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**Cambridge**  
**Intermediate Mathematics**

**GEOMETRY**

**PART I**

*With Answers*

**Cambridge University Press**

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Cambridge  
Intermediate Mathematics

by  
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GEOMETRY

PART I

*With Answers*



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## PREFACE

The Cambridge Intermediate Mathematics series, consisting of text-books in Arithmetic, Algebra and Geometry, each in two parts, has been designed to meet the needs of pupils in the newly organised Modern Schools and Senior Classes recommended by the Hadow Report.

“The first work of teachers and administrators is,” to use the words of *The New Prospect in Education*, just published by the Board of Education, “to think out their goal, feeling their way towards an appropriate curriculum.” It is impossible to predict with any certainty the form which the curriculum will ultimately take; indeed, it is improbable that any such limitations as are imposed upon the Secondary School can ever be applied to the diversified types of senior schools which are about to spring up. But probably all are agreed that these schools must not become “an anaemic reflection of the present Secondary School.”

The underlying notion on which the treatment in the books of this series has been based is that the aim of the Modern Schools, whether selective or non-selective, is to fit the pupils to take their places in the industrial and commercial rather than in the professional walks of life. For the latter the academic course of the Secondary School is a more or less fitting preparation; for the former it is decidedly out of place. It is assumed that the mathematical work of the newly organised schools will have a practical bias; their scholars need to be able to apply principles rather than to be able to derive them. Hence in these books theoretical explanations have been reduced to a minimum, and the use of the results has been emphasised.

GEOMETRY, PART I, deals in a simple manner with the more elementary principles of the subject, emphasising throughout the portions which lead directly to problems involving practical constructions and numerical calculations. It has not been thought necessary to deviate widely from the generally accepted sequence of proofs, but it is not intended that the proofs themselves shall be memorised. This book, with PART II, will introduce the pupil

to the elementary principles on which the more advanced parts of Geometry and Trigonometry are based; if he is transferred at any stage of the course to a Secondary School, or if he wakes up his studies again in later life, he will not feel at a loss. PART I will provide a complete course of Geometry for the average non-selective senior school, and may even be found to be sufficient as a preparation for the simpler examinations.

A feature of the book is the introduction of exercises which can be solved either without any working on paper or with the aid of a rough diagram; for want of a better term these exercises have been described as *Mental*. The author believes that there is as much justification for mental geometry as for mental arithmetic, and for the same reason, namely to fix firmly and to revise rapidly the basic principles.

A further feature is the interpolation of a series of sectional revision exercises, each consisting of a *Mental* and a *Written* section; a general revision exercise concludes the book. These two features have also been adopted in the companion volumes of the series.

I am indebted to Mr E. F. Partridge, B.Sc., for his valuable assistance in obtaining solutions to the questions, to the Cambridge Local Examinations Syndicate for permission to use four pages from their *Cambridge Four-Figure Mathematical Tables*, and to the following authorities for their courtesy in allowing me to make use of examination papers: the Oxford Local Examinations Delegacy, the Cambridge Local Examinations Syndicate, the Royal Society of Arts, the East Midland Educational Union (E.M.E.U.), the Union of Educational Institutions, and the Union of Lancashire and Cheshire Institutes. Special attention is directed to the first papers set in the Royal Society of Arts Junior Schools Certificate Examination and in the E.M.E.U. Central Schools Examination, both instituted in 1927; copies of these papers will be found at the end of the book.

H. J. L.

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# GEOMETRY

## PART I

1. Suppose you wanted to fix a position on one of the picture-rails of a room. You could do this by starting at either end of the rail and making *one measurement* from the end to the position in question.

If, however, you wished to fix a point on the floor of the room, it would be necessary to make *two measurements*; you could do it by starting at the point and measuring "straight" to a wall, and then from that point on the wall to a corner of the floor.

Finally, suppose you wanted to fix a point anywhere within the room, that is neither on wall, floor nor ceiling. In this case *three measurements* would be needed: you could measure from the point to be fixed straight to the floor, then from that point on the floor straight to a wall, and lastly from that point to the corner of the floor.

All objects existing by themselves in nature have three dimensions. You can imagine a *surface* of a "solid," but it is difficult to imagine a surface existing by itself, for it would have no thickness at all. Even the thinnest piece of paper has some thickness; you can calculate it by placing some hundreds of sheets together and dividing the total thickness by the number of sheets.

Still less can a *line* exist as an object by itself: the "line" drawn on paper is really a solid having length, breadth and thickness. Our "line" drawn on paper is the nearest representation we can give to it.

Even less still can a *point* exist by itself: the so-called point fixed with a pencil point is really a collection of particles of blacklead each having three dimensions.

This should give you some idea of:

- (1) *No dimensions*—a point,
- (2) *one dimension*—a line,
- (3) *two dimensions*—a surface,
- (4) *three dimensions*—an object or a space having volume.

2. It will be assumed that you are already acquainted with such "solids" as can be constructed out of cardboard or plasticine, e.g. the cube, rectangular solid, pyramid, cylinder, cone, and sphere. Each of these has surfaces, sometimes curved, sometimes flat. Some of the flat surfaces are rectangles, others triangles, pentagons, hexagons, circles and so on. All the surfaces except the sphere have edges, sometimes straight, sometimes curved. In most of the solids the edges meet in "corners" or points.

You will understand then that:

When two lines meet, they meet at a *point*.

Two surfaces meet in a *line*.

### Lines.

Lines may be straight or curved. They may be tested by placing a "straight-edge" against them: if the line is straight, there will be no space between it and the straight-edge placed against it.

### Surfaces.

Surfaces may be plane or curved, e.g. all the surfaces of a rectangular solid or pyramid are plane, whereas in a cylinder one surface is curved and two are plane.

Surfaces are bounded by lines, sometimes curved and sometimes straight, e.g. a circle has a boundary of one curved line, a triangle a boundary of three straight lines, a semi-circle a boundary of one curved and one straight line.

Note that what is generally called a circle is merely the circumference of a circle, and when we say, "Describe a semi-circle on a given straight line," we really mean, "Describe a semi-circumference."

### EXERCISE 1.

1. Draw a line  $OX$  of any length. Along it measure off  $OA\ 2\cdot3''$ ,  $AB\ 1\cdot1''$ ,  $BC\ 1\cdot3''$ . Measure the lengths of  $OC$  and  $AC$ .

Do the same as above with

2.  $OA\ 3\cdot1''$ ,  $AB\ 1\cdot3''$ ,  $BC\ \cdot9''$ . 3.  $OA\ 1\cdot8''$ ,  $AB\ 1\cdot5''$ ,  $BC\ 1\cdot7''$ .

4.  $OA\ 1\frac{1}{8}''$ ,  $AB\ 1\frac{3}{4}''$ ,  $BC\ 1\frac{5}{8}''$ . 5.  $OA\ 3\frac{1}{8}''$ ,  $AB\ 1\frac{1}{4}''$ ,  $BC\ 1\frac{3}{16}''$ .

6.  $OA\ 2\frac{1}{2}''$ ,  $AB\ 1\frac{1}{12}''$ ,  $BC\ 1\frac{1}{12}''$ .

By drawing a line and measuring off suitable distances upon prove that:

7.  $5 \cdot 1 - 1 \cdot 4 = 3 \cdot 7$ .

8.  $2\frac{5}{8} - 1\frac{1}{4} = 1\frac{7}{12}$ .

9.  $1\frac{1}{2} + 1\frac{1}{3} + 1\frac{1}{4} = 4\frac{1}{12}$ .

10.  $2\frac{1}{3} + 1\frac{5}{8} + 1\frac{3}{8} = 5\frac{1}{3}$ .

Draw the straight lines of which the dimensions are given in inches, and find their lengths in centimetres to the nearest first place of decimals:

11.  $2 \cdot 4''$ .

12.  $1 \cdot 9''$ .

13.  $2 \cdot 1''$ .

14.  $3\frac{5}{16}''$ .

15.  $2\frac{7}{8}''$ .

16.  $1\frac{5}{12}''$ .

Draw the straight lines of which the dimensions are given in centimetres etc., and find their lengths in inches to the nearest first place of decimals:

17.  $9 \cdot 5$  cm.

18.  $8$  cm.  $9$  mm.

19.  $12$  cm.  $1$  mm.

20.  $156$  mm.

21.  $108$  mm.

22.  $139$  mm.

Using the relation  $1$  cm. =  $\cdot 3937''$  calculate:

23. The number of feet and inches in  $256$  cm., to the nearest inch.

24. The number of cm. and mm. in  $5 \cdot 97''$ , to the nearest mm.

25. The number of cm. and mm. in  $1\frac{3}{16}''$ , to the nearest mm.

Make rough diagrams showing:

26. A rectangular solid.

27. A triangular pyramid.

28. A crystal composed of two triangular pyramids base to base.

29. A crystal composed similarly of two square pyramids.

30. A pyramid on a base of 5 equal sides.

31. A cone.

32. A cylinder.

33. A hemisphere.

34. A cylinder surmounted by a cone.

Describe the surfaces, and give the number of each:

35. A cube.

36. A rectangular solid on a square base.

37. A regular square pyramid.

38. A cone.

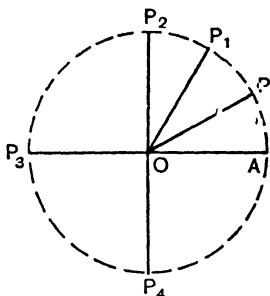
39. A cylinder.

40. A sphere.

**Angular Measurement.**

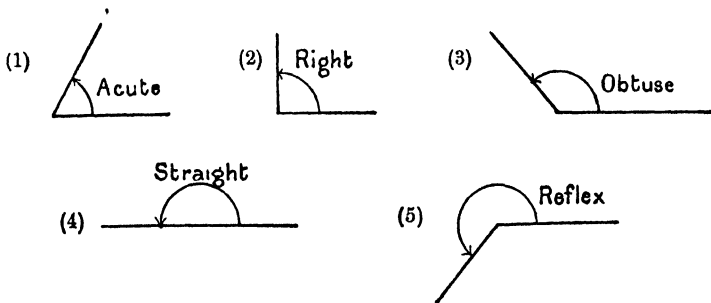
3. The idea of a *right angle* is familiar to everyone, though not everyone can satisfactorily define it. Some will tell you that it is an angle of  $90^\circ$ ; others that it is the angle formed when one line is perpendicular to another. Neither is a definition: each is merely another description.

Suppose a line  $OA$  revolves in a direction opposite to that taken by the hands of a clock, i.e. in a counter-clockwise or anti-clockwise direction. It takes in succession the positions  $OP$ ,  $OP_1$ ,  $OP_2$ ,  $OP_3$ ,  $OP_4$ , etc., and returns to the original position  $OA$ .



It has performed one complete revolution, and has clearly traced out four right angles. This gives us material for a definition: *a right angle is the angle traced out by a line revolving round its extremity after a quarter of a complete revolution.*

4. Anything less than a right angle is called an *acute angle*, whilst an angle less than two right angles but more than one is called an *obtuse angle*. An angle of two right angles is called a *straight angle*. Finally an angle greater than two right angles but less than four is called a *reflex angle*.



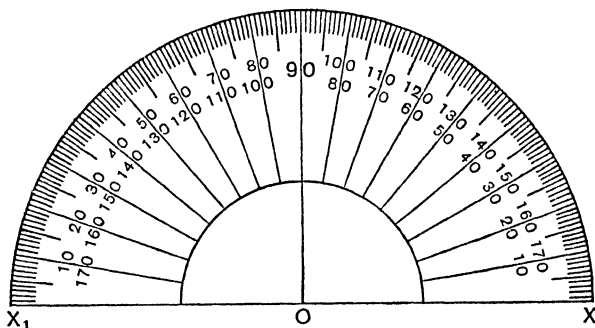
The complete revolution, then, gives four right angles, and this angle is divided up into 360 equal parts called *degrees*. Hence a right angle is one of 90 degrees—always written

$90^\circ$ . A straight angle is  $180^\circ$ . An obtuse angle is more than  $90^\circ$  but less than  $180^\circ$ . A reflex angle lies between  $180^\circ$  and  $360^\circ$ .

5. Two angles together equal to a right angle are said to be *complementary* to one another, and each is said to be the *complement* of the other: e.g.  $30^\circ$  and  $60^\circ$  are complementary; so are  $x^\circ$  and  $(90 - x)^\circ$ .

### The Protractor.

6. The *Protractor* is an instrument used in constructing and measuring angles. The following illustration shows one form of the instrument:

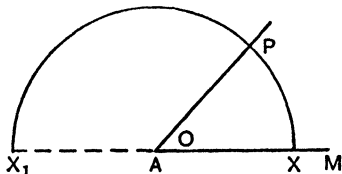


*The Protractor is used as follows in constructing an angle:*

Suppose  $\angle MAP$  is to be  $47^\circ$ .

Place the protractor so that  $OX$  lies along  $AM$ , and  $O$  lies on the point  $A$  at which the angle is to be made.

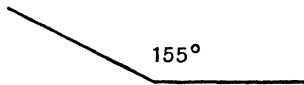
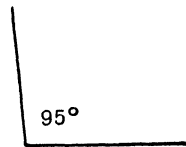
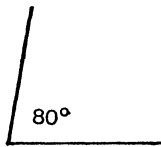
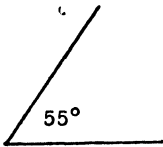
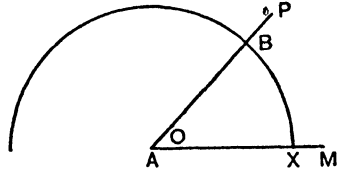
Count along the semi-circle from  $X$  until the  $40^\circ$  mark is reached, then count until  $47^\circ$  is found. Mark the position of the point  $P$ . Join  $AP$ . Then  $\angle MAP$  is  $47^\circ$ .



In making an angle just less or just more than  $90^\circ$ , be careful to use the proper sequence of angles, or you may discover that you have made  $83^\circ$  instead of  $97^\circ$ , or vice versa.

The use of the Protractor in measuring an angle is as follows:

Placing the protractor so that  $O$  lies on  $A$  and  $OX$  lies along  $AM$ ; note where  $AP$  cuts the semi-circle and read off the size of the angle. An eye estimate of the angle will always help you to decide whether you are using the appropriate sequence of angles, e.g.:



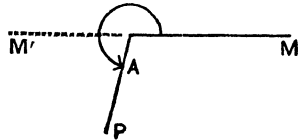
For *reflex* angles it is necessary to make an angle  $180^\circ$  less, with  $M'A$  for one arm, as shown in the diagram.

E.g. suppose  $257^\circ$  is to be made.

$$257^\circ - 180^\circ = 77^\circ.$$

$\therefore M'AP$  must be made  $77^\circ$ ,

and the reflex  $M'AP$  will then be  $257^\circ$ .



### EXERCISE 2.

1. Draw an angle  $AOB$  of  $37^\circ$ . Produce  $BO$  to any point  $C$ . Measure  $\angle AOC$  with protractor. Find the sum of  $\angle$ s  $AOB$ ,  $AOC$ .

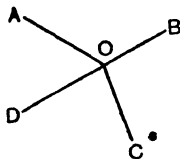
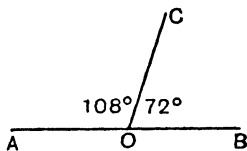
2. Make an angle  $AOB$  of  $115^\circ$ . Produce  $BO$  to any point  $C$ . Measure  $\angle AOC$ . Find the sum of  $\angle$ s  $AOB$ ,  $AOC$ .

3. Draw any line  $BOC$  and another line  $AO$  meeting it at  $O$ . Measure  $\angle$ s  $AOB$ ,  $AOC$  and find their sum.

4. Draw any line  $BOC$  with  $AO, DO$ , two other lines meeting it at  $O$ . Measure the  $\angle$ s  $BOD, DOA, AOC$  and find their sum.

5. Make  $\angle AOC 108^\circ$ , and  $\angle BOC 72^\circ$ .

What do you notice concerning the lines  $AO, OB$ ?



6. Make  $\angle AOB 121^\circ, \angle BOC 98^\circ, \angle COD 83^\circ$ . Measure  $\angle AOD$ . Add to the total of the other three.

7. Draw any four lines as in the figure of Question 5. Measure the four angles and find their sum.

8. Through what angle does the minute hand of a clock pass in half an hour? Give the answer in two forms, right angles and degrees.

9. Through what angle does the hour hand of a clock pass in half an hour? Give the answer in two forms.

10. A line  $AB$  runs due E. and W.,  $A$  being E. of  $B$ .  $AC$  is in a N.W. direction from  $A$ . What is  $\angle CAB$  in degrees and also as a fraction of a right angle?

11.  $OA$  runs from W. to E. and  $B$  is N.N.W. of  $O$ . Find  $\angle AOB$  in degrees.

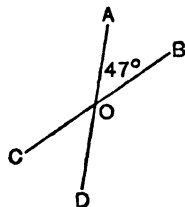
12. What angle is formed between the directions S.S.W. and W.S.W. from a point?

13. Draw an angle  $AOB 47^\circ$ . Produce  $AO$  and  $BO$  as shown in the figure. Measure  $\angle$ s  $AOC, COD, DOB$ . What do you notice?

14. Do the same with  $\angle AOB 128^\circ$ .

15. Do the same with  $\angle AOB 90^\circ$ .

16. Draw any two lines cutting one another and measure all four angles. What do you notice?



**7. Summary of Observations.**

1. If  $CO$  meets a straight line  $AOB$  at  $O$ ,

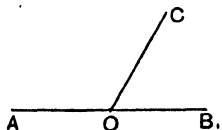
$$\angle COA + \angle COB = 2 \text{ right angles,}$$

and  $\angle$ s  $AOC, BOC$  are said to be *supplementary*, each being the *supplement* of the other.

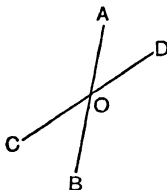
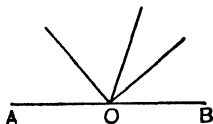
2. If  $CO$  meets two lines  $AO, OB$  so that

$$\angle COA + \angle COB = 2 \text{ right angles,}$$

$AOB$  is one straight line.



3. If any number of lines are drawn meeting a given line  $AB$  at one point the sum of the separate angles so formed is equal to two right angles.



4. When two straight lines  $AB, CD$  cut at  $O$ , the *vertically opposite*  $\angle$ s  $AOD, COB$ , and also  $\angle$ s  $AOC, BOD$  are equal.

It will be sufficient to indicate the proof for one angle  $AOD$  of  $37^\circ$ .

- (1)  $\angle COD = 180^\circ$ ,  
 $\angle AOD = 37^\circ$ ,  $\therefore \angle AOC = 143^\circ$ .
- (2)  $\angle AOB = 180^\circ$ ,  
 $\angle AOC = 143^\circ$ ,  $\therefore \angle BOC = 37^\circ$ .
- (3)  $\angle COD = 180^\circ$ ,  
 $\angle BOC = 37^\circ$ ,  $\therefore \angle BOD = 143^\circ$ .

**Learn:**

*If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.*

*If a straight line meets two straight lines at the same point, so as to make the sum of the two adjacent angles so formed equal to two right angles, the two straight lines are in the same straight line.*

*If two straight lines intersect, the vertically opposite angles are equal.*

**Simple Constructions.**

**1. To bisect a given line.**

1. Let  $AB$  be the line to be bisected.

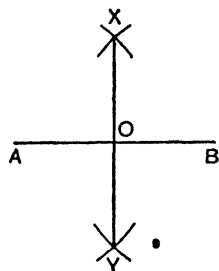
With centre  $A$  and any convenient radius describe arcs above and below the line.

With centre  $B$  and the same radius cut those arcs at  $X$  and  $Y$ .

Join  $XY$ .

This cuts  $AB$  at  $O$ , and  $AB$  is bisected at  $O$ .

No proof is required at this stage.



2. If the length of the line is given, a distance  $AO$  equal to half of this can be cut off from  $A$ .

3. If tracing paper is used, the paper can be folded so that  $A$  coincides with  $B$ , and the fold will pass through the point of bisection.

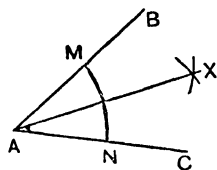
**2. To bisect a given angle.**

1. Let  $BAC$  be the angle to be bisected.

With centre  $A$  and any radius describe an arc cutting  $AB$ ,  $AC$  at  $M$  and  $N$ .

With centres  $M$  and  $N$  and any radius describe arcs cutting at  $X$ .

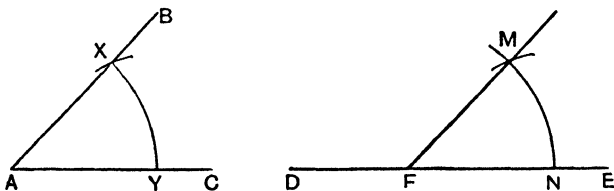
Then  $AX$  bisects  $\angle BAC$ .



2. If the number of degrees in  $\angle BAC$  is known, an angle  $BAX$  equal to half of this can be constructed with the protractor.

3. If tracing paper is used, the paper can be folded through  $A$  so that  $AB$  coincides with  $AC$ , and the fold will bisect the angle.

**3. To copy a given angle.**



Let  $BAC$  be the angle to be copied at  $F$  in the line  $DE$ .

With centre  $A$  and any radius describe an arc cutting  $AB$  and  $AC$  at  $X$  and  $Y$ .

With centre  $F$  and the same radius describe the arc  $MN$ .

Measure the distance  $YX$ .

Cut off  $NM$  equal to it and join  $FM$ .

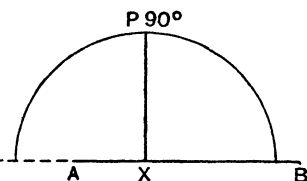
Then  $\angle MFN = \angle BAC$ .

## 11. Construction of Perpendiculars.

### 1. From a given point in a line.

(1) *With protractor.*

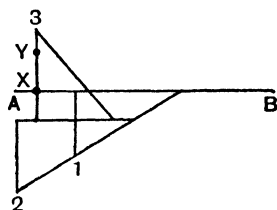
The protractor is placed as shown in the figure and  $XP$  is drawn to the point indicating  $90^\circ$ .



(2) *With set-squares;* this is the commonest method.

One of the set-squares is placed against the line as shown by the sketch of the set-square with vertex 1.

We then slide this set-square along a straight-edge, away from  $AB$ , until the vertex takes any other position 2.



The second set-square is then placed as indicated by the triangle with vertex 3, the edge passing through  $X$ .

Note that this method is also used when the given point  $Y$  is outside the line.

(3) *With ruler and compasses.*

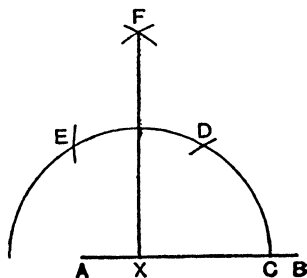
With centre  $X$  and any radius make an arc cutting  $AB$  at  $C$ .

With centre  $C$  and the same radius cut this arc at  $D$ .

With centre  $D$  and the same radius make arcs at  $E$  and  $F$ .

With centre  $E$  and the same radius cut the former arc at  $F$ .

Join  $FX$ .



The reason will be seen later. We are actually making an angle  $DXC$   $60^\circ$ , and  $DXE$   $60^\circ$ , and bisecting  $\angle DXE$ .

$$\begin{aligned} \therefore \angle FXC &= 60^\circ + \frac{1}{2} \text{ of } 60^\circ \\ &= 60^\circ + 30^\circ \\ &= 90^\circ. \end{aligned}$$

**2. From a point outside the line.**

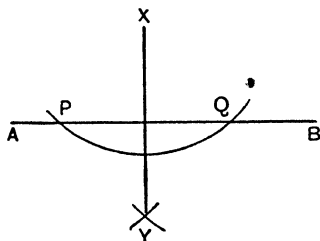
**12.** (1) *With set-squares.* See previous section.

(2) *With ruler and compasses.*

With centre  $X$  and any suitable radius cut the line  $AB$  at  $P$  and  $Q$ .

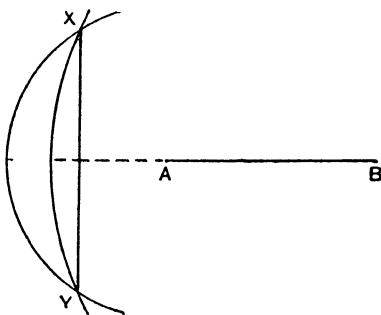
With centres  $P$  and  $Q$  describe arcs cutting at  $Y$ .

Join  $XY$ . This is the required perpendicular.



(3) *Second method with ruler and compasses.*

With centres  $A$  and  $B$  describe arcs through  $X$  cutting again at  $Y$ . Join  $XY$ .



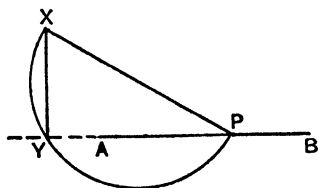
(4) *Third method with ruler and compasses.*

Join  $X$  to any point  $P$  in the line  $AB$ .

On  $XP$  describe a semi-circle cutting  $AB$  at  $Y$ .

Join  $XY$ .

Methods (3) and (4) can be used in certain cases where method (2) would be impossible.



### 13. Parallel Straight Lines.

Though you may never have attempted to give a definition of parallel lines, you will be able to give many examples of such *pairs of lines* from objects you see about you; e.g., opposite edges of a page, a picture frame, a ceiling, a wall, or a ruler. The essential feature of parallel lines is that they keep the same distance apart, and hence never meet.

The following are three methods of drawing a line parallel to another through a given point:

#### Method 1.

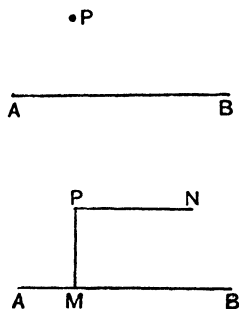
Let  $AB$  be the given line and  $P$  the given point outside. The parallel to  $AB$  is to be drawn through  $P$ .

It is well known that the opposite sides of a rectangle are parallel, and this is assumed in the method which follows:

Draw  $PM$  at right angles to  $AB$  by one of the methods shown in the previous paragraph.

Then draw  $PN$  at right angles to  $PM$ .

We now have three sides of a rectangle, and  $PN$  is parallel to  $AB$ .



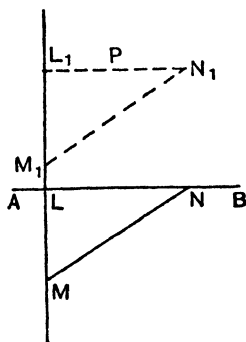
#### Method 2.

Place a straight-edge in such a position that one edge of a set-square lies against it at  $LM$ , and another edge lies along the line  $AB$  at  $LN$ .

Slide  $LM$  along the straight-edge until the line  $LN$  passes through  $P$  as at  $L_1N_1$ .

Then  $L_1N_1$  will be parallel to  $AB$ .

The proof will be easy to discover later.

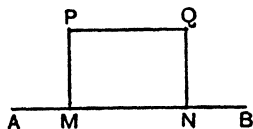


**Method 3.**

Draw  $PM$  perpendicular to  $AB$ .

Draw another perpendicular  $QN$  of the same length.

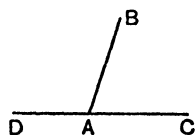
Then  $PQ$  will be parallel to  $AB$ .



**EXERCISE 3.**

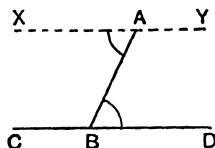
1. Draw a line 3.25" long and bisect it.
2. Divide a line 9.3 cm. long into four equal parts by the use of ruler and compasses.
3. Make an angle of  $35^\circ$  and bisect it.
4. Divide an angle of  $127^\circ$  into four equal parts by the use of ruler and compasses.
5. Construct this figure with  $\angle BAC 73^\circ$ .

Bisect both angles at  $A$  and measure the angle between the bisectors  $AX, AY$ .

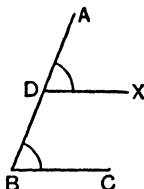


6. In Question 5, how could you prove that  $\angle XAY$  is a right angle?

7. Draw any parallelogram  $ABCD$  and bisect the angles  $A$  and  $B$ . Let the bisectors be  $AX$  and  $BX$ . Measure  $\angle AXB$ .



8. Make any angle  $ABD$ , and produce  $DB$  to  $C$ . At  $A$  make  $\angle XAB$  equal to  $\angle ABD$ , using ruler and compasses. Measure  $\angle$ s  $YAB, ABC$ .



9. Make any  $\angle ABC$ . In  $AB$  take a point  $D$ . At  $D$  make  $\angle ADX$  equal to  $\angle ABC$ . What do you notice about  $DX$  and  $BC$ ?

10. Draw any triangle  $ABC$ . Make another triangle with the same angles on a base  $XY$ , by the use of ruler and compasses.

11. Draw any line  $AB$  and fix a point  $P$  about 1" away from it. Through  $P$  draw a parallel to  $AB$  in three different ways:

(a) By drawing  $PM$  perpendicular to  $AB$  with a set-square and making a right angle at  $P$ .

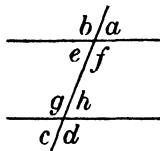
(b) By sliding a set-square.

(c) By drawing a second perpendicular to  $AB$  equal to  $PM$ .

12. Draw any line  $AB$  and draw another parallel to it and 1" away. How many such lines could you draw? [Note. First draw a perpendicular to  $AB$  1" long.]

13. Draw any two lines  $AB, CD$  cutting one another at  $P$ . Draw 2 lines parallel to  $AB$  and 1" away, and 2 lines parallel to  $CD$  and 1" away.

14. Draw two lines  $AB, AC$  making an angle of  $43^\circ$ . Using the method of Question 3, find *one* point  $1\frac{1}{2}$ " away from both lines. How many such points could be found, if  $BA$  and  $CA$  were produced?



15. Draw any two parallel lines and a line cutting them, as shown in the diagram.

Measure angles  $a$  and  $h$ .

What do you find?

16. Measure angles  $d$  and  $f$ . 17. Measure angles  $c$  and  $e$ .

18. Measure angles  $b$  and  $g$ .

19. Draw any line  $AB$ .

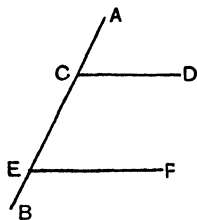
At any point  $C$  make the  $\angle ACD$   $38^\circ$ .

At any other point  $E$  make the  $\angle CEF$   $38^\circ$ .

Place one set-square with an edge along  $EF$ , and a second set-square along another edge of the first.

Slide the first up until it reaches  $CD$ .

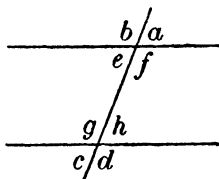
What do you find?



20. Repeat this with an  $\angle ACD$  of  $102^\circ$ , and  $CEF$   $102^\circ$ .

14. Observations.

When a straight line cuts any two straight lines as shown in the figure, the following pairs of angles are said to be *corresponding* angles:



$a, h$                        $b, g$   
 $c, e$                        $d, f$ .

- Note that (1)  $a$  is *exterior*,  $h$  *interior*.  
 (2) They are on the *same* side of the cutting line.  
 (3) They are *not adjacent* to one another.

In some of the examples just worked you should have discovered that, when the two lines are parallel,

$$a = h; \quad b = g; \quad c = e; \quad d = f.$$

That is, the corresponding angles are equal.

In others you should have discovered that if the lines were drawn such that either

$$a = h, \text{ or } b = g, \text{ or } c = e, \text{ or } d = f,$$

the two lines cut by the *cutting line*, or *transversal*, are parallel.

Learn:

1. *If a straight line cuts two parallel straight lines, the corresponding angles are equal.*
2. *If a straight line cuts two straight lines, so as to make a pair of corresponding angles equal, the two straight lines are parallel.*
3. *Corresponding angles are such that:*  
*One is exterior and the other interior.*  
*They are on the same side of the cutting line.*  
*They are not adjacent.*

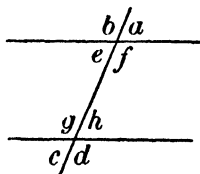
EXERCISE 4.

1. Draw two parallel lines and any line cutting them.

Measure and compare angles  $e$  and  $h$ .

What do you notice?

2. Measure  $f$  and  $g$ . Do you find a corresponding result?



3. Make an angle  $ABC$  of  $49^\circ$ .

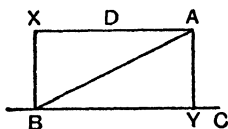
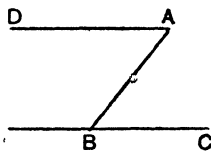
At  $A$  make an angle  $DAB$   $49^\circ$ .

4. Repeat this with an angle  $ABC$  of  $127^\circ$ .

Test  $DA$  and  $BC$  with set-squares and see if they are parallel.

5. Repeat this with an angle  $ABC$  of  $90^\circ$ .

6. Repeat with an angle  $ABC$  of  $28^\circ$ .



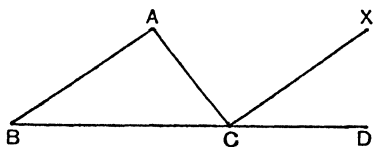
Draw perpendiculars  $BX$  and  $AY$  and measure them.

7. Draw  $ABC$  of any dimensions and produce  $BC$  to  $D$ .

Draw  $CX$  parallel to  $BA$ .

Measure  $\angle$ s  $BAC$ ,  $ACX$  and compare them.

Measure  $\angle$ s  $ABC$  and  $XCD$  and compare them.



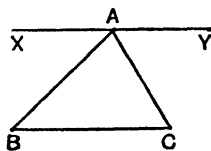
8. Draw any triangle  $ABC$ .

Draw  $XAY$  parallel to  $BC$ .

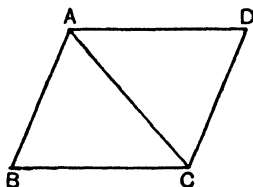
Measure and compare  $\angle$ s  $XAB$ ,  $ABC$ .

Also compare  $\angle$ s  $YAC$ ,  $ACB$ .

What do you notice?



9. Draw any parallelogram  $ABCD$ .



Measure and compare  $\angle$ s  $BAC$ ,  $ACD$ .

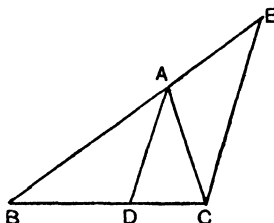
Also measure and compare  $\angle$ s  $DAC$ ,  $ACB$ .

10. Draw any triangle  $ABC$ .

Bisect  $\angle BAC$  by  $AD$ .

Draw  $CE \parallel$  to  $AD$ .

Measure  $\angle$ s  $ACE$ ,  $AEC$  and compare with  $\angle$ s  $BAD$ ,  $DAC$ .



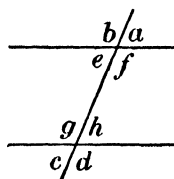
**15. Observations.**

When a straight line cuts any two straight lines as shown in the figure,

Angles  $e$  and  $h$  are said to be *alternate*;

Angles  $f$  and  $g$  are also *alternate*.

I.e. Alternate angles are two *non-adjacent interior* angles on *opposite sides* of the cutting line.



When two straight lines are parallel we might have expected to find that

$$\angle e = \angle h,$$

$$\angle f = \angle g.$$

For we have proved

$$\angle a = \angle h.$$

We know

$$\angle a = \angle e.$$

$$\therefore \angle e = \angle h,$$

and similarly

$$\angle b = \angle g,$$

$$\angle b = \angle f.$$

$$\therefore \angle f = \angle g.$$

*The converse is also true.*

If  $\angle e$  is known to be equal to  $\angle h$ , since we know  $\angle e = \angle a$ ,

$$\therefore \angle a = \angle h.$$

But these are corresponding angles.

Therefore the lines cut by the *transversal* (or cutting line) are parallel.

**Learn:**

1. *If a straight line cuts two parallel straight lines, the alternate angles are equal.*
2. *If a straight line cuts two straight lines so as to make the alternate angles equal, the two straight lines are parallel.*
3. *Alternate angles are such that:*  
*Both are interior.*  
*They are on opposite sides of the cutting line.*  
*They are not adjacent.*

**EXERCISE 5.**

1. Draw any two parallel lines, and a line cutting them. Measure angles  $f$  and  $h$  with the protractor. Find their sum.

2. Measure angles  $e, g$  and find their sum.

3. Repeat with the cutting line at right angles to the parallels.

4. Make an angle  $BAC$  of  $52^\circ$ .

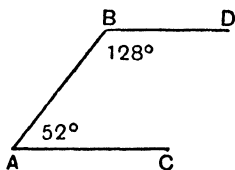
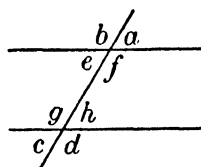
At  $B$  make an angle  $ABD$  of  $128^\circ$ .

Note that  $128^\circ + 52^\circ = 180^\circ$ .

See if  $AC, BD$  are parallel by using set squares.

5. Repeat with angles of  $140^\circ, 40^\circ$ .

6. Repeat with angles of  $90^\circ, 90^\circ$ .

**Learn:**

*If a straight line cuts two parallel straight lines, the two interior angles on the same side of the cutting line are together equal to two right angles.*

*If a straight line cuts two straight lines so as to make two interior angles on the same side of the cutting line equal to two right angles, the two straight lines are parallel.*

## ANGLES OF TRIANGLES AND RECTILINEAL FIGURES

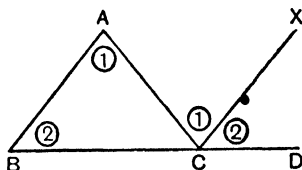
**16. Relation between exterior angle and interior angles of a triangle.**

$\angle ACD$  is exterior.

$\angle ACB, \angle ABC, \angle BAC$  are interior.

We can prove that

$$\angle ACD = \angle ABC + \angle BAC.$$



**Proof.**

Draw  $CX \parallel$  to  $AB$ .

The angles marked (1) are equal alternate angles; those marked (2) are equal corresponding angles.

$$\therefore \angle ACD = \angle ABC + \angle BAC.$$

**Learn:**

*The exterior angle of a triangle is equal to the sum of the two interior opposite angles.*

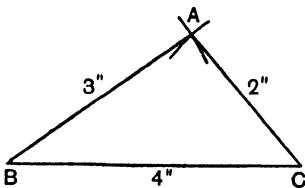
Hence it is clear that an exterior angle of a triangle is greater than either of the two interior opposite angles.

**17. Construction of Triangles given the lengths of three sides.**

Suppose the sides are 3", 2", 4".

Let  $ABC$  be a rough figure. On it mark  $BC$  4",  $CA$  2",  $AB$  3".

Now you are ready to start the construction. Make  $BC$  4". With the point of the compasses on  $B$  and with a radius 3", obtained by measuring from a ruler, make an arc. With the point of the compasses on  $C$  and with a radius 2", make another arc cutting the first at  $A$ . Join  $AB, AC$ . Then the triangle  $ABC$  has been made with  $AB$  3",  $BC$  4",  $CA$  2".



## EXERCISE 6.

1. Make a triangle  $ABC$  in paper or cardboard. Cut it into 3 portions as shown in Fig. 1.

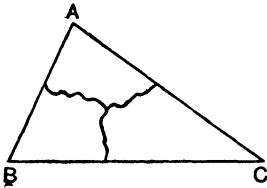


Fig. 1

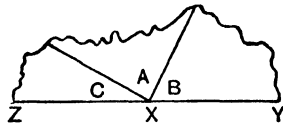


Fig. 2

Fit the angles together as shown in Fig. 2.

What do you notice about the new arms  $XY$ ,  $XZ$ ?

What does this show?

2. Make any triangle and measure its angles with the protractor. Find the sum.

3. Make a triangle with sides  $3''$ ,  $4''$ ,  $4\frac{1}{2}''$ . Measure the angles and find the sum.

4. Repeat with sides  $4''$ ,  $4''$ ,  $4\frac{1}{2}''$ . Find the sum. What do you notice besides the total value of the angles?

5. Repeat with sides  $3''$ ,  $3''$ ,  $3''$ . Find the sum. What else do you notice?

6. If two angles of a triangle are  $50^\circ$  and  $70^\circ$ , what is the third angle?

7. If two angles of a triangle are  $103^\circ$  and  $43^\circ$ , what is the third angle?

8. Try to make a triangle with sides  $3''$ ,  $2''$  and  $5''$ . What do you discover?

9. Try to make a triangle with sides  $3''$ ,  $2''$ ,  $6''$ .

10. If the angles of a triangle are represented by  $x^\circ$ ,  $2x^\circ$  and  $3x^\circ$ , find the size of each angle.

11. If one angle of a triangle is  $24^\circ$  and the other two are equal, how many degrees are there in each?

12. Find the angles of a triangle if each of the angles at the base is double the vertical angle.

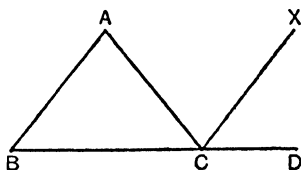
13. Find the angles of a triangle if each angle at the base is half the vertical angle.

14.  $ABC$  is a triangle with one side  $BC$  produced to  $D$ .

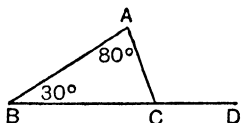
$CX$  is drawn parallel to  $AB$ .

By finding a pair of equal corresponding angles and a pair of equal alternate angles, prove

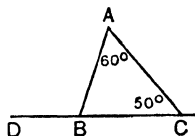
$$\angle ACD = \angle ABC + \angle BAC.$$



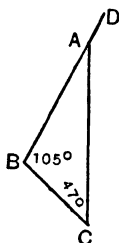
15. Calculate  $\angle ACD$ .



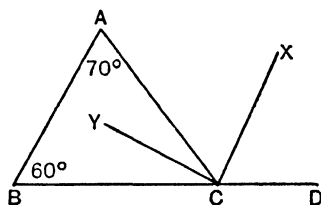
16. Calculate  $\angle ABD$ .



17. Calculate  $\angle DAC$ .



18. If  $CX, CY$  bisect the angles  $ACD, ACB$  calculate the angle  $XCY$ .



### 18. Angles of a Triangle.

In the preceding exercise you have used the fact that the sum of the angles of a triangle is equal to two right angles.

The following is a further proof:

Draw a line through the vertex parallel to the base.

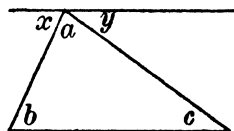
Then  $\angle x = \angle b$  alternate,

$\angle y = \angle c$  alternate.

$$\therefore x + a + y = b + a + c.$$

$$\text{But } x + a + y = 2 \text{ rt } \angle \text{s.}$$

$$\therefore b + a + c = 2 \text{ rt } \angle \text{s.}$$



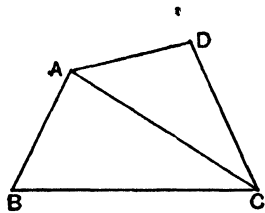
**Learn:**

*The sum of the angles of a triangle is equal to two right angles.*

**19. (i) Angles of a Quadrilateral, i.e. a four-sided figure.**

By dividing it into two triangles we see that all the angles of the quadrilateral are together equal to all the angles of the two triangles.

$\therefore$  the sum is 4 right angles.

**Learn:**

*The sum of the angles of a quadrilateral is equal to four right angles.*

**(ii) Angles of a Five-sided Figure, i.e. a pentagon.**

Here we can divide into 3 triangles.

$\therefore$  the sum is 6 right angles.

**(iii) The following table will help us to arrive at a general solution:**

Sides	Sum of angles	
3	2	or $2 \times 3 - 4$ rt $\angle$ s
4	4	or $2 \times 4 - 4$ „
5	6	or $2 \times 5 - 4$ „
6	8	or $2 \times 6 - 4$ „
$n$	$2n - 4$ rt $\angle$ s.	

**20. Angle of a Regular Polygon.**

Hence in a regular polygon each of the  $n$  angles is equal to  $\frac{2n-4}{n}$  right angles, or to  $\frac{90}{n}(2n-4)$  degrees.

**Ex. 1.** *Suppose there are 12 sides.*

$$\begin{aligned} 12 \text{ angles} &= 24 - 4 \text{ rt } \angle \text{s} \\ &= 20 \text{ rt } \angle \text{s} \\ &= 1800^\circ. \end{aligned}$$

$$\therefore \text{ each } \angle = 150^\circ.$$

Ex. 2. Suppose there are 240 sides.

Using formula:

$$\begin{aligned} \text{Each angle} &= \frac{90}{240} (480 - 4) \text{ degrees} \\ &= \frac{3}{240} \times 476 \\ &= 178\frac{1}{2}^\circ. \end{aligned}$$

Ex. 3. Find how many sides there are in a regular polygon if each angle =  $135^\circ$ .

$$\begin{aligned} 135 &= \frac{90}{n} (2n - 4). \\ 135n &= 180n - 360. \\ 45n &= 360. \\ n &= 8. \end{aligned}$$

### EXERCISE 7.

1. Draw any quadrilateral and find the sum of its angles by measurement.
2. Three angles of a quadrilateral are  $50^\circ$ ,  $70^\circ$  and  $108^\circ$ . Find the fourth angle.
3. Two angles of a quadrilateral are right angles and one of the others is  $70^\circ$ . Find the fourth angle.
4. The angles of a quadrilateral are  $x^\circ$ ,  $2x^\circ$ ,  $3x^\circ$ ,  $4x^\circ$ . Find the number of degrees in each.
5. The angles of a quadrilateral are  $x + 20^\circ$ ,  $x + 30^\circ$ ,  $x + 50^\circ$  and  $80^\circ$ . Find  $x$ .
6. Can you make a quadrilateral with angles of  $50^\circ$ ,  $70^\circ$ ,  $95^\circ$ ,  $145^\circ$ ?
7. Draw any five-sided figure and produce the sides in order. Measure the exterior angles and find their sum.
8. Repeat this with a quadrilateral.

Find the number of right angles in the sum of the angles of the following figures:

9. A 10-sided figure.

10. A 24-sided figure.

In the following regular figures, i.e. having all sides and all angles equal, find the number of degrees in each angle:

11. 5 sides.

12. 6 sides.

13. 7 sides.

14. 10 sides.

15. 100 sides.

16. 120 sides.

Find the number of sides in the regular polygons having angles of

17.  $90^\circ$ .

18.  $108^\circ$ .

19.  $150^\circ$ .

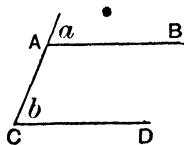
20.  $177\frac{3}{8}^\circ$ .

## SECTIONAL REVISION A

## EXERCISE 8 (a). MENTAL.

1. If a solid has 4 triangular faces, how many edges has it?
2. How many degrees are there in  $\frac{2}{3}$  of a right angle?
3. Two angles of a triangle are  $78^\circ$  and  $42^\circ$ . What is the third?

4. Under what conditions are angles  $a$  and  $b$  equal?

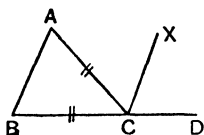


5. How many triangles can be constructed on one side of a given line  $2''$  long, having two other sides of  $1.9''$  and  $1.5''$ ?

6.  $ABC$  is an isosceles triangle having  $CA$  equal to  $CB$  and  $BC$  is produced to  $D$ .

$\angle ACD$  is bisected by  $CX$ .

Write down all equal angles.



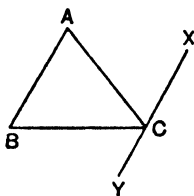
7. If from a line  $AB$  a distance  $1.75$  inches is cut off and the rest is found to be  $2.35$  inches long, what is the length of  $AB$ ?

8. What is the size of the angle of a regular pentagon?

## EXERCISE 8 (b).

1. Draw a diagram representing a rectangular solid of which one vertical face is only just seen.
2. How many vertices has a pyramid on a 5-sided base? Draw a rough diagram to illustrate your answer.
3. Draw a line  $5.7$  cm. long and bisect it with the use of ruler and compasses.
4. Construct a triangle with sides  $5.3$  cm.,  $5.9$  cm. and  $7.3$  cm. Measure the largest angle.

5. Draw any  $\triangle ABC$  as shown. Through  $C$  draw  $XCY$  parallel to  $AB$ .



Prove that the three angles of your triangle are equal to two right angles.

6. Draw an angle  $BAC$  of  $59^\circ$ . Produce the arms  $BA$  and  $CA$  to  $D$  and  $E$ . Bisect angles  $BAC$ ,  $DAE$ . What do you notice?

7. Draw any parallelogram and bisect two opposite angles. What do you notice about the bisectors?

8. Draw any triangle  $ABC$  and bisect all three sides,  $AB$  at  $X$ ,  $BC$  at  $Y$  and  $CA$  at  $Z$ . Join  $CX$ ,  $AZ$ ,  $BY$ . What do you notice?

### EXERCISE 9 (a). MENTAL.

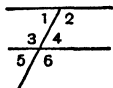
1. How many vertices has a cube?

2. What is the angle formed by the hands of a clock at 2 o'clock?

3. A quadrilateral has three of its angles right angles. Why must the fourth angle also be a right angle?

4. If two angles of a triangle are  $85^\circ 30'$  and  $45^\circ 30'$ , what is the third?

5. Name the pairs of alternate angles in this figure.



6. If two adjacent angles, formed when one line meets another line, are equal, why must the two lines be at right angles to one another?

7. What is the complement of  $28^\circ 50'$ ?

8. Why is it impossible to make a triangle with sides  $4''$ ,  $2''$ ,  $1\frac{1}{2}''$ ?

## EXERCISE 9 (b).

1. Construct a triangle with sides 6 cm., 8 cm., 10 cm. and measure the largest angle.

2. Draw  $AB$  3'' long. Find a point  $X$  in it so that  $AX = 1.2''$ . At  $X$  draw a perpendicular to  $AB$ , with the use of ruler and compasses.

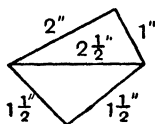
3. Make an angle of  $117^\circ$  with the protractor and bisect it.

4. Make a triangle with sides 2'', 3'', 4'', and draw perpendiculars from the vertices to the opposite sides. What do you notice?

5. One angle of a triangle is one-third of a right angle, and another is half a right angle. How many degrees are there in the third angle?

6. Draw a rough figure representing two regular tetrahedra on a common base. Name all its edges. How many vertices has this object?

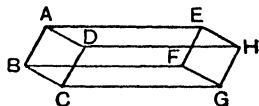
7. Construct the figure shown in the rough diagram.



8. A pavement is made up of regular tiles. Why will tiles with angles of  $60^\circ$ ,  $90^\circ$ ,  $120^\circ$  be suitable for this, and not a tile with an angle of  $108^\circ$ ?

## EXERCISE 10 (a). MENTAL.

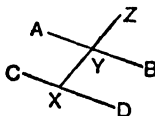
1. Name the parallelograms in this solid, assuming that each face is a parallelogram.



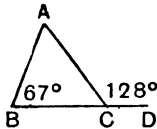
2. If in the parallelogram  $ADHE$  two opposite angles are  $120^\circ$  each and the other two are equal, how large is each?

3. What is the angle between the hands of a clock at 5 o'clock?

4. Name two pairs of alternate angles in this figure.



5. What is the supplement of  $(120 - x)^\circ$ ?
6. If from a line 9.05 cm. a distance 35 mm. is cut off, how many millimetres remain?
7. How many right angles are there in 1350 degrees?
8. What is the size of the angle  $BAC$  in this figure?



## EXERCISE 10 (b).

1. Construct a triangle with two sides 8.3 cm. and one side 4.1 cm. From the ends of the base draw perpendiculars to the two sides.

2. Draw a line  $AB$  3.1 in. long. Bisect it at right angles. On the right bisector take a point  $P$  1.7" from the line. Measure  $PA$  and  $PB$ .

3. If the angles of a quadrilateral are  $(90 - x)^\circ$ ,  $(180 - x)^\circ$ ,  $60^\circ$  and  $130^\circ$ , find the value of  $x$ .

4. How many sides are there in a regular polygon each angle of which is  $144^\circ$ ?

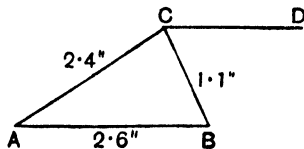
5. Construct a reflex angle of  $315^\circ$  with the help of the protractor.

6. Make a triangle with sides 3.1", 2.8", 2.5".

Construct a triangle with equal angles and with the longest side 1.7".

7. In a line 3.9" long fix a point 1.1" from one end. At this point draw a perpendicular to the given line with the help of ruler and compasses.

8. Construct this figure in which  $CD$  is parallel to  $AB$ .



## EXERCISE 11 (a). MENTAL.

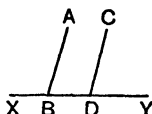
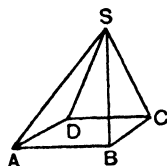
1. Write down the names of the edges in this solid.

2. If the above is a regular pyramid and  $\angle BSC = 42^\circ$ , what is the size of  $\angle SBC$ ?

3. If  $SB = 3''$  and  $SC = 3''$  and  $\angle BSC = 42^\circ$ , construct the triangle  $SBC$ .

4. What angle does the minute hand of a clock traverse in 10 minutes?

5. If  $AB$  and  $CD$  are parallel, name a pair of supplementary angles which are not adjacent.



6. What is the complement of an angle of  $\frac{1}{6}$  of a right angle in degrees?

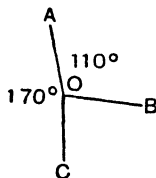
7. What is the size of each angle of a regular hexagon?

8. Why is it impossible to describe a triangle with angles of  $105^\circ$ ,  $75^\circ$  and  $25^\circ$ ?

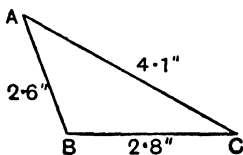
## EXERCISE 11 (b).

1. Construct an equilateral triangle with sides 2.9 in. long. Bisect any one of the angles and measure the segments of the side cut by the bisector.

2. Construct this figure with arms of any length and measure  $\angle BOC$ .



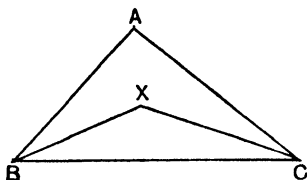
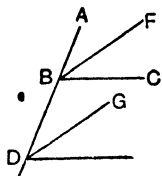
3. Construct this figure, and then with the help of ruler and compasses draw a line from  $A$  perpendicular to the direction of  $BC$ .



4. Make an angle of  $37^\circ$  with the aid of the protractor and then construct an angle of  $217^\circ$ , naming the angle so constructed, and indicating it with an arc.

5. Draw any parallelogram. Bisect two adjacent angles and measure the angle between the bisectors.

6. If  $\angle ABC = \angle BDE$  and these angles are bisected by  $BF$ ,  $DG$ , why is  $BF$  parallel to  $DG$ ?



7. If  $\angle ABC = 48^\circ$ , and  $\angle ACB = 38^\circ$ , and these are bisected by  $BX$  and  $CX$ , calculate  $\angle BXC$ .

8. Draw a rough diagram showing a pyramid resting on a face of a cube.

## CONGRUENCE OF TRIANGLES

### 21. 1st Case.

Suppose we have to construct accurately the triangle indicated in the rough figure.

Here we are given two sides and the included angle. We first make  $BC$   $3\frac{1}{2}$ " long.

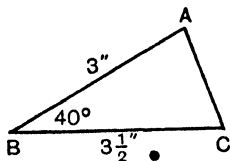
Next make  $\angle ABC$   $40^\circ$  with protractor.

Make the arm  $BA$   $3$ " long.

Join  $AC$ .

Then  $ABC$  is the triangle required.

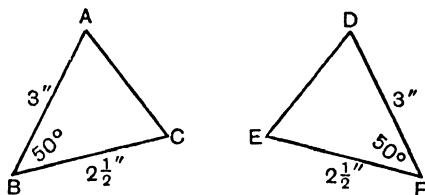
Suppose all the triangles made correctly to these dimensions by a class were cut out and put together. It would be found that any two of them would fit exactly together. They are all said to be congruent.



### Learn :

*If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.*

*Example.* These are congruent because  $AB$ ,  $BC$  and  $\angle ABC$  are equal to  $DF$ ,  $FE$  and  $\angle DFE$ .



Note that, if they are congruent, all parts must fit together.

$\therefore$  (1)  $AC = DE$ .

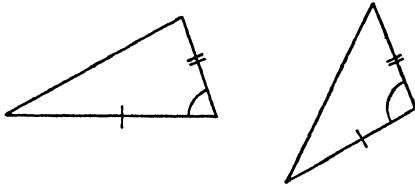
(2)  $\angle ACB = \angle DEF$  .....opposite to  $3$ " sides.

(3)  $\angle BAC = \angle EDF$  .....opposite to  $2\frac{1}{2}$ " sides.

(4) Area of  $\triangle ABC =$  area of  $\triangle DEF$ .

*It is usual to mark off similarly on the rough figure the parts that are known to be equal, e.g.*

The markings indicate 2 sides and the contained angle equal in each.



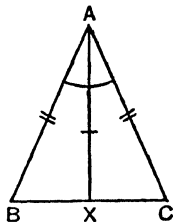
**22.** The following exercise gives an application of the principle.

Suppose  $ABC$  is a triangle with two sides equal, viz.  $AB = AC$ .

Let  $AX$  bisect the angle  $BAC$ , i.e. let  $\angle BAX = \angle CAX$ .

Now we have two triangles  $BAX$ ,  $CAX$  in which

$$\begin{cases} AB = AC. \text{ Given.} \\ AX \text{ is the same in both } \triangle s. \\ \angle BAX = \angle CAX. \end{cases}$$



These triangles have two sides and the contained angle equal in each. They are therefore congruent.

$$\therefore BX = XC \dots\dots\dots(1),$$

$$\angle ABX = \angle ACX \dots\dots\dots(2),$$

$$\angle AXB = \angle AXC \dots\dots\dots(3).$$

*These three results, obtained when the vertical angle of an isosceles triangle is bisected, are worth examining.*

- (1) *The bisector of the vertical angle of an isosceles triangle also bisects the base.*
- (2) *The angles opposite to the equal sides of an isosceles triangle are equal.*
- (3) *Since  $\angle AXB + \angle AXC = 2 \text{ rt } \angle s$ , and these are equal, it follows that each is a right angle. Therefore the bisector of the vertical angle of an isosceles triangle is at right angles to the base.*

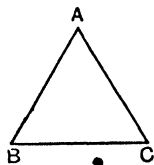
The second of these results must be known :

*The angles at the base of an isosceles triangle are equal.*

**23. Equilateral Triangles.**

Since the angles at the base of an Isosceles Triangle are equal, it follows that all the angles of an Equilateral Triangle are equal:

$$\begin{aligned}
 AB &= AC, \\
 \therefore \angle ABC &= \angle ACB. \\
 AC &= BC, \\
 \therefore \angle ABC &= \angle BAC. \\
 \therefore \angle A &= \angle B = \angle C.
 \end{aligned}$$



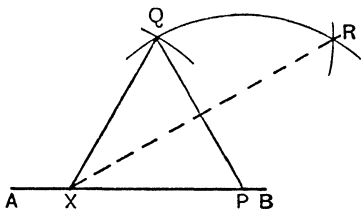
- Again, since  $\angle A + \angle B + \angle C = 180^\circ$ ,  
 $\therefore$  each angle =  $60^\circ$ .

**24. Geometrical Construction of angles of  $60^\circ$  and  $30^\circ$ ,  $120^\circ$ ,  $150^\circ$ .**

The above gives us a method of constructing an angle of  $60^\circ$  at any point  $X$  in the line  $AB$ .

With centre  $X$  and any radius describe arcs at  $P$ , in  $AB$ , and  $Q$ .

With centre  $P$  and the same radius describe an arc cutting the former arc at  $Q$ .



Then  $QXP$  is an equilateral triangle and  $\angle QXP = 60^\circ$ .

To construct  $30^\circ$ , take the same radius, and with  $P$  and  $Q$  as centres describe arcs cutting at  $R$ . Join  $XR$ .

Then  $\angle RXP = 30^\circ$ .

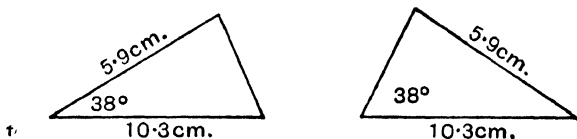
$\angle AXQ = 120^\circ$  and  $\angle AXR = 150^\circ$ .

**EXERCISE 12.**

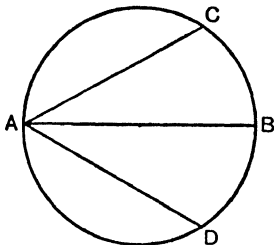
1. Make a triangle with one side  $2\frac{3}{4}$ " , a second side  $3\frac{1}{4}$ " and a contained angle  $49^\circ$ . Measure the third side and the other two angles.
2. Repeat with sides  $2\cdot7$ " ,  $3\cdot3$ " and contained angle  $102^\circ$ .
3. Repeat with sides  $9\cdot3$  cm.,  $8\cdot7$  cm. and contained angle  $45^\circ$ .

4. Make a triangle  $ABC$  in which  $a$  (the side opposite to the angle  $A$ ) =  $2\cdot5''$ ,  $b$  (opposite to the angle  $B$ ) =  $2\cdot9''$  and the angle  $C = 83^\circ$ . Make another  $\triangle ABC$  in which  $a = 2\cdot9''$ ,  $b = 2\cdot5''$ ,  $\angle C = 83^\circ$ . Are these congruent?

5. Are these triangles congruent? Give reasons. (The diagrams are purposely inaccurate.)



6.  $AC$  and  $AD$  are equal chords of a circle and  $AB$  bisects the  $\angle CAD$ .

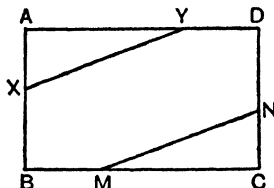


Show that the chords  $BC$ ,  $BD$  are equal.

7. Show that the bisector of any angle of an equilateral triangle is at right angles to the opposite side.

8. Two right-angled triangles, i.e. triangles with one angle a right angle, have sides containing the right angles  $8\cdot7$  cm. and  $9\cdot5$  cm. Show that they are congruent.

9.  $ABCD$  is a rectangle of which, of course, the opposite sides are equal.



$AY$  and  $CM$  are cut off equal to one another, and  $AX = CN$ . Prove that  $XY = MN$ .

10.  $AB = AC$  and the  $\angle BAC$  is bisected by  $AX$ . Any point  $D$  is taken on  $AX$ . (Fig. (1).)

Prove that  $BD = CD$ .

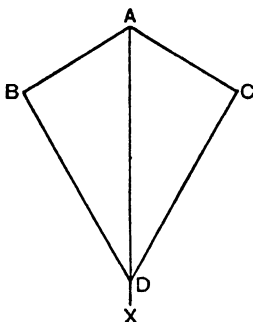


Fig. (1)

11.  $ABCDE$  is a regular pentagon (i.e. all its sides and angles are equal). (Fig. (2).)

Prove that  $BED$  is an isosceles triangle.

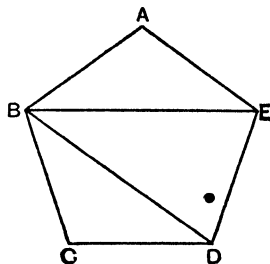


Fig. (2)

12.  $ABCD$  is a rectangle (and therefore its opposite sides are equal, and its angles are right angles). (Fig. (3).)

Prove that  $AC = BD$ .

*Note.* Imagine the triangles  $ABC$ ,  $DCB$  taken away from the figure and drawn separately.

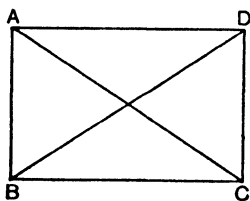


Fig. (3)

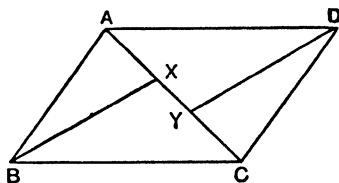


Fig. (4)

13.  $ABCD$  is a parallelogram (opposite sides equal) and equal distances  $AX$ ,  $CY$  are cut off on its diagonal. (Fig. (4).)

Prove that  $BX = DY$ .

14.  $ABCDE$  is a regular pentagon.  $X$  is the mid-point of  $AB$  and  $Y$  the mid-point of  $ED$ . (Fig. (5).)

Prove that  $CX = CY$ .

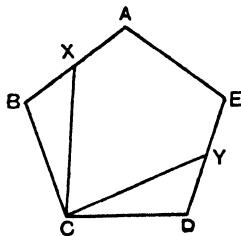
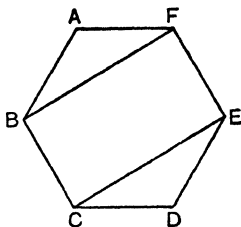


Fig. (5)

15.  $ABCDEF$  is a regular hexagon.

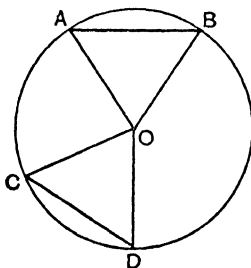


Prove that  $BF = CE$ .

16.  $ABC$  is a triangle having its sides  $AB, AC$  equal.  $D, E, F$  are the middle points of the sides  $BC, CA, AB$  respectively. Prove that  $DE = DF$ .

17. Prove that the triangle formed by joining the mid-points of the sides of an isosceles triangle is also isosceles.

18.  $AB$  and  $CD$  are two chords of a circle and  $O$  the centre, and  $\angle AOB = \angle COD$ .



Prove that the chords must be equal.

Note that  $AOB$  is said to be the angle *subtended* by  $AB$  at the centre.

### 25. 2nd Case of Congruence of Triangles.

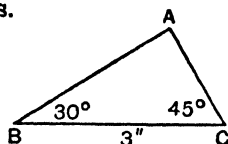
Suppose we had to construct accurately the triangle of which the rough figure is given.

Notice that if we know two angles, we also know the third, for

$$A + B + C = 180^\circ. \quad \therefore A = 105^\circ.$$

First make  $BC$  3". Then make an angle of  $30^\circ$  at  $B$  and  $45^\circ$  at  $C$ .

Here again if a whole class of scholars made accurately a triangle with one side and the angles adjacent to it of given



dimensions, all the triangles would fit together. It might be necessary to turn some of the triangles over, or to turn them round, but there could not be triangles of different dimensions satisfying the given conditions.

In Fig. (1) are shown two triangles with *two angles equal in each* (and therefore three) and with *one side equal in each*. Are they congruent? Yes, because it would be possible to cut out

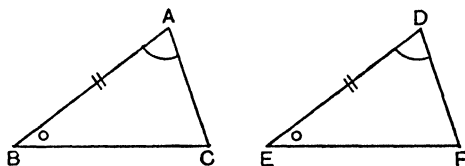


Fig. (1)

$DEF$  and fit it exactly on  $ABC$ , for  $DE$  and  $AB$  would fit together, and the arms of the angles would fit also.

It is not so easy to see that the triangles in Fig. (2) would fit together.

But if  $DFE$  were cut out and turned over, it would take the same shape as  $DEF$  in Fig. (1). Hence these triangles also are congruent.

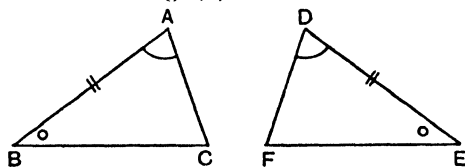


Fig. (2)

All pairs of triangles which have one side and the two angles adjacent to that side in one equal to the corresponding features in the other are congruent.

Are the triangles in Fig. (3) congruent? To find out we must first turn  $DEF$  over, when it takes the form shown in Fig. (4). (Both diagrams are purposely inaccurate.)

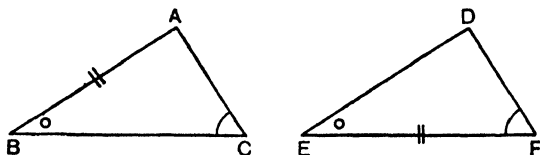


Fig. (3)

The third angles are also equal; that is  $\angle D = \angle A$ . If we now attempt to place  $DEF$  on  $ABC$ , we may make  $FE$  fit on  $AB$ , and the arm  $ED$  will be in the direction  $BC$ . But  $FD$  may not be in the direction  $AC$  because  $\angle F$  is not given equal to  $\angle A$ .

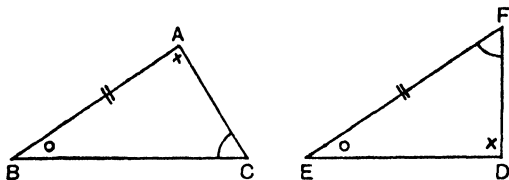


Fig. (4)

The point to notice is that the sides equal in each must be *opposite* to angles which are known to be equal in each. These are called *corresponding sides*.

### Learn :

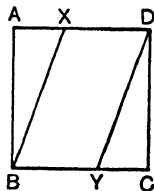
*If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.*

This result is used in the following exercises.

Ex. 1.  $ABCD$  is a square.

$BX, DY$  are drawn so that  $\angle ABX = \angle CDY$ .

Prove that  $BX = DY$ .



**Proof.** In the  $\Delta$ s  $ABX, CDY$ ,

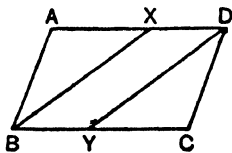
$$\begin{cases} \angle BAX = \angle DCY \dots\dots\dots \text{rt. } \angle \text{s,} \\ \angle ABX = \angle CDY \dots\dots\dots \text{given,} \\ AB = CD \dots\dots\dots \text{sides of a sq.} \end{cases}$$

I.e. Two angles and a corresponding side are equal in each.

$\therefore$  the  $\Delta$ s are congruent,

$\therefore BX = DY$ .

Ex. 2.  $ABCD$  is a parallelogram, and  $BX, DY$  bisect the angles at  $B$  and  $D$ . Assuming that the opposite sides and angles of a parallelogram are equal, prove that  $BX = DY$ .



• **Proof.** In the  $\Delta$ s  $ABX$ ,  $CDY$ ,

$$\begin{cases} \angle BAX = \angle DCY \dots\dots\dots \text{opp. } \angle \text{ s of a parm.,} \\ \angle ABX = \angle CDY \dots\dots\dots \text{halves of opp. } \angle \text{ s,} \\ AB = CD \dots\dots\dots \text{opp. sides of a parm.} \end{cases}$$

$\therefore$  the  $\Delta$ s are congruent and  $BX = DY$ .

**EXERCISE 13.**

1. Make the triangle of which the rough figure (Fig. (1)) is shown.

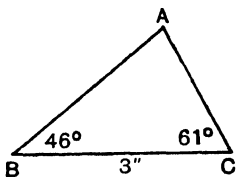


Fig. (1)

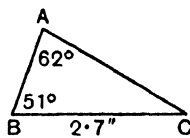


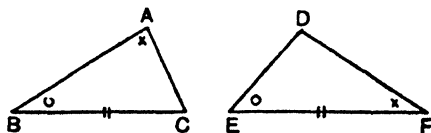
Fig. (2)

2. Calculate  $\angle ACB$  and then construct the triangle  $ABC$  (Fig. (2)).

Construct triangles, having given the following parts:

- |                            |                      |                   |
|----------------------------|----------------------|-------------------|
| 3. $a = 3.1''$ ,           | $A = 49^\circ$ ,     | $B = 52^\circ$ .  |
| 4. $a = 2.9''$ ,           | $A = 59^\circ$ ,     | $C = 40^\circ$ .  |
| 5. $a = 8.3$ cm.,          | $B = 82^\circ$ ,     | $C = 31^\circ$ .  |
| 6. $b = 6.9$ cm.,          | $A = 29^\circ$ ,     | $B = 48^\circ$ .  |
| 7. $b = 3.6''$ ,           | $B = 33^\circ$ ,     | $C = 89^\circ$ .  |
| 8. $c = 2.3''$ ,           | $C = 103^\circ$ ,    | $A = 41^\circ$ .  |
| 9. $c = 2.8''$ ,           | $A = 22^\circ$ ,     | $B = 127^\circ$ . |
| 10. $c = 2\frac{3}{4}''$ , | $A = C = 50^\circ$ . |                   |

11. Are the triangles indicated in the rough figure congruent? Give reasons. (The diagrams are purposely inaccurate.)



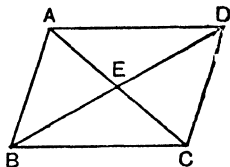
12. The  $\triangle ABC$  has  $\angle B = \angle C$ , and the vertical angle is bisected by  $AD$ .



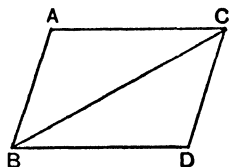
Prove that  $AB = AC$ .

13.  $ABCD$  is a parallelogram. Suppose we know that its opposite sides are equal as well as parallel.

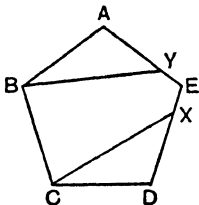
Prove that the diagonals bisect one another (i.e. prove the triangles  $AED$ ,  $CEB$  congruent).



14. Two  $\triangle$ s  $ABC$ ,  $DCB$  with  $BA = CD$ ,  $AC = BD$  and the  $\angle BAC = \angle BDC$ , are fitted together as shown. Prove that  $ABDC$  is a parallelogram (i.e. prove that the opposite sides are parallel).



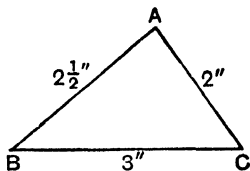
15.  $ABCDE$  is a regular pentagon.



$CX$  is drawn so that  $\angle XCD = 30^\circ$  and  $BY$  so that  $\angle ABY = 30^\circ$ .  
Prove that  $CX = BY$ .

### 26. 3rd Case of Congruence of Triangles.

You have learnt that, to construct a triangle, given the lengths of the three sides, you first construct one side, e.g.  $BC$ ; then with centre  $B$  and radius  $2\frac{1}{2}$ ", and with centre  $C$  and radius  $2$ ", describe arcs cutting at  $A$ .



You have discovered that no triangle can be constructed if one side is equal to or greater than the sum of the other two sides. It will probably be clear that a triangle of only one size and shape can be constructed with the dimensions given above.

Hence it seems as if two triangles with three sides equal in each must be congruent. Suppose we are merely told that the base is 3" and the other two sides  $2\frac{1}{2}$ " and 2". Let us try to construct two different triangles.

In Fig. (1) we are assuming that  $BA$  must be  $2\frac{1}{2}$ " and  $CA$  2". With centre  $B$  and radius  $2\frac{1}{2}$ " make a circle.

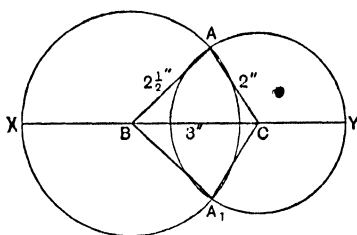


Fig. (1)

Every point  $2\frac{1}{2}$ " from  $B$  must be on the circumference of that circle.

With centre  $C$  and radius 2" make another circle.

Every point 2" from  $C$  must be on the circumference of that circle.  $A$  and  $A_1$  are the only points on both circles.  $\therefore ABC$  and  $A_1BC$  are the only two triangles that can be made on the base  $BC$  with  $BA$  and  $AC$   $2\frac{1}{2}$ " and 2".

It is easy to see that the bottom part of the figure can be folded over the diameters, which lie on  $XBCY$ , so that it exactly coincides with the top part.

Hence  $\triangle A_1BC$  must be congruent with  $\triangle ABC$ .

But we might have made  $AB$  2" and  $AC$   $2\frac{1}{2}$ ", as in Fig. (2).

It seems at first as if we may have found a triangle not congruent with the first.

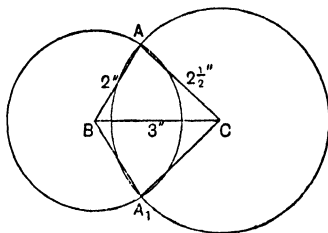


Fig. (2)

But if Fig. (2) is turned over it can be made to coincide exactly with Fig. (1). We have therefore proved that two triangles with three sides equal in each must be congruent.

**Learn :**

*If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.*

## EXERCISE 14.

Make triangles with sides as follows:

1.  $a = 3''$ ,  $b = 4''$ ,  $c = 5''$ .
2.  $a = 2 \cdot 8''$ ,  $b = 3 \cdot 1''$ ,  $c = 4 \cdot 9''$ .
3.  $a = 8 \cdot 9$  cm.,  $b = 5 \cdot 8$  cm.,  $c = 6 \cdot 3$  cm.
4.  $a = 10 \cdot 8$  cm.,  $b = 6 \cdot 4$  cm.,  $c = 7 \cdot 9$  cm.
5.  $a = b = c = 2 \cdot 3''$ .
6.  $a = b = 2 \cdot 9''$ ,  $c = 3 \cdot 2''$ .
7.  $a = 2b = 2c$ ,  $c = 1 \frac{1}{2}''$ .
8.  $a = b = 1 \frac{1}{2}c$ ,  $a + b + c = 9''$ .
9.  $a = 8 \cdot 3$  cm.,  $b = 4 \cdot 2$  cm.,  $c = 3 \cdot 6$  cm. What do you notice?
10.  $a = 3''$ ,  $b = 1 \frac{1}{2}''$ ,  $c = 1 \frac{1}{2}''$ . What do you notice?

11. These triangles have three sides of one equal to three sides of the other (Fig. (1)). Hence they are congruent. Name the pairs of equal angles.

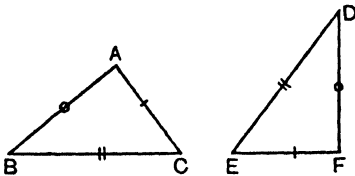


Fig. (1)

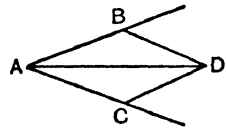


Fig. (2)

12. In bisecting an angle  $BAC$ , we arrange that  $AB = AC$ , and  $BD = CD$  (Fig. (2)).

Prove that the angle is bisected.

13.  $ABCD$  is a rhombus, i.e. all its sides are equal (Fig. (3)).

Prove that the diagonal  $AC$  bisects two opposite angles.

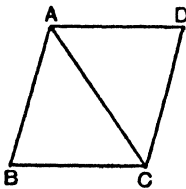


Fig. (3)

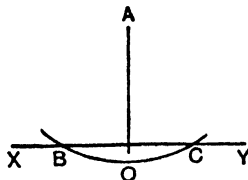


Fig. (4)

14.  $A$  is the centre of a circle which cuts a line  $XY$  at  $B$  and  $C$ .  $BC$  is bisected at  $O$  (Fig. (4)).

Prove that  $AO$  is perpendicular to  $XY$ .

15.  $ABC$  is a triangle in which  $AB = AC$ . The bisector of the angle  $ABC$  meets  $AC$  in  $E$ , and the bisector of the angle  $ACB$  meets  $AB$  in  $F$ . Prove that  $BE = CF$ .

16. Show that two equal chords of a circle subtend equal angles at the centre (i.e. if  $O$  is the centre and  $AB, CD$  the equal chords, prove  $\angle AOB = \angle COD$ ).

**27. A Fourth Case of Congruence of Triangles.**

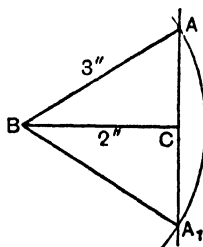
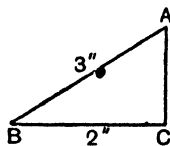
Ex. 1. To construct a right-angled triangle having given the length of the hypotenuse and that of one side.

Suppose  $ABC$  is our rough figure, and that the hypotenuse is to be  $3''$  and  $BC$   $2''$ .

We start by making  $BC$   $2''$  and raising a perpendicular at  $C$ .

$BA$  has to be made  $3''$  long, so with centre  $B$  and radius  $3''$  we make an arc of a circle cutting the perpendicular at  $A$  and  $A_1$ . It cannot cut the perpendicular at any other points.

Also it is obvious that these two triangles are congruent.



**Learn :**

*If two right-angled triangles have the hypotenuse and one side equal in each, the triangles are congruent.*

Ex. 2.  $AB$  is a chord of a circle and  $OX$  is drawn from the centre of the circle perpendicular to  $AB$ .

Prove that  $OX$  bisects  $AB$ .

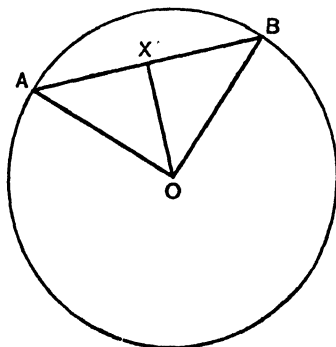
The following is the statement required:

**Construction.** Join  $OA, OB$ .

**Proof.** In the right-angled triangles  $OXA, OXB$ ,

$\left\{ \begin{array}{l} OA = OB, \text{ radii,} \\ OX \text{ is common to both triangles.} \end{array} \right.$

$\therefore$  the  $\Delta$ s are congruent and  $AX = XB$ .



The method of statement is important. We *must* say the triangles are right angled, but we *must not* state the proof as follows:

In the  $\Delta$ s  $OXA, OXB,$

$$\begin{cases} \angle OXA = \angle OXB, \\ OA = OB, \\ OX \text{ is common,} \end{cases}$$

$\therefore$  the triangles are congruent,

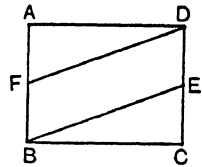
or we shall be saying that two triangles are congruent because they have two sides and one angle (*not contained by the two sides*) equal in each.

### EXERCISE 15.

Make right-angled triangles according to the following requirements:

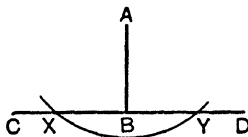
1.  $A = 90^\circ$ ,  $a = 8.5$  cm.,  $b = 5.1$  cm.
2.  $C = 90^\circ$ ,  $a = 3''$ ,  $b = 1.8''$ .
3.  $B = 90^\circ$ ,  $a = 2.8''$ ,  $b = 3.2''$ .
4.  $A = 90^\circ$ ,  $b = 3.7''$ ,  $c = 2.7''$ .
5.  $B = 90^\circ$ ,  $b = 4''$ ,  $c = 2''$ .

6.  $ABCD$  is a rectangle, and equal lines  $BE, DF$  are drawn to cut  $CD$  at  $E$  and  $AB$  at  $F$ . Prove that  $AF = CE$  and hence that  $BF = DE$ .



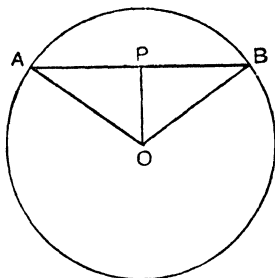
7.  $ABCD$  is a rectangle.  $E, F$  are points on  $CD$  such that  $BE = AF$ . Prove that  $CE = DF$ .

8.  $AB$  is perpendicular to  $CD$ , and with centre  $A$  and any radius a circle is drawn cutting  $CD$  at  $X$  and  $Y$ . Prove that  $BX = BY$ .



9. Assuming that the diagonals of a rhombus bisect the opposite angles, prove that the diagonals bisect at right angles.

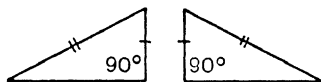
10. From  $O$  the centre of a circle a line  $OP$  is drawn bisecting  $AB$ . Prove that  $OP$  is at right angles to  $AB$ .



11. Using the previous figure: from  $O$  the centre of a circle a line  $OP$  is drawn at right angles to the chord. Prove that  $AP = PB$ .

12. Prove that the line drawn from the vertex of an isosceles triangle perpendicular to the base bisects it.

13. Are the triangles indicated in the rough figure congruent and why?



**28. A Case of Conditional Congruence.**

Suppose we have to copy the triangle  $ABC$  (Fig. (1)), given two sides and the angle opposite to one of them.

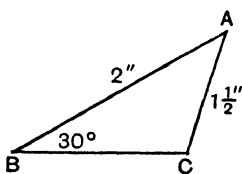


Fig. (1)

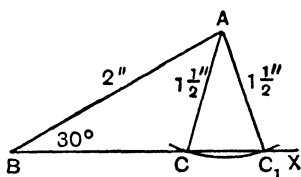


Fig. (2)

We can easily make  $\angle ABC$  and cut off  $BA$   $2''$  on one of the arms (Fig. (2)). As yet we do not know where the point  $C$  is situated, but we know that it is  $1\frac{1}{2}''$  from  $A$ .

Hence with centre  $A$  and radius  $1\frac{1}{2}''$  we make an arc cutting  $BX$ . In this case it cuts  $BX$  at two points  $C$  and  $C_1$ . We therefore have two triangles  $ABC, ABC_1$  satisfying the given conditions.

This is a case in which two triangles have two sides and an angle equal in each, but the triangles are not congruent. The angle in question is not *contained* by the sides equal in each.

If  $AC$  happened to be  $1''$ , we should find our arc just touching  $BX$  at one point  $C$ , and only one triangle would be possible.

If  $AC$  were more than  $2''$ , we should find that the arc with centre  $A$  and radius  $2''$  cuts  $BX$  at two points, but one point would be to the left of  $B$  and this triangle would not have an angle  $B$  of  $30^\circ$ . There would only be one possible triangle in this case also.

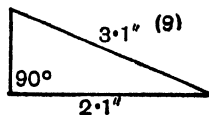
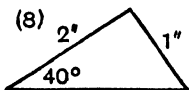
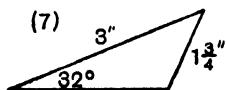
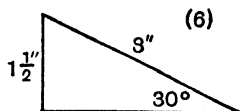
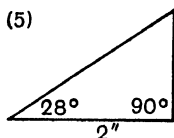
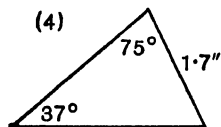
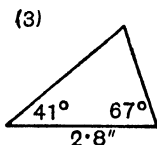
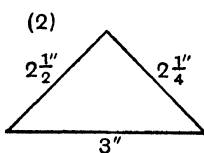
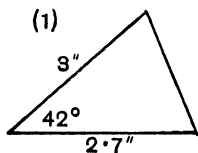
Finally if  $AC$  were less than  $1''$ , the arc would neither touch nor cut  $BX$ . No triangle would be possible.

We therefore have to consider four cases:

- (1) Where the arc touches  $BX$ ...1 solution.
- (2) Where the arc cuts  $BX$  at two points, one to the left of  $B$ ...1 solution.
- (3) Where the arc cuts  $BX$  at two points to the right of  $B$ ... 2 solutions.
- (4) Where the arc neither touches nor cuts  $BX$ ...no solution.

### EXERCISE 16.

Construct triangles, where possible, with the dimensions indicated in the rough diagrams:



Construct triangles, where possible, with the following dimensions:

10.  $A = 50^\circ$ ,  $B = 60^\circ$ ,  $a = 2.3''$ .
11.  $A = 43^\circ$ ,  $b = 1.9''$ ,  $c = 2.4''$ .
12.  $a = 12.3$  cm.,  $b = 11.9$  cm.,  $c = 10.3$  cm.
13.  $A = 102^\circ$ ,  $B = 38^\circ$ ,  $c = 1.1''$ .
14.  $a = 2.8''$ ,  $b = 1.8''$ ,  $A = 55^\circ$ .
15.  $a = 2.9''$ ,  $b = 1.3''$ ,  $B = 37^\circ$ .

16.  $ABC$  is a triangle having its sides  $AB, AC$  equal.  $E$  and  $F$  are the middle points of the sides  $CA$  and  $AB$  respectively. Prove that  $BE = CF$ .

17.  $ABCD$  is a rectangle (i.e. a quadrilateral with its opposite sides equal and parallel and all its angles right angles), and  $P$  is any point equidistant from  $A$  and  $B$ . Prove that  $PC = PD$ .

18. Draw any triangle  $ABC$ . Produce  $BA$  to  $D$  so that  $AD = AC$ , and  $CA$  to  $E$  so that  $AE = AB$ . Prove that  $DE = BC$ , and  $CD$  is parallel to  $BE$ .

19. If in a triangle  $AB = AC$ , prove that the bisector of the angle  $A$  bisects the base  $BC$  at right angles.

20. Two circles have the same centre  $O$ . Radii  $OA, OB$  of the smaller circle are produced to meet the larger in  $C$  and  $D$ . Prove that  $CD$  is parallel to  $AB$ .

21. "The parts of a triangle being its 3 angles and its 3 sides, two triangles are *always* congruent when 3 corresponding parts are equal, each to each." State, *without proof*:

- (a) Three sets of 3 corresponding parts which corroborate this statement;
- (b) Two sets of 3 corresponding parts which contradict it.

In cases (b) indicate with the help of sketches the relations between the triangles.

22. Construct the triangle  $ABC$  in which  $AB = 9.8$  cm., the angle  $A = 37^\circ$ , and the angle  $C = 99^\circ$ . Find a point  $P$  equidistant from  $A$  and  $B$ , and also equidistant from  $AB$  and  $AC$ . Measure  $AP$ .

23. If the perpendiculars on two sides of a triangle from the extremities of the base are equal, prove that the given triangle must be isosceles.

24. Two places  $A$  and  $B$  lie east and west of one another. From  $A$  a man goes due south 8 miles and then due west 4 miles. From  $B$  another man goes due north 4 miles and then due east 8 miles. Prove that the second is now as far from  $B$  as the first is from  $A$ .

25. From a given point  $X$  draw a straight line  $XY$  making an angle of  $30^\circ$  with a given straight line  $AB$  and another line  $XZ$  making an angle of  $55^\circ$  with it. Measure  $YZ$ .

26. An aeroplane  $A$  is just in the vertical plane through two ground stations  $B$  and  $C$  which are 1500 yards apart. If the angle  $CBA = 25^\circ$ , and the angle  $BCA = 48^\circ$ , draw the triangle  $ABC$  to a scale of 1000 feet = 1 inch, and find the vertical height of the aeroplane above the ground.

27. A man observes two objects at the same time, one exactly behind the other, and both in the north-west direction. He walks 600 yards due west, and then finds that one object is due north and the other north-east. Find how far he was from each object at first.

28. Construct a triangle  $PQR$  having  $PQ = 6.7$  cm.,  $PR = 2.9$  cm., and  $RQ = 5.6$  cm. Draw a line  $RS$  cutting  $QP$  produced in  $S$ , so that the angle  $PRS$  is equal to the angle  $PQR$ .

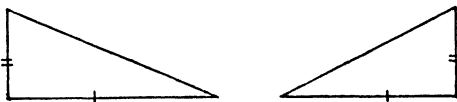
29. A ladder,  $17\frac{1}{2}$  feet long, leans against a wall 12 feet high. When the foot of the ladder is in position  $A$  on the ground, a length of 3 feet 6 inches of the ladder extends beyond the top of the wall, and when in position  $B$  there is a length of 1 foot 6 inches above the wall. Make a scale drawing showing the two positions of the ladder.

30. Five towns  $A, B, C, D, E$  are situated such that the distance from  $B$  to  $C$  is 45 miles,  $C$  to  $D$  50 miles,  $C$  to  $A$  80 miles,  $D$  to  $B$  70 miles,  $D$  to  $E$  50 miles,  $D$  to  $A$  70 miles,  $E$  to  $A$  50 miles. Show the positions of the towns by a scale drawing in which 1" represents 20 miles.

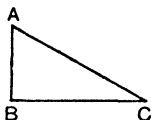
## SECTIONAL REVISION B

## EXERCISE 17 (a). MENTAL.

1. Would these right-angled triangles be congruent if accurately constructed? Give reasons.



2.  $ABC$  is a right-angled triangle with  $\angle BAC$   $60^\circ$ . If a second triangle of the same size were made on the other side of  $BC$ , what figure would be formed?



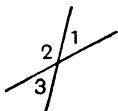
3. One road runs north and south, and another runs north-east and south-west. What is the angle between them?

4. The figure shows two lines drawn perpendicular to another line. Why are they parallel?



5. The angles of a quadrilateral are  $2x$ ,  $2x$ ,  $3x$  and  $5x$  degrees. What is the value of  $x$ ?

6. What are angles 1 and 3 called with respect to one another, and what are angles 1 and 2 called?



7. Express an angle of  $1\frac{3}{4}$  right angles in degrees.

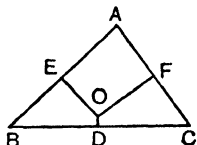
8. The distance from  $A$  to  $B$  is 150 miles, from  $B$  to  $C$  120 miles, and from  $C$  to  $A$  270 miles. How do you know that  $A$ ,  $B$  and  $C$  are in the same straight line?

## EXERCISE 17 (b).

1. Two triangles  $ABC$  and  $DEF$  have  $\angle A = \angle D$ . Prove that  $\angle B + \angle C = \angle E + \angle F$ .

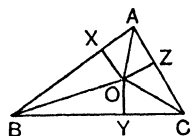
2. Prove that the diagonal of a rectangle divides it into two triangles equal in area.

3.  $OE, OF$  bisect  $AB, AC$  at right angles. Prove that  $OD$  must bisect  $BC$  at right angles.



4.  $OB$  bisects  $\angle ABC$ . Prove that  $OX = OY$  where these are perpendicular to the sides.

$OA$  bisects  $\angle BAC$ . Prove similarly that  $OX = OZ$ .



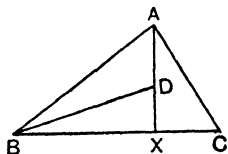
5. Using the above results, prove that  $OC$  bisects  $\angle ACB$ .

6. What angle is formed at the centre of a circle by radii from the extremities of one side of a regular 12-sided figure described in the circle?

7.  $AB, BC, CD$  are three sides of a many-sided regular polygon. Prove that  $AC = BD$ .

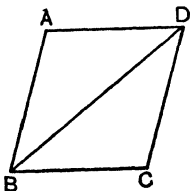


8. If in the figure  $\angle ABC$  is bisected by  $BD$  and  $AX$  is perpendicular to  $BC$ , calculate  $\angle ADB$  if  $\angle ABC = 40^\circ$ .

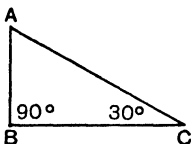


## EXERCISE 18 (a). MENTAL.

1.  $ABCD$  is a rhombus. How do we know that  $ADB, CDB$  are congruent triangles?



2. If  $\frac{AB}{AC}$  is always the same for an angle of  $30^\circ$ , whatever the lengths of  $AB$ ,  $AC$ , find  $AB$  if  $AC = 5.9$  cm.



3. If I turn from a direction  $10^\circ$  east of north to a direction  $20^\circ$  west of north, through what angle have I turned?

4. How many pairs of parallel edges are there in all the faces of a cube?

5. If one angle of a triangle is  $\frac{3}{4}$  of a right angle and another is a right angle, how many degrees are there in the third angle?

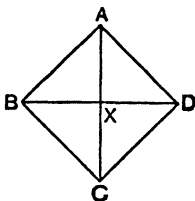
6. The angles of regular polygons gradually increase in size as the number of sides is increased. What is the value of the angle which is not quite reached, when the number of sides is infinitely great?

7. How many corners has a regular prism on a base of 12 sides?

8. If in a triangle  $ABC$ ,  $\angle B = \angle C$ , which two sides are equal?

### EXERCISE 18 (b).

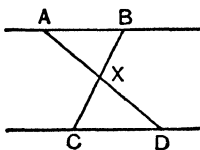
1.  $ABCD$  is a rhombus. Prove that  $\angle BAC = \angle CAD$ .



2. Hence prove that  $\Delta$ s  $AXB$ ,  $AXD$  are congruent and therefore  $BX = XD$ .

3.  $ABC$  is an isosceles triangle and  $XY$  is drawn parallel to the base  $BC$  to cut the sides  $AB$ ,  $AC$  at  $X$  and  $Y$ . Prove that  $AXY$  is an isosceles triangle.

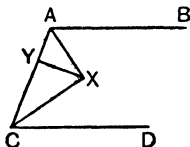
4.  $AB$  and  $CD$  are parallel lines and  $AD$ ,  $BC$  are any two lines cutting at  $X$ . Prove that the triangles  $AXB$ ,  $CXD$  are equiangular.



5. Construct a triangle of which the longest side is  $2\cdot9''$ , the angles being such that the largest is three times the smallest and the other twice the smallest.

6. Construct a parallelogram having diagonals  $8\cdot2$  cm. and  $5\cdot9$  cm.

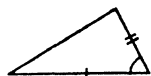
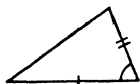
7. Angles  $BAC$  and  $ACD$  are bisected by  $AX$  and  $CX$ , and  $XY$  is perpendicular to  $AC$ . If  $AB$  and  $CD$  are parallel and  $\angle CAB = 124^\circ$ , calculate  $\angle AXY$ .



8. Two sides of an isosceles triangle are  $3\cdot2''$  each and one of the base angles is  $70^\circ$ . Construct the triangle and draw its altitude.

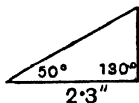
### EXERCISE 19 (a). MENTAL.

1. If the triangles indicated in the rough figures were drawn accurately, why would they be congruent?



2. What is the ratio of the smallest side of a triangle with two angles of  $30^\circ$  and  $60^\circ$  to the longest side?

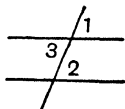
3. Why is it impossible to construct a triangle with these dimensions?



4. If the sides of an isosceles triangle with base angles of  $60^\circ$  each are produced, what is the size of each angle on the other side of the base?

5. Two lines cut at an angle of  $87^\circ$ . At what other angle might they be said to cut?

6. What are angles 1 and 2 called with respect to one another, and what are angles 2 and 3 called?



7. Express an angle of  $112\frac{1}{2}^\circ$  as a fraction of a right angle.

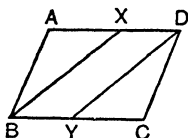
8. What is the perimeter of a square if each edge is  $5.85$  cm. long?

### EXERCISE 19 (b).

1. Prove that any point on the perpendicular bisector of a given line is equidistant from the two ends.

2. Two straight lines  $AB$ ,  $CD$  cut at  $X$ . If each is bisected at  $X$ , prove that  $AC = BD$ .

3.  $ABCD$  is a parallelogram and the angles at  $B$  and  $D$  are bisected by  $BX$ ,  $DY$ . Prove that  $BX = DY$ .



4. If the opposite angles of a parallelogram are together equal to two right angles, prove that the figure must be a rectangle.

5. Calculate the angle subtended at the centre of a circle by one side of a regular five-sided figure inscribed in it. Hence construct a regular pentagon in a circle of  $1.8''$  radius.

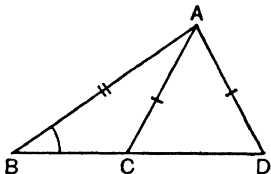
6. Construct a parallelogram with two sides  $8.5$  cm. long,  $4.8$  cm. apart, and with one angle of  $108^\circ$ .

7. How many triangles can be constructed with two sides  $3.3''$  and  $1.1''$  and an angle of  $36^\circ$  opposite the shorter of these sides?

8. Bisect an angle and prove by means of congruent triangles that you have bisected it.

## EXERCISE 20 (a). MENTAL.

1.  $ABC, ABD$  are two triangles with two sides and one angle equal in each. Why is  $\angle ACB$  supplementary to  $\angle ADR$ ?

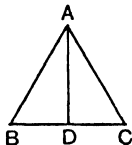


2. What name is given to the triangle  $ACD$  in the above figure?

3. What is the angle between the hands of a clock at 3.30 o'clock?

4. How many pairs of parallel edges are there in the faces of a hexagonal prism?

5.  $ABC$  is an equilateral triangle and  $AD$  bisects the vertical angle. Make a rough copy and mark on it the equal parts which prove  $\Delta s ADB, ADC$  congruent.



6. I go 5.27 miles N., then 2.3 miles E., then 1.7 miles S., and finally 1.1 miles W. How many miles am I then N. and E. of my starting-point?

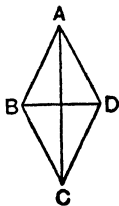
7. How many edges has a regular pyramid on a base of 10 sides?

8. What do you understand by the term congruent triangles?

## EXERCISE 20 (b).

1.  $ABD$  and  $BCD$  are two equal isosceles triangles. Prove that  $\Delta s ABC, ADC$  are congruent.

2. Hence prove that  $BD$  bisects  $AC$  at right angles.



3. Construct a quadrilateral  $ABCD$  in which  $AB = BC = 2.5''$ ,  $AD = CD = 1.8''$ , and  $\angle ABC = 90^\circ$ .

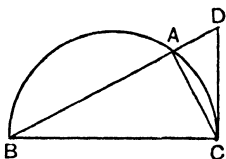
4. Using the figure of Question 3, prove that  $\angle ADB = \angle BDC$ .

5. Describe two triangles which are not congruent, having one side  $3.8''$ , another  $2.7''$ , and the angle opposite to the latter  $28^\circ$ .

6. What angle is subtended at the centre of a circle by one side of a regular hexagon described in it? Hence construct such a figure on a base of  $1.7$  inches.

7. Prove that the perpendicular from the centre of a circle to a chord bisects the chord.

8. Construct this figure if  $BC = 3.9''$ ,  $BAC$  is a semicircle and  $DC$  at right angles to  $BC$ .



## AREAS OF TRIANGLES AND QUADRILATERALS

### 29. Some Kinds of Quadrilaterals.

**Parallelogram.**

Opposite sides parallel.

**Trapezium or Trapezoid.**

One pair only of parallel sides.

**Rectangle or Oblong.**

Parallelogram with one angle a right angle. *Hence all its angles are right angles.*

**Square.**

Rectangle with two adjacent sides equal. *Hence all its sides are equal.*

**Rhombus.**

All sides equal but none of its angles a right angle.

### 30. Construction of Quadrilaterals.

Ex. 1. Here we make  $CD$   $2''$ , then  $\angle$  s  $BCD$  and  $ADC$  (Fig. (1)).

$BC$  is made  $1''$ ,  $AD$   $2''$ .

Finally  $A, B$  are joined.

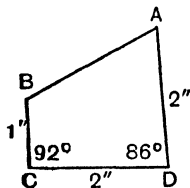


Fig. (1)

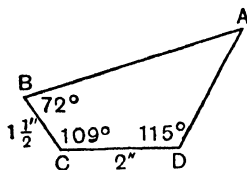
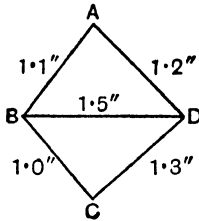


Fig. (2)

Ex. 2.  $BC, CD$  and  $\angle BCD$  are made first (Fig. (2)).

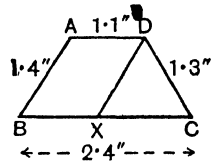
Next we construct  $\angle$  s at  $B$  and  $D$ , and the arms  $BA, DA$  meet at  $A$ .

Ex. 3. When the four sides and the length of a diagonal are given we first make one of the two triangles, and then the other on the diagonal.



Ex. 4. To construct a trapezium given the lengths of the four sides, with an indication as to which sides are parallel.

In the rough figure draw  $DX$  parallel to  $AB$ .  
 $\therefore ABXD$  is a parallelogram, and  $DX = 1.4''$ .  
 $BX$  now equals  $1.1''$ .  $\therefore XC$  is  $1.3''$ .



First construct  $\triangle DXC$  with sides  $1.4''$ ,  $1.3''$ ,  $1.3''$ .

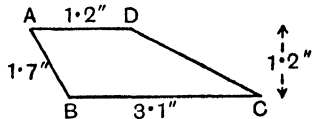
Produce  $CX$  to  $B$  making  $CB$   $2.4''$ . Draw  $DA$  parallel to  $BC$  and  $1.1''$  and join  $AB$ . Then  $ABCD$  is the trapezium required.

Ex. 5. To construct a trapezium given the lengths and the distance apart of the parallel lines, and the length of one side.

Make  $BC$   $3.1''$ .

Draw a line parallel to  $BC$  and  $1.2''$  away.

With centre  $B$  and radius  $1.7''$  cut this parallel at  $A$ .

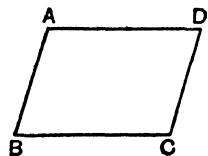


Make  $AD$   $1.2''$  long. Then  $ABCD$  is the trapezium.

Notice that the circle with centre  $B$  and radius  $1.7''$  cuts the parallel to  $BC$  at two points. Hence there are two solutions of the problem.

Ex. 6. To construct a parallelogram given two sides and the angle contained by them.

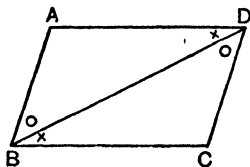
The  $\angle ABC$ ,  $AB$  and  $BC$  are drawn to given dimensions;  $AD$  is then drawn parallel to  $BC$  and  $CD$  parallel to  $BA$ ; or with centre  $A$  and radius equal to  $BC$ , and with centre  $C$  and radius equal to  $BA$ , arcs are drawn cutting at  $D$ .



### 31. Parallelograms.

The definition of a parallelogram merely states that it is a four-sided figure with its opposite sides parallel.

But we can also prove that the opposite sides and angles are equal, and that either diagonal divides it into two exactly equal portions.



**Proof.**  $ABD$  and  $DBC$  are congruent because

$$\left\{ \begin{array}{l} \text{Alternate } \angle \text{s } ADB, DBC \text{ are equal,} \\ \text{Alternate } \angle \text{s } ABD, CDB \text{ are equal,} \\ BD \text{ is common to both triangles.} \end{array} \right.$$

Hence

- (1)  $AB = CD$ ,
- (2)  $AD = CB$ ,
- (3)  $\angle BAD = \angle DCB$ ,

and since  $\angle ADB = \angle DBC$ , and  $\angle CDB = \angle ABD$ ,

$$\therefore (4) \angle ADC = \angle CBA.$$

**Learn:**

*The opposite sides of a parallelogram are equal.*

*The opposite angles of a parallelogram are equal.*

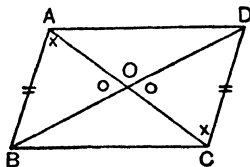
*Each diagonal of a parallelogram divides the figure into two equal triangles.*

### 32. The diagonals of a parallelogram bisect one another.

The  $\Delta$ s  $ABO$ ,  $CDO$  are congruent because

$$\left\{ \begin{array}{l} \angle AOB = \angle COD. \text{ Vertically opposite.} \\ \angle BAO = \angle DCO. \text{ Alternate.} \\ AB = CD, \text{ as proved above.} \end{array} \right.$$

$$\therefore AO = OC, \\ BO = DO.$$



**Learn:**

*The diagonals of a parallelogram bisect one another.*

### 33. Construction of Parallelograms and Trapezoids.

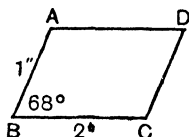
The construction of these figures to given dimensions will become simple if the following principles are remembered:

1. A rough figure should always be drawn, and the given dimensions marked upon it.
2. A parallelogram has two pairs of parallel sides, and a trapezoid one pair.
3. The diagonals of a parallelogram bisect one another.

Consider the following hints for construction, the rough figures indicating the dimensions.

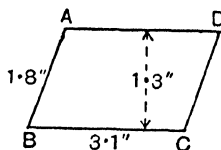
#### Parallelograms.

1. Here we copy  $AB$ ,  $BC$  and  $\angle ABC$ . Then  $AD$  is drawn parallel to  $BC$ , and  $CD$  parallel to  $BA$ .



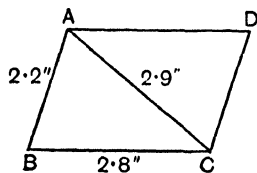
2. The parallels are drawn  $1.3''$  apart.  $BC$  is made  $3.1''$ .

With centre  $B$  and radius  $1.8''$  an arc is drawn cutting the parallel  $AD$ .



3. Here we make the triangle  $ABC$ , of which the three sides are given.

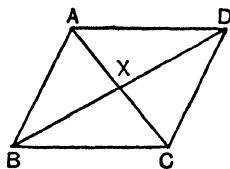
The rest is obvious.



4. Given diagonals  $3''$ ,  $2''$  and  $BC$   $1.7''$ .

The diagonals bisect.  $\therefore BX = 1.5''$ ,  $CX = 1''$ .

Make  $BXC$  with the three sides  $1.7''$ ,  $1.5''$ ,  $1.0''$ .



Produce the half-diagonals so as to complete them, and join  $AB$ ,  $AD$ ,  $CD$ .

**Trapezoids.**

1.  $AB$ ,  $BC$  and  $\angle ABC$  are made first.

Then  $AD$  is drawn parallel to  $BC$  and made  $2\cdot1''$ .

2. The triangle  $DBC$  is made first.

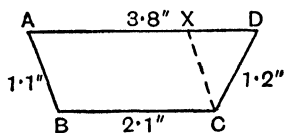
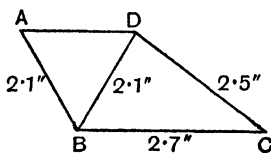
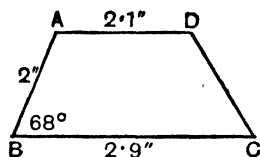
$DA$  is drawn parallel to  $BC$ .

Then  $BA$  is made  $2\cdot1''$ .

Note that two different figures are sometimes possible.

3. If the four sides are given, we divide the figure into a parallelogram and a triangle.

First construct the triangle with sides  $1\cdot2''$ ,  $1\cdot1''$ , and  $3\cdot8'' - 2\cdot1''$ .

**EXERCISE 21.**

1.  $ABCD$  is a parallelogram (Fig. (1).) Write down four pairs of equalities.

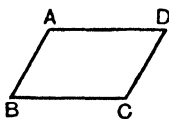


Fig. (1)

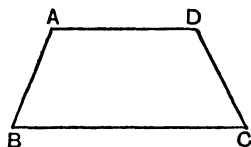


Fig. (2)

2.  $ABCD$  is a trapezoid (Fig. (2).) Show how you would divide it into a parallelogram and a triangle.

Construct parallelograms having:

3. Adjacent sides  $2\cdot3''$ ,  $1\cdot2''$  and one angle  $62^\circ$ .
4. Adjacent sides  $8\cdot8$  cm.,  $6\cdot2$  cm. and one angle  $98^\circ$ .
5. Adjacent sides  $3\cdot7''$ ,  $2\cdot2''$ , with the longer sides  $1\cdot3''$  apart.
6. Adjacent sides  $8\cdot3$  cm.,  $10\cdot8$  cm., with the shorter sides  $7\cdot9$  cm. apart.
7. Adjacent sides  $3\cdot1''$ ,  $3\cdot2''$  and a diagonal  $4\cdot5''$ .
8. Adjacent sides  $9\cdot2$  cm.,  $9\cdot2$  cm. and a diagonal  $9\cdot2$  cm.

9. Diagonals  $2\cdot9''$ ,  $3\cdot3''$ , containing an angle of  $123^\circ$ .
10. Diagonals  $7\cdot5$  cm.,  $7\cdot2$  cm., containing an angle of  $52^\circ$ .
11. \*Diagonals  $3\cdot1''$ ,  $3\cdot0''$  and one side  $4\cdot0''$ .

Construct trapezoids  $ABCD$  having  $AD$ ,  $BC$  parallel and:

12.  $AB\ 2''$ ,  $BC\ 3\cdot8''$ ,  $BD\ 3\cdot5''$ ,  $AD\ 2''$ .
13.  $AB\ 2\cdot6''$ ,  $BC\ 3\cdot5''$ ,  $AD\ 1\cdot1''$ ,  $\angle ABC\ 60^\circ$ .
14.  $AB\ 1\cdot7''$ ,  $BC\ 3\cdot4''$ ,  $CD\ 1\cdot9''$ ,  $AD\ 1\cdot5''$ .
15.  $BC\ 2\cdot8''$ ,  $CD\ 1\cdot3''$ ,  $AD\ 1\cdot1''$ ,  $\angle BCD\ 35^\circ$ .

Construct quadrilaterals  $ABCD$  with the following dimensions:

16.  $AB = 1\cdot8''$ ,  $BC = 2\cdot9''$ ,  $CD = 1\cdot5''$ ,  $\angle B = 60^\circ$ ,  $\angle C = 85^\circ$ .
17.  $AB = 5\cdot7$  cm.,  $CD = 5\cdot9$  cm.,  $AD = 8\cdot2$  cm.,  $\angle A = 71^\circ$ ,  $\angle D = 58^\circ$ .
18.  $AB = 2''$ ,  $BC = 2''$ ,  $CD = 3''$ ,  $DA = 3''$ ,  $BD = 3''$ .
19.  $AB = 4\cdot1$  cm.,  $BC = 4\cdot0$  cm.,  $CD = 5\cdot5$  cm.,  $DA = 5\cdot6$  cm.,  $\angle A = 85^\circ$ .

20. Construct a quadrilateral  $ABCD$  with

$$AB = AC = AD = CD = 2 \text{ in.},$$

and the angle  $BAC = 30^\circ$ .

- (i) Measure, (ii) calculate the angles  $ABC$ ,  $ABD$ .

If  $AC$  and  $BD$  intersect at  $E$ , prove that  $BC = BE$ .

21. Construct a parallelogram in which the diagonals are  $6''$  and  $4''$  and the shorter sides are each  $3''$ . State your method, and measure the longer sides.

22. Construct a quadrilateral  $ABCD$  in which

$$AB = CD = 2'', \quad BC = 2\cdot5'',$$

the diagonal  $AC = 4''$ , and the angle  $CAD = 30^\circ$ .

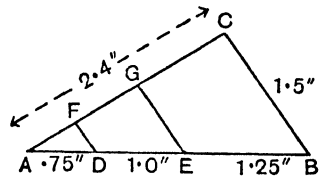
23. Construct a rhombus  $ABCD$  in which the angle  $ADC = 120^\circ$  and the diagonal  $BD$  is  $2\cdot5$  inches long. If  $P$  is a point on  $BD$  produced such that  $PD = DB$ , prove that  $PAB$  is a right angle.

24. Construct a pentagon  $ABCDE$ , having  $AB = 5\cdot0$  cm.,  $BC = 4\cdot1$  cm.,  $CD = 3\cdot6$  cm.,  $DE = 7\cdot4$  cm., and  $EA = 4\cdot5$  cm.; also angle  $ABC = 105^\circ$ , and the angle  $EAB = 143^\circ$ .

25. A man walks  $4\frac{1}{2}$  miles to the north, then  $3\frac{1}{2}$  miles south-east, then 5 miles south, and finally 2 miles north-west. Represent this journey by a drawing to a scale of  $\frac{3}{4}$  inch to a mile. How far is the man from his starting-point at the end of the journey?

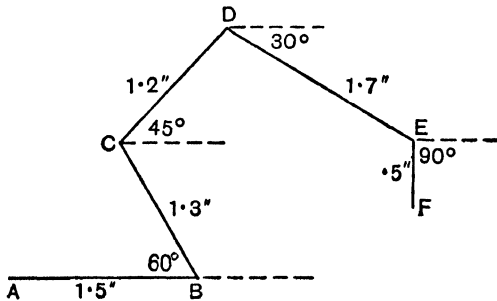
26. Construct geometrically the triangle  $ABC$  to the given dimensions, and by using set-squares draw  $DF$ ,  $EG$  parallel to  $BC$ .

Measure the lengths of  $AF$ ,  $FG$ ,  $GC$  and compare  $\frac{AF}{FG}$ ,  $\frac{FG}{GC}$  with  $\frac{AD}{DE}$ ,  $\frac{DE}{EB}$ .



27.  $ABCD$  is a quadrilateral in which  $AB = 2.9''$ ,  $BC = 3.1''$ ,  $CD = 3.5''$ ,  $DA = 2.5''$ , and  $AC = 3.8''$ . Bisect the angles at  $B$  and  $D$  and find the length of the section of  $AC$  between the intersections with it of the bisectors.

28. Starting from the point  $A$ , draw the figure  $ABCDEF$  to the given dimensions. Measure the length of the line  $AF$  and the magnitudes of the angles  $BAF$  and  $CDE$ .



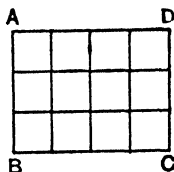
## AREAS

### 34. The Area of a Rectangle.

Suppose the rectangle  $ABCD$  has sides  $AD$  4", and  $AB$  3".

Let  $AD$  be divided into four equal parts of 1" each, and  $AB$  into three of 1" each.

Draw lines parallel to the sides through the points of division, thus dividing the figure into 12 equal squares of 1" side.



It is seen from the figure that the area is 12 square inches. Similarly, if the sides are  $2\frac{1}{2}$  inches and  $2\frac{1}{4}$  inches, the area is  $2\frac{1}{2} \times 2\frac{1}{4}$  square inches, for the figure can be divided into  $10 \times 9$ , i.e. 90 quarter-inch squares, and

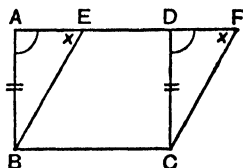
$$2\frac{1}{2}'' \times 2\frac{1}{4}'' = \frac{5}{2}'' \times \frac{9}{4}'' = \frac{45}{8}'' \text{ or } \frac{90}{16} \text{ square inches.}$$

In general the area of a rectangle = base  $\times$  height, or  $A = bh$ .

### The Area of a Parallelogram.

Let  $EBCF$  be a parallelogram on the same base as the rectangle  $ABCD$  and of the same height.

The triangles  $ABE$ ,  $DCF$  are congruent, for the reasons indicated in the diagram.



If we add the trapezoid  $EBCD$  to each of these, we see that

$$ABCD = EBCF.$$

But

$$ABCD = bh,$$

$$\therefore EBCF = bh,$$

since both have the same base and the same height.

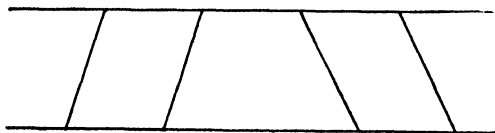
**Learn:**

*The area of a parallelogram is found by multiplying base by height, both being given in the same units; or  $A = bh$ .*

*All parallelograms having equal bases and heights are equal in area.*

The latter is obvious, since  $b$  and  $h$ , and therefore  $b \times h$ , will be the same for all.

Note:



The above parallelograms lie between the same parallels. The heights of the parallelograms must be the same, since parallels are always the same distance apart.

### 35. The Area of a Triangle.

In Fig. (1) the triangle  $ABC$  is half the rectangle  $DBCE$  of the same height and on the same base, since

$$\triangle ABX = \frac{1}{2} \text{ the rectangle } DX,$$

and

$$\triangle ACX = \frac{1}{2} \text{ the rectangle } EX.$$

By adding we get

$$ABC = \frac{1}{2} \text{ the rectangle } DBCE.$$

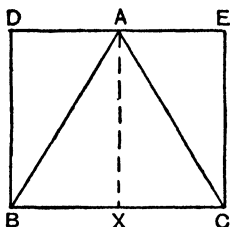


Fig. (1)

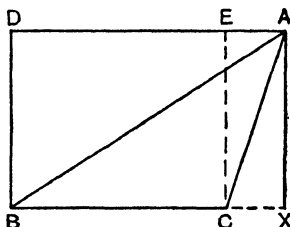


Fig. (2)

In Fig. (2) we must subtract the results, and once again we get

$$\triangle ABC = \frac{1}{2} DBCE.$$

But

$$DBCE = bh, \quad \therefore \triangle ABC = \frac{1}{2} bh,$$

since bases and heights of triangle and rectangle are equal.

Learn:

*The area of a triangle is half that of a rectangle with the same base and height, or  $\Delta = \frac{1}{2} bh$ .*

*All triangles having equal bases and heights are equal in area.*

*All equal triangles having equal bases have equal heights.*

*All equal triangles having equal heights have equal bases.*

Note also that triangles lying between the same parallels must have equal heights.

**36. The Area of a Trapezium or Trapezoid.**

All trapezoids have one pair of parallel sides.

Here  $AD$  is parallel to  $BC$ .

Divide it into two triangles  $ABC, ADC$  by joining  $AC$ .

Call  $BC$  the base of  $\triangle ABC$ . Then  $AX$  is the height, and  $ABC = \frac{1}{2} BC \cdot AX$ .

Call  $AD$  the base of  $\triangle ADC$ . Then  $CY$  is the height, and  $ADC = \frac{1}{2} AD \cdot CY$ .

But  $AX = CY$ , because these are distances between parallel lines.

Adding results we get:

$$\begin{aligned} ABCD &= ABC + ADC \\ &= \frac{1}{2} BC \cdot AX + \frac{1}{2} AD \cdot AX \\ &= \frac{1}{2} AX (BC + AD) \\ &= \frac{1}{2} (\text{sum of parallel sides}) \times \text{distance between them.} \end{aligned}$$

**Learn:**

*The area of a trapezium is found from the formula  $\text{Area} = \frac{1}{2}h(a + b)$ , where  $a$  and  $b$  are the lengths of the parallel sides and  $h$  the distance between them.*

**37. To make a triangle equal in area to a given quadrilateral.**

This is done by joining any two alternate vertices  $A$  and  $C$ , drawing  $DX$  parallel to  $AC$  and joining  $A$  to  $X$ , the point where  $DX$  meets  $BC$  produced.

Then  $ABX = ABCD$ .

**Proof.**  $ADC, AXC$  are on the same base and of the same height—because they are between the same parallels.

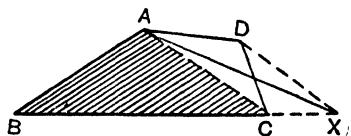
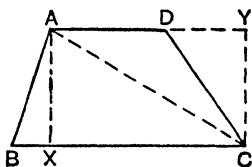
$$\therefore ADC = AXC.$$

Add the shaded part  $ABC$ .

$$\therefore ABC + ADC = ABC + AXC,$$

or

$$ABCD = ABX.$$



38. To reduce any rectilinear figure to a figure of equal area having one side less.

The process is the same as above.

Join any two alternate vertices, e.g.  $EC$ .

Draw  $DX$  parallel to  $EC$ , and join  $EX$  as before.

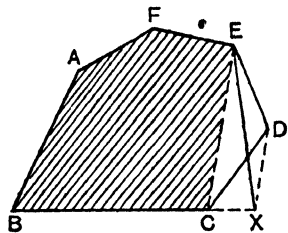
Then  $ABXEF = ABCDEF$ .

**Proof.**  $EDC = EXC$ , as before.

Add the shaded part  $ABCEF$ .

$$\therefore ABCEF + EDC = ABCEF + EXC,$$

$$\text{or} \quad ABCDEF = ABXEF.$$



By continuing the process we can reduce any rectilinear figure to an equal triangle.

39. To make a rectangle equal in area to a given triangle.

Since the area of a  $\Delta = \frac{1}{2} b \cdot h$  and a rectangle  $= b \cdot h$ , there are two simple ways of constructing a rectangle equal in area to a triangle:

(1) Keep the same base and halve the height;

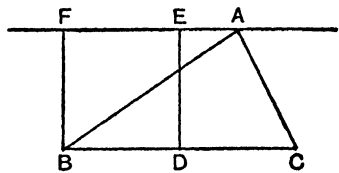
or (2) Keep the same height and halve the base.

Adopting the latter process, first draw through  $A$  a line parallel to  $BC$ .

Bisect  $BC$  at  $D$ .

Draw perpendiculars to  $BC$  through  $B$  and  $D$ .

Then  $FBDE$  is a rectangle equal to  $ABC$ .

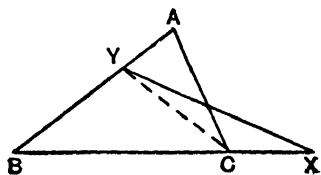


40. To make a triangle on a given base equal to a given triangle.

In the rough figure let  $ABC$  represent the given  $\Delta$  and  $BX$  the base of the new triangle.

Obviously the new  $\Delta$  must be of a less height than that of  $ABC$ .

Let  $YBX$  be the new triangle.



• Join  $YC$ .

Now if  $ABC = YBX$ , then  $AYC + YBC = YBC + YCX$ .

$\therefore AYC = YCX$ .

But these are on the same base.

$\therefore$  they are of the same height.  $\therefore AX$  is parallel to  $CY$ .

The process of working back from this for the construction of the triangle is left as an exercise to the student.

#### 41. The Area of a Triangle by Formula.

The proof of the formula for the area of a triangle, given the lengths of the three sides, is too difficult at this stage, but the formula should be learnt and used where required:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where  $s$  = half the sum of the sides, and  $a, b, c$  are the lengths of the sides.

Ex. Find the area of a triangle with sides 7, 8 and 11 inches.

$$s = \frac{7 + 8 + 11}{2} = \frac{26}{2} = 13.$$

$$s - a = 6, \quad s - b = 5, \quad s - c = 2.$$

$$\therefore \Delta = \sqrt{13 \times 6 \times 5 \times 2}$$

$$= \sqrt{780} = \text{approximately } 28 \text{ sq. inches.}$$

#### 42. Area of any Rectilinear Figure.

(1) *Irregular.*

Divide into triangles and find the sum of the areas of the triangles.

(2) *Regular.*

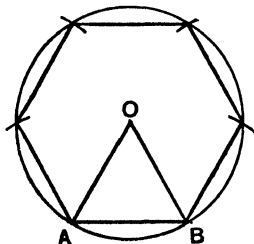
Every regular polygon can be described in a circle.

The centre is found by bisecting any two sides at right angles.

Let  $O$  be the centre of such a polygon.

Join  $OA, OB$ .

Here there are 6 sides,  $\therefore$  6 triangles equal to  $OAB$  will be formed.

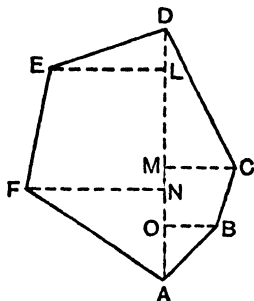


Calculate the area of  $OAB$ , either by formula as above, or by using the altitude of  $\triangle OAB$ .

Multiply the result by the number of sides, i.e. by the number of triangles.

### Further Examples:

Ex. 1. Find the area of the field illustrated in the given diagram, the offsets and their projections on the base line being of the lengths indicated.



Yards		
	D	
	400	
200	300	
	200	120
250	150	
	100	100
	A	

From the Field-Book figures we get:

$AO = 100$ ;  $OM = 100$ ;  $AN = 150$ ;  $NL = 150$ ;  $MD = 200$ ;  $LD = 100$ .

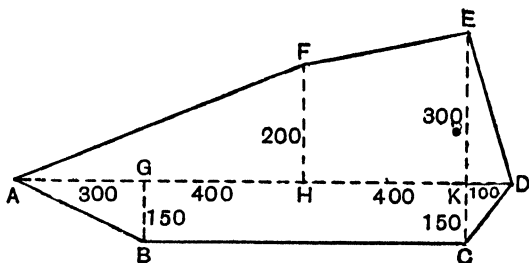
Total area =  $\triangle ELD$  + trapezoid  $EFNL$  +  $\triangle FNA$  +  $\triangle OAB$   
+ trapezoid  $OMCB$  +  $\triangle DMC$

$$\begin{aligned}
 &= \frac{1}{2} \cdot 200 \cdot 100 + \frac{1}{2} (200 + 250) \cdot 150 + \frac{1}{2} \cdot 250 \cdot 150 \\
 &\quad + \frac{1}{2} \cdot 100 \cdot 100 + \frac{1}{2} (120 + 100) \cdot 100 + \frac{1}{2} \cdot 200 \cdot 120 \\
 &= 200 \times 50 + 225 \times 150 + 125 \times 150 + 100 \times 50 \\
 &\quad + 110 \times 100 + 60 \times 200 \\
 &= 200 (50 + 60) + 150 (225 + 125) + 100 (50 + 110) \\
 &= 22,000 + 52,500 + 16,000 \\
 &= 90,500 \text{ sq. yds.} \\
 &= \text{approximately } 19 \text{ acres.}
 \end{aligned}$$

Ex. 2. Interpret the following Field-Book dimensions, draw a rough diagram of the field, and calculate its area.

The base line is  $AD$ , 1200 links, and this is measured from West to East.

Links		
	D	
	1200	
300	1100	150
200	700	
	300	150
	A	G o E.



The areas can now be calculated as before.

### EXERCISE 22.

Find, correct to one place of decimals, the areas of rectangles having sides:

1.  $3\cdot72''$ ,  $2\cdot3''$ .
2.  $8\cdot72$  cm.,  $7\cdot62$  cm.
3.  $4\cdot15$  ft.,  $2\cdot75$  ft. (sq. ft. and to the nearest sq. inch).
4.  $1\cdot035$  metres,  $\cdot865$  metre (sq. cm. to the nearest first place of decimals).

What are the areas of the following parallelograms?

5. Base  $5\cdot6''$ , height  $2\cdot9''$ .
6. Base  $8\cdot75$  cm., height  $5\cdot4$  cm.

Find the areas of the following triangles:

7. Base  $5\cdot4''$ , height  $2\cdot5''$ .
8. Base  $3\cdot9''$ , height  $7\cdot2''$ .

What are the areas of the following trapezia?

9. Parallel sides  $1\cdot93''$ ,  $1\cdot07''$ ; distance apart  $2\cdot5''$ .
10. Parallel sides  $5\cdot6$  cm.,  $2\cdot4$  cm.; distance apart  $5$  cm.
11. Construct a parallelogram with two adjacent sides  $2\cdot8''$ ,  $2\cdot1''$ , and a contained angle  $73^\circ$ . Find its height by measurement and calculate the area.

12. Find by measurement and calculation the area of a parallelogram with two sides 5.6 cm., 5.2 cm., and a diagonal 6.8 cm.

Draw the following triangles as accurately as you can, and by measuring one altitude calculate the areas:

13. Sides 3.5", 3.5", 3.5".

14. Sides 3.7", 3.1", and the angle contained by them  $49^\circ$ .

15. Two angles  $43^\circ$ ,  $87^\circ$ , and a side adjacent to these angles 8.75 cm.

16. Two angles  $30^\circ 30'$ ,  $48^\circ 30'$ , and the smallest side 1.9".

17. Draw a trapezium  $ABCD$  with sides  $AB$  2",  $BC$   $3\frac{1}{2}"$ ,  $CD$   $2\frac{1}{2}"$ ,  $DA$  2",  $AD$ ,  $BC$  being parallel. By measurement and calculation, find the area.

18. Repeat the trapezium in Question 17 and reduce it to an equal triangle.

19. The base  $BC$  of a triangle is trisected at  $D$  and  $E$ , and  $AD$  is joined (Fig. (1)). Why is  $\triangle ADC$  half the area of  $\triangle ABD$ ?

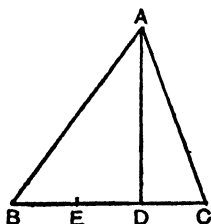


Fig. (1)

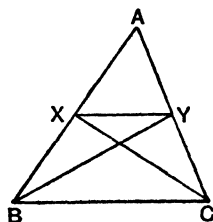


Fig. (2)

20. The sides  $AB$ ,  $AC$  of a  $\triangle ABC$  are bisected at  $X$ ,  $Y$ .  $BY$ ,  $CX$  are joined (Fig. (2)). Name two triangles equal in area to  $\triangle AXY$ .

21. Since the triangles named in your answer to Question 20 are equal, what follows?

22. Draw a triangle with sides 4", 3",  $3\frac{1}{2}"$ . Make a rectangle equal to it.

23. Draw any five-sided figure and construct a quadrilateral equal to it.

24. Draw any quadrilateral. Reduce it to an equal triangle. Construct a rectangle equal to this triangle.

25. Draw a triangle with sides 2·8", 2·5", 2·3". On a base 3·1" construct a triangle equal in area to it.

26. Draw to a scale of 1 inch to 100 yards the field represented by the following extracts from the Field-Book, and calculate the area of the field to the nearest acre.

Yards		
	D	
	800	
150	500	
	350	200
250	150	
	A	Go N. W.

27. A field *ABCDE* is surveyed with the following results:

Yards to D		
	400	
	350	360 to C
420 to E	320	
	110	460 to B
	From A	

Draw the plan of the field to a scale of 1 inch to 200 yards, and determine the area of the field in acres and square yards.

28. Draw a plan of the field *ABCDE* from the Field-Book entry below, to a scale of 1 inch to 2 chains, and find the area of the field in acres.

Links		
	To D	
	350	
250 to C	300	
	280	240 to E
210 to B	210	
	From A	

29. Draw a plan, and find the area in acres of a field with straight hedges from the following data:

Links	
	To C
	350
240 to D	250
	150
180 to E	100
	From A

120 to B

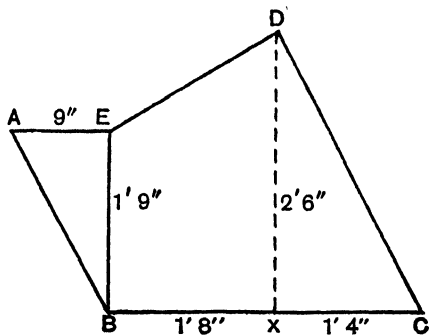
30. Draw a plan, and find the area of a field with straight hedges from the following data:

Links	
	To C
	700
20 to D	650
	520
92 to E	340
70 to F	
	From A

48 to B

31. Find the area of the piece of sheet copper shown in the sketch. If 1 sq. ft. of the metal weighs 5.6 lb., find its weight in lb.

( $\angle$ s  $AEB$ ,  $EBC$ ,  $DXC$  are right angles.)



32. A quadrilateral  $ABCD$  has the following dimensions:  $AB = 240$  cm.;  $BC = 150$  cm.;  $AD = 110$  cm.;  $\angle ABC = 85^\circ$ ;  $\angle BAD = 90^\circ$ . Draw a plan of the quadrilateral to a suitable scale and find its area to the nearest square metre.

33. A plot of land  $ABCD$  has the following dimensions:  $AB = 48$  ft.;  $AD = 26$  ft.;  $BC = 22$  ft.;  $CD = 36$  ft.; and  $\angle A = 75^\circ$ . Draw the plot to the scale of  $\frac{1}{10}$  inch = 1 ft. Reduce the quadrilateral to a triangle having the same area, and find its area.

## SIMILAR TRIANGLES\*

43. In an earlier section it was discovered that two triangles need not necessarily be congruent when any three parts are equal in each. Thus two triangles with all angles equal, each to each, might be very different in size, though they would be similar in shape. Such triangles are known as similar triangles.

But although the corresponding sides of these triangles are not equal, they are definitely related to one another.

Let  $ABC, DBE$  be two such similar triangles with the smaller placed so as to occupy a corner of the larger (Fig. (1)). Suppose it is known that  $BE = \frac{3}{5}$  of  $BC$ , or  $\frac{BE}{BC} = \frac{3}{5}$ .

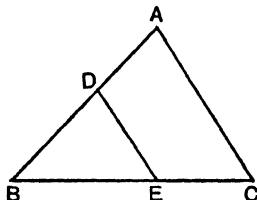


Fig. (1)

Divide  $BC$  into five equal parts as shown in Fig. (2) and draw parallels to  $AC$  through the points.  $DE$  will be one of these parallels, since  $\angle DEB = \angle ACB$ .

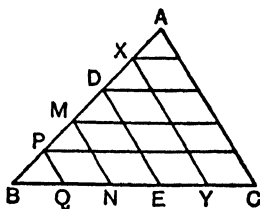


Fig. (2)

These four parallels will cut  $AB$  at four points. Through these four points draw parallels to  $BC$ .

We have now divided the figure  $ABC$  into a number of triangles and parallelograms, and all these parallelograms have one side equal to one of the divisions of  $BC$ .

\* Paragraphs 43 to 48 and Exercise 23 may be omitted if desired.

The triangles are all equal because one side is either a side of one of these parallelograms or one of the divisions of  $BC$ .

$\therefore AB$  is divided into 5 equal parts, and  $BD = 3$  of them.

$$\therefore BD = \frac{3}{5} \text{ of } AB, \text{ or } \frac{BD}{AB} = \frac{3}{5}.$$

Similarly  $AC$  is divided into 5 equal parts, and  $DE = 3$  of them.  $\therefore DE = \frac{3}{5}$  of  $AC$ , or  $\frac{DE}{AC} = \frac{3}{5}$ .

Hence we find that the sides of these similar triangles are proportional to one another, i.e. corresponding sides are the same fractions of one another. This is always true, whatever the ratio between any pair of corresponding sides.

**Learn:**

*The corresponding sides of similar triangles are proportional.*

#### 44. Areas of Similar Triangles.

Consider Fig. (2) again. It will be seen that each small parallelogram is twice the area of each small triangle, being of equal bases and equal heights.

$DBE$  contains 3 parallelograms and 3 triangles; this is equivalent to 9 triangles.

$ABC$  contains 10 parallelograms and 5 triangles; or 25 triangles.

$$\therefore \frac{DBE}{ABC} = \frac{9}{25} = \left(\frac{3}{5}\right)^2; \text{ and } \frac{BE}{BC} = \frac{3}{5}.$$

Consider  $MBN$  and  $XY$ .

$$MBN = 1 \text{ parm.} + 2 \text{ triangles} = 4 \text{ triangles,}$$

$$XY = 6 \text{ parm.} + 4 \text{ triangles} = 16 \text{ triangles.}$$

$$\therefore \frac{MBN}{XY} = \frac{4}{16} = \left(\frac{2}{4}\right)^2, \text{ or } \left(\frac{1}{2}\right)^2; \text{ and } \frac{BN}{XY} = \frac{1}{2}.$$

These relations will be found to be true, whatever the ratio between two pairs of corresponding sides.

**Learn:**

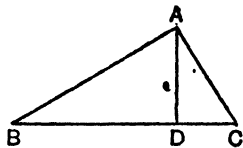
*The areas of similar triangles are proportional to the squares of corresponding sides.*

**45. An Important Exercise.**

Let  $ABC$  be a triangle with a right angle at  $A$ .  $BC$  is called the *hypotenuse*.

Draw  $AD$  at right angles to  $BC$ .

The triangles  $ABD$ ,  $ACD$ ,  $ABC$  are all similar, with angles and sides corresponding as follows:



	Right angle	Smallest angle	Middle angle	Hypotenuse	Smallest side	Middle side
$ABD$	$ADB$	$ABD$	$BAD$	$AB$	$AD$	$BD$
$ACD$	$ADC$	$DAC$	$ACD$	$AC$	$DC$	$AD$
$ABC$	$BAC$	$ABC$	$ACB$	$BC$	$AC$	$AB$

$\angle BAD = \angle ACB$ , because each is complementary to  $\angle CAD$ .

$\angle DAC = \angle ABC$ , because each is complementary to  $\angle BAD$ .

$$\therefore \frac{ABD}{ABC} = \frac{AB^2}{BC^2} \quad \text{and} \quad \frac{ACD}{ABC} = \frac{AC^2}{BC^2}.$$

$$\therefore \frac{ABD + ACD}{ABC} = \frac{AB^2 + AC^2}{BC^2}.$$

But  $ABD + ACD = ABC$ .  $\therefore AB^2 + AC^2 = BC^2$ .

**Learn:**

*In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.*

**46. Relations between the sides of a right-angled triangle.**

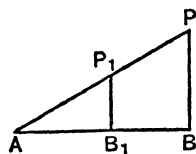
Let  $PAB$  be any angle.

Draw  $PB$  perpendicular to  $AB$ .

Then  $PBA$  is a right-angled triangle.

Let  $P_1B_1$  be another such perpendicular.

Then  $P_1AB_1$  and  $PAB$  are similar triangles.



We have proved that the sides of similar triangles are proportional:

$$\therefore (1) \frac{P_1B_1}{AP_1} = \frac{PB}{AP}, \quad (2) \frac{AB_1}{AP_1} = \frac{AB}{AP}, \quad (3) \frac{P_1B_1}{AB_1} = \frac{PB}{AB}.$$

That is to say  $\frac{PB}{AP}$ ,  $\frac{AB}{AP}$  and  $\frac{PB}{AB}$  always have the same value as long as the angle  $A$  remains the same.

**Note that:**  $AP$  is the hypotenuse,  
 $PB$  is *opposite* the angle  $A$ ,  
 $AB$  is *adjacent* to the angle  $A$ .

The above fractions, or ratios, are known as the sine, cosine and tangent of the angle  $A$ .

I.e. The sine of the angle  $A$  (written  $\sin A$ , but called sine  $A$ )  

$$= \frac{\text{opposite side}}{\text{hypotenuse}}.$$

The cosine of the angle  $A$  (written and called  $\cos A$ )  

$$= \frac{\text{adjacent side}}{\text{hypotenuse}}.$$

The tangent of the angle  $A$  (written and called  $\tan A$ )  

$$= \frac{\text{opposite side}}{\text{adjacent side}}.$$

The mnemonic **sohcahtoa** will help you to remember these ratios, the initial letters giving the words sine, opposite, hypotenuse; cosine, adjacent, hypotenuse; tangent, opposite, adjacent.

**47. Use of Sin, Cos, and Tan Tables.**

The values of Sin, Cos, and Tan of any acute angle can readily be obtained from books of tables. The following are extracts from a page of tan tables:

									Differences								
	0'	10'	20'	30'	40'	50'	60'		1'	2'	3'	4'	5'	6'	7'	8'	9'
45°	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44°	6	12	18	24	30	36	41	47	53
46	1.0355	.0416	.0477	.0538	.0599	.0661	.0724	43	6	12	18	25	31	37	43	49	55
47	1.0724	.0786	.0850	.0913	.0977	.1041	.1106	42	6	13	19	26	32	38	45	51	57
48	1.1106	.1171	.1237	.1303	<b>.1369</b>	.1436	.1504	41	7	13	20	27	<b>33</b>	40	46	53	60
49	1.1504	.1571	.1640	.1708	.1778	.1847	.1918	40°	7	14	21	28	34	41	48	55	62

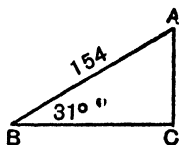
From these tables we read  $\tan 48^\circ 45'$  as follows:

- (1) Look for  $48^\circ$  in the first column.
- (2) Look along the horizontal line containing  $48^\circ$  until the vertical column for  $40'$  is met. Place the finger under **.1369**. Note that this stands for  $1.1369$ .
- (3) Look further along this horizontal line until the vertical column for the remaining  $5'$  is met. The number found is **33**.
- (4)  $1369 + 33 = 1402$ .  
 $\therefore \tan 48^\circ 45' = 1.1402$ .

## 48. Use of the values of these ratios.

Ex. 1.  $\angle ABC = 31^\circ$ ,  $\angle ACB = 90^\circ$  and  $AB = 154$  feet.

Find  $AC$  and  $BC$ .



$$(1) \sin 31^\circ = \frac{o}{h} = \frac{AC}{AB} = \cdot 5150.$$

$$\therefore \frac{AC}{154} = \cdot 5150,$$

$$\begin{aligned} AC &= 154 \times \cdot 5150 \\ &= 79\cdot 3 \text{ ft.} \end{aligned}$$

$$(2) \cos 31^\circ = \frac{a}{h} = \frac{BC}{AB} = \cdot 6009.$$

$$\therefore \frac{BC}{154} = \cdot 8572,$$

$$\begin{aligned} BC &= 154 \times \cdot 8572 \\ &= 132 \text{ ft.} \end{aligned}$$

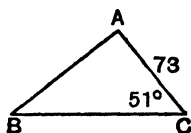
Ex. 2. Here  $\angle ACB = 51^\circ$ ,  $\angle BAC = 90^\circ$  and  $AC = 73$  ft.

Find  $AB$ .

$$\tan 51^\circ = \frac{o}{a} = \frac{AB}{AC} = 1\cdot 2549.$$

$$\therefore \frac{AB}{73} = 1\cdot 2549,$$

$$\begin{aligned} AB &= 73 \times 1\cdot 2549 \\ &= 91\cdot 6 \text{ ft.} \end{aligned}$$

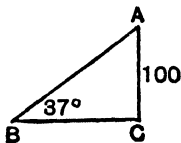


Ex. 3. The angle of elevation of the top of a tower is  $37^\circ$  from a point on the ground. If the height of the tower is 100 feet, find the distance of the point from the foot of the tower.

$$\text{Method 1. } \tan 37^\circ = \frac{o}{a} = \frac{100}{BC} = \cdot 7536.$$

$$\therefore BC = \frac{100}{\cdot 7536}$$

$$= 133 \text{ ft. to nearest foot.}$$



*Method 2.* It is clear that  $\angle BAC = 53^\circ$ .

$$\tan 53^\circ = \frac{o}{a} = \frac{BC}{100}.$$

$$\therefore \frac{100}{BC} = 1.3270,$$

$$\therefore BC = 132.7 \text{ ft.}$$

EXERCISE 23 (a). MENTAL.

1.  $XY$  is parallel to  $AC$ .  $BY = 3''$ ,  $YC = 2''$ . If  $AB = 6.5''$ , find  $BX$ .

2. In the figure of Question 1, if  $BY = 3 \text{ cm.}$ ,  $YC = 2 \text{ cm.}$ ,  $BX = \frac{3}{4}''$ , find  $AX$ .

3.  $ABC$  is a right-angled triangle with sides  $3''$ ,  $4''$ ,  $5''$ ,  $C$  being the right angle. Find the values of  $\sin B$ ,  $\cos B$ ,  $\tan B$ .

4. In this figure find also  $\cos A$ ,  $\sin A$ ,  $\tan A$ .

5. What can you deduce about the  $\sin$  and  $\cos$  of the complementary angles  $A$  and  $B$ ?

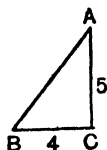
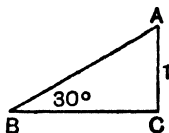
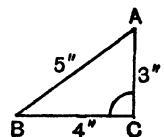
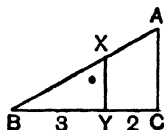
6.  $\sin 30^\circ$  is known to be  $\frac{1}{2}$  and  $AC = 1''$ . What is the length of  $AB$ ?

7. Write down from tables the values of:  $\sin 71^\circ$ ,  $\sin 52^\circ$ ,  $\sin 18^\circ$ .

8. Write down  $\cos 21^\circ$ ,  $\cos 43^\circ$ ,  $\cos 85^\circ$ .

9. Write down  $\tan 45^\circ$ ,  $\tan 63^\circ$ ,  $\tan 78^\circ$ .

10. If  $AC = 5$ ,  $BC = 4$  and  $\angle ACB = 90^\circ$ , find from tables the value of the angle  $A$  to the nearest degree.



EXERCISE 23 (b).

1. Draw any two triangles with corresponding angles equal. Find by measurement and calculation the ratios of the longest and shortest sides in each, and verify that they are equal.

2. Draw an angle of  $56^\circ$ . Measure the opposite side and the hypotenuse of a right-angled triangle containing this angle and hence find the sine of  $56^\circ$ . Compare this with the result obtained from tables.

3. Draw an angle of  $73^\circ$  and find its cosine in the same way.

4. Draw an angle of  $15^\circ$  and find its tangent.

5. A ladder is placed so that its foot is 12 feet from a wall. The angle which the ladder makes with the wall is  $32^\circ$ . Find the length of the ladder.

6. Show how to construct an angle whose cosine is  $\frac{3}{8}$  and find by measurement its tangent.

7. Find by construction the sin, cos and tan of the angle  $60^\circ$ .

8. What is the cos of an angle whose sin is  $\frac{5}{13}$ ?

9. The string of a kite is 150 feet in length. If the string, supposed to be stretched tight, makes with the ground an angle of  $53^\circ$ , how high is the kite above the ground?

10. At a distance of 150 feet from the foot of a church tower, the elevation of the top of the tower was observed to be  $43^\circ$ . Find the height of the tower.

11. The sides of a rectangle are 8 and 10 inches. Find the angles made by a diagonal with a shorter side.

12. Find by tables the angles between the diagonals of a rectangle with sides 5 and 3 inches.

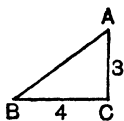
13. The sides of a right-angled triangle  $ABC$  containing the right angle are 3 and 4 units.

(1) Find  $\angle B$  from tan tables.

(2) Find the cos of this angle.

(3) Hence find  $AB$ .

(4) Find the value of  $AC^2 + BC^2$  and compare it with  $AB^2$ .

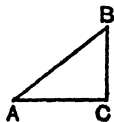


14. Make any acute angle  $A$ . Find the sin and cos of the angle from tables, and hence obtain the value of  $\sin^2 A + \cos^2 A$ .

15. Make any acute angle  $A$ . Find the sin and cos by measurement and calculation and hence obtain the value of

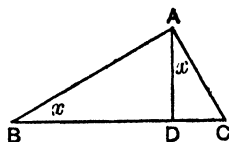
$$\sin^2 A + \cos^2 A.$$

16. Obtain the result  $\sin^2 A + \cos^2 A = 1$  by using the result  $AC^2 + BC^2 = AB^2$  and dividing both sides by an appropriate quantity.



17.  $\angle BAC$  is a right angle, and  $AD$  is perpendicular to  $BC$ .

Note that  $\angle ABD = \angle CAD$ .



- (1) Find the values of  $\sin x$  in  $\triangle ADC$  and  $\triangle ABC$  in terms of the sides. Hence prove  $DC \cdot BC = AC^2$ .
- (2) Find the values of  $\cos x$  in  $\triangle ABD$  and  $\triangle ABC$  in terms of the sides. Hence prove  $BD \cdot BC = AB^2$ .
- (3) Add results of (1) and (2) and hence prove that  $AC^2 + AB^2 = BC^2$ .

**Note.**

This gives an additional proof of the fact that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

49. The following gives an alternative proof of the fact proved in paragraph 45 and in the last example of the previous exercise.

Make the squares on the sides  $BC$ ,  $AB$ ,  $AC$ .

Join  $KC$ ,  $AD$ .

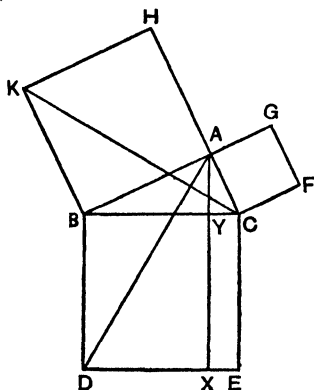
Draw  $AYX$  parallel to  $BD$ .

*Steps in the proof.*

- (1) Square  $HB =$  twice  $\triangle KBC$ . Why?
- (2) Rect.  $BX =$  twice  $\triangle ABD$ . Why?
- (3)  $\triangle KBC = \triangle ABD$ .—See below.
- (4)  $\therefore$  Sq.  $HB =$  rect.  $BX$ .

Similarly

- (5) Sq.  $AF =$  rect.  $YE$ .
- (6)  $\therefore AB^2 + AC^2 = BC^2$ .



Why is  $\triangle KBC$  equal to  $\triangle ABD$ ? Imagine the  $\triangle ABD$  pivoted round about  $B$  until  $AB$  takes the position  $KB$ . Then  $BD$  will take the position  $BC$  because we have turned these sides through a right angle.

The following are familiar examples of right-angled triangles:

(1) With sides 3, 4, 5 units, since  $5^2 = 3^2 + 4^2$ .

(2) With sides 5, 12, 13 units, since  $13^2 = 5^2 + 12^2$ .

Other combinations giving right-angled triangles are 7, 24, 25; 9, 40, 41.

This is called the *Theorem of Pythagoras* from its discoverer. The converse is equally true. If, in a triangle,  $c^2 = a^2 + b^2$ , where  $a$ ,  $b$  and  $c$  are the lengths of the sides, then the triangle is right-angled, and  $C$  is the right angle.

Ex. 1. The sides of a right-angled triangle which contain the right angle are 8·4 in. and 1·3 in. Find the length of the hypotenuse.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (8\cdot4)^2 + (1\cdot3)^2 \\ &= 70\cdot56 + 1\cdot69 \\ &= 72\cdot25, \\ \therefore c &= \sqrt{72\cdot25} = 8\cdot5 \text{ in.} \end{aligned}$$

Ex. 2. The hypotenuse of a right-angled triangle is 6·1 in., and one side is 1·1 in. Find the length of the other side.

$$\begin{aligned} c^2 &= a^2 + b^2, \\ \therefore a^2 &= c^2 - b^2 \\ &= (6\cdot1)^2 - (1\cdot1)^2 \\ &= 37\cdot21 - 1\cdot21 \\ &= 36, \\ \therefore a &= 6 \text{ in.} \end{aligned}$$

50. To obtain an approximate answer by construction we proceed as follows:

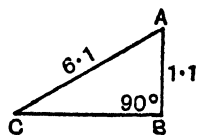
Draw a rough figure as shown.

To construct the correct figure first make  $AB$  1·1".

Draw  $BC$  at right angles to it.

With centre  $A$  and radius 6·1" cut  $BC$  at  $C$ .

Then  $ABC$  is the required triangle, and  $BC = 6$  in.



Ex. 3. Find graphically  $\sqrt{7}$ .

We require to find two squares of which the sum or difference is 7.

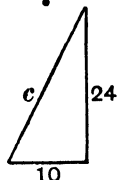
$$7 = 16 - 9.$$

If therefore we make a triangle with hypotenuse 4 and one side 3, the square of the other side will be 7,

$$\therefore \text{the other side} = \sqrt{7}.$$

Ex. 4. A ladder with its foot 10 feet from the wall of a house just reaches a window 24 feet above the level of the ground. Find its length.

$$\begin{aligned} c^2 &= 24^2 + 10^2 \\ &= 576 + 100 \\ &= 676, \\ \therefore c &= \sqrt{676} = 26. \end{aligned}$$



The ladder is therefore 26 feet long.

The following example illustrates the arithmetical method of finding the square root of a number when the factor method is impossible.

Ex. 5. Find  $\sqrt{357}$  to 2 places.

$$\begin{array}{r} 3\overline{)57\cdot00\mid00\mid00}(18\cdot894 \\ \underline{1} \\ 28\overline{)257} \\ \underline{224} \\ 368\overline{)3300} \\ \underline{2944} \\ 3769\overline{)35600} \\ \underline{33921} \\ 37784\overline{)167900} \end{array}$$

Sq. root = 18.89 to 2 places.

*Method.*

(1) Mark off the figures in groups of 2, starting from the decimal point.

(2) Find the sq. root of the first group, viz. 3. This is 1.

(3) Square it and subtract result from 3. Bring down the next group. The result is 257.

(4) Double the number now in the sq. root, i.e. 1, and place it to the left of 257. By the side of this result, viz. 2, and by the side of the first figure in the quotient place such a number—here 8—that  $28 \times 8$  will give the largest possible quantity to take from 257. Subtract.

(5) Repeat process (4), i.e. double 18 and place 36 to the left of the second remainder 3300. By the side of 36 and 18 place 8 because  $8 \times 368$  gives the largest possible quantity to take from 3300.

(6) Proceed until the required number of places is obtained.

#### EXERCISE 24.

If  $a$ ,  $b$  and  $c$  are the sides of right-angled triangles in which  $C$  is the right angle, find to the first place of decimals the missing side.

	$a$	$b$	$c$
1.	$2\frac{1}{2}''$	$6''$	
2.	$1\frac{1}{2}''$		$2\frac{1}{2}''$
3.		$1\frac{1}{4}''$	$3\frac{1}{4}''$
4.		$12''$	$12\frac{1}{2}''$
5.	3 cm.	$13\frac{1}{2}$ cm.	
6.	$8''$	$9''$	
7.		3.3 cm.	3.9 cm.
8.	$2.3''$	$3.5''$	
9.	231 mm.	309 mm.	
10.		5.37	$10.74''$ .

Construct graphically using  $1''$  for unit:

11.  $\sqrt{3}$ .      12.  $\sqrt{5}$ .      13.  $\sqrt{7}$ .      14.  $\sqrt{11}$ .

15. The sides of a rectangle are 20 ft. and 25 ft. Find the length of the diagonal.

16. The side of a square is 13 cm. What is the length of the diagonal to the nearest mm.?

17. Find the side of a square which has a diagonal of 10 in., to the nearest tenth of an inch.

18. A ladder 29 ft. long just reaches a window 20 ft. above ground. How far out is the foot?

19. Find the height of an isosceles  $\Delta$  which has a base 5" and two sides  $6\frac{1}{2}$ ".

20. What is the slant edge of a cone which has a vertical height of 17" and a base 9" in diameter?

21. Which of the following  $\Delta$ s are right-angled?

(a) Sides 2",  $2\frac{1}{2}$ ",  $1\frac{1}{2}$ ".

(b) Sides 5 cm., 12 cm., 14 cm.

(c) Sides 6",  $1\frac{3}{4}$ ",  $6\frac{1}{4}$ ".

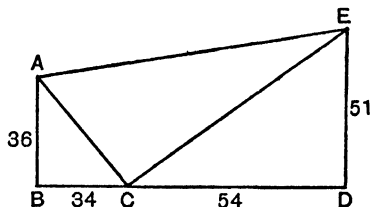
22. Construct a line whose length is  $2\sqrt{3}$  cm.

23. The diagonals of a rhombus are 5" and 8" in length. Calculate the length of a side.

24. The sides of a quadrilateral  $ABCD$  are  $AB$  16 cm.,  $BC$  12 cm.,  $CD$  29 cm. and  $DA$  21 cm. If  $\angle ABC$  is a right angle, prove that  $\angle CAD$  is also a right angle.

25. From the vertex  $A$  of a triangle  $ABC$  a perpendicular  $AX$  is drawn to the base. Prove that  $AB^2 - AC^2 = BX^2 - XC^2$ .

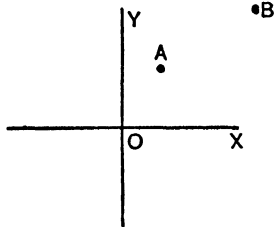
26. In the figure, which is not drawn to scale,  $AB$  and  $ED$  are perpendicular to  $BD$ . The sides being as shown, calculate  $AC^2$ ,  $CE^2$ ,  $AE^2$  and then prove that  $\angle ACE$  is a right angle.



27.  $A$  and  $B$  are points plotted on a graph.  $A$  is the point 2, 3 and  $B$  the point 7, 6.

Calculate the length of the line  $AB$ .

28. Similarly if  $A$  is the point  $-3, 4$  and  $B$  the point  $4, -3$ , calculate  $AB$ .



29. Is it possible to describe a quadrilateral with sides 8", 9", 1", 12" and with opposite angles right angles? If it is, calculate the length of one diagonal.

30.  $ABCD$  is a quadrilateral in which the angles at  $B$  and  $C$  are right angles,  $AB$  is  $1\frac{1}{2}$ ",  $BC$  is 2", and  $CD$  is 3". Calculate the length of  $AD$ .

31. A ladder rests with one end on the ground and the other projecting over the top of a wall 36 ft. high. The end on the ground is 27 ft. away from the wall, and the ladder projects 15 ft. of its length beyond the top of the wall. Find by calculation the greatest distance from the wall at which one end may rest on the ground so that the other reaches to the top of the wall.

32. The squares on the sides of a triangle are of areas 1 sq. in., 1 sq. in. and 2 sq. in. respectively. *Calculate* (without drawing an accurate figure) (i) the angles, (ii) the area of the triangle.

33. A rectangular field with an area of 30,618 sq. yd. has a width of 94.5 yd. What is the length of a straight path crossing it from corner to corner?

34. A ladder 52 ft. long was placed so as to reach a window 48 ft. high. On turning the top of the ladder over to the other side of the street, keeping its foot in the same place, it reaches a window 20 ft. high. Find the width of the street.

35. The perimeter of a square field is a quarter of a mile. Draw a plan of the field to a scale of 1 inch to 1 chain. Calculate the length of the diagonal.

## SECTIONAL REVISION C

## EXERCISE 25 (a). MENTAL.

1. What is the formula for the area of a parallelogram?
2. The hypotenuse of a rt.-angled triangle is  $2\frac{1}{2}$ " long and one side is  $1\frac{1}{2}$ ". Find the other side.
3. Calculate the area of the quadrilateral from the dimensions given (Fig. (1)).

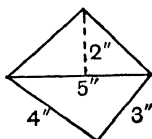


Fig. (1)

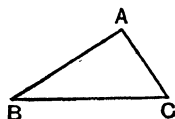
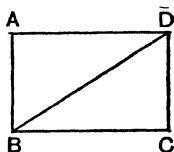
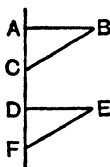


Fig. (2)

4. How would you bisect the triangle  $ABC$  by a line drawn through the vertex? (Fig. (2).)
5. What facts do you know about the diagonals of a rhombus?
6. Why is  $\triangle DBC$  half the area of the rectangle  $ABCD$ ?

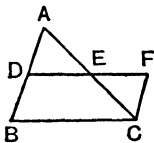


7. If half the vertical angle of an isosceles triangle =  $25^\circ$ , what is each base angle?
8. A set-square  $ABC$  is slid down a ruler to the position  $DEF$ . Why is  $BC$  parallel to  $EF$ ?



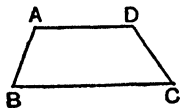
## EXERCISE 25 (b).

- Construct a triangle with sides 3", 3.5", 4.5", and find its area.
- Calculate the area of a trapezium with parallel sides 8.52 cm. and 5.48 cm. which are 5 cm. apart.
- Make a parallelogram with sides 4", 3", and an angle of 78°. Then construct a rectangle equal to it in area.
- Copy the parallelogram in Question 3 and reduce it to an equal triangle.
- Bisect the two sides  $AB$ ,  $AC$  of a  $\triangle$  at  $D$  and  $E$ .  
Produce  $DE$  to  $F$  making  $EF = DE$ . Join  $FC$ .  
Prove  $\triangle$ 's  $ADE$ ,  $CFE$  congruent.
- From the results in Question 5, how do you know that  $DBCF$  is a parallelogram?
- What is the slant edge of a cone if the vertical angle is 60° and the diameter of the base is 4"?
- State the three main sets of conditions for the congruence of two triangles.



## EXERCISE 26 (a). MENTAL.

- Give the formula for the area of a triangle with base  $b$  and vertical height  $h$ . In how many ways can the area of a triangle thus be calculated?
- The sides of a triangle are in the ratio  $\frac{3}{4} : 1 : 1\frac{1}{4}$ . How do you know it is right-angled?
- What is the area of a trapezium in which the parallel sides are 1.56" and 2.44" and the distance between them 4"?
- How would you divide the trapezium  $ABCD$  into a parallelogram and a triangle?
- What facts do you know about the sides and angles of a parallelogram?



6. If  $\angle BAD = 118^\circ$  and  $\angle ADC = 108^\circ$ , and  $\angle$ s at  $B$  and  $C$  are bisected by  $BX$  and  $CX$ , calculate  $\angle BXC$ . (Fig. (1).)

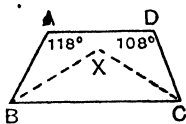


Fig. (1)

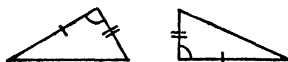


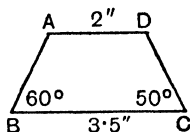
Fig. (2)

7. If these  $\Delta$ s are accurately constructed, are they congruent and why? (Fig. (2).)

8. What is the angle between the hands of a clock at 6.30?

## EXERCISE 26 (b).

1. Find by measurement and calculation the area of the trapezium shown in the rough diagram.



2. Calculate the area of a parallelogram with sides  $5''$ ,  $4''$ , and one angle  $150^\circ$ .

3. Make any parallelogram, and then construct another equal to it in area with a side twice as great as the longer side of the first.

4. Copy the trapezium in Question 1 and reduce it to an equal triangle.

5. If  $\angle ABC = \angle DCB$  and  $\angle BAD = \angle ADC$  (Fig. (3)), prove  $AB = DC$ .

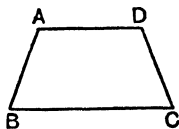


Fig. (3)

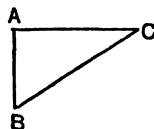


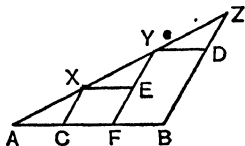
Fig. (4)

6.  $ABC$  is a rt.-angled triangle (Fig. (4)).  $AB = \frac{1}{3}BC$  and  $AC = 4''$ . Calculate  $AB$  to 2 places of decimals.

7. Prove that the bisectors of the angles of a triangle meet at a point.

8. Prove the following method for the trisection of  $AB$ .

Draw any line  $AZ$  and along it cut off equal portions  $AX$ ,  $XY$ ,  $YZ$ . Draw  $XC$ ,  $YF$  parallel to  $ZB$ .



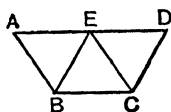
### EXERCISE 27 (a). MENTAL.

1. Give the formula for the area of a trapezium if  $x$  and  $y$  are the lengths of the parallel sides and  $d$  the distance between them.

2. If the side of an equilateral triangle is 4 inches long, find the area of the square on the altitude.

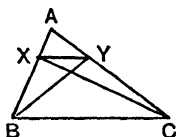
3. If  $\sqrt{3} = 1.73$  to 2 places, calculate the area of the triangle in Question 2 correct to 1 place of decimals.

4. If  $ABCE$ ,  $EBCD$  are parallelograms and  $AED$  is a straight line, how do you know at once that  $ABCE = EBCD$ ?



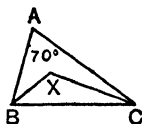
5. What fact do you know about the exterior angle of any triangle?

6.  $XY$  is drawn parallel to  $BC$ .



How do you know that  $\triangle AYB = \triangle AXC$ ?

7.  $\angle BAC = 70^\circ$  and  $\angle s$  at  $B$  and  $C$  are bisected by  $BX$ ,  $CX$ . Calculate  $\angle BXC$ .



8. What conditions must hold between the parts of two right-angled triangles if they are to be congruent?

## EXERCISE 27 (b).

1. Find the area of the parallelogram which has a base 8.5 cm., one side 4.7 cm. and base angles  $115^\circ$  and  $65^\circ$ .

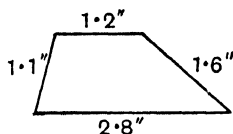
2. Calculate the area of a triangle with sides 5.2", 5.7", 6.1".

3. Make the triangle indicated in Question 2 and then construct a triangle equal to it on a base twice as great.

4. Make a triangle with a base 4" and base angles  $65^\circ$ ,  $78^\circ$  and construct a rectangle equal to it in area.

5. A right-angled triangle has an angle of  $45^\circ$ , and one side is 4" long. Calculate the length of the hypotenuse to two places of decimals.

6. Construct the trapezium indicated in the diagram and measure its altitude.



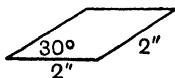
7. Reduce the figure to an equal triangle and then make a rectangle equal to it.

8. What is the size of each angle of a regular polygon of 120 sides?

## EXERCISE 28 (a). MENTAL.

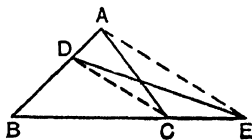
1. Give the formula for finding the area of a triangle given the length of the three sides, and say what each letter stands for.

2. What is the area of this rhombus?



3. The hypotenuse of a right-angled triangle is 13 cm., and one side is 12 cm. What is the length of the other side?

4. If  $AE$  is parallel to  $DC$ , why is the triangle  $ABC$  equal in area to the triangle  $DBE$ ?



5. What fact do you know about the diagonals of any parallelogram?

6. If these  $\Delta$ s are constructed accurately, are they congruent and why?



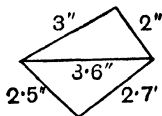
7. What is the vertical angle of a triangle if half the sum of the base angles is  $55^\circ$ ?

8. What are  $\angle$ s 1 and 2 called with respect to one another?



### EXERCISE 28 (b).

1. Find by measurement and calculation the area of the quadrilateral shown in the rough diagram.



2. Calculate the area of a rhombus which has a side of  $5''$  and one diagonal  $6''$ .

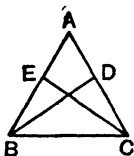
3. Make a triangle with sides  $2.5''$ ,  $2.9''$ ,  $3.7''$ , the last being the base, and then construct a triangle equal to it in area on a base  $4.5''$ .

4. Construct any five-sided rectilinear figure and reduce it step by step to an equal triangle.

5. An isosceles triangle has two sides  $5.5$  cm. long and a base  $4.8$  cm. Calculate the altitude and hence the area.

6. Construct a rectangle with sides  $3.5''$  and  $2.8''$ . Make a rectangle equal to it on a base half the length.

7.  $ABC$  is equilateral and  $BD$ ,  $CE$  two altitudes. Prove that  $BD = CE$ .



8. The angles of a quadrilateral are  $2x - 5^\circ$ ,  $3x + 25^\circ$ ,  $4x + 75^\circ$ , and  $175^\circ$ . Find  $x$ .

## THE CIRCLE

**51.** The essential feature of a circle is that every point on the circumference is the same distance from the centre. This distance is the radius.

The perpendicular to a chord from the centre of a circle bisects the chord.

Let  $XY$  be the chord,  $O$  the centre,  $OP$  the perpendicular.

We want to prove  $PX = PY$ .

At a glance we see that the hypotenuse and one side are equal in the two triangles.

$$\therefore PX = PY.$$

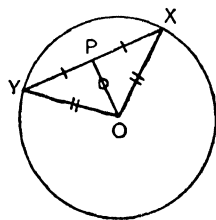
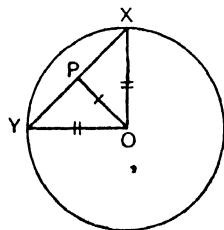
**Conversely** if a line  $OP$  from the centre bisects  $XY$  then  $OP$  is at right angles to  $XY$ .

In this case the  $\Delta$ s  $OPX$ ,  $OPY$  have 3 sides equal in each.

$$\therefore \angle OPX = \angle OPY.$$

But together they make two right angles,

$$\therefore \text{each is a right angle.}$$



**Learn:**

*A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord.*

*The perpendicular to a chord from the centre bisects the chord.*

**52. Ex. 1.** A chord  $3''$  long is drawn in a circle of radius  $2\frac{1}{2}''$ . Find the distance from the centre.

$OP$  bisects  $AB$ .

$$\therefore PB = 1\frac{1}{2}''$$

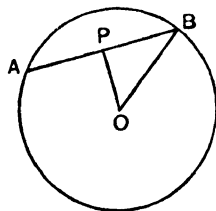
$$OB = 2\frac{1}{2}''.$$

$$OP^2 = OB^2 - PB^2$$

$$= (2\frac{1}{2})^2 - (1\frac{1}{2})^2$$

$$= \frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4.$$

$$\therefore OP = 2''.$$



Ex. 2. A chord of a circle of radius  $6\frac{1}{2}$ " is 6" from the centre. Find its length.

Using the former figure,

$$OB = 6\frac{1}{2}" , \quad OP = 6"$$

$$\therefore PB^2 = OB^2 - OP^2$$

$$= (6\frac{1}{2})^2 - (6)^2 = \frac{169}{4} - 36 = \frac{169 - 144}{4} = \frac{25}{4} .$$

$$\therefore PB = \frac{5}{2} .$$

Hence the chord = 5".

Ex. 3. A chord 6" long is 4" from the centre. Find the radius.

$$\text{Here } PB = 3, \quad OP = 4 .$$

$$\therefore OB^2 = 9 + 16 = 25 ,$$

$$OB = 5 .$$

From Ex. 1 we see that wherever the chord of length 3" is drawn in the circle the distance from the centre is the same.

From Ex. 2 we see that all chords 6" from the centre must be of the same length.

**Learn:**

*Equal chords in a circle are equally distant from the centre.*

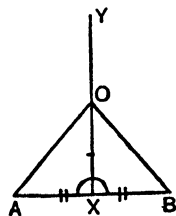
*Chords equally distant from the centre of a circle are equal.*

53. The centres of all circles passing through two points lie on the perpendicular bisector of the line joining the two points. This is proved as follows:

Let  $AB$  be the line joining the two points and  $XY$  the perpendicular bisector.

Take any point whatever on  $XY$  and join  $OA$ ,  $OB$ .

The triangles  $OXA$ ,  $OXB$  are congruent, having 2 sides and the contained angle equal in each.



$$\therefore OA = OB .$$

This is true for any point  $O$ .

$\therefore$  with centre  $O$  and radius  $OA$  a circle can be described passing through  $A$  and  $B$ .

**Learn:**

*The centres of all circles passing through any two points lie on the perpendicular bisector of the line joining the two points.*

54. Ex. 1. Describe a circle passing through three given points  $A, B, C$ .

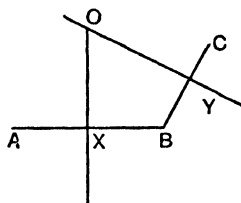
The circle must pass through  $A$  and  $B$ .

$\therefore$  the centre lies on the perp.-bisector  $OX$ .

Also the circle passes through  $B$  and  $C$ .

$\therefore$  the centre lies on  $OY$ .

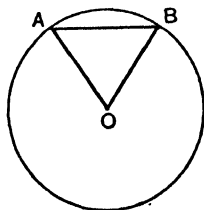
Hence the centre must be at  $O$ , and there is only one such centre.



This gives us a method of constructing a circle passing through the vertices of a triangle.

Ex. 2. Construct a circle of radius  $1''$  passing through two points  $\frac{1}{2}''$  apart.

From the rough figure we see that in order to find the centre of the circle we must make the triangle  $ABO$  with  $AB \frac{1}{2}''$ ,  $AO 1''$ ,  $BO 1''$ .



Then with centre  $O$  and radius  $1''$  the circle can be described passing through  $A$  and  $B$ .

Ex. 3. Construct a circle passing through two points and having its centre on a given line.

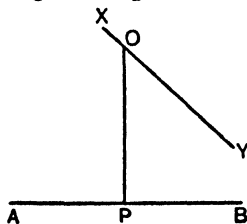
Let  $A, B$  be the points,  $XY$  the line.

The centre must lie on the perpendicular bisector of  $AB$ .

i.e. it lies on  $OP$ .

This line cuts  $XY$  at  $O$ .

$\therefore O$  is the centre of the circle.



## EXERCISE 29.

1. A chord 5" long is 6" from the centre. Find the radius of the circle.

2. How far from the centre of a circle of radius 17 cm. is a chord 15 cm. in length?

3. What is the length of a chord in a circle if the radius is 18 cm. and the distance of the chord from the centre 8 cm.?

4.  $XY$  is a chord 5" long in a circle of radius  $6\frac{1}{2}$ ". If  $O$  is the centre of the circle, what is the area of the triangle  $OXY$ ?

5. Two points  $A$  and  $B$  are 5 cm. apart. Draw a circle of radius 3 cm. passing through  $A$  and  $B$ .

6. A chord of a circle is 2.3" long and the distance from the centre is 1.2". Construct the circle and measure and calculate the radius.

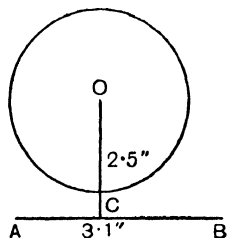
7. Three points  $A, B, C$  are such that  $AB = 2.3"$ ,  $BC = 1.8"$ , and  $CA = 1.7"$ . Describe a circle passing through the three points.

8. Describe a triangle with sides 3, 4 and 5 inches and construct the circumscribing circle.

9. Draw a straight line  $AB$  3.1" and fix a point  $O$  so that  $OC$  is perpendicular to  $AB$  and of length 2.5", and  $AC$  1.2".

With centre  $O$  and radius 1.8" describe a circle.

Construct a circle passing through  $A$  and  $B$  with its centre on the given circle.



10. Draw a circle of radius 1.8". Fix a point  $A$  1.3" from the centre and through  $O$  draw a chord 3" long.

11. The length intercepted on a certain straight line by a circle of radius 3" is 5". What will be the length intercepted on the same straight line by a concentric circle of radius 5"?

12.  $O$  is the mid-point of the line joining the centres of two equal intersecting circles, and  $P$  is one of the points in which they intersect. If the radius of each is 3" and the line joining the centres is 5", calculate the length of  $OP$ .

13.  $O$  is the centre of a circle circumscribing the triangle  $ABC$ ; prove that if  $AO$  bisects  $BC$ , the angle  $ABO$  equals the angle  $ACO$ .

14. Prove that the curve joining all the mid-points of equal chords of a circle is a concentric circle, and say how you would discover its radius.

**55. Angles in a Circle.**

$AOB$  is an angle at the centre of a circle standing on the arc  $AB$ .  
 $ACB$  is an angle at the circumference standing on the same arc.  
 It can be proved that  $\angle AOB = \text{twice } \angle ACB$ .

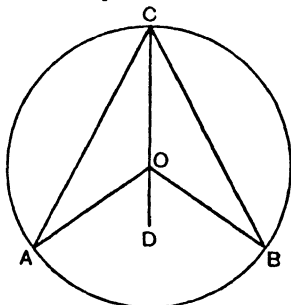


Fig. (1)

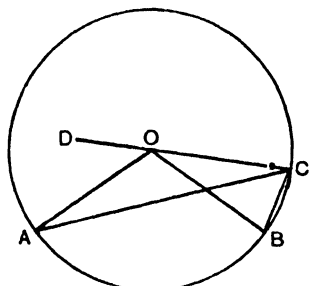


Fig. (2)

**Proof.**  $OAC$  is an isosceles triangle.

$$\therefore \angle OAC = \angle OCA.$$

Hence any angle equal to the sum of these two angles is twice either of them.

$\angle AOD$  is such an angle, for the exterior angle of any triangle equals the sum of the two interior opposite angles.

$$\therefore \angle AOD = 2 \angle ACO. \quad \dots\dots(a)$$

In the same way  $\angle BOD = 2 \angle BCO. \quad \dots\dots(b)$

In Fig. (1), by adding (a) and (b) we get  $\angle AOB = 2 \angle ACB$ .

In Fig. (2) we subtract and get the same results.

**Learn:**

*The angle at the centre of a circle is twice the angle at the circumference standing on the same arc.*

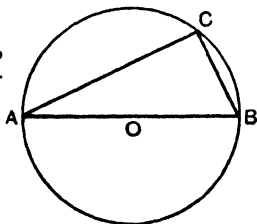
**56. Particular Cases and Extensions.**

(1) Suppose  $\angle AOB$  is a straight angle, i.e.  $AOB$  is a diameter and  $ACB$  is a semi-circle.

It is still true that  $\angle AOB = 2 \angle ACB$ .

$$\text{But } \angle AOB = 2 \text{ rt. } \angle \text{ s.}$$

$$\therefore \angle ACB \text{ is a right angle.}$$



**Learn:**

*The angle in a semi-circle is a right angle.*

(2) In Fig. (1)  $\angle ACB$  is obtuse and the reflex  $\angle AOB$ , i.e. an angle more than 2 rt.  $\angle$ s, is twice  $\angle ACB$ .

The same proof as before will hold good.

$$\text{For } \angle AOD = 2 \angle ACD.$$

$$\text{And } \angle BOD = 2 \angle BCD.$$

$$\therefore \text{Reflex } \angle AOB = 2 \angle ACB.$$

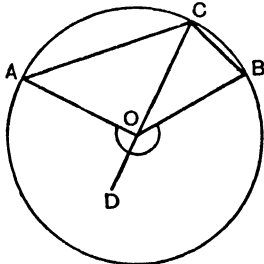


Fig. (1)

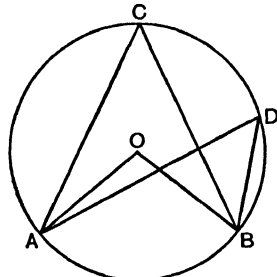
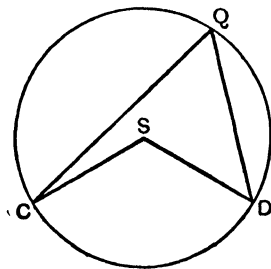
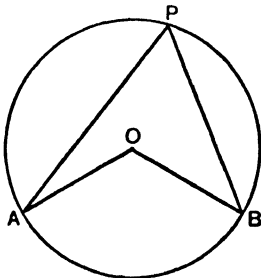


Fig. (2)

(3) In Fig. (2) we have two  $\angle$ s  $ACB, ADB$  on the same arc  $AB$ , i.e. in the same segment  $ACDB$ .

$$\angle AOB = \text{twice either angle.} \quad \therefore \angle ACB = \angle ADB.$$

(4) The two circles in the figures given below are of equal radii and arc  $AB = \text{arc } CD$ .



Hence  $\angle APB$  and  $\angle CQD$  stand on equal arcs, or are in equal segments.

It is possible to place one circle on the other so that arcs  $CD, AB$  coincide, and the radii also coincide.

$\angle APB$  and  $\angle CQD$  will be in the same segment.

$$\therefore \angle APB = \angle CQD.$$

(5) Let  $AB$  and  $CD$  be equal arcs in the same circle.

We can imagine  $OC$  to be revolved about the centre  $O$  so that the equal arcs  $CD$ ,  $AB$  coincide.

$$\therefore \angle AOB = \angle COD.$$

Similarly the angles at the circumference standing on these equal arcs will be equal.

(6)  $ABCD$  is a quadrilateral in a circle.

$$\angle BOD = 2 \angle BCD.$$

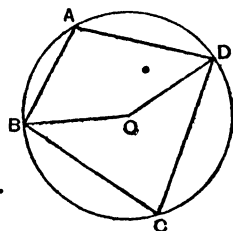
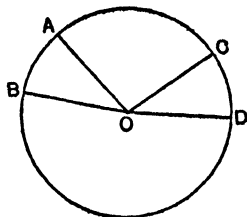
$$\text{Reflex } \angle BOD = 2 \angle BAD.$$

Adding, we get:

$$\begin{aligned} \angle BOD + \text{Reflex } \angle BOD \\ = 2 (\angle BCD + \angle BAD). \end{aligned}$$

$$\therefore 4 \text{ rt. } \angle s = 2 (\text{opp. } \angle s \text{ of the quadl.).}$$

$\therefore$  Opposite angles of a quadrilateral inscribed in a circle (or cyclic quadrilateral) are equal to two right angles.



**Learn :**

1. *Angles in the same segment of a circle are equal.*

**Or** *Angles at the circumference of a circle which stand on the same arc are equal.*

2. *In equal circles angles, whether at the centre or the circumference, which stand on equal arcs are equal.*

3. *Angles at the centre or circumference of a circle which stand on equal arcs are equal.*

4. *The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.*

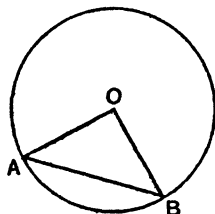
**EXERCISE 30.**

1. If  $\angle AOB$  at the centre is  $102^\circ$ , calculate

(1)  $\angle OAB$ ,

(2) Reflex  $\angle AOB$ ,

2. Using the figure of Question 1, what is the size of the angle  $APB$  at the circumference standing on the arc  $AB$ ?



3. If the point  $P$  is in the minor segment of this circle, what is the angle  $APB$ ?

4. If  $\angle APB$  in Fig. (1) is  $72^\circ$ , find

(1) Reflex  $\angle AOB$ ,  $O$  being the centre.

(2)  $\angle ARB$ .

(3)  $\angle OAB$ .

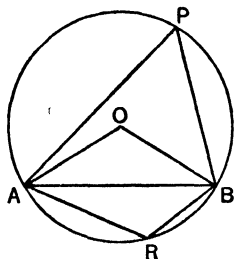


Fig. (1)

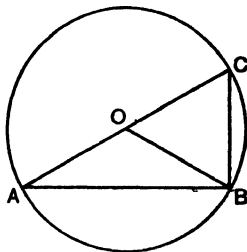


Fig. (2)

5. If  $\angle AOB$  in Fig. (2) is  $118^\circ$ , and  $AOC$  is a straight line, find

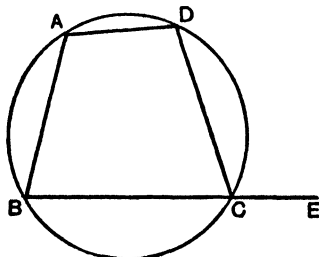
(1)  $\angle OAB$ .

(2)  $\angle OBC$ .

6. Using the figure of Question 5, if  $\angle OBA$  is  $27^\circ$  find  $\angle ACB$ .

7. Using the same figure, if  $\angle OBC$  is  $61^\circ$  find  $\angle AOB$ .

8. If  $\angle BAD$  in this quadrilateral is  $105^\circ$ , find  $\angle DCE$ .



9. Two adjacent angles of a cyclic quadrilateral are  $71^\circ$  and  $89^\circ$ . What are the other two?

10. If the median (i.e. the line from the vertex bisecting the base) of a triangle is half the base, show that the triangle must be right-angled.

11.  $AB$  and  $CD$  are parallel chords in a circle. Show that  $AD = BC$ .

12.  $AB$  is a diameter of a circle and  $CD$  a chord at right angles to it cutting it at  $E$ . If another chord  $AFG$  is drawn cutting  $CD$  at  $F$ , prove that a circle can be drawn to pass through  $B, E, F$ , and  $G$ .

13. Show how to construct a triangle  $ABC$ , given the length of  $BC$ , the angle at  $A$ , and the area of the triangle. (Use the fact that the angle  $BOC$  at the centre of the circumscribing circle is twice  $\angle A$ .)

14. Construct a quadrilateral  $ABCD$ , which can be inscribed in a circle, and is such that  $AB$  is 2", the angle  $ABC$  is a right angle, the angle  $BAD$  is  $70^\circ$ , and the side  $DC$  produced meets  $AB$  beyond  $B$  at a point distant 1" from  $B$ .

15. Construct a right-angled triangle with a hypotenuse 4 inches long and a side 2 inches long. Explain your construction and calculate the area of the triangle.

16. A hexagon is inscribed in a circle; prove that the sum of three alternate angles of the hexagon is equal to four right angles.

17.  $ABCDE$  is a pentagon inscribed in a circle and all its angles are equal; prove that all its sides are equal.

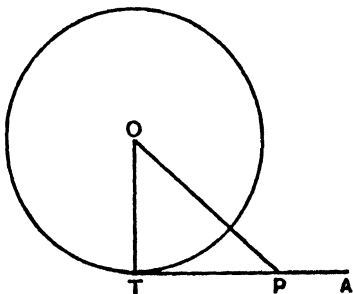
### 57. Tangents.

A tangent to a circle touches it at one point only.

Let  $AT$  be such a tangent.

$OT$  is obviously the shortest distance from  $O$  to  $AT$ , for any other line  $OP$  must be greater than a radius.

But the shortest distance from a point to a line is the perpendicular distance.



$\therefore \angle OTA$  is a rt.  $\angle$ .

Hence to draw a tangent at  $T$  we simply make  $\angle OTA$  a rt.  $\angle$ .

### Learn:

*The tangent at any point of a circle and the radius through the point are perpendicular to one another.*

**58.** To draw a tangent to a circle from a point outside it.

Draw a rough figure showing the tangent  $AT$  and the radius  $OT$ . We know

(1) That  $\angle ATO$  must be a rt.  $\angle$ .

(2) That the angle in a semi-circle is a right angle.

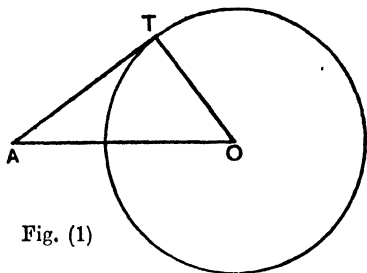


Fig. (1)

Hence we make a semi-circle on  $AO$ , and where this cuts the circle at  $T$  we get the point of contact of a tangent  $AT$ . (Fig. (1).)

In the diagram Fig. (2), it is seen that the semi-circle may be constructed on either side of the diameter.

Hence both  $S$  and  $T$  can be points of contact of tangents from  $A$ .

Note too that these tangents are equal, because  $\Delta s AOT, AOS$  are congruent. Why?

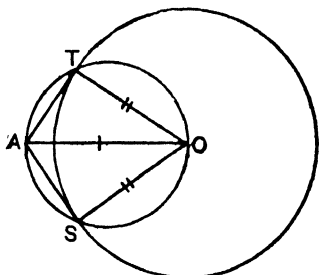


Fig. (2)

**Learn:**

- (1) *Two tangents can be drawn from a point outside a circle.*
- (2) *The tangents to a circle from an external point are equal.*

**Ex. 1.** Find the length of the tangent to a circle of radius 3'' from a point 5'' from the centre.

Draw a rough figure.

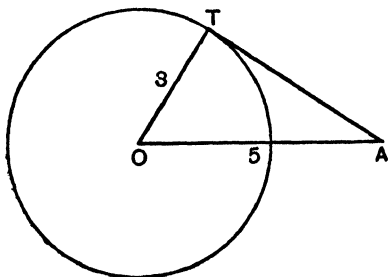
$\angle OTA$  is a rt.  $\angle$ .

$$\therefore AT^2 = OA^2 - OT^2$$

$$= 25 - 9$$

$$= 16.$$

$$\therefore AT = 4 \text{ inches.}$$



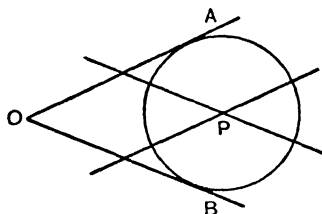
Ex. 2. If a tangent to a circle from an external point is 7'' long and the radius is 3'', how far is the point from the centre?

Here  $OA$  is required.

$$\begin{aligned} OA^2 &= OT^2 + TA^2 \\ &= 3^2 + 7^2 = 9 + 49 = 58. \end{aligned}$$

$$\therefore OA = \sqrt{58} = 7.6''.$$

Ex. 3. Describe a circle of 1'' radius touching two given lines.



Draw lines parallel to  $OA$  and  $OB$  1'' away. Then  $P$  is the centre.

### 59. Common Tangents to Two Circles.

#### *Direct common tangents.*

We see from the rough figure that at least two common tangents can be drawn to two circles, as long as one does not lie inside the other.

Those in the figure are called Direct Common Tangents.

Consider the tangent  $AB$ .

We have  $AX$  and  $BY$  both at rt.  $\angle$ s to  $AB$ .

By drawing  $SY$  at rt.  $\angle$ s to  $AS$ , we form a rectangle and a right-angled  $\triangle XSY$ .

In the latter

$$\begin{aligned} SY &= \text{the common tangent } AB, \\ XY &\text{ is the line joining the centres of the circles,} \\ SX &= AX - AS \\ &= AX - BY \\ &= \text{the difference of the radii.} \end{aligned}$$

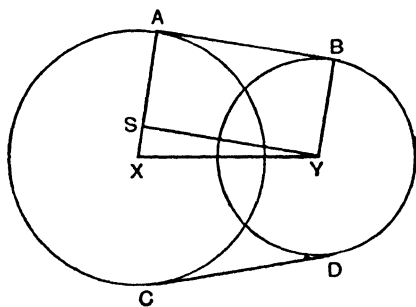


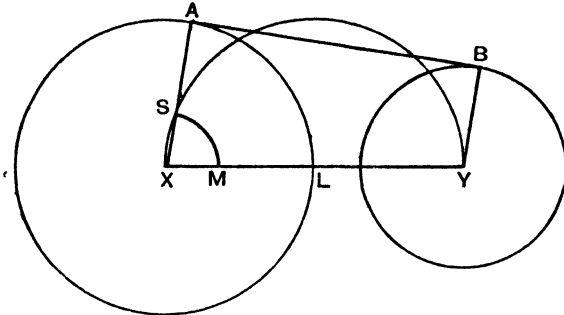
Fig. (1)

This gives us the method of construction.

Join  $XY$ , and on it describe a semi-circle.

The point  $S$  in the former figure must lie on this semi-circle.

Cut off  $XS$  equal to the difference of the radii.



Note that this can be done as follows:

(1) Make  $LM$  equal to the smaller radius.

(2) With centre  $X$  and radius  $XM$  cut the semi-circle at  $S$ .

Join  $XS$  and produce till it meets the circle at  $A$ .

This is one point on the common tangent.

Draw  $YB$  parallel to  $XA$ .

This gives the other point  $B$ .

Join  $AB$ .

No proof is required, because we have seen that the construction must inevitably produce the desired result.

*Transverse common tangents.*

From the rough figure we see that circles which do not touch or cut can have two transverse as well as two direct common tangents.

Consider the tangent  $AB$ .

It is at right angles to  $AX$  and  $BY$ .

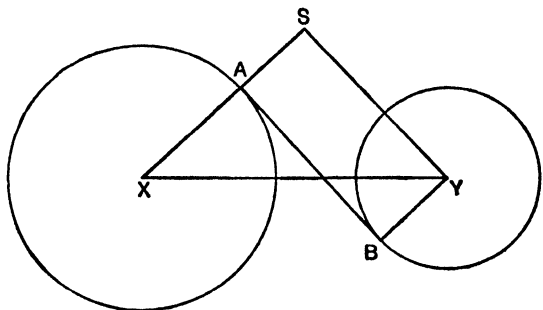


Fig. (2)

By drawing  $SY$  at right angles to  $XA$  produced, we form a rectangle and a right-angled  $\triangle XSY$ .

In the latter

$$\begin{aligned} SY &= \text{the common tangent } AB, \\ XY &\text{ is the line of centres,} \\ SX &= AX + AS \\ &= AX + BY \\ &= \text{the sum of the radii.} \end{aligned}$$

This gives us the method of construction, which is left as an exercise to the student.

### Lengths of Common Tangents.

#### 1. Direct common tangent.

In Fig. (1)

$$\begin{aligned} AB &= SY, \\ SY^2 &= XY^2 - SX^2 \text{ in the rt. } \angle \text{d. } \triangle XSY, \\ SY &= \sqrt{XY^2 - SX^2}, \\ t_1 &= \sqrt{c^2 - (a - b)^2}, \end{aligned} \quad \dots\dots(a)$$

where  $c$  is the line of centres and  $a, b$  the radii.

#### 2. Transverse common tangent.

In Fig. (2)

$$\begin{aligned} AB &= SY, \\ SY^2 &= XY^2 - SX^2, \\ SY &= \sqrt{XY^2 - SX^2}, \\ t_2 &= \sqrt{c^2 - (a + b)^2}. \end{aligned} \quad \dots\dots(b)$$

Results (a) and (b) should be learnt by heart.

### 60. Tangential Circles.

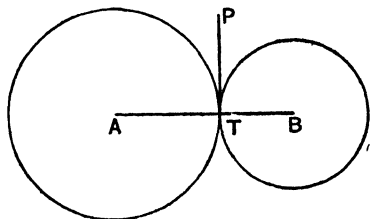
Let the two circles touch at  $T$ . There can only be one transverse common tangent.

Let it be  $TP$ .

$TP$  is at right angles to  $TB$  and to  $TA$ .

$$\therefore \angle ATP + \angle BTP = 2 \text{ rt. } \angle \text{s.}$$

$\therefore ATB$  is a straight line.



#### Learn:

*If two circles touch, the point of contact lies on the straight line through the centres.*

Ex. 1. Two circles of radii 8" and 3" have centres 10" apart. Find the lengths of the common tangents.

(1) *Direct.*

$$\begin{aligned} t &= \sqrt{c^2 - (a - b)^2} \\ &= \sqrt{100 - 25} \\ &= \sqrt{75} = 5\sqrt{3} = 8.7". \end{aligned}$$

(2) *Transverse.*

$$\begin{aligned} t &= \sqrt{c^2 - (a + b)^2} \\ &= \sqrt{100 - 121} \\ &= \sqrt{-21}. \end{aligned}$$

This is not a real quantity and so we cannot draw transverse common tangents. An approximate figure will show why this is so.

Ex. 2. Two circles of radii 6" and 2" have centres 3" apart. Can any tangents be drawn?

$$\begin{aligned} \text{(1) Try} \quad t &= \sqrt{c^2 - (a - b)^2} \\ &= \sqrt{9 - 16} \\ &= \sqrt{-7}. \end{aligned}$$

∴ No direct tangents can be drawn.

$$\begin{aligned} \text{(2) Try} \quad t &= \sqrt{c^2 - (a + b)^2} \\ &= \sqrt{9 - 64} \\ &= \sqrt{-55}. \end{aligned}$$

∴ No transverse tangents can be drawn. Of course not, because one circle lies wholly inside the other.

Ex. 3. How far apart are the centres of two circles of radii 5" and 4" if the direct common tangent is 10" long?

$$\begin{aligned} t &= \sqrt{c^2 - (a - b)^2}, \\ 10 &= \sqrt{c^2 - 1}, \\ 100 &= c^2 - 1, \\ c^2 &= 101, \\ c &= \sqrt{101}. \end{aligned}$$

## EXERCISE 31.

1. Draw a circle of radius  $1\cdot5''$ . Find a point  $P$   $3\cdot1''$  from the centre. From  $P$  draw a tangent to the circle.

2. Measure and calculate the length of this tangent.

3. What is the radius of a circle if the length of a tangent from a point  $2''$  from the centre is equal to  $1\cdot2''$ ?

4. Find the distance from the centre of a circle of the point from which a tangent  $2\cdot5''$  long can be drawn, if the radius of the circle is  $\cdot8''$  long. Verify by construction.

5. Two concentric circles are drawn of radii  $1\cdot5''$  and  $1\cdot9''$ . Find the length of a chord of the larger circle which touches the inner. Verify by construction.

6. The vertical angle of an isosceles triangle is  $60^\circ$ , and a circle of radius  $3''$  is inscribed in it. Find the distance of the centre of the circle from the vertex of the triangle.

7. Two lines  $AB, AC$ , each  $2\cdot8''$  long, are drawn enclosing an angle of  $78^\circ$ . Describe a circle which touches these lines at  $B$  and  $C$ .

8. If  $AP$  and  $AQ$  are tangents to a circle with centre  $O$ , prove  $\angle OAP = \angle OAQ$ .

9.  $O$  is the centre of a circle with radius  $2\cdot3''$ .  $P$  is an external point  $1\cdot2''$  radially from the circumference. Calculate to the nearest hundredth of an inch the length of the tangents from  $P$  to the circle.

10. Two circles touch at  $A$  and from  $A$  a common tangent is drawn to the two circles. If any point  $P$  is taken on this line and tangents are drawn to the two circles, prove that these tangents are equal.

11. Construct two direct common tangents to two circles of radii  $2''$  and  $1''$  with centres  $2\cdot5''$  apart.

12. Calculate the lengths of these tangents. Could transverse common tangents be drawn to these circles?

13. Construct two transverse common tangents to two circles of radii  $1\cdot5''$  and  $1\cdot2''$  with centres  $3\cdot5''$  apart.

14. Calculate the lengths of these tangents.

15. What is the length of the direct common tangents to circles of radii  $1\cdot5''$  and  $3\cdot5''$  if the circles are  $2''$  apart radially?

16. If the length of the transverse common tangent to two circles of radii  $3''$  and  $4''$  is  $1\cdot2''$ , calculate how far the centres are apart.

17. Two engine wheels on the same side of an engine have their axles 10 feet apart, and their radii are 2' 9" and 2' 2". What length of railway line do they cover?

18. A see-saw is constructed from a 10 ft. plank and a portion of a tree-trunk which has a radius of 1' 2". If the plank is fastened at its middle point to the trunk, what is the distance from the end of the plank to the centre of the trunk?

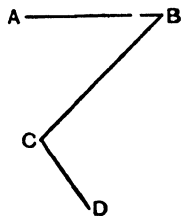
19. A sphere of radius 3 cm. is fitted into a cone whose semi-vertical angle is  $30^\circ$ . Find the distance of the centre of the sphere from the vertex of the cone.

20. Construct a circle of radius  $1\frac{1}{2}$ " to touch two circles whose radii are 2" and  $2\frac{1}{2}$ " and whose centres are  $3\frac{1}{2}$ " apart.

21. Draw a line  $AB$  4" long and at  $A$  erect a perpendicular  $AC$  3" long. Describe a circle to pass through  $A$ ,  $B$  and  $C$ . Produce  $AB$  to  $D$ , making  $BD$  2". From  $D$  draw tangents to the circle and measure them.

22. Construct a circle with radius 3.75 cm. Draw a diameter  $AOB$  where  $O$  is the centre of the circle. Bisect  $AO$  at  $C$ . Through  $C$  draw a line  $DC$  at right angles to  $AB$  and meeting the circle at  $D$  and  $E$ . Measure the length of the chord  $DE$ . At  $D$  and  $E$  draw tangents to the circle.

23. Draw three lines  $AB$ ,  $BC$ ,  $CD$  as shown in the figure, so that  $BC = 2$ ",  $\angle ABC = 70^\circ$ ,  $\angle BCD = 100^\circ$ . Connect  $AB$  and  $BC$  by a tangential arc of  $\frac{1}{2}$ " radius, and  $BC$ ,  $CD$  by a tangential arc of  $\frac{3}{4}$ " radius; and line in clearly the resulting figure.



### 61. The Circumference of a Circle.

It can be discovered by measurement that the ratio of the circumference of a circle to its diameter is always the same, however large the circle may be. This ratio is known as  $\pi$  (pi). Its value is approximately  $3\frac{1}{7}$ , or, more closely 3.14159.

$\therefore C = \pi D$ , where  $D$  is the diameter,

or  $C = 2\pi r$ , where  $r$  is the radius.

Ex. 1. Find the circumference of a circle of radius 21 inches.

$$C = 2\pi r$$

$$= 2 \times \frac{22}{7} \times 21 \text{ inches}$$

$$= 132 \text{ sq. in.}$$

Ex. 2. Find the radius of a circle of circumference 45 ft. 9 in., using  $\pi = 3.142$ .

$$C = 2\pi r.$$

$$\therefore r = \frac{c}{2\pi}$$

$$= \frac{45.75}{2 \times 3.142}$$

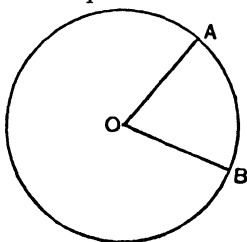
or

$$\begin{aligned} \text{Log } 45.75 &= 1.6604 \\ \text{Log } 6.284 &= .7983 \\ \text{Log } 45.75 - \text{Log } 6.284 &= .8621 = \log r. \\ \therefore r &= 7.28 \text{ ft.} \end{aligned}$$

$$\begin{array}{r} 45.75 \div 6.284 \\ \underline{7.280} \\ 6.284)45.750 \\ \underline{43.988} \\ 17620 \\ \underline{12568} \\ 50520 \\ \underline{50272} \\ 2480. \end{array}$$

### 62. The Meaning of a Radian.

Imagine an arc  $AB$  which, if stretched out into a straight line, would be found to be equal to the radius.



The angle between the two radii  $OA$  and  $OB$  is known as a *radian*.

We know that the circumference =  $2\pi r$ . Also  $AB$  has the same ratio to the circumference as the angle  $AOB$  has to four right angles.

$$\therefore \frac{\text{arc } AB}{\text{circumference}} = \frac{\angle AOB}{4 \text{ rt. } \angle s} = \frac{1 \text{ radian}}{4 \text{ rt. } \angle s}$$

or

$$\frac{r}{2\pi r} = \frac{1 \text{ radian}}{4 \text{ rt. } \angle s}$$

$$\therefore \frac{1}{2\pi} = \frac{1 \text{ radian}}{4 \text{ rt. } \angle s}$$

or

$$\frac{1}{\pi} = \frac{1 \text{ radian}}{2 \text{ rt. } \angle s}$$

$$\therefore \pi \text{ radians} = 180^\circ \text{ or } 2 \text{ rt. } \angle \text{s.}$$

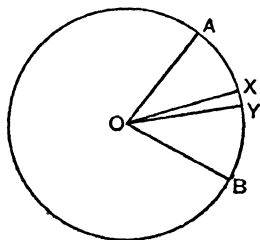
$$\begin{aligned} \therefore 1 \text{ radian} &= \frac{180}{3.1416} \\ &= 57.29^\circ \text{ approximately.} \end{aligned}$$

### 63. The Area of a Circle.

Let  $\angle AOB$  be one radian.

Imagine this split up into a very large number of triangles, such as  $OXY$ , in which  $OX$  and  $OY$  are radii very close together.

The area of  $OXY$  is almost exactly equal to  $\frac{1}{2} XY$  times the radius, and if all the triangles making up the sector  $AOB$  are added together, we get half the arc  $AB$  times the radius as the area.



$$\begin{aligned} \therefore \text{sector } AOB &= \frac{1}{2} r \times r, \text{ for the arc } AB = r, \\ &= \frac{1}{2} r^2. \end{aligned}$$

$$\text{But } \frac{\text{sector } AOB}{\text{circle}} = \frac{\angle AOB}{4 \text{ rt. } \angle \text{s}} = \frac{1 \text{ radian}}{2\pi \text{ radians}} = \frac{1}{2\pi}.$$

$$\therefore \frac{\text{circle}}{\text{sector } AOB} = \frac{2\pi}{1}.$$

$$\begin{aligned} \therefore \text{circle} &= 2\pi \times \text{sector } AOB \\ &= 2\pi \times \frac{1}{2} r^2 \\ &= \pi r^2. \end{aligned}$$

**Learn:**

*The area of a circle is  $\pi r^2$ , where  $r$  is the radius and  $\pi = \text{approximately } 3\frac{1}{4} \text{ or } 3.1416.$*

**Ex. 1.** Find the area of a circle of radius 7 inches.

$$A = \pi r^2 = 2\frac{1}{4} \times 7 \text{ sq. in.} = 22 \text{ sq. in.}$$

**Ex. 2.** Find the diameter of a circle of area 456 sq. in.

$$A = \pi r^2; \quad r^2 = \frac{A}{\pi}; \quad r = \sqrt{\frac{A}{\pi}}; \quad d = 2 \sqrt{\frac{A}{\pi}}.$$

$$\therefore d = 2 \sqrt{\frac{456}{\pi}} \text{ inches.}$$

The result can be found by arithmetical computation or by using logs.

## EXERCISE 82.

Find the circumferences of the following circles, using  $\pi = 3\frac{1}{2}$ .

1.  $r = 14''$ .      2.  $r = 4\cdot2''$ .      3.  $D = 14$  ft.  
 4.  $D = 32$  ft.      5.  $r = 15$  ft. 6 in.      6.  $D = 13\cdot12$  in.

Find the circumferences of the following circles, using  $\pi = 3\cdot1416$  and giving answers correct to the nearest hundredth of the unit:

7.  $r = 15''$ .      8.  $r = 1\cdot85''$ .      9.  $D = 73$  ft.

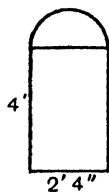
Find the diameter and radius in each of the following circles, the area being given in each case, using  $\pi = 3\cdot142$ .

10. 25 sq. ft.      11. 19 sq. ft.      12. 1 sq. ft. 3 sq. in.  
 13. 4\cdot73 sq. in.      14. \cdot159 sq. ft.      15. \cdot087 sq. in.

16. A right-angled triangle has two sides 5 and 12 inches. What is the area of the semi-circle described on the hypotenuse as diameter?

17. Construct a circle which has an area half that of a circle with diameter  $3\frac{1}{2}$  in.

18. Find the area and weight of the piece of steel plate shown in the sketch. The upper portion is semi-circular, and one square foot of the plate weighs 5\cdot1 lb.



19. Draw a right-angled isosceles triangle  $ABC$  whose hypotenuse  $AB$  is 12 cm. Bisect the hypotenuse in  $D$ , and draw a circle passing through  $B$ ,  $C$  and  $D$ . Calculate the area of the circle.

20. Find the radius of a circle which is equal in area to the sum of the areas of two circles of radii 2 ft. and 5 ft. 10 in. respectively.

21. Draw an equilateral triangle with base  $1\frac{1}{2}''$ . On each side as diameter, draw a semi-circle outside the triangle. Calculate the distance round the outside of the figure so obtained.

## GENERAL REVISION

## EXERCISE 33 (a). MENTAL.

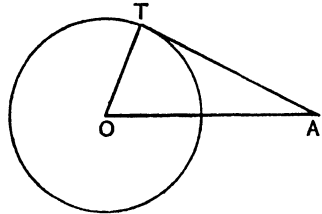
1. If  $AT$  is a tangent to the circle and  $\angle TAO = 27^\circ$ , what is  $\angle TOA$ ?

2. If in the figure of Question 1,  $OT = 2\frac{1}{2}''$  and  $OA = 6\frac{1}{2}''$ , find  $AT$ .

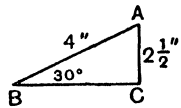
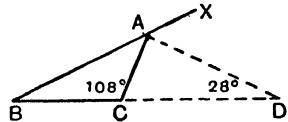
3. If the extremities of all tangents 1" long drawn to a given circle were joined, what figure would be formed?

4. If a parallelogram can be inscribed in a circle, why must it be a rectangle?

5.  $AD$  bisects the exterior angle  $CAX$  of the  $\triangle ABC$ . If  $\angle ACB = 108^\circ$  and  $\angle ADC = 28^\circ$ , what is  $\angle BAC$ ?



6. How many different triangles can be made with the dimensions shown in the rough diagram?



7. The angles of a triangle are in the ratio 4 : 5 : 6. Find them.

8. If the sum of two opposite angles of a parallelogram is  $240^\circ$ , what is each of the other two angles?

## EXERCISE 33 (b).

1. Find the number of degrees in each angle of a regular hexagon  $ABCDEF$ . Prove that  $ACE$  is an equilateral triangle.

2. Draw a straight line  $AB$  3 inches long, and take a point  $O$  in  $AB$  1 inch from  $A$ . Without using a protractor or set-square draw through  $O$  a straight line  $OC$  perpendicular to  $AB$  and 2 inches long. Join  $AC$  and  $BC$ . Measure the length of  $AC$  and the angle  $ABC$ .

3. A man stands at the centre of the top of a building which is in the form of a cylinder 20 feet high and of radius 10 feet. His eyes are 5 feet above the top of the tower. Draw a figure, taking a length of 1 inch to represent 10 feet, and hence find the least possible distance from the base of the cylinder to the nearest points in the ground which he can see.

4.  $A$  is a point outside a given straight line  $XY$ .  $B$  is a point on the line  $XY$ . Find a point on the line  $XY$  which is equidistant from  $A$  and  $B$ . State and prove your construction.

5. Construct a quadrilateral  $ABCD$  in which the diagonal  $AC$  is 7.7 cm. long,  $BD$  cuts  $AC$  at  $O$  so that  $AO$  is 3.3 cm.,  $AB = 4.4$  cm.,  $\angle AOB = 47^\circ$ , and  $\angle BCD = 59^\circ$ .

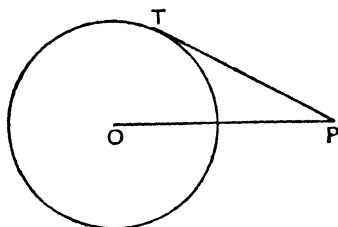
6. Draw a rectangle  $ABCD$ , in which  $AB = 10$  cm. and the diagonal  $AC = 11$  cm. Within this rectangle construct a crescent composed of two circular arcs, given that the crescent touches  $CD$ , has its horns at  $A$  and  $B$ , and is of maximum thickness 2 cm.

7. Show how to construct a parallelogram, given the length of each diagonal and the sum of the angles which one side subtends at the other two vertices.

8.  $ABCD$  is a quadrilateral in which  $AB = 3.12$  in.,  $BC = 2.34$  in.,  $CD = 3.60$  in.,  $DA = 1.50$  in., and the diagonal  $AC = 3.90$  in. Prove that two of the angles of the quadrilateral are right angles, and find its area in square inches.

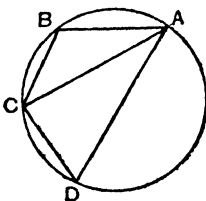
EXERCISE 34 (a). MENTAL.

1.  $PT$  is a tangent 12 cm. long and  $OP = 13$  cm. Find the diameter of the circle.



2. Give a formula for the length of a tangent to a circle of radius  $r$  inches from a point  $c$  inches from the centre.

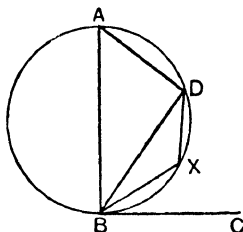
3. If  $\angle BAD$  is bisected by  $AC$ , how do you know that  $BC = CD$ ?



4. What angle is subtended at the centre of a circle by a side of a regular 12-sided figure inscribed in the circle?

5. Two angles of a triangle are  $28^\circ$  and  $108^\circ$ . What are the sizes of the three exterior angles?

6.  $BC$  is a tangent to a circle and  $AB$  is a diameter. How do we know that  $\angle DBC = \angle BAD$ ?



7. If in this figure  $\angle DBC = 30^\circ$ , what is the angle  $DXB$ , and why?

8. What is the area of a triangle with sides 5, 12 and 13 ins.?

### EXERCISE 34 (b).

1. Construct a parallelogram  $ABCD$  having  $AB$  2.5 in., angle  $ABC$   $50^\circ$ , and the area equal to the area of a rectangle whose adjacent sides are 3 in. and 2 in. long. Describe your construction.

2. The perpendicular bisector of the side  $BC$  of a triangle  $ABC$  meets the circumference of the circumscribed circle in  $D$  on the side remote from  $A$ . Prove that  $AD$  bisects the angle  $BAC$ .

3. Draw a circle of radius 5.3 cm., and take a point  $P$  at a distance of 11 cm. from its centre. Construct the tangents from  $P$  to the circle, and measure the angle between them.

4. On a line  $AB$  of length 4 inches as base, construct a triangle  $ABC$  having its area 6 square inches and  $AC$   $3\frac{1}{2}$  inches long. Give reasons for your construction.

5. A chord  $AB$  of a circle subtends an angle of  $60^\circ$  at the centre  $O$ .  $AO$  meets the circle again at  $D$  and  $C$  is a point in  $AB$  produced such that  $AB = BC$ . Prove that the circle on  $CD$  as diameter passes through  $O$ .

6.  $A$  is a point 300 ft. due west of a flagstaff 100 ft. high. Find by careful drawings to a scale of 2 in. = 100 ft. the distance from the foot of the flagstaff of a point  $B$  200 ft. south-east of  $A$ , and the angle of elevation of the top of the flagstaff when viewed from  $B$ .

7. Without using a set-square construct a triangle  $ABC$  with the angle  $B$  a right angle,  $AB$  2 inches long and  $BC$  1.4 inches long.

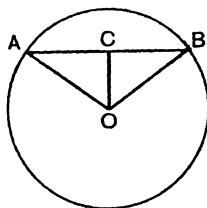
Through  $C$  draw a straight line to pass between the points  $A$  and  $B$  and be equidistant from them.

Measure the distance of  $A$  or  $B$  from the straight line which you have drawn.

8. Draw two triangles in each of which two sides are 7 cm. and 4.8 cm. in length, and the angle opposite the shorter side is  $35^\circ$ . Show that the sum of the angles opposite the greatest sides of these two triangles is two right angles.

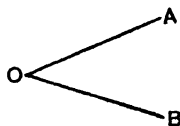
### EXERCISE 35 (a). MENTAL.

1.  $O$  is the centre of the circle and  $OC$  is perpendicular to  $AB$ . If  $OB = 5$  cm. and  $OC = 3$  cm., what is  $AB$ ?

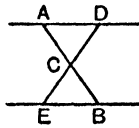


2. Find the area of the triangle  $OAB$ .

3. If all possible circles were drawn touching  $OA$  and  $OB$ , what line do you think would be formed by joining the centres?



4.  $AD$ ,  $EB$  are parallel lines and  $C$  is the middle point of  $AB$ . How do you know that  $EC = CD$ ?



5. How many circles in all can be drawn touching the three sides of a triangle, or one of the sides and the other two produced?

6. What is the length of the diagonal of a rectangle if the sides are 5" and 12"?

7. If I go N. 10 miles, then E. 3 miles, then S. 2 miles and finally W. 8 miles, how far N. and W. of my starting-point am I then?

8. The diagonals of a rhombus are 8" and 5". What is the area?

### EXERCISE 85 (b).

1.  $AB$ ,  $BC$ ,  $CD$  are three consecutive sides of a regular five-sided figure.  $AB$  and  $DC$  when produced meet at  $E$ . Calculate the number of degrees in the angle  $AED$ .

2. Construct a parallelogram with adjacent sides 5.9 cm. and 4.8 cm. long, and including an angle of  $40^\circ$ . Construct a triangle of the same area as this parallelogram.

3. Two lines  $AB$  and  $AC$ , each 5.6 cm. long, are at right angles to each other. Find two points, each equidistant from  $A$  and  $B$  and 3.4 cm. from  $C$ .

4. A cone, the radius of whose base is 7 cm. and whose height is 15 cm., lies on its side on a horizontal plane. Find from a figure the distance of the highest point of the cone from the plane.

5. Draw two lines  $OA$  and  $OB$  each 3 inches long, including an angle of  $70^\circ$ . From  $A$ , and on the same side of  $OA$  as  $B$ , draw a straight line  $AP$  such that the angle  $OAP$  is  $100^\circ$ . Find two points in  $AP$  each  $2\frac{3}{4}$  inches from  $B$ .

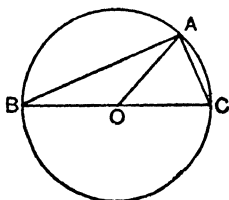
6.  $A, B$  are two points one mile and a half apart on a straight coast-line and  $A$  is north-east of  $B$ . An observer at  $A$  sees a ship at sea  $30^\circ$  west of north, and at the same instant an observer at  $B$  sees the ship  $20^\circ$  east of north. Find, by careful drawing to a scale of 4 inches to the mile, the distance of the ship from the coast.

7. Construct a trapezium  $ABCD$  in which the parallel sides  $AB, CD$  are 3 inches long and 5 inches long respectively, and the slant sides  $BC, AD$  are each 2 inches long. Prove that the area of the trapezium is four times the area of an equilateral triangle whose sides are 2 inches long.

8. Calculate (correct to one-tenth of an inch) the length of each side of a right-angled isosceles triangle whose area is 8 square inches.

### EXERCISE 36 (a). MENTAL.

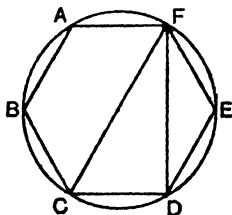
1. If  $BC$  is a diameter of a circle and  $\angle ABO = 20^\circ$ , find  $\angle AOC$  and  $\angle OAC$ .



2. Give the value of  $\pi$  to 4 places of decimals.

3. How do you describe a circle circumscribing a triangle?

4.  $ABCDEF$  is a regular hexagon described in a circle. What is the size of the angle  $CFD$ , and why?



5. If two triangles are such that the sum of the base angles of one is equal to the sum of the base angles of the other, why are the vertical angles equal?

6. Three angles of a quadrilateral are  $84^\circ$ ,  $85^\circ$ ,  $86^\circ$ . What is the fourth?

7. If in the figure of Question 4  $CD = 1''$ , what is  $CF$ ?

8. How many congruent triangles can be made in different positions on a given base, if the three sides are given equal in each?

### EXERCISE 36 (b).

*Being the paper set in December 1926 at the Junior Local Examination of the University of Oxford.*

1. Prove that if the sides of a convex polygon are produced in order, the sum of the angles thus formed is equal to four right angles.

Three angles of a hexagon are right angles and the other three angles are equal to each other. Find the number of degrees in each of the equal angles.

2. Prove that the parallelogram whose diagonals are at right angles has all its sides equal (i.e. is a rhombus).

Construct a rhombus with a diagonal 6 in. long and an area of 10 sq. in. Explain your construction.

3. The hypotenuse of a right-angled triangle is 8 in. and one side is 3 in. long. Calculate the length of the other side to the nearest hundredth of an inch, and the area of the triangle to the nearest tenth of a square inch.

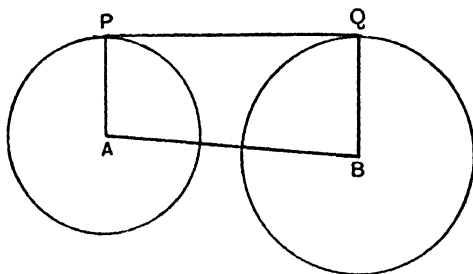
4.  $AB$ ,  $CD$  are two chords of a circle which are parallel to one another. Prove that the triangles  $ACD$ ,  $BDC$  are equal in all respects.

5. Construct a triangle  $ABC$  in which  $AB = 4$  in.,  $AC = 5$  in., and  $BC = 6$  in. Construct the perpendicular  $AD$  from  $A$  to  $BC$  and draw the circles circumscribing the triangles  $ADB$ ,  $ADC$ .

6. The sloping faces of a square pyramid standing on a base 3 in. square are all equilateral triangles. Find, by calculation, or by careful drawing, the height of the vertex of the pyramid above the base.

## EXERCISE 37 (a). MENTAL.

1. If  $PQ$  is a common tangent to the two circles, why is  $PA$  parallel to  $QB$ ?

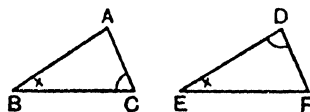


2. If all the parallel chords of a circle are bisected, what line would be formed by joining up all the middle points?

3. Give a formula connecting the length of the direct common tangents to two circles, the radii and the distance between the centres.

4. If two radii of a circle contain an angle of  $60^\circ$ , what is the length of the chord joining the extremities?

5. If  $\angle ABC = \angle DEF$  and  $\angle ACB = \angle EDF$ , what else must we know before the triangles can be said to be congruent?



6. What is the area of an equilateral triangle with a side 2 in.?

7. If two tangents to a circle from an external point contain an angle of  $60^\circ$  and the radius of the circle is 3'', what is the distance from the point to the centre?

8. The area of a triangle is 10 sq. cm. and the height is 5 cm. What is the length of the base?

## EXERCISE 37 (b).

*Being the whole of Part I of the paper set in December 1926 at the Junior Local Examination of the University of Cambridge. In this Examination candidates can pass by doing sufficiently well in Part I.*

## PART I.

1. Draw a square  $ABCD$  of side 2 inches, and on the side  $AD$  construct an equilateral triangle  $ADE$  outside the square. Construct a triangle of area equal to that of the figure  $ABCDE$  with one vertex at  $E$  and with its base on the line of which  $BC$  forms a part. Measure the base of this triangle.

2. Prove that the sum of the angles of a triangle is equal to two right angles.

Show that two triangles can be drawn in each of which two sides are 8 cm. and 5.5 cm. in length, and the angle opposite the shorter side is  $42^\circ$ . Show that the sum of the angles opposite the greatest sides of these two triangles is two right angles.

3. Prove that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

$O$  is the centre of the circle circumscribing the triangle  $ABC$ ;  $AO$ ,  $BO$ ,  $CO$  are drawn, and  $CO$  and  $BO$  are produced to meet  $AB$  and  $AC$  respectively in  $F$  and  $E$ . Prove that the sum of the angles  $BFC$ ,  $BEC$  is equal to three times the angle  $BAC$ .

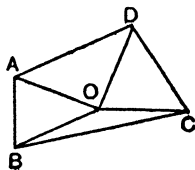
4. Prove that in a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.

$P$  is a point outside a rectangle  $ABCD$  (for convenience  $P$  may be taken above and to the right of the rectangle); prove that

$$PA^2 + PC^2 = PB^2 + PD^2.$$

## EXERCISE 38 (a). MENTAL.

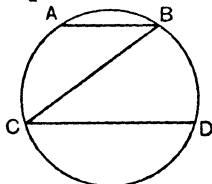
1. If  $\angle BAD = 108^\circ$  and  $O$  is the centre of a circle passing through  $A$ ,  $B$ ,  $C$  and  $D$ , calculate  $\angle BOD$  and  $\angle BCD$ .



2. If all possible circles were described passing through  $A$  and  $B$ , what line would be  $A$  —————  $B$  formed by joining the centres?

3. When is it impossible to construct a circle to pass through two given points and have its centre on a given line?

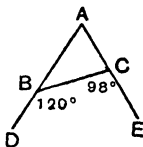
4. If  $AB$  and  $CD$  are parallel chords, how do you know that  $AC = BD$ ?



5. If I go 8 miles due N. and then 6 miles due W., how far am I from my starting-point?

6. The angles at the base of an isosceles triangle are each twice the vertical angle. What is the size of the latter?

7. The exterior angles  $DBC$  and  $BCE$  of the triangle  $ABC$  are  $120^\circ$  and  $98^\circ$ . What is  $\angle BAC$ ?

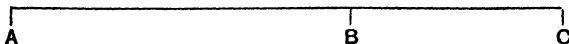


8. The distance from the centre of a circle to each of 6 equal chords forming a hexagon is  $1.73''$  and the chords are  $2''$  in length. What is the area of the hexagon?

EXERCISE 38 (b).

*Being the Geometry Questions of the 1927 Junior School Commercial Certificate of the Royal Society of Arts.*

1. Distances measured from left to right, such as  $AB$  on the line below, are positive; distances measured from right to left, such as  $BA$ , are negative. Find in inches as accurately as you can the distances  $AB$ ,  $BC$ ,  $CA$ , and write them down in your answer book with the proper sign.



What is the magnitude of  $AB + BC + CA$ ? If the length  $BC$  is called  $x$ , what is  $CA$  in terms of  $x$ ?

2. Prove that if  $O$  is the centre of a circle,  $BC$  is any chord and  $K$  is the middle point of the chord, then  $OK$  is perpendicular to  $BC$ .

3. Prove that if the angle  $S$  of the triangle  $RST$  is a right angle, then  $RT^2 = RS^2 + ST^2$ .

The wall of a house is 25 feet high. Find, either by calculation or by drawing and measurement, the length of the shortest ladder that will reach to the top of the wall, if the foot of the ladder cannot stand less than 15 feet from the foot of the wall.

4. Draw a triangle  $ABC$  having the side  $BC$  9.4 centimetres long, the angle  $ABC$  equal to  $62^\circ$  and the angle  $ACB$  equal to  $53^\circ$ .

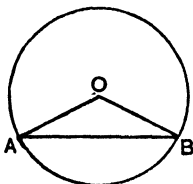
By a geometrical construction draw a line bisecting the angle  $ABC$  and prove that your construction is correct.

The bisector of the angle  $ABC$  meets  $AC$  in  $E$ . Work out in degrees the sizes of the angles  $BAC$ ,  $AEB$ ,  $CEB$ . Check your results by measurement with a protractor and write your measurements in your answer book.

**Note.** *These were four of the six questions, for which two hours were allowed, the other two questions being purely Algebraical.*

### EXERCISE 39 (a). MENTAL.

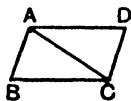
1. If  $\angle OAB = 28^\circ$ , find  $\angle AOB$ .



2. What is the size of each angle of a regular polygon of 12 sides?

3. If the middle points of all equal chords of a circle were joined, what figure do you think would be formed?

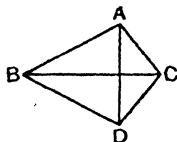
4. If  $AB = CD$  and  $AD = BC$ , why is the quadrilateral  $ABCD$  bound to be a parallelogram?



5. If a rhombus and a square have equal sides, which has the greater area and why?

6. If the angle  $(2x - 45)^\circ$  is the complement of  $(90 - x)^\circ$ , find  $x$ .

7. If  $AB = BD$  and  $AC = CD$ , how do you know that  $\angle BAC = \angle BDC$ ?



8. If in the above figure  $AC = 3$ ,  $AB = 4$ ,  $BC = 5$ , what is the area of the quadrilateral?

### EXERCISE 39 (b).

*Being Section III of the East Midland Educational Union Central Schools Examination 1927.*

1. Draw a line  $OP = 1\frac{1}{2}$  inches. On  $OP$  as diameter describe a circle. With centre  $O$  and radius  $OP$  describe a circle. In the circumference of the larger circle take any point  $T$ . Join  $PT$  cutting the circumference of the smaller circle at  $Q$ . Join  $OQ$ . What do you discover by measuring angle  $OQP$ , and lines  $TQ$ ,  $QP$ ?

Give reasons for each result.

2. Construct a right-angled triangle  $ABC$ . Make the hypotenuse  $AC = 6.8$  cm.; make one side  $AB = 3$  cm. Join the point  $B$  to the middle point of  $AC$ . Measure the angle at  $A$  with protractor. Also measure the line from  $B$  to the middle point of  $AC$ , give its length in centimetres and mention anything you discover respecting it.

3. On  $AB$ , a line  $1\frac{1}{2}$  inches long, describe an equilateral triangle  $ABC$ . In  $AB$  take any point  $D$ ; join  $CD$ .

Prove that the angle  $BDC$  equals the sum of the angles  $DAC$  and  $ACD$ .

							Differences.										
	0'	10'	20'	30'	40'	50'	60'	1'	2'	3'	4'	5'	6'	7'	8'	9'	
0°	0'0000	0'0029	0'0058	0'0087	0'0116	0'0145	0'0175	80°	3	6	9	12	15	17	20	23	26
1	'0175	'0204	'0233	'0262	'0291	'0320	'0349	88	3	6	9	12	15	17	20	23	26
2	'0349	'0378	'0407	'0436	'0465	'0494	'0523	87	3	6	9	12	15	17	20	23	26
3	'0523	'0552	'0581	'0610	'0640	'0669	'0698	86	3	6	9	12	15	17	20	23	26
4	'0698	'0727	'0756	'0785	'0814	'0843	'0872	85	3	6	9	12	15	17	20	23	26
5	0'0872	0'0901	0'0929	0'0958	0'0987	0'1016	0'1045	84	3	6	9	12	14	17	20	23	26
6	'1045	'1074	'1103	'1132	'1161	'1190	'1219	83	3	6	9	12	14	17	20	23	26
7	'1219	'1248	'1276	'1305	'1334	'1363	'1392	82	3	6	9	12	14	17	20	23	26
8	'1392	'1421	'1449	'1478	'1507	'1536	'1564	81	3	6	9	11	14	17	20	23	26
9	'1564	'1593	'1622	'1650	'1679	'1708	'1736	80°	3	6	9	11	14	17	20	23	26
10°	0'1736	0'1765	0'1794	0'1822	0'1851	0'1880	0'1908	79	3	6	9	11	14	17	20	23	26
11	'1908	'1927	'1965	'1994	'2022	'2051	'2079	78	3	6	9	11	14	17	20	23	26
12	'2079	'2108	'2136	'2164	'2193	'2221	'2250	77	3	6	9	11	14	17	20	23	26
13	'2250	'2278	'2306	'2334	'2363	'2391	'2419	76	3	6	8	11	14	17	20	23	25
14	'2419	'2447	'2476	'2504	'2532	'2560	'2588	75	3	6	8	11	14	17	20	23	25
15	0'2588	0'2616	0'2644	0'2672	0'2700	0'2728	0'2756	74	3	6	8	11	14	17	20	22	25
16	'2756	'2784	'2812	'2840	'2868	'2896	'2924	73	3	6	8	11	14	17	20	22	25
17	'2924	'2952	'2979	'3007	'3035	'3062	'3090	72	3	6	8	11	14	17	19	22	25
18	'3090	'3118	'3145	'3173	'3201	'3228	'3256	71	3	6	8	11	14	17	19	22	25
19	'3256	'3283	'3311	'3338	'3365	'3393	'3420	70°	3	5	8	11	14	16	19	22	25
20°	0'3420	0'3448	0'3475	0'3502	0'3529	0'3557	0'3584	69	3	5	8	11	14	16	19	22	25
21	'3584	'3611	'3638	'3665	'3692	'3719	'3746	68	3	5	8	11	14	16	19	22	24
22	'3746	'3773	'3800	'3827	'3854	'3881	'3907	67	3	5	8	11	13	16	19	21	24
23	'3907	'3934	'3961	'3987	'4014	'4041	'4067	66	3	5	8	11	13	16	19	21	24
24	'4067	'4094	'4120	'4147	'4173	'4200	'4226	65	3	5	8	11	13	16	19	21	24
25	0'4226	0'4253	0'4279	0'4305	0'4331	0'4358	0'4384	64	3	5	8	10	13	16	18	21	24
26	'4384	'4410	'4436	'4462	'4488	'4514	'4540	63	3	5	8	10	13	16	18	21	23
27	'4540	'4566	'4592	'4617	'4643	'4669	'4695	62	3	5	8	10	13	15	18	21	23
28	'4695	'4720	'4746	'4772	'4797	'4823	'4848	61	3	5	8	10	13	15	18	20	23
29	'4848	'4874	'4899	'4924	'4950	'4975	'5000	60°	3	5	8	10	13	15	18	20	23
30°	0'5000	0'5025	0'5050	0'5075	0'5100	0'5125	0'5150	59	3	5	8	10	13	15	18	20	23
31	'5150	'5175	'5200	'5225	'5250	'5275	'5299	58	2	5	7	10	12	15	17	20	22
32	'5299	'5324	'5348	'5373	'5398	'5422	'5446	57	2	5	7	10	12	15	17	20	22
33	'5446	'5471	'5495	'5519	'5544	'5568	'5592	56	2	5	7	10	12	15	17	19	22
34	'5592	'5616	'5640	'5664	'5688	'5712	'5736	55	2	5	7	10	12	14	17	19	22
35	0'5736	0'5760	0'5783	0'5807	0'5831	0'5854	0'5878	54	2	5	7	9	12	14	17	19	21
36	'5878	'5901	'5925	'5948	'5972	'5995	'6018	53	2	5	7	9	12	14	16	19	21
37	'6018	'6041	'6065	'6088	'6111	'6134	'6157	52	2	5	7	9	12	14	16	18	21
38	'6157	'6180	'6202	'6225	'6248	'6271	'6293	51	2	5	7	9	11	14	16	18	20
39	'6293	'6316	'6338	'6361	'6383	'6406	'6428	50°	2	4	7	9	11	13	16	18	20
40°	0'6428	0'6450	0'6472	0'6494	0'6517	0'6539	0'6561	49	2	4	7	9	11	13	15	18	20
41	'6561	'6583	'6604	'6626	'6648	'6670	'6691	48	2	4	7	9	11	13	15	17	20
42	'6691	'6713	'6734	'6756	'6777	'6799	'6820	47	2	4	6	9	11	13	15	17	19
43	'6820	'6841	'6862	'6883	'6905	'6926	'6947	46	2	4	6	8	11	13	15	17	19
44	'6947	'6967	'6988	'7009	'7030	'7050	'7071	45	2	4	6	8	10	12	15	17	19
	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'

	Sines						Differences.										
	0'	10'	20'	30'	40'	50'	60'	1'	2'	3'	4'	5'	6'	7'	8'	9'	
45°	0.7071	0.7092	0.7112	0.7133	0.7153	0.7173	0.7193	44°	2	4	6	8	10	12	14	16	18
46	.7193	.7214	.7234	.7254	.7274	.7294	.7314	43	2	4	6	8	10	12	14	16	18
47	.7314	.7333	.7353	.7373	.7392	.7412	.7431	42	2	4	6	8	10	12	14	16	18
48	.7431	.7451	.7470	.7490	.7509	.7528	.7547	41	2	4	6	8	10	12	13	15	17
49	.7547	.7566	.7585	.7604	.7623	.7642	.7660	40°	2	4	6	8	9	11	13	15	17
50°	0.7660	0.7679	0.7698	0.7716	0.7735	0.7753	0.7771	39	2	4	6	7	9	11	13	15	17
51	.7771	.7790	.7808	.7826	.7844	.7862	.7880	38	2	4	5	7	9	11	13	14	16
52	.7880	.7898	.7916	.7934	.7951	.7969	.7986	37	2	4	5	7	9	11	12	14	16
53	.7986	.8004	.8021	.8039	.8056	.8073	.8090	36	2	3	5	7	9	10	12	14	16
54	.8090	.8107	.8124	.8141	.8158	.8175	.8192	35	2	3	5	7	8	10	12	14	15
55	0.8192	0.8208	0.8225	0.8241	0.8258	0.8274	0.8290	34	2	3	5	7, 8	10	12	13	15	15
56	.8290	.8307	.8323	.8339	.8355	.8371	.8387	33	2	3	5	6	8	10	11	13	14
57	.8387	.8403	.8418	.8434	.8450	.8465	.8480	32	2	3	5	6	8	9	11	13	14
58	.8480	.8496	.8511	.8526	.8542	.8557	.8572	31	2	3	5	6	8	9	11	12	14
59	.8572	.8587	.8601	.8616	.8631	.8646	.8660	30°	1	3	4	6	7	9	10	12	13
60°	0.8660	0.8675	0.8689	0.8704	0.8718	0.8732	0.8746	29	1	3	4	6	7	9	10	11	13
61	.8746	.8760	.8774	.8788	.8802	.8816	.8829	28	1	3	4	6	7	8	10	11	12
62	.8829	.8843	.8857	.8870	.8884	.8897	.8910	27	1	3	4	5	7	8	9	11	12
63	.8910	.8923	.8936	.8949	.8962	.8975	.8988	26	1	3	4	5	6	8	9	10	12
64	.8988	.9001	.9013	.9026	.9038	.9051	.9063	25	1	3	4	5	6	8	9	10	11
65	0.9063	0.9075	0.9088	0.9100	0.9112	0.9124	0.9135	24	1	2	4	5	6	7	8	10	11
66	.9135	.9147	.9159	.9171	.9182	.9194	.9205	23	1	2	3	5	6	7	8	9	10
67	.9205	.9216	.9228	.9239	.9250	.9261	.9272	22	1	2	3	4	6	7	8	9	10
68	.9272	.9283	.9293	.9304	.9315	.9325	.9336	21	1	2	3	4	5	6	7	9	10
69	.9336	.9346	.9356	.9367	.9377	.9387	.9397	20°	1	2	3	4	5	6	7	8	9
70°	0.9397	0.9407	0.9417	0.9426	0.9436	0.9446	0.9455	19	1	2	3	4	5	6	7	8	9
71	.9455	.9465	.9474	.9483	.9492	.9502	.9511	18	1	2	3	4	5	6	6	7	8
72	.9511	.9520	.9528	.9537	.9546	.9555	.9563	17	1	2	3	4	4	5	6	7	8
73	.9563	.9572	.9580	.9588	.9596	.9605	.9613	16	1	2	2	3	4	5	6	7	7
74	.9613	.9621	.9628	.9636	.9644	.9652	.9659	15	1	2	2	3	4	5	5	6	7
75	0.9659	0.9667	0.9674	0.9681	0.9689	0.9696	0.9703	14	1	1	2	3	4	4	5	6	7
76	.9703	.9710	.9717	.9724	.9730	.9737	.9744	13	1	1	2	3	3	4	5	5	6
77	.9744	.9750	.9757	.9763	.9769	.9775	.9781	12	1	1	2	3	3	4	4	5	6
78	.9781	.9787	.9793	.9799	.9805	.9811	.9816	11	1	1	2	2	3	3	4	5	5
79	.9816	.9822	.9827	.9833	.9838	.9843	.9848	10°	1	1	2	2	3	3	4	4	5
80°	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2	3	3	4	4
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8	0	1	1	2	2	3	3	3	4
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7	0	1	1	2	2	2	3	3	3
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6	0	1	1	1	2	2	2	3	3
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5	0	1	1	1	1	2	2	2	3
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4	Differences are so small here that tabulation is unnecessary.								
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3									
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2									
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1									
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0°									
	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'

							Differences.										
	0'	10'	20'	30'	40'	50'	60'	1'	2'	3'	4'	5'	6'	7'	8'	9'	
0°	0'0000	0'0029	0'0058	0'0087	0'0116	0'0145	0'0175	80°	3	6	9	12	15	17	20	23	26
1	'0175	'0204	'0233	'0262	'0291	'0320	'0349	88	3	6	9	12	15	17	20	23	26
2	'0349	'0378	'0407	'0437	'0466	'0495	'0524	87	3	6	9	12	15	18	20	23	26
3	'0524	'0553	'0582	'0612	'0641	'0670	'0699	86	3	6	9	12	15	18	20	23	26
4	'0699	'0729	'0758	'0787	'0816	'0846	'0875	85	3	6	9	12	15	18	21	23	26
5	0'0875	0'0904	0'0934	0'0963	0'0992	0'1022	0'1051	84	3	6	9	12	15	18	21	24	26
6	'1051	'1080	'1110	'1139	'1169	'1198	'1228	83	3	6	9	12	15	18	21	24	27
7	'1228	'1257	'1287	'1317	'1346	'1376	'1405	82	3	6	9	12	15	18	21	24	27
8	'1405	'1435	'1465	'1495	'1524	'1554	'1584	81	3	6	9	12	15	18	21	24	27
9	'1584	'1614	'1644	'1673	'1703	'1733	'1763	80°	3	6	9	12	15	18	21	24	27
10°	0'1763	0'1793	0'1823	0'1853	0'1883	0'1914	0'1944	79	3	6	9	12	15	18	21	24	27
11	'1944	'1974	'2004	'2035	'2065	'2095	'2126	78	3	6	9	12	15	18	21	24	27
12	'2126	'2156	'2186	'2217	'2247	'2278	'2309	77	3	6	9	12	15	18	21	24	27
13	'2309	'2339	'2370	'2401	'2432	'2462	'2493	76	3	6	9	12	15	18	22	25	28
14	'2493	'2524	'2555	'2586	'2617	'2648	'2679	75	3	6	9	12	16	19	22	25	28
15	0'2679	0'2711	0'2742	0'2773	0'2805	0'2836	0'2867	74	3	6	9	13	16	19	22	25	28
16	'2867	'2899	'2931	'2962	'2994	'3026	'3057	73	3	6	9	13	16	19	22	25	28
17	'3057	'3089	'3121	'3153	'3185	'3217	'3249	72	3	6	10	13	16	19	22	26	29
18	'3249	'3281	'3314	'3346	'3378	'3411	'3443	71	3	6	10	13	16	19	23	26	29
19	'3443	'3476	'3508	'3541	'3574	'3607	'3640	70°	3	7	10	13	16	20	23	26	29
20°	0'3640	0'3673	0'3706	0'3739	0'3772	0'3805	0'3839	69	3	7	10	13	17	20	23	27	30
21	'3839	'3872	'3906	'3939	'3973	'4006	'4040	68	3	7	10	13	17	20	24	27	30
22	'4040	'4074	'4108	'4142	'4176	'4210	'4245	67	3	7	10	14	17	20	24	27	31
23	'4245	'4279	'4314	'4348	'4383	'4417	'4452	66	3	7	10	14	17	21	24	28	31
24	'4452	'4487	'4522	'4557	'4592	'4628	'4663	65	4	7	11	14	18	21	25	28	32
25	0'4663	0'4699	0'4734	0'4770	0'4806	0'4841	0'4877	64	4	7	11	14	18	21	25	29	32
26	'4877	'4913	'4950	'4986	'5022	'5059	'5095	63	4	7	11	15	18	22	25	29	33
27	'5095	'5132	'5169	'5206	'5243	'5280	'5317	62	4	7	11	15	18	22	26	30	33
28	'5317	'5354	'5392	'5430	'5467	'5505	'5543	61	4	8	11	15	19	23	26	30	34
29	'5543	'5581	'5619	'5658	'5696	'5735	'5774	60°	4	8	12	15	19	23	27	31	35
30°	0'5774	0'5812	0'5851	0'5890	0'5930	0'5969	0'6009	59	4	8	12	16	20	24	27	31	35
31	'6009	'6048	'6088	'6128	'6168	'6208	'6249	58	4	8	12	16	20	24	28	32	36
32	'6249	'6289	'6330	'6371	'6412	'6453	'6494	57	4	8	12	16	20	25	29	33	37
33	'6494	'6536	'6577	'6619	'6661	'6703	'6745	56	4	8	13	17	21	25	29	33	38
34	'6745	'6787	'6830	'6873	'6916	'6959	'7002	55	4	9	13	17	21	26	30	34	39
35	0'7002	0'7046	0'7089	0'7133	0'7177	0'7221	0'7265	54	4	9	13	18	22	26	31	35	40
36	'7265	'7310	'7355	'7400	'7445	'7490	'7536	53	5	9	14	18	23	27	32	36	41
37	'7536	'7581	'7627	'7673	'7720	'7766	'7813	52	5	9	14	18	23	28	32	37	42
38	'7813	'7860	'7907	'7954	'8002	'8050	'8098	51	5	10	14	19	24	29	33	38	43
39	'8098	'8146	'8195	'8243	'8292	'8342	'8391	50°	5	10	15	20	24	29	34	39	44
40°	0'8391	0'8441	0'8491	0'8541	0'8591	0'8642	0'8693	49	5	10	15	20	25	30	35	40	45
41	'8693	'8744	'8796	'8847	'8899	'8952	'9004	48	5	10	16	21	26	31	36	41	47
42	'9004	'9057	'9110	'9163	'9217	'9271	'9325	47	5	11	16	21	27	32	37	43	48
43	'9325	'9380	'9435	'9490	'9545	'9601	'9657	46	6	11	17	22	28	33	39	44	50
44	'9657	'9713	'9770	'9827	'9884	'9942	1'0000	45	6	11	17	23	29	34	40	46	51
	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'

	TANGENTS						Differences.										
	0'	10'	20'	30'	40'	50'	60'	1'	2'	3'	4'	5'	6'	7'	8'	9'	
45°	1'0000	1'0058	1'0117	1'0176	1'0235	1'0295	1'0355	44°	6	12	18	24	30	36	41	47	53
46	'0355	'0416	'0477	'0538	'0599	'0661	'0724	43	6	12	18	25	31	37	43	49	55
47	'0724	'0786	'0850	'0913	'0977	'1041	'1106	42	6	13	19	26	32	38	45	51	57
48	'1106	'1171	'1237	'1303	'1369	'1436	'1504	41	7	13	20	27	33	40	46	53	60
49	'1504	'1571	'1640	'1708	'1778	'1847	'1918	40°	7	14	21	28	34	41	48	55	62
50°	1'1918	1'1988	1'2059	1'2131	1'2203	1'2276	1'2349	39	7	14	22	29	36	43	50	58	65
51	'2349	'2423	'2497	'2572	'2647	'2723	'2799	38	8	15	23	30	38	45	53	60	68
52	'2799	'2876	'2954	'3032	'3111	'3190	'3270	37	8	16	24	31	39	47	55	63	71
53	'3270	'3351	'3432	'3514	'3597	'3680	'3764	36	8	16	25	33	41	49	58	66	74
54	'3764	'3848	'3934	'4019	'4106	'4193	'4281	35	9	17	26	35	43	52	60	69	78
55	1'4281	1'4370	1'4460	1'4550	1'4641	1'4733	1'4826	34	9	18	27	36	45	54	63	73	82
56	'4826	'4919	'5013	'5108	'5204	'5301	'5399	33	10	19	29	38	48	57	67	76	86
57	'5399	'5497	'5597	'5697	'5798	'5900	'6003	32	10	20	30	40	50	60	71	81	91
58	'6003	'6107	'6212	'6319	'6426	'6534	'6643	31	11	21	32	43	53	64	75	85	96
59	'6643	'6753	'6864	'6977	'7090	'7205	'7321	30°	11	23	34	45	56	68	79	90	102
60°	1'732	1'744	1'756	1'767	1'780	1'792	1'804	29	1	2	4	5	6	7	8	10	11
61	'804	'816	'829	'842	'855	'868	'881	28	1	3	4	5	6	8	9	10	12
62	'881	'894	'907	'921	'935	'949	'963	27	1	3	4	5	7	8	10	11	12
63	'963	'977	'991	2'006	2'020	2'035	2'050	26	1	3	4	6	7	9	10	12	13
64	2'050	2'066	2'081	2'097	2'112	2'128	2'145	25	2	3	5	6	8	9	11	13	14
65	2'145	2'161	2'177	2'194	2'211	2'229	2'246	24	2	3	5	7	8	10	12	14	15
66	'2246	'2264	'2282	'2300	'2318	'2337	'2356	23	2	4	5	7	9	11	13	15	16
67	'2356	'2375	'2394	'2414	'2434	'2455	'2475	22	2	4	6	8	10	12	14	16	18
68	'2475	'2496	'2517	'2539	'2560	'2583	'2605	21	2	4	6	9	11	13	15	17	20
69	'2605	'2628	'2651	'2675	'2699	'2723	'2747	20°	2	5	7	9	12	14	17	19	21
70°	2'747	2'773	2'798	2'824	2'850	2'877	2'904	19	3	5	8	10	13	16	18	21	23
71	'2904	'2932	'2960	'2989	3'018	3'047	3'078	18	3	6	9	12	14	17	20	23	26
72	3'078	3'108	3'140	3'172	3'204	3'237	3'271	17	3	6	10	13	16	19	23	26	29
73	3'271	3'305	3'340	3'376	3'412	3'450	3'487	16	4	7	11	14	18	22	25	29	32
74	3'487	3'526	3'566	3'606	3'647	3'689	3'732	15	4	8	12	16	20	24	29	33	37
75	3'732	3'776	3'821	3'867	3'914	3'962	4'011	14	5	9	14	19	23	28	33	37	42
76	4'011	4'061	4'113	4'165	4'219	4'275	4'331	13	5	11	16	21	27	32	37	43	48
77	4'331	4'390	4'449	4'511	4'574	4'638	4'705	12	6	12	19	25	31	37	44	50	56
78	4'705	4'773	4'843	4'915	4'989	5'066	5'145	11	7	15	22	29	37	44	51	59	66
79	5'145	5'226	5'309	5'396	5'485	5'576	5'671	10°	9	18	26	35	44	53	61	70	79
80°	5'671	5'769	5'871	5'976	6'084	6'197	6'314	9	The differences change so rapidly here that they cannot be tabulated.								
81	6'314	6'435	6'561	6'691	6'827	6'968	7'115	8									
82	7'115	7'269	7'429	7'596	7'770	7'953	8'144	7									
83	8'144	8'345	8'556	8'777	9'010	9'255	9'514	6									
84	9'514	9'788	10'078	10'385	10'712	11'059	11'430	5									
85	11'43	11'83	12'25	12'71	13'20	13'73	14'30	4	The cotangent of a small angle of $n$ minutes of arc or the tangent of $90^\circ$ minus $n$ minutes is very nearly equal to $3438$ divided by $n$ .								
86	14'30	14'92	15'60	16'35	17'17	18'07	19'08	3									
87	19'08	20'21	21'47	22'90	24'54	26'43	28'64	2									
88	28'64	31'24	34'37	38'19	42'96	49'10	57'29	1									
89	57'29	68'75	85'94	114'59	171'89	343'77	$\infty$	0°									
	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'



## NUMERICAL ANSWERS

### EXERCISE 1.

- | 1.                              | 2.                               | 3.                    | 4.                 | 5.                           | 6.                         |
|---------------------------------|----------------------------------|-----------------------|--------------------|------------------------------|----------------------------|
| $OC=4.7$ in.                    | 5.3 in.                          | 5 0 in.               | $5\frac{1}{4}$ in. | $5\frac{1}{8}$ in.           | 5 in.                      |
| $AC=2.4$ in.                    | 2.2 in.                          | 3.2 in.               | $3\frac{3}{8}$ in. | $2\frac{1}{8}$ in.           | $2\frac{1}{2}$ in.         |
| 11. 6.1 cm.                     | 12. 4.8 cm.                      | 13. 5.3 cm.           | 14. 8.4 cm.        | 15. 7.3 cm.                  | 16. 3.6 cm.                |
| 17. 3.7 in.                     | 18. 3.5 in.                      | 19. 4.8 in.           | 20. 6.1 in.        | 21. 4.3 in.                  | 22. 5.5 in.                |
| 23. 8 ft. 5 in.                 | 24. 15 cm. 2 mm.                 | 25. 3 cm.             | 35. 6 squares.     | 36. 2 squares, 4 rectangles. | 37. 1 square, 4 triangles. |
| 38. 1 circle, 1 curved surface. | 39. 2 circles, 1 curved surface. | 40. 1 curved surface. |                    |                              |                            |

### EXERCISE 2.

- |                                       |  |   |
|---------------------------------------|--|---|
| 1. $143^\circ, 180^\circ$ .           | 2. $65^\circ, 180^\circ$ .               | 3. 2 rt $\angle$ s or $180^\circ$ .       |
| 4. $180^\circ$ .                      | 6. $58^\circ, 360^\circ$ .               | 7. 4 rt $\angle$ s or $360^\circ$ .       |
| 8. $180^\circ, 2$ rt $\angle$ s.      | 9. $15^\circ, \frac{1}{3}$ rt $\angle$ . | 10. $45^\circ, \frac{1}{2}$ rt $\angle$ . |
| 11. $112\frac{1}{2}^\circ$ .          | 12. $45^\circ$ .                         | 13. $133^\circ, 47^\circ, 133^\circ$ .    |
| 14. $52^\circ, 128^\circ, 52^\circ$ . | 15. $90^\circ, 90^\circ, 90^\circ$ .     |   |

### EXERCISE 3.

- |                 |                 |        |        |
|-----------------|-----------------|--------|--------|
| 5. $90^\circ$ . | 7. $90^\circ$ . | 12. 2. | 14. 4. |
|-----------------|-----------------|--------|--------|

### EXERCISE 5.

- |                  |                  |                  |
|------------------|------------------|------------------|
| 1. $180^\circ$ . | 2. $180^\circ$ . | 3. $180^\circ$ . |
|------------------|------------------|------------------|

### EXERCISE 6.

- |                                      |                   |                                      |                                      |
|--------------------------------------|-------------------|--------------------------------------|--------------------------------------|
| 6. $60^\circ$ .                      | 7. $34^\circ$ .   | 8. Impossible.                       | 9. Impossible.                       |
| 10. $30^\circ, 60^\circ, 90^\circ$ . | 11. $78^\circ$ .  | 12. $36^\circ, 72^\circ, 72^\circ$ . | 13. $45^\circ, 45^\circ, 90^\circ$ . |
| 15. $110^\circ$ .                    | 16. $110^\circ$ . | 17. $152^\circ$ .                    | 18. $90^\circ$ .                     |

### EXERCISE 7.

- |                       |                              |                     |   |
|-----------------------|------------------------------|---------------------|---|
| 1. $360^\circ$ .      | 2. $132^\circ$ .             | 3. $110^\circ$ .    | 4. $36^\circ, 72^\circ, 108^\circ, 144^\circ$ . |
| 5. $60^\circ$ .       | 7. 4 rt $\angle$ s.          | 8. 4 rt $\angle$ s. | 9. 16 rt $\angle$ s.                            |
| 10. 44 rt $\angle$ s. | 11. $108^\circ$ .            | 12. $120^\circ$ .   | 13. $128\frac{1}{2}^\circ$ .                    |
| 14. $144^\circ$ .     | 15. $176\frac{2}{3}^\circ$ . | 16. $177^\circ$ .   | 17. 4 sides.                                    |
| 18. 5 sides.          | 19. 12 sides.                | 20. 150 sides.      |   |

**EXERCISE 8 (a).**

1. 6 edges.                      2.  $36^\circ$ .                      3.  $60^\circ$ .                      5. 2 triangles.  
 7. 4.1 in.                      8.  $108^\circ$ .

**EXERCISE 8 (b).**

2. 6.                      4.  $81^\circ$ .

**EXERCISE 9 (a).**

1. 8 vertices.                      2.  $60^\circ$ .                      4.  $49^\circ$ .                      7.  $61^\circ 10'$ .

**EXERCISE 9 (b).**

1.  $90^\circ$ .                      5.  $105^\circ$ .                      6. 5 vertices.  
 8.  $108^\circ$  is not a factor of  $360^\circ$

**EXERCISE 10 (a).**

2.  $60^\circ$ .                      3.  $150^\circ$ .                      5.  $(x+60)^\circ$ .  
 6. 55.5 mm.                      7. 15 rt  $\triangle$ s.                      8.  $61^\circ$ .

**EXERCISE 10 (b).**

2. 2.3 in.                      3.  $50^\circ$ .                      4. 10 sides.

**EXERCISE 11 (a).**

2.  $69^\circ$ .                      4.  $60^\circ$ .                      6.  $72^\circ$ .                      7.  $120^\circ$ .

**EXERCISE 11 (b).**

1. 1.45 in.                      2.  $80^\circ$ .                      7.  $137^\circ$ .

**EXERCISE 12.**

1. 2.5 in.;  $76^\circ, 55^\circ$ .                      2. 4.7 in.;  $35^\circ, 43^\circ$ .                      3. 6.9 cm.;  $72^\circ, 63^\circ$ .

**EXERCISE 16.**

22. 4.95 cm. approximately.                      26. 490 ft.                      27. 424 ft., 848 ft.

**EXERCISE 17 (a).**

3.  $45^\circ$ .                      5.  $x=30^\circ$ .                      7.  $144^\circ$ .

**EXERCISE 17 (b).**

6.  $30^\circ$ .                      8.  $110^\circ$ .

EXERCISE 18 (a).

2. 2.95 cm.      3. 30°      4. 12 pairs.      5.  $22\frac{1}{2}^\circ$ .  
6. 180°      7. 24.

EXERCISE 18 (b).

7. 28°.

EXERCISE 19 (a).

2. 1 : 2.      4. 120°.      5. 93°.      7.  $1\frac{1}{4}$  rt  $\Delta$ s.      8. 23.4 cm.

EXERCISE 19 (b).

5. 72°.

EXERCISE 20 (a).

3. 75°      4. 18.      6. 3.57 miles N.; 1.2 miles E.      7. 20.

EXERCISE 20 (b).

6. 60°.

EXERCISE 21.

20.  $\angle ABC = 75^\circ$ ,  $\angle ABD = 45^\circ$ .      21. 4.1 in.      25. 1.6 miles.  
27. .2 in.      28.  $AF = 3.25$  in.,  $\angle BAF = 10\frac{1}{2}^\circ$ ,  $\angle CDE = 105^\circ$ .

EXERCISE 22.

1. 8.6 sq. in.      2. 66.4 sq. cm.      3. 11 sq. ft. 59 sq. in.  
4. 8593 sq. cm.      5. 16.24 sq. in.      6. 47.25 sq. cm.  
7. 6.75 sq. in.      8. 14.04 sq. in.      9. 3.75 sq. in.  
10. 20 sq. cm.      11. 5.6 sq. in.      12. 28.5 sq. cm.  
13. 5.3 sq. in.      14. 4.3 sq. in.      15. 34.1 sq. cm.  
16. 2.6 sq. in.      17. 5.5 sq. in.      26. 40 acres.  
27. 44 ac. 3740 sq. yd.      28. .91 acre.      29. .735 acre.  
30. .5336 acre.      31. 32.8 lb.      32. 3 sq. m.  
33. 975 sq. ft.

EXERCISE 23 (a).

1. 3.9 in.      2.  $\frac{1}{2}$  in.      3.  $\sin B = \frac{2}{3}$ ,  $\cos B = \frac{4}{5}$ ,  $\tan B = \frac{3}{4}$ .  
4.  $\cos A = \frac{3}{5}$ ,  $\sin A = \frac{4}{5}$ ,  $\tan A = \frac{4}{3}$ .      6.  $AB = 2$  in.  
7.  $\sin 71^\circ = .9455$ ;  $\sin 52^\circ = .7880$ ;  $\sin 18^\circ = .3090$ .  
8.  $\cos 21^\circ = .9336$ ;  $\cos 43^\circ = .7314$ ;  $\cos 85^\circ = .0872$ .  
9.  $\tan 45^\circ = 1$ ;  $\tan 63^\circ = 1.963$ ;  $\tan 78^\circ = 4.705$ .      10.  $A = 39^\circ$ .

## EXERCISE 23 (b).

2. From tables,  $\sin 56^\circ = \cdot 8290$ .      3.  $\cos 73^\circ = \cdot 2924$ .  
 4.  $\tan 15^\circ = \cdot 2679$ .      5. 22.6 ft.  
 6. 1.333.      7.  $\sin 60^\circ = \cdot 8660$ ;  $\cos 60^\circ = \cdot 5$ ;  $\tan 60^\circ = 1.7320$ .  
 8.  $\frac{1}{3}$  or .923.      9. 119.8 ft.      10. 139.88 ft.  
 11.  $51^\circ 21'$ .      12.  $61^\circ 56'$ ;  $119^\circ 4'$ .      13.  $\angle B = 36^\circ 52'$ ;  $\cos B = \cdot 8$ ;  $AB = 5$ .

## EXERCISE 24.

1.  $c = 6.5$  in.      2.  $b = 2$  in.      3.  $a = 3$  in.  
 4.  $a = 3.5$  in.      5.  $c = 13.7$  cm.      6.  $c = 12$  in.  
 7.  $a = 2.1$  cm.      8.  $c = 4.2$  in.      9. 385.8 mm.  
 10.  $a = 9.3$  in.      11.  $\sqrt{3} = 1.73$ .      12.  $\sqrt{5} = 2.24$ .  
 13.  $\sqrt{7} = 2.65$ .      14.  $\sqrt{11} = 3.32$ .      15. 32 ft.  
 16. 18.4 cm.      17. 7.1 in.      18. 21 ft.  
 19. 6 in.      20. 17.6 in.      21. (a), (c).  
 23. 4.7 in.      26.  $AC^2 = 1296 + 1156 = 2452$ ;  $EC^2 = 2916 + 2601 = 5517$ ;  
 $AE^2 = 225 + 7744 = 7969$ ;  $AC^2 + EC^2 = 2452 + 5517 = 7969$ .  
 27.  $\sqrt{34}$  or 5.83.      28.  $\sqrt{98}$  or 9.9.      29. Yes; 12.04 in.  
 30.  $2\frac{1}{2}$  in.      31. 48 ft.      32.  $45^\circ, 45^\circ, 90^\circ$ ; .5 sq. in.  
 33. 337.5 yd.      34. 68 ft.      35. 155.5 yd.

## EXERCISE 25 (a).

2. 2 in.      3. 11 sq. in.      7.  $65^\circ$ .

## EXERCISE 25 (b).

1. 5.24 sq. in.      2. 35 sq. cm.      7. 4 in.

## EXERCISE 26 (a).

1. 3 ways.  $A = \frac{1}{2}bh$ .      2. Ratio = 3 : 4 : 5;  $5^2 = 4^2 + 3^2$ .  
 3. 8 sq. in.      6.  $113^\circ$ .      8.  $15^\circ$ .

## EXERCISE 26 (b).

1. 2.85 sq. in.      2. 10 sq. in.      6. 1.41 in.

## EXERCISE 27 (a).

1.  $\frac{1}{2}d(x+y)$ .      2. 12 sq. in.      3. 6.9 sq. in.      7.  $125^\circ$ .

EXERCISE 27 (b).

1. 36.2 sq. cm.      2. 13.7 sq. in.      5. 5.66 in.  
 6. 1.03 in.      8. 177°.

EXERCISE 28 (a).

1.  $A = \sqrt{s(s-a)(s-b)(s-c)}$ .      2. 2 sq. in.      3. 5 cm.  
 7. 70°.

EXERCISE 28 (b).

1. 6.4 sq. in.      2. 24 sq. in.      5. 4.95 cm.; 11.88 sq. cm.  
 8.  $x = 10^\circ$ .

EXERCISE 29.

1.  $6\frac{1}{2}$  in.      2. 15.26 cm.      3. 32.25 cm.      4. 15 sq. in.  
 6. 1.66 in.      11. 9.43 in.      12. 3.32 in.

EXERCISE 30.

1.  $\angle OAB = 39^\circ$ ; Reflex  $\angle AOB = 258^\circ$ .      2.  $51^\circ$ .      3.  $129^\circ$ .  
 4. Reflex  $\angle AOB = 216^\circ$ ;  $\angle ARB = 108^\circ$ ;  $\angle OAB = 18^\circ$ .  
 5.  $\angle OAB = 31^\circ$ ;  $\angle OBC = 59^\circ$ .      6.  $63^\circ$ .      7.  $122^\circ$ .  
 8.  $105^\circ$ .      9.  $109^\circ, 91^\circ$       15. 3.46 sq. in. or  $2\sqrt{3}$ .

EXERCISE 31.

2. 2.71 in.      3. 1.6 in.      4. 2.62 in.      5. 2.33 in.  
 6. 6 in.      9. 2.64 in.      12. 2.29 in.      14. 2.23 in.  
 15. 6.71 in.      16. 7.1 in.      17. 9 ft. 11.8 in.      18. 5 ft. 1.6 in.  
 19. 6 cm.      21. 3.46 in.      22.  $DE = 3.25$  in.

EXERCISE 32.

1. 88 in. or 7 ft. 4 in.      2. 26.4 in.      3. 44 ft.  
 4.  $100\frac{1}{4}$  ft.      5.  $97\frac{3}{4}$  ft.      6. 41.36 in.  
 7. 94.25 in.      8. 11.62 in.      9. 229.34 ft.  
 10.  $r = 2.82$  ft.;  $d = 5.64$  ft.      11.  $r = 2.46$  ft.;  $d = 4.91$  ft.  
 12.  $r = 6.84$  in.;  $d = 13.68$  in.      13.  $r = 1.23$  in.;  $d = 2.45$  in.  
 14.  $r = .23$  ft.;  $d = .45$  ft.      15.  $r = .17$  in.;  $d = .33$  in.  
 16. 66.4 sq. in.      17.  $r = 1.24$  in.  
 18. 58.5 lb.      19.  $56\frac{1}{4}$  sq. cm.  
 20. 6 ft. 2 in.      21.  $14\frac{1}{2}$  in.

## EXERCISE 33 (a).

1.  $63^\circ$ .      2. 6 in.      5.  $20^\circ$ .      6. Two.  
 7.  $48^\circ, 60^\circ, 72^\circ$ .      8.  $60^\circ$ .

## EXERCISE 33 (b).

1.  $120^\circ$ .    2.  $AC=2.24$  in.,  $\angle ABC=45^\circ$ .    3. 40 ft.    4.  $6.3504$  sq. in.

## EXERCISE 34 (a).

1. 10 cm.      2.  $t=\sqrt{c^2-r^2}$ .      4.  $30^\circ$ .      5.  $152^\circ, 72^\circ, 136^\circ$ .  
 7.  $150^\circ$ .      8. 30 sq. in.

## EXERCISE 34 (b).

3. About  $57\frac{1}{2}^\circ$ .  
 6. Distance= $212.5$  ft.,  $\angle$  of elevation = between  $24\frac{1}{2}^\circ$  and  $25^\circ$ .  
 7.  $.81$  in.

## EXERCISE 35 (a).

1. 8 cm.      2. 12 sq. cm.      5. 4.      6. 13 in.  
 7. 8 N., 5 W.      8. 20 sq. in.

## EXERCISE 35 (b).

1.  $36^\circ$ .      4.  $12.7$  cm.      6.  $.8$  mile.      8. 4 in., 4 in.,  $5.7$  in.

## EXERCISE 36 (a).

1.  $\angle AOC=40^\circ$ ;  $\angle OAC=70^\circ$ .      2. 3.1416.      4.  $30^\circ$ .  
 6.  $105^\circ$ .      7. 2 in.      8. 4; 2 on each side of the base.

## EXERCISE 36 (b).

1.  $150^\circ$ .    3. Other side= $7.42$  in., area= $11.1$  sq. in.    6.  $2.12$  in.

## EXERCISE 37 (a).

3.  $T=\sqrt{c^2-(a-b)^2}$ .      4. Equal to the radius.  
 6.  $\sqrt{3}$  sq. in., or  $1.73$  sq. in.      7. 6 in.      8. 4 cm.

## EXERCISE 37 (b).

1.  $3.07$  in.

## EXERCISE 38 (a).

1.  $\angle BOD=144^\circ$ ,  $\angle COD=72^\circ$ .      5. 10 miles.      6.  $36^\circ$ .  
 7.  $38^\circ$ .      8.  $10.38$  sq. in.

## EXERCISE 38 (b).

- $AB=1.76$  in.,  $BC=1.09$  in.,  $CA=-2.85$  in.;  
 $AB+BC+CA=0$ ;  $CA=-2.61x$ .
- $29.2$  ft.      4.  $\angle BAC=65^\circ$ ,  $\angle AEB=84^\circ$ ,  $\angle CEB=96^\circ$ .

## EXERCISE 39 (a).

- $124^\circ$ .
- $150^\circ$ .
- $x=45^\circ$ .
- $12$  sq. units.

## EXERCISE 39 (b).

- $\angle OQP=1$  rt.  $\angle$ ,  $TQ=QP$ .
- $\angle A$  = about  $63.8^\circ$ ;  $Bx=3.4$  cm.;  $Bx=\frac{1}{2}AC$ .

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