dot{x}_1 &= \pm y_1 + C(x_1) \\ \dot{y}_1 &= - \left[y_1^2 - 2A(x_1)y_1 + B(x_1) \right] E(x, y), \quad E(0, 0) = 1. \end{aligned}$$

It is sufficient actually to discuss the system with the $+$ sign, since

the other may be deduced from it by changing t into $-t$. Writing now again x, y for x_1, y_1 , the system that we shall discuss is

$$(5) \quad \begin{aligned} \dot{x} &= y + C(x), \\ \dot{y} &= -[y^2 - 2A(x)y + B(x)] E(x, y). \end{aligned}$$

We notice in passing that $A = [x]_1$, $B = [x]_2$, $C = [x]_2$.

Now clearly the following system

$$(6) \quad \begin{aligned} \dot{x} &= [y^2 - 2A(x)y + B(x)] E(x, y) \\ \dot{y} &= y + C(x) \end{aligned}$$

which is merely the Bendixson system in the form (19.2) of [1] Chapter X, is orthogonal to (5). This system differs in no essential way from the system (19.4) (loc. cit.) which has been fully discussed there.

The Bendixson system in both of its above mentioned forms (19.2) and (19.4) has the y axis as trajectory or tangent to a trajectory, and it is proved (loc. cit.) that all the possible phase-portraits are reduced to the following three types: Node, Saddle-point (four hyperbolic sectors with no fans between them) and a node coalescent with two hyperbolic sectors. In all the cases each right or left half plane is filled with a node or two hyperbolic sectors.

By Puiseux's theorem, the right hand side of (6) can be decomposed in two factors of the form

$$y = x^{r/n} E(x^{1/n}).$$

Each such factor represents part of the geometrical locus in which $\dot{x}$ vanishes, and hence, where the tangents of the trajectories are vertical. We will denote these branches by Γ_V . If the factors are complex, there are no Γ_V branches at all. The locus in which $\dot{y}$ vanishes, is a branch denoted by Γ_H (a single one, since $\dot{y}$ has a linear term in y) and is the locus where the trajectories have horizontal tangent.

At this point it is interesting to observe that if we have two mutually orthogonal phase-portraits near $(0, 0)$, the Γ_H branches associated with one singularity become the Γ_V branches associated with the orthogonal phase-portrait (i.e., to the other singularity) and vice versa.

This simple property almost enables us to construct at once, from the Bendixson system all the possible local phase-portraits of the system

(6), which we know from Lefschetz [1'], and taking orthogonal trajectories.

In the Bendixson system, in the form (19.4) the Γ_V branches are symmetrical with respect to the x axis. For (19.2) however, it does not necessarily happen, and we will consider, as above stated the orthogonal to this latter system.

Changing t in $-t$ in (5) we see at once that near the y axis, $\dot{y}$ will be positive and $\dot{x}$ of the opposite sign to y ; we do this merely to fix a sense to the trajectories.

From all the above considerations we easily obtain the "representative" or "typical" halves of Plate I. These halves are typical in the sense that all the other possible configurations are equivalent to those of Plate I. In certain cases orthogonality does not give us a complete information of the behaviour of the trajectories. For example, let us consider the right half numbered II, Plate I. The Bendixson phase-portrait does not enable us to decide if there are "TO-curves" (trajectories tending to or away from the origin) between the Γ_H branches or between the Γ_V branch and the x axis. The curves in Plate I, II, in fact, do not occur. In this connection we refer the reader to a forthcoming paper by Lefschetz.

Then, by matching all the right and left halves of Plate I, we obtain the eight different (not, however, topologically) local phase-portraits of Plate II.

In each drawing of Plate II there is a chemical like formula denoting the "composition" of the phase-portrait.

The meaning of the letters are the following:

- C - center;
- F - focus;
- H - hyperbolic sector;
- N - node;
- O - nested oval sector.

The phase-portraits orthogonal to those of Plate II, i.e., the Bendixson ones are drawn in Plate III with numbers corresponding to their orthogonals in Plate II are

- $2H$ - N orthogonal to 1, 2, and 5.
- N - N orthogonal to 4, 6, 7 and 8.
- $4H$ - N orthogonal to 3.

This shows at once that two topologically equivalent phase-portraits may have topologically non-equivalent orthogonal phase-portraits.

The only stable phase-portraits are the focus or center Plate II, Figure 6 and node Plate II, Figure 8, both of them orthogonal to nodes.

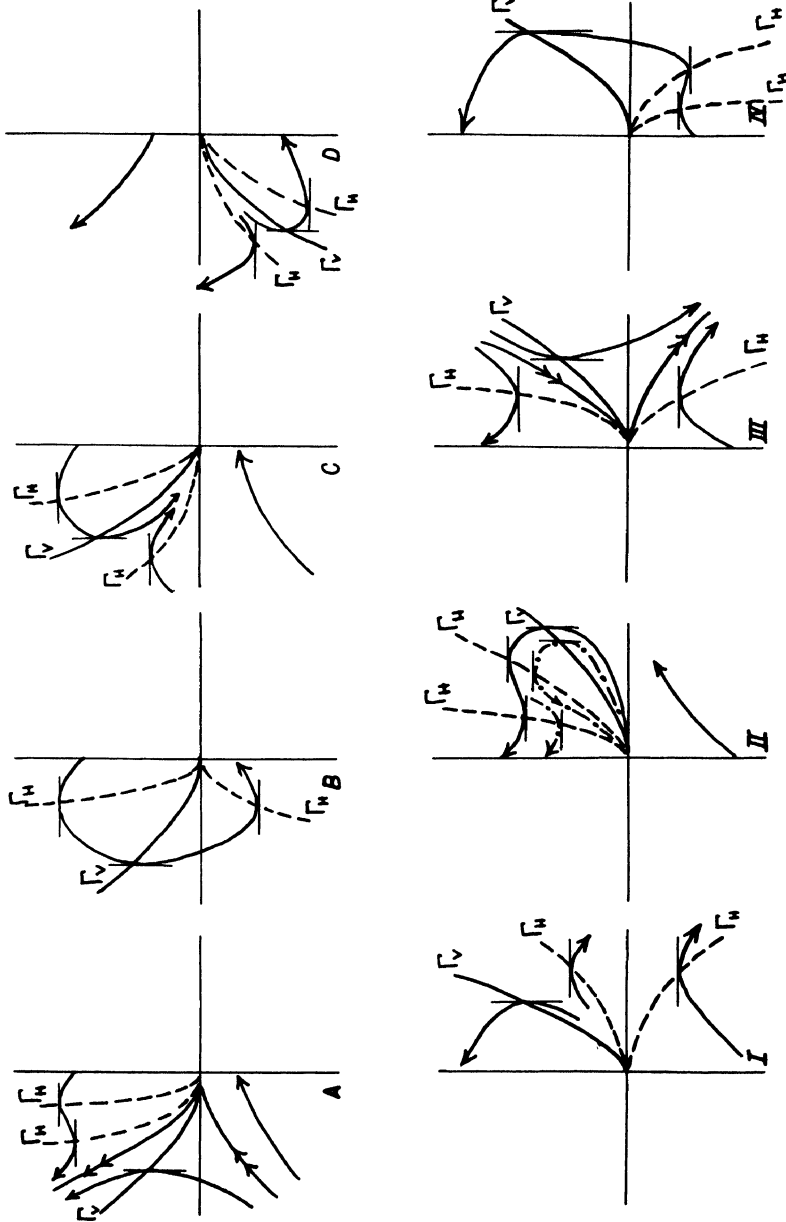


PLATE I

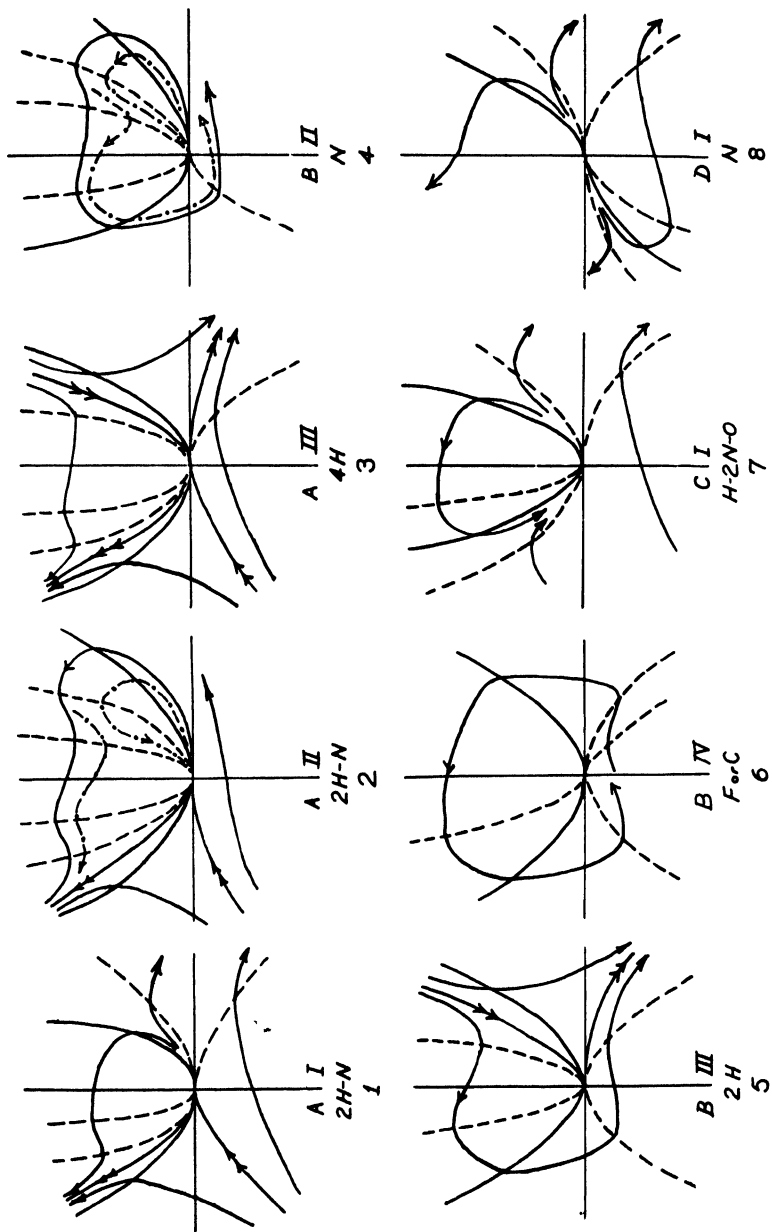
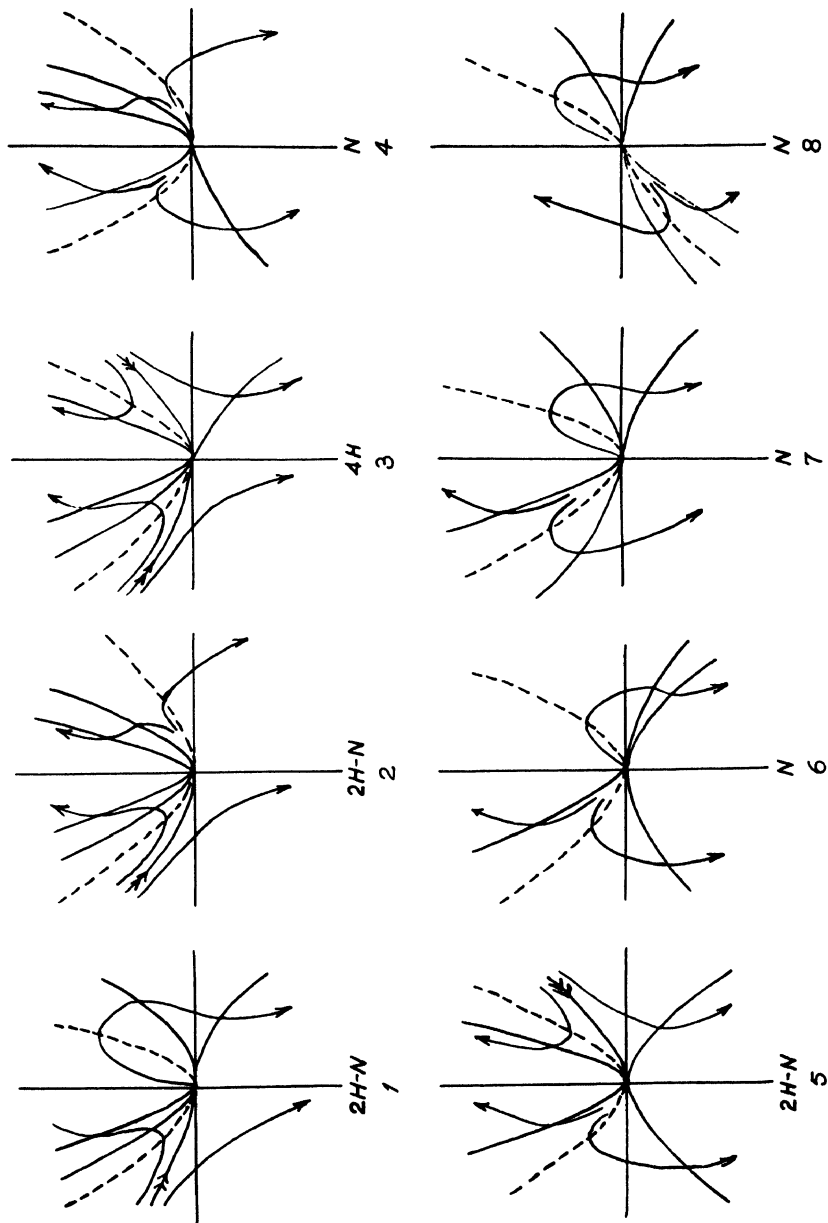


PLATE II



Then, stability in the orthogonal system implies stability in the Bendixson system. The converse is not true (see Plate II, Figure 7).

In the description given here there may arise fans between consecutive hyperbolic sectors. As we do not know in all cases whether these fans exist or not, we say that our description is complete "modulo fans".

Now, we may summarize our results in the following:

THEOREM. In the system where both characteristic roots are equal to zero, but with non null matrix, there may arise modulo fans all the possible phase-portraits (consistent with the Poincaré index formula) made up of zero, two or four hyperbolic sectors and in addition of one hyperbolic sector and a nested oval sector. The correspondence with the Bendixson system is indicated below:

<u>Bendixson System</u>	<u>Orthogonal System</u>
2H-N	2H-N; 2H
4H	4H
N	H-2N-0; N; F or C.

Observe that corresponding configurations have the same Poincaré index. If one endeavours to determine a priori the configurations compatible with the preservation of the index one finds types like 3H-0 or 2H-20. However, in the note of Lefschetz already mentioned it is proved that these configurations cannot arise and that the only scheme under which a nested oval sector may occur is the combination H-0 (or H-2N-0).

In Plate I, Figure II, and Plate II, Figures 2 and 4, configurations a priori possible (but actually not occurring) are indicated by lines in point and dash.

In certain cases we may dispose of the fans between consecutive hyperbolic sectors. In the forthcoming book of S. Lefschetz (Chapter X, (21.7)) it is proved that the system (6) does not have inflexions in the zone between the Γ_V and Γ_H branches and in the zone below the x axis. On the other hand, in these zones there arises the possibility of a fan in the orthogonal system to (6).

We will show here by a simple geometrical argument that this situation excludes the possibility of fans in our particular regions.

Let us suppose that there exists a fan orthogonal to a hyperbolic sector in a region where all the curves of the family (other than the separatrices and the origin) are convex toward the origin, Fig. a.

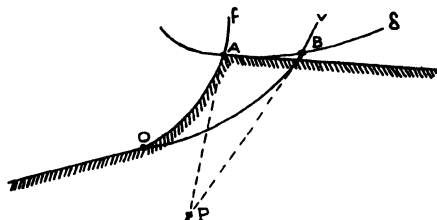


FIGURE a

Consider one fixed trajectory f of the assumed fan, and another variable one v ; and take their intersection A and B with an orthogonal trajectory s sufficiently near to the origin. Then, for a small arc $\widehat{AB}$ the curvatures of OA and OB are not of opposite signs, and the tangents to f and v in the points A and B intersect in a point P of the region limited by the tangent to f at the origin, the arc $\widehat{OA}$ and the tangent to s at A .

As B tends to A the point P must tend to a limiting position (or infinity contained in the shaded region).

However, since the arc $\widehat{AB}$ is convex towards O , the center of curvature at A (which is also the limit of P) is in the region above the tangent at A .

This contradiction proves that no fan such as assumed may exist in our regions. The fan is thus reduced indeed to a single separatrix.

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- [4] (Added in the proofs) The author has just learned of the existence of the following paper in Russian (which he cannot read) in which the same questions are also discussed:
 ANDREEV, A. F., "Investigations of the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point," Annals of the University of Leningrad, (1955), pp. 43-55.

VII. ON THE TOTAL NUMBER OF SINGULAR POINTS AND LIMIT CYCLES OF A DIFFERENTIAL EQUATION¹

Felix Haas

The object of the present paper is to study the relation between the number of singular points and limit cycles of a differential equation on one hand and the Betti numbers of a manifold on which this differential equation is realized on the other hand. If the vector field defining the differential equation is the gradient of a potential function and if the limit sets consist of singular points only the Morse inequalities for the critical points of the potential function immediately give inequalities for the singular points of the vector field in terms of the Betti numbers (1, 2). From the point of view of differential equations, however, it is desirable to derive similar inequalities, even for vector fields which do not come from potential functions, and to include limit cycles in the consideration.

There is a fair amount of opportunity to use such inequalities in the study of concrete differential equations. Existence theorems are sometimes proved by using the theorem that the sum of the singular point indices equals the Euler characteristic. The Morse inequalities are sharper and should be used more frequently. The surfaces involved are generally phase spaces of various kinds or invariant surfaces of a differential equation defined in a larger space.

El'sgol'c established certain inequalities for the case where the limit set consists of singular points of the simplest kind only³. The proof given there is somewhat sketchy and in places difficult to understand. In Part I, therefore, we state the result for the simplest singularities and supply a proof. In Part II complicated singular points are analysed in preparation for subsequent parts. The classification, valid also for non-analytic systems, goes back to Brower^{4,5}. In Part III inequalities are proved which relate the total number of hyperbolic, elliptic, stable, and

¹ Work done under Office of Naval Research Contract No. N6ori-105-Task V Project No. NRD43-942.

unstable sectors (see Part II) to the Betti numbers. In Part IV limit sets are allowed which may include any kind of singular point, no matter how complicated, limit cycles, stable, unstable or stable-unstable, and limiting configurations involving singular points and various connecting trajectories. These last mentioned limiting configurations may be fairly complicated since complicated singular points are included. Inequalities involving the Betti numbers are again obtained. In the same part the requirement of uniqueness of trajectory is removed at least partially. In Part V the theory is applied to certain types of differential equations and the existence of various kinds of limit cycles proved.

I. Inequalities in the Simplest Cases

1. In this part very stringent conditions will be imposed on the differential equation to be discussed. However, in later parts all these restrictions (uniqueness of trajectory, simplicity of singular points, no limit cycles, etc.) will be removed.

HYPOTHESIS $\#_1$: M is a union of disjoint, twice-differentiable, two-dimensional, closed manifolds. However, we shall speak of M as a manifold and of the component parts as closed sub-manifolds. V is a vector field on M defining a differential equation. V has a finite number of singular points only, all of a simple type. V is sufficiently regular for the differential equation defined by V to have a unique trajectory through each ordinary point. The limit set of the differential equation defined by V is to consist of singular points only. The last restriction implies, in particular, that there are no closed curve solutions. It is also assumed that there exist no closed curves consisting of saddle points and connecting trajectories.

NOTATION: If A is a manifold we shall denote by $B_0(A)$, $B_1(A)$, and $B_2(A)$ the zeroth, first, and second Betti-Number of A , respectively. Let p_i , $i = 1, 2, \dots, N_0$, be the unstable singular points of the differential equation on M . Let q_i , $i = 1, 2, \dots, N_2$, be the stable singular points of the differential equation on M . Finally, let r_i , $i = 1, 2, \dots, N_1$, be the saddle points of the differential equation on M .

THEOREM I (El'sgol'c). Under hypothesis $\#_1$.

$$B_0(M) \leq N_0,$$

$$B_1(M) \leq N_1,$$

$$B_2(M) \leq N_2.$$

REMARK. Theorem I was first proved in the case where V comes from a potential function by Morse. In its present form the theorem is due to El'sgol'c.

2. PROOF OF THEOREM 1. We first construct a system of neighborhoods covering M . If p_1 is an unstable singular point, $U(p_1)$ will be an open spherical neighborhood of p_1 which is sufficiently small

- a) to exclude all other singular points
- b) to exclude the closure of any neighborhood, $U(p_j)$, previously constructed
- c) to be bounded by a simple closed curve which has the additional property that every trajectory emerging from p_1 intersects this closed curve exactly once and from the inside to the outside.

If q_1 is a stable singular point $U(q_1)$ will be an open spherical neighborhood of q_1 having the same properties as the neighborhood of the unstable singular points, only this time with respect to negative time. $U(q_1)$ is assumed sufficiently small to exclude the closure of any neighborhood, $U(q_j)$, previously constructed.

Let $C_{1,1}$ and $C_{1,2}$ be the two trajectories emerging from the saddle point r_1 . Then $U(C_{1,j})$ an open spherical neighborhood of $C_{1,j}$ is sufficiently small so that

- a) $U(C_{1,j}) \cap \overline{U(p_k)}$ is empty
- b) $U(C_{1,j}) \cap \overline{U(q_k)}$ is empty unless $C_{1,j}$ tends towards q_k in which case $U(C_{1,j})$ is chosen so that the intersection with $U(q_k)$ consists of a single open spherical neighborhood which does not include q_k .
- c) $U(C_{1,j}) \cap U(C_{k,1})$ is empty unless $C_{1,j}$ and $C_{k,1}$ share at least one of their end points. At a saddle point which is an end point of both $C_{1,j}$ and $C_{k,1}$ the intersection is to be an open spherical neighborhood and

$$U(C_{1,j}) - \sum_{k,1} \left[U(C_{1,j}) \cap U(C_{k,1}) \right]$$

is to be a non-empty spherical neighborhood. The same is to hold for $U(C_{k,1})$.

3. It is clear that there exists a set U obtained from $\sum_1 U(p_1)$ by isotopic deformation such that M is covered by

$$U + \sum U(C_{1,j}) + \sum U(q_1).$$

We indicate the deformation: Solution curves starting at the unstable singular points are parametrized so that the parametric value on the boundary of $U(p_1)$ is the same for all curves and all i . Now all solution

curves have to tend to the stable singular points and saddle points. Since there are only a finite number of singular points there exists a value T of the parameter t , which was used in parametrizing the solution curves, such that all solution curves remain in the previously chosen neighborhoods of the stable singular points and saddle points for $t \geq T$. The set $\Sigma U(p_1)$ can be deformed to include all points with parametric value $t < T$ on the trajectories starting at one of the unstable singular points p_1 . If we call this new set U it is clear that $U + \Sigma U(q_1) + \Sigma[U(C_{1,j})]$ cover M .

LEMMA 3.1. The Betti-Numbers of U are as follows:

$$B_0(U) = N_0,$$

$$B_1(U) = 0,$$

$$B_2(U) = 0.$$

PROOF OF LEMMA 3.1. The Lemma follows from the construction of U .

LEMMA 3.2. The Betti-Numbers of $U + U(C_{1,j})$ are as follows:

$$B_0(U + \Sigma[U(C_{1,j})]) \leq N_0,$$

$$B_1(U + \Sigma[U(C_{1,j})]) \leq N_1,$$

$$B_2(U + \Sigma[U(C_{1,j})]) = 0.$$

PROOF OF LEMMA 3.2. We may assume that for each i $U(C_{1,i}) + U(C_{1,2})$ is an open spherical neighborhood. We shall use the following notation:

$$[U(C_{1,1}) + U(C_{1,2})] = U(r_1),$$

$$U + \sum_{j < 1} [U(C_{j,1}) + U(C_{j,2})] = P(r_1),$$

$$B_k[U(r_1) + P(r_1)] - B_k[P(r_1)] = d_1^k.$$

The proof of the Lemma will use the Meyer-Vietoris Sequence with integer coefficients:

$$\begin{array}{ccccccc} 0 \longrightarrow & H_2[P(r_1)U(r_1)] & \longrightarrow & H_2[P(r_1)] + H_2[U(r_1)] & \longrightarrow & H_2[P(r_1) + U(r_1)] \\ & & & & & \downarrow \\ & H_1[P(r_1) + U(r_1)] & \longleftarrow & H_1[P(r_1)] + H_1[U(r_1)] & \longleftarrow & H_1[P(r_1)U(r_1)] \\ & \downarrow & & & & \\ & H_0[P(r_1)U(r_1)] & \longrightarrow & H_0[P(r_1)] + H_0[U(r_1)] & \longrightarrow & H_0[P(r_1) + U(r_1)] \longrightarrow 0 \end{array}$$

We next consider the various patterns in which $P(r_1)$ and $U(r_1)$ can intersect. In each case the pattern will hold for sufficiently small $U(r_1)$:

CASE 1: $C_{1,1}$ and $C_{1,2}$ tend to stable singular points or to saddle points not contained in $P(r_1)$. The two trajectories which tend to r_1 come from saddle points in $P(r_1)$. See Figure 1.

CASE 2: This case is identical with Case 1 except that $C_{1,1}$ or $C_{1,2}$, but not both, tends to a singular point in $P(r_1)$. See Figure 2.

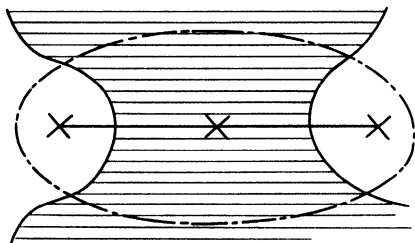


FIGURE 1

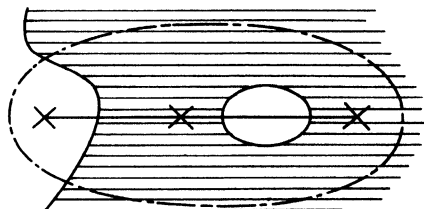


FIGURE 2

CASE 3: This case is identical with Case 1 except that now both $C_{1,1}$ and $C_{1,2}$ tend to singular points of $P(r_1)$. See Figure 3.

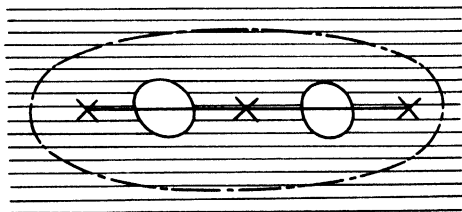


FIGURE 3

CASE 4: $C_{1,1}$ and $C_{1,2}$ tend to singular points not in $P(r_1)$. One of the trajectories which tends to r_1 comes from a saddle point in $P(r_1)$ and one comes from an unstable singular point (which, of course, is also in $P(r_1)$). The figure would be the same as in Case 1.

CASE 5: This case is identical with Case 4 except that either $C_{1,1}$ or $C_{1,2}$, but not both, tends to a singular point in $P(r_1)$. The

figure would be the same as in Case 2.

CASE 6: This case is identical with Case 4 except that now both $C_{1,1}$ and $C_{1,2}$ tend to singular points in $P(r_1)$. The figure would be the same as in Case 3.

CASE 7: $C_{1,1}$ and $C_{1,2}$ tend to singular points which are not contained in $P(r_1)$. Both trajectories which tend towards r_1 come from unstable singular points. See Figure 4.

CASE 8: This case is identical with Case 7 except that now either $C_{1,1}$ or $C_{1,2}$, but not both, tends to a singular point in $P(r_1)$. See Figure 5.

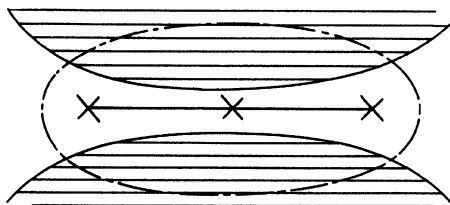


FIGURE 4

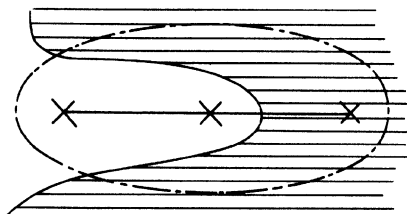


FIGURE 5

CASE 9: This case is identical with Case 7 except that now both $C_{1,1}$ and $C_{1,2}$ tend to singular points of $P(r_1)$. See Figure 6.

CASE 10: $C_{1,1}$ and $C_{1,2}$ tend to singular points not contained in $P(r_1)$. One of the trajectories which tends to r_1 comes from a saddle point in $P(r_1)$ and one comes from a saddle point not contained in $P(r_1)$. See Figure 7.

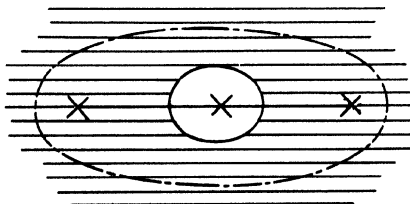


FIGURE 6

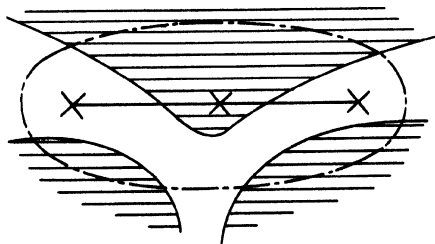


FIGURE 7

CASE 11: This case is identical with Case 10 except that now either $C_{1,1}$ or $C_{1,2}$, but not both, tends to a singular point in $P(r_1)$. See Figure 8.

CASE 12: This case is identical with Case 10 except that now both $C_{1,1}$ and $C_{1,2}$ tend to singular points in $P(r_1)$. See Figure 9.

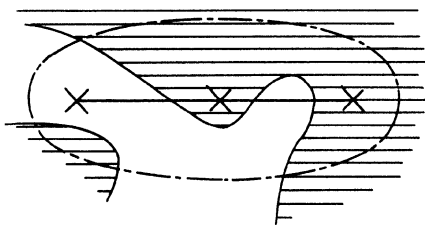


FIGURE 8

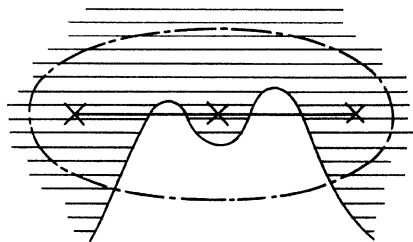


FIGURE 9

CASE 13: $C_{1,1}$ and $C_{1,2}$ tend to singular points not contained in $P(r_1)$. One of the trajectories tending to r_1 comes from an unstable singular point and one comes from a saddle point not in $P(r_1)$. The figure would be the same as in Case 10.

CASE 14: This case is identical with Case 13 except that now either $C_{1,1}$ or $C_{1,2}$, but not both, tends to a singular point contained in $P(r_1)$. The figure would be the same as in Case 11.

CASE 15: This case is identical with Case 13 except that now both $C_{1,1}$ and $C_{1,2}$ tend to singular points contained in $P(r_1)$. The figure would be the same as in Case 12.

CASE 16: $C_{1,1}$ and $C_{1,2}$ tend to singular points not contained in $P(r_1)$. Both trajectories which tend to r_1 come from singular points outside $P(r_1)$. See Figure 10.

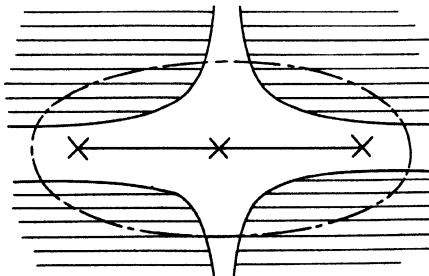


FIGURE 10

CASE 17: This case is identical with Case 16 except that now either $C_{1,1}$ or $C_{1,2}$, but not both, tends to a singular point contained in $P(r_1)$. See Figure 11.

CASE 18: This case is identical with Case 16 except that now both $C_{1,1}$ and $C_{1,2}$ tend to singular points of $P(r_1)$. See Figure 12.

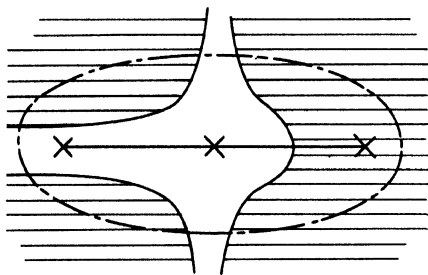


FIGURE 11

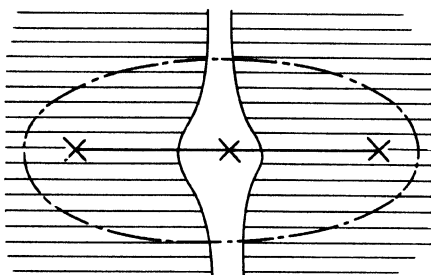


FIGURE 12

4. We next study the Meyer-Vietoris Sequence for the various possible cases. We shall use the symbol I to denote the additive group of integers. We shall use the symbol d_1^k defined at the start of the proof of Lemma 3.2.

CASES 7, 11, 14, and 18:

$$B_0[P(r_1)U(r_1)] = 2,$$

$$B_1[P(r_1)U(r_1)] = 0,$$

$$B_2[P(r_1)U(r_1)] = 0.$$

The following sequence is obtained:

$$0 \rightarrow H_2[P(r_1)] \rightarrow H_2[P(r_1) + U(r_1)] \rightarrow 0 \rightarrow H_1[P(r_1)] \rightarrow H_1[P(r_1) + U(r_1)] \\ 0 \leftarrow H_0[P(r_1) + U(r_1)] \leftarrow H_0[P(r_1)] \leftarrow I \xleftarrow{f} 2I \xleftarrow{g}$$

That the map denoted by f cannot have a trivial image follows from the definition of the Meyer-Vietoris Sequence. It follows that the image of g has at most one generator. Hence $d_1^1 \leq 1$. We can also immediately read off the above sequence that $d_1^2 = 0$ and $d_1^0 \leq 0$.

CASES 3 AND 6:

$$B_0[P(r_1)U(r_1)] = 1,$$

$$B_1[P(r_1)U(r_1)] = 2,$$

$$B_2[P(r_1)U(r_1)] = 0.$$

The following sequence is obtained:

$$\begin{aligned} 0 \rightarrow H_2[P(r_1)] \rightarrow H_2[P(r_1) + U(r_1)] \rightarrow 2I \rightarrow H_1[P(r_1)] \searrow \\ 0 \leftarrow H_0[P(r_1) + U(r_1)] \leftarrow H_0[P(r_1)] + I \xleftarrow{f} I \xleftarrow{g} H_1[P(r_1) + U(r_1)] \end{aligned}$$

Again f is not trivial. The same argument which was used in the preceding case to show that h was not trivial can now be used to show that the image of h has at least two generators. Since f is not trivial g is trivial. It follows that $d_1^0 = 0$, $d_1^1 = 2$, and $d_1^2 = 0$.

CASES 10, 13, AND 17:

$$B_0[P(r_1)U(r_1)] = 3,$$

$$B_1[P(r_1)U(r_1)] = 0,$$

$$B_2[P(r_1)U(r_1)] = 0.$$

The following sequence is obtained:

$$\begin{aligned} 0 \rightarrow H_2[P(r_1)] \rightarrow H_2[P(r_1) + U(r_1)] \rightarrow 0 \rightarrow H_1[P(r_1)] \rightarrow H_1[P(r_1) + U(r_1)] \\ 0 \leftarrow H_0[P(r_1) + U(r_1)] \leftarrow H_0[P(r_1)] + I \xleftarrow{f} 4I \swarrow g \end{aligned}$$

Again f is not trivial; hence the image of g has at most three generators. It follows that $d_1^0 \leq 0$, $d_1^1 \leq 3$, and $d_1^2 = 0$.

5. We are now prepared to prove Lemma 3.2. Since $d_1^0 \leq 0$ and $d_1^2 = 0$ in all possible cases it follows immediately from Lemma 3.1 that

$$B_0[U + \Sigma U(C_{1,j})] = B_0[U + \Sigma U(r_1)] \leq N_0,$$

and

$$B_2[U + \Sigma U(C_{1,j})] = B_2[U + \Sigma U(r_1)] = 0.$$

To study the changes in the first Betti-Number we note that d_1^1 may be greater than one only in Cases 10, 13, 17 (where $d_1^1 \leq 2$) and in Case 16 (where $d_1^1 \leq 3$). But it follows from the definition of Case 16 that if there exists a saddle with a neighborhood which leads to Case 16 there must also exist either one saddle with a neighborhood leading to one of the Cases 2, 3, 5, 6, 9 or two saddles each with a neighborhood leading to one of the Cases 1, 4, 8, 12, 15.

Now $d_1^1 \leq 0$ for the Cases 1, 4, 8, 12, 15 and $d_1^1 \leq -1$ for the Cases 2, 3, 5, 6, 9. The total change in the first Betti-Number due to two saddles, one of which has a neighborhood which leads to Case 16 and one of which has a neighborhood which leads to one of the Cases 2, 3, 4, 5, 6, 9 is therefore ≤ 2 . Similarly, the total change in the first Betti-Number due to three saddles, one with a neighborhood leading to Case 16 and two, each with a neighborhood leading to one of the Cases 1, 4, 8, 12, 15, would be ≤ 3 .

On the basis of a similar argument for saddles with neighborhoods which led to Cases 10, 13, and 17 we can state the over-all result. The addition of the N_1 neighborhoods of saddle points does not increase the first Betti-Number by more than N_1 . This fact together with Lemma 3.1 implies that

$$B_1[U + \Sigma U(C_{1,j})] = B_1[U + \Sigma U(r_1)] \leq N_1.$$

This completes the proof of Lemma 3.2.

6. We next add the neighborhoods of the various stable singular points. We denote by $P(q_1)$ the set

$$U + \Sigma U(C_{1,j}) + \sum_{j < 1} U(q_j).$$

d_1^k will now denote $B_k[P(q_1) + U(q_1)] - B_k[P(q_1)]$. Then the following Lemma holds

LEMMA 6.1.

$$B_0[P(q_1) + U(q_1)] - B_0[P(q_1)] = 0,$$

$$B_1[P(q_1) + U(q_1)] - B_1[P(q_1)] \leq 0,$$

$$B_2[P(q_1) + U(q_1)] - B_2[P(q_1)] \leq 1.$$

PROOF OF LEMMA 6.1. By construction, the intersection of $U(q_1)$ with $P(q_1)$ consists of exactly one open annular region. It follows that the second Betti-Number of the intersection is zero, and that the zeroth and

first Betti-Numbers of the intersection are equal to one. The first and second Betti-Numbers of $U(q_1)$ are, of course, zero and the zeroth Betti-Number of $U(q_1)$ equals one. The following sequence is obtained:

$$0 \rightarrow H_2[P(q_1)] \rightarrow H_2[P(q_1) + U(q_1)] \rightarrow I \xrightarrow{h} H_1[P(q_1)] \rightarrow H_1[P(q_1) + U(q_1)] \\ 0 \leftarrow H_0[P(q_1) + U(q_1)] \leftarrow I + H_0[P(q_1)] \xleftarrow{f} I \quad \swarrow g$$

That f cannot be trivial follows from the construction of the Meyer-Vietoris Sequence. Hence g is trivial. If h is trivial $d_1^2 = 1$ and $d_1^1 = 0$; if h is not trivial $d_1^2 = 0$ and $d_1^1 = -1$. $d_1^0 = 0$ in all cases. The Lemma follows from the above.

Since $M = U + \Sigma U(C_{1,j}) + \Sigma U(q_1)$ Theorem I follows from Lemmas 3.2 and 6.1.

II. Analysis of Complicated Singular Points

7. We briefly discuss a classification of singular points. The methods are due to Brouwer^{4,5}. This particular classification is also valid for non-analytic systems; in fact, not even uniqueness of trajectory is required in most cases. On the other hand, the results obtained are weaker than those for analytic systems⁶. We shall give the classification in this generality even though our hypothesis in Part II is more restrictive, since such a general approach will be needed in the last parts of the paper. The only restriction we impose is that there be no limit points of singular points on M .

Let p be the singular point under consideration. We distinguish between Type A and Type B singular points: a singular point is of Type A if and only if there exists an open neighborhood of p which is free of closed curve solutions except for those which pass through p . Type B is the converse case. A singular point of Type B will be called a center point. See Figure 13.

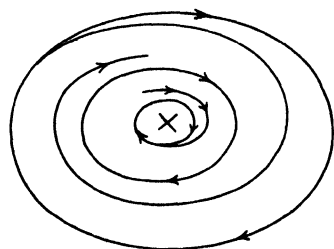


FIGURE 13

If p is of Type A let c be a circle around p of sufficiently small radius so that the closed neighborhood bounded by c contains only one singular point (p) and no closed curve trajectories which enclose p but do not pass through p . Let U be the open neighborhood bounded by c . $\bar{U}$ contains no closed trajectories, since the theorem that a closed trajectory contains a singular point does not depend on uniqueness

of trajectories. We distinguish among nine classes of points on c . If q is a point of c let C_q^+ denote a positive semi-characteristic through q . Let C_q^- denote a negative semi-characteristic through q . Furthermore, let the exterior of c be denoted by E . The interior is, of course, just U . The definition of the nine classes follows:

q belongs to Class I	aC_q^+ enters U and aC_q^- enters U ,
q belongs to Class II	aC_q^+ " E " " E ,
q belongs to Class III	" c " " c ,
q belongs to Class IV	" U " " c ,
q belongs to Class V	" E " " c ,
q belongs to Class VI	" c " " U ,
q belongs to Class VII	" c " " E ,
q belongs to Class VIII	" U " " E ,
q belongs to Class IX	" E " " U .

We note that unless uniqueness of trajectory is assumed a point may belong to several classes. However, if uniqueness of trajectory is assumed the class of a point is uniquely determined. Now p is said to be of Type A_1 (source point) if we can choose c so that none of its points belong to Classes I, II, IV, VII, VIII. p is said to be of Type A_2 (sink point) if we can choose c so that none of its points belong to Classes I, II, V, VI, IX. p is of Type A_3 (sector point) if it is of Type A and not of Type A_1 or A_2 .

In the case of a sink point every trajectory through a point of c tends to p . This follows from the following modified version of the Poincaré-Bendixson Theorem for a differential equation not satisfying a uniqueness condition:

THEOREM [Brouwer, 5]. If a positive, semi-characteristic, C^+ , remains in a bounded region of the plane and if it does not have a singular point in its limit set then one of the following will take place:

- C^+ is a closed curve trajectory,
- C^+ spirals towards a closed curve trajectory,
- C^+ runs into a closed curve trajectory.

Figure 14 shows a sink point. The ordinary stable node, stable focus, and stable spiral point are examples of sink points. However, these are by no means the only kind of sink points in non-analytic systems. What is missing is the theorem of Bendixson's that in the case of an analytic system a trajectory tends towards a singular point either along a limiting direction or assumes all possible directions arbitrarily late

in its path. The same discussion and figure with the direction of time reversed applies to source points.

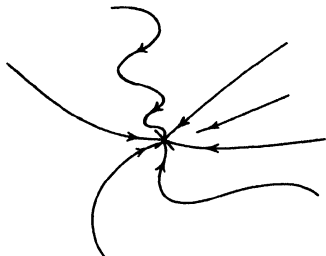


FIGURE 14

We next construct the sectors of a sector point and prove that there is only a finite number of them. In this process we shall need the following four superclasses of points belonging to c .

Superclass 1, $\bar{S}_1$, will contain a point $q \iff C_q^+$ remains in $U + c = \bar{U}$ and there exists a point r on $C_q^- \cap E$ which precedes on C_q^- any point in U .

Superclass 2, $\bar{S}_2$, will contain a point $q \iff C_q^-$ remains in $U + C = \bar{U}$ and there exists a point r on $C_q^+ \cap E$ which precedes on C_q^+ any point in U .

Superclass 3, $\bar{S}_3$, will contain a point $q \iff q$ belongs to Class VI or IX.

Superclass 4, $\bar{S}_4$, will contain a point $q \iff q$ belongs to Class IV or VIII.

We now modify c to eliminate all points of class I and II. This can always be done without increasing the length of c by more than a factor of three in the following way: If q belongs to Class I or II we replace the arc of c through q in a sufficiently small neighborhood of q by the trajectory through q and two suitable arcs on orthogonal trajectories. Once this modification has been carried out the closures of S_1 and S_3 on c , $\bar{S}_1$ and $\bar{S}_3$, are disjoint, for $\bar{S}_1$ can contain only points of Classes III, IV, VII, and IX, while $\bar{S}_3$ can contain only points of Classes II, V, VI, and VIII. Similarly, it can be shown that $\bar{S}_2$ and $\bar{S}_4$ are disjoint. Let $d > 0$ be the distance between $\bar{S}_1$ and $\bar{S}_3$ on c and let $d_2 > 0$ be the distance of $\bar{S}_2$ and $\bar{S}_4$ on c .

Let $\{s_{1,1}\}$ be the set of arc components of $\bar{S}_1$ on c . We claim that there exists a point of $\bar{S}_3$ between any two of these arc components. Since $\bar{S}_3$ is at a non-zero distance d_1 from $\bar{S}_1$, and c

(even the modified c) is of finite arc length, this will establish that $\bar{S}_1$ has only a finite number of components. Let $s_{1,1}$ and $s_{1,j}$ be two distinct components of $\bar{S}_1$. Suppose there is no point of $\bar{S}_3$ between them on at least one side. Let q_1 and q_2 be the end points between which there exists no point of $\bar{S}_3$. It follows from the definition of S_3 that the positive semi-characteristic through each point of the closed arc $q_1 q_2$ either enters U or runs into one of the components $s_{1,1}$ or $s_{1,j}$. If a positive semi-characteristic enters U it must remain in $\bar{U}$, since the part of U which it enters is bounded by the arc $q_1 q_2$ and the positive semi-characteristics through q_1 and q_2 , which in turn could leave $\bar{U}$ only through the arc $q_1 q_2$ on c , which is impossible. Similarly, the negative semi-characteristics through the points of the arc $q_1 q_2$ either enter E before entering U or run into $s_{1,1}$ or $s_{1,j}$. Hence, all points between q_1 and q_2 would belong to $\bar{S}_1$, which is a contradiction. Thus there was a point of $\bar{S}_3$ between $s_{1,1}$ and $s_{1,j}$. It follows that $\bar{S}_1$ only has a finite number of arc components on c .

The components of $\bar{S}_1$ are closed intervals or single points. The interior of a closed interval of $\bar{S}_1$ together with the positive trajectories through the interior points of this closed interval will be called a stable sector based on p . Since $\bar{S}_1$ only has a finite number of components there exist only a finite number of stable sectors.

By an argument identical with the one of the preceding two paragraphs it can be shown that there exists a point of $\bar{S}_4$ between any two points of $\bar{S}_2$ and hence only a finite number of arc components of $\bar{S}_2$. An unstable sector based on p will be the interior of such an arc component of $\bar{S}_2$ together with the negative trajectories through the interior points of the arc component. At this point we modify c so that the trajectories of an unstable sector actually cross c from the inside to the outside and the trajectories of a stable sector if pursued in the negative direction actually cross c from the inside to the outside. This can be accomplished by using arcs on the trajectories through the boundary points of $\bar{S}_2$ and $\bar{S}_1$ respectively as parts of c .

9. $\{c - (\text{Interior of } \bar{S}_1) - (\text{Interior of } \bar{S}_2)\}$ consists of a finite number of closed intervals $F_1, F_2, \dots, F_m$. We focus our attention on a particular F_1 and further subdivide it. A point q of F_1 will be called distinguished if either C_q^+ or C_q^- tends to p and the entire arc (qp) on the trajectory is contained in U . We shall speak of stable and unstable distinguished points and stable and unstable distinguished arcs. We note that the change in c which eliminated points of Class I and II (Section 8) has the additional effect that there exists a distance $d_3 > 0$ between the stable and unstable distinguished points of F_1 .

We claim that the stable and unstable distinguished points have

to alternate on F_1 . For suppose q_1 and q_j are unstable distinguished points of F_1 . Points on the arc $(q_1 q_j)$ sufficiently near q_1 are of Class IX. Since we have no unstable interval as part of F_1 the negative trajectories through such points can not tend to p and hence have to leave $\bar{U}$ through points of the arc $q_1 q_j$. Hence there exists a point r on the open arc $q_1 q_j$ with the following two properties: C_r^- leaves $\bar{U}$ at r ; C_r^+ does not tend to p and reaches a point of U before it reaches a point of E . Let J be an arc component around r on $q_1 q_j$ of points which have the same properties as r . At least one of the two end points of J will have to be a distinguished stable point.

It follows from the last two paragraphs that there exists only a finite number of stable and unstable points on F_1 . The distinguished stable and unstable arcs divide $(\bar{U} - \text{stable sectors} - \text{unstable sectors})$ into a finite number of sectors which we shall call stable-unstable sectors until they are further classified. We note that the end-points of F_1 are always distinguished points so that the stable and unstable sectors are divided from the stable-unstable ones by distinguished arcs. When we speak of a stable-unstable sector we shall include the bounding distinguished arcs. Such an arc then may belong to two stable-unstable sectors and each stable-unstable sector contains in its boundary one stable and one unstable distinguished arc.

A stable-unstable sector will be called a hyperbolic sector based on p if there exist points r_1 arbitrarily close to p in the sector such that both $C_{r_1}^+$ and $C_{r_1}^-$ contain points outside $\bar{U}$. A stable-unstable sector which is not hyperbolic will be called an elliptic sector based on p . In the case of a hyperbolic sector it is easy to show that actually all points in the sector have the behavior of the sequence r_1 . In the case of an elliptic sector it is always possible to modify c so that the part of c which belongs to the boundary of the sector is a trajectory and that in this way the whole sector is bounded by a single trajectory which tends in both the positive and negative direction to p . It is clear then that the trajectories inside an elliptic sector all have to tend to p in both the positive and the negative direction. Figure 15 exhibits the four possible types of sectors based on a sector point (Type A_3 Singular Point).

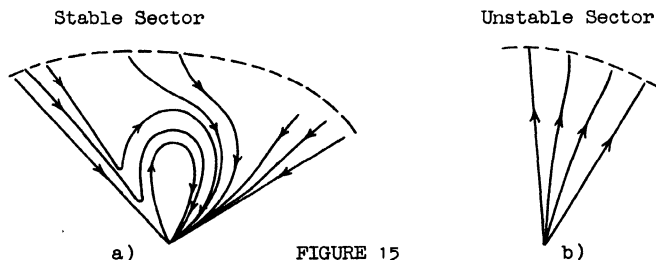
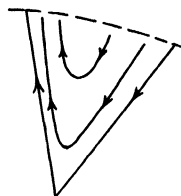


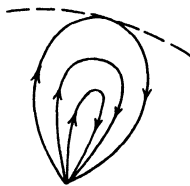
FIGURE 15

Hyperbolic Sector



c)

Elliptic Sector



d)

FIGURE 15

Summarizing then, a singular point is either a center point (Type B) or a source point (Type A_1) or a sink point (Type A_2) or a sector point (Type A_3); in the last case it consists of a finite number of stable, unstable, hyperbolic, and elliptic sectors. A sector point without elliptic sectors will be called a saddle point.

We note at this point that the notion of center point, sink point, source point, and sector point does not depend on the curve c chosen. The number of sectors does depend on c , but in subsequent considerations we shall always assume that a curve c was used which, in addition to satisfying our other requirements, gave rise to the least number of sectors.

III. Inequalities for Complicated Singular Points

10. In this part the restrictions of the first part are somewhat relaxed. The results obtained in Part III can be derived from the results of Part IV but the proofs here are much simpler. In particular, we shall prove certain inequalities under the following hypothesis:

HYPOTHESIS $\#_2$: The conditions are the same as in $\#_1$ with the following exceptions. The limit sets may contain any kind of singular point different from a center point and the positive limit set may also contain trajectories connecting sector points. More generally, there may be any kind of closed curve which contains both sector points and connecting trajectories provided this closed curve intersects the negative limit set only in its sector points.

NOTATIONS AND DEFINITIONS: Let

$p_1, 1 = 1, 2, \dots, N_0$ be the source points

$q_1, 1 = 1, 2, \dots, N_2$ be the sink points

$r_1, 1 = 1, 2, \dots, N_1$ be the sector points.

Let $n_{1,3}$ be the number of hyperbolic sectors and $n_{1,4}$ be the number of unstable sectors based on the sector point r_1 .

Let $N_3 = \Sigma_{n_1,3}$ and $N_4 = \Sigma_{n_1,4}$.

In what is to follow the notion of a cycle will be important: A cycle of the differential equation is a closed curve made up of sector points and trajectories with all the trajectories having the same positive direction in relation to the curve as a whole. A closed sub-manifold of M will be called special if it carries a cycle and no sink point of the differential equation.

Let N_5 be the number of special closed sub-manifolds of M .

11. We start by constructing a system of neighborhoods covering M . The neighborhoods $U(p_1)$ and $U(q_1)$ of the source and sink points respectively are constructed in the same way in which the neighborhoods of the stable and unstable singular points were constructed in Section 2. Let $Y_1, Y_2, \dots$ be the interiors of the various unstable sectors and $W_1, W_2, \dots$ be the interiors of the various stable sectors.

If r_1 is a sector point $U(r_1)$ is an open spherical neighborhood of r_1 covering r_1 , its elliptic sectors and all those trajectories which tend in both directions to r_1 and are entirely contained in one of the stable sectors of r_1 . We assume $U(r_1)$ is sufficiently small

- a) to exclude all singular points different from r_1 ,
- b) to exclude all $\overline{U(p_1)}$, $\overline{U(q_1)}$, and all $\overline{U(r_j)}$ previously constructed $j \neq 1$,
- c) to exclude the curves c used in the construction of the sectors except, of course, for the c used to construct the sectors of r_1 if r_1 has elliptic sectors,
- d) to exclude all Y_j and W_j not based on r_1 ,
- e) to intersect each Y_j and W_j based on r_1 in a single open spherical neighborhood but not include any of the W_j or Y_j completely.

DEFINITION: There exists a certain number of distinguished unstable arcs which emerge from r_1 (see Section 9) and which do not form part of the boundary of an elliptic sector based on r_1 . Let $C_{1,j}$ be the set of trajectories which start as distinguished unstable arcs at r_1 . (We shall speak of distinguished trajectories.) A distinguished trajectory will be said to be of the first kind if it tends to a sink. A distinguished trajectory will be said to be of the second kind if it tends to a sector point (which may be r_1 itself). A distinguished trajectory will be said to be of the third kind if it winds towards a stable limit cycle. This last kind of trajectory will sometimes be referred to as a winding trajectory.

If $C_{1,j}$ is of the first kind with q_k as end point then there exist points $s_{1,j}$ and $t_{1,j}$ on $C_{1,j}$ such that the arcs $[r_1, s_{1,j}]$ and $[t_{1,j}, q_k]$ on $C_{1,j}$ belong to $U(r_1)$ and $U(q_k)$ respectively. $U(C_{1,j})$ then will be an open spherical neighborhood of the arc $s_{1,j}t_{1,j}$ which is sufficiently small

- a) to exclude all singular points and all distinguished trajectories except $C_{1,j}$ and those winding $C_{k,1}$ which have a positive limit set which includes $C_{1,j}$
- b) to exclude all $\overline{U(p_1)}$, all $\overline{U(q_j)}$ $j \neq k$, and all $\overline{U(r_j)}$ $j \neq 1$
- c) to exclude all $U(C_{k,1})$ previously constructed
- d) to exclude all W_1 and all stable and unstable distinguished arcs except for the unstable distinguished arc from which $C_{1,j}$ started
- e) to exclude all Y_1 not based on r_1 and among those Y_1 based on r_1 all those which are not partially bounded by the distinguished arc from which $C_{1,j}$ started
- f) for $U(r_1) \cap U(C_{1,j})$ and $U(q_k) \cap U(C_{1,j})$ each to consist of a single open spherical neighborhood which excludes r_1 and q_k .

Note that d) and f) together imply that the intersection of $U(C_{1,j})$ with the closure of any sector based on r_1 is empty unless the sector is one of the two sectors which were separated by the distinguished unstable arc from which $C_{1,j}$ starts. We assume that the neighborhoods of all the distinguished trajectories of the first kind have been constructed.

If $C_{1,j}$ is of the second kind with the sector point r_k as its positive limit set there exist points $s_{1,j}$ and $t_{1,j}$ on $C_{1,j}$ such that the arcs $[r_1, s_{1,j}]$ and $[t_{1,j}, r_k]$ are entirely contained in $U(r_1)$ and $U(r_k)$ respectively. $U(C_{1,j})$ then will be an open spherical neighborhood of the arc $[s_{1,j}, t_{1,j}]$ on $C_{1,j}$ which is sufficiently small

- a) to exclude all singular points and all distinguished trajectories except $C_{1,j}$ and those winding $C_{k,1}$ which have a positive limit set which includes $C_{1,j}$
- b) to exclude all $U(p_1)$, all $U(q_j)$, and all $U(r_j)$ unless j equals 1 or k
- c) to exclude all $U(C_{1,j})$ previously constructed
- d) to exclude all W_1 except those based on r_k and among those W_1 based on r_k all those which together with their boundary do not intersect $C_{1,j}$ arbitrarily close to r_k
- e) to exclude all stable and unstable arcs except for the

- unstable distinguished arc from which $C_{1,j}$ started and the stable arc to which $C_{1,j}$ may give rise to near r_k
- f) to exclude all Y_1 not based on r_1 and among those Y_1 which are based on r_1 all those which are not partially bounded by the distinguished arc from which $C_{1,j}$ started
 - g) for $U(r_1) \cap U(C_{1,j})$ and $U(r_k) \cap U(C_{1,j})$ each to consist of a single open spherical neighborhood which excludes r_1 and r_k ; if $r_1 = r_k$ we may assume that $U(r_1)$ was sufficiently small so that the intersection consists of two spherical neighborhoods, one at each end.

Note that d) and f) together imply that the intersection of $U(C_{1,j})$ with the closure of any sector based on r_1 and r_k is empty unless the closure of this sector is actually intersected by $C_{1,j}$. We now assume that the neighborhoods of all the distinguished trajectories of the first and second kind have actually been constructed.

If $C_{1,j}$ is of the third kind it tends toward a closed curve consisting of sector points and trajectories. If such a closed curve is part of the stable limit set of some trajectory it must be made up of sector points, distinguished trajectories of the second kind or boundaries of elliptic sectors. In general, there may be curves, of course, connecting sector points which are not distinguished trajectories of the second kind or boundaries of elliptic sectors, but these are not part of the stable limit set. Thus, the neighborhoods of the sector points and distinguished trajectories of the second kind cover the closed curve toward which $C_{1,j}$ winds. There exists a point $s_{1,j}$ on $C_{1,j}$ such that the arc $r_1 s_{1,j}$ on $C_{1,j}$ is entirely contained in $U(r_1)$. There also exists a point $t_{1,j}$ on $C_{1,j}$ such that the part of the trajectory $C_{1,j}$ beyond $t_{1,j}$ remains in previously chosen neighborhoods which cover the closed curve toward which $C_{1,j}$ winds. $U(C_{1,j})$ is now an open spherical neighborhood of the arc $s_{1,j} t_{k,j}$ sufficiently small so that

- a) $U(C_{1,j})$ intersects $U(r_1)$ in a single open spherical neighborhood which does not include r_1 ,
- b) $U(C_{1,j})$ intersects one of the neighborhoods covering the closed curve toward which it winds in a single open spherical neighborhood (this may require some modification of the neighborhoods previously constructed) which includes no singular point,
- c) With exceptions a) and b) $U(C_{1,j})$ does not intersect any of the $\overline{U(r_j)}$ or any of the $\overline{U(r_{k,l})}$ previously constructed,

- d) $U(C_{1,j})$ does not intersect any of the other distinguished trajectories.
- e) $U(C_{1,j})$ does not intersect any of the $\overline{U(p_1)}$ or $\overline{U(q_1)}$,
- f) $U(C_{1,j})$ excludes all Y_1 not based on r_1 and among those Y_1 which are based on r_1 all those which are not partially bounded by the distinguished arc from which $C_{1,j}$ started
- g) $U(C_{1,j})$ excludes all W_1 and all stable and unstable distinguished arcs except for the distinguished unstable arc from which $C_{1,j}$ started.

12. Let $V = \Sigma U(p_1) + \Sigma Y_1$. It is clear that there exists a set U obtained from V by isotopic deformation such that the manifold M is entirely covered by $U + \Sigma U(r_1) + \Sigma U(q_1) + \Sigma U(C_{1,j})$. The construction of such a deformation was described in Section 3. We note that the W_1 are not needed since an open neighborhood of each sector point is covered by the $U(r_1)$ and this neighborhood was sufficiently large to include those trajectories which are entirely contained in a stable sector and which tend in both directions to r_1 .

We now proceed to relate the number of singular points to the topological structure.

LEMMA 12.1.

$$B_0(U) = N_0 + N_4$$

$$B_1(U) = 0$$

$$B_2(U) = 0$$

PROOF OF LEMMA 12.1. Lemma 12.1 is a direct consequence of the construction of U .

13. In this section we add the various $U(C_{1,j})$. In this connection we shall use the following notation: When considering the addition of $U(C_{1,j})$, $P(C_{1,j})$ will denote U plus whatever $U(C_{1,j})$ had previously been added. $d_{1,j}^k$ will denote $B_k[P(C_{1,j}) + U(C_{1,j})] - B_k[P(C_{1,j})]$. We start by adding the neighborhoods of the $C_{1,j}$ of the third kind.

LEMMA 13.1. If $C_{1,j}$ is of the third kind (winding $C_{1,j}$)

$$B_0[P(C_{1,j}) + U(C_{1,j})] - B_0[P(C_{1,j})] \leq 0$$

$$B_1[P(C_{1,j}) + U(C_{1,j})] - B_1[P(C_{1,j})] \leq 1$$

$$B_2[P(C_{1,j}) + U(C_{1,j})] - B_2[P(C_{1,j})] \leq 0$$

PROOF OF LEMMA 13.1. We shall again use the Meyer-Vietoris Sequence described in Section 3. $P(C_{1,j})$ intersects $U(C_{1,j})$ in two spherical neighborhoods if $U(C_{1,j})$ was originally chosen sufficiently small. The following sequence is obtained:

$$0 \rightarrow H_2[P(C_{1,j})] \rightarrow H_2[P(C_{1,j}) + U(C_{1,j})] \rightarrow 0 \rightarrow H_1[P(C_{1,j})] \\ 0 \leftarrow H_0[P(C_{1,j}) + U(C_{1,j})] \leftarrow I + H_0[P(C_{1,j})] \xleftarrow{f} \mathbb{Z}I \xleftarrow{f} H_1[P(C_{1,j}) + U(C_{1,j})]$$

Where I again stands for the additive group of integers. That f can not be trivial follows from the construction of the Meyer-Vietoris Sequence. Hence the image of g has at most one generator. It then follows by inspection of the sequence that $d_{1,j}^0$ and $d_{1,j}^2$ are smaller than or equal to zero and that $d_{1,j}^1$ is smaller than or equal to one.

We now assume that all the neighborhoods of the distinguished trajectories of the third kind have been added. We next add the neighborhoods of the $C_{1,j}$ of the first and second kind.

LEMMA 13.2. If $C_{1,j}$ is of the first or second kind and if $P(C_{1,j})$ includes all the neighborhoods of the $C_{k,1}$ of the third kind

$$B_0[P(C_{1,j}) + U(C_{1,j})] - B_0[P(C_{1,j})] \leq 0 \\ B_1[P(C_{1,j}) + U(C_{1,j})] - B_1[P(C_{1,j})] \leq 1 \\ B_2[P(C_{1,j}) + U(C_{1,j})] - B_2[P(C_{1,j})] \leq 0$$

PROOF OF LEMMA 13.2. The proof of Lemma 13.2 is identical with the proof of Lemma 13.1.

Lemma 13.3 follows immediately from Lemmas 12.1, 13.1, and 13.2 and the fact that the number of distinguished unstable arcs at a sector point is smaller than or equal to the number of hyperbolic sectors at this sector point.

LEMMA 13.3.

$$B_0[U + \Sigma U(C_{1,j})] \leq N_0 + N_4 \\ B_1[U + \Sigma U(C_{1,j})] \leq N_3 \\ B_2[U + \Sigma U(C_{1,j})] \leq 0$$

14. We next add the neighborhoods of the sector points, $U(r_1)$. In this connection $P(r_1)$ will denote at each stage $U + \Sigma U(C_{1,j})$ plus whatever $U(r_j)$ have previously been added.

LEMMA 14.1.

$$B_0[P(r_1) + U(r_1)] - B_0[P(r_1)] \leq 0$$

$$B_1[P(r_1) + U(r_1)] - B_1[P(r_1)] \leq -1$$

$$B_2[P(r_1) + U(r_1)] - B_2[P(r_1)] \leq 0$$

unless there exists a special closed submanifold of M entirely covered by $P(r_1) + U(r_1)$ but not entirely covered by $P(r_1)$ in which case the change in the first and zeroth, Betti number is less than or equal to 0 and the change in the second Betti number equals one.

PROOF OF LEMMA 14.1. The intersection of $U(r_1)$ with $P(r_1)$ consists of a single open annular region if $U(r_1)$ was chosen sufficiently small which we may assume. The following Meyer-Vietoris Sequence is obtained:

$$\begin{array}{ccccccc} 0 \rightarrow H_2[P(r_1)] & \rightarrow & H_2[P(r_1) + U(r_1)] & \rightarrow & I & \xrightarrow{h} & H_1[P(r_1)] \\ & & & & & & \downarrow \\ 0 \leftarrow H_0[P(r_1) + U(r_1)] & \leftarrow & H_0[P(r_1)] & \leftarrow & I & \xleftarrow{f} & I \xleftarrow{g} H_1[P(r_1) + U(r_1)] \end{array}$$

Again f cannot be trivial. Hence g is trivial. We first assume that h is not trivial. Then it follows from the sequence that the second and zeroth Betti number can not increase and that the first Betti number actually has to decrease.

If h is not trivial the non-bounding cycle of the intersection bounds in $P(r_1)$. It follows that there exists a closed submanifold entirely covered by $P(r_1) + U(r_1)$ but not entirely covered by $P(r_1)$. There are no sink points on this submanifold since the $U(q_1)$ have not been added yet. It is easy to show that the submanifold carries a cycle of the differential equation. If there exist elliptic sectors or winding distinguished trajectories on the submanifolds we are finished. Hence we may assume that there are none. Since there are no sink points and no winding trajectories on the submanifold a distinguished trajectory leaving r_1 has to tend to some other sector point (there has to be a distinguished trajectory $C_{1,j}$ leaving r_1 since there are no elliptic sectors). But by the same argument there has to be a distinguished trajectory leaving the sector point to which $C_{1,j}$ tends. This new trajectory again has to tend to a sector point and so forth. Since there is only a finite number of sector points this process

$$B_2(M) \leq N_5 + N_2$$

REMARK 15.1. The result seems unsymmetric since one would expect to get similar results for B_0 and B_2 which are, after all, the same. Actually it is possible to get more symmetric results by deforming in the direction of $(-t)$ even though $\mathcal{H}_2$ was not symmetric. But we shall not try such an approach at this time and instead postpone it to the next part where we shall have a symmetric hypothesis. However, we note that the estimate of B_2 is better than the estimate of B_0 .

REMARK 15.2. Something like the number N_5 is essential to the theory. The two examples which follow show vector fields on the sphere and on the

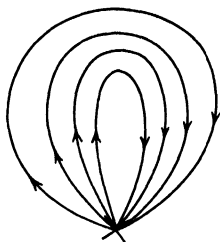


FIGURE 16

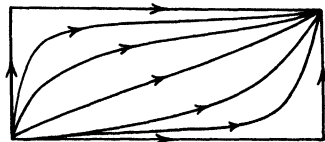


FIGURE 17

torus which give rise to neither a sink nor a stable limit cycle. In fact, there does not seem to be anything else which characterizes these vector fields except that they give rise to a cycle (not a limit cycle) and no sink points. Of course, instead of N_5 we could have used the number of stable limit cycles plus the number of stable sectors, but this estimate is much worse. Figure 16 shows the construction in one half of the sphere only; the construction in the other half is to be identical. Figure 17 is to be made into a torus by double identification.

IV. Inequalities for Limit Cycles and Complicated Singular Points

16. In this part we change the hypothesis to include limit cycles and center points.

HYPOTHESIS $\mathcal{H}_3$. Hypothesis $\mathcal{H}_3$ is identical with hypothesis $\mathcal{H}_1$ except that we now permit the limit sets to contain any kind of singular point, any kind of closed curve consisting of singular points and trajectories, and any kind of closed curve trajectory. The restriction on closed curve solutions and closed curves consisting of singular points and trajectories are completely dropped except that we require the number of closed trajectories to be bounded in the following sense: For each center point z_1 there exists a closed trajectory C_1 such that

- a) C_1 is contractible to z_1
- b) C_1 is a limit cycle from the outside
- c) there exists an open spherical neighborhood around z_1 which includes C_1 and is free of singular points
- d) there exists only a finite number of closed curve trajectories in $(M - (\Sigma \text{ Interior of } C_1) - (\Sigma C_1))$.

NOTATION. In what is to follow the various symbols will have the same meaning as in Part III.

$P_1, P_2, \dots, P_{N_0}$ are the source points.

$r_1, r_2, \dots, r_{N_1}$ are the sector points.

$q_1, q_2, \dots, q_{N_2}$ are the sink points.

N_3, N_4 and N_5 are the total number of hyperbolic sectors, stable sectors and special closed submanifolds respectively, where a submanifold now is considered special only if in addition to the previously stated properties it carries no stable or stable-unstable limit cycle. (See Section 10.)

Certain new definitions will also be needed. A center point, z_1 will be called stable or unstable depending on whether C_1 (See #3.) is stable or unstable from the outside. Let $z_1, z_2, \dots, z_{N_6}$ be the stable center points and let $z_{N_6+1}, z_{N_6+2}, \dots, z_{N_6+N_7}$ be the unstable center points.

By hypothesis there exists only a finite number of closed curve trajectories in $(M - \Sigma \text{ Interior of } C_1 - \Sigma C_1)$. We shall call these simple limit cycles, reserving the term limit cycle for a larger class. A simple limit cycle is stable $\iff$ it has two sides and is stable on both, or if it has only one side and is stable on that one side. The terms stable-unstable simple limit cycle and unstable simple limit cycle are defined similarly.

Let $E_1, E_2, \dots, E_{N_8}$ be the stable simple limit cycles.

Let $F_1, F_2, \dots, F_{N_9}$ be the stable-unstable simple limit cycles.

Let $G_1, G_2, \dots, G_{N_{10}}$ be the unstable simple limit cycles.

An alternating sequence of trajectories and sector points may form the complete limit set for certain other trajectories. We call such a sequence a limiting configuration. If such a limiting configuration has two sides (as in the case of an orientable manifold) both sides cannot form complete limit sets. Otherwise, the sector points involved would not be singular points. So whether a limiting configuration has two sides or not there is always a preferred side associated with it, this being the side

on which it forms a complete limit set. Any part of a limiting configuration or in fact the whole limiting configuration may be part of some other limiting configuration on the non-preferred side, of course. See Figure 18.

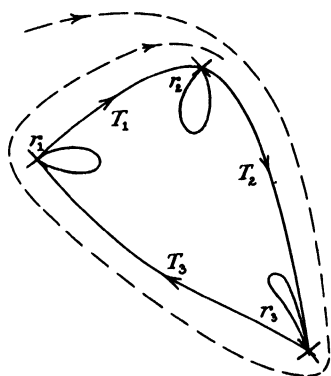


FIGURE 18

In Figure 18 the sector points r_1 , r_2 , and r_3 and the trajectories T_1 , T_2 , T_3 form a limiting configuration from the outside only. The inside may be part of a limiting configuration but cannot form a complete limiting configuration. Let $H_1, H_2, \dots, H_{N_{11}}$ be the unstable limiting configurations and $K_1, K_2, \dots, K_{N_{12}}$ be the stable limiting configurations.

The term limit cycle will be used for either simple limit cycles or limiting configurations.

17. We next construct a system of neighborhoods covering M . The neighborhoods of the sink and source points, $U(q_1)$ and $U(p_1)$, are constructed as before, only we now assume that they are sufficiently small so that their closure does not intersect any limit cycle or any of the curves C_1 which enclosed the center points. (See Sections 11 and 16.) The $Y_1, Y_2, \dots$, and $W_1, W_2, \dots$ will be as in Section 11 the interiors of the various unstable and stable sectors.

If z_1 is an unstable center point C_1 was unstable from the outside. There exists then a closed curve D_1 such that

- D_1 is outside C_1 and homotopic to C_1
- the closed cell bounded by D_1 is at a non-zero distance from all the singular points other than z_1 , all the $U(p_1)$, $U(z_j)$ previously constructed, $Y_1, W_1, U(q_1)$, C_j if $j \neq 1$, all the limit cycles, and all the trajectories which tend in both directions to singular points. (We recall that all the simple limit cycles were contained in $(M - \Sigma \text{ Interior of } C_j - \Sigma C_j)$ by definition and that there was only a finite number of them.)
- the solution curves through points of D_1 all cross from the inside to the outside.

Let $U(z_1)$ be the inside of the closed curve D_1 .

If z_1 is a stable center point $U(z_1)$ is constructed in the

same way except that we now require that solution curves through each point of the closed curve D_1 cross from the outside to the inside.

If E_1 is a stable simple limit cycle $U(E_1)$ will be an open annular region which is sufficiently small to exclude all singular points, other limit cycles, $U(p_1)$, $U(q_1)$, $U(z_1)$, Y_1 , W_1 , and all trajectories which tend in both directions to singular points.

If F_1 is a stable-unstable limit cycle we shall associate with it two neighborhoods, to be defined presently, $V(F_1)$ and $U(F_1)$. If F_1 is stable-unstable there always exists a closed curve, d , homotopic to F_1 on the side on which F_1 is unstable such that all solution curves through points of d cross d from the open annular region between d and the unstable side of F_1 to the other side of d . We shall mean by $V(F_1)$ the open unstable region between the unstable side of F_1 and d where d is taken sufficiently close to F_1 so that

- a) $V(F_1)$ is at a non-zero distance from all the singular points, all the limit cycles except F_1 , and all the trajectories which tend in both directions to singular points,
- b) the intersection of $V(F_1)$ with all the $U(p_1)$, $U(q_1)$, $U(E_1)$, $U(z_1)$, W_1 , Y_1 , and all the $V(F_j)$ previously constructed is empty.

$U(F_1)$ will be an open annular neighborhood of F_1 which does not contain all of $V(F_1)$ and intersects $V(F_1)$ in a single open annular region. It is assumed that $U(F_1)$ is sufficiently small for the intersection of $U(F_1)$ with all singular points, all limit cycles except F_1 , all trajectories which tend in both directions to singular points, all $U(p_1)$, all $U(q_1)$, all $U(z_1)$, all Y_1 , all W_1 , all $U(E_1)$, all $V(F_j)$ $j \neq 1$, and all $U(F_j)$ previously constructed to be empty.

If G_1 is an unstable simple limit cycle let $U(G_1)$ be an open annular neighborhood of G_1 bounded by one or two simple closed curves, depending on whether G_1 has one or two sides, which is sufficiently small so that

- a) the solution curves through points of the boundary of $U(G_1)$ cross from $U(G_1)$ to $M - U(G_1)$,
- b) the intersection of $U(G_1)$ with all singular points, all limit cycles except G_1 , all trajectories which tend in both directions to singular points, all $U(p_1)$, all $U(q_1)$, all $U(z_1)$, all Y_1 , all W_1 , all $U(E_1)$, all $U(F_1)$, all $V(F_1)$, and all $U(G_j)$ previously constructed to be empty.

If H_1 is an unstable limiting configuration there exists a

closed curve d homotopic to the limiting configuration on its preferred side such that all solution curves through points of d cross d from the open annular region between d and the preferred side of H_1 to the other side of d . $U(H_1)$ will then be the open annular region between H_1 and the appropriate d where d is taken sufficiently close to H_1 so that

- a) the intersection of $U(H_1)$ with all simple limit cycles, all $U(p_1)$, all $U(q_1)$, all $U(E_1)$, all $U(z_1)$, all $U(F_1)$, all $V(F_1)$, all $U(G_1)$, all Y_1 , all W_1 , and all r_1 not part of H_1 is empty.
- b) the intersection of $U(H_1)$ with those trajectories which tend in both directions to singular points is empty unless such a trajectory is part of H_1 in which case the intersection consists of the whole trajectory, but is not part of $U(H_1)$ itself
- c) the intersection of $U(H_1)$ with the sector points which are part of H_1 consists of those sector points, but again the intersection is not part of $U(H_1)$ itself
- d) the intersection of $U(H_1)$ with any $U(H_j)$ previously constructed is empty unless H_1 and H_j share certain trajectories and singular points in which case the intersection again consists of those trajectories and singular points but is not contained in either $U(H_1)$ or $U(H_j)$.

The neighborhoods of the sector points r_1 and the distinguished trajectories $C_{1,j}$ are constructed as in Section 11. We shall again speak of distinguished trajectories of the first, second, and third kind. Certain slight modifications are needed.

$U(r_1)$ is assumed sufficiently small so that in addition to fulfilling the requirements of Section 11 the intersection of $U(r_1)$ with the closures of the neighborhoods of center points, simple limit cycles, and unstable limiting configurations not containing r_1 is empty. The $U(C_{1,j})$ are sufficiently small now so that in addition to fulfilling the requirements of Section 11 the intersection of $U(C_{1,j})$ with the closures of the neighborhoods of center points, simple limit cycles and unstable limiting configurations not containing $C_{1,j}$ is empty. If an unstable limiting configuration H_k contains $C_{1,j}$ let the intersection of $U(H_k)$ with $U(C_{1,j})$ consist of a single open spherical neighborhood.

18. Let $V = \{EU(p_1) + \Sigma Y_1 + \Sigma U(G_1) + \Sigma U(H_1) + \Sigma V(F_1) + \Sigma U(z_1)\}$ where the last sum extends only over the unstable center points. It is clear that there exists a set U obtained from V by isotopic deformation

such that

$$U + \Sigma U(q_1) + \Sigma U(r_1) + \Sigma U(C_{1,j}) + \Sigma U(E_1) + \Sigma U(F_1) + \Sigma U(z_1)$$

cover M where the last sum extends only over the stable center points.

We indicate briefly how the deformation is carried out. In the case of a source point, unstable center point, or unstable simple limit cycle we always reparametrize so that the parametric value of the solution curves is constant on the boundary of the chosen neighborhoods. In the case of an unstable sector we reparametrize so that the parametric value is constant on the part of the boundary of the sector which is not a distinguished arc or a singular point (we recall that the unstable sectors were modified so that all solution curves crossed this part of the boundary from the inside to the outside). In the case of the neighborhood of a stable-unstable simple limit cycle or an unstable limiting configuration we reparametrize so that the value of the parameter remains constant on the curves which we called d in the construction of the neighborhoods of stable-unstable simple limit cycles and unstable simple limiting configurations. (See Section 17.) V can then be expanded homotopically to arbitrarily large parametric values.

LEMMA 18.1.

$$B_0(U) \leq N_0 + N_4 + N_7 + N_9 + N_{10} + N_{11}$$

$$B_1(U) \leq N_9 + N_{10} + N_{11}$$

$$B_2(U) = 0$$

PROOF OF LEMMA 18.1. Lemma 18.1 is a consequence of the definition of U .

LEMMA 18.2.

$$B_0[U + \Sigma U(r_1) + \Sigma U(C_{1,j})] \leq N_0 + N_4 + N_7 + N_9 + N_{10} + N_{11}$$

$$B_1[U + \Sigma U(r_1) + \Sigma U(C_{1,j})] \leq N_9 + N_{10} + N_{11} + N_3 + N_5 - N_1$$

$$B_2[U + \Sigma U(r_1) + \Sigma U(C_{1,j})] \leq N_5$$

PROOF OF LEMMA 18.2. The proof of this lemma is identical with the proofs of Lemmas 13.1, 13.2, 13.3, 14.1, and 14.2 in Sections 13 and 14.

It remains to add the neighborhoods of the stable limit cycles, sinks, stable-unstable limit cycles and stable center points. We shall first add the neighborhoods of the stable limit cycles, $U(E_1)$. In this

connection $P(E_1)$ will denote $U + \Sigma U(r_1) + \Sigma U(C_{1,j})$ plus whatever $U(E_j)$ have previously been added.

LEMMA 18.3. If a certain number or none of the $U(E_j)$ have been added to $U + \Sigma U(r_1) + \Sigma U(C_{1,j})$, but $U(E_1)$ has not been added previously

$$B_0[P(E_1) + U(E_1)] - B_0[P(E_1)] \leq 0$$

$$B_1[P(E_1) + U(E_1)] - B_1[P(E_1)] \leq 1$$

$$B_2[P(E_1) + U(E_1)] - B_2[P(E_1)] \leq 1$$

PROOF OF LEMMA 18.3. $U(E_1)$ and $P(E_1)$ intersect in two open annular regions. The following Meyer-Vietoris Sequence (see Section 3) is obtained. I again stands for the additive group of integers.

$$\begin{array}{ccccccc} 0 \longrightarrow H_2[P(E_1)] & \longrightarrow & H_2[P(E_1) + U(E_1)] & \longrightarrow & 2I & \xrightarrow{h} & I + H_1[P(E_1)] \\ & & & & & & \searrow \\ 0 \longleftarrow H_0[P(E_1) + U(E_1)] & \longleftarrow & I + H_0[P(E_1)] & \xleftarrow{f} & 2I & \xleftarrow{g} & H_1[P(E_1) + U(E_1)] \end{array}$$

That f can not be trivial follows from the construction of the Meyer-Vietoris Sequence; hence the image of g has at most one generator. The image of h has one or two generators since the non-bounding cycle of the intersection does not bound in $U(E_1)$. If the image of h has two generators none of the Betti numbers can increase; if the image of h has one generator, only the first and second Betti numbers can increase (by at most one each). The Lemma follows.

We assume now that all the neighborhoods of the stable simple limit cycles have been added. We next add the chosen neighborhoods of the stable-unstable simple limit cycles. When adding a neighborhood $U(F_1)$ $P(F_1)$ will denote $U + \Sigma U(C_{1,j}) + \Sigma U(r_1) + \Sigma U(E_1)$ plus whatever $U(F_j)$ have previously been added.

LEMMA 18.4. If a certain number of the $U(F_j)$ or none have been added to $U + \Sigma U(C_{1,j}) + \Sigma U(r_1) + \Sigma U(E_1)$ but $U(F_1)$ has not been previously added

$$B_0[P(F_1) + U(F_1)] - B_0[P(F_1)] \leq 0$$

$$B_1[P(F_1) + U(F_1)] - B_1[P(F_1)] \leq 1$$

$$B_2[P(F_1) + U(F_1)] - B_2[P(F_1)] \leq 1$$

PROOF OF LEMMA 18.6. The proof of Lemma 18.6 is identical with the proof of Lemma 18.5.

19. Lemmas 18.2, 18.3, 18.4, 18.5, and 18.6 together imply the following theorem:

THEOREM III_A. Under Hypothesis $\mathcal{H}_3$

$$B_0(M) \leq N_0 + N_4 + N_7 + N_9 + N_{10} + N_{11}$$

$$B_1(M) \leq N_8 + 2N_9 + N_{10} + N_{11} + N_3 + N_5 - N_1$$

$$B_2(M) \leq N_5 + N_2 + N_6 + N_9 + N_8$$

By reversing the direction of all trajectories it is possible to prove the symmetric Theorem III_B of which we shall note only two of the relations

THEOREM III_B. Under Hypothesis $\mathcal{H}_3$

$$B_1(M) \leq N_8 + 2N_9 + N_{10} + N_{12} + N_5' - N_1$$

$$B_2(M) \leq N_5' + N_0 + N_7 + N_9 + N_{10}$$

where N_5' , defined symmetrically to the old N_5 , is the number of closed submanifolds of M which carry a cycle (see Section 10) of the differential equation but on which there are no source points, unstable center points, unstable simple limit cycles, or stable-unstable simple limit cycles.

It is clear that on each of the submanifolds just defined there exists either an unstable sector or an unstable limiting configuration. Hence, $N_5' \leq N_4 + N_{11}$. Since the second and zeroth Betti number are the same we may therefore substitute N_5' for $N_4 + N_{11}$ in the estimate of B_0 of Theorem III_A. It also follows from the definitions that each of the closed submanifolds counted under N_5 or N_5' contains a sector point; thus either N_5 or N_5' is $\leq N_1$ which gives a simpler but weaker form for the estimate of the first Betti number. The following theorem then follows.

THEOREM III. Under Hypothesis $\#_3$

$$B_0(M) \leq N_0 + N_7 + N_9 + N_{10} + N_5^I \leq N_0 + N_7 + N_9 + N_{10} + N_4 + N_{11}$$

$$\left. \begin{array}{l} B_1(M) \leq N_8 + 2N_9 + N_{10} + N_3 + N_{11} + N_5 - N_1 \\ \text{or} \\ B_1(M) \leq N_8 + 2N_9 + N_{10} + N_3 + N_{12} + N_5^I - N_1 \end{array} \right\} \leq N_8 + 2N_9 + N_{10} + N_3 + \min(N_{11}, N_{12})$$

$$B_2(M) \leq N_2 + N_6 + N_9 + N_8 + N_5 \leq N_2 + N_6 + N_9 + N_8 + N_{13} + N_{12}$$

where all the constants except N_{13} were defined in Section 16. N_{13} is the number of stable sectors.

OBSERVATION ON UNIQUENESS. It is clear from the proof of Theorem IV that this theorem holds even if the differential equation under consideration does not have unique solutions in sufficiently small neighborhoods of the sink points, source points, center points, and simple limit cycles. In Section 17 a series of neighborhoods were constructed. Some of these neighborhoods were subsequently deformed and some were not deformed. If a neighborhood was not deformed it does not matter from the point of view of the proof if the differential equation did not have unique trajectories in this neighborhood. If a neighborhood does get deformed we can permit non-uniqueness in a sufficiently smaller sub-neighborhood, without affecting the deformation process.

REMARK ON THEOREM III. We shall show in a series of examples that the constants on the right hand side of the inequalities represent roughly the best possible estimates. In particular, we shall show that something like N_5 and N_5^I is required and that counting the stable-unstable limit cycles twice in the estimate of the first Betti number is essential. The first two examples were previously given in Section 15 but the interpretation was different.

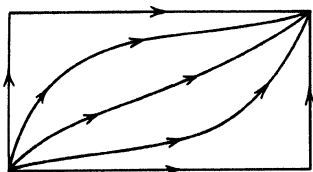


FIGURE 19

EXAMPLE 1. A vector field on the torus is obtained by double identification of the sides of the rectangle of Figure 19. In this case $N_3 = 2$, $N_1 = 1$, $N_5 = 1$, $N_5^I = 1$, $N_4 = 1$, and $N_{13} = 1$; all other $N_i = 0$.

Hence Theorem III gives

$$B_0 \text{ (torus)} \leq 1$$

$$B_1 \text{ (torus)} \leq [2 + 1 - 1]$$

$$B_2 \text{ (torus)} \leq 1$$

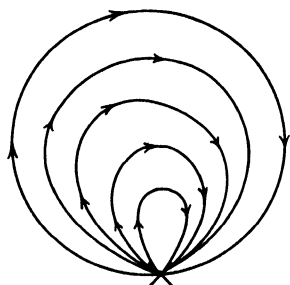


FIGURE 20

EXAMPLE 2. A vector field is constructed on the sphere of which we only show the front half. The behavior on the rear half is assumed identical. There exists a single singular point which is a sector point with two elliptic, one stable, and one unstable sector. Hence, $N_1 = 1$, $N_2 = 1$, $N_3 = 1$, $N_4 = 1$, and $N_{13} = 1$ while all the other $N_i = 0$. Theorem III then takes the following form:

$$B_0 \text{ (sphere)} \leq 1$$

$$B_1 \text{ (sphere)} \leq 1 - 1$$

$$B_2 \text{ (sphere)} \leq 1.$$

EXAMPLE 3. It is easy to construct a vector field on the torus such that the limit set consists of a single stable-unstable simple limit cycle. In this case, $N_9 = 1$ and all the other $N_i = 0$. Theorem III then takes the following form:

$$N_0 \text{ (torus)} \leq 1$$

$$N_1 \text{ (torus)} \leq 2$$

$$N_2 \text{ (torus)} \leq 1.$$

V. Applications of the Inequalities

20. We assume throughout this part that Hypothesis $\#_3$ is satisfied.

- a) Suppose there exists a component of M without a cycle, i.e., no closed curve consisting of both singular points and trajectories can be given a direction which coincides with the

direction of all its trajectories. Suppose there also exist no center points on this component of M . Then N_5 and N_5' are zero. It follows from the inequality involving the second Betti number that there has to be a stable simple limit cycle, or a stable-unstable simple limit cycle, or a sink point on this component. (The important point here is that the stable sectors do not enter the picture.) Similarly, there has to be an unstable simple limit cycle, or a stable-unstable simple limit cycle, or a source point on this component.

- b) Suppose M is a k -handled sphere and suppose there are no sector points on M . Then N_1, N_3, N_5 and N_{11} are zero. The inequality for the first Betti number implies that the number of limit cycles is greater than or equal to k .
- c) Suppose M is again a k -handled sphere. Suppose none of the sector points has hyperbolic sectors. Suppose also that there exists either a sink point or a source point or a center point. Then either N_5 or N_5' is zero. It then follows from one of the two inequalities for $B_1(M)$ that the number of limit cycles is greater than or equal to $(k + \text{one half times the number of sector points})$. This inequality improves if we assume that all the simple limit cycles are either stable or unstable. In this case the number of limit cycles is greater than or equal to $(2k + \text{the number of sector points})$.
- d) Suppose M is again a k -handled sphere. N_5 is either zero or 1. If k is large it follows from the inequality for B_1 that there are either many hyperbolic sectors or many limit cycles.

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VIII. BANACH SPACES AND THE PERTURBATION OF ORDINARY DIFFERENTIAL EQUATIONS

George Hufford¹

1. Introduction

The terminology and theory of Banach spaces can be a powerful tool for the study of ordinary differential equations. A striking example is the way in which the fundamental existence theorem is very elegantly proved using Schauder's fixed point theorem (1). A second example, one which we should like to present here, concerns an autonomous system of differential equations having a periodic solution and the results of giving this system a periodic perturbation. This is a problem which has been previously treated by Krylov and Bogoliubov (2) and by Levinson (3); but, as will be seen, our treatment is more general and, in a sense, simpler.

I wish to acknowledge here my indebtedness to Professor S. P. Diliberto of the University of California who suggested the problem, and to Professor S. Lefschetz who was a constant encouragement.

Let us begin, then, with an autonomous system of n differential equations which we write in the vector form

$$(1.1) \quad \frac{dx}{dt} = X(x)$$

where, say, the second derivatives of $X(x)$ satisfy a Lipschitz condition. Let there exist a solution

$$(1.2) \quad x = u(t)$$

of period 1.

¹ This is a dissertation presented to the faculty of Princeton University in candidacy for the degree of Doctor of Philosophy. It was done in connection with an Office of Ordinance Research Project, OOR Contract No. DA-36-034-ORD-640.

Then we can write the equation of first variation

$$(1.3) \quad \frac{dv}{dt} = \frac{\partial X(u(t))}{\partial x} v$$

where the right hand side is meant to represent the $n \times n$ Jacobian matrix acting on the n -vector v . This equation is a linear homogeneous equation with periodic coefficients, and if V is a matrix consisting of n columns of linearly independent solutions, then it is well known that V may be written as

$$(1.4) \quad V(t) = P(t)e^{At}$$

where A is a constant (complex) matrix and $P(t)$ a non-singular matrix of period 1. The real parts of the characteristic roots of A are Liapounov's characteristic numbers. Their values are decisive in the analysis that follows.

In order to emphasize the special role that the trajectory (1.2) plays, we shall want to transform our equations to coordinates $(y_1, \dots, y_{n-1}, \theta)$ defined by some such equation as

$$(1.5) \quad x = u(\theta) + S(\theta)y$$

where $S(\theta)$ is a suitable $n \times (n-1)$ matrix of period 1 and of, say, class C^4 . That such a transformation can be satisfactorily made, at least within a neighborhood of the trajectory (1.2), has been shown by Levinson (3). It might well be profitable to present here a paraphrase of his proof.

We remark firstly that (1.5) will always define a new set of coordinates in some neighborhood of the curve (1.2) provided that at all points on this curve the Jacobian satisfies the inequality

$$(1.6) \quad \frac{\partial(x_1 \dots x_n)}{\partial(y_1 \dots y_{n-1}, \theta)} = |S(\theta) \quad X(u(\theta))| \neq 0$$

Thus we must look for a matrix $S(\theta)$ for which this is true. Note that the regularity of $S(\theta)$ need not concern us, since as long as we can find a continuous matrix satisfying (1.6), we can always find a second matrix having any required number of derivatives but whose elements differ from those of the original matrix by as little as desired, thus maintaining the inequality (1.6).

Now if the dimension of our space $n = 2$, then put

$$(1.7) \quad S(\theta) = \begin{pmatrix} X_1(u(\theta)) \\ -X_2(u(\theta)) \end{pmatrix}$$

and (1.6) follows immediately. Suppose, therefore, $n > 2$, and consider the curve in n -space defined by $x = X(u(\theta))$. It is of class C^2 and so cannot be space filling. Thus there exists a ray extending from the origin which does not intersect this curve, and we may, without loss of generality, suppose that this ray coincides with the x_1 -axis. In other words, we can assume

$$\sum_{i=2}^n X_i(u(\theta))^2 > 0.$$

Then put

$$(1.8) \quad S(\theta) = \begin{pmatrix} KX_2(u(\theta)) & KX_3(u(\theta)) & \dots & KX_n(u(\theta)) \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \dots & 1 \end{pmatrix}$$

The Jacobian (1.6) becomes

$$(1.9) \quad \begin{vmatrix} KX_2(u(\theta)) & KX_3(u(\theta)) & \dots & KX_n(u(\theta)) & X_1(u(\theta)) \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & \dots & 1 & X_n(u(\theta)) \end{vmatrix} \\ = -(-1)^n \left[X_1(u(\theta)) - K \sum_{i=2}^n X_i(u(\theta))^2 \right]$$

and this, if K is only large enough, will never equal zero.

To continue, after we have made the coordinate transformation (1.5) the trajectory (1.2) is defined by $y = 0$, $\theta = t$, and the equation (1.1) becomes

$$(1.10) \quad \frac{dy}{dt} = Y(y, \theta) \\ \frac{d\theta}{dt} = \theta(y, \theta)$$

where $Y(y, \theta)$ is, of course, an $(n-1)$ -vector. Note that Y and θ both have period 1 in θ , that both have second derivatives which satisfy a Lipschitz condition, and that

$$\begin{aligned} Y(0, \theta) &= 0 \\ \theta(0, \theta) &= 1 \end{aligned} \quad (1.11)$$

Now consider a new, perturbed system of differential equations

$$\frac{dx}{dt} = X(x, t, \epsilon) \quad (1.12)$$

where ϵ is a real number (or, with only very slight modifications in what follows, an element of any locally compact topological space), where $X(x, t, \epsilon)$ is a function whose second derivatives with respect to x and t are continuous in ϵ and satisfy a Lipschitz condition in x and t , where $X(x, t, \epsilon)$ is of period T in t , and where finally $X(x, t, 0) = X(x)$.

The curve defined by (1.2) is, of course, not a trajectory of our perturbed system, but there is nothing to prevent us from continuing to describe points in its neighborhood, and the motion of such points, by means of the coordinates (y, θ) ; and this we shall henceforth do. Thus, our perturbed system (1.12) must be rewritten in the form

$$\begin{aligned} \frac{dy}{dt} &= Y(y, \theta, t, \epsilon) \\ \frac{d\theta}{dt} &= \theta(y, \theta, t, \epsilon) \end{aligned} \quad (1.13)$$

where the functions Y and θ satisfy the same conditions as does X and in addition have period 1 in θ .

Now, let us suppose that at $t = 0$ we start at a point (y_0, θ_0) and follow its motion until $t = T$. Assuming that ϵ is small enough and (y_0, θ_0) close enough to the curve (1.2) so that this motion does not leave the domain of definition of our new coordinate system, the point (y_0, θ_0) will have been carried to a point given by

$$\begin{cases} y = F(y_0, \theta_0, \epsilon) \\ \theta = \psi(y_0, \theta_0, \epsilon) \end{cases} \quad (1.14)$$

where, of course, $F(y, \theta, \epsilon)$ is an $(n-1)$ -dimensional function and $\psi(y, \theta, \epsilon)$ a simple one dimensional function. Note, too, that since the

differential equation (1.13) has the period T , it is also true that the point (y_0, θ_0) at $t = kT$, k being any integer, will be carried into the point (1.14) at $t = (k + 1)T$.

Some properties of the functions F and ψ are noteworthy. Since, in a sense, they are the general solution to (1.13), and since the right hand side of (1.13) has second derivatives which satisfy a Lipschitz condition, it follows that the second derivatives of $F(y, \theta, \epsilon)$, $\psi(y, \theta, \epsilon)$ with respect to y, θ exist and are continuous in y, θ, ϵ . Furthermore, F has the period 1 in θ , while $\psi(y, \theta + 1, \epsilon) = \psi(y, \theta, \epsilon) + 1$. And finally, since $y = 0, \theta = t, \epsilon = 0$, is a solution to (1.13) we see that

$$(1.15) \quad \begin{aligned} F(0, \theta, 0) &= 0 \\ \psi(0, \theta, 0) &= \theta + T \end{aligned}$$

Now let us wander afield a bit to consider the Banach space $\mathcal{C}^1$ whose elements α are the $(n-1)$ -dimensional, real valued functions $\alpha(\theta)$ of class C^1 and of period 1. The norm is defined as

$$(1.16) \quad \|\alpha\| = \max\{|\alpha_j(\theta)|, |d\alpha_j(\theta)/d\theta|; j = 1 \dots n-1, 0 \leq \theta \leq 1\}$$

Clearly, if the element α lies in a small enough neighborhood of 0, it may be interpreted as a closed curve

$$(1.17) \quad \alpha: y = \alpha(\theta)$$

in n -space lying in a neighborhood of the curve (1.2).

Consider the mapping of n -space into itself which is obtained by following along for a given length of time the trajectories resulting from (1.13). Since it carries closed curves of class C^1 into closed curves of class C^1 , it is very natural to think of this mapping as the transformation of an element α of $\mathcal{C}^1$ into some other element of $\mathcal{C}^1$. In particular, if we consider the curve $y = \alpha(\theta)$ at $t = kT$, then at $t = (k + 1)T$ it will have been carried into a curve which we can define parametrically by

$$(1.18) \quad f(\alpha, \epsilon): \begin{cases} y = F(\alpha(\phi), \phi, \epsilon) \\ \theta = \psi(\alpha(\phi), \phi, \epsilon) \end{cases}$$

By this notation we mean that $f(\alpha, \epsilon)$ is the new element of $\mathcal{C}^1$ and that it corresponds to a certain closed curve in n -space whose coordinates (y, θ) are prescribed in terms of the parameter ϕ by means of the equations (1.18).

Thus for each value of ϕ we begin at $t = kT$ with the initial point $y_0 = \alpha(\phi)$, $\theta_0 = \phi$ which is located on the curve α , and at $t = (k+1)T$ this point will, because of (1.14), have been carried into the corresponding point of (1.18).

One further provision, however, must be made before we can say that this new curve is actually an element of $\mathcal{C}^1$: we must require y to be a single valued function of θ , which in turn requires θ to be a monotonic function of ϕ . But

$$(1.19) \quad \frac{d\theta}{d\phi} = \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial y} \frac{d\alpha}{d\phi}$$

and since, because of (1.15), $\partial \psi(0, \theta, 0)/\partial \theta = 1$, the derivative will always be positive provided that ϵ and $\|\alpha\|$ are small enough.

To summarize, for each ϵ small enough we have defined a transformation of a certain open set $\mathcal{D} \subset \mathcal{C}^1$ which contains the origin, into the space $\mathcal{C}^1$. It is with these transformations that we are to be concerned.

Note especially, that if there exists an element α such that $f(\alpha, \epsilon) = \alpha$, in other words, if there exists a fixed point, then any point of n -space which at $t = 0$ is on the curve represented by α , will always return to this same curve at $t = kT$. The motion of such points is therefore described as motion on a two dimensional torus imbedded in n -space, and the analysis of Poincaré and Denjoy (4) will be applicable. From this and the fact that $\psi(\alpha(\phi), \phi, \epsilon)$ is twice differentiable, it follows immediately that the motion is almost periodic.

Our purpose, then, will be to show that such fixed points do exist under proper conditions. To be specific, we shall prove the following theorem.

THEOREM 1.1. If the only characteristic number of the equation of first variation (1.3) which is zero is the one associated with the solution $v = X(u(t))$, then for any ϵ small enough there exists a unique closed curve of class C^2 which after a time T is carried by the motion resulting from the perturbed equation (1.12), into itself.

2. The Implicit Function Theorems

We turn now to a few theorems involving functions on arbitrary Banach spaces. These theorems, together with their proofs, are very similar to theorems proved by Lusternik and Sobolev (5). However, we include them

here both because of slight differences in their formulation and because the above reference is rather inaccessible to the English reading public.

Let $f(\alpha)$ be a function from an open set $\mathfrak{D}$ of a real Banach space $\mathfrak{X}$ into another Banach space $\mathfrak{Y}$. We say that $f(\alpha)$ is differentiable at α if for any $\eta \in \mathfrak{X}$ the limit

$$(2.1) \quad \lim_{s \rightarrow 0} \frac{1}{s} [f(\alpha + s\eta) - f(\alpha)]$$

exists and is a bounded linear transformation in η . This bounded linear transformation is called the derivative and is denoted by $f'(\alpha)$ or, variously, by $f_\alpha(\alpha)$. Thus the limit (2.1) is written $f'(\alpha)\eta$ or $f_\alpha(\alpha)\eta$.

Now, let β^* be any element of the adjoint space $\mathfrak{Y}^*$. Notice then that for small enough s the expression $\beta^*f(\alpha + s\eta)$ is a function of the real variable s , and furthermore, that it is differentiable at $s = 0$,

$$(2.2) \quad \left. \frac{d}{ds} \beta^*f(\alpha + s\eta) \right|_{s=0} = \lim_{s \rightarrow 0} \frac{1}{s} [\beta^*f(\alpha + s\eta) - \beta^*f(\alpha)] = \\ = \beta^*f'(\alpha)\eta$$

THEOREM 2.1. Let α_1, α_2 be two points such that the straight line joining them $L(\alpha_1, \alpha_2) = \{\alpha | \alpha = \alpha_1 + s(\alpha_2 - \alpha_1), 0 \leq s \leq 1\}$ lies wholly within $\mathfrak{D}$. Suppose that the function $f(\alpha)$ is differentiable at every point of this line and that the derivative $f'(\alpha)$ is continuous along it. Then

$$(2.3) \quad \|f(\alpha_1) - f(\alpha_2)\| \leq \sup_{\alpha \in L(\alpha_1, \alpha_2)} \|f'(\alpha)\| \cdot \|\alpha_2 - \alpha_1\|$$

PROOF. Let β^* be any element of $\mathfrak{Y}^*$. Then the expression $\beta^*f(\alpha_1 + s(\alpha_2 - \alpha_1))$ is a continuously differentiable function of the real variable s , and it follows from the mean value theorem that there exists a corresponding s^* , $0 \leq s^* \leq 1$, such that

$$(2.4) \quad \beta^*f(\alpha_2) - \beta^*f(\alpha_1) = \frac{d}{ds} \beta^*f(\alpha_1 + s(\alpha_2 - \alpha_1))$$

Thus from (2.2)

$$\begin{aligned}
 (2.5) \quad & \|\beta^*[f(\alpha_2) - f(\alpha_1)]\| \leq \|\beta^*\| \cdot \|f'(\alpha_1 + s^*(\alpha_2 - \alpha_1))\| \cdot \|\alpha_2 - \alpha_1\| \\
 & \leq \|\beta^*\| \sup_{\alpha \in L(\alpha_1, \alpha_2)} \|f'(\alpha)\| \cdot \|\alpha_2 - \alpha_1\|.
 \end{aligned}$$

But it is a known theorem that there exists one β^* , $\|\beta^*\| = 1$, for which the left hand side attains the maximum value of $\|f(\alpha_2) - f(\alpha_1)\|$. The theorem follows.

THEOREM 2.2. The implicit function theorem. Let $f(\alpha, \mu)$ be a continuous function on $\mathfrak{D} \times \mathfrak{E}$ to $\mathfrak{Y}$, where $\mathfrak{E}$ is any topological space. And suppose

- i) for every $\alpha \in \mathfrak{D}$, $\mu \in \mathfrak{E}$, $f(\alpha, \mu)$ is differentiable and in addition the derivative $f_\alpha(\alpha, \mu)$ is continuous in both α and μ ;
- ii) there exist α_0 and μ_0 such that $f(\alpha_0, \mu_0) = 0$;
- iii) the inverse $f_\alpha(\alpha_0, \mu_0)^{-1}$ exists and is a bounded linear transformation.

Then there exists a neighborhood N_0 of μ_0 such that for any $\mu \in N_0$ there is a corresponding unique $\alpha(\mu)$ for which $f(\alpha(\mu), \mu) = 0$. Furthermore, the function $\alpha(\mu)$ is continuous.

PROOF. Define the function

$$(2.6) \quad g(\alpha, \mu) = f_\alpha(\alpha_0, \mu_0)^{-1} [f_\alpha(\alpha_0, \mu_0)\alpha - f(\alpha, \mu)].$$

This is a continuous function on $\mathfrak{D} \times \mathfrak{E}$ to $\mathfrak{X}$, and for every $\alpha \in \mathfrak{D}$, $\mu \in \mathfrak{E}$ it has a derivative

$$(2.7) \quad g_\alpha(\alpha, \mu) = f_\alpha(\alpha_0, \mu_0)^{-1} [f_\alpha(\alpha_0, \mu_0) - f_\alpha(\alpha, \mu)]$$

which is also continuous in both α and μ . Furthermore, we have the particular values

$$\begin{aligned}
 (2.8) \quad & g(\alpha_0, \mu_0) = \alpha_0 \\
 & g_\alpha(\alpha_0, \mu_0) = 0.
 \end{aligned}$$

From here we proceed almost exactly as in the proof of the ordinary implicit function theorem. Choose any positive $k < 1$; then there exists

a ρ and a neighborhood N_1 of μ_0 such that $\|\alpha - \alpha_0\| < \rho$ and $\mu \in N_1$ imply

$$(2.9) \quad \|g_\alpha(\alpha, \mu)\| < k$$

Again, there is also a neighborhood N_2 such that $\mu \in N_2$ implies

$$(2.10) \quad \|g(\alpha_0, \mu) - \alpha_0\| < \rho(1 - k)$$

Let $N_0 \subset N_1 \cap N_2$. Then $\mu \in N_0$, $\|\alpha_1 - \alpha_0\| < \rho$, $\|\alpha_2 - \alpha_0\| < \rho$ imply

$$\sup_{\alpha \in L(\alpha_1, \alpha_2)} \|g_\alpha(\alpha, \mu)\| < k;$$

and therefore by Theorem 2.1 and from the fact that $g(\alpha, \mu)$ is continuously differentiable along $L(\alpha_1, \alpha_2)$ it follows that

$$(2.11) \quad \|g(\alpha_2, \mu) - g(\alpha_1, \mu)\| < k\|\alpha_2 - \alpha_1\|$$

Now, define by iteration

$$(2.12) \quad \alpha_{n+1} = g(\alpha_n, \mu)$$

This forms a sequence which is well defined as long as $\|\alpha_n - \alpha_0\| < \rho$, and we must show that this inequality is indeed true for all n . Suppose, first that $\|\alpha_1 - \alpha_0\| < \rho \dots \|\alpha_n - \alpha_0\| < \rho$; then we have for $j = 0, \dots, n$,

$$(2.13) \quad \begin{aligned} \|\alpha_{j+1} - \alpha_j\| &= \|g(\alpha_j, \mu) - g(\alpha_{j-1}, \mu)\| < k\|\alpha_j - \alpha_{j-1}\| \\ &< k^2\|\alpha_{j-1} - \alpha_{j-2}\| < \dots < k^j\|\alpha_1 - \alpha_0\| \\ &= k^j\|g(\alpha_0, \mu) - \alpha_0\| < \rho(1 - k)k^j. \end{aligned}$$

Thus

$$(2.14) \quad \begin{aligned} \|\alpha_{n+1} - \alpha_0\| &\leq \|\alpha_{n+1} - \alpha_n\| + \|\alpha_n - \alpha_{n-1}\| + \dots + \|\alpha_1 - \alpha_0\| \\ &< \rho(1 - k)(k^n + k^{n-1} + \dots + k^0) < \rho \end{aligned}$$

Whence $\|\alpha_n - \alpha_0\| < \rho$ for all n .

Furthermore, for any n, p we have

$$\begin{aligned}
 \|\alpha_{n+p} - \alpha_n\| &\leq \|\alpha_{n+p} - \alpha_{n+p-1}\| + \dots + \|\alpha_{n+1} - \alpha_n\| \\
 (2.15) \qquad &< \rho(1-k)(k^{n+p-1} + \dots + k^n) < \rho k^n.
 \end{aligned}$$

And since this last tends to zero as n goes to infinity, $\{\alpha_n\}$ is a Cauchy sequence and so converges to some element $\alpha(\mu)$. Because $g(\alpha, \mu)$ is continuous, $g(\alpha_n, \mu) \rightarrow g(\alpha(\mu), \mu)$ and $\alpha(\mu) = g(\alpha(\mu), \mu)$ which is to say $f(\alpha(\mu), \mu) = 0$.

To prove uniqueness, suppose there were a second element α' such that $\|\alpha' - \alpha_0\| < \rho$ and $f(\alpha', \mu) = 0$. Then

$$(2.16) \qquad \|\alpha(\mu) - \alpha'\| = \|g(\alpha(\mu), \mu) - g(\alpha', \mu)\| < k\|\alpha(\mu) - \alpha'\|$$

from which it follows that $\alpha' = \alpha(\mu)$.

Finally, if $\mu, \mu' \in N_0$, we have

$$\begin{aligned}
 \|\alpha(\mu) - \alpha(\mu')\| &= \|g(\alpha(\mu), \mu) - g(\alpha(\mu'), \mu')\| \\
 (2.17) \qquad &\leq \|g(\alpha(\mu), \mu) - g(\alpha(\mu), \mu')\| + \|g(\alpha(\mu), \mu') - g(\alpha(\mu'), \mu')\| \\
 &< \|g(\alpha(\mu), \mu) - g(\alpha(\mu), \mu')\| + k\|\alpha(\mu) - \alpha(\mu')\|
 \end{aligned}$$

or

$$(2.18) \qquad \|\alpha(\mu) - \alpha(\mu')\| < \frac{1}{1-k} \|g(\alpha(\mu), \mu) - g(\alpha(\mu), \mu')\|$$

so that continuity follows. Thus the theorem is completely proved.

Using the same notation we have the formal statement

$$\begin{aligned}
 f_\alpha(\alpha, \mu)^{-1} &= \sum_{n=0}^{\infty} \left\{ f_\alpha(\alpha_0, \mu_0)^{-1} \left[f_\alpha(\alpha_0, \mu_0) - f_\alpha(\alpha, \mu) \right] \right\}^n f_\alpha(\alpha_0, \mu_0)^{-1} \\
 (2.19) \qquad &= \sum_{n=0}^{\infty} \left[g_\alpha(\alpha, \mu) \right]^n f_\alpha(\alpha_0, \mu_0)^{-1}.
 \end{aligned}$$

But from (2.11) it follows that for $\|\alpha - \alpha_0\| < \rho$ and $\mu \in N_0$, the series converges and hence truly represents the inverse. In particular, we have shown that for $\mu \in N_0$ the inverse $f_\alpha(\alpha(\mu), \mu)^{-1}$ exists and is bounded.

THEOREM 2.3. If $\mathfrak{E}$ is itself a Banach space or an open subset of a Banach space, and if, besides the properties

set forth in Theorem 2.2, there is a point $\mu \in N_0$ such that when $\alpha = \alpha(\mu)$ the function $f(\alpha, \mu)$ has a partial derivative with respect to μ , then the function $\alpha(\mu)$ is also differentiable at μ and

$$(2.20) \quad \alpha'(\mu) = -f_{\alpha}(\alpha, \mu)^{-1} f_{\mu}(\alpha, \mu)$$

PROOF. Let η be any element of $\mathfrak{S}$ (or of the Banach space in which $\mathfrak{S}$ lies). Then we know already from Theorem 2.2 that $\alpha(\mu + s\eta)$ is a continuous function in s . But we can show even more than this; for from (2.18) there follows

$$(2.21) \quad \begin{aligned} \left\| \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] \right\| &< \frac{1}{1-K} \left\| \frac{1}{s} [g(\alpha, \mu + s\eta) - g(\alpha, \mu)] \right\| \\ &\leq \frac{1}{1-K} \|f_{\alpha}(\alpha_0, \mu_0)^{-1}\| \cdot \left\| \frac{1}{s} [f(\alpha, \mu + s\eta) - f(\alpha, \mu)] \right\| \\ &\rightarrow \frac{1}{1-K} \|f_{\alpha}(\alpha_0, \mu_0)^{-1}\| \cdot \|f_{\mu}(\alpha, \mu)\| \end{aligned}$$

so that the quantity $\left\| \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] \right\|$ is bounded. Then also

$$(2.22) \quad \begin{aligned} &\left\| \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] + f_{\alpha}(\alpha, \mu)^{-1} f_{\mu}(\alpha, \mu)\eta \right\| \\ &\leq \|f_{\alpha}(\alpha, \mu)^{-1}\| \cdot \|f_{\alpha}(\alpha, \mu) \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] + f_{\mu}(\alpha, \mu)\eta\| \\ &\leq \|f_{\alpha}(\alpha, \mu)^{-1}\| \cdot \|f_{\mu}(\alpha, \mu)\eta - \frac{1}{s} [f(\alpha, \mu + s\eta) - f(\alpha, \mu)]\| \\ &\quad + \|f_{\alpha}(\alpha, \mu)^{-1}\| \cdot \|f_{\alpha}(\alpha, \mu) \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] \\ &\quad + \frac{1}{s} [f(\alpha, \mu + s\eta) - f(\alpha, \mu)]\|. \end{aligned}$$

In this last, the first term goes to zero because of the definition of $f_{\mu}(\alpha, \mu)\eta$. And as for the second term, since $f(\alpha, \mu) = f(\alpha(\mu + s\eta), \mu + s\eta) = 0$, we have

$$(2.23) \quad \begin{aligned} &\|f_{\alpha}(\alpha, \mu) \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] + \frac{1}{s} [f(\alpha, \mu + s\eta) - f(\alpha, \mu)]\| \\ &= \left\| \frac{1}{s} [h(\alpha(\mu + s\eta)) - h(\alpha(\mu))] \right\| \end{aligned}$$

where

$$(2.24) \quad h(\beta) = f_{\alpha}(\alpha, \mu)\beta - f(\beta, \mu + s\eta).$$

Now at all points $\beta \in \mathfrak{B}$, $h(\beta)$ is differentiable, and

$$(2.25) \quad h'(\beta) = f_{\alpha}(\alpha, \mu) - f_{\alpha}(\beta, \mu + s\eta).$$

Furthermore, we have on the one hand that $\beta = \alpha(\mu)$ and $s = 0$ gives $h'(\beta) = 0$, and on the other hand that $h'(\beta)$ is continuous in both β and s . Therefore it follows from Theorem 2.1 that

$$(2.26) \quad \left\| \frac{1}{s} [h(\alpha(\mu + s\eta)) - h(\alpha(\mu))] \right\| \leq \sup \{ \|h'(\beta)\|, \beta \in L(\alpha(\mu + s\eta), \alpha(\mu)) \} \\ \cdot \left\| \frac{1}{s} [\alpha(\mu + s\eta) - \alpha(\mu)] \right\| \rightarrow 0.$$

And so the theorem is proved.

3. The Proof of Theorem 1.1

As it turns out, the proof of Theorem 1.1 is accomplished best not by treating directly the function $f(\alpha, \epsilon)$ of (1.18), but instead by considering the function

$$(3.1) \quad g(\beta, \gamma, \epsilon): \quad \begin{cases} y = F(\beta(\phi), \phi, \epsilon) \\ \theta = \psi(\gamma(\phi), \phi, \epsilon) \end{cases}$$

Defining $\mathfrak{D}$ to be the same subset of $\mathbb{G}^1$ which was introduced in Section 1, it is clear that $g(\beta, \gamma, \epsilon)$ is defined for all $\beta, \gamma \in \mathfrak{D}$ and ϵ small enough. And what we now intend to show is that there is an ϵ_0 such that for any $|\epsilon| < \epsilon_0$ there exists a corresponding $\alpha \in \mathfrak{D}$ for which $\alpha = g(\alpha, \alpha, \epsilon)$.

First of all, we remark that for any β, γ, ϵ within the range of definition $g(\beta, \gamma, \epsilon)$ is differentiable with respect to β . Thus for any $\eta \in \mathbb{G}^1$

$$(3.2) \quad g_{\beta}(\beta, \gamma, \epsilon)\eta: \quad \begin{cases} y = \frac{\partial F(\beta(\phi), \phi, \epsilon)}{\partial y} \eta(\phi) \\ \theta = \psi(\gamma(\phi), \phi, \epsilon) \end{cases}.$$

In particular, we have the very important special case

$$(3.3) \quad g_{\beta}(0, 0, 0)\eta: \quad y = \frac{\partial F(0, \theta-T, 0)}{\partial y} \eta(\theta - T).$$

Furthermore, note that $g_{\beta}(\beta, \gamma, \epsilon)$ is continuous in all three of its variables.

Now consider the equation $0 = \beta - g(\beta, \gamma, \epsilon)$. We want to solve

it for β in terms of γ, ϵ . Since we have already $g(0, 0, 0) = 0$, all the conditions of Theorem 2.2 are satisfied if we can only show that $[I - g_\beta(0, 0, 0)]^{-1}$ exists and is bounded.

To do this we must consider the exact form that the derivatives $\partial F(0, \theta, 0)/\partial y$ take. But as we mentioned before, the functions $F(y, \theta, \epsilon)$, $\psi(y, \theta, \epsilon)$ are, in a sense, the general solution to (1.10), and therefore their derivatives are solutions to the equation of first variation

$$(3.4) \quad \frac{dw}{dt} = \frac{\partial(Y, \theta)}{\partial(y, \theta)} w$$

where, of course, we suppose the Jacobian matrix evaluated at $y = 0, \theta = t, \epsilon = 0$.

Now, since the transformation (1.5) is periodic and since its Jacobian vanishes nowhere, the solutions to (3.4) have characteristic numbers which are identical with those of the solutions to the original equation of first variation (1.3). Indeed, if W is a matrix of linearly independent solutions to (3.4), we can write

$$(3.5) \quad W(t) = Q(t)e^{At}$$

where A is the same constant matrix introduced in (1.4) and where $Q(t)$ is some non-singular matrix of period 1.

It follows from (1.11) that the last column of the Jacobian matrix in (3.4) consists solely of zeros. Therefore the n -vector

$$(3.6) \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is a solution, and if we suppose that initially $W(t)$ is the unit matrix then the last column is exactly this solution and so will always remain constant. Thus

$$(3.7) \quad e^A = W(1) = \begin{pmatrix} \tilde{W}(1) & 0 \\ x(1) & 1 \end{pmatrix}$$

where $\tilde{W}(t)$ is an $(n-1) \times (n-1)$ matrix and $x(t)$ an $(n-1)$ row vector. But from the form of (3.7) it is easy to see that A may be chosen so that its last column, too, consists solely of zeros, and henceforward we shall suppose

that this is so. Note finally that this implies that the last column of $Q(t)$ is also equal to (3.6).

Thus it follows that at $y = 0$, $\theta = \theta$, $\epsilon = 0$,

$$(3.8) \quad \begin{aligned} \frac{\partial(F, \psi)}{\partial(y, \theta)} &= W(\theta + T)W(\theta)^{-1} \\ &= Q(\theta + T)e^{AT}Q(\theta)^{-1}. \end{aligned}$$

But all the matrices of this equation have the same form, inasmuch as their last columns are always equal to the column vector (3.6). Thus, if we wish to determine the first $n-1$ rows and the first $n-1$ columns in this matrix equation, we see that neither the last rows nor the last columns have any effect, so that

$$(3.9) \quad \frac{\partial F(0, \theta, 0)}{\partial y} = \tilde{Q}(\theta + T)e^{\tilde{A}T}\tilde{Q}(\theta)^{-1}$$

where $\tilde{A}$, $\tilde{Q}$ are the matrices formed from the first $n-1$ rows and the $n-1$ columns of the corresponding $n \times n$ matrices A , Q .

Now we can proceed to prove the existence of $[I - g_\beta(0, 0, 0)]^{-1}$. Actually what we shall do is to prove its existence on the (complex) Banach space of all complex $(n-1)$ -dimensional functions $\alpha(\theta)$ of class C^1 and period 1. But then the transformation $I - g_\beta(0, 0, 0)$ defined in the obvious way over this new complex space cannot carry any function with non-zero imaginary part into purely real ones, because this would be tantamount to the existence of purely imaginary functions which are carried into zero, a thing which is forbidden since the inverse exists. It follows that the inverse must carry purely real functions into purely real functions, and therefore defines a bounded inverse on our original space $\mathcal{C}^1$.

Define, therefore, transformations on the complex space

$$(3.10) \quad \begin{aligned} \tilde{Q}\eta: \quad y &= \tilde{Q}(\theta)\eta(\theta) \\ \tilde{B}\eta: \quad y &= e^{\tilde{A}T}\eta(\theta - T). \end{aligned}$$

There follows immediately

$$(3.11) \quad \tilde{Q}^{-1}\eta: \quad y = \tilde{Q}(\theta)^{-1}\eta(\theta)$$

and

$$(3.12) \quad g_\beta(0, 0, 0) = \tilde{Q}\tilde{B}\tilde{Q}^{-1}$$

So that the inverse exists if and only if $(I - \tilde{B})^{-1}$ exists.

LEMMA 3.1. If none of the characteristic roots of the matrix $e^{\tilde{A}T}$ is of absolute value 1, then the transformation

$$(3.13) \quad (I - \tilde{B})\eta: \quad y = \eta(\theta) - e^{\tilde{A}T}\eta(\theta - T)$$

has a bounded inverse.

PROOF. Let R be the non-singular matrix for which $R^{-1}e^{\tilde{A}T}R$ is in canonical form. Then $(I - \tilde{B})$ has a bounded inverse if and only if

$$(3.14) \quad R^{-1}(I - \tilde{B})R\eta: \quad y = \eta(\theta) - R^{-1}e^{\tilde{A}T}R\eta(\theta - T)$$

has a bounded inverse. But this transformation acts independently on certain linear subspaces of $\mathbb{C}^1$, these subspaces being characterized as the r -dimensional, complex valued functions of class C^1 and period 1. Thus (3.14) can be written as the direct product of transformations each having the form

$$(3.15) \quad (I - \Lambda)\eta: \quad y = \eta(\theta) - \Lambda\eta(\theta - T)$$

where Λ is an $r \times r$ matrix of the form

$$(3.16) \quad \Lambda = \begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & & & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & & 1 & \lambda \end{pmatrix}$$

and $|\lambda| \neq 1$. It follows that $(I - \tilde{B})$ has a bounded inverse if and only if each of the subtransformations $(I - \Lambda)$ have bounded inverses. Let us consider the two cases that arise.

CASE I. $|\lambda| < 1$. Let

$$(3.17) \quad S = \begin{pmatrix} \delta & & & \\ & \delta^2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \delta^r \end{pmatrix}$$

be a "shear" transformation such that $|\lambda| + |\delta| < 1$. Then

$$(3.18) \quad SAS^{-1}\eta: \quad y = \begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ \delta & \lambda & & & \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ 0 & 0 & & \delta & \lambda \end{pmatrix}$$

It follows from the definition (1.16) of the norm that $\|SAS^{-1}\| = |\lambda| + |\delta| < 1$; and thus

$$(3.19) \quad (I - SAS^{-1})^{-1} = S(I - A)^{-1}S^{-1} = \sum_{n=0}^{\infty} (SAS^{-1})^n$$

converges and so defines a bounded transformation.

CASE II. $|\lambda| > 1$. In this case the matrix A has an inverse A^{-1} whose characteristic root is $1/\lambda$, and, of course, $|1/\lambda| < 1$. Then

$$(3.20) \quad A^{-1}\eta: \quad y = A^{-1}\eta(\theta + T)$$

is a well defined bounded linear transformation. From the previous case the inverse of

$$(3.21) \quad (I - A^{-1})\eta: \quad y = \eta(\theta) - A^{-1}\eta(\theta + T)$$

exists. Thus $-(I - A^{-1})^{-1}A^{-1} = (I - A)^{-1}$, and in this case, too, the inverse exists and is bounded.

Thus we have shown that all the subtransformations $I - A$ have bounded inverses, and the Lemma is proved.

But the conditions of Theorem 1.1 assure us that the only characteristic root of the matrix A which has a zero real part is the trivial zero which our machinations have eliminated from $\tilde{A}$. Thus, for every characteristic root λ of $e^{\tilde{A}T}$ there is a corresponding root ρ of $\tilde{A}$ and $|\lambda| = e^{\operatorname{Re} \rho T} \neq 1$. It follows that under the conditions of Theorem 1.1, $(I - \tilde{B})$, and hence $[I - g_\beta(0, 0, 0)]$, always has a bounded inverse.

Thus, all the conditions of Theorem 2.2 as applied to the function $\beta - g(\beta, \gamma, \epsilon)$ are satisfied, and it follows that there is a continuous function $\beta(\gamma, \epsilon)$ defined on a certain neighborhood $\|\gamma\| < \rho_1$, $|\epsilon| < \epsilon_1$, such that $\beta(\gamma, \epsilon) = g(\beta(\gamma, \epsilon), \gamma, \epsilon)$.

LEMMA 3.2. If the $(n-1)$ -dimensional function $\gamma(\theta)$ is of class C^2 , i.e., if $d\gamma/d\theta \in \mathfrak{C}^1$, and if $\|\gamma\|$, $\|d\gamma/d\theta\|$, $|\epsilon|$ are all small enough, then the function $\beta(\theta)$ corresponding to $\beta(\gamma, \epsilon)$ is also of class C^2 . Furthermore, $d\beta/d\theta$ is a continuous function of γ , $d\gamma/d\theta$, ϵ to the space $\mathfrak{C}^1$.

PROOF. From the definition of $\beta(\gamma, \epsilon)$

$$(3.22) \quad \begin{cases} \beta(\theta) = F(\beta(\phi), \phi, \epsilon) \\ \theta = \psi(\gamma(\phi), \phi, \epsilon) \end{cases}$$

for all θ . Since $\beta(\theta) \in C^1$ we can differentiate this, finding

$$(3.23) \quad \frac{d\beta}{d\theta} = \left[\frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \gamma} \frac{d\beta}{d\phi} \right] \left[\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \gamma} \frac{d\gamma}{d\phi} \right]^{-1}$$

where the derivatives are to be evaluated at the obvious points.

Now define the element

$$(3.24) \quad \xi: \quad \begin{cases} \gamma = \frac{\partial F}{\partial \theta} \left[\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \gamma} \frac{d\gamma}{d\phi} \right]^{-1} \\ \theta = \psi(\gamma(\phi), \phi, \epsilon) \end{cases}$$

and the linear transformation

$$(3.25) \quad D\eta: \quad \begin{cases} \gamma = \frac{\partial F}{\partial \gamma} \left[\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \gamma} \frac{d\gamma}{d\phi} \right]^{-1} \eta(\phi) \\ \theta = \psi(\gamma(\phi), \phi, \epsilon) \end{cases}$$

Then (3.23) can be written

$$(3.26) \quad \frac{d\beta}{d\theta} = \xi + D \frac{d\beta}{d\phi}.$$

And this is a valid equation in the space $\mathfrak{C}^0$ of continuous $(n-1)$ -dimensional functions of period 1. But D is a bounded linear transformation from $\mathfrak{C}^0$ to $\mathfrak{C}^0$ which is continuous in γ , $d\gamma/d\theta$, ϵ . Furthermore, at $\gamma = d\gamma/d\theta = 0$, $\epsilon = 0$, we have $D = g_\beta(0, 0, 0)$. Now Lemma 3.1 and its proof are equally valid in $\mathfrak{C}^0$, and therefore it follows that for $\|\gamma\|$,

$\|d\gamma/d\theta\|, \epsilon$ small enough, $(I - D)^{-1}$ exists and is bounded. Thus

$$(3.27) \quad \frac{d\beta}{d\theta} = (I - D)^{-1}\zeta$$

is the unique element of $\mathcal{C}^0$ which satisfies (3.26).

But ζ is also an element of $\mathcal{C}^1$, and furthermore it is continuous in $\gamma, d\gamma/d\theta, \epsilon$. And D is a bounded linear transformation from $\mathcal{C}^1$ to $\mathcal{C}^1$ and is again continuous in $\gamma, d\gamma/d\theta, \epsilon$. It follows that $(I - D)^{-1}\zeta$, and hence $d\beta/d\theta$, is likewise an element of $\mathcal{C}^1$ and is continuous in $\gamma, d\gamma/d\theta, \epsilon$.

Let us stop here to examine more closely the way in which ζ depends on the three variables $\gamma, d\gamma/d\theta, \epsilon$. At $\gamma = d\gamma/d\theta = 0, \epsilon = 0$, we have $\zeta = 0$, because $\beta(0, 0) = 0$ and $\partial F(0, \theta, 0)/\partial\theta = 0$. And so for $\|\gamma\|, \|d\gamma/d\theta\|, |\epsilon|$ all small enough ζ will become as small as desired. But this result is, to some extent, independent of $d\gamma/d\theta$; indeed we contend that $\zeta \rightarrow 0$ with γ, ϵ uniformly for all bounded $d\gamma/d\theta$. Now, to say that $\|\gamma\|$ goes to zero while $\|d\gamma/d\theta\|$ remains bounded is to say that $|\gamma_1(\theta)|, |d\gamma_1(\theta)/d\theta|$ go to zero for all θ while $|d^2\gamma_1(\theta)/d\theta^2|$ remains bounded. But if this is true, it can be seen (again because $\partial F(0, \theta, 0)/\partial\theta = 0$) that both $|\zeta_1(\theta)|$ and $|d\zeta_1(\theta)/d\theta|$ go to zero, so that indeed $\zeta \rightarrow 0$ in the topology of $\mathcal{C}^1$.

Furthermore, the same can be said about $d\beta/d\theta$. For suppose that σ is a positive number such that for $\|d\gamma/d\theta\| < \sigma$ and $\|\gamma\|, |\epsilon|$ small enough, Lemma 3.2 is valid and $d\beta/d\theta \in \mathcal{C}^1$; then certainly $\|(I - D)^{-1}\|$ is bounded and from (3.27) it follows that $d\beta/d\theta \rightarrow 0$ with γ, ϵ uniformly for $\|d\gamma/d\theta\| < \sigma$. In particular, we can infer the existence of values $\rho_2 \leq \rho_1, \epsilon_2 \leq \epsilon_1$ such that $\|d\gamma/d\theta\| < \sigma, \|\gamma\| < \rho_2, |\epsilon| < \epsilon_2$ imply that $d\beta/d\theta$ exists and

$$(3.28) \quad \|d\beta/d\theta\| < \sigma$$

To continue, if β, γ are both of class C^2 , the derivative of $g(\beta, \gamma, \epsilon)$ with respect to γ exists, and

$$(3.29) \quad g_\gamma(\beta, \gamma, \epsilon)\eta: \quad \begin{cases} \gamma = - \left[\frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \gamma} \frac{d\beta}{d\theta} \right] \left[\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \gamma} \frac{d\gamma}{d\theta} \right]^{-1} \frac{d\psi}{d\gamma} \eta(\phi) \\ \theta = \psi(\gamma(\phi), \phi, \epsilon) \end{cases}$$

But from Lemma 3.2, if γ is of class C^2 , and if $\|d\gamma/d\theta\| < \sigma, \|\gamma\| < \rho_2, |\epsilon| < \epsilon_2$ then $\beta(\gamma, \epsilon)$ is of class C^2 , and it follows from Theorem 2.3 that $\beta_\gamma(\gamma, \epsilon)$ exists and

$$(3.30) \quad \beta_\gamma(\gamma, \epsilon) = - \left[I - g_\beta(\beta, \gamma, \epsilon) \right]^{-1} g_\gamma(\beta, \gamma, \epsilon)$$

This again is continuous in the three variables $\gamma, d\gamma/d\theta, \epsilon$. And again since $d\beta/d\theta$ goes to zero with γ, ϵ uniformly for $\|d\gamma/d\theta\| < \sigma$ so also does $g_\gamma(\beta, \gamma, \epsilon)$ and hence $\beta_\gamma(\gamma, \epsilon)$. Thus for any positive $k < 1$, there exist values $\rho_3 \leq \rho_2, \epsilon_3 \leq \epsilon_2$ such that $\|d\gamma/d\theta\| < \sigma, \|\gamma\| < \rho_3, |\epsilon| < \epsilon_3$ all imply

$$(3.31) \quad \|\beta_\gamma(\gamma, \epsilon)\| < k$$

Now the set of elements γ for which $\|d\gamma/d\theta\| < \sigma, \|\gamma\| < \rho_3$ is a convex set, and it follows from Theorem 2.1 that if γ_1, γ_2 are any two elements of this set

$$(3.32) \quad \|\beta(\gamma_1, \epsilon) - \beta(\gamma_2, \epsilon)\| < k\|\gamma_1 - \gamma_2\|$$

Finally, define $\epsilon_0 \leq \epsilon_3$ in such a way that $|\epsilon| < \epsilon_0$ implies

$$(3.33) \quad \|\beta(0, \epsilon)\| < \rho_3(1 - k)$$

Then for any $|\epsilon| < \epsilon_0$ define by iteration

$$(3.34) \quad \begin{aligned} \alpha_0 &= 0 \\ \alpha_{n+1} &= \beta(\alpha_n, \epsilon) \end{aligned}$$

This, as we shall show, defines in the limit the function $\alpha(\epsilon)$ for which we are looking.

First, assume $\|\alpha_0\| < \rho_3, \dots, \|\alpha_n\| < \rho_3$; and assume that $d\alpha_0/d\theta, \dots, d\alpha_n/d\theta$ exist as elements of $\mathbb{C}^1$ while $\|d\alpha_0/d\theta\| < \sigma, \dots, \|d\alpha_n/d\theta\| < \sigma$. Then we are assured that α_{n+1} is defined, and from (3.28) we know that $d\alpha_{n+1}/d\theta$ is an element of $\mathbb{C}^1$ and $\|d\alpha_{n+1}/d\theta\| < \sigma$. Furthermore, because (3.32) is valid for $\alpha_0, \dots, \alpha_n$, we know that we can obtain expressions analogous to (2.13), (2.14), (2.15), thus showing that $\|\alpha_{n+1}\| < \rho_3$. It follows by induction that $\|\alpha_n\| < \rho_3, \|d\alpha_n/d\theta\| < \sigma$ for all n . Therefore we have actually defined a sequence which is a Cauchy sequence and hence converges to some element $\alpha(\epsilon)$ which satisfies the relationship

$$(3.35) \quad \alpha(\epsilon) = \beta(\alpha(\epsilon), \epsilon) = g(\alpha(\epsilon), \alpha(\epsilon), \epsilon)$$

LEMMA 3.3. If $\alpha \in \mathbb{C}^1$ satisfies (3.35) and if $\|\alpha\|$ and

$|\epsilon|$ are small enough, then the corresponding function $\alpha(\theta)$ is of class C^2 , and furthermore $d\alpha/d\theta$ is a continuous function of α , ϵ to $\mathbb{C}^1$.

PROOF. We proceed along lines similar to those used in the proof of Lemma 3.2. We have

$$(3.36) \quad \begin{cases} \alpha(\theta) = F(\alpha(\phi), \phi, \epsilon) \\ \theta = \psi(\alpha(\phi), \phi, \epsilon) \end{cases}$$

and since $\alpha(\theta) \in C^1$ we can differentiate this obtaining

$$(3.37) \quad \begin{cases} \frac{d\alpha}{d\theta} = \left[\frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \psi} \frac{d\alpha}{d\phi} \right] \left[\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \psi} \frac{d\alpha}{d\phi} \right]^{-1} \\ \theta = \psi(\alpha(\phi), \phi, \epsilon) \end{cases}$$

Now consider the function

$$(3.38) \quad d(\delta, \alpha, \epsilon): \begin{cases} \gamma = \left[\frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \psi} \delta(\phi) \right] \left[\frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \psi} \delta(\phi) \right]^{-1} \\ \theta = \psi(\alpha(\phi), \phi, \epsilon) \end{cases}$$

This can be considered as a function in δ either on $\mathbb{C}^0$ to $\mathbb{C}^0$ or as well on $\mathbb{C}^1$ to $\mathbb{C}^1$. In both spaces it is differentiable with respect to δ , the derivative $d_\delta(\delta, \alpha, \epsilon)$ is continuous in all three variables, and $d_\delta(0, 0, 0) = g_\beta(0, 0, 0)$. Thus the equation $\delta = d(\delta, \alpha, \epsilon)$ has a unique solution for δ in terms of α, ϵ . But in the space $\mathbb{C}^0$ this solution must, because of (3.37), be exactly $d\alpha/d\theta$, so that the solution in $\mathbb{C}^1$ must also be $d\alpha/d\theta$. It follows that $\delta(\alpha, \epsilon) = d\alpha/d\theta \in \mathbb{C}^1$ and is a continuous function of α, ϵ .

Finally, to complete the proof of Theorem 1.1 we must prove uniqueness. But from Lemma 3.3 it follows that any solution of (3.35) is of class C^2 , and that moreover there exists a $\rho_0 \leq \rho_3$ such that $\|\alpha\| < \rho_0$ implies $\|d\alpha/d\theta\| < \sigma$. Thus if there is a second solution α' , $\|\alpha'\| < \rho_0$, then from (3.32)

$$(3.39) \quad \|\alpha - \alpha'\| = \|\beta(\alpha, \epsilon) - \beta(\alpha', \epsilon)\| < k\|\alpha - \alpha'\|$$

and $\alpha = \alpha'$.

4. The Analytic Case

When the functions $X(x, t, \epsilon)$ of (1.12) are all analytic, then the trajectories are analytic and it might well be supposed that the fixed curve α whose existence we have demonstrated is also analytic. Indeed, Levinson (3) has stated that when the original periodic solution (1.2) is asymptotically stable, then this is in fact true. Unfortunately, there is a counter-example which seems to disprove not only the general statement, but also Levinson's more specialized statement.

This counter-example is based on the following considerations. If we were going to attempt the proof of some such statement, we should want to consider functions $\alpha(\zeta)$, $\zeta = \theta + js$, which are of period 1 and are analytic in some suitable region $|s| \leq \delta$; and we should want to define a transformation $f(\alpha, \epsilon)$ on some suitably restricted class of functions α to give an analytic function $\beta(\zeta)$ belonging to the same restricted class. But this is not in general possible, because the transformation $f(\alpha, \epsilon)$ will be forced to use values of $\alpha(\zeta)$ which lie outside of the region $|s| \leq \delta$. And it is not possible to control the behavior of such values by putting any but the most excessive restrictions on $\alpha(\zeta)$ inside this region.

To proceed to the counter-example itself, we suppose that the transformation $f(\alpha, \epsilon)$ of (1.18) can be written in the form

$$(4.1) \quad f(\alpha, \epsilon): \quad \begin{cases} y = k\alpha(\xi) + \epsilon F(\xi) \\ \zeta = \psi(\xi, \epsilon) \end{cases}.$$

where we have used the letter ξ for the parametric representation to indicate that it can take on complex values, and where k is a constant, $k < 1$, and F and ψ are analytic functions of their variables. We can solve the last of these equations, finding $\xi = \xi(\zeta, \epsilon)$; and it is quickly seen that the solution to the equation

$$(4.2) \quad \alpha(\zeta) = k\alpha(\xi) + \epsilon F(\xi)$$

is simply

$$(4.3) \quad \alpha(\zeta) = \epsilon \sum_{n=0}^{\infty} k^n F \left[\xi^{(n+1)}(\zeta) \right]$$

where $\xi^{(n)}(\zeta)$ is the n th iteration of $\xi(\zeta)$.

In particular, suppose that for every integer m , $F(\xi)$ has a

pole at $m \pm 1$; and that $\psi(\xi, \epsilon) = \xi + 1 - \epsilon^2 \sin \xi$. Then $\alpha(\xi)$ as defined in (4.3) will have poles at every point for which $\xi^{(n)}(\xi) = \pm 1$; i.e., for every $\xi = \psi^{(n)}(\pm 1, \epsilon)$. But

$$(4.4) \quad \psi(1s, \epsilon) \equiv 1(s - \epsilon^2 \sinh s) \pmod{1}$$

which, for small enough s, ϵ , is nearer to the real axis than was the original $1s$. Indeed, it follows that $\psi^{(n)}(\pm 1, \epsilon) \rightarrow 0$. In other words, the origin is a point of accumulation of poles and $\alpha(\xi)$ cannot be analytic there.

5. Final Remarks

One or two generalizations of Theorem 1.1 are immediately evident. For instance, the period T of (1.12) may change with ϵ . If this is so and if the function $T(\epsilon)$ is continuous, then we can carry through unchanged the arguments of our proof, so that the conclusion is still valid.

Again, if the functions $X(x, t, \epsilon)$ of (1.12) have greater regularity than we have required, one would expect that the resulting fixed curve whose existence is assured by our theorem, will also have a greater regularity. Indeed, we can immediately state that if $X(x, t, \epsilon)$ has the k th derivatives ($k \geq 2$) with respect to x and t which are continuous in ϵ and satisfy a Lipschitz condition in x and t , then the fixed curve α will be of class C^k . That this is true can be most simply seen by substituting everywhere in our previous proof the spaces $\mathcal{E}^{k-1}$ and $\mathcal{E}^{k-2}$ for $\mathcal{E}^1$ and $\mathcal{E}^0$, where always $\mathcal{E}^p$ is the Banach space of $(n-1)$ -dimensional functions $\alpha(\theta)$ of period 1 and of class C^p with the norm

$$(5.1) \quad \|\alpha\| = \max \left\{ |d^m \alpha_j(\theta)/d\theta^m|, j = 1 \dots n-1, m = 0 \dots p, 0 \leq \theta \leq 1 \right\}$$

An interesting example of the kind of equations to which the results of this paper apply is given by the well-known van der Pol equation with a small forcing term

$$(5.2) \quad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = \epsilon \cos \omega t$$

At $\epsilon = 0$ this equation has the unstable solution $x = 0$ as well as a stable periodic orbit encircling the origin of the $x, \dot{x}$ plane. Here the words "unstable" and "stable" are used to imply that the corresponding Liapounov characteristic numbers are in the first instance positive and in the second negative. Thus for small enough ϵ we can say this about the solutions to (5.2):

- (1) the unstable critical point at the origin is changed into an unstable periodic solution with period $2\pi/\omega$; and
- (2) the stable periodic orbit is changed into a stable set of almost periodic functions.

Furthermore, by analogy with the autonomous case one would suspect that all other solutions tend towards the latter almost periodic solutions. However, as Cartwright (6) has already pointed out, this is still an open problem.

Actually, we are being a bit glib when we brush off the second set of solutions as almost periodic functions. A better description is this: There is a closed curve $\alpha(t, \epsilon)$ which always remains near the original periodic orbit and encircles the unstable periodic solution near the origin. This closed curve $\alpha(t, \epsilon)$ forms a stable set inasmuch as any point near enough will approach it asymptotically. And, finally, $\alpha(t, \epsilon)$ is periodic, by which we mean that the entire curve repeats itself whenever t increases by $2\pi/\omega$. It follows that the solutions which go to make up $\alpha(t, \epsilon)$ are described as solutions on a torus and so, depending on ϵ , fall into one of two cases. They may all be almost periodic and have two fundamental periods which are irrationally related. Or there may be one or more solutions which are periodic having a least period $2\pi m/\omega$ where m is some suitable integer depending on ϵ ; all the other solutions will tend asymptotically towards these periodic ones.

Added in Press. W. T. Kyner has pointed out to me that the proof of Theorem 1.1 as presented in Section 3 is fallacious. The theorems of Section 2 and several lemmas of Section 3 require that $g_p(\beta, \gamma, \epsilon)$ be continuous in γ and ϵ in the uniform topology, which is not true for even the simplest examples of such functions. However, Kyner, elsewhere in this volume, has shown that Theorem 1.1 is still valid. As he demonstrates, Lemma 3.1, which is unaffected, together with the consequence that $[I - g_p(0, 0, 0)]^{-1}$ exists, are sufficient to make the present theorem a special case of his own more general result.

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IX. A FIXED POINT THEOREM*

Walter T. Kyner

SECTION 1

The purpose of this paper is to prove a fixed point theorem that is useful in the Diliberto perturbation theory. The (non-degenerate) perturbation theorems of G. Hufford [1] and M. Marcus [2] are implied by this result.

I wish to thank Professor S. P. Diliberto for suggesting this problem and for his continued interest in it. In addition, I wish to thank Professor L. Nirenberg for pointing out a mistake in an earlier "proof" of this theorem.

We are interested in a family of transformations defined on a subset of the space of real multiply periodic continuously differentiable m -dimensional vector valued functions of s real variables. If $f = (f_1, \dots, f_m)$ is such a function, then $f(\theta_1, \dots, \theta_s)$ must be periodic with period ω_k in θ_k .

We use

$$\|f\| = \max \left[\|f\|_0, \left\| \frac{\partial f}{\partial \theta_k} \right\|_0 \right],$$

where

$$\|f\|_0 = \max_1 \max_{\theta} |f_1(\theta)|.$$

For a fixed real number γ , the transformation $g = T_\gamma(f)$ is given by

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$$(1) \quad g(\bar{\theta}) = N(\theta)f(\theta) + W(f(\theta), \theta, \gamma)$$

where θ is a function of $\bar{\theta}$, f , and γ defined implicitly by

$$(2) \quad \bar{\theta} = V_{f\gamma}(\theta) = \theta + \pi + U(f(\theta), \theta, \gamma).$$

π is a constant s -vector. $N(\theta)$ is an $m \times m$ matrix whose entries are C^2 functions of θ . $U(y, \theta, \gamma)$ and $W(y, \theta, \gamma)$ are vector valued functions of $m + s + 1$ real variables whose second partial derivatives with respect to y and θ are continuous in y, θ , and γ . U, W , and N are periodic in θ . We also require that

$$W(0, \theta, 0) \equiv 0,$$

$$U(0, \theta, 0) \equiv 0,$$

and

$$\frac{\partial W(0, \theta, 0)}{\partial y} \equiv 0.$$

The transformations used by Marcus and Hufford are of this form.

To insure that equation (2) can be solved for θ , we restrict γ and $\|f\|$ so that

$$\left(\frac{\partial \bar{\theta}_i}{\partial \theta_j} \right)$$

is non-singular.

For each γ sufficiently small, we hope to find a function that is invariant under T_γ . It is necessary to make an additional restriction on the transformations considered. Hufford (for $s = 1$) assumed that all but one of the characteristic exponents of a variational equation had non-zero real parts. This hypothesis determines the form of $N(\theta)$. For his many variable result, Marcus assumed that the norm of $N(\theta)$ was less than one. Marcus' theorem does not imply Hufford's. It will be shown below that their hypotheses imply the following:

(H): For each f and γ such that $\|f\| \leq r_0$, $|\gamma| \leq \gamma_0$, define the linear transformation $L_{f\gamma}$ by

$$(3) \quad [L_{f\gamma}g](\theta) = g(V_{f\gamma}(\theta)) - N(\theta)g(\theta).$$

We assume that $L_{f\gamma}$ is invertible and that its inverse is uniformly bounded, i.e., $\|L_{f\gamma}^{-1}\| \leq A$ for all $\|f\| \leq r_0$, $|\gamma| \leq \gamma_0$.

SECTION 2

THEOREM. If T_γ is defined as above, and if the hypothesis (H) is satisfied, then there exists γ_0 such that if $|\gamma| \leq \gamma_0$, T_γ has a unique fixed point f_γ . f_γ is continuous in γ , and as $\gamma \rightarrow 0$, $f_\gamma \rightarrow 0$. Furthermore

$$\left\{ -\frac{\partial f_\gamma}{\partial \theta_k} \right\}$$

satisfy a uniform Lipschitz condition.

PROOF. If f is invariant under T_γ , then

$$(4) \quad f(V_{f\gamma}) = N(\theta)f(\theta) + W(f(\theta), \theta, \gamma).$$

Therefore

$$(5) \quad f = L_{f\gamma}^{-1}W(f, \gamma)$$

where

$$[W(f, \gamma)](\theta) = W(f(\theta), \theta, \gamma).$$

It is sufficient to show that $L^{-1}W$ has a fixed point. This will be done by restricting its domain to a compact convex set. If this set and γ_0 are properly selected, then $L^{-1}W$ is continuous on this set and maps it into itself. By Schauder's theorem, $L^{-1}W$ has a fixed point. Uniqueness and continuity in γ follow from a separate computation.

Let

$$C(r, u) = \left\{ g \in C^2 \mid \|g\| \leq r, \left\| \frac{\partial g}{\partial \theta_k} \right\| \leq u, \quad k = 1, \dots, s \right\}.$$

The proof depends on the careful selection of the numbers r , u , and γ . To emphasize this a separate numbering system will be used.

(o) We pick r_0 , γ_0 so that if $|r| \leq r_0$, $|\gamma| \leq \gamma_0$, then T_γ is well defined and $L_{f\gamma}^{-1}$ is uniformly bounded by A .

If f is in $C(r, u)$ and $g = L_{f\gamma}^{-1}W(f, \gamma)$, then $\|g\| \leq A\|W(f, \gamma)\|$. We want $\|g\| \leq r$.

If we expand $W(f(\theta), \theta, \gamma)$, we get

$$(6) \quad \|W(f, \gamma)\| \leq \|W(0, \gamma)\| + \left\| \frac{\partial W}{\partial \gamma} \right\| \cdot \|f\|.$$

To control the size of $\left\| \frac{\partial W}{\partial \gamma} \right\|$, it is sufficient to limit $\left\| \frac{\partial W_1}{\partial \gamma_j} \right\|$.

$$(7) \quad \left\| \frac{\partial}{\partial \theta_k} \left(\frac{\partial W_1}{\partial \gamma_j} \right) \right\|_0 \leq \sum_{n=1}^m \left\| \frac{\partial^2 W_1}{\partial \gamma_n \partial \gamma_j} \right\|_0 r + \left\| \frac{\partial^2 W_1}{\partial \theta_k \partial \gamma_j} \right\|_0.$$

(i) Since $\frac{\partial^2 W_1}{\partial \gamma_n \partial \gamma_j}$ is uniformly bounded, and since $\frac{\partial W(0, \theta, 0)}{\partial \gamma} = 0$, we can pick $r_1 \leq r_0$ and $\gamma_1 \leq \gamma_0$, so that $A \left\| \frac{\partial W}{\partial \gamma} \right\| \leq \frac{1}{2}$.

(ii) We then pick $\gamma_2 \leq \gamma_1$, so that $A \|W(0, \gamma)\| \leq (\frac{1}{2})r_1$, if $|\gamma| \leq \gamma_2$. As we further restrict the size of r , it will be necessary to select γ so that (ii) is still satisfied.

We now have that if f is in $C(r_1, u)$ and $|\gamma| \leq \gamma_2$, then $\|L_{f\gamma}^{-1}W(f, \gamma)\| \leq r_1$. If $g = L_{f\gamma}^{-1}W(f, \gamma)$, then $L_{f\gamma}g = W(f, \gamma)$. Therefore

$$(8) \quad \begin{aligned} \sum_{i=1}^s \frac{\partial g}{\partial \theta_i} \frac{\partial V_1}{\partial \theta_k} - \frac{\partial N(\theta)}{\partial \theta_k} g(\theta) - N(\theta) \frac{\partial g(\theta)}{\partial \theta_k} \\ = \sum_{n=1}^m \frac{\partial W}{\partial \gamma_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial W}{\partial \theta_k}. \end{aligned}$$

To obtain a bound on $\frac{\partial g}{\partial \theta_k}$, we form a product space and consider elements $\bar{h} = (h^1, \dots, h^s)$. For fixed f and γ , a linear transformation $\bar{M}_{f\gamma}$ is defined on this space by

$$(9) \quad [\bar{M}_{f\gamma} \bar{h}]^k(\theta) = \sum_{i=1}^s h^i(V_{f\gamma}) \frac{\partial V_1}{\partial \theta_k} - N(\theta) h^k(\theta).$$

Defining $\bar{L}_{f\gamma}$ by $[\bar{L}_{f\gamma}] = (\delta_{ik} L_{f\gamma})$, we see that

$$(10) \quad [(\bar{M}_{f\gamma} - \bar{L}_{f\gamma}) \bar{h}]^k(\theta) = \sum_{i=1}^s h^i(V_{f\gamma}) \left[\sum_{n=1}^m \frac{\partial U_1}{\partial \gamma_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial U_1}{\partial \theta_k} \right], \quad k = 1, \dots, s.$$

$L_{f\gamma}$ invertible implies that $\bar{L}_{f\gamma}$ is invertible. To invert $\bar{M}_{f\gamma}$, we must make $\|\bar{M}_{f\gamma} - \bar{L}_{f\gamma}\|$ sufficiently small. To do this, we see that

$$(11) \quad \left\| \sum_{n=1}^m \frac{\partial U_1}{\partial y_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial U_1}{\partial \theta_k} \right\|_0 \leq \sum_{n=1}^m \left\| \frac{\partial U_1}{\partial y_n} \right\|_0 r + \left\| \frac{\partial U_1}{\partial \theta_k} \right\|_0.$$

$$(12) \quad \left\| \frac{\partial}{\partial \theta_t} \left[\sum_{n=1}^m \frac{\partial U_1}{\partial y_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial U_1}{\partial \theta_k} \right] \right\|_0 \leq \sum_{n=1}^m \left[\sum_{j=1}^m \left\| \frac{\partial^2 U_1}{\partial y_n \partial y_j} \right\|_0 r^2 + \left\| \frac{\partial^2 U_1}{\partial \theta_t \partial y_n} \right\|_0 r + \left\| \frac{\partial U_1}{\partial y_n} \right\|_0 u \right] + \left\| \frac{\partial^2 U_1}{\partial \theta_t \partial \theta_k} \right\|_0,$$

$$(13) \quad \|h(V_{f\gamma})\|_0 = \|h\|_0,$$

and

$$\left\| \frac{\partial}{\partial \theta_k} h(V_{f\gamma}) \right\|_0 \leq \|h\|_0 \left[1 + \sum_{n=1}^m \left\| \frac{\partial U_1}{\partial y_n} \right\|_0 r + \left\| \frac{\partial U_1}{\partial \theta_k} \right\|_0 \right].$$

$U(0, \theta, 0) = 0$, and $\frac{\partial U}{\partial y_n}, \frac{\partial^2 U}{\partial y_n \partial y_j}$ are uniformly bounded.

(111) We can pick $r_2 \leq r_1$, $\gamma_3 \leq \gamma_2$, and u_1 , so that if f is in $C(r_2, u_1)$ and $|\gamma| \leq \gamma_3$, then $\bar{M}_{f\gamma}$ is invertible and $\|\bar{M}_{f\gamma}^{-1}\| \leq B$.

We now have (from equation (8))

$$(14) \quad \frac{\partial g}{\partial \theta_k} = \bar{M}_{f\gamma}^{-1} \left[\frac{\partial N}{\partial \theta_k} g + \frac{\partial W}{\partial y} \frac{\partial f}{\partial \theta_k} + \frac{\partial W}{\partial \theta_k} \right].$$

Therefore

$$(15) \quad \left\| \frac{\partial g}{\partial \theta_k} \right\| \leq B \left[\left\| \frac{\partial N}{\partial \theta_k} \right\| \|g\| + \left\| \frac{\partial W}{\partial y} \right\| \left\| \frac{\partial f}{\partial \theta_k} \right\| + \left\| \frac{\partial W}{\partial \theta_k} \right\| \right].$$

(1v) We pick $r_3 \leq r_2$, $\gamma_4 \leq \gamma_3$, so that $B \left\| \frac{\partial W}{\partial y} \right\| \leq \left(\frac{1}{2}\right)$, and

so that

$$B \left[\left\| \frac{\partial N}{\partial \theta_k} \right\| r_3 + \left\| \frac{\partial W}{\partial \theta_k} \right\| \right] \leq \left(\frac{1}{2} \right) u_1.$$

It follows that if f is in $C(r_3, u_1)$, and if $|\gamma| \leq \gamma_4$, then

$$(16) \quad B \left[\left\| \frac{\partial N}{\partial \theta_k} \right\| r_3 + \left\| \frac{\partial W}{\partial \theta_k} \right\| \right] \leq \left[1 - B \left\| \frac{\partial W}{\partial y} \right\| \right] u_1.$$

This implies that $\left\| \frac{\partial g}{\partial \theta_k} \right\| \leq u_1$.

We have selected a conditionally compact set that is mapped into itself by $L^{-1}W$. We will now show that $L^{-1}W$ also maps $\bar{C}$, the closure of $C(r_3, u_1)$, into itself and is continuous on $\bar{C}$.

If f_n is in $C(r_3, u_1)$, and if $\lim f_n = f_0$, then we can assume that $\lim g_n = g_0$, where g_n is the image of f_n . f_0 and g_0 are in $\bar{C}$. Let $L_n = L_{f_n \gamma}$, and $V_n = V_{f_n \gamma}$. Since $L_n g_n = W(f_n, \gamma)$, we have

$$(17) \quad g_n(V_n(\theta)) = W(f_n(\theta), \theta, \gamma) + N(\theta)g_n(\theta).$$

The limit of the right hand side of the equation defines a function g^* .

$$(18) \quad |g^*(\theta) - g_0(V_0(\theta))| \leq |g^*(\theta) - g_n(V_n(\theta))| + |g_n(V_0(\theta)) - g_0(V_0(\theta))| \\ + |g_n(V_n(\theta)) - g_n(V_0(\theta))|.$$

Since g_n is in $C(r_3, u_1)$ (and therefore in $\bar{C}$),

$$(19) \quad |g_n(V_n(\theta)) - g_n(V_0(\theta))| \leq sr_3 \|U(f_n, \gamma) - U(f_0, \gamma)\|_0.$$

By taking n large, we see that $\|g^* - g_0(V_0)\|_0 = 0$. Therefore $g^* = g_0(V_0)$, and $g_0 = L_0^{-1}W(f_0, \gamma)$. We have shown that $\bar{C}$ is mapped into itself. If we duplicate the argument, taking f_n in $\bar{C}$, we see that $L^{-1}W$ is continuous.

For fixed γ , $L^{-1}W$ is a continuous map of a compact convex set into itself. By Schauder's theorem, there is at least one fixed point. We will complete the proof of our theorem by showing that this fixed point is unique, and that it is a continuous function of γ .

Suppose we have two invariant functions f and h . Then $L_{f\gamma}f = W(f, \gamma)$,

and $L_{h\gamma}h = W(h, \gamma)$. Following Diliberto, we write

$$(20) \quad \|L_{f\gamma}f - L_{f\gamma}h\| \leq \|L_{h\gamma}h - L_{f\gamma}h\| + \|W(f, \gamma) - W(h, \gamma)\|.$$

Since

$$\|L_{f\gamma}^{-1}\| \leq A, \quad \left(\frac{1}{A}\right) \|f - h\| \leq \|L_{f\gamma}(f - h)\|.$$

$L_{h\gamma}h(\theta) - L_{f\gamma}h(\theta) = h(V_{h\gamma}(\theta)) - h(V_{f\gamma}(\theta))$. h is an element of $\bar{C}$, therefore h is Lipschitzian with constant r_3 , and $\frac{\partial h}{\partial \theta_k}$ is Lipschitzian with constant u_1 . Using this plus the big O properties of W and U ,

(v) we pick $r_4 \leq r_3$, $u_2 \leq u_1$, and $\gamma_5 \leq \gamma_4$, so that the right hand side of (20) is less than or equal to $\left(\frac{1}{2A}\right) \|f - h\|$.

We now have that if $L^{-1}W$ is restricted to the closure of $C(r_4, u_2)$ and if $|\gamma| \leq \gamma_5$, the fixed points h and f satisfy the inequality

$$(21) \quad \left(\frac{1}{A}\right) \|f - h\| \leq \left(\frac{1}{2A}\right) \|f - h\|.$$

Therefore $f = h$.

If we let $L_{f\gamma}f = W(f, \gamma)$, and $L_{h\gamma}h = W(h, \gamma')$, then using (20), it is easy to show that $\|f - h\|$ tends to zero as $|\gamma - \gamma'|$ tends to zero. This completes the proof of the theorem.

SECTION 3

It was stated earlier that M. Marcus' result is included in the above theorem. Marcus assumed that $\|N\|_0 < 1$, and obtained an invariant function that is continuous and satisfies a Lipschitz condition. In order to obtain invariant C^1 functions whose derivatives satisfy a Lipschitz condition, we will make the slightly stronger assumption that $\|N\| < 1$.

If this is done, $L_{f\gamma}^{-1}$ exists and is uniformly bounded. This is established by taking $\theta = G_{f\gamma}(\bar{\theta})$, the inverse of $\bar{\theta} = V_{f\gamma}(\bar{\theta})$, and then picking r_0 and γ_0 so that for all h , $\|N(G_{f\gamma})h(G_{f\gamma})\| \leq q\|h\|$. q is a constant such that $\|N\| < q < 1$.

SECTION 4

We now take $s = 1$. All functions considered will have period 1 in θ . Let B be a constant matrix, $Q(\theta)$ a non-singular periodic matrix, and let

$$(22) \quad N(\theta) = Q(\theta + \pi)BQ^{-1}(\theta), \quad \text{so} \quad N(\theta - \pi) = Q(\theta)BQ^{-1}(\theta - \pi).$$

We take $L_{f\gamma}$ as a transformation on the space of complex valued functions. If $L_{f\gamma}^{-1}$ exists and is uniformly bounded, it induces a transformation on the space of real functions that is also uniformly bounded. We must use the complex space since the eigenvalues of B play an important role.

Hufford showed that his $N(\theta)$ had the above form. He assumed that if λ is an eigenvalue of B , then $|\lambda| \neq 1$. We will use the equation $B = RJR^{-1}$, where R is a constant matrix and J is the Jordan canonical form of B .

If $N(\theta)$ has this special form, then $L_{f\gamma}^{-1}$ is uniformly bounded. To show this, we will find transformations acting independently on certain subspaces. Since none of the eigenvalues of B have absolute value equal to 1, these transformations will have inverses. This will imply that $L_{f\gamma}^{-1}$ exists and is uniformly bounded.

Let $P(\theta) = Q(\theta)R$. We define linear transformations $\bar{P}$, π^{-1} , $\bar{V}_{f\gamma}$ and $\bar{J}$ by

$$(23) \quad \begin{aligned} [\bar{P}h](\theta) &= P(\theta)h(\theta) \\ [\pi^{-1}h](\theta) &= h(\theta - \pi) \\ [\bar{V}_{f\gamma}h](\theta) &= h(V_{f\gamma}(\theta)) \\ [\bar{J}h](\theta) &= Jh(\theta). \end{aligned}$$

Then

$$(24) \quad \pi^{-1}L_{f\gamma} = \pi^{-1}\bar{V}_{f\gamma} - \bar{P}\bar{J}\pi^{-1}\bar{P}^{-1} = \bar{P}[\bar{P}^{-1}\pi^{-1}\bar{V}_{f\gamma}\bar{P} - \pi^{-1}\bar{J}]\bar{P}^{-1}$$

Let

$$(25) \quad K_{f\gamma} = \bar{P}^{-1}\pi^{-1}\bar{V}_{f\gamma}\bar{P} - \bar{J}\pi^{-1}.$$

$$(26) \quad [K_{f\gamma}h](\theta) = P^{-1}(\theta)P(V_{f\gamma}(\theta - \pi))h(V_{f\gamma}(\theta - \pi)) - Jh(\theta - \pi).$$

$$(27) \quad V_{f\gamma}(\theta - \pi) = \theta + U(f(\theta - \pi), \theta - \pi, \gamma).$$

Because of this, if

$$(28) \quad P(V_{f\gamma}(\theta)) - P(\theta) = P_{f\gamma}(\theta),$$

then $\|P_{f\gamma}\|$ can be made small by restricting $\|f\|$ and $|\gamma|$.

$$(29) \quad [K_{f\gamma}h](\theta) = h(V_{f\gamma}(\theta - \pi)) - Jh(\theta - \pi) + P^{-1}(\theta)P_{f\gamma}(\theta)h(V_{f\gamma}(\theta - \pi)).$$

The first two terms define a linear transformation that acts independently on the subspaces determined by the canonical form of J . The third term can be made small in norm. A slight modification of the argument of Section 3 plus the proof used by Hufford in Lemma 3.1 of his thesis [1] shows that $\pi^{-1}V_{f\gamma} - \pi^{-1}\bar{J}$ has a uniformly bounded inverse. This implies that $L_{f\gamma}^{-1}$ is uniformly bounded.

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X. PERTURBATION THEOREMS FOR NON-LINEAR
ORDINARY DIFFERENTIAL EQUATIONS

S. P. Diliberto and G. Hufford*

1. Introduction

Consider the equation

$$(1.1) \quad \frac{dx}{dt} = f(x)$$

where x and $f(x)$ are n -vectors; and suppose that $x = u(t)$ is a periodic solution of period, say, ω_1 , of this equation. The usual perturbation problems connected with this situation can be classified into these three types:

TYPE I. (Poincaré). Does the equation

$$(1.2) \quad \frac{dx}{dt} = f(x) + \epsilon R(x, \epsilon)$$

have any periodic solutions "near" $u(t)$?

TYPE II. (Poincaré). If $R(x, t, \epsilon)$ is of period ω_1 in t , does the equation

$$(1.3) \quad \frac{dx}{dt} = f(x) + \epsilon R(x, t, \epsilon)$$

have any period solutions of period ω_1 "near" $u(t)$?

TYPE III. (Krylov-Bogoliubov). If $R(x, t, \epsilon)$ is of period ω_2 , say, in t , does the equation (1.3) have any almost periodic solutions "near" $u(t)$?

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The present study is principally concerned with problems of the last type.

Krylov and Bogoliubov [5], who first considered such problems, dealt with the very specific equation of a harmonic oscillator with a small periodic force function:

$$(1.4) \quad \frac{d^2x}{dt^2} + \omega_1^2 x = \epsilon R\left(x, \frac{dx}{dt}, t, \epsilon\right)$$

where here x is a real variable and R is an analytic function of its four variables. Furthermore, they go on to state, without a detailed proof, that the general problem of Type III, as given above, can be answered affirmatively with these restrictions: f and R must be analytic functions of their variables, and $x = u(t)$ must be an asymptotically stable solution of the unperturbed equation.

Using a more geometric approach, Levinson [6] supplied the details to this latter problem and in so doing, showed that instead of requiring f and R to be analytic, it was enough if they were both of class C^3 . And most recently, Hufford [4] has shown how the stability condition on the solution $x = u(t)$ can be relaxed.

It is not true, however, that the first problem treated by Krylov and Bogoliubov, that of equation (1.4), is a special case of these later results. For in the unperturbed equation formed by setting $\epsilon = 0$ in (1.4), a kind of degeneracy is introduced in that the periodic solution $x = u(t)$ is one of a whole family of similar periodic solutions. One of our objectives in this paper is to extend the results of Krylov and Bogoliubov to more general equations having such a degeneracy; but to do this we must decide in which direction to generalize: for which is the more pertinent fact; that the unperturbed equation has a family of periodic solutions, or that it has an integral? We have chosen the latter direction; but our solution is certainly not the only one. Friedrichs [3], for example, has proposed a second attack on the problem of degeneracy.

In this connection, the work of Malkin [7] is interesting. He has treated the two-dimensional dynamical problem in which the unperturbed equations have the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} \quad \frac{dy}{dt} = - \frac{\partial H}{\partial x} .$$

For such equations, of course, H is an integral.

In all these problems, we are essentially dealing with the behavior of a differential equation in the neighborhood of a given closed

curve. The natural coordinates for us to use are obviously not the original x -coordinates, but rather an "angular" coordinate measured along the closed curve, and $n-1$ "normal" coordinates measured in hyperplanes perpendicular to the curve. Notwithstanding the great number of places in the literature where such coordinate systems have been used, we have been unable to find a proof of their actual existence. Therefore, although such proofs are elementary, we have made bold to supply one here.

Finally, since a part of our study consisted of systematizing the arguments used in these perturbation problems, it seemed advisable also to reexamine the older perturbation problems of Poincaré, the problems we have labeled here Types I and II.

Throughout this paper we use a matrix notation, a fact which will seldom be mentioned explicitly. Thus any entity appearing will be a rectangular matrix, although the determination of its size will ordinarily be left to the reader. If $u(y)$ is an n -vector, y an m -vector, then $\partial u/\partial y$ is an $(n \times m)$ matrix. By the letter I , we mean a unit matrix which must, of course, be square. Where necessary we shall indicate the components of a vector by superscripts. And finally, $\|y\|$ or $\|W\|$ are meant to indicate any of the standard norms of vectors or matrices.

We should like to acknowledge here our indebtedness to Professor Lefschetz for his friendly encouragement and sound advice.

2. Definitions

As before, let $x = u(t)$ be a solution of period ω_1 of equation (1.1). If we make the substitution $x = u(t) + v$ and then subsequently discard all but the terms linear in v , we are led to the so-called variational equation

$$(2.1) \quad \frac{dv}{dt} = G(t)v$$

where $G(t)$ is the $n \times n$ matrix

$$(2.2) \quad G(t) = \left(\frac{\partial f_1(u(t))}{\partial x_j} \right)$$

which, of course, has the period ω_1 in t .

Now, as is well known, there exists a certain linear transformation $w = B(t)v$, having the period ω_1 in t , by means of which equation (2.1) is transformed into a set of linear equations with constant (possibly complex) coefficients

$$(2.3) \quad \frac{dw}{dt} = Aw$$

The characteristic roots of A are called the characteristic exponents of the solution $x = u(t)$. (It should be noted here that some writers use the term "characteristic exponents" to refer to the characteristic roots of the matrix $\exp(\omega, A)$.)

The order of a periodic solution is the number of characteristic exponents whose real parts are different from zero. The degree of degeneracy is one less than the number of characteristic exponents whose real parts are zero. It is defined as one less, because there is always, whether degeneracy is present or not, the trivial solution $v = f(u(t))$ to (2.3) which corresponds to a characteristic exponent equal to zero.

A function $g(x)$ defined in a neighborhood N of $x = u(t)$ is a regular integral through $u(t)$ provided:

- (1) g is an integral of the equation (1.1) in that each member of the one-parameter family of $(n-1)$ -surfaces $g(x) = c$ is an integral surface,
- (2) g is continuously differentiable and the gradient $(\partial g / \partial x_1, \dots, \partial g / \partial x_n)$ is never zero in N ,
- (3) g has been normalized so that $g(u) = 0$,
- (4) the separate branches of the surface $g(x) = c$ (if there are more than one) do not intersect and after following along a given branch once around u , one arrives again on the same branch.

Now, let K be an $(r-1)$ -dimensional analytic manifold and I the real line. Let $x = F(y, t)$ be a map of $K \times I$ into E^n with these two properties:

- (1) its first derivatives satisfy a Lipschitz condition and
- (2) for each fixed t , F is a homeomorphism of K onto $M_t = F(K, t)$. Then $M = F(K, I)$ is an r -dimensional invariant surface of the equation

$$(2.4) \quad \frac{dx}{dt} = g(x, t)$$

if, for any solution $v(t)$ of (2.4), $v(t_0) \in M_{t_0}$ implies $v(t) \in M_t$ for all t . Of course, if g is independent of t such an invariant surface can have no proper self-intersections, but we have

worded the definition specifically so that in the general case such proper self-intersections are entirely possible. The purpose of requiring regularity conditions on the map F is to enable us to consider solutions $v(t)$ lying in M as defined by a differential equation on the original manifold K .

Here are two examples of how we may use this notion of invariant surfaces. First, let $x = u(t)$ be a solution of period ω_1 of equation (1.1). Then clearly $x = u(\theta)$ defines in parametric form an invariant curve. And the purpose of Poincaré's investigation of equations (1.2) and (1.3) was to find a whole family of invariant curves $x = u(\theta, \epsilon)$ such that

$$u(\theta, \epsilon) \implies u(\theta)$$

as $\epsilon \longrightarrow 0$.

Again, if in (1.3) $R(x, t, \epsilon)$ has period ω_2 in t , then the equation will almost never possess periodic solutions. What we can do, however, is to prove that with the proper restrictions, (1.3) does possess a family of invariant tori $x = T(\phi_1, \phi_2, \epsilon)$ such that $T(\phi_1, \phi_2, \epsilon) \implies u(\phi_1 + \phi_2)$ as $\epsilon \longrightarrow 0$. In this way we have reformulated the Krylov-Bogoliubov-Levinson theorem. The nature of the solutions on the tori will depend on the smoothness of T and f . If they are smooth enough then Denjoy's results [2] will imply either that each solution is almost periodic and is everywhere dense on T , or else that there are certain periodic solutions on T whose period is a multiple of ω_2 and towards which every other solution converges.

A periodic solution $x = u(t)$ of (1.1) is commonly said to be asymptotically stable if the real parts of its characteristic exponents (always excepting the trivial zero) are all negative. This is true if and only if, letting M_t be the point $u(t)$, there exist constants δ, A, λ , all positive, such that if $v(t)$ is any solution of (1.1),

$$(2.5) \quad \text{dist}(v(0), M_0) < \delta$$

implies

$$(2.6) \quad \text{dist}(v(t), M_t) < Ae^{-\lambda t} \text{dist}(v(0), M_0)$$

for all $t \geq 0$. In the same way, any invariant surface of (2.4) is said to be asymptotically stable if there exist constants δ, A, λ such that if $v(t)$ is a solution, then (2.5) implies (2.6) for all $t \geq 0$.

3. Normal Coordinates

Let $x = u(\theta)$ be a simple closed curve in E^n where u is differentiable and has the least period ω_1 . Let $\eta_1(\theta) \dots \eta_{n-1}(\theta)$ be $n-1$ linearly independent vectors, each of period ω_1 , and each perpendicular to $u'(\theta)$. Then every point p has at least one vector representation

$$(3.1) \quad p = u(\theta) + \sum_1 y^1 \eta_1(\theta)$$

The n -tuple of real numbers $(\theta, y^1, \dots, y^{n-1}) = (\theta, y)$ are called normal coordinates. If each of the vectors $\eta_1(\theta) \dots \eta_{n-1}(\theta)$ is sufficiently near to being perpendicular to $u'(\theta)$ we give the n -tuple (θ, y) the name almost normal coordinates. And if for some neighborhood N of $x = u(\theta)$, the representation (3.1) is unique in $\theta \pmod{\omega_1}$ and in y , we say that the normal coordinates are unique.

The primary use of normal coordinates is to enable us to rewrite (1.1) so that in a neighborhood of the solution $x = u(t)$ it appears in a more natural form. Thus, if the normal coordinates are unique we can speak of the map $F: x \rightarrow (\theta, y)$; differentiating this map with respect to t , using (1.1) to eliminate dx/dt , and then (3.1) to eliminate x , we arrive at the transformed equations

$$(3.2) \quad \frac{d\theta}{dt} = \theta(\theta, y)$$

$$\frac{dy}{dt} = Y(\theta, y)$$

where θ and Y have period ω_1 in θ . Clearly, the solution $x = u(t)$ is transformed into $\theta = t, y = 0$. Thus

$$(3.3) \quad \theta(\theta, 0) \equiv 1$$

$$Y(\theta, 0) \equiv 0$$

Similarly, of course, the perturbed equations (1.2) and (1.3) can be transformed into normal coordinates.

Now, any periodic solution of (1.1) or of a perturbation of (1.1) which has the period T , will correspond to a solution in normal coordinates $\theta = \psi(t), y = u(t)$ having this property: $\psi(t + mT) = \psi(t) + m\omega_1$, m being any integer, while $u(t)$ is periodic, having the period T . Because of the behavior of ψ , such a solution is not, strictly speaking, periodic, although it has most of the qualitative characteristics one attributes to

a periodic solution. We call such a solution $\theta = \psi(t)$, $y = u(t)$ a repeating solution.

THEOREM 3.1. If the first derivative of $u(\theta)$ satisfies a Lipschitz condition then normal coordinates exist which are unique and for which the $\eta_1(\theta)$ are continuous.

THEOREM 3.2. Let $x = u(t)$ be a periodic solution of (1.1) and suppose that f is of class C^s , $s \geq 1$. Then almost normal coordinates exist which are unique and for which the transformed equations (3.2) are also of class C^s .

THEOREM 3.3. Let $x = u(t)$ be a periodic solution of (1.1) and let $g_1(x) \dots g_k(x)$ be k independent regular integrals. Then there exist normal coordinates $(\theta, y^1 \dots y^{n-k-1}, z^1 \dots z^k) = (\theta, y, z)$ such that the surface $g(x) = c$ becomes $z = c$.

It follows from this last theorem that in terms of these normal coordinates, (1.1) becomes

$$\begin{aligned} \frac{d\theta}{dt} &= \theta(\theta, y, z) \\ \frac{dy}{dt} &= Y(\theta, y, z) \\ \frac{dz}{dt} &= 0 \end{aligned} \quad (3.4)$$

where θ and Y are of period ω_1 in θ . Thus, the equations have the form of a system of $n-k$ equations with k parameters.

Now let us continue with the proofs of the above three theorems. First we establish the following lemmata.

LEMMA 3.1. Let $v(\theta)$ be a unit vector in E^n , $n \geq 3$, which is of period ω_1 and which satisfies a Lipschitz condition. Then there exists a constant unit vector ξ such that $v(\theta) \neq \pm \xi$ for any θ .

PROOF. Clearly if S is any set on the unit sphere in E^n of diameter less than d , there is a spherical cap S_c and an absolute constant K such that $S \subset S_c$ and the area of S_c is less than Kd^{n-1} . Since $\|v(\theta) - v(\theta')\| < M|\theta - \theta'|$, if we divide $0 \leq \theta \leq \omega_1$ into l intervals of equal length, then $v(\theta)$ may be covered by spherical caps whose total

area is less than $\#K(M\omega_1/\#)^{n-1}$. But this, because $n \geq 3$, tends to zero as $\#$ tends to infinity. Thus let ξ be a vector chosen from "almost anywhere" on the sphere.

LEMMA 3.2. Let $v(\theta)$ be a vector in E^n , which is of period ω_1 , which nowhere vanishes, and which satisfies a Lipschitz condition. Then there exist continuous vectors $\eta_1(\theta) \dots \eta_{n-1}(\theta)$ of period ω_1 , such that the set $\{v(\theta), \eta_1(\theta), \dots, \eta_{n-1}(\theta)\}$ is orthogonal for all θ .

PROOF. By Lemma 3.1 there exists a unit vector ξ_1 which is never parallel to $v(\theta)$. Thus $v(\theta)$ has a non-vanishing projection $\xi_1(\theta)$ on the hyperplane perpendicular to ξ_1 . In that plane there exists a unit vector ξ_2 which is never parallel to $\xi_1(\theta)$. Repeating this process we obtain a set of $n-2$ unit vectors $\xi_1 \dots \xi_{n-2}$ which, together with the vector $v(\theta)$, form a linearly independent set. By the Gram-Schmidt orthogonalization process one obtains a set of $n-1$ orthogonal vectors $v(\theta), \eta_1(\theta) \dots \eta_{n-2}(\theta)$. Finally, let $\eta_{n-1}(\theta)$ be one of the two vectors perpendicular to the plane determined by this set.

LEMMA 3.3. Let $x = u(\theta)$ be a regular (i.e., $u'(\theta)$ never vanishes) simple closed curve of period ω_1 whose first derivative satisfies a Lipschitz condition. Then there exist $\delta > 0$, $\epsilon > 0$ such that for any point x and any two distinct values θ_1, θ_2 , if $\|x - u(\theta_1)\| < \delta$, $\|x - u(\theta_2)\| < \delta$, $|\theta_1 - \theta_2| < \epsilon$, then the vectors $x - u(\theta_1)$ and $x - u(\theta_2)$ cannot both be normal to the curve $u(\theta)$.

PROOF. Suppose, indeed, that $x - u(\theta_1)$ is normal to $u(\theta)$; then $[x - u(\theta_1)]u'(\theta_1) = 0$. Now, since $u'(\theta)$ satisfies a Lipschitz condition we have $\|u'(\theta_1) - u'(\theta_2)\| < K|\theta_1 - \theta_2|$. And since $u'(\theta)$ never vanishes, $\|u'(\theta)\|^2$ is bounded away from zero. Let ϵ be so small that $|\theta_1 - \theta| < \epsilon$ implies $u'(\theta_1)u'(\theta) \geq m > 0$ for some value m . Then

$$\begin{aligned} & \left| [x - u(\theta_2)]u'(\theta_2) \right| = \left| [x - u(\theta_2)]u'(\theta_2) - [x - u(\theta_1)]u'(\theta_1) \right| \\ & \geq \left| [u(\theta_1) - u(\theta_2)]u'(\theta_1) \right| - \left| [x - u(\theta_2)][u'(\theta_1) - u'(\theta_2)] \right| \\ & \geq \left| \int_{\theta_1}^{\theta_2} u'(\theta_1)u'(\theta)d\theta \right| - \delta K|\theta_1 - \theta_2| \geq (m - \delta K)|\theta_1 - \theta_2|. \end{aligned}$$

Thus, if $\delta < m/K$ and $\theta_1 \neq \theta_2$, we have $[x - u(\theta_2)]u'(\theta_2) \neq 0$.

We are now ready to complete the proof of Theorem 3.1. Since Lemma 3.2 guarantees the existence of normal coordinates all we have left to prove is their uniqueness. Suppose that every neighborhood of $u(\theta)$ contains a point whose representation in the form (3.1) is not unique. Thus there exists a sequence of points $\{x_1\}$ to each of which corresponds two distinct values $\theta_1, \bar{\theta}_1$ such that $x_1 - u(\theta_1)$ and $x_1 - u(\bar{\theta}_1)$ are both normal to $u(\theta)$ while their lengths tend to zero. Now, by choosing a suitable subsequence we may suppose that $\theta_1 \rightarrow \theta$ and $\bar{\theta}_1 \rightarrow \bar{\theta}$. Furthermore, $\|u(\theta_1) - u(\bar{\theta}_1)\| \leq \|x - u(\theta_1)\| + \|x - u(\bar{\theta}_1)\| \rightarrow 0$, which implies $\|u(\theta) - u(\bar{\theta})\| = 0$ or $\theta = \bar{\theta}$. But this clearly contradicts Lemma 3.3.

Since the $\eta_1(\theta) \dots \eta_{n-1}(\theta)$ are all continuous, there exist trigonometric polynomials which are as close to these as one could desire and hence as close to being perpendicular to $u'(\theta)$ as one desires. Thus there exist almost normal coordinates for which the $\eta_1(\theta)$ are all trigonometric polynomials. In these terms the map $F: x \rightarrow (\theta, y)$ is of class C^{S+1} , and from this Theorem 3.2 follows.

To establish Theorem 3.3, let $\zeta_1 = \partial g_1 / \partial x = (\partial g_1 / \partial x^1 \dots \partial g_1 / \partial x^n)$, $i = 1 \dots k$, and determine $\eta_1(\theta) \dots \eta_{n-k-1}(\theta)$ as before.

4. Perron's Lemma

In what follows we shall often make use of the following:

LEMMA. Consider the n -dimensional linear equation

$$(4.1) \quad \frac{dy}{dt} = C(t)y$$

where $C(t)$ is of period ω_1 . If the characteristic exponents of its solutions all have negative real parts, then there exist a non-singular periodic change of variables $x = B(t)y$ and a constant $\lambda > 0$, such that

$$(4.2) \quad \frac{dx}{dt} = Ax$$

is an equation with constant coefficients while if $W(t)$ is its matrix solution, $W(0) = I$, then

$$(4.3) \quad \|W(t)\| < e^{-\lambda t}$$

The proof is simple: By Floquet's theorem we may assume that a non-singular periodic transformation $z = B^*(t)y$ exists such that $dz/dt = A^*z$ where A^* is a constant matrix in canonical form. The

transformation

$$(4.4) \quad x^i = \sum_j \rho^j z^j$$

where ρ is a positive number, leads to the equation $dx/dt = Cx$ where $C_{ij} = \rho^{i-j} C_{ij}^*$. Since $C_{ij}^* = 0$ for $i > j$ and $C_{11} = C_{11}^*$, the diagonal terms remain constant, while as $\rho \rightarrow \infty$ all the other terms tend to zero. From this the lemma follows.

5. Perturbation Problems of Type I

THEOREM 5.1. If $g_i(x)$, $i = 1, \dots, k$, are k independent regular integrals, $x = u(t)$ being of order $n - k - 1$, then if f is of class C^2 equation (1.1) possesses a continuous k -parameter family of periodic solutions $x = u(t, \lambda)$, $\lambda = (\lambda_1, \dots, \lambda_k)$, and $u(t, 0) = u(t)$.

The case $k = 1$ was proved by Poincaré, and $k = n - 1$ by Reeb [8].

PROOF. Using the normal coordinates guaranteed by Section 3, (1.1) becomes

$$(5.1) \quad \begin{aligned} \frac{d\theta}{dt} &= \theta(\theta, y, z) \\ \frac{dy}{dt} &= Y(\theta, y, z) \\ \frac{dz}{dt} &= 0 \end{aligned}$$

Let the general solution which at $t = 0$ has the initial value $\theta = 0$, $y = y_0$, $z = z_0$, be written in the form

$$(5.2) \quad \begin{aligned} \theta &= \psi(t, y_0, z_0) \\ y &= \mu(t, y_0, z_0) \\ z &= v(t, y_0, z_0) \equiv z_0 \end{aligned}$$

This solution corresponds to a periodic solution of period T if at $t = T$ it has the value $\theta = \omega_1$, $y = y_0$, $z = z_0$. Hence we must find continuous functions $T(z_0)$, $y_0(z_0)$ such that $T(0) = \omega_1$, $y_0(0) = 0$ and

$$(5.3) \quad \begin{aligned} \psi(T, y_0, z_0) - \omega_1 &= 0 \\ \mu(T, y_0, z_0) - y_0 &= 0 \end{aligned}$$

But since $T = \omega_1$, $y_0 = 0$, $z_0 = 0$ is one solution, the theorem will follow from the implicit function theorem if only we can prove the Jacobian is not zero. That is, we must prove

$$(5.4) \quad \begin{vmatrix} \frac{\partial \psi}{\partial t} & \frac{\partial \psi}{\partial y_0} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial y_0} - I \end{vmatrix} \neq 0$$

where all the functions are to be evaluated at $T = \omega_1$, $y_0 = 0$, $z_0 = 0$. But $\psi(t, 0, 0) = t$, $u(t, 0, 0) = 0$, so that $\partial \psi / \partial t = 1$, $\partial u / \partial t = 0$. Thus (5.4) becomes

$$(5.5) \quad \left| \frac{\partial u}{\partial y_0} - I \right| \neq 0$$

And this is equivalent to saying that the original periodic solution $x = u(t)$ is of order $n - k - 1$.

REMARK. From this point on, whenever we assume that a system has k integrals, we shall also assume that the k -parameter family of periodic solutions given by this theorem, are given. When this is true, the equations in normal coordinates may be simplified: if $y_0(z_0)$ is the function defined above, define $\bar{\theta}$, $\bar{y}$ by means of the equations

$$(5.6) \quad \begin{aligned} y &= u(\bar{\theta}, y_0(z), z) - \bar{y} \\ \theta &= \psi(\bar{\theta}, y_0(z), z) \end{aligned}$$

Since $\bar{\theta} = t$, $\bar{y} = 0$ is exactly the periodic solution corresponding to $z_0 = z$, it follows that changing the coordinates to $\bar{\theta}$, $\bar{y}$, z will give equations similar to (5.1) with $\theta(\theta, 0, z) \equiv 1$, $Y(\theta, 0, z) \equiv 0$.

Conversely, the assumption that $\theta(\theta, 0, z) \equiv 1$, $Y(\theta, 0, z) \equiv 0$, implies that the k -parameter family of periodic solutions is given simply by $\theta = t$, $y = 0$, $z = z_0$.

THEOREM 5.2. When $\epsilon = 0$, let $\theta = t$, $y = 0$, $z = 0$ be a solution of order $n - k - 1$ of the equations

$$(5.7) \quad \frac{d\theta}{dt} = \theta(\theta, y, z) + \epsilon \theta_\epsilon(\theta, y, z, \epsilon)$$

$$\frac{dy}{dt} = Y(\theta, y, z) + \epsilon Y_{\epsilon}(\theta, y, z, \epsilon)$$

$$\frac{dz}{dt} = \epsilon Z_{\epsilon}(\theta, y, z, \epsilon)$$

in which all the functions are of class C^2 and of period ω_1 in θ , and in which $s(\theta, 0, z) = 1$, $Y(\theta, 0, z) = 0$. Then for some neighborhood of $\epsilon = 0$, there exists a family of repeating solutions provided:

$$(a) \quad \int_0^{\omega_1} Z_{\epsilon}(\tau, 0, 0, 0) d\tau = 0$$

$$(b) \quad \left| \int_0^{\omega_1} \frac{\partial}{\partial z^j} Z_{\epsilon}^1(\tau, 0, 0, 0) d\tau \right| \neq 0$$

PROOF. Let the general solution which at $t = 0$ has the initial values $\theta = 0$, $y = y_0$, $z = z_0$, be written in the form

$$(5.8) \quad \begin{aligned} \theta &= \psi(t, y_0, z_0, \epsilon) \\ y &= u(t, y_0, z_0, \epsilon) \\ z &= v(t, y_0, z_0, \epsilon) \end{aligned}$$

Then we will have proved the theorem if we can find a continuous functions $T(\epsilon)$, $y_0(\epsilon)$, $z_0(\epsilon)$ such that $T(0) = \omega_1$, $y_0(0) = 0$, $z_0(0) = 0$ and

$$(5.9) \quad \begin{aligned} \psi(T, y_0, z_0, \epsilon) - \omega_1 &= 0 \\ u(T, y_0, z_0, \epsilon) - y_0 &= 0 \\ v(T, y_0, z_0, \epsilon) - z_0 &= 0 \end{aligned}$$

A simple calculation shows that the Jacobian of this system with respect to (T, y_0, z_0) evaluated at $(\omega_1, 0, 0)$ is identically zero, so that we are momentarily stopped. However, define

$$(5.10) \quad h(t, y_0, z_0, \epsilon) = \frac{1}{\epsilon} \left[v(t, y_0, z_0, \epsilon) - z_0 \right]$$

Clearly,

$$(5.11) \quad h(t, y_0, z_0, \epsilon) = \int_0^t Z_{\epsilon}(\psi, u, v, \epsilon) d\tau$$

which is obviously well-defined and differentiable, even at $\epsilon = 0$. Thus (5.9) will be uniquely solvable if and only if the same system but with $v(T, y_0, z_0, \epsilon) - z_0 = 0$ replaced by $h(T, y_0, z_0, \epsilon) = 0$, is uniquely solvable. In particular, we must have

$$(5.12) \quad h(\omega_1, 0, 0, 0) = \int_0^{\omega_1} Z_\epsilon(\tau, 0, 0, 0) d\tau = 0$$

which is condition (a). Thus the theorem will follow if we can prove the Jacobian is not zero. That is, if we can prove

$$(5.13) \quad \begin{vmatrix} \frac{\partial \psi}{\partial t} & \frac{\partial \psi}{\partial y_0} & \frac{\partial \psi}{\partial z_0} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial y_0} - I & \frac{\partial u}{\partial z_0} \\ \frac{\partial h}{\partial t} & \frac{\partial h}{\partial y_0} & \frac{\partial h}{\partial z_0} \end{vmatrix} \neq 0$$

where all the derivatives are to be evaluated at $t = \omega_1, y_0 = 0, z_0 = 0, \epsilon = 0$. But since $\psi(t, 0, z_0, 0) = t, u(t, 0, z_0, 0) = 0$, we have $\partial \psi / \partial t = 1, \partial \psi / \partial z_0 = 0, \partial u / \partial t = 0, \partial u / \partial z_0 = 0$. Thus (5.3) becomes

$$(5.14) \quad \left| \frac{\partial u}{\partial y_0} - I \right| \cdot \left| \frac{\partial h}{\partial z_0} \right| \neq 0$$

But the first of these terms cannot be zero since the original repeating solution is of order $n - k - 1$. And that the second cannot be zero is exactly condition (b).

6. Levinson's Theorem

THEOREM 6.1. Let $x = u(t)$ be an asymptotically stable periodic solution of equation (1.1). Let $R(x, t, \epsilon)$ be of period ω_2 in t , and suppose that both f and R are of class C^3 . Then there exists a neighborhood of $\epsilon = 0$ such that for any ϵ within this neighborhood, equation (1.3) possesses an asymptotically stable invariant torus $x = T(t, \theta, \epsilon)$, where T is of period ω_1 in θ and ω_2 in t , and has first derivatives which satisfy a Lipschitz condition.

Furthermore, $T(t, \theta, \epsilon) \xrightarrow{\epsilon \rightarrow 0} u(t + \theta)$ as $\epsilon \rightarrow 0$, while solutions of (1.3) on T have the form $x = T(t, \theta(t), \epsilon)$ where $\theta(t)$ satisfies the equation

$$(6.1) \quad \frac{d\theta}{dt} = h(\theta, t, \epsilon)$$

h having periods ω_1 in θ and ω_2 in t .

From this theorem and from Denjoy's work there immediately follows the

COROLLARY. Let $\mu(\epsilon)$ be the rotation number for the solutions of (6.1). Then

(i) if $\mu(\epsilon)$ is irrational,

$$(6.2) \quad \theta(t) = \theta_0 + \frac{\mu\omega_1 t}{\omega_2} + P\left(\theta_0 + \frac{\mu\omega_1 t}{\omega_2}, t, \epsilon\right)$$

where $P(\theta, t, \epsilon)$ has periods ω_1 in θ and ω_2 in t ;

(ii) if $\mu(\epsilon)$ is rational

$$(6.3) \quad \theta(t) = \theta_0 + \frac{\mu\omega_1 t}{\omega_2} + g(\theta_0, t, \epsilon)$$

where $g(\theta_0, t, \epsilon) \rightarrow 0$ as $t \rightarrow \infty$, and where $g(\theta_0, t, \epsilon) = 0$ for some (perhaps many) values of θ_0 .

To begin the proof of Theorem 6.1, let us construct normal coordinates as in Section 3, so that (1.3) becomes

$$(6.4) \quad \frac{d\theta}{dt} = \theta(\theta, y) + \epsilon \theta_\epsilon(\theta, y, t, \epsilon)$$

$$\frac{dy}{dt} = Y(\theta, y) + \epsilon Y_\epsilon(\theta, y, t, \epsilon)$$

where all the functions involved are of class C^3 and have period ω_1 in θ and ω_2 in t . Furthermore, θ and Y can be written in the form

$$(6.5) \quad \begin{aligned} \theta(\theta, y) &= 1 + \theta_r(\theta, y) \\ Y(\theta, y) &= A(\theta)y + Y_r(\theta, y) \end{aligned}$$

where:

- (1) All functions are of class C^3 and of period ω_1 in θ
- (2) θ_r is of first and Y_r of second order in y ; in particular there exist $R > 0$, $K > 0$ such that $\|y\| < R$ implies

$$(6.6) \quad \|Y_r(\theta, y)\| \leq K \|y\|^2$$

$$(6.7) \quad \|Y_r(\theta_1, y_1) - Y_r(\theta_2, y_2)\| \leq K \|y_1 + y_2\| \left[|\theta_1 - \theta_2| + \|y_1 - y_2\| \right]$$

further, we assume

$$\|Y_\epsilon(\theta, y, t, \epsilon)\| \leq K$$

- (3) We may assume, without loss of generality, that $A(\theta)$ is a constant matrix A . For, as we know, there exists a linear transformation $L(t)$ reducing the linear differential equation $\dot{y} = A(t)y$ to one with constant coefficients. Then the transformation $\theta \rightarrow \theta$, $y \rightarrow L(\theta)y$, since $d\theta/dt = 1 + O(\|y\|)$, reduces (6.5) to one where the corresponding $A(\theta)$ is constant. This amounts to merely a new choice of the normal coordinates (θ, y) . Unfortunately, as we also know, the transformation $L(\theta)$ may be complex. This is an annoying detail which we dispose of in the following way. Consider the $(n-1)$ -dimensional complex Euclidean space. The transformation $L(\theta)$ carries the $(n-1)$ -dimensional real Euclidean space into a real sub-manifold of the complex space. In what follows, therefore, we treat the y 's as elements of this sub-manifold, and in it will all our further operations be carried out.

Furthermore, the hypothesis of stability means, according to Perron's lemma, that, with perhaps one more linear transformation of the y -coordinates, we can assume there exists a $\lambda > 0$ such that if $W(t)$ is the matrix for which

$$(6.8) \quad \frac{dW}{dt} = AW, \quad W(0) = I$$

then

$$(6.9) \quad \|W(t)\| \leq e^{-\lambda t}$$

The Transformation T_ϵ . Let

$$(6.10) \quad \begin{aligned} \theta(t) &= \Psi(t, t_0, \theta_0, y_0, \epsilon) \\ y(t) &= U(t, t_0, \theta_0, y_0, \epsilon) \end{aligned}$$

be those solutions of (6.4) for which

$$\begin{aligned} \theta_0 &= \Psi(t_0, t_0, \theta_0, y_0, \epsilon) \\ y_0 &= U(t_0, t_0, \theta_0, y_0, \epsilon) \end{aligned}$$

The functions Ψ and U are of class C^2 and have these further properties:

$$(6.11) \quad \begin{aligned} (i) \quad & \Psi(\theta_0 + \omega_1) = \Psi(\theta_0) + \omega_1 \\ & U(\theta_0 + \omega_1) = U(\theta_0) \\ (ii) \quad & \Psi(t + \omega_2, t_0 + \omega_2) = \Psi(t, t_0) \\ & U(t + \omega_2, t_0 + \omega_2) = U(t, t_0) \\ (iii) \quad & \Psi(t, t_0, \theta_0, 0, 0) = \theta_0 + t - t_0 \\ & U(t, t_0, \theta_0, 0, 0) = 0 \end{aligned}$$

These relations are established by showing that they are true for $t = t_0$ and that both sides are solutions; the uniqueness theorem then asserts that they hold for all t .

Let C denote the space of all continuous curves $y = \alpha(\theta)$, α having period ω_1 , and values in the real sub-manifold of the complex Euclidean space mentioned above. And let the norm $\|\alpha\|$ be defined in the usual way by $\|\alpha\| = \max_\theta \|\alpha(\theta)\|$. Then define a transformation $T_\epsilon(b, a)$ of C into itself in the following way: for $\alpha \in C$,

$$(6.12) \quad [T_\epsilon(b, a)\alpha](\theta) = U(b, a, \bar{\theta}, \alpha(\bar{\theta}), \epsilon)$$

where $\bar{\theta}$ is defined by the equation

$$(6.13) \quad \theta = \Psi(b, a, \bar{\theta}, \alpha(\bar{\theta}), \epsilon)$$

In this way, the curve $T_\epsilon(b, a)\alpha$ is merely the set of points which were on the curve α at time a and which have proceeded along the trajectories of (6.4) until time b . Note that $T_\epsilon(a, a) = I$, the identity transformation, while $T_\epsilon(c, b)T_\epsilon(b, a) = T_\epsilon(c, a)$, $T_\epsilon(b + \omega_2, a + \omega_2) = T_\epsilon(b, a)$, and finally $T_0(b, a) = 0$.

One consideration must be made, however, before $T_\epsilon(b, a)$ is well-defined. For given α and θ there must be defined a unique $\bar{\theta}$.

But for $\alpha = 0$ and $\epsilon = 0$, $\bar{\theta} = \theta - b + a$ is such a unique value. Furthermore, as we shall show in Lemma 6.2, if both $\|\alpha\|$ and $\|d\alpha/d\theta\|$ are small enough, then the derivative $\partial\bar{\Psi}/\partial\bar{\theta}$ is approximately 1. Thus it follows that $T_\epsilon(b, a)\alpha$ is well-defined, at any rate if $\|\alpha\|$ and $\|d\alpha/d\theta\|$ are small enough. We are, however, concerned only with such α ; therefore we redefine $T_\epsilon(b, a)$ by restricting its domain, so that it is well-defined.

Now put $T_\epsilon = T_\epsilon(\omega_2, 0)$. Then our principal object is to show that T_ϵ has a fixed point. In other words that there exists a curve α depending on ϵ such that $T_\epsilon\alpha = \alpha$. For suppose that $y = \alpha(\theta, \epsilon)$ is such a curve; then $y = [T_\epsilon(t, 0)\alpha](\theta)$, or more explicitly

$$\begin{aligned} \theta &= \Psi(t, 0, \bar{\theta}, \alpha(\bar{\theta}, \epsilon), \epsilon) \\ (6.14) \quad y &= U(t, 0, \bar{\theta}, \alpha(\bar{\theta}, \epsilon), \epsilon) \end{aligned}$$

is exactly the invariant torus for which we are looking. If, as we shall show, $\alpha(\bar{\theta}, \epsilon) \Rightarrow 0$ as $\epsilon \rightarrow 0$, then

$$\begin{aligned} \Psi(t, 0, \bar{\theta}, \alpha(\bar{\theta}, \epsilon), \epsilon) &\Rightarrow t + \bar{\theta} \\ (6.15) \quad U(t, 0, \bar{\theta}, \alpha(\bar{\theta}, \epsilon), \epsilon) &\Rightarrow 0 \end{aligned}$$

which is just the original periodic solution $x = u(t + \bar{\theta})$.

Properties of T_ϵ . The necessary facts about T_ϵ which we need are contained in the following lemmata.

LEMMA 6.1. Let $\eta(t)$ be an $(n-1)$ -vector function which satisfies the differential equation

$$(6.16) \quad \dot{\eta} = A\eta + G(\eta, t)$$

where G is of class C^1 and satisfies this property: there exist $\kappa > 0$, $T > 0$ such that $\|\eta\| < \kappa$ implies $\|G(\eta, t)\| < \lambda\kappa$, for all $0 \leq t \leq T$. Then $\|\eta(0)\| < \kappa$ implies $\|\eta(t)\| < \kappa$, for all $0 \leq t \leq T$.

Again, put

$$\begin{aligned} \rho &= e^{-\lambda\omega_2/2} \\ (6.17) \quad \mu &= \frac{\rho}{1+\rho} \end{aligned}$$

both of which are positive and strictly less than one. And suppose that $\|\eta\| < \kappa$ implies $\|G(\eta, t)\| < \lambda\mu\kappa$

for all $0 \leq t \leq \omega_2$. Then $\|\eta(0)\| < \kappa$ implies
 $\|\eta(\omega_2)\| < \rho\kappa$.

PROOF. Let $W(t)$ be the matrix defined above. Then we have

$$(6.18) \quad \eta(t) = W(t)\eta(0) + \int_0^t W(t-\tau)G(\eta(\tau), \tau)d\tau$$

or

$$\|\eta(t)\| < e^{-\lambda t}\|\eta(0)\| + e^{-\lambda t} \int_0^t e^{\lambda\tau} \|G(\eta(\tau), \tau)\| d\tau.$$

Now suppose the lemma false. Then there exists a first t_1 , $0 < t_1 < T$, such that $\|\eta(t_1)\| = \kappa$. But for $0 < t < t_1$ the inequality does hold, and

$$(6.19) \quad \|\eta(t_1)\| < e^{-\lambda t_1}\kappa + e^{-\lambda t_1} \int_0^{t_1} e^{\lambda\tau} \lambda\kappa d\tau = \kappa$$

which is a contradiction.

If the stronger condition $\|G(\eta, t)\| < \lambda\mu\kappa$ holds, then surely $\|\eta(t)\| < \kappa$ for all $0 \leq t \leq \omega_2$, and

$$(6.20) \quad \begin{aligned} \|\eta(\omega_2)\| &< e^{-\lambda\omega_2}\kappa + e^{-\lambda\omega_2} \int_0^{\omega_2} e^{\lambda\tau} \lambda\mu\kappa d\tau \\ &= \kappa \left[\rho^2 + (1 - \rho^2) \frac{\rho}{1+\rho} \right] = \rho\kappa. \end{aligned}$$

LEMMA 6.2. Let $(\theta(t), y(t))$ be the solution of (6.4) which has the initial value (θ_0, y_0) . Then there exists $\delta_0 > 0$, such that for any $0 < \delta < \delta_0$, $\|y_0\| < \delta$, $|\epsilon| < \delta^2$ imply $\|y(t)\| < \delta$ for all $t \geq 0$. In particular if $\|\alpha\| < \delta$, $|\epsilon| < \delta^2$, then $\|T_\epsilon \alpha\| < \delta$.

PROOF. Put $\eta(t) = y(t)$ and $G(\eta, t) = Y_\eta(\theta(t), y) + \epsilon Y_\epsilon(\theta(t), y, t, \epsilon)$. Then (6.16) becomes exactly the second part of (6.4). From (6.7) it follows that if $\|y\| < R$ then $\|G(y, t)\| \leq K[\|y\|^2 + |\epsilon|]$. Now put $\delta_0 = \min(R, \lambda/2K)$; if $\|y\| < \delta < \delta_0 \leq R$, and if $|\epsilon| < \delta^2$, then $\|G(y, t)\| < 2K\delta^2 < 2K\delta_0 \leq \lambda\delta$ for all $t \geq 0$. The lemma then follows from Lemma 6.1.

LEMMA 6.3. There exists $\delta_1 > 0$ such that for any

$$0 < \delta < \delta_1, \quad \|\alpha/d\theta\| < \delta, \quad \|\alpha\| < \delta, \quad |\epsilon| < \delta^2 \quad \text{imply}$$

$$(6.21) \quad \left\| \frac{d}{d\theta} (T_\epsilon \alpha) \right\| < \delta$$

PROOF. By definition

$$(6.22) \quad \begin{aligned} (T_\epsilon \alpha)(\theta) &= U(\bar{\theta}, \alpha(\bar{\theta})) \\ \theta &= \Psi(\bar{\theta}, \alpha(\bar{\theta})); \end{aligned}$$

and so

$$(6.23) \quad \frac{d}{d\theta} (T_\epsilon \alpha) = \frac{dU}{d\bar{\theta}} \left(\frac{d\Psi}{d\bar{\theta}} \right)^{-1}.$$

We shall proceed to calculate these terms.

First we dispense with $d\Psi/d\bar{\theta}$ by noting that $\Psi(\omega_2, 0, \theta, y, \epsilon)$ is of class C^2 , while at $y = 0, \epsilon = 0$ we have, by (6.11(iii)), $\partial\Psi/\partial\theta = 1$. Thus

$$(6.24) \quad \frac{d\Psi}{d\bar{\theta}} = \frac{\partial\Psi}{\partial\theta} + \frac{\partial\Psi}{\partial y} \frac{d\alpha}{d\theta} = 1 + o\left(|\epsilon| + \|\alpha\| + \left\| \frac{d\alpha}{d\theta} \right\| \right)$$

or under our hypotheses,

$$(6.25) \quad \frac{d\Psi}{d\bar{\theta}} = 1 + o(\delta).$$

Now let $(\theta(t), y(t))$ be the solution of (6.4) with the initial value $(\bar{\theta}, \alpha(\bar{\theta}))$, and put $\phi(t) = d\Psi(t, 0, \bar{\theta}, \alpha(\bar{\theta}), \epsilon)/d\bar{\theta}$, $\eta(t) = dU(t, 0, \bar{\theta}, \alpha(\bar{\theta}), \epsilon)/d\bar{\theta}$. Then η satisfies the variational equation

$$(6.26) \quad \dot{\eta} = A\eta + \left(\frac{\partial Y_r}{\partial \theta} + \epsilon \frac{\partial Y_\epsilon}{\partial \theta} \right) \phi + \left(\frac{\partial Y_r}{\partial y} + \epsilon \frac{\partial Y_\epsilon}{\partial y} \right) \eta$$

where the derivatives are to be evaluated at $(\theta(t), y(t))$. But this is just (6.16) where $G(\eta, (t))$ consists of the last four terms. Using the notation of Lemma 6.2, let us require $\delta_1 < \delta_0$. Then $\|\alpha\| < \delta < \delta_1$, $|\epsilon| < \delta^2$ imply, by Lemma 6.2, $\|y(t)\| < \delta$. Thus

$$(6.27) \quad \begin{aligned} \|G(\eta, t)\| &= o(\|y\|^2) + |\epsilon| |\phi| + o(\|y\| + |\epsilon|) \|\eta\| \\ &= o(\delta^2) + o(\delta) \|\eta\| \end{aligned}$$

and $\|\eta\| < \delta$ implies $\|G(\eta, t)\| = o(\delta^2)$. In other words, there exists δ_1 such that $\|\eta\| < \delta < \delta_1$ implies $\|G(\eta, t)\| < \lambda_0 \delta$ for all $0 \leq t \leq \omega_2$.

Then by Lemma 6.1, $\|\eta(0)\| = \|\alpha/d\theta\| < \delta < \delta_1$ implies $\|\eta(\omega_2)\| = \|dU/d\bar{\theta}\| < \rho\delta$. Thus finally,

$$(6.28) \quad \left\| \frac{d}{d\theta} (T_\epsilon \alpha) \right\| < \left\| \frac{dU}{d\bar{\theta}} \right\| \cdot \left| \frac{d\psi}{d\bar{\theta}} \right|^{-1} < \rho\delta(1 + o(\delta))$$

from which the lemma follows.

LEMMA 6.4. There exists a δ_2 such that $\|\alpha\| < \delta_2$, $\|\alpha/d\theta\| < \delta_2$, $|\epsilon| < \delta_2^2$, and $\|d^2\alpha/d\theta^2\| < 1$ imply

$$(6.29) \quad \left\| \frac{d^2}{d\theta^2} (T_\epsilon \alpha) \right\| < 1$$

(the value 1 here has no significance).

PROOF. From (6.22)

$$(6.30) \quad \frac{d^2}{d\theta^2} (T_\epsilon \alpha) = \frac{d^2 U}{d\bar{\theta}^2} \left(\frac{d\psi}{d\bar{\theta}} \right)^{-2} - \frac{dU}{d\bar{\theta}} \left(\frac{d\psi}{d\bar{\theta}} \right)^{-3} \frac{d^2 \psi}{d\bar{\theta}^2}$$

Put $\phi(t) = d^2\psi(t, 0, \bar{\theta}, \alpha(\bar{\theta}), \epsilon)/d\bar{\theta}^2$, $\eta(t) = d^2U(t, 0, \bar{\theta}, \alpha(\bar{\theta}), \epsilon)/d\bar{\theta}^2$. Then η satisfies the variational equation

$$(6.31) \quad \begin{aligned} \dot{\eta} = & A\eta + \left(\frac{\partial Y_r}{\partial \theta} + \epsilon \frac{\partial Y_\epsilon}{\partial \theta} \right) \phi + \left(\frac{\partial Y_r}{\partial y} + \epsilon \frac{\partial Y_\epsilon}{\partial y} \right) \eta + \\ & + \left(\frac{\partial^2 Y_r}{\partial \theta^2} + \epsilon \frac{\partial^2 Y_\epsilon}{\partial \theta^2} \right) \left(\frac{d\psi}{d\bar{\theta}} \right)^2 + \left(\frac{\partial^2 Y_r}{\partial y^2} + \epsilon \frac{\partial^2 Y_\epsilon}{\partial y^2} \right) \left(\frac{dU}{d\bar{\theta}} \right)^2 + \\ & + \left(\frac{\partial^2 Y_r}{\partial \theta \partial y} + \epsilon \frac{\partial^2 Y_\epsilon}{\partial \theta \partial y} \right) \left(\frac{d\psi}{d\bar{\theta}} \right) \left(\frac{dU}{d\bar{\theta}} \right) \end{aligned}$$

which again is (6.16). By Lemma 6.2, 6.3, $\|y\|$, $|d\psi/d\bar{\theta} - 1|$, $\|dU/d\bar{\theta}\|$ are all $o(\delta_2)$, while since $d^2\alpha/d\theta^2$ is bounded so is ϕ . Thus with $G(\eta, t)$ defined as the remainder terms in (6.31) we have $\|G(\eta, t)\| = o(\delta_2^2) + o(\delta_2)\|\eta\|$. Therefore we can find δ_2 small enough so that $\|\eta\| < 1$ implies $\|G(\eta, t)\| < \lambda\mu$. Then from Lemma 6.1, $\|\eta(0)\| = \|d^2\alpha/d\theta^2\| < 1$ implies $\|\eta(\omega_2)\| = \|d^2U/d\bar{\theta}^2\| < \rho$. Since the second term in (6.29) is in norm $o(\delta_2)$, the lemma follows easily.

LEMMA 6.5. Let $(\theta_1(t), y_1(t)), (\theta_2(t), y_2(t))$ be solutions of (6.4) for which $y_1(0) = \bar{y}_1$, $y_2(0) = \bar{y}_2$, $\theta_1(\omega_2) = \theta_2(\omega_2) = \theta_0$. Then there exists $\delta_3 > 0$ such

that for any $\gamma > 0$, $\|y_1\| < \delta_3$, $\|y_2\| < \delta_3$, $|\epsilon| < \delta_3^2$,
 $\|y_1 - y_2\| < \gamma$ implies $\|y_1(\omega_2) - y_2(\omega_2)\| < \rho\gamma$.

PROOF. First, it is clear that such solutions do exist and are unique, since it is only necessary to solve the equations

$$(6.32) \quad \theta_0 = \Psi(\omega_2, 0, \bar{\theta}_1, \bar{y}_1, \epsilon) \quad i = 1, 2$$

for the initial values $\bar{\theta}_1$; and this can be done in only one way provided y_1, ϵ are small enough. Secondly, if $\delta_3 \leq \delta_0$, then it follows from Lemma 6.2 that $\|y_1(t)\| < \delta_3$ for all $t \geq 0$.

Now from (6.32) there follows

$$(6.33) \quad \frac{\partial \Psi}{\partial \theta} (\bar{\theta}_1 - \bar{\theta}_2) + \frac{\partial \Psi}{\partial y} (\bar{y}_1 - \bar{y}_2) = 0$$

the derivative to be evaluated at suitable points "between" $(\bar{\theta}_1, \bar{y}_1)$ and $(\bar{\theta}_2, \bar{y}_2)$. Thus $|\bar{\theta}_1 - \bar{\theta}_2| = o(\gamma)$, and since $\theta_1(t) = \Psi(t, 0, \bar{\theta}_1, \bar{y}_1, \epsilon)$

$$(6.34) \quad |\theta_1(t) - \theta_2(t)| = \left\| \frac{\partial \Psi}{\partial \theta} (\bar{\theta}_1 - \bar{\theta}_2) - \frac{\partial \Psi}{\partial y} (\bar{y}_1 - \bar{y}_2) \right\| = o(\gamma)$$

uniformly for $0 \leq t \leq \omega_2$.

Finally put $\eta(t) = y_1(t) - y_2(t)$. Then from (6.4)

$$(6.35) \quad \begin{aligned} \frac{d\eta}{dt} = & A\eta + \left[Y_r(\theta_1, y_1) - Y_r(\theta_2, y_2) \right] + \\ & + \left[Y_\epsilon(\theta_1, y_1) - Y_\epsilon(\theta_2, y_2) \right] \end{aligned}$$

which once again is in the form (6.16). Setting $G(\eta, t)$ equal to the remainder terms and using (6.7)

$$(6.36) \quad \begin{aligned} \|G(\eta, t)\| \leq & K[\|y_1\| + \|y_2\|] \left[|\theta_1 - \theta_2| + \|\eta\| \right] + \\ & + K\epsilon \left[|\theta_1 - \theta_2| + \|\eta\| \right] \end{aligned}$$

so that $\|\eta\| < \gamma$ implies $\|G(\eta, t)\| = o(\delta_3\gamma)$; or if δ_3 is small enough, $\|G(\eta, t)\| < \lambda\mu\gamma$. Thus, from Lemma 6.1, it follows that $\|\eta(0)\| = \|y_1 - y_2\| < \gamma$ implies $\|\eta(\omega_2)\| = \|y_1(\omega_2) - y_2(\omega_2)\| < \rho\gamma$.

LEMMA 6.6. There exists a δ_4 such that $\|\alpha_1\| < \delta_4$,
 $\|d\alpha_1/d\theta\| < \delta_4$, $|\epsilon| < \delta_4^2$, $i = 1, 2$, imply
 $\|T_\epsilon \alpha_1 - T_\epsilon \alpha_2\| < \sqrt{\rho} \|\alpha_1 - \alpha_2\|$.

PROOF. To calculate $[T_{\epsilon}\alpha_1 - T_{\epsilon}\alpha_2](\theta)$ we must find solutions $(\theta_1(t), y_1(t))$, $i = 1, 2$, to (6.4) such that $\theta_1(\omega_2) = \theta$, while the initial points lie on α_1 . As in Lemma 6.5, the initial values of these solutions will be $\bar{\theta}_1, \bar{y}_1 = \alpha_1(\bar{\theta}_1)$, then

$$\begin{aligned} \|\bar{y}_1 - \bar{y}_2\| &= \|\alpha_1(\bar{\theta}_1) - \alpha_2(\bar{\theta}_2)\| \\ (6.37) \quad &\leq \|\alpha_1(\bar{\theta}_1) - \alpha_2(\bar{\theta}_1)\| + \|\alpha_2(\bar{\theta}_1) - \alpha_2(\bar{\theta}_2)\| \\ &\leq \|\alpha_1 - \alpha_2\| + \|d\alpha_2/d\theta\| \cdot |\bar{\theta}_1 - \bar{\theta}_2|. \end{aligned}$$

Thus $\|\alpha_1 - \alpha_2\| < \gamma$ implies $\|\bar{y}_1 - \bar{y}_2\| < \gamma + o(\delta_4\gamma)$; or if δ_4 is small enough, $\|\bar{y}_1 - \bar{y}_2\| < \gamma/\sqrt{\rho}$. From Lemma 6.5, therefore, $\|[T_{\epsilon}\alpha_1 - T_{\epsilon}\alpha_2](\theta)\| < \sqrt{\rho}\gamma$.

With these six lemmas we are now in a position to complete the proof of Theorem 6.1. For if we put $\delta_5 = \min(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$ and define D to be those elements α of C such that $\|\alpha\| < \delta_5$, $\|d\alpha/d\theta\| < \delta_5$, $\|d^2\alpha/d\theta^2\| < 1$, and if we suppose $|\epsilon| < \delta_5^2$, then by Lemmas 6.2, 6.3, 6.4, the transformation T_{ϵ} carries D into itself; by Lemma 6.6 it is, within D , a distance decreasing transformation; thus it follows from the Cacciopoli fixed point theorem [1] that a unique curve $\alpha(\theta, \epsilon)$ exists in $\bar{D}$ which is invariant under T_{ϵ} . Moreover, since $\bar{D}$ consists of curves whose first derivatives satisfy a Lipschitz condition, our theorem is proved.

7. A Generalization of the Krylov-Bogoliubov Theorem

Consider now the system

$$\begin{aligned} \frac{d\theta}{dt} &= 1 + \theta_T(\theta, y, z) + \epsilon \theta_{\epsilon}(\theta, y, z, t, \epsilon) \\ (7.1) \quad \frac{dy}{dt} &= Ay + Y_T(\theta, y, z) + \epsilon Y_{\epsilon}(\theta, y, z, t, \epsilon) \\ \frac{dz}{dt} &= B(\theta, t)y + C z + \epsilon Z_T(\theta, y, z, t, \epsilon) \end{aligned}$$

where y and z are vectors of dimension $n-k-1$ and k , respectively; and where the functions on the right hand sides have the periods ω_1 in θ and ω_2 in t . Thus it is a system of k -fold degeneracy, having a periodic perturbation. The theorem of Krylov and Bogoliubov is concerned with the special case $n = 2, k = 1$.

THEOREM 7.1. Let the functions on the right hand sides of (7.1) be all of class C^3 , and let A, C be constant matrices whose characteristic roots all have negative real parts. Then there exists $\epsilon_0 > 0$ such that to each $\epsilon, 0 < \epsilon < \epsilon_0$, there corresponds an asymptotically

stable invariant torus $y = T_1(\theta, t, \epsilon)$, $z = T_2(\theta, t, \epsilon)$, T_1 and T_2 being of period ω_1 in θ and ω_2 in t , and having first derivatives which satisfy a Lipschitz condition. Furthermore, as $\epsilon \rightarrow 0$, we have $T_1(\theta, t, \epsilon) \rightarrow 0$, $T_2(\theta, t, \epsilon) \rightarrow 0$.

The proof of this theorem and that of Theorem 6.1 are very similar. The added complexity is taken care of by treating separately the y -coordinates and the z -coordinates. Thus we define two function spaces C_y , C_z which consist of continuous functions $y = \alpha(\theta)$, $z = \beta(\theta)$ all of period ω_1 . Let

$$(7.2) \quad \begin{aligned} \theta &= \Psi(t, t_0, \theta_0, y_0, z_0, \epsilon) \\ y &= U(t, t_0, \theta_0, y_0, z_0, \epsilon) \\ z &= V(t, t_0, \theta_0, y_0, z_0, \epsilon) \end{aligned}$$

be the solution of (7.1) which at $t = t_0$ has the values $\theta = \theta_0$, $y = y_0$, $z = z_0$. Then for each pair of elements (α, β) , α in C_y , β in C_z , we can define a second pair $S_\epsilon(b, a)(\alpha, \beta)$ in C_y and $T_\epsilon(b, a)(\alpha, \beta)$ in C_z in this way:

$$(7.3) \quad \begin{aligned} S_\epsilon(b, a)(\alpha, \beta)(\theta) &= U(b, a, \bar{\theta}, \alpha(\bar{\theta}), \beta(\bar{\theta}), \epsilon) \\ T_\epsilon(b, a)(\alpha, \beta)(\theta) &= V(b, a, \bar{\theta}, \alpha(\bar{\theta}), \beta(\bar{\theta}), \epsilon) \end{aligned}$$

where in both cases, $\bar{\theta}$ is defined by

$$\theta = \Psi(b, a, \bar{\theta}, \alpha(\bar{\theta}), \beta(\bar{\theta}), \epsilon)$$

Thus, putting $S_\epsilon = S_\epsilon(\omega_2, 0)$, $T_\epsilon = T_\epsilon(\omega_2, 0)$, it is our aim to find a pair (α, β) such that $S_\epsilon(\alpha, \beta) = \alpha$ and $T_\epsilon(\alpha, \beta) = \beta$.

Now let us consider the right hand sides of (7.1). From the remarks following Theorem 5.1, it is clear that we can always assume $\|e_r(\theta, y, z)\| = O(\|y\|)$, $\|Y_r(\theta, y, z)\| = O(\|y\|(\|y\| + \|z\|))$. Furthermore, $\|Z_\epsilon(\theta, y, z, t, \epsilon)\| = O(\|y\| + \|z\|^2 + \epsilon)$. Now, if $W_y(t)$, $W_z(t)$ are matrix solutions of the linear equations

$$(7.4) \quad \frac{dW_y}{dt} = AW_y \quad \frac{dW_z}{dt} = CW_z$$

which are equal initially to the identity matrices, then there exists a $\lambda > 0$ such that

$$(7.5) \quad \|W_y(t)\| < e^{-\lambda t} \quad \|W_z(t)\| < e^{-\lambda t}$$

Finally, note that none of the above considerations are affected if we make the homologous transformation $z \rightarrow \sigma z$. What does happen, however, is that the matrix $B(\theta, t)$ becomes $B(\theta, t)/\sigma$; and by making σ large enough we can make this matrix as small as we please. In what follows, therefore, we shall assume

$$(7.6) \quad \|B(\theta, t)\| < \frac{\lambda \mu_2}{2}$$

where μ_2 is the number defined below in (7.8).

Armed with these facts we prove the following lemmata, all of them following very closely the six of Section 6.

LEMMA 7.1. Let $\eta(t)$, $\zeta(t)$ be $(n-k-1)$ - and k -vector functions, respectively, which satisfy the differential equations

$$(7.7) \quad \begin{aligned} \dot{\eta} &= A\eta + G(\eta, \zeta, t) \\ \dot{\zeta} &= \epsilon C \zeta + \epsilon H(\eta, \zeta, t) \end{aligned}$$

where $\epsilon > 0$ and where G and H are of class C^1 and satisfy this property: there exist $\kappa_1 > 0$, $\kappa_2 > 0$, $T > 0$ such that $\|\eta\| < \kappa_1$, $\|\zeta\| < \kappa_2$ imply $\|G(\eta, \zeta, t)\| < \kappa_1 \lambda$, $\|H(\eta, \zeta, t)\| < \kappa_2 \lambda$ for all $0 \leq t \leq T$. Then $\|\eta(0)\| < \kappa_1$, $\|\zeta(0)\| < \kappa_2$ imply $\|\eta(t)\| < \kappa_1$, $\|\zeta(t)\| < \kappa_2$ for all $0 \leq t \leq T$.

Again, put

$$(7.8) \quad \begin{aligned} \rho_1 &= e^{-\lambda \omega_2 / 2} & \rho_2 &= e^{-\lambda \omega_2 \epsilon / 2} \\ \mu_1 &= \frac{\rho_1}{1 + \rho_1} & \mu_2 &= \frac{\rho_2}{1 + \rho_2} \end{aligned}$$

all of which are positive and strictly less than one. And suppose that $\|\eta\| < \kappa_1$, $\|\zeta\| < \kappa_2$ imply $\|G(\eta, \zeta, t)\| < \kappa_1 \mu_1 \lambda$, $\|H(\eta, \zeta, t)\| < \kappa_2 \mu_2 \lambda$ for all $0 \leq t \leq \omega_2$. Then $\|\eta(0)\| < \kappa_1$, $\|\zeta(0)\| < \kappa_2$ imply $\|\eta(\omega_2)\| < \rho_1 \kappa_1$, $\|\zeta(\omega_2)\| < \rho_2 \kappa_2$.

PROOF. We see immediately

$$\begin{aligned} \|\eta(t)\| &< e^{-\lambda t} \|\eta(0)\| + e^{-\lambda t} \int_0^t e^{\lambda \tau} \|G(\eta(\tau), \zeta(\tau), \tau)\| d\tau \\ (7.9) \quad \|\zeta(t)\| &< e^{-\epsilon \lambda t} \|\zeta(0)\| + \epsilon e^{-\epsilon \lambda t} \int_0^t e^{\epsilon \lambda \tau} \|H(\eta(\tau), \zeta(\tau), \tau)\| d\tau. \end{aligned}$$

Thus if the lemma were false there would exist a first t_1 such that either $\|\eta(t_1)\| = \kappa_1$ or $\|\zeta(t_1)\| = \kappa_2$. Then for $0 < t < t_1$ we have $\|\eta(t)\| < \kappa_1$, $\|\zeta(t)\| < \kappa_2$ and hence

$$\begin{aligned} \|\eta(t_1)\| &< e^{-\lambda t_1} \kappa_1 + e^{-\lambda t_1} \int_0^{t_1} e^{\lambda \tau} \lambda \kappa_1 d\tau = \kappa_1 \\ (7.10) \quad \|\zeta(t_1)\| &< e^{-\epsilon \lambda t_1} \kappa_2 + \epsilon e^{-\epsilon \lambda t_1} \int_0^{t_1} e^{\epsilon \lambda \tau} \lambda \kappa_2 d\tau = \kappa_2 \end{aligned}$$

which is a contradiction.

If the stronger condition holds, then surely $\|\eta(t)\| < \kappa_1$, $\|\zeta(t)\| < \kappa_2$ for all $0 < t < \omega_2$, and

$$\begin{aligned} \|\eta(\omega_2)\| &< e^{-\lambda \omega_2} \kappa_1 + e^{-\lambda \omega_2} \int_0^{\omega_2} e^{\lambda \tau} \lambda \mu_1 \kappa_1 d\tau = \rho_1 \kappa_1 \\ (7.11) \quad \|\zeta(\omega_2)\| &< e^{-\epsilon \lambda \omega_2} \kappa_2 + \epsilon e^{-\epsilon \lambda \omega_2} \int_0^{\omega_2} e^{\epsilon \lambda \tau} \lambda \mu_2 \kappa_2 d\tau = \rho_2 \kappa_2. \end{aligned}$$

LEMMA 7.2. Let $(\theta(t), y(t), z(t))$ be the solution of (7.1) which has the initial value (θ_0, y_0, z_0) .

Then there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, $\|y_0\| < \delta$, $\|z_0\| < \delta$, $0 < \epsilon < \delta^2$ imply $\|y(t)\| < \delta$, $\|z(t)\| < \delta$ for all $t \geq 0$. In particular, if $\|\alpha\| < \delta$, $\|\beta\| < \delta$, $0 < \epsilon < \delta^2$, then $\|S_\epsilon(\alpha, \beta)\| < \delta$, $\|T_\epsilon(\alpha, \beta)\| < \delta$.

PROOF. Equations (7.1) have exactly the form (7.7) with $\eta(t) = y(t)$, $\zeta(t) = z(t)$. Furthermore, $\|y\| < \delta$, $\|z\| < \delta$, $0 < \epsilon < \delta^2$, imply $\|G\| = \|Y_r + \epsilon Y_\epsilon\| = O(\|y\|[\|y\| + \|z\|] + \epsilon) = O(\delta^2)$ and

$\|H\| = \|By + Z_r\| < \frac{1}{2} \lambda \mu_2 \|y\| + o(\|y\| + \|z\|^2 + \epsilon) < \frac{1}{2} \lambda \mu_2 \delta + o(\delta^2)$. Thus for δ small enough, $\|G\| < \lambda \delta$, $\|H\| < \lambda \delta$ and the lemma follows from Lemma 7.1.

LEMMA 7.3. There exists $\delta_1 > 0$ such that for any $0 < \delta < \delta_1$, $\|\alpha\| < \delta$, $\|\beta\| < \delta$, $\|d\alpha/d\theta\| < \delta$, $\|d\beta/d\theta\| < \delta$, $0 < \epsilon < \delta^2$ imply

$$(7.12) \quad \left\| \frac{d}{d\theta} S_\epsilon(\alpha, \beta) \right\| < \delta, \quad \left\| \frac{d}{d\theta} T_\epsilon(\alpha, \beta) \right\| < \delta$$

PROOF. As before,

$$(7.13) \quad \begin{aligned} \frac{d}{d\theta} S_\epsilon(\alpha, \beta) &= \frac{dU}{d\bar{\theta}} \left(\frac{d\bar{y}}{d\bar{\theta}} \right)^{-1} \\ \frac{d}{d\theta} T_\epsilon(\alpha, \beta) &= \frac{dV}{d\bar{\theta}} \left(\frac{d\bar{y}}{d\bar{\theta}} \right)^{-1} \end{aligned}$$

where all the functions are evaluated at $(\omega_2, 0, \bar{\theta}, \alpha(\bar{\theta}), \beta(\bar{\theta}), \epsilon)$. Since $\bar{y}(t, 0, \theta_0, 0, z_0, 0) = \theta_0 + t$, we have immediately

$$(7.14) \quad \frac{d\bar{y}}{d\bar{\theta}} = 1 + o(\|\alpha\| + \|d\alpha/d\theta\| + \epsilon) = 1 + o(\delta)$$

Now let $(\theta(t), y(t), z(t))$ be the solution of (7.1) with the initial value $(\bar{\theta}, \alpha(\bar{\theta}), \beta(\bar{\theta}))$, and let $\phi(t), \eta(t), \zeta(t)$ be the derivatives of $\bar{y}, U, V$ with respect to $\bar{\theta}$. Then

$$(7.15) \quad \begin{aligned} \dot{\eta} &= A\eta + \left(\frac{\partial Y_r}{\partial \theta} + \epsilon \frac{\partial Y_\epsilon}{\partial \theta} \right) \phi + \left(\frac{\partial Y_r}{\partial y} + \epsilon \frac{\partial Y_\epsilon}{\partial y} \right) \eta + \\ &\quad + \left(\frac{\partial Y_r}{\partial z} + \epsilon \frac{\partial Y_\epsilon}{\partial z} \right) \zeta \end{aligned}$$

$$\dot{\zeta} = \epsilon B\eta + \epsilon C \zeta + \epsilon \left(\frac{\partial z_r}{\partial \theta} \phi + \frac{\partial z_r}{\partial y} \eta + \frac{\partial z_r}{\partial z} \zeta \right)$$

This has the form (7.7) where

$$(7.16) \quad \begin{aligned} \|G\| &= o(\|y\|(\|y\| + \|z\|) + \epsilon) + o(\|y\| + \|z\| + \epsilon)\|\eta\| + \\ &\quad + o(\|y\| + \epsilon)\|\zeta\| \end{aligned}$$

$$\|H\| < \frac{1}{2} \lambda \mu_2 \|\eta\| + o(\|y\| + \|z\| + \epsilon) + o(\|y\| + \|z\| + \epsilon)(\|\eta\| + \|\zeta\|).$$

Now if we require $\delta_1 \leq \delta_0$, then by Lemma 7.2, $\|\alpha\| < \delta$, $\|\beta\| < \delta$, $0 < \epsilon < \delta^2$

imply $\|\gamma\| < \delta$, $\|z\| < \delta$. Thus if further, $\|\eta\| < \delta$, $\|\xi\| < \delta$, then $\|G\| = O(\delta^2)$, $\|H\| < \frac{1}{2} \lambda_{\mu_2} \delta + O(\delta^2)$. For δ small enough, therefore, $\|G\| < \lambda_{\mu_1} \delta$, $\|H\| < \lambda_{\mu_2} \delta$, and hence by Lemma 7.1, $\|\eta(0)\| = \|\alpha/d\theta\| < \delta$, $\|\xi(0)\| = \|\beta/d\theta\| < \delta$ imply $\|\eta(\omega_2)\| = \|dU/d\theta\| < \rho_1 \delta$, $\|\xi(\omega_2)\| = \|dV/d\theta\| < \rho_2 \delta$.

LEMMA 7.4. There exists $\delta_2 > 0$ such that $\|\alpha\| < \delta_2$, $\|\beta\| < \delta_2$, $\|\alpha/d\theta\| < \delta_2$, $\|\beta/d\theta\| < \delta_2$, $0 < \epsilon < \delta_2^2$, and $\|d^2\alpha/d\theta^2\| < 1$, $\|d^2\beta/d\theta^2\| < 1$ imply

$$(7.17) \quad \left\| \frac{d^2}{d\theta^2} S_\epsilon(\alpha, \beta) \right\| < 1 \quad \left\| \frac{d^2}{d\theta^2} T_\epsilon(\alpha, \beta) \right\| < 1$$

PROOF. The proof follows along the same lines. We derive formulas similar to (6.29) and (6.30) and find that we have differential equations in the form (7.7). Then under our hypotheses, we show $\|G\| = O(\delta_2)$, $\|H\| < \frac{1}{2} \lambda_{\mu_2} + O(\delta_2)$, and the result follows easily.

LEMMA 7.5. Let $(\theta_1(t), y_1(t), z_1(t), (\theta_2(t), y_2(t), z_2(t)))$ be solutions of (7.1) for which $y_1(0) = \bar{y}_1$, $z_1(0) = \bar{z}_1$, $\theta_1(\omega_2) = \theta_0$, $i = 1, 2$. Then there exists $\delta_3 > 0$ such that for any $\gamma > 0$, $\|\bar{y}_1\| < \delta_3$, $\|\bar{z}_1\| < \delta_3$, $0 < \epsilon < \delta_3^2$, $\|\bar{y}_1 - \bar{y}_2\| < \gamma$, $\|\bar{z}_1 - \bar{z}_2\| < \gamma$ imply $\|y_1(\omega_2) - y_2(\omega_2)\| < \rho_1 \gamma$, $\|z_1(\omega_2) - z_2(\omega_2)\| < \rho_2 \gamma$.

PROOF. Again the proof follows along the same lines. We need only derive formulas corresponding to (6.31), (6.32), (6.33), (6.34) whereupon we shall have differential equations of the form (7.7). Then under our hypotheses, $\|G\| = O(\delta_{3\gamma})$, $\|H\| < \frac{1}{2} \lambda_{\mu_2\gamma} + O(\delta_{3\gamma})$, and the lemma follows.

LEMMA 7.6. There exists $\delta_4 > 0$ such that $\|\alpha_1\| < \delta_4$, $\|\beta_1\| < \delta_4$, $\|\alpha_1/d\theta\| < \delta_4$, $\|\beta_1/d\theta\| < \delta_4$, $0 < \epsilon < \delta_4^2$, $i = 1, 2$, imply $\|S_\epsilon(\alpha_1, \beta_1) - S_\epsilon(\alpha_2, \beta_2)\| < \sqrt{\rho_1} \delta_4$, $\|T_\epsilon(\alpha_1, \beta_1) - T_\epsilon(\alpha_2, \beta_2)\| < \sqrt{\rho_2} \delta_4$.

PROOF. Just as in the proof of Lemma 6.6, we must find solutions $(\theta_1(t), y_1(t), z_1(t))$, $i = 1, 2$, to (7.1) such that $\theta_1(\omega_2) = \theta$. If the initial values of these solutions are $\bar{\theta}_1$, $\bar{y}_1 = \alpha_1(\bar{\theta}_1)$, $\bar{z}_1 = \beta_1(\bar{\theta}_1)$, then it is easy enough to see that $\|\bar{y}_1 - \bar{y}_2\| < \|\alpha_1 - \alpha_2\| + \|d\alpha_2/d\theta\| \cdot |\bar{\theta}_1 - \bar{\theta}_2|$, $\|\bar{z}_1 - \bar{z}_2\| < \|\beta_1 - \beta_2\| + \|d\beta_2/d\theta\| \cdot |\bar{\theta}_1 - \bar{\theta}_2|$. Thus if δ_4 is small enough, $\|\alpha_1 - \alpha_2\| < \gamma$, $\|\beta_1 - \beta_2\| < \gamma$ imply $\|\bar{y}_1 - \bar{y}_2\| < \gamma/\sqrt{\rho_1}$, $\|\bar{z}_1 - \bar{z}_2\| < \gamma/\sqrt{\rho_2}$. Then our result follows from Lemma 7.5.

In this way, we have proved the six lemmas which we need. The

completion of the proof of Theorem 7.1 is a trivial modification of the last paragraph of Section 6.

8. Reduction to Normal Form

We have proved Theorem 7.1 under the assumption that the differential equation is in the form, which here we call normal, of (7.1). The natural question is now, given a differential system of k -fold degeneracy having a periodic perturbation, is it possible by means of coordinate transformations to reduce it to the normal form (7.1)? The answer is not, unfortunately, an unqualified yes; and this is admittedly a shortcoming of our procedure. What we have done essentially is to follow the notions of Krylov and Bogoliubov, which seem in this respect to be overly restrictive. Continuing, however, with these notions we find that we can say a little more.

First of all, given any differential system of k -fold degeneracy Theorem 3.3 guarantees the existence of normal coordinates so that we may assume that the unperturbed equation is given in the form (3.4). On the other hand, the most general periodic perturbation will include, instead of the third equation of (7.1), the equation

$$(8.1) \quad \begin{aligned} \frac{dz}{dt} = & \epsilon Z_0(\theta, t) + \epsilon B(\theta, t)y + \epsilon C(\theta, t)z \\ & + \epsilon Z_r(\theta, y, z, t, \epsilon). \end{aligned}$$

Now since we can no longer toy with the θ or y coordinates, we want to find a transformation of the z coordinates which

- (1) eliminates the function $Z_0(\theta, t)$, and
- (2) reduces $C(\theta, t)$ to a constant matrix.

Note that the most general transformation which we can employ for this purpose will be of the form $\bar{z} = f(z, \theta, t)$ where f is of period ω_1 in θ and ω_2 in t . Still following Krylov and Bogoliubov we shall restrict ourselves to a transformation of the form

$$(8.2) \quad \bar{z} = z - \epsilon[v(\theta, t) + D(\theta, t)z]$$

where v and D have the periods ω_1 in θ and ω_2 in t . Thus

$$(8.3) \quad \frac{d\bar{z}}{dt} = \epsilon \left[Z_0 - \frac{\partial v}{\partial \theta} - \frac{\partial v}{\partial t} \right] + \epsilon \bar{B}y + \epsilon \left[C - \frac{\partial D}{\partial \theta} - \frac{\partial D}{\partial t} \right] \bar{z} + \epsilon Z_r$$

where $\bar{B} = \bar{B}(\theta, t)$, $\bar{Z}_r = \bar{Z}_r(\theta, y, z, t, \epsilon)$ are two new functions whose exact form is immaterial.

Then the solution to our problem can be stated in the following theorem.

THEOREM 8.1. There exists a transformation of the form (8.2) which carries (8.1) into a differential system in the normal form (7.1) if and only if there exist functions $v(\theta, t)$, $D(\theta, t)$ with periods ω_1 in θ and ω_2 in t such that

$$(8.4) \quad Z_0 = \frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial t}$$

and such that

$$(8.5) \quad \bar{C} = C(\theta, t) - \frac{\partial D}{\partial t} - \frac{\partial D}{\partial \theta}$$

is a constant matrix.

In particular we must have

$$(8.6) \quad \int_0^{\omega_2} \int_0^{\omega_1} Z_0(\theta, t) d\theta dt = 0$$

while the constant matrix will be equal to

$$(8.7) \quad \bar{C} = \frac{1}{\omega_1 \omega_2} \int_0^{\omega_2} \int_0^{\omega_1} C(\theta, t) d\theta dt$$

In regard to the existence of v and D we have the following lemma, which makes no pretense to being as general as is possible.

LEMMA 8.1. Given a function $f(\theta, t)$ of class C^3 and having periods ω_1 in θ and ω_2 in t , it can be expanded in a Fourier series:

$$(8.8) \quad f(\theta, t) = \sum_{pq} a_{pq} \exp \left\{ Z \pi i \left(\frac{p\theta}{\omega_1} + \frac{qt}{\omega_2} \right) \right\}$$

Then there exists a function $\phi(\theta, t)$ of class C^4 which is of period ω_1 in θ and ω_2 in t , and which satisfies the equation

$$(8.9) \quad \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial t} = f$$

if and only if

$$(8.10) \quad \phi(\theta, t) = \sum \frac{a_{pq}}{z\pi i \left(\frac{p}{\omega_1} + \frac{q}{\omega_2} \right)} \exp \left\{ z\pi i \left(\frac{p\theta}{\omega_1} + \frac{qt}{\omega_2} \right) \right\}$$

exists and is such a function. In particular, we must have $a_{00} = 0$.

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XI. A NOTE ON THE EXISTENCE OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS*

S. P. Diliberto and M. D. Marcus

1. Use of Borsuk's Antipodal Point Theorem

The existence of a periodic solution of the vector differential equation

$$(1) \quad \frac{dx}{dt} = f(x, t) = f(x, t + \omega),$$

f sufficiently smooth and defined for all x and t , having period ω is completely equivalent to the existence of a fixed point for a certain transformation, T , of Euclidean n -space, E^n , into itself. Namely, for any point x let $v(t, x)$ be that solution of (1) for which $v(0, x) = x$, and define $T(x) = v(\omega, x)$.¹ By defining at each point x the vector $V(x) = T(x) - x = v(\omega, x) - x$, the problem becomes that of finding where $V(x) = 0$. One manner of doing this is to find an $n-1$ sphere, S , on which the degree of V , $d(V, S)$, is non-zero.² In such an event, there must exist an x inside S^{n-1} at which $V(x) = 0$.

There are several criteria for the non-vanishing of the degree. We mention three:

- (1) E being the closed n -cell whose boundary is S ,
then $T(E) \subset E^3$

* This work was done in connection with Office of Naval Research, Contract No. 041-134.

¹ If $x(t)$ is a solution of (1), so also is $x(t + \omega)$. Consequently, if $x(0) = x(0 + \omega)$ the uniqueness theorem asserts $x(t) = x(t + \omega)$. One must know that the solutions are defined for $0 \leq t \leq \omega$ i.e., that $\|x\| \not\rightarrow \infty$ as $t \rightarrow t^1 \leq \omega$.

² More precisely: Let $V^* = \frac{V(x)}{\|V(x)\|}$ and S_1^{n-1} be the unit $n-1$ sphere, then let $d(V, S^{n-1})$ be the degree of the mapping $V^* : S^{n-1} \rightarrow S_1^{n-1}$.

³ Brouwer's Theorem.

- (ii) $-x$ being the point antipodal to x on S ,
 $V(-x) = -V(x)$ for $x \in S^4$
 (iii) $U(x)$ being a second vector distribution on S
 with $d(U, S) \neq 0$, either $\cos(V, U) \neq -1$
 or $\cos(V, U) \neq 1$ ⁵

In linear or "nearly" linear problems one should expect (ii) to yield better results than (i) since it is a linear condition. We shall illustrate this by sharpening a result of Antosiewicz. Let

$$f(x, t, \lambda) = f(x, t + \omega, \lambda) = A(t)x + p(x, t) + q(x, t, \lambda).$$

where f, x, p , and q are (column) n -vectors and $A(t) = (a_{ij}(t))$ is an n -square matrix. We shall suppose that $A(t)x, p(x, t)$, and $q(x, t, \lambda)$ are continuous in (x, t, λ) for all $x \in \mathcal{E}^n, t \in \mathcal{E}^1, \lambda \in \mathcal{E}^1$; and that they are locally Lipschitzian in x . This will insure that the standard theorems on existence, uniqueness, and continuity with respect to parameters are applicable.

THEOREM. Let

$$(2) \quad \frac{dx}{dt} = f(x, t, \lambda) = f(x, t + \omega, \lambda) = A(t)x + p(x, t) + q(x, t, \lambda).$$

If $\|p(x, t)\| \leq K\|x\|^2$ for $\|x\| \leq R, q(x, t, 0) \equiv 0$,
 and the characteristic exponents of solutions of

$$(3) \quad \frac{dy}{dt} = A(t)y$$

have non-zero real parts, there exists $\lambda_0 > 0$ such
 that for $|\lambda| < \lambda_0$ equation (2) has a periodic solution of period ω .

Antosiewicz's theorem required in addition that the real parts of the characteristic roots be negative.

2. Proof 1. (Diliberto)

Let $v(t, \lambda, x)$ denote the solution of (2) such that $v(0, \lambda, x) = x$, and assume for the moment that $v(t, \lambda, x)$ may be defined in the interval $0 \leq t \leq \omega$. Let S_r denote the sphere $\|x\| = r$, and let $V_\lambda(x) = v(\omega, \lambda, x) - x$. $V_\lambda(x)$ is continuous in both x and λ . If we can show

⁴ Borsuk's Theorem. [B].

⁵ The condition guarantees that V is homotopic to U via non-zero vectors.

that there is an r_0 for which $d(V, S_{r_0}) \neq 0$, the theorem will follow from the continuity of V_λ on λ and of $d(V_\lambda, S_{r_0})$ on V_λ .

Our problem has been reduced to the study of

$$(4) \quad \frac{dx}{dt} = A(t)x + p(x, t) = g(x, t).$$

Put $v(t, x) = v(t, 0, x)$ and $V_0(x) = V(x)$. Let $u(t, x)$ denote the solution of (3) such that $u(0, x) = x$, and let $U(x) = u(\omega, x) - x$. We shall show, using (11), that $d(U, S_r) \neq 0$ for all r ; and that there is an r_0 such that if $r \leq r_0$ then $\cos(U, V) \neq -1$ on S_r . This will prove the theorem.

As to the degree $d(U, S_r)$: Let $Y(t)$ be the matrix solution of

$$(5) \quad \frac{dy}{dt} = A(t)y$$

such that $Y(0) = I$. The solution of $u(t, x)$ of (3) may be written $u(t, x) = Y(t)u(0, x) = Y(t)x$. Now,

$$(6) \quad U(x) = u(\omega, x) - x = [Y(\omega) - I]x \neq 0$$

or else (3) would have a solution with a purely imaginary characteristic root. Since $U(x) \neq 0$, $d(U, S_r)$ is defined. From (6) it follows that $U(-x) = -U(x)$, and so $d(U, S_r) \neq 0$. From (6) also it follows that for some $c > 0$

$$(7) \quad \|U(x)\| \geq c\|x\|.$$

Since $x \equiv 0$ (defined for all t) is a solution of (4), it follows that there is an r_1 such that if $\|x\| \leq r_1$ then $\|v(t, x)\| \leq R$.

To show that $\cos(U, V) \neq -1$ on S_{r_1} , it follows from the triangle with vertices $x, x + U, x + V$ that it is sufficient to show that

$$(8) \quad \|U - V\| < \|U\|.$$

From equations (3) and (4) we have

$$\begin{aligned} u(t, x) &= x + \int_0^t A(s)u \, ds \\ v(t, x) &= x + \int_0^t (A(s)v + p(v, s)) \, ds. \end{aligned}$$

Put $K_a = \max_{0 \leq t \leq \omega} \|A(t)\|$, then for $\|x\| \leq r_1$

$$(9) \quad \|v\| \leq \|x\| + \int_0^t [K_a + KR] \|v\| \, ds$$

$$(10) \quad \|u - v\| \leq \int_0^t \|p(v, s)\| \, ds + \int_0^t K_a \|u - v\| \, ds$$

From a well known inequality⁶ we conclude from (9) that

$$\|v(t, x)\| \leq \|x\| e^{t[K_a + KR]}$$

and for a suitable constant K_1 that

$$\|v(t, x)\| \leq K_1 \|x\|.$$

Using this in (10) we find

$$\|u(t, x) - v(t, x)\| \leq \omega K_1^2 \|x\|^2 e^{tK_a}.$$

From this, for $t = \omega$, and suitable constant K_2

$$(11) \quad \|U(x) - V(x)\| = \|u(\omega, x) - v(\omega, x)\| \leq K_2 \|x\|^2.$$

From the inequalities (11) and (7) one finds that if

$$\|x\| \leq \frac{c}{K_2}$$

(in addition to $\|x\| \leq r_1$) inequality (8) will hold.

3. Proof 2. (Marcus)

We assume p is a C^3 function in its domain of definition. Since $\|p(x, t)\| \leq k\|x\|^2$ we expand p to second order terms and make the linearizing change of variable $x = \lambda y$. Equation (2) then becomes

$$(12) \quad \frac{dy}{dt} = A(t)y + \bar{q}(t) + \lambda G(y, t, \lambda), \quad \bar{q}(t) = \frac{\partial q}{\partial \lambda}(0, t, 0).$$

It clearly suffices to obtain a family of solutions, $v(b, t, \lambda)$, of (12) defined for $\lambda \leq \lambda_0$, $\lambda_0 > 0$, which has period ω .

⁶ T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Annals of Mathematics*, Vol. 20, (1918), pp. 292-296.

Let $Y(t)$ be the fundamental matrix solution of

$$\frac{dY}{dt} = A(t)Y, \quad Y(0) = I.$$

If $v(b, t, \lambda)$ is any solution of (12) then

$$v(b, t, \lambda) = Y(t)b + Y(t) \int_0^t Y^{-1}(s) (\bar{q}(s) + \lambda G(v(b, s, \lambda), s, \lambda)) ds.$$

Now $v(b, \omega, \lambda) = b$ if and only if

$$(Y(\omega) - I)b + Y(\omega) \int_0^\omega Y^{-1}(s)$$

(13)

$$(\bar{q}(s) + \lambda G(v(b, s, \lambda), s, \lambda)) ds = 0.$$

Let $L(b, \lambda)$ be the left side of (13). The condition on the characteristic exponents is equivalent to the statement that

$$\det(Y(\omega) - I) \neq 0.$$

Thus $L(b, 0) = 0$ has a solution $b = b_0$. Since $\det(L_b(b, 0)) = \det(Y(\omega) - I) \neq 0$ we conclude, by the Implicit Function Theorem, that there exists $\lambda_0 > 0$ and such a continuous function $b(\lambda)$ that $b(0) = b_0$ and for $\lambda \leq \lambda_0$, $L(b(\lambda), \lambda) = 0$. Hence $v(b(\lambda), t, \lambda)$ is the desired solution. Note that $v(b(\lambda), t, \lambda)$ is continuous in λ . We define $x(x^0, t, \lambda) = \lambda v(b(\lambda), t, \lambda)$ where $x^0(\lambda) = x(x^0, 0, \lambda) = \lambda b(\lambda)$. Hence $x^0(\lambda)$ is continuous for $\lambda \leq \lambda_0$ and $x^0(0) = 0$. Thus $x(x^0(\lambda), t, \lambda)$ depends continuously on λ and reduces to the trivial periodic solution $x = 0$ for $\lambda = 0$.

XII. AN INVARIANT SURFACE THEOREM FOR A NON-DEGENERATE SYSTEM*

Marvin D. Marcus

1. Three Dimensional Systems. Introduction

The notion of an invariant surface for systems of differential equations has been defined and investigated in a concurrent paper by Professor S. P. Diliberto [1]. An invariant surface, M , of dimension r for

$$(1) \quad \frac{dx}{dt} = f(x, t)$$

x , f n -vectors, $f \in C^1$ is a C^r manifold in E^n with the property that if $U(t)$ is a solution to (1) and $U(t_0) \in M$ then $U(t) \in M$ for all t .

Here we investigate the existence of a family of invariant tori for the holonomic 3-dimensional system.

$$\frac{d\theta}{dt} = 1 + zH(\theta, \varphi, z) + \lambda P(\theta, \varphi, z)$$

$$(2) \quad \frac{d\varphi}{dt} = 1 + zK(\theta, \varphi, z) + \lambda Q(\theta, \varphi, z)$$

$$\frac{dz}{dt} = A(\theta, \varphi)z + z^2L(\theta, \varphi, z) + \lambda R(\theta, \varphi, z).$$

It is assumed that $H, K, L, P, Q, R, \in C^3$ and are doubly periodic in θ and φ of periods ω_1 and ω_2 respectively. It is clear that for $\lambda = 0$ the solutions with initial values on the θ and φ axes generate a torus $\Gamma_0 = \Gamma(\theta, \varphi)$ of period ω_1 in θ and ω_2 in φ by identifying the planes $\varphi = \pm n\omega_1, \theta = \pm n\omega_2, n = 0, 1, \dots$.

The system (2) is of interest since it arises after transformation to normal coordinates from the equation

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$$\frac{dx}{dt} = f(x) + \epsilon R(x, t, \epsilon)$$

x, f 3-vectors $\mathbb{R} \times \mathbb{C}^3$, R period ω , in t , which is assumed to have an invariant surface $\Delta(\theta_1, \theta_2)$ of period ω_1 in θ_1 . A reduction of Duffing's equation to equation (2) will be found in a concurrent paper [2].

In this paper we establish sufficient conditions for the existence of a family of invariant tori for (2) by use of the Schauder fixed point theorem [3]. Let

$$W(a, b, t, \lambda) = (\theta(a, b, t, \lambda), \varphi(a, b, t, \lambda), z(a, b, t, \lambda))$$

be the solution of (2) with initial values $W(a, b, 0, \lambda) = (a, 0, b)$.

REMARK 1. $W(a + n\omega_1, b, t, \lambda) = W(a, b, t, \lambda) + (n\omega_1, 0, 0)$ where n is any integer. That is, the solution in the phase space starting at $(a + n\omega_1, 0, b)$ is obtained from the solution starting at $(a, 0, b)$ by a translation parallel to the θ axis through a distance $n\omega_1$. This is an immediate consequence of periodicity and uniqueness.

REMARK 2. There exists $b_0 > 0$ and $\lambda_0 > 0$ such that if $|\lambda| \leq \lambda_0$ and $|b| \leq b_0$ then $W(a, b, t, \lambda)$ has a unique intersection with the plane $\varphi = \omega_2$. That is, there exists a unique function $T(a, b, \lambda)$ such that

$$\varphi(a, b, T(a, b, \lambda), \lambda) = \omega_2$$

and

$$T(a, 0, 0) = \omega_2.$$

PROOF. Let $a \in [0, \omega_1]$. Then $W(a, 0, t, 0) = (t + a, t, 0)$ which clearly intersects $\varphi = \omega_2$ for $t \leq \tau$. By continuity in λ and b choose b_0 and λ_0 such that for $|\lambda| \leq \lambda_0$, $|b| \leq b_0$

$$|z(a, b, t, \lambda)K(W(a, b, t, \lambda)) + \lambda Q(W(a, b, t, \lambda))| < \frac{1}{2}.$$

Then $\varphi_t(a, b, t, \lambda) \geq \frac{1}{2}$. Thus there exists a unique $T(a, b, \lambda)$ such that

- (i) $T(a, 0, 0) = \omega_2$ for $a \in (0, \omega_1)$;
- (ii) $\varphi(a, b, T(a, b, \lambda), \lambda) = \omega_2$.

Now by Remark 1, any solution $W(a, b, t, \lambda)$ can be obtained by translation from those for which $a \in [0, \omega_1]$.

Thus it is clear from Remarks 1 and 2 that for λ and b sufficiently small $W(a, b, t, \lambda)$ well defines a mapping from the $\varphi = 0$ plane to the $\varphi = \omega_2$ plane which carries curves of period ω_1 into such. More specifically let $\alpha(a)$ be periodic of period ω_1 in $\varphi = 0$ plane.

Define $F_\lambda : \alpha \longrightarrow \bar{\alpha}$ where $\bar{\alpha}$ is given by the formula

$$\bar{\alpha}(\theta(a, \alpha(a), T(a, \alpha(a), \lambda), \lambda)) = z(a, \alpha(a), T(a, \alpha(a), \lambda), \lambda)$$

Remarks 1 and 2 show that $\bar{\alpha}$ is well defined. We define two families of periodic curves: $\mathfrak{M} = E\alpha$ [α is continuous of period ω_1]. $\mathfrak{M}_{r\eta} = E\alpha$ [$\alpha \in \mathfrak{M}$ and $\|\alpha\| \leq r$ and α is Lipschitzian with constant η]. Note that F_λ is continuous on $\mathfrak{M}_{r\eta}$ to $\mathfrak{M}$.

REMARK 3. $\mathfrak{M}_{r\eta}$ is a convex compact subset of the function space $\mathfrak{M}$ under the topology of uniform convergence.

PROOF. Convexity is clear and compactness follows from Ascoli's theorem. We will show that for each $|\lambda| \leq \lambda^*$ that $F_\lambda(\mathfrak{M}_{r\eta}) \subset \mathfrak{M}_{r\eta}$ for appropriate choices of η and r depending on λ and thereby apply the fixed point theorem.

THEOREM 1. If $H(\theta, \varphi, 0) = 0$ and $K(\theta, \varphi, 0) = 0$ and

$$\int_0^{\omega_2} A(t + a, t) dt < 0 \quad \text{for all } 0 \leq a \leq \omega_1$$

then there exists $\lambda^* > 0$ such that for each $|\lambda| \leq \lambda^*$ there exists a curve $\alpha^\lambda \in \mathfrak{M}$ such that $F_\lambda(\alpha^\lambda) = \alpha^\lambda$.

PROOF. By Remark 2 obtain λ_0 and b_0 such that $T(a, b, \lambda)$ and F_λ are well defined. Now

$$\begin{aligned} \frac{dz}{dt}(a, b, t, 0) &= A(\theta(a, b, t, 0), \varphi(a, b, t, 0))z(a, b, t, 0) \\ &\quad + z^2(a, b, t, 0)L(W(a, b, t, 0)). \end{aligned}$$

Hence using the formula for variation of parameters

$$z(a, b, t, 0) = be^{\int_0^t A dt} + e^{\int_0^t A dt} \int_0^t e^{-\int_0^s A dt} z^2 L(W) dt$$

where we systematically leave out the arguments of the functions occurring.

$$\left| z(a, b, t, 0) e^{-\int_0^t A dt} \right| \leq |b| + \int_0^t \left| e^{-\int_0^s A dt} z \right| |zL(W)| dt$$

applying Gronwall's Lemma [4] and multiplying both sides by

$$e^{\int_0^t A dt}$$

we obtain

$$|z(a, b, t, 0)| \leq |b| e^{\int_0^t A dt + \int_0^t |z(a, b, t, 0)| |L(W)| dt}$$

Now to shorten writing we introduce some notation: Let

$$M(a, b, \lambda) = z(a, b, T(a, b, \lambda), \lambda),$$

$$I_1(a, b, \lambda) = \int_0^{T(a, b, \lambda)} A(\theta, \varphi) dt + \int_0^{T(a, b, \lambda)} |z(a, b, t, 0)| |L(W)| dt.$$

Then

$$|M(a, b, 0)| = |z(a, b, T(a, b, 0), 0)| \leq |b| e^{I_1(a, b, 0)}$$

and

$$\begin{aligned} I_1(a, b, 0) &= \int_0^{T(a, b, 0)} A(\theta, \varphi) dt + \int_0^{T(a, b, 0)} |z(a, b, t, 0)| |L(W)| dt \\ &\leq \int_0^{T(a, b, 0)} A(\theta, \varphi) dt + |b| k_1 \end{aligned}$$

where k_1 is some constant. Now

$$\int_0^{T(a, b, 0)} A(\theta(a, b, t, 0), \varphi(a, b, t, 0)) dt \leq \int_0^{\omega_2} A(t + a, t) dt + |b| k_2.$$

Since

$$\int_0^{\omega_2} A(t + a, t) dt < 0$$

for $0 \leq a \leq \omega_1$ and A has period ω_1 in θ we conclude there exist positive constants Δ, k such that

$$\begin{aligned}
|M(a, b, \lambda)| &\leq |M(a, b, 0)| + |\lambda|k \leq |b| e^{I_1(a, b, 0)} + |\lambda|k \\
&\leq |b| e^{-\Delta+k|b|} + |\lambda|k.
\end{aligned}$$

Now if $\|\alpha\| \leq r$ we want

$$(3) \quad \|F_\lambda(\alpha)\| \leq r.$$

Since $\|F_\lambda(\alpha)\| = \|M(a, b, \lambda)\| \leq r e^{-\Delta+k\eta} + |\lambda|k$ condition (3) will be satisfied if

$$(4) \quad r e^{-\Delta+k\eta} + |\lambda|k \leq r.$$

Set $r = |\lambda|^\mu$, $0 < \mu < 1$. Then (4) becomes, upon dividing by $|\lambda|^\mu$.

$$(5) \quad g(\lambda) = e^{-\Delta+k|\lambda|^\mu} + |\lambda|^{(1-\mu)k} < 1.$$

Since $g(0) = e^{-\Delta} < 1$, (3) holds.

We next show that $F_\lambda(\alpha) = \bar{\alpha}$ is Lipschitzian of constant η for an appropriate choice of η .

To shorten notation let

$$\theta(i) = \theta(a_i, \alpha(a_i), T(a_i, \alpha(a_i), \lambda), \lambda) \quad \text{for } i = 1, 2.$$

Then, by repeated applications of Taylor's Theorem

$$\begin{aligned}
|\bar{\alpha}(\theta(1)) - \bar{\alpha}(\theta(2))| &\leq |z_a(-; \lambda)| |a_1 - a_2| \\
&\quad + |z_b(-; \lambda)| \eta |a_1 - a_2| \\
(6) \quad &\quad + |z_T(-; \lambda)| |T(a_1, \alpha(a_1), \lambda) - T(a_2, \alpha(a_2), \lambda)| \\
&\leq \left\{ |z_a(-; \lambda)| + |z_b(-; \lambda)| \right. \\
&\quad + |z_T(-; \lambda)| |T_a(-; \lambda)| \\
&\quad \left. + \eta |z_T(-; \lambda)| |T_b(-; \lambda)| \right\} |a_1 - a_2|
\end{aligned}$$

and

$$\begin{aligned}
|\theta(1) - \theta(2)| &= |\theta_a(-; \lambda)(a_1 - a_2) + \theta_b(-; \lambda)(\alpha(a_1) - \alpha(a_2)) \\
&\quad + \theta_T(-; \lambda)(T(a_1, \alpha(a_1), \lambda) - T(a_2, \alpha(a_2), \lambda))| \\
(7) \quad &\geq |a_1 - a_2| \left\{ |\theta_a(-; \lambda)| - (|\theta_b(-; \lambda)| \eta \right. \\
&\quad \left. + |\theta_T(-; \lambda)| |T_a(-; \lambda)| + |\theta_T(-; \lambda)| |T_b(-; \lambda)| \eta) \right\}
\end{aligned}$$

where the dash indicates evaluation at an intermediate point. The remainder of the proof is devoted to obtaining magnitude approximations on

$\theta_a, \theta_b, \theta_T, T_a, T_b, z_a, z_b, z_T$. We remark that θ_T and z_T are bounded in the finite region we consider. We thus investigate the variational equations for $\lambda = 0$.

$$\begin{aligned} \frac{du}{dt} &= z(H_\theta u + H_\phi v + H_z n) + Hn \\ (8) \quad \frac{dv}{dt} &= z(K_\theta u + K_\phi v + K_z n) + Kn \\ \frac{dn}{dt} &= z(A_\theta u + A_\phi v) + An + z^2(L_\theta u + L_\phi v + L_z n) + 2znL. \end{aligned}$$

Now b occurs as a parameter on the right side of (8). Since $z(a, 0, t, 0) \equiv 0$ (8) becomes, for $b = 0$,

$$\begin{aligned} \frac{du}{dt} &= H(t + a, t, 0)n = 0 \\ \frac{dv}{dt} &= K(t + a, t, 0)n = 0 \\ \frac{dn}{dt} &= A(t + a, t)n. \end{aligned}$$

Hence, for $\lambda = 0, b = 0$

$$\begin{aligned} u &= C_1 \\ v &= C_2 \\ n &= C_3 e^{\int_0^t A(s+a, s)ds}. \end{aligned}$$

Since $W(a, 0, t, 0) = (t + a, t, 0)$ we conclude

$$\begin{aligned} \theta_a(a, 0, t, 0) &= 1 & \theta_b(a, 0, t, 0) &= 0 \\ \varphi_a(a, 0, t, 0) &= 0 & \varphi_b(a, 0, t, 0) &= 0 \\ (9) \quad z_a(a, 0, t, 0) &= 0 & z_b(a, 0, t, 0) &= e^{\int_0^t A(s+a, s)ds}. \end{aligned}$$

Also note

$$T(a, 0, 0) = \omega_2 \quad \text{implies} \quad T_a(a, 0, 0) = 0,$$

$$\varphi(a, b, T(a, b, \lambda), \lambda) = \omega_2 \quad \text{implies} \quad T_b(a, 0, 0) = 0$$

and

$$z_b(a, 0, T(a, 0, 0), 0) = e^{\int_0^{\omega_2} A(t+a, t)dt} < e^{-\Delta}.$$

By use of relations (9) we secure a constant $\Gamma > 0$ such that

$$\begin{aligned}
 (10) \quad & \left. \begin{aligned}
 |\theta_a(a, b, T(a, b, \lambda), \lambda)| &\geq 1 - \Gamma(|b| + |\lambda|) \\
 |\varphi_a(a, b, T(a, b, \lambda), \lambda)| &\leq \Gamma(|b| + |\lambda|) \\
 |z_a(a, b, T(a, b, \lambda), \lambda)| &\leq \Gamma(|b| + |\lambda|) \\
 |\theta_b(a, b, T(a, b, \lambda), \lambda)| &\leq \Gamma(|b| + |\lambda|) \\
 |\varphi_b(a, b, T(a, b, \lambda), \lambda)| &\leq \Gamma(|b| + |\lambda|) \\
 |z_b(a, b, T(a, b, \lambda), \lambda)| &\leq e^{-\Delta+\Gamma|b|} + \Gamma(|\lambda| + |b|) \\
 |T_a(a, b, \lambda)| &\leq \Gamma(|b| + \lambda) \\
 |T_b(a, b, \lambda)| &\leq \Gamma(|b| + \lambda)
 \end{aligned} \right\}
 \end{aligned}$$

By use of relations (7) and (10)

$$\begin{aligned}
 (11) \quad & |\theta(1) - \theta(2)| \geq |a_1 - a_2| \left\{ 1 - \Gamma(|b| + |\lambda|) \right. \\
 & \left. - \left(\Gamma(|b| + |\lambda|)\eta + |\theta_T| \Gamma(|b| + |\lambda|) + |\theta_T| \Gamma(|b| + |\lambda|)\eta \right) \right\} \\
 & = |a_1 - a_2| \left\{ 1 - \Gamma(|b| + |\lambda|) \left(1 + |\theta_T| + \eta(1 + |\theta_T|) \right) \right\} \\
 & \geq |a_1 - a_2| \left\{ 1 - \Gamma_1(|b| + |\lambda|)(\eta + 1) \right\}
 \end{aligned}$$

where Γ_1 is some constant.

Similarly, by use of relations (6) and (10)

$$\begin{aligned}
 (12) \quad & |\tilde{\alpha}(\theta(1)) - \tilde{\alpha}(\theta(2))| \leq \left\{ \Gamma(|b| + |\lambda|) \right. \\
 & \left. + \left(e^{-\Delta+\Gamma|b|} \Gamma(|b| + |\lambda|)\eta + |z_T| \Gamma(|b| + |\lambda|) \right) \right. \\
 & \left. + \eta |z_T| \Gamma(|b| + |\lambda|) \right\} |a_1 - a_2| \\
 & \leq \left\{ \Gamma(|b| + |\lambda|) \left(1 + |z_T| + \eta(|z_T| + 1) \right) \right. \\
 & \left. + e^{-\Delta+\Gamma|b|}\eta \right\} |a_1 - a_2| \\
 & \leq \left(\Gamma_2(|b| + |\lambda|)(1 + \eta) + e^{-\Delta+\Gamma|b|}\eta \right) |a_1 - a_2|.
 \end{aligned}$$

We simplify notation by setting $\Gamma_1 = \Gamma_2 = \Gamma$. Now the calculation will be complete if we show that

$$(13) \quad 1 - \Gamma(\Gamma + |\lambda|)(\eta + 1) > 0$$

and

$$(14) \quad \frac{\Gamma(r+|\lambda|)(1+\eta)+e^{-\Delta+\Gamma r}}{1-\Gamma(r+|\lambda|)(\eta+1)} \leq \eta.$$

Now $r = |\lambda|^\mu$ where $0 < \mu < 1$ so set

$$\eta = |\lambda|^{\mu/2}.$$

Then (13) and (14) are satisfied if

$$1 - \Gamma(|\lambda|^\mu + |\lambda|)(|\lambda|^{\mu/2} + 1) > 0$$

$$\frac{\Gamma(|\lambda|^\mu + |\lambda|)(1+|\lambda|^{\mu/2}) + |\lambda|^{\mu/2} e^{-\Delta+\Gamma|\lambda|^\mu}}{1-\Gamma(|\lambda|^\mu + |\lambda|)(|\lambda|^{\mu/2} + 1)} \leq |\lambda|^{\mu/2}$$

dividing the second inequality by $|\lambda|^{\mu/2}$ we obtain

$$(15) \quad \frac{\Gamma(|\lambda|^{\mu/2} + |\lambda|^{1-\mu/2})(1+|\lambda|^{\mu/2}) + e^{-\Delta+\Gamma|\lambda|^\mu}}{1-\Gamma(|\lambda|^\mu + |\lambda|)(|\lambda|^{\mu/2} + 1)} \leq 1.$$

Setting $\lambda = 0$ on the left side of (13) and (15) we obtain the obvious relations

$$1 > 0$$

$$e^{-\Delta} < 1.$$

Thus it is clear that we can obtain $\lambda^* > 0$ such that for $|\lambda| < \lambda^*$ if $r = |\lambda|^{\mu/2}$ and $\eta = |\lambda|^{\mu/2}$ then $F_\lambda(\mathfrak{N}_{r\eta}) \subset \mathfrak{N}_{r\eta}$, thereby completing the proof.

THEOREM 2. If $H(\theta, \varphi, 0) = 0$ and $K(\theta, \varphi, 0) = 0$ and

$$\int_0^{\omega_1} A(t, t+a) < 0$$

for all $0 \leq a \leq \omega_2$ then there exists $\lambda^* > 0$ such that for $|\lambda| \leq \lambda^*$ there exists a periodic curve $\alpha_1^{\omega_1}$ of period ω_2 that is mapped onto itself by the solutions starting in the $\theta = 0$ plane and extending to the $\theta = \omega_2$ plane.

This is obviously obtained by simply interchanging the roles of θ and φ in Theorem 1.

THEOREM 3. Under the combined hypotheses of Theorems 1 and 2 there exists an invariant torus $\Gamma_\lambda(\theta, \varphi)$ of (2) for $|\lambda| \leq \lambda^*$ of periods ω_1 in θ and ω_2 in φ such that

$$\lim_{\lambda \rightarrow 0} \text{dist}(\Gamma_\lambda, \Gamma_0) = 0.$$

These tori are obtained after the identification mentioned in the Introduction.

PROOF. That we actually have a torus generated by the integral curves is an immediate consequence of Theorems 1, 2 and Remark 1. Since $\|\alpha^\lambda\| \leq |\lambda|^\mu$, by continuity in initial values we conclude that

$$\lim_{\lambda \rightarrow 0} \text{dist}(\Gamma_\lambda, \Gamma_0) = 0.$$

2. n-Dimensional Systems

We extend the results obtained for the system (2) to a system in which z is an $n-2$ vector. We use precisely the same device to obtain a fixed periodic curve under the mapping F_λ .

$$\frac{d\theta}{dt} = 1 + H(\theta, \varphi, z)z + \lambda P(\theta, \varphi, z)$$

$$\frac{d\varphi}{dt} = 1 + K(\theta, \varphi, z)z + \lambda Q(\theta, \varphi, z)$$

$$\frac{dz}{dt} = A(\theta, \varphi)z + L(\theta, \varphi, z)z + \lambda R(\theta, \varphi, z)$$

θ, φ, L, P, Q are scalars, z an $(n-2)$ column vector, H, K, R $(n-2)$ vectors, A an $(n-2)$ square matrix. The periodicity conditions are as before and $\tilde{\alpha}, F_\lambda, \mathfrak{M}_{r\eta}$ have the same meaning.

THEOREM 4. If $H, K, L = \theta(\|z\|)$ and the fundamental matrix solution to

$$(16) \quad \frac{dz}{dt} = A(t + a, t)z$$

satisfies the inequality

$$(17) \quad \|Z(a, \omega_2)\| < 1$$

for $0 \leq a \leq \omega_1$, then there exists $\lambda^* > 0$ such that for each λ , $|\lambda| \leq \lambda^*$, there exists a periodic hyper-curve α^λ of period ω_1 such that

$$F_\lambda(\alpha^\lambda) = \alpha^\lambda.$$

PROOF. Let $Z(a, b, t)$ be the fundamental matrix of the homogeneous equation

$$(18) \quad \frac{dz}{dt}(a, b, t, 0) = A(\theta(a, b, t, 0), \varphi(a, b, t, 0), z(a, b, t, 0)) \\ + L(W)z$$

then precisely as before

$$\|z(a, b, t, 0)\| \leq \|b\| \|Z(a, b, t)\| e^{\int_0^t \|Z^{-1}\| \|L\| dt};$$

since $Z(a, 0, t) = Z(a, t)$. If

$$M(a, b, \lambda) = z(a, b, T(a, b, \lambda), \lambda)$$

we see that

$$\|M(a, b, 0)\| \leq \|b\| \|Z(a, b, T(a, b, 0))\| e^{\int_0^{T(a, b, 0)} \|Z^{-1}(-)\| \|L\| dt}.$$

Now $\|Z(a, b, T(a, b, 0))\| \leq \|Z(a, 0, \omega_2)\| + k_1 \|b\|$ and since $L = O(\|z\|)$ we know that

$$\int_0^{T(a, b, 0)} \|Z^{-1}(a, b, t)\| \|L(W)\| dt = O(\|b\|).$$

Thus

$$\|M(a, b, 0)\| \leq \|b\| e^{k_1 \|b\|} (\|Z(a, \omega_2)\| + k_1 \|b\|)$$

so

$$\|M(a, b, \lambda)\| \leq \|M(a, b, 0)\| + |\lambda| k_2 \\ \leq \|b\| e^{k_1 \|b\|} (\|Z(a, \omega_2)\| + k_1 \|b\|) + |\lambda| k_2.$$

If $\|\alpha\| \leq r$ we want

$$(19) \quad \|F_\lambda(\alpha)\| \leq r$$

and (19) will be clearly satisfied if

$$(20) \quad \begin{aligned} \|F_\lambda(\alpha)\| &= \|M(a, b, \lambda)\| \\ &\leq r e^{k_1 r} (\|Z(a, \omega_2)\| + k_1 r) + |\lambda| k_2 \leq r. \end{aligned}$$

Setting $r = |\lambda|^\mu$, $0 < \mu < 1$, dividing through (20) by r and setting $\lambda = 0$ we obtain as a sufficient condition for (19) that

$$\|Z(a, \omega_2)\| < 1.$$

We proceed now to the invariance of the Lipschitz constant under F_λ . Denote $\theta(1)$, $z(1)$, $i = 1, 2$, as before,

$$z_b(a, b, t, \lambda) = \left(\frac{\partial z^1}{\partial b_j} \right)$$

an $(n-2)$ square matrix,

$$z_a(a, b, t, \lambda) = \left(\frac{\partial z^1}{\partial a}, \dots, \frac{\partial z^{n-2}}{\partial a} \right)$$

$$T_{ab} = \left(\frac{\partial T}{\partial a}, \frac{\partial T}{\partial b_1}, \dots, \frac{\partial T}{\partial b_{n-2}} \right).$$

Then

$$(21) \quad \begin{aligned} \|\bar{\alpha}(\theta(1)) - \bar{\alpha}(\theta(2))\| &= \|z(1) - z(2)\| \\ &= \left\| \left[z_a(-)(a_1 - a_2), z_b(-)(a_1) - \alpha(a_2), z_T(-)(T(a_1, \alpha(a_1), \lambda) \right. \right. \\ &\quad \left. \left. - T(a_2, \alpha(a_2), \lambda)) \right] \right\| \\ &\leq \|z_a(-)\| |a_1 - a_2| + \|z_b(-)\|_\eta |a_1 - a_2| \\ &\quad + \|z_T(-)\| \|T_{ab}(-)\| (|a_1 - a_2| + \eta |a_1 - a_2|) \\ &= \left\{ \|z_a(-)\| + \|z_b(-)\|_\eta + \|z_T(-)\| \|T_{ab}(-)\| (1 + \eta) \right\} |a_1 - a_2|. \end{aligned}$$

We establish one further inequality:

$$\begin{aligned}
 (22) \quad |\theta(1) - \theta(2)| &= |\theta_a(a_1 - a_2) + \theta_b(\alpha(a_1) - \alpha(a_2)) \\
 &\quad + \theta_T(T(a_1, \alpha(a_1), \lambda) - T(a_2, \alpha(a_2), \lambda))| \\
 &\geq |\theta_a| |a_1 - a_2| - \left\{ \|\theta_b\| |a_1 - a_2| \eta \right. \\
 &\quad \left. + |\theta_T| \|T_{ab}\| (|a_1 - a_2| + \eta |a_1 - a_2|) \right\} \\
 &= \left\{ |\theta_a| - (\|\theta_b\| \eta + |z_T| \|T_{ab}\| (1 + \eta)) \right\} |a_1 - a_2|.
 \end{aligned}$$

Consider now the variational equations for $\lambda = 0$. For the variation in a we have:

$$\frac{d\theta_a}{dt} = O(\|z\|) + H(\theta, \varphi, z)z_a$$

$$\frac{d\varphi_a}{dt} = O(\|z\|) + k(\theta, \varphi, z)z_a$$

$$\frac{dz_a}{dt} = O(\|z\|) + A(\theta, \varphi)z_a + L(\theta, \varphi, z)z_a$$

where b occurs as a parameter on the right. Since $z(a, 0, t, 0) \equiv 0$, $\theta(a, 0, t, 0) = t + a$, $\varphi(a, 0, t, 0) = t$, and $H, K, L = \theta(\|z\|)$ we conclude

$$\theta_a(a, 0, t, 0) = 1$$

$$\varphi_a(a, 0, t, 0) = 0$$

$$z_a(a, 0, t, 0) = 0$$

for the variation in b :

$$\frac{d\theta_b}{dt} = H_b z + z_b H$$

$$\frac{d\varphi_b}{dt} = K_b z + z_b K'$$

$$\frac{dz_{b_j}}{dt} = \frac{\partial A(\theta, \varphi)}{\partial b_j} z + A(\theta, \varphi) z_{b_j} + \frac{\partial L}{\partial b_j} z + L z_{b_j}$$

where

$$H_b = \left(\frac{\partial H^1}{\partial b_j} \right)$$

an $(n-2)$ square matrix,

$$K_b = \left(\frac{\partial K^1}{\partial b_j} \right)$$

an $(n-2)$ square matrix, and

$$\theta_b = \left(\frac{\partial \theta}{\partial b_1}, \dots, \frac{\partial \theta}{\partial b_{n-2}} \right), \quad \varphi_b = \left(\frac{\partial \varphi}{\partial b_1}, \dots, \frac{\partial \varphi}{\partial b_{n-2}} \right),$$

$$z_{b_j} = \left(\frac{\partial z^1}{\partial b_j}, \dots, \frac{\partial z^{n-2}}{\partial b_j} \right).$$

Since b occurs as a parameter on the right, set $b = 0$, readily obtaining

$$\theta_b(a, 0, t, 0) \equiv 0, \quad \varphi_b(a, 0, b, 0) \equiv 0$$

and

$$\frac{dz_{b_j}}{dt}(a, 0, t, 0) = A(t + a, t)z_{b_j}.$$

Thus, since $z(a, b, 0, 0) = b$, $z_{b_j}(a, b, 0, 0) = \epsilon_j$ the j^{th} unit vector. Hence, $z_{b_j}(a, 0, t, 0) = Z(a, t)\epsilon_j$ and also $z_T(a, 0, T, 0) \equiv 0$, $z_b(a, 0, t, 0) = Z(a, t)$. We conclude then that

$$z_{ab}(a, 0, t, 0) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ Z(a, t) \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Thus $\|z_{ab}(a, 0, \omega_2, 0)\| < 1$. We also note that

$$\theta(a, b, T(a, b, \lambda), \lambda) = \omega_2$$

and hence $\varphi_b + \varphi_T T_b = 0$ and $T_b(a, 0, 0) = 0$. Compiling these results we have:

$$(23) \quad \left. \begin{array}{ll} \theta_a(a, 0, t, 0) = 1 & \theta_b(a, 0, t, 0) = 0 \\ \varphi_a(a, 0, t, 0) = 0 & \varphi_b(a, 0, t, 0) = 0 \\ z_a(a, 0, t, 0) = 0 & z_b(a, 0, t, 0) = Z(a, t) \\ T_a(a, 0, 0) = 0 & T_b(a, 0, 0) = 0 \end{array} \right\}$$

$$\|z_{abT}(a, 0, \omega_2, 0)\| < 1.$$

Now, establishing relations (23) which correspond to relations (9) is the essential step in establishing the invariance of the Lipschitz constant. It is clear that we may proceed as before to obtain

$$F(\mathfrak{M}_{r\eta}) \subset \mathfrak{M}_{r\eta}.$$

Theorems 2 and 3 carry over with no difficulty.

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XIII. AN APPLICATION OF PERIODIC SURFACES (SOLUTION OF A SMALL DIVISOR PROBLEM)*

Stephen P. Diliberto

Section 1

We propose to discuss the equation

$$(1) \quad \ddot{x} + r\dot{x} + \omega_1^2 x = p(t) + \epsilon f(x, \dot{x})$$

where $p(t)$ is almost periodic. For $\epsilon = 0$ and $r > 0$ the equation may be solved by quadratures and produces a certain (one only) almost periodic solution, say, $q(t)$; one has the additional fact that every other solution approaches $q(t)$ exponentially. For a long time nothing was known about the case $\epsilon \neq 0$. In 1946 Friedrichs [2] showed that if

$$p(t) = \sum_{pq} c_{pq} e^{2\pi_1(\omega_1 p + \omega_2 q)t}$$

$$f(x, \dot{x}) = x^3$$

where $\sum |c_{pq}| < +\infty$. Then for ϵ small enough (he gives precise estimates in terms of r and $\|p\| = \sum |c_{pq}|$) equation 1 has solutions of the form

$$\sum_{pq} B_{pq} e^{2\pi_1(\omega_1 p + \omega_2 q)t}$$

(where $\sum |B_{pq}| < +\infty$).

The notion of periodic surface can produce somewhat stronger results but under the restriction that $p(t)$ is an almost periodic polynomial.

THEOREM. Let $x = q(t)$ be the almost periodic solution (periodic if $\omega_2 = \omega_3$) of

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$$(2) \quad \ddot{x} + r\dot{x} + \omega_1^2 x = A_2 \sin \omega_2 t + A_3 \sin \omega_3 t + \epsilon f(x, \dot{x})$$

for $r > 0$, $\epsilon = 0$, $f \in C^2$. For $|\epsilon|$ small enough equation (2) possesses one solution

$$(3) \quad x = q(t) + g(t, t, \epsilon) = U(t, \epsilon)$$

where $g(\theta_2, \theta_3, \epsilon)$ has period $\frac{2\pi}{\omega_1}$ in θ_1 , where g is continuous in t, ϵ , and where $g(\theta_2, \theta_3, 0) = 0$.

Furthermore, every other solution of (2) near $U(t, \epsilon)$ tends to it exponentially.

There are several points that are worth noticing about this result: First, since $g(\theta_2, \theta_3, \epsilon)$ is doubly periodic it follows that $g(t, t, \epsilon)$ is almost periodic (provided of course that $m\omega_1 \neq n\omega_2$). Secondly, the solution $U(t, \epsilon)$ converges uniformly to $q(t)$ over $-\infty < t < +\infty$. This is surprising because initially there seems to be no way of excluding that $U(t) = g(t + \delta(t, \epsilon)) + h(t, \epsilon)$ where $h(t, \epsilon) \rightarrow 0$ uniformly in ϵ but where

$$\delta(t, \epsilon) = O(\epsilon, t);$$

in other words, that there may be a phase shift. Third, there are no restrictions on the form of the function f . Fourth, that here for the first time a fairly standard form of the classical small-divisor problem has been solved by a geometric device (this will be clear from the proof)¹. Fifth, the above result clearly generalizes to the case where $p(t)$ is any trigonometric polynomial.

There are no results for the case $r = 0$. It seems highly probable, however, that periodic surface will enter the problem. For $r = 0$, $\epsilon = 0$, and $p(t) \equiv 0$ all solutions are periodic, i.e., in the phase space there is a one parameter family of periodic solutions. For $r = 0$, $\epsilon \neq 0$, $f(x, \dot{x}) = x^2$, say, $p(t) \equiv 0$, there is again a one parameter family of periodic solutions in the phase plane. For $r = 0$, $\epsilon = 0$, $p(t) = A_2 \sin \omega_2 t$ the equation possesses a one parameter family of periodic 2-surfaces. We conjecture that for $r = 0$, $\epsilon \neq 0$, $f(x, \dot{x}) = x^2$, $p(t) = A_2 \sin \omega_2 t$ the equation will also possess a one parameter family of periodic two surfaces. We do not at present have any good methods for establishing the existence of bands of periodic surfaces.

¹ These small-divisor difficulties are inherent in methods which attempt to settle existence questions by means of series expansions, and if they are resolved are done so by an ingenious analytic device -- the most recent example being Friedrich's proof.

2. Proof of Theorem

Replace equation (2) by the equations

$$(4) \quad \left. \begin{aligned} \ddot{x} + r\dot{x} + \omega_1^2 x &= A_2 \sin \omega_2 \varphi_2 + A_3 \sin \omega_3 \varphi_3 + \epsilon f(x, \dot{x}) \\ \frac{d\varphi_1}{dt} &= 1 \quad 1 = 1, 2. \end{aligned} \right\}$$

Let $x = q(\varphi_2, \varphi_3)$ be the solution (for $\varphi_2 = t + \text{const}$ and $\varphi_3 = t + \text{const}$) of equation (4) which is almost periodic (periodic if for some integers m_2 and m_3 , $m_2 \omega_2 = m_3 \omega_3$).

The change of variables

$$(5) \quad \left. \begin{aligned} x_1 &= x + q(\varphi_2, \varphi_3) \\ x_2 &= \dot{x} + \frac{\partial q}{\partial \varphi_2} + \frac{\partial q}{\partial \varphi_3} = \dot{x} + r(\varphi_2, \varphi_3) \end{aligned} \right\}$$

leads to the system

$$(6) \quad \left. \begin{aligned} \frac{d\varphi_1}{dt} &= 1 \quad 1 = 2, 3 \\ \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -rx_2 - \omega_1^2 x_1 - \epsilon f(x_1 + q, x_2 + r) \end{aligned} \right\}.$$

Since the right members have period $\frac{2\pi}{\omega_1}$ in φ_1 ($1 = 2, 3$) and the characteristic roots of the linear terms in x_1 have negative real parts Theorem [4] of [1] applies. Hence there is a curve $x_1 = \alpha_1(\varphi_2, t)$ which is a permanent repeating curve for say $|\epsilon| < \epsilon_0$. This repeating curve generates a surface $x_1 = \beta_1(\varphi_2, \varphi_3, \epsilon)$ ($1 = 1, 2$) (period $\frac{2\pi}{\omega_1}$ in φ_1 of course). On this surface solutions are, of course, $\varphi_1 = t + \text{const}$. This, plus the fact that nearby solutions approach the surface exponentially, proves the theorem.

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XIV. REPEATING SOLUTIONS FOR A DEGENERATE SYSTEM*

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1. Introduction

We consider in this paper theorems concerning the existence of repeating solutions to the equations

$$\begin{aligned} \frac{d\theta}{dt} &= 1 + \lambda P(\theta, \varphi, z, \lambda) \\ (1) \quad \frac{d\varphi}{dt} &= 1 + \lambda Q(\theta, \varphi, z, \lambda) \\ \frac{dz}{dt} &= \lambda(A(\theta, \varphi)z + R(\theta, \varphi, z, \lambda)) \end{aligned}$$

$P, Q, A, R \in C^3$ and period ω_1 in θ , ω_2 in φ . Let $\theta(a, b, t, \lambda)$, $\varphi(a, b, t, \lambda)$, $z(a, b, t, \lambda)$ be solutions to (1) with starting values for $t = 0$, $a, 0, b$ respectively. The initial remarks in [1] concerning the transformation from the $\varphi = 0$ to the $\varphi = \omega_2$ plane are readily seen to be true for (1). We recall that the existence of repeating solutions of (1) is equivalent to the existence of a curve $\alpha(a)$ of period ω_1 in a and a function $T(a, b, \lambda)$ such that

$$\begin{aligned} (2) \quad \varphi(a, b, T(a, b, \lambda), \lambda) &= \omega_2 \\ z(a, \alpha(a), T(a, \alpha(a), \lambda)) &= \alpha(\theta(a, \alpha(a), T(a, \alpha(a), \lambda), \lambda), T(a, b, 0) = \omega_2. \end{aligned}$$

As in [1] this device is used to obtain a family of invariant tori of (1). The idea here is as in [1] to make the proof depend on the Schauder fixed point theorem for mappings in a function space [2]. We will first define several sets occurring in the theorems. Let $\mathcal{M}$ be the set of all continuous periodic functions of period ω_1 with the topology of uniform

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convergence Let $\mathbb{M}_{r\eta} = E\alpha)[\alpha \in \mathbb{M}$ and α is differentiable and $\|\alpha\| \leq r$, $\|D\alpha\| \leq \eta]$, and $\mathbb{M}_{r\eta}^1 = E\alpha)[\alpha \in \mathbb{M}$ and $\|\alpha\| \leq r$ and α is Lipschitzian with constant $\eta]$. We do not specify the dimension of the range of the curves $\alpha \in \mathbb{M}$. There will be two theorems in the sequel for z a scalar, and z a 2-vector. However, we will try to avoid duplication by not specifying this until it is necessary to do so. We note several properties of $\mathbb{M}$, $\mathbb{M}_{r\eta}$ and $\mathbb{M}_{r\eta}^1$:

- (i) $\mathbb{M}$ is a Banach space.
- (ii) $\mathbb{M}_{r\eta}^1$ is a compact, convex subset of $\mathbb{M}$.
- (iii) $\mathbb{M}_{r\eta} \subset \mathbb{M}_{r\eta}^1$ and $\mathbb{M}_{r\eta}$ is convex; thus $\mathbb{M}_{r\eta} \subset \mathbb{M}_{r\eta}^1$ and $\mathbb{M}_{r\eta}$ is convex and compact.
- (iv) If $F: \mathbb{M} \rightarrow \mathbb{M}$ is continuous on $\mathbb{M}_{r\eta}^1$ and $F(\mathbb{M}_{r\eta}) \subset \mathbb{M}_{r\eta}$ then F has a fixed point in $\mathbb{M}_{r\eta}$.

(i) is clear; (ii) follows from Ascoli's theorem; (iii) follows from (ii); and since the continuity of F implies $F(\mathbb{M}_{r\eta}) \subset F(\overline{\mathbb{M}_{r\eta}^1}) \subset \mathbb{M}_{r\eta}$ we see that (iv) follows from (iii) and the Schauder fixed point theorem [2].

2. Statement and Proof of the Theorem

We will state and prove theorems for the cases in which z is a one and two dimensional vector. Let

$$C(a) = \int_0^{\omega_2} A(t + a, t) dt$$

and let $L_\lambda(a) = I + \lambda C(a)$. We let the norm of any matrix function $M(a)$ that occurs in the sequel be $\sup_{a \in [0, \omega_1]} \|M(a)\|$ where $\|M(a)\|$ is the usual

Hilbert space norm. We state and prove a preliminary result in which an assumption is made regarding $\|L_\lambda^{-1}\|$. Sufficient conditions to guarantee this hypothesis will be subsequently investigated.

THEOREM 1. If $P, Q = O(\|z\| + |\lambda|)$ and $R = O(\|z\|^2 + |\lambda|)$

$$\|L_\lambda^{-1}\| \leq \frac{1}{1+\lambda\Delta}, \quad \Delta > 0,$$

then there exists $\lambda_0 > 0$ such that for $0 \leq \lambda \leq \lambda_0$ equations (1) have a repeating solution which reduces to the solution $\theta = t + a$, $\varphi = t$, $z = 0$ for $\lambda = 0$.

PROOF. It is clear that for λ sufficiently small $\varphi_T(a, b, T, \lambda) \neq 0$. Thus we may obtain a function $T(a, b, \lambda)$ satisfying (2₁). Now, by the formula for variation of parameters

$$(3) \quad z(a, b, T(a, b, \lambda), \lambda) = b + \lambda \int_0^{T(a, b, \lambda)} (Az + R) dt.$$

Let

$$I(a, b, \lambda) = \int_0^{T(a, b, \lambda)} \left[A(\theta(a, b, t, \lambda), \varphi(a, b, t, \lambda)) z(a, b, t, \lambda) + R(\theta(a, b, t, \lambda), \varphi(a, b, t, \lambda), z(a, b, t, \lambda)) \right] dt.$$

Then

$$\begin{aligned} I(a, b, \lambda) &= I(a, b, 0) + \lambda H(a, b, \lambda) \\ &= \int_0^{\omega_2} A(t + a, t) b dt + \int_0^{\omega_2} R(t + a, t, b, 0) dt \\ &\quad + \lambda H(a, b, \lambda). \end{aligned}$$

Then equation (2₁) becomes

$$\begin{aligned} L_\lambda(a) \alpha(a) &= \alpha(\theta(a, \alpha(a), T(a, \alpha(a)), \lambda), \lambda) \\ &\quad - \lambda \int_0^{\omega_2} R(t + a, t, \alpha(a), \lambda) dt - \lambda^2 H(a, \alpha(a), \lambda). \end{aligned}$$

Since $L_\lambda^{-1}(a)$ obviously exists for λ sufficiently small we obtain for (2₁)

$$(4) \quad \alpha(a) = L_\lambda^{-1}(a) \left(\alpha(\theta) - \lambda \int_0^{\omega_2} R dt - \lambda^2 H \right).$$

We will systematically leave out the arguments of these functions since they have previously been indicated. We define a continuous function $F_\lambda : \mathbb{R}_\eta^1 \rightarrow \mathbb{R}$ by the formula

$$F_\lambda(\alpha)(a) = L_\lambda^{-1}(a) \left(\alpha(\theta) - \lambda \int_0^{\omega_2} R dt - \lambda^2 H \right).$$

We will show that for each $\lambda > 0$ there exists $r(\lambda) > 0$ and $\eta(\lambda) > 0$

such that

$$(5) \quad \begin{aligned} F_\lambda(\mathfrak{A}_{r\eta}) \subset \mathfrak{A}_{r\eta} \cdot \|F_\lambda(\alpha)\| \leq \|L_\lambda^{-1}\| \left\{ \|\alpha\| \right. \\ \left. + \lambda \left\| \int_0^{\omega_2} R(t + a, t, \alpha, 0) dt \right\| + \lambda^2 \|H\| \right\}. \end{aligned}$$

Since

$$R = O(\|z\|^2 + |\lambda|) \quad \text{and} \quad \|L_\lambda^{-1}\| \leq \frac{1}{1+\lambda\Delta},$$

$$\|F_\lambda(\alpha)\| \leq \frac{1}{1+\lambda\Delta} \left\{ r + \Gamma(\lambda r^2 + \lambda^2) \right\}$$

whenever $\|\alpha\| \leq r$ and where Γ is a positive constant. We seek r such that

$$(6) \quad \frac{1}{1+\lambda\Delta} \left\{ r + \Gamma(\lambda r^2 + \lambda^2) \right\} \leq r.$$

It is clear $r = \lambda^\mu$, $0 < \mu < 1$, satisfies (6) for each $\lambda > 0$.

We must now consider the magnitude of the derivative, $DF_\lambda(\alpha)$, of the transformed curve $F_\lambda(\alpha)$.

$$\begin{aligned} DF_\lambda(\alpha)(a) &= DL_\lambda^{-1}(a) \left[\alpha(\theta) - \lambda \int_0^{\omega_2} R dt - \lambda^2 H \right] \\ &+ L_\lambda^{-1}(a) \left\{ D\alpha(\theta)(\theta_a + \theta_b D\alpha(a) + \theta_T(T_a + T_b D\alpha(a)) \right. \\ &\quad - \lambda \int_0^{\omega_2} R_\theta(t + a, t, \alpha(a), 0) dt \\ &\quad - \lambda \int_0^{\omega_2} R_z(t + a, t, \alpha(a), 0) D\alpha(a) dt \\ &\quad \left. - \lambda^2 (H_a(a, \alpha(a), \lambda) + H_b(a, \alpha(a), \lambda) D\alpha(a)) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} \|DF_\lambda(\alpha)\| &\leq \|L_\lambda^{-1}\| \|\theta_b + \theta_T T_b\| \|\alpha\|^2 \\ &+ \left\{ \|L_\lambda^{-1}\| \left(\|\theta_a + \theta_T T_a\| + \lambda \left\| \int_0^{\omega_2} R_z \right\| + \lambda^2 \|H_b\| \right) \right\} \|D\alpha\| \\ &+ \left\{ \|L_\lambda^{-1}\| \left(\lambda \left\| \int_0^{\omega_2} R_\theta \right\| + \lambda^2 \|H_a\| \right) \right. \\ &\left. + \|DL_\lambda^{-1}\| \left(\|\alpha\| + \lambda \left\| \int_0^{\omega_2} R \right\| + \lambda^2 \|H\| \right) \right\}. \end{aligned}$$

We note the following:

$$\begin{aligned} \left\| \int_0^{\omega_2} R_z(t + a, t, \alpha(a), 0) dt \right\| &= O(\|\alpha\|) \\ \left\| \int_0^{\omega_2} R_\theta(t + a, t, \alpha(a), 0) dt \right\| &= O(\|\alpha\|) \\ \left\| \int_0^{\omega_2} R(t + a, t, \alpha(a), 0) dt \right\| &= O(\|\alpha\|) \\ \|\theta_b + \theta_T T_b\| &= O(|\lambda|). \end{aligned}$$

Thus if $\|\alpha\| \leq r$ we obtain

$$\begin{aligned} \|\mathcal{D}\mathcal{F}_\lambda(\alpha)\| &\leq M \|L_\lambda^{-1}\| |\lambda| \|\mathcal{D}\alpha\|^2 \\ &+ \left\{ \|L_\lambda^{-1}\| \left(\|\theta_a + \theta_T T_a\| + M(\lambda r + \lambda^2) \right) \right\} \|\mathcal{D}\alpha\| \\ &+ \|\mathcal{D}L_\lambda^{-1}\| \left\{ r + M(\lambda r + \lambda^2) \right\} \end{aligned}$$

where M is an appropriate constant. Note that since $L_\lambda(a) L_\lambda^{-1}(a) = I$ that

$$\mathcal{D}L_\lambda L_\lambda^{-1} + L_\lambda \mathcal{D}L_\lambda^{-1} = 0 \quad \text{and} \quad \|\mathcal{D}L_\lambda^{-1}\| \leq \|L_\lambda^{-1}\|^2 \|\mathcal{D}L_\lambda\|.$$

Since

$$\|L_\lambda^{-1}\| \leq \frac{1}{1+\lambda\Delta} \quad \text{and} \quad \|\mathcal{D}L_\lambda\| = O(|\lambda|)$$

we conclude that $\|\mathcal{D}L_\lambda^{-1}\| = O(|\lambda|)$. Also by straightforward calculations we verify that

$$\|\theta_a\| = 1 + O(\lambda^2 + |\lambda|r).$$

$$\|T_a\| = O(\lambda^2 + |\lambda|r).$$

Thus

$$\begin{aligned} \|\mathcal{D}\mathcal{F}_\lambda(\alpha)\| &\leq \lambda M \|\mathcal{D}\alpha\|^2 + \left\{ \frac{1+M(\lambda^2+\lambda r)}{1+\lambda\Delta} \right. \\ (7) \quad &\left. + M(\lambda r + \lambda^2) \right\} \|\mathcal{D}\alpha\| + K(\lambda r + \lambda^2). \end{aligned}$$

If $\|\mathcal{D}\alpha\| \leq \eta$ then we want

$$(8) \quad \|\mathcal{D}\mathcal{F}_\lambda(\alpha)\| \leq \eta.$$

By (7) after transposing and dividing through by λ condition (8) becomes

$$g(\lambda, \eta) = M\eta^2 + \left\{ \frac{m(\lambda+r)-\Delta}{1+\lambda\Delta} + M(r+\lambda) \right\} \eta + K(\lambda r + \lambda^2) \leq 0.$$

Since $r = \lambda^k$ it is elementary to show that there exists a continuous function $\eta(\lambda)$ such that $g(\lambda, \eta(\lambda)) \leq 0$ for λ sufficiently small and non-negative.

We thus see that (5) is satisfied and hence there exists a curve $\alpha = \alpha^\lambda \in \mathbb{N}$ satisfying (2₁). Hence (1) has a repeating solution and since $r(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$ we conclude the result.

We now investigate the condition

$$(9) \quad \|L_\lambda^{-1}\| \leq \frac{1}{1+\lambda\Delta}.$$

If the z equation is simply 1-dimensional then

$$L_\lambda^{-1}(a) = \frac{1}{1+\lambda C(a)},$$

so if

$$\int_0^{\omega_2} A(t+a, t) dt > \Delta > 0$$

for all a and if $\lambda > 0$ then

$$\|L_\lambda^{-1}\| \leq \frac{1}{1+\lambda\Delta}.$$

To discuss the two dimensional situation we use the following formula for the Hilbert space norm of any 2-square matrix M .

$$(10) \quad \|M\| = \left(\frac{N(M) + (N^2(M) - 4D(M)^2)^{\frac{1}{2}}}{2} \right)^{\frac{1}{2}}$$

where $N(M) = \sum |m_{ij}|^2$ and $D(M)$ = determinant of M . The formula (10) may be calculated by simply maximizing the function of two variables $f(x) = \|Mx\|$ with the condition $\|x\| = 1$. From (10) and

$$M^{-1} = \frac{Adj M}{D(M)}$$

we see that

$$\|M^{-1}\| = \left(N(M) + (N(M)^2 - 4D(M)^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} / \sqrt{2D(M)}.$$

Setting $M = L_\lambda(a) = I + \lambda C(a)$ we see that

$$N(M) = 2 \left(1 + \lambda \operatorname{tr} C(a) + \frac{\lambda^2}{2} N(C(a)) \right)$$

$$D(M) = 1 + \lambda \operatorname{tr} C(a) + \lambda^2 D(C(a)),$$

and

$$(11) \quad \|L_\lambda^{-1}(a)\| = \left(\frac{1 + \lambda(\operatorname{tr} C(a) + \sqrt{E_2(\lambda)(a)}) + \lambda/2 N(C(a))}{(1 + \lambda \operatorname{tr} C(a))^2 + \lambda^2 E_2(\lambda)(a)} \right)^{\frac{1}{2}}$$

where

$$E_1(\lambda)(a) = \left(\frac{N(C(a))}{2} - D(C(a)) \right) \left\{ 2(1 + \lambda \operatorname{tr} C(a)) \right. \\ \left. + \lambda^2 \left(\frac{N(C(a))}{2} + D(C(a)) \right) \right\}$$

and $E_2(\lambda)(a) = 2D(C(a))(1 + \lambda \operatorname{tr} C(a)) + \lambda^2 D^2(C(a))$. Let $H(\lambda)$ be the quantity inside the large parentheses in (11). Condition (9) is equivalent then with

$$H(\lambda) \leq \frac{1}{(1 + \lambda \Delta)^2},$$

that is,

$$(12) \quad s(\lambda) = H(\lambda)(1 + \lambda \Delta)^2 - 1 \leq 0.$$

Now $s(0) = H(0) - 1 = 0$ and

$$s'(0) = 2\Delta + H'(0) = 2\Delta + \sqrt{E_1(0)} - \operatorname{tr} C.$$

Now there will exist $\Delta > 0$ such that (12) obtains if $\sqrt{E_1(0)} < \operatorname{tr} C$ where $E_1(0) = N(C(a)) - 2D(C(a))$. Thus if for all $a \in [0, \omega_1]$

$$\left(N(C(a)) - 2D(C(a)) \right)^{\frac{1}{2}} < \operatorname{tr} C(a)$$

we may conclude that (1) has a repeating solution according to the above theorem.

We note that in case A is a constant matrix with distinct positive roots that we may assume by a linear change of variables that A has the form

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

and $(N(C) - 2D(C))^{\frac{1}{2}} = |\mu_1 - \mu_2| < \mu_1 + \mu_2 = \text{tr} C$ and the result hold. Similar statements can be made about special cases.

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XV. BOUNDS FOR PERIODS OF PERIODIC SOLUTIONS*

Stephen P. Diliberto

1. The Problem

Let $x = u(t)$ be a periodic solution, of period ω , of the two dimensional system of real first order ordinary differential equations

$$(1) \quad \frac{dx}{dt} = X(x)$$

where $x = (x_1, x_2)$ and $X(x) \in C^1$ for all x . Let A be a region which is bounded both from infinity and the singular points of X . We shall show in this case that if the periodic solution $u(t)$ lies in A there is an upper bound for ω , the period of u , which depends solely on $\|X\|$, certain curvatures, and some measure of the two boundedness properties of A .

We shall use the following device for characterizing that A is bounded both from infinity and the singular points of X : There exists a region B of finite area B ; free of singular points, and containing A in its interior -- specifically we shall suppose that $p \in A, q \notin B$ implies $\text{dist}(p, q) \geq d > 0$.

Let $H(x)$ and $H^*(x)$ denote, respectively, the curvature of the trajectory and orthogonal trajectory passing through x . In terms of the constants let

$$m = \min_{x \in A} \|X\| > 0$$

$$K = \max_{x \in B} \left[\max (H(x), H^*(x)) \right]$$

$$\rho = \min \left[\frac{d}{2}, \frac{\pi}{8K\sqrt{2}} \right].$$

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We shall show that

$$(2) \quad \omega \leq \frac{4B}{\pi \rho^2} + 2 \frac{\rho \sqrt{2}}{m}.$$

In the last section we shall use this boundedness result to establish a convergence theorem.

2. A Formula for $H_\phi(x)$

At a point x , the vectors $X = (X_1, X_2)$ and $X^\perp = (-X_2, X_1)$ define a coordinate system. Let ϕ be the angle, measured counterclockwise, from X to a segment, L , through x . Let L be parametrized by the distance u along L with $u = 0$ corresponding to the point x . Let $x(u)$ denote the point on L determined by u and let $X[u] = X(x(u))$. We define $H_\phi(x)$ as the rate of change (at $u = 0$) of the direction of $X[u]$ with respect to u . The following formula will be established

$$(3) \quad H_\phi(x) = H(x) \cos \phi + H^*(x) \sin \phi.$$

Let $X^\phi = (X_1^\phi, X_2^\phi) = (X_1 \cos \phi - X_2 \sin \phi, X_1 \sin \phi + X_2 \cos \phi)$.

Then

$$x(u) = x + u \frac{1}{\|X\|} X^\phi.$$

In obtaining $H_\phi(x)$ we must first find $\theta(u)$ the angle between $X[u]$ and a fixed direction and then calculate

$$\left. \frac{d}{du} \theta(u) \right|_{u=0}.$$

Since this will be the same for any fixed direction, the most convenient choice for this direction is $X^\perp$. Clearly,

$$\begin{aligned} H_\phi(x) &= \left. \frac{d}{du} \theta(u) \right|_{u=0} = \left. \frac{d}{du} \cos^{-1} \left[\frac{(X^\perp, X[u])}{\|X[u]\| \|X^\perp\|} \right] \right|_{u=0} \\ &= \left(\frac{d}{d[\]} \cos^{-1}[\] \right) \left(\frac{d[\]}{du} \right)_{u=0} \\ &= \frac{1}{\sin \frac{\pi}{2}} \left. \frac{d}{du} \left[\frac{(X^\perp, X[u])}{\|X[u]\| \|X^\perp\|} \right] \right|_{u=0} \end{aligned}$$

$$= \frac{(X^1, \frac{d}{du} X[u])_{u=0}}{\|X[0]\|}.$$

Consequently,

$$H_\varphi(x) = \frac{\cos \varphi}{\|X\|^3} \left\{ -\frac{\partial X_1}{\partial x_1} X_1 X_2 - \frac{\partial X_1}{\partial x_2} X_2^2 + \frac{\partial X_2}{\partial x_1} X_1^2 + \frac{\partial X_2}{\partial x_2} X_1 X_2 \right\} \\ + \frac{\sin \varphi}{\|X\|^3} \left\{ \frac{\partial X_1}{\partial x_1} X_2^2 - \frac{\partial X_1}{\partial x_2} X_1 X_2 - \frac{\partial X_2}{\partial x_1} X_1 X_2 + \frac{\partial X_2}{\partial x_2} X_1^2 \right\}.$$

The formula (3) follows from this by observing that $H_0(x) = H(x)$ is obvious; and that since $H_{\pi/2}(x)$ can be obtained from $H(x)$ by substituting $-X_2$ for X_1 and X_1 for X_2 , one has $H_{\pi/2} = H^*$.

3. Proof of Inequality (2)

Let $S_\rho(p)$ denote the circle with center p and radius ρ . We shall say a solution $u(t)$ "enters a region just once" if $u(t)$ passes through p and meets the boundary of $S_\rho(p)$ in exactly two points. The principle idea of our proof is contained in the following

LEMMA. A periodic solution $u(t)$ enters $S_\rho(p)$ just once.

PROOF. From the formula of the last section, it follows that if $q \in S_{2\rho}(p)$ the angle between $X(p)$ and $X(q)$ is less than (or equal)

$$\left(\frac{\pi}{4K\sqrt{2}} \right) (\max |H_\varphi|) \leq \frac{\pi}{4}.$$

As a result, the diameter L of $S_{2\rho}(p)$ through p and perpendicular to $X(p)$ is a segment without contact. We shall show (below) that if $q \in S_\rho(p)$ then the solution through q cuts L . This, plus the fact that a periodic solution meets a segment without contact just once, implies that if q lies on $u(t)$ it must lie on the arc of $u(t)$ through p : $u(t)$ enters $S_\rho(p)$ just once.

We shall show that $q \in S_\rho(p)$ implies the solution through q cuts L . At p , $X(p)$ and $X^1(p)$ determine a coordinate system. We shall suppose that q lies in the second or third quadrant, the other case reduces to this by reversing t . Suppose that q is in the second quadrant

(the other case is similar). The two lines $L +$ and $L -$ through q with slope $+1$ and -1 , respectively, cut L at $q +$ and $q -$ on L . At each point r in the triangle $qq+q-$, $X(r)$ has a component $\geq \frac{1}{\sqrt{2}}m$ in the $X(p)$ direction. This, plus the fact that the solution through q cannot leave the triangle on the sides $qq+$ or $qq-$ at any point r (or else the angle between $X(r)$ and $X(p)$ exceeds $\frac{\pi}{4}$), imply that it must leave through $q+q-$ (in a time less than $\rho \frac{m}{\sqrt{2}}$ and so cuts L . This proves the lemma.

We shall now derive the inequality (2). We shall construct a sequence of points $p_1 = u(t_1)$ according to this rule: Let $p_0 = u(0)$. Let $t_1 > 0$ be the smallest value of $t > 0$ such that $u(t)$ lies on the boundary of $S_\rho(p_0)$. When $p_0, \dots, p_k$ have been determined let t_{k+1} be the smallest value of $t > t_k$ such that $p_{k+1} = u(t_{k+1})$ lies on the boundary of $S_\rho(p_k)$ but not inside any $S_\rho(p_j)$ for $j < k$. The important property of the sequence is that $\text{dist}(p_i, p_j) \geq \rho$ for $i \neq j$. Consequently the circles $S_{\frac{1}{2}\rho}(p_i)$ are disjoint and lie entirely inside B . Thus the number r of such circles $S_{\frac{1}{2}\rho}(p_i)$ can not exceed $\frac{4B}{\pi\rho^2}$.

By elementary properties of curvature the length of arc from p_1 to p_{i+1} is at most $\rho\sqrt{2}$. The time required to travel from p_1 to p_{i+1} is then at most $\frac{\sqrt{2}}{m}\rho$. Including the two arcs from p_{r-1} to p_r and p_r to p_0 we have

$$(4) \quad \omega \leq \left(\frac{4B}{\pi\rho^2} + 2 \right) \frac{\rho\sqrt{2}}{m}$$

4. A Uniform Convergence Theorem

THEOREM. Let

$$(5) \quad \frac{dx}{dt} = X(x, \lambda)$$

be a system of real first order ordinary differential equations where X is C^1 in $x = (x_1, x_2)$ for all x , C^0 in λ for $0' \leq \lambda \leq 1$, and where the singular points are the same for all λ .

If for $\lambda > 0$, the equations have a family of periodic solutions $x = u(t, \lambda)$ lying in a region A which is bounded and strictly interior to a region B , not including singular points, in which the curvatures are uniformly bounded then the equation

$$(6) \quad \frac{dx}{dt} = X(x, 0)$$

has a periodic solution.

The proof is clear: Since the periods are bounded, choose a subsequence λ^1 so that as $1 \rightarrow \infty$, $\lambda^1 \rightarrow 0$, and ω^1 (the period of $x = u(t, \lambda^1)$) converges to ω . Because of the curvature bounds, the $x(t, \lambda^1)$ are equi-uniformly continuous. By Arzela's Theorem there is a subsequence $x = u(t, \lambda^{1j})$ which converges uniformly to $u(t)$. $u(t)$ is obviously periodic and a solution of equation (5).

5. Rotated Vector Fields

The uniform convergence theorem of the last section gives a very direct way of establishing the principal theorem on rotated vector fields. This concerns a result about the periodic solutions of the equations

$$(7) \quad \frac{dx}{dt} = X(x, \alpha),$$

where $X(x, 0) = X(x)$ is C^1 for all $x = (x_1, x_2)$ and the vector $X(x, \alpha)$ is obtained by rotating X through the angle α counterclockwise. Note that the singular points remain fixed as α varies.

By an annular ring of curves in the plane we shall mean a set of curves generated by a topological mapping of the annulus $a \leq r \leq b$ (or $a < r < b$ if the annulus is "open") into the plane and such that the circle $r = \text{constant}$, $0 \leq \phi < 2\pi$ are mapped into the curves. Including infinity in the set of singular points the theorem in question is:

FUNDAMENTAL THEOREM ON ROTATED VECTOR FIELDS. Each periodic solution of (6) is contained in an open annular ring of periodic solutions having singular points on both the inner and outer boundary. The angle α is a continuous function of r the radial parameter.

It is to be observed in what follows that $\alpha(r)$ is not monotone. The first step in proving the theorem is to characterize periodic solutions in some local manner. Let $x = u(t, \alpha_0)$ be a periodic solution of equation (6) for $\alpha = \alpha_0$; and suppose that the solution is traversed counterclockwise by t . (The ensuing discussion requires only minor changes if u is oriented clockwise.) Let L be a segment perpendicular to u at some point, say $u(0)$, and parametrized by a real number λ ; let $\lambda = 0$ correspond to $u(0)$ and $\lambda < 0$ be outside of u . If L is

short enough it intersects $u(t)$ in the one point. For each value λ let $f(\lambda, \alpha)$ be the value of λ determined by the next intersection with L of the solution of (6) through λ . $f(\lambda, \alpha)$ is defined only for values of λ near zero and for α near α_0 .

LEMMA 1. In a neighborhood of $\lambda = 0$, $\alpha = \alpha_0$, $f(\lambda, \alpha)$ is continuous and $\alpha_1 > \alpha_2 \implies f(\lambda, \alpha_1) > f(\lambda, \alpha_2)$.

The continuity of $f(\lambda, \alpha_0)$ is classical and its strict monotonicity is obvious.

Now the periodic solution $u(t)$ corresponds to $\lambda = 0$, where $f(0, \alpha_0) = 0$; and may be characterized by the behavior of $f(\lambda, \alpha_0)$ near $\lambda = 0$. In particular periodic solutions correspond to values of λ such that $f(\lambda, \alpha_0) = \lambda$.

Let $g(\lambda, \alpha) = f(\lambda, \alpha) - \lambda$. Then $g(\lambda, \alpha_1) > g(\lambda, \alpha_2)$ if $\alpha_1 > \alpha_2$, as for f ; and periodic solutions correspond to the zeros of $g(\lambda, \alpha)$. The monotone properties of g imply that for some δ and $\alpha_1 < \alpha_0 < \alpha_2$: $g(\lambda, \alpha_1) < 0$ and $g(\lambda_1, \alpha_2) > 0$ if $|\lambda| \leq \delta$. Clearly for each λ with $|\lambda| \leq \delta$ there exists $\alpha(\lambda)$, continuous in λ , with $\alpha_1 < \alpha < \alpha_2$ such that $g(\lambda, \alpha(\lambda)) = 0$. Thus each periodic solution $x = u(t, \alpha_0)$ is a member of a radial band (actually an interior member).

From the above it follows that each periodic solution $x = u(t, \alpha_0)$ belongs to some maximal open annulus, A , of periodic solutions. Let $a < r < b$ be the annulus A . We shall show that if the annulus A has no singular point on its boundary it is not maximal. Choose a sequence r_1 so that $r_1 \rightarrow b$ and $\alpha(r_1) \rightarrow \alpha_0 \bmod 2\pi$. If the annulus has no singular points on its outer boundary, the conditions of Theorem 1 are satisfied, hence $u(t, \alpha(r_1)) \implies u(t, \alpha_0)$, a periodic solution. But we have shown in the previous paragraph that each periodic solution is an interior curve for some annulus, hence A is not maximal. This contradiction establishes the theorem.

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XVI. ONE-DIMENSIONAL REPEATING CURVES IN THE NON-DEGENERATE CASE*

Paul Koosis

1. Introduction

We shall consider solutions of the system

$$\begin{aligned} \dot{\theta}_1 &= 1 + zF(\theta_1, \theta_2, z) + \epsilon P(\theta_1, \theta_2, z) \\ (1) \quad \dot{\theta}_2 &= 1 + zG(\theta_1, \theta_2, z) + \epsilon Q(\theta_1, \theta_2, z) \\ \dot{z} &= A(\theta_1, \theta_2)z + z^2 H(\theta_1, \theta_2, z) + \epsilon R(\theta_1, \theta_2, z). \end{aligned}$$

Here ϵ is a small parameter. The functions $A, F, G, H, P, Q,$ and R are all periodic in θ_1 and θ_2 with periods ω_1 and ω_2 respectively. Also, we assume that these functions have continuous first partial derivatives in $\theta_1, \theta_2,$ and z .

When $\epsilon = 0$, a family of solution curves of these equations is:

$$\begin{aligned} \theta_1 &= a + t \\ \theta_2 &= t \\ z &= 0. \end{aligned}$$

Throughout this paper, we shall consider (θ_1, θ_2, z) as Cartesian Coordinates, and a point (a, b, c) will denote the values $\theta_1 = a, \theta_2 = b, z = c$. If we do this, the aforementioned solution curves generate a surface Σ_0 , the $\theta_1 - \theta_2$ plane, which is invariant to all translations of the form $\theta_1 \rightarrow \theta_1 + n\omega_1, \theta_2 \rightarrow \theta_2 + m\omega_2, n$ and m integers.

We inquire as to under what conditions does there exist, for $\epsilon \neq 0$, a surface Σ_ϵ consisting of solution curves to the system (1) so

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that the Σ_ϵ tend to Σ_0 as $\epsilon \rightarrow 0$ and Σ_ϵ is also invariant to these translations. In our discussion, we shall use, besides the standard existence and continuity theorems for differential equations [1], the Schauder-Leray fixed point theorem for Banach spaces, [2].

LEMMA 1.1. Let Γ_1 be a solution curve to the system (1) passing through the point (a, b, c) and Γ_2 a solution curve passing through $(a + \omega_1, b, c)$. Then, on making the translation $\theta_1 \rightarrow \theta_1 + \omega_1$, $\theta_2 \rightarrow \theta_2$, $z \rightarrow z$, the curve Γ_1 is brought into congruence with Γ_2 .

This follows from the fact that A, F, G, H, P, Q , and R all have period ω_1 in θ_1 , and the standard uniqueness theorem for differential equations.

REMARK. Solution curves to (1) passing through $(a, 0, c)$ cut the plane $\theta_2 = \omega_2$ for $|\epsilon|$ and $|c|$ small enough. For if $c = \epsilon = 0$, the solution curves $\theta_1 = a + t$, $\theta_2 = t$, $z = 0$ cut the plane $\theta_2 = 2\omega_2$, so by the theorem concerning continuous change of parameters, if $|c|$ and $|\epsilon|$ are small enough, the solution curves passing through points $(a, 0, c)$ will come so close to the plane $\theta_2 = 2\omega_2$ that they will also cut the plane $\theta_2 = \omega_2$.

For $|c|$ and $|\epsilon|$ small enough, the (unique) solution curve r passing through $(a, 0, c)$ cuts the plane $\theta_2 = \omega_2$ in a point which we shall denote by $(\bar{a}, \omega_2, \bar{c})$. Let us write $\phi^\epsilon(a, c) = (\bar{a}, \bar{c})$. The transformation ϕ^ϵ is then defined for all points (a, c) with $|c| \leq h$ provided that $|\epsilon| \leq h$, if h is a small positive number. ϕ^ϵ is one-one by the uniqueness theorems, and continuous, by the theorem on continuous change of parameters. We therefore have:

LEMMA 1.2. If h is small enough and $|\epsilon| \leq h$, the transformation ϕ^ϵ is topological on the strip $((a, c) \mid |c| \leq h)$.

*

Let γ be a curve lying in the strip $|c| \leq h$ of Lemma 1.2. Suppose γ is given in the form $c = g(a)$ and suppose furthermore that $g(a + \omega_1) = g(a)$, i.e., γ has period ω_1 in a . Then the transformation ϕ^ϵ maps γ on some curve $\bar{\gamma}$ consisting of points $(\bar{a}, \bar{c})$. By Lemma 1.1 $(\bar{a}, \bar{c})$ will lie on the curve $\bar{\gamma}$ if and only if $(a + \omega_1, \bar{c})$ lies on $\bar{\gamma}$, i.e., $\bar{\gamma}$ has period ω_1 in $\bar{a}$ if γ has period ω_1 in a .

2.

We now make a more detailed study of the system (1). For our purposes, it is best to rewrite these equations as follows:

$$(2) \quad \begin{aligned} \frac{d\theta_1}{d\theta_2} &= \frac{1+ZF+\epsilon P}{1+ZG+\epsilon Q} \\ \frac{dz}{d\theta_2} &= \frac{AZ+Z^2H+\epsilon R}{1+ZG+\epsilon Q} \end{aligned}$$

For c small enough in absolute value there exists a unique solution passing through $(a, 0, c)$ which we shall write in the form

$$\begin{aligned} \theta_1 &= \theta_1^\epsilon(a, c, \theta_2) \\ z &= Z^\epsilon(a, c, \theta_2). \end{aligned}$$

Here the functions on the right have continuous first partial derivatives in a, c , and θ_2 and are analytic in ϵ near $\epsilon = 0$. Also, by the discussion of Section 1, $\theta_1^\epsilon(a, c, \theta_2)$ and $Z^\epsilon(a, c, \theta_2)$ are defined on the whole interval $0 \leq \theta_2 \leq \omega_2$ if $|c|$ and $|\epsilon|$ are small enough. In the notation of Section 1 we have:

$$\begin{aligned} \bar{a} &= \theta_1^\epsilon(a, c, \omega_2) \\ \bar{c} &= Z^\epsilon(a, c, \omega_2). \end{aligned}$$

We now make the

ASSUMPTION. The quantity

$$-M_a = \int_0^{\omega_2} A(a + \theta_2, \theta_2) d\theta_2$$

is < 0 for all a .

Since $A(\theta_1 + \omega_1, \theta_2) = A(\theta_1, \theta_2)$, the numbers M_a are uniformly bounded away from zero.

Using this assumption we obtain estimates for the four quantities

$$\frac{\partial \bar{a}}{\partial a}, \quad \frac{\partial \bar{a}}{\partial c}, \quad \frac{\partial \bar{c}}{\partial a} \quad \text{and} \quad \frac{\partial \bar{c}}{\partial c}.$$

Let $\epsilon = 0$ and $c = 0$. Then the solution to the system (2) is

$$\theta_1 = a + \theta_2$$

$$z = 0$$

whence $\bar{a} = a + \omega_2$ and therefore

$$\frac{\partial \bar{a}}{\partial a} = 1, \quad \frac{\partial \bar{a}}{\partial c} = 0$$

for c and $\epsilon = 0$. We determine the other two quantities for ϵ and $c = 0$. If u denotes either of the quantities a or c , and we use subscripts to denote partial differentiation in the usual way, we obtain from the second of the equations (2):

$$\begin{aligned} \frac{dz_u}{d\theta_2} &= \frac{A}{1+zG} + \frac{Z}{1+zG} \left(A_{\theta_1} \frac{\partial \theta_1}{\partial u} + A_{\theta_2} \frac{\partial \theta_2}{\partial u} + zH_z z_u \right. \\ &\quad \left. + 2z_u H + zH_{\theta_1} \frac{\partial \theta_1}{\partial u} + zH_{\theta_2} \frac{\partial \theta_2}{\partial u} \right. \\ &\quad \left. - \frac{A+zH}{1+zG} \left\{ z_u G + zG_z z_u + zG_{\theta_1} \frac{\partial \theta_1}{\partial u} + zG_{\theta_2} \frac{\partial \theta_2}{\partial u} \right\} \right) \end{aligned}$$

in case $\epsilon = 0$. If also $c = 0$, $z = 0$ for all θ_2 and we obtain

$$\frac{dz_u}{d\theta_2} = Az_u$$

so that

$$z_u(\omega_2) = z_u(0) \exp \int_0^{\omega_2} A(a + \theta_2, \theta_2) d\theta_2 = z_u(0) e^{-M_a}.$$

Now clearly

$$z_a(0) = \frac{\partial c}{\partial a} = 0$$

$$z_c(0) = \frac{\partial c}{\partial c} = 1$$

so for ϵ and $c = 0$,

$$\frac{\partial \bar{c}}{\partial a} = 0, \quad \frac{\partial \bar{c}}{\partial c} = e^{-Ma} < 1.$$

Since the four partial derivatives

$$\frac{\partial \bar{a}}{\partial a}, \quad \frac{\partial \bar{a}}{\partial c}, \quad \frac{\partial \bar{c}}{\partial a}, \quad \text{and} \quad \frac{\partial \bar{c}}{\partial c}$$

are continuous in a and c (in fact, uniformly continuous in a because of the periodicity of the functions on the right of (2)) and analytic in ϵ we obtain:

LEMMA 2.1. For $\delta > 0$ there exists $h > 0$ so that if $|\epsilon|$ and $|c|$ are $\leq h$ we have

$$\left| \frac{\partial \bar{a}}{\partial a} - 1 \right| < \delta, \quad \left| \frac{\partial \bar{a}}{\partial c} \right| < \delta$$

$$\left| \frac{\partial \bar{c}}{\partial a} \right| < \delta, \quad 0 \leq \frac{\partial \bar{c}}{\partial c} < e^{-M} + \delta$$

where $M = \inf_a M_a$ is > 0 .

LEMMA 2.2. If h is small enough, then for $|\epsilon| \leq h$, $|\bar{c}| \leq h$ for all a if $|c| \leq h$.

Because

$$\left| \frac{\partial \bar{c}}{\partial c} \right| < 1$$

by Lemma 2.1 if h is small enough.

LEMMA 2.3 Let $\gamma : c = g(a)$ be a curve lying in $|c| \leq h$ such that $g'(a) = \frac{dc}{da}$ exists and is bounded for all a . Then, if h is small enough and $|\epsilon| \leq h$, $\bar{\gamma} = \phi^\epsilon \gamma$ is a differentiable curve lying in $|\bar{c}| \leq h$. Moreover, if p is a point of γ for which $\left| \frac{dc}{da} \right| \leq 1$, then $\left| \frac{d\bar{c}}{d\bar{a}} \right| \leq 1$ at the corresponding point $\bar{p} = \phi^\epsilon p$ of $\bar{\gamma}$.

PROOF.

$$\frac{d\bar{c}}{d\bar{a}} = \frac{\frac{\partial \bar{c}}{\partial a} + \frac{\partial \bar{c}}{\partial c} \left(\frac{dc}{da} \right)}{\frac{\partial \bar{a}}{\partial a} + \frac{\partial \bar{a}}{\partial c} \left(\frac{dc}{da} \right)}$$

exists if the denominator on the right side is different from zero. But by Lemma 2.1,

$$\left| \frac{\partial \bar{a}}{\partial a} - 1 \right| < \delta, \quad \left| \frac{\partial \bar{a}}{\partial c} \right| < \delta,$$

so the denominator is different from zero if $\frac{dc}{da}$ is bounded, and δ , i.e., h , is small enough. That $\bar{\gamma}$ lies in $|\bar{c}| \leq h$ follows from Lemma 2.2. Using the inequalities of Lemma 2.1,

$$\left| \frac{d\bar{c}}{d\bar{a}} \right| \leq \frac{\delta + (e^{-M+\delta}) \left| \frac{dc}{da} \right|}{1-\delta \left(1 + \left| \frac{dc}{da} \right| \right)} \leq \frac{e^{-M+2\delta}}{1-2\delta}$$

if $\left| \frac{dc}{da} \right| \leq 1$. If h is so small that $4\delta \leq 1 - e^{-M}$, then $\left| \frac{d\bar{c}}{d\bar{a}} \right| \leq 1$.

3.

We now combine the results of Sections 1 and 2 in order to establish the result of this paper.

LEMMA 3.1. For h sufficiently small, if $|\epsilon| \leq h$ and γ is a curve in $|c| \leq h$ for which $|c_1 - c_2| \leq |a_1 - a_2|$, (a_1, c_1) on γ , then $\bar{\gamma} = \phi^\epsilon \gamma$ lies in $|\bar{c}| \leq h$ and $|\bar{c}_1 - \bar{c}_2| \leq |\bar{a}_1 - \bar{a}_2|$ for $(\bar{a}_1, \bar{c}_1)$ on $\bar{\gamma}$.

PROOF. Let h be chosen so small that the conclusion of Lemma 2.3 holds. Let (a_1, c_1) be a point of γ so that for any other point (a, c) of γ , $|c_1 - c| \leq |a_1 - a|$. Let us consider $(\bar{a}_1, \bar{c}_1)$ on $\bar{\gamma}$; we wish to show that if $(\bar{a}, \bar{c})$ is on $\bar{\gamma}$, $|\bar{c}_1 - \bar{c}| \leq |\bar{a}_1 - \bar{a}|$.

Let us denote by R the region

$$\{(a, c) \mid |c - c_1| \leq |a - a_1|, \quad |c| \leq h\}.$$

R is bounded by the two lines

$$M_+ : c - c_1 = a - a_1$$

$$M_- : c - c_1 = a_1 - a,$$

which meet in the point (a_1, c_1) . Consider the curves: $\bar{M}_+ = \phi^\epsilon M_+$ and $\bar{M}_- = \phi^\epsilon M_-$. They intersect in the point $(\bar{a}_1, \bar{c}_1)$. By Lemma 1.2, ϕ^ϵ maps R topologically on a region $\bar{R}$. Since M_+ and M_- form the boundary of R , $\bar{R}$ is one of the two regions in $|\bar{c}| \leq h$ bounded by $\bar{M}_+$ and $\bar{M}_-$. By Lemma 2.3 the curves $\bar{M}_+$ and $\bar{M}_-$, as well as any curve $\bar{l} = \phi^\epsilon l$, where l is any line in R , all have slopes of absolute value less than one everywhere. Therefore $\bar{R}$ is that region bounded by $\bar{M}_+$ and $\bar{M}_-$ which is contained in the set

$$\{(\bar{a}, \bar{c}) \mid |\bar{c} - \bar{c}_1| \leq |\bar{a} - \bar{a}_1|, \quad |\bar{c}| \leq h\}.$$

Since γ lies in R , $\bar{\gamma}$ lies in $\bar{R}$, and the result is proved.

REMARK. If γ is a curve having the same property as in the above Lemma, then for some function g , γ is the set of points (a, c) for which $c = g(a)$. For if (a_1, c_1) and (a_2, c_2) lie on γ , $a_1 = a_2$ implies $c_1 = c_2$.

Consider now the Banach space B of all continuous functions $f(a)$ on the real line for which $f(a + \omega_1) = f(a)$, using the uniform norm. Consider the subset D of B , consisting of functions g for which $|g(a)| \leq h$ for all a , and $|g(a_1) - g(a_2)| \leq |a_1 - a_2|$. D is a bounded convex subset of B , and is compact in B by Ascoli's theorem. If h is small enough and $|\epsilon| \leq h$, the function ϕ^ϵ induces naturally a continuous mapping T_ϵ of D into itself by Lemma 3.1. Namely, to $g(a)$ in D corresponds a curve $\gamma : c = g(a)$; and if $\bar{\gamma} = \phi^\epsilon \gamma$; $\bar{\gamma}$ is given by some function $\bar{c} = \bar{g}(\bar{a})$, and $\bar{g}(a)$ also lies in D , by this Lemma. We write $\bar{g} = T_\epsilon g \cdot T_\epsilon : D \rightarrow D$. The continuity of T_ϵ follows from the theorems on continuous change of parameters in a differential equation.

By the Schauder-Leray theorem, T_ϵ has a fixed point in D ; i.e., for some g , $T_\epsilon g = g$. Therefore, if γ is the curve $c = g(a)$, the curves γ and $\bar{\gamma} = \phi^\epsilon \gamma$ are identical. Now consider the surface Σ_ϵ consisting of all solution curves to equations (1) passing through the points $(a, 0, c)$ with (a, c) on γ . The surface Σ_ϵ intersects the plane $\theta_2 = \omega_2$ in a curve $\bar{\gamma}$, and by the above discussion, γ and $\bar{\gamma}$ are brought into congruence on making the translation

$$\begin{aligned}\theta_1 &\longrightarrow \theta_1 \\ \theta_2 &\longrightarrow \theta_2 + \omega_2 \\ z &\longrightarrow z.\end{aligned}$$

It follows that Σ_ϵ is invariant to translations of the form $\theta_2 \longrightarrow \theta_2 + \omega_2$, and hence also to translations of the form $\theta_2 \longrightarrow \theta_2 + m\omega_2$, m an integer, by the uniqueness theorem for differential systems, since A, F, G, H, P, Q , and R have period ω_2 in θ_2 . By Lemma 1.1, Σ_ϵ is also invariant to translations of the form $\theta_1 \longrightarrow \theta_1 + n\omega_1$, n an integer, since the curve γ has period ω_1 in a . We have therefore found, for $|\epsilon|$ small enough, a surface Σ_ϵ consisting of solution curves of (1) which is invariant to all translations

$$\begin{aligned}\theta_1 &\longrightarrow \theta_1 + n\omega_1 \\ \theta_2 &\longrightarrow \theta_2 + m\omega_2 \\ z &\longrightarrow z.\end{aligned}$$

For each h , h small enough, find two such invariant surfaces Σ_ϵ for

$\epsilon = \pm h$. Then, for $|\epsilon|$ small enough, a family of invariant surfaces Σ_ϵ so that Σ_ϵ tends to Σ_0 as $\epsilon \rightarrow 0$ has been found, where Σ_0 is the $\theta_1 - \theta_2$ plane.

We have proved:

THEOREM. Consider the system

$$\dot{\theta}_1 = 1 + zF(\theta_1, \theta_2, z) + \epsilon P(\theta_1, \theta_2, z)$$

$$\dot{\theta}_2 = 1 + zG(\theta_1, \theta_2, z) + \epsilon Q(\theta_1, \theta_2, z)$$

$$\dot{z} = A(\theta_1, \theta_2)z + z^2H(\theta_1, \theta_2, z) + \epsilon R(\theta_1, \theta_2, z)$$

where all the functions on the right have continuous first partial derivatives, and have periods ω_1 in θ_1 and ω_2 in θ_2 . If

$$\int_0^{\omega_2} A(a + \theta_2, \theta_2) d\theta_2 < 0$$

for all a , then there exists for any ϵ small enough in absolute value a surface Σ_ϵ consisting of solution curves to the system, such that Σ_ϵ is invariant to all translations of the form

$$\theta_1 \rightarrow \theta_1 + n\omega_1$$

$$\theta_2 \rightarrow \theta_2 + m\omega_2$$

$$z \rightarrow z,$$

n and m integral. The surfaces Σ_ϵ tend to the $\theta_1 - \theta_2$ plane as $\epsilon \rightarrow 0$.

REMARK. The theorem is also true if

$$\int_0^{\omega_2} A(a + \theta_2, \theta_2) d\theta_2 > 0$$

for all a , as is seen by using the plane $\theta_2 = -\omega_2$ instead of $\theta_2 = \omega_2$ in Section 2. So we may replace the condition

$$\int_0^{\omega_2} A(a + \theta_2, \theta_2) d\theta_2 < 0 \quad \text{by} \quad \int_0^{\omega_2} A(a + \theta_2, \theta_2) d\theta_2 \neq 0$$

for all a .

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