The Concepts and Principles of Physics

Volume I

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NOTE

This volume is a preliminary version of the first half of a textbook to be entitled *The Concepts and Principles of Physics*. It is our belief that by introducing vector methods at the outset and by drawing on the knowledge of calculus the student has already acquired and is concurrently expanding, a course in physics for science and engineering majors can treat the subject in a more fundamental way than would otherwise be possible. Owing to the fact that there is little obsolescence in physics, we have had to limit ourselves to those topics that are part of the main stream of classical and modern physics in order to develop them fully. Accordingly, such traditional components of an elementary physics course as sound, hydrodynamics, geometrical optics, and various engineering applications have had to be severely condensed or eliminated. The present preliminary edition is deliberately abbreviated in order that we may learn through our experience and that of others in teaching from it of the most appropriate configuration the final work should assume.

S. B.

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Chapter 1

INTRODUCTION

The purpose of this book is to provide a modern introduction to the study of physics. By physics we mean the science whose concern is the properties of matter—a vast subject indeed, but one capable of being reduced to certain fundamental ideas. It is these ideas, rather than the details of their application, which we shall emphasize.

The domain of physics is actually narrower than the above statement implies, since biology and chemistry also deal with properties of matter. Biology concentrates on living matter, however, while chemistry deals with those processes in which the chemical composition of the matter involved is significant. The three sciences overlap, and as a result such disciplines as biophysics, biochemistry, physical chemistry and chemical physics have arisen. Whenever there is a biological activity in which the "living" aspects are secondary, for example the flow of blood through the arteries, physics can supply important information. Whenever chemistry touches directly on the interactions of the elementary particles of which all matter is constituted, for example how two hydrogen atoms form a molecule, it must call upon physics for an explanation. The basic laws of physics, such as the first and second laws of thermodynamics and the exclusion principle, which we shall discuss in detail, have had profound effects on the thinking of biologists and chemists.

Physics is the most successful of the sciences. The reason for this is the unique ability of physics both to arrive at general principles that apply to all systems despite their complexity, and to isolate various aspects of the behavior of matter from one another so that they may be individually analyzed. The latter method of attack is made possible by the large differences in relative magnitude exhibited by different physical phenomena; thus when we deal with the forces within an atomic nucleus we can usually ignore any other forces present, such as electrical and gravitational ones, because they are less pronounced in this case. In atoms and molecules electrical forces are the most significant, while in treating large bodies gravitational forces predominate and we are justified in neglecting nuclear and electrical ones. In this manner we are able to investigate the properties of matter one by one, and then try to put the pieces together to explain the exceedingly complicated workings of nature.

1.1. The Scientific Method. A science differs from other scholarly ways of analyzing nature in that it employs the *scientific method* in pursuing its objectives. There are three steps in the scientific method. The first is the inductive formulation of a *hypothesis:* we observe some events that occur in the natural world and try to discover causal relations among them. If we succeed, we summarize our ideas as to the connection between cause and effect in the form of a hypothesis.

Next, we try to infer all possible conclusions from this hypothesis, and compare the conclusions with experience. In testing a hypothesis we do not always wait for an appropriate natural event to take place, but may deliberately arrange a situation which permits such a test to be made. The latter is called an *experiment.* If the experiment contradicts the hypothesis we discard the hypothesis. Even if the conclusions drawn from our hypothesis agree with the results of an experiment, though, we cannot conclude that the hypothesis is correct. The reason is that we can never affirm an antecedent here the hypothesis by affirming its consequence, although the antecedent may be denied by denying the consequence. The classic if hoary example of this begins with the statement that all men are mortal. If Socrates is a man, then he assuredly is mortal. However, if all we know is that Socrates is mortal, he is then not necessarily a man: he may with equal likelihood be a cat or a goat. But if Socrates is *not* mortal, then he cannot be a man. Thus agreement between hypothesis and experiment does not establish the certainty of the hypothesis, and while repeated successful experiments make it more and more probable that the hypothesis is on the right track, they can never establish it beyond question. There have been instances in physics and other sciences where theories that withstood testing for hundreds of years have had to be abandoned when contradicted by a new, more precise experiment.

The third step in the scientific method is reformulating our hypothesis in the light of experiment. This is done when the experimental results disagree with the original hypothesis, and the revised hypothesis must then be checked against further experiments. The scientific method is thus a self-correcting procedure.

To compensate for its inherent lack of certainty, the scientific method has the supreme virtue that if we adhere strictly to it we will never go astray in our conclusions about the workings of nature. Laymen often wonder why scientists bother to retest accepted theories whenever improved instruments are invented; the answer is that only such continual review permits us to retain and use knowledge inherited from the past.

1.2. Units. Physics is a quantitative science. It is therefore necessary that all measurements be made in terms of standard *units,* for example the foot in the case of distance, in order that there be no question about the exact magnitude of the quantity involved. If we say that a certain object is 8 long, the statement is meaningless unless we specify the name of the unit involved. There is a considerable difference between 8 inches and 8 miles.

A great many systems of units have been developed in the thousands of years that measurements have been made, but today two of them predominate in physics. These are the *British* system, in which the unit of length is the foot (ft), the unit of weight the pound (lb), and the unit of time the second (sec), and the *metric* system, in which the unit of length is the meter (m), the unit of mass the kilogram (kg), and the unit of time once more the second (sec). The various multiples of these units in common use, and the conversion factors that enable translating measurements made in one system to the units of the other, appear in Table 1-1. (The important distinction between weight and mass will be explained in Chapter 4.)

Table 1-1

Units

1 meter = 3.281 feet	1 foot = 0.3048 meter
1 kilogram (force) = 2.205 pounds	1 pound = 0.4536 kilograms (force)

1 meter = 100 centimeters = 1000 millimeters = 0.001 kilometers
1 kilogram = 1000 grams = 1,000,000 milligrams

1 foot = 12 inches = 1/3 yard = 1/5280 mile
1 pound = 16 ounces = 1/2000 ton

The precise magnitudes of the above units are embodied in *standards*. The standard meter is the distance between two marks on a certain carefully preserved platinum-iridium bar, and the length of the foot is, by agreement, 1200/3937 meter. The standard kilogram is the mass of a certain cylinder of platinum-iridium, and the pound mass is, again by agreement, 0.4535924277 kilogram. The second is defined a little differently; it is 1/86,400 of a mean solar day, which is the average time required by the earth to rotate once with respect to the sun.

In recent years the standards of length and time have been accurately compared with quantities found in nature whose magnitudes are believed to be invariant. Thus the length of the meter may be specified in terms of one of the characteristic wavelengths of light emitted by a particular isotope of cadmium, and the duration of the second may be specified in terms of one of the characteristic frequencies of vibration of the ammonia molecule. In this way every laboratory in the world has direct access to the fundamental standards of measurement.

It is remarkable that almost all of the diverse quantities that occur in physics can be expressed as combinations of length, mass, and time measurements. The specific combination of the latter is called the *dimension* of a particular quantity. To give an example, "work", in the sense that this term is used in physics, has the dimensions of (mass) $(\text{length})^2/(\text{time})^2$, which would be $\text{kg m}^2/\text{sec}^2$ in the metric system. This set of units is called the *joule* as an abbreviation, but, like most of its fellow specialized units, it can always be reduced to its equivalent in units of length, mass, and time. A useful check on the correctness of a calculation or derivation is to see whether the dimensions throughout are consistent; in this way we can learn at once if we have omitted some pertinent quentity or made an incorrect conversion from one size unit to another. Many of the illustrative examples in this book have been worked out with the units given for each important step to show how this is done. After a little practice the subject of dimensions will be much less frightening than it perhaps seems at this point.

Two quantities not normally specified in terms of length, mass, and time are electrical charge and temperature. Although it is possible to formulate electrical theory without invoking additional units, and in fact the subject was so treated until perhaps fifty years ago, the existence in nature of a basic unit of charge makes it convenient to use electrical charge as a separate dimensional quantity. Temperature, as we shall see, is in a category by itself, and reference must be made to certain properties of matter in order to deal with it adequately.

1.3. Measurements. There is a limit to the accuracy with which physical measurements can be made. When we express the results of weighing a man on a particular scale as 183 lb, be are not saying that this is his *exact* weight. Upon using a better scale it might turn out that he weighs 182.8 lb; a still more accurate determination might yield 182.81 lb. In each case we may not use more *significant figures* than are justified by the particular means of making the measurement.

All numerical statements contain significant figures, which are the essential results of the measurement, plus some means of indicating the size of the number. An elephant weighs 3500 lb. If the scale is accurate to ±100 lb, only the first two digits are significant. The zeros serve only to indicate the position of the decimal point. If the scale is accurate to ±10 lb, the first three digits are significant. It is frequently difficult to tell just how many significant figures are present in a given number.

Significant figures are important when various quantities are to be combined arithmetically. The elephant weighs 3500±100 lb, and on its head sits a mahout weighing 182.81 lb. What is the total weight? It is absurd to write 3682.81±100 lb; a correct statement under the circumstances would be 3700 ± 100 lb. In general, the result of a numerical computation contains no more significant figures than the poorest known figure that went into it, although one should carry along an additional figure during the arithmetic.

Very large and very small numbers are not only cumbersome to write, but also may be misleading as to the number of significant figures they contain. The mean distance from the earth to the sun is 92,900,000 miles; certainly the 929 is significant, but how many of the zeros are? Should we write it as $92{,}900{,}000 \pm 100{,}000$ miles? A better procedure makes use of *exponential notation* involving powers of ten. The number 45 may be written 4.5×10; 679 may be written 6.79×100 or 6.79×10^2; 0.02 may be written 2×0.01 or 2×10^{-2}; and so on. The mean distance between the earth and the sun in this notation is therefore $9.29 \pm 0.01 \times 10^7$ miles, and usually it is sufficient to write just 9.29×10^7 miles since it is understood that we employ only as many digits as are justified by the accuracy of the measurement and leave the magnitude to be set by the exponent of the 10.

1.4. Mathematics and Physics. The laws of physics are expressed most compactly and conveniently as mathematical relationships among quantities appropriate in describing the particular system involved. There is a great distinction between mathematics and physics, however. Mathematics is essentially the study of certain relationships among symbols, with no reference to nature at all. Physics, on the other hand, is concerned exclusively with nature, and uses mathematics only as an aid in stating its findings. To the physicist mathematics is a tool, and his work consists of much more than manipulating equations.

While all of the material in this book could have been presented using no more than the most elementary geometry and algebra, such a procedure would be very wasteful. An equation in physics is an abstraction of reality, and its concise form helps us to grasp the meaning of the things it describes. The more advanced the mathematics we employ, generally speaking, the briefer the equation and the more clear the statement. A mathematical technique, for instance integration, may be complicated in itself, but mastering it permits us to absorb a good deal of information less painfully than otherwise would be possible. For this reason more mathematics is used in what follows than perhaps is conventional, and as a result we are able to penetrate a little more deeply into the fundamental principles governing the physical world. The next chapter, in fact, is a discussion of *vector analysis*, a branch of mathematics especially useful in the study of physics.

Chapter 2

VECTOR ANALYSIS

Before going into the actual subject matter of physics we must learn how to deal with the various measurable quantities to be encountered. The most familiar of them are simply magnitudes, usually with a unit included so that there is no ambiguity in what is meant. Such a quantity is called a *scalar,* an example of which is "12 feet." Scalars are treated mathematically in the same way that ordinary numbers are. When we add 3 tons and 9 tons we have 12 tons, and when we divide 72 miles by 3 hours we obtain 24 miles/hour.

Other quantities present more complications. A boat travels 45 miles, a scalar statement. However, for many purposes this amount of information is inadequate. The boat may have gone 45 miles to the north, it may have gone 20 miles west and then 25 miles south, or it may have gone 22.5 miles east and then 22.5 miles west to end up at its starting point. By adding a *direction* to a magnitude, for instance by saying that the boat went 45 miles southwest, we have a more complete description of its displacement. Quantities that involve directions as well as magnitudes are called *vectors,* and they play an important role in physics. Vectors require a different kind of arithmetic from scalars, one that takes directions into account as well as numbers, and we shall encounter many examples of them in this book.

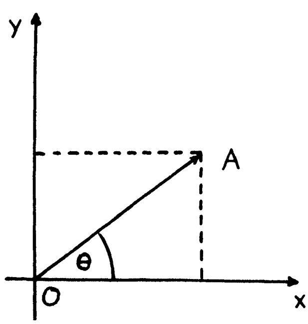


Fig. 2-1a

2-1. Vector Addition. The most obvious of all vectors is a displacement. If an object is originally at the point O in Fig. 2-1a and is then moved to the point A, its change in position may be represented by the line segment OA whose arrow indicates the sense of the displacement. Since this line segment has both a magnitude (its length) and a direction, it is a vector. If we wanted to specify the displacement OA without the aid of Fig. 2-1a we would have to state three things: (1) the plane it lies in, here the plane of the paper; (2) the distance from O to A; and (3) the angle between OA and some reference direction, say the x-axis. There are other ways of describing OA. A common one is to have O as the origin of a coordinate system and give the corresponding coordinates of the point A, which would be A_x, A_y, A_z in rectangular coordinates, (Fig. 2-1b), A_r, θ, ϕ in spherical polar coordinates (Fig. 2-1c), and so on. No matter what means we employ to describe OA, though, we require three separate quantities. (In the example of the boat given above it is implied that the boat is on the surface of the earth, providing the third part of the vector description of its displacement.)

Let us now move the object through two successive displacements, OA followed by AB (Fig. 2-2). The net result is the displacement OB, which is defined as the *sum* of OA and AB. In general, if we wish to add the vector **A** (bold-face type is customarily used to represent vectors) to the vector **B**, we draw **A** and at its end draw **B** as shown. The sum of the two, the *resultant* **R** is the vector extending from the beginning of **A** to the terminus of **B**. The same procedure is followed if the third vector **C** is to be added; **C** is drawn at the end of **R**, and **R'** is the final resultant. In this way any number of vectors may be added together. In the special case where the vectors involved are parallel, the magnitude of their resultant is the same as the scalar sum of the individual magnitudes, but otherwise it is smaller.

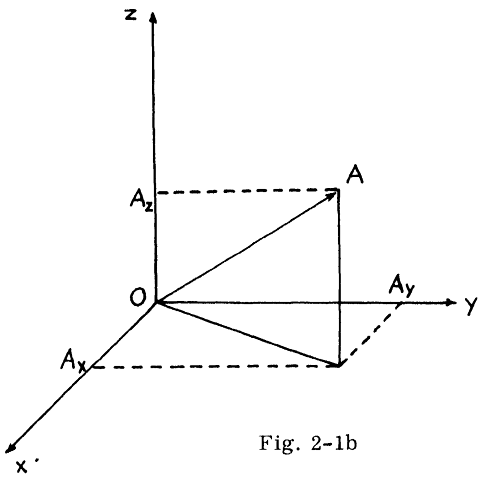


Fig. 2-1b

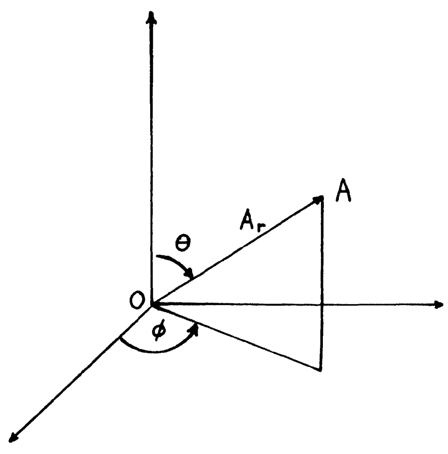


Fig. 2-1c

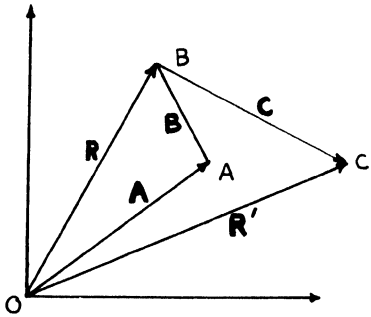


Fig. 2-2

Example. What is the resultant of a vector of magnitude 12 that is at an angle of 30° from the x-axis (angles specified in this manner are normally to be taken counterclockwise from the +x direction) and another of magnitude 8 that is at an angle of 100° from the x-axis?

Solution. The problem is sketched in Fig. 2-3. The magnitude of the resultant is the side of the triangle whose other side of the triangle whose other sides are of lengths 12 and 8 with an included angle of 110°. From the law of cosines,

$$R = \sqrt{8^2 + 12^2 - 2x8x12\cos 110^\circ}$$
$$= 16.5 \ .$$

The angle between **R** and the side of length 12 may be found from the law of sines, yielding

$$\frac{\sin\phi}{8} = \frac{\sin 110^\circ}{16.5}$$
$$\phi = 27^\circ \ .$$

The resultant therefore has the magnitude 16.5 and makes an angle of 57° with the x-axis.

Where no great accuracy is required vector addition may be performed graphically with the help of a ruler and protractor. The procedure is simply to make a scale drawing of the vectors placed head-to-tail, and then to measure the length and orientation of the resultant. Even when a numerical calculation is made a careful sketch is useful as a check. Of course, the graphical method can only be used in the case of three or more vectors when all of them lie in a plane.

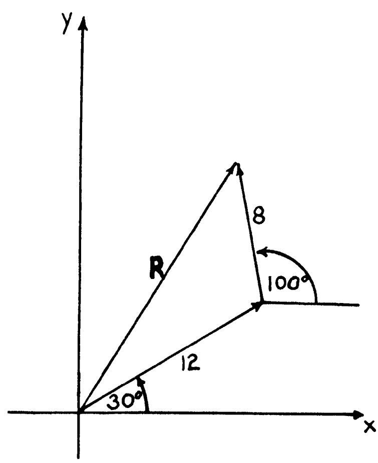


Fig. 2-3

While it is perhaps not as obvious as it is for scalars, vectors may be added together in any order. They obey both commutative and distributive rules; that is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

and (2.1)

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

The method by which one vector is subtracted from another follows from the relationship

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}). \tag{2.2}$$

Vectors that lie along the same straight line add in the same way that scalars do; therefore the negative of a vector, here $-\mathbf{B}$, is a vector of the same magnitude as $\mathbf{B}$ drawn along the same line but pointing in the opposite direction. Fig. 2-4 shows how Eq. (2.2) is to be applied. First $-\mathbf{B}$ is constructed, and then it is combined with $\mathbf{A}$ by ordinary vector addition.

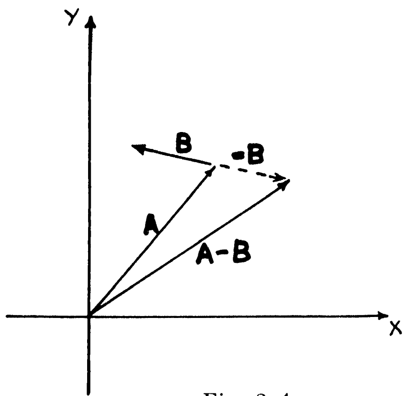


Fig. 2-4

2.2. Resolution into Components. We can take further advantage of the fact that when adding and subtracting vectors lying on the same line only their magnitudes need be considered. What is done is to establish certain fixed directions in space, usually the x, y and z coordinate axes, and to express each vector we encounter in terms of three vectors along these directions. The breaking down of a vector into an equivalent set of other vectors in fixed directions is called the resolution of the vector into its components.

Let us first consider vectors in a plane. In Fig. 2-5 the vector $\mathbf{C}$ is evidently the sum of the vectors $\mathbf{C}_x$ and $\mathbf{C}_y$, which are the components of $\mathbf{C}$. Their magnitudes are

$$C_x = C \cos \theta$$

$$C_y = C \sin \theta.$$

A vector in a plane may equivalently be specified either by its magnitude and direction or by its components along two perpendicular axes, and we can go from one method to the other by using simple trigonometry.

In adding two vectors together, we begin by resolving each one into components. Then the components along the same axes are added together as though they were ordinary numbers, and the

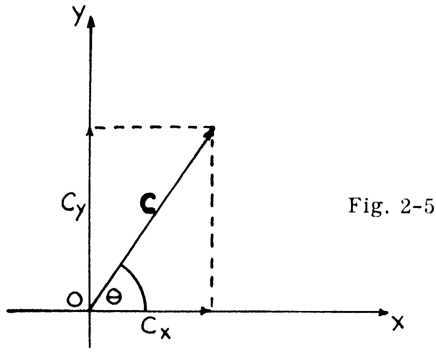


Fig. 2-5

resultant is the vector determined by the new set of components. The resultant may be given in terms of these components or its magnitude and direction may be computed from them. If we write

$$\mathbf{A} = (A_x, A_y)$$

to express the fact that the vector **A** is completely defined by its components A_x and A_y, and similarly with **B**, the resultant of **A** + **B** is

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x, A_y) + (B_x, B_y)$$
$$= (A_x + B_x,\ A_y + B_y)\ .$$

This statement can be generalized in an obvious way to cover more than two vectors in space. Adding the three-dimensional vectors **A**, **B**, and **C** together thus yields in this notation

$$\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$
$$= (A_x, A_y, A_z) + (B_x, B_y, B_z) + (C_x, C_y, C_z)$$
$$= (A_x + B_x + C_x,\ A_y + B_y + C_y,\ A_z + B_z + C_z) \qquad (2.3)$$
$$= (R_x, R_y, R_z)\ .$$

Example. Find by the method of components the resultant of a vector of magnitude 12 that is at an angle of 30° from the x-axis and another of magnitude 8 that is at an angle of 100° from the x-axis.

Solution. The first step is to resolve each of the vectors into x and y components. Calling them **A** and **B** respectively, from Fig. 2-6 we see that

$$A_x = 12\cos 30^\circ = 10.4$$
$$A_y = 12\sin 30^\circ = 6.0$$
$$B_x = -8\cos 80^\circ = -1.4$$
$$B_y = 8\sin 80^\circ = 7.9\ .$$

The components of the resultant **R** are

$$R_x = A_x + B_x = 9.0$$

$$R_y = A_y + B_y = 13.9 ,$$

which specifies **R** completely. The alternate description in terms of magnitude and direction is not difficult to obtain. From the Pythagorean theorem,

$$R = \sqrt{R_x^2 + R_y^2} = 16.5 ,$$

and the angle made by **R** with the x-axis is determined by

$$\tan \theta = \frac{R_y}{R_x} = 1.54 ,$$

so that

$$\theta = 57^\circ .$$

In all but the simplest problems the use of vector components in addition and subtraction is preferable, and it is well to become familiar with them.

2.3. Scalar Product. When ordinary numbers are multiplied together, the process is actually one of addition. There is no such direct relationship between the addition and "multiplication" of vectors, though. Two methods of combining vectors which have been found to be very useful in treating physical situations are customarily called vector multiplication; in one of them the result is a scalar quantity, and in the other it is a vector quantity. The two are quite distinct, and will be discussed separately. The techniques of vector multiplication do not provide any new information when applied to a particular problem, but by abbreviating the purely mathematical aspects they make possible a clearer perspective of the physical principles involved.

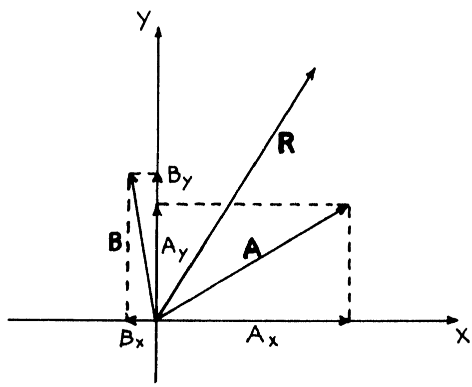


Fig. 2-6

The *scalar product* of the two vectors **A** and **B** is written

$$\mathbf{A} \cdot \mathbf{B} ,$$

to be read "A dot B." It is sometimes called the "dot product." The scalar product is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta, \tag{2.4}$$

where θ is the angle between **A** and **B**. Depending upon the value of θ, the magnitude of $\mathbf{A}\cdot\mathbf{B}$ can vary from AB to -AB, the former occurring when **A** and **B** are parallel and the latter when they are antiparallel. $\mathbf{A}\cdot\mathbf{B}=0$, no matter what the magnitudes of **A** and **B** separately, when **A** and **B** are perpendicular, since in that case $\theta=90^\circ$ and $\cos\theta=0$.

There is an interpretation of the scalar product that is often quite helpful. Let us rewrite Eq. (2.4) as

$$\mathbf{A}\cdot\mathbf{B} = (A\cos\theta)\,B \tag{2.5a}$$

or

$$\mathbf{A}\cdot\mathbf{B} = A\,(B\cos\theta)\ . \tag{2.5b}$$

Now $A\cos\theta$ is the component of **A** in the direction of **B** (Fig. 2-7), and similarly $B\cos\theta$ is the component of **B** in the direction of **A**. Hence we can say that the scalar product of two vectors is the component of one of them in the direction of the other, multiplied (in the ordinary sense)by the magnitude of the latter. It makes no difference which of the two forms of Eq. (2.5) is used.

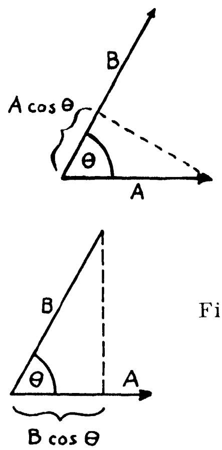


Fig. 2-7

Example. Work is defined as the scalar product of a force and the displacement through which it acts. If a box is moved a distance of 5 ft along an inclined plane which makes an angle of 60° with the horizontal by means of a horizontal force of 10 lb, how much work is done?

Solution. Since $\mathbf{W} = \mathbf{F}\cdot\mathbf{d}$,

$$\begin{aligned} W &= Fd\cos\theta \\ &= 10\text{ lb} \times 5\text{ ft} \times \cos 60^\circ \\ &= 25\text{ ft-lb}\ . \end{aligned}$$

As we have seen, vectors can also be completely described by their components, and we shall now determine the form of the scalar product in terms of them. Fig. 2-8 shows the vectors **A** and **B** that make the angles ϕ and ψ respectively with the +x axis. The angle θ included between them is

$$\theta = \phi - \psi\ ,$$

and therefore

$$\cos\theta = \cos\phi\cos\psi + \sin\phi\sin\psi\ .$$

Hence the scalar product of **A** and **B** becomes

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$$

$$= AB\cos\phi\cos\psi + AB\sin\phi\sin\psi$$

$$= (A\cos\phi)(B\cos\psi) + (A\sin\phi)(B\sin\psi) .$$

However,

$$A\cos\phi = A_x$$

$$A\sin\phi = A_y$$

$$B\cos\psi = B_x$$

$$B\sin\psi = B_y ,$$

which leads to the interesting result that

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y . \qquad (2.6)$$

Eq. (2.6) may be generalized to

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z . \qquad (2.7)$$

To obtain the scalar product of two vectors whose components are known, then, we simply multiply together each pair of components that lie along the same axis and add the results for the three axes together.

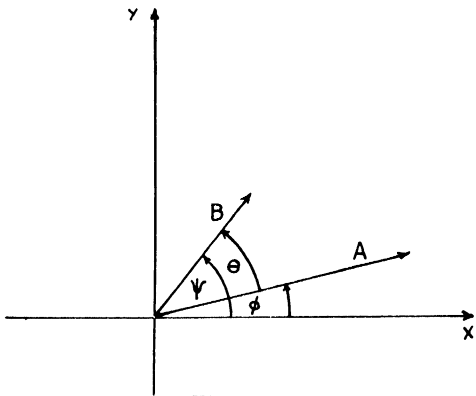


Fig. 2-8

Example. What is the angle between two vectors which both originate at (0,0,0) and which terminate respectively at (1,2,3) and (3,2,1)?

Solution. From Eq. (2.7),

$$A\cdot B = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z .$$

Hence

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$$

$$= \frac{3 + 4 + 3}{\sqrt{1^2+2^2+3^2}\sqrt{1^2+2^2+3^2}}$$

$$= \frac{10}{14} = 0.7143$$

and

$$\theta = 45° 35' .$$

2.4. Vector Product. Taking the scalar product of two vectors yields a magnitude only, with no direction associated with it. In the other type of vector multiplication the product is itself a vector, having a direction as well as a magnitude. The vector product of **A** and **B** is written

$$\mathbf{A} \times \mathbf{B} ,$$

to be read as "A cross B". It is sometimes called the "cross product." The magnitude of the vector product **A** x **B** is

$$|\mathbf{A} \times \mathbf{B}| = AB\sin\theta, \qquad (2.8)$$

where again θ is the angle included between **A** and **B**. The direction of **A** x **B** is more complicated to describe. **A** and **B** together define a plane, as in Fig. 2-9, and **A** x **B** lies perpendicular to this plane as shown.

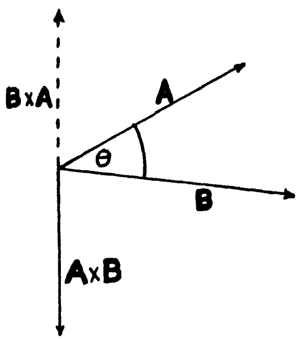


Fig. 2-9

To determine the sense of the direction, that is, whether **A** x **B** is pointing up or down in the drawing, we must imagine the line representing **A** being rotated about the origin through the angle θ until it reaches **B**. The direction in which an ordinary screw (known as a "right-handed" screw) would move under this rotation is the direction of **A** x **B**. If the two vectors lie in the plane of the page, for example, a clockwise rotation of **A** into **B** means that **A** x **B** points down into the paper, while a counterclockwise rotation means that **A** x **B** points upward out of the paper. An important consequence of the above definition is that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \qquad (2.9)$$

or, in other words, the vector product is non-commutative (Fig. 2-10).

The vector product can also be expressed in terms of the components of **A** and **B**, in which case the direction of **A** x **B** appears automatically. For simplicity we shall again start with **A** and **B** in the x,y plane with the components (A_x, A_y) and (B_x, B_y) respectively, as illustrated in Fig. 2-8. The vector product **R** of **A** and **B** must lie along the z direction, and is

$$|\mathbf{R}| = R_z = AB\sin\theta.$$

Evaluating $\sin\theta$ in terms of ϕ and ψ gives

$$\begin{aligned}\sin\theta &= \sin(\phi - \psi) \\ &= \sin\phi\cos\psi - \cos\phi\sin\psi ,\end{aligned}$$

so that

$$\begin{aligned}R_z &= |\mathbf{A} \times \mathbf{B}| = A\sin\phi B\cos\psi - A\cos\phi B\sin\psi \\ &= |A_yB_x - A_xB_y| . \qquad (2.10)\end{aligned}$$

In order to determine whether R_z is positive or negative, which gives us the direction of R_z, we note from Fig. 2-8 that $A_x > B_x$ and $A_y < B_y$. Hence the quantity within the absolute value signs on the right-hand side of Eq. (2.10) is negative, and since $\mathbf{A} \times \mathbf{B}$ as illustrated is in the +z direction, the correct expression for R_z is

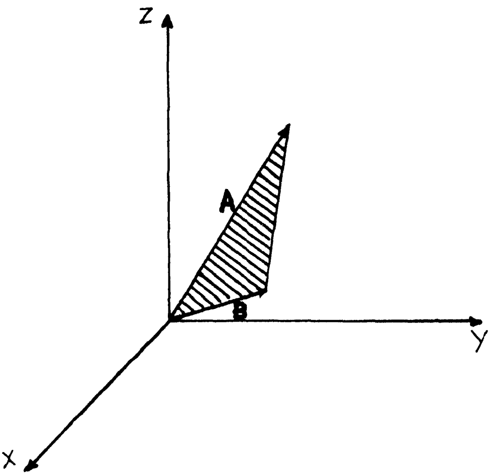


Fig. 2-10

$$R_z = A_xB_y - A_yB_x . \qquad (2.11)$$

Similarly, if $\mathbf{A}$ and $\mathbf{B}$ were in the y,z plane we would find that

$$R_x = A_yB_z - A_zB_y , \qquad (2.12)$$

and if they were in the x,z plane

$$R_y = A_zB_x - A_xB_z . \qquad (2.13)$$

The general result for the vector product in three dimensions is obtained from Eqs. (2.11), (2.12), and (2.13), and is

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y,\ A_zB_x - A_xB_z,\ A_xB_y - A_yB_x). \qquad (2.14)$$

We have not yet justified introducing vector multiplication, which at this point may seem difficult and confusing. The usefulness of these procedures will become apparent in the next few chapters, however, and they are used throughout the book as a means of clarifying the physical content of the theoretical material. We may note that vector "division" plays no part in physics.

Example. Find the area of the triangle contained between the vectors **A** and **B** (Fig. 2-10) which both originate at the origin and terminate respectively at the points (1,2,3) and (3,2,1).

Solution. The area of the triangle is

$$S = \tfrac{1}{2}AB\sin\theta = \tfrac{1}{2}\,|\mathbf{A} \times \mathbf{B}|\,.$$

Instead of finding the value of θ, we can use the coordinates of the termini of **A** and **B** directly to find the area S, which is

$$S = \tfrac{1}{2}\sqrt{R_x^2 + R_y^2 + R_z^2}\;.$$

We have

$$R_x = A_yB_z - A_zB_y = -4$$

$$R_y = A_zB_x - A_xB_z = 8$$

$$R_z = A_xB_y - A_yB_x = -3\,,$$

so that

$$S = \tfrac{1}{2}\sqrt{16 + 64 + 9} = 4.7\;.$$

Chapter 3

KINEMATICS

Kinematics is the description of motion without regard to its causes. The behavior of a bullet when it has left a rifle barrel falls in the province of kinematics. The origin of motion, why the bullet leaves the rifle initially, is the subject of *dynamics,* which will be discussed subsequently. In this chapter we are interested in analyzing the properties of motion so that, besides arriving at some useful mathematical formulas, we will know what questions to ask of dynamics. Kinematics is largely the creation of Galileo Galilei (1564-1642), the first major figure in modern physical thought.

Linear Motion

3.1. Speed and Velocity. Let us begin with a train moving forward along a straight track, and measure at regular intervals the distance it has travelled. Our results might be the following:

Time (t)	*Distance (s)*
0 sec.	0 ft.
1	40
2	80
3	120
4	160

These measurements are plotted in Figure 3-1 as a series of points joined by a smooth curve. Strictly speaking, only the points themselves have meaning; in drawing a straight line through them we have assumed that nothing untoward occurred within each time interval. This kind of approximation is called *interpolation,* and is often (although not always) a perfectly valid procedure. Extending the line beyond the measured points, called *extrapolation,* is a more risky affair, since we have no way of knowing whether the train continues on after 4 seconds, or comes to a stop or otherwise behaves differently from before.

We now assume that the (s,t) curve in Figure 3.1 accurately represents the motion of the train, and define a property of this motion called *average speed.* The average speed, $\bar{v}$, of a body is

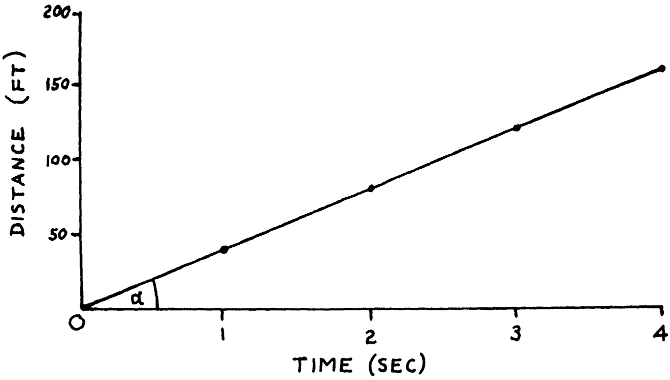


Fig. 3-1

the distance it covers in a certain period of time divided by that amount of time. That is,

$$\bar{v} = \frac{\Delta s}{\Delta t} = s_{final} - s_{initial} \Big/ t_{final} - t_{initial} \tag{3.1}$$

Here the (s, t) curve is a straight line, and we get the same velocity for any time interval to which we apply Eq. (3.1). The average speed is therefore constant and equal to the slope of the line, tan α. Average speed is measured in units of distance /time, such as km./sec., miles/hour, etc.

Suppose next that the train moves in a more complicated way, still in a straight line, and that our measurements are:

Time (t)	*Distance (s)*
0 sec.	0 ft.
1	40
2	70
3	90
4	100

Figure 3-2a is a graph of the above data, again with the individual points connected by a curve. Here, however, applying Eq. (3.1) to any time interval does not indicate the true state of affairs. For instance, in the interval 0-1 sec. we find from the formula v=40 ft/sec., which is plotted in Figure 3-2a as a dotted line of slope 40 ft / sec. Apparently something has gone awry, since the slope of the solid line corresponding to the actual motion is changing continuously while the slope of the dotted line obtained from Eq. (3.1) is constant.

To clarify the situation we must define a quantity called the *instantaneous speed*. If we wish to learn the speed of the train at

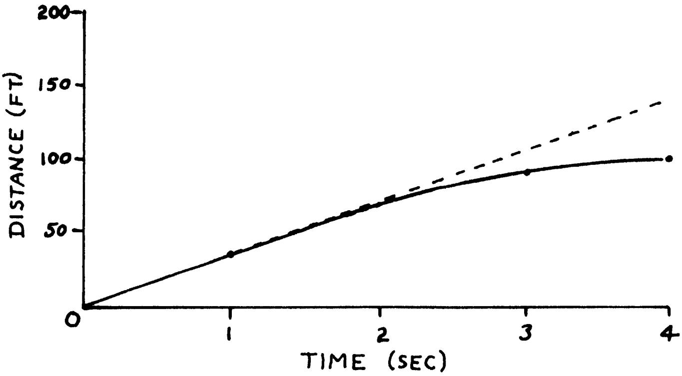


Fig. 3-2a

the precise moment t between 0 and 1 sec., instead of letting $t_{initial}$ and t_{final} be 0 and 1 sec. respectively, we can choose a *smaller* interval, say t_1 and t_2, as in Figure 3-2b. The speed is now $\tan \alpha$, which is evidently closer to the actual motion. Next we take the still smaller interval $t_1'-t_2'$, giving the still more accurate speed, $\tan \alpha'$. As the size of the time intervals Δt is decreased, the computed speed at the instant t becomes closer to the actual speed at that time. We now define the instantaneous speed, v, as the limiting value of $\Delta s/\Delta t$ as Δt approaches 0, that is,

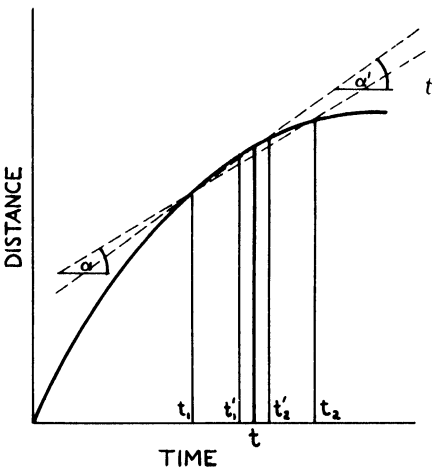


Fig. 3-2b

$$v = \underset{\Delta t \to 0}{L} \frac{\Delta s}{\Delta t} \quad . \qquad (3.2)$$

This limit is the derivative of s with respect to t, and is written

$$v = \frac{ds}{dt} \quad . \qquad (3.2a)$$

Let us now see how the instantaneous speed is related to the average speed defined by Eq. (3.2). From Eq. (3.2),

$$ds = v\,dt \quad .$$

Integrating the left-hand side from $s_{initial}$ to s_{final} and the right-hand side from $t_{initial}$ to t_{final},

$$s_f - s_i = v(t_f - t_i),$$

if v *is constant.* Rewriting this result,

$$v = s_f - s_i / t_f - t_i,$$

so that when the instantaneous speed is constant over an interval it equals the average speed in that interval. When the instantaneous speed varies there is still a single average speed that describes the motion in any particular finite interval.

Before going on it is necessary to make a rather important distinction between *speed* and a related quantity, *velocity*. Speed is always used in the sense of the *scalar distance actually travelled* divided by the time required. If a boat goes 3 miles in an hour and then turns around and returns to its starting point in another hour, its average speed is 3 miles/hour. *Velocity* is *vector displacement* divided by time and the net displacement of the boat is 0, since it has gone nowhere. Hence the boat's average velocity is 0. Velocity is a vector quantity, and both a magnitude and a direction must be specified. The term speed is also used for the magnitude of the velocity, and the two uses of this word are closely related. Thus the minute hand of a clock has a constant speed, but its velocity is not constant since the direction of motion is constantly changing.

3.2. Acceleration. As we have seen, when the distance covered by a moving body is plotted against time it is possible to find the speed of the body at any instant from the slope of the curve at that instant. Now let us plot the *speed* of the body as a function of time. Figure 3-3 shows two curves of this kind: A, which is a straight line, and B, which is not. *Acceleration,* a, is the rate at which the speed of a body changes, and the instantaneous acceleration is defined by

$$a = dv/dt \tag{3.3}$$

In other words, the acceleration is the slope of the (v,t) curve at a particular point. In curve A the slope is constant and therefore the acceleration is also constant. This means that the speed is increasing at a fixed rate. In curve B the instantaneous acceleration varies with time, and the speed of the body therefore changes irregularly. However, we can still define an *average acceleration* a over a finite time interval,

$$a = \frac{v_f - v_i}{t_f - t_i} \tag{3.3a}$$

that can cover this situation.

Acceleration, like velocity, is a vector quantity characterized by both a magnitude and a direction. In this book we will be concerned

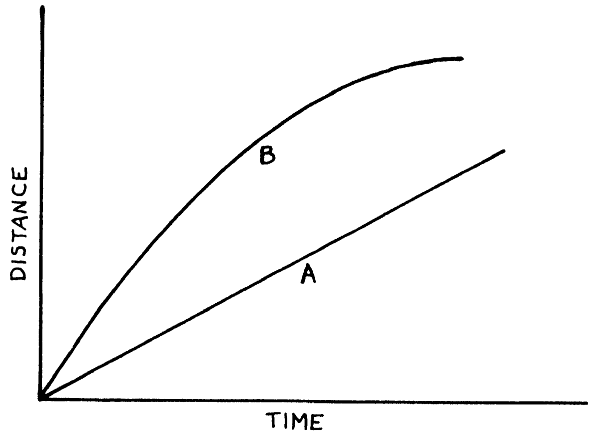


Fig. 3-3

only with accelerations whose magnitudes are constant, although in actual practice this is not always true.

The units of acceleration follow from its definition: a velocity unit divided by a time unit. In the metric system a typical unit of acceleration is "kilometers per second per second" (km/sec/sec) which is usually written km/sec^2. The time units need not be the same. Thus a common unit of acceleration in the British system is (miles/hr)/sec.

Acceleration is a familiar concept. When an automobile starts moving, its velocity is changing with time, and it is therefore being accelerated. When it reaches its final speed and continues moving with this speed, its acceleration is 0. When it slows down it is again being accelerated, this time negatively. A negative acceleration, corresponding to a decreasing velocity, is sometimes called *deceleration.* We will use only the term acceleration, sometimes adding the adjectives positive or negative to clarify the meaning where it is necessary.

When the acceleration is constant, Eq. 3.3 can be integrated easily. We begin by rewriting it

$$dv = a\ dt ,$$

and, integrating both sides,

$$v = at + v_0 . \tag{3.4}$$

Here v is the final velocity of the body after it has been accelerated for a time t. The constant of integration v_0 is the initial velocity of the body, that is, its velocity just before it was accelerated.

According to Eq. (3.2a), $v=ds/dt$. If we substitute ds/dt for v in Eq. (3.4) we have

$$ds/dt = at + v_0 \ ,$$

which can be integrated further to give

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0 \quad . \tag{3.5}$$

The symbols in this equation require some attention. As before, v_0 is the initial velocity and a is the value of the constant acceleration. The quantity s_0 refers to the original position of the body, and s is its position after a time t; thus $s-s_0$ is the net displacement from the starting point. If the body is stationary at the start, $v_0 = 0$, and

$$s-s_0 = d = \tfrac{1}{2}at^2 \tag{3.5a}$$

gives the distance d it covers in the time t. From Eq. (3.5a) we see that a uniformly accelerated body starting from rest goes 4 times as far in 2 sec as it did in 1 sec, 9 times as far in 3 sec, and so on.

3.3. Gravity. The most important example of uniformly accelerated motion in a straight line is the effect of gravitation on bodies near the earth. Galileo discovered experimentally, with the aid of balls dropped from the Leaning Tower of Pisa, that all objects in the vicinity of the earth fall with a constant acceleration of 32 ft/sec^2. In metric units this acceleration, abbreviated g, is 9.8 m/sec^2. What Galileo demonstrated was that the distance a body travels is proportional to the square of the elapsed time, and that it is in-independent of the weight of the body. The latter conclusion, especially, led to difficulties with his contemporaries, who held to the contrary notion of Aristotle.

Example. An object has an initial velocity of 20cm/sec and is being accelerated at the rate of 10cm/sec^2. What is its velocity after $\frac{1}{4}$ sec and $\frac{1}{2}$ sec have elapsed? What distance did it travel in $\frac{1}{4}$ sec and $\frac{1}{2}$ sec?

Solution. From Eq. (3.4), after $\frac{1}{4}$ sec

$$v = 10 \times \tfrac{1}{4} + 20 = 22.5 \text{ cm/sec}$$

and after $\frac{1}{2}$ sec

$$v = 10 \times \tfrac{1}{2} + 20 = 25 \text{ cm/sec} \quad .$$

The distance travelled can be found from Eq. (3.5). When $t=\frac{1}{4}$ sec,

$$d = s-s_0 = \tfrac{1}{2} \times 10 \times (\tfrac{1}{4})^2 + 20 \times \tfrac{1}{4} = 5\tfrac{5}{16} \text{ cm}$$

and when $t=\frac{1}{2}$ sec,

$$d = s-s_0 = \tfrac{1}{2} \times 10 \times (\tfrac{1}{2})^2 + 20 \times \tfrac{1}{2} = 11\tfrac{1}{4} \text{ cm.}$$

Example. An object is thrown vertically upward with a velocity of

49 m/sec. How high will it rise? What is its position after it has travelled for 2 sec, 8 sec, 11 sec?

Solution. In this problem it is clear that the initial velocity, which is upward, is in the opposite direction to the acceleration, which is downward. Hence the speed at first decreases to zero, at which time the object is at the top of its path, and then increases as it falls down. Since both acceleration and velocity are vector quantities, it is easy to deal with situation by defining positive and negative directions. Customarily up is positive and down negative; therefore $v_0 = +49$ m/sec and $a = g = -9.8 \text{ m/sec}^2$. To solve the first part of the problem, we note that at the top of its path the object has 0 velocity. Using Eq. (3.4),

$$\begin{aligned} v &= 0 = at + v_0 \\ &= -9.8t + 49 \quad , \end{aligned}$$

and

$$t = 5 \text{ sec.}$$

Choosing the starting point $s_0 = 0$, Eq. (3.5) yields for the maximum height

$$\begin{aligned} s &= \tfrac{1}{2}at^2 + v_0 t + s_0 \\ &= -\tfrac{1}{2} \times 9.8 \times 5^2 + 49 \times 5 + 0 \\ &= 122.5 \text{ m.} \end{aligned}$$

For the second part of the problem we again use Eq. (3.5). The results are:

$$\begin{aligned} &t=2 \text{ sec: } s = -\tfrac{1}{2} \times 9.8 \times 2^2 + 49 \times 2 = 78.4 \text{ m} \\ &t=8 \text{ sec: } s = -\tfrac{1}{2} \times 9.8 \times 8^2 + 49 \times 8 = 78.4 \text{ m} \\ &t=11 \text{ sec: } s = -\tfrac{1}{2} \times 9.8 \times 11^2 + 49 \times 11 = -53.9 \text{ m.} \end{aligned}$$

We notice that the position of the object is the same after 8 sec as it was after 2 sec, and that after 11 sec it is negative. Figure 3-4 illustrates the motion. In 2 sec the object has risen to 78.4 m. Three seconds later it is at its maximum height of 122.5 m, and in another 3 sec, making a total of 8 sec since it was thrown, it has fallen down to 78.4 m above the starting point. The motion is symmetrical about the highest point—you can verify if you wish that the speeds at 2 sec and 8 sec are the same, although the velocities are in opposite directions. After 11 sec the object has fallen so far that it is below the position from which it was thrown, as the minus sign indicates. The quantity s is the actual *displacement* of the object, not the total distance it has travelled. For instance, after 11 sec the total distance is 122.5 m up, 122.5 m back down to the starting position, and then 53.9 m farther down, making a total of 298.9 m. The net displacement, however, is -53.9 m.

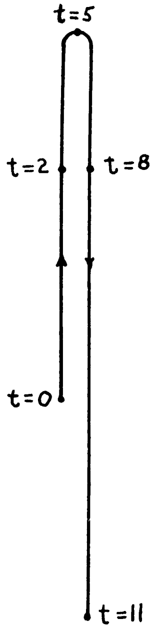


Fig. 3-4

3.4. Projectile Motion. Most moving bodies do not travel in straight lines. Very often, however, the motion lies in a plane, and in such cases the vector properties of displacement s, velocity v, and acceleration a become of great importance.

Suppose we have two balls at the edge of a table and, as in Figure 3-5, allow A to drop down while we push B off horizontally with the velocity v_x. The experimental result is that both balls reach the ground at the same instant, showing that the vertical and horizontal motions are independent. This is true regardless of the magnitude of v_x, which remains constant* because the acceleration g due to gravity acts only in the vertical direction. The vertical component of velocity v_y obeys Eq. (3.4), and is

$$v_y = gt.$$

These velocity components may be added vectorially to give an instantaneous velocity v at any time t after the balls are released, of magnitude

$$v = \sqrt{v_x^2 + v_y^2}$$
$$= \sqrt{v_x^2 + g^2t^2} \quad .$$

It is not difficult to derive the equation of the path that ball B follows. (Ball A, of course, falls vertically downward.) In the absence of a horizontal acceleration, the horizontal displacement s_x of the ball at the time t is

$$s_x = v_x t$$

while from Eq. (3.5)

$$s_y = \tfrac{1}{2} gt^2 \quad .$$

The edge of the table is taken as the origin of coordinates here. Eliminating t from these equations,

$$s_y = (\tfrac{1}{2}g/v_x^2)\, s_x^2 \quad ,$$

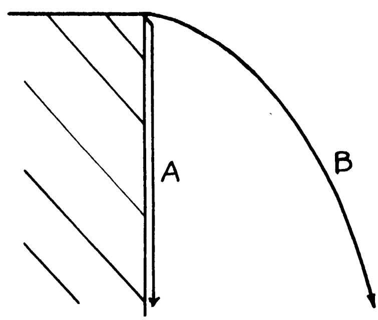


Fig. 3-5

which, since the quantities in the

*In this example and elsewhere in this chapter it is assumed that there is no air resistance, which would ordinarily provide both horizontal and vertical accelerations.

parenthesis are constant, is a parabola. This conclusion is verified experimentally, indicating that our initial assumption that velocity is a vector is correct.

As a further application of these ideas, we will consider the motion of a projectile. In Figure 3-6 a body is released with a velocity v_0 that makes an angle θ with the ground. We wish to know (a) its total travel time; (b) the maximum height it will reach; and (c) how far it will go. Since we are neglecting air friction the answers to these questions can only approximate reality, but the problem makes a useful exercise and, as a matter of historical interest, was one of the most important problems first solved by Galileo.

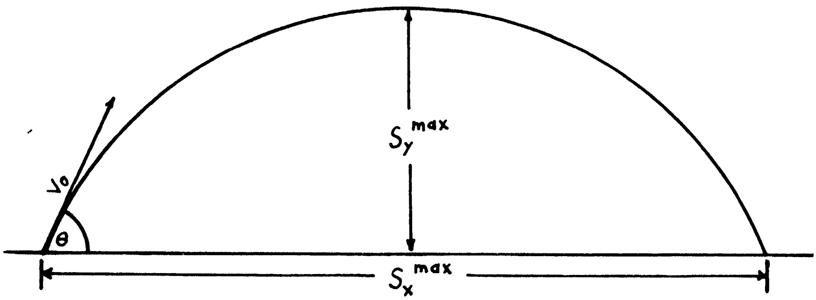


Fig. 3-6

In question (a), we are given v, θ, and g, and we are to find t when the body returns to the ground, that is, at $s_y = 0$. The x and y components of the motion are independent, and we confine ourselves to the latter. We know that

$$v_{0y} = v \sin \theta$$

and, in general,

$$s_y = v_{0y}t + \tfrac{1}{2}gt^2 \ .$$

Setting $s_y = 0$ we find that

$$t = \frac{-2v_0 \sin \theta}{g} \quad ,$$

and, since g is negative according to our sign convention, the time turns out to be positive.

For (b) we make use of the fact that at the top of its trajectory the projectile has no vertical velocity. Thus we wish to find s_y when $v_y = 0$. We begin by eliminating t from the fundamental formulas for uniformly accelerated motion,

$$v = v_0 + at$$

$$s = v_0 t + \tfrac{1}{2}at^2$$

This gives

$$v^2 = v_0{}^2 + 2as$$

Therefore we have

$$v_y{}^2 = v_{0y}{}^2 + 2gs_y$$

and

$$v_{0y} = v \sin \theta$$

and when $v_y = 0$,

$$s_y^{max} = \frac{-v^2 \sin^2 \theta}{2g}$$

Again, g being negative yields a positive value for the maximum height.

Question (c) requires that we find s_x when $s_y = 0$. With no horizontal acceleration,

$$s_x = v_x t = (v\cos\theta)\ t\ .$$

Substituting for t the expression obtained in part (a) for the case of $s_y = 0$,

$$s_x^{max} = \frac{-2v^2 \cos \theta \sin\theta}{g}$$

$$= \frac{-v^2 \sin 2\theta}{g} \quad .$$

The horizontal distance covered by a projectile is called its *range*, and from the above equation we see that the maximum range is reached when $\theta = 45°$. Any other range may be obtained by either of two initial angles (Figure 3-7), a fact empirically known by artillery experts in the time of Galileo but unexplained until his work.

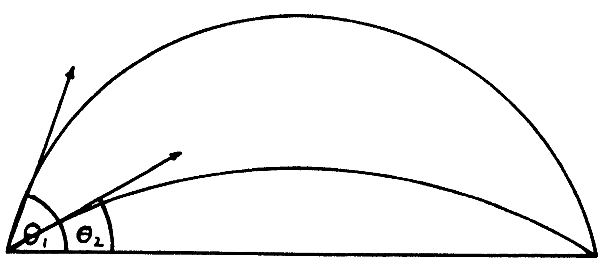


Fig. 3-7

3.5. Vector Summary. Thus far we have seen how the motion of a body can be conveniently described by the three vectors s, v, and a, which are related by

$$\mathbf{v} = d\mathbf{s}/dt$$

$$\mathbf{a} = dy/dt = d^2\mathbf{s}/dt^2 \ .$$

In the case of uniformly accelerated motion these equations can be integrated to give

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t$$

$$\mathbf{s} = \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2 \ .$$

Eliminating the time,

$$\mathbf{v}\cdot\mathbf{v} = \mathbf{v}_0\cdot\mathbf{v}_0 + 2\mathbf{a}\cdot\mathbf{s}$$

in vector notation. The dot products in the above expression represent the scalar products of two vectors.

To generalize from the two dimensions of motion in a plane to the three dimensions of motion in space requires no change in these equations; it is only necessary to give $\mathbf{s}$, $\mathbf{v}$, and $\mathbf{a}$ three components each instead of two. Vector notation always permits a very succinct statement of what might otherwise be a complicated set of equations, a considerable advantage when trying to understand a physical problem in more than one dimension.

We can now derive the relationship between speed and velocity that was stated at the beginning of this chapter. The velocity vector $\mathbf{v}$ has three cartesian components, dx/dt, dy/dt, and dz/dt. The magnitude of $\mathbf{v}$ is equal to

$$v = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$$

$$= \frac{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}{dt}$$

But $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ is just the arc length along the curve on which the body moves. Hence the magnitude of the velocity equals the distance traversed along the actual path divided by the time taken, which is by definition the speed of the body.

3.6. Circular Motion. A body moving in a circular path is a special case of motion in a plane. Such motion is quite common: for instance, every point on a rotating object such as a wheel travels in a circle. There are two ways of mathematically describing the circular motion of a particular body. The first is to specify its velocity and acceleration at every point in its path, as we did for projectiles. This is unnecessarily complicated here though, because even when the body is travelling in a circle with uniform speed the velocity and acceleration, being vectors, are not constant.

An alternative method takes advantage of the fact that the motion is restricted to lie on a circle. Only a single number is needed therefore to specify the position of the body in its path, instead of the two coordinates s_x and s_y otherwise required. For convenience we use the angle θ made by a radial line drawn from the center of the circle to the body with a reference radius, as in Figure 3-8. θ is called the angular displacement of the body, and can be measured in degrees, radians, even revolutions if we like. The radian, equal to the ratio between arc length s and radius R, is almost always the preferable unit and will be used here.

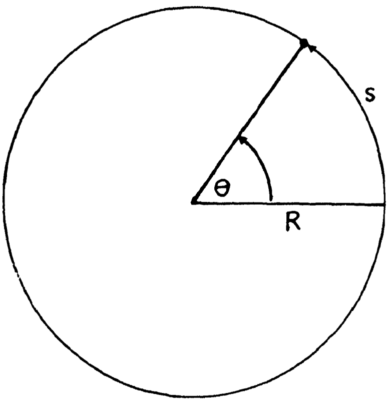


Fig. 3-8

We can define angular velocity and angular acceleration in terms of the angular displacement θ in a manner analogous to Eqs. (3.2) and (3.3) for the linear case. Thus

angular velocity

$$= \omega = d\theta/dt \tag{3.7}$$

angular acceleration

$$= \alpha = d\omega/dt = d^2\theta/dt^2 \tag{3.8}$$

When the angular acceleration is constant, these equations may be integrated just as the linear ones were, yielding

$$\omega = \omega_0 + \alpha t \tag{3.9}$$

$$\theta = \tfrac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \quad . \tag{3.10}$$

The various formulas obtained for linear motion with a = constant may be applied to angular motion with α = constant by replacing s, v, and a by θ, ω, and α. Thus Eq. (3.6) becomes here

$$\omega^2 = \omega_0^2 + 2\alpha\theta \tag{3.11}$$

We can also use the distance the body travels along its circular path as a displacement coordinate. There are very simple relationships between the speed of the body and its angular velocity and between the magnitude of its linear acceleration and its angular acceleration. *With θ expressed in radians,*

$$\theta = s/R \quad .$$

In a circular path R is constant, and differentiating θ gives

$$d\theta/dt = \omega = \frac{1}{R}\frac{ds}{dt} = \frac{v}{R} \quad .$$

Differentiating again,

$$\frac{d^2\theta}{dt^2} = \alpha = \frac{1}{R}\frac{dv}{dt} = \frac{a}{R} \quad .$$

If we multiply the above equations by R we find that

$$v = \omega R \tag{3.12}$$

$$a = \alpha R \tag{3.13}$$

Eqs. (3.12) and (3.13) hold for circular motion regardless of whether ω stays constant or not, and are often very helpful.

Example. A 12-inch diameter phonograph record revolves at $33\frac{1}{3}$ rpm. What is the linear velocity of a point on its rim?

Solution. The angular velocity corresponding to $33\frac{1}{3}$ rpm is

$$\omega = \frac{60 \times 2\pi}{33\frac{1}{3}} = 11.3 \text{ radians/sec} \quad .$$

Since the radius of the record is 6 inches,

$$v = \omega R = 67.8 \text{ in/sec} \quad ,$$

which is equal to 38.5 miles/hour!

Example. A phonograph turntable requires 5 sec to reach its operating speed of $33\frac{1}{3}$ rpm. What is its average angular acceleration, and how many revolutions does it make until it has reached its operating speed?

Solution. From Eq. (3.9) we have

$$\alpha = \frac{11.3 \text{ rad/sec}}{5 \text{ sec}} = 2.3 \text{ rad/sec}^2$$

for the angular acceleration. From Eq. (3.10), since the initial velocity is 0,

$$\theta = \tfrac{1}{2}\alpha t^2 = 28.8 \text{ radians} = 4.58 \text{ revolutions.}$$

3.7. Vector Properties of Angular Velocity and Acceleration. As we have seen, even one-dimensional motion along a straight line has vector properties. It is necessary to distinguish between the two directions possible (for instance, right or left, up or down), by using + and - signs. Similarly, we must now make a distinction between clockwise and counterclockwise rotations in circular motion.

In order to associate a vector with angular velocity, both its magnitude and direction must be specified. We already know the magnitude, which is $\omega = d\theta/dt$. The direction is a more complicated matter. It is taken with reference to a line drawn perpendicular to the plane of the circle in which the motion takes place. The direction is now defined as the one along which a right-handed screw moves* when it is turned in the sense of the actual motion (Figure 3-9). If we are describing the motion of the minute hand of a clock, the vector would point *into* the clock face and have a magnitude of $2\pi/60 \times 60 = 0.0175$ rad/sec.

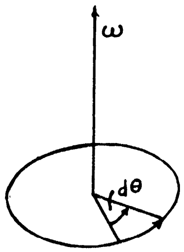


Fig. 3-9

Even when the particle does not move in a circular path, it is possible to speak of an *instantaneous* angular velocity about the center of curvature of a segment of its path. If the particle undergoes a displacement ds in the time dt (Figure 3-10), the radius vector to the particle undergoes a displacement $d\theta$. The instantaneous angular velocity ω is a vector of magnitude $d\theta/dt$ whose direction lies along the perpendicular to the plane defined by R and ds with the sense given by the right-hand screw rule.

To find the direction of the angular acceleration, we begin with its vector definition

$$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt. \tag{3.14}$$

Now we calculate $\boldsymbol{\omega}$ at the time $t + dt$ and vectorially subtract from it the value of $\boldsymbol{\omega}$ at the time t. The limit of this difference divided by dt as dt approaches 0 gives both the magnitude and direction of the angular acceleration. In plane circular motion the angular acceleration about the center is in the same direction as the angular velocity about the center.

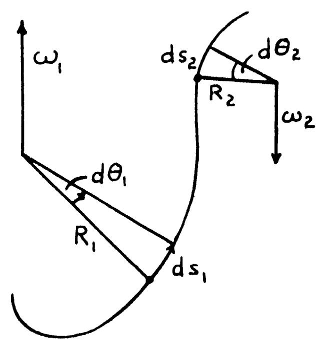


Fig. 3-10

There are also vector analogs of Eqs. (3.12) and (3.13), which we will not derive:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} \tag{3.15}$$

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{R} \tag{3.16}$$

In the case of motion in a circle $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are perpendicular to R and (3.15) and (3.16) reduce to the scalar equations $v = \omega R$ and $a = \alpha R$. The direction of v at any point is along the tangent to the circle at that point. The acceleration given by Eq. (3.16) refers to changes in the speed of the moving body and its direction is also along the tangent to the circle. As we

*All common screws are right-handed; that is, when turned clockwise by a screw-driver they move further into a piece of wood, while when turned counterclockwise they move out.

will learn later, there is another acceleration experienced by a body moving in a circle even when its speed remains constant.

While we have assumed that angular velocity is a vector because both a magnitude and a direction are required to describe it, there is more to a vector than that. Specifically, we must be sure that combining angular velocities by vector addition gives results that agree with experiment. This conclusion is not obvious, as we can see by defining angular displacement the way we defined angular velocity. Let us have the magnitude of the "vector" equal to the magnitude of the displacement, and its direction perpendicular to the plane in which the rotation takes place according to the right-hand screw rule. Suppose we are dealing with a child's block as in Figure 3-11, and turn it first by 90° about the z axis and then by

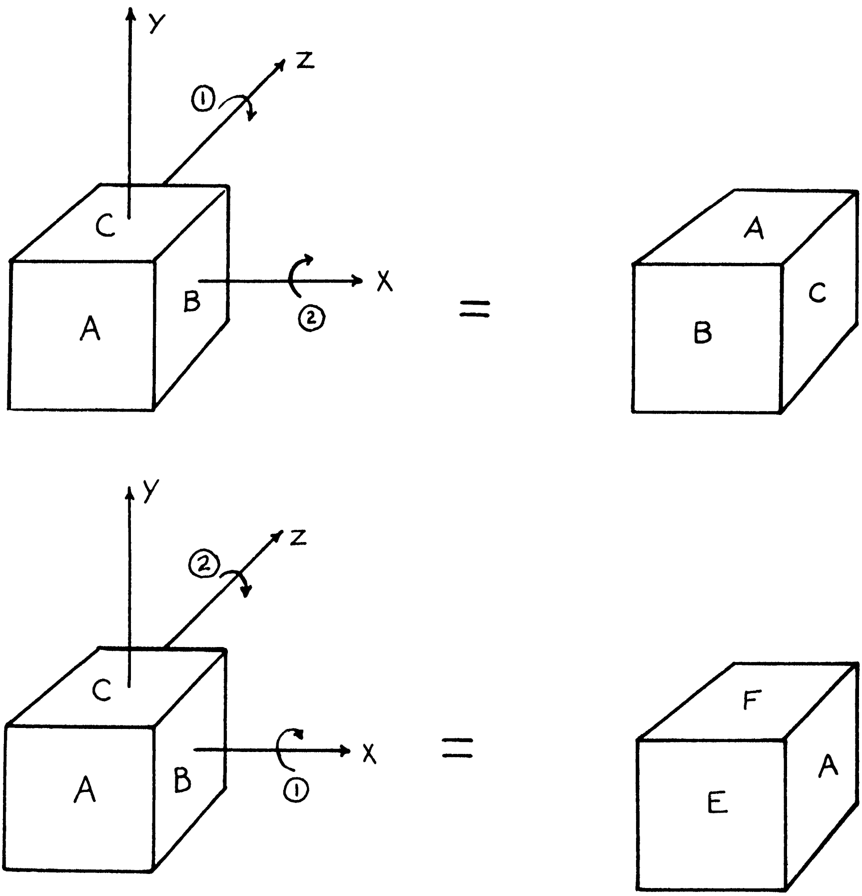


Fig. 3-11

90° about the x axis. The final position of the block is different from what it is when we first turn it about the x axis and then about the z axis, as we observe. But vectors must satisfy the commutative law of addition

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

in other words, the order of the addition should make no difference. Since this is not true here, the conclusion is that finite rotations are not vectors. However, a similar experiment in which angular *velocities* are added vectorially would indicate that they are true vectors, as we assumed.

3.8. Centripetal Acceleration When a particle travels in a circle with uniform *angular* velocity, its *linear* velocity is not constant. In Fig. 3-12 the velocity vector of such a particle is shown at two points in its path. The average acceleration the particle experiences between those points is

$$\bar{\mathbf{a}}_c = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1},$$

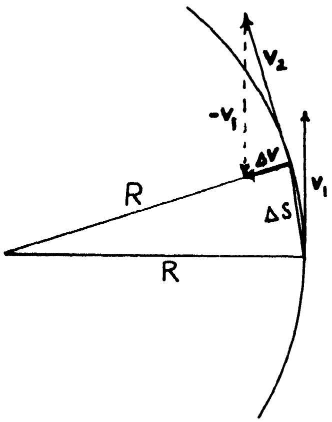


Fig. 3-12

as usual. From the drawing we see that the triangles bounded by R, Δs, R and by $\mathbf{v}_1, \Delta\mathbf{v}, \mathbf{v}_2$ are similar (the magnitudes of $\mathbf{v}_1$ and $\mathbf{v}_2$ are both the same, of course, and equal to v), so that

$$\frac{\Delta v}{v} = \frac{\Delta s}{R}.$$

Hence

$$\bar{a}_c = \frac{v}{R} \frac{\Delta s}{\Delta t} \tag{3.17}$$

is the magnitude of the average acceleration of the particle. Since $\Delta\mathbf{v}$ is directed inward toward the center of the circular path, $\bar{\mathbf{a}}_c$ is in this direction also. The instantaneous acceleration is given by

$$a_c = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{R} \tag{3.18}$$

$$= \omega^2 R.$$

This is called the *centripetal acceleration* of the particle.

If the particle does not have a constant angular velocity, its total acceleration is the vector sum of the centripetal acceleration, which is radially inward, and the linear acceleration α R, which at any instant is tangent to the path of the particle at that instant.

Example. The earth's radius is approximately 6400 kilometers, and it revolves on its axis approximately once over 24 hours. What is the centripetal acceleration of a body on the earth's equator?

Solution. The angular velocity of the earth is $2\pi/86{,}400 = 7.2\text{x}10^{-5}$ radians/sec. From Eq. (3-18)

$$a_c = \omega^2 R = 3.3\text{x}10^{-2}\ \text{m/sec}^2.$$

The apparent value of the acceleration due to gravity g at the equator will be reduced by this amount.

Example. A 12-inch diameter phonograph record revolves at $33\frac{1}{3}$ rpm. What is the centripetal acceleration of a point on its rim?

Solution. As we calculated earlier, this record has an angular velocity of 11.3 radians/sec. Hence

$$a_c = \omega^2 R = 766\ \text{in/sec}^2 = 63.8\ \text{ft/sec}^2\ ,$$

which is almost twice g.

Chapter 4

DYNAMICS OF PARTICLES

The discovery by Galileo of the laws of kinematics provided the foundation for the development of dynamics. Kinematics deals with the behavior of a body that is accelerated; dynamics treats of the circumstances under which acceleration takes place. Isaac Newton (1642-1727), who fittingly was born in the year of Galileo's death, supplied the basic ideas of dynamics in his three Laws of Motion. These laws apply to *particles,* which are defined as material bodies that have no extension in space, simply points with no volume but otherwise the same as ordinary bodies. Particles are abstractions, and the dynamics of particles is only an approximation to the dynamics of real objects. Often it is possible to consider an actual object as though it were a particle for convenience in solving a specific problem, but we must keep in mind that this is not always the case.

4.1. Newton's First Law. The First Law of Motion states that a particle moves with uniform velocity or remains at rest ($\mathbf{v} = 0$) unless there is a net force acting on it. If acceleration is present, there must be a net force as well. If there is no acceleration, there is no net force. Thus we can always determine the presence of a force by observing the motion of the particle involved.

This law seems simple enough, but sometimes it can lead to an unexpected conclusion. For instance, in order to drive an automobile at a constant speed we must supply a force with the help of the motor, apparently contradicting the First Law. Actually, however, there are frictional forces that act to retard the automobile's motion, and the motor just balances these forces when the speed is constant. The *net* force is zero. Only when we change the speed of the automobile or its direction, in other words accelerate it, is an additional force required beyond that needed to compensate for friction.

In using the concept of net force, we must make two tacit assumptions. The first is that the vector addition of two or more forces gives the net force acting, and the second is that the effect of the net force by itself is the same as the sum of the effects of the individual forces acting separately. Both assumptions had to be experimentally verified before the Laws of Motion could be accepted.

Until Newton's time it was believed that the orbits of the moon and planets are "natural" ones, with no forces necessary to hold these bodies in their orbits. In most people's minds forces are pushes and pulls directly applied. Since there is no visible contact between the heavenly bodies and anything else, the conclusion was that no forces are acting on them. The First Law of Motion, however, states that when there is acceleration there must be a force also present, and here the existence of an acceleration is revealed by the curved paths taken by the moon and planets. This revolutionary idea led Newton to formulate the law of gravitation, in which a force was proposed for the first time that acted between two distant bodies without any physical contact between them. The fundamental forces of nature all share the property of acting at a distance.

4.2. Newton's Second Law. The Second Law of Motion states that the force exerted on a particle is proportional to its mass multiplied by its acceleration. This law enables us to determine the magnitudes of forces.

We begin by clarifying what we mean by *mass*. From experience we know that all objects tend to resist changes in their motions. This characteristic behavior, sometimes called inertia, arises from the property of matter known as mass. It will be assumed here that the mass of a body does not change when it moves. (The Theory of Relativity, discussed in a later chapter, predicts an increase in mass with very high velocity, but such variations are not significant in ordinary macroscopic phenomena.)

The mathematical statement of the Second Law is

$$\mathbf{F} = km\mathbf{a} \; , \tag{4.1}$$

where $\mathbf{F}$ represents force, m mass, $\mathbf{a}$ acceleration, and k the constant of proportionality whose value depends on the units we select. If we apply two forces in succession to the same body and each time it exhibits the same acceleration, the two forces are equal. If the accelerations are different, say a_1 and a_2, we have, considering magnitudes only,

$$F_1 = kma_1$$

$$F_2 = kma_2 \; .$$

Dividing the first equation by the second,

$$\frac{F_1}{F_2} = \frac{a_1}{a_2} \; . \tag{4.2}$$

The ratio of the forces equals the ratio of the accelerations they produce. We might apply the same force to two different bodies of masses m_1 and m_2. Now we find that

$$\frac{m_1}{m_2} = \frac{a_2}{a_1} \tag{4.3}$$

The masses of the bodies are *inversely* proportional to the accelerations produced by the same force acting on them, just as ordinary experience suggests.

Newton's Second Law may be used to compare masses. By selecting a particular object, such as a piece of metal, as the unit mass, the masses of all other bodies can be found in terms of the unit mass with the aid of Eq. (4.1). There are two methods of accomplishing this comparison through the use of a spring whose elongation is proportional to the force applied. (Many springs have this property, and those that do are said to obey Hooke's Law.) The first method employs Galileo's discovery that all bodies near the earth experience the same acceleration g. When two different masses are hung from the spring, the forces they exert on the spring are

$$F_1 = km_1 g$$

$$F_2 = km_2 g \ .$$

Dividing one equation by the other,

$$\frac{F_1}{F_2} = \frac{m_1}{m_2} \ . \tag{4.4}$$

Since the ratio of the forces equals the ratio of the spring elongations, we can determine m_1/m_2. When one of the masses is the unit mass, the other can be expressed as a multiple of it. This procedure, with a balance sometimes substituted for the spring, is usually employed in practice.

The other method is a dynamic one. We attach a body to the end of a horizontal spring and pull on it until the acceleration is a certain amount. Next the unit mass is attached to the spring, which is stretched exactly as much as before so that the force is the same. By comparing the acceleration of the unit mass with that of the body, Eq. (4.3) gives us the mass of the latter. The results obtained by the two methods turn out to be the same, provided that a correction is made via the theory of relativity for the fact that in one case relative motion is involved while the other is a static experiment.

The application of the Second Law of Motion will be postponed until the Third Law and systems of units are discussed.

4.3. Newton's Third Law. The Third Law of Motion states that if a particle A exerts a force on another particle B, then particle B exerts an equal force on particle A but in the opposite direction. In vector notation,

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \ . \tag{4.5}$$

This law is important in analyzing the behavior of systems of particles, and also enable us to measure masses without resorting to calibrated springs. Before going any further, though, we must be sure that we understand the subtleties contained in Eq. (4.5).

Let us apply the Third Law, also known as the law of action and reaction, to the situation of a horse pulling a cart. For the sake of simplicity we shall consider both of them as particles which is reasonable since all we are interested in is their effect on each other. According to the Third Law the force the horse exerts on the cart is matched by an equal and opposite force exerted by the cart on the horse. At first glance the forces, being equal and opposite, seem to cancel out and leave no net force at all. Further thought reveals that it is only the force on the cart which determines its motion, and that the force exerted by the cart on the horse affects the horse alone. Hence if the horse is able to apply enough force to overcome the frictional forces present, the cart will accelerate.

Another example is that of a body near the earth's surface. The earth acts on it with a force causing it to experience the acceleration g of 32 ft./sec^2 or 9.8 m/sec^2. Simultaneously the body exerts the same force on the earth. From Eq. (4.3) the ratio of the accelerations produced is inversely proportional to the ratio of the masses of the body and the earth. Because the earth is so much larger, the reaction force on it causes too small an acceleration to be detected. In general, the forces of action and reaction do not balance out since they act on different objects.

Suppose we permit two masses to collide with one another. According to the Third Law the magnitudes of the forces exerted on them by the collision are the same. Eq. (4.3) applies here, and states that the ratio of their accelerations, which can be measured, is inversely proportional to their mass ratio. As before, by having one of the colliding bodies be a unit mass, the other mass can be compared with it. Here the comparison is direct, and no spring or balance is needed.

Newton's three Laws of Motion have been presented here as axioms, more like the axioms of geometry than in the spirit of a living science. This has been done only to make their meaning clear. The formulation of these laws by Newton actually is a classic example of the application of the scientific method. Newton found that the Laws of Motion, together with the law of gravitation we shall consider later, could account for all of the planetary data then in existence. He then made some predictions which were subsequently verified by experiment and observation. No discrepancies were uncovered between theory and practise until about fifty years ago, when certain sub-atomic and large-scale astronomical phenomena apparently contradicted the Laws of Motion and of gravitation as Newton had formulated them. These laws therefore had to be revised in ways to be described shortly. For normal terrestrial phenomena they are still entirely adequate, however, and we shall assume their correctness unless we state otherwise.

4.4. Units. Before we can apply Newton's laws to problems we must have an appropriate system of units in which force, mass, and acceleration can be expressed. Acceleration has the dimensions of length/(time)^2, and needs no special unit of its own. Mass units depend on what we choose to be our standard mass. From the Second Law of Motion force is given in terms of mass and acceleration, so force units may be defined as a particular combination of mass, length, and time units. The question of suitable units probably seems trivial the first time it is raised, but a clear knowledge of the different systems in use and how they are related is as essential to the physicist and engineer as a clear knowledge of currency systems is essential to the international businessman.

By general agreement the unit of time is the *second* (sec). The "universal" second is experimentally determined from the period of rotation of the earth; since this period is observed to fluctuate slightly, the "ephemeris" second has been defined in such a way that it always equals the value the "universal" second had in 1900. The distinction is a very small one, but owing to the importance of invariant units the "ephemeris" second is to be preferred.

Units of length and mass have simpler definitions, and there are several competing sets throughout the world. The two most common ones are the Metric system, based on a standard of length called the *meter* (m) and a standard of mass called the *kilogram* (kg), and the British system, in which the standard of length is the *yard* and the standard of mass is the *pound**(lb). The mass standards are simply pieces of metal whose masses can be compared with those of other bodies by the methods described earlier. The length standards are metal bars with a scratch at either end to mark the limits of the unit they represent. Metric units are nearly always used for scientific work, while British units find application in engineering in English-speaking countries. Approximate conversions between the two systems are

$$\begin{aligned} 1 \text{ kilogram} &= 0.4536 \text{ pound} \\ 1 \text{ pound} &= 2.205 \text{ kilograms} \\ 1 \text{ meter} &= 3.281 \text{ feet} \\ 1 \text{ foot} &= 0.3048 \text{ meters} \end{aligned}$$

4.5. Absolute Systems of Units. The meter-kilogram-second (mks) system is the one we will emphasize in this book. The unit of force in the mks system is defined by Newton's Second Law, Eq. (4.1), with the constant of proportionality k taken equal to 1. Thus

$$F = ma \quad , \tag{4.6}$$

and the unit of force is the kg m/sec^2. This unit has been given the name *newton* (n), so that

*The American pound is heavier by about one part in five million than the English pound.

$$1 \text{ newton} = 1 \frac{\text{kg m}}{\text{sec}^2} . \tag{4.7}$$

The newton is that force which, when applied to a particle having a mass of 1 kilogram, imparts to it an acceleration of 1 m/sec^2.

Another absolute Metric system is the centimeter-gram-second (cgs) system, which is still extensively used. Mks units are gradually replacing cgs ones because of the advantages of the former in electricity and magnetism. The conversions between the cgs and mks systems are

$$\begin{aligned} 100 \text{ centimeters (cm)} &= 1 \text{ meter} \\ 1000 \text{ grams (g)} &= 1 \text{ kilogram} . \end{aligned}$$

In the cgs system $k=1$ in Eq. (4.1) also, resulting in a unit of force called the *dyne* equal to 1 g cm/sec^2. We easily find that 1 newton $=10^5$ dynes.

There is an absolute system of British units, but it is seldom employed and is only mentioned here for the sake of completeness. The unit of length is the *foot* (ft), equal to 1/3 yard, the unit of mass is the pound, and the unit of time is the second. The force unit in this system is the *poundal*, where 1 poundal=1 lb ft/sec^2.

4.6. Gravitational Systems of Units. In gravitational systems we begin with the units of force, length, and time, and from Newton's Second Law obtain a unit of mass. In the British gravitational system, the unit of length is the foot, the unit of time is the second, and the unit of force is that force which the earth exerts on a mass of one pound, which is its "weight." This unit of force is itself called the *pound*, an ambiguity that has been so firmly established that it is impossible to eliminate it. However, it is usually clear from the particular situation involved whether the pounds mentioned are of force of mass. We shall write "lb (force)" or "lb (mass)" whenever there is any question as to which unit we are using.

Newton's Second Law, again with $k=1$, is now used to define a mass unit called the *slug*. From $F=ma$ we find that

$$1 \text{ slug} = 1 \frac{\text{lb (force)}}{\text{ft/sec}^2} , \tag{4.8}$$

which means that

$$1 \text{ lb (force)} = 1 \text{ slug ft/sec}^2 .$$

We can convert pounds of mass to slugs by noting that a 1 lb force always gives a 1 lb mass an acceleration of g. Hence we have

$$\frac{1 \text{ lb (mass)}}{32.2 \text{ ft/sec}^2} = 1 \text{ slug} . \tag{4.9}$$

In order to use

$$F = ma \tag{4.6}$$

in British gravitational units, masses given in lb (mass) must first be changed to slugs with the help of Eq. (4.9). The force unit will then automatically be lb (force) as it should be. Because of their practical importance we shall use British gravitational units frequently in order to promote familiarity with their application.

The Metric gravitational systems use force units equal to the kilogram (force) or gram (force), respectively the amount of force exerted by the earth on a kilogram mass and on a gram mass. Length and time have the same units as before, and mass in the proper units is calculated by dividing the mass expressed in kilograms or grams by 9.8 m/sec^2 or 980 cm/sec^2 respectively. There is no special name for these mass units, largely because Metric gravitational systems are very rarely used.

There is a basic difference between absolute and gravitational systems of units. In the gravitational systems force is the standard unit, and mass is obtained by taking into account the value of g. But g is observed to vary somewhat over the earth's surface, implying in these systems that the mass of a body varies in the same way. Since we like to think of mass as an intrinsic property of a body, absolute systems in which mass is primary are logically preferable. To avoid confusion in the use of gravitational systems, g is conventionally assumed to have the constant value 32.2 ft/sec^2 corresponding to 9.80 m/sec^2.

Example. A particle of mass 5 kilograms is acted on by a force of 0.03 newtons. What is its acceleration?

Solution. From Newton's Second Law $F=ma$, and so

$$a = \frac{F}{m} = \frac{0.03\ \text{n}}{5\ \text{kg}} = 0.006\ \text{m/sec}^2.$$

Example. A particle of mass 4 lb is acted on by a force of 6 lb. What is its acceleration?

Solution. First we convert the mass value from lb (mass) to slugs. From Eq. (4.9),

$$\frac{4\ \text{lb (mass)}}{32\ \text{ft/sec}^2} = \frac{1}{8}\ \text{slug}\ .$$

Now we proceed as in the previous example, and find that

$$a = \frac{F}{m} = \frac{6\ \text{lb (force)}}{1/8\ \text{slug}} = 48\ \text{ft/sec}^2.$$

Example. A particle at rest having a mass of 2.5 kilograms is acted on by a force of 8 newtons. What is its velocity 3 seconds after the force is applied, and how far has it gone in that time?

Solution. We begin by finding the acceleration of the particle, which is

$$a = \frac{F}{m} = \frac{8 \text{ n}}{2.5 \text{ kg}} = 3.2 \text{ m/sec}^2.$$

Referring back to Eq. (3.4) of Chapter 3, we have, since $v_0 = 0$,

$$v = at = 3.2 \text{ m/sec}^2 \times 3 \text{ sec} = 9.6 \text{ m/sec}.$$

From Eq. (3.5a),

$$d = \tfrac{1}{2}at^2 = \tfrac{1}{2} \times 3.2 \text{ m/sec}^2 \times (3 \text{ sec})^2 = 14.4 \text{ m}.$$

4.7. Transmitting Forces With Ropes. A rope is a common means of transmitting a force from its origin to the object it is to act on. Ropes can convey "pulls" but not "pushes;" to compensate for this disadvantage, ropes are able to change the direction of a force, even to reverse it, with the help of a pulley, while leaving the magnitude of the force unaltered.

Let us examine the situation of a rope attached to a wall and having a force applied to its other end, as in Fig. 4-1. If we consider

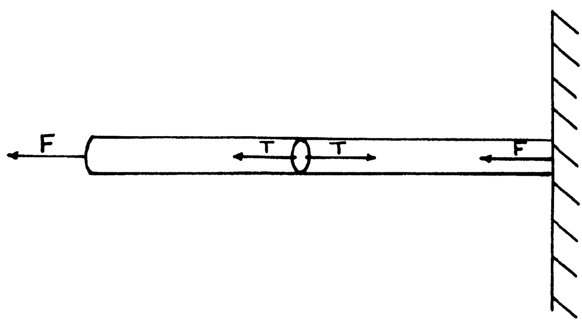


Fig. 4-1

a thin slice of the rope anywhere along its length, as shown, by Newton's Third Law equal and opposite forces must act across the slice since it is stationary. *Either one of these forces is called the tension in the rope.* At the left-hand end of the rope the force F is applied. In the rope this force becomes the tension T. At the right-hand end the tension T reappears as the force F. The rope is not accelerated, and therefore the net force acting on it is zero; hence the wall must exert a force -F on the end of the rope in order to keep it stationary.

To further clarify the meaning of tension, suppose we connect spring scales between the left-hand end of the rope and the hand pulling on it and between the right-hand end and the wall. Each scale has the same reading, and the value of this reading equals the tension T in the rope. By thinking of tension in this way it is possible to avoid the confusion that sometimes arises in solving problems involving ropes.

Example. A particle of mass 50 grams is attached to a rope and is being accelerated upward with an acceleration of 10 cm/sec^2. What is the upward force?

Solution. There are two forces acting on the particle. The first is its weight **W**, which is the downward force exerted by the earth on a 50 gram mass. The second is the upward force we are to find, which is equal to the tension **T** in the rope. **T** must be sufficiently greater than **W** to produce the observed acceleration. The resultant force is in the direction of the acceleration, and is

$$F = T - W = ma$$
$$= 0.05 \text{ kg} \times 0.1 \text{ m/sec}^2 = 0.005 \text{ n} \ .$$

The weight of the body is

$$W = mg = 0.05 \text{ kg} \times 9.8 \text{ m/sec}^2 = 0.49 \text{ n} \ ,$$

and so

$$T = 0.495 \text{ n} \ .$$

Example. A particle of mass 50 grams is attached to a rope and is being accelerated *downward* with an acceleration of 10 cm/sec^2. What is the upward force?

Solution. Since the resultant force F is in the direction of the acceleration, the weight of the particle must exceed the upward force provided by the tension in the rope. Therefore

$$F = W - T = ma = 0.005 \text{ n} \ ,$$

and since $W = 0.49$ as before,

$$T = 0.485 \text{ n}.$$

Example. Two particles of masses 50 and 100 grams are joined by a string and placed over a frictionless pulley (Fig. 4-2). This is called an Atwood's Machine, and it is assumed that the length of the string does not change and that the masses of the string and pulley are so small that they can be neglected. Find the acceleration of each particle.

Solution I. Since the pulley is frictionless, the tension in the string is constant throughout its length. To solve the problem, we consider each particle separately and set the net force acting on it equal to its mass multiplied by its acceleration. We begin by noting that the larger mass will move *downward;* hence its weight must exceed the upward tension T provided by the rope. Thus the net force on it is $F_{100} = W_{100} - T$. W_{100} is given by

$$W_{100} = mg = 0.1 \text{ kg} \times 9.8 \text{ m/sec}^2 = 0.98 \text{ n},$$

and so the net force is

$$(0.98 - T) \text{ n} = 0.1 \text{ kg} \times a \text{ m/sec}^2 , \qquad \text{(A)}$$

where a is the acceleration. The smaller mass is accelerated *upward* with the same acceleration a, and its weight

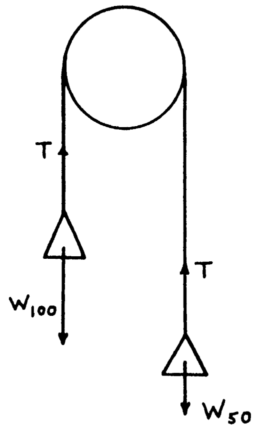


Fig. 4-2

$$W_{50} = mg = 0.05 \text{ kg} \times 9.8 \text{ m/sec}^2 = 0.49 \text{ n}$$

is therefore less than T. The net force on the 50 gram mass is

$$(T - 0.49) \text{ n} = 0.05 \text{ kg} \times a \text{ m/sec}^2 . \qquad \text{(B)}$$

Solving Eqs. (A) and (B) simultaneously, we find that

$$T = 0.653 \text{ newtons}$$

and

$$a = 3.27 \text{ m/sec}^2 .$$

Solution II. In the above solution we have considered the problem in detail, and were able to obtain the tension in the rope in addition to the acceleration. However, since all we are asked for is the acceleration, there is a much simpler method of attack. The net force acting on the *system of both particles* is the difference between the two weights

$$F = W_{100} - W_{50} = 0.49 \text{ n} .$$

The mass that this force acts on is the *total mass of the system,* 150 grams, and from Newton's Second Law

$$F = ma$$

$$0.49 \text{ n} = 0.15 \text{ kg} \times a \text{ m/sec}^2 ,$$

we obtain

$$a = 3.27 \text{ m/sec}^2$$

as before. This approach, in which a *system* as an entity is investigated rather than the component particles separately, is often employed in physics, and we shall make use of it in the next chapter.

4.8. Rotational Motion. As we have seen, even though a particle moving with constant angular velocity has a constant linear speed, its motion is nevertheless accelerated. This centripetal acceleration has the constant magnitude

$$a_C = \frac{v^2}{r} ,$$

and it always points toward the center of the circle on which the particle's path lies. The force that supplies the acceleration is called the *centripetal force*, and its magnitude is

$$F_C = \frac{mv^2}{r} . \tag{4.10}$$

Centripetal forces are quite common since they must be present whenever rotational motion occurs. Gravitation provides the centripetal force that keeps the planets moving around the sun and the moon around the earth. Friction between the tires and the road provides the centripetal force needed by an automobile in rounding a curve. Cohesive forces are the centripetal forces that keep a rotating flywheel together. Whirl an object at the end of a string; the tension in the string provides the centripetal force. If the string breaks there is no longer any tension, hence no centripetal force, and the object flies off.

Example. A ball is whirled in a vertical circle at the end of a string 0.5 m long. What is the minimum number of rotations per second required to keep the string taut even at the top of the circle?

Solution. In order that the string be just taut at the top of its path (Fig. 4-3), the gravitational downward force mg must provide the exact amount of centripetal force mv^2/r needed to keep the ball moving in a circle. Hence

$$mg = \frac{mv^2}{r} = m\omega^2 r$$

$$\omega = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.8 \text{ m/sec}^2}{0.5 \text{ m}}} = 4.43 \text{ rad/sec}$$

$$= \frac{4.43 \text{ rad/sec}}{2\pi \text{ rad/rev}} = 0.71 \text{ rev/sec.}$$

4.9. Rotating Coordinate Systems. There is another way of looking at the forces acting on a rotating particle. Suppose that the particle is at rest with reference to a laboratory which is rotating, and that the inhabitants of the laboratory are unaware of the rotation. (This is not an unlikely situation; hundreds of thousands of years passed before man realized that the earth revolves daily on its axis.) From outside the laboratory a stationary observer would say that in order for the particle (Fig. 4-4) to keep rotating so that it does not change its position in the laboratory, a centripetal force mv^2/r must be acting in the direction of the center of rotation 0 a distance r from the particle. In the diagram the centripetal force is supplied by a string attached between 0 and the particle.

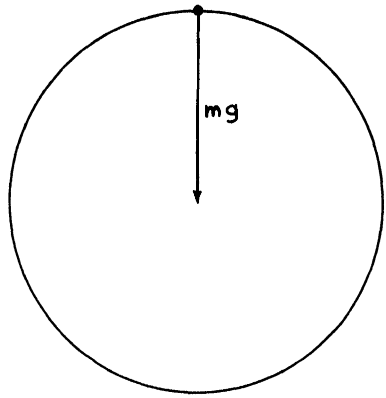


Fig. 4-3

To a person in the laboratory things seem quite different. He observes that a tension must be maintained in the string in order that the particle stay at rest (with respect to him, of course). However, by the first law of motion, a particle can be at rest only when the sum of the forces acting on it are zero. The man in the laboratory must therefore postulate a force

$$F_c' = -\frac{mv^2}{r} \qquad (4.11)$$

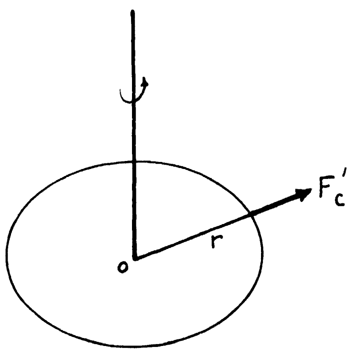


Fig. 4-4

to balance the tension in the string. This fictitious force is called *centrifugal force,* and always seems to act *away* from the center of rotation. Nonexistent forces of this kind which come into play when the observer is himself rotating are called *inertial forces*. When a passenger in a car is "thrown" to one side when a sharp turn is made, the centrifugal force he thinks is pushing him outward is actually his tendency to keep on moving in a straight line despite the path the car takes.

Another variety of inertial force appears when a body moves with respect to a rotating coordinate system, for example an airplane over the surface of the earth. This is the *Coriolis force*, given by

$$F_C = 2m(\mathbf{v} \times \boldsymbol{\omega}) \ , \qquad (4.12)$$

where $\mathbf{v}$ is the velocity of the body and $\boldsymbol{\omega}$ the angular velocity of the rotating system. The direction of the Coriolis force is, from the definition of the vector product, perpendicular to the plane of $\mathbf{v}$ and $\boldsymbol{\omega}$. The Coriolis force must be introduced by an observer in the rotating system to account for the independence the moving body exhibits of the rotation; to an observer on the earth the airplane's path appears to be bent as a result of the Coriolis acceleration, which is simply another way of expressing the fact that the observer himself is moving with the earth underneath the true trajectory of the airplane.

Masses of air moving over the earth's surface also experience Coriolis accelerations, and consequently when air flows into a low-pressure region it is deflected into counterclockwise rotation in the northern hemisphere and into clockwise rotation in the southern (Fig. 4-5). Hurricanes, typhoons, tornadoes, and similar storms owe

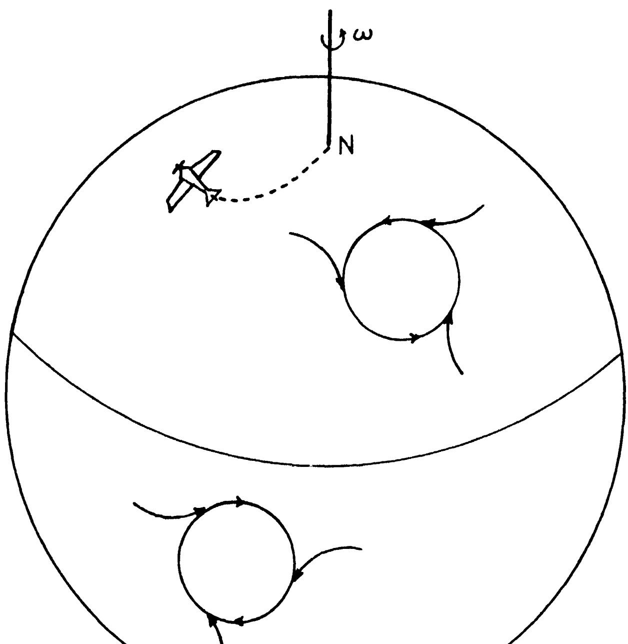


Fig. 4-5

their origin to Coriolis forces and therefore exhibit these rotations. Within the earth it is believed that Coriolis forces affect the motion of electrically conducting fluids, which helps, in a complicated way, to maintain the terrestrial magnetic field. In terms of their effects on individual objects such as birds and airplanes, though, Coriolis forces arising from the earth's rotation are usually negligible.

Example. What is the Coriolis acceleration of an airplane starting from the North Pole with a velocity of 1000 miles/hr?

Solution. Since the direction of $\boldsymbol{\omega}$ is along the earth's axis and at the Pole $\mathbf{v}$ is perpendicular to the axis, the magnitude of the Coriolis acceleration is

$$a_C = \frac{F_C}{m} = 2v\omega$$

and its direction is to the airplane's right (Fig. 4-5). The angular velocity is that of the earth, which is

$$\omega = \frac{2\pi}{24x60x60} \frac{\text{rad}}{\text{sec}} = 7.3 \times 10^{-5} \text{ rad/sec} \quad ,$$

and

$$v = 1000 \frac{\text{mi}}{\text{hr}} \times 5280 \frac{\text{ft}}{\text{mi}} \times \frac{1}{3600} \frac{\text{hr}}{\text{sec}}$$

$$= 1.5 \times 10^3 \text{ ft/sec.}$$

Hence

$$a_C = 0.22 \text{ ft/sec}^2 \quad ,$$

so that the Coriolis acceleration here is less than one per cent of the gravitational acceleration.

7.3. Inertial Systems and Reference Frames. We have seen that the laws of motion are different in different frames of reference. In a stationary frame the force in the equation

$$\mathbf{F} = m\,\mathbf{a} \tag{4.13}$$

is the total external real force, while if the phenomena are observed from a rotating frame fictitious inertial forces such as centrifugal and Coriolis forces must be introduced in order to account for the apparent accelerations. The choice of a coordinate system from which to make measurements is evidently of great importance if the results are to be assessed properly.

The problem of finding a coordinate system in which Eq. (4.13) holds exactly as it stands troubled Newton. This system cannot be attached to the earth, since it is rotating. But how do we know that the earth rotates without direct reference to a fixed set of coordinates? The answer is that since inertial forces must be invented to account for the behavior of bodies on the earth, it must be rotating. Any statement about motion implies the existence somewhere of a fixed set of coordinates, and therefore Newton could legitimately say that there must be such a set in the universe in which Eq. (4.13) is valid.

This fixed frame is not the only one in which the laws of motion hold, however. These laws describe forces which can be measured only in terms of the accelerations they produce; hence experiments performed in other frames of reference moving with constant velocities with respect to the fixed frame yield the same laws. Consider the axes x', y', z' moving with the velocity v parallel to the x axis of the fixed frame x,y,z (Fig. 4-6). The coordinates of a point in the fixed frame are related to its coordinates in the moving frame by the equations

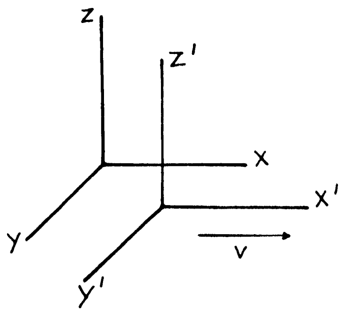


Fig. 4-6

$$x' = x - vt$$

$$y' = y$$

$$z' = z \ .$$

Taking the second derivative with respect to time on both sides,

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}$$

$$\frac{d^2y'}{dt^2} = \frac{d^2y}{dt^2}$$

$$\frac{d^2z'}{dt^2} = \frac{d^2z}{dt^2} \ ,$$

so that

$$\mathbf{a}' = \mathbf{a} .$$

Because accelerations are the same whether measured from one frame or the other, Newton's laws have the same form in both. Frames of reference in which the laws of motion hold true without modification, namely the fixed set of coordinates postulated for the universe and all other sets moving at constant velocities relative to it, are called *inertial* frames.

Newton's analysis, while profound, was unfortunately not quite correct. The modifications in it that must be made require a background in the ideas of electricity and magnetism for their understanding, and therefore will be postponed until a later chapter, where the theory of relativity is discussed.

Chapter 5

CONSERVATION LAWS OF PARTICLE DYNAMICS

It would be helpful in understanding the behavior of particles and systems of particles if we knew of some quantity or quantities which remain constant during whatever motion is taking place. We might even be able to draw some conclusions about the motion without investigating it in detail, basing our analysis simply on the conservation of a particular quantity. As we will see, this procedure turns out to be a very powerful one that enables us to attack problems whose solution would be difficult or impossible by other means.

5.1. Conservation of Linear Momentum. Newton found it convenient to define something he called the "quantity of motion" of a particle, which is now known as its *momentum*. Momentum, whose symbol is $\mathbf{p}$, is the product of the mass of a particle and its velocity, that is,

$$\mathbf{p} = m\mathbf{v}. \tag{5.1}$$

Since $\mathbf{p}$ is the product of a scalar, m, and a vector, $\mathbf{v}$, it is a vector.

We can restate Newton's second law of motion in terms of momentum provided that we restrict ourselves to bodies whose mass does not change with time. We have

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}}, \tag{5.2}$$

where we have used a dot above the p to represent its time derivative. (Similarly, two dots above a symbol stands for the second derivative with respect to time, so that $\ddot{x} = d^2x/dt^2$, etc.) When the total external force $\mathbf{F}$ acting on a particle is zero, from Eq. (5.2) we have that $\dot{\mathbf{p}} = 0$ also. This means that $\mathbf{p}$ does not change with time. We have therefore obtained a conservation law of the kind discussed above: *when the external force acting on a particle is zero, the momentum of the particle is a constant.* Note that the above is a *vector* statement, and applies both to the magnitude of $\mathbf{p}$ and to its direction.

For single particles this law does not lead to any important consequences. Where more than one particle is involved, however, conservation of momentum has a more significant role. Suppose we have a particle of mass m, initially at rest, suddenly breaking up into two particles of masses m_1 and m_2 which fly off. The forces acting on the particle that caused it to break up were internal ones, and no external force was present. Since m had the initial momentum $\mathbf{p}=0$, the final momentum of m_1 and m_2, taken together, must be zero. Hence

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = 0$$

$$\mathbf{v}_2 = -\frac{m_1}{m_2} \mathbf{v}_1 ,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the final velocities of the two particles. We notice immediately that these velocities must be in opposite directions along the same line.

Example. A rifle weighing 6 lb fires a bullet weighing $\frac{1}{2}$ oz at a muzzle velocity of 2000 ft/sec. What is the recoil velocity of the rifle?

Solution. From the above equation,

$$v_{\text{rifle}} = -\left(\frac{m_{\text{bullet}}}{m_{\text{rifle}}}\right) v_{\text{bullet}}$$

$$= -\left(\frac{1/32}{6}\right) 2000 \text{ ft/sec}$$

$$= -10.4 \text{ ft/sec.}$$

The law of conservation of momentum is indispensible in dealing with a collision between particles. In such cases no external forces act on the participants, and therefore the total momentum of the particles before they collide (added vectorially) equals the total momentum afterward. The essential effect of the collision is to redistribute the momentum of the particles. In the next chapter we will study problems of this kind.

5.2. Torque and Angular Momentum. Another quantity often conserved is *angular momentum.* If we consider a particle moving along an arbitrary path, as in Fig. 5-1, its angular momentum $\mathbf{L}$ about some point 0 is defined as

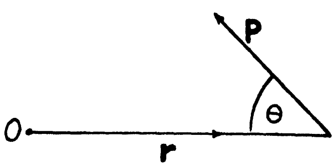


Fig. 5-1

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{5.3}$$

The radius vector $\mathbf{r}$ is drawn from 0 to the position of the particle at a particular instant, and $\mathbf{p}$ is the linear momentum of the particle at that instant. The simplest case is that of a particle moving in a circle with a uniform speed v; here $\mathbf{v}$ is always perpendicular to $\mathbf{r}$, and L=pr. In general, the magnitude of $\mathbf{L}$ is rp $\sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{p}$. From the definition of the vector product in Chapter 2, the angular momentum vector $\mathbf{L}$ is perpendicular to the plane in which $\mathbf{r}$ and $\mathbf{p}$ lie.

The angular momentum of a particle evidently depends on the choice of the point 0 about which it is taken. If we select another point 0' (Fig. 5-2) both $\mathbf{r}$ and θ are different from before, and a different value of $\mathbf{L}$ is obtained. There are even points about which the angular momentum vanishes; for example, computing $\mathbf{L}$ about the position of the particle itself yields $\mathbf{L}=0$, since the radius vector $\mathbf{r}$ is then zero. If the point chosen lies along the direction of $\mathbf{p}$, such as 0'' in Fig. 5.2, the angle $\theta=0$ and $\mathbf{L}=0$. The angular momentum (taken about any fixed point) of a particle moving along a complicated path varies as the particle changes its position.

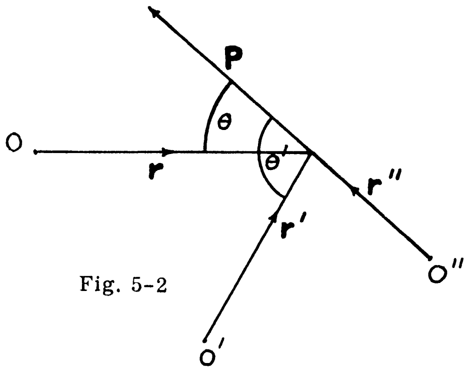


Fig. 5-2

We shall now obtain a quantity called *torque* whose relationship to angular momentum is the same as the relationship between force and linear momentum. We start with the latter, which is Newton's second law

$$\mathbf{F} = \dot{\mathbf{p}} , \tag{5.2}$$

and take the cross product of both sides with $\mathbf{r}$. This gives

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \dot{\mathbf{p}} . \tag{5.3}$$

We can write $\mathbf{r} \times \dot{\mathbf{p}}$ in another way by noting that

$$\frac{d(\mathbf{r} \times \mathbf{p})}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} . \tag{5.4}$$

(Cross products of vectors are differentiated in the same way as ordinary products except that the original order of the quantities must be preserved.) The second term of the above equation is

$$\frac{d\mathbf{r}}{dt} \times \mathbf{p} = m\mathbf{v} \times \mathbf{v} = 0 ,$$

since $d\mathbf{r}/dt=\mathbf{v}$ and $\mathbf{p}=m\mathbf{v}$ and the cross product of any vector with itself vanishes. Therefore, with the help of Eq. (5.3),

$$\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \dot{\mathbf{p}} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} = \frac{d\mathbf{L}}{dt} = \dot{\mathbf{L}} ,$$

and Eq. (5.4) may be rewritten

$$\mathbf{r} \times \mathbf{F} = \dot{\mathbf{L}} \ . \tag{5.5}$$

The left-hand side of (5.5) is called torque, $\mathbf{N}$, so that

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \tag{5.6}$$

and

$$\mathbf{N} = \dot{\mathbf{L}} \ . \tag{5.7}$$

The rate of change of angular momentum is equal to the applied torque.

From its definition, torque is to be calculated about the same point that angular momentum is taken about. Torque is also called *moment of a force,* and it is a measure of the amount of "turning" that the force $\mathbf{F}$ provides about a particular point. In Fig. 5-3 a rod is pivoted at point 0 while a force $\mathbf{F}$ is applied at one end. The rod tends to turn under the influence of the torque $\mathbf{N}$ whose magnitude is $rF\sin\theta$. Displacing the line of action of $\mathbf{F}$ changes the value of $\mathbf{N}$, and therefore when dealing with torques we can no longer think of force as a "free" vector which may be shifted parallel to itself at will. Similarly, the position of the pivot point must be specified precisely.

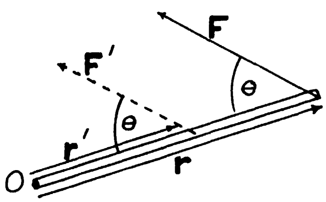


Fig. 5-3

A conservation theorem for angular momentum follows from Eq. (5.7) in the same way that a conservation theorem for linear momentum followed from Eq. (5.2), $\mathbf{F} = \dot{\mathbf{p}}$. When $\mathbf{N}=0$, $\mathbf{L}$=constant: *when the external torque applied to a particle about a point is equal to zero, the angular momentum of the particle about that point is a constant.*

The most important application of the conservation of angular momentum is to the case of particle motion in the presence of what is known as *central forces.* A central force acting on a particle is one which is always directed toward a particular point regardless of the position of the particle. The gravitational field of the sun causes a central force to act on the planets, holding them in fixed orbits. In the Bohr model of the hydrogen atom the electric field of the proton exerts a central force on the electron, again compelling motion in an orbit. We can describe a central force very simply in vector notation by saying that

$$\mathbf{F} = c\mathbf{r} \ , \tag{5.8}$$

where c is some scalar constant. Let us calculate the moment of this force on a particle a distance $\mathbf{r}$ away from its center S, as as in Fig. 5-4. It is

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} = c\mathbf{r} \times \mathbf{r} = 0,$$

since the vector product of a vector with itself is zero. Hence the angular momentum of the particle about S must be a constant.

Angular momentum is a vector, and and the statement that it is constant implies that neither its magnitude nor its direction varies. Because $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ the direction of $\mathbf{L}$ is perpendicular to the plane defined by $\mathbf{r}$ and $\mathbf{p}$. When $\mathbf{L}$ is constant, then, the motion described by the radius vector $\mathbf{r}$ and the momentum $\mathbf{p}$ lies in a plane at all times. This observation is a direct consequence of the conservation of angular momentum.

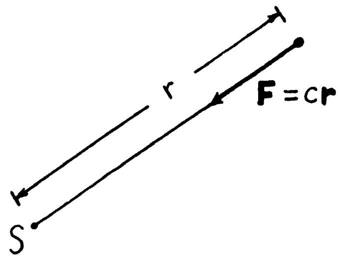


Fig. 5-4

The magnitude of $\mathbf{L}$,

$$L = mvr \sin\theta ,$$

is also conserved. In the infinitesimal time dt the area swept out by the radius vector to the moving particle from S is (Fig. 5-5)

$$dA = \tfrac{1}{2} rds \sin\theta ,$$

and the rate at which $\mathbf{r}$ sweeps out area is, neglecting the second-order contribution due to the change in $\mathbf{r}$,

$$\frac{dA}{dt} = \tfrac{1}{2} r \frac{ds}{dt} \sin\theta$$

Since $ds/dt=v$,

$$\frac{dA}{dt} = \tfrac{1}{2} rv \sin\theta = \frac{L}{2m} = \text{constant} \tag{5.9}$$

because m and L do not change with time. Thus we conclude that the radius vector to a particle moving under the influence of a central force sweeps out equal areas in equal times *regardless of the particular path it follows.*

We may now state that the planets, which are acted upon by a central force originating at the sun, move in orbits that each lie in a plane, and that the areas swept out in equal times by each planet are the same for the planet. These conclusions were first obtained by Johannes Kepler, a predecessor of Newton, who arrived that them by analyzing an enormous mass of astronomical data. They played an important part in Newton's discovery of the law of gravitation, as we shall see in Chapter 6.

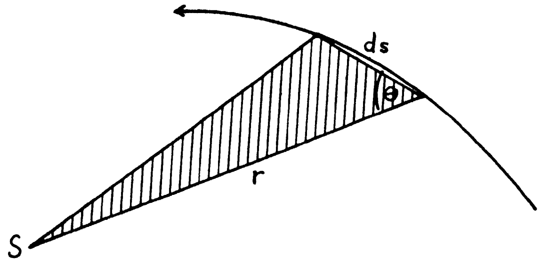


Fig. 5-5

5.3. Work. Perhaps the most important conservation law is that concerning *work* and *energy*. These words are used by everybody, and everybody has his own concept of what they mean. Unfortunately the technical definitions of work and energy are quite explicit and leave no room for individual interpretation; for instance, while it may seem like "hard work" to hold a 50 lb weight over one's head for five minutes, to a physicist no work at all is being done.

Work is easy to define as long as we restrict ourselves to one dimension. When a constant force F is applied to a body in, let us say, the $+x$ direction, and the body moves a distance d in this direction, the work W done on the body by the force is

$$W = Fd \ , \tag{5.10}$$

the product of the force and the distance through which it acts.

In general the force varies as the body moves. Here we can plot F as a function of displacement x (Fig. 5-6), and divide the total

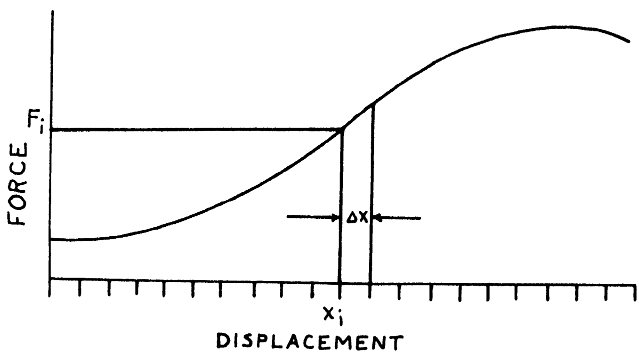


Fig. 5-6

distance through which the body moves into a large number of intervals Δx long. If Δx is small enough, F is nearly constant within each interval even though it changes from one interval to the next.

The work done by the force in propelling the body from the position x_i to the position $x_i + \Delta x$ is therefore

$$W = F(x_i)d = F(x_i)[(x_i + \Delta x)-(x_i)] = F(x_i)\Delta x \ ,$$

where $F(x_i)$ is the value of the force at x_i. The total amount of work done in going from A to B is the sum of the work done in each of the intervals Δx, namely

$$W = \sum_{i=1}^{N} F(x_i)\Delta x \ .$$

Since F actually varies continuously in Fig. 5-6, the smaller we make Δx the more accurate our result will be. Finally, as Δx becomes smaller and smaller and the number of intervals i approaches infinity, the finite sum above becomes an integral,

$$W = \lim_{N\to\infty} \sum_{i=1}^{N} F(x_i)\Delta x$$

$$= \int_{B}^{A} F(x)dx \ . \qquad (5.11)$$

Work is a scalar quantity, and in the British gravitational system the appropriate unit is the ft-lb. In the cgs system work is expressed in dyne-cm, which are called *ergs*, while in the mks system it is expressed in newton-meters, which are called *joules*. Evidently

$$1 \text{ joule} = 10^7 \text{ ergs}.$$

Example. A spring requires a force of 4 lb/ft to extend it. How much work is done in stretching the spring 10 ft? (We assume that the force needed to extend the spring is *linear*, so that the spring obeys the force law $F(x)=kx$ regardless of how large x may be.)

Solution. From Eq. (5.11), since $F(x) = kx$,

$$W = \int_{A}^{B} F(x)dx$$

$$= \int_{0}^{L} kx\,dx = \frac{kL^2}{2} \ .$$

Here the spring constant k is 4 lb/ft, so that

$$W = 4\,\frac{\text{lb}}{\text{ft}} \times (10 \text{ ft})^2 \times \frac{1}{2} = 200 \text{ ft-lb}.$$

Let us now remove the restriction that the applied force and the displacement lie along the same line. The first generalization is to motion in a plane, as in Fig. 5-7. There the force is constant, with the constant components F_x and F_y, and the particle moves from O to P. This displacement is equal to the sum of the two displacements OX and OY. Since work is a scalar quantity, the total work done here is the work done by F_x in moving the particle the distance X plus the work done by F_y in moving the body the distance Y, which is

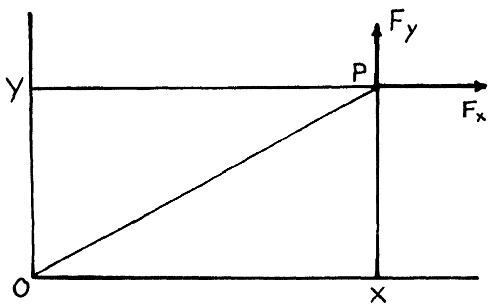


Fig. 5-7

$$W = F_xX + F_yY.$$

This result can be extended to the situation where the applied force is not constant in the same way as was done in the one-dimensional case. In Fig. 5-8 the work dW done on the particle as it goes from (x,y) to (x+dx, y+dy) is

$$dW = F_x(x,y)dx + F_y(x,y)dy.$$

(F_x and F_y may be different functions of x and y, and for that reason are written $F_x(x,y)$ and $F_y(x,y)$.) The total work is found by integrating dW, yielding

$$W = \int_A^B [F_x(x,y)dx + F_y(x,y)dy] .$$

It is obvious that in three dimensions we have

$$W = \int_A^B [F_x(x,y,z)dx + F_y(x,y,z)dy + F_z(x,y,z)dz]. \quad (5.12)$$

The integrand of Eq. (5.12) should be familiar; it is just the scalar product of the vectors **F** and **ds**, whose components are F_x, F_y, F_z and dx, dy, dz respectively. Hence we can rewrite Eq. (5.12) in the form

$$W = \int_A^B \mathbf{F}\cdot\mathbf{ds} . \quad (5.13)$$

Still another way of expressing W follows from Eq. (5.13). With F the magnitude of the force, ds the magnitude of the displacement, and θ the angle between the direction of the vectors **F** and **ds** ,

$$W = \int_A^B F \cos\theta \, ds . \quad (5.14)$$

These three equations are identical mathematically, and the appropriate one to use depends upon the problem at hand.

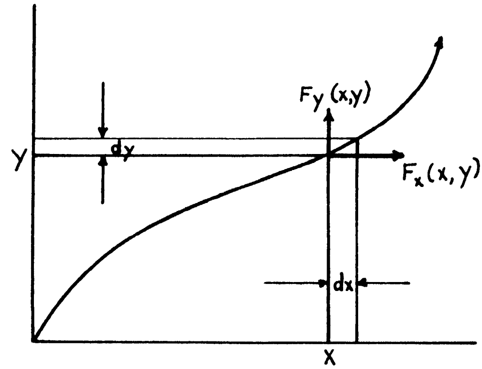


Fig. 5-8

Example. A boy pulls a sled of mass 0.5 slug with a force of 10 lb exerted at an angle of 30° above the horizontal (Fig. 5-9). How much work does he do in moving the sled 100 ft?

Solution. There is no need to integrate here because the force is constant, and so

$$W = Fs\cos\theta = 10 \text{ lb} \times 100 \text{ ft} \times \cos 30°$$
$$= 866 \text{ ft-lb}.$$

Note that the mass of the sled is irrelevant for this calculation.

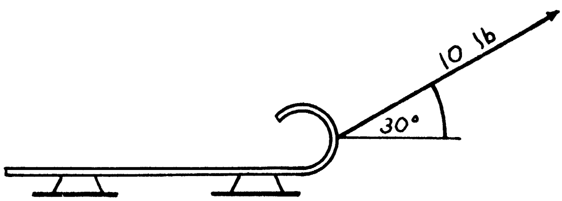


Fig. 5-9

Example. How much work is done in moving a body of mass m to a height h up a frictionless plane of angle θ with the horizontal? Assume that just enough work is done for the lifting, so that the body has effectively zero velocity at the height h.

Solution. Let us use Eq. (5.12) to solve this problem. The only force that acts on the particle at the foot of the plane that we must consider is its weight, mg, which is in the y direction, and the applied force must equal this. (A greater force would result in an acceleration and hence a velocity greater than zero at h.) F_x and F_z both are zero. If ds is an infinitesimal displacement along the plane (Fig. 5-10),

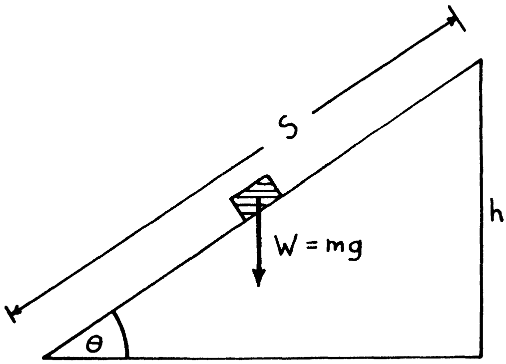


Fig. 5-10

$$dy = ds \sin\theta$$

and

$$W = \int_0^S F_y dy = mgS\sin\theta ,$$

where S is the distance the body moves. But $h/S = \sin\theta$, and therefore

$$W = mgh .$$

This interesting result means that the work done does not depend upon the angle of the plane (in the absence of friction) but only on the total height through which the weight is lifted. Even if the weight is raised straight up the same amount of work is involved.

Example. A particle is acted on by a central force in the x,y plane. The direction of the force is toward the origin, and its magnitude is proportional to the distance from the origin to the particle, that is,

$$\mathbf{F} = k\mathbf{r} .$$

How much work is done in "just" moving the particle from the origin (0,0) to the point (1,1)?

Solution. Here the precise path to be followed by the particle in going from (0,0) to (1,1) is not specified. The simplest path is directly along the radius vector (Fig. 5-11), in which case

$$W = \int_A^B \mathbf{F}\cdot\mathbf{ds} = k\int_0^{\sqrt{2}} r\,dr = \frac{kr^2}{2}\Big|_0^{\sqrt{2}} = k.$$

Will W be different if another path is followed? Let us take the particle from (0,0) to (1,0), and then from (1,0) to (1,1). The work done in the first step is

$$W_1 = \int_A^B F_x dx = k\int_0^1 x\,dx = \frac{k}{2}\ .$$

For the second step we note that the magnitude of **F** is

$$F = kr = k\sqrt{x^2 + y^2}\ ,$$

so that

$$F_y = F\sin\theta = F\frac{y}{\sqrt{x^2 + y^2}} = ky\ .$$

Hence

$$W_2 = \int_0^1 F_y dy + k\int_0^1 y\,dy = \frac{k}{2}\ ,$$

and the total work done in going from (0,0) to (1,1) by this path is

$$W = W_1 + W_2 = k.$$

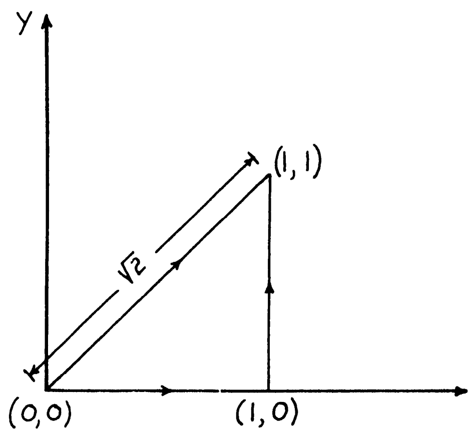


Fig. 5-11

The amount of work done is the same regardless of the path taken, which is characteristic of what are called *conservative forces*. When only conservative forces are acting, the work involved in moving a body from one place to another depends only on the initial and final positions, and not on the details of the displacement, that is, on the particular path. In the above example an amount k of work had to be done *against* the central force to carry it to (1, 1). In going from (1,1) back to (0,0), exactly the same work k would have to be done, this time by the central force.

Example. A particle is acted on by a frictional retarding force that is proportional to the normal (perpendicular) force holding it against a surface. In the horizontal x,y plane the force is

$$F_f = \mu N\ ,$$

where the normal force is the particle's weight in this case. How much work is done in "just" moving the particle from (0,0) to (1,1)?

Solution. We shall start with the first of the paths used in the previous example. The work done is

$$W = \int_A^B F_f dr = \mu N \int_0^{\sqrt{2}} dr = \sqrt{2}\,\mu N .$$

The work done against the retarding force when the particle traverses the second path is

$$W = W_1 + W_2 = \int_0^1 F_f dx + \int_0^1 F_f dy$$

$$= 2\,\mu N ,$$

which is greater than the work done in the shorter path! Furthermore, in going from (1,1) to (0,0) work would still have to be done *against* the retarding force, the amount depending on the length of the path. In the presence of *nonconservative* forces of this kind (which are also called *dissipative* forces), the work required to move a body depends upon the details of the motion and not on its endpoints.

5.4. Energy. Returning for the moment to one dimension, let us determine the effect that doing an amount of work W on a particle of mass m has on its motion. In going from x_1 to x_2 under the influence of the applied force F,

$$W = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} ma dx = m\int_{x_1}^{x_2} \frac{dv}{dt} dx$$

$$= m\int_{x_1}^{x_2} \frac{vdx}{dt} = m\int_{v_1}^{v_2} vdv = \frac{m}{2} v^2 \Big|_{v_1}^{v_2}$$

$$= \tfrac{1}{2}mv_2{}^2 - \tfrac{1}{2}mv_1{}^2 \quad . \tag{5.15}$$

The work done on the particle appears in the form of the difference between $\frac{1}{2}mv^2$ evaluated at the final position x_2, where the particle's velocity is v_2, and at the initial position x_1, where its velocity is v_1. The applied force need not be constant for this result to hold true.

In the general case of three dimensions we have

$$W = \int_1^2 [F_x dx + F_y dy + F_z dz]$$

$$= \int_1^2 [v_x dv_x + v_y dv_y + v_z dv_z]$$

$$= \frac{m}{2}\left[v_x^2 + v_y^2 + v_z^2\right]\Big|_1^2 = \frac{m}{2}v^2\Big|_1^2$$

$$= \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \qquad (5.15)$$

just as before, since $(v_x^2 + v_y^2 + v_z^2) = v^2$. The quantity $\frac{1}{2}mv^2$ evidently plays an important role in describing the motion of a body. It is called *kinetic energy*, and Eq. (5.15) states that the work W done by a force in taking a particle from point 1 to point 2 is equal to the difference between the kinetic energies of the particle at the two points.

Example. A particle of mass 20 kg is at rest when a force of 5 newtons is applied. What is the velocity of the particle after it has gone a distance of 10 meters?

Solution. From Eq. (5.15), since the initial velocity is 0,

$$W = \tfrac{1}{2}mv^2 = Fs$$

$$v = \sqrt{\frac{2Fs}{m}} = \sqrt{\frac{2 \times 5 \times 10}{20}} = \sqrt{5}\ \text{m/sec.}$$

This same result, of course, can be obtained by the methods we have developed in kinematics and dynamics. For $v_0 = 0$,

$$s = \tfrac{1}{2}at^2$$

and

$$v = at\ .$$

Hence

$$v = \sqrt{2as}\ ,$$

and since $F=ma$

$$v = \sqrt{\frac{2Fs}{m}} = \sqrt{5}\ \text{m/sec}$$

as before. Even in this simple example the advantages of using kinetic energy as an aid in solving problems are clear.

In one of the illustrative examples of the preceding section we encountered a situation in which the work done on a particle could be expressed as a function *only* of its initial and final positions, with no reference to the path followed in going from one to the other. This property is characteristic of *conservative* forces. Suppose we have such a force acting in a region of space, and transport a particle from some position A in this region to another position B *against* the force by doing an amount of work W. The particle was stationary at A and is stationary at B; we have done just enough work to move the particle without imparting to it any kinetic energy. Where has the work gone?

The key to the question is that work done against a conservative force is recoverable. If we release the particle at B, we will have to perform no work in order to return it to its original position at A. The force is now acting to accelerate the particle, rather than opposing its motion, and at A the particle will have gained an amount of kinetic energy equal to the work W needed to carry it from A to B initially. In order to describe this state of affairs mathematically, a quantity called *potential energy* must be introduced. Potential energy is a function of position only, and we may therefore say that

$$
\begin{aligned}
W &= V(A) - V(B) \\
&= -\int_A^B dV \quad , \qquad (5.16)
\end{aligned}
$$

where V denotes potential energy. In moving the particle from A to B we have increased its potential energy by an amount equal to the work done, and this potential energy reappears as kinetic energy if we allow the particle to return to A. The potential energy V depends only on the endpoints of the motion.

Example. A brick weighing 6 lb is raised to a height of 10 ft, where it is dropped. What is its velocity when it reaches a height 6 ft from the ground?

Solution. The work required to lift the brick is

$$W = \int_0^h F ds = Wh \quad ,$$

since the force we must apply is its weight. Hence the potential energy that the brick has at the height of 10 ft is

$$\text{potential energy (10 ft)} = Wh = 60 \text{ ft-lb} \quad ,$$

and its potential energy 6 ft from the ground, similarly, is

$$\text{potential energy (6 ft)} = 36 \text{ ft-lb.}$$

In falling from 10 ft to 6 ft the change in potential energy is 24 ft-lb, which is converted to kinetic energy $\frac{1}{2}mv^2$. Therefore

$$
\begin{aligned}
\tfrac{1}{2}mv^2 &= \Delta V \\
v &= \sqrt{\frac{2\Delta V}{m}} = \sqrt{\frac{2V}{W/g}} = \sqrt{\frac{2 \times 24 \text{ ft-lb}}{6 \text{ lb}/32 \text{ ft/sec}^2}} \\
&= 16 \text{ ft/sec.}
\end{aligned}
$$

We have emphasized that potential energy is a concept that has meaning only in the presence of conservative forces. Let us now inquire into the kinds of forces that are conservative. The simplest instance is that of a force which is constant in direction and magnitude. The work done in carrying a particle from A to B with this force acting is

$$W = \int_A^B \mathbf{F}\cdot d\mathbf{s} = \int_A^B F ds_f = \int_A^B F dl ,$$

where ds_f represents the projection of the actual path element ds along the direction of $\mathbf{F}$, which is more conveniently called dl (Fig. 5-12). Integrating,

$$W = Fl,$$

where l is the straight-line distance between A and B, thus depending only upon the positions of A and B and not on the path taken between them. In Fig. 5-13 a much more complicated path from A to B is shown, including a section where the path bends back on itself. Since we are concerned only with the projections of $\mathbf{ds}$ along $\mathbf{F}$, we still find that $W=Fl$. If we take the particle from B to A, dl is now in the opposite direction to F and the work done is $W=-Fl$. When the path bends back upon itself the overlapping portions cancel as a consequence of this. An example of a force that is constant in magnitude and direction is that exerted by the earth on bodies near its surface, a force that manifests itself as "weight."

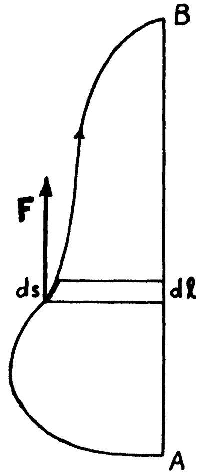


Fig. 5-12

Another important class of conservative forces are *central forces*, whose direction, as we have already seen, is always toward a particular point and whose magnitude is some function of the distance from this point. Such forces may be represented F(r), where r is the distance from the origin of the force to the body it is acting on. Consider the length element $ds=\widehat{AB}$ in Fig. 5-14. At A and B let us draw circular arcs that intersects OP at A' and B'. The work done by the force $\mathbf{F}_1$ in going from A to B is $F_1 ds \cos\theta$ where θ is the angle between $\mathbf{F}_1$ and $\mathbf{ds}$. However, $ds \cos\theta = ds' = \widehat{A'B'}$, and since the force depends only on r and the arcs are circular, $F_1' = F_1$. The total amount of work done in going from (1) to (2) in Fig. 5-14 is

$$W = \int_1^2 \mathbf{F}_1\cdot d\mathbf{s} = \int_1^2 F_1'\cdot ds' = \int_{r_1}^{r_2} F(r)dr .$$

The latter integral depends only upon the coordinates of the endpoints of the path and not on the details of the path, and so central forces are conservative.

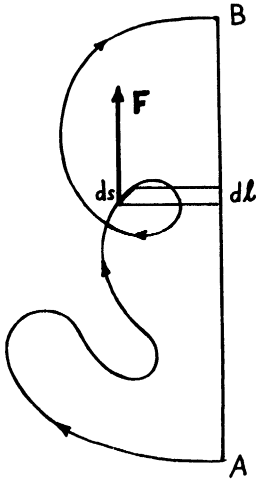


Fig. 5-13

Two significant central forces are the "linear restoring force" and the "inverse square force." In the former

$$F(r) = -kr,$$

and the work done by it in moving a particle from r_1 to r_2 is

$$W = -k\int_{r_1}^{r_2} r\,dr = \tfrac{1}{2}k(r_1{}^2 - r_2{}^2).$$

The potential energy of a point r from the center of force is the indefinite integral

$$V = -k\int r\,dr = \tfrac{1}{2}kr^2 + C_1 \,. \qquad (5.17)$$

The constant of integration disappears when we compute the difference in potential energy between two points, which is either the work done *against* the force, in which case the final position has the higher V, or the work done *by* the force, in which case the final position has the lower V. Sometimes it is convenient to give the constant C_1 an arbitrary value, for instance assigning it at the value of 0 at the center of force, so that $V(0)=0$. This procedure enables us to speak of the potential energy at a point, although strictly speaking only ΔV can ever be measured and hence it alone is significant.

The second central force mentioned above is that which varies as the inverse square of the distance from the origin,

$$F(r) = \frac{k}{r^2} \,.$$

Here

$$W = k\int_{r_1}^{r_2} \frac{dr}{r^2} = k\left[\frac{1}{r_1} - \frac{1}{r_2}\right] ,$$

and

$$V = \frac{k}{r} + C_2 \,. \qquad (5.18)$$

In this case it is usual to set the constant so that $V(\infty)=0$. We shall encounter both of these forces subsequently.

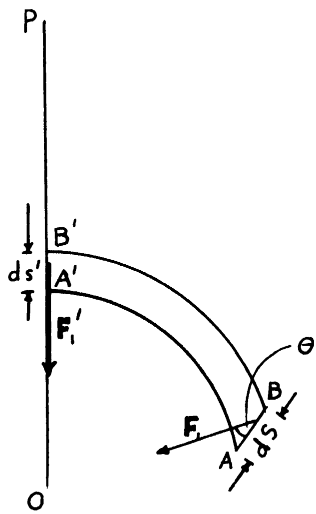


Fig. 5-14

Chapter 6

SYSTEMS OF PARTICLES

When we discussed Newton's laws of motion in the past few chapters, it was with the understanding that they applied to abstractions called particles which, strictly speaking, do not exist. The supposition was made, though, that under appropriate circumstances these laws also apply to extended bodies composed of a great many particles acting together. In this chapter we shall show that Newton's laws can indeed be so applied, and shall try to state just what the circumstances are under which this is true. The latter problem troubled Newton himself, in fact, to such an extent that he delayed publication of the law of gravitation for many years until he was satisfied that even so enormous an object as the Earth behaved as though it were a particle from certain points of view.

To begin with, we require a model of an extended body so that we know exactly what we are talking about. The definition we shall adopt states that an extended body consists of a very large number of particles held together by forces acting among the particles which obey Newton's third law of motion. The first body we will treat is one consisting of three particles, and the results we obtain will then be generalized to include any number of particles held together by mutually attractive forces.

6.1. System of Three Particles.

All of the forces acting on each of the three particles shown in Fig. 6-1 may be divided into two classes: (1) *external* forces that act from outside the system, and (2) *internal* forces exerted by the other particles of the system. Consider the forces acting on particle 1. The resultant of all of the external forces is $\mathbf{F}^e_1$. The internal forces are $\mathbf{F}^i_{12}$ and $\mathbf{F}^i_{13}$, exerted respectively by particle 2 and particle 3 on particle 1. Applying the second law of motion to particle # 1,

$$\mathbf{F}^e_1 + \mathbf{F}^i_{12} + \mathbf{F}^i_{13} = m_1 \mathbf{a}_1 \quad .$$

For the other two particles we find similar relations,

$$\mathbf{F}^e_2 + \mathbf{F}^i_{21} + \mathbf{F}^i_{23} = m_2 \mathbf{a}_2$$

and

$$\mathbf{F}^e_3 + \mathbf{F}^i_{31} + \mathbf{F}^i_{32} = m_3 \mathbf{a}_3 \ .$$

Adding these three equations,

$$\mathbf{F}^e_1 + \mathbf{F}^e_2 + \mathbf{F}^e_3 + \mathbf{F}^i_{12} + \mathbf{F}^i_{21} + \mathbf{F}^i_{13} + \mathbf{F}^i_{31} + \mathbf{F}^i_{23} + \mathbf{F}^i_{32}$$
$$= m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 + m_3 \mathbf{a}_3 \ . \qquad (6.1)$$

At this point we make use of the fact that, according to our definition of an extended body, the various internal forces all obey Newton's third law. This means that the force exerted by one particle on another is equal in magnitude and opposite in direction to the force exerted by the latter on the former, so that

$$\begin{aligned} \mathbf{F}^i_{12} &= -\mathbf{F}^i_{21} \\ \mathbf{F}^i_{13} &= -\mathbf{F}^i_{31} \qquad (6.2) \\ \mathbf{F}^i_{23} &= -\mathbf{F}^i_{32} \ . \end{aligned}$$

Combining Eqs. (6.1) and (6.2) yields

$$\mathbf{F}^e_1 + \mathbf{F}^e_2 + \mathbf{F}^e_3 = \mathbf{F} = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 + m_3 \mathbf{a}_3 \ , \qquad (6.3)$$

where $\mathbf{F}$ is the sum of all of the external forces that act on the system of the three particles. The internal forces have disappeared from the analysis, and only the external ones have any effect on the motion of the system.

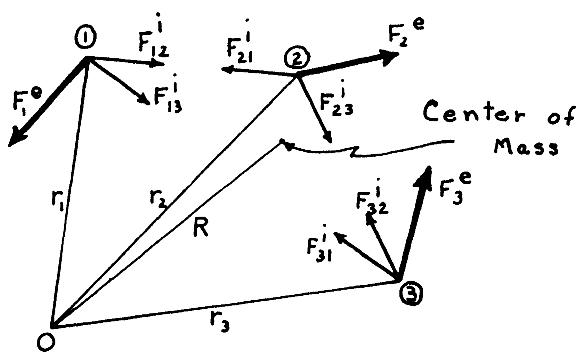


Fig. 6-1

Eq. (6.3) can be simplified further by rewriting its right-hand side as

$$\frac{d^2}{dt^2} (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3) \ ,$$

where, as usual, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ represent the position vectors of the particles. Now we define a vector $\mathbf{R}$ such that

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_1 + m_2 + m_3} \tag{6.4}$$

$$= \frac{1}{M}(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3) \ ,$$

in which we have abbreviated the total mass of the system $m_1+m_2+m_3$ by M. The new vector $\mathbf{R}$ is called the position vector of the *center of mass* of the system of particles. In terms of this vector Eq. (6.3) becomes

$$\mathbf{F} = M\frac{d^2\mathbf{R}}{dt^2} = M\,\mathbf{A}.$$

Here A is the acceleration of the center of mass, In other words, we have found that the *motion of the system of particles as a whole is exactly the same as if the entire mass of the system were concentrated at its center of mass, with the net force acting on it restricted to the total external force only.*

If instead of only three particles the extended body consists of n particles, exactly the same reasoning as we used above has the result that

$$\mathbf{F} = M\frac{d^2\mathbf{R}}{dt^2} = M\mathbf{A} \ , \tag{6.5}$$

where now $\mathbf{F}$, M, and $\mathbf{R}$ are defined more generally as

$$\mathbf{F} = \sum_{j=1}^{n} \mathbf{F}_j^e$$

$$M = \sum_{j=1}^{n} m_j \tag{6.6}$$

$$\mathbf{R} = \frac{1}{M}\sum_{j=1}^{n} m_j \mathbf{r}_j \ .$$

6.2. Center of Mass. Before we can apply Eq. (6.5) to actual objects and benefit from its elegant simplicity, we must be able to calculate the positions of the centers of mass of the objects. The last formula in (6.6) tells us everything there is about how to find the center of mass, of course, but a few examples are nevertheless in order.

Example. Where is the center of mass of a system of two particles of masses m_1 and m_2 a distance d apart?

Solution. First we have to establish the coordinate system to be used in expressing the location of the center of mass. It makes no real difference where we place the origin of coordinates, but certain positions make matters easier. For instance, in the present example, by placing the x axis along the line joining the two particles (Fig. 6-2), the y and z coordinates may be ignored. If we select the position of particle a as the origin, the center of mass has the x coordinate

$$R_x = X = \frac{m_a x_a + m_b x_b}{m_a + m_b}$$

$$= \frac{m_b}{m_a + m_b} \quad d$$

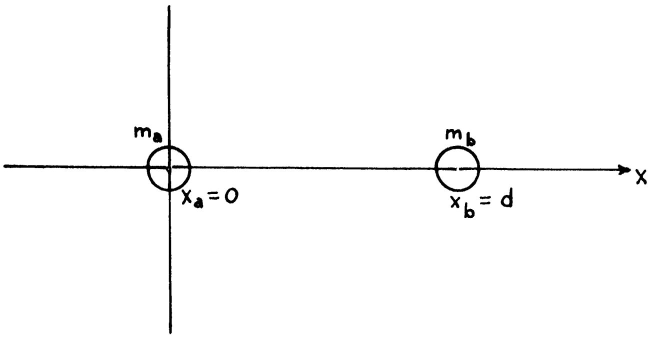


Fig. 6-2

since x_a=0 and x_b=d. The larger m_b is compared with m_a, the closer the center of mass lies to b. (A balanced see-saw, with the heavier child nearer the point of support in order for the center of mass to be directly above it, is an illustration of this result.) As a check, we note that in the limit of m_a=0, X=d, and in the limit of m_b=0, X=0. Locating the origin elsewhere would give a different equation for X, but the actual location in space of the center of mass would be unchanged.

Example. Three particles of the same mass are situated in a plane at the positions (0,0), (1,2), and (2,0). Where is their center of mass?

Solution. Applying Eq. (6.4) we have for each of the components of R

$$R_x = X = \frac{m_a x_a + m_b x_b + m_c x_c}{m_a + m_b + m_c}$$

$$= \frac{m(0) + m(1) + m(2)}{3m}$$

$$= 1$$

$$R_y = Y = \frac{m_a y_a + m_b y_b + m_c y_c}{m_a + m_b + m_c}$$

$$= \frac{m(0) + m(2) + m(0)}{3m}$$

$$= 2/3$$

$$R_z = Z = 0.$$

Hence the center of mass has the coordinates (1,2/3) as in Fig. 6-3.

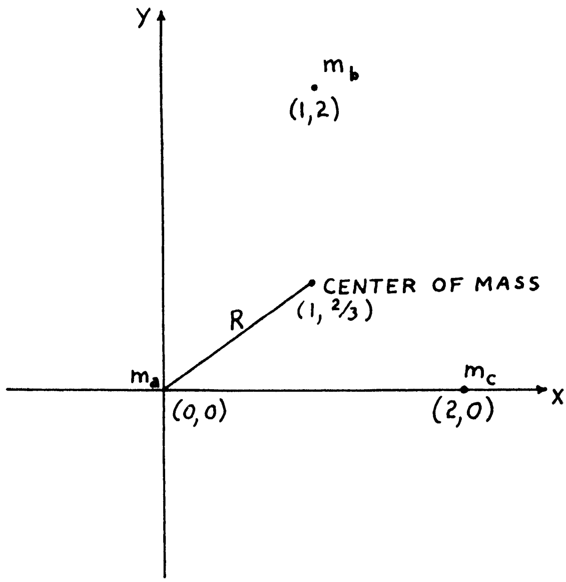


Fig. 6-3

If there are so many particles that the system may be thought of as a continuous distribution of mass, the finite sum of Eq. (6.6) may be replaced by the integral

$$\mathbf{R} = \frac{1}{m} \int \mathbf{r}\, dm \ . \tag{6.7}$$

A more convenient form for this equation involves the density (mass per unit volume or, if an essentially two-dimensional object is being considered, mass per unit area) ρ of the body, where ρ is some function of position in the body which we may write as $\rho(r)$. The mass dm of the volume element dV is ρdV, and

$$\mathbf{R} = \frac{1}{M} \int \rho(r)\, \mathbf{r}\, dV \tag{6.8}$$

where

$$M = \int \rho \, dV \ .$$

Example. Find the location of the center of mass of a thin uniform sheet of metal in the shape of an equilateral triangle.

Solution. From symmetry we know that the center of mass lies along the altitude of the triangle, leaving only the y coordinate to find. With ρ the constant density per unit area of the metal, the mass of the strip lying between y and y+dy is ρdA (Fig. 6-4). From the geometry involved we note that the area dA of the strip is the difference between the areas of the triangles DB'C' and DBC. Since $BC=2y/\sqrt{3}$ and $B'C'=2(y+dy)/\sqrt{3}$,

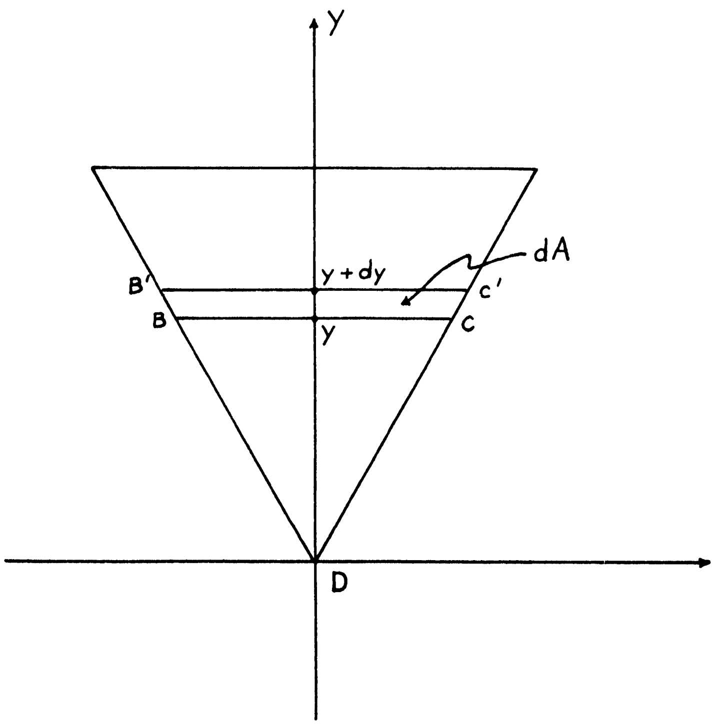


Fig. 6-4

$$dA = \tfrac{1}{2}(y+dy)\cdot\frac{2(y+dy)}{\sqrt{3}} - \tfrac{1}{2}\frac{2y}{\sqrt{3}}\cdot y$$

$$= \frac{2ydy}{\sqrt{3}}$$

where we have dropped higher-order differentials. Using Eq. (6.8) we have

$$R_y = Y = \frac{1}{M}\int_0^h \rho y\,\frac{2ydy}{\sqrt{3}} = \frac{2}{3}\,\frac{h^3\rho}{\sqrt{3}\,M}\,.$$

The area of an equilateral triangle is $h^2/\sqrt{3}$, so that the mass of the triangle is

$$M = \rho A = \frac{h^2\rho}{\sqrt{3}}\,.$$

Substituting this value of M into the expression for Y yields

$$Y = \frac{2h}{3}\,,$$

so that the center of mass is 2/3 of the distance from the apex to the base along the altitude joining them.

When the density of the body is constant throughout its volume, the center of mass is at the geometrical center if the body has a regular shape. The center of mass of a sphere, for instance, is at its center, the center of mass of a uniform parallelipiped is at the intersection of its diagonals, and so on.

Example. Where is the center of mass of a cylindrical metal rod of length 12 ft and radius 3 inches that has a flywheel of the same metal 2 ft long and 1 ft in radius at one end?

Solution. The simplest procedure here is to find first the centers of mass of the rod and flywheel separately, and then to compute the center of mass of the system of a particle at the center of mass of the rod having the mass of the rod and a particle at the center of mass of the flywheel having the mass of the flywheel. As Fig. 6-5

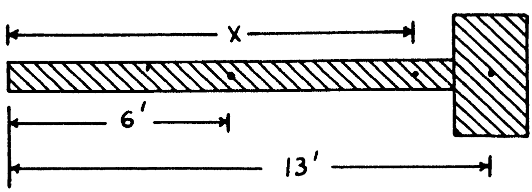


Fig. 6-5

shows, these latter centers of mass are located at the geometrical centers of the rod and flywheel, respectively 6 ft and 13 ft from the left-hand end of the rod. The mass of the rod is

$$m_r = \rho \pi (0.25)^2 12 = 2.36 \rho ,$$

and that of the flywheel is

$$m_f = \rho\pi (1)^2 2 = 6.28 \rho .$$

Hence the center of mass of the combination is at the distance

$$X = \frac{(2.36\rho)6 + (6.28\rho)13}{2.36\rho + 6.28\rho} = 11.1 \text{ ft.}$$

from the left-hand end of the rod.

6.3. Conservation of Linear Momentum. The fact that we have been able to express Newton's second law for systems of particles in a form (Eq. 6.5) closely resembling that holding for single particles implies that momentum is conserved in such systems. By defining the total momentum $\mathbf{P}$ of an object as the vector sum of the momenta of its constituent particles,

$$\mathbf{P} = \sum_{j=1}^{n} m_j \mathbf{v}_j , \tag{6.9}$$

we can rewrite Eq. (6.5) as

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} . \tag{6.10}$$

When the sum of the external forces acting on a system of particles is zero, its total momentum $\mathbf{P}$ remains constant. This form of the law of conservation of linear momentum is more useful than its counterpart for individual particles since it is able to treat such sometimes complex problems as collisions and disintegrations.

In collision problems the requirement that momentum be conserved in the absence of external forces always permits us to write down an equation that the various participants must obey. Consider a particle of mass m_a and velocity $\mathbf{v}_a$ striking a particle of mass m_b and velocity $\mathbf{v}_b$. Two particles, let us suppose, emerge from the collision, with masses m_c and m_d and the respective velocities $\mathbf{v}_c$ and $\mathbf{v}_d$. Since the initial momentum is unchanged in the collision,

$$m_a \mathbf{v}_a + m_b \mathbf{v}_b = m_c \mathbf{v}_c + m_d \mathbf{v}_d .$$

Equations of this kind, though, are generally insufficient by themselves. Even if we know m_a, m_b, m_c, m_d, $\mathbf{v}_a$ and $\mathbf{v}_b$, which seems sporting enough, we are still left with the two unknowns $\mathbf{v}_c$ and $\mathbf{v}_d$ and only one equation to find them with. Additional information is needed, and is often provided by energy considerations. Thus collisions are customarily described in terms of the "elasticity" of the interaction: elastic collisions are those in which the participants rebound with the same total kinetic they had initially (although it may be differently distributed among them), while in inelastic collisions some of the kinetic energy is converted into other forms, such as heat and sound. Collisions in which the various particles adhere to one another on impact are termed completely inelastic; here, of course, only the velocity of the single final body need be determined, and conservation of momentum by itself is adequate.

Example. A 12 g lump of clay moving with a velocity of 7 cm/sec strikes a 10 g lump of clay which is moving perpendicular to the first one with a velocity of 10 cm/sec. The two lumps stick together after the collision. What is the velocity of the final body? How much kinetic energy is lost?

Solution. In the absence of externalforces the total momentum of the two lumps in unchanged by the collision. Let us call the direction of the first piece the x direction and that of the second the y direction. The initial momentum therefore has the components

$$P_x = 12\text{ g} \times 7\text{ cm/sec} = 84\text{ g cm/sec}$$

$$P_y = 10\text{ g} \times 10\text{ cm/sec} = 100\text{ g cm/sec}$$

$$P_z = 0 \ .$$

The final momentum has the same components, and so its magnitude is

$$P = \sqrt{84^2 + 100^2} = 131\text{ g cm/sec.}$$

The direction may be specified in terms of the angle θ made by **P** with the x axis, so that

$$\tan\theta = \frac{P_y}{P_x} = 1.19$$

and

$$\theta = 50^\circ \ .$$

The final velocity of the composite piece of clay is in this direction (Fig. 6-6) with the speed

$$v_f = \frac{P}{m_1+m_2} = \frac{131\text{ g cm/sec}}{22\text{g}} = 5.94\text{ cm/sec.}$$

The initial kinetic energy of the two lumps is

$$KE^i = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$
$$= 794 \text{ ergs,}$$

while afterward it has decreased to

$$KE^f = \tfrac{1}{2}(m_1+m_2)v_f^2$$
$$= 388 \text{ ergs.}$$

Hence 406 ergs of kinetic energy, over half the original amount, has disappeared.

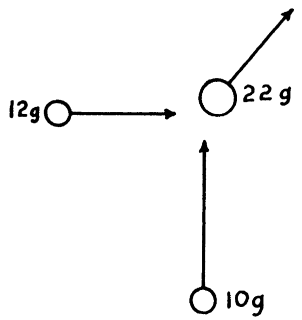


Fig. 6-6

6.4. Gravitational Interactions. Newton was the first to demonstrate quantitatively that the same laws of nature apply in the universe as are valid on the surface of the earth. His laws of motion, so successful in explaining terrestrial phenomena, also turn out to be capable of interpreting events on a cosmic scale. In particular, the gravitational force which caused the apple of the legend to fall on Newton's head was shown to be of the same kind as the one holding the planets in their courses around the sun—a hardly self-evident concept, and one of Newton's great achievements.

For many centuries before Newton astronomical observations of a surprisingly precise nature had been carried out. From the planetary data then in existence Kepler was able to infer his three famous laws: (1) the planets move about the sun in elliptical orbits with the sun at one focus; (2) the radius vector from the sun to a planet sweeps out equal areas in equal times; and (3) the cubes of the major axes of the orbits are proportional to the squares of the periods of revolution of the respective planets. Let us see how the law of gravitation may be obtained from these statements together with Newton's laws of motion.

In order to avoid mathematical complications which have no physical significance, it is convenient to replace the actual elliptical orbits with circular ones. In this case Kepler's second law, which expresses conservation of momentum, means that the speed v of a planet of orbital radius r is constant, so that its centripetal acceleration is

$$a = \frac{v^2}{r}$$

in magnitude and directed toward the sun, which is the center of force. In terms of the period of revolution T of the planet (i.e the time required for it to revolve once about the sun) the velocity is

$$v = \frac{2\pi r}{T} \quad ,$$

and the acceleration becomes

$$a = \frac{4\pi^2 r}{T^2} \quad .$$

From Kepler's third law, however,

$$\frac{r^3}{T^2} = C \quad ,$$

where C is some constant holding for all of the planets, and this can be combined with the expression for a to give

$$a = \frac{4\pi^2 C}{r^2} \quad . \tag{6.11}$$

Hence the acceleration of each planet depends solely upon its distance from the sun, and not on such other properties as size or mass. Eq. (6.11) is also obtained when the ellipticity of the orbits is taken into account, in which event r denotes the semi-major axis of the orbit in question.

Now we turn to the Earth as a center of force. The gravitational force that produces the acceleration of Eq. (6.11) is, since $\mathbf{F} = m\mathbf{a}$, an inverse square one, and it should apply to the moon's motion about the earth (with the constant C assuming a different value). That is, we assume that these bodies act as though they are particles in their interaction in the same way that this assumption is made in the case of the planets and the sun. What we want is to explain the attraction between the Earth and objects at its surface in terms of the same force that holds the moon in its orbit about the Earth. Here we seem to run into trouble, because it is not obvious that a relatively small object near the Earth and the Earth itself can be thought of in their effects on one another as being simply point masses. This was the problem faced by Newton. The method of attacking it is to consider the inverse square force required by Eq. (6.11) as applying between each particle of the earth and a single particle at the position of the object whose acceleration is to be calculated, and then to integrate the force over the volume of the Earth. We shall perform this integration in Sec. 10.6 with the help of Gauss' theorem, and here only state the result for the present situation: a spherically-symmetric system of particles exerting a gravitational force on an external body acts as though a single center of force were present whose location is the center of mass of the system.

With this information in hand, Eq. (6.11) indicates that the acceleration g due to gravity experienced by an object at the Earth's surface should be

$$g = \frac{4\pi^2 C'}{R_e^2} , \tag{6.12}$$

where R_e is the radius of the Earth. The constant C' may be found by applying Kepler's third law to the moon's motion, so that

$$C' = \frac{R_m^3}{T^2} ,$$

where R_m is the radius of the moon's orbit and T its period. Thus

$$g = \frac{4\pi^2 R_m^3}{T^2 R_e^2} . \tag{6.13}$$

All of the quantities on the right-hand side of this equation can be measured; the values that have been obtained are R_e=6.37x10^6 m, R_m=3.83x10^8 m, and T=2.36x10^6 sec. Hence the theoretical figure is

$$g = 9.81 \text{ m/sec}^2 ,$$

in complete agreement with the experimental value.

Having shown that the phenomenon of gravitation obeys the same law everywhere, we are in a position to derive the equation governing that law. From Eq. (6.11) the force the Earth exerts on the moon is

$$F = m_m a_m = \frac{4\pi^2 m_m C'}{R_m^2} ,$$

where m_m is the moon's mass. According to the third law of motion the moon exerts a force of the same magnitude on the Earth, in this case given by

$$F = m_e a_e = \frac{4\pi^2 m_e C''}{R_m^2} ,$$

where m_e is the Earth's mass. The constant C'' depends upon the relevant properties of the moon. Equating the two expression for F,

$$\frac{C'}{m_e} = \frac{C''}{m_m} ,$$

which entitles us to suppose that C/m is a universal constant. If we let

$$\frac{C}{m} = \frac{G}{4\pi^2} , \tag{6.14}$$

the gravitational force law holding between two bodies of masses m_1 and m_2 that are separated by the distance r is

$$F = \frac{G m_1 m_2}{r^2} . \tag{6.15}$$

The "constant of gravitation" G may be found from astronomical data or by performing an experiment to measure directly the gravitational force between two objects of known mass. Both methods yield the value

$$G = 6.67\text{x}10^{-11} \text{ n m}^2/\text{kg}^2 \ .$$

Example. A large number of minor planets, called asteroids, revolve about the sun in orbits lying between those of Mars and Jupiter. Assuming that they once constituted a single planet of mass $2\text{x}10^{24}$ g and orbital radius $4.3\text{x}10^{13}$ cm, and that their densities are all 3.5 g/cm^3, find the period of revolution T_a and the surface gravitational acceleration g_a of the original planet.

Solution. From Kepler's third law,

$$\frac{R_a{}^3}{R_e{}^3} = \frac{T_a{}^2}{T_e{}^2}$$

where R_e is the earth's orbital radius $1.5\text{x}10^{13}$ cm and T_e its period of 365 days. Hence $T_a = 1.78\text{x}10^3$ days, or almost 5 years. The gravitational acceleration is

$$g_a = \frac{Gm_a}{r_a{}^2} \ ,$$

where r_a is the planet's radius. Since we know its mass m_a and density ρ_a, and

$$\rho V = m_a = \rho\frac{4}{3}\pi r_a{}^3 \qquad ,$$

we find that $r_a = 5.1\text{x}10^7$ cm, a little less than a third the radius of the moon. In cgs units $G = 6.67\text{x}10^{-8}$ dyne cm^2/g^2, so that

$$g_a = 51.4 \text{ cm/sec}^2.$$

Chapter 7

ROTATIONAL AND HARMONIC MOTION

The dynamics of rotational motion is in many respects similar to the more familiar dynamics of linear motion. In the first part of this chapter we shall try to develop these similarities as far as they can be taken, especially with reference to the rotation of rigid bodies. Later in the chapter we shall take up the study of harmonic motion, a topic of great importance with applications throughout physics.

7.1. Angular Momentum. In the last chapter we found that whenever the translational motion of a rigid body is being considered, we can think of the entire mass of the body as being concentrated at its center of mass with the total force acting being the vector sum of the external forces only. This is not the complete picture, though. Suppose that we apply two equal and opposite forces to the flywheel shown in Fig. 7-1. The vector sum of these forces is 0, and therefore the center of mass of the flywheel undergoes no acceleration. However, the wheel rotates, and all of its constituent particles except the ones on its axis of rotation are accelerated. In order fully to describe the behavior of rigid bodies we must investigate rotational as well as translational motion.

As long as we restrict ourselves to rigid bodies rotating about fixed axes of rotation it is possible to take over many of the results of Chapter 5. (By axis of rotation is meant the line of particles in a *rotating* body which do not move.) If we consider any plane section in the body perpendicular to the axis of rotation, each of the particles in this section moves in a circle lying in the plane (Fig. 7-2. Since all of the particles are rigidly attached to one another, they move around their circular paths in the same period of time, or, in other words, they all have the same angular velocity ω. The *linear* velocities are different, of course. We may speak of the common angular velocity ω as the angular velocity of the entire rigid body.

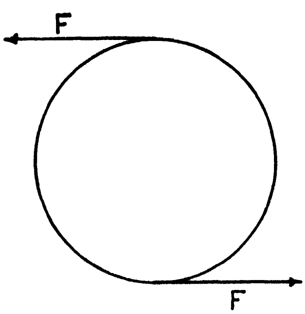


Fig. 7-1

The angular momentum of a system of particles in general is the vector sum of the angular momenta of the individual particles,

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i . \qquad (7.1)$$

It is extremely difficult to carry out this summation if the motion of the particles is not subject to any restrictions. For the case of rigid body rotation, with fixed axis of rotation, however, the velocity of each particle is

$$v_i = r_i \omega ,$$

and since the velocity is perpendicular to $\mathbf{r}$ at all times,

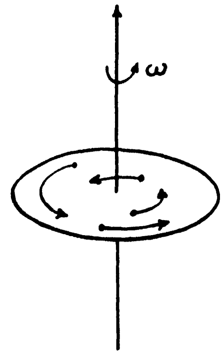


Fig. 7-2

$$\mathbf{r}_i \times \mathbf{v}_i = r_i v_i \sin 90° = r_i v_i .$$

Thus the angular momentum of a rigid body reduces to

$$\mathbf{L} = \sum_i m_i r_i^2 \boldsymbol{\omega}$$

$$= \boldsymbol{\omega} \sum_i m_i r_i^2 . \qquad (7.2)$$

The ω can be removed from the summation because it has the same value for all of the particles. If we abbreviate the purely geometrical factor in Eq. (7.2) by I, that is, if we let

$$I = \sum_i m_i r_i^2 , \qquad (7.3)$$

we have for the angular momentum of a rigid body

$$\mathbf{L} = I \boldsymbol{\omega} . \qquad (7.4)$$

The quantity I is called the *moment of inertia* of the body in question, and it plays a part in rotational motion analogous to that of mass in linear motion. This is evident upon comparing Eq. (7.4) with $\mathbf{p} = m \mathbf{v}$, the definition of linear momentum.

7.2. Moment of Inertia. We can see that the moment of inertia of a body depends both upon the distribution of mass in the body and upon where the axis of rotation, from which r is measured, is located. The first of these points is clear when the two discs of Fig. 7-3 are compared; one is a disc of iron surrounded by a wooden ring, while the other, of identical total mass and dimensions, consists of a wooden core surrounded by an iron ring. The latter has the greater moment of inertia because, since each mass element is weighted by

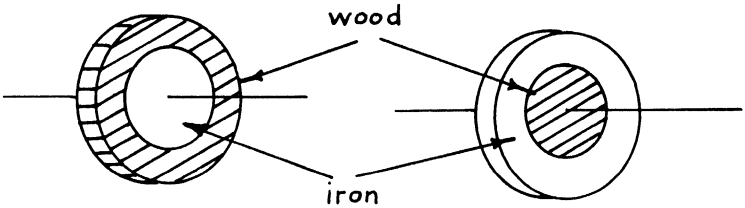


Fig. 7-3

the square of its distance from the axis, the heavier iron on the outside has a greater effect on I than when it forms the core of the body. The dependence of I upon the position of the axis is illustrated by the examples below.

Example. Find the moment of inertia of a weightless rod 4 ft long that has 10 lb iron balls at each end, with the axis perpendicular to the rod and passing through its midpoint (Fig. 7-4).

Solution. From Eq. (7.3),

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$= 2 \times \left(\frac{10}{32} \times 2^2\right) \text{ slug ft}^2 = 2.5 \text{ slug ft}^2 .$$

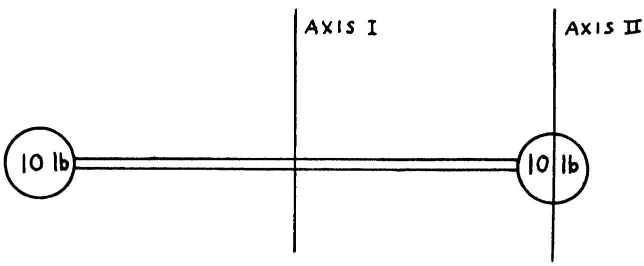


Fig. 7-4

Example. Find the moment of inertia of the object described above about an axis perpendicular to the rod but passing through one end.

Solution. Here the iron ball through which the axis passes has $r=0$, so that

$$I = m_1 r_1^2 = \frac{10}{32} \times 4^2 = 5 \text{ slug ft}^2 .$$

The moment of inertia is twice what it was before, even though only one of the weights contributes to it!

In dealing with a continuous distribution of mass the sum in Eq. (7.3) is replaced by an integral,

$$I = \int r^2 dm . \tag{7.5}$$

In carrying out the integration it is usually convenient to make use of the density of the body, provided, naturally, that it is uniform or varies in a known manner.

Example. Find the moment of inertia of a uniform rod of length L and mass m about an axis perpendicular to the rod and passing through its midpoint (Fig. 7-5).

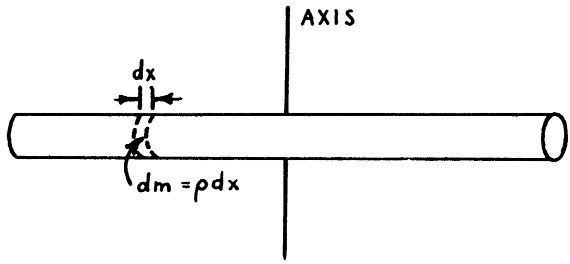


Fig. 7-5

Solution. By calling the mass per unit length of the rod ρ,

$$m = \rho L$$

and

$$dm = \rho dx.$$

Thus Eq. (7.5) becomes

$$I = \int_{-L/2}^{+L/2} x^2 \rho \, dx$$

$$= \rho \frac{x^3}{3} \Big|_{-L/2}^{+L/2} = \rho \frac{L^3}{12} .$$

Since $\rho = m/L$, the final result is

$$I = \frac{mL^2}{12.}$$

Table 7.1 is a collection of moments of inertia for various regularly-shaped bodies.

Table 7.1

Moments of Inertia

Body	Moment I
Rectangular sheet of sides A and B, about axis through center perpendicular to plane of sheet	$\frac{m}{12}(A^2+B^2)$
Circular sheet of radius R, about axis through center perpendicular to plane of sheet	$\frac{m}{2}R^2$
Circular sheet of radius R, about axis along a diameter	$\frac{m}{4}R^2$
Sphere of radius R, about axis along a diameter	$\frac{2m}{5}R^2$
Right circular cylinder of radius R, about axis of symmetry	$\frac{m}{2}R^2$
Right cone of base radius R, about axis of symmetry	$\frac{3m}{10}R^2$
Ellipsoid of semi-axis A,B,C, about axis A	$\frac{m}{5}(B^2+C^2)$

It is occasionally convenient to express the moment of inertia of an object in terms of the distance its mass, contracted to a point, must be placed from the axis of rotation in order to give the same value of I. This distance is called the *radius of gyration* R_G. In terms of R_G the moment of inertia is

$$I = mR_G^2 \ , \tag{7.6}$$

so that when we know I and m we can always compute the radius of gyration.

Example. What is the radius of gyration of the uniform rod of the previous example?

Solution. Rewriting Eq. (7.6),

$$R_G = \sqrt{\frac{I}{m}} = \sqrt{\frac{mL^2}{12m}}$$

$$= \frac{L}{2\sqrt{3}} \ .$$

7.3. Conservation of Angular Momentum. To find the rate of change of angular momentum with time for any system of particles, we need only differentiate Eq. (7.1), so that

$$\frac{dL}{dt} = \frac{d}{dt} \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{7.7}$$

In the case of a rigid body $\mathbf{r}_i$ does not change with time, hence, carrying out the differentiation,

$$\frac{dL}{dt} = \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i$$

$$= \sum_i \mathbf{r}_i \times \mathbf{F}_i \quad .$$

This follows from the fact that $\mathbf{F}_i = \dot{\mathbf{p}}_i$. Recalling the definition of torque $\mathbf{N}$,

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \ , \tag{5.6}$$

we see that

$$\frac{dL}{dt} = \sum_i \mathbf{N}_i = \mathbf{N} \ . \tag{7.8}$$

Internal forces among the particles do not contribute to $\mathbf{N}$. This is clear when we consider that the forces between any pair of particles must be equal in magnitude and opposite in direction, with a common line of action (see Fig. 6-4). Thus the torques due to internal forces about any arbitrary point cancel out in pairs, and we may express Eq. (7.8) in words by saying that *the rate of change of the angular momentum of a rigid body is equal to the sum of the external torques on it. If the external torque is 0, the angular momentum of the body is a constant.*

The conservation of angular momentum is employed by a skater doing a spin. The skater starts turning with his arms extended and then pulls them in close to his sides, whereupon he spins much faster. By drawing in his arms the skater is actually decreasing his moment of inertia, and to keep the angular momentum the same his angular velocity increases. The same is true of a tumbler who brings his knees against his chest in the course of a somersault.

A spinning football is another example of conservation of angular momentum. Angular momentum is given a football by throwing it so that it spins about its long axis. (Fig. 7-6). External torques are virtually absent and the angular momentum remains constant during the football's flight. $\mathbf{L}$ is a vector and its direction therefore may not vary here, which prevents the ball from travelling end over end since that would require a different $\mathbf{L}$. In Fig. 7-6 $\mathbf{L}'$ is the angular momentum vector corresponding to the ball going end over end.

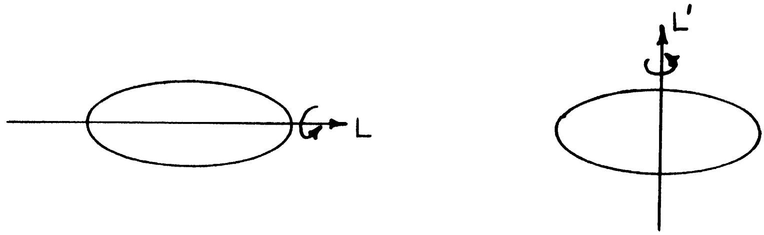


Fig. 7-6

We can obtain further insight into the rotational motion of rigid bodies by rewriting Eq. (7.7) in the form

$$\frac{dL}{dt} = \frac{d}{dt} \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i m_i \mathbf{r}_i \times \frac{d\mathbf{v}_i}{dt}$$

$$= \sum_i m_i \mathbf{r}_i \times \mathbf{a}_i \quad ,$$

where $\mathbf{a}_i$ is the acceleration of the ith particle. For a rigid body rotating about a fixed axis the acceleration of any particle is perpendicular to its radius vector $\mathbf{r}_i$, and since $a_i = r_i x$,

$$\mathbf{r}_i \times \mathbf{a}_i = r_i^2 \boldsymbol{\alpha} \ .$$

Thus

$$\frac{dL}{dt} = \sum_i m_i r_i^2 \boldsymbol{\alpha}$$

$$= I \boldsymbol{\alpha} \, ,$$

and

$$\mathbf{N} = I \boldsymbol{\alpha}. \tag{7.10}$$

The product of the moment of inertia and the angular acceleration of a rigid body equal the external torque applied to it. The analog of this equation in linear motion we recall to be $\mathbf{F} = m\,\mathbf{a}$.

Relationships such as Eq. (7.10) are very helpful, but is should be kept in mind that they were obtained from the basic laws of motion together with the definition of a rigid body. These derived laws make it possible to avoid summing the laws applicable to individual particles over all of the particles of a specific rigid body, since they are themselves the results of such summations; the point is that they are not fundamental principles of physics, and we must be careful not to apply them in situations not consistent with the assumptions used in their derivation.

Example. How much torque must be applied to the earth in order to increase the length of the day by 1 sec per year? Assume that the earth is a perfect sphere of uniform density.

Solution. From Table 7.1 we note that for a sphere

$$I = \frac{2}{5}\ mr^2\ .$$

Since the earth's mass is approximately $6.0\text{x}10^{24}$ kg and its radius approximately $6.4\text{x}10^6$ m,

$$I_{earth} = \frac{2}{5} \text{ x } 6.0\text{x}10^{24} \text{ x } (6.4\text{x}10^6)^2 \text{ kg m}^2$$

$$= 9.8\text{x}10^{37} \text{ kg m}^2\ .$$

Assuming that a normal day is exactly 24 hours or 86,400 sec long, a change in the length of the day of 1 sec means an angular velocity change of

$$\Delta\omega = \frac{2\pi \text{ radians}}{86{,}400 \text{ sec}} - \frac{2\pi \text{ radians}}{86{,}401 \text{ sec}}$$

$$= 8.4\text{x}10^{-10} \text{ rad/sec}\ .$$

This change, spread over a year, implies an angular acceleration of

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{8.4\text{x}10^{-10} \text{ rad/sec}}{365\text{x}86{,}400 \text{ sec}} = 2.7\text{x}10^{-17} \text{ rad/sec}^2.$$

From Eq. (7.10) we find that the required torque is

$$N = I\alpha = 9.8\text{x}10^{37} \text{ x } 2.7\text{x}10^{-17} \text{ kg m}^2/\text{sec}^2$$

$$= 2.6\text{x}10^{21} \text{ newton-meters,}$$

a rather large figure.

7.4. Rotational Kinetic Energy. It is not difficult to express the kinetic energy of a rotating rigid body in terms of rotational quantities. Summing the kinetic energies of the constituent particles,

$$T = \sum_i \tfrac{1}{2} m_i v_i^2\ .$$

Substituting ωr_i for v_i,

$$T = \sum_i \tfrac{1}{2} m_i r_i^2 \omega^2$$

$$= \tfrac{1}{2} L\omega^2\ . \qquad (7.11)$$

We note that the units in which the kinetic energy given by Eq. (7.11) are expressed are the same as those of the usual form, $T = \frac{1}{2}mv^2$.

Table 7.2 contains a listing of some of the various analogies between linear and angular quantities and relationships that may be helpful.

Table 7.2

Comparable linear and angular quantities.

Linear		Angular	
displacement	$x = vt + \frac{1}{2}at^2$	angular displacement	$\theta = \omega t + \frac{1}{2}\alpha t^2$
velocity	$v = at$	angular velocity	$\omega = \alpha t$
acceleration	a	angular acceleration	α
mass	m	moment of inertia	I
force	$F = ma$	torque	$N = I\alpha$
momentum	$p = mv$	angular momentum	$L = I\omega$
kinetic energy	$T = \frac{1}{2}mv^2$	kinetic energy	$T = \frac{1}{2}I\omega^2$

When a rigid body is both moving through space with velocity v and rotating simultaneously about an axis perpendicular to v its kinetic energy is

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \ , \tag{7.12}$$

the sum of its translational and rotational kinetic energies. In computing the first term the velocity is taken to be the velocity of the center of mass, and in the second term the values of I and ω are determined on the basis of an axis of rotation passing through the center of mass. As usual, utilizing the law of conservation of energy makes it possible to solve apparently complex problems in a straightforward manner. In treating these problems the potential energy of the body is considered to be that of a particle having the mass of the entire body that is located at the center of mass.

Example. A cylinder of radius R and mass M rolls without slipping down an inclined plane. If its started from rest at the top of the plane, what is its velocity at the bottom?

Solution. At the top of the inclined plane the cylinder has potential energy only, while at the bottom it has kinetic energy only. Applying the law of conservation of energy,

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \ .$$

For a cylinder

$$I = \frac{1}{2}MR^2 \ ,$$

and $\omega R=v$ if there is no slippage; therefore

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2 \ ,$$

and the velocity at the bottom is

$$v = \sqrt{\frac{4}{3} gh}\ .$$

If the body slid down the inclined plane instead of rolling down,

$$Mgh = \tfrac{1}{2}Mv^2$$

and

$$v = \sqrt{2gh}$$

Hence the cylinder travels more slowly when it rolls owing to the fact that the rotation absorbs some of the available energy (Fig. 7-7).

7.5. Simple Harmonic Motion.

Simple harmonic motion is encountered so often in physics that it is appropriate for us to discuss it in some detail. A particle exhibits this type of motion whenever a force acts on it whose magnitude is proportional to the displacement of the particle from a suitably located origin and whose direction is always toward that origin. We can specify this force mathematically by writing it in the form

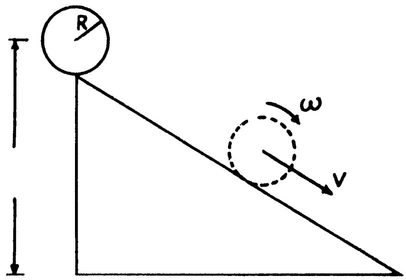


Fig. 7-7

$$\mathbf{F} = -k\mathbf{r}\ , \qquad (7.13)$$

where k is a constant. Coil springs obey Eq. (7.13) fairly well; thus we might expect that attaching an object to a spring will enable us to discover some of the characteristics of simple harmonic motion.

We know from experience that the more a spring is compressed or extended, the harder it is to further compress or extend it. This behavior is contained in Eq. (7.13), with a large value of k corresponding to a stiff spring and a small value to a weak spring. Let us start by placing the spring on a frictionless horizontal surface with one end fixed in place and a weight of some sort at the other end (Fig. 7-8). When the spring is at rest no force acts on the weight, so, by Eq. (7.13) the weight is at $\mathbf{r}=0$. Now we elongate the spring a distance A and release it. The initial force acting is $F = -kA$, tending to pull the weight back toward the origin. As the weight moves it is accelerated, and even though the force on it decreases as it approaches the origin, its velocity increases, but with a diminishing acceleration. Finally, back at the origin, $F=0$, but the moving weight continues going on the other side of the origin (Fig. 7-8). There it encounters a force (due to the compression of the spring) in the opposite direction, and it is slowed down. Ultimately

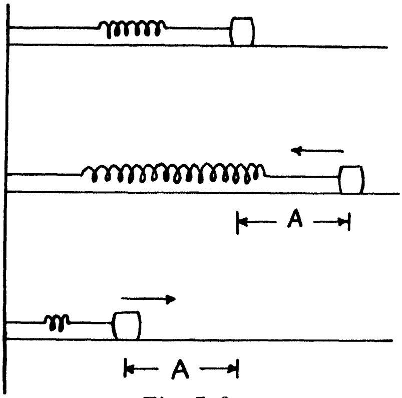


Fig. 7-8

the weight has zero velocity, and under the influence of the compressed spring, begins to accelerate back toward the origin. In the absence of friction the weight oscillates back and forth between A and -A indefinitely.

Evidently simple harmonic motion is a complex affair, what with constantly changing force, acceleration, velocity, and position. There are certain regularities, though: the motion is confined to a definite region of space and it is periodic, that is, it repeats itself at regular intervals. We shall now define three characteristics of simple harmonic motion that are needed in describing it: (1) *amplitude*—the maximum displacement the particle undergoes from the origin. In Fig. 7-8 the amplitude is A (not 2A); (2) *period*—the time T required by the particle to go through a complete cycle and return to its starting point. This would be the time involved in going from A to -A and back to A once more; and (3) *frequency*—the number f of complete cycles the particle experiences per unit time. The frequency is related to the period by $f=1/T$.

7.6. The Mathematics of Simple Harmonic Motion. Applying the second law of motion to Eq. (7.13) we have

$$\mathbf{F} = m\,\mathbf{a}$$

$$-k\mathbf{r} = m\,\frac{d^2\,\mathbf{r}}{dt^2}\quad . \tag{7.14}$$

By solving Eq. (7.14) we will determine the precise manner in which the position of the particle varies with time, and from a knowledge of $\mathbf{r}\,(t)$ we can then find equations for $\mathbf{v}$ and $\mathbf{a}$ merely by differentiating. It is convenient to consider one component of $\mathbf{r}$ at a time, say the x component. Rewriting Ew. (7.14),

$$m\,\frac{d^2x}{dt^2} + kx = 0.$$

To find the solution we begin by multiplying both sides of this equation by dx/dt to obtain

$$m\,\frac{dx}{dt}\,\frac{d^2x}{dt^2} + k\,\frac{dx}{dt}\,x = 0,$$

which is equivalent to

$$\frac{m}{2}\,\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}\,\frac{d}{dt}\,(x^2) = 0$$

or

$$\frac{d}{dt}\left[\frac{m}{2}\,\frac{dx^2}{dt} + \frac{k}{2}\,x^2\right] = 0.$$

Integrating,

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}\,x^2 = C\ , \tag{7.15}$$

which we may recognize as the condition for the conservation of energy. The first term is the kinetic energy of the oscillating particle, and the second is the potential energy of the spring when displaced by the distance x from its equilibrium position (see Eq. 5.17).

Now we solve Eq. (7.15) for dx/dt, and find

$$\frac{dx}{dt} = \left(\frac{2C - kx^2}{m}\right)^{\frac{1}{2}} .$$

Separating variables and letting $2C/m = A^2$,

$$\frac{dx}{\left(A^2 - \frac{k}{m}\,x^2\right)^{\frac{1}{2}}} = dt.$$

To integrate we refer to a table of integrals for help with the left-hand side, and have as a result

$$\sin^{-1}\frac{x}{A} = \sqrt{\frac{k}{m}}\ t + \phi\ .$$

Solving for x,

$$x = A\sin\left(\sqrt{\frac{k}{m}}\ t + \phi\right) . \tag{7.16}$$

It is not difficult to determine the amplitude and period of the simple harmonic motion described by Eq. (7.16). Since the value of the sine function varies from -1 to $+1$, the particle oscillates between $-A$ and $+A$, and thus the amplitude is A. For the period T, let us consider two times, t_0 and t_1, that differ by exactly T. That is,

$$t_1 - t_0 = T\ .$$

By the definition of the period, the position of the particle must be the same at t_1 as it was at t_0, which means that the sine must have the same value at both times. This can happen only if the argument $(\sqrt{k/m}\ t + \phi)$ increases by exactly 2π in the interval, and we have therefore that

$$\sqrt{\frac{k}{m}}\ t_1 + \phi = \sqrt{\frac{k}{m}}\ t_0 + \phi + 2\pi\ .$$

Hence

$$\sqrt{\frac{k}{m}}\ (t_1-t_2) = \sqrt{\frac{k}{m}}\ T = 2\pi\ ,$$

so that the period is

$$T = 2\pi\sqrt{\frac{m}{k}}\quad . \tag{7.17}$$

We note that the period depends only upon the mass of the particle and the constant k, and not upon the amplitude of the motion. The time required for a complete oscillation is the same whether the particle barely moves or describes wide excursions on either side of the equilibrium position. This behavior finds practical application in the regulation of a clock by a pendulum or of a watch by a balance wheel attached to a spring.

The frequency of a particle in simple harmonic motion is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ , \tag{7.18}$$

and in terms of f we can write in place of Eq. (7.16) that

$$x = A \sin (2\pi\, ft + \phi)\ . \tag{7.19}$$

The quantity ϕ is called the *phase* of the motion. It tells us the position at which the particle was located at the time t=0 when we began making measurements. For example, if the particle of Fig. 7-8 is at the right-hand end of its trajectory when t=0, we have that

$$A = A \sin\phi$$

$$\phi = \pi/2\ .$$

We have therefore for the complete description of the particle's motion

$$\begin{aligned} x &= A \sin (2\pi\ ft + \pi/2) \\ &= A \cos\ 2\pi\, ft\ . \end{aligned}$$

If at t=0 the particle is at the origin,

$$0 = A \sin\phi$$

$$\phi = 0$$

and

$$x = A \sin 2\pi\, ft.$$

The velocity of a particle executing simple harmonic motion may be found by differentiating Eq. (7.19);

$$v = \frac{dx}{dt} = 2\pi\ A \cos (2\pi\, ft + \phi)\ . \tag{7.20}$$

Since $\sin^2\theta + \cos^2\theta = 1$, this may be rewritten

$$v = 2\pi f\sqrt{A^2 - A^2\sin^2(2\pi ft + \phi)}$$

$$= 2\pi f\sqrt{A^2 - x^2}\ . \tag{7.21}$$

Differentiating once more yields the acceleration,

$$a = \frac{dv}{dt} = -4\pi^2 f^2 x\ . \tag{7.22}$$

According to Eq. (7.22) the acceleration of the particle, and therefore the force on it, is proportional to the displacement and in the opposite direction. This, we remember, is the definition of simple harmonic motion with which we began.

Example. An uncalibrated spring scale is found to have a 1 sec period of oscillation when a 20 lb weight is suspended from it. How much does the spring elongate when a 50 lb weight is suspended from it?

Solution. Since the period is given by

$$T = 2\pi\sqrt{\frac{m}{k}}\ ,$$

we find that

$$k = 25 \text{ slug/ft}\ .$$

The elongation when a 50 lb weight is attached is therefore

$$x = \frac{F}{k} = 2 \text{ ft}\ .$$

Note the importance of keeping units correct when using the British system.

Example. What is the velocity of the 20 lb weight in the above example when it passes through the equilibrium position? The amplitude of the oscillation is 6 in.

Solution. From Eq. (7.21),

$$v = 2\pi f\sqrt{A^2 - x^2}\ ,$$

and since $x=0$ at the equilibrium position,

$$v = 2\pi fA = \pi \text{ ft/sec}$$

there.

7.7. Energy Considerations. At any instant the total energy of the system of particle + spring consists of the kinetic energy of the former and the potential energy of the latter. As we saw in Eq. (7.15),

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 ,$$

which we may rewrite

$$\begin{aligned} E &= \tfrac{1}{2}m4\pi^2 f^2 (A^2 - x^2) + \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}kA^2 \end{aligned} \qquad (7.23)$$

since $f=\sqrt{k/m}/2\pi$. This equation states that the total energy equals the potential energy at the maximum displacement $x=\pm A$, when the kinetic energy is zero. This comes as no surprise, of course, but it is encouraging to find our theoretical picture selfconsistent. As the particle moves toward the origin the potential energy decreases, until at the origin there is only kinetic energy. Past the origin on the other side the potential energy grows at the expense of the kinetic energy. We may regard simple harmonic motion, then, as a constant interchange of a fixed amount of energy between the moving particle and the medium in which it travels, which is the spring.

Example. A simple pendulum consists of a point mass (called the "bob") m suspended by a weightless string L long. Find the frequency of vibration of the pendulum for small displacements of the bob from the vertical.

Solution. We begin by calculating the potential energy V of the bob when the string is displaced from the vertical by the angle θ (Fig. 7-9). This is

$$V = mgh = mgL\,(1 - \cos\theta).$$

Since

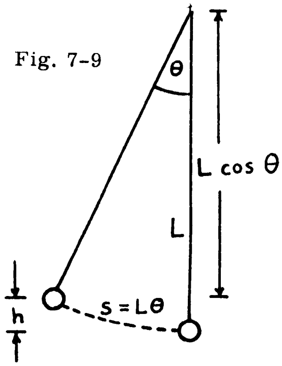
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots ,$$

if θ is small we have that

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

approximately. Hence

Fig. 7-9



$$V = \tfrac{1}{2}mgL\,\theta^2 ,$$

or, since the displacement s of the bob is equal to $L\theta$,

$$V = \frac{\tfrac{1}{2}mg}{L}\, s^2 .$$

This is in the same form as Eq. (7.23), and we infer from a comparison of the two that the restoring force constant for the simple pendulum is

$$K = \frac{mg}{L} .$$

Hence, from Eq. (7.18),

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad .$$

The frequency of vibration of a simple pendulum depends only upon its length for oscillations of small amplitude.

7.8. Relation to Circular Motion. The simple harmonic motion of a particle turns out to be exactly the same as that of the projection on a line of the motion of a particle moving in a circular path with uniform angular velocity. Suppose, as in Fig. 7-10, that the radius of the circle is A and the angular velocity is ω. If the particle begins moving, at t=0, when its radius vector makes the angle ϕ with the vertical, in a time t the angle will have increased to

$$\theta = \omega t + \phi .$$

But

$$x = A \sin\theta$$

in Fig. 7-10, so that the projection on the x-axis at any time is

$$x = A \sin(\omega t + \phi). \qquad (7.24)$$

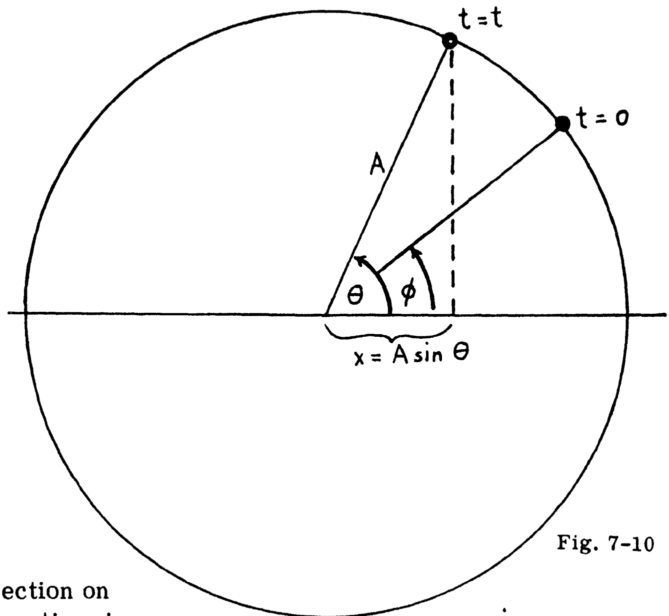


Fig. 7-10

If T is the time needed for one complete revolution,

$$\omega T = 2\pi$$

and

$$\omega = \frac{2\pi}{T} = 2\pi f \; . \qquad (7.25)$$

Eq. (7.24) is identical to Eqs. (7.16) and (7.19), and is often employed instead of them in discussions of simple harmonic motion. When used for this purpose the quantity ω is called the *angular frequency* of the motion.

Chapter 8

KINETIC THEORY OF MATTER

We have already noticed cases where mechanical energy is not conserved. When a block is pulled up an inclined plane with friction present, for example, more work must be done than the amount required to simply raise the potential energy of the block to its final value. Does this mean that conservation of energy is not a generally valid concept?

There is a way in which we can rescue conservation of energy, however. What we do is assume that other kinds of energy exist than the ones we know of thus far, so that, in the above instance, the energy "lost" to friction is in reality merely converted into a form whose nature is not immediately evident. This position has been universally adopted by physicists, and with the discovery of a variety of these alternate guises the theorem of the conservation of energy has been confirmed whenever brought to an experimental test. There is no guarantee that an experiment will never be performed where the initial and final energies of a system undergoing a particular process do not exactly balance, but until now that has not happened. In the case of the block and the inclined plane it is customary to state that the excess work went into the *internal energy* of the block and the plane, and the precise nature of internal energy in such situations is the subject of this chapter.

8.1. Kinetic Theory of Matter. All specimens of matter, whether solid, liquid, or gas, can be divided into smaller and smaller parts. Ultimately we reach the *molecule*, the smallest quantity of a substance that possesses the characteristics of that substance. We can subdivide molecules still farther into atoms, the atoms into nuclei and electrons, and so on, but in doing so the properties of the material we started with are lost. If we have a particular physical problem that cannot be explained on the macroscopic level, the solution may appear when we consider the behavior of the individual molecules rather than the gross behavior of the substances they compose. Sometimes this approach is not adequate and we must descend to the atomic or nuclear domains in order to interpret a particular experimental observation. What determines how far down it is necessary to go is conservation of energy; we stop at the first level at which the various energies involved in a given process balance properly. To explain stellar radiation, for example, it is necessary to consider nuclear energy sources. It the case of the block and the plane, though, we need only go to the molecular level to find out where the missing energy has gone. Molecules in solids possess kinetic and potential energies, and the excess work that had to be done in pulling the block up the plane adds to these internal molecular energies. As a result of the increased internal energy the block and plane become warmer, a phenomenon we shall explore shortly.

The analysis of the behavior of matter from the point of view of the individual particles of which it consists is called the *kinetic theory of matter*. Unfortunately the kinetic theory of solids is in a rudimentary state and we cannot analyze the problem of the block and plane much beyond identifying the reservoir of the energy lost to frictional forces. The kinetic theory of liquids is even less satisfactory. However, the kinetic theory of gases is well developed and in considerable agreement with experiment. We shall discuss here the kinetic theory of the simplest of all gases, the fictitious *ideal gas* whose behavior is approximated by many actual gases. In doing so we will make use of some of the techniques of kinetic theory and discover the type of conclusions that can be drawn from it.

8.2. Kinetic Theory of An Ideal Gas. Let us begin with a reasonable hypothesis for the composition of a gas and then try to develop a theory from this model that can be compared with experiment. We assume the following:

1. The gas consists of molecules. These molecules can experience *elastic* collision with one another and with the walls of their container. (At this level there are no frictional or other nonconservative forces);
2. The molecules of the gas are sufficiently far apart most of the time so that they do not exert any forces on each other. Therefore they cannot have any potential energy (other than gravitational, which is so small that we can neglect it), but have only kinetic energy;
3. The distribution of velocities among the gas molecules at equilibrium does not change with time. The same number of molecules always has a particular velocity, although the specific molecules having this velocity may be different at different times. More precisely, on the average as many molecules of some initial velocity v_1 emerge from collisions with the higher velocity v_2 as there are molecules of initial velocity v_2 which are slowed down to v_1 by other collisions. This assumption is characteristic of the kinetic theory of matter. While in dealing with large numbers of identical particles there is the disadvantage that we cannot follow the motion of each particle individually, the fact that the numbers are so great permits us to use a statistical treatment to discuss the *average* behavior of the material as a whole.

We now place N molecules of the ideal gas in a cubical box L long on each edge (Fig. 8-1). What is the force on the walls of the box? Consider a molecule of velocity v whose components are v_x, v_y, v_z. If we are interested in the force on walls ABCD and EFGH, only v_x will contribute to this force. The molecule has the momentum in the x direction mv_x when it strikes ABCD and $-mv_x$ when it rebounds after the collision, a momentum change of $2mv_x$. Since force is equal to the rate of change of momentum, the force on ABCD may be found by dividing the momentum change $2mv_x$ by the time required for the molecule to go from EFGH to ABCD and back, a distance of 2L. (Sec. 5.1). The time is $2L/v_x$, and the resulting average force on ABCD (or EFGH) is

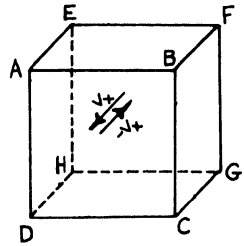


Fig. 8-1

$$f = \frac{mv_x^2}{L} \quad . \tag{8.1}$$

This equation gives the contribution to the total force on one of the sides of the box due to a single molecule whose x component of velocity is v_x.

Not all of the gas molecules have the same x velocity component. A number N_1 of them has the velocity v_{1x}, N_2 the velocity v_{2x}, etc. Eq. (8.1) makes use of the square of the x component, so let us form an average of this square for all N of the molecules:

$$\overline{v_x^2} = \frac{N_1 v_{1x}^2 + N_2 v_{2x}^2 + N_3 v_{3x}^2 + \cdots}{N_1 + N_2 + N_3 + \cdots}$$

$$= \frac{N_1 v_{1x}^2 + N_2 v_{2x}^2 + N_3 v_{3x}^2 + \cdots}{N} \quad . \tag{8.2}$$

The total force on ABCD is accordingly given by

$$F = \frac{Nm\overline{v_x^2}}{L} \quad . \tag{8.3}$$

The *pressure* (defined as the force per unit area acting perpendicular to a surface) on ABCD is in turn F divided by the area L^2 of ABCD, or

$$p = \frac{Nm\overline{v_x^2}}{L^3} = \frac{Nm\overline{v_x^2}}{V} \quad ,$$

where V is the volume of the box. It will be convenient to write this equation in the form

$$pV = Nm\overline{v_x^2} = 2N\cdot\tfrac{1}{2}m\overline{v_x^2} \quad . \tag{8.4}$$

Identical reasoning gives $pV = 2N\cdot\frac{1}{2}m\overline{v_y^2}$ for the sides BCHG and ADEF and $pV = 2N\cdot\frac{1}{2}m\overline{v_z^2}$ for the sides ABGF and CDEH.

While we have ignored the possibility that the various molecules collide with each other and thereby alter their momenta without touching the walls of the box, the above analysis does not change when this possibility is taken into account. Recall our third assumption, that the *distribution* of velocities does not change despite changes in the velocities of the individual molecules. There is also no "preferred" direction in the box which the molecules might tend to follow. Consequently

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} \quad ,$$

and since the mean square velocity of the molecules is

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} \quad ,$$

we find that

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\ \overline{v^2} \ .$$

Eq. (8.4) now takes the form

$$pV = \frac{2}{3}N\cdot\frac{1}{2}m\overline{v^2} = \frac{2}{3}NU \quad , \tag{8.5}$$

where U represents the average kinetic energy of each molecule. In our original model we permitted the molecules to have only kinetic energy; thus NU is the total internal energy of a sample of ideal gas containing N molecules.

We derived Eq. (8.5) for a box of cubical shape, but it is possible, although more complicated, to obtain the same equation for a container of any shape whatever. It is often helpful to assume a specific geometrical situation in investigating a problem, and if the result contains no geometrical reference in most cases it is a perfectly general one restricted only by the other initial conditions.

Example. Two grams of hydrogen are in a container under "standard" laboratory conditions (0°C temperature and 1.017×10^6 dynes/cm^2 pressure). What is the *root mean square* velocity, $\sqrt{\overline{v^2}}$, of the molecules?

Solution. According to Avogadro's hypothesis, a mass of any gas equal to its chemical molecular weight expressed in grams contains 6.02×10^{23} molecules. Since the molecular weight of hydrogen is 2, there are 6.02×10^{23} hydrogen molecules in the container. At the standard laboratory conditions, a gram-molecular weight of a gas occupies a volume of 22,400 cm^3. Substituting these figures into Eq. (8.5) we have

$$\frac{1.017\times10^6 \times 22{,}400}{2/3 \times 6.02\times10^{23}} = \frac{1}{2}m\overline{v^2} \ .$$

The mass of a hydrogen molecule is 3.3×10^{-24} grams, so that

$$\sqrt{\overline{v^2}} = 6\times10^4 \text{ cm/sec,}$$

which is over 1300 miles/hour! We could not hope to measure this velocity directly, but from certain experimental results and the kinetic theory of gases we have obtained at least an estimate of its value.

8.3. Temperature. Before going on with our analysis of an ideal gas we must discuss the concept of *temperature*. This is not as obvious a matter as it may seem; while our senses can readily distinguish things that are hot from things that are cold, they provide no means for determining hotness or coldness in a precise way. What we require is a measurable property of matter that changes when the temperature changes. A familiar example of such a thermometric property is the increase in volume of a liquid when its temperature increases, which underlies the operation of mercury and alcohol thermometers. Other thermometric properties of matter include the dimensions of a solid, the pressure of a gas kept at constant volume, the volume of a gas kept at constant pressure, the electrical resistance of a metal, the electromotive force of a junction of dissimilar metals (known as a *thermocouple*), and the color of a body when it is hot enough to glow.

Once we select a particular thermometric property, we have reduced the measurement of temperature to the measurement of that property. Now we need a unit of temperature for expressing temperature differences, and a reference temperature with which to compare our readings. From experience it is known that there are certain natural systems which remain at constant temperatures. The most important of these from a practical point of view are a mixture of ice and water and an open container of boiling water. How can we be sure that their temperatures do not change? The answer is that none of the thermometric properties we know of vary in any way in these systems, and for this reason and because they are so readily duplicated ice and water mixtures and boiling water are convenient to use as fixed points in constructing a scale of temperature.

In the Centigrade scale the temperature of melting ice is, by definition, the zero point. The unit of temperature, the Centigrade degree, is specified as 1/100 of the difference in a linear thermometric property of matter between the temperatures of melting ice and boiling water. (By a linear property is meant one that is directly proportional to the temperature. The pressure of a gas at constant volume varies linearly with temperature, for example, while the electromotive force of a thermocouple does not.) Hence the steam point is 100°C.

In the Fahrenheit scale the temperature of melting ice is taken as 32. The unit of temperature, the Fahrenheit degree, is 1/180 of the difference in a linear thermometric property between the ice and steam points, which puts the steam point at 212°F.

The Centigrade degree is evidently larger than the Fahrenheit degree; in fact,

$$1°\text{C} = \frac{180}{100}\ °\text{F} = \frac{9}{5}\ °\text{F} .$$

In converting temperature measurements from one scale to the other we must also take into account the fact that the zero points of the two do not coincide. If t_C and t_F are the Centigrade and Fahrenheit readings respectively of the same temperature, the relationship between them is

$$t_C = \frac{5}{9}\ (t_F - 32) ,$$

or

$$t_F = \frac{9}{5}\ t_C + 32 \quad .$$

Example. What temperature produces the same numerical value on both Centigrade and Fahrenheit thermometers?

Solution. Setting t_C equal to t_F,

$$\begin{aligned} t_C = t_F &= \frac{5}{9}\ (t_F - 32) \\ &= -40°. \end{aligned}$$

8.4. Properties of Gases. Some of the earliest scientific research in the modern sense was carried out by Robert Boyle (1627-1691) on the properties of gases. His experiments yielded the conclusion that, when a gas sample is maintained at a constant temperature, its pressure and volume are not independent of one another but obey the law

$$pV = \text{const. (constant temperature).} \tag{8.6}$$

At different temperatures the value of the constant is different. The striking thing about Boyle's Law is that it holds reasonably well for *all* gasses. (More precise data indicate that, while quite accurate at pressures less than atmospheric, deviations from Eq. (8.6) occur at higher pressures.) We note that Eq. (8.5), which we derived from the kinetic theory of an ideal gas, predicts that

$$pV = \frac{2}{3}\ NU ,$$

which reduces to Boyle's Law provided that the internal energy NU of the gas sample remains constant when the temperature does not change. Gases obeying Boyle's Law are called *ideal gases,* and we shall confine ourselves to them in most of what follows.

At this point we might suspect that there is some connection between the temperature of a gas and its internal energy. To establish this we must study the behavior of gases when their temperature is permitted to vary. One method is illustrated in Fig. 8-2. A cylinder containing some gas is fitted with a tight piston, so that the pressure on the gas is atmospheric pressure (which is due to the weight of the total amount of air above the piston) plus the pressure due to the weight of the piston itself, and is constant. When the container is heated the gas temperature increases, and we find that the piston moves upward. Cooling the container causes the temperature to drop and the gas volume to contract. The results of the experiment, which was first performed by Gay-Lussac in the late 18th Century, show that

$$V = V_0(1 + \beta t) \text{ (constant pressure)}, \tag{8.7}$$

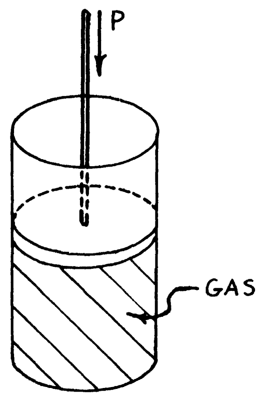


Fig. 8-2

where V is the gas volume at the temperature t, V_0 its volume at zero temperature, and β a constant, called the coefficient of volume expansion. Provided that the temperature is not too high, β is the same for *all* gases, with the value 1/273.18 per °C.

We are now in a position to relate the pressure, volume, and temperature of an ideal gas with a single equation. Let us consider a gas sample whose volume is V_0 at atmospheric pressure p_0 and temperature 0°C. We heat the gas while keeping its pressure constant at p_0, and its volume increases to V_1 and its temperature to t. (Fig. 8-3 is a p versus V graph of this process.) Then the gas is compressed at the constant temperature t until its volume is V and its pressure p (point 2 in Fig. 8-3). By Boyle's Law

$$p_0V_1 = pV \quad ,$$

and by Gay-Lussac's Law

$$V_1 = V_0(1 + \beta t).$$

Multiplying the latter of these equations by p_0,

$$p_0V_1 = p_0V_0\,(1 + \beta t),$$

and combining it with the former one,

$$pV = p_0V_0\,(1 + \beta t)\ .$$

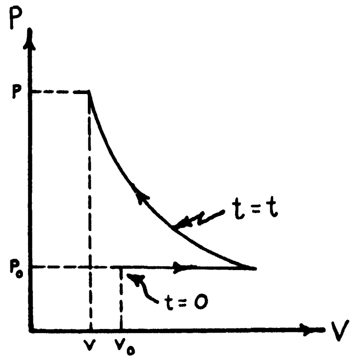


Fig. 8-3

This equation can be rewritten

$$pV = p_0V_0\beta(t + \frac{1}{\beta}). \qquad (8.8)$$

The value of $p_0V_0\beta$ may be computed in the following way. We know that the volume v_0 occupied by one mole of an ideal gas at 0°C. at the standard atmospheric pressure of

$$p_0 = 1.013\text{x}10^6 \text{ dynes/cm}^2$$
$$= 1.013\text{x}10^5 \text{ newtons/m}^2$$

is

$$v_0 = 22{,}400 \text{ cm}^3 = 0.224 \text{ m}^3 .$$

Under these conditions of pressure and temperature n moles of the gas occupies the volume

$$V_0 = nv_0 .$$

Thus

$$p_0V_0\beta = np_0v_0\beta = nR , \qquad (8.9)$$

where

$$R = \frac{1.013\text{x}10^5 \text{ x } 0.224}{273.18} = 8.31 \text{ joules/mole}^\circ\text{C}.$$

The value of R is the same for all gases, and is called the universal gas constant. Substituting Eq. (8.9) into (8.8),

$$pV = nR(t + \frac{1}{\beta}) = nR(t + 273.18^\circ\text{C}).$$

If we define a new temperature scale by letting

$$T = t_C + 273.18^\circ\text{C} , \qquad (8.10)$$

which merely shifts the zero point of the Centigrade scale, we have

$$pV = nRT , \qquad (8.11)$$

relating the pressure, volume, and temperature of n moles of an ideal gas. Eq. (8.11) is known as the *equation of state* of an ideal gas.

The temperature scale established by Eq. (8.10) is called the absolute Centigrade or Kelvin scale, and the temperature $T=0^\circ$K (or 0°A) is called absolute zero. (Note that, by Gay-Lussac's Law, the volume of an ideal gas at absolute zero is zero.) The absolute scale using Fahrenheit degrees is called the Rankine scale, and is defined by

$$T_R = t_F + 460^\circ F.$$

We shall discuss the meaning of absolute zero, and the reason for its name, in the next chapter.

8.5. Specific Heats of Gases. We are now in a position to compare the empirical equation of state of an ideal gas,

$$pV = nRT , \qquad (8.11)$$

with the equation of state derived from kinetic theory,

$$pV = \frac{2}{3} NU ,$$

where U is the kinetic energy per molecule and N the number of molecules present. If the latter is correct, then

$$\frac{2}{3} NU = nRT ,$$

and

$$U = \frac{3}{2} \frac{RT}{(N/n)} .$$

The quantity (N/n) is the number of molecules per mole of any substance, which is Avogadro's Number N_0 ($=6.12\text{x}10^{23}$). Hence

$$U = \frac{3}{2} \frac{R}{N_0} T$$

$$= \frac{3}{2} kT , \qquad (8.12)$$

where k, known as Boltzmann's Constant, is the universal gas constant per molecule rather than per mole. The value of k is

$$k = \frac{R}{N_0} = 1.38\text{x}10^{-23} \text{ joule/molecule } ^\circ K.$$

Eq. (8.12) is a very interesting one, since it states that the total kinetic energy of an ideal gas molecule is directly proportional to the *absolute* temperature.

We have still not proved that the ideal gas envisioned in kinetic theory is the same ideal gas whose properties are summarized by the formula pV=nRT. To do this we must calculate the consequences of the theory and compare them with the corresponding experimental results. For example, we might determine the amount of energy required to raise the temperature of a given quantity of an ideal gas by 1°C (which is the same as 1°K because the two scales use the same size degree). This energy is called the *specific heat* of the gas. If we hold the volume of the gas fixed and permit the pressure alone to vary, we have the specific heat at constant volume, denoted c_V, which is given by the partial derivative

$$c_V = \left(\frac{\partial U}{\partial T}\right)_V . \tag{8.13}$$

Referring to Eq. (8.12) we find that, if the theory is correct, $U=3/2$ kT per molecule, so that

$$c_V = \frac{3}{2}\ k \text{ joule/molecule °C} .$$

(The specific heat at constant pressure c_p is larger than this, as we shall see in the next chapter, owing to the work that must be done to push back the container walls in the expansion.) Because there are N_0 molecules per mole,

$$c_V = \frac{3}{2}\ N_0 k = \frac{3}{2}\ R$$

$$= 12.5 \text{ joules/mole °C} ,$$

which should hold for *all* gases. This figure indeed agrees very well with the experimental values obtained for monatomic gases (Table 9-3), indicating that the kinetic theory is valid for such gases. However, while the agreement between theory and experiment is strikingly good for monatomic gases, it is poor for diatomic and polyatomic ones. Before being able to completely accept kinetic theory, therefore, we must find some explanation for this discrepancy.

One of the postulates of the kinetic theory of an ideal gas is that the internal energy of the gas consists exclusively of the kinetic energy of its molecules. This kinetic energy was considered as entirely due to the translational motion of the molecules, which were assumed to be essentially mass points. Each molecule has three *degrees of freedom* in this picture, another way of stating that the motion has three *independent* components, along the x, y, and z axes. Because the total energy per molecule is 3/2 kT, we may think of each degree of freedom as having associated with it $\frac{1}{2}$kT of kinetic energy. This scheme seems reasonably valid for molecules consisting of individual atoms, but it apparently fails when dealing with more complex molecules.

Let us examine Table 9-3 once more. We note that the specific heats of diatomic and polyatomic molecules are all higher than that of a monatomic molecule, indicating that more energy per mole per °C is needed to raise the temperature of such gases. This suggests that there might be more degrees of freedom in the motion of the more complex molecules, enabling them to absorb a greater amount of energy without much change in the average molecular velocity, Let us examine the diatomic molecule more closely. If we think of it as shown in Fig. 8-4 with the two atoms separated by a small distance, we see that when the molecule rotates about either of the two mutually perpendicular axes (y and z in the diagram) that are perpendicular to the line joining the atoms, it will have kinetic energy of rotation. About the line joining the atoms (x in the diagram) there is no kinetic energy since we consider each atom as a mass point. Thus we have for a diatomic molecule a total of 5 degrees of freedom, three of translation referring to the motion of the molecule as a whole and two of rotation. Assigning $\frac{1}{2}kT$ of energy to each degree of freedom for each molecule, which is $\frac{1}{2}R$ per degree of freedom per mole of gas, we have

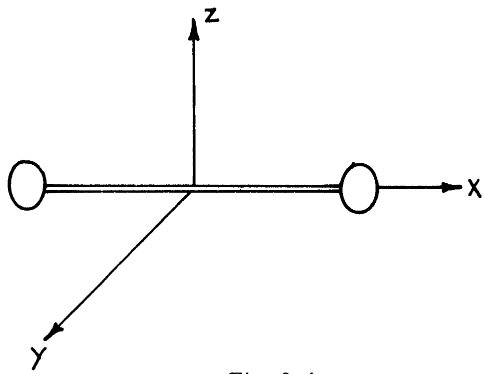


Fig. 8-4

$$c_V = \frac{5}{2} R = 20.8 \text{ joules/mole °C (diatomic molecules)},$$

in agreement now with the data listed in Table 9-3. In a similar way we can account for the specific heats of polyatomic molecules. The conclusion is that the kinetic theory of gases, and its result that each degree of freedom of a molecule possesses $\frac{1}{2}kT$ of kinetic energy, is in accord with our empirical knowledge of this state of matter. (At extremely low temperatures the theory must be modified somewhat, as we shall see in a later chapter.)

Chapter 9

THERMODYNAMICS

The nature of heat provided scientists of the past with a splendid subject for speculation. Just what is it that causes a body to change its temperature? To the learned men of the 17th and 18th centuries it was reasonable that when something, say a pan of water, is placed in contact with something else at a higher temperature, say a flame, the water becomes hot because a mysterious and invisible fluid called *caloric* flows from the flame into it. Similarly, when a pan of hot water is placed on a block of ice, the water grows cooler owing to the flow of caloric away from it and into the ice. There is, according to these ideas, just so much caloric in the universe, and hot objects contain more of it than cold ones.

Caloric as an explanation for heat met its downfall in the observations of Count Rumford (1753-1814) of Bavaria, who found that whenever a cannon was being bored it became very hot. In fact, by using a dull drill so that little or no metal was actually removed, there was apparently no limit to the heat, and hence caloric, that could be produced. This violated the supposed conservation of caloric. Rumford concluded that the work done against friction in the boring process was converted into heat, an interpretation that has remained valid. As we saw in the previous chapter, the temperature of a gas sample can be raised by the addition of energy to it, which strongly suggests that heat is simply a form of energy and can be included in a generalized conservation theorem.

9.1. Heat Units. There is no reason why the energy units we are already familiar with—the joule and the ft-lb—cannot be used to measure heat. However, a different set of units for heat has become established, and it is customary to use them whenever appropriate. The units are based on the experimental observation that the same amount of heat, or very close to it, is always required to raise the temperature of a given quantity of a particular substance through a certain temperature interval. The same heat is needed to raise 12 lb of copper from 30°F to 40°F as from 200°F to 210°F. In both cases the temperature interval of 10°F is identical even though the initial and final temperatures are not. In establishing heat units the standard substance is chosen to be water.

The *calorie* is the unit of heat in the metric system. It is that quantity of heat necessary to raise 1 gram of water through 1°C.

The *British thermal* unit (Btu) is the unit of heat in the English system. It is that quantity of heat necessary to raise 1 lb of water through 1°F.

We observe that in the metric system the *mass* of water is specified, while in the English system the *weight* is used instead. These different usages are traditional and, despite their inconsistency, are universally employed. For convenience both will be denoted by m in those portions of the text dealing with heat.

To be precise, we must take into account the slight variation in the amount of heat needed to change the temperature of a unit mass of water at different parts of the temperature scale. This variation is small, however, and we shall disregard it here.

Example. How many calories are equivalent to 1 Btu?

Solution. We note that 1 lb = 454 g and that 1°F = 5/9°C. Therefore

$$1 \text{ Btu} = \frac{5}{9} \times 454 = 252 \text{ calories.}$$

We now require appropriate conversion factors to relate the heat units of the calorie and the Btu with the energy units of the joule and the ft-lb. The British physicist James Prescott Joule performed an experiment in the middle of the 19th century to determine these factors, and, by showing that they are constant, demonstrated quantitatively for the first time that Rumford's hypothesis was correct. His apparatus consisted of a weight attached through a string to a set of paddle wheels that rotated in an insulated container filled with water. When the weight descended the wheels turned, stirring the water and, through the agency of friction, raising its temperature. By comparing the work $W = mgh$ done by the falling weight with the heat Q produced by the paddle wheels, Joule found that

$$\frac{W}{Q} = J = \text{constant.}$$

Modern measurements give for the value of J, the *mechanical equivalent of heat,* 4.186 joules/calorie or 778 ft-lbs/Btu. In other words,

$$\begin{aligned} 1 \text{ calorie} &= 4.186 \text{ joules} \\ 1 \text{ Btu} &= 778 \text{ ft-lb.} \end{aligned} \tag{9.1}$$

9.2. Calorimetry. If we take 1 g samples of a variety of different substances and add just enough heat to each of them to raise their temperatures by 1°C, we will find that the heats Q needed by the samples are not the same. A gram of water, by definition, absorbs

1 calorie in going up 1°C, but a gram of aluminum absorbs 0.217 calorie and a gram of silver 0.056 calorie in the same situation. These figures are of considerable usefulness in *calorimetry*, the measurement of heat, and are called the *specific heats* of the substances (cf. Sec. 8.5). That is, the specific heat C of some material is the quantity of heat required to increase the temperature of 1 g of the material by 1°C, and is the same numerically as the quantity of heat required to increase the temperature of 1 lb of the material by 1°F. In equation form,

$$\begin{aligned} C &= \frac{Q}{m\Delta T} \quad \frac{\text{calories}}{\text{g °C}} \\ &= \frac{Q}{m\Delta T} \quad \frac{\text{Btu}}{\text{lb °F}} , \end{aligned} \tag{9.2}$$

where ΔT is the temperature change. Table 9.1 is a list of the specific heats of some common substances, together with the range of temperatures in which these values hold.

Table 9.1

Specific Heats

Substance	Specific Heat	Approximate Temperature Range (°C)
aluminum	0.21	0-50
iron	0.11	0-100
lead	0.030	0-50
silver	0.056	0-100
glass (crown)	0.117	10-50
ice	0.55	0-10
alcohol (ethyl)	0.55	0-40

Example. How much heat must be added to a 3 lb iron frying pan in order to raise its temperature from 70°F to 300°F?

Solution. From Eq. (9.2),

$$\begin{aligned} Q &= mC\Delta T = 3 \text{ lb} \times 0.11 \frac{\text{Btu}}{\text{lb°F}} \times 230\text{°F} \\ &= 78 \text{ Btu}. \end{aligned}$$

Example. If the 78 Btu had been added to a 3 lb aluminum frying pan at 70°F, what would its final temperature have been?

Solution. We rewrite Eq. (9.2) in the form

$$\Delta T = \frac{Q}{mC} ,$$

from which

$$T = \frac{78 \text{ Btu}}{3 \text{ lb} \times 0.21 \text{ Btu/lb °F}}$$

$$= 124 \text{ °F}.$$

The final temperature is the initial temperature, 70°F, plus the temperature change of 124°F, which is 194°F.

Another factor involved in calorimetry is the heat absorbed or evolved when a substance undergoes a *change of phase*. It is observed that ordinarily certain characteristic amounts of energy per unit mass are needed to convert a given material from a solid to a liquid at its melting temperature and from a liquid to a gas at its vaporization temperature, and that these same amounts of energy are liberated when the liquid turns into a solid or the gas into a liquid. For example, the heat required to convert 1 g of ice at 0°C to 1 g of water at 0°C, called its *latent heat of fusion* (L), is 80 calories, and 80 calories is given off whenever 1 g of water is converted to 1 g of ice at this temperature. To change 1 g of water at 100°C to 1 g of steam at 100°C requires 540 calories, the *latent heat of vaporization* of water (also denoted by L), and again this heat is given off by 1 g of steam going into 1 g of water. In English units the latent heat of fusion of water is 144 Btu/lb and the latent heat of vaporization is 970 Btu/lb. Table 9.2 contains the latent heats of fusion and vaporization of some common substances together with their melting and boiling points. Certain substances, such as glass and sealing wax, are in reality "supercooled liquids", and exhibit no specific melting points or latent heats of fusion.

Table 9.2

Latent Heats of Fusion and Vaporization

Substance	Latent Heat	Temperature
Fusion		
aluminum	77 cal/g	658°C
carbon dioxide	45	-56
lead	6	327
silver	21	961
water	80	0
Vaporization		
helium	6	-269
alcohol (ethyl)	204	78
lead	175	1170
water	540	100

Example. A piece of ice at 0°C is dropped into a lake containing water at 0°C. What is the minimum height in feet from which the ice should be dropped if it is to melt completely when it hits the water?

Solution. The ice has a kinetic energy of mgh when it reaches the water after having been dropped from a height h, and the heat required to melt the ice is mL. Therefore

$$mgh \text{ ft-lb} = mL \text{ Btu} \times 778 \frac{\text{ft-lb}}{\text{Btu}}$$

which yields, after cancelling m on both sides,

$$h = \frac{778\ L}{g} = \frac{778 \times 144}{32} = 3500 \text{ ft.}$$

This is a case in which internal energy is increased without a change in temperature.

Example. Three hundred grams of ice at -10°C are added to a 50 g glass jar containing 200 g of water at 10°C. What is the final temperature of the mixture? How much ice, if any, remains?

Solution. We begin by noting that the amount of heat required to bring the ice to its melting point of 0°C is

$$Q = mC\Delta T = 300 \times 0.55 \times 10 = 1650 \text{ calories.}$$

Supplying this heat to the ice reduces the temperature of the jar and the water. To find the exact temperature change we begin with the equation

$$Q = m_g C_g \Delta T + m_w C_w \Delta T \ ,$$

from which

$$\Delta T = \frac{Q}{m_g C_g + m_w C_w} = \frac{1650}{10 + 200} = 7.86°\text{C}.$$

Thus, when the ice has reached 0°C the jar and the water are still at 2.14°C, and in order for equilibrium (when all of the constituents of the mixture are at the same temperature) to be reached some ice must melt. The heat absorbed by the ice that melts $m_1 L$ equals the heat evolved by the jar and the water, so that

$$m_1 L = m_g C_g \Delta T + m_w C_w \Delta T$$

$$m_1 = \frac{(m_g C_g + m_w C_w)\Delta T}{L}$$

$$= \frac{210 \times 2.14}{80}$$

$$= 5.62 \text{ g.}$$

The answer, then is that the final temperature of the mixture is 0°C with 294.38 g of ice left unmelted.

9.3. The First Law of Thermodynamics. Energy, as we know, is always conserved. In order to write an equation expressing this conservation rule where heat is involved it is convenient if we think in terms of a "system" whose interactions with its environment may be measured. The system may be a container of a gas, an automobile, the solar system, and so forth, the important thing being its existence as an entity separate from its surroundings. A system may give out or absorb heat and do external work or have work done on it. The usual way of expressing the relationship between these quantities in any process is called the *first law of thermodynamics,* and is written

$$Q = U - U_0 + W. \tag{9.3}$$

In this equation Q represents the heat *added* to the system, U_0 is its initial internal energy and U its final internal energy, and W is the work done *by* the system in the process. Should the system *give out* heat, Q in Eq. (9.3) is taken as negative, and similarly work done *on* the system is taken as negative. All of the quantities in Eq. (9.3) must, of course, be expressed in the same units, whether calories, Btu, joules, or ft-lb. ●

The first law of thermodynamics in the above form is of great importance in the abstract since it applies to *any* system undergoing *any* process, but it is difficult to apply in its present form to a particular situation. We shall be interested in discussing the behavior of an ideal gas here, and will begin by considering a sample of such a gas confined in a cylinder with a movable piston (Fig. 9-1). The pressure p on the piston is kept constant. When we heat the gas it expands against the piston in order to increase the volume it occupies. During the expansion the gas does an amount of work dW on the piston equal to the force it exerts multiplied by the distance dx the piston moves. Since the applied force is the pressure on the piston times the latter's area A,

$$dW = Fdx = pAdx = p\,dV\ , \tag{9.4a}$$

since Adx is the change in volume dV of the gas. The work done in a finite expansion is

$$W = \int_{V_{\text{initial}}}^{V_{\text{final}}} pdV\ . \tag{9.4b}$$

If the volume increases, meaning that work has been done *by* the system, W is positive, while if the volume decreases, meaning that work has been done *on* the system, W is negative. This agrees with the sign convention we used before with the first law of thermodynamics.

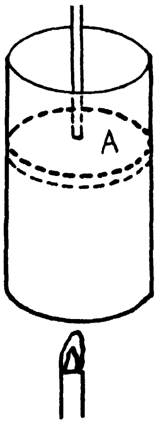


Fig. 9-1

In terms of infinitesimal changes in Q, U, and W,

$$dQ = dU + dW$$

$$= dU + pdV \quad (9.5)$$

This equation yields the interesting result that when the volume of a gas sample does not change in some process, all heat added to it goes into its internal energy, and, conversely, all heat removed from a constant-volume gas sample comes from its internal energy. Changes in internal energy, as we know, are manifested by changes in temperature if there is no change of state involved.

9.4. Specific Heats of Gases. We have already seen that changes in the temperature of a substance are related to the amount of heat dQ added to or subtracted from it by the relation

$$dQ = mCdT, \quad (9.6)$$

where m is the mass of the substance and C its specific heat. In point of fact there are two different specific heats of importance, the specific heat when the volume of the sample is held constant, denoted C_V, and the specific heat when the pressure on the sample is held constant, denoted C_p. The difference between C_p and C_V is negligible for most solids and liquids, but it is significant in the case of gases.

At constant volume we have that

$$dQ = dU = mC_V dT,$$

which may be written

$$\left(\frac{\partial U}{\partial T}\right)_V = mC_V . \quad (9.7a)$$

Here the specific heat is expressed in calories/g °C or Btu/lb °F as usual. In dealing with gases it is often more convenient to speak in terms of the number of moles n that are present, in which case we would denote the specific heat at constant volume by c_V (calories/mole °C or Btu/mole °F) and have

$$\left(\frac{\partial U}{\partial T}\right)_V = nc_V . \quad (9.7b)$$

If we add heat to a gas while keeping the pressure on it constant, the gas is able to expand and do work. In this situation we must not only add heat to increase the internal energy of the gas by raising its temperature but must also supply heat which will be converted into the external work. Accordingly the specific heat at constant pressure is always greater than the specific heat at constant volume. The exact difference is important because it is much easier to measure c_p accurately, while c_V is what is calculated from the theoretical expression for internal energy for a particular model. This difference can be derived for an ideal gas in the following way.

We start with one mole of an ideal gas and add the amount of heat dQ_1, holding the volume constant, which produces an internal energy change of dU. To another mole of the same gas we add dQ_2 at constant pressure so as to cause the same internal energy change dU as before. In equation form,

$$dQ_1 = c_V dT = dU$$

$$dQ_2 = c_p dT = dU + pdV ,$$

and, subtracting,

$$(c_p - c_V)\, dT = pdV \quad .$$

The equation of state of an ideal gas is $pV=nRT$. For one mole $n=1$, and so, differentiating and keeping in mind that p is constant here,

$$pdV = RdT \; .$$

Substituting this result into the above expression for pdV,

$$(c_p - c_V)\, dT = RdT ,$$

or

$$c_p - c_V = R = 8.31 \text{ joules/mole °C.} \qquad (9.8)$$

Table 9.3 shows how this theoretical figure, obtained for an ideal gas, agrees with experimental measurements on real gases.

Table 9.3

Specific Heats of Gases

Gas	C_p	C_V	$C_p - C_V$	γ	atoms/molecule
helium	2.52R	1.52R	1.00R	1.66	1
argon	2.52	1.51	1.01	1.67	1
hydrogen	3.42	2.44	0.98	1.41	2
carbon monoxide	3.50	2.49	1.01	1.40	2
carbon dioxide	4.40	3.38	1.02	1.30	3
ammonia	4.48	3.42	1.06	1.31	4

9.5. Adiabatic Processes. A further application of the first law of thermodynamics concerns the behavior of an ideal gas in a process in which no heat is transferred between the gas and its surroundings. Processes of this kind are called *adiabatic*, and they can be realized in practice either by insulating the system under study or by performing the experiment so rapidly that no appreciable heat exchange takes place.

The condition for an adiabatic process, then, is that

$$dQ = 0 = dU + pdV \ .$$

Since we know from before that, for one mole of ideal gas,

$$dU = c_V dT \ ,$$

$$pV = RT \ ,$$

and

$$c_p - c_V = R \ ,$$

we have

$$c_V dT = (c_p - c_V) \, T \, \frac{dV}{V} = 0 \ .$$

Dividing through by $c_V T$,

$$\frac{dT}{T} + \left(\frac{c_p}{c_V} - 1\right) \frac{dV}{V} = 0 \ ,$$

which becomes

$$\frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0$$

if we abbreviate the ratio of specific heats c_p/c_V by γ. By integrating we obtain

$$\log T + (\gamma - 1) \log V = \text{constant}$$

$$\log TV^{\gamma-1} = \text{constant},$$

which reduces to

$$TV^{\gamma-1} = \text{constant}. \qquad (9.9)$$

A more useful form of this equation is

$$pV^{\gamma} = \text{constant} \ , \qquad (9.10)$$

which follows from $pV=RT$, where the new constant differs by a factor of R from the one in (9.9).

Example. A 5 ft^3 tank contains helium at a pressure of 200 lb/in^2 which is later used to fill a balloon. Assuming that the filling process is rapid enough to be considered adiabatic, what is the initial volume of the balloon? What is its volume after the helium has reached thermal equilibrium with the atmosphere? (The ratio of specific heats γ is 1.66 for helium.)

Solution. Considering the expansion as adiabatic,

$$p_1 V_1^\gamma = p_2 V_2^\gamma \quad .$$

Here $p_1 = 200\ \mathrm{lb/in^2}$, $V_1 = 5\ \mathrm{ft^3}$, and p_2 is atmospheric pressure, $14.7\ \mathrm{lb/in^2}$. Hence

$$V_2 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} V_1 \ .$$

Solving this equation with the help of logarithms,

$$V_2 = 24\ \mathrm{ft^3} \ .$$

When the helium in the balloon has reached the temperature of the atmosphere once more we may consider the overall process as isothermal. In this case

$$p_1 V_1 = nRT = p_2 V_2' \ ,$$

and

$$V_2' = 68\ \mathrm{ft^3}.$$

9.6. The Second Law of Thermodynamics. While the first law of thermodynamics, like its counterpart in dynamics, the law of conservation of energy, is invaluable for understanding certain aspects of the behavior of various systems, it does not by itself go far enough. There are many processes that are entirely possible according to the first law, yet which never occur. For example, a piece of ice that is dropped to the ground from a sufficiently great height melts, the required heat coming from the conversion of its initial potential energy to kinetic energy and then into internal energy. Reversing this process would have a puddle of water spontaneously rising from the ground while turning into ice, acquiring kinetic energy from its latent heat of fusion as it freezes. Even though an event of this kind is hardly likely, what we know thus far about thermodynamics does not prohibit it. The general theorem that has been found to govern the possibility of occurrence of hypothetical processes is called the *second law of thermodynamics,* and it occupies a vital position in the heirarchy of physical principles.

There are a number of equally valid statements of the second law of thermodynamics. One, due to Kelvin, deals with cyclic processes, those that take some "working substance" through successive changes in pressure, volume, and temperature that periodically bring it back to its initial state. This states that *it is impossible to operate a cyclic process in which heat is taken from a single source and completely converted into work.* Entirely equivalent is Clausius' statement of the second law: it is impossible to operate a cyclic process in which heat is taken from a source and *completely* transferred to a reservoir at a higher temperature than the source. Evidently a whole class of processes is explicitly forbidden, such at the one above, but it requires a more subtle analysis to uncover the very much wider implications of the second law. We shall explore in the remainder of this section the interesting and unexpected conclusions that may be drawn about the efficiency of engines by applying this law.

In the early part of the 19th century a French engineer named Carnot devised a working cycle for an engine that can be proved, with the help of the second law of thermodynamics, to be the most efficient possible under a given set of circumstances. The Carnot cycle is illustrated in the p-V diagram in Fig. 9-2. The working substance undergoes an isothermal expansion at the initial temperature T_1 during which an amount of heat Q_1 is added to it. Then the substance is allowed to expand adiabatically, from b to c in the diagram, while its temperature drops to T_2. Now an isothermal compression is performed, with the result that the amount of heat Q_2 is given off. To bring the substance back to its starting point at a it is compressed adiabatically starting at point d, which raises its temperature to T_1 while also restoring the original pressure and volume. Fig. 9-3 shows how one might attempt to carry out this process in the laboratory; it must be kept in mind that the theory of the Carnot engine does not take into account friction, imperfect insulators, and similar factors that are involved in actual engines.

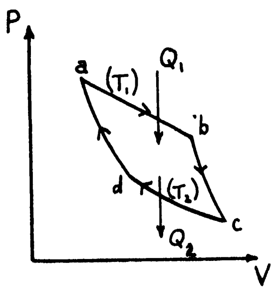


Fig. 9-2

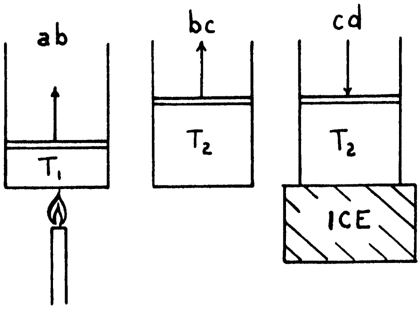


Fig. 9-3

The efficiency of any engine is

$$E = \frac{\text{work output}}{\text{energy input}} \quad .$$

In the Carnot cycle the energy input is Q_1, the heat added during the isothermal expansion from a to b. The work output W is, by the first law of thermodynamics,

$$W = Q_1 - Q_2 ,$$

the difference between the heat put into the engine at the temperature T_1 and the heat rejected at the lower temperature T_2. (W is equal to the area enclosed by abcd in Fig. 9-2.) Thus we have for the efficiency

$$E = 1 - \frac{Q_2}{Q_1} . \tag{9.11}$$

Since external energy is not involved in the adiabatic portions bc and da of the cycle,

$$Q_1 = \int_a^b pdV$$

and

$$Q_2 = \int_d^c pdV .$$

Assuming an ideal gas as the working substance, the substitution of nRT/V for p in the above equation gives

$$Q_1 = nRT_1 \int_a^b \frac{dV}{V} = nRT_1 \log \frac{V_b}{V_a}$$

and

$$Q_2 = nRT_2 \int_d^c \frac{dV}{V} = nRT_2 \log \frac{V_c}{V_d} .$$

and

$$E = 1 - \frac{Q_2}{Q_1}$$

$$= 1 - \frac{T_2}{T_1} \left[\frac{\log \frac{V_c}{V_d}}{\log \frac{V_b}{V_a}} \right] .$$

We can evaluate the bracketed quantity in the above equation by considering the adiabatic portions of the cycle. From Eq. (9.9) we have that

$$T_1 V_b^{\gamma-1} = T_2 V_c^{\gamma-1}$$

and

$$T_2 V_d^{\gamma-1} = T_1 V_a^{\gamma-1}$$

for the adiabatic expansion and compression respectively. This pair of relations yields

$$V_b = \left(\frac{T_2}{T_1}\right)^{1-\gamma} V_c$$

$$V_d = \left(\frac{T_1}{T_2}\right)^{1-\gamma} V_a ,$$

and so

$$\frac{\log \frac{V_c}{V_d}}{\log \frac{V_b}{V_a}} = \frac{\log \frac{V_c}{V_a}\left(\frac{T_2}{T_1}\right)^{1-\gamma}}{\log \frac{V_c}{V_a}\left(\frac{T_2}{T_1}\right)^{1-\gamma}} = 1 .$$

Our result for the efficiency of a Carnot engine is therefore

$$E = 1 - \frac{T_2}{T_1} , \tag{9.12}$$

which depends only on the ratio of the *absolute temperatures* of the two heat reservoirs, and not upon the specific details of the operating cycle. Owing to inevitable losses due to friction, imperfect insulators and conductors of heat, and the like, actual efficiencies are usually much less than Eq. (9.12) would predict.

9.7. Stefan's Law. The principles of thermodynamics may be used to investigate some properties of radiant energy. (Radiant energy is carried by electromagnetic waves, of which some examples are visible light, radio waves, and X-rays.) Let us consider an insulated cylinder with reflecting walls which contains a certain amount of radiant energy as the working substance, and take it through a Carnot cycle. In Fig. 9-4 AB is the isothermal expansion at temperature T, BC is an adiabatic expansion in which the temperature is reduced to T-dT, CD is an isothermal compression at T-dT, and DA is an adiabatic compression that brings the engine back to its starting point. The pressure that radiant energy exerts on a reflecting surface equals one-third of the energy density e of the radiation. That is,

$$p = \frac{e}{3} ,$$

where e is the energy per unit volume within the cylinder containing the radiant energy. Hence the work done in going from A to B is

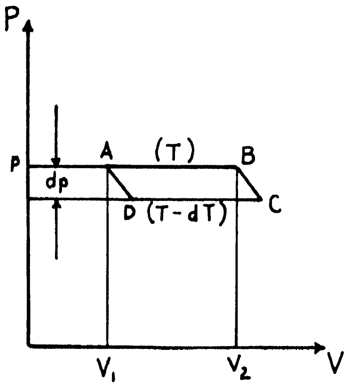


Fig. 9-4

$$W = p(V_2-V_1) = \frac{e}{3}(V_2-V_1).$$

The energy density depends only upon T, and so the increase in the internal energy of the cylinder of radiation is

$$\Delta U = e(V_2-V_1) \ .$$

The heat input Q in the expansion AB is, by the first law of thermodynamics, the sum of ΔU and W, and is given by

$$Q = \frac{4}{3}e(V_2-V_1) \ .$$

The work dW done by the engine in the entire cycle is the area ABCD,

$$dW = dp(V_2-V_1)$$

$$= \frac{de}{3}(V_2-V_1) \ .$$

From Eq. (9.12) the efficiency here is

$$E = \frac{dW}{Q} = \frac{dT}{T} = \frac{de}{4e} \ ,$$

and we have

$$\frac{de}{e} = 4\frac{dT}{T} \ .$$

Integrating,

$$\log e = 4 \log T + \text{const.}$$

$$e = cT^4 \ , \tag{9.13}$$

where c is a constant. Thus the energy density of radiation within a closed container having reflecting walls is proportional to T^4, where T is the absolute temperature of the container. If we bore a small hole in the container, the amount of radiation that leaks out per unit time is proportional to e and hence to T^4, a conclusion verified by experiments with "perfect radiators" (which we shall discuss later on). Eq. (9.13) is known as Stefan's law. As a matter of experience, the more something is heated, the brighter it becomes —for instance an iron poker in a fire—which follows from Stefan's law.

Chapter 10

THE ELECTRIC FIELD

We turn now to a field of physics rather different from any we have encountered thus far. In the study of mechanics we saw that there is a fundamental force, known as gravitational force, existing in nature which causes all material objects to attract one another. This force acts at a distance from the seat of the force itself, so that even widely separated bodies experience it. Another way of describing gravitation is to say that there is a gravitational *field* around every material body, and that this field exerts an attractive force on every other body. There are other fundamental force fields that exist in space, and we shall consider one of these, the electric field, in this chapter.

The importance of electrical forces follows from the facts that (1) particles having electrical properties are the basic constituents of all matter, and (2) that electrical forces, on the whole, are much stronger than gravitational ones. While gravity significantly affects the behavior of only relatively large quantities of matter, electrical forces predominate in the world of microscopic phenomena. Magnetism, which is closely related to electricity and which we shall shortly come to, manifests itself from the interior of the atom to the vast reaches of galactic space. Electricity and magnetism, while complex subjects, are essential elements in the store of basic knowledge a physicist must possess if he is to understand the workings of the universe.

10.1. Conservation of Charge. Despite the fact that electrified particles compose all matter, electrical forces seldom spontaneously display themselves. Something must be done to matter before its electrical properties become evident. Since the time of Thales it has been known that by rubbing a piece of amber with some fur the amber can be made to attract light objects such as bits of paper or pith balls. By rubbing a glass rod with a silk cloth a similar effect can be produced. The amber and glass in their new states are said to be *electrified,* and they can communicate some of their electrification to other bodies by direct contact. Thus we can perform experiments to determine the properties of electrification. For example, we might touch an amber rod that has been rubbed with some fur to each of two pith balls, and then bring the balls close together. What we observe is that the two balls repel one another strongly, and if they are attached to strings tied together they will stay apart despite the gravitational force tending to pull them down and hence together (Fig. 10-1a). If we electrify two similar balls with a glass rod that has been rubbed with silk, the same thing happens. However, if one pith ball is electrified by the amber and the other one by the glass, the two are found to *attract* one another (Fig. 10-1c).

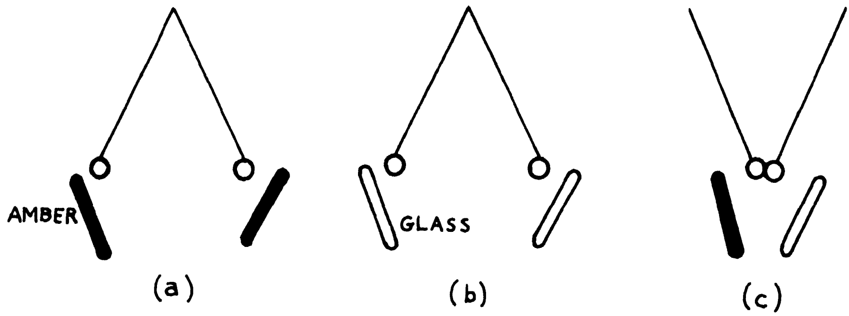


Fig. 10-1

Experiments such as the ones described above were carried out in the eighteenth century, and led to the conclusion that there are two different kinds of electricity. No matter how electrification was produced, it always behaved as if it had originated either in a glass or in an amber rod, and these types were called vitreous and resinous electricity respectively. There is a repulsive force acting between bodies similarly electrified, while if they are oppositely electrified the force is an attractive one. (In contrast, gravitational forces are always attractive.)

Benjamin Franklin arrived at a highly significant conclusion on the basis of some experiments he performed. He noted that when amber is rubbed with fur, the fur as well as the amber becomes electrified. Furthermore, the fur is electrified vitreously, while the amber becomes electrified resinously. The same effect, although reversed, occurs in the case of glass and silk cloth. Franklin surmised that equal amounts of each kind of electricity are always produced, and that the effect of the rubbing was to separate them. Actually, Franklin's own experiments were hardly quantitative enough to permit so sweeping a statement, but more precise measurements have proven his hypothesis correct. We may summarize this discovery by saying that *electricity can be neither created nor destroyed, but equal amounts of the two different types can be separated.*

Today we speak of an electrified body as being *charged,* with *positive* charge corresponding to vitreous electrification and *negative* charge to resinous electrification. This usage is very convenient, since we can determine the exact amount and kind of charge present somewhere merely by adding the various charges together algebraically. When the net charge on a body is zero, it is called *neutral.* A neutral body remains that way when equal amounts of positive and negative charge are either added or removed. We may restate the conclusion of the previous paragraph by saying that *charge is conserved.* Together with the other conservation laws we have already learned and are about to learn, the conservation of charge is one of the foundation stones underlying all physical theory.

In dealing with electrical phenomena, it is customary to classify all substances as either conductors or insulators. The distinction between them is that charges are free to move about in a conductor, while they cannot do so in an insulator. When a charged rod is touched to a conductor, the entire surface of the conductor becomes electrified—not necessarily in a uniform manner, but the essential fact is that the charge distributes itself until it is in equilibrium. (We shall discuss in Sec. 10.6 why the charge remains on the surface of the conductor.) On the other hand, when a charged rod is brought in contact with an insulator, any charge transferred to the latter stays in the precise location where it was placed initially. With very few exceptions, and those occurring only under unusual circumstances, no material is either a perfect conductor or a perfect insulator. Instead there are degrees of charge mobility, and while we may think in terms of ideal insulators and conductors in trying to analyze electrical experiments in general terms, we must remember that nature is less arbitrary.

10.2. Coulomb's Law. By means of a direct measurement, Coulomb discovered that the force in air between two charges is directly proportional to the product of their magnitudes and inversely proportional to the square of their separation. That is,

$$\mathbf{F} = k \frac{q_1 q_2}{r^2} \mathbf{r}_1 , \tag{10.1}$$

where k is a constant of proportionality and r is the distance between the two charges q_1 and q_2; $\mathbf{r}_1$ is, as usual, a unit vector in the $\mathbf{r}$ direction. Eq. (10.1) is called Coulomb's law, and it expresses the fact that the force between q_1 and q_2 is in the $+\mathbf{r}_1$ direction, which implies repulsion, when q_1 and q_2 have the same sign, while it is in the $-\mathbf{r}_1$ direction, which implies attraction, when the two charges have opposite signs.

We must now provide units for the various quantities appearing in Eq. (10.1) and establish the value of k. In the cgs system, F is in dynes and r in cm, and the unit of charge is defined by taking $k=1$. The resulting *electrostatic unit of charge*, or esu, is that charge on each of two bodies that results in a force between them of 1 dyne when their separation is 1 cm. The esu, although buttressed by tradition and still often used, is far too small for practical applications. To supplement it a much larger unit of charge called the *coulomb* was introduced, where by experiment

$$1 \text{ coulomb} = 2.9979 \times 10^9 \text{ esu} .$$

The coulomb forms the basis of the "practical" system of electrical units, which was used side-by-side with esu units. The modern mks system has simplified the situation by adopting the coulomb as its sole unit of charge. Expressing F in newtons, r in meters, and q in coulombs, the constant k turns out to equal 8.9874×10^9 n m^2/coul^2, but for most purposes it is adequate to let

$$k = 9 \times 10^9 \text{ n m}^2/\text{coul}^2.$$

For various reasons, largely of mathematical convenience, k is often written

$$k = \frac{1}{4\pi\epsilon_0} ,$$

where ϵ_0, called the permittivity of free space, is approximately 8.85×10^{-9} coul^2/n m^2. We shall adopt this usage, rewriting Coulomb's law as

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \mathbf{r}_1 . \qquad (10.2)$$

Example. Three charges, $q_1=+7\times10^{-5}$ coul, $q_2=-4\times10^{-5}$ coul, and $q_3=-5\times10^{-5}$ coul, are placed as shown in Fig. 10-2a at the vertices of a triangle whose sides measure 3, 4, and 5 cm. Determine the magnitude and direction of the force on q_3.

Solution. Let us call $\mathbf{F}_1$ the force exerted on q_3 by q_1 and $\mathbf{F}_2$ the force exerted on q_3 by q_2. Then, by Eq. (10-.2), the magnitudes of the forces are

$$F_1 = \frac{q_1 q_3}{4\pi\epsilon_0 r_1{}^2} = -1.26\times10^{-4} \text{n}$$

$$F_2 = \frac{q_2 q_3}{4\pi\epsilon_0 r_2{}^2} = 1.13\times10^{-4} \text{n}.$$

The directions of $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel to the 4 and 5 cm sides of the triangle as shown in Fig. 10-2b. In order to evaluate the net force $\mathbf{F}$ on q_3 we shall resolve $\mathbf{F}_1$ and $\mathbf{F}_2$ into x and y components. The results are

$$F_{1y} = F_1 \sin\theta = \frac{3}{5} F_1 = 7.56\text{x}10^{-5}\ \text{n}$$

$$F_{1x} = F_1 \cos\theta = \frac{4}{5} F_1 = -1.01\text{x}10^{-4}\text{n}$$

$$F_{2y} = 0$$

$$F_{2x} = 1.13\text{x}10^{-4}\text{n} .$$

Hence

$$F = \sqrt{F_x^2 + F_y^2} = 7.65\text{x}10^{-5}\text{n}$$

and the angle between F and the x axis is

$$\phi = \tan^{-1} \frac{F_y}{F_x} = \tan^{-1} 6.30 = 81° .$$

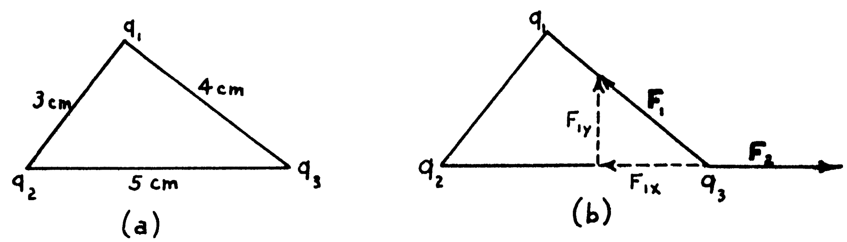


Fig. 10-2

10.3. Electric Field. We may think of *electric field* as an alteration in the properties of space produced by the presence of a charge such that other charges experience forces when present in the field. There are two principal reasons for introducing the concept of electric field as an entity apart from the charge that gives rise to it. The first is that it is possible to exert an electrical force on a charge by means other than placing another charge in its vicinity. In other words, static electrification is not the only source of electric field. The second reason is that if we were to instantaneously "create" a charge in a certain place, a finite time would be required for the effects of this charge to be felt at some other place. We shall elaborate both of these aspects of electric field later.

The electric field intensity $\mathbf{E}$ at a point in space is defined as the ratio between the force $\mathbf{F}$ on a positive charge q placed at that point and the magnitude of q as q approaches 0. In symbols,

$$\mathbf{E} = \lim_{q\to 0} \frac{\mathbf{F}}{q} . \tag{10.3}$$

This peculiar definition is required in place of just

$$\mathbf{E} = \frac{\mathbf{F}}{q} \tag{10.4}$$

because the presence of a finite charge q may induce a separation of charge in nearby matter which would modify **F**. Strictly speaking Eq. (10.3) can never be evaluated since there is a certain minimum electric change in nature (see next section), and letting $q \to 0$ therefore cannot be accomplished. As a practical matter, the smallest charge obtainable is so minute that defining macroscopic electric fields by (10.3) usually presents no problems. If we confine ourselves for the present to charges in vacuum or, to a good approximation, in air, Eq. (10.4) is sufficient. Electric field in the mks system evidently is expressed in newtons/coulomb.

Coulomb's law together with Eq. (10.4) enables us to calculate the electric field of a charge or of a distribution of charges. The procedure is to place an imaginary positive test charge q' wherever we want to determine **E**, and then calculate the force on q'. For example, to find the electric field a distance r from a charge q, we have for the force on q'

$$\mathbf{F} = \frac{qq'}{4\pi\epsilon_0 r^2} \mathbf{r}_1 \quad .$$

Hence

$$\mathbf{E} = \frac{\mathbf{F}}{q'} = \frac{q}{4\pi\epsilon_0 r^2} \mathbf{r}_1 \ . \tag{10.5}$$

Should more than one charge contribute to **E**, it is necessary to add vectorially the forces exerted on q' by the various charges.

Example. Find the electric field intensity at the location of charge q_3 in the previous example.

Solution. The force acting on q_3 is $7.65\text{x}10^5$n, so that the magnitude of **E** is

$$E = \frac{F}{q_3} = -1.53\text{x}10^5 \text{ n.}$$

E is parallel to **F** but in the opposite direction, which is to be expected from the use of a negative charge as the test charge q'.

We may readily generalize the formula for finding the electric field around a single charge to that around a continuous distribution of charge. The result is the vector integral

$$E = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r^2} \quad , \tag{10.6}$$

where dq is the charge in an infinitesimal region a distance r from the point we are interested in and the integral is taken over the entire charged region. In applying Eq. (10.6) the vector nature of $\mathbf{E}$ must be taken into account, that is, we must determine its components.

Example. Determine the electric field of a thin, uniformly-charged ring lying in the xy plane with its center at the origin at any point on the z axis.

Solution. We start by computing the contribution $d\mathbf{E}$ to the electric field at the point P due to the element of charge dq on the portion of the ring dL shown in Fig. 10-3. If the linear charge density in the ring is λ,

$$dq = \lambda\, dL$$

and

$$dE = \frac{\lambda\, dL}{4\pi\epsilon_0 r^2} \quad .$$

We need take into account only the component of $d\mathbf{E}$ along the z axis, since the components perpendicular to z cancel out when we integrate around the ring. Hence, with R the radius of the ring,

$$\mathbf{E} = \int_0^{2\pi R} \frac{dL\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\lambda RZ}{2\,(R^2+Z^2)^{3/2}}\ \mathbf{z}_1$$

is the electric field intensity at the point $z=Z$.

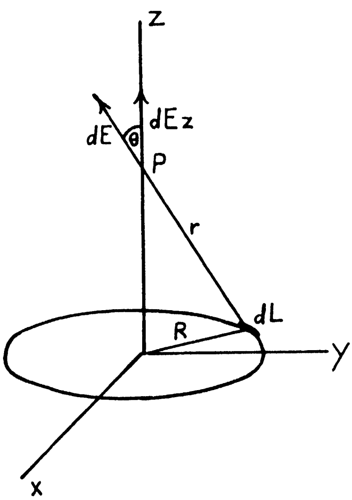


Fig. 10-3

10.4. Charge of the Electron.

Given the electric field $\mathbf{E}$ at a point, the force a charge q would experience if placed there is simply $q\mathbf{E}$. R. A. Millikan employed this fact in measuring the smallest quantity of charge found in nature. He used a chamber in which a constant electric field was maintained, and sprayed minute drops of oil inside with an atomizer. When illuminated from the side, these drops could be studied with a microscope. X-rays directed into the chamber caused some of the drops to become charged. By adjusting the electric field $\mathbf{E}$, Millikan was able to balance the downward force of gravity mg on a particular drop by an upward electrical force qE, where q was the unknown charge on the drop. Then, letting the drop fall freely by removing the electric field, the terminal velocity of the drop was measured. From this velocity and the known coefficient of viscosity of air the drop radius was calculated, leading to a value for m, the mass of the drop. The charge on each drop examined could be determined from the equation

$$q = mg/E \ .$$

Millikan found that *all* of the charges he measured in this way were multiples of a particular value. He concluded that this value, designated e, is the basic unit of electrical charge. Modern measurements give

$$e = 1.602 \times 10^{-19} \text{ coulomb.}$$

Electrons ($m_e = 9.11 \times 10^{-31}$ kg), protons ($m_p = 1.67 \times 10^{-27}$ kg), and neutrons ($m_n = 1.67 \times 10^{-27}$ kg), the basic constituents of all matter, possess the intrinsic charges $-e$, $+e$, and 0 respectively. Furthermore, all other known elementary particles are either neutral or carry charges of $+e$, so that this quantity is one of fundamental constants of nature. It is usually referred to as the "charge of the electron" or "electronic charge."

10.5. Gauss' Theorem. An electric field is a vector field, which means that at every point within it the field strength has a magnitude and a direction. Generally we have represented individual vectors by arrows pointing in the direction of the vector which have lengths proportional to its magnitude. In trying to represent a vector field, however, it is more appropriate to use continuous lines in place of arrows, with the direction of the lines indicating the direction of the field and the *density* of lines indicating the field intensity. In this way we can readily illustrate the essential features of a specific vector field. The lines themselves are called *lines of force*. In Fig. 10-4 are pictured the lines of force surrounding two charges of equal magnitude and opposite sign separated by a small distance, and Fig. 10-5 shows the same situation except that the charges here have the same sign.

The concept of lines of force, which was introduced by Faraday, may be related quantitatively to the electric field strength at a certain point. Let us consider an infinitesimal area dA perpendicular to the lines of force dN at that point (Fig. 10-6). Since we have

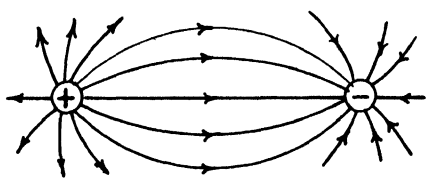


Fig. 10-4

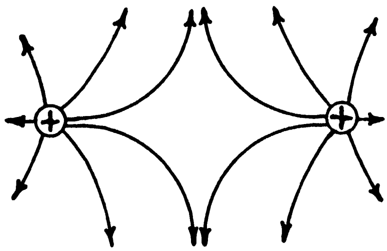


Fig. 10-5

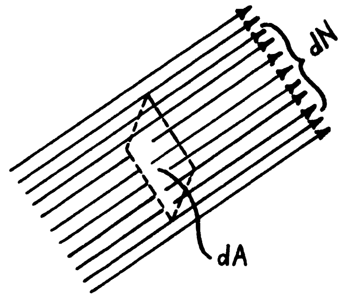


Fig. 10-6

drawn the lines of force so that their density is proportional to E,

$$E = \alpha \frac{dN}{dA}, \tag{10.7}$$

where α is an arbitrary constant.

Let us calculate the number of lines of force that emerge from a charge q. According to Eq. (10.7),

$$\alpha N = \int E_n dA, \tag{10.8}$$

where E_n is the component of **E** normal to dA. In order to intercept all of the lines of force leaving q we shall perform the integration over the surface of a sphere of radius R having q at its center (Fig. 10-7). In this case the angle between **E** and dA is always 90°, and

$$E_n = E = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} .$$

Hence

$$\alpha N = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \cdot 4\pi R^2 = \frac{q}{\epsilon_0} .$$

If we now assign the constant α the value

$$\alpha = \frac{1}{\epsilon_0} ,$$

we have that the number N of lines of force leaving the charge q is

$$N = q . \tag{10.9}$$

Eq. (10.8) becomes, with the new definition of α,

$$N = \epsilon_0 \int E_n dA . \tag{10.10}$$

The idea of lines of force, while helpful in thinking about electric fields, is nevertheless a completely artificial way of describing the situation. However, because we have specified what we mean by N in a precise mathematical form, we are entitled to make use of any conclusions we have obtained with the help of lines of force. Let us combine Eqs. (10.9) and (10.10), yielding

$$q = \epsilon_0 \oint_{\text{closed surface}} E_n dA \ . \qquad (10.11)$$

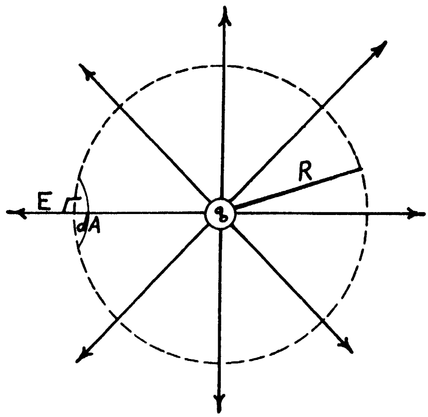


Fig. 10-7

This equation states that when the electric field intensity normal to some closed surface is integrated over that surface and multiplied by ϵ_0, the result is exactly equal to the charge enclosed within the surface. While we have not actually proved that this is correct except for the situation where the surface we integrate E_n over is a sphere with q at its center, it is possible to show that the statement of Eq. (10.11) is indeed true no matter what the shape of the surface A or the location within A of the various charges whose sum is q. Eq. (10.11) is called *Gauss' theorem.*

Gauss' theorem is chiefly useful in determining the electric fields surrounding symmetrical charge distributions. Consider, for example, the electric field of an infinite uniform plane sheet of charge covering the entire xy plane (Fig. 10-8). The charge density is σ coulombs/m^2. Let us choose a cylinder as the closed Gaussian surface to integrate E_n over. The electric field intensity is the same on both sides of the sheet of charge, and is everywhere directed parallel to the z axis because the components of **E** perpendicular to this axis cancel out. **E** has no component perpendicular to the sides of the cylinder, and we need integrate only over its ends. The result is that

$$q = 2\epsilon_0 \int_0^A E_n dA = 2\epsilon_0 EA \ ,$$

where A is the area on one of the ends. Since the charge enclosed by the cylinder is σA,

$$E = \frac{\sigma}{2\epsilon_0} \ . \qquad (10.12)$$

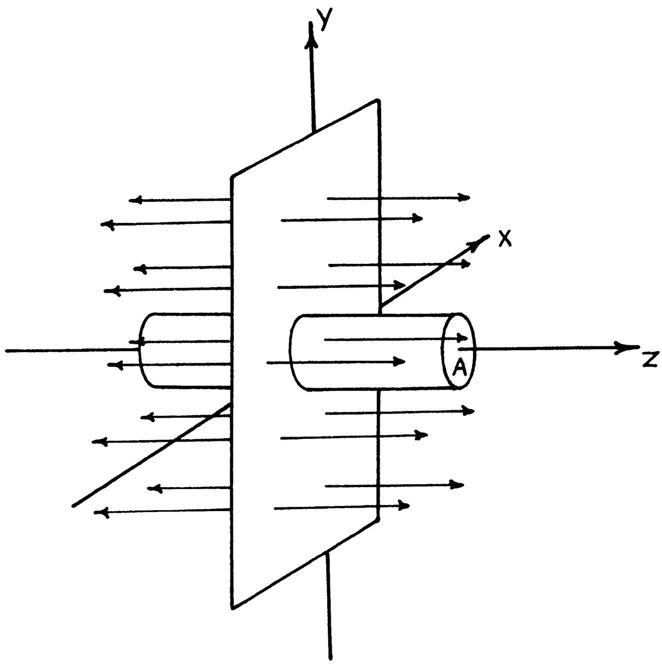


Fig. 10-8

The electric field of an infinite plane sheet of charge is proportional to the charge density only, and is constant everywhere regardless of distance from the sheet. Eq. (10.12) is approximately valid near a finite charge sheet provided that the perpendicular distance from the point in question to the sheet is small compared with the distance from that point to the edges of the sheet.

10.6. Charged Conductors. We can use Gauss' theorem to investigate the final location of charges placed on a conducting body. Ultimately the charges must come to rest somewhere in or on the body. This conclusion follows from the experimental facts that charge is always associated with mass (the moving charges in a conductor are electrons, and a deficiency of electrons acts as a positive charge and a surplus as a negative charge), and that, as was mentioned earlier, even the best natural conductors exhibit some resistance to the motion of charges in them. Under these circumstances the principle of conservation of energy would be violated if the charges were to continue to move about indefinitely. When the charges slow down and stop, their final arrangement must be one in which there is no force on any of them. But if $\mathbf{F}$ is zero, since

$$E = \frac{F}{q}$$

there can be no electric field within the conducting body. Now we draw a Gaussian surface just underneath the surface of the conductor (Fig. 10-9). $E=0$ at this surface, and therefore there can be no charge inside it. It can only be inferred from this that when charge is present in a conductor it must reside on the *outside surface* of the conductor.

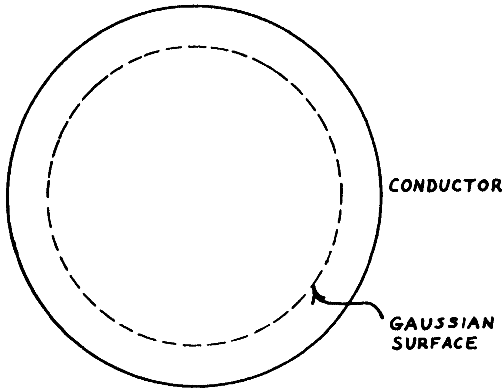


Fig. 10-9

A single thin sheet of charge, such as the one discussed in the previous section, can be prepared only by depositing charge on an insulator. If we electrify a thin infinite plane conductor, the charge distributes itself evenly over *both* sides. In effect, what we now have are two sheets of charge, each of density σ, and each giving rise to an electric field of magnitude $\sigma/2\epsilon_0$. Thus the electric field on either side of the conductor is that due to two sheets of charge, and

$$E = 2\frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} . \qquad (12.13)$$

Eq. (12.13) may be used to determine the electric field intensity near any charged conductor provided, as before, that the point we are interested in is sufficiently close to the conductor to permit disregarding any deviation from an infinite plane surface.

Example. Find the electric field intensity around a charged conducting sphere.

Solution. Here the charge distributes itself evenly over the sphere's surface and the surrounding electric field is, from symmetry together with the vector nature of **E**, everywhere directed radially. (It is outward when the charge is positive and inward when it is negitive.) We draw our Gaussian surface as a sphere of radius r outside the charged sphere. Because **E** is radial it is perpendicular to this surface, and so

$$q = \epsilon_0 \int_0^{4\pi r^2} E_n dA = 4\pi\epsilon_0 r^2 E$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} ,$$

where q is the total charge on the sphere. This result means that a charged conducting sphere sets up an external electric field identical with that set up by a single charge equal to the total charge on the sphere located at the center of the sphere. The radius of the sphere is not relevant, except that, of course, inside the sphere **E**=0. A calculation identical with this one but applied to gravitational fields shows that the gravitational field of a sphere is the same as that of a particle of equal mass located at the center of the sphere.

10.7. Potential Difference. Problems involving vector fields are often complicated to deal with. In the case of electric fields, **E** is seldom constant over the region in which we might, for instance, wish to investigate the motion of a charged particle, and in order to determine the effect of the field on the particle we would have to perform an integration over the varying field taking into account its vectorial properties. In fact, as we have seen, merely computing the magnitude and direction of the electric field due to a distribution of charges is involved enough when treating a symmetrical system, and may become exceedingly laborious in more general situations. What we shall do in order to overcome this difficulty is analogous to our prodedure in mechanics, were we introduced the *scalar* concept of energy. In a gravitational field we expressed the interaction of the field and a body within it in terms of the *potential energy* of the body; work must be done in order to increase the potential energy of the body, while any decrease in the potential energy manifests itself as kinetic energy of motion or as external work. When any force field is *conservative*, all work done on a body against the field is recoverable as kinetic energy which in turn can do precisely the initial amount of work. Electric fields are conservative, and we shall now reformulate our discussion of them in terms of a scalar quantity closely related to potential energy.

We define the potential difference V_{AB} between two points A and

B in an electric field as the work that must be done in order to move a unit positive charge from A to B. That is,

$$V_{AB} = \frac{W_{AB}}{q} \quad . \tag{10.13}$$

If the potential difference is positive, we have done work on the charge and its potential energy at B is greater than it was at A; if the potential difference is negative, the electric field has done work on the charge and its potential energy at B is lower than it was at A. V is expressed in joule/coulomb, from its definition, and is so often used that it has been given the special unit of the *volt,* where

$$1 \text{ volt} = 1 \text{ joule/coulomb.}$$

Let us calculate the potential difference between two locations, A and B, near a point charge q (Fig. 10-10). Our procedure is to take a test charge q' from A to B and evaluate

$$W_{AB} = \int_A^B \mathbf{F}\cdot d\mathbf{s} \quad .$$

The electrical force on q' is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2}\,\mathbf{r}_1 \quad ,$$

and the force we must use in finding W is $-\mathbf{F}$, the one we must actually supply. This force is radial, and since the component of $d\mathbf{s}$ in the radial direction is dr the scalar product $\mathbf{F}\cdot d\mathbf{s}$ is just Fdr. Consequently

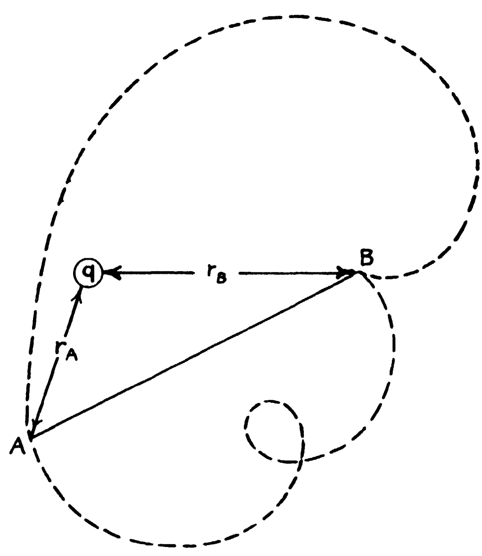


Fig. 10-10

$$W_{AB} = -\int_{r_A}^{r_B} \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2}\,dr$$

$$= \frac{qq'}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \quad ,$$

and the potential difference between A and B is

$$V_{AB} = \frac{W_{AB}}{q'} = \frac{q}{4\pi \epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) . \qquad (10.14)$$

Because V_{AB} depends only upon r_A and r_B we could have used *any* other path from A to B, even the ones represented by dashed lines in Fig. 10-10, which is characteristic of conservative force fields.

Example. An electron having the initial velocity 10^3 cm/sec is directed at another electron, whose position is fixed, from a distance of 1 cm. How close to the stationary electron will the other one approach before it stops and reverses its path?

Solution. As the moving electron travels toward the stationary one its kinetic energy is transformed into potential energy. In effect, i does work on the electric field of the fixed electron. Finally, when all of the initial kinetic energy is gone, the electron stops and, like a stone thrown upward, begins moving in the direction it came from. Letting r_A be the initial separation of the electrons and r_B the final one, and being careful to have all quantities in mks units,

$$eV_{AB} = m_e v^2/2$$

$$\frac{e^2}{4\pi \epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) = m_e v^2/2$$

$$r_B = \frac{e^2 r_A}{2\pi \epsilon_0 m_e v^2 r_A + e^2}$$

$$= 3.4 \times 10^{-1} \text{ m} = 0.34 \text{ cm}.$$

10.8. Potential. When we were dealing with the gravitational field of the earth, it often proved helpful to define the potential energy of any object at the earth's surface as zero. This enabled us to express the potential energy of an elevated object as simply mgh, instead of the more general expression $mg(h-h_0)$ in which we would always have to specify a reference height h_0. We can do the same thing in the case of electric fields, where it is most convenient to choose as the zero of potential its value at infinity, that is, its value an infinite distance from all charges. Adopting this convention, we may now speak of the potential at a point, with the understanding that what we really mean is the difference between the potential there and at infinity. For the potential of a point charge, letting $r_A = \infty$ and $r_B = r$,

$$V = \frac{q}{4\pi \epsilon_0 r} . \qquad (10.15)$$

The potential at a point due to a number of charges is just the algebraic sum of the individual potentials there, since V, unlike **E**, is a scalar, so that

$$V = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} \quad , \tag{10.16}$$

where r_i is the distance from the ith charge to the point in question.

It is interesting to investigate the different aspects of an electric field revealed by the field intensity **E** and the potential V. Consider first a point midway between two identical positive charges a distance d apart (Fig. 10-11). Here

$$E = 0$$

$$V = \frac{q}{\pi\epsilon_0 d} \quad .$$

Fig. 10-11

If we replace one of the charges by a negative charge of the same magnitude,

$$E = \frac{2q}{\pi\epsilon_0 d^2}$$

$$V = 0 \quad .$$

In the first case a test charge placed at the specified point would feel no force, but work must have been done in bringing it to that position. Because electric fields are conservative, the same amount of work is recoverable. In the second case a test charge would certainly feel a force, but owing to its location no *net* work had to be done to bring it there.

E is always directly proportional to the force a test charge would experience, and V to the potential energy the charge would possess. We can therefore express both sides of the equation

$$dW = \mathbf{F}\cdot d\mathbf{s}$$

in electrical terms as

$$q\,dV = -\,q\,\mathbf{E}\cdot d\mathbf{s} \quad ,$$

where the minus sign enters because work is being done against the electric field. Writing E_s for the component of **E** in the direction of **ds**,

$$dV = -\,E_s ds$$

and

$$E_s = -\frac{\partial V}{\partial s} \quad . \tag{10.17}$$

The component of electric field intensity in any direction is the negative of the partial derivative of the potential in that direction. (In calculating a partial derivative with respect to a particular variable, the rules for ordinary differentiation are followed except that the other variables are treated as constants.) In a Cartesian coordinate system the entire field E is given by

$$E_x = - \frac{\partial V}{\partial x}$$

$$E_y = - \frac{\partial V}{\partial y} \qquad (10.18)$$

$$E_z = - \frac{\partial V}{\partial z} \quad .$$

Example. One of the most important charge configurations in physics is the *dipole*, which consists of two equal and opposite charges a certain distance apart. Find the potential at a point whose distance from the dipole is large compared with the separation of the charges, and from this determine the electric field of the dipole.

Solution. In Fig. 10-12 the charges have been placed at $x=-d$ and $x=+d$, and the point P is in the xy plane. (The results will be no different if P has a z coordinate as well, but the calculation is more complicated in that case.) The potential at P due to the two charges is

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right) ,$$

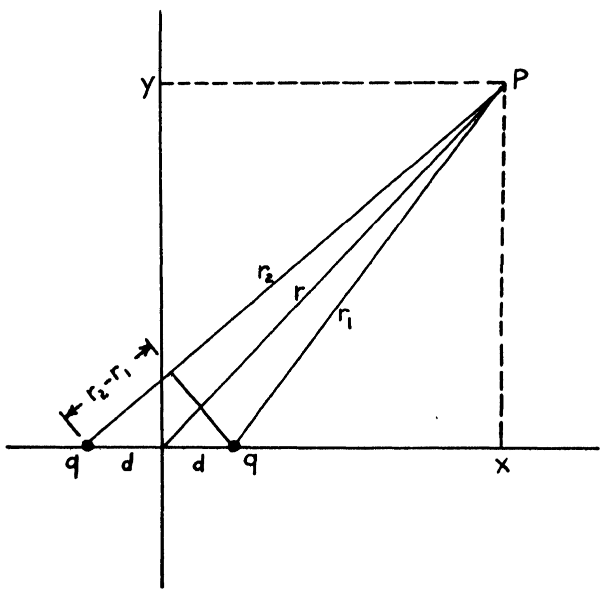


Fig. 10-12

which is

$$V = \frac{q}{4\pi\epsilon_0}\,\frac{(r_2 - r_1)}{r_1\, r_2} \quad .$$

When $2d$ is much smaller than r_1 and r_2,

$$\frac{r_2 - r_1}{r_1 r_2} \approx \frac{r_2 - r_1}{r^2} \quad ,$$

where the symbol $\approx$ means "is approximately equal to" and r is the distance from the center of the dipole to P. From the diagram

$$r_2 - r_1 \approx 2d\cos\theta \ ,$$

and

$$V = \frac{2dq\cos\theta}{4\pi\epsilon_0 r^2} \quad .$$

The quantity $2dq$, which is the product of the magnitude of one of the charges and the distance between them, is called the *dipole moment* p. In terms of p,

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$= \frac{px}{4\pi\epsilon_0 (x^2+y^2)^{3/2}} \quad .$$

The components of the electric field at P are therefore

$$E_x = -\frac{\partial V}{\partial x} = \frac{p}{4\pi\epsilon_0}\,\frac{2x^2 - y^2}{(x^2+y^2)^{5/2}}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{p}{4\pi\epsilon_0}\,\frac{xy}{(x^2+y^2)^{5/2}}$$

$$E_z = -\frac{\partial V}{\partial z} = 0 \ .$$

An imaginary surface in space all of whose points are at the same potential is called an *equipotential surface*. The surface of a conductor, for example, is always an equipotential surface because the charges on it have distributed themselves to that the electric field is perpendicular to the surface at the surface (Sec. 10.6). If a test charge is moved about on the surface of a conductor, then, the work done is

$$W = q\int \mathbf{E}\cdot d\mathbf{s} = 0$$

since $\mathbf{E}$ is perpendicular to $d\mathbf{s}$. Equipotential surfaces are always everywhere perpendicular to the lines of force of $\mathbf{E}$. Fig. 10-13 shows several field configurations, with the lines of force shown as solid lines and the intersections of the equipotential surfaces with the plane of the paper as dashed lines.

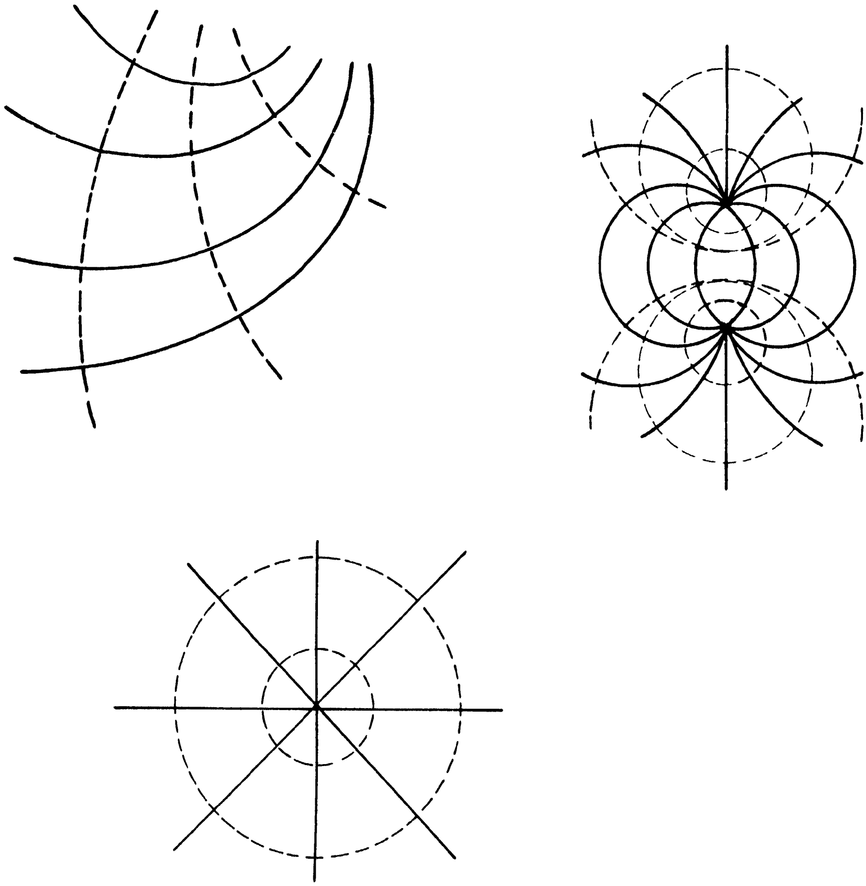


Fig. 10-13

Chapter 11

ELECTRIC CURRENT

An electric current consists of moving charges. Previously we have discussed various ways of describing the electric field of static distributions of charge, a subject in itself of little significance. However, the interaction of electric fields with charges that are free to move is of great importance from a practical point of view as well as from the point of view of understanding physical phenomena, and a substantial part of the remainder of this book is devoted, in one way or another, to discussing electric currents and their effects. In this chapter we shall restrict ourselves to steady currents in conductors.

11.1. Electromotive Force. The magnitude i of an electric current is, by definition,

$$i = \frac{dq}{dt} \quad , \tag{11.1}$$

where dq/dt is the rate at which charges move through some specified surface in space. In the case of a conducting wire, where the charges are restricted to the wire, we may speak of current as the rate at which charge travels past a point in the wire. From Eq. (11.1) the unit of current is the coulomb/second, which is called the *ampere*. The direction of the current i is conventionally taken as the direction of *positive* current flow, that is, the direction in which positive charges would have to move to produce the observed current. Actually, of course, currents in metallic conductors consist of the flow of negative electrons, so that we must be careful to distinguish between "conventional current" and "electron current." The use of conventional current, which we shall call simply current from now on, is so ingrained in present science and technology that there seems little hope of dislodging it in favor of the more realistic electron current.

What causes a current to flow? According to the way we have defined potential, positively-charged particle that are free to move travel from regions of high potential to regions of low potential. A conductor is a source of charges which are free to move, and therefore if opposite ends of a conducting object are initially at different potentials, current will flow in order to equalize them. Unless the difference in potential between the ends of the conductor is maintained by some external means, the entire conductor ultimately reaches the same potential and the current will cease. However, if we remove the positive charges from the low-potential end as fast as they arrive and do work on them by bringing them back to the high-potential end to replenish the supply there (Fig. 11-1), a steady-state situation results in which the current is constant and does not change with time.

Every time a charge is taken around the circuit work is done on it. An electrostatic field cannot provide this work since it is conservative; the net work done by a conservative force field over a closed path is zero. To maintain a steady current, then, we require non-electrostatic sources of electric field, which are called sources of electromotive force. Such sources are commonly called *batteries*, which convert chemical energy into electrical energy, and *generators*, which convert mechanical energy into electrical energy. (Other sources, for example *thermocouples* which convert heat into electrical energy, also exist, but are of less practical importance.) Electromotive force is usually abbreviated emf. The electromotive force in a circuit is the amount of work done per unit positive charge by the sources of emf on the charges in a circuit.

Even though work is done on charges in a circuit by the sources of emf, we know from experience that energy does not accumulate there. The energy supplied by a battery to an external circuit is dissipated in the circuit, partly as heat and partly as electromagnetic radiation (both of which we shall discuss subsequently). As in electrostatics, the amount of energy per unit positive charge dissipated between two points in the external circuit is called the *potential difference* between those points. The distinction between emf and potential difference is that in the former case work is done on the charge, whereas in the latter case the charge does work on the circuit. Both of these quantities are measured in *volts* since they are work per unit charge.

This distinction is important owing to the fact that energy dissipation may occur *within* the source of emf as well as outside it.

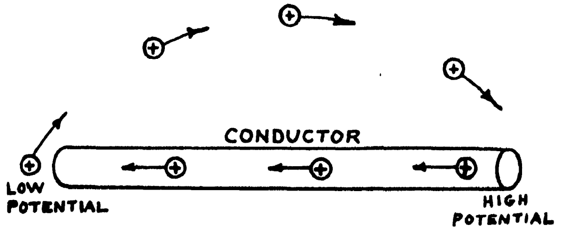


Fig. 11-1

When a battery causes a current to flow, its emf is equal to the entire potential drop in the complete circuit; however, because the battery itself is part of the circuit, the potential difference between the terminals of the battery is less than the emf of the battery.

11.2. Ohm's Law. There is a simple empirical relationship, discovered by Ohm, between the potential difference V across the ends of a conductor and the current i flowing through that conductor. It is

$$V = iR , \tag{11.2}$$

where R is a constant of proportionality called the *resistance* whose value depends upon the characteristics of the conductor. When V is in volts and i in amperes, R is expressed in *ohms*.

The resistance of a piece of wire (the usual conducting body used in electrical circuits) depends upon both the geometry of the wire and the material of which it is composed. We can separate these factors by considering an infinitesimal cube of the conductor $dxdydz$ with current flowing through it in the x direction. If dR is the resistance of the cube, the *resistivity* ρ of the material is defined as

$$\rho = dR \frac{dydz}{dx} , \tag{11.3}$$

and is found to be constant for a given material. Table 11-1 gives the resistivities of some common substances. In terms of resistivity, the resistance of a wire of length L and uniform cross-section A is

$$R = \rho \frac{L}{A} . \tag{11.4}$$

In terms of resistivity, Ohm's law for an infinitesimal cube is (Fig. 11-2)

$$-dV = i \frac{\rho}{dydz} dx .$$

Table 11.1

Resistivities of Metals at 20°C.

metal	resistivity
aluminum	2.83×10^{-6} ohm-cm
copper	1.72
gold	2.44
lead	22
mercury	96
silver	1.63

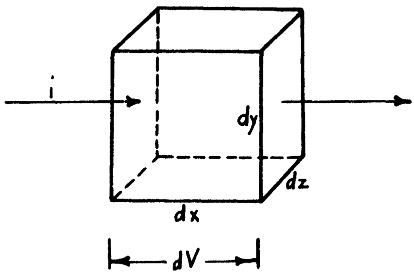


Fig. 11-2

The minus sign is inserted since V is decreasing in the direction of current flow. We have then

$$-\frac{dV}{dx} = \frac{i}{dydz}\,\rho.$$

We now define the *current density* j as the vector of magnitude

$$j = \frac{i}{dydz}$$

and with the direction of the current flow. Since $dV/dx = -\mathbf{E}$,

$$\mathbf{j} = \mathbf{E}/\rho$$

or defining the *conductivity* of the material as $\sigma = 1/\rho$,

$$\mathbf{j} = \sigma\,\mathbf{E}. \tag{11.5}$$

Thus the current density at any point within a conductor is equal to its conductivity multiplied by the electric field intensity there.

In some substances the resistivity increases with increasing temperature, while in others it decreases. The approximate formula for the resistivity ρ of a conductor or a semiconductor at a temperature T above some reference temperature at which the resistivity is ρ_0 is

$$\rho = \rho_0(1 + \alpha T), \tag{11.6}$$

where α, which may be either positive or negative, is independent of temperature within a certain range of temperatures.

In metals ρ is positive, so that resistivity increases with temperature. The reason for this is that the higher the temperature, the larger the amplitude of the vibrations of the atoms constituting the metal. These vibrating atoms impede the flow of the electrons that carry the current, increasing the resistivity. Semiconductors, on the other hand, are characterized by a far smaller supply of free electrons than metals. In them the principal effect of a higher temperature is to increase the number of electrons available for carrying current, which more than counterbalances the increased interference of the vibrating atoms. Hence the resistivity goes down with temperature.

11.3. Resistors in Series and Parallel. For most purposes it is convenient to consider the various resistive elements in an electrical circuit in the form of individual resistors connected together by resistanceless wires. There are many ways in which resistors can be put together, but almost all of them consist of a combination of *series* (Fig. 11-3) and *parallel* (Fig. 11-4) arrangements. It is often helpful to know the *equivalent resistance* of a group of resistors in some particular arrangement, that is, the value of the single resistor that can be substituted for the group without changing the characteristics of the circuit.

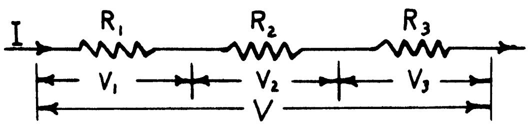


Fig. 11-3

Let us first examine three resistors, R_1, R_2, and R_3, connected in series as in Fig. 11-3. The potential differences across the resistors are V_1, V_2, and V_3 respectively, and V, the potential difference across the combination, is

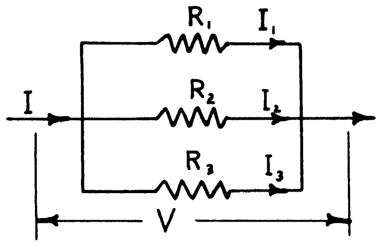


Fig. 11-4

$$V = V_1 + V_2 + V_3 \ .$$

The same current I flows through all of the resistors. Therefore, by Ohm's law, $V_1 = IR_1$, $V_2 = IR_2$, and $V_3 = IR_3$.
If the equivalent resistance of the combination is R_S, the total potential difference across the three resistors in series is $V = IR$. Hence

$$IR_S = IR_1 + IR_2 + IR_3 \ ,$$

and, dividing by the common factor I,

$$R_S = R_1 + R_2 + R_3 \ . \tag{11.7}$$

Hence in a series arrangement the equivalent resistance is the sum of the individual resistances.

In Fig. 11-4 these resistors are connected in parallel. Here they all have the same potential difference V across their terminals, but each carries a different current. The total current I is the sum of the individual currents, that is,

$$I = I_1 + I_2 + I_3 \ .$$

Again applying Ohm's law to each resistor, $I_1 = V/R_1$, $I_2 = V/R_2$, and $I_3 = V/R_3$. In terms of the equivalent resistance R_p the total current I is $I = V/R_p$. Combining these equations,

$$\frac{V}{R_p} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} ,$$

so that

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} . \tag{11.8}$$

In a parallel arrangement the reciprocal of the equivalent resistance is the sum of the reciprocals of the individual resistances. Here the equivalent resistance is always less than the lowest resistance present.

Example. An 11 ohm resistor is connected to a battery having an emf of 6 volts and an internal resistance of 1 ohm. How much current flows in the circuit, and what is the potential difference across the terminals of the battery?

Solution. The total resistance in this circuit consists of the resistor R and the internal resistance r in series. Hence

$$I = \frac{E}{R + r} = 0.5 \text{ amp.}$$

The potential difference V across the terminals of the battery may be found in two ways. We can determine V from the iR drop in the external resistor, so that

$$V = iR = 5.5 \text{ volts,}$$

or we can subtract from the battery emf of 6 volts the ir drop in its internal resistance, so that

$$V = E - ir = 5.5 \text{ volts.}$$

Note that the potential difference is less than the emf.

Example. Find the equivalent resistance of the circuit in Fig. 11-5.

Solution. The method of attacking this problem is shown schematically in Fig. 11-6. First we compute the equivalent resistance R' of R_2 and R_3 in parallel:

$$R' = \frac{R_2 R_3}{R_2 + R_3} = \frac{6}{7} = 0.86 \text{ ohms.}$$

R' and R_1 are in series, and their equivalent resistance R'' is just

$$R'' = R' + R_1 = 3.86 \text{ ohms.}$$

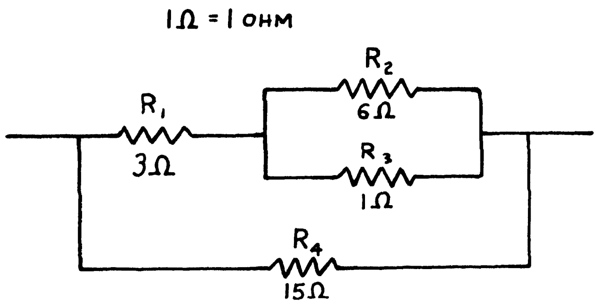


Fig. 11-5

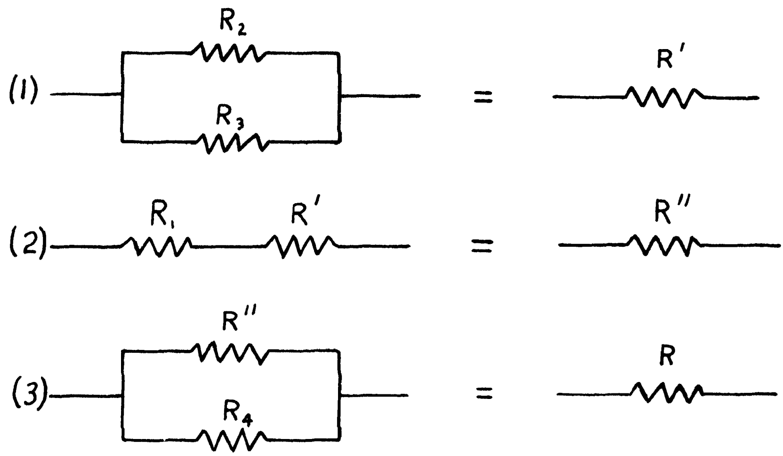


Fig. 11-6

Now, in step (3), R'' and R_4 are in parallel, so the equivalent resistance R of the entire circuit is

$$R = \frac{R''R_4}{R'' + R_4} = 3.06 \text{ ohms.}$$

Example. Find the current through R_3 in the above example if the entire circuit has a potential difference of 10 volts placed across it.

Solution. The current through R'' is, by Ohm's Law,

$$I'' = \frac{V}{R''} = \frac{10}{3.86} = 2.59 \text{ amperes.}$$

This current goes through both R_1 and r' and so the voltage drop V' across the equivalent resistance R' is

$$V' = I'R' = 2.59 \times 0.86 = 2.23 \text{ volts.}$$

Hence the current through R_3 is given by

$$I_3 = \frac{V'}{R_3} = \frac{2.23}{1} = 2.23 \text{ amperes.}$$

In solving this problem we have, in effect, gone backward from the single equivalent resistance to the combination of R_1 and R', and then to the original circuit. This procedure is characteristic when currents through individual resistors in a circuit are to be found.

11.4. Power. Energy is dissipated by the moving charges that constitute a current in an electrical circuit as rapidly as the sources of emf in the circuit do work on them. The energy dW lost by a charge dq in moving through a potential difference V is

$$dW = Vdq.$$

Hence the rate of energy loss dW/dt, which equals the power input P, is

$$P = \frac{dW}{dt} = V\frac{dq}{dt} = Vi.$$

When V is in volts and i in amperes, P is in watts, the mks unit of power. Applying Ohm's law, Eq. (11.9) may be expressed in the three equivalent forms

$$P = Vi = i^2R = \frac{V^2}{R}, \tag{11.9}$$

in which R is the resistance of the part of the circuit under consideration.

Example. An incandescent lamp of resistance 144 ohms is placed across a potential difference of 120 volts. How much heat, in calories, does it produce in 1 hour?

Solution. Since

$$P = V^2/R,$$

the rate at which the lamp develops heat is 100 watts. There are 3600 sec/hr, and so

$$W = \int Pdt = 3.6 \times 10^5 \text{ joules.}$$

The mechanical equivalent of heat is 4.186 joules/calorie, with the result that

$$W = \frac{3.6 \times 10^5 \text{ joules}}{4.186 \text{ joules.cal}} = 8.6 \times 10^4 \text{ calories.}$$

11.5. Kirchhoff's Laws. It is not always easy to determine the currents that flow in various parts of a complicated circuit. By applying the principles of conservation of charge and conservation of energy to a circuit in a systematic manner, however, even the most obdurate such problem can be made tractable.

Let us being with the fact that the total charge in the universe is conserved. A consequence of this is that if we take a volume element somewhere, the difference between the current that flows into it in a given time and the current that flows out of it in the same time equals the net charge that has accumulated there. From experiment it is known that charge never accumulates at any point in a circuit. Thus *the total current flowing into a junction of two or more wires is equal to the total current flowing out of the junction.* This statement is called Kirchhoff's first law.

We have learned that when a current passes through a resistor a certain amount of energy is dissipated. In order to maintain the current at a constant value in a closed loop, then, the total energy lost in the resistances must be equal to the work done by the sources of emf in the loop. From this we conclude that in going around a *closed loop the sum of the potential differences is equal to the sum of the emf's in the loop,* which is Kirchhoff's second law. In tracing a circuit it is essential to keep in mind the sign conventions: (1) a potential difference is taken as positive in the direction the current is assumed to flow, and negative in the direction opposite to the current; and (2) an emf is taken as positive if we meet the negative terminal in going around the loop, and negative if we meet the positive terminal.

In actually applying Kirchhoff's laws, the first step is to *assume* a current direction in each loop. (If we have chosen the correct one, when we solve for its value a positive figure will result. If we have made an incorrect choice, we will find a negative figure of the correct magnitude in which the minus sign reveals our mistake.) Having assumed currents in each loop, we now apply Kirchhoff's first law to each junction, which provides a number of equations relating the various currents. Then, applying Kirchhoff's second law to each loop, we have additional equations, this time relating the currents, emf's, and resistances in the circuit. In all there will always be as many equations as there are unknown currents in the typical problem in which the emf's and resistances are given, and solving the problem is merely a matter of algebra.

Example. In the circuit shown in Fig. 11-7, find the current through each resistor, (External resistances are labelled R_i and internal ones r_i.)

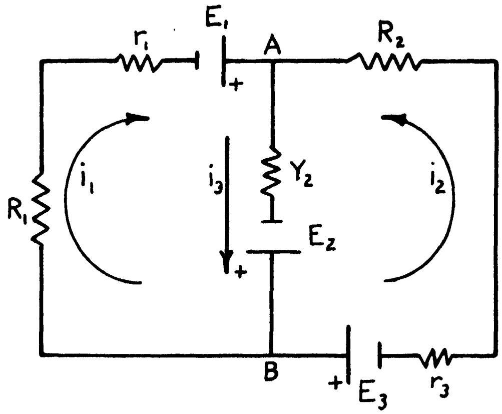


Fig. 11-7

Solution. Applying Kirchhoff's first law to junction A or B,

$$i_1 + i_2 = i_3 \ .$$

In the left-hand loop Kirchhoff's second law gives

$$i_1 R_1 + i_1 r_1 + i_3 r_2 = E_1 + E_2 ,$$

and in the right-hand loop

$$i_3 r_2 + i_2 r_3 + i_2 R_2 = E_2 - E_3 \ .$$

These equations may be solved for i_1 and i_2 with no difficulty.