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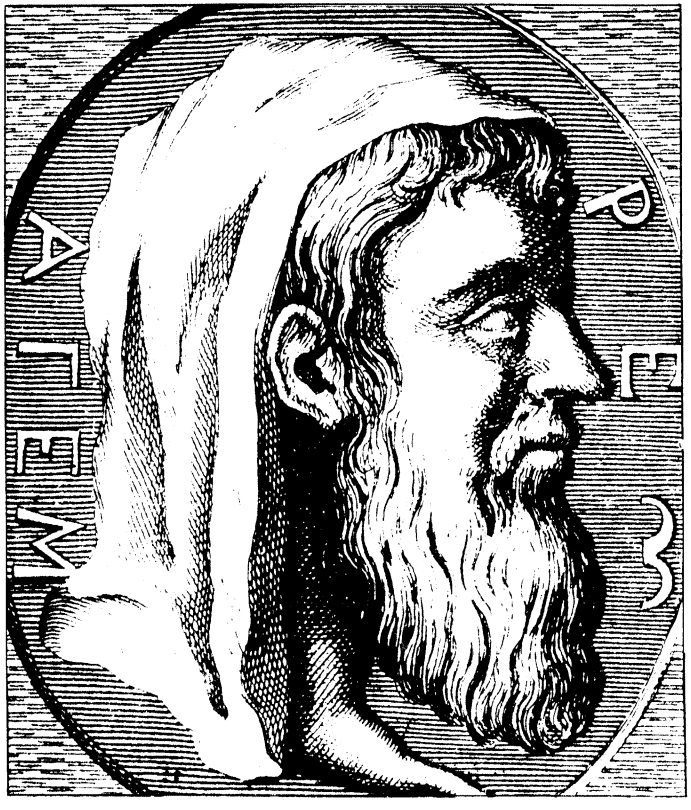

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GEOMETRY

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PREFACE

It is a common belief that plane geometry was completed by Euclid 2000 years ago and that nothing has been added to it or taken from it since. This is simply not true. That there are logical gaps in Euclid's presentation has been known for a long time. Means to remedy these deficiencies have been known for about sixty years, but strangely enough a mathematically adequate and yet elementary treatment of plane geometry in the spirit of Euclid has not appeared in print. This text represents an earnest effort to do just this. The current interest in improvement of the secondary school curriculum makes this an appropriate time for such an attempt.

This book, in earlier unpublished versions, has been taught to *normal* classes in *average* high schools for five years. Many of the classes were taught by one of us in the Burris School, the laboratory school of Ball State Teachers College, but the material has been taught in other schools by teachers who had had no previous special training in geometry. Our experience is that both student and teacher find this treatment of geometry stimulating. The good students become excited about mathematics and the students of average ability learn the basic geometric facts.

Besides its mathematical completeness, an attribute that makes for clarity and ease of understanding, there are several features which we think important.

There is a full chapter on logic early in the text. It pays the teacher to return to this logic again and again in order to clarify the principles underlying deductive proof. For the nonscience student, this chapter on logic may be the item of greatest permanent value.

The postulates are presented in groups, as needed, with a resulting gain in clarity and simplicity. Although it is true that at the start most students are hardly "provers," by the end of the year a considerable part of the class will exhibit real mathematical perception. In the early stages the small number of postulates makes proofs easier to construct.

Definitions are given explicitly; that is, they are formally labeled as such. It is one of our prime objectives to get the student to see that precise definitions are necessary for proofs.

There is plenty of material for a full year's work for all kinds of classes. More difficult sections, theorems, or exercises are marked with an asterisk, and some of the more difficult theorems may be treated as postulates. The chapters on analytic geometry, locus, solid geometry, and philosophy of mathematics provide supplementary material beyond the standard course. The chapter on solid geometry alone could occupy a good portion of a semester if desired.

There is a review of important concepts at the end of each chapter. In addition, there are algebra review sections, designed not only to refresh skills but also to bring out the logical structure of algebra.

The Appendix lists all the postulates, plus the principal definitions and theorems. Besides being a convenient reference, the Appendix is worth studying during the year, for it gives, in brief, the step-by-step construction of the full geometry.

We believe that the text raises students' reading ability. There is little verbiage. Each section has a definite purpose, and the text is directed to this purpose. Good classroom discussion, with careful attention to the use of words, will add much to each student's ability to express himself clearly.

A separate Teachers' Manual gives greater detail on selection of topics, emphasis, and time schedule. There is also an Answer Book for the exercises.

It is a great pleasure to thank all those who have helped us with encouragement or constructive criticism. Special thanks are due to Dr. Curtis Howd, Principal of the Burris Laboratory School, who encouraged us and made possible the experimental sections; to Professor Jack Forbes, who observed many of the classes in various schools and who insisted that the mathematics be correct; to teachers at the Burris School, and many others, who by teaching the material have improved the exposition; to The National Science Foundation, which, through a grant for in-service institutes at Ball State, enabled us to reach a wider audience and see the text taught by many teachers in a variety of school situations; and finally, to our wives, who have waited impatiently for the publication day.

January 1960 C. B., R. E., M. S.

FOREWORD TO THE STUDENT

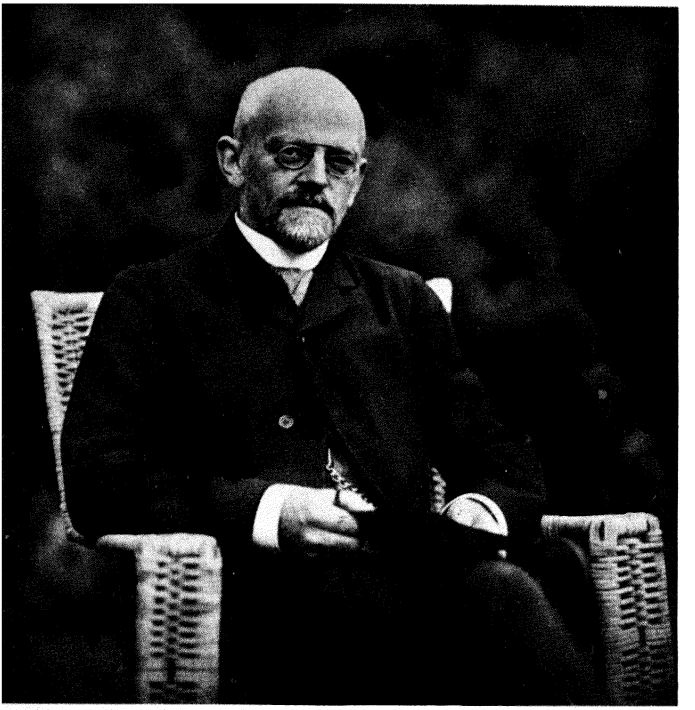
Mathematics is much more than a collection of interesting facts and useful tools for solving special problems. Mathematics is primarily concerned with the *development* of important ideas. Frequently mathematicians develop an interesting mathematical theory that is apparently useless so far as the solution of practical problems is concerned. Later someone finds that this mathematics fits very nicely into the pattern of some complicated practical problem. The mathematical theory then becomes a useful tool.

As a branch of mathematics, geometry is first of all concerned with the development of geometrical theory. As you study geometry you will see again and again how the geometrical concepts fit into patterns in the real world and become helpful in the solution of practical problems.

Mathematics requires thought, and thinking deserves your best effort. Do not hesitate to ask questions.

Good luck,

The Authors



DAVID HILBERT
(*1862–1943*)
Professor of Mathematics,
University of Göttingen, Germany
Perhaps the world's foremost mathematician during the first half of the twentieth century.
Courtesy of the New York Public Library.

CHAPTER 1

INTRODUCTION

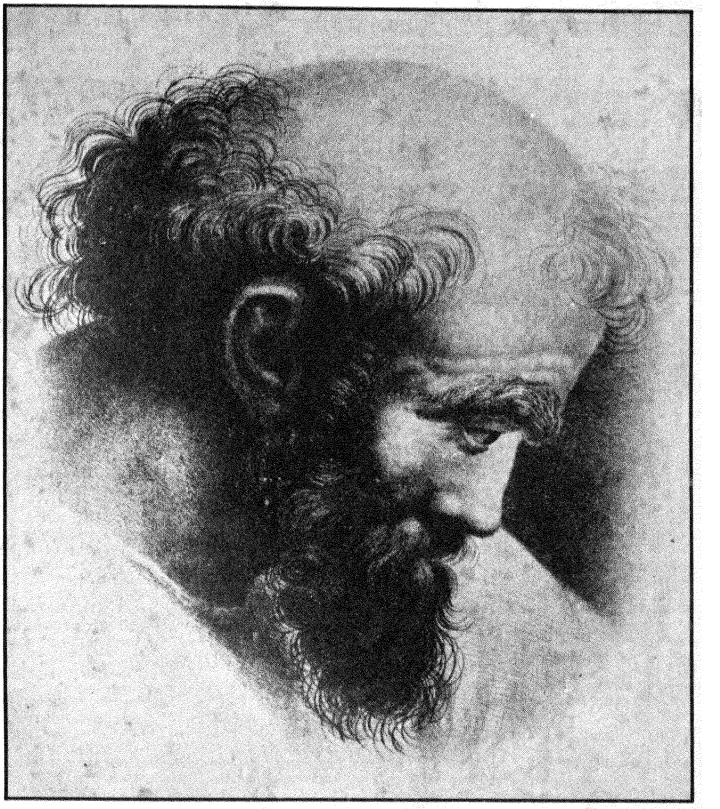
1-1 WHY STUDY GEOMETRY?

There are at least four reasons why geometry is studied by so many students in the high schools of the United States and, in fact, throughout the world. The first of these reasons is that the logical beauty and precision of geometry has fascinated men since the time of the ancient Greeks. Geometry is one of the great achievements of the human mind, and for 2000 years men have regarded its study as a necessary part of the making of a truly educated person. The feeling has been that here is at least one thing whose truth is clear, because every statement can be demonstrated beyond a doubt. People also felt that because the reasoning in geometry had to be careful and accurate, the study of it would help one to be careful and accurate in other activities.* Because this great invention of the mind has value as an introduction to correct logical reasoning, geometry is a required subject for entrance to many colleges regardless of what one plans to study.

A second important reason for studying geometry is its practical importance. People in all kinds of occupations have occasional need for a bit of geometry, and in some fields the study of geometry is a most important step in professional training. Fields in which it is required are physics, chemistry, engineering, pure mathematics, statistics, some of the biological sciences, and, more recently, certain branches of economics and psychology.

Perhaps most students start the study of geometry because of its practical value or because they are told it is required. But a third reason is that after you have studied geometry you will have the knowledge that is necessary to give you a deeper insight into the wonderful details and complexities of our world, both the natural world and the constructions

* The Dutch philosopher Benedictus de Spinoza (1632–1677) was so impressed by the clarity of geometrical reasoning that he arranged his reasoning on religion in a "geometric style."



PYTHAGORAS
From a fresco by Raphael.
Courtesy of *Scripta Mathematica*.

THE DEATH OF ARCHIMEDES
From a mosiac found in the ancient city of Pompeii.
Courtesy of *Scripta Mathematica.*

One Greek geometer in particular deserves mention here because his fame is so slight compared with his worth. This man is Eudoxus of Cnidus (4th century B.C.), one of the foremost mathematicians of all time. His fame rests upon his general theory of proportion, certain geometric constructions, and his "method of exhaustion." He lived during the time of Plato.*

We find that the Greeks organized and extended geometry for their own mental enjoyment. Deductive geometry (that is, with proofs from simple assumptions) is truly a Greek invention. These men were not seeking practical applications at all; in fact, an educated Greek regarded applications of geometry as beneath his dignity. By the time of Plato (4th century B.C.), Greek scholars had a fairly well-organized body of geometry and perhaps even a geometric text written for members of Plato's academy.

By far the best known of the ancient Greek geometers is Euclid, who wrote the *Elements* about 300 B.C. This work is the most successful textbook of all time and was used throughout the world until well into the present century. It consists of 13 "books," the first six of which are concerned with plane geometry. The others deal with arithmetic and solid geometry. The *Elements* has not only been widely used as written (or as translated); it has also been the model for innumerable other books. Isaac Newton, the renowned English mathematician and physicist, wrote his great book, the *Principia*, in the "geometric style" even though this often tended to hide the path he took in making his discoveries. Most of the geometry books used in American high schools today are somewhat altered and simplified versions of Euclid's *Elements*. It is astonishing, and a great tribute to Euclid, that this book has retained its value for over 2000 years.

Little is known of Euclid's life except that he probably learned his mathematics from pupils of Plato's academy and that he later founded his own school at Alexandria.† Undoubtedly he made full use of earlier writings, but the organization of the *Elements* is his own, as are many significant improvements.

One of the many amusing stories concerning Euclid is the following.

* Plato and Aristotle are the two most renowned of the Greek philosophers. Although Eudoxus is less well known, he was a much better mathematician than either of these men. However, both Plato and Aristotle have had tremendous influence on later thought. It is said that above the entrance to Plato's academy were inscribed the words, "Let no one ignorant of geometry enter here."

† The city of Alexandria was founded by Alexander the Great in 332 B.C. Alexander was a supporter of learning, as was his successor, Ptolemy I. It was Ptolemy who started the great library at Alexandria, and it was during his rule that Euclid opened a school there.

Upon being asked by a student of what profit was the study of geometry, Euclid directed a slave to "give the man a coin that he may profit from what he learns."

Many able Greek mathematicians followed Euclid, among whom were Apollonius (3rd century B.C.) and Archimedes (3rd century B.C.). Archimedes, who had perhaps the greatest mind of ancient times, did fine work in mathematics, physics, and engineering. He was employed as a kind of technical consultant in warfare by the king of Syracuse and was killed by a soldier in the siege of that city in 212 B.C.

With the rise of the Roman Empire, Greece declined as a political power but retained its eminence as a center of learning and culture well into the Christian era. Rome added little (compared with Greece) to the arts, science, or literature. It is in the realm of practical government, warfare, and architecture that the Romans made their contributions. But when Rome fell to the barbarians from the north in the 5th century A.D., the "dark ages" began. During the dark ages, mathematical research was largely in the hands of the Arabs. (Our word *algebra* is of Arabic origin.) Serious scientific work in Europe was almost at a standstill until the Renaissance, or revival of learning, in the 14th century.

Beginning with the 16th century, mathematical activity was primarily in fields other than classical geometry, particularly after the invention of calculus* by Newton and Leibniz in the 17th century. Nevertheless, significant studies in geometry were made which culminated in the discovery, in the 19th century by K. F. Gauss, N. Lobachevski, and J. Bolyai, independently, that geometries other than Euclid's are possible and just as true (that is, logically correct). It was during the 19th century that men became aware that there were certain assumptions made by Euclid which he did not actually state. Therefore Euclidean geometry was re-examined and made complete and precise according to modern standards. The first really complete treatment of Euclidean geometry was published by the German mathematician David Hilbert in 1899. Our book is based upon Hilbert's treatment (with some simplifications). Thus you will be studying geometry that is almost 2300 years old but dressed up in modern form.

Hilbert (who died in 1943 at the age of 81) had wide-ranging mathematical interests and made outstanding contributions to many fields. His influence on American mathematics has been considerable, because at the time when American mathematicians were beginning to achieve eminence, during the latter part of the 19th century, many of them studied under Hilbert at the University of Göttingen in Germany.

---

* Calculus is a powerful mathematical tool useful in solving certain kinds of problems. It is a subject required of most science majors in college.

Suggested Discussion Topics

1. Discuss the four reasons listed for studying geometry. Can you think of any others?
2. What do you know about Greek civilization other than what has been mentioned in this chapter?
3. What were some of the major scientific developments of the 19th century? In what countries did they occur?
4. Look up and report on contributions of the Arabs to mathematics.
5. Look up Euclid in an encyclopedia.
6. Look up Pythagoras in an encyclopedia.
7. Look up Archimedes in *The World of Mathematics*, by J. R. Newman.

1–3 FAMILIAR GEOMETRIC OBJECTS

In your previous study of arithmetic and algebra you have met many "geometric objects," such as the ones shown in Fig. 1–1, where the examples are all *plane* figures. Some *solid* figures that you may have encountered are pictured in Fig. 1–2.

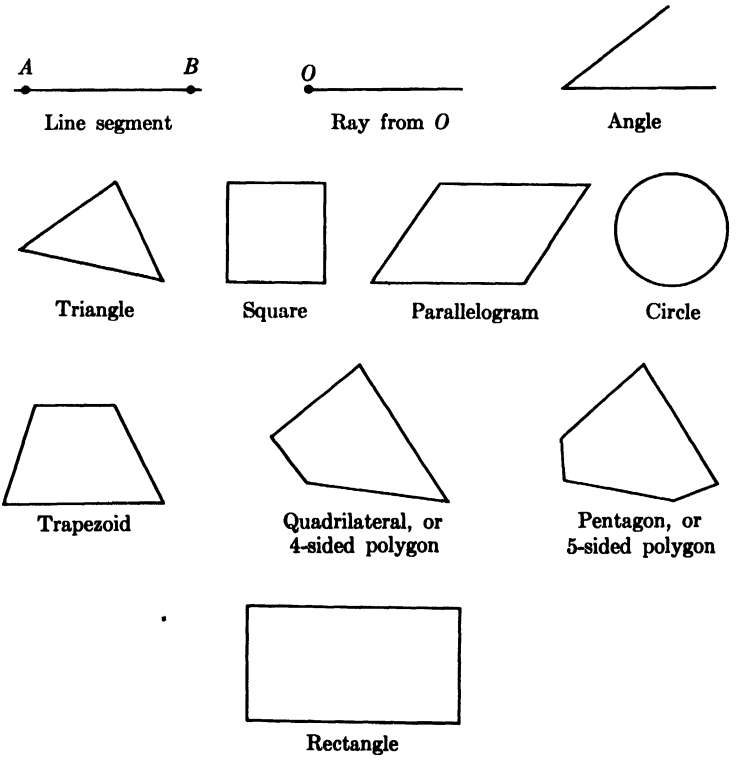
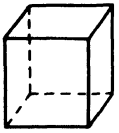
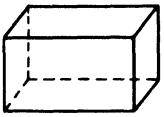


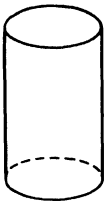
FIGURE 1–1



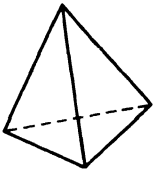
Cube



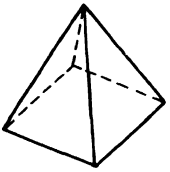
Rectangular parallelepiped or box



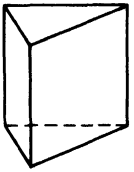
Circular cylinder



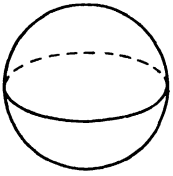
Tetrahedron



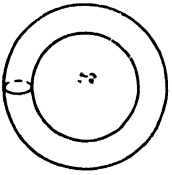
Pyramid



Prism



Sphere



Torus, or doughnut, or anchor ring

Cone

FIGURE 1–2

All these shapes are easily recognized as familiar; you would have no difficulty in recognizing one of them on sight. However, you may have noticed that we have not *defined* the shapes, but only drawn them. Part of our job later in this book will be to give accurate definitions. As you study geometry, it will become more and more clear that we cannot prove anything without a clear definition. But in this chapter, we shall most often use drawings to introduce the things you will study later.

We have used the words "geometric object" or "geometric figure," but have only given examples by drawings on a page of the book. A precise definition is the following:

Definition 1–1. A geometric figure is any *set of points.*

Some explanation of the words "set of points" is necessary. A set of points is just a "bunch" of points (or "collection" of points, or "aggregate"

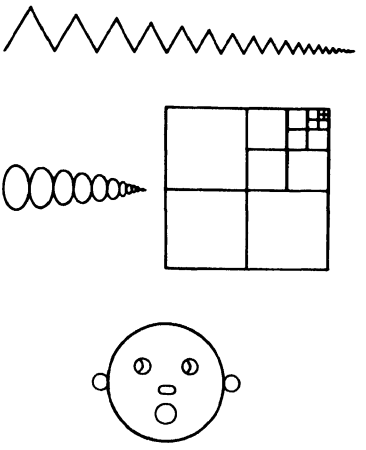


FIGURE 1-3

of points). This definition allows some rather strange sets of points to be called geometric figures, for example those in Fig. 1–3. However, we usually deal with simpler sets of points than these, namely, with lines, line segments, triangles, circles, etc. A word of caution is necessary regarding our use of the word "line." By "line" we mean a *straight line* indefinite in extent in both directions. We shall see in Chapter 3 exactly how to state what we need to know about lines in order to prove other things about them.

Exercise Group 1–1

1. Describe in your own words what is meant by a *set* of points.

2. Invent some geometric figures of your own and then describe these sets of points.

3. Mark a point on a sheet of paper and consider the set of all points on your paper such that each point in the set is two inches from this first point. Describe this set of points in another way.

4. Some of the shapes drawn in Fig. 1–3 are intended to represent infinitely many line segments or circles. Which are they? Can you invent other figures with infinitely many segments or circles?

1–4 WHAT ARE POINTS?

Up to now we have been talking about points just as if we knew what they are. But no one has ever seen a point. The smallest dot you can make on a sheet of paper is certainly not a point. The dot would look huge to a microbe (if it could see), and the microbe, which can be seen under a microscope, is very large compared with the molecules of chemical

substances of which it is composed. What, then, is a point? The answer is that points are an *invention* of the human mind. We *imagine* that there *ought to be* points, which themselves cannot be split up into smaller pieces. A given "hunk" or portion of space can be split into two hunks. And each of these can be split in two, and so on indefinitely. A point is what *should* be left when our hunk has no extension in space or, to use other words, when it is "infinitely small." Euclid, in fact, gave something like that as a definition of a point. But such a "definition" is not really a help in geometry because from it we cannot derive anything more. All it does is make us feel more or less happy about the existence of the things we *think* we are talking about. The modern point of view (and really Euclid's too) is that points are undefined. The things that we do define are certain relations between points or sets of points.

It is essentially the same regarding lines and planes. No one has ever seen a line or a plane. A thin ray of light or a thin line on the page is only an approximation to what we imagine that a line should be like. And a very flat surface, like a mirror, is but an approximation to what we imagine a plane to be. Therefore right at the beginning of geometry we have to make a strong effort of the imagination. We must imagine a large set of things, called points, which we call a plane. Certain parts of this plane, that is, certain special sets of points in the plane, are called straight lines, or just lines. The study of geometry begins when we tell *how these points and lines are related to each other.*

Exercise Group 1–2

1. Draw two intersecting lines on the blackboard. You will then have some chalk hanging on the board. How many reasons can you give to explain that what you have drawn are not lines? Does the chalk on both your drawn lines correspond to what you think of as a point?

2. Mark two dots on the blackboard. Are they points? Is a "blob" of chalk a point? How many lines do you feel there should be which lie on two given points? If points were blobs of chalk, could you draw more than one line through them?

3. When you draw a figure to illustrate a geometric idea, what is the relation of the drawn figure to the geometric idea you have in mind?

4. Is it reasonable to expect a drawing to give a proof of something in geometry?

5. If an object had length or width, would you call it a point? Can you see points and lines?

6. Can you describe what you mean by straightness of a line?

1–5 TOOLS YOU WILL NEED

Even though points and lines are creations of the mind it is necessary for us to draw pictures of them in order to see what is "going on." Thus you will need a *ruler* to draw pictures of straight lines. A ruler without

a marked scale on it is called a *straightedge*, and for the constructions we shall make this is all that is absolutely necessary. But it is very convenient to be able to *measure* the length of a line segment, and for this reason an ordinary foot ruler, or a six-inch one, will do. However, we shall not actually deal with measurement* of lengths until Chapter 4.

Angles will be discussed more fully later, and we shall consider measurement of angles in Chapter 12. But you are already familiar with a device, called the *protractor*, for measuring pictures of angles drawn on paper or on the blackboard.

You will also need a device, called a *compass*, for drawing pictures of circles. Undoubtedly you are familiar with its use.

With these simple tools you can make a number of "constructions." These constructions will seem to be obviously correct. However, we will prove later in the book that they actually are correct.

You have learned in your previous mathematics how to use these tools. The exercises below are for review of definitions and practice in the use of ruler, compass, and protractor. However, throughout this course your best tool will be your ability to think straight.

Exercise Group 1–3

1. Draw a picture of a line, using your straightedge. Now if you sharpened your pencil *very* sharp, could you draw a picture of a line inside the first one? Was your drawing really what you intended a line to be?

2. Draw a picture of a circle with your compass. How many differences can you find between your drawing and an imaginary perfect circle?

3. Draw a picture of an angle. Measure it with your protractor. Have a friend measure the angle. Do you get *exactly* the same answers? How accurate do you think your measurement is?

4. How long are the sides of an angle? When you draw a picture of an angle, you cannot draw the sides infinitely long because your straightedge and paper are limited in size.

5. Draw a picture of a ray from a point A. Draw another ray from A. What we call the *angle* consists of the points on the two rays. The points between the rays form the *interior* of the angle. The other points of the plane form the *exterior* of the angle. (Fig. 1–4.)

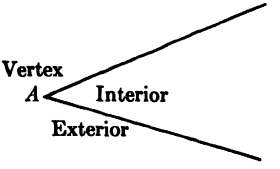


FIGURE 1–4

6. A *right angle* has a measure of 90°. Construct a right angle with your protractor.

* It is interesting and surprising that Euclid in his *Elements* did not discuss measurement, that is, the process of getting a *number* for the distance between two points.

7. Draw an angle of 37°, of 120°, of 85°.

8. An angle is *bisected* if a ray from the vertex splits the angle into two angles of the same size. In Fig. 1–5, the ray AC bisects $\angle BAD$ because $\angle CAB$ and $\angle DAC$ are the same size. The ray AC is the bisector of the angle. Draw an angle of 45° and bisect it, using your protractor.

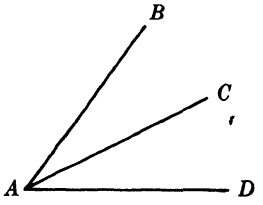


FIGURE 1–5

9. Draw a picture of a triangle. Measure each of the angles of the triangle. What is their sum? Repeat for several different triangles. Can you guess something that is true for all triangles?

FIGURE 1–6

10. As in Fig. 1–6, an *acute angle* is an angle of less than 90°; an *obtuse angle* is an angle of more than 90°. Draw two acute angles and two obtuse angles with your protractor.

11. Draw a triangle and measure the sides and the angles. Which side is opposite the largest angle? opposite the smallest angle? Draw another triangle and repeat the measurements. Can you guess something which now seems to be true for all triangles?

12. A triangle that has one right angle is called a *right triangle*. Draw a picture of a right triangle, as in Fig. 1–7. Measure the two acute angles of the triangle. What is their sum? Draw another right triangle and make the same measurements. Is the sum of the two angles the same as before?

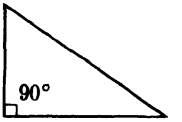


FIGURE 1–7

13. One angle is said to be *complementary* to another if their sum is 90°. Construct two such angles with your protractor. In Fig. 1–8, if $\angle AOB$ is a right angle, what can you say about $\angle COA$? about $\angle COB$?

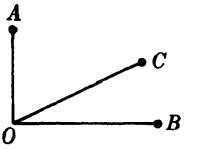


FIGURE 1–8

14. One angle is said to be *supplementary* to another if their sum is 180°. Construct two such angles with your protractor. What can you say about $\angle AOB$ and about $\angle BOC$ in Fig. 1–9?

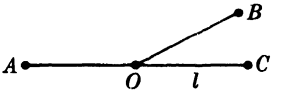


FIGURE 1–9

15. Two distinct rays OA and OB which lie on one straight line form a *straight angle* (Fig. 1–10). What is the measure of a straight angle?

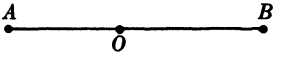


FIGURE 1–10

16. A *quadrilateral* (Fig. 1–11) is a figure with four sides, just as a triangle is a figure with three sides. Draw a picture of a quadrilateral and measure its angles. What is their sum?

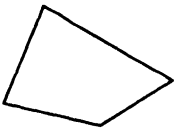


FIGURE 1–11

17. Add the following:
(a) 12° 10′ 25″, 31° 48′ 30″, and 43° 50′ 40″.
(b) 17° 18′ 19″, 90° 9′ 9″, and 41° 48′ 48″.
(c) 121° 18′ 17″, 40° 40′ 40″, and 18° 1′ 3″.
(d) 66° 55′ 44″, 55° 44′ 33″, and 44° 33′ 22″.

18. Subtract:
(a) 41° 15′ 13″ from 63° 28′ 44″.
(b) 41° 15′ 13″ from 50° 18′ 7″.
(c) 61° 28′ 37″ from 80° 18′ 23″.

19. What is the complement of an angle of
(a) 51° 17′ 18″?
(b) 42° 38′ 29″?
(c) $A°$?

20. What is the supplement of an angle of
(a) 51° 17′ 18″?
(b) 138° 4′ 5″?
(c) $A°$?

21. If an angle is twice as large as its complement, what is its degree measure? [*Hint:* Set up an equation to solve.]

22. Two angles have a sum of 56° 28′ and a difference of 8° 46′. Find the angles.

23. Find the complement and supplement of an angle of 56° 28′ 31″. What is the difference between the supplement and complement? Can you guess something that is true for all acute angles?

24. What kind of angle is the supplement of an acute angle? of an obtuse angle? Does an obtuse angle have a complement?

1–6 RULER AND COMPASS

During their study of geometry, the ancient Greeks found that many interesting figures could be drawn with a straightedge and compass alone. Drawings needing only these tools are called *ruler and compass constructions.* The Greeks tried to solve all construction problems using only these tools, but they encountered some that could not be done this way.

The exercises below involve some simple ruler and compass constructions that the Greeks knew. Exercise 2, in fact, is the first proposition in Euclid's *Elements.* We shall return to these constructions again in Chapter 7, where we shall *prove* that some of them are correct.

Exercise Group 1–4

1. A triangle having two sides of equal length is called an *isosceles triangle.* Draw a picture of an isosceles triangle. Measure the angles opposite the equal sides. Draw several isosceles triangles, each with the equal sides three inches long.

2. A triangle with all sides the same length is called an *equilateral triangle* (Fig. 1–12). Draw a picture of an

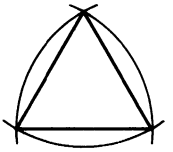


FIGURE 1–12

equilateral triangle. This is easy to do with your compass and straightedge. What is the degree measure of each angle in an equilateral triangle? A triangle with all angles equal is called an *equiangular triangle.* Are equilateral triangles equiangular? Are equiangular triangles equilateral?

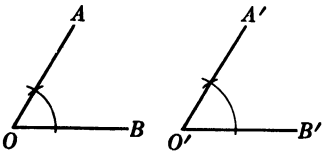


FIGURE 1–13

3. Draw an angle like that in Fig. 1–13. Use your compass to copy $\angle AOB$ as $\angle A'O'B'$ in the figure. Measure the two angles with your protractor to see if they are the same size.

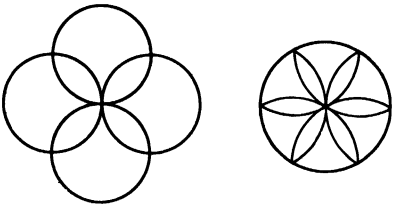


FIGURE 1–14

4. Draw the figures shown in Fig. 1–14.

5. Use your straightedge and compass to draw and bisect an angle (Fig. 1–15). Check your work by measurement with a protractor. State in your own words why you think the construction is correct.

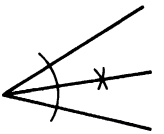


FIGURE 1–15

6. Angles which have a common vertex and a common side and have no interior points in common are said to be *adjacent.* Thus, in Fig. 1–16, $\angle AOB$ is adjacent to $\angle BOC$. Is $\angle AOC$ adjacent to $\angle BOC$? Are the angles in Fig. 1–17 adjacent to one another?

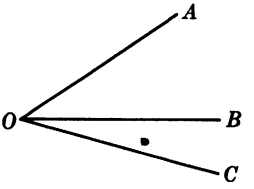


FIGURE 1–16

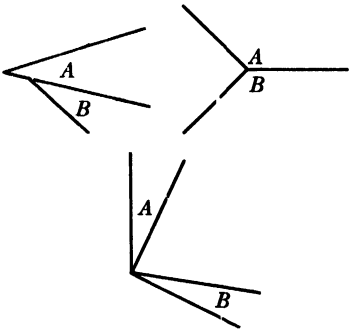


FIGURE 1–17

7. Draw a picture of a segment. From some point A on a line, construct with your compass a segment AB as long as the segment you first drew. (See Fig. 1–18.)

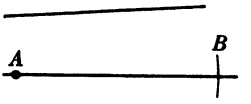


FIGURE 1–18

8. Two lines are *perpendicular* if they intersect at right angles. Another way of defining perpendicular would be to say that all the adjacent angles formed by the intersecting lines are the same size. To indicate that l and m are perpendicular, we write $l \perp m$.

A line l is called the *perpendicular bisector* of the segment AB if the line l is perpendicular to AB and intersects the segment AB in a point C such that AC and CB have equal lengths. The point C is called the *midpoint* of AB. (See Fig. 1–19.)

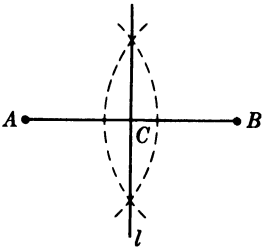


FIGURE 1–19

Draw a line segment and construct its perpendicular bisector with straightedge and compass. Check by measurement. State why you think the construction is correct.

9. Draw a line l and mark a point O on it. With straightedge and compass, construct a line $m \perp l$ and passing through O. (See Fig. 1–20.)

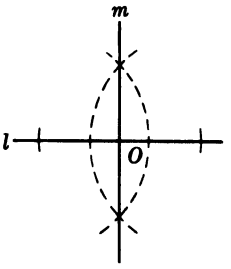


FIGURE 1–20

10. Draw a line l and mark a point P not on the line. With your straightedge and compass, construct a line m through P perpendicular to l. Check by measurement with your protractor. State why you think the construction is correct. (See Fig. 1–21.)

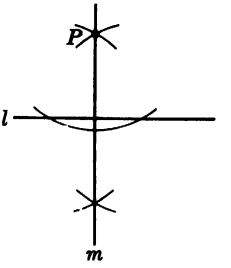


FIGURE 1–21

11. A line segment drawn from a vertex of a triangle to the midpoint of the opposite side is called a *median* of the triangle. A triangle has three medians that meet in one point O (see Fig. 1-22). We shall prove this later.

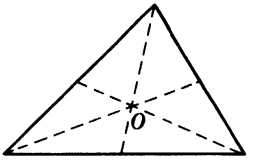


FIGURE 1–22

Draw a triangle and its medians. Can you describe where the medians intersect? Try to find how to describe O by measurement. Make this construction for several triangles.

12. Cut a triangle out of cardboard. Find where the medians intersect. If your work is accurate you will find that the triangle will balance on your finger at the place where the medians intersect (Fig. 1–23).

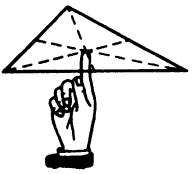


FIGURE 1–23

13. An *altitude* of a triangle is a line segment drawn from a vertex perpendicular to the line *containing* the opposite side [see Fig. 1–24, (a) and (b)]. A triangle has three altitudes. The lines containing the altitudes of a triangle always intersect in one point [see Fig. 1–24(c)]. Draw several triangles and their altitudes.

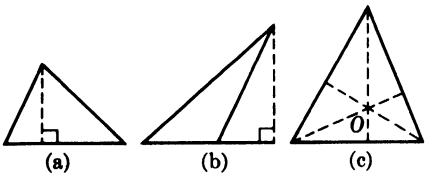


FIGURE 1–24

14. Draw a triangle and bisect each of the angles of the triangle. The three bisectors will meet in one point (Fig. 1–25).

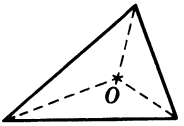


FIGURE 1–25

15. Two lines, l and m, are said to be *parallel* if they do not intersect. We indicate "l is parallel to m" by writing $l \parallel m$. (Remember that lines are infinitely long.) If another line n intersects the parallel lines, then the angles 1 and 2 in Fig 1–26 are the same size. Conversely, if angles 1 and 2 are the same size, then the lines are parallel.

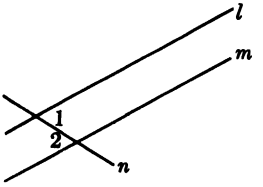


FIGURE 1–26

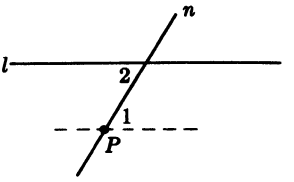


FIGURE 1–27

As in Fig. 1–27, draw a line l and a line n intersecting it. Choose a point P on n but not on l. At P, copy angle 2 in order to get a line parallel to l. Make this construction with your protractor and then repeat it with your straightedge and compass.

We shall prove later that this construction with straightedge and compass is correct.

16. Draw a line l and mark a point A not on l. Construct a line through the point A parallel to l.

17. Draw a right triangle (Fig. 1–28). The side opposite the right angle is called the *hypotenuse*. The other two sides are called the *legs* of the triangle. Measure the lengths of the sides in your right triangle. If a, b, and

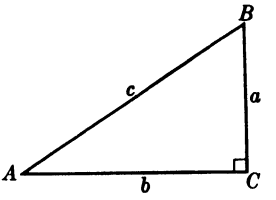


FIGURE 1–28

c are the lengths of the sides, where c is the hypotenuse, you will find that $a^2 + b^2 = c^2$. This formula is called the *Pythagorean theorem.* It was discovered (probably) by the Greek Pythagoras in the 6th century B.C. We will prove this important theorem later in the book. Test the truth of the Pythagorean theorem by measuring several different right triangles.

REVIEW OF CHAPTER 1

1. Make sure that you are familiar with all the following terms:

line segment	perpendicular
angle	right angle
angle bisector	equilateral
triangle	equiangular
degree, minute, second	isosceles
midpoint	median
complementary	altitude
supplementary	hypotenuse
straight angle	diameter
obtuse angle	radius
acute angle	adjacent angles
ray	intersect
set	contain

2. Make sure that you can do the following straightedge and compass constructions:

(a) Copy a segment.
(b) Draw the bisector of a segment.
(c) Draw a line $\perp$ to a given line from a point not on the given line.
(d) Draw a line $\perp$ to a given line from a point on a given line.
(e) Draw a line $\parallel$ to a given line, through a point not on the line.
(f) Bisect an angle.
(g) Copy an angle.

Brief Review Tests

Decide whether the following statements are true or false. If in doubt, look up the answer. Do not guess.

I. *History*

1. Euclid lived before Pythagoras.
2. Euclid collected, organized, and added to the geometry of his day.
3. Euclid's geometry was incomplete as he wrote it. It was completed in the 19th century.
4. Archimedes lived at the same time as Euclid.
5. Thales lived before Euclid.
6. Greece was superior to Rome in all ways.
7. Rome was a great center for the study of geometry.

II. *Geometry*

1. A straightedge is used for measurement.
2. A line may have finite length.
3. A straight angle contains 90°.
4. If angles have a common vertex and a common side, they are adjacent.
5. Complementary angles have a sum of 180°.
6. In a right triangle the side opposite the right angle is called the hippopotamus.
7. The altitude of a triangle is its distance from the center of the earth.
8. If two lines intersect they are parallel.

ARITHMETIC AND ALGEBRA REFRESHER

1. Carry out the arithmetic operations indicated:

(a) $[(12)(8)] \div 6$ (b) $2992 \div 68$ (c) $326 + (-257)$ (d) $\frac{1}{2} + \frac{1}{4}$
(e) $\frac{1}{3} \div \frac{11}{3}$ (f) $56789 \div 357$ (g) $\frac{2}{3} - \frac{2}{5}$ (h) $3.762 \div 5.7$
(i) $\frac{1}{2} \cdot \frac{1}{3}$ (j) $\frac{1}{2} \div \frac{1}{3}$ (k) $(-24) \div (-6/5)$
(l) $[(-6.1) + (-8.5)] - 2.7$ (m) $3/7 \cdot 35/24$ (n) $2.5679/0.1372$
(o) $[3.4 - 8.5] + (-2.7)$ (p) $(5 + \frac{1}{2})/33$

2. Write the answers to the following. If you take more than a minute and miss any part of this problem, you need more review.

(a) $2 \times 2 =$ (b) $2 \div 2 =$ (c) $2 \times 0.2 =$
(d) $2 \div 0.2 =$ (e) $2 \div \frac{1}{2} =$ (f) $2 \times \frac{1}{2} =$
(g) $\frac{1}{2} + \frac{1}{3} =$

3. Write formulas for the following. Be sure that you tell what each letter stands for. For example: If A is the area of a rectangle l its length and w its width, then $A = lw$.

(a) The volume of a box.
(b) The area of a circle.
(c) The circumference of a circle.
(d) The area of a triangle.
(e) The product of a number z, by one more than the number.
(f) Twice a number plus 5 is equal to 13.
(g) One number is 4 times another.

4. Find x if

(a) $2x - 5 = 3$ (b) $3x - 5 = x + 7$
(c) $2x = 3$ (d) $ax = b, a \neq 0$
(e) $x^2 = 4$ (f) $\frac{1}{3} \cdot (12x) = 44$
(g) $2x/3 + 1 = 5$ (h) $4x + 5x - x = 96$
(i) $x \div 6 = 5 \div 24$ (j) $6.4 = 2x - 0.4$

5. If x and y are numbers, what do we call xy? $x + y$?

6. Write equations for the following statements:

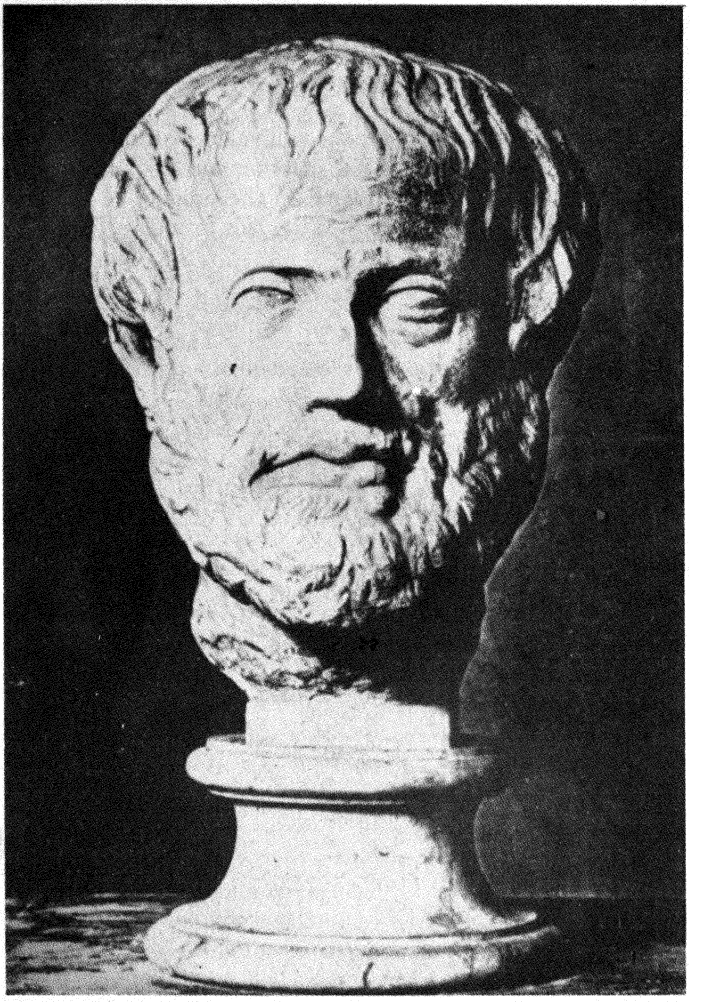
(a) The sum of two numbers is 17 and their difference is 7.

(b) A number z is equal to the sum of the product of x and y, and the product of a and b.

(c) The number π is equal to the quotient of the circumference of a circle divided by its diameter.

(d) The volume of a sphere is equal to $\frac{4}{3}$ of the product of π and the cube of the radius.

(e) Einstein's famous formula for the equivalence of mass and energy is as follows: If E is the energy of a mass m, and c is the velocity of light, in suitable units, then E is equal to the product of the mass by the square of the velocity of light. Write this formula.



ARISTOTLE

A view of a statue in a Vienna museum.
Courtesy of *Scripta Mathematica*.

CHAPTER 2

LOGIC

2–1 STATEMENTS

Everyone thinks he knows what a *statement* is. A politician issues a *statement* when he concedes defeat during an election. The plumber sends your father a *statement* after installing a new sink. But in mathematics we use the term in a narrow technical sense. We mentioned in the first chapter that clear, precise definitions are necessary. Accordingly, we make the following definition, which turns out to be very useful.

Definition 2–1. A statement is a sentence which is either true or false, but not both.

Consider the following:

(a) Go back home.
(b) $2 \times 2 = 5$.
(c) $2 \times 2 = 4$.
(d) New York is a suburb of Los Angeles.
(e) Thursday up tar paper.
(f) The silliness of speedily.

Some of the above are statements and some are not. Before reading on, decide for yourself which ones satisfy the definition.

Let us look at some slightly different examples.

(a) I am the most beautiful girl in town.
(b) If I don't get a good night's sleep, I can't pass the examination.
(c) $x + 2 = 5$.

All of these are sensible sentences. The only question is whether or not they satisfy our condition of being either true or false but not both. Before we can decide whether (a) is a true statement or a false statement we must know who "I" is and what town is meant. Then we can decide upon the truth or falsity. (We could hold a beauty contest!) In (c) if we replace the letter x by "3," we have a true statement. If we replace x by "2" the

resulting statement is false. In statement (b), we must again know who "I" is and what examination is meant in order to decide whether it is a true or false statement.

Besides rather simple statements, we can have more complicated ones, such as:

(a) I cannot go to the dance because I have nothing to wear, my feet hurt, and I can't dance.
(b) A triangle has three equal sides if and only if it has three equal angles.
(c) Jones is a mathematician and Smith is an engineer, while Brown doesn't work at all.
(d) There is one and only one line through two points.
(e) Either I will marry you or get a new job.
(f) Two lines either intersect or are parallel.

In some of these examples the full statement is made up of simpler statements. We see that statements may be combined to give more complicated statements. For example:

"Mr. Baker is a barber," "Mr. Barber is a baker,"

are two statements. We can combine them into a single statement by joining them with "and."

"Mr. Baker is a barber and Mr. Barber is a baker."

Here are some examples of combining two statements. Make up some others of your own.

(a) We are having a picnic and it is not raining.
(b) It is night and I can hear the wind.
(c) Mary is going with Fred and Suzie is going with John.
(d) I like spinach and I dislike candy.
(e) You are rich and I am poor.
(f) Thursday is April second and the moon is made of green cheese.

To be perfectly clear about combining statements, we indicate statements by letters, using Greek letters like α (alpha), β (beta), γ (gamma). Now if we let α indicate the statement "Mr. Barber is a baker" and let β indicate "Mr. Baker is a barber," then the statement "Mr. Barber is a baker and Mr. Baker is a barber" may be written "α and β."

Problem. In each of the combined statements (a) through (f) above, pick out the two statements which form it. Note that each statement has the form "α and β."

The combination of two statements with "and" to get a new statement has a name. We emphasize this by making the definition:

Definition 2–2. If α and β are statements, the statement "α and β" is called the *conjunction* of α and β.

You should not accept definitions blindly. It is possible to sneak a false statement into a definition! Such a definition would be worthless. From the way Definition 2–2 is worded it is clear that we have *taken for granted* that "α and β" will always be a statement. But a sentence is not a statement (according to Definition 2–1) unless it is either true or false. Therefore, we must decide upon some *rule* which will enable us to say when "α and β" is true and when it is false. If we cannot agree upon such a rule, Definition 2–2 will be meaningless.

The following statements are all of the form "α and β." Decide which ones are true and which ones are false, and then try to state a general rule for deciding upon the truth of such statements.

(a) $2 \times 2 = 4$ and $2 + 2 = 5$.
(b) $3 + 2 = 5$ and there are at least 52 Sundays in every year.
(c) A triangle has three sides and a pentagon has six sides.
(d) Every rectangle is a square and all men have purple hair.
(e) The positive square root of 16 is 4 and the negative square root of 16 is -4.

You will probably have no trouble coming to the same conclusion that mathematicians have agreed upon. That is, "α and β" is considered true when both α and β are true. If one of the statements is false (or both are), then "α and β" is false. We can describe this rule most clearly by the *truth table* below.

α	β	α and β
T	T	T
T	F	F
F	T	F
F	F	F

The table is self-explanatory. It points out that "α and β" is true (T) in only one case and false (F) in all others.

The next step in our development of logic will be harder. We can combine statements in another way by putting the word "or" between them. For example:

I will go to the movies *or* I will stay home.
I will receive an *A* in geometry *or* Latin.
State University will not admit you unless you have had a year of Spanish *or* a year of Latin.

Study the three statements above carefully. Everyone would agree that in the first statement the speaker means that he will go to the movie

or he will stay home *but he will not do both.* In the second statement does the speaker mean that he will get *either* an *A* in geometry *or* an *A* in Latin *but not both*? Or does he mean that he is sure of getting one *A* and perhaps he will get two? Of course there is no question about the last statement. State University will not refuse to admit you because you had one year of Spanish *and* one year of Latin.

We are pointing out that in everyday speech we use the word "or" ambiguously. Sometimes when we say "α or β" we mean that *one and only one* of the statements is true. Sometimes we mean that at least one is true and possibly both are. We cannot use the word "or" with this double meaning in mathematics. We must explain precisely what we mean when we say "α or β." Mathematicians have agreed to call statements of the form "α or β" true in all cases except when α and β are both false. Of course this differs from the ordinary usage of the word "or." But we use everyday language too carelessly for our purpose in mathematics. We must be able to express ourselves precisely and know that our meaning will not be misunderstood. You must remember that when we say "α or β" we mean that either α is true and β is false, *or* α is false and β is true, *or* α is true and β is true. We make the following formal definition.

Definition 2–3. The statement "α or β" is called the *disjunction* of α and β. It is false when both α and β are false and true in all other cases.

Problem. Construct the truth table for the statement "α or β."

Exercise Group 2–1

In each exercise below there are two statements. Form their conjunction and disjunction. Discuss the resulting statements with respect to truth and falsity.

1. Some roses are red. Some violets are blue.
2. September has 30 days. There are eight days in a week.
3. 3 is less than 4. 4 is less than 3.
4. Napoleon was an emperor of Japan. Marco Polo discovered America.
5. $2 + 2 = 5$; $6 + 3 = 9$.
6. $7 + 4 = 11$; $8 + 0 = 9$.
7. $1 + 1 = 2$; $1 \times 1 = 2$.
8. $5 + 1 = 5$; $5 \times 1 = 6$.

2–2 NEGATION

Sometimes we make statements about other statements. One of the simplest and most useful statements of this type has the form "α is false." At times you may make a statement that you believe true, and someone else may show his disagreement by saying, "That is not true." Just as we use the terms *conjunction* and *disjunction* to describe the statements "α and β" and "α or β," so also we have a particular name for the statement "α is false."

Definition 2–4. The statement "α is false" is called the *negation* of α (and will be written, and read, "not α").

Problem. State a rule for determining when "not α" is true and when it is false. Make a truth table for "not α."

When we form the negation of a statement in words we do not usually place "not" in front of it. Logically it would be correct to do so, but the resulting sentence might sound awkward. For example, let α be "a square is a rectangle." Anyone untrained in logic would have a right to complain that he did not understand our meaning if we said, "not a square is a rectangle." The examples below indicate how negations may be formed. After studying these, make up several examples of your own.

(a) α: A square is a rectangle.
not α: A square is not a rectangle.

(b) α: $2 + 2 = 4$.
not α: $2 + 2 \neq 4$.

(c) α: A triangle has five sides.
not α: A triangle does not have five sides.

(d) α: It is false that my donkey has three legs.
not α: My donkey has three legs.

Exercise Group 2–2

Form negations of the following statements.

1. The grass in my yard is green.
2. This rose is black.
3. I am tired.
4. The temperature of the sun is more than 1000 degrees.
5. Geometry is interesting.
6. $5 + 4 = 9$.
7. $6 + 3 = 7$.
*8. $x = 2$ and $y = 3$.
*9. $x = 2$ or $y = 3$.
*10. All pigs are fat.
*11. Some Americans are liars.

Sometimes we wish to state the negation of a statement like "All cats are black." If we can find *one* cat which is not black we will have proved this statement false. Hence we form the negation of "All cats are black" by saying "There is at least one cat which is not black" or "Some cat is not black." The most common error made in forming the negation of a statement like "All students in this class like geometry" is to say "No student in this class likes geometry." These statements certainly do not agree but they are definitely not negations of each other. Why? Remember that if α is false then "not α" must be true, for "not α" states that α is false. Show that the statements, "All students in this class like

* Starred exercises, sections, theorems, etc., are optional.

geometry" and "No student in this class likes geometry," could both be false.

Another common error is to negate "All cats are black" by saying "All cats are not black." Although in everyday speech this sentence might be understood to mean "Not all cats are black," we do not interpret it so. If we say "All cats are red," we mean *every cat has the property of being red.* So we interpret "All cats are not black" to mean *every cat has the property of being not black.* This is a subtle distinction, and again it points out that the precise language of mathematics differs from careless, everyday speech.

Exercise Group 2–3

Give the negation of each statement.

1. All men in Georgia are at least six feet tall.
2. All students in France like geometry.
3. All students in Russia hate geometry.
4. Every rectangle is a square.
5. Every triangle is equilateral.
6. All gadgets are widgets.
7. All miggles are maggles.
*8. For every pair of numbers a and b, $a + b = b + a$.
9. Every circle has a center.

Consider the statement "*Some* cats are black." We usually understand the word "some" to mean "more than one." But in logic it is more convenient to agree that it means "one or more." (We must have precise definitions!) The negation of "Some cats are black" then becomes "No cat is black." We could also say, "Every cat is a color different from black" or "Every cat is not black." Notice again that this last statement, "Every cat is not black," means something altogether different from "Not every cat is black."

Exercise Group 2–4

Give negations of the following statements. After doing this, make up some examples of your own.

1. Some men play golf.
2. Some boys play football.
3. There is at least one equilateral triangle.
4. For some numbers x, $x^2 + 2 = 5$.
5. There is at least one number x such that $x + 3 = 7$.
6. There is at least one number x such that $x^2 + 2 = 0$.
7. Some dogs have fewer than four legs.
*8. Some numbers are negative.
9. Some fishermen always tell the truth.
10. Some people work hard.

When we have two statements, α and β, and wish to form the *negation* of their *conjunction*, we must think a little harder than before. Thus if we wish to form the negation of "Smith is a dentist and Jones is a teacher," we must say that the statement is false. By our agreement

in Section 2–1, we must say that at least one of the statements "Smith is a dentist" and "Jones is a teacher" is false. We do this by saying, "Smith is *not* a dentist *or* Jones is *not* a teacher."

As another illustration, suppose one student says, "We will win our game with Lincoln *and* we will win our game with Bay City." To call his statement false is to say, "We will *not* win our game with Lincoln *or* we will *not* win our game with Bay City." It should be clear now that the negation of "α and β" is the statement "not α *or* not β."

Problem. Construct a truth table for each of the statements "α and β" and "not α *or* not β." Show that when one statement is true the other is false, and vice versa.

Exercise Group 2–5

Give the negations of the statements below. Make up some additional exercises of your own.

1. Today is Monday and the weather is cold.
2. $x = 2$ and $y = 5$.
3. $2 \times 2 = 4$ and $3 \times 5 = 15$.
4. $\triangle ABC$ is isosceles and $\triangle ABC$ is a right triangle.
5. The point A is on the line l and the point B is on the line m.

*6. All cats are black and all men are mortal.

*7. All cats are black and some birds are blue.

*8. Some rectangles are squares and some triangles are equilateral.

*9. Some widgets are gadgets and no wiggles are waggles.

We may wish to form the *negation* of the *disjunction* of two statements. For example, we may wish to deny the statement "Johnson is a tailor or Brown is a doctor." You should have no trouble expressing the negation of this statement.

Problem. State the general rule for forming the negation of the disjunction of two statements. Do so by completing the sentence, "The negation of 'α or β' is _____." Construct a truth table for each of the statements "α *or* β" and "not α *and* not β."

Exercise Group 2–6

Give the negation of each statement below. Invent some more examples.

1. I will go to the game or I will go to the movie.
2. I will study hard or I will fail.
3. We will start at 6:30 or we will be late.
4. A donkey has three legs or a rabbit has long ears.
5. $2 + 2 = 5$ or $2 + 2 = 6$.
6. $3 \times 7 = 21$ or $4 \times 5 = 20$.
7. Billings is right or Jones is wrong.

*8. Some will work or all will starve.

*9. Every triangle is isosceles or every triangle is right.

*10. No roses are red or some violets are blue.

2–3 WHAT ARE PROOFS?

In everyday life when trying to convince a person that something is true, we may say, "*This* is true because *that* is true." For such a statement to be convincing, both sides in the argument must agree that the *that* is true and the *this* must then *necessarily* follow. In other words, there must be agreement as to what information we start with and how we draw conclusions from this information. *Logic is the study of the rules for making correct statements.* Proof always consists of making statements of the following type: If so and so is true, then such and such must also be true. For example, consider the statement, "If the sky is blue (no clouds at all), then it is not raining." Everyone would agree to this statement because from our experience we know that rain invariably comes from clouds. However, the following statement, which you might hear in a political speech, is not logically correct. "The United States is the best country because it is a democracy." That we are a democracy is true; that we are the best country (whatever that means) may be true although it would not be accepted by all. The logical error comes from the fact that there are other democratic countries, and hence they too by our careless reasoning should be best. Therefore, that the United States is best does not follow from the fact that we are a democracy.

Here are simple examples from algebra:

$$\text{"If } 3x = 4, \text{ then } x = \tfrac{4}{3}\text{."}$$

$$\text{"If } a = 2 \text{ and } b = 3, \text{ then } a^2 + ab = 10\text{."}$$

Both of these statements are correct and both are of the form, "If α then β." *All mathematical proofs employ statements of this type.* We express such a statement another way by saying

$$\alpha \textit{ implies } \beta.$$

For example, let α be the statement "Today is Monday" and let β be the statement "Tomorrow is Tuesday." Then the statement "if α then β" is certainly true. However, the statement "If the sun comes up tomorrow, then it will not rise the next day" is very probably false. At least we shall hope that it turns out to be false. It is fortunate that statements like "If he takes her to the dance, then I'll never speak to him again" and "If our party loses the election, then the country will be ruined" are usually false. It is convenient to have a symbolic way to indicate that α implies β. We shall write this as

$$\alpha \rightarrow \beta,$$

where you should read the arrow as the word "implies." We call the statement "α implies β" an *implication.*

Exercise Group 2–7

Put the following statements in an "if α then β" form so that α implies β.

1. If I eat strawberries I get hives.
2. The picnic will be canceled in case of rain.
3. We will win the game if Jonescevitch plays.
4. Numbers larger than 5 are positive.
5. $x^2 = y^2$ because $x = y$.
6. $3x^2 - 2 = 10$ when $x = 2$.
7. A geometric figure which is a square is not a circle.
8. Squares are rectangles.
9. The area of a rectangle with sides of length a and b is ab.
10. With a million dollars, I'd eat steak for breakfast.
11. People who don't study mathematics don't have good sense.
12. When it rains my rheumatism bothers me.
13. When I study I do well in school.
14. People who do not vote are poor citizens.
15. Two lines which intersect are not parallel.
16. I know that $x = 2$ because $3x = 6$.
17. Satisfaction guaranteed or your money back.
18. I know $3x = 6$ because $x = 2$.
19. I will not go to the party if Jimmy is coming.
20. This shirt will not shrink or your money will be refunded.
21. This shirt will shrink or your money will be refunded.
22. To remain free we must vote in elections.
23. The crops will be saved if we have rain this week.
24. The crops will be saved only if we have rain this week.
25. I will go to the dance if Joe asks me.
26. I will go to the dance only if Joe asks me.
27. It will rain tomorrow if it turns cold.
28. It will rain tomorrow only if it turns cold.

Working with statements of the form "$\alpha \rightarrow \beta$" is called *deductive reasoning* or, in everyday language, *drawing conclusions*. The statement α is called the *hypothesis* and the statement β is called the *conclusion*. We use this method of reasoning in daily life when we try to prove to someone that some statement β is true. We try to find another statement α which implies β, and which is true. Then we say, "α is true, therefore β is true."

Exercise Group 2–8

In Exercises 1 through 7, a statement $\alpha \rightarrow \beta$ is given which is to be accepted as true. What conclusion can you draw if (a) is true? if (b) is true?

1. If Fred is on the team he must go to bed early.
   (a) Fred goes to bed early. (b) Fred does not go to bed early.
2. If a man is a postman he has foot trouble.
   (a) This man has perfect feet. (b) That man has foot trouble.
3. You will be given a ticket if you are a sophomore.
   (a) Jim was not given a ticket. (b) Joe was given a ticket.

4. You will be given a ticket only if you are a sophomore.
   (a) Jean was given a ticket. (b) George is a sophomore.
5. If two triangles have equal bases and heights then they have equal areas.
   (a) These triangles have equal areas. (b) These triangles do not have equal areas and they do have equal heights.
6. The house will burn down if we don't get some water.
   (a) We did not get any water. (b) We did get some water.
7. If a man is beheaded in June then he will die before July 4.
   (a) This man died before July 4. (b) Smith was not beheaded in June.

Is the reasoning in Exercises 8 through 18 correct? Invent some other examples illustrating both correct and incorrect reasoning.

8. All men are mortal. Socrates is a man.
   Conclusion: Socrates is mortal.
9. All widgets are gadgets. This is a gadget.
   Conclusion: This is a widget.
10. All widgets are gadgets. This is not a gadget.
    Conclusion: This is not a widget.
11. Only widgets are gadgets. This is a gadget.
    Conclusion: This is a widget.
12. Only widgets are gadgets. This is a widget.
    Conclusion: This is a gadget.
13. Every equilateral triangle is isosceles. $\triangle ABC$ is isosceles.
    Conclusion: $\triangle ABC$ is equilateral.
14. Parallel lines do not intersect. Lines r and s intersect.
    Conclusion: Lines r and s are not parallel.
15. All of these dresses will shrink if washed in hot water. This dress did shrink.
    Conclusion: It was washed in hot water.
16. All sweet food has honey in it. This food is sweet.
    Conclusion: It has honey in it.
17. All sweet food has honey in it. This food doesn't have honey in it.
    Conclusion: It is not sweet.
18. All sweet food has honey in it. This food is not sweet.
    Conclusion: It does not have honey in it.

2–4 THE TRUTH TABLE FOR IMPLICATION

If α and β are statements, then "$\alpha \rightarrow \beta$" should be another statement. In other words, "$\alpha \rightarrow \beta$" must be either true or false. In that case we must agree on a way of determining whether "$\alpha \rightarrow \beta$" is true or false.

The definition we give may seem a bit strange at first, but keep in mind two things that may help you. In the first place the definition below is

precise. In the second place we want to use the implication $\alpha \rightarrow \beta$ only when the hypothesis α is presumed true.

Definition 2–5. $\alpha \rightarrow \beta$ is true whenever the statement "(not α) or β" is true. It is false when the statement "(not α) or β" is false.

The truth table, then, is as follows.

α	β	$\alpha \rightarrow \beta$	(not α) or β
T	T	T	T
T	F	F	F
F	F	T	T
F	T	T	T

The last line may surprise you. Just remember that $\alpha \rightarrow \beta$ means that β is true or α is false and that β is true *if* α is true.

Exercise Group 2–9

The exercises referred to below are those in Exercise Group 2–8.

1. Write each statement $\alpha \rightarrow \beta$ in Exercises 1 through 7 in the "(not α) or β" form.

2. Re-examine the validity of the conclusions in Exercises 8 through 18.

3. A statement $\alpha \rightarrow \beta$ is also sometimes expressed as "α is sufficient for β," or "α suffices for β," or again, "for β to be true it is sufficient that α be true." Thus for x^2 to equal 4 it is sufficient that $x = -2$. Express the implications in Exercises 1 through 7 in this language.

4. A statement $\alpha \rightarrow \beta$ is also sometimes expressed as "β is necessary for α" or "in order that α be true it is necessary that β be true." Thus for x to equal -2 it is necessary that $x^2 = 4$. Express the implications in the Exercises 1 through 7 in this language.

2–5 THE CONVERSE OF AN IMPLICATION

From one implication "$\alpha \rightarrow \beta$" we can form others which may or may not be true. One way of doing this gives what we call the *converse*.

Definition 2–6. The *converse* of "$\alpha \rightarrow \beta$" is "$\beta \rightarrow \alpha$."

Example 1. $\alpha \rightarrow \beta$: If an animal is a man then it is a human being.

Converse: $\beta \rightarrow \alpha$: If an animal is a human being then it is a man.

In this example "$\alpha \rightarrow \beta$" is true but "$\beta \rightarrow \alpha$" is false.

Example 2. $\alpha \rightarrow \beta$: If $\triangle ABC$ is equilateral then $\triangle ABC$ is equiangular.

Converse: $\beta \rightarrow \alpha$: If $\triangle ABC$ is equiangular, then $\triangle ABC$ is equilateral.

In this example both "$\alpha \rightarrow \beta$" and "$\beta \rightarrow \alpha$" are true.

Exercise Group 2–10

Write the *converses* of the following statements. In each exercise decide whether the statement is true or false. Do the same thing for the converse.

1. Every man born in Indiana is an American citizen.
2. If $x - y = 3$, then x is greater than y.
3. If two rectangles have equal bases and equal altitudes, they have equal areas.
4. If $x^2 = 9$, then $x = 3$ or $x = -3$.
5. If a man is a Polynesian, he is at least six feet tall.
6. If plants are green, they must have had sunshine.
7. If a point is on the perpendicular bisector of a line segment, it is equidistant from the endpoints of the segment.
8. If I am sleeping, then I am breathing.
9. If $2 + 4 = 5$, then $4 + 7 = 10$.
10. If $2 + 4 = 5$, then $4 + 7 = 11$.
11. If $2 + 4 = 6$, then $4 + 7 = 10$.
12. If $2 + 4 = 6$, then $4 + 7 = 11$.

2–6 EQUIVALENT STATEMENTS

When two people give the same mathematical proof they probably will not make exactly the same statements. But it is very likely that at certain places in their proofs they will make statements which we say are *equivalent* to one another. In order to understand precisely what this means we make the following definition.

Definition 2–7. Two statements α, β are said to be *equivalent* to one another if *both* the statement $\alpha \rightarrow \beta$ and the converse $\beta \rightarrow \alpha$ are true. If α and β are equivalent statements, we indicate this by writing $\alpha \leftrightarrow \beta$, which we read as "$\alpha$ implies β and β implies α."

This definition may be difficult to understand. An explanation is that "equivalent statements present the same information—they may simply be stated differently."

An example of a pair of equivalent statements is

α: Line l is perpendicular to line m.
β: Line m is perpendicular to line l.

Exercise Group 2–11

In 1 through 7 below decide whether the statements are equivalent to one another. Invent a few examples of pairs of statements which are equivalent to one another and some which are not.

1. α: Point P lies on the circle with center at O and radius five inches.
   β: The length of the line segment OP is five inches.
2. α: Line l is parallel to line m.
   β: Line n is parallel to line m.

3. α: x is greater than y.
   β: y is less than x.

4. α: $2x + 3 = 17$.
   β: $x = 7$.

5. α: This geometric figure is a square.
   β: This geometric figure is a special kind of parallelogram.

6. α: $\triangle ABC$ is equilateral.
   β: $\triangle ABC$ is equiangular.

7. α: $x = 3$ or $x = -3$.
   β: $x^2 = 9$.

*8. A statement $\alpha \leftrightarrow \beta$ is also sometimes expressed by saying, "In order that α be true it is necessary and sufficient that β be true." In each of the above problems state the equivalence of α and β in this language.

Exercises 4 and 7 above illustrate that equivalent statements have an important place in algebra. See if you can prove that the following statement is true:

If two statements α, β are equivalent to each other, then either both are true or both are false.

An easy way to prove this is to complete the following truth table. Can you understand how this table really presents a proof?

α	β	$\alpha \rightarrow \beta$	$\beta \rightarrow \alpha$	$\alpha \leftrightarrow \beta$
T	T	T	T	?
T	F	F	T	?
F	T	T	F	?
F	F	T	T	?

This truth table must be used together with Definition 2–7. By this definition we know whether to mark $\alpha \leftrightarrow \beta$ as true or false. Studying the table, we see that according to Definition 2–7, the statement $\alpha \leftrightarrow \beta$ must be labeled true on lines 1 and 4 and labeled false on lines 2 and 3. It should be easy for you to complete the proof.

Exercise Group 2–12

Form statements equivalent to those listed below. If you are not sure of the equivalance, make a truth table and check your work.

1. $x + y = 7$ and $x - y = 3$.
2. x is greater than y.
3. Line segments AB and CD do not intersect.
4. There is a point P which lies on both of the lines l and m.
5. Whether you are a smoker or whether you are not, whether you are married or single, whether you are a church member or not, you will like Dum-Dums.
6. This geometric figure has four sides, and pairs of opposite sides are parallel to one another.
7. That figure is a triangle.
8. This geometric figure consists of two points A and B and all the points between A and B.
9. The points A and B lie on the same side of the line l.
10. The points A and B lie on opposite lines of the line l.

2–7 THE CONTRAPOSITIVE OF AN IMPLICATION

A second important implication related to $\alpha \rightarrow \beta$ is its *contrapositive.*

Definition 2–8. The statement "not $\beta \rightarrow$ not α" is called the *contrapositive* of the statement "$\alpha \rightarrow \beta$."

Examples:

(a) $\alpha \rightarrow \beta$: $x = 3 \rightarrow x^2 = 9$.
not $\beta \rightarrow$ not α: $x^2 \neq 9 \rightarrow x \neq 3$.

(b) $\alpha \rightarrow \beta$: If two lines intersect they are not parallel.
not $\beta \rightarrow$ not α: If two lines are parallel they do not intersect.

(c) $\alpha \rightarrow \beta$: If I make an A on the examination I will receive an A in the course.
not $\beta \rightarrow$ not α: If I do not receive an A in the course then I will not have made an A on the examination.

(d) $\alpha \rightarrow \beta$: If I go to the game I will not go to the movie.
not $\beta \rightarrow$ not α: If I go to the movie I will not go to the game.

The most interesting property of the statement "not $\beta \rightarrow$ not α" is that it is equivalent to $\alpha \rightarrow \beta$. We can prove this by showing that these two implications have identical truth tables.

Problem. Complete the following truth table and show that "$\alpha \rightarrow \beta$" and "not $\beta \rightarrow$ not α" are equivalent to each other.

α	β	not α	not β	$\alpha \rightarrow \beta$	not $\beta \rightarrow$ not α
T	T	F	F	—	—
T	F	F	T	—	—
F	T	T	F	—	—
F	F	T	T	—	—

Exercise Group 2–13

Form the contrapositive of each statement in 1 through 8 below.

1. If $2 \times 2 = 4$, then oranges are composed of uranium.

2. If you can run the mile in four minutes, then I'll eat my hat.

3. If the ground hog sees his shadow, then we will have a cold spring.

4. If there is no atmosphere on Mars, then there is no life there.

5. If point P is on line r, then it is not on line s.

6. If you are not satisfied, then your money will be returned.

7. If food is sweet, then it has honey in it.

8. If you always obey traffic laws, then you will not be given a ticket.

*9. Using the word *necessary*, state the contrapositive of the implication of Exercise 4.

*10. Using the word *sufficient*, state the contrapositive of the implication of Exercise 4.

2–8 "IF AND ONLY IF"

Consider the two statements "α if β" and "α only if β." Our experience in working with statements like these tells us that "α if β" means "$\beta \rightarrow \alpha$" and that "α only if β" means "$\alpha \rightarrow \beta$." If both of these statements are true, we have "$\alpha \leftrightarrow \beta$." If both of these statements are true, their conjunction is true.

Definition 2–9. "α if and only if β" means "$\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$," that is, "$\alpha \leftrightarrow \beta$."

Consider the following two statements. "Two lines are parallel if they do not intersect." "Two lines do not intersect if they are parallel." According to Definition 2–9 we may write the conjunction of these two statements as "Two lines are parallel if and only if they do not intersect."

Exercise Group 2–14

In each exercise below there are two statements. Formulate a single statement, using "if and only if," that has the same meaning as the conjunction of these statements.

1. If the weather is bad we will stay home. If we stay home the weather is bad.
2. If you are a sophomore then you will get a ticket. If you get a ticket then you are a sophomore.
3. If the weather is bad we will stay home. If the weather is not bad we will not stay home.
4. Everyone who receives more than 50% on this test will pass the course. All others will fail.
5. Your dog will obey you if you treat him well. If you do not treat him well he will not obey you.
6. I will marry you if you buy me a mink coat. If you don't buy me a mink coat I won't marry you.
7. If food is sweet then it has honey in it. Food with honey in it is sweet.

2–9 THEOREMS AND PROOFS

In Section 2–3 we said logic is a study of the rules for making correct statements. The Greek philosopher Aristotle was the first great student of logic. His work was not improved upon until modern times. In fact, all the logical concepts we have studied in this chapter belong to what we call *Aristotelian logic*, which is over 2000 years old and is just as useful today as it ever was. The logical principles of this chapter are all that you will need for many years. If you have found these ideas interesting, you may wish to take courses in logic later on in college. Whether or not you continue your study of logic, the work of this chapter should help you become careful in both speech and writing. In particular,

during the work in geometry this year, do not tolerate false statements. Remember that if one false statement is accepted as true then strange things can be proved. Let us strive then to express ourselves precisely.

Most of our geometrical work will consist of *proving theorems.* Theorems are statements. To *prove a theorem* is to show that it follows logically from our assumptions. To do so we employ our principles of logic. For example, *if* we know that the statement $\alpha \rightarrow \beta$ is true *and if* we know that α is true, *then* we can infer that β is true. In symbols,

$$\alpha \text{ and } (\alpha \rightarrow \beta) \rightarrow \beta.$$

Another technique of proof is the following: *If* we know that $\alpha \rightarrow \beta$ is true *and if* we know that β is false, *then* we can infer that α is false. Symbolically,

$$\text{not } \beta \text{ and } (\alpha \rightarrow \beta) \rightarrow \text{not } \alpha.$$

You need only look at the truth table for $\alpha \rightarrow \beta$ to realize that the two rules above are correct.

α	β	$\alpha \rightarrow \beta$
T	T	T
T	F	F
F	T	T
F	F	T

Problem. Explain how this truth table justifies the two methods of proof described above.

Let us illustrate how these proof techniques may be employed (although you have already reasoned in this way in many exercises in this chapter).

Example 1. The notice said, "If it rains on commencement day then the graduation ceremony will be held in the gymnasium." ($\alpha \rightarrow \beta$.) Today is commencement day and it is raining. (α is true.) Therefore, since both of the statements α and $\alpha \rightarrow \beta$ are true, we know that β is true. That is, the graduation ceremony will be held in the gymnasium.

Example 2. Jim says that -3 is a root of the equation $x^2 - x - 6 = 0$. If -3 is a root of the equation $x^2 - x - 6 = 0$, then $(-3)^2 - (-3) - 6 = 0$. ($\alpha \rightarrow \beta$.) But, $(-3)^2 - (-3) - 6 = 6$. (β is false.) Now, since $\alpha \rightarrow \beta$ is true and β is false, I know that α is false and Jim was wrong. The number -3 is not a root of the equation $x^2 - x - 6 = 0$.

In these examples we would say that we have proved the theorems,

1. The graduation ceremony will be held in the gym.
2. The number -3 is not a root of $x^2 - x - 6 = 0$.

We might have written out our proofs as follows, in a "statement-reason" form.

STATEMENTS	REASONS
1. Rain on commencement day → ceremony in gym.	1. Given.
2. It is raining on commencement day.	2. Given.
3. The ceremony will be in the gym.	3. Statements 1 and 2 and laws of logic.

STATEMENTS	REASONS
1. -3 is a root of $x^2 - x - 6 = 0 \rightarrow (-3)^2 - (-3) - 6 = 0$.	1. Definition of root.
2. $(-3)^2 - (-3) - 6 \neq 0$.	2. Arithmetic.
3. -3 is not a root of $x^2 - x - 6 = 0$.	3. Statements 1 and 2 and laws of logic.

Exercise Group 2–15

In each problem below you are given certain statements which we shall call true. Then a theorem is stated. In some cases the theorem may be false; in others there may not be enough information given to decide whether the theorem is true or false. Write out proofs. We refer to our given true statements in each problem as our *hypothesis* (hyp).

1. *Hyp:* Uncle George will take us skating today if Mother says we may go. Mother has said we may go.
   Theorem: Uncle George will take us skating today.
2. *Hyp:* If there is no oxygen upon the moon then there can be no animal life there. Tests have shown conclusively that there is no oxygen upon the moon.
   Theorem: There is no animal life on the moon.
3. *Hyp:* People with green eyes cannot be trusted. Joan has green eyes.
   Theorem: Joan cannot be trusted.
4. *Hyp:* People with blue eyes are honest. Henry is dishonest.
   Theorem: Henry does not have blue eyes.
5. *Hyp:* If Joe parts his hair on the left then he does not like spinach. Joe does not like spinach.
   Theorem: Joe parts his hair on the left.
6. *Hyp:* Cats with green whiskers have long tails. Cats who do not like milk do not have long tails. Simba is a cat who does not like milk.
   Theorem: Simba does not have green whiskers.
7. *Hyp:* A student who attends M.I.T. will spend more than \$1200 during the school year. Kate spent \$980 last year in school.
   Theorem: Kate did not attend M.I.T. last year.
8. *Hyp:* $x + y = 20$. $x - y = 4$.
   Theorem: $x \neq 11$.

9. *Hyp:* $2x - 3y = 7$. $x + 2y = 3$.
Theorem: $3x - y = 10$.

10. *Hyp:* Joe has 11 marbles: four blue ones and seven yellow ones. Joe gave his sister six marbles. Joe did not give all his blue marbles away.
Theorem: Joe does not have more than four yellow marbles left.

11. *Hyp:* Only those students whose test grades average more than 90% will receive A's for their semester's work in geometry. Jim's test grades have been 89, 84, 92, 85, and 100. Sally's scores are 91, 90, 89, 94, and 87.
Theorem: Sally will receive an A and Jim will not.

12. *Hyp:* The lovely actress Amanda Thistle has promised to marry her honest admirer Silas Moneybags only if he buys her a mink stole, a yacht, and the great Calcutta Emerald. Silas has purchased for his Amanda these items and many others. Amanda always keeps her promises.
Theorem: Amanda will wed Silas.

13. *Hyp:* $x + y = 10$. x and y are positive whole numbers. x is greater than y. $y + 4$ is greater than x.
Theorem: $y = 4$.

14. *Hyp:* The square of a positive number is positive. The square of a negative number is positive. The square of zero is zero. Zero is neither positive nor negative. k is a number. $k^2 = -1$.
Theorem: There are numbers which are not positive or negative or equal to zero.

15. *Hyp:* The product of two positive numbers is positive. The number a is positive. The product of numbers a and b is not positive.
Theorem: The number b is negative.

*16. *Hyp:* If a is any number, then $1 \cdot a = a$. If a, b, c are any numbers, then $ba + ca = (b + c)a$. $1 + 1 = 2$.
Theorem: For every number a, it is true that $a + a = 2 \cdot a$.

*17. *Hyp:* $4 + (-4) = 0$. $6 = 2 + 4$. For all numbers a, b, c, it is true that $(a + b) + c = a + (b + c)$. We always have $x + 0 = x$.
Theorem: $6 + (-4) = 2$.

*18. *Hyp:* If $8 + x = 0$ then $x = -8$. $8 = 3 + 5$. Numbers can be rearranged when we add them. $3 + (-3) = 0$. $5 + (-5) = 0$. $0 + 0 = 0$.
Theorem: $(-3) + (-5) = -8$.

*19. *Hyp:* If neither of two numbers is equal to zero then their product is not zero. The product of the two numbers a and b is zero. $b \neq 0$.
Theorem: $a = 0$.

*20. *Hyp:* Dogs with long ears have short tails. Dogs who chase rabbits never get measles. Dogs who do not chase rabbits have long tails. My dog Rover died of the measles.
Theorem: Rover did not have long ears.

REVIEW OF CHAPTER 2

The ideas in this chapter are not to be memorized as you may have memorized the addition and multiplication tables when you studied arithmetic. It is very important that you realize that *all* our complicated mathematical statements may be built up from simple statements using only the connectives "and," "or," "not," and "if . . . then." You should relate the precise meaning of other connectives like "only if" and "if and only if" to these four fundamental terms. *Do not* make any effort to memorize the meanings of these latter terms. You should remember the meaning of conjunction, disjunction, negation, implication, and equivalence. You should be able to form the converse or contrapositive of an implication. You should be able to give examples showing that two statements, "$\alpha \rightarrow \beta$," and the converse, "$\beta \rightarrow \alpha$," are not necessarily equivalent to each other, that is, that one might be true and the other false. You should see clearly that the implication "$\alpha \rightarrow \beta$" and its contrapositive, "not $\beta \rightarrow$ not α," really assert the same thing; hence they are equivalent statements. You should understand and be able to use the important rules of logic employed so often in proofs, namely,

1. α and $(\alpha \rightarrow \beta) \rightarrow \beta$.
2. not β and $(\alpha \rightarrow \beta) \rightarrow$ not α.

You should understand how to set up and use truth tables. For example, to set up a truth table for the statement "α and $\alpha \rightarrow \beta$," we could list the following:

α	β	$\alpha \rightarrow \beta$	α and $(\alpha \rightarrow \beta)$
T	T	T	?
T	F	F	?
F	T	T	?
F	F	T	?

Problem. Fill in the last column in the truth table above and show that when the statement "α and $\alpha \rightarrow \beta$" is true, then β is true also. Make a similar truth table to illustrate the second rule of logic listed above. Can you invent a rule of logic like these two and show that it is correct by setting up a truth table?

Review Exercises

1. Under what condition is "α and β" a true statement? Give examples.
2. Give an example to show why we call "α or β" true when both α and β are true.
3. What is the negation of not α? Give examples.
4. Make up a statement of the form "α and β." State its negation.
5. Make up a statement of the form "α or β." State its negation.
6. What is meant by "$\alpha \rightarrow \beta$"? Give two statements having this form, one of which is true and the other false.
7. Why is the statement "α or not α" true no matter whether α is true or false? Give examples to illustrate this.
*8. Construct a truth table to show under what condition "(α and β) or γ" is a true statement. Invent an example of this type.
9. Give an example of an implication "$\alpha \rightarrow \beta$" whose converse is true and one whose converse is false.
10. Form the converse of:
    (a) If a man is from Chicago then he is rich.
    (b) If she sings in the choir then I will not go to church.
    (c) If Smith is elected president he will get ulcers.
    (d) You must be a sophomore if you are taking geometry.
    (e) Only the brave are free.
11. Form the contrapositive of each implication in Exercise 10.
12. Write a statement equivalent to:
    (a) Sweet food always has honey in it.
    (b) My rheumatism never bothers me in warm weather.
    (c) In order to be healthy you must drink milk.
    (d) A grade of 80% on the final examination will be sufficient to pass the course.
    (e) Sophomores may come to the dance, but no one else will be permitted to do so.
13. Write the negation of:
    (a) Honesty is the best policy.
    (b) All math teachers are brutes.
    (c) The potato crop will fail and everyone will starve.
    (d) Roses are red and violets are blue, sugar is sweet and I love you.
    (e) Some cats can swim but no chickens can.

There are two statements in each exercise below. Calling the first statement α and the second one β, rewrite them in the symbolic form "α and β," "α or β," "$\alpha \rightarrow \beta$," "$\beta \rightarrow \alpha$," "$\alpha \rightarrow$ not β," "$\alpha \leftrightarrow \beta$," etc., as the case may be. For example, the statement, "A man is honest if he has big feet" would be written "$\beta \rightarrow \alpha$," not "$\alpha \rightarrow \beta$."

14. Vacation begins May 12 but we are not going any place.
15. A man on the faculty must park his car in the main parking lot.

16. You will not receive an A in geometry if you do not work this example correctly.

17. Mother will let me go to the dance if and only if I wash the dishes for a week.

18. Students who laugh at his jokes are the only ones who get good grades.

19. It is necessary that we have rain this week if the melon crop is to be good.

20. You will write a term paper or you will fail.

21. This is the last exercise and I am glad that it is.

ALGEBRA REVIEW

In the exercises below we indicate some of the ways that logical reasoning is used in algebra. First we shall list some statements that we agree to call *true* in algebra. Then we shall set up algebraic problems which provide opportunities to use these true statements and reason logically to correct conclusions. In what follows, all letters refer to integers, that is, whole numbers in the set $(\ldots, -3, -2, -1, 0, 1, 2, \ldots)$. Let us agree that the following statements are true:

1. For all integers a, b, c it is true that if $a + b = a + c$, then $b = c$.
2. For all integers a, b, c it is true that if $ab = ac$ and $a \neq 0$, then $b = c$.

Let us prove the theorem: If $x = 5$ then $3x + 4 = 19$.

Proof. If $x = 5$ then $3x + 4 = 3 \cdot 5 + 4 = 15 + 4 = 19$.

Let us prove the converse of the above theorem, namely: *If* $3x + 4 = 19$ *then* $x = 5$.

Proof. $3x + 4 = 19 \rightarrow 3x + 4 = 15 + 4$. Now, by statement (1) above we know that $3x + 4 = 15 + 4 \rightarrow 3x = 15$ and that $3x = 15 \rightarrow 3x = 3 \cdot 5$. Using statement (2) above, we have $3x = 3 \cdot 5 \rightarrow x = 5$. We could present this proof briefly in the form:

$$3x + 4 = 19 \rightarrow 3x + 4 = 3 \cdot 5 + 4 \rightarrow 3x = 3 \cdot 5 \rightarrow x = 5.$$

Exercises

1. Prove that if $x = 7$ then $4x + 5 = 33$.

2. Prove the converse of the statement in Exercise 1.

3. Prove that if $7x + 9 = 65$ then $x = 8$, and conversely.

4. Prove that $5x + (-3) = 27$ only if $x = 6$.

5. Prove that $x = 11$ is a necessary condition for $5x + 4 = 59$.

6. Prove that if $x = 11$, $5x + 4 = 59$.

7. Prove that $2x + 7 = 19$ if and only if $4x - 3 = 21$.

8. Prove that there is no number x such that $3x + 5 = 11$ and $4x - 1 = 3$.

9. What can you infer if it is known that $a \neq 0$ and $a(3b - 8) = a$?

10. What can you prove if it is known that $r(7b - 5) = r$?

11. It is given that $(3a + 12)(2b - 8) = 0$ and $b \neq 4$. What can be proved?

*12. It is known that $(2a - 3)(b - 3) = 2a - 2$. Prove that if $a = 1$ then $b = 3$. Prove that if $a \neq 1$ then $b \neq 3$. What are the values of a and b if $a \neq 1$? (Remember that a and b are whole numbers.)

CHAPTER 3

CONCERNING LINES

3–1 INTRODUCTION

We have seen in Chapter 1 how from our everyday experience it is natural to imagine the existence of things called points and lines. Even though one cannot "see" a point or a line, the study of such imaginary objects turns out to be practically useful as well as interesting.

Because the objects of study in geometry are inventions of the mind, we must state very carefully the properties we wish them to possess. Then if we start with enough information about points and lines, other properties of points and lines can be *proved*. Many of these provable properties will be surprising, and not at all obvious at the beginning. It is, of course, impossible to prove anything about points and lines unless we agree in advance about some properties that they are to have. These agreed-upon properties, or assumed properties, are called *postulates*, or *axioms*.* We begin the proper study of plane geometry in this chapter when we list our beginning postulates in Section 3–3. The postulates should be simple enough to seem almost obvious and yet must be sufficient for us to prove geometric facts that seem to be true in our everyday life.

A different method of procedure would be to list all the known geometric facts and assume that they are true. Then when we had memorized all these facts, our study of geometry would be completed. There are at least three objections to such a plan. In the first place, we might not have listed all the facts needed and we would have no experience in finding new ones. In the second place, it would be a terrible strain upon the memory. To avoid memorizing everything, we need to study how the different facts are related and how the more complicated ones depend upon other simpler ones. When we prove theorems from postulates or other theorems, we are studying the way the geometric facts are related

* The word "postulate" is derived from a Latin word meaning "demand." 'Axiom" comes from a Greek word meaning "worthy."

to each other; this helps us to remember them. In the third place, we would lose the great amount of pleasure that we can get out of proving from very simple and obvious assumptions that certain other things are true. It is because of this pleasure that geometry has been so attractive to men since the time of Euclid. The fact that all geometry can be established from a small list of postulates gives to geometry a kind of beauty that has appealed to men as different as Abraham Lincoln and the Greek philosopher Plato.

3–2 STRAIGHTNESS

Soon, when we list our postulates, we shall be talking about *lines*, and of course what we have in mind are things which are like the familiar *straight* lines of our experience. But we will call them simply *"lines."* There is really nothing to be gained by putting in the extra word "straight" because it would add nothing to our use of the postulates. It will turn out much later that our lines *are* straight in the sense that any path between two points, other than a line, would be longer than the line.

When we draw pictures of lines we will, of course, draw them to *appear* straight, for this will help us see how the points and lines are related. But we will not be concerned with the straightness of the lines because the figures are meant to be only a guide to the proofs and to help us see what is going on. *The proofs are all completely independent of the figures.* By this we mean that we must not say something is true because of the appearance of a figure; rather, we must give reasons for its truth. The things that are true in our geometry are only those things which can be proved by our use of our assumptions and laws of logic. But as we progress in our study of geometry we will gradually use figures more and more freely when it is quite clear how to fill in a complete argument.

To understand why it is that we are not concerned with straightness at all, at least to begin with, we shall look at a plane in two different ways. First, imagine a sheet of paper with lines drawn on it so that some of the lines might look like Fig. 3–1. Now suppose you have a pane of uneven glass between your eyes and the paper, so that when you look at the drawing it looks like Fig. 3–2. Note that in looking at the drawing, with or without the glass, you are seeing the same lines. In one case they appear

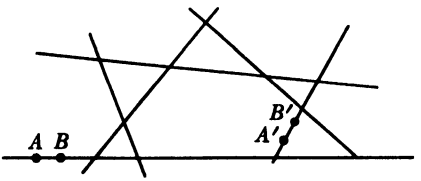


FIGURE 3–1

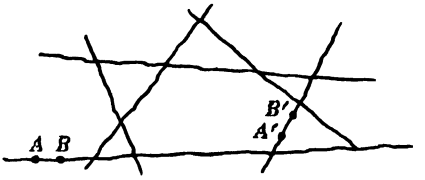


FIGURE 3–2

straight; in the other case they do not. In the first figure A and B may appear to be the same distance apart as A' and B', but this may not be so in the second figure. Therefore, *if it were possible to see the plane only through the glass, you would have to be told which sets of points are "straight" lines, and whether or not A and B are the same distance apart as A′ and B′.*

In our study of geometry we shall be talking about imaginary things—points and lines. We cannot really *see* our lines. Hence we shall not call them "straight" but refer to them only as *lines.* All the properties that we need to know about these lines will be stated clearly in the postulates.

3–3 THE POSTULATES

We imagine that there is a *set* (or "bunch," or "collection") of things which we call *points.* This set of points will be called *the plane.* In the plane certain other sets of points are called *lines.* We represent points by capital letters and lines by small letters. When a point P is one of the points of the line l (Fig. 3–3), we say that "P is on l" and also that the line "goes through P," or "passes through P," or "l contains P." If two lines l and m have a point P in common (Fig. 3–4), that is, if P is on both l and m, then we say that the lines *intersect.* Sometimes we will label two points with the same letter but different *subscripts,* such as P_1 and P_2. We read P_1 as "P sub-one" and P_2 as "P sub-two."

FIGURE 3–3

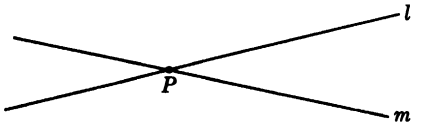


FIGURE 3–4

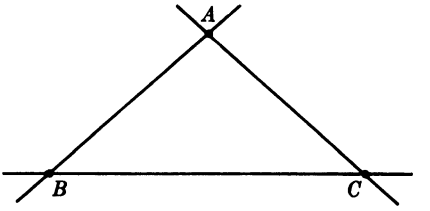


FIGURE 3–5

The postulates will be divided into six groups, of which only the first group will be given in this section. We shall give the other groups as we need them. It is extremely important at the beginning of your study of geometry that you understand the relationship between our mathematical postulates and the real world. We decide *what* we are going to assume about the imaginary points and lines of our geometry by *looking* at the real world. As a matter of fact, in order to choose the postulates for our geometry, we need only to look at some simple drawings. For example, the postulates in our first group are suggested by Fig. 3–5. Study this figure to see if you can decide what these postulates will be.

Of course, after we have decided what postulates to use, we must prove our theorems by *using these postulates and not by looking at other figures.*

Postulate Group I. This group contains three postulates, indicated by I–1, I–2, and I–3.

I–1 *The three-point postulate.* There are at least three points which do not all lie on one line.

I–2 *The point-line postulate.* For any two different points there is exactly one line containing these points.

I–3 *The line-point postulate.* Every line contains at least two points.

The point-line postulate tells us one way we want lines to behave. The line-point postulate *starts* us on the way to getting points on a line. The three-point postulate tells us that not all points are on one line. It gets us off the line.

With only these postulates we cannot as yet prove very much. The following two theorems are about all that can be formulated.

Theorem 3–1. *Two different lines do not intersect in more than one point.*

Proof. The theorem is either true or false. We shall prove that it cannot be false. Hence it must be true. We therefore suppose that the lines intersect in at least two points and then show that this is impossible. Suppose the lines are l and m and that they intersect in the two points A and B (Fig. 3–6). We would then have two different lines, l and m, through A and B. This contradicts the point-line postulate. Therefore our assumption that l and m intersect in more than one point is false and our theorem is true.

Remark. Several ideas from logic are involved in this proof. Comment on them.

Theorem 3–2. *There are at least three lines in the plane.*

FIGURE 3–6

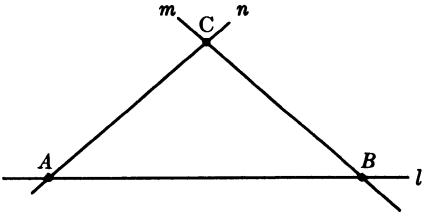


FIGURE 3–7

Proof.

STATEMENTS	REASONS
1. There are at least three points in the plane not all on one line. Call them A, B, and C.	1. Three-point postulate.
2. Let l be the line through A and B, m the line through B and C, and n the line through C and A.	2. Point-line postulate.
3. The three lines l, m, and n are all different from one another.	3. If any two were the same line, all three points A, B, and C would be on one line, contrary to the way they were chosen.

The three points and lines are shown in Fig. 3–7. Of course, it looks just like the drawing (Fig. 3–5) that suggested our first postulates.

Remark. At this point we cannot prove from our postulates that there are any more points or lines than shown in Fig. 3–7.

Exercise Group 3–1

1. Restate the point-line and the three-point postulates in other words.

2. Prove that if l is any line, there is at least one point not on l.

3. Explain why both a and b in Fig. 3–8 cannot represent lines if P and Q represent points.

FIGURE 3–8

4. Prove that if l is a line and P is on l then there is another point on l.

5. Why can we not prove there are more than three lines in our plane?

6. State what "intersect" means.

7. Suppose that l is a line. List all the things you can say about this line and points of the plane. Give your reasons.

8. Given Fig. 3–9, what can you conclude about l and m? Why?

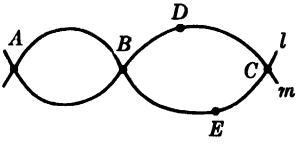


FIGURE 3–9

9. A line l contains the points P and Q. The line m also contains P and Q. What can you conclude?

10. Two lines have a point A in common. What can you say about the lines?

11. Prove that there are at least two lines on each point.

12. Why, when looking at Fig. 3–10, can we not prove that there are at least five points in the plane?

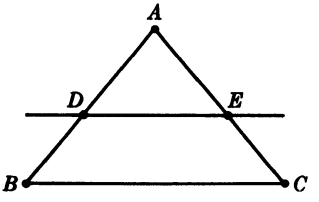


FIGURE 3–10

3–4 POSTULATES OF BETWEENNESS

We come now to our assumptions about the order, or arrangement, of points on a line. It is interesting that Euclid did not state these assumptions, but used them, without comment, wherever necessary. It was the German mathematician Moritz Pasch (pronounced "posh") who first, in the latter part of the 19th century, made a study of the properties of betweenness actually needed in geometry. The postulates given below are a modification of those of Pasch. You will have observed that the postulates of Group I seemed very obvious, and perhaps those of Group II will also. But remember that we are trying to state clearly what we imagine to be true about a plane, and hence we must be careful to state *everything* we are going to use.

Postulate Group II. Betweenness. There is a relation, called *between*, such that one of three points is sometimes between the other two. Of course, if we consider *any* three points, it does not *have* to happen that any one is between the others. But if for three points A, B, and C it is true that B is between A and C, we shall indicate this by writing "ABC" or "CBA," that is, with B in the middle.

II–1 *The ABC betweenness postulate.* If ABC, that is, if B is between A and C, then A, B, and C are three different (distinct) points on a line. (A drawing of the line and points might look like Fig. 3–11.)

FIGURE 3–11

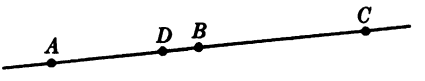


FIGURE 3–12

II–2 *The three-point betweenness postulate.* For every three points on a line, exactly one of them is between the other two. That is, if R, S, and T are on one line, then exactly one of the statements RST, STR, and TRS is true.

II–3 *The four-point betweenness postulate.* Any four points on a line may be labeled A_1, A_2, A_3, A_4 so that the only betweenness relations are the same as the order of the subscript numbers. That is, the only betweenness relations are $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, $A_2A_3A_4$. Of course these may all be reversed as $A_3A_2A_1$, $A_4A_2A_1$, $A_4A_3A_1$, $A_4A_3A_2$. A similar statement is to hold for any five points on a line, and in fact for any finite number of points on a line.

II–4 *The line-building postulate.* If A and B are two points, then there is at least one point C such that ABC, and at least one point D such that ADB. (See Fig. 3–12.)

Before stating the last of the betweenness postulates, we need some definitions.

Definition 3–1. Two points not on a line l are said to be *on the same side of* l if no point of l is between the two points. (See Fig. 3–13.) Two points not on a line l are said to be *on opposite sides of* l if there is a point of l between the two points. (See Fig. 3–14.)

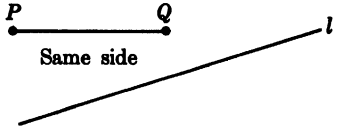


FIGURE 3–13

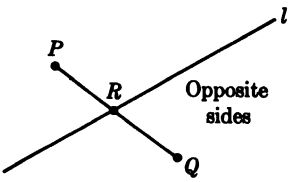


FIGURE 3–14

II–5 *The two-side postulate.* Each line separates the plane. By this we mean that all points of the plane not on the line are divided into two sets, called the two sides of the line, such that if P and Q belong to one of these sets, then P and Q are on the same side of the line, whereas if P belongs to one set and Q belongs to the other, then P and Q are on opposite sides of the line. (Briefly, this says that a line has two sides.)

Before using these postulates, let us first discuss their meaning in everyday life. If three men are standing in a line (Fig. 3–15) and A cannot see C because B is in the way, then we say that B is between A and C. On the other hand, B and C can see each other perfectly. Hence A is not between B and C. That is, we do not have BAC.

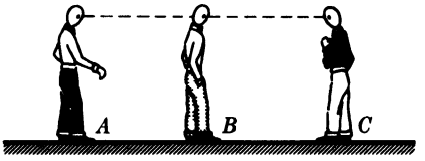


FIGURE 3–15

The ABC betweenness postulate says that to talk about betweenness, we must have the men "lined up." The three-point betweenness postulate says that when three men are lined up, exactly one of them is between the other two.

The four-point betweenness postulate gives a definite meaning to what we expect to see when we say, "Line up these four people." It tells exactly who is between whom. An easy way to remember the content of this postulate is to observe that of four points on a line, two of them

are never between any others. These are called the *endpoints*. The other two points are between points exactly twice.

The line-building postulate tells us first that there is always a point C "beyond" B from A. In fact, Euclid stated (vaguely) something very much like this as a postulate. But our postulate also tells us that there are points between any two. Euclid never stated this. It is this postulate that gives the existence of more than two points on a line, and therefore it is called the line-building postulate. You might think about the relation between this postulate and the real world. When you draw pictures of points, can you always mark a picture of a third point between any two?

The two-side postulate says that a line has two sides. It merely expresses the way we tell whether or not two of us are on the same or on different sides of a fence. If, when we look at each other, we see the fence between us, we know we are on opposite sides of it. If we do not see the fence between us, we are on the same side.

Now, using these postulates, we can define some familiar geometric objects.

Definition 3–2. If A and B are any two points, then the set of points consisting of A and B and all the points between A and B will be called the *line segment* (or *segment*) determined by A and B. The segment will be indicated by "AB." The points A and B are called the *ends* of the segment. In Fig. 3–16, P is a point of the segment AB.

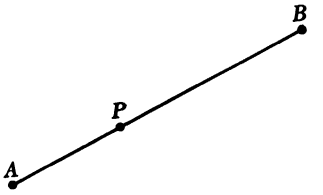


FIGURE 3–16

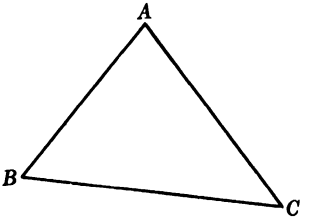


FIGURE 3–17

Definition 3–3. If A, B, and C are three points not all on one line,* the set of points of the segments AB, BC, AC is called a *triangle* (Fig. 3–17). We call the points A, B, and C *vertices* of the triangle and the segments AB, AC, BC *sides* of the triangle. To indicate "triangle ABC," we write "$\triangle ABC$."

Before working the exercises, read all the postulates again to be sure that you understand them.

---

* Points all lying on a line are said to be *collinear*. Hence in the definition we could have said, "If A, B, and C are not collinear . . .".

Exercise Group 3–2

1. Draw a line and mark five points on it. Write out all the betweenness relations.

2. If ABC and ACD, prove that the four points A, B, C, and D lie on one line.

3. Give in your own words the definitions of "segment," "triangle," and "two points are on the same side of a line."

4. If A, B, C, and D are four points of a line and we know that CAD and ADB, what are the other betweenness relations? Draw a picture. Is the picture proof?

*5. If A, B, C, and D are four points of a line and we know that CAD and CAB, what are the other betweenness relations?

6. If E, F, G, and H are four points on a line and we know that EFG, how many possible positions are there for H that can be described in terms of our betweenness relations?

7. A and B are points on a line l. Write three true statements about points either on l or not on l. Give reasons, as below.

STATEMENTS	REASONS
1. There is a point C such that ACB.	1. The line-building postulate.
2. _____	2. _____
3. _____	3. _____
4. _____	4. _____

8. If RST and VRS, relabel the points R, S, T, and V as A_1, A_2, A_3, and A_4 (in some order) in accordance with II–3. If CAD, BDE, and BCA, relabel the points A_1, A_2, A_3, A_4, and A_5 in accordance with II–3.

9. Prove that there are at least four lines on each point.

10. If DEF and EGF, then the four-point betweenness postulate implies that _____ and _____.

11. If RST, then what postulate tells us that R, S, and T are distinct points on a line?

*12. Draw a line and choose two points on it, A and B (Postulate I–3). Mark a point C such that ABC (Postulate II–4). Mark a point D such that BCD (Postulate II–4). Is it true that ACD? Why? Use II–3 to prove that ACD.

*13. Draw a line and choose two points on it, A and B. Choose a point C such that ACB (Postulate II–4). Now choose a point D such that CDB. Is it true that ACD? that ADB? Use II–3 to prove this.

Now choose a point E such that DEB. What are all the betweenness relations between these five points? Can this process be indefinitely repeated? Describe, in your own words, the meaning of "this process." How many points are there on a line segment?

*14. Draw a line and choose a point A on one side and a point B on the other side. Now mark a point C on the opposite side of the line from A. Points B and C will be on the same side of the line. How does this follow from II–5?

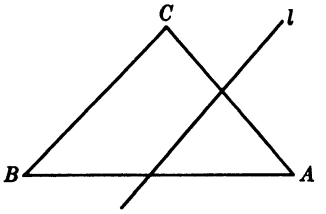


FIGURE 3–18

*15. A line l (Fig. 3–18) does not contain any of the vertices of a triangle. The line l does intersect the side AB, but does not intersect the side BC. Prove that l must intersect AC.

*16. Draw a line and choose two points A and B on one side of the line. Choose C so that ACB. Can you prove that C and A are on the same side of the line?

*17. If A is on line l and B is not on l and ACB, prove that C and B are on the same side of l.

*18. Prove that for every line l there is a point on each side of l.

3–5 RAYS AND ANGLES

It is a familiar fact from experience that a point O on a line l divides the other points of the line into two sets—those on one side and those on the other side of the point O on the line l. (See Fig. 3–19.) From our postulates on betweenness we can now make precise definitions of this observation. First we need to know that a point on a line separates the line.

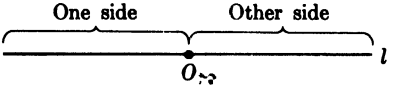


FIGURE 3–19

Theorem 3–3 The two-side theorem. *Let O be any point of a line l. The points of l, different from O, are separated into two sets such that:*
(1) *If P and Q are in the same set, then either OPQ or OQP.*
(2) *If P is in one set and Q is in the other set, then POQ.*

**Proof.* Choose a point A not on l and consider the line m through O and A (see Fig. 3–20). We are using I–1 and I–2. By the two-side postulate, II–5, the line m separates the plane into the two sides of m. Let one of the sets, which we seek, consist of the points of l which are on one side of m, and let the other set consist of the points of l on the other side of m.

If P and Q are in one of these sets, they are by definition on the same side of m, and therefore O is not between them. Hence by II–2 we must have either OPQ or OQP. This proves (1).

Now suppose that P is in one set and Q in the other set (see Fig. 3–21). Then by definition of the sets, P and Q are on different sides of m. By

* Starred proofs are difficult and may be omitted. However, the meaning of the theorems must be understood.

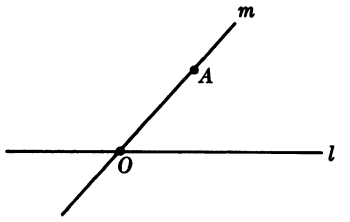


FIGURE 3–20

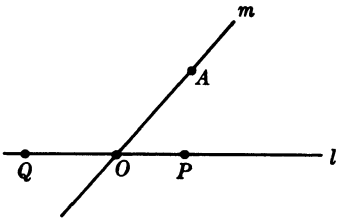


FIGURE 3–21

the two-side postulate, II–5, there is a point of m between P and Q. But l and m have only point O in common. Hence the point between must be O, and we have POQ. This proves (2).

Remark. The two sets of the theorem are not dependent on how we choose the line m, because we end up with the same betweenness relations.

Definition 3–4. The two sets of Theorem 3–3 are called the two *sides* of O on the line l.

Definition 3–5. A *ray* from O, or a ray with end O, is a set of points consisting of O and all the points on one side of O of a line l through O (see Fig. 3–22).

If P is any point of the ray, we may speak of "ray OP." Note that we may describe the ray as the set of points consisting of O and P and all points Q such that OQP and all points R such that OPR. Refer back to Theorem 3–3.

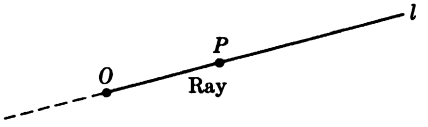


FIGURE 3–22

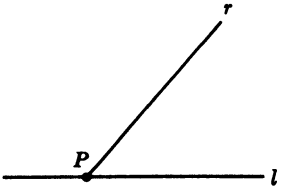


FIGURE 3–23

Exercise 3–3

Consider a line l and a ray r, not on l, from a point P on l, as in Fig. 3–23. What do we mean by "r's side of l"?

Definition 3–6. The set of all points on two rays from a point is called an *angle.* The point is called the *vertex* of the angle, and the two rays are called the *sides* of the angle. If the two rays are on the same line, the angle is called a *straight angle.* If O is the vertex and A and B are points other than O on the sides of the angle, we shall speak of "angle AOB" and write this as "$\angle AOB$." (See Fig. 3–24.)

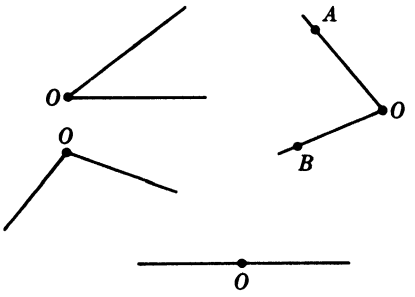


FIGURE 3–24

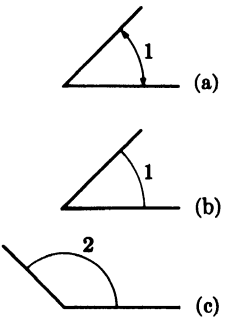


FIGURE 3–25

Remark. Note that what we call the angle is the set of points in the *sides* of the angle. Points "between" the rays are *not* part of the angle. Later we discuss the inside and the outside of an angle. It will be convenient to have a way to clearly indicate the sides of the angles. This can be done as in Fig. 3–25(a), and the angle can be numbered. Since the arrowheads indicating the sides are not vital, we can as well use the simpler notation shown in Fig. 3–25(b) and (c). *Remember that this notation is just a way of indicating the sides of the angle.*

Definition 3–7. If two lines l and m intersect at O and the four rays from O are as indicated in Fig. 3–26, then angles 1 and 2 are said to be *vertical* with respect to each other. For brevity, we say that 1 and 2 are a pair of vertical angles. Angles 3 and 4 are also vertical to each other.

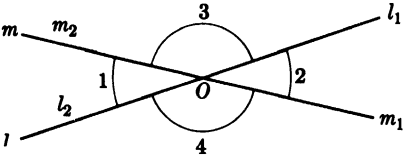


FIGURE 3–26

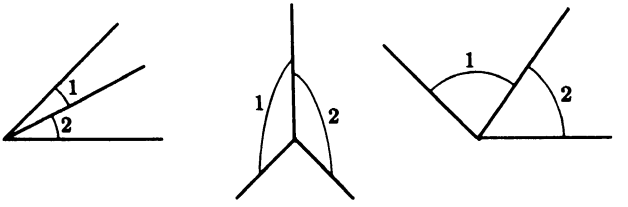


FIGURE 3–27

Definition 3–8. If two angles have a common vertex and a common side and if the other sides are on opposite sides of the line on the common side, then the angles are said to be *adjacent* to each other (Fig. 3–27).

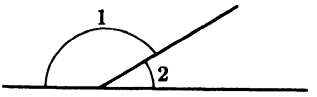


FIGURE 3–28

Definition 3–9. Two adjacent angles whose noncommon sides form a straight angle are said to be *supplementary and adjacent* to each other (Fig. 3–28).

Exercise Group 3–4

1. State in your own words the definitions of "ray," of "angle," and of "straight angle."
2. How "long" are the sides of an angle?
3. If $\angle AOB$ and $\angle DOC$ are vertical to each other with B, O, D collinear, what kind of angle is $\angle AOC$?
4. If $\angle AOC$ is a straight angle and OR is a ray from O, how are angles AOR and ROC related?
5. Two lines intersect at O. With the four rays from O, how many different angles with vertex O can be found? Pick out the pairs of vertical angles.
6. Three lines intersect at O. With the six rays from O, how many different angles with vertex O can be formed? How many pairs of vertical angles are there?
7. Draw two angles with a common vertex but which are not adjacent to each other.
8. Draw two angles with a common vertex and side but which are not adjacent.
9. State Definitions 3-7, 3–8, and 3–9 in your own words.
10. If $\angle AOB$ is not a straight angle, how many angles are there that are vertical with respect to $\angle AOB$?
11. Why is it not sensible to say, "$\angle AOB$ is a vertical angle"?

3–6 INTERIOR AND EXTERIOR

When we draw a picture of an angle (not a straight angle) on a sheet of paper, we can tell by looking at it which points of the plane are inside and which are outside the *picture* of the angle. But in our development of geometry starting from but a few postulates, we must *define* what these sets are. (Why?) Definition 3–10 is one that can be understood by a blind person provided he knows what an angle is and that a line separates the plane. To appreciate the definition, consider Fig. 3–29. The point D lies on one side of l, which side has been shaded with vertical lines. The point C lies on one side of m, which side has been shaded with horizontal lines. The points in the plane which are shaded twice (i.e., where the vertical and horizontal lines cross) fill up the *interior* of $\angle COD$.

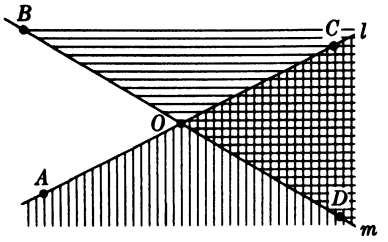


FIGURE 3–29

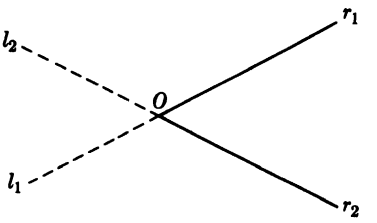


FIGURE 3–30

Exercise Group 3–5

1. Copy the lines l and m and the points A, B, C, D on a sheet of paper and shade all the points of the plane which are (a) on the same side of l as B and the same side of m as A, (b) on the same side of l as B and on the same side of m as C, (c) on the same side of l as D and on the same side of m as A.

2. In each case name the angle whose interior has been described.

Definition 3–10. Consider an angle, not a straight angle, with vertex O and whose sides are the rays r_1 and r_2 on the lines l_1 and l_2, respectively. The *interior* of the angle is the set of all points on the same side of l_1 as r_2 and on the same side of l_2 as r_1. The *exterior* of the angle is the set of all points in the plane which are not points of the angle and are not in the interior of the angle. (Fig. 3–30.)

There are several properties of the interior and exterior of angles that we can now prove. These properties are all quite obvious when we look at a figure. But even when something seems obvious from a picture, we must be sure that we have the right postulates so that our geometry agrees with our feelings about what should be true—that is, with the picture. It is really not at all obvious that our *geometry* agrees with our *picture*. Let us see that this is so by proving some "obvious" theorems.

In all these theorems we shall be concerned with an angle whose vertex is O, which is not a straight angle, and whose sides are the rays r_1 and r_2.

***Theorem 3–4.** *If P and Q are both in the interior of $\angle O$, then the segment PQ is in the interior of $\angle O$.*

Proof. We must show that if A is any point between P and Q, then A is in the interior of the angle (Fig. 3–31). This is easy if we use the

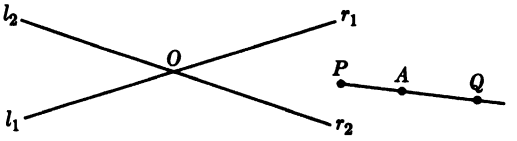


FIGURE 3–31

result of Exercise 16 in Exercise Group 3–2. By that exercise, since P and Q are on one side of l_1, then A is also on that same side of l_1. Similarly, A is on the same side of l_2, as are both P and Q. Hence, by the definition of *interior* of an angle, A is in the interior of the angle formed by rays r_1 and r_2.

***Theorem 3–5.** *If r is a ray from O other than r_1 and r_2, then either all the points of r other than O lie in the interior of $\angle O$ or they lie in the exterior of $\angle O$.*

Proof. Let r_1' and r_2' be the rays from O on l_1 and l_2 other than r_1 and r_2. (See Fig. 3–32). If $r = r_1'$ or $r = r_2'$, then r is in the exterior. If r is neither r_1' nor r_2', then r lies on a definite side of l_1 and a definite side of l_2 because of the way a ray is defined. Hence r is in either the interior or exterior of $\angle O$.

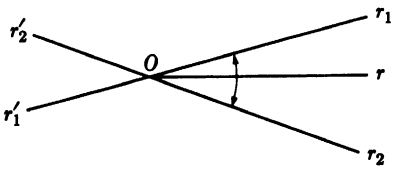


FIGURE 3–32

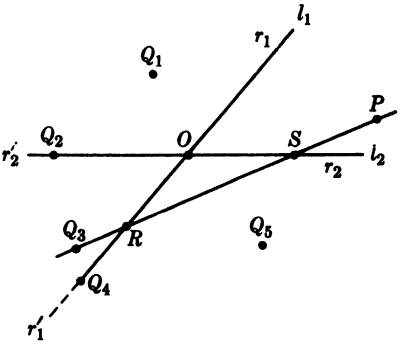


FIGURE 3–33

***Theorem 3–6.** *If P is in the interior of $\angle O$ and Q is in the exterior of $\angle O$, then the segment PQ contains a point of the angle.*

Proof. There are five possible locations of the point Q, represented by the points $Q_1, \ldots, Q_5$ in Fig. 3–33. (Describe these positions in terms of sides of lines.) For positions Q_1, Q_2, Q_4, Q_5 the proofs of the theorem are quite easy and are assigned as exercises below. Hence we restrict our attention here to the position Q_3.

Since Q_3 is on opposite sides of both l_1 and l_2 from P (why?), the segment Q_3P must contain a point R of l_1 and a point S of l_2. We assert that either R or S is a point of the angle, that is, a point of either r_1 or r_2. If R is on r_1, our segment contains a point of the angle. Suppose that R is not on r_1. Then R is on r_1' and hence has the same relative position as Q_4. (Why?) But then the segment RP contains a point of the angle which must be S. (Why?) Since Q_3RP and RSP, we have Q_3SP. (Why?)

Definition 3–11. The *interior* (or *inside*) of a triangle is the set of all points which are interior to each of the angles of the triangle. The *exterior* of a triangle is the set of points which are neither on the triangle nor in the interior.

If the triangle is ABC (see Fig. 3–34), an alternative definition of interior would be "the set of all points on the same side of AB as C, on the same side of BC as A, and on the same side of CA as B." Show that these two definitions say the same thing.

We can prove the following remark by using Theorem 3–6. The remark is an "obvious" theorem that is hard to prove. We will not need this result later.

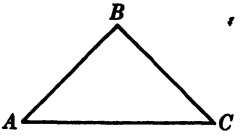


FIGURE 3–34

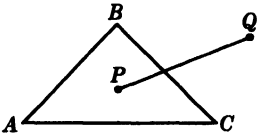


FIGURE 3–35

Remark. If P is in the interior of $\triangle ABC$ and Q is in the exterior, then the segment PQ contains a point of the triangle (Fig. 3–35).

Exercise Group 3–6

1. Referring to Fig. 3–36, draw an angle with vertex O. Pick a point A on one side of $\angle O$. Draw an angle with vertex A with one side the ray AO and the other side a ray intersecting the other side of $\angle O$ in B. Shade the interior of $\angle O$ and the interior of $\angle A$. Are points in both interiors also in the interior of $\triangle OBA$?

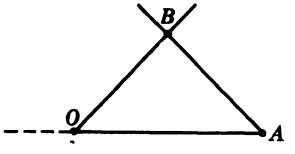


FIGURE 3–36

2. Draw a $\triangle ABC$. If a point P is on the same side of AB as C and on the same side of BC as A, show that P is not necessarily on the same side of AC as B.

3. Draw two angles whose interiors have no points in common.

4. A straight angle does not have an interior. Tell why we do not try to define interior for a straight angle.

5. State in your own words the definitions of "interior of an angle" and "interior of a triangle."

6. For a $\triangle ABC$, the point P is in the interior of $\angle A$ and not in the interior of $\angle B$. Can you be sure that P is in the exterior of $\triangle ABC$?

7. Consider the angle of Fig. 3–37, with sides l_1 and m_1 indicated on lines l and m. Tell where the point P is if:

(a) P is on the opposite side of m from l_1.

(b) P is on the opposite side of l from m_1.

(c) P is on l_1.

(d) P is on m_1.

(e) P is on the same side of l as m_1.

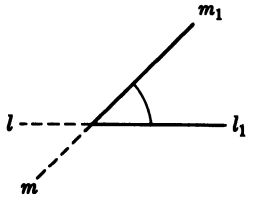


FIGURE 3–37

(f) P is on the same side of l as m_1 and on the same side of m as l_1.

8. Prove that the following definition is equivalent to Definition 3–11: The interior of $\triangle ABC$ is the set of all points in the interior of $\angle A$ and also in the interior of $\angle B$. (With this definition it is not too difficult to prove the truth of the remark preceding this group of exercises.)

9. In $\angle AOB$, not a straight angle, point P is between A and B. Prove that P is in the interior of the angle.

10. In triangle ABC we have the relations AXB, AYC, and XZY. Prove that Z is in the interior of the triangle.

11. If $\angle AOB$ is not a straight angle and we have the relations XOB and XYA, prove that Y is in the exterior of $\angle AOB$.

*12. In Theorem 3–6 suppose Q is in position Q_1, that is, on the same side of l_2 as r_1 and on the opposite side of l_1 as r_2 (see Fig. 3–38). Show that there is a point R of r_1 between Q and P. [*Hint:* Certainly there is a point R of l_1 between Q and P. Show that it is impossible that R be on r_1'.]

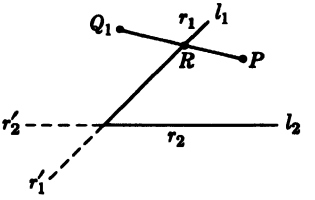


FIGURE 3–38

*13. In Theorem 3–6 suppose Q is in position Q_2, that is, on r_2' (see Fig. 3–39). Show that there is a point R of r_1 between Q and P.

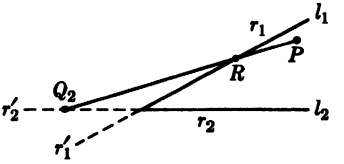


FIGURE 3–39

*14. Tell why the positions Q_4 and Q_5 of Fig. 3–33 are much the same as positions Q_2 and Q_1, respectively. If you have done positions Q_1 and Q_2, why are positions Q_5 and Q_4 easy?

REVIEW OF CHAPTER 3

1. The fundamental relations of betweenness are expressed in the postulates. Study them again to be sure you know them.

2. Definitions of the following concepts were given in this chapter. Be sure that you know all of them.

two sides of a line (see II–5)	straight angle
segment	vertical angles
triangle	adjacent angles
collinear (see footnote to Def. 3–3)	supplementary angles
	interior of an angle
two sides of a point on a line	exterior of an angle
ray	interior of a triangle
angle	exterior of a triangle

3. The following statements have been proved in this chapter. First re-read them, and then study their proofs again.

(a) Two different lines intersect in at most one point.
(b) There are at least three lines.
(c) A point O on a line separates the line into two sides.
(d) If P and Q are in the interior of $\angle O$, then the segment PQ is in the interior of $\angle O$.
(e) A ray from the vertex of an angle lies, except for the vertex, either in the interior or in the exterior of the angle.
(f) If P is in the interior of $\angle O$ and Q is in the exterior, then the segment PQ intersects the angle.

Important Remark

By assuming a few properties of points and lines, we have been able to *define* many familiar types of figures and to *prove* that these figures have properties that may seem obvious. *Therefore we have proved that some of our figures agree with our geometry.* From now on in the book, we will often determine properties of betweenness by just looking at figures. The definitions and theorems of this chapter will enable us to use our figures in the confident belief that we could always go back and supply proofs. Therefore, we have completed the first stage in our development of geometry. The theorems proved in this chapter need not be memorized because the facts are all familiar from drawings. However, the postulates and definitions of this chapter are very important, and therefore should be memorized.

Review Questions

1. If A and B are distinct points, what can you say about a point P which is on the segment AB? Why?
2. What postulate did we use to define the sides of a point on a line?
3. A line l contains the distinct points R and S. A line m also contains these points. What do you conclude?
4. If ABC, and B is on line l but A is not on l, is the point C on l? Why?

5. Tell whether the following statements are true or false, and give explanations for your answers.

(a) For $\triangle ABC$, if the point S is in the interior of $\angle A$, $\angle B$, and $\angle C$, then it is a theorem that S is in the interior of the triangle.
(b) If APB, then P belongs to the segment AB, by definition.
(c) If ARQ and AGR, then AGQ.
(d) A point P not on an angle must be in either the interior or exterior.
(e) We postulate that a point on a line separates the line.
(f) If AQB, then A and B are on a ray from Q.
(g) A line segment contains a definite number of points.

6. What does it mean to say that three or more points are noncollinear? How can we be sure that there are such sets of points?

7. If ADB and BAC, what are the other betweenness relations?

8. Lines s and t intersect at Q forming rays s_1, s_2 and t_1, t_2. Draw a figure and indicate the pairs of vertical angles. Which pairs of angles are adjacent? supplementary?

9. If A and B are on opposite sides of a line l, and B and C are on opposite sides of l, then A and C are on the same side of l. Why is this true?

10. Explain in your own words what it means for a point to be in the interior of an angle.

Review Test

1. Define the following: (a) angle, (b) segment, (c) when two points are on the same side of a line, (d) triangle.

2. Tell whether true or false:

   (a) A, B, C are three points on a line and ABC and CAB.
   (b) If PRU, then P, R, and U are distinct points on a line.
   (c) It is a theorem that a line separates the plane.
   (d) It is a theorem that a point separates a line.
   (e) It is a theorem that there is a point between any two points.
   (f) Betweenness is an undefined relation.
   (g) If ABC, then CBA.
   (h) If E, F, G are not on l and such that the segment EF does not intersect l, while the segment FG does intersect l, then the segment EG intersects l.

3. State in your own words:

   (a) The point-line postulate.
   (b) The ABC betweenness postulate.
   (c) The line-building postulate.

ALGEBRA REVIEW

In this review you are supposed to use the properties of addition and multiplication of numbers (both positive and negative), along with the definitions of subtraction and division, to make simple proofs. All letters here refer to rational numbers (fractions). These exercises show how definitions are used in algebra.

Definition. If $a = b + x$, then x is called *"a minus b"* and is written *"$x = a - b$."* It is also called *"b subtracted from a."*

1. If $c = p - q$, what is p? Why?

2. Why does $7 - 3 = 4$? Write $7 - 3 = x$, and ask what property x must have.

3. Using the definition, what can you say about these numbers: $\frac{9}{5} - \frac{3}{5}$? $\frac{1}{2} - \frac{1}{3}$? $\frac{7}{8} - \frac{3}{4}$? $4.678 - 2.589$?

Definition. $-b$ (called "*negative b*") is defined to be $0 - b$. That is, $-b = 0 - b$ and is a number such that $b + (-b) = 0$.

4. Why does $-7 = 0 - 7$? $-3 = 0 - 3$? $-\frac{1}{2} = 0 - \frac{1}{2}$?

5. Show that $9 + (-3) = 9 - 3$, $11 + (-8) = 11 - 8$, $\frac{5}{6} + (-\frac{1}{3}) = \frac{5}{6} - \frac{1}{3}$.

*6. Prove that $a + (-b) = a - b$. [*Hint:* Let $a - b = x$ so that $a = b + x$. Then $a + (-b) = b + x + (-b)$.]

7. Show that $-(7 + 3) = (-7) + (-3)$, $-(2 + 7) = (-2) + (-7)$.

8. Show that $-(a + b) = (-a) + (-b)$.

Definition. If $a = bx$ and $b \neq 0$, then x is called "*a divided by b*" and is written "$x = a \div b$" or "$x = a/b$."

9. Explain why in this last definition we required that $b \neq 0$. [*Hint:* Would x be definite otherwise?]

10. According to the definition, $\frac{1}{2}$ is a number such that $1 = 2(\frac{1}{2})$. What then are the numbers $\frac{3}{4}$, $-\frac{2}{3}$, $4/-5$, $\frac{7}{16}$, $\frac{5}{10}$, $\frac{1}{1}$, $\frac{2}{2}$?

11. Why does $b \cdot a/b = a$?

12. Prove that $a/b + c/b = (a + c)/b$.

13. Show that $\frac{1}{2} = \frac{2}{4}$. [*Hint:* Let $\frac{1}{2} = x$. Then $1 = 2 \cdot x$. Why? Hence $2 = 4x$. Why? Hence $\frac{2}{4} = x$. Why?]

14. Prove that if $b \neq 0$ and $c \neq 0$, then $a/b = ac/bc$.

CHAPTER 4

CONGRUENCE OF SEGMENTS

4–1 INTRODUCTION

In this chapter we shall be concerned with comparing the "sizes" of segments. All of us have learned to understand the terms "distance" and "length" as used in everyday life. For example, we may say that it is 12 miles to the nearest town, or that a fence is 153 feet long. In each case we give a number to indicate a distance or length. To give a number is a most convenient means of describing the distance between two points or, in other words, of telling how far apart they are.

But even before men learned to count, they must have wished at times to tell how far apart points were, or to decide if two objects had the same length. For example, a hunter visiting a friend in a neighboring village might have said to his friend, "My son is as tall as this mark on my spear." Then the men could see, by comparing with the mark, whether the host's son was as tall as the visitor's son. In this example the marked spear served as a kind of "yardstick" for comparing the height of the hunter's son with the height of others. Even in very primitive societies men must have had a means for comparing the sizes of objects.

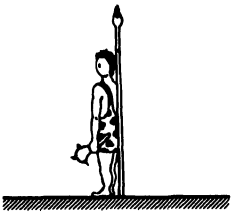


FIGURE 4–1

Problem. Describe other examples where objects are compared in size without the use of measurement, that is, a number. Some of your examples may use a marked stick (or "yardstick") and some not.

We are so used to speaking of distance or of length in terms of numbers that it is hard for us to find other words to describe how far apart two points are. Although we usually associate numbers with the words "distance" and "length," what we study in this chapter is so simple that it would be understandable to a person who knew nothing of numbers at all. We will be talking about the "size" of a segment and whether

two segments are "the same size." As we have seen with the example of the hunter, a practical method for comparing segments is to use a marked stick and to carry it from one place to another. We shall see in the Section 4–2 how in geometry we can avoid carrying such sticks.

Exercise Group 4–1

1. Your foot is a natural "yardstick" you carry with you. Describe several examples of using your foot for comparing lengths.

2. Think of several ways to use your hands as "yardsticks."

4–2 CONGRUENCE

In daily life we are often interested in size, and we have ways of testing whether or not two things are the same size. But our geometry so far has in it *no notion of size at all.* Because our geometry deals with imaginary concepts (points and lines), to talk about size it is necessary to assume (postulate) properties that we desire. We do not wish to talk about "carrying a segment around in the plane," because this involves the idea of movement, which is not a part of geometry proper. Consequently *we will just suppose that by some means (it does not matter how) we can tell whether or not two segments are the same size.* In other words, we will assume that we can pick out all the segments of the same size.

We have been using the words "the same size." We now use another word which is to have exactly the same intuitive significance. The word is *congruent,* and whenever you see it you can, if you like, say to yourself, "the same size." Congruence refers to the *relation* of being the same size. The word comes from a Latin word meaning "agree" or "conform."

Euclid used the word "equal" instead of congruent. However, *in this book we will use the word "equal" (and the sign "=") to mean "identically the same."* Thus, if we write $A = B$ we mean that A and B are one and the same point. Or, if x and y are numbers, $x = y$ means that they are one and the same number. And in dealing with segments, if we write $AB = CD$, we mean that $A = C$ and $B = D$ or $A = D$ and $B = C$, that is, AB and CD are one and the same segment.

4–3 POSTULATE GROUP III. LINEAR CONGRUENCE

We assume that there is a relation between segments called *congruence.* This relation is to have the following properties.

III–1 *The existence postulate for segments.* Given points A and B on a line l and a point A' on a line l' (Fig. 4–2), then on each side of A' on l' there is exactly one point B' such that AB is congruent to $A'B'$. To indicate that AB is congruent to $A'B'$, we write "$AB \cong A'B'$."

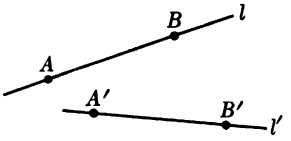


FIGURE 4–2

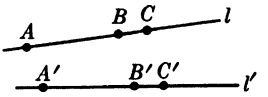


FIGURE 4–3

In congruence the order of the points does not matter, that is, if $AB \cong A'B'$, then also $AB \cong B'A'$, $BA \cong A'B'$, and $BA \cong B'A'$.

III–2 *The three-part segment postulate.* Congruence also satisfies the following:

(a) $AB \cong AB$. (A segment is congruent to itself; that is, congruence of segments is reflexive.)

(b) If $AB \cong A'B'$, then $A'B' \cong AB$. (Congruence of segments is symmetric.)

(c) If $AB \cong A'B'$ and $A'B' \cong A''B''$, then $AB \cong A''B''$. (Congruence of segments is transitive.)

Problem. Prove that if $AB \cong CD$ and $EF \cong CD$, then $AB \cong EF$. [*Hint:* Use parts (b) and (c) of III–2.]

III–3 *The addition and subtraction of segments postulate.* Suppose that B is between A and C on a line l, and that B' is between A' and C' on a line l' (Fig. 4–3).

(a) If $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.

(b) If $AC \cong A'C'$ and $BC \cong B'C'$, then $AB \cong A'B'$.

These postulates are somewhat like our earlier ones. They seem so obvious that you might feel they do not need to be stated. As before, however, you must remember that points and lines are inventions of the mind, and so we must state clearly what properties they are to have. Euclid did not state them as clearly as we have. The postulates given in this book are easier to use than those of Euclid because they tell you exactly what you can do. They are similar to ones given by David Hilbert (see Section 1–2).

In less exact language, the postulates say the following things. The first says that on the line l' there are two points B' and B'', one on each side of A', such that AB is the same size as both $A'B'$ and $A'B''$. The second one says three things: that a segment is the same size as itself; that if AB is the same size as $A'B'$, then $A'B'$ is the same size as AB; that if two segments are the same size as a third segment, then the two segments are the same size. The third postulate tells how to combine segments by what we call *addition and subtraction of segments.*

Exercise Group 4–2

1. What does it mean to say $RS = TU$?

2. If $RS = TU$, is $RS \cong TU$? Justify your answer.

3. Is it true for every pair of segments RS and TU that if $RS \cong TU$ then $RS = TU$?

4. Two segments AB and CD are not congruent to each other. (We write this as $AB \ncong CD$.) Prove that AB and CD are not equal to each other ($AB \neq CD$).

5. If ABC, what postulates must you use to prove that $AB \ncong AC$?

6. Is the following statement true or false? If AB is any segment and A' is any point on a line l', then there is one and only one point B' on l' such that $AB \cong A'B'$.

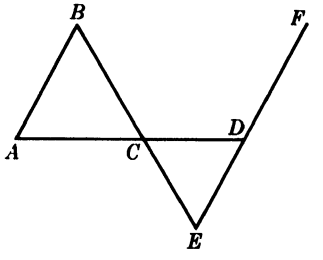


FIGURE 4–4

7. In Fig. 4–4 the following congruences hold: $AB \cong BC$, $CD \cong CE$, $CE \cong ED$, $EF \cong EB$. List all other congruences that you recognize.

8. The sign = was defined in this section. State the definition in your own words.

9. State the three postulates of linear congruence in your own words, using the phrase "the same size" instead of "congruent." Then go back and read again the paragraph preceding this group of exercises.

10. Show (using postulates) that if A, B, C, D are points, in this order, on a line l, and if A', B', C', D' are points, in this order, on a line l', and if $AB \cong A'B'$, $BC \cong B'C'$, $CD \cong C'D'$, then $AD \cong A'D'$. [*Hint:* Draw a figure. You will have to use part (a) of Postulate III-3 twice.]

11. Assuming that the points have the same order as in Exercise 10, prove that ($AD \cong A'D'$ and $CD \cong C'D'$ and $BC \cong B'C'$) $\rightarrow AB \cong A'B'$.

12. Suppose E, F, G and E', F', G' are *any* points on lines l and l', respectively, such that $EF \cong E'F'$ and $FG \cong F'G'$. Show, by drawing a figure, that it does not need to be true that $EG \cong E'G'$. What do we need to know about the position of the points? Does III-3 take care of this?

*13. Part (b) of Postulate III–3 can be *proved* from the other postulates.

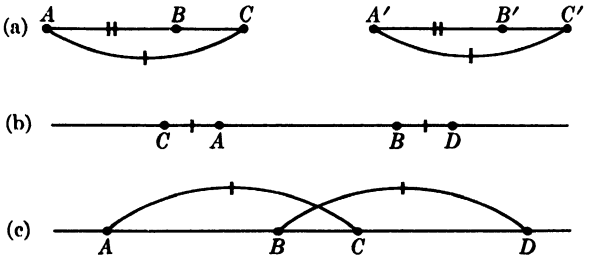


FIGURE 4–5

We have included this as a postulate to make things easier. Try to prove (b). You will have to use (a) and Postulate III–1.

*14. Try to reword III–3 in terms of length.

15. Each part of Fig. 4–5 indicates certain congruence relations. The marks show which segments are congruent. What other relations can be deduced? Point out how the postulates are used.

*16. Show how to relabel the points in parts (b) and (c) of Fig. 4–5 as A, B, C and A', B', C' so that the application of Postulate III–3 is clearly shown.

4–4 COMPARING SEGMENTS

The postulates in Group III tell us about segments that are the same size or congruent to each other. We can now *define*, in terms of congruence and betweenness, the ideas of larger and smaller segments. The definition will simply describe what the hunter did when he compared the boy with the mark on the spear.

We need a symbol to stand for "is greater than" and another symbol to stand for "is less than." These symbols are: $>$ and $<$. Whenever you see $>$, read "is greater than," and whenever you see $<$, read "is less than."

Definition 4–1. Let AB be any segment and $A'B'$ any other segment on line l' (Fig. 4–6). On the same side of A' as B', let B'' be the point such that $AB \cong A'B''$. If $A'B'B''$ (that is, if B' is between A' and B''), we say that $AB > A'B'$. If $A'B''B'$ (that is, B'' is between A' and B'), we say that $AB < A'B'$.

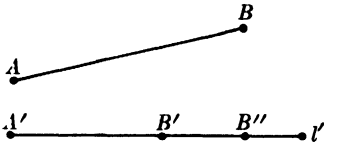


FIGURE 4–6

Remark. Because of III–1 there is only one point B'' on the same side of A' as the point B' and such that $AB \cong A'B''$. The three points A', B', and B'' are therefore in some definite order (unless $B' = B''$, in which case $AB \cong A'B'$). We cannot have A' between B' and B'' because B' and B'' are on the same side of A'. Therefore if $AB \ncong A'B'$, we must have either $A'B'B''$ or $A'B''B'$. In other words, either $AB > A'B'$ or $AB < A'B'$. This remark proves the following theorem.

Theorem 4–1. *If AB and $A'B'$ are any two segments, then either $AB > A'B'$, or $AB \cong A'B'$, or $AB < A'B'$, and exactly one of these is true.*

Exercise Group 4–3

1. Does Fig. 4–7 indicate that $AB < CD$ or that $CD > AB$? Read Definition 4–1 again.

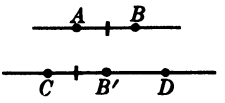


FIGURE 4–7

2. What relationships are indicated by Fig. 4–8? Use Definition 4–1.

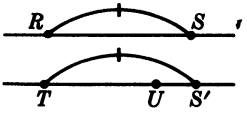


FIGURE 4–8

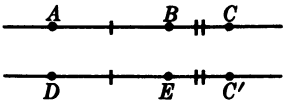


FIGURE 4–9

3. What relationships are indicated by Fig. 4–9? Use Definition 4–1.

4. If we know that $AB \cong CD$ and ABX, what can we conclude? Use Definition 4–1.

5. It is known that $RS \cong UT$ and RVS. What conclusion can be drawn? Use Definition 4–1.

6. Does Definition 4–1 say that if $AB > CD$ then $CD < AB$? Does looking at a figure prove that this is so?

The exercises above are meant to prepare you for the main theorem of this section, namely: if one segment is greater than a second, then the second is less than the first. Of course, we know that if we draw two pencil marks so that the first is longer than the second, then the second is shorter than the first. But we have not yet defined *length of a segment.* As a matter of fact we shall need to introduce a new postulate into our geometry before length can be defined. Since our definitions of "greater than" and "less than" have been formulated in terms of betweenness, the truth of any statement about these relations ("greater than" and "less than") must be established by using properties of betweenness.

†Theorem 4–2. *If $AB > A'B'$ then $A'B' < AB$, and conversely, if $A'B' < AB$ then $AB > A'B'$.*

Proof. Figure 4–10 illustrates what must be proved in order to establish the first part of the theorem.

(1) $AB > A'B'$ means $AB \cong A'C'$ and $A'B'C'$.
(2) $A'B' < AB$ means $A'B' \cong AC$ and ACB.

To show that $AB > A'B' \rightarrow A'B' < AB$, we must prove that if C' falls "beyond" B', then C falls between A and B.

From (1) above we have $AB \cong A'C'$ and $A'B'C'$. We know also that on a given side of A there is a point C such that $AC \cong A'B'$. To estab-

† Although this theorem is intuitively obvious, its proof is quite difficult. If the teacher wishes, it may be treated as a postulate.

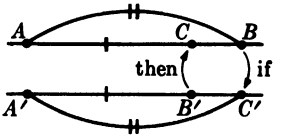


FIGURE 4–10

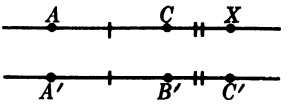


FIGURE 4–11

lish that we do indeed have ACB, consider the point X in Fig. 4–11 such that ACX and $CX \cong B'C'$. Now, by addition of segments, $AX \cong A'C'$. But $AB \cong A'C'$, so $X = B$. (Why?) And since ACX, we have ACB; and from (2) above we have $A'B' < AB$.

The proof of the converse is similar to the above proof and is left as an exercise.

From now on we shall freely interchange the "greater than" and "less than" terminology since the statements $AB > CD$ and $CD < AB$ are equivalent to each other.

Theorem 4–3. *If $AC \cong A'C'$ and $A'B'C'$ and B is the point on C's side of A such that $AB \cong A'B'$, then ABC* (Fig. 4–12).

Proof. Since $AC > A'B'$, $A'B' < AC$, and so by Definition 4–1 we have B between A and C.

This theorem is proved at this time because we shall need it in the next chapter when we discuss the length of a segment. We can then show that congruent segments have the same length. It is also a very convenient tool to use in many subsequent proofs.

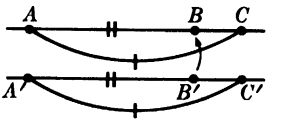


FIGURE 4–12

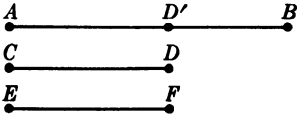


FIGURE 4–13

Definition 4–2. If we wish to say "$AB \cong A'B'$ or $AB > A'B'$," we write "$AB \geqq A'B'$" and read the symbol $\geqq$ as "is greater than or congruent to." Similarly, if we wish to say "$AB \cong A'B'$ or $AB < A'B'$," we write "$AB \leqq A'B'$" and read the symbol $\leqq$ as "is less than or congruent to."

Theorem 4–4. *If $AB > CD$ and $CD \cong EF$, then $AB > EF$.*

Proof. Since $CD < AB$ (why?) there is a point D' (Fig. 4–13) on AB such that $CD \cong AD'$. But then, by III–2, $EF \cong AD'$. Hence, by Definition 4–1, $EF < AB$ and $AB > EF$.

Exercise Group 4–4

1. Using the definition of $>$, explain what must be done in order to prove that $AB > CD$. Draw a figure to illustrate this.

2. What must one show in order to prove $RS < TU$? Draw a figure to illustrate this.

3. What basic undefined relations are used to define the phrases "is greater than" and "is less than"?

4. Is it correct to say that $AB \ncong CD \rightarrow AB \ncong CD$? Is the converse of this implication true?

5. Is it correct to say that $AB > CD \rightarrow AB \ncong CD$? State both the converse and contrapositive of this statement. Which one of these statements is true?

6. What conclusion can you draw if you know that A, B, C, and D are points on line l and that $AB > CD$ and $A = C$?

7. If AB is neither equal to nor greater than CD, can you conclude $AB < CD$?

8. If AB is neither congruent to nor less than CD, can you conclude $AB > CD$?

*9. Prove that if $AB < EF$ and $EF \cong CD$, then $AB < CD$. State the converse and contrapositive of this theorem.

*10. Prove that if $AB > CD$ and $CD > EF$, then $AB > EF$.

*11. Prove that if $AB < CD$ and $CD < EF$, then $AB < EF$.

*12. Prove that if B is between A and C, and B' is between A' and C', and $AB \cong A'B'$, and $BC > B'C'$, then $AC > A'C'$.

13. If $AB > CD$ and $CD < EF$, can you prove that $AB > EF$?

*14. Prove that if ABC and $A'C'B'$ and $AC \cong A'C'$, then $AB < A'B'$.

REVIEW OF CHAPTER 4

Several important ideas have been developed in this chapter. You should realize that *congruence* is really an undefined relation. Of course, you are so accustomed to comparing the sizes of things by comparing numbers that it is probably quite difficult to see clearly that numbers are not actually needed. We know that a 6′8″ basketball center is taller than a 6′2″ center. We know that a 240-lb football tackle is larger than a 180-lb tackle. But such comparisons as these can be made *without using numbers at all.* Surely men were able to recognize differences in sizes long before they had a number system to measure them. Historically, the Greeks developed their geometry without the use of number concepts.

In the next chapter, when we define what is meant by the length of a segment, it will naturally turn out that congruent segments have the same length. That is, if $AB \cong CD$, and x and y are numbers which are the lengths of these segments, then $x = y$.

Again, the postulates of this chapter are so obvious to our intuition that you might wonder why we bother to make them. All we can say is that postulates should be obvious. The amazing thing about mathe-

matics is that out of such simple assumptions as we are making in our study of geometry one can construct theorems of marvelous complexity which provide the foundation for all work in science.

The proofs in this chapter are quite difficult. There is no need to memorize the results of the theorems since they merely show that our geometry agrees with the figures that we draw.

In working with Definition 4–1, you should have learned an important mathematical lesson, namely, that definitions must be clearly understood. It would be very easy to read Definition 4–1 carelessly and to assume that if $AB > CD$ then $CD < AB$. This is not stated in the definition, but of course we were able to use the definition, our postulates, and the rules of logic to prove that it is indeed true.

The exercises below will help you review the concepts of this chapter.

Label the following statements true or false.

1. If $AB \cong CD$, then $A = C$ and $B = D$.
2. If $A = C$ and $B = D$, then $AB \cong CD$.
3. If $AB \cong CD$ or $AB < CD$, then $AB \leqq CD$.
4. If A, B, and C are points on a line l and A', B', and C' are points on a line l', and $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.
5. If AB is any segment and A' is any point on a line l', then there are exactly two points B' and B'' on l' such that $A'B'$ and $A'B''$ are congruent to AB.
6. If AB is not less than CD, then it must be greater than CD.
7. If RS is neither greater than nor less than TU, then $RS \cong TU$.
8. If A, B, C, and D are points in this order on a line l and $AC \cong BD$, then $AB \cong CD$.
9. If A, B, C, and D are points in this order on a line l and $AC > BD$, then $AB < CD$.
10. If A, B, C, and D are points in this order on a line l and $AC < BD$, then $AB \leqq CD$.

This chapter has included three postulates, two definitions, and four theorems. The following statements are all true. Designate a postulate or definition or theorem which could be used in justifying each statement.

11. For every line segment AB, it is true that $AB \cong AB$.
12. If $AB \cong CD$ and ABX, then $AX > CD$.
13. If AB is not less than CD, then $AB \geqq CD$.
14. If $AB \cong CD$ and $CD \cong EF$ and $XY > AB$, then $XY > EF$.
15. If $RS \cong TU$ and $XY \cong TU$, then $RS \cong XY$.
16. If ABC and BCD, then $AC < AD$.
17. If $AB \leqq CD$, then there is no point E such that CDE and $AB \cong CE$.
18. If in triangle ABC we have side AC greater than side AB, then there is a point E on side AC such that $AE \cong AB$.
19. If ABC and $AB \cong BC$ and XYZ and $XY \cong YZ \cong AB$, then $AC \cong XZ$.
20. If $A = X$ and $B = Y$, then $AY \cong BX$.

ALGEBRA REVIEW

In algebra the relations of "greater than" and "less than" are defined for pairs of numbers. We say that $5 > 3$ because $5 = 3 + 2$, and 2 is a positive number. More generally we make the following definitions:

(1) If a, b, and c are numbers and $a = b + c$ and c is positive, then we say that a is greater than b and write $a > b$.

(2) If a, b, and c are numbers and $a = b - c$ and c is a positive number, then we say that a is less than b and write $a < b$.

For example, since $-\frac{3}{2} = -\frac{5}{2} + \frac{2}{2}$, we say $-\frac{3}{2} > -\frac{5}{2}$; and since $0 = \frac{3}{4} - \frac{3}{4}$ we say that $0 < \frac{3}{4}$.

1. Use the first definition above to prove that $0 > -\frac{4}{3}$.
2. Use the second definition to prove that $-\frac{4}{3} < 0$.
3. Use the two definitions to prove that if $a > b$ then $b < a$.
4. If $a = b + 1$ and $b = c + 2$, prove that $a > c$.
5. Prove that if $a > b$ and $b > c$, then $a > c$.
6. If $a = b - 2$ and $b = c - 3$, then prove that $a < c$.
7. Prove that if $a < b$ and $b < c$, then $a < c$.
8. Show that if $a = b + 3$, then $3a > 3b$ and $-2a < -2b$.
9. Prove that if $a > b$, then $ca > cb$ if c is positive and $ca < cb$ if c is negative.
10. Prove that if $a < b$, then $ca < cb$ if c is positive and $ca > cb$ if c is negative.

CHAPTER 5

MEASUREMENT OF SEGMENTS

5–1 INTRODUCTION

In this chapter we are going to get a number for the length of a segment, that is for the distance between two points. The way we get the length will be exactly the same as the method we use in practical work. We will see that finding a length is based upon simple *counting*. For example, when you pace off the length of a building (Fig. 5–1), you count how many steps you take. Or when you say that the room is 15 feet wide, you mean that a foot ruler can be laid down 15 times, one after the other, on a line. If the width of the room is between 14 and 15 feet, the ruler can be laid down 14 times, but 15 times would put the end of the ruler past the wall. To measure the room more accurately, you must choose a smaller unit of measure, say an inch, so that if the width is 14 feet and three inches, the foot ruler can be laid down 14 times and beyond that the inch ruler can be laid down three times. If even then you have not come exactly to the wall of the room, you must choose an even smaller unit of measure, some fraction of an inch, and so on.

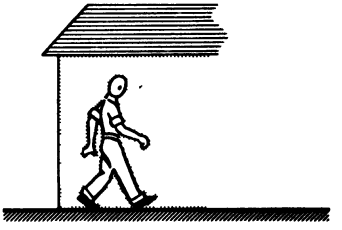


FIGURE 5–1

Exercise Group 5–1

1. Measure the length of your desk. Describe the process of measurement in terms of counting.
2. Measure the length of a page. Describe the counting you do to get the length.
3. Think of other examples of measurement and describe them in terms of counting.

5–2 NUMBERS

Because we express length in terms of numbers, we need certain facts about numbers that you learned in algebra and arithmetic. You have studied different kinds of numbers. In the beginning grades you first used whole numbers 1, 2,... and learned how to add, subtract, and multiply them. These numbers we now call the *positive integers* instead of "whole numbers." When you tried to divide integers you found that it was necessary to have other numbers, which you called *fractions*, for example, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{8}$, $\frac{7}{37}$. These are numbers obtained by dividing one integer by another integer. *The numbers which are either positive integers or positive fractions are called positive rational numbers.*

You also wrote numbers as *decimals*, for example, 1.25, 2.537, 173.612. Sometimes a rational number when written as a decimal "comes out even," as for example, $\frac{1}{2} = 0.5$, $\frac{3}{8} = 0.375$. Sometimes it does not come out even, so that the decimal goes on forever, for example, $\frac{1}{3} = 0.333\ldots$, $\frac{2}{3} = 0.6666\ldots$, $\frac{1}{11} = 0.090909\ \ldots$ You also met with other numbers which did not come out even, for example, $\sqrt{2} = 1.4142\ldots$, $\sqrt{3} = 1.7321\ldots$. These numbers are different from $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{11}$ because they not only do not come out even, but *the decimal never repeats.* In other words, no matter to how many decimal places you were to calculate $\sqrt{2}$, you would never reach a place where, from then on, one group of digits would occur over and over again in the same order.

Decimals that, after some place, repeat one group of digits over and over are called *repeating decimals.* Decimals which come out even are called *terminating decimals*, and may be considered as repeating with zeros.

Exercise 5–2

Write the following rational numbers as decimals to see that they are repeating decimals. If they come out even (terminate), the repeating part is all zeros. 1/6, 5/6, 122/99, 1/13, 1/7, 6/5, 3/2.

It is proved in advanced algebra that rational numbers are always repeating decimals, and that, conversely, repeating decimals are rational numbers. (These proofs are hard, so don't be discouraged if you cannot construct them; your teacher may show you.) Hence, rational numbers and repeating decimals are the same. What, then, about the decimals which never repeat? These numbers are not rational numbers, and so they have been called *irrational numbers.** Irrational numbers are de-

* The word "irrational" is not a very pleasant choice because it sounds like "not sensible." What it really means is this. Rational numbers are obtained by dividing one integer by another; that is, a rational number is the ratio of two integers. Hence the word "rational" in "rational number" means that the number is a *ratio* of integers. Therefore, an *irrational* number is one which is *not* the ratio of two integers.

fined to be *those numbers which are not rational.* Examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{17}$, and, in fact, any root of an integer which is not itself an integer.

All the decimals together are defined to be the *real numbers.* Thus a real number is a decimal. If the decimal repeats, the number is a rational number. If the decimal does not repeat, it is an irrational number.

Exercise Group 5–3

1. Write the symbols for several rational numbers and for several irrational numbers.

2. Is 3.131313..., where the 1's and 3's keep alternating, a rational number?

3. Is 0.234234234..., where 2, 3, and 4 keep following each other, a rational number?

*4. The number 1.010010001... is formed from just 0's and 1's. After each 1 there is one more 0 than the previous time. Is this number rational?

5. Give the definitions of rational number, irrational number, real number.

6. A real number is either rational or irrational. Why?

*7. There is a rule for deciding when a fraction written as a decimal will have its repeating part all zeros and when it will not. For example, each of the fractions 3/20, 17/250, 5/16, and 4/25 has its repeating decimal part all zeros. The fractions 7/15, 5/12, and 11/23 do not. Can you discover the rule?

When we write a decimal, for example 1.23, we mean $1 + 0.2 + 0.03$ or $1 + 2/10 + 3/100$. Similarly, $4.383 = 4 + 3/10 + 8/100 + 3/1000$. Thus, any terminating decimal (one which comes out even) may be written as a sum of ordinary fractions with denominators 10, 100,.... If the decimal is repeating, this sum is an *infinite* one, for example, $1/3 = 0.333... = 3/10 + 3/100 + 3/1000 + 3/10000 + \cdots$.

Exercise Group 5–4

1. Write the following decimals as sums of rational numbers: 4.82, 1.29, 2.75, 7.122, 0.8875, 2.40. In order to express each number as a single fraction, add the rational numbers.

2. Write the following numbers as decimals: $3 + 2/10 + 4/100$, $2 + 3/10 + 5/100 + 6/1000$, $4 + 3/100$, $1 + 1/10 + 1/10000$.

3. Write the following repeating decimals as sums of rational numbers: 0.6666..., 0.7777..., 0.121212....

4. Write the following numbers as decimals: $1 + 2/10 + 2/100 + 2/1000 + \cdots$, $3 + 1/10 + 4/100 + 1/1000 + 4/10000 + \cdots$, $8/10 + 8/100 + 8/1000 + 8/10000 + \cdots$.

5–3 INEQUALITIES

In this book you are expected to know how to do arithmetic, that is, how to add, subtract, multiply, and divide real numbers. But there is one part of arithmetic you have not yet studied systematically. That

part is concerned with comparing the sizes of real numbers. Since you already know what it means to say that the number x is larger than the number y, all we need is a symbol to indicate this. We therefore make the definition: $x > y$ means "x is greater than y." Another way to say this is "$x - y$ is a positive number." We also define $x < y$ to mean "x is less than y," or, in other words "y is greater than x." Listed below are several properties of real numbers. These properties can be established as indicated in the review of Chapter 4.

(a) If $x > y$ and $y = z$, then $x > z$.
(b) If $x > y$ and $y > z$, then $x > z$.
(c) x is a positive number if and only if $x > 0$.
(d) x is a negative number if and only if $x < 0$.
(e) If $x > y$, then $x - y > 0$, and conversely.

We define $x \geqq y$ to mean "either $x > y$ or $x = y$," and we read "$\geqq$" as "is greater than or equal to." We make a similar definition for $x \leqq y$ ("x is less than or equal to y").

(f) If $x > y$ and $u \geqq v$, then $x + u > y + v$.

Exercise Group 5–5

1. Substitute numbers for the variables in properties (a), (b), (c), and (f) above, and thus verify their truth for certain cases.

2. Write in symbols: a is greater than b; a is greater than or equal to b; u is less than w; $a + c$ is less than or equal to $r + s$; $(a + b)^2$ is positive; $\overleftrightarrow{z}$ is a negative number.

3. If x is any real number, $x^2 \geqq 0$. Why?

4. Why is it true that if $x > 0$ and $y > 0$, then $xy > 0$?

5. Tell why $xy < 0$ if $x > 0$ and $y < 0$.

5–4 MEASUREMENT

To measure a line segment we must first tell what our unit of length is. Some units of length often used in practical work are inch, foot, yard, mile, meter, centimeter, etc. Having picked a unit of length, we will describe the geometric process whereby we get a real number, that is, a decimal for the length of a line segment.

Choose any segment CD as a unit (Fig. 5–2). This segment and all segments congruent to it are defined to have length 1. Suppose AB is any segment on a line l. On the same side of A as B there is a point A_1 such that $AA_1 \cong CD$. If $A_1 = B$, then the length of AB is 1. If B

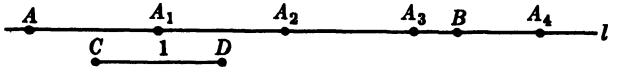


FIGURE 5–2

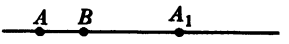


FIGURE 5–3

lies between A and A_1 (Fig. 5–3), then our unit was too large to measure the length of AB in integers (whole numbers). We will return to this possibility later. If A_1 lies between A and B, then there is a point A_2, on the same side of A_1 as B, such that either $A_2 = B$, or B is between A_1 and A_2, or A_2 is between A_1 and B. If $A_2 = B$, the length of AB is 2. If B is between A_1 and A_2, then our unit will not measure the length in integers. If A_2 is between A_1 and B, there is a point A_3, on the same side of A_2 as B, such that $A_2A_3 \cong CD$.

The process described above is what we actually follow when we measure. If we lay out segments congruent to CD from A toward B, we know from experience that we eventually either reach B exactly, so that the length of AB is an integer, or reach a point *beyond* B. Thus in Fig. 5–2, A_4 is beyond B but A_3 is still between A and B. The length in this case will be a number between 3 and 4. If the above process can always be carried out in our geometry, then we have described the beginning of the process of measurement.

The only part of the above process that does not follow from our postulates is the proof that we can always get to or beyond B. Because we need to do this, we now add a new postulate to our list. It was the Greek mathematician Archimedes (287–212 B.C.) who first stated this postulate. He apparently did not see how to get it from Euclid's postulates and, needing it, assumed it was true. It is one of the things that Euclid forgot to state as a postulate. It is called *Archimedes' axiom* or *Archimedes' postulate* in spite of its use by Eudoxus much earlier. This is the first of the two new postulates in Postulate Group IV which we need for measurement. The other one will be given later.

IV–1 *Archimedes' postulate.* If AB is any segment on a line l and CD any other segment, then there is a definite number n of points $A_1, A_2, \ldots, A_n$ on l such that the points $A = A_0, A_1, A_2, \ldots, A_n$ are arranged in this order, and all the segments $AA_1, A_1A_2, \ldots, A_{n-1}A_n$ are congruent to CD, and either $B = A_n$ or B lies between A_{n-1} and A_n.

The integer n is called *Archimedes' integer.* When n is 4, an illustration would look like Fig. 5–4 or Fig. 5–5.

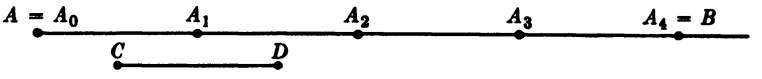


FIGURE 5–4

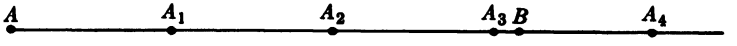


FIGURE 5–5

Exercise Group 5–6

1. If AB is this segment A_____B and CD is this segment C_____D, find how many points are needed to get beyond B.

2. Draw a segment AB and a segment CD. Find the integer n of Archimedes' axiom for your segments.

Using Archimedes' postulate, we find that the length of AB is either an integer or is between two integers. If the length lies between two integers, the next step in the process is to divide the unit interval into tenths (Fig. 5–6). (We would not have to keep to tenths if we were not trying to come out with a decimal.) Then we count "tenths" until either we reach B exactly or B lies between two of the tenth marks. If we reach B exactly, then the length of AB comes out exactly in tenths. If B is not reached exactly, then we must divide the unit interval into one-hundredths and repeat the process. In this way we obtain a real number or decimal, for the length of AB. In Fig. 5–7 the length of AB is about 2.3.

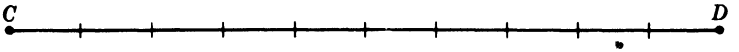


FIGURE 5-6

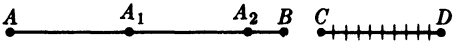


FIGURE 5–7

We have indicated a segment by writing the two endpoints of the segment as AB. At this point we also need a notation for the *length* of the segment. For this we will use $\overline{AB}$, that is $\overline{AB}$ = *the length of the segment* AB. Whenever you see the symbol $\overline{AB}$ you must remember that it represents a *number*, while the symbol AB stands for a *set of points*.

Except for dividing a segment into tenths, the entire procedure of this section can be carried out (that is, described) in our geometry. At the moment we do not have the means for dividing a segment into tenths. As soon as we have studied parallel lines (Chapter 8), we will be able to make this construction. Since we do not need the results of this chapter until after we have studied parallel lines, our procedure is logically correct.

Exercise Group 5–7

1. Draw a line segment. Then draw a segment that will be your unit. Now measure the first segment, describing how you use Archimedes' postulate and relations of betweenness in the process.

2. Draw a line segment. Measure it with your ruler to the nearest 1/16 of an inch. Describe your measurement process, using Archimedes' postulate and betweenness.

5–5 POINTS ON A LINE

We have just described the measurement process for getting a number for the length of a segment. We now turn the question around and ask whether, given a number, there is a segment with that number for its length. Of course, we would like this to be true, but our postulates so far do not tell us that it is true. The process of measurement describes a method for "sneaking up" on the point B, but if we start out with a number we are not assured that there is a point to sneak up on! To have this property, we must postulate it. This postulate tells us that there are enough points on the line.

IV–2. *The completeness postulate.* For every line l and any point A on l, and for any positive real number x, there is, on a given side of A, a point B such that $\overline{AB} = x$.

The name of the last postulate is meant to signify that our lines are complete, that is, that they have all the points on them we expect them to have.

Exercise Group 5–8

1. Draw a segment that is to be the unit segment. Draw another segment that is 1.3 units in length.

2. With the unit segment of Exercise 1, draw a segment of length $2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ units.

5–6 LENGTH AND CONGRUENCE

When we started talking about congruent segments we had in mind a notion of size. Now that we have lengths for segments it should turn out that congruent segments have the same length, and conversely that segments of the same length are congruent. We now prove these theorems. To do so we need the following theorem (4–3) that we proved in Chapter 4. *If $AB \cong A'B'$ and ACB and if C' is on the same side of A' as B' and $AC \cong A'C'$, then $A'C'B'$.*

***Theorem 5–1.** *If $AB \cong A'B'$, then $\overline{AB} = \overline{A'B'}$.*

Proof. To get the length of segment AB, we establish a sequence of points *between* A and B. Let us designate this sequence by $C_1, C_2, C_3, \ldots$. Now, consider the corresponding sequence, $C_1', C_2', C_3', \ldots$, where $AC_1 \cong A'C_1'$, $AC_2 \cong A'C_2'$, $AC_3 \cong A'C_3', \ldots$, and all the points $C_1', C_2', C_3', \ldots$ are on the same side of A' as B'. Using Theorem 4–3, we see that all the points $C_1', C_2', C_3', \ldots$, are between A' and B'; therefore, this sequence determines the same real number (decimal) as does the sequence $C_1, C_2, C_3, \ldots$.

***Theorem 5–2.** *If* $\overline{AB} = \overline{A'B'}$, *then* $AB \cong A'B'$.

Proof. This theorem is the converse of Theorem 5–1. We will prove it by choosing a point B'', on the same side of A as B, such that $AB'' \cong A'B'$, and then proving that $B'' = B$ (Fig. 5–8). This will show that $AB \cong A'B'$. Either $B'' = B$ or $B'' \neq B$. Since $AB'' \cong A'B'$, then from Theorem 5–1 $\overline{AB''} = \overline{A'B'}$. We are given that $\overline{AB} = \overline{A'B'}$. Hence $\overline{AB''} = \overline{AB}$. Thus we must show that if $B'' \neq B$, we cannot have $\overline{AB''} = \overline{A'B'}$. But if $B'' \neq B$, then, if the unit segment is smaller than BB'', the corresponding Archimedes integers for AB and AB'' would be different. This would lead to different lengths for AB and AB''. Hence B'' must be the same as B and $AB \cong A'B'$.

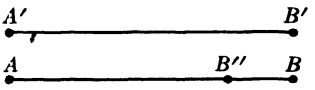


FIGURE 5–8

These two theorems give us the right to use the words "the same length" in place of "congruent." Therefore we can use everyday language in talking about congruence of segments.

Exercise Group 5–9

1. $AB \cong CD$ and $CD \cong EF$, and $\overline{AB} = 7.26$ inches. What is $\overline{EF}$?
2. If $\overline{PQ}$ is 1.23 ft, and $VU \cong RS$, and $\overline{RS} = 1.23$ ft, is $QP \cong VU$?
3. If $AB \cong CD$, $CD \cong EF$, $EF \cong GH$, is $\overline{AB} = \overline{GH}$?
4. An isosceles triangle can be defined now in two ways. State the two definitions.

5–7 MORE ABOUT INEQUALITIES

In Chapter 4 we defined $>$ and $<$ for segments. In this chapter (Section 5–3) we defined $>$ and $<$ for numbers. Since "same length" means "congruent," it should also be true that if $AB > CD$, then $\overline{AB} > \overline{CD}$, and conversely.

***Theorem 5–3.** *If* $AB > CD$ *then* $\overline{AB} > \overline{CD}$, *and conversely, if* $\overline{AB} > \overline{CD}$ *then* $AB > CD$.

Proof. The theorem could also be stated as "$AB > CD$ if and only if $\overline{AB} > \overline{CD}$." Or, again, as "If $AB > CD$ then $\overline{AB} > \overline{CD}$, and conversely." We first prove that $AB > CD$ implies $\overline{AB} > \overline{CD}$. The hypothesis, $AB > CD$, means that if E is on the same side of C as D, and $AB \cong CE$, then D is between C and E. (See Fig. 5–9.) By Theorem 5–1, $\overline{AB} = \overline{CE}$. In the measurement process, if the unit is chosen smaller

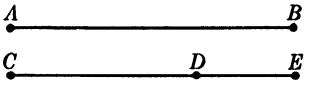


FIGURE 5–9

than DE, the corresponding Archimedes' integers for CD and CE will be different, and in fact the integer for CD will be smaller than that for CE. Hence $\overline{CD} < \overline{CE}$ and, since $\overline{CE} = \overline{AB}$, $\overline{CD} < \overline{AB}$.

The converse is proved in a similar manner. We have given that $\overline{AB} > \overline{CD}$. Choose E, on the same side of C as D, such that $AB \cong CE$. By Theorem 5–1, $\overline{AB} = \overline{CE}$. Now either CDE, $D = E$, or CED. (Why?) We cannot have $D = E$. (Why?) An argument like the first part of the proof will show we cannot have CED. Hence we must have CDE. It follows from the definition of "less than" for segments that $CD < CE \cong AB$.

Exercise Group 5–10

1. If ABC and $\overline{AB} = 5$, what can you say about $\overline{AC}$?
2. If $AB > EF$ and $\overline{EF} = 4$, what can you say about $\overline{AB}$?
3. If $AB > CD > EF$, what can you say about $\overline{AB}$ and $\overline{EF}$?
4. If $ABCD$, what can you say about $\overline{AD}$ and $\overline{BC}$?
5. If $\overline{AB} > \overline{AC}$, and C is a point on the line containing A and B, and C lies on the same side of A as B, where is the point C?

5–8 CHANGE OF SCALE

In measurement we choose a fixed segment for a unit of length. What happens if a different segment is chosen for the unit? This is a familiar situation because we are all used to dealing with inches, feet, yards, or miles to measure length. In Fig. 5–10, if CD is the unit, the length of AB is 3. If ED is the unit, the length of AB is 6. This illustrates the following theorem.

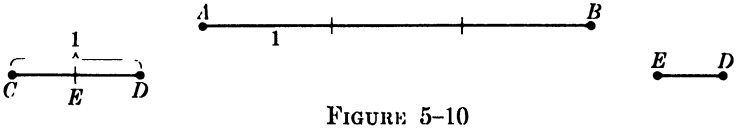


FIGURE 5–10

Theorem 5–4. *If $\overline{AB} = x$ when CD is the unit, and if $\overline{CD} = y$ when EF is the unit, then $AB = xy$ when EF is the unit.*

We will not give the proof of this theorem for arbitrary x and y because it is long to write out and quite hard if y is an irrational number. But if y is an integer, the proof is very easy and is really clear from Fig. 5–10, where $y = 2$.

Exercise Group 5–11

1. $\overline{AB} = 5$ when CD is the unit, and $\overline{CD} = 2.3$ when EF is the unit. What is $\overline{AB}$ when EF is the unit?

2. The width of the room is 14 ft. What is the width in inches? How does this follow from Theorem 5–4?

3. One kilometer is 1000 meters. One meter is 100 centimeters. How many centimeters in a kilometer? Show how this follows from Theorem 5–4.

4. $\overline{PQ} = 6.5$ when $\overline{RS} = 1$. $\overline{RS} = \frac{1}{2}$ when $\overline{UV} = 1$. What is $\overline{PQ}$ when $\overline{UV} = 1$?

5. If $CD < EF$, how does $\overline{AB}$ with unit CD compare with $\overline{AB}$ with unit EF? In other words, if we use two different units and compute two lengths for one segment, does the smaller unit give the smaller or the larger length?

REVIEW OF CHAPTER 5

1. In this chapter we have given definitions of the following: real number, rational number, irrational number, positive integer, greater than, less than, the process of measurement. Be sure you know these definitions.

2. This chapter has two new postulates, Archimedes' postulate and the completeness postulate. Go over them again and be sure you see what they mean.

3. In this chapter we have proved the following theorems. Study them again to be sure you know what they say. You should be able to state clearly the facts upon which each proof depends.

$$(AB \cong CD) \leftrightarrow \overline{AB} = \overline{CD},$$
$$(AB > CD) \leftrightarrow \overline{AB} > \overline{CD},$$
$$(AB < CD) \leftrightarrow \overline{AB} < \overline{CD}.$$

Note. In this chapter we have shown that congruence is equivalent to equality of length. From this point on in our development of geometry you may use the ordinary language of length in speaking of congruence.

ALGEBRA REVIEW

1. Segment XY has a length of 28 units. Point T is between X and Y and $\overline{XT} = \overline{YT} + 9$. Determine the lengths of XT and YT.

2. Points R, T, and S are in this order on a line. Segment RT is twice as long as segment TS. The difference in their lengths is 11. Find the length of RS.

3. Points A, B, and C are on one line. The length of AC is 12 more than that of AB. The length of BC is four times the length of AB. Find the lengths of AB and AC if (a) A is between B and C, and (b) B is between A and C.

4. Points A, B, C, and D are in this order on one line. $\overline{BC} = \overline{AB} + 4$; $\overline{CD} = \overline{AB} + \overline{BC}$; $\overline{AD} = 32$. Find $\overline{AB}$.

5. The length of XY is $\frac{2}{3}$ of the length of XZ. The length of XZ is $\frac{3}{5}$ of the length of YZ. If these three points are collinear: (a) What is the order of the

points on l? (b) Express $\overline{XY}$ as a fraction of $\overline{YZ}$. (c) Determine $\overline{XY}$ if $\overline{XZ} - \overline{XY} = 6$.

6. The point M is the midpoint of segment AB, N is the midpoint of MB, and O is the midpoint of NB. If $\overline{AN} - \overline{NO} = 20$, find the length of AB.

7. $\overline{AC} = 2\overline{AB}$, $\overline{AB} + \overline{BC} = \overline{AC} + 2$. If $\overline{AB} > 1$, prove that the three points A, B, and C are not on one line.

8. A square and equilateral triangle have equal perimeters. The length of a side of the triangle exceeds the length of a side of the square by six inches. Find the perimeter of each.

9. The sum of the perimeters of a square and a rectangle is 36 ft. The rectangle is twice as long as it is wide, and the width of the rectangle is 1 ft more than the length of a side of the square. Find the dimensions of each.

10. The point B is the midpoint of segment AC. Point X is between B and C, and $\overline{BX} = 3$ inches. Show that a square having a side congruent to AB has an area of 9 in^2 more than a rectangle with sides congruent to AX and XC.

CHAPTER 6

CONGRUENCE OF ANGLES AND TRIANGLES

6–1 INTRODUCTION

In the last two chapters we have been concerned with congruence of segments only. But we also have a feeling for size of an angle—and, in fact, in Chapter 1 we used a protractor to measure angles. If our geometry is to be like the world of our experience, we must include in it postulates that will enable us to compare the sizes of angles. These postulates form Postulate Group V. There are four postulates in the group, and we take them up one at a time. As in Chapter 4, we shall use the word "congruent" to give the idea of *the same size.*

6–2 POSTULATE V–1, THE EXISTENCE POSTULATE FOR ANGLES

This first postulate of Group V tells of the existence of angles congruent to a given angle. Remember to think of the word "congruent" as meaning *the same size.*

V–1 If $\angle A$ is not a straight angle, and if r' is a ray from A' on a line l', then on a given side of l', there is exactly one ray s' from A' such that the angle A', with sides r' and s', is congruent to angle A (Fig. 6–1). We write this as $\angle A' \cong \angle A$. If $\angle A$ is a straight angle, then the straight angle at A', with one side r', is the only angle with one side r' congruent to angle A.

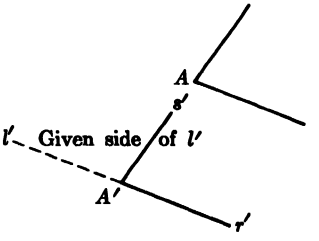


FIGURE 6–1

Exercise Group 6–1

1. If $\angle A$ is not a straight angle, how many angles with side r' are congruent to $\angle A$? How many if $\angle A$ is a straight angle?

2. In a figure like 6–2 draw the possible positions of the side s' such that $\angle A' \cong \angle A$.

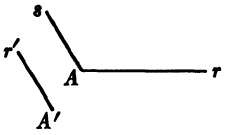


FIGURE 6–2

3. In a figure like 6–3, draw all possible positions of s' such that $\angle A' \cong \angle A$.

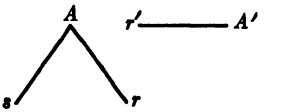


FIGURE 6–3

4. If in Fig. 6–4, A is a straight angle and r' is a ray from A', draw an angle, with one side r', that is congruent to A. Is it the only such angle?

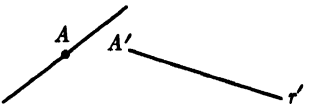


FIGURE 6–4

5. If $\angle O$ is a straight angle and $\angle O' \cong \angle O$, is $\angle O'$ a straight angle? Why?

6–3 POSTULATE V–2, THE THREE-PART ANGLE POSTULATE

The next postulate is for angles what Postulate III–2 is for segments. It may seem odd to have to state such seemingly obvious things, but remember we are dealing with those imaginary things called points and lines, and so we must state *all* the properties we need.

V–2 (a) $\angle A \cong \angle A$. (Congruence of angles is reflexive.)

(b) If $\angle A \cong \angle B$, then $\angle B \cong \angle A$. (Congruence of angles is symmetric.)

(c) If $\angle A \cong \angle B$, and $\angle B \cong \angle C$, then $\angle A \cong \angle C$. (Congruence of angles is transitive.)

Exercise Group 6–2

1. State each part of Postulate V–2 in words.

2. If $\angle E \cong \angle F$, and $\angle F \cong \angle G$, and $\angle G \cong \angle H$, show, by using part (c) of Postulate V–2, that $\angle E \cong \angle H$.

3. If $\angle R \cong \angle S$ and $\angle T \cong \angle S$, prove that $\angle R \cong \angle T$.

*4. Use Postulate V–1 and (b) and (c) of Postulate V–2 to prove that for every $\angle A$, $\angle A \cong \angle A$.

6–4 POSTULATE V–3, THE ADDITION AND SUBTRACTION OF ANGLES POSTULATE

The next postulate does for angles what Postulate III–3 does for segments. It enables us to "add" and "subtract" angles. To avoid a complicated statement in words, we explain the postulate by using Fig. 6–5, where the side common to $\angle B$ and $\angle C$, except the vertex, is in the interior of $\angle A$, and the same for $\angle A'$, $\angle B'$, and $\angle C'$.

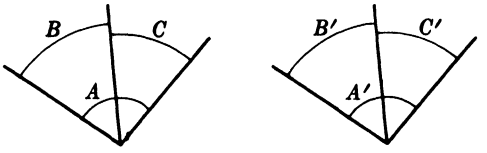


FIGURE 6–5

V–3 If the angles in any two of the pairs $\angle A$ and $\angle A'$, $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$ are congruent to each other, then the angles of the third pair are also congruent to each other. In other words:

(a) If $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$, then $\angle C \cong \angle C'$.

(b) If $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$, then $\angle A \cong \angle A'$.

(c) If $\angle C \cong \angle C'$ and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$.

The angle A may be a straight angle.

Exercise Group 6–3

1. We have not yet in our geometry studied the measurement of angles in degrees. However, in this exercise, use the fact (which will be proved later) that angles congruent to each other have the same measure in degrees, and add or subtract degree measures as you are accustomed to do. Refer to Fig. 6–6.

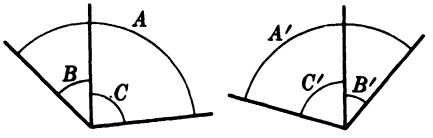


FIGURE 6–6

(a) If $\angle A$ is 130° and $\angle B$ is 45°, how many degrees in $\angle C$?

(b) If $\angle B'$ is 41° and $\angle C'$ is 75°, how many degrees in $\angle A'$?

(c) If $\angle A \cong \angle A'$ and $\angle A$ is 150°, how many degrees in $\angle A'$?

(d) If $\angle A \cong \angle A'$ and $\angle C \cong \angle C'$, are $\angle B$ and $\angle B'$ congruent to each other?

(e) If $\angle A$ is 87° and congruent to $\angle A'$, and if $\angle C \cong \angle C'$ and $\angle C'$ is 33°, how many degrees in $\angle B$?

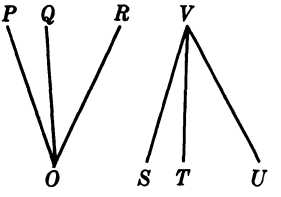


FIGURE 6–7

2. In Fig. 6–7, if $\angle POR \cong \angle SVU$, and $\angle QOR \cong \angle TVU$, what can you say about $\angle POQ$?

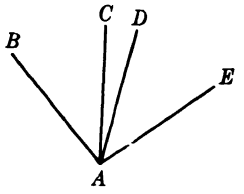


FIGURE 6–8

3. In Fig. 6–8, if $\angle BAC \cong \angle DAE$, what can you say about $\angle BAD$?

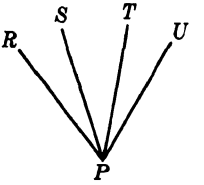


FIGURE 6–9

4. In Fig. 6–9, if $\angle SPU \cong \angle RPT$, what can you say about $\angle RPS$?

5. Is Exercise 2 an example of "adding" or "subtracting" angles? What about Exercises 3 and 4?

Supplementary and adjacent angles were defined in Definition 3–9. The statement that $\angle A$ is *supplementary* to $\angle B$ shall mean that $\angle A$ is congruent to an angle that is *supplementary and adjacent* to $\angle B$.

Theorem 6–1. *If two angles are congruent to each other, then their supplements are also congruent to each other.*

Proof. Let the given angles be A and A', and let their supplements be B and B', respectively, as in Fig. 6–10. We are given that $\angle A \cong \angle A'$. Since any two straight angles are congruent to each other, it follows at once from Postulate V–3 that $\angle B \cong \angle B'$. (Why?)

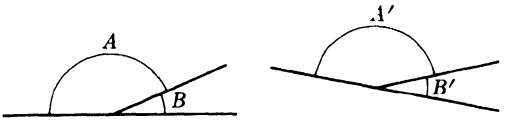


FIGURE 6–10

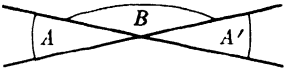


FIGURE 6–11

Theorem 6–2. *Angles that are vertical to each other are also congruent to each other.*

Proof. Let the angles A and A' be vertical to each other, as in Fig. 6–11, and consider $\angle B$, one of the other angles formed by the lines. Then $\angle B$ is the supplement of $\angle A$ and also the supplement of $\angle A'$. Since $\angle B \cong \angle B$, it follows from Theorem 6–1 that $\angle A \cong \angle A'$.

Definition 6–1. An angle (Fig. 6–12) that is congruent to its supplement is called a *right angle.*

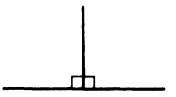


FIGURE 6–12

Remark. Note that the words *congruent, vertical, equal,* and *supplementary* refer to relations between two objects, not to their properties. That is, it makes no sense to say that $\angle A$ is a vertical angle, or a congruent angle, or a supplementary angle. At times, we refer to two angles A and B as equal angles, or vertical angles, or congruent angles, etc. What we really mean is that $\angle A$ is equal to $\angle B$, $\angle A$ is vertical to $\angle B$, $\angle A$ is congruent to $\angle B$, etc. It is important to distinguish between properties and relations.

Exercise Group 6–4

1. If an angle is a right angle, what is its supplement?

2. If $\angle 1 \cong \angle 2$ in Fig. 6–13, prove that $\angle EAC \cong \angle DAB$.

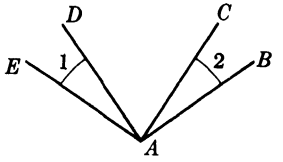


FIGURE 6–13

3. In Fig. 6–14 it is given that $\angle B$ is congruent to the supplement of $\angle C$ and $\angle C \cong \angle D$. Prove that $\angle A \cong \angle E$.

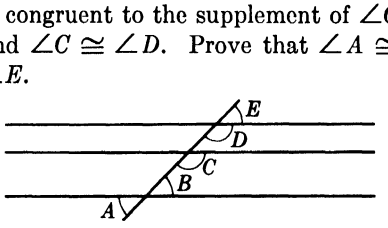


FIGURE 6–14

*4. An angle is congruent to a right angle. Show that it, too, is a right angle.

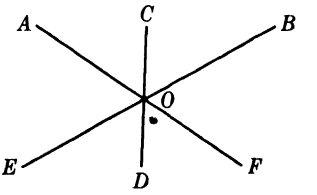


FIGURE 6–15

5. In Fig. 6–15, the line CD bisects $\angle AOB$. Show that it also bisects $\angle EOF$. (By "bisect' we mean that $\angle AOC \cong \angle COB$.)

6. Decide whether the following words refer to properties or relations: perpendicular, parallel, vertex, altitude, greater than, angle, segment, median, right (angle), straight (angle), bisector.

6–5 POSTULATE V–4, THE SIDE-ANGLE-SIDE POSTULATE

This postulate (abbreviated S.A.S.) is one that combines congruence of segments with congruence of angles. Up to this point, we have the two kinds of congruence but no relationship between them. This is the connecting link that will enable us to prove many theorems.

V–4 If two triangles, $\triangle ABC$ and $\triangle A'B'C'$, are such that $AB \cong A'B'$, $AC \cong A'C'$, and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$ (Fig. 6–16).

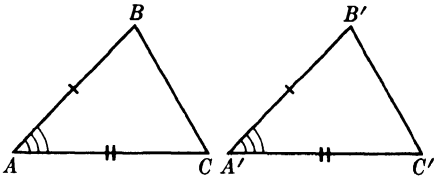


FIGURE 6–16

This postulate tells us that if two triangles have two sides of one congruent to two sides of the other, and if the angles formed by these two sides (called the *included* angles) are congruent to each other, then the other pairs of corresponding angles are congruent.

Exercise Group 6–5

1. In Fig. 6–17, if $AB \cong DF$, $BC \cong DE$, and $\angle B \cong \angle D$, what angle is congruent to $\angle A$? to $\angle C$?

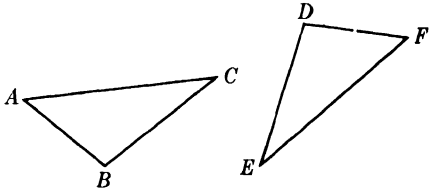


FIGURE 6–17

2. In Fig. 6–18, if $PQ = US$, $QR = UT$, and $\angle Q \cong \angle U$, what congruences hold?

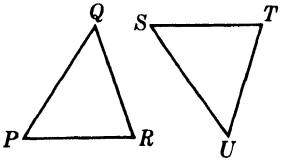


FIGURE 6–18

3. Draw figures to show that if $AB \cong A'B'$, and $\angle A \cong \angle A'$, but $AC \ncong A'C'$ (the symbol $\ncong$ is read "is not congruent to"), then it need not be true that $\angle C \cong \angle C'$.

4. Draw figures to show that if $AB \cong A'B'$, and $AC \cong A'C'$, but $\angle A \ncong \angle A'$, then it need not be true that $\angle B \cong \angle B'$.

5. If, in the figure $AC \cong AD$, and $\angle CAB \cong \angle BAD$, prove that $\angle C \cong \angle D$. (Fig. 6–19)

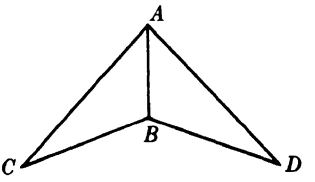


FIGURE 6–19

6. If, in the figure, $DO \cong OB$, and $AO \cong OC$, prove that $\angle A \cong \angle C$. (Fig. 6–20)

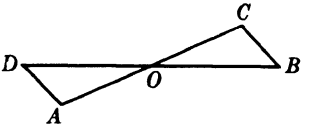


FIGURE 6–20

7. If, in the figure, $AB \cong CD$, and $\angle CBA \cong \angle BCD$, prove that $\angle CAB \cong \angle BDC$. (Fig. 6–21)

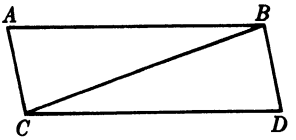


FIGURE 6–21

8. If, in the figure, $AD \cong BD$, and $\angle ADC$ is a right angle, prove that $\angle A \cong \angle B$. (Fig. 6–22)

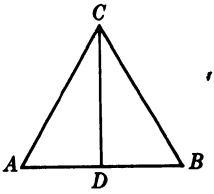


FIGURE 6–22

9. Given: $\angle EBC \cong \angle ABD$,
$AB \cong BC$,
$BE \cong BD$.
Prove: $\angle A \cong \angle C$.
(Fig. 6–23)

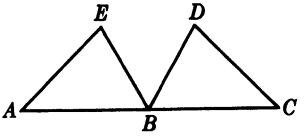


FIGURE 6–23

10. Given: $AD \cong DC$,
$\angle 1 \cong \angle C$,
$\angle 2 \cong \angle C$.
Prove: $\angle 3 \cong \angle 4$.
(Fig. 6–24)

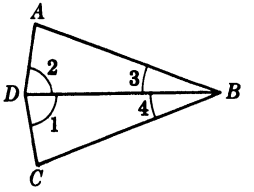


FIGURE 6–24

11. Given: $\angle 1 \cong \angle 2$,
$\angle BCF \cong \angle ADF$,
$CG \cong HD$,
$CF \cong DF$.
Prove: $\angle 3 \cong \angle 4$.
(Fig. 6–25)

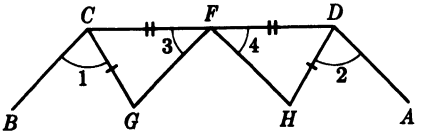


FIGURE 6–25

12. Given: $\angle 1 \cong \angle 2$,
$AE \cong CD$,
$AB \cong BC$.
Prove: $\angle E \cong \angle D$.
(Fig. 6–26)

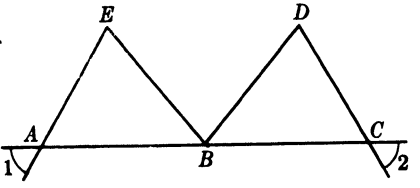


FIGURE 6–26

13. Given: $\angle 1 \cong \angle 2$,
$AC \cong BC$,
$AM \cong BM$.
Prove: $\angle 3 \cong \angle 4$.
(Fig. 6–27)

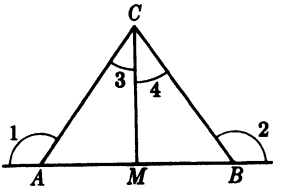


FIGURE 6–27

14. Given: $\angle ADC \cong \angle ABC$,
$\angle 1 \cong \angle 2$,
$AD \cong BC$.
Prove: $\angle A \cong \angle C$.
(Fig. 6–28)

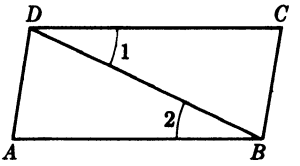


FIGURE 6–28

15. Given: $AD \cong DC$,
$\angle 1 \cong \angle 2$.
Prove: $\angle A \cong \angle C$.
(Fig. 6–29)

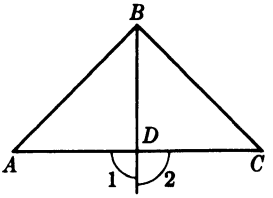


FIGURE 6–29

*16. Given: $BF \cong ED$,
$AE \cong CF$,
$\angle 1 \cong \angle 2$.
Prove: $\angle 3 \cong \angle 4$,
$\angle 5 \cong \angle 6$.
(Fig. 6–30)

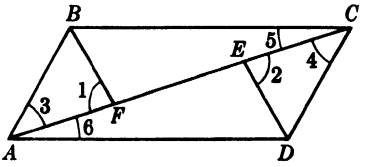


FIGURE 6–30

*17. Given: $AB \cong AD$,
$BO \cong OD$,
$\angle OAB \cong \angle OAD$.
Prove: $\angle 1 \cong \angle 2$.
(Fig. 6–31)

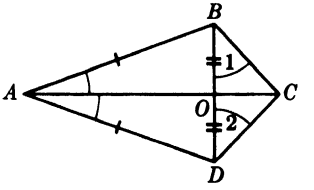


FIGURE 6–31

6–6 ISOSCELES TRIANGLES

We can now prove a theorem about isosceles triangles, for which we need a definition. The definition of isosceles triangle is the same as that given in Chapter 1, Exercise Group 1–4:

Definition 6–2. A triangle is called *isosceles* if two of its sides are congruent (of equal length). The *base angles* of the triangle are those opposite the congruent sides.

Theorem 6–3. *In every isosceles triangle, the base angles are congruent to each other.*

Proof. Consider any isosceles triangle, and label the vertices in two different ways, as shown in Fig. 6–32. Let $AB \cong AC$. Let us look at

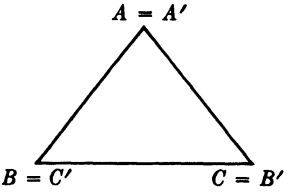


FIGURE 6-32

the triangle both as $\triangle ABC$ and as $\triangle A'B'C'$. We have $AB \cong AC \cong A'B'$, and $AC \cong AB = A'C'$, and $\angle A \cong \angle A'$. Therefore, by the S.A.S. postulate, $\angle B \cong \angle B'$, that is, $\angle B \cong \angle C$.

This proof is quite interesting because it illustrates very well how good notation can help. In the proof we look at the same triangle in two different ways, and once we have done this, the proof is very short.

Definition 6-3. A *midpoint* of a segment AB is a point M of the segment, such that $AM \cong MB$.

Exercise Group 6-6

1. Using Theorem 6-3, prove that an equilateral triangle is equiangular.

2. If $PQ = PR$, and $\angle Q$ is 44°, how many degrees in $\angle R$? (We have not yet studied degrees in our geometry, but for this problem you may use what you already know from Chapter 1.) (Fig. 6-33)

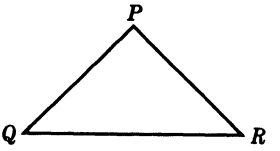


FIGURE 6-33

3. If, in the figure, $AB \cong AC$ and $BD \cong DC$, prove that

$\angle ABC \cong \angle ACB$,
$\angle CBD \cong \angle BCD$,
$\angle ABD \cong \angle ACD$.

(Fig. 6-34)

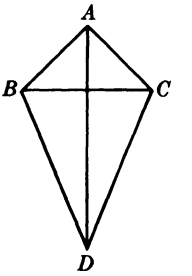


FIGURE 6-34

4. Given: $AC \cong BC$,
$AD \cong DB$.
Prove: $\angle 1 \cong \angle 2$.
(Fig. 6-35)

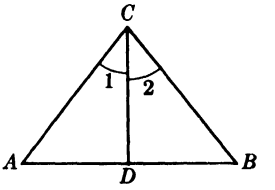


FIGURE 6-35

5. Given: $\angle ABC \cong \angle ABE$,
$CB \cong BE$.
Prove: $\angle C \cong \angle E$.
(Fig. 6-36)

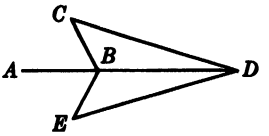


FIGURE 6–36

6. Given: $AC \cong DF$, $BD \cong CE$, $\angle 1 \cong \angle 2$.
Prove: $\angle A \cong \angle F$.
(Fig. 6–37)

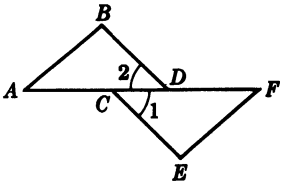


FIGURE 6–37

7. In the figure $AD \cong EC$, and $BE \cong BD$. Show that $\angle A \cong \angle C$. (Fig. 6–38)

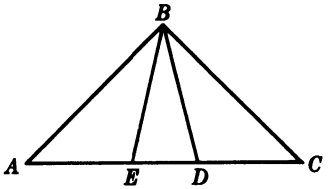


FIGURE 6–38

8. Given: $RS \cong RT$, $US \cong UT$.
Prove: $\angle RSU \cong \angle RTU$.
(Fig. 6–39)

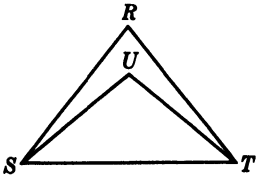


FIGURE 6–39

9. Given: $AC \cong BC$, $CD \cong CE$.
Prove: $\angle 1 \cong \angle 2$.
(Fig. 6–40)

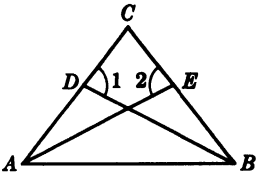


FIGURE 6–40

10. Given: $AO \cong OD$, $AF \cong FD$, $BO \cong OC$, $BF \cong FC$.
Prove: $\angle EAB \cong \angle EDC$.
(Fig. 6–41)

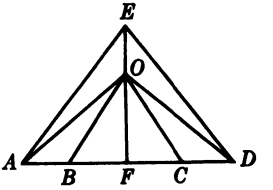


FIGURE 6–41

11. In the figure, $AC \cong BC$, $CG \cong CF$, and $\angle 1 \cong \angle 2$. Prove that $\angle 3 \cong \angle 4$. (Fig. 6–42)

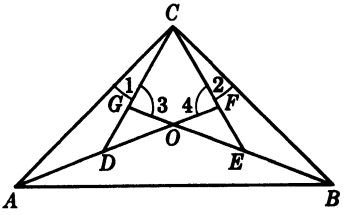


FIGURE 6–42

6–7 CONGRUENCE OF TRIANGLES

We now define congruence for triangles and prove some theorems.

Definition 6–4. Two triangles are said to be *congruent* to each other if they can be labeled $\triangle ABC$ and $\triangle A'B'C'$, such that $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. We then write $\triangle ABC \cong \triangle A'B'C'$ and read this as "triangle ABC is congruent to triangle $A'B'C'$." $\angle A$ corresponds to $\angle A'$, $\angle B$ to $\angle B'$, side AC to side $A'C'$, etc.

Exercise 6–7

Prove that congruence of triangles is transitive.

Theorem 6–4. *If the vertices of two triangles may be labeled A, B, C and A', B', C', such that $AB \cong A'B'$, $AC \cong A'C'$, $\angle A \cong \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Remark. This is the first of the three congruence theorems for triangles. It may also be stated as follows: *if two triangles are such that two sides and the included angle of one are congruent, respectively, to two sides and the included angle of the other, then the triangles are congruent.* This theorem is often called the "*side-angle-side* theorem" and is abbreviated S.A.S.

Proof. From the S.A.S. postulate, we know that $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$. It only remains to be shown that the third sides are congruent, that is, that $BC \cong B'C'$. We prove this by considering the point D' on ray $C'B'$, such that $CB \cong C'D'$, as in Fig. 6–43, and then showing that $D' = B'$.

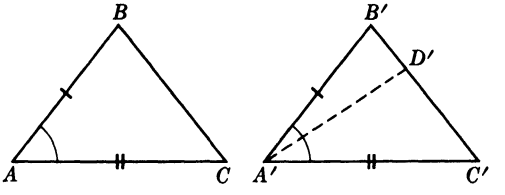


FIGURE 6–43

STATEMENTS	REASONS
1. There is a point D' on the ray $C'B'$ such that $CB \cong C'D'$.	1. Existence postulate for segments.
2. $AC \cong A'C'$, and $AB \cong A'B'$.	2. Given.
3. $\angle B'A'C' \cong \angle BAC$.	3. Given.
4. $\angle C \cong \angle C'$, and $\angle B \cong \angle B'$.	4. S.A.S. postulate.

5. $\angle BAC \cong \angle D'A'C'$.

6. Ray $A'B'$ is the same as the ray $A'D'$.

7. $B' = D'$.

8. $BC \cong B'C'$.

9. $\triangle ABC \cong \triangle A'B'C'$.

5. By S.A.S. postulate, considering $\triangle ACB$ and $\triangle A'C'D'$.

6. Existence postulate for angles.

7. The lines $A'B'$ and $B'C'$ intersect in exactly one point by Theorem 3-1. This point is B' and is also D'.

8. Statements 1 and 7.

9. Definition 6-4.

In the exercises that follow we will make use of the congruence theorems. In some exercises it will be convenient to know that *"halves" of congruent segments are congruent to each other.* That is, if $AB \cong CD$ and M and N are midpoints of AB and CD respectively, then $AM \cong CN$. To see this we note that $\overline{AB} = \overline{CD}$ and by the properties of numbers $\overline{AM} = \overline{CN}$ so that $AM \cong CN$.

Exercise Group 6–8

1. Given the figure, let $\angle 1 \cong \angle 2$ and $AB \cong BC$. Prove that $AD \cong DC$. (Fig. 6–44)

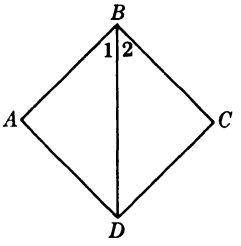


FIGURE 6–44

2. Given the figure, let $\angle 1 \cong \angle 2$ and $EF \cong GH$. Prove that $\triangle EGH = \triangle HFE$. (Fig. 6–45)

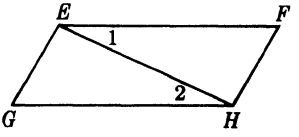


FIGURE 6–45

3. Given the figure, let $AC \cong AD$ and AB bisect $\angle A$. Prove that $\angle CBA \cong \angle DBA$. (Fig. 6–46)

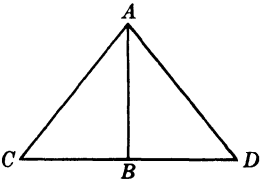


FIGURE 6–46

4. Given the figure, let $AB \cong AD$ and $BC \cong CD$. Prove that $\triangle ABC \cong \triangle ADC$. (Fig. 6–47)

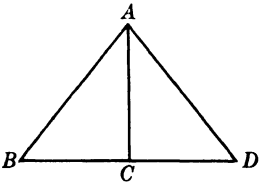


FIGURE 6–47

5. If, in the figure, $AB \cong AC$, D is the midpoint of AB, and E is the midpoint of AC, prove that $\triangle ABE \cong \triangle ACD$. (Fig. 6–48)

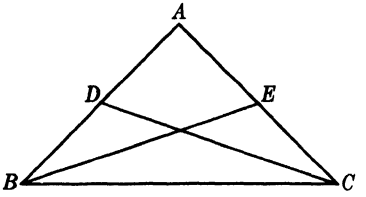


FIGURE 6–48

6. If, in the figure, $AB \cong AD$, and $BC \cong CD$, prove that $\angle ABC \cong \angle ADC$ and $\angle BAC \cong \angle DAC$. (Fig. 6–49)

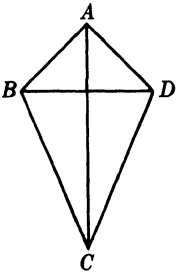


FIGURE 6–49

7. In the figure, $AB \cong BC$, and D, E are the midpoints of AB and BC respectively. Prove that $AE \cong CD$. (Fig. 6–50)

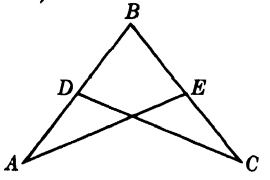


FIGURE 6–50

8. Given: $EB \cong BD$, B is the midpoint of AC, and $\angle EBC \cong \angle DBA$. Prove that $AE \cong CD$. (Fig. 6–51)

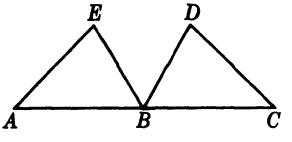


FIGURE 6–51

9. Prove that $BD \cong AC$ if $\angle DAB \cong \angle CBA$ and $AD \cong BC$. (Fig. 6–52)

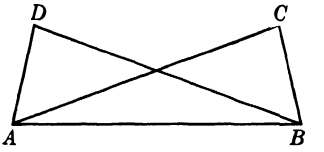


FIGURE 6–52

10. In the figure, $AD \cong EC$, and $BE \cong BD$. Prove that $\triangle ABC$ is isosceles. (Fig. 6–53)

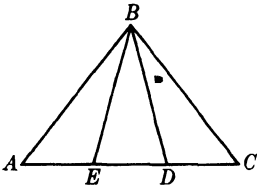


FIGURE 6–53

11. A farmer, wishing to find the distance between posts at A and B on opposite sides of a barn, does the following. He places a stake at C, and on the line through B and C places a stake at D, such that $\overline{BC} = \overline{DC}$. Then on the line through A and C, he places a stake at E, such that $\overline{AC} = \overline{CE}$. He measures DE and finds the length is 78 ft. Prove that $\overline{AB} = 78$ ft. (Fig. 6–54)

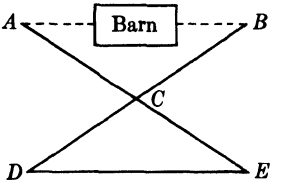


FIGURE 6–54

6–8 THE SECOND CONGRUENCE THEOREM

The first congruence theorem was the side-angle-side theorem (S.A.S.). The second one says that two triangles are congruent if two angles and the included side of one are, respectively, congruent to two angles and the included side of the other. This theorem is called the "angle-side-angle theorem" and is abbreviated A.S.A.

Theorem 6–5. *If two triangles* $\triangle ABC$ *and* $\triangle A'B'C'$ *are such that* $\angle A \cong \angle A'$, $\angle C \cong \angle C'$, *and* $AC \cong A'C'$, *then* $\triangle ABC \cong \triangle A'B'C'$.

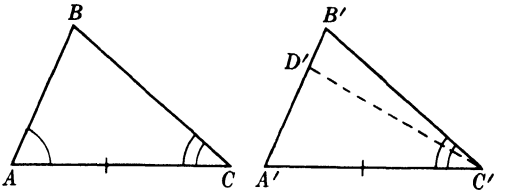


FIGURE 6–55

Proof. We consider the point D' on the ray $A'B'$ (Fig. 6–55), such that $AB \cong A'D'$. Then it is true that $\triangle ABC \cong \triangle A'D'C'$. (Why?) Then, if $D' \neq B'$, we have both $\angle D'C'A'$ and $\angle B'C'A'$ congruent to $\angle C$. This is impossible, so $B' = D'$.

STATEMENTS	REASONS
1. There is a point D' on the ray $A'B'$, such that $AB \cong A'D'$.	1. Existence postulate for segments.
2. $\angle A \cong \angle A'$, $AC \cong A'C'$.	2. Given.
3. $\angle C \cong \angle D'C'A'$.	3. S.A.S. postulate, applied to $\triangle ABC$ and $\triangle A'D'C'$.
4. $\angle C \cong \angle B'C'A'$.	4. Given.
5. Ray $C'D'$ = ray $C'B'$.	5. Existence postulate for angles.
6. $D' = B'$.	6. Two lines intersect in at most one point.
7. $\triangle ABC \cong \triangle A'B'C'$.	7. S.A.S. theorem.

Exercise Group 6–9

1. If, in the figure, $\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$, prove that $\triangle ABD \cong \triangle CBD$. (Fig. 6–56)

2. If, in the figure, $\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$, prove that $AB = CD$. (Fig. 6–57)

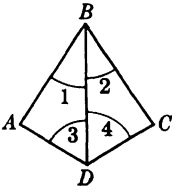


FIGURE 6–56

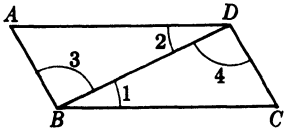


FIGURE 6–57

3. If, in the figure, $\angle 1 \cong \angle 2$, and $AB \cong BC$, prove that $AE \cong DC$. (Fig. 6–58)

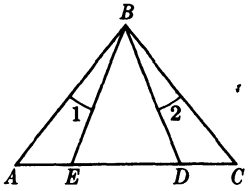


FIGURE 6–58

4. Draw a triangle. Let one angle be 31°, a second angle be 53°, and their included side be 7 inches.

5. If, in the figure, $\angle ABC \cong \angle ABE$, and $\angle CDB \cong \angle EDB$, prove that $CD \cong ED$. (Fig. 6–59)

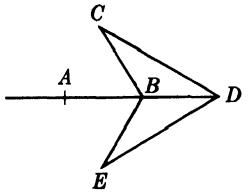


FIGURE 6–59

6. If, in the figure, $AB \cong BC$, and $\angle A \cong \angle C$, prove that $\triangle ABD \cong \triangle CBE$. (Fig. 6–60)

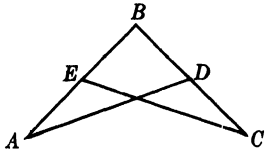


FIGURE 6–60

7. If, in the figure, $\angle A \cong \angle D$ and $AB \cong BD$, prove that $AE \cong CD$. (Fig. 6–61)

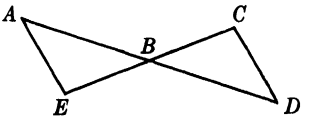


FIGURE 6–61

8. If, in the figure, $AF \cong CD$, $\angle A \cong \angle D$, and $\angle ACB \cong \angle DFE$, prove that $\triangle ABC \cong \triangle DEF$. (Fig. 6–62)

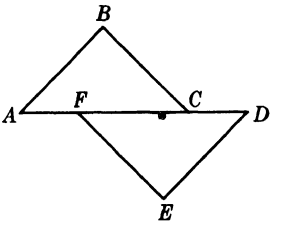


FIGURE 6–62

9. Draw triangles ABC whose parts are:

(a) $\angle A = 36°$; $\angle B = 45°$; $AB = 3$ in.

(b) $\angle A = 60°$; $\angle C = 30°$; $AC = 4$ in.

(c) $\angle C = 33°$; $\angle B = 33°$; $BC = 5/2$ in.

10. Given: $\angle 3 \cong \angle 4$,
$AB \cong AD$,
AC bisects $\angle BAD$.
Prove: $BC \cong CD$.
(Fig. 6–63)

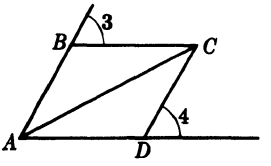


FIGURE 6–63

11. In the figure, $AF \cong FB$, and $AE \cong DB$. Prove that $\angle 3 \cong \angle 4$. (Fig. 6–64)

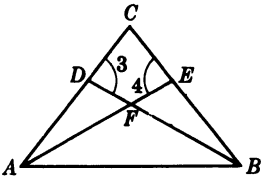


FIGURE 6–64

12. Prove that $\angle 1 \cong \angle 2$ if $AB \cong BC$ and $AD \cong DC$. (Fig. 6–65)

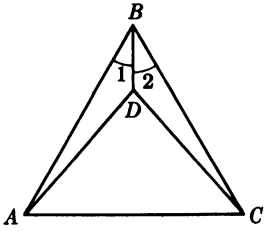


FIGURE 6–65

13. If $AH \cong HB$, and $AG \cong GB$, prove that $\angle 3 \cong \angle 4$. (Fig. 6–66)

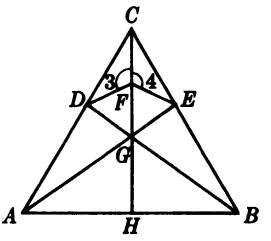


FIGURE 6–66

14. Given: $\angle 1 \cong \angle 2$,
$AC \cong BC$,
$DF \cong EG$.
Prove: $\triangle HIJ$ is isosceles.
(Fig. 6–67)

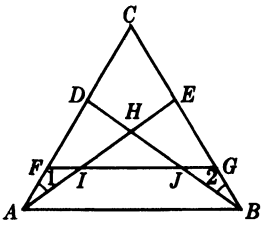


FIGURE 6–67

In Section 6–6 we proved that the base angles of an isosceles triangle are congruent. We can now easily prove the converse.

Theorem 6–6. *If a triangle has two angles congruent to each other, then the sides opposite these angles are congruent.*

Proof. Suppose that the triangle is $\triangle ABC$, and that $\angle A \cong \angle C$. Let the vertices be labeled A', B', C', as in Fig. 6–68. Then we have $\angle A \cong \angle C \cong \angle A'$, and $\angle C \cong \angle A \cong \angle C'$, and $AC \cong CA \cong A'C'$.

Hence by the A.S.A. congruence theorem, $\triangle ABC \cong \triangle A'B'C'$. Therefore $AB \cong A'B' \cong CB$.

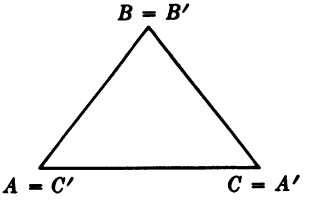


FIGURE 6–68

Exercise 6–10

Prove that an equiangular triangle is equilateral.

6–9 THE THIRD CONGRUENCE THEOREM

The two previous congruence theorems were the S.A.S. and A.S.A. theorems. The third of the three fundamental congruence theorems states that two triangles are congruent if the sides of one are congruent to the sides of the other. It is called the "side-side-side theorem" and is abbreviated S.S.S.

Theorem 6–7. *If the sides of one triangle are congruent, respectively, to the sides of another triangle, then the triangles are congruent.*

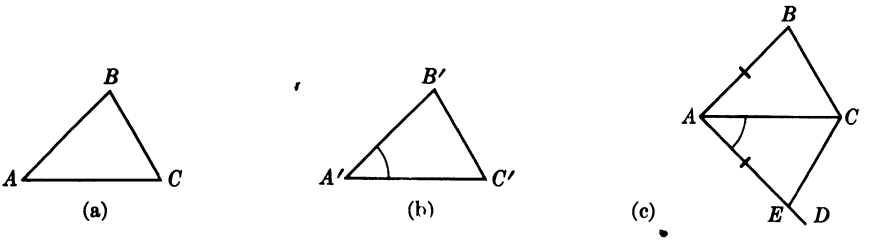


FIGURE 6-69

Proof. Suppose that the triangles are labeled $\triangle ABC$ and $\triangle A'B'C'$, as in Fig. 6–69(a) and 6–69(b), and that $AB \cong A'B'$, $AC \cong A'C'$, and $BC \cong B'C'$. We have only to prove that the angles are congruent. There is a ray AD, as in Fig. 6–69(c), from A on the opposite side of AC from B, such that $\angle DAC \cong \angle A'$. On AD, there is a point E, such that $AE \cong A'B'$.

STATEMENTS	REASONS
1. $\angle EAC \cong \angle A'$.	1. The ray AE was chosen this way.
2. $AE \cong A'B'$.	2. E was so chosen.
3. $AC \cong A'C'$.	3. Given.
4. $\triangle AEC \cong \triangle A'B'C'$.	4. S.A.S. theorem.

Now consider the segment BE and the triangles ABE and BEC, as in Fig. 6–70.

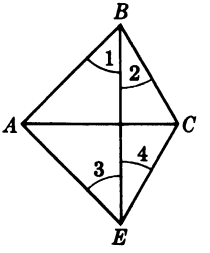


FIGURE 6–70

STATEMENTS	REASONS
5. $\triangle ABE$ is isosceles.	5. $AB \cong A'B' \cong AE$.
6. $\angle 1 \cong \angle 3$.	6. Statement 5 and Theorem 6–3.
7. $\angle 2 \cong \angle 4$.	7. Same argument as used for $\angle 1$ and $\angle 3$.
8. $\angle B \cong \angle E$.	8. Addition of angles postulate.
9. $\triangle ABC \cong \triangle AEC$.	9. S.A.S. theorem.
10. $\triangle ABC \cong \triangle A'B'C'$.	10. Statements 4 and 9.

Remark. In our proof, we have tacitly assumed that BE intersects the segment AC at a point between A and C. This need not be the case. We could have either Fig. 6–71(a) or 6–71(b).

Problem. Prove both cases.

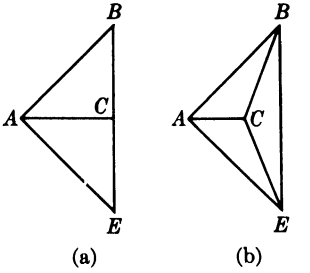


FIGURE 6–71

Exercise Group 6–11

1. You can conclude from Theorem 6–7 that a triangle is *rigid*. Why? Why is a brace put on a gate? (Fig. 6–72)

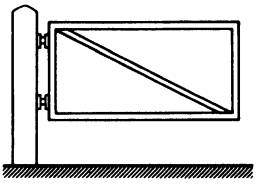


FIGURE 6–72

2. If, in the figure, $\overline{AB} = \overline{BC}$, and $\overline{AD} = \overline{DC}$, prove that $\angle ABD \cong \angle DBC$. (Fig. 6–73)

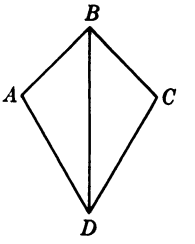


FIGURE 6–73

3. If, in the figure, $\overline{AB} = \overline{DC}$, and $AD \cong BC$, prove that $\angle CAB \cong \angle ACD$ and $\angle D = \angle B$. (Fig. 6–74)

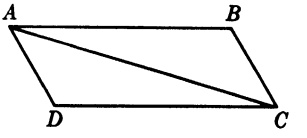


FIGURE 6–74

4. If, in the figure, $AB \cong BC$, and $AD \cong DC$, prove that BD bisects $\angle ABC$. (Fig. 6–75)

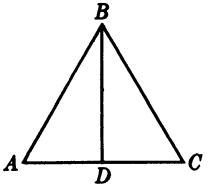


FIGURE 6–75

5. If, in the figure, $AB \cong AD$, and $BC \cong CD$, prove that $\angle 1 \cong \angle 2$, $\triangle ABE \cong \triangle ADE$, and $\angle AEB$ is a right angle. (Fig. 6–76)

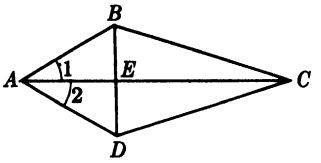


FIGURE 6–76

6. If, in the figure, $\overline{AB} = \overline{CD}$, and $\overline{BC} = \overline{AD}$, and $\overline{AF} = \overline{EC}$, prove that $\overline{BF} = \overline{ED}$. (Fig. 6–77)

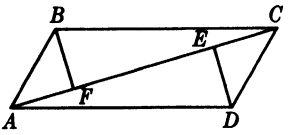


FIGURE 6–77

7. If, in the figure, $\triangle ABC$ is isosceles with vertex B, and $AD \cong EC$, prove that $BE \cong BD$. (Fig. 6–78)

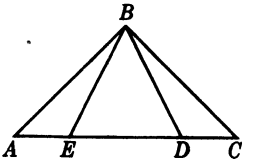


FIGURE 6–78

8. If, in the figure, $AB \cong AD$, and $BC \cong CD$, prove that $\angle B \cong \angle D$. (Fig. 6–79)

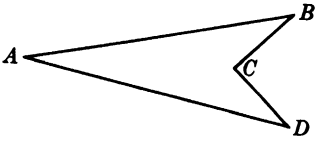


FIGURE 6–79

9. Given: $CE \cong CF$,
$AE \cong BF$,
$AD \cong BD$.
Prove: $\angle 1 \cong \angle 2$.
(Fig. 6–80)

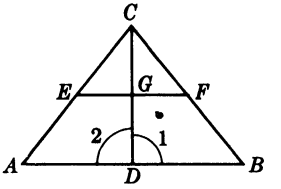


FIGURE 6–80

10. Given: $\angle 1 \cong \angle 2$,
$\angle DCE \cong \angle DCF$,
$AC \cong BC$.
Prove: $AD \cong BD$.
(Fig. 6–81)

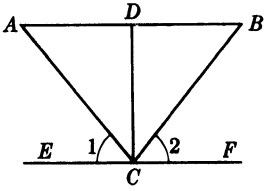


FIGURE 6–81

11. If $AC \cong BD$, and $AD \cong BC$, show that $\angle C \cong \angle D$. (Fig. 6–82)

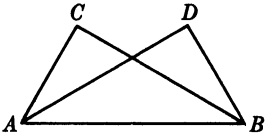


FIGURE 6–82

12. If $AB \cong BC$, and $BD \cong BE$, show that $\angle A \cong \angle C$. (Fig. 6-83)

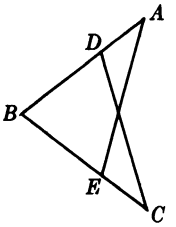


FIGURE 6-83

13. Given: $AC \cong BC$,
$AD \cong DB$.
Prove: $AO \cong BO$.
(Fig. 6-84)

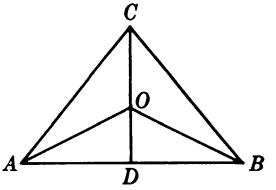


FIGURE 6-84

14. Prove that $AD \cong BE$ if $AC \cong BC$ and $\angle 1 \cong \angle 2$. (Fig. 6-85)

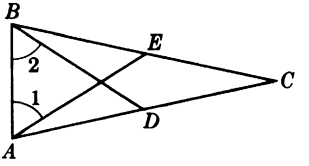


FIGURE 6-85

15. Given: $AD \cong BC$,
$AB \cong DC$,
$AM \cong MC$.
Prove: $EM \cong MF$.
(Fig. 6-86)

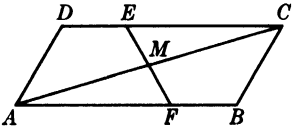


FIGURE 6-86

16. Given: $\angle 1 \cong \angle 2$,
$AE \cong DB$.
Prove: $\angle 3 \cong \angle 4$.
(Fig. 6-87)

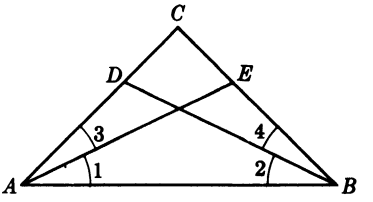


FIGURE 6-87

17. Given: $BE \cong CE$,
$\angle 1 \cong \angle 2$.
Prove: $\angle 3 \cong \angle 4$.
(Fig. 6-88)

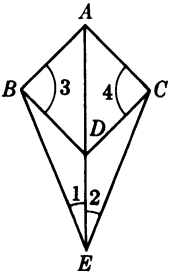


FIGURE 6-88

18. Prove that the medians from the base angles of an isosceles triangle are congruent.

19. Given: $AD \cong BC$,
$AB \cong CD$,
$\angle 1 \cong \angle 2$.
Prove: $DE \cong BF$.
(Fig. 6-89)

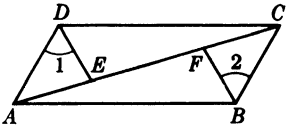


FIGURE 6-89

20. Given: $AD \cong CB$, $BD \cong AC$.
Prove: $\triangle DEC$ is isosceles. (Fig. 6–90)

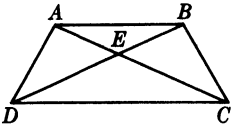


FIGURE 6–90

21. Given: $AB \cong CD$, $AE \cong CF$, $\angle 3 \cong \angle 4$.
Prove: $\angle 1 \cong \angle 2$. (Fig. 6–91)

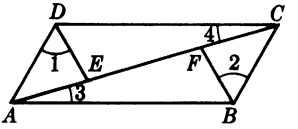


FIGURE 6–91

22. If triangles ABC and ADE are isosceles with vertex A, and $\angle 1 \cong \angle 2$, prove that $\angle 3 \cong \angle 4$, $FG \cong FH$, and $GI \cong HJ$. (Fig. 6–92)

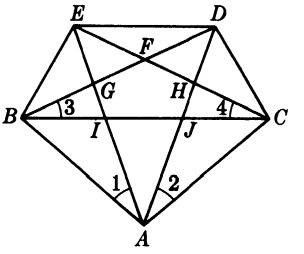


FIGURE 6–92

23. Prove that the line joining the vertex of an isosceles triangle to the midpoint of the base bisects the vertex angle, and conversely.

6–10 RIGHT ANGLES

In Definition 6–1 we defined right angles, but so far *we do not know that there are such angles.* We now show how to use our congruence theorem (S.A.S.) to show that there is a right angle in our geometry.

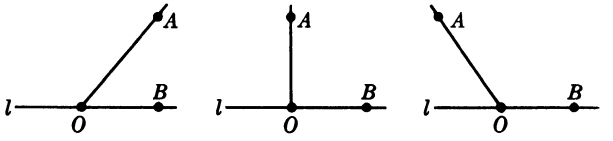


FIGURE 6–93

Proof. On a line l, choose a point O and let $\angle AOB$ be an angle (not straight) with one side on l (Fig. 6–93). Now consider the angle congruent to $\angle AOB$, with vertex O, one side OB on the line l and the other on the opposite side of l from A. Choose the point A' on this last side, such that $OA \cong OA'$. Because A and A' are on opposite sides of l, the segment AA' intersects the line in some point C. There are three cases to consider, as shown in Fig. 6–94.

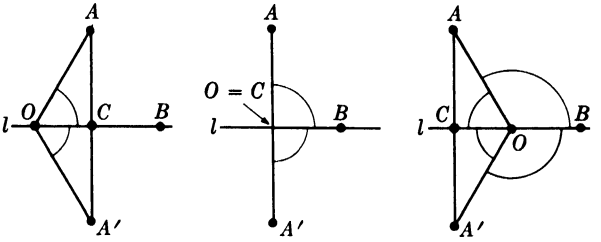


FIGURE 6–94

If C is on the same side of O as B, then by the S.A.S. theorem, $\triangle AOC \cong \triangle A'OC$. Hence $\angle OCA \cong \angle OCA'$. Since these angles are congruent and supplementary to each other, they are right angles. In case $C = O$, we have $\angle AOB \cong \angle A'OB$, and hence they are right angles. If C is on the opposite side of O from B, then the angles AOC and $A'OC$ are congruent. (Why?) We may apply the S.A.S. congruence theorem. $\angle ACO \cong \angle A'CO$, and these are right angles.

Exercise Group 6–12

1. Perform a construction to draw a right angle.

2. If, in the figure, $PQ \cong PS$, and $\angle QPR \cong \angle SPR$, prove that $\angle PRQ$ is a right angle. (Fig. 6–95)

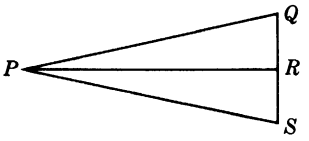


FIGURE 6–95

3. If, in the figure, $AB \cong AD$, and $BC \cong CD$, prove that $\angle AEB$ is a right angle. (Fig. 6–96)

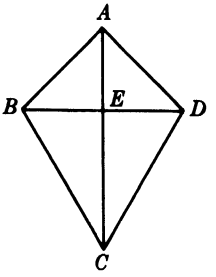


FIGURE 6–96

6–11 INEQUALITIES FOR ANGLES

Even before studying geometry you used angles of the same size and compared angles of different sizes (by saying, for example, that one angle is larger than another). In this section, we see how to compare the sizes of angles. To do this, we have to define $>$ (is greater than) and $<$ (is less than) for angles. The idea we use in forming this definition is illustrated in Fig. 6–97. If $\angle B'$ is congruent to $\angle B$, and $\angle B'$ and $\angle A$ are as drawn (that is, the other side of $\angle B'$ is *in the interior* of $\angle A$),

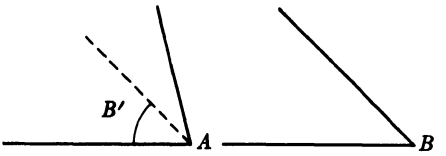


FIGURE 6–97

then we would say that $\angle B < \angle A$. In Fig. 6–97, we have, so to speak, "moved" $\angle B$ to the position of $\angle B'$ for comparison. Notice how the definition below parallels the definition of "greater than" and "less than" for segments.

Definition 6–5. Let A and B be any angles not straight angles. Let $\angle B'$ be an angle congruent to $\angle B$, such that one side of $\angle B'$ is the same as the first side of $\angle A$, and the other side of $\angle B'$ and the second side of $\angle A$ both lie on the same side of the first side of $\angle A$.

(a) If the second side of $\angle B'$ is interior to $\angle A$, we say that $\angle B < \angle A$. (See Fig. 6–98.)

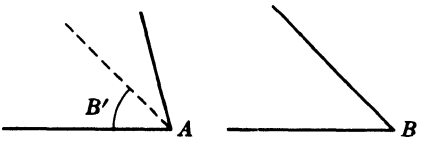


FIGURE 6–98

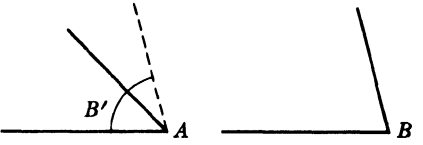


FIGURE 6–99

(b) If the second side of $\angle B'$ is exterior to $\angle A$, we say that $\angle B > \angle A$. (See Fig. 6–99.)

(c) If $\angle A$ is a straight angle, and $\angle B$ is not, we say that $\angle A > \angle B$ and that $\angle B < \angle A$.

Our intuition tells us that if $\angle B < \angle A$, then $\angle A > \angle B$, and if $\angle B > \angle A$, then $\angle A < \angle B$. These statements require proof. Since the proofs of these statements are essentially the same, we prove only the first.

Theorem 6–8. *If $\angle B$ and $\angle A$ are two angles, such that $\angle B < \angle A$, then $\angle A > \angle B$.*

**Proof.* Figure 6–100 shows that $\angle B < \angle A$. To show that $\angle A > \angle B$, we must show that the second side of $\angle A'$ is in the exterior of $\angle B$, as in Fig. 6–101.

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* This proof is difficult. If the teacher wishes, Theorem 6–8 may be treated as a postulate.

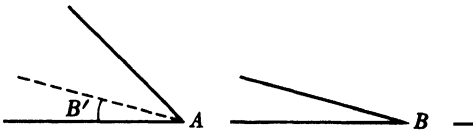


FIGURE 6–100

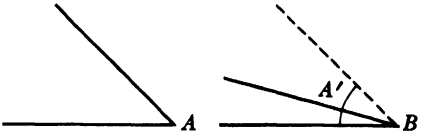


FIGURE 6–101

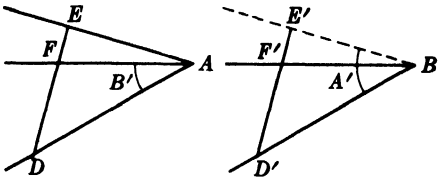


FIGURE 6–102

Choose points D and E on the sides of $\angle A$ and points D' and E' on the sides of $\angle A'$ such that $AD \cong BD'$ and $AE \cong BE'$. (See Fig. 6–102.) DE intersects the second side of $\angle B'$ in some point F between D and E because the second side of $\angle B'$ is in the interior of $\angle A$. $\triangle ADE \cong \triangle BD'E'$ by the S.A.S. theorem, and hence $DE \cong D'E'$. Choose F' between D' and E', such that $DF \cong D'F'$. This is possible because $D'E' \cong DE > DF$. The ray BF' is in the interior of $\angle A'$. (Why?) $\angle D \cong \angle D'$, and hence $\triangle FDA \cong \triangle F'D'B$. $\angle DAF \cong \angle D'BF'$, and $\angle DAF$ was given congruent to $\angle B$. By the three-part angle postulate, $\angle B \cong \angle D'BF'$; it follows that the ray BF' is the same as the second side of $\angle B$. Therefore, the second side of $\angle B$ is interior to $\angle A'$ or, what is the same, the second side of $\angle A'$ is in the exterior of $\angle B$. This completes the proof that $\angle A > \angle B$.

Remark. Theorem 6–8 tells us that we get the same result regardless of whether we "move" B to compare with A, or "move" A to compare with B.

Theorem 6–9. *For any two angles $\angle A$ and $\angle B$, either $\angle A < \angle B$, or $\angle A \cong \angle B$, or $\angle A > \angle B$; and exactly one of these is true.*

Proof. If $\angle A$ is a straight angle, either $\angle A \cong \angle B$ and $\angle B$ is also a straight angle or, by Definition 6–5, $\angle A > \angle B$. Similarly, if $\angle B$ is a straight angle, either $\angle B \cong \angle A$ or $\angle B > \angle A$. If neither $\angle A$ nor $\angle B$ is straight, then the second side of $\angle B'$ coincides with the second side of $\angle A$, and $\angle B \cong \angle A$; *or* the second side of $\angle B'$ is interior to $\angle A$, and $\angle A > \angle B$; *or* the second side of $\angle B'$ is exterior to $\angle A$, and $\angle A < \angle B$. Since only one of these possibilities can hold, the proof is complete.

Some simple properties of inequalities are collected in the following theorem.

Theorem 6–10.
(a) *If* $\angle A \cong \angle B$, *and* $\angle B < \angle C$, *then* $\angle A < \angle C$.
(b) *If* $\angle A < \angle B$, *and* $\angle B \cong \angle C$, *then* $\angle A < \angle C$.
(c) *If* $\angle A \cong \angle B$, *and* $\angle B > \angle C$, *then* $\angle A > \angle C$.
(d) *If* $\angle A < \angle B$, *and* $\angle B < \angle C$, *then* $\angle A < \angle C$.
(e) *If* $\angle A \geqq \angle B$, *and* $\angle B \geqq \angle C$, *then* $\angle A \geqq \angle C$.*

Proof of (a). Since $\angle B < \angle C$, on "moving" $\angle B$ to compare with $\angle C$, the second side of $\angle B'$ is interior to $\angle C$. Since $\angle A \cong \angle B$, the second sides of $\angle A'$ and $\angle B'$ would be the same. Hence $\angle A < \angle C$.

The proofs in all the other cases are essentially the same.

Exercise Group 6–13

1. Can you think of any other properties not included in Theorem 6–10? There are several. Write them down. Draw figures to illustrate them.

2. Draw figures to indicate that parts (b) through (e) of Theorem 6–10 are true. Be sure you understand how to use Definition 6–5.

Definition 6–6. An angle less than a right angle is called an *acute* angle. An angle greater than a right angle is called an *obtuse* angle.

6–12 RIGHT ANGLES AGAIN

We can now use Theorem 6–9 to prove that all right angles are congruent to each other. It is interesting, from a historical point of view, to know that Euclid *postulated* this fact. The proof we give below could have been formulated by Euclid just as well as he formulated other proofs. Not only did Euclid ignore postulates that were needed (like the betweenness postulates), but he also introduced postulates that were not needed.

Theorem 6–11. *All right angles are congruent to each other.*

Proof. Suppose that two right angles are $\angle A$ and $\angle B$, and that their supplements are $\angle A'$ and $\angle B'$, as in Fig. 6–103(a) and 6–103(b). There are only three possibilities, according to Theorem 6–9. Either $\angle A < \angle B$, or $\angle A \cong \angle B$, or $\angle A > \angle B$, and one of these must be true. We will show that $\angle A < \angle B$ and $\angle A > \angle B$ are false. Suppose that $\angle A < \angle B$, and that $\angle B''$ is congruent to $\angle B$ and located as in Fig. 6–103(c). Let B''' be the supplement of $\angle B''$.

* The symbol $\geqq$ means "greater than or congruent to," as in Chapter 4.

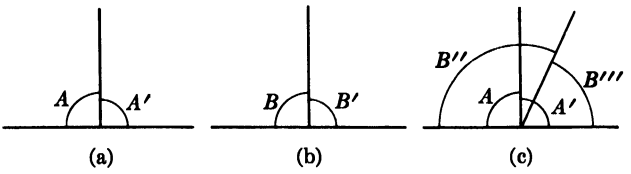


FIGURE 6–103

STATEMENTS	REASONS
1. $\angle B'' \cong \angle B$.	1. By definition of $\angle B''$.
2. $\angle B'''$ is the supplement of $\angle B''$, and $\angle B'$ is the supplement of $\angle B$.	2. By definition of supplement.
3. $\angle B' \cong \angle B'''$.	3. If angles are congruent, their supplements are congruent.
4. $\angle B \cong \angle B'$.	4. $\angle B$ is a right angle.
5. $\angle B \cong \angle B'''$.	5. Angle congruence is transitive.
6. $\angle A < \angle B$.	6. Hypothesis.
7. $\angle A < \angle B'''$.	7. $\angle A < \angle B \cong \angle B'''$.
8. $\angle B''' < \angle A'$.	8. The "second" side of $\angle B'''$ is in the interior of $\angle A'$.
9. $\angle B''' < \angle A$.	9. $\angle B''' < \angle A' \cong \angle A$.
10. $\angle A < \angle A$.	10. Statements 7 and 9.

Since $\angle A \cong \angle A$, we know by Theorem 6–9 that it is impossible for $\angle A$ to be less than itself. Therefore, it is impossible that $\angle A < \angle B$. In the same way, it can be shown that it is impossible that $\angle B < \angle A$. Therefore, we must have $\angle A \cong \angle B$. This completes the proof.

Definition 6–7. Two lines, l and m, are said to be *perpendicular* to each other if they intersect and the angles formed are right angles, as in Fig. 6–104(a). To indicate that lines are perpendicular we write "$l \perp m$."

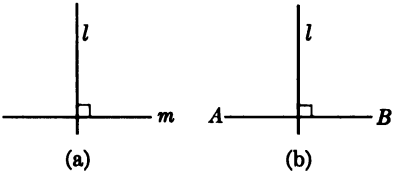


FIGURE 6–104

The line l is called the *perpendicular bisector* of the segment AB if $l \perp AB$ and l passes through the midpoint of AB, as in Fig. 6–104(b). This is written "$l \perp$ bis AB," and read "l is the perpendicular bisector of AB."

Exercise Group 6–14

1. Prove the theorem that if the two legs of one right triangle are congruent, respectively, to the two legs of another right triangle, then the triangles are congruent.

2. In the figure, angles ADC and BCD are right, $AD \cong BC$, and $AB \cong CD$. Prove that $\triangle ADC \cong \triangle BCD$ and $\triangle ABD \cong \triangle BAC$. (Fig. 6–105)

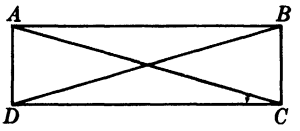


FIGURE 6–105

3. If, in the figure, $AD \cong DC$, and $BD \perp AC$, prove that $\angle BAD \cong \angle BCD$. (Fig. 6–106)

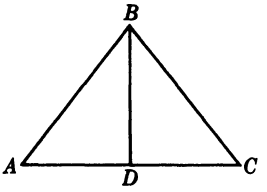


FIGURE 6–106

4. If, in the figure, $AD \cong BC$, angles ADC and BCD are right, and $DE \cong EC$, prove that $AE \cong BE$. (Fig. 6–107)

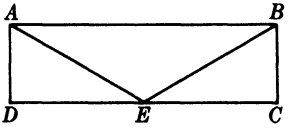


FIGURE 6–107

5. Given: $BD \perp AC$,
$\angle ABD \cong \angle DBC$.
Prove: $\angle A \cong \angle C$.
(Fig. 6–108)

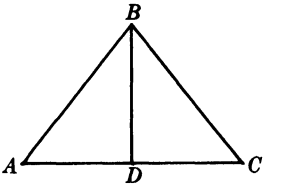


FIGURE 6–108

6. Using Fig. 6–108, prove that $\angle A \cong \angle C$ if $BD \perp$ bis AC.

7. In the figure, $AB \perp BD$, $DE \perp BD$, and $BC \cong CD$. Prove that $AB \cong DE$. (Fig. 6–109)

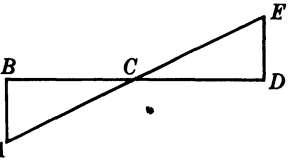


FIGURE 6–109

8. Given: $AB \perp DC$,
$BE \cong BC$,
$\angle DEB \cong \angle C$.
Prove: $DE \cong AC$.
(Fig. 6–110)

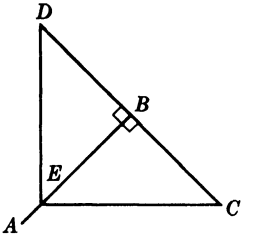


FIGURE 6–110

9. Prove that the bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

10. Prove that the median from the vertex of an isosceles triangle is perpendicular to the base.

11. Given: $AC \cong AD$,
$AC \perp CB$,
$AD \perp DB$.
Prove: $\triangle ACB \cong \triangle ADB$.
(Fig. 6–111)

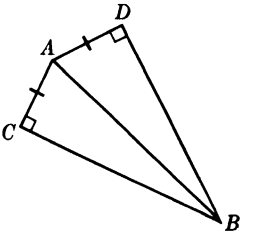


FIGURE 6–111

REVIEW OF CHAPTER 6

In this chapter there are many important ideas and geometric facts. It is most important that you review the topics listed below. Each topic contains many ideas, so study each one separately. Then study the whole list to be sure that you understand how the topics are related.

1. First in importance are the four postulates of Group V. It is not necessary to memorize the postulates word for word, but you must understand the meaning of each one so that you will know when you can use it.

2. We also gave several important definitions. Be sure that you know definitions for the following: right angle, congruent triangles, $>$ and $<$ for angles, acute angle, obtuse angle.

3. Below we list the most important theorems proved in this chapter. You should be able to prove many of them. In reviewing the proofs of these theorems, first see whether you can formulate a proof without looking at the text.

If two angles are congruent, their supplements are congruent.

Vertical angles are congruent.

In an isosceles triangle, the base angles are congruent.

The first congruence theorem (S.A.S.).

For any two angles $\angle A$ and $\angle B$, exactly one of the following is true: $\angle A > \angle B$, $\angle A \cong \angle B$, $\angle A < \angle B$.

If $\angle A \cong \angle B$, and $\angle B < \angle C$, then $\angle A < \angle C$, etc.

All right angles are congruent.

The second congruence theorem (A.S.A.).

The third congruence theorem (S.S.S.).

Review Exercises

1. Draw an $\angle A$. Draw a ray r from a point O. Draw the two angles congruent to A that have the ray r as one side. See Postulate V–1.

2. If $\angle A \cong \angle D$, and $\angle D \cong \angle E$, and $\angle E \cong \angle F$, prove that $\angle A \cong \angle F$. See Postulate V–2.

3. If in Fig. 6–112, $\angle A_1 \cong \angle R$, and $\angle A_2 \cong \angle S$, prove that $\angle B \cong \angle T$. See Postulate V–3.

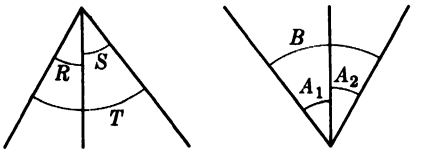


FIGURE 6–112

4. If in Fig. 6–113, $\angle A \cong \angle B$, prove that $\angle C \cong \angle D$. What is the converse of this? Is the converse true? What is the contrapositive of this? Is it true?

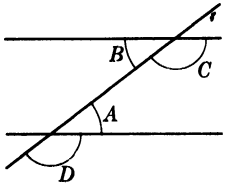


FIGURE 6–113

5. If in Fig. 6–114, $\angle AOB \cong \angle BOC$, what is $\angle AOB$ called?

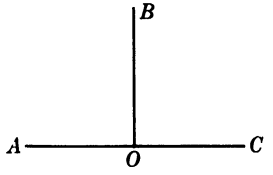


FIGURE 6–114

6. If (Fig. 6–115) $FE \cong ST$, $EG \cong RT$, and $\angle E \cong \angle T$, to what angle is $\angle F$ congruent?

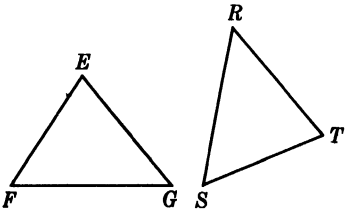


FIGURE 6–115

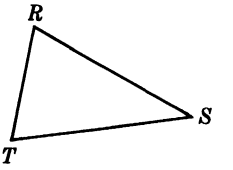


FIGURE 6–116

7. If (Fig. 6–116) $\overline{RS} = \overline{TS}$, prove that $\angle R \cong \angle T$.

8. Using Fig. 6–115, prove that the triangles are congruent.

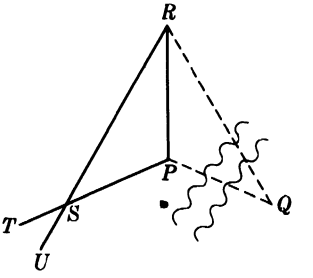


FIGURE 6–117

9. Figure 6–117 shows a way to measure the width $\overline{PQ}$ of a river. From any point R not on the line PQ, RU is sighted so that $\angle URP \cong \angle PRQ$. From P a line PT is sighted so that $\angle TPR \cong \angle QPR$. The point S, where PT and RU intersect, is determined, and PS is measured. Why is this the distance from P to Q?

10. The first congruence theorem says that if two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent. What is the converse of this theorem? Is it true?

11. If a triangle is equilateral, then it is isosceles. What is the converse of this statement? Is it true? What is the contrapositive? Is it true?

12. If in Fig. 6–118, $\triangle ABC$ and $\triangle ADC$ are isosceles, with vertices B and D, prove that BD bisects $\angle ABC$.

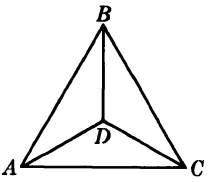


FIGURE 6–118

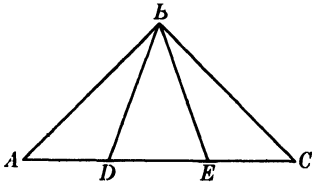


FIGURE 6–119

13. If (Fig. 6–119) $\triangle ABC$ is isosceles, and $\angle ABD \cong \angle EBC$, prove that $\overline{AD} = \overline{EC}$.

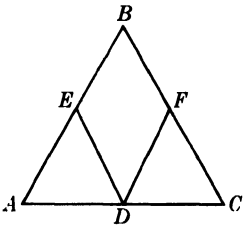


FIGURE 6–120

14. If (Fig. 6–120) $\overline{AB} = \overline{BC}$, $\overline{AD} = \overline{DC}$, and $\angle ADE \cong \angle CDF$, prove that $BE \cong BF$.

15. When we build a house, the subflooring and the sheathing boards are usually laid diagonally. Why?

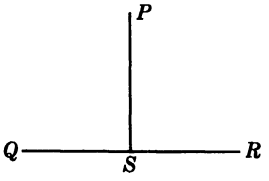


FIGURE 6–121

16. If (Fig. 6–121) $QS \cong SR$, and $PS \perp QR$, prove that $PQ \cong PR$.

17. If the point O bisects the segments AB and CD, prove that $\angle A \cong \angle B$. See Fig. 6–122.

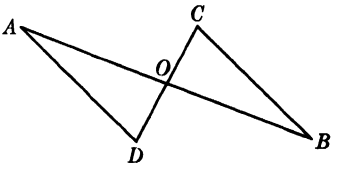


FIGURE 6–122

18. Show that if two isosceles triangles have congruent vertex angles, and their congruent sides have the same length, then the triangles are congruent.

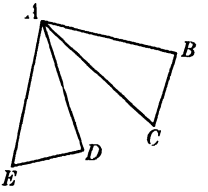


FIGURE 6–123

19. If in Fig. 6–123, $AB \cong AD$, $BC \cong DE$, and $AC \cong AE$, to what angle is $\angle B$ congruent? Why?

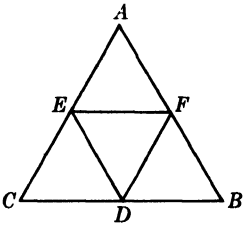


FIGURE 6–124

20. If in Fig. 6–124 $\triangle ABC$ is equilateral, and D, E, F bisect the sides, prove that $\triangle EFD$ is equilateral.

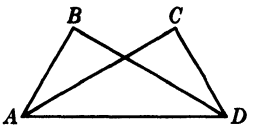


FIGURE 6–125

21. If (Fig. 6–125) $\angle CAD \cong \angle BDA$, and $\angle CDA \cong \angle BAD$, prove that $\overline{AB} = \overline{CD}$.

22. Prove that if two points are equidistant from the ends of a segment, they determine the perpendicular bisector of the segment.

23. Prove that if the bisector of an angle of a triangle is perpendicular to the opposite side, then it bisects that side, and the triangle is isosceles.

24. Prove that any point on the perpendicular bisector of a segment is equidistant from the ends of the segment.

25. Given: $\angle 1 \cong \angle 2$,
$AC \cong BC$,
$IJ \cong IK$.
Prove: $GJ \cong HK$,
$GF \cong FH$.

(Fig. 6–126)

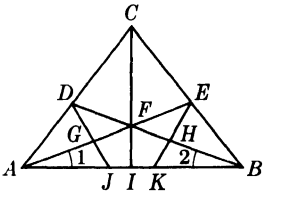


FIGURE 6–126

Test 1
True-False

1. If a theorem is true, its contrapositive is true.
2. The supplement of a 40° angle is a 50° angle.
3. If $\angle A < \angle B$, and $\angle B < \angle C$, and C is a 50° angle, then $\angle A$ is less than a 49° angle.
4. If ABC (B is between A and C), DEF, $AC \cong DF$, and $AB > DE$, then $BC > EF$.
5. If $\triangle ABC$ is isosceles, with vertex A, and D, E are the midpoints of AB, AC respectively, then $BE \cong CD$.
6. If in triangles ABC and $A'B'C'$, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, and $AB \cong A'C'$, then $\triangle ABC \cong \triangle A'B'C'$.
7. If $\angle ABC$ is 40°, and $\angle DBC$ is 50°, then $\angle DBA$ is a right angle.
8. If $\angle ABC$ is 60°, and $\angle DBC$ is 120°, then either $\angle DBA \cong \angle ABC$, or $\angle DBA$ is a straight angle.
9. If segments AB and CD intersect at O, then $\angle AOB \cong \angle BOD$.
10. For all statements α, β, γ, such that $\alpha \rightarrow \beta$ is true and $\beta \rightarrow \gamma$ is false, the implication $\alpha \rightarrow \gamma$ is false.

Test 2

1. Lines l and m intersect so that a pair of adjacent angles differ by 18°. Determine the number of degrees in each angle.
2. If $\overline{AB} = 24$, and ACB, and $\overline{AC} = 3 \cdot \overline{CB}$, compute $\overline{CB}$.
3. If $\overline{AB} = 24$, ABC, and $\overline{AC} = 3 \cdot \overline{CB}$, compute $\overline{CB}$.

4. Line segments AB and CD are congruent. The length of AB as measured by EF is 12. The length of CD as measured by GH is 36. What is the length of GH as measured by EF?

5. The length of AB as measured by CD is $\frac{3}{4}$. What is the length of CD as measured by AB?

6. Let AB be a unit segment. Designate the midpoint of AB by A_1, the midpoint of AA_1 by A_2, the midpoint of AA_2 by A_3, etc. What is the length of A_5A_3?

ALGEBRA REVIEW

1. The perimeter of an isosceles triangle is 75 ft. One side is 15 ft longer than the second side. Prove that the shortest side of the triangle is either 15 ft long or 20 ft long.

2. A triangle has two equal angles. The length of the included side is 7 in. The perimeter of the triangle is 43 in. Determine the lengths of the other two sides of the triangle.

3. $\triangle ABC$ is equilateral. Points X, Y, and Z are the midpoints of sides AB, BC, and CA, respectively. The perimeter of $\triangle XYZ$ is 30 in. Determine the length of each side of $\triangle ABC$.

4. In Fig. 6–127, $\overline{AE} = \overline{AD}$, and $\overline{AB} = \overline{AC}$. $\overline{BE} = 18$ in., and $\overline{DO} = \frac{1}{2}\overline{OC}$. Determine the length of segment OC.

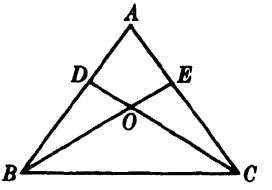


FIGURE 6–127

5. In $\triangle ABC$, the bisector of vertex angle A is perpendicular to the base BC at point D. $\overline{BD} = \overline{CD}$. $\overline{AB} - \overline{BD} = 8$ in., and the perimeter of $\triangle ABC$ is 40 in. Determine the length of each side of $\triangle ABC$.

6. Prove that if $(x + 2)(x - 3) = 0$, and x is positive, then $x = 3$.

7. Prove that if $(x + 2)(x - 3) = 0$, and x is negative, then $x = -2$.

8. A businessman bought several suits of clothes. He paid \$40 each for some of the suits and \$50 each for the rest. If he paid a total of \$1500 for these suits, prove that he could not possibly have bought 22 of the \$40 suits.

*9. Prove that the following problem has no solution: Find two numbers whose sum is 32, such that if the larger of the numbers is divided by 6, and the smaller is divided by 5, then the sum of the quotients is 6.

*10. Describe all possible pairs of numbers a and b such that $(a - 2)/(2b - 9) = a - 2$.

CHAPTER 7

USE OF THE CONGRUENCE THEOREMS

7–1 INTRODUCTION

In this chapter we do two things. In the first part we prove several inequalities involving sides and angles of a triangle, ending up with the *triangle inequality*. This enables us to prove that a line segment is the shortest path between two points. In the second part we prove that some of the constructions we made with ruler and compass in Chapter 1 are correct.

7–2 EXTERIOR ANGLES

In this section we prove a theorem about triangles which Euclid found to be very useful. We need a definition.

Definition 7–1. If an angle is adjacent and supplementary to an angle of a triangle, it is called an *exterior angle* of the triangle. The other two angles of the triangle are called the *opposite interior angles*. For example, $\angle DBC$ in Fig. 7–1 is an exterior angle and $\angle A$ and $\angle C$ are interior angles and opposite to $\angle DBC$.

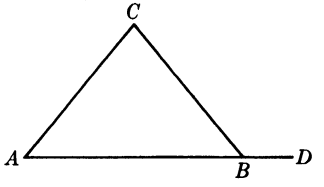


FIGURE 7–1

Exercise Group 7–1

1. In $\triangle ABC$, how many exterior angles are there having vertex B? How are these exterior angles related to $\angle B$?

2. How many exterior angles does a triangle have?

3. Do the words *opposite*, *interior*, and *exterior*, used in Definition 7–1, refer to properties or to relations?

Theorem 7–1. *An exterior angle of a triangle is greater than either opposite interior angle.*

We give two separate proofs of this theorem. The first proof may be found in Hilbert's work, the second in Euclid's *Elements.*

First Proof. We know that, for any two angles, the first angle is either $>$, or $<$, or $\cong$ the second. Therefore, we need only show that the exterior angle cannot be $\cong$ or $<$ an opposite interior angle, and it must follow that the exterior angle is $>$ an opposite interior angle. Suppose that the triangle is labeled as in Fig. 7–2.

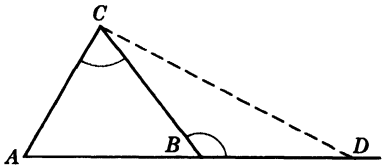


FIGURE 7–2

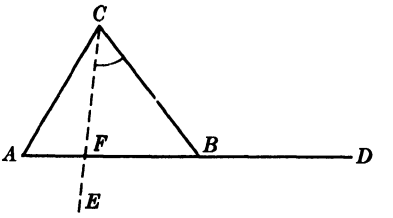


FIGURE 7–3

We first show that the exterior angle cannot be congruent to $\angle C$. Choose D, as in the figure, such that $BD \cong AC$, and suppose that $\angle DBC \cong \angle C$. Then $\triangle ACB \cong \triangle DBC$ because of the S.A.S. theorem. Hence $\angle CBA \cong \angle BCD$. Therefore, because of the addition of angles postulate, $\angle ACD \cong \angle ABD$. But $\angle ABD$ is a straight angle, and so $\angle ACD$ is a straight angle. This is impossible. (Why?) Hence $\angle DBC$ is *not* congruent to $\angle C$.

We now show that it is impossible that $\angle DBC < \angle ACB$. Assume that $\angle DBC < \angle ACB$. With C as a vertex, let $\angle ECB$ be the angle, as drawn in Fig. 7–3, that is congruent to $\angle DBC$. (Why is there such an angle?) Then the ray CE is in the interior of $\angle ACB$. (Why?) Hence the segment AB contains a point F in the ray CE. (This is obvious from the figure, and it may be proved with what we know about betweenness and interior of angles.) Now consider $\triangle FCB$. $\angle DBC$ is an exterior angle of this triangle which is congruent to the opposite interior angle FCB. This has already been shown to be impossible. Hence we cannot have $\angle DBC < \angle ACB$. The only possibility then is $\angle DBC > \angle ACB$. This completes the proof.

Second Proof. Using the same labeling for the triangle, let G be the point on BC, such that $BG \cong GC$. Let H be the point on the line AG that is on the other side of G from A, such that $AG \cong GH$. Then $\triangle GCA \cong \triangle GBH$. (Why?) Hence $\angle C \cong \angle GBH$. The ray BH is interior to $\angle DBC$. (This is obvious from Fig. 7–4, but it needs proof.

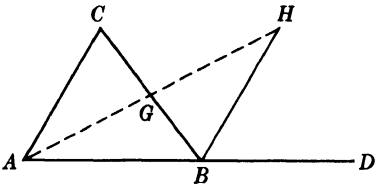


FIGURE 7–4

It is interesting to note that Euclid neglects to prove this. See Exercise 3 below.) Therefore, $\angle DBC > \angle HBC$, and so $\angle DBC > \angle C$. (Why?)

Remark. In Chapter 8, when we study parallel lines, we shall be able to prove that an exterior angle is congruent to the "sum" of the opposite interior angles, and from this fact, Theorem 7–1 follows easily. What is interesting is that Theorem 7–1 may be proved without any knowledge of properties of parallel lines.

Exercise Group 7–2

1. What kind of angles are the exterior angles of a right triangle?

2. Use Theorem 7–1 to prove that the angles opposite the legs of a right triangle are acute angles.

*3. Prove that the ray BH is interior to $\angle DBC$. [*Hint:* first show that H is on the same side of line BC as D. Since G is on the same side of BD as C, if H were on the opposite side of BD from C, then the segment GH would contain a point of the line BD. This is absurd.]

4. Prove the following corollary to Theorem 7–1. There is at most one line perpendicular to a given line l from a given point P. [*Hint:* Suppose that there were two perpendiculars, as in Fig. 7–5.]

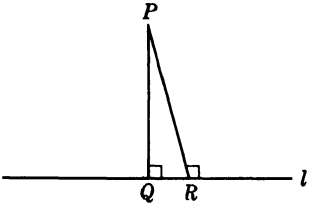


FIGURE 7–5

7–3 INEQUALITIES IN TRIANGLES

In this section we prove some theorems about the relative sizes of sides and angles in a triangle. We arrive at a proof of a theorem that, in effect, says that a line segment is the shortest path between two points.

Theorem 7–2. *If, in* $\triangle ABC$, $AB > AC$, *then* $\angle C > \angle B$, *and conversely, if* $\angle C > \angle B$, *then* $AB > AC$.

Remark. The meaning of this theorem can be stated briefly as "the greater side is opposite the greater angle."

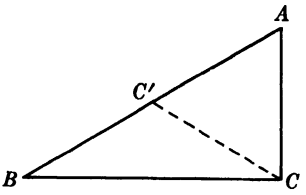


FIGURE 7–6

Proof. To prove the first part of the theorem, suppose that $AB > AC$, as shown in Fig. 7–6. Choose C' on AB, such that $AC' \cong AC$. Then by Theorem 7–1, $\angle AC'C > \angle B$. But $\angle AC'C < \angle C$. Hence $\angle C > \angle B$.

The converse follows readily from the fact that any two segments, are either congruent to each other or one is greater than the other. The argument is given below.

Given that $\angle C > \angle B$, we wish to show that $AB > AC$. Exactly one of the relations $AB < AC$, $AB \cong AC$, or $AB > AC$ is true. If $AB \cong AC$, then we would have $\angle C \cong \angle B$, which contradicts the given fact that $\angle C > \angle B$. If $AB < AC$, then by the first part of the theorem, we would have $\angle C < \angle B$, which also contradicts the given fact that $\angle C > \angle B$. Hence the only possibility is that $AB > AC$.

Exercise Group 7–3

1. Which angle in Fig. 7–7 is the smallest? Which is the largest? The numbers are lengths of the sides.

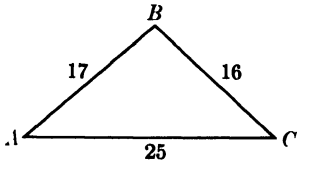


FIGURE 7–7

2. If in $\triangle EFG$, $\angle E$ has a measure of 61°, and $\angle F$ a measure of 54°, which side is the longer, FG or EG? Can you say anything yet about the length of EF?

3. In $\triangle PQR$, S is a point of PR, such that QS bisects $\angle PQR$ (see Fig. 7–8). Prove that $PQ > PS$. [*Hint:* Prove first that $\angle PSQ > \angle PQS$.]

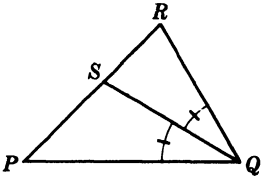


FIGURE 7–8

We now come to a very important and familiar inequality for triangles, called the *triangle inequality.*

Theorem 7–3. *In any* $\triangle ABC$, $\overline{AB} + \overline{BC} > \overline{AC}$.

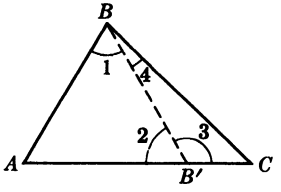


FIGURE 7–9

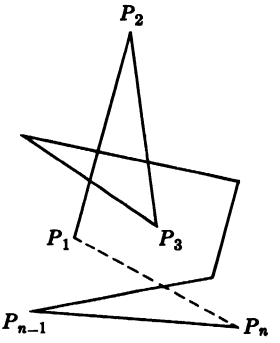


FIGURE 7–10

Proof. If AC is not the longest of the sides, the inequality is obvious. Therefore we may confine our attention to the case in which $\overline{AC} > \overline{AB}$ and $\overline{AC} > \overline{BC}$, as in Fig. 7–9.

STATEMENTS	REASONS
1. There is a point B' on AC, such that $AB \cong AB'$.	1. Why?
2. $\angle 1 \cong \angle 2$.	2. Why?
3. $\angle 3 > \angle 1$.	3. Why?
4. $\angle 2 > \angle 4$.	4. Why?
5. $\angle 3 > \angle 4$.	5. This follows from Statements 2, 3, and 4. Why?
6. $\overline{B'C} < \overline{BC}$.	6. Why?
7. $\overline{AB'} + \overline{B'C} < \overline{AB} + \overline{BC}$.	7. Statements 1 and 6 and properties of numbers.
8. $\overline{AC} < \overline{AB} + \overline{BC}$.	8. $\overline{AC} = \overline{AB'} + \overline{B'C}$.

Remark. This theorem is sometimes loosely stated as "a straight line is the shortest distance between two points." In fact, this last statement is sometimes used as a postulate, although incorrectly, since one must first have a notion of distance or of length of a segment.

Having proved the triangle inequality, we now see that the lines we have been talking about all along are, after all, "straight" in the precise sense of the triangle inequality. Perhaps this can be made clearer by defining *polygonal path* and *length of a polygonal path.*

Definition 7–2. Let $P_1, P_2, \ldots, P_n$ be any set of n points (Fig. 7-10). The set of points on all the segments P_1P_2, $P_2P_3, \ldots, P_{n-1}P_n$ is called a *polygonal path.* The path is said to join, or connect, the *endpoints* P_1 and P_n. The points $P_1, \ldots, P_n$ are called the *vertices.* In particular, if $P_1 = P_n$, the polygonal path is called a *polygon* and designated as *polygon* $P_1P_2 \ldots P_{n-1}$.

Definition 7–3. The *length of the polygonal path* $P_1 \ldots P_n$ is defined to be $\overline{P_1P_2} + \overline{P_2P_3} + \cdots + \overline{P_{n-1}P_n}$.

Corollary 7–3–1. The length of a polygonal path is greater than, or equal to, the length of the segment joining its ends.

Exercise Group 7–4

1. Two sides of a triangle have lengths of 5 and 7 in. What can you say about the third side?

2. Can the numbers 6, 8, and 15 be the lengths of the three sides of a triangle?

3. Prove that the difference between the lengths of two sides of a triangle is less than the length of the third side.

*4. Use the triangle inequality and Fig. 7–11 to show that if $\angle C > \angle B$, then $AB > AC$.

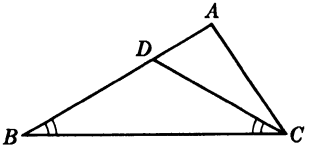


FIGURE 7–11

*5. In Fig. 7–12, $AC \cong A'C'$, $BC \cong B'C'$, and $\angle ACB > \angle A'C'B'$. Prove that $\overline{AB} > \overline{A'B'}$. [*Hint:* Show how to find a point Q on AB, such that $QB \cong QB'$, and use $\overline{AB} = \overline{AQ} + \overline{QB} = \overline{AQ} + \overline{QB'}$.]

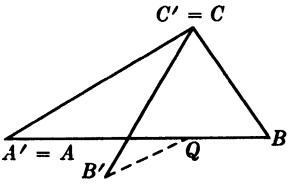


FIGURE 7–12

*6. Use the result of Exercise 5 to prove that if, in two triangles, $\triangle ABC$ and $\triangle A'B'C'$, $AC \cong A'C'$, $BC \cong B'C'$, and $AB > A'B'$, then $\angle C > \angle C'$. Show that the statements $\angle C \cong \angle C'$ and $\angle C < \angle C'$ are false.

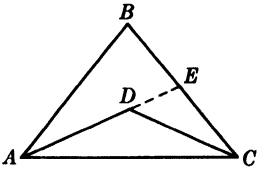


FIGURE 7–13

*7. If D in Fig. 7–13 is a point inside $\triangle ABC$, show that $\overline{AD} + \overline{DC} < \overline{AB} + \overline{BC}$. [*Hint:* Consider first $\triangle ABE$ and $\triangle DEC$.] Show also that $\angle ADC > \angle ABC$.

8. By using Theorem 7–2, prove that an equiangular triangle is equilateral.

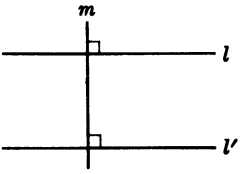


FIGURE 7–14

9. If, in Fig. 7–14, $l \perp m$, and $l' \perp m$, prove that l and l' do not intersect. [*Hint:* Suppose they do.]

10. Show that a polygonal path connecting points A and B could have four vertices and still have the same length as AB.

7–4 CIRCLES

There are two ways that we could define a circle. One is to use the idea of congruent segments, and the other is to use the idea of distance. Because two segments are congruent if and only if they have the same length, these two ways are equivalent.

Definition 7–4. A circle is determined by a point O, called the *center*, and a positive number r. The *circle* is the set of all points A of the plane, such that $\overline{OA} = r$. If A is a point of the circle, then the segment OA is called a *radius* (plural, *radii*). If A and B are on the circle, and the segment AB contains O, then AB is called a *diameter*.

Remark 1. You will note that we have not said that a circle is a "curve," because nowhere in our geometry have we defined what "curve" might mean. The circle is a set of points, and in the definition, we have told only how to decide whether any point A is, or is not, a point of the circle.

Remark 2. According to the definition, a radius is a segment, that is, a set of points. It is common usage to also call the real number r the radius. We will also use this language, so that the circle can be described as the "circle with center O and radius r." Similarly, if AB is a diameter, the length $\overline{AB} = 2r$ will be called the diameter. No confusion will arise, because it will always be clear whether we are talking about a set of points or a number.

Remark 3. We have not mentioned the inside or outside of the circle as yet, because these sets have not yet been defined.

Definition 7–5. If a circle (Fig. 7–15) has center O and radius r, the *interior* of the circle is the set of all points B, such that $\overline{OB} < r$. The *exterior* of the circle is the set of all points C, such that $\overline{OC} > r$.

A *chord* is a segment determined by two points E and F on the circle; the line through E and F is a *secant*.

A *central angle* of a circle is an angle whose vertex is at the center of the circle. An *inscribed angle* (such as $\angle EFG$) is an angle whose vertex is on the circle, and having points of the circle on each of the two sides of the angle.

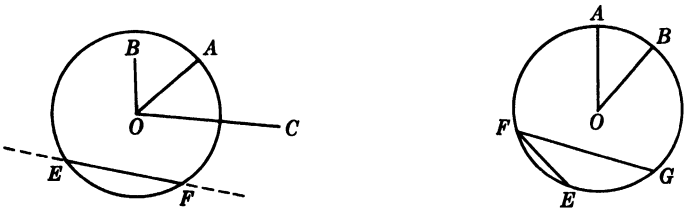


FIGURE 7–15

Exercise Group 7–5

1. A circle has center A and radius 5. The points B, C, D, E, are such that $\overline{AB} = 4$, $\overline{AC} = 5$, $\overline{AD} = 17$, and $\overline{AE} = 5$. Where are these points? That is, in what sets are they?

2. If AB is a diameter of a circle with center C, what is $\angle ACB$?

3. In Fig. 7–16, the circle has center O. What kind of angles are $\angle A$, $\angle B$, $\angle C$, $\angle D$, and $\angle E$?

4. Give a definition of circle in terms of congruence instead of distance.

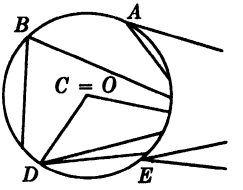


FIGURE 7–16

5. If A is a point of a circle and O is the center, what can be said about a point B, such that $\overline{OB} \leqq \overline{OA}$?

7–5 RULER AND COMPASS

In the preceding section we defined a circle in terms of distance. When drawing a circle on a piece of paper, we use the fact, familiar from experience, that the distance between the two points of the compass stays fixed. In this section we define those drawings that are called "ruler and compass constructions." Do not think that these are the only constructions possible. They are only the simplest kind. The ruler used in these constructions is just an unmarked straightedge.

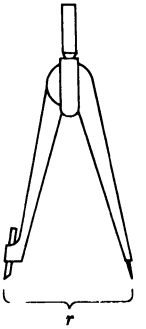


FIGURE 7–17

Definition 7–6. To make a *ruler and compass* (RC) *construction* is to draw a picture of a set of points, using only a ruler and compass. The ruler can only be used to draw a line through two points. The compass can only be used to draw a circle with a given center and radius.

Remark 1. RC constructions are sets that can be illustrated with certain mechanical devices. Circles and lines exist in our geometry independently of these mechanical devices. Interest in RC constructions originates in the simplicity of the tools needed to make them.

Remark 2. To draw a line or circle in an RC construction, two points are needed. We may not slide the ruler, mark it, or guess the opening of the compass. We must have two points already constructed in order to draw a line or circle.

Remark 3. The study of RC constructions at this point is not really necessary for the logical development of our geometry, but it does help us to visualize geometric figures, and it demonstrates the usefulness of what has already been studied.

Remark 4. It will be clear that the knowledge of RC constructions is quite useful to a draftsman.

7–6 HISTORICAL NOTE

The ancient Greeks were much attracted by RC constructions and attempted to solve all sorts of problems with these two tools. However, they encountered several problems that they could not handle using only ruler and compass. This does not mean that these problems are not solvable—only that a ruler and compass are not enough. Three of the problems that puzzled the Greeks have had a long and interesting history, which we cannot pursue here. They are:

(a) *The trisection of an angle.* Given any $\angle A$, the problem is to construct angles B, C, and D, as in Fig. 7–18, such that $\angle B \cong \angle C \cong \angle D$.

(b) *The duplication of a cube.* Suppose that a cube has side of length x. The problem is to construct the side of a cube having twice the volume of the given cube.

(c) *Squaring the circle.* Given a circle, the problem is to construct a square with the same area as the circle (Fig. 7–19).

All these problems have been completely resolved in modern times, and *all are impossible of solution with ruler and compass alone.* This is a fact

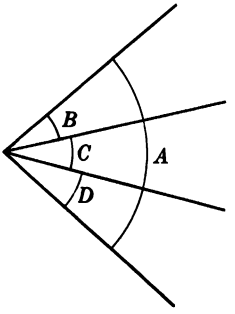


FIGURE 7–18

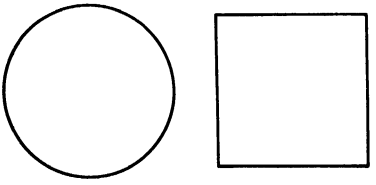


FIGURE 7–19

beyond dispute. It does not mean that in the trisection problem, for example, angles B, C, and D do not exist. It means that one cannot draw them using ruler and compass alone. When you think about this a moment, it may not be so surprising. No one is surprised that it is impossible to build a large building with only a tack hammer. A ruler and compass alone are like a tack hammer in so far as these problems are concerned.

There are, however, many constructions possible with ruler and compass. We will study some of them.

7–7 COPYING ANGLES AND TRIANGLES

$\angle A$ and a ray from point B are given, as in Fig. 7–20. We wish to construct an angle at B congruent to $\angle A$. Of course, from the existence postulate for angles, there are two such angles, as shown by the dotted lines in the figure. When we draw the angle at B congruent to $\angle A$, we say that we have "copied" $\angle A$.

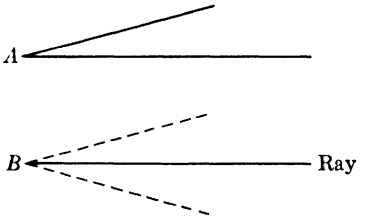


FIGURE 7–20

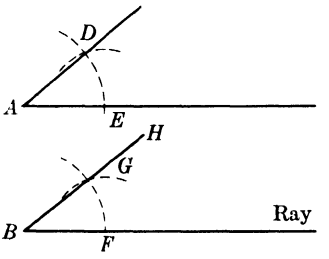


FIGURE 7–21

Construction 7–1. Draw a circle of radius r, with center A. This circle will intersect the sides of $\angle A$ in two points D and E (Fig. 7–21). (Why?) Draw a circle of the same radius r with center B. This circle will intersect the ray from B in a point F. Draw a circle, with center F and radius $\overline{DE}$. On a given side of the line BF, the two circles will intersect in a point G. $\angle GBF$ is congruent to $\angle A$.

You should recognize that there are two things to prove. Because the entire construction is centered around finding a point in which our circles intersect, we must first prove that such a point exists in our geometry. Of course, when we draw the picture, it is clear that there should be such a point. The real question, however, is whether we can take the assumptions we have made and prove that there is such a point. After we establish the existence of this point, we must prove further that the angles are congruent to each other. These two proofs should be considered separately.

Proof. (a) (The existence of a point of intersection of the circles.) By the existence postulate for angles, there is exactly one ray BH, on a given side of BF, such that $\angle FBH \cong \angle A$. The existence of the *ray* is thus assured. The existence of the points D, E, and F follows from theorems of measurement and congruence. There is a point G on the ray BH, such that $BG \cong BF \cong AE \cong AD$. Since $\angle A \cong \angle B$, $\triangle ADE \cong \triangle BGF$ by the S.A.S. theorem. Therefore, $GF \cong DE$. Hence, the point G is on the circle with radius $\overline{DE}$ and center F. Since G is also on the circle with center B and radius $\overline{BF}$, it is therefore on both circles and is a point of intersection of the two circles.

Proof. (b) (The congruence of the angles.) We now consider any point G on both circles. $AD \cong AE \cong BG \cong BF$. Since both circles have the same radius, $DE \cong GF$, and $\triangle AED \cong \triangle BFG$ by the S.S.S. theorem. Hence $\angle A \cong \angle B$.

In a similar manner we can copy a triangle.

Construction 7–2. Given $\triangle ABC$ (Fig. 7–22) and a ray from a point A', use the previous construction to copy $\angle A$. Then draw a circle, with center A' and radius $\overline{AB}$, which intersects the other side of $\angle A'$ at B'. Then draw a circle, with center A' and radius $\overline{AC}$, which intersects the given ray in C'. Then $\triangle ABC \cong \triangle A'B'C'$.

Proof. The proof is left to the student.

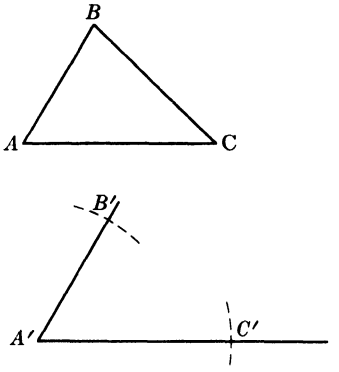


FIGURE 7–22

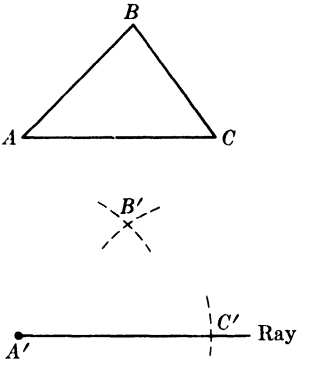


FIGURE 7–23

Construction 7–3. Given $\triangle ABC$ (Fig. 7–23) and a ray from a point A', draw a circle, with center A' and radius $\overline{AC}$, which intersects the ray in C'. Then draw circles, with centers at A' and C' and radii $\overline{AB}$ and $\overline{BC}$, respectively. These two circles will intersect, on a given side of the ray, in exactly one point B'. We will then have $\triangle ABC \cong A'B'C'$.

Proof. The proof is left to the student. Remember, you must first prove that B' exists and then prove $\triangle ABC \cong \triangle A'B'C'$.

7–8 OTHER CONSTRUCTIONS

It should now be clear that proving the validity of certain constructions involves first the proof of the existence of certain points and lines, and secondly the proof that the construction produces the desired result. In the remainder of our work on constructions, we will concentrate our efforts on the second part. The proof of existence of certain points can actually be quite hard at times. For example, when we try to prove that the construction for the perpendicular bisector of a segment is valid, we need a theorem, Theorem 7–4, that is very difficult to prove at this time.

Theorem 7–4. *If AB is a segment, and circles are drawn, with centers at A and B and radius $r > \frac{1}{2}\overline{AB}$, then these circles intersect in exactly two points P and Q, on opposite sides of the line through A and B* (Fig. 7–24).

Remark. From what we know about measurement of length, we know that every line segment AB has exactly one midpoint C, that is, a point C such that $\overline{AC} = \overline{BC} = \frac{1}{2}\overline{AB}$. Likewise, it is easy to show that every angle has a bisector, that is, from the vertex of the angle, there is a ray that forms, with the sides of the given angle, two congruent angles.

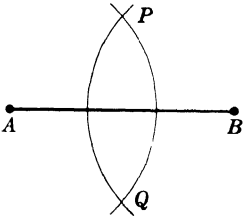


FIGURE 7–24

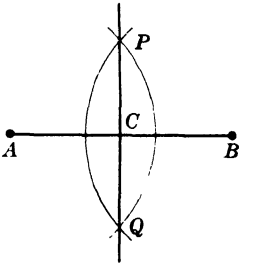


FIGURE 7–25

The constructions of this section show how to make RC constructions for midpoints of segments and bisectors of angles; but the *existence* of these midpoints and bisectors has been obtained independently of these constructions.

Construction 7–4. Draw circles, with centers A and B and equal radii greater than $\frac{1}{2}\overline{AB}$. These circles intersect at points P and Q. The line PQ is the desired perpendicular bisector (Fig. 7–25).

Proof.

STATEMENTS	REASONS
1. $AP \cong PB \cong AQ \cong QB$.	1. Construction.
2. $\triangle APQ \cong \triangle BPQ$.	2. S.S.S. theorem.

3. PQ intersects line AB in C.	3. P and Q are on opposite sides of AB.
4. $\triangle APC \cong \triangle BPC$.	4. S.A.S. theorem.
5. $AC \cong CB$, and $\angle ACP \cong \angle BCP$.	5. Statement 4.
6. PQ is the $\perp$ bis AB.	6. Statement 5.

Exercise Group 7–6

1. Prove that each segment has exactly one perpendicular bisector.
2. Prove that every point on the perpendicular bisector of AB is equidistant from A and B.
3. Prove that if a point R is equidistant from A and B, then it lies on the perpendicular bisector of AB.
4. Construct the perpendicular bisector of a segment by constructing two points on the same side of the line and equidistant from the ends of the segment.

Construction 7–5. To construct the perpendicular to a line at a point P on the line (Fig. 7–26), draw a circle, with center P and arbitrary radius, intersecting the line in points A and B. Then construct the perpendicular bisector of AB.

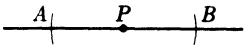


FIGURE 7–26

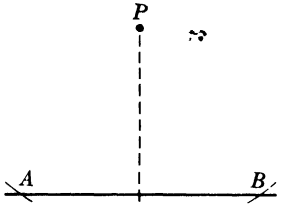


FIGURE 7–27

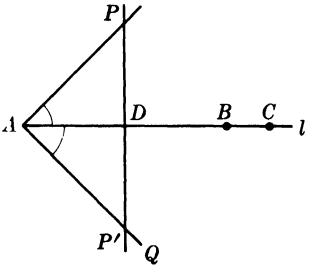


FIGURE 7–28

Construction 7–6. To construct a perpendicular to a line from a point not on the line, choose a point A on the line, and draw the circle with center P and radius AP (Fig. 7–27). The circle will intersect the line in a second point B (if A is properly chosen). Then construct the perpendicular bisector of AB, which passes through P.

Proof. We first show that there actually is a line through P perpendicular to the line. Then we show that the RC construction produces this line.

First choose any point A on l and any other point C on l (Fig. 7–28). Let AQ be the ray, on the opposite side of l from P, such that $\angle PAC \cong \angle QAC$. (There is only one such ray, by the existence postulate for angles.) Choose P' on AQ, such that $AP' \cong AP$. Then PP' intersects l in a point D. (Why?) If $A \neq D$, then triangles are formed, and by S.A.S.,

$\triangle APD \cong \triangle AP'D$. Therefore, $\angle ADP$ and $\angle ADP'$ are right angles, and $PP' \perp l$. If $A = D$, there are no triangles, but then, because of the way AQ was selected, $PP' \perp l$. (Why?)

Having established that a perpendicular from P exists, we now show that the RC construction gives us this perpendicular. If $D \neq A$, there is a point B on the opposite side of D from A, so that $AD \cong BD$. It follows that the circle with center P and radius AP does intersect the line in a second point B, and this was the main part to be proved in the construction. Now the perpendicular bisector of AB can be drawn, and P lies on it. (Why?)

Exercise Group 7–7

1. Construct a right triangle, having been given the lengths of the two legs.

2. Construct points D, E, and F on the segment AB, such that $\overline{AD} = \overline{DE} = \overline{EF} = \overline{FB} = \frac{1}{4}\overline{AB}$.

*3. Suppose that P is not on a line l, and that a given circle with center at P intersects l in exactly one point Q. Prove that PQ is perpendicular to l. [*Hint:* If PQ were not perpendicular, then the construction in the proof of Construction 7–6 would lead to a contradiction.]

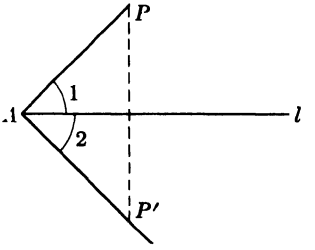


FIGURE 7–29

4. Construct the perpendicular from P by choosing A, as in Fig. 7–29, constructing $\angle 2 \cong \angle 1$, and point P', such that $AP \cong AP'$.

Construction 7–7. To bisect $\angle A$, draw a circle, with center A, intersecting the sides of $\angle A$ in points B and C (Fig. 7–30). Then construct a point Q on the perpendicular bisector of BC and in the interior of $\angle A$. The ray AQ bisects $\angle A$.

Proof. The point Q can be found by a previous construction. $\triangle ABQ \cong \triangle ACQ$, by S.S.S. Hence $\angle BAQ \cong \angle CAQ$.

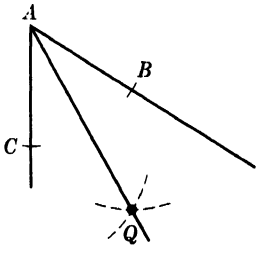


FIGURE 7–30

Exercise Group 7–8

1. Draw an acute angle, and construct its bisector.
2. Construct and bisect a right angle.
3. Bisect an obtuse angle.
4. Draw a triangle, and construct the bisectors of its angles. (They should meet in one point.)
5. If P is on the bisector of $\angle A$, and B and C are points on the sides of $\angle A$, such that $AB = AC$, prove that $PB \cong PC$.
6. Prove that an angle has only one bisector.
7. Suppose that B and C are points on the two sides of $\angle A$, such that $AB \cong AC$. Show that if M is the midpoint of BC, then AM bisects $\angle A$.

*8. Figure 7–31 shows another construction for the angle bisector. Consider two circles, with center A and radii r and r', such that $r' > r$. These circles intersect the sides of A in four points C, C', D, and D'. The lines through these points intersect in a point P on the bisector. (Why?)

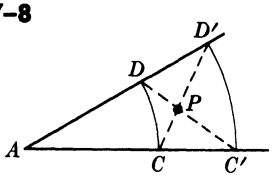


FIGURE 7–31

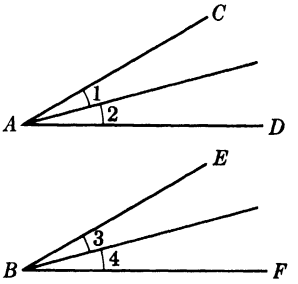


FIGURE 7–32

*9. If, in Fig. 7–32, $\angle CAD \cong \angle EBF$, $\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$, then $\angle 2 \cong \angle 4$. *Note:* This theorem is often stated as "halves of congruent angles are congruent to each other." Prove this theorem.

REVIEW OF CHAPTER 7

In this chapter we have used the congruence theorems to study inequalities in triangles and verify constructions.

(a) Make a list of all the theorems on inequalities in triangles, then study their proofs.
(b) Make a list of the different RC constructions. Prove them correct without looking at the book.

Review the following definitions, which you met for the first time in this chapter: exterior angle, circle, interior of circle, exterior of circle, radius, diameter, chord, secant, central angle, inscribed angle, RC construction.

Review Exercises

1. Construct an angle of 45°.
2. Construct an angle of 60°, of 30°.
3. Construct an angle of 15°, of 75°.
4. Construct an equilateral triangle.

5. Construct a right triangle having acute angles of 60° and 30°.

6. Draw an obtuse triangle, and construct the bisectors of its angles.

7. Draw an angle. Copy it, using ruler and compass.

8. Draw three separate segments. Construct a triangle having sides congruent to the given segments.

9. Draw a line, and mark a point P not on the line. Construct the perpendicular to the line, passing through P. How do you know that there is only this one perpendicular?

10. Draw a triangle. Construct the altitudes of the triangle.

11. Draw a triangle. Construct the perpendicular bisectors of its sides.

12. Construct an isosceles triangle if the vertex angle and the congruent sides are given.

13. Draw a segment. Construct a circle having this segment as a diameter.

14. In a right triangle, which side is longest? Why?

15. Prove that the sum of the lengths of three sides of a quadrilateral is greater than the length of the fourth side.

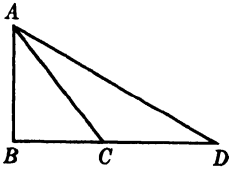


FIGURE 7–33

16. $\angle ABC$ is a right angle, and C is between B and D. Prove that $AD > AC$. (Fig. 7–33)

17. Prove that $\overline{AD} + \overline{CB} > \overline{AB} + \overline{CD}$. (Fig. 7–34)

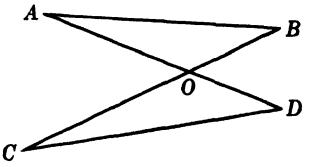


FIGURE 7–34

18. AE bisects $\angle DAF$, and AC bisects $\angle DAB$. (Fig. 7–35) Prove that $\angle CAE$ is right. [*Hint:* Extend AE.]

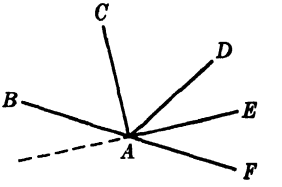


FIGURE 7–35

19. AC bisects $\angle DAB$, and $\angle CAE$ is right. Prove that AE bisects $\angle DAF$. (Fig. 7–36)

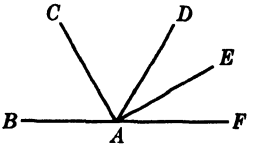


FIGURE 7–36

ALGEBRA REVIEW

1. In $\triangle ABC$, $AB < BC < CA$. The greatest angle in this triangle has a measure of 32° more than the measure of the smallest angle. The sum of the measures of these two angles is 120°. Determine the number of degrees in the smallest angle in this triangle.

2. An exterior angle at A of $\triangle ABC$ has a measure of 65°. The sum of the measures of angles A and B is 145°, and the sum of the measures of angles A and C is 150°. Determine the number of degrees in each angle of the triangle.

3. Points O, A, and B are collinear. $\overline{OA}$ is 7 in. more than $\overline{BA}$. A is not between O and B. Determine the length of OB.

4. $\overline{AB} = 17$, and $\overline{AC} = 12$. The number $\overline{BC}$ is a root of the equation $(x - 6)(x - 4) = 0$. Determine $\overline{BC}$.

5. If B is between A and C, $\overline{AC}/\overline{BC} = \frac{5}{3}$, and $\overline{AB} = 10$ in., determine the lengths of AC and BC.

6. B is between A and C, and $\overline{AB}/\overline{AC} = \frac{2}{5}$. The midpoint of AC is M, and $\overline{MB} = 3$. Determine $\overline{AC}$.

*7. A, B, C and D are points in this order on a line. The length of AC is twice the length of BD. The length of AB is four times the length of CD. The length of BC is 4. Determine the lengths of AB and AD.

*8. The difference of the squares of two numbers is 33. The sum of these numbers is 11. Determine the numbers.

*9. Analyze the following mathematical trick. Choose any number; multiply this number by 2; add 4 to this product; multiply by 5; divide by 10; subtract from this result the original number. The result will be 2, no matter what number was originally chosen.

CHAPTER 8

PARALLEL LINES

8–1 DEFINITION OF PARALLEL LINES

In earlier mathematics courses, you learned some facts about parallel lines. Before reading this chapter, try to recall some properties of parallel lines that you have accepted as true. The following exercises will help your memory and prepare you for proofs.

Exercise Group 8–1

1. What do you mean when you say, "Line l is *parallel* to line m"?
2. Name some geometric figures that have pairs of *parallel* sides.
3. Explain why two sides of a triangle are not *parallel*.
4. Two rocks are dropped, one from the Brooklyn Bridge, and the other from the Golden Gate Bridge. Would you say that these rocks fall along *parallel* paths?
5. If you drop two rocks, one from each hand, do they traverse *parallel* paths?
6. Roads r and s run "due north and south." Are they *parallel*?
7. Roads t and u run "due east and west." Are they *parallel*?
8. How many different pairs of *parallel* edges does a cube have?
9. Both lines l and $m \perp$ line n. Does this suggest a theorem about *parallels*?
10. Each of the lines l and m is *parallel* to line n. What theorem does this suggest?
11. Lines p and q are *parallel*. Line $r \perp p$. What theorem does this suggest?
12. Draw two angles, $\angle AOB$ and $\angle A'O'B'$, such that the lines AO and $A'O'$ are *parallel*, and also lines BO and $B'O'$ are *parallel*. Does this figure suggest a theorem? Be careful! Draw more than one figure.
13. Draw a line l, and mark a point A not on l. What might it mean to say that a ray from A is *parallel* to line l?
14. Which of the above exercises do not strictly belong to plane geometry?

If you discussed Exercises 6 and 7, someone probably pointed out that roads on the surface of the earth are not quite the same as lines. Also, in connection with Exercises 4 and 5, the paths along which objects fall

are hardly the imaginary lines of our geometry. Since the earth rotates about its axis and at the same time revolves around the sun, an object falling toward the center of the earth follows a path extremely hard to visualize. Our task is to choose a definition of parallel lines, such that we can use the definition in our plane geometry along with our postulates to prove the truth of statements that concern parallel lines and that we feel ought to be true. The definition below was perhaps suggested by someone in the class during the discussion of Exercise Group 8–1.

Definition 8–1. Two lines l and m are parallel if and only if they do not intersect. We indicate that two lines are parallel by writing "$l \parallel m$."

Exercise Group 8–2

1. Give definitions of the following. In later sections, it will be convenient to have these definitions. Good definitions are not hard to make, and so are given as exercises here.
   (a) A ray parallel to a line.
   (b) Parallel rays.
   (c) A line segment parallel to a line.
   (d) Parallel line segments.
2. Draw a figure of a line not intersecting a segment and not parallel to the segment.
3. Can a ray and a line not intersect and yet not be parallel?

Now that we have parallel lines, it is possible for us to talk about rays that have the same direction. This requires another definition.

Definition 8–2. Two rays have the same *direction* if and only if they are parallel and both rays are on the same side of the line through their vertices (Fig. 8–1).

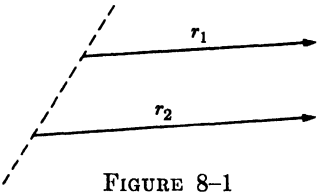


FIGURE 8–1

Exercise 8–3

What would have been wrong had we defined lines to be parallel if they had the same direction?

8–2 EXISTENCE OF PARALLELS

Perhaps it will seem strange to you that we are going to prove that parallel lines actually exist. We could add an extra postulate and assume that there are parallels, but the proof is not difficult. We give three different proofs of the following theorem.

Theorem 8–1. *If l is any line, and A any point not on l, then there exists at least one line through A parallel to l.*

First Proof. Referring to Fig. 8–2, let B be any point on l, and let n be the line through A and B. Let E be another point on l, and let D be a point on the same side of n as E, such that $\angle DAC \cong \angle EBA$. Lines l and m cannot intersect, for if they did, say at P, we should have $\triangle PBA$ with exterior $\angle PAC \cong \angle PBA$. But we know this is impossible, for an exterior angle of a triangle is always greater than either opposite interior angle.

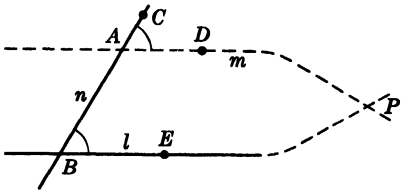


FIGURE 8–2

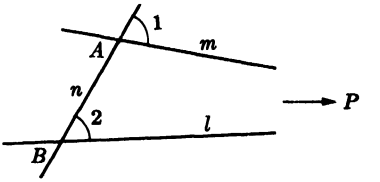


FIGURE 8–3

**Second Proof. Lemma.* Let line l, point A, and line n be as in Theorem 8–1. Now, referring to Fig. 8–2, if m is not parallel to l, then $\angle 1$ is not congruent to $\angle 2$. (For "not parallel to," we use the symbol $\nparallel$.) *Proof.* If m is not parallel to l, these lines intersect, a triangle is formed, and $\angle 1 \ncong \angle 2$. Now Theorem 8–1 follows quickly from the lemma. The lemma states that $l \nparallel m \rightarrow \angle 1 \ncong \angle 2$. The contrapositive of the lemma is $\angle 1 \cong \angle 2 \rightarrow l \parallel m$, and the theorem obviously follows.

**Third Proof.* Referring to Fig. 8–4, let l, m, and n refer to the lines of the preceding proofs with n cutting m at A, and l at B, and the indicated angles at A and B congruent to each other. If l and m intersected at P, we could choose Q on m, as above, so that $AQ \cong BP$. It follows at once, by the S.A.S. theorem, that $\triangle BAQ \cong \triangle ABP$. Hence, $\angle ABQ \cong \angle BAP$.

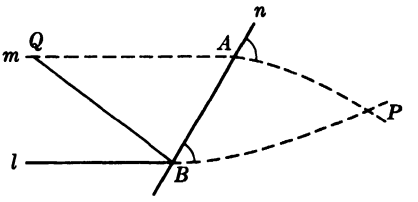


FIGURE 8–4

* If the teacher wishes, the second and third proofs may be omitted. Note that the second proof employs a lemma, or helping theorem.

Since also, $\angle ABP \cong \angle BAQ$, we have $\angle QBP \cong \angle QAP$. $\angle QAP$ is a straight angle, so $\angle QBP$ is a straight angle. It follows that Q lies on line l, and so if lines l and m intersect in one point, they have a second point in common. This is impossible, so lines l and m do not intersect.

Exercise Group 8–4

1. Draw a line, and mark a point not on it. Use the first proof of Theorem 8–1 to draw a line, parallel to the first line, through your point.
2. Prove that if $l \perp m$, and $n \perp m$, then, if $l \neq n$, $l \parallel n$.
3. Is a line parallel to itself?
4. In the first proof of Theorem 8–1, several details were omitted. First a point B was chosen and then a point E. How was C selected? How would we have argued if l and m were to intersect on the other side of AB?
5. The second proof of Theorem 8–1 is really just a reworded version of the first proof. Explain why this is so.
6. Without pointing at Fig. 8–4, describe how the point Q is chosen in the third proof. What postulate guarantees the existence of Q?
7. Prove that if $l_1 \parallel l_2$, then all the points of l_2 are on the same side of l_1.
8. Use the result of Exercise 7 to give a definition of "the set of points between l_1 and l_2, where $l_1 \parallel l_2$."

8–3 THE PARALLEL POSTULATE

We proved in Section 8–2 that through a point A not on line l, there is *at least one line parallel to* l. Now we raise the question, *is there more than one such line*? Our geometric intuition tells us that there is only the one, and it is natural to attempt to prove this from our other postulates. For 2000 years, many fine mathematicians tried and failed to find such a proof. At last, in the 19th century, it was shown that *it is impossible to prove from the postulates we have already listed that there is no more than one line through a given point parallel to a given line.* Euclid himself assumed that there was no more than one such parallel, and we shall make the same assumption.

Postulate VI–1. The parallel postulate. Through a point not on a line l, there is no more than one line parallel to l.

Historical note. It may seem strange that it is possible to prove that the parallel postulate is not a consequence of the other postulates. The understanding of the fact that the parallel postulate is independent of the others came about in the following way.

After vainly trying to prove the truth of the parallel postulate by using the other postulates of Euclid, a young Hungarian mathematician, Johann Bolyai (Yo′-hahn Bull′-yī), decided to assume that *a second parallel could be drawn through* A. Using this postulate, he proved many strange and beautiful theorems. Later it was shown that Bolyai's geometry was just as acceptable as ordinary Euclidean geometry. We call this geometry

KARL FRIEDRICH GAUSS
(*1777–1855*)
Probably the first mathematician to recognize the existence of non-Euclidean geometries.
Courtesy of *Scripta Mathematica.*

NICOLAI IVANOVICH LOBACHEVSKY
(*1793–1856*)
Co-inventor, with Johann Bolyai, of
non-Euclidean geometry.
Courtesy of *Scripta Mathematica*.

of Bolyai's a *non-Euclidean* geometry to contrast it with our Euclidean geometry which employs the parallel postulate of Euclid. Whenever you encounter the term non-Euclidean geometry, you can be sure that some postulate different from the parallel postulate is used. All the theorems we have proved up to now are true in the non-Euclidean geometry of Bolyai.

When these things are first explained, many students ask which postulate is right. Can only one parallel be drawn, or is it possible to draw several? While these questions are quite natural ones to ask, they do not have the simple answers that one might expect. Our postulates are statements that we invent in order to prove interesting and useful theorems. The theorems of Euclidean geometry are useful in carpentry, drafting, surveying, navigation, astronomy, and in many other fields. The theorems of some non-Euclidean geometries have been found to be extremely useful in astronomy and physics. Today mathematicians study many different geometries. Each of the geometries has a postulate system different from that of any other geometry. All are interesting. Many have proved to be of practical use. It does not make sense to ask which geometry is true. Mathematicians design geometries as architects design houses. You might as well ask which house is true. It *is* appropriate to ask which house is better adapted to one's living requirements. And so it is also appropriate to ask which geometry is most convenient to describe the universe.*

8–4 CONSEQUENCES OF THE PARALLEL POSTULATE

We can now prove several interesting theorems quite easily.

Theorem 8–2. *If l, m, and n are distinct lines, with $l \parallel m$ and $m \parallel n$, then $l \parallel n$* (Fig. 8–5).

Proof. If l were not parallel to n, then l and n would intersect, say at P in Fig. 8–5. Two lines, namely l and n, would then pass through P, parallel to m, thus contradicting the parallel postulate.

Theorem 8–3. *If $l \parallel m$, and line $n \neq l$ intersects l at P, then n intersects m* (Fig. 8–6).

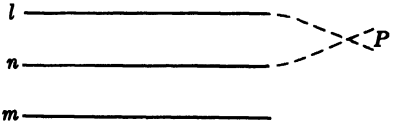


FIGURE 8–5

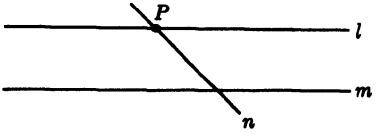


FIGURE 8–6

* See, for example, the discussion of Gauss' experiment in *The World of Mathematics*, by J. R. Newman, Vol. 3, p. 1644.

Proof.

STATEMENTS	REASONS
1. $l \parallel m$.	1. Given.
2. P is on n and l, and $n \neq l$.	2. Given.
3. l is the only line on P parallel to m.	3. Statements 1, 2, and the parallel postulate.
4. n intersects m.	4. Statements 2 and 3.

Exercise Group 8–5

1. Prove that the construction of a parallel to l through a given point A is an RC construction.

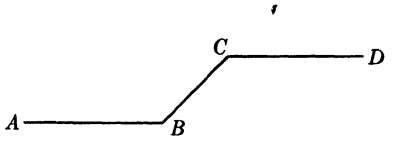


FIGURE 8–7

5. If $l \parallel m$, $k \parallel m$, and l and k intersect at P, what can you say?

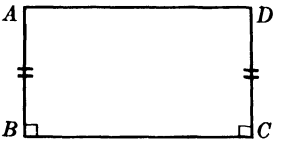


FIGURE 8–8

2. In Fig. 8–7, $\angle ABC \cong \angle BCD$. Prove that $AB \parallel CD$.

3. Prove that if $l \perp n$, $n \perp m$, and $l \neq m$, then $l \parallel m$. Have you used the parallel postulate?

4. If $l \parallel m$, and $l \perp n$, prove that $n \perp m$. Did you use the parallel postulate?

*6. In Fig. 8–8, $\overline{AB} = \overline{CD}$, and $\angle ABC$ and $\angle BCD$ are right. Prove that $AD \parallel BC$. [*Hint:* Prove that $\angle A \cong \angle D$.]

*7. If a rectangle is defined as a quadrilateral with four right angles, prove that rectangles actually exist.

8–5 CORRESPONDING AND ALTERNATE ANGLES

When a line intersects two other lines, we often wish to refer to certain pairs of the angles formed. In order to be precise, we need a definition.

Definition 8–3. A line l (Fig. 8–9) that intersects lines l_1 and l_2 in distinct points is called a *transversal* of l_1 and l_2. We also say that l_1 and l_2 are *cut* by the transversal l. Angles 3, 4, 5, and 6 are called *interior angles*. Angles 1, 2, 7, and 8 are *exterior angles*. These eight angles are grouped into four pairs referred to as pairs of *corresponding angles*. Angles 1 and 5 form such a pair of corresponding angles, as do also 2 and 6, 3 and 7, and 4 and 8. Angles such as 3 and 5 are called *alternate angles*. Other pairs of alternate angles are 2 and 8, 1 and 7, and 4 and 6. We speak of the pair 3 and 5 as *alternate interior angles*. We call the pair 2 and 8, *alternate exterior angles*. We observe that *alternate* angles have the points

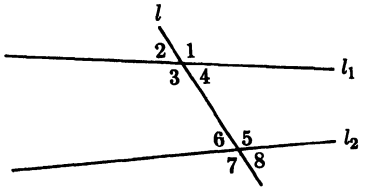


FIGURE 8–9

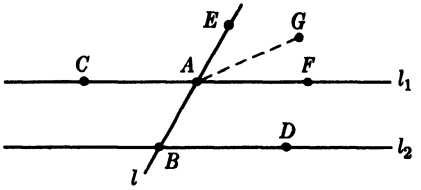


FIGURE 8–10

of their interiors on *opposite* sides of l, while *corresponding* angles have their interiors on the *same* side of l. In a pair of corresponding angles, one angle is always exterior and one interior.

Theorem 8–4. *Let l_1 and l_2 be lines cut by line l. If $l_1 \parallel l_2$, then two alternate interior angles are congruent to each other, and conversely, if two alternate interior angles are congruent, then $l_1 \parallel l_2$* (Fig. 8–10).

Proof. The last part of the theorem follows at once from the first proof of Theorem 8–1, for if $\angle CAB \cong \angle ABD$, then $\angle ABD \cong \angle EAF$, and $l_1 \parallel l_2$.

The first part of the theorem requires the parallel postulate, and may be proved as follows. Choose G, located in the interior of $\angle DBE$, such that $\angle EAG \cong \angle ABD$. Then line $AG \parallel l_2$. (Why?) But A is also on l_1, and $l_1 \parallel l_2$. Hence the line through A and G is the same as l_1, so G is on l_1. Now, $\angle CAB \cong \angle EAG \cong \angle ABD$. Thus the alternate interior angles are congruent.

As important consequences of Theorem 8–4, we state the following corollaries, which you can easily prove.

Corollary 8–4–1. *Let l cut l_1 and l_2 at distinct points. If $l_1 \parallel l_2$, then alternate exterior angles are congruent, and conversely.*

Corollary 8–4–2. *Let l cut l_1 and l_2 at distinct points. If $l_1 \parallel l_2$, then corresponding angles are congruent, and conversely.*

Exercise Group 8–6

1. Prove that if a line is perpendicular to one of two parallel lines, it is perpendicular to the other.

2. Prove that if $l \parallel m$, $n \perp l$, $p \perp m$, and $n \neq p$, then $n \parallel p$.

3. In Fig. 8–11, $\overline{AO} = \overline{CO}$, and $\overline{BO} = \overline{OD}$. Prove that $AB \parallel CD$.

4. Show that $\overline{BC} = \overline{AD}$, and $BC \parallel AD$ in Fig. 8–11.

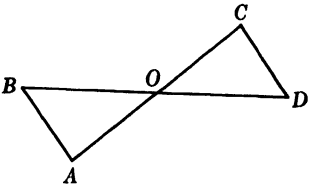


FIGURE 8–11

5. Show that if $AB \cong CD$, and $AB \parallel CD$, then $BO \cong OD$, and $BC \parallel AD$ in Fig. 8–11.

6. If $l \parallel m$, and A and B are any two points on l, show that the perpendicular distances from A and B to m are equal. We may state this result in the form: if $l \parallel m$, then l and m are "everywhere equidistant." State and prove the converse of this theorem. State also the contrapositive.

8–6 ANGLE SUMS

One of the most interesting consequences of the parallel postulate is a theorem relating to the angles of a triangle. We need a definition.

Definition 8–4. If $\angle AOB$ and $\angle BOC$ are adjacent angles (Fig. 8–12), and either $\angle AOC$ is a straight angle, or OB is in the interior of $\angle AOC$, then we say that $\angle AOC$ is a *sum* of $\angle AOB$ and $\angle BOC$. Likewise, if $\angle A'O'B' \cong \angle AOB$, and $\angle B''O''C'' \cong \angle BOC$, then we say that $\angle AOC$ is a *sum* of $\angle A'O'B'$ and $\angle B''O''C''$.

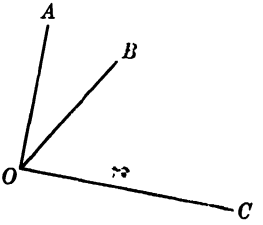


FIGURE 8–12

Remark. At this stage in our geometry, we cannot yet "add" angles by adding their measures in degrees, because we have not as yet described how angles are measured. The measurement of angles is more complicated than measurement of length, and will be described in a later chapter.

Draw a figure to illustrate the last part of Definition 8–4.

Definition 8–5. Two angles are said to be *complementary* if their sum is a right angle.

Definition 8–6. Two angles are said to be *supplementary* if their sum is a straight angle.

Exercise Group 8–7

1. Draw an obtuse angle. Draw an angle that is supplementary to this angle and is not adjacent to it. Draw one that is adjacent and supplementary.

2. Draw an acute angle. Draw an angle that is complementary to this angle and is adjacent to it. Draw one that is not adjacent, but is complementary, to the acute angle.

3. Prove that if two acute angles are congruent, then their complements are congruent.

4. Prove that if the sides of one angle are parallel, respectively, to the sides of a second angle, then either the angles are congruent or they are supplementary.

Theorem 8–5. *The sum of the angles of a triangle is a straight angle.*

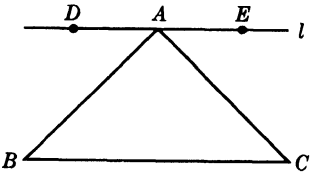


FIGURE 8–13

Proof. In Fig. 8–13, let l be the line through A and parallel to the side BC of $\triangle ABC$. Let D be a point on l that is on the opposite side of AB from C. Let E be a point of l on the same side of AB as C. Then $\angle DAB \cong \angle ABC$, and $\angle EAC \cong \angle ACB$. The straight angle DAE is the sum of angles A, B, and C.

Exercise Group 8–8

1. Show that the acute angles of a right triangle are complementary.

2. Two angles of a triangle are 40° 17′ 30″ and 78° 31′ 38″, respectively. Determine the number of degrees in the third angle.

3. How many degrees are there in each angle of an equilateral triangle?

4. Show that a median of an equilateral triangle divides the triangle into two right triangles, each of which has acute angles of 30° and 60°.

5. Prove that in a right triangle, with acute angles of 30° and 60°, the side opposite the 30° angle is half as long as the hypotenuse.

6. In $\triangle ABC$, an exterior angle at B is 124°. $\angle A$ is 58°. How many degrees in $\angle C$?

7. Prove that if two angles of one triangle are congruent, respectively, to two angles of another triangle, then the third angles are congruent.

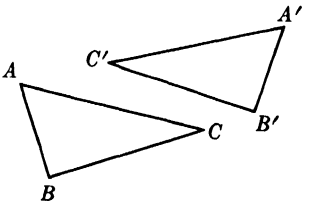


FIGURE 8–14

8. In Fig. 8–14, $AB \cong A'B'$, $\angle A \cong \angle A'$, and $\angle C \cong \angle C'$. Prove that $\triangle ABC \cong \triangle A'B'C'$.

9. Is the following true? Two triangles are congruent to each other if one side and two angles of the one triangle are congruent to one side and two angles of the other.

10. Prove that two right triangles are congruent if the hypotenuse and an acute angle of one triangle are congruent, respectively, to the hypotenuse and acute angle of the other.

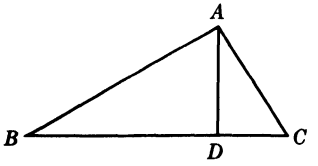


FIGURE 8–15

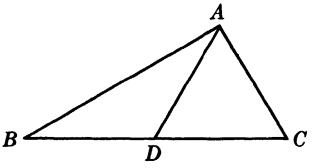


FIGURE 8–16

11. In $\triangle ABC$ (Fig. 8–15), $\angle A$ is right, and $AD \perp BC$. Prove that $\angle BAD \cong \angle ACD$, and $\angle ABD \cong \angle DAC$.

*12. In Fig. 8–16, D is the midpoint of BC, and $BD \cong AD$. Prove that $\angle BAC$ is right.

*13. Prove that two right triangles are congruent if the hypotenuse and one side of one triangle are congruent to the hypotenuse and one side of the other.

8–7 POLYGONS

In this and the remaining sections, we will be studying certain polygons. First, we need to recall the definition of a polygonal path.

A *polygonal path* (Fig. 8–17) is determined by a number of points, $P_1, P_2, \ldots, P_n$, called the *vertices*, given in a definite order. The path is the set of points on the segments P_1P_2, $P_2P_3, \ldots, P_{n-1}P_n$. The point P_1 is called the *beginning* of the path, and P_n is called the *end*. The segments P_1P_2, etc., are called the *sides* of the path.

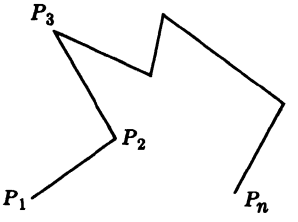


FIGURE 8–17

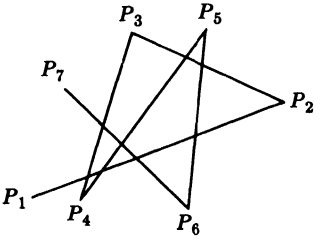


FIGURE 8–18

Remark. There is no requirement that the sides of a polygonal path not intersect. Hence the path can be quite complicated. See Fig. 8–18 for a six-sided path.

Exercise Group 8–9

1. Draw a polygonal path with five sides that does not cross itself. Draw one that does.

2. If the length of a polygonal path is the sum of the lengths of its sides, how does the length of a polygonal path $P_1 \ldots P_n$ compare with $\overline{P_1P_n}$?

Definition 8–7. A *polygon* is a polygonal path whose beginning and end coincide. If the vertices are all different and no two sides have a point in common (other than a vertex of two adjacent sides), then the polygon is called *simple* (Fig. 8–19).

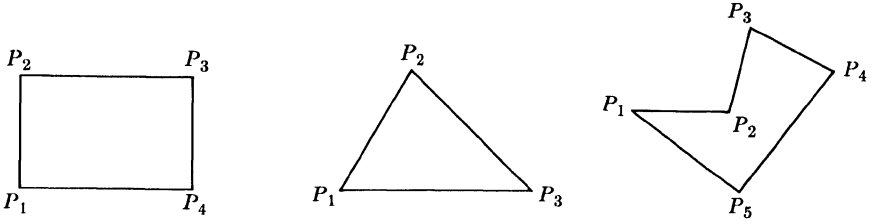
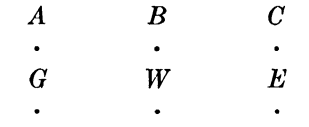


FIG. 8–19. Simple polygons.

Each simple polygon separates the points of the plane, other than the polygon itself, into two sets called the *interior* and *exterior* of the polygon. If A is a point of the interior, and B is a point of the exterior, every polygonal path connecting A to B contains at least one point of the polygon. If A and B both belong to either the interior or exterior, then there exist polygonal paths joining A to B that contain no point of the polygon. The proofs of these statements are quite difficult, and will not be given here. It would be instructive to carry out the proofs in the case in which the polygon is a triangle. The student interested in doing so should restudy Section 3–6.

**Problem.* The theorem that a simple polygon separates the plane is powerful and has surprising applications. It can be used to show that the well-known puzzle below has no solution.

It is desired to connect each of three houses A, B, and C with three outlets, one for gas G, one for water W, and one for electricity E.



Is it possible to run the lines, one from each house to each outlet, in such a way that no lines cross? In other words, can polygonal paths from A to G, A to W, etc., be drawn so that no two paths cross?

Definition 8–8. Vertices of a polygon that are endpoints of the same side are called *adjacent* vertices (Fig. 8–20). Angles whose vertices are adjacent to each other are called *consecutive* angles. A *diagonal* of a polygon is a segment joining any two nonadjacent vertices. Sides with a common vertex are called *adjacent* sides.

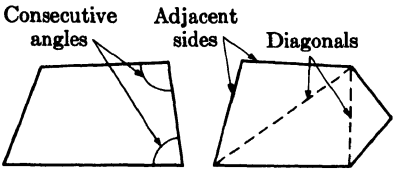


FIGURE 8–20

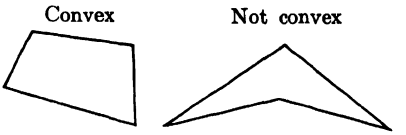


FIGURE 8–21

Definition 8–9. A simple polygon is called *convex* if, whenever A and B are points of the polygon, every point of the segment AB lies either on the polygon or in its interior (see Fig. 8–21).

Remark. Unless otherwise stated, we will confine our attention to simple convex polygons.

Definition 8–10. If all sides of a polygon have the same length, the polygon is *equilateral*. If all angles are congruent, it is *equiangular*. A convex polygon that is equilateral and equiangular is called *regular*. Polygons are also classified according to the number of sides. Polygons of 3, 4, 5, 6, 7, 8, 9, and 10 sides are called *triangles, quadrilaterals, pentagons, hexagons, heptagons, octagons, nonagons,* and *decagons,* respectively. A polygon of n sides will be called an *n-gon.*

Exercise Group 8–10

1. What do we call a regular polygon of three sides? of four sides?

2. Draw a convex polygon of five sides. Draw a quadrilateral that is not convex. Draw a polygon that is not simple.

*3. Is it true that every interior point of a convex polygon is also interior to every angle of the polygon?

4. We can divide a convex polygon into triangles by drawing the diagonals from any vertex. Into how many triangles does this procedure divide a convex polygon of four sides? of ten sides? of n sides?

*5. How many diagonals has a convex polygon of four sides? five sides? ten sides? n sides?

6. Show that an equiangular polygon need not be equilateral.

*7. Find an example of an equilateral, equiangular polygon that is not regular.

A diagonal of a convex quadrilateral divides the quadrilateral into two triangles. Since the sum of the angles in each triangle is a straight angle, we shall say that *the sum of the (four) angles of the quadrilateral is two straight angles.* The two diagonals from one vertex of a convex pentagon divide the pentagon into three triangles, and we say that *the sum of the (five) angles of the pentagon is three straight angles.*

When we have defined degree measure for angles, we will see that the sum of the degree measures of the angles of a convex quadrilateral is

$2 \cdot 180° = 360°$, of a pentagon, $3 \cdot 180° = 540°$, and in general for a polygon of n sides, the sum of degree measures is $(n - 2) \cdot 180°$.

Do not misunderstand the statement that the sum of the angles of a pentagon is three straight angles. This is using the word "sum" in a different sense than when we speak of the sum of *two angles*. Recall that when we add two angles, the sum is less than or congruent to a straight angle.

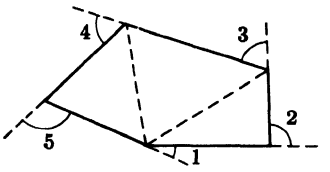


FIG. 8–22. Pentagon (three triangles).

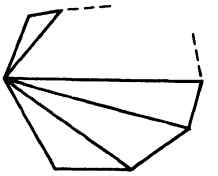


FIG. 8–23. n-gon (n–2 triangles).

We observe that the sum of the five interior angles and the five indicated exterior angles in Fig. 8–22 is precisely five straight angles. Since the interior angles account for three straight angles, the sum of the exterior angles must be two straight angles. Similar reasoning shows that for any convex polygon of n sides, the sum of the n exterior angles (one at each vertex) is exactly two straight angles (Fig. 8–23).

Exercise Group 8–1

1. A quadrilateral has three 95° angles. How many degrees are there in the fourth angle?

2. How many degrees are there in each angle of a regular hexagon?

3. How many degrees are there in each exterior angle of a regular pentagon?

4. A convex quadrilateral has three angles of the same size. The fourth angle is a 120° angle. How many degrees are there in each of the other angles?

5. Explain why it is impossible for a convex polygon to have as many as six angles smaller than 120°.

*6. What is the sum of the angles at the vertices of a regular five-pointed star? (The points of the star are vertices of a regular pentagon.)

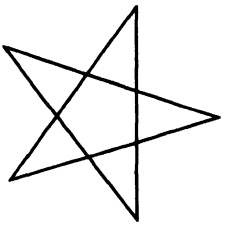


FIGURE 8–24

7. An exterior angle of a regular polygon is 20°. How many sides has the polygon?

8. Pentagon $ABCDE$ is regular. Prove that a line l, bisecting the angle at A, is perpendicular to side CD.

9. How many sides does a regular polygon have if each angle is 120°? 140°? 160°? 179°?

10. Make up some theorems of your own relating to regular pentagons or regular hexagons.

8–8 PARALLELOGRAMS

The following definitions make certain familiar terms precise.

Definition 8–11. A *parallelogram* is a quadrilateral having two pairs of parallel sides. A *rectangle* is a parallelogram with a right angle. A *square* is a rectangle with two congruent perpendicular sides. A *rhombus* is an equilateral parallelogram.

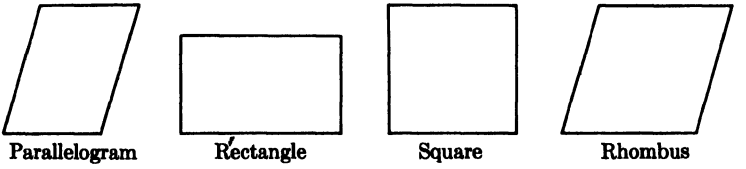


FIGURE 8–25

Our knowledge of parallels and these definitions make it possible to prove many results, as the following exercises will demonstrate.

Exercise Group 8–12

The exercises below present many interesting theorems concerning polygons. Give proofs for these theorems.

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. Opposite sides of a parallelogram are congruent to each other.
3. The diagonals of a parallelogram bisect each other.
4. Opposite angles of a parallelogram are congruent.
5. Consecutive angles of a parallelogram are supplementary to each other.
6. Every angle of a rectangle is right.
7. All sides of a square are congruent.
8. The diagonals of a rectangle are congruent.
9. The diagonals of a square are perpendicular to each other.
10. The diagonals of a rhombus are perpendicular.
11. If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.
12. If the diagonals of a rectangle are perpendicular, the rectangle is a square.
13. The quadrilateral formed by connecting midpoints of the sides of a parallelogram is also a parallelogram.
14. The parallelogram formed by connecting midpoints of the sides of a square is also a square.
15. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
16. If a quadrilateral has a pair of sides congruent and parallel to each other, it is a parallelogram.
17. A parallelogram has one angle of 70°. Give the number of degrees in each angle.
18. One angle of a parallelogram is twice as large as a second angle. Determine the number of degrees in each angle of the parallelogram.

19. A diagonal of a rhombus bisects two angles of the rhombus.

20. If a rectangle is not a square, a diagonal does not bisect an angle of the rectangle.

21. If each angle of a quadrilateral is congruent to the angle opposite it, then the quadrilateral is a parallelogram.

22. Is a rhombus a regular polygon?

23. For every quadrilateral, it is true that either the bisectors of two consecutive angles of the quadrilateral are perpendicular, or the quadrilateral is not a parallelogram.

24. If a diagonal of a parallelogram bisects one angle of the parallelogram, it bisects a second angle, and the parallelogram is a rhombus.

25. If a parallelogram is not a rhombus, then the four bisectors of the angles of the parallelogram are not concurrent.

26. If a parallelogram is not a rhombus, then the four bisectors of the angles intersect to form a rectangle.

27. The four bisectors of the angles of a rectangle that is not a square intersect to form a square.

28. How many degrees are there in the angle formed by the two diagonals drawn from one vertex of a regular pentagon?

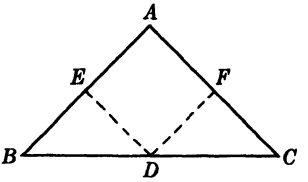


FIGURE 8–26

29. In Fig. 8–26, $DE \parallel AC$, and $DF \parallel AB$. Use the figure to prove that the sum of the angles of $\triangle ABC$ is a straight angle.

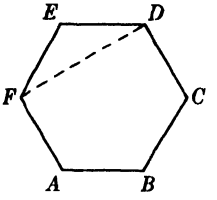


FIGURE 8–27

30. In the regular hexagon $ABCDEF$, prove that diagonal FD is perpendicular to side DC (Fig. 8–27).

31. Prove that $\overline{FC} = 2 \cdot \overline{DC}$ in Fig. 8–27. Show that the diagonals FC and AD bisect each other.

REVIEW OF CHAPTER 8

This chapter has dealt with consequences of the parallel postulate, by far the most famous of all the postulates of Euclid. (Euclid did not state it exactly as we have given it.) It was through the careful investigation of this postulate that non-Euclidean geometries were invented early in the 19th century. This investigation also led indirectly to the first set of correct postulates from which the theorems of Euclidean geometry could be rigorously proved. These postulates, presented about 1900 by the German mathematician David Hilbert, are equivalent to the ones that we have listed in this text. We list below some of the most important consequences of the parallel postulate shown in this chapter.

If two lines are parallel to one line, they are parallel to each other.

When parallel lines are cut by a line, alternate interior angles are congruent.

The sum of the angles of any triangle is a straight angle.

The sum of the interior angles of any convex polygon is $(n - 2)$ straight angles.

Several new terms were defined in this chapter. In addition, precise definitions of some familiar concepts were formulated. (For example, though we have referred freely to squares and rectangles in earlier chapters, they had not yet been exactly defined. Indeed, it is not even possible to prove that squares and rectangles exist until the parallel postulate is available.) You should know the definitions given in this chapter for the following:

alternate interior angles	octagon
corresponding angles	n-gon
polygon	regular polygon
simple polygon	parallelogram
convex polygon	rectangle
parallel lines	square
pentagon	rhombus
hexagon	diagonal
equilateral polygon	equiangular polygon

ALGEBRA REVIEW

1. Two angles of a triangle are congruent, and the measure of each exceeds that of the third angle by 10°. Determine the number of degrees in each angle of the triangle.

2. In $\triangle ABC$, the sum of $\angle B$ and an exterior angle at A has a measure of 170°. The sum of this exterior angle and $\angle C$ is 160°. Determine the measure of each angle of the triangle.

3. An exterior angle of a triangle has a measure of 117°. One angle of this triangle has a measure of 42°. Determine the degree measure of each angle of the triangle.

4. Prove that there is no $\triangle ABC$ such that the measure of an exterior angle at A is 103° and an exterior angle at B is 74°.

5. Diagonal AC of parallelogram $ABCD$ forms with sides AB and AD two angles whose measures differ by 10°. The measure of $\angle B$ is 110°. Determine the measure of $\angle A$.

6. Two angles of a triangle are complementary. Prove that the measure of the third angle is 90°.

7. The measure of the vertex angle of an isosceles triangle is 84°. Find the measure of each angle of the triangle.

8. The first angle of a triangle has a measure three-fourths of the second. The first angle is also one-third of the sum of the other two. Compute each angle.

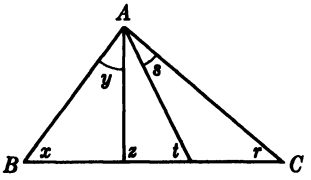


FIGURE 8–28

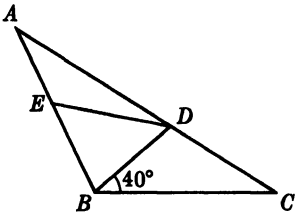


FIGURE 8–29

9. In Fig. 8–28 the letters represent the degree measures of the indicated angles. Prove that $x + y + t = r + s + z$.

10. In Fig. 8–29, $AB \cong BC$, $BE \cong BD$, and the measure of $\angle CBD$ is 40°. Determine the measure of $\angle ADE$.

REVIEW EXERCISES, CHAPTERS 1 THROUGH 8

1. Give truth values of α, β, and γ for which a statement of the form (α or β) and γ is true; is false.

2. Can you find truth values of α, β, and γ for which (α or β) and γ is false, and α or (β and γ) is true? Can you find truth values for which (α or β) and γ is true, and α or (β and γ) is false? Is (α or β) and $\gamma \rightarrow \alpha$ or (β and γ) true? Is α or (β and γ) $\rightarrow$ (α or β) and γ true?

3. Formulate an "if..., then..." statement equivalent to (a) no fishermen lie, (b) only fishermen lie, (c) fishermen necessarily lie.

4. Give the contrapositive of each of your implications in Exercise 3.

5. One set of points contains, among others, the points A, B, C, and D. A second set of points contains A, B, and D, but does not contain the point C. Prove that these two sets of points cannot both be lines.

6. Point C is in the interior of $\angle AOB$. Is every point of ray OC in the interior of this angle?

7. Define *interior of a triangle* in at least two ways.

8. Explain what is meant by saying that *congruence of segments* has the *reflexive* property; the *symmetric* property; the *transitive* property.

9. Explain why one of these statements is true, and the other false:
(a) $AB \cong CD \rightarrow AB = CD$, (b) $AB = CD \rightarrow AB \cong CD$.

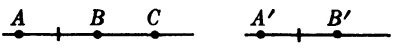


FIGURE 8–30

10. Figure 8–30 indicates that $AB \cong A'B'$, and ABC. Using only the definitions of *greater than* and *less than*, does the figure indicate that $AC > A'B'$, or that $A'B' < AC$?

11. The length of AB, measured with CD as unit, is 17.23. What is Archimedes' integer?

12. The length of CD in Exercise 11, as measured by EF, is 10. If AB is measured by EF, what is its length? What is Archimedes' integer in this case?

13. Express the rational number $\frac{3}{19}$ as a repeating decimal. Find the rational number whose decimal representation is the repeating decimal 0.272727...; 0.027027027....

14. State the S.A.S. postulate for triangles.

15. Two lines that intersect to form congruent adjacent angles are said to be perpendicular to each other. Does this definition in itself prove that there is a pair of perpendicular lines in the plane? How do we know that right angles exist?

16. Are there points A, B, and C, such that $\overline{AB} = 5$, $\overline{BC} = 7$, and $\overline{AC} = 13$?

17. What can be said about the position of the points R, S, and T, if $\overline{RS} = 8$, $\overline{ST} = 19$, and $\overline{RT} = 11$?

18. Consider a circle whose center is A and which contains the point B.
    (a) If $\overline{AX} > \overline{AB}$, what can be said about point X?
    (b) If P and Q are on the circle, how do we refer to $\angle PAQ$? to segment PQ?
    (c) If O and C are on the circle, how do we refer to $\angle AOC$?
    (d) If R and S are points of the circle, and A is on the segment RS, what do we call segment RS?

19. What does it mean to say that two rays have the same direction?

*20. Without using the parallel postulate or any of its consequences, prove the following theorem.

If in triangles ABC and $A'B'C'$, $AB \cong A'B'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$, then $\triangle ABC \cong \triangle A'B'C'$.

*21. Without using the parallel postulate, prove the following theorem, which relates to Fig. 8–31.

If $\angle CAB$ and $\angle ABD$ are right, $CA \cong DB$, and M and N are midpoints of CD and AB, respectively, then MN is perpendicular both to AB and to CD.

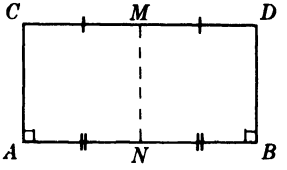


FIGURE 8–31

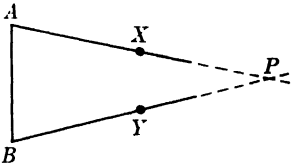


FIGURE 8–32

*22. Carry through the proofs of Exercises 20 and 21 using the parallel postulate.

*23. Euclid's formulation of the parallel postulate is different from ours. The following statement is essentially the postulate that Euclid used:

If the points X and Y are on one side of line AB, and if the sum of $\angle XAB$ and $\angle YBA$ is less than a straight angle, then the rays AX and BY intersect.

Figure 8–32 illustrates Euclid's parallel postulate. (a) Prove Euclid's parallel postulate as a theorem, using our parallel postulate. (b) Assuming the truth of Euclid's parallel postulate, prove ours as a theorem.

CHAPTER 9

SIMILARITY OF TRIANGLES AND POLYGONS

9–1 INTRODUCTION

In this chapter we study relations between two polygons that have "the same shape." (Later on we explain precisely what we mean by "the same shape," and say that such figures are *similar to each other.*) Then we prove some theorems about similar triangles, and using these theorems we prove the Pythagorean theorem, which states that if a and b are lengths of the sides of a right triangle, and c is the length of the hypotenuse, then $c^2 = a^2 + b^2$. We see as we study this chapter, that the theorems on similar triangles depend very much on the properties of parallel lines. For this reason, we first continue our study of parallel lines.

9–2 TRANSVERSALS

The results of this chapter are based on the important theorem about parallel lines given below. In the theorem, one of the lines is *between* the other two. You should be able to tell what this means in terms of the ideas of betweenness that you have studied.

Exercise 9–1

If l, l', and l'' are three parallel lines, what should it mean to say that l' is between l and l''?

Theorem 9–1. *If l, l', and l'' are parallel lines, and m is a transversal intersecting the lines in points A, B, and C, respectively, such that $AB \cong BC$, and if m' is any other transversal intersecting l, l', and l'' in A', B', and C', respectively, then $A'B' \cong B'C'$.*

Remark. A brief way of stating the theorem is to say that if parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.

Proof.

STATEMENTS	REASONS
1. Consider lines m'' and m''' through A and B, respectively, and parallel to m' in Fig. 9–1.	1. Which postulate?
2. m'' intersects l' in D, and m''' intersects l'' in E.	2. Why?
3. A, A', B', and D are vertices of a parallelogram, as also are B, B', C', and E.	3. Since l, l', and l'' are parallel, as also are m', m'', and m'''.
4. $AD \cong A'B'$, $BE \cong B'C'$.	4. Why?
5. $\angle ABD \cong \angle BCE$, $\angle BAD \cong \angle CBE$.	5. Why?
6. $AB \cong BC$.	6. Given.
7. $\triangle ABD \cong \triangle BCE$.	7. A.S.A.
8. $AD \cong BE$.	8. Statement 7.
9. $A'B' \cong B'C'$.	9. Statements 4 and 8 and properties of congruence.

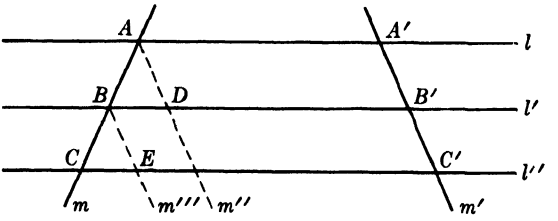


FIGURE 9–1

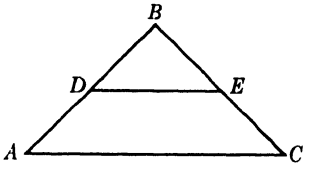


FIGURE 9–2

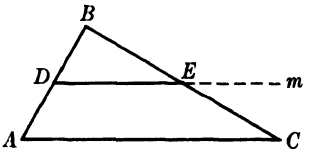


FIGURE 9–3

Corollary 9–1–1. *If a line is parallel to one side of a triangle and bisects another side, then it bisects the third side.*

Corollary 9–1–2. *If a line bisects two sides of a triangle, it is parallel to the third side, and the segment formed is half as long as the third side.* (In Fig. 9–2, $DE \parallel AC$, and $\overline{DE} = \frac{1}{2}\overline{AC}$.)

Proof. It is given that $\overline{AD} = \overline{DB}$, and $\overline{BE} = \overline{EC}$. Through D, let m be a line parallel to AC, as in Fig. 9–3. By Corollary 9–1–1, m meets

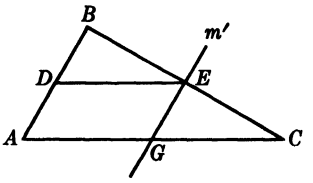


FIGURE 9–4

BC in a point F, which bisects BC. Since a segment has only one bisector, $F = E$. To see that $\overline{DE} = \frac{1}{2}\overline{AC}$, consider a line m' through E, parallel to AB, as in Fig. 9–4. Then m' meets AC in G, which bisects AC. It is left for the student to show that $\overline{AC} = 2\overline{DE}$.

Exercise Group 9–2

1. The three sides of a triangle have lengths 5, 12, and 13. What are the lengths of the segments joining the midpoints of the sides?

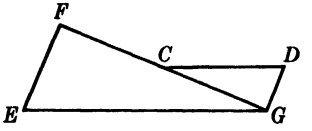


FIGURE 9–5

2. In Fig. 9–5, $FC \cong CG$, $CD \parallel EG$, and $DG \parallel EF$. Prove that $\overline{EG} = 2\overline{CD}$.

3. Prove Corollary 9–1–1.

4. In Fig. 9–6, if $\overline{AB} = 2\overline{BC}$, how are $\overline{A'B'}$ and $\overline{B'C'}$ related? Suppose that $\overline{AB} = 3\overline{BC}$; $3\overline{AB} = 5\overline{BC}$. Prove your statements.

5. In Fig. 9–6, suppose that $\overline{AC} = 10$ in., $BC = 2$ in., and $B'C' = 3$ in. How long is $A'B'$?

6. In Fig. 9–6, if $\overline{AB} = 6$ in., $\overline{A'B'} = 8$ in., and $\overline{AC} = 12$ in., what is $\overline{B'C'}$?

7. In Fig. 9–7, if $\overline{EF} = 2$, $\overline{FG} = 3$, and $\overline{EP} = 3$, what is $\overline{EQ}$?

8. In the proof of Theorem 9–1, we assumed that m was not parallel to m'. Prove Theorem 9–1 if $m \parallel m'$.

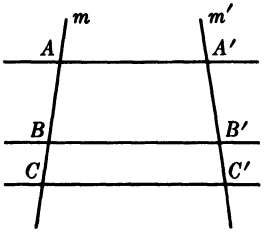


FIGURE 9–6

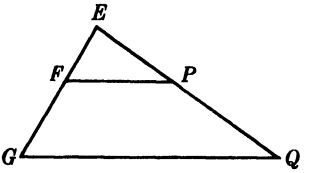


FIGURE 9–7

9–3 A CONSTRUCTION

In this section we show how to divide a segment into any number of congruent segments. The construction can be made with ruler and compass.

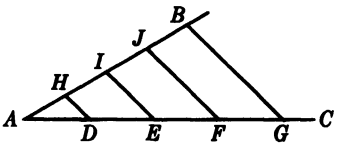


FIGURE 9–8

Construction 9–1. Given the segment AB, the problem is to divide it into a number of congruent segments. (In Fig. 9–8, the number is four.)

STATEMENTS	REASONS
1. Draw a ray AC, forming an angle with AB that is not a straight angle.	1. How do you know that such a ray exists?
2. With a compass, mark off points D, E, F, and G, in order $ADEFG$, such that $AD \cong DE \cong EF \cong FG$.	2. This can be done because of the congruence postulate and betweenness postulates.
3. Draw BG.	3. Postulate I.
4. Draw lines through D, E, and F, $\parallel BG$.	4. These lines exist by Theorem 8–1. The construction is RC. Why?
5. The parallel lines through D, E, and F intersect AB in points H, I, and J, respectively.	5. Why?
6. $AH \cong HI \cong IJ \cong JB$.	6. Theorem 9–1 on transversals.

Remark. In discussing measurement in Chapter 5, we had to divide the unit segment into tenths, and then hundredths, etc. The above construction shows that, with the parallel postulate, such a division of the unit segment is possible. At no time between Chapter 5 and now have we needed measurement for the development of our geometry.

Exercise Group 9–3

1. Draw a segment, and divide it into three segments of equal length. This is called *trisecting* a segment.
2. Draw a segment, and divide it into five congruent segments.
3. In Fig. 9–8, suppose that points I and J have been found. State two different ways to find H.
4. Can you divide a segment into four equal parts without using the above construction? eight equal parts? Is this an RC construction?
5. If, in Fig. 9–9, $\overline{AB} = 10$, $\overline{AD} = 2$, and $\overline{EC} = 3$, what is $\overline{BC}$?

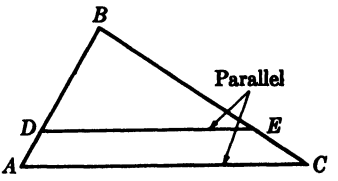


FIGURE 9–9

6. If, in Fig. 9–9, $\overline{AD} = 6$, $\overline{DB} = 8$, and $\overline{EC} = 2$, what is $\overline{BC}$?

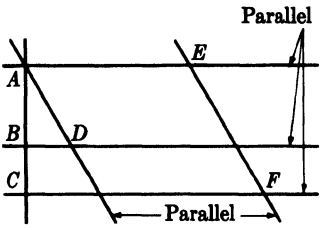


FIGURE 9–10

7. In Fig. 9–10, $\overline{AB} = 18$, $\overline{BC} = 3$, and $\overline{AD} = 24$. What is $\overline{EF}$?

8. Using Fig. 9–10, find $\overline{EF}$ if $\overline{BC} = 1$, $\overline{AB} = 2$, and $\overline{AD} = 3$.

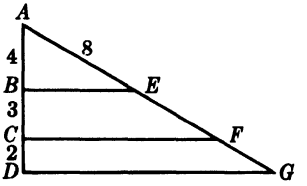


FIGURE 9–11

9. In Fig. 9–11, if $BE \parallel CF \parallel DG$, and the lengths are as shown, how long is AG?

10. Solve Exercise 9 if everything is the same except that $\overline{AE} = 6$.

11. In Fig. 9–12, how long is AB?

*12. In Fig. 9–13, $ABCD$ is a parallelogram, and E and F are midpoints of AB and CD, respectively. Prove that AC intersects ED and BF in G and H, and that G and H trisect AC. Prove also that $\overline{EG} = \frac{1}{3}\overline{ED}$.

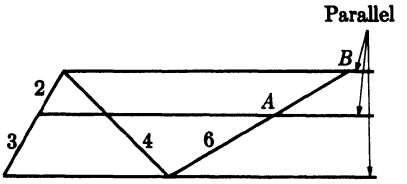


FIGURE 9–12

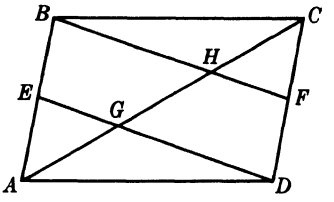


FIGURE 9–13

*13. In Fig. 9–14, x, y, z, and 1 refer to the lengths of the segments. If x is a whole number, show that $z = xy$.

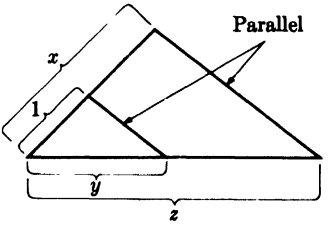


FIGURE 9–14

9–4 PROPORTION

Before discussing similar triangles, we recall some facts about proportions that you learned in arithmetic and algebra.

Definition 9–1. If x and y are numbers, with $y \neq 0$, the ratio of x to y is the number x/y, that is, x divided by y. A *proportion* is a statement that two (or more) ratios are equal. For example, $a/b = c/d$ is called

a proportion. In this proportion, we say that a and c are proportional to b and d, respectively, and also that "a is to b as c is to d." More generally, the statement that $x, y, z, u, \ldots$ are proportional to $x', y', z', u', \ldots$ means that $x/x' = y/y' = z/z' = u/u' = \cdots$. In all proportions, we must require that the denominators are not zero, because division by zero is impossible.

You may recall that in algebra you learned a number of theorems about proportions, some of which are troublesome to remember. All statements about proportions are simple consequences of the basic laws of algebra. The theorems on proportions in this chapter are not to be memorized, but rather their proofs are to be understood.

Exercise Group 9–4

1. Give other ratios equal to the following:
(a) 9/12, (b) $2x^2/3x$, (c) $12abc/27c^2$, (d) $(x^2 - x - 6)/(4x^2 + 8x)$, (e) $(x^2 - 5x + 6)/(x^2 - 4)$.

2. Are the following proportions true?

(a) $\frac{a}{b} = \frac{a^2}{ab}$, (b) $\frac{x}{y} = \frac{x^2}{y^2}$,

(c) $\frac{3}{4} = \frac{9}{16}$, (d) $x = \frac{6x}{6}$.

3. Any number can be regarded as a ratio because $N = N/1$. Hence, can $x + 2$ be regarded as a ratio for every number x?

4. Determine x, so that the following proportions are true:

(a) $\frac{2x}{8} = \frac{5}{3}$,

(b) $\frac{(3 - x)}{x} = \frac{5}{3}$,

(c) $\frac{2x + 3}{5} = \frac{x - 3}{2}$,

(d) $\frac{ax}{5} = \frac{a}{10}$,

*(e) $\frac{2x - 7}{2} = \frac{3x - 5}{3}$,

(f) $\frac{9}{x} = \frac{24}{8}$, (g) $\frac{3}{x} = \frac{x}{12}$,

(h) $\frac{a}{x} = \frac{x}{b}$.

5. Determine the ratio of u to v if (a) $4u = 5v$, (b) $v = 3u$, (c) $au = bv$, (d) $u = 3v$, (e) $4u - 18v = 0$, (f) $(2u - v)/(u + v) = \frac{1}{2}$.

6. If x is to 5 as 5 is to x, find x.

We list below some theorems about proportions that are simple consequences of the laws of algebra. Since a ratio is a number obtained by division, we first recall the meaning of division.

Definition 9–2. If $b \neq 0$, then a/b (or $a \div b$) is the number q, such that $a = qb$. In other words, for $b \neq 0$, $a/b = q$ if and only if $a = qb$.

Theorem 9–2 *If $b \neq 0$, and $c \neq 0$, then $a/b = (ac/bc)$.*

Proof. If $q = a/b$, then $a = qb$. Therefore, $ac = q \cdot bc$. Whence $q = (ac/bc)$.

Theorem 9–3. *If neither b nor $d = 0$, $a/b = c/d$ if and only if $ad = bc$.*

Proof. The statement $a/b = c/d$ says that two numbers are equal, that is, they are the same number. Multiplying this number by bd, we have $bd \cdot a/b = bd \cdot c/d$ or $da = bc$.

Conversely, if $ad = bc$ and $bd \neq 0$, then dividing this number by bd, we have $(ad/bd) = (bc/bd)$ or $a/b = c/d$.

Theorem 9–4. *If $c \neq 0$, then $a/b = c/d$ if and only if $a/c = b/d$.*

Proof. Both proportions are equivalent to $ad = bc$, by Theorem 9–3.

Exercise Group 9–5

1. If $dx = ey$, what is the ratio of x to y?

2. If $3u = 7v$, what is the ratio of u to v? of v to u?

3. If $hx - ky = ry + ty$, what is the ratio of x to y?

*4. Prove that if $x/y = u/v$, $x = u$, and $x \neq 0$, then $v = y$.

*5. Prove that if $a/b = c/d$, and $b + d \neq 0$, then $a/b = (a+c)/(b+d)$.

*6. Prove that if $a/b = c/d$, then $(a + b)/b = (c + d)/d$.

7. Use Exercise 6 to form a new proportion from $x/3 = z/7$.

8. Use Exercise 5 to form a new proportion from $6/15 = 10/25$.

*9. State and prove the converse of Exercise 5.

*10. State and prove the converse of Exercise 6.

9–5 RATIO AND PROPORTION IN GEOMETRY

The objects of study in geometry are sets of points. Some of these objects have numbers attached to them; for example, a segment has a *length*. Later, we also discuss area of polygons and measurement of angles. In geometry, when we speak of the ratio of two segments, we mean the *ratio of the numbers* that are the lengths of the segments. It will be much the same when we speak of the ratio of any other geometric objects.

Definition 9–3. If AB and CD are segments, the ratio of AB to CD is the number $\overline{AB}/\overline{CD}$. The statement that the segments AB and CD are *proportional* to the segments $A'B'$ and $C'D'$ means that $\overline{AB}/\overline{A'B'} = \overline{CD}/\overline{C'D'}$. We say that the sides of $\triangle ABC$ are proportional to the sides of $\triangle A'B'C'$ if $\overline{AB}/\overline{A'B'} = \overline{BC}/\overline{B'C'} = \overline{CA}/\overline{C'A'}$.

Exercise Group 9–6

1. Draw two segments. By measurement, determine the ratio of one segment to the other.

2. In Fig. 9–15, the sides of $\triangle EFG$ are proportional to the sides of $\triangle E'F'G'$. What is the ratio of EG to

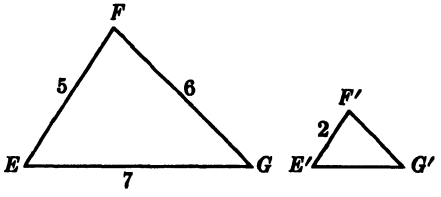


FIGURE 9–15

$E'G'$, if $\overline{E'F'} = 2$, and the lengths of the sides of $\triangle EFG$ are as shown? What is the ratio of $F'G'$ to $E'F'$?

3. If $\overline{AB} = 5, \overline{CD} = 7$, and $\overline{EF} = 3$, what is the ratio of AB to CD? of CD to EF? of AB to FE?

4. The sides of $\triangle ABC$ are proportional to the sides of $\triangle A'B'C'$ and $\overline{AB} = 5$, $\overline{BC} = 6$, $\overline{CA} = 8$. If $\overline{C'A'} = 9$, find $\overline{A'B'}$.

5. If the sides of $\triangle EFG$ are proportional to the sides of $\triangle PQR$ and $\overline{EF} = 7, \overline{FG} = 8, \overline{QR} = 6, \overline{RP} = 8$, find $\overline{GE}$ and $\overline{PQ}$.

Before proceeding, we should observe that a question might be raised concerning Definition 9–3. The number that we get for the length of a segment depends on what we choose for a segment of unit length. Thus a foot ruler has length 1 if the unit length is one foot, but has length 12 if the unit is one inch. We now justify Definition 9–3 by proving that the ratio does not depend on the unit of length.

Theorem 9–5. *No matter what unit of length is chosen for measuring two segments, the ratio of these segments is the same number.*

Proof. Suppose that the two segments are AB and $A'B'$, and suppose that we have two different unit segments CD and EF, as in Fig. 9–16.

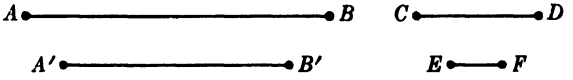


FIGURE 9–16

Suppose the lengths of AB and $A'B'$, when CD is the unit, are l and l' respectively, and their lengths, when EF is the unit, are L and L' respectively. We have seen, in Theorem 5–4, that changing the unit of measure just multiplies the length by a number. That is, if the length of CD, when EF is the unit, is x, then $L = l \cdot x$, and $L' = l' \cdot x$. Hence, if we form the ratio of AB to $A'B'$ when CD is the unit, we get l/l'; and if we form the ratio when EF is the unit, we get

$$\frac{L}{L'} = \frac{l \cdot x}{l' \cdot x} = \frac{l}{l'}.$$

Hence, the two ratios are the same, and the theorem is proved.

Exercise Group 9–7

1. Write the ratios above when CD is twice as long as EF; three times as long.

2. One inch is approximately 2.54 centimeters. What is the length of a foot rule in centimeters?

3. Two sides of a triangle have lengths 12 in. and 16 in. What would their ratio be if the sides were measured in miles?

4. If the ratio of segment AB to segment CD is 0.75, what is the ratio of CD to AB?

5. Is the ratio of one segment to another ever the number zero?

We have defined a ratio to be a number, and numbers can be either *rational* or *irrational*. When we encounter these two kinds of ratios in geometry, we use the words *commensurable* and *incommensurable*. We inherit this language from the Greeks, who discovered irrational numbers and were much disturbed by their discovery. We make the language clear in the following definition.

Definition 9–4. One segment is said to be *commensurable* with another if the ratio of the first to the second is a rational number. They are said to be *incommensurable* if this ratio is an irrational number.

Theorem 9–6. *Two segments, AB and CD, are commensurable with each other if and only if there is a segment EF which, if used as a unit of measure, makes $\overline{AB}$ and $\overline{CD}$ whole numbers (integers).*

Remark. Such a segment EF is a common unit of measure for the two segments, hence the word *commensurable.*

Proof. We have to show that if such a segment EF exists, then $\overline{AB}/\overline{CD}$ is a rational number, and conversely, that if $\overline{AB}/\overline{CD}$ is a rational number, then such a segment EF exists.

Suppose first that such a segment EF exists. Then $\overline{AB} = m$, and $\overline{CD} = n$, where m and n are integers. Therefore $\overline{AB}/\overline{CD} = m/n$, and is a rational number.

To prove the converse, suppose that $\overline{AB}/\overline{CD}$ is a rational number, that is, $\overline{AB}/\overline{CD} = m/n$, where m and n are integers. Divide the segment CD into n equal subintervals, and call one of these subintervals EF (Fig. 9–17). With EF as a unit of measure, $\overline{CD} = n$, and $\overline{AB} = m$.

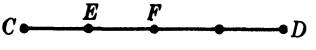


FIGURE 9–17

Note. In this book, our theorems deal only with commensurable pairs of segments, and we assume their truth for incommensurable cases. Proofs

for incommensurable cases require some knowledge of *limits*. The ideas involved are difficult and we avoid limits whenever possible.

Historical remark. Pythagoras (6th century B.C.) believed that all quantitative relations could be described by pure numbers. By this, he meant whole numbers, or integers. He was (so the story goes) profoundly shocked when he discovered that the side and diagonal of a square are incommensurable. Because of this discovery, later Greeks (notably Eudoxus and Euclid) invented an elaborate theory of ratio and proportion that is a model of ingenuity and precision. The interesting fact about this theory, from our point of view, is that *there were no numbers in Greek geometry, that is, no measurement*, and hence the meaning of the words "two ratios are equal" had to be defined in a special way that we cannot consider here.

9–6 SIMILAR TRIANGLES AND POLYGONS

In this section we are primarily concerned with similar triangles, but we also define similarity for polygons, and solve some problems concerning pairs of similar polygons. Remember that we are trying to make precise what we mean when we say that two polygons *have the same shape.*

Definition 9–5. Two *triangles* are said to be *similar* to each other if corresponding angles are congruent and corresponding sides are proportional. For example, in Fig. 9–18, $\triangle ABC$ is similar to $\triangle A'B'C'$ if $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, and $\overline{AB}/\overline{A'B'} = \overline{BC}/\overline{B'C'} = \overline{CA}/\overline{C'A'}$. To indicate that triangles are similar, we write $\triangle ABC \sim \triangle A'B'C'$.

Two *convex polygons* are said to be *similar* to each other if the corresponding angles are congruent and the corresponding sides are proportional. For example, the polygons in Fig. 9–19 are similar if $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, $\angle D \cong \angle D'$, $\angle E \cong \angle E'$, and $\overline{AB}/\overline{A'B'} = \overline{BC}/\overline{B'C'} = \overline{CD}/\overline{C'D'} = \overline{DE}/\overline{D'E'} = \overline{EA}/\overline{E'A'}$.

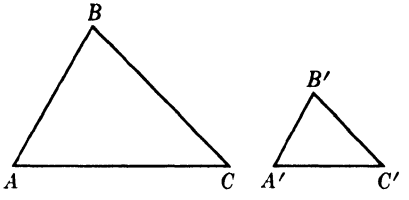


FIGURE 9–18

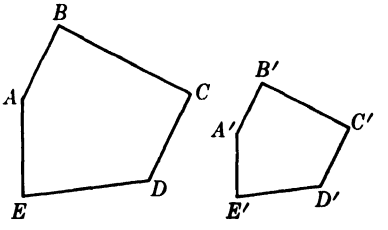


FIGURE 9–19

Exercise Group 9–8

1. Two triangles are similar, and two sides of one triangle are 7 and 11 in. In the other triangle, the side corresponding to the 7-in. side has length 12 in. Find the length of the side corresponding to the 11-in. side.

2. Two triangles are similar, and the sides of one triangle have lengths 4, 6, and 8. In the other triangle, the shortest side has length 6. Find the lengths of the other sides.

3. If $\triangle ABC \sim \triangle DEF$ and $\overline{AC} = 21$, $\overline{DF} = 15$, $\overline{FE} = 18$, and $\overline{DE} = 21$, find the lengths of the other sides of $\triangle ABC$.

4. Two quadrilaterals are similar, and one quadrilateral has sides of 5, 7, 12, and 16 in. The other quadrilateral has its longest side 12 ft. How long are the other sides?

5. The sides of a triangle are 4, 5, and x in. The corresponding sides of a triangle similar to the first triangle are 6, y, and 8 in. What are the sides of each triangle?

6. A pentagon has sides 4, 6, 10, 15, and x ft. In a pentagon similar to the first pentagon, the corresponding sides are u, v, 8, w, and 16. Find u, v, w, and x.

7. Two right triangles are similar, and in the first, the ratio of the shorter to the longer leg is $\frac{2}{3}$. In the other triangle, the shorter leg is 13 in. long. How long is the longer leg of this triangle?

8. Find as many examples of similar polygons from the everyday world as you can.

9. Prove that if $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle PQR$, then $\triangle ABC \sim \triangle PQR$.

10. Are all squares similar? Why?

11. Are all rectangles similar? Why?

12. State conditions under which two parallelograms are similar.

13. Prove that congruent polygons are similar polygons. Is the converse true?

14. Prove that all equilateral triangles are similar.

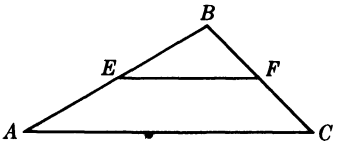


FIGURE 9–20

15. In Fig. 9–20, E and F are the midpoints of AB and BC. Prove that $\triangle BEF \sim \triangle BAC$.

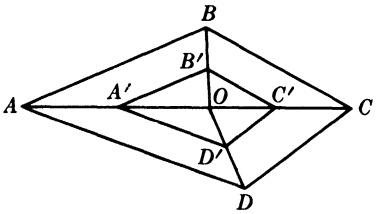


FIGURE 9–21

16. If, in Fig. 9–21, A', B', C', and D' bisect the corresponding segments from O, prove that the quadrilaterals $ABCD$ and $A'B'C'D'$ are similar.

9–7 THEOREMS ON SIMILAR TRIANGLES

In this section we give the fundamental theorems on similar triangles. They are extremely important because they have wide application in mathematics, other sciences, and engineering.

Theorem 9–7. *If the angles of one triangle are congruent, respectively, to the angles of another triangle, then the triangles are similar.*

Proof. Suppose that the triangles are $\triangle ABC$ and $\triangle A'B'C'$, as in Fig. 9–22, and that $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. We must prove that the sides are proportional, that is, $\overline{AB}/\overline{A'B'} = \overline{BC}/\overline{B'C'} = \overline{CA}/\overline{C'A'}$.

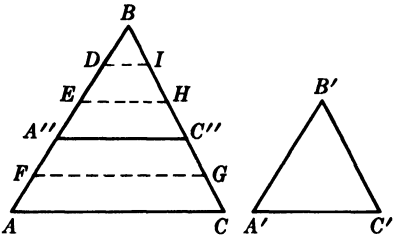


FIGURE 9–22

In forming the proof, we will suppose that the segments AB and $A'B'$ are commensurable. In other words, $\overline{AB}/\overline{A'B'}$ is a rational number. The proof for the incommensurable case is difficult, since it involves knowing about *limits*. We therefore suppose that, with a suitable unit, the lengths of AB and $A'B'$ are positive integers. The figure is drawn with $\overline{AB} = 5$ and $\overline{A'B'} = 3$. We may suppose that $AB > A'B'$, for if $AB < A'B'$, the argument can be reversed, and if $AB \cong A'B'$, the triangles are congruent (why?), and hence similar.

STATEMENTS	REASONS
1. There is a point A'' on AB, such that $A''B \cong A'B'$, and $\overline{A''B} = \overline{A'B'} = 3$.	1. Congruence postulate.
2. A'' is between A and B.	2. We have assumed that $AB > A'B'$.
3. There is a line through A'' parallel to AC which intersects BC in C''.	3. Why?
4. $\triangle BA''C'' \cong \triangle B'A'C'$.	4. A.S.A.
5. $BC'' \cong B'C'$.	5. Statement 4.
6. There are points D, E, and F, on AB, such that the points A, F, A'', E, D, and B, in that order, are at unit distance from their neighbors.	6. It is given that $\overline{AB} = 5$. By Statement 1, $\overline{A''B} = \overline{A'B'} = 3$.
7. There are lines through D, E, and F, parallel to AC. These lines intersect BC in points, I, H, and G, respectively.	7. Why?

8. $CG \cong GC'' \cong C''H \cong HI \cong IB$.	8. Theorem 9–1 on transversals.
9. If CG is chosen as unit of length, $\overline{BC} = 5$, and $\overline{BC''} = \overline{B'C'} = 3$.	9. Definition of length and Statement 5.
10. $\overline{BC}/\overline{B'C'} = 5/3$.	10. Statement 9.
11. $\overline{AB}/\overline{A'B'} = 5/3$.	11. Given.
12. $\overline{AB}/\overline{A'B'} = \overline{BC}/\overline{B'C'}$.	12. Statements 10 and 11.
13. In an analogous way, by considering a line parallel to BC, it may be shown that $\overline{AB}/\overline{A'B'} = \overline{AC}/\overline{A'C'}$.	13. Same reasons as in Statements 1 through 12.
14. $\triangle ABC \sim \triangle A'B'C'$.	14. Statements 12 and 13.

Remark. It should be clear that the proof is essentially the same if AB and $A'B'$ are any other positive integers besides 5 and 3.

Corollary 9–7–1. *If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.*

Corollary 9–7–2. *If an acute angle of one right triangle is congruent to an acute angle of another right triangle, then the triangles are similar.*

The proofs are left to the student.

Exercise Group 9–9

1. If, in Fig. 9–23, $BE \parallel CD$, $\overline{AE} = 4$, and $\overline{ED} = 6$, what is $\overline{AB}/\overline{AC}$?

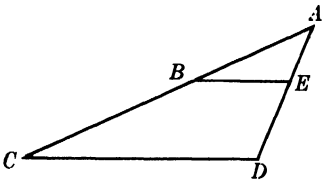


FIGURE 9–23

2. A church steeple casts a shadow 100 ft long, and a 6-ft post casts, at the same time, a shadow 7 ft long. How high is the steeple?

3. A line from the top of a cliff to the ground just passes over the top of a pole 8 ft high and meets the ground at a point 5 ft from the base of the pole. If it is 60 ft from this point to the base of the cliff, how high is the cliff?

4. As in Fig. 9–24, a lake lies between A and C. If lines AB and DE run north-south, $\overline{AB} = 8$ mi, $\overline{DE} = 1.2$ mi, and $\overline{DC} = 1.5$ mi, how far is it from A to C?

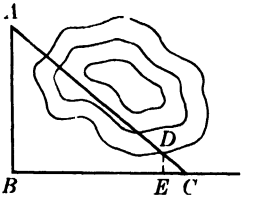


FIGURE 9–24

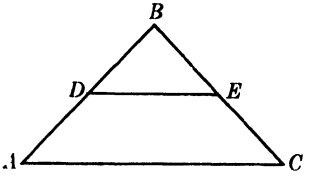


FIGURE 9–25

5. In Fig. 9–25, $DE \parallel AC$. Prove that $\triangle ABC \sim \triangle DBE$.

6. If in $\triangle DEF$, G and H are points on sides EF and DF, respectively, and $\angle GHF \cong \angle EDF$, prove that $\triangle DEF \sim \triangle HGF$.

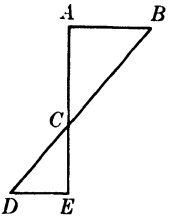


FIGURE 9–26

7. In Fig. 9–26, $AB \perp AE$, and $DE \perp AE$. Prove that (a) $\overline{AB}/\overline{DE} = \overline{AC}/\overline{CE}$; (b) $\overline{AB}/\overline{BC} = \overline{DE}/\overline{DC}$.

8. In $\triangle ABC$, let E, F, and G be the midpoints of the sides AB, BC, and CA, respectively. Prove that $\triangle ABC \sim \triangle FGE$.

9. If $\triangle ABC \sim \triangle A'B'C'$, and BD and $B'D'$ are the altitudes from B and B' respectively, prove that $\overline{AC}/\overline{A'C'} = \overline{BD}/\overline{B'D'}$.

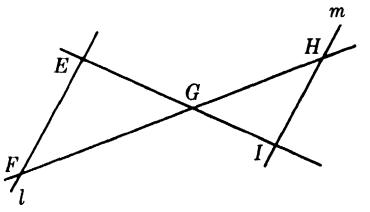


FIGURE 9–27

10. In Fig. 9–27, $l \parallel m$. Prove that $\triangle EFG \sim \triangle IHG$.

11. If, in Fig. 9–28, $\triangle PQR \sim \triangle P'Q'R'$, and PS and $P'S'$ bisect angles P and P', prove that $\overline{PQ}/\overline{PS} = \overline{P'Q'}/\overline{P'S'}$.

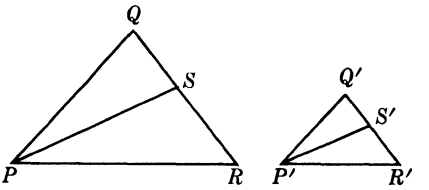


FIGURE 9–28

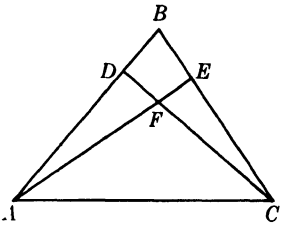


FIGURE 9–29

12. In $\triangle ABC$, CD and AE are altitudes, as in Fig. 9–29. Prove that (a) $\triangle BEA \sim \triangle BDC$; (b) $\triangle ADF \sim \triangle CEF$.

Before proving the next of the fundamental theorems on similar triangles, we prove a theorem that we need later and which is interesting in itself.

Theorem 9–8. *A line that divides two sides of a triangle proportionally is parallel to the third side. This means that if* $\overline{AD}/\overline{AB} = \overline{AE}/\overline{AC}$ *in Fig. 9–30, then* $DE \parallel BC$.

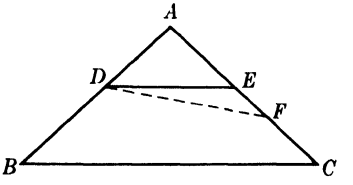


FIGURE 9–30

Proof.

STATEMENTS	REASONS
1. Let DF be a line parallel to BC and meeting AC in a point F.	1. Why is there such a line?
2. The point F is on the segment AC.	2. Why?
3. $\triangle ABC \sim \triangle ADF$.	3. Explain how Theorem 9–7 is used here.
4. $\overline{AF}/\overline{AC} = \overline{AD}/\overline{AB}$.	4. From the definition of similar triangles.
5. $\overline{AD}/\overline{AB} = \overline{AE}/\overline{AC}$.	5. Given.
6. $\overline{AF} = \overline{AE}$.	6. Statements 4 and 5.
7. $F = E$.	7. $AF \cong AE$ and congruence postulates.
8. $DE \parallel BC$.	8. Statements 7 and 1.

Theorem 9–9. *Given* $\triangle ABC$ *and* $\triangle A'B'C'$ *with* $\angle A \cong \angle A'$ *and* $\overline{AB}/\overline{A'B'} = \overline{AC}/\overline{A'C'}$, *then* $\triangle ABC \sim \triangle A'B'C'$.

Proof. We show that $\angle B \cong \angle B'$, and use Corollary 9–7–1. We may suppose that $AB > A'B'$. (Why?)

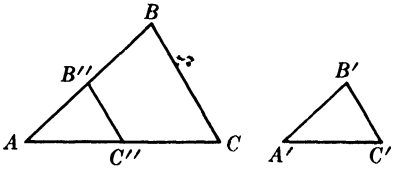


FIGURE 9–31

STATEMENTS	REASONS
1. In Fig. 9–31, there are points B'' on AB and C'' on AC such that $A'B' \cong AB''$ and $A'C' \cong AC''$.	1. Why?
2. $\triangle AB''C'' \cong \triangle A'B'C'$.	2. Why?
3. $\overline{AB}/\overline{AB''} = \overline{AC}/\overline{AC''}$.	3. Why?
4. $B''C'' \parallel BC$.	4. Theorem 9–8.
5. $\angle AB''C'' \cong \angle B$.	5. Why?
6. $\angle B \cong \angle B'$.	6. Statements 2 and 5.
7. $\triangle ABC \sim \triangle A'B'C'$.	7. Why?

Exercise Group 9–10

1. Prove that if the legs of one right triangle are proportional to the legs of another right triangle, then the triangles are similar.

2. In Fig. 9–32, $\overline{AO} = 10$, $\overline{BO} = 12$, $\overline{OD} = 5$, and $\overline{OC} = 6$. Show that $\triangle AOB \sim \triangle DOC$.

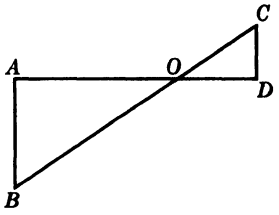


FIGURE 9–32

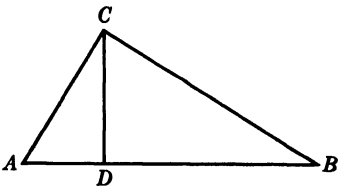


FIGURE 9–34

3. Prove that two isosceles triangles are similar if their vertex angles are congruent; likewise if their base angles are all congruent to one another.

4. In Fig. 9–33, if $\overline{AD} = x \cdot \overline{AB}$, and $\overline{AE} = x \cdot \overline{AC}$, prove that $\triangle ABC \sim \triangle ADE$.

5. If $\triangle ABC$ is a right triangle, with the right angle at C, and if, in Fig. 9–34, $\overline{AD}/\overline{AC} = \overline{AC}/\overline{AB}$, prove that $CD \perp AB$.

*6. In Fig. 9–35, if $\triangle ABC$ is a right triangle, with right angle at C, and if $CD \perp AB$, prove that $\triangle ABC \sim \triangle ACD$.

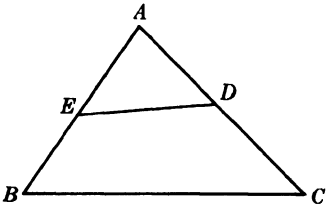


FIGURE 9–33

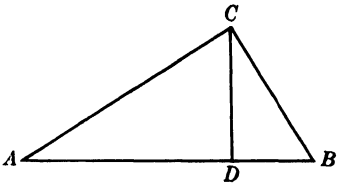


FIGURE 9–35

The next theorem is the last of the three basic theorems on similar triangles.

Theorem 9–10. *If the sides of one triangle are proportional to the sides of another triangle, then the triangles are similar. In other words, referring to Fig.* 9–36, *if* $\overline{AB}/\overline{A'B'} = \overline{AC}/\overline{A'C'} = \overline{BC}/\overline{B'C'}$, *then* $\triangle ABC \sim \triangle A'B'C'$.

Proof. It is sufficient to show that $\angle B \cong \angle B'$. Why? We may assume that $AB > A'B'$, for if $AB < A'B'$, the argument can be reversed, and if $\overline{AB} = \overline{A'B'}$, then the triangles are congruent (why?), and hence similar.

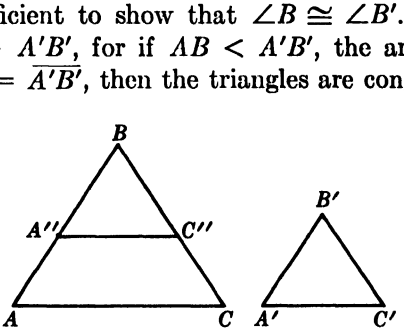


FIGURE 9–36

STATEMENTS	REASONS
1. Choose A'' on AB and C'' on BC, such that $\overline{A''B} = \overline{A'B'}$ and $\overline{C''B} = \overline{C'B'}$.	1. Why possible?
2. $\overline{AB}/\overline{A''B} = \overline{BC}/\overline{BC''}$.	2. The hypothesis of Theorem 9–10 and Statement 1.
3. $\triangle ABC \sim \triangle A''BC''$.	3. Theorem 9–9.
4. $\overline{AC}/\overline{A''C''} = \overline{AB}/\overline{A''B} = \overline{AB}/\overline{A'B'}$.	4. Statements 3 and 1.
5. $\overline{AC}/\overline{A''C''} = \overline{AC}/\overline{A'C'}$.	5. Statement 4 and the given fact $\overline{AB}/\overline{A'B'} = \overline{AC}/\overline{A'C'}$.
6. $\overline{A''C''} = \overline{A'C'}$.	6. Statement 5.
7. $\triangle A''BC'' \cong \triangle A'B'C'$.	7. Statements 1 and 6 and S.S.S.
8. $\angle B \cong \angle B'$.	8. Statement 7.
9. $\triangle ABC \sim \triangle A'B'C'$.	9. Theorem 9–9.

Exercise Group 9–11

1. State, in your own words, the three basic theorems on similar triangles, Theorems 9–7, 9–9, and 9–10.

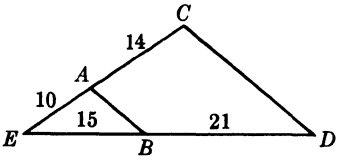


FIGURE 9–37

2. In Fig. 9–37, the numbers are the lengths of the segments. How are the lines AB and CD related? Why?

3. If the hexagons of Fig. 9–38 are similar, prove that (a) $\triangle BCD \sim \triangle B'C'D'$; (b) quadrilateral $ABCD \sim$ quadrilateral $A'B'C'D'$.

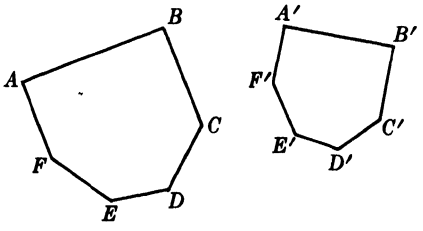


FIGURE 9–38

4. In Fig. 9–39, $BE \perp AC$, and $CD \perp AB$. Find two similar triangles. How many proportions can you write?

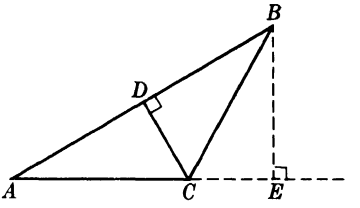


FIGURE 9–39

5. If $\triangle ABC \sim \triangle A'B'C'$, and AD and $A'D'$ are medians, prove that the ratio of the medians is equal to the ratio of any two corresponding sides. Restate this using the word *proportion* (or *proportional*).

6. Prove that if two triangles are similar, then the ratio of their perimeters is the same as the ratio of two corresponding sides.

7. Is a statement like that of Exercise 6 true for similar polygons? How would you prove this?

9–8 THE PYTHAGOREAN THEOREM

With our knowledge of similar triangles, we are able to prove a theorem associated with the Greek mathematician Pythagoras, who lived in the 6th century B.C. It is believed that he first gave a proof of the theorem that bears his name. First, we state a theorem that is useful for the proof of the Pythagorean theorem.

Theorem 9–11. *The altitude to the hypotenuse of a right triangle forms two right triangles, which are similar to the given triangle. For example, in Fig.* 9–40, $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.

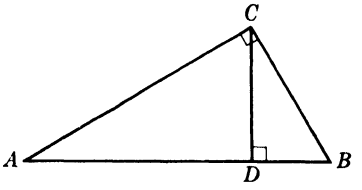


FIGURE 9–40

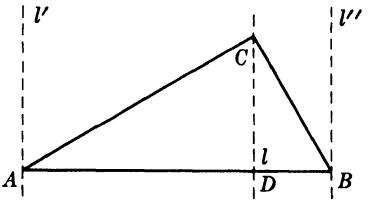


FIGURE 9–41

Proof. In the first place, a line through C perpendicular to AB must intersect the segment AB. This fact may be seen as follows: Consider lines l, l', and l'' in Fig. 9–41, through C, A, and B, and perpendicular to AB. Then $l' \parallel l \parallel l''$. The point C is between l' and l'', because angles A and B are acute angles. Therefore, l is between l' and l'', and hence l intersects AB in a point D. $\triangle ACD \sim \triangle ABC$, since each triangle has a right angle, and the triangles have the acute angle A in common. By the same reasoning, $\triangle BCD \sim \triangle BAC$.

Corollary 9–11–1. *The altitude to the hypotenuse is a mean proportional* between the two segments formed on the hypotenuse.*

Proof. We have to show that, using Fig. 9–40, $\overline{AD}/\overline{CD} = \overline{CD}/\overline{DB}$. Since $\triangle ADC \sim \triangle CDB$, this is an immediate consequence of the definition of similar triangles.

Corollary 9–11–2. *A leg of a right triangle is the mean proportional between the hypotenuse and the adjacent segment on the hypotenuse formed by the altitude from the right angle. That is, in Fig.* 9–40, $\overline{AC}^2 = \overline{AD} \cdot \overline{AB}$.

---

* x is a "mean proportional" between a and b if $a/x = x/b$, or, which is the same, $x^2 = ab$.

Exercise Group 9–12

1. In Fig. 9–42, $EF \perp FG$, and $FH \perp EG$. If $\overline{EF} = 8$, and $\overline{FG} = 5$, what is $\overline{FH}/\overline{EH}$? $\overline{FH}/\overline{HG}$?

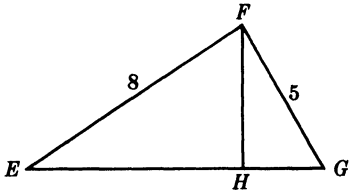


FIGURE 9–42

2. In Fig. 9–43, the sides of a right triangle are 5, 12, and 13. How long is the altitude on the hypotenuse?

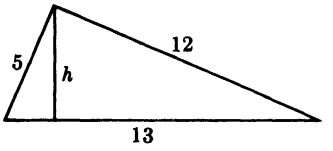


FIGURE 9–43

3. In Fig. 9–44, Q is a right angle, and $QS \perp PR$. Find $\overline{PQ}$, if $\overline{PS} = 4$, and $\overline{SR} = 5$.

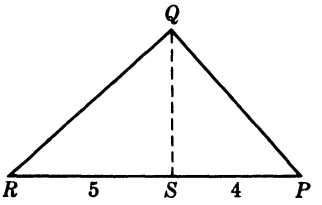


FIGURE 9–44

4. Using Fig. 9–44, if $\overline{QS} = 7$, what is the product $\overline{PS} \cdot \overline{SR}$?

5. In Fig. 9–45, $\angle C$ is a right angle, as is $\angle ADC$. Express h in terms of the lengths a, b, and c of the sides. Obtain $\overline{AD}$ in terms of a, b, and c.

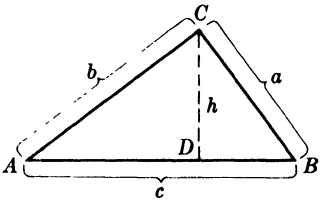


FIGURE 9–45

Theorem 9–12. Pythagorean theorem. *The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the legs.*

If the triangle is $\triangle ABC$, with the right angle at C, and if the lengths of the sides opposite angles A, B, and C are a, b, and c, respectively, as in Fig. 9–46, then we are able to show that $c^2 = a^2 + b^2$.

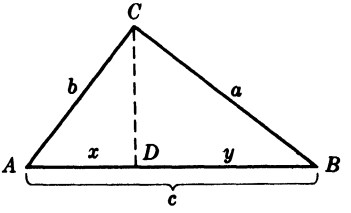


FIGURE 9–46

Proof.

STATEMENTS	REASONS
1. Let CD be the altitude and $x = \overline{AD}$ and $y = \overline{DB}$. Then $x/b = b/c$ and $y/a = a/c$.	1. Corollary 9–11–2.
2. $x + y = c$.	2. Properties of length.
3. $b^2/c + a^2/c = c$.	3. Statements 1 and 2.
4. $a^2 + b^2 = c^2$.	4. Statement 3 and algebra.

Theorem 9–13. Converse to Pythagorean theorem. *If a triangle has sides of length a, b, and c, such that* $c^2 = a^2 + b^2$, *then the triangle is a right triangle, and the side of length c is the hypotenuse.*

Proof. The method of proof is to show that there is a right triangle that is congruent to the given triangle. Suppose that the given triangle is $\triangle ABC$.

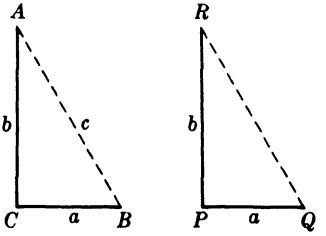


FIGURE 9–47

STATEMENTS	REASONS
1. There is a segment $PQ \cong BC$, as in Fig. 9–47.	1. Congruence postulates.
2. There is a ray perpendicular to PQ at P.	2. From the congruence postulates, there is an angle at P congruent to a right angle.
3. There is a point R on this ray such that $PR \cong AC$.	3. Congruence postulates.
4. $\overline{PQ} = a$, and $\overline{PR} = b$.	4. Statements 1 and 3.
5. $\overline{RQ}^2 = a^2 + b^2$.	5. Pythagorean theorem.
6. $c^2 = a^2 + b^2$.	6. Given.
7. $\overline{RQ}^2 = c^2$, so $\overline{RQ} = c$.	7. Statements 5 and 6 and algebra.
8. $RP \cong AC$, $PQ \cong CB$, and $RQ \cong AB$.	8. By construction and Statement 7.
9. $\triangle ABC \cong \triangle RQP$.	9. S.S.S.
10. $\angle C$ is a right angle.	10. $\angle P$ is a right angle, by construction, and $\angle C \cong \angle P$, by Statement 9.

Exercise Group 9–13

1. It is said that the ancient Egyptians constructed right angles for surveying purposes by forming a triangle using ropes of length 3, 4, and 5 units. Why would this be correct? Can you think of other integers for the sides of a right triangle?

2. In a right triangle, the hypotenuse is 7 and one leg is 3. How long is the other leg?

3. Write a formula for the length of one leg of a right triangle in terms of the lengths of the hypotenuse and other leg.

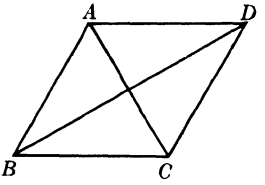


FIGURE 9–48

4. In Fig. 9–48, $AB \parallel CD$, $BC \parallel AD$, and $AC \perp BD$. If $\overline{BD} = 24$, and $\overline{AD} = 13$, find $\overline{AC}$.

5. In a right triangle, the two legs are 6 ft and 9 ft. How long is the hypotenuse?

6. The two legs of a right triangle are equal, and the hypotenuse is 4 in. long. How long are the legs?

7. The base of an isosceles triangle is 12 in. long. The two congruent sides have a combined length of 13 in. How long is the altitude to the base?

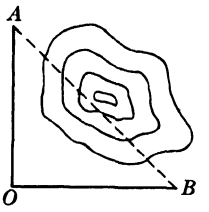


FIGURE 9–49

8. Point A in Fig. 9–49 is 21 mi due north of O, while point B is 20 mi due east of O. What is the distance from A to B, across the lake shown?

9. Find the altitude of an equilateral triangle, the length of whose side is a.

10. A rope 100 ft long reaches from the top of a pole to a point 40 ft from the foot of the pole. How tall is the pole?

11. Prove that the median to the hypotenuse of a right triangle is half as long as the hypotenuse.

12. Prove that in a 30°-60° right triangle, the shorter leg is half as long as the hypotenuse.

13. Prove that in an isosceles right triangle, the ratio of leg to hypotenuse is $1/\sqrt{2}$.

14. Construct a segment of length $\sqrt{5}$ in.

15. What is the length of a diagonal of a square of side 1? of side a?

16. Construct a segment of length $\sqrt{6}$.

*17. If n is any positive integer, can you construct (RC) a segment of length $\sqrt{n}$?

9–9 ON CONSTRUCTIONS

In our previous study of RC constructions, we assumed that two circles with equal radii would intersect if the distance between their centers were less than the sum of their radii. We can easily prove this by using the Pythagorean theorem. To do this, we first calculate where they *should* intersect, and then prove that this point *is* on both circles.

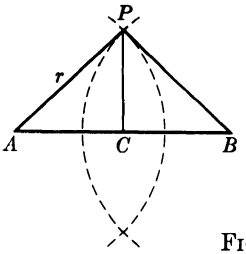


FIGURE 9–50

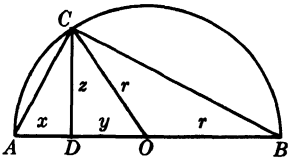


FIGURE 9–51

Consider Fig. 9–50, where r is the radius of the circles, and $d = \overline{AB}$ is the distance between their centers. If the circles intersect at P, and C is the midpoint of AB, then $\triangle PAC \cong \triangle PBC$ (why?), and hence $\angle C$ is a right angle. Therefore, $r^2 = \overline{PC}^2 + \overline{AC}^2$, so that $\overline{PC} = \sqrt{r^2 - (\frac{1}{2}\overline{AB})^2}$. We now know "where" the point of intersection P must be, and we can prove that such a point of intersection exists.

Theorem 9–14. *Suppose that circles, with centers A and B, have equal radii, $r > \frac{1}{2}AB$. These circles intersect.*

Proof. Let C be the midpoint of AB. Through C there is a line perpendicular to AB. On this line, let P be a point at the distance $PC = \sqrt{r^2 - (\frac{1}{2}\overline{AB})^2}$ from C. (How do we know there is such a point P?) Because $\triangle PAC$ and $\triangle PBC$ are right triangles, it follows from the Pythagorean theorem that $\overline{PA} = \overline{PB} = r$. Hence P lies on both circles.

Remark. It also follows from the above proof that the two circles can intersect in two, and only two, points, for the following reason. On the line through C, perpendicular to AB, there are only two points at the correct distance from C, namely one on each side of the line AB.

Theorem 9–15. *An angle inscribed in a semicircle is a right angle. That is, if in Fig. 9–51, AB is a diameter of the circle, with center at O, and C is any point on the circle other than A and B, then $\angle ACB$ is a right angle.*

Proof outline. We drop a perpendicular from C to AB, and prove that the right triangles ADC and CDB are similar. It then follows that $\angle C$ is a right angle.

STATEMENTS	REASONS
1. Let CD be the perpendicular from C to AB, and suppose that D is between A and O.	1. The point D must be between A and B. (Why?) D, then, either is O, or is between A and O, or O and B. If $D = O$, the proof that $\angle C$ is a right angle is easy. We discuss only the case in which D is between A and O. If D is between O and B, a similar proof can be made.

2. Let x, y, z, and r be the lengths of the segments in Fig. 9–51. Then $x = r - y$, and $z = \sqrt{r^2 - y^2}$.

2. Why?

3. $\dfrac{x}{z} = \dfrac{r - y}{\sqrt{r^2 - y^2}} = \sqrt{\dfrac{r - y}{r + y}}$.

3. Statement 2 and algebra.

4. $\dfrac{z}{r + y} = \dfrac{\sqrt{r^2 - y^2}}{r + y} = \sqrt{\dfrac{r - y}{r + y}}$.

4. Statement 2 and algebra.

5. $\triangle ACD$ and $\triangle CDB$ are right triangles.

5. By construction.

6. $\triangle ACD \sim \triangle CDB$.

6. Statements 3, 4, and 5 and Theorem 9–9.

7. $\angle ACD \cong \angle CBD$.

7. Statement 6.

8. $\angle ACB$ is a right angle.

8. Why?

Exercise Group 9–14

1. A circle passes through the vertices of a right triangle whose sides are 3, 4, and 5 in. What is the radius of the circle?

2. If PQ is a diameter of a circle of radius 13, and R is a point on the circle, such that $\overline{PR} = 10$, how long is $\overline{RQ}$?

3. In Fig. 9–52, AB is a diameter of the circle, and $CD \perp AB$. If $\overline{AB} = 30$ and $\overline{EO} = 12$, find $\overline{CD}$.

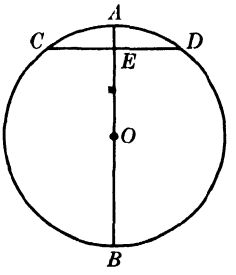


FIGURE 9–52

REVIEW OF CHAPTER 9

Be sure you know the definitions given in this chapter of the following concepts: transversal, ratio, proportion, ratio of segments, proportional sides, commensurable, similar triangles.

Some of the important theorems of the chapter were concerned with these definitions. Try to recall the theorems.

The main ideas of the chapter are listed briefly below. Be sure that you know them, and review the proofs.

1. If parallel lines cut off congruent segments on one transversal, they do so on every transversal.

This theorem depended upon what facts about parallel lines and triangles?

This theorem enables one to divide a segment into m congruent parts with ruler and compass. Show how to do this.

2. What is a proportion? What is the ratio of x to y if x and y are numbers? if x and y are segments?

If neither b nor d is zero, $a/b = c/d$ if and only if $ad = bc$. Illustrate this theorem by numerical examples.

3. The ratio of two segments does not depend on the unit of length.

Illustrate this fact by getting the ratio of depth to width of your desk, first measuring in feet and then in inches.

On what theorem about measurement of length does this fact depend?

*4. Two segments are defined to be commensurable with each other if their ratio is a rational number. What is a rational number? Give an example of one. Give an example of an irrational number. What are incommensurable segments?

State a theorem concerning commensurable segments.

5. What is the definition of similar triangles and polygons? Draw two similar triangles; pentagons.

What do an architect's drawings have to do with similar figures?

What does a map have to do with similarity?

6. Two triangles with corresponding angles congruent are similar.

Illustrate the proof of this theorem, using two similar triangles ABC and $A'B'C'$, with $AB = 12$, and $A'B' = 8$.

7. If two triangles have their corresponding sides proportional, they are similar.

What must be established to prove this theorem?

8. The Pythagorean theorem is proved, using a theorem about similar triangles in a right triangle. State this theorem.

Prove the Pythagorean theorem.

State its converse.

Review Exercises

1. $\overline{AB} = 24$, and C is a point on the line AB. If the ratio of AC to $CB = 3/5$, determine $\overline{AC}$ (a) if C is between A and B; (b) if A is between C and B.

2. Explain how to locate the point C in Exercise 1(a) by an RC construction.

*3. Can you give an RC construction for the point C in Exercise 1(b)?

4. A flagpole throws a 20-ft shadow when a yardstick throws a 10-in. shadow. Determine the height of the flagpole.

5. The length of the sides of a triangle are 4, 5, and 6. A triangle similar to this one has a perimeter of 30 in. Give the length of each side of this triangle.

6. In $\triangle ABC$, the altitude from vertex A divides the triangle into two similar triangles. $\triangle ABC$ is not isosceles. What conclusion may be drawn?

7. In $\triangle ABC$, $\angle A = 40°$, and $\angle B = 80°$. In $\triangle A'B'C'$, $\angle B' = 40°$, and $\angle C' = 60°$. Are the two triangles similar?

8. What is the mean proportional of 4 and 9? of 2 and 4? of 6 and 6?

9. A triangle whose sides have lengths 3, 4, and 5 is a right triangle. So also is the triangle with sides of 5, 12, and 13; 7, 24, and 25; 9, 40, and 41; 11, 60, and 61; etc. Explain the pattern underlying the formation of these triples of numbers.

10. On a certain map, a distance of $\frac{5}{8}$ in. represents 80 mi. On this map, the points representing two cities are $5\frac{7}{16}$ in. apart. What is the airline distance between these cities?

11. The parallel sides of a trapezoid have lengths 18 and 12, as in Fig. 9–53. The altitude is 8, and one of the nonparallel sides has length 10. If the nonparallel sides are extended, a triangle is formed by these extended sides and the greater base. (a) Show that this triangle is right. (b) Compute the lengths of the sides of this triangle!

12. If, in Exercise 11, the altitude is 6, rather than 8, and the other dimensions are as given, compute the length of the altitude of the triangle from the vertex opposite the side of length 18.

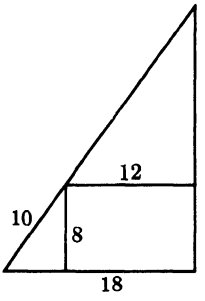


FIGURE 9–53

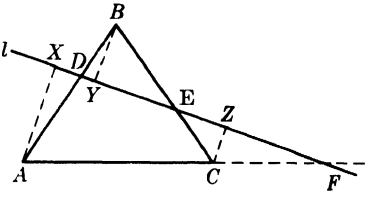


FIGURE 9–54

13. A parallelogram has adjacent sides of lengths 12 and 8, and these sides form a 60° angle. Find the lengths of the diagonals of the parallelogram. Find the lengths of the diagonals, if the included angle is 30°.

14. Draw three segments, and label them AB, CD, and EF. Locate by a ruler and compass construction a point X on AB, such that the ratio of AX to XB is equal to the ratio of CD to EF.

*15. In Fig. 9–54, line l cuts sides AB and BC of $\triangle ABC$ in points D and E, and cuts AC extended in point F. Prove that the product of the three ratios, AD to DB, BE to EC, and CF to FA, is 1. [*Hint:* Consider perpendiculars AX, BY, and CZ from A, B, and C to l. Then $\overline{AD}/\overline{DB} = \overline{AX}/\overline{BY}$, etc.]

16. Construct a triangle similar to a given triangle and having a given perimeter.

17. Upon a given segment as side, construct a triangle similar to a given triangle.

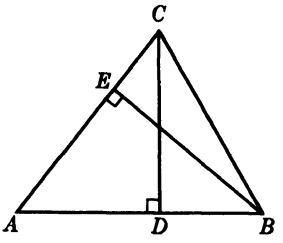


FIGURE 9–55

18. In $\triangle ABC$, CD and BE are altitudes, as in Fig. 9–55. Prove that $\overline{CD}/\overline{EB} = \overline{AC}/\overline{AB}$.

19. Prove that if, in triangles ABC and $A'B'C'$, $AB \parallel A'B'$, $BC \parallel B'C'$, and $CA \parallel C'A'$, then $\triangle ABC \sim \triangle A'B'C'$.

20. Prove that the line joining the midpoints of two opposite sides of a parallelogram is parallel to the two other sides and passes through the point of intersection of the diagonals.

CHAPTER 10

AREA

10–1 INTRODUCTION

You already have an intuitive understanding of the concept of *area*. In this chapter we will describe *area* in much the same way as we described length in Chapter 5. There we determined a number that we called the length of a line segment by choosing an arbitrary *unit segment* and *counting*. Here, for a given geometric figure, we determine a number that we call the *area* of the figure by first choosing an arbitrary *unit square* and then *counting*.

It is much more difficult to describe the *counting process* for area than it is for length. We content ourselves with a crude description of the counting process as it relates to finding areas of rectangles. Once the area formula for rectangles is determined, we employ it to derive area formulas for other simple polygons.

10–2 AREA BY COUNTING

We understand by *polygon*, of course, the set of points on the sides. When we refer to the *area* of a *polygon*, we really are considering the set of interior points. It would be more descriptive to speak of the *area* of this *interior set*. But for brevity, we simply speak of the area of a polygon. First, we study rectangular polygons.

We need to explain what we mean by the area of a rectangle. It will help to consider a few examples.

Having chosen a unit length, a square 1 unit of length on a side is said to have 1 unit of area. Clearly, if we had two squares side by side, as in Fig. 10–1, we would want to count the area as 2. Suppose we had a rectangle with sides of lengths 3 and 2. Figure 10–1 shows that we would want the area to be 6 units.

If we had a square of side $\frac{1}{2}$ unit, then placing it in a unit square as in Fig. 9–2, we see that the area should be only $\frac{1}{4}$ unit.

If we had a rectangle with sides $2\frac{1}{2}$ and 3 units, what would the area be?

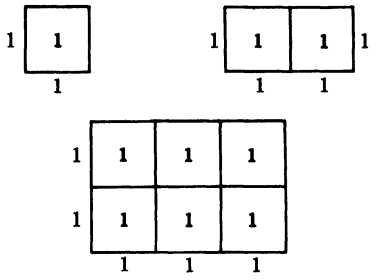


FIGURE 10–1

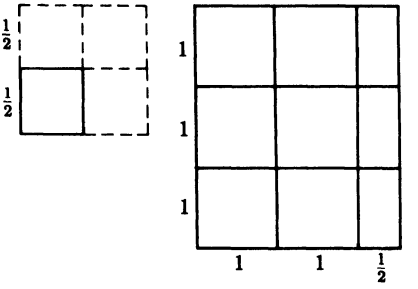


FIGURE 10–2

Exercise Group 10–1

1. Draw a rectangle whose sides are 3 and 5 units. Determine its area by decomposing it into unit squares and counting.

2. Sketch a figure to illustrate why the area of an m by n rectangle is mn units. Here, m and n are positive integers.

3. A rectangle is $3\frac{1}{3}$ units long and 3 units wide. Find its area. What is the largest size square, such that the rectangle can be covered by an integral number of squares? Suppose that the rectangle were $3\frac{1}{3}$ by 2. What is the answer then?

*4. Suppose that the length of a rectangle is a, and its width is b/c, where a, b, and c are positive integers. Obtain the area by dividing the rectangle into squares $1/c$ units on a side and then counting.

5. A rectangle has sides 2.2 and 3.3 in. Obtain its area by dividing it into small squares and counting.

6. A rectangle has sides 3.24 and 4.83 ft. Describe how to obtain the area by counting squares.

*7. The sides of a rectangle are a/b and c/b units, where a, b, and c are positive integers. Show how its area is obtained by counting squares.

10–3 DEFINITION OF RECTANGLE AREA

In the previous section we saw how to obtain the area of a rectangle by counting squares if the rectangle has sides whose lengths are integers. Remember that, so far, we have not explicitly defined what area is, but have used only our previous experience.

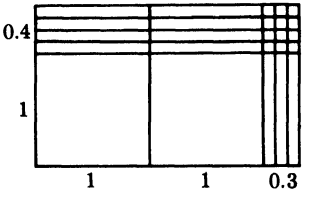


FIGURE 10–3

Suppose the lengths of the sides of a rectangle are l and w, where l and w are terminating decimals. If l and w end in 10ths, then we chop up the rectangle into squares 0.1 on a side. If l and w end in 100ths, we chop the rectangle into squares 0.01 on a side, and so on. In Fig. 10–3, we have started to chop up the rectangle, which is 1.4 by 2.3. Is it necessary to draw more of the small squares in order to count them?

It is rather obvious that the method of counting could be applied to a rectangle whose sides are any rational numbers.

Exercise 10–2

Find, by counting, the area of the rectangle whose sides are (a) 2.5 and 3.3, (b) 2.23 and 1.77, (c) 5/3 and 7/3, (d) 5/3 and 1/2, (e) 5/3 and 3/4.

In all our problems so far, we have found, by counting squares, that the area of the rectangle is lw.

Now suppose that at least one of the sides of the rectangle is not a rational number, that is, this side and the unit are not commensurable. To be definite, let us consider a rectangle whose length is $\sqrt{2}$ and whose width is 1. (Remember that $\sqrt{2}$ is not rational, and that *approximately* $\sqrt{2} = 1.4142$.) Examination of Fig. 10–4 shows that the area of the rectangle is greater than 1.4 and less than 1.5. We may see this by counting squares. If we were to subdivide the length into 100ths, we would also see that the area is greater than 1.41 and less than 1.42. As we approximate to $\sqrt{2}$ more and more closely by rational numbers, for example, by 1.4, 1.41, 1.414, 1.4142,..., we will get rectangles whose areas are ever closer to $\sqrt{2}$.

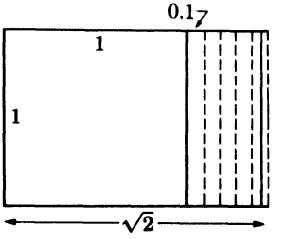


FIGURE 10–4

Therefore, the situation in which we find ourselves is as follows. If the sides of the rectangle are rational numbers, we can get the area by counting. If the sides are not rational, then simple counting does not give us the area. In this latter case, we must approximate to the sides by rational numbers and consider the *limit** of the areas of the rectangles

* The word "limit" has a precise mathematical sense, which need not concern us now. Roughly, the idea is that the areas of the approximating rectangles get close to a number that we call the *area* of the given rectangle.

whose sides have rational numbers for lengths. It is possible to prove that this limit exists and is equal to the product lw.

You see that to define the area of a rectangle by starting with counting may be quite troublesome. Therefore, in this book we do not give a detailed definition of area of a rectangle, but only state the following theorem, which tells us how to calculate the area.

Theorem 10–1. *The area of a rectangle whose sides have lengths l and w is lw.*

Exercise Group 10–3

1. A square has side x yards. What is its area in square feet?

2. What is the area of a rectangle $3\sqrt{2}$ units long and $2\sqrt{2}$ units wide?

10–4 AREA OF POLYGONS

So far, the only polygons whose areas we have studied are rectangles. In discussing areas of other polygons, and in particular triangles, perhaps we should define area in terms of some kind of counting process. If we did, we would be forced to concern ourselves with limits, so we will determine areas of polygons by a method used by the Greek geometers. Our basic formula for the area of a rectangle will enable us to find areas of triangles, and other figures, without any counting of squares.

Remember that although the Greeks used numbers in commerce, they had no numbers other than integers in their geometry. They did not compute numerical measures of angles, lines, or areas. Just as in a primitive society a hunter might compare his son's height with that of the son of his friend by marking his spear, so the Greeks devised a method for comparing the sizes of polygons. Of course, the Greeks had no word in their geometry corresponding to our term "area," since area is a *number*. Instead of saying that two geometric figures had *equal areas*, they merely remarked that the figures were *equal*. The Greeks showed that two polygons were *equal* by cutting them up into congruent triangles. Note that the word *equal* meant *the same size* to the Greeks. This is different

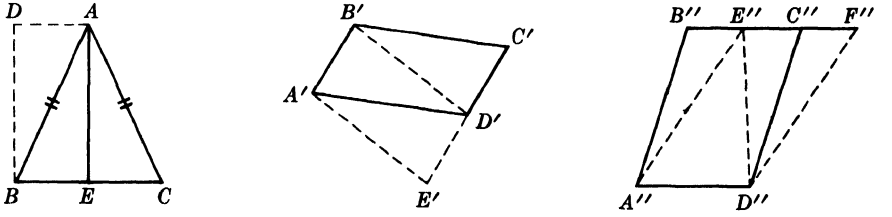


FIGURE 10–5

from the word's meaning in modern mathematics, where *equal* means *identical*. Figure 10–5(a) illustrates what is meant. The Greek geometers would call rectangle $ADBE$ and isosceles triangle ABC *equal* because $\triangle ABE \cong \triangle ABE$ and $\triangle ABD \cong \triangle ACE$.

Exercise 10–4

In Fig. 10–5(b), $A'B'C'D'$ is a parallelogram, and $A'B' \cong D'E'$. Show that $A'B'D'E'$ and $A'B'C'D'$ are *equal* in the sense of Greek geometry. Do the same for parallelograms $A''B''C''D''$ and $A''E''F''D''$ in Fig. 10–5(c).

We use the methods of the Greek geometers to develop area formulas. But before doing so, we need a few theorems about areas of geometric figures. These theorems can be proved, using properties of numbers and our geometric postulates, but their proofs are long and difficult. Accordingly, we treat them as postulates even though proofs can be given.

Theorem 10–2. *Congruent triangles have equal areas.*

Theorem 10–3. *If two simple polygons have one side and only the points of this side in common, and if, moreover, they have no interior points in common, then the simple polygon formed by the sides that remain after the common side is deleted has an area equal to the sum of the areas of the original polygons.*

Theorem 10–2 is intuitively obvious, since any method of counting squares to get the area of one triangle can be duplicated for the other triangle. Draw a figure to illustrate this. Figure 10–6 will make the meaning of Theorem 10–3 clear. The theorem tells us that the area of polygon $ABCDGFE$ is the sum of the areas of polygons $ABCDE$ and $DEFG$.

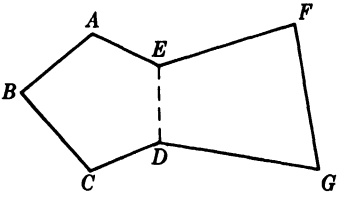


FIGURE 10–6

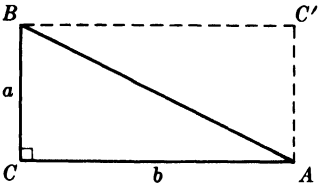


FIGURE 10–7

With these two theorems and our rectangle formula, we can establish theorems on the areas of triangles and parallelograms.

Theorem 10–4. *If a and b are the lengths of the legs of a right triangle, then the area of the triangle is $\frac{1}{2}ab$.*

Proof. Let the right triangle be $\triangle ABC$, as in Fig. 10–7, and let C' be chosen as indicated in the figure. (How is C' chosen?)

STATEMENTS	REASONS
1. $\triangle ABC \cong \triangle BAC'$.	1. Why?
2. Area $(\triangle ABC)$ = area $(\triangle ABC')$.	2. Theorem 10–2.
3. Area $(\triangle ABC)$ + area $(\triangle ABC')$ = area $(\square AC'BC) = ab$.	3. Theorem 10–3 and formula for area of rectangle.
4. 2 area $(\triangle ABC) = ab$.	4. Statements 2 and 3.
5. Area $(\triangle ABC) = \frac{1}{2}ab$.	5. Statement 4.

In a parallelogram, if we call the length of one side the *base*, then the corresponding *altitude* (or *height*) is the perpendicular distance from this side to the line containing the opposite side. From Theorem 10–4, we can deduce the area of a parallelogram.

Theorem 10–5. *The area of a parallelogram is equal to the product of its base and its height.*

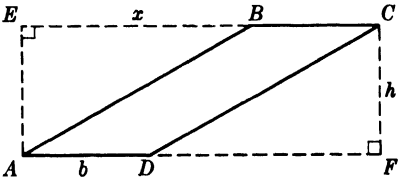


FIGURE 10–8

Proof. Let the parallelogram be $ABCD$, as in Fig. 10–8, its base $b = \overline{AD}$, and its height h. Consider, as in the figure, the rectangle $AECF$. (How can the vertices E and F be found?) Let $\overline{EB} = x$.

STATEMENTS	REASONS
1. $\triangle AEB \cong \triangle CFD$.	1. Why?
2. Area $(\triangle AEB)$ = area $(\triangle CFD) = \frac{1}{2}hx$.	2. Why?
3. Area $(\square AECF)$ = area $(\triangle AEB)$ + area $(\square ABCD)$ + area $(\triangle CFD)$.	3. Why?
4. $\overline{AF} = \overline{EC} = b + x$.	4. Why?
5. $h(b + x) = \frac{1}{2}hx +$ area $(\square ABCD) + \frac{1}{2}hx$.	5. Statements 3 and 4.
6. $hb + hx$ = area $(\square ABCD) + hx$.	6. Statement 5 and algebra.
7. Area $(\square ABCD) = bh$.	7. Statement 6 and algebra.

We can now quite easily derive the familiar formula, $A = \frac{1}{2}b \cdot h$, for computing the area of a triangle. Here, b (base) is the length of a side of a triangle, and h (height) is the perpendicular distance from the opposite vertex to the line containing the base.

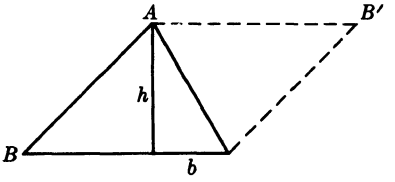


FIGURE 10–9

Theorem 10–6. *The area of a triangle is given by the formula* $A = \frac{1}{2}b \cdot h$ (Fig. 10–9).

The proof is left to the student.

From these last two theorems, we may conclude that parallelograms (or triangles) with equal bases and heights have equal areas. The area formula for triangles provides us with a tool for determining areas of polygons in general, for the area of any polygon can be computed as the sum of the areas of certain triangles.

Exercise Group 10–5

1. A triangle has an area of 30 and a base of 6. Determine its height.

2. Two adjacent sides of a parallelogram have lengths of 12 and 8 in. These sides determine a 30° angle. Compute the area of the parallelogram.

3. A triangle has area 50 and altitude 8. What is the length of its base?

4. A rhombus has one angle of 45° and is 2 ft on a side. Find its area.

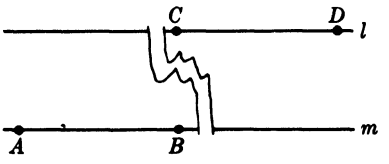


FIGURE 10–10

5. In Fig. 10–10, l and m are parallel and 3 ft apart. $\overline{AB} = \overline{CD} = 5$ ft. If the distance from B to C is 1 mi, what is the area of the quadrilateral $ABDC$?

6. What is the area of $\square ABCD$ if the sides are as shown in Fig. 10–11 and $BD \perp AD$?

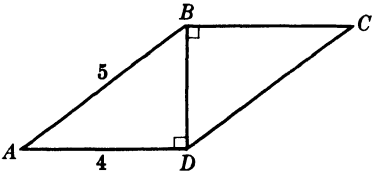


FIGURE 10–11

7. The area of a triangle is A, and its base is x. Find the altitude in terms of A and x.

8. The base of a parallelogram can be expressed in terms of its area and height. What is this formula?

9. Two triangles have the same side for base. The height of one triangle exceeds that of the other by 4 in. The areas differ by 40 in^2. What is the length of the common base?

10. The midpoints of the sides of a parallelogram determine a polygon. Compare the area of this polygon to the area of the original parallelogram.

*11. The midpoints A_1, B_1, and C_1 of the sides of $\triangle ABC$ are joined to form a second triangle. Consider also the midpoints A_2, B_2, and C_2 of the sides of triangle $A_1B_1C_1$. This process is continued. Compare the area of $\triangle A_5B_5C_5$ with that of $\triangle ABC$.

*12. Triangles ABC and $A'B'C'$ are similar. The lengths of two corresponding sides are s and s', where $s = 5s'$. If A and A' represent the respective areas of the triangles, prove that $A = 25A'$. Generalize this problem, and establish the theorem that *if two triangles are similar, the ratio of their areas is equal to the ratio of the squares of the lengths of two corresponding sides.*

13. $\triangle ABC$ is similar to $\triangle A'B'C'$, and $AB = 2A'B'$. How do their areas compare? Draw a figure.

*14. Plans for a house are drawn, so that $\frac{1}{4}$ in. on the plan corresponds to 1 ft in the house. If the plans cover an area of 105 in^2, how many square feet in the house?

15. Two equilateral triangles have sides of 8 and 10 in., respectively. How do their areas compare?

16. A rhombus has sides 10 in. long. If one angle is 30°, what is its area? if 60°? if 120°?

17. Find the area of an isosceles right triangle if one leg is 20 ft long.

18. In Fig. 10–12, $AD \parallel BC$. Compute the area of the figure.

*19. Generalize Exercise 18, designating the lengths of the parallel sides, or

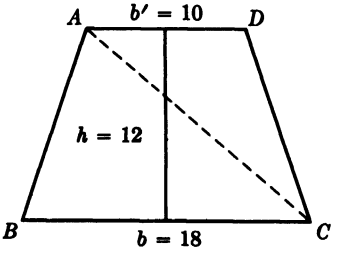


FIGURE 10–12

bases, by b and b' and the height by h. Of course, you are developing a formula for the area of a trapezoid.

20. Besides the proof suggested by the dotted line in Fig. 10–12, Fig. 10–13 also suggests a proof. Here, it is assumed that $b > b'$. Make a proof.

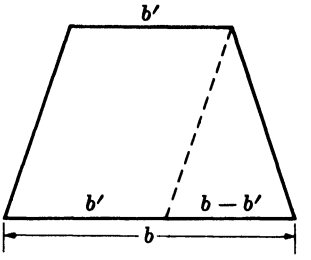


FIGURE 10–13

21. The bases of a trapezoid are 3 and 12, and the altitude is 6. What is its area?

22. The area of a trapezoid of height 6 in. is 39 in^2. If one base is 6 in. what is the other?

23. One base of a trapezoid is twice as long as the other. If the area is 30 ft^2 and the altitude is 5 ft, what are the two bases?

24. An equilateral triangle has sides 18 in. long. Find its area in square feet.

25. An equilateral triangle has an area of $(9\sqrt{3})/4$. How long is its side?

26. In Fig. 10–14, $ABCD$ is a parallelogram. Prove that the four small

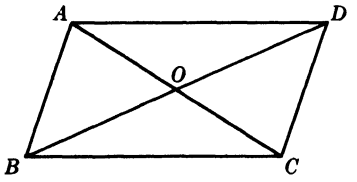


FIGURE 10–14

triangles, $\triangle AOB$, $\triangle BOC$, etc., all have equal areas.

27. The areas of two similar triangles are 360 and 250 square units. If a side of the first triangle is 8 units, how long is the corresponding side of the second triangle?

*28. Prove that the ratio of the areas of similar polygons is equal to the square of the ratio of corresponding sides.

29. Find the area of a regular hexagon 6 in. on a side. What will be the area of a regular hexagon whose side is 8 in.?

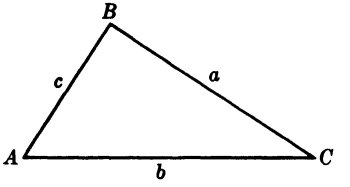


FIGURE 10–15

*30. If a, b, and c are the lengths of the sides of any triangle (Fig. 10–15), and $s = \frac{1}{2}(a + b + c)$, then the area is given by

$$A = \sqrt{s(s - a)(s - b)(s - c)}.$$

This is "Heron's formula," named after an Alexandrian Greek by that name of about 100 B.C., possibly later. Show that this formula is correct.

31. The sides of a triangle are 8, 11, and 13 in. Find its area.

32. The sides of a triangle are 17, 25, and 26 ft. Find its area. Find the height to the 25-ft side.

33. Find the area of an isosceles triangle if the base is 12 in. and the sides 10 in.

34. Find the area of an equilateral triangle if one side is 6 in.

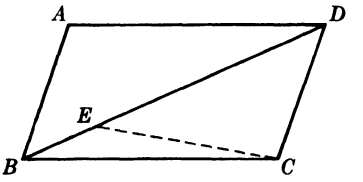


FIGURE 10–16

35. If, in Fig. 10–16, $ABCD$ is a parallelogram, with $\overline{BE} = 2$ and $\overline{ED} = 6$, what is the number

$$\frac{\text{Area } (BEC)}{\text{Area } (ABCD)}?$$

36. A square has a diagonal of length 12 in. Compute the area of the square. Determine a formula for the area of a square in terms of the length d of its diagonal.

37. The diagonals of a rhombus are 10 in. and 6 in., respectively. What is its area? Generalize this problem by formulating and proving a theorem.

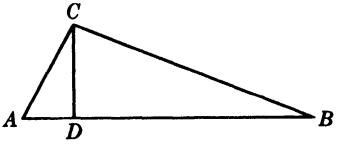


FIGURE 10–17

38. In Fig. 10–17, $\angle C$ is right, $CD \perp AB$, $\overline{AC} = 5$, and $\overline{CB} = 12$. Compute the areas of triangles ABC, ACD, and BCD. Check your work by verifying that the sum of the area of the latter two triangles is equal to the area of $\triangle ABC$.

10–5 AREA PROOF OF THE PYTHAGOREAN THEOREM

The Greek geometers thought of the Pythagorean theorem, not as a numerical relationship between lengths of the sides of a right triangle, but rather as a relationship between areas. Viewed in this light, the theorem may be stated as:

Theorem 10–7. *For every right triangle, it is true that the square on the hypotenuse has an area equal to the sum of the areas of the squares on the other two sides* (see Fig. 10–18).

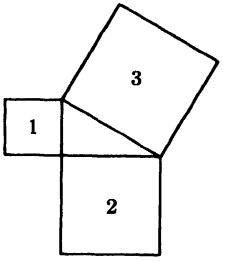


FIGURE 10–18

Area 1 + area 2 = area 3.

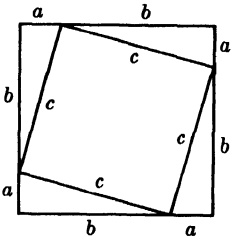


FIGURE 10–19

Probably more different proofs have been given for the Pythagorean theorem than for any other theorem in mathematics. Our earlier proof did not refer to area at all. It was essentially an "algebraic style" proof. We give a proof below that is a kind of algebraic and area proof. If the sides of the right triangle are a, b, and c, consider Fig. 10–19.

Proof. Evidently, the area of the large square, of side $a + b$, is equal to the area of the square of side c plus the areas of the four triangles. That is,

$$(a + b)^2 = c^2 + 4(\tfrac{1}{2}ab).$$

But $(a + b)^2 = a^2 + 2ab + b^2$, so that we get

$$a^2 + 2ab + b^2 = c^2 + 2ab.$$

From algebra then,

$$a^2 + b^2 = c^2.$$

There are many more proofs of this theorem. Your teacher can show you some if you are interested.

Exercise Group 10–6

1. Triangles ABC and $A'B'C'$ are similar. $\overline{AB} = 3$, $\overline{A'B'} = 5$, and area $(ABC) = 18$. Compute area $(A'B'C')$.

2. Determine the area of an equilateral triangle if the length of one side is 10.

3. Show that the area of an equilateral triangle is given by the formula $A = (s^2/4) \cdot \sqrt{3}$, where s is the length of a side.

4. Develop a formula for the area of a regular hexagon if each side has length c.

5. The lengths of two adjacent sides of a parallelogram are 12 in. and 8 in. The included angle is 45°. Determine the area of the parallelogram.

6. The altitude of a rectangle is 12 ft. A diagonal has length 20 ft. Compute the area of the rectangle.

*7. A square and a regular hexagon have equal perimeters. Find the ratio of their areas.

*8. It is said that the Hindu mathematician Bhaskara, born about 1100 A.D., discovered a proof of the Pythagorean theorem and presented the figure like that in Fig. 10–20 with the single word, "Behold." Give a proof. In Fig. 10–20 $DF \parallel AC$, and $EG \parallel BC$. [*Hint:* $\overline{FC} = a - b$.]

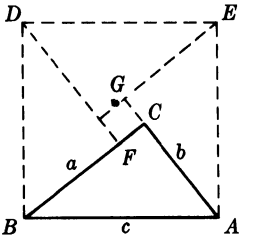


FIGURE 10–20

*10–6 CONSTRUCTION PROBLEMS

There are many RC constructions related to area concepts. It is interesting to note that it is possible to construct a triangle whose area is equal to the area of any given polygon. We indicate in Fig. 10–21 a procedure for constructing a triangle equal in area to a quadrilateral. DE has been drawn parallel to diagonal AC. Show that area (ABE) = area $(ABCD)$. [*Hint:* Show that area (ADC) = area (AEC).]

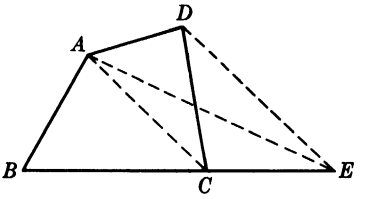


FIGURE 10–21

Exercise Group 10–7

1. Construct a triangle whose area is equal to that of pentagon $ABCDE$ (Fig. 10–22). Use the method employed above to construct a triangle equal in area to a given quadrilateral.

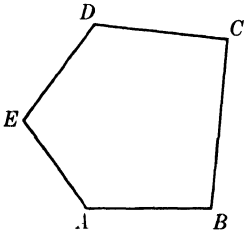


FIGURE 10–22

2. Show how to construct a square whose area is equal to the sum of the areas of two given squares.

3. Solve Exercise 2 with "sum" replaced by "difference."

4. Construct a square whose area is $\frac{3}{4}$ of the area of a given square.

5. Construct a square whose area is equal to the area of a given rectangle.

6. Construct a rectangle whose area is equal to the area of a given triangle.

7. Construct a square whose area is the same as the area of a given triangle.

8. Construct a triangle similar to a given triangle and having exactly twice the area of the given triangle.

9. Show how to construct a polygon of $n - 1$ sides whose area is the same as that of an n-gon. Assume $n > 3$.

*10. Prove that if P is any point in the interior of an equilateral triangle (Fig. 10–23), the sum of the lengths of the perpendiculars from P to the sides of the triangle is equal to the length of an altitude of the triangle. [*Hint:* Consider the areas of triangles APB, APC, and BPC.]

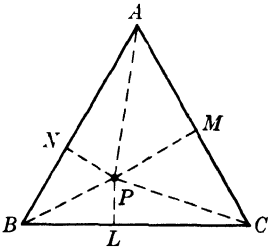


FIGURE 10–23

REVIEW OF CHAPTER 10

1. In this chapter we have studied methods for finding the areas of rectangles, triangles, and polygons. All area formulas were derived from the formula for the area of a rectangle.

Area $\square$ = length · width = lw.
Area $\square$ = base · altitude = bh.
Area $\triangle$ = $\frac{1}{2}$ base · altitude = $\frac{1}{2}bh$.
Area $\trapezoid$ = $\frac{1}{2}$ altitude · sum of bases = $h(b + b')/2$.

2. Review how the above area formulas were obtained.

3. Areas of similar figures are proportional to the squares of corresponding sides.

Review Exercises

1. $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. If area $(A_2B_2C_2) = 72$, and $\overline{A_1B_1}/\overline{A_2B_2} = \frac{3}{4}$, find area $(A_1B_1C_1)$.

2. Prove that a median of a triangle divides the triangle into two equal areas.

*3. Prove that the medians divide a triangle into six triangles of equal areas (Fig. 10–24).

4. Prove that the area of a square is half the area of a square whose side is the diagonal of the given square.

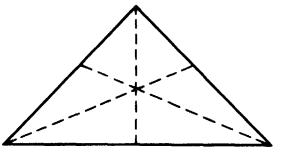


FIGURE 10–24

5. Two similar triangles have areas of 108 and 48 ft^2. The first has altitude $7\frac{1}{2}$ ft; find the altitude of the second.

6. Prove that the area of a trapezoid is equal to the altitude times the length of the segment joining the midpoints of the nonparallel sides.

7. A triangle has one angle of 30°. Prove that the area is $\frac{1}{4}$ of the product of the lengths of the sides adjacent to the 30° angle.

8. In $\triangle ABC$, if D is the foot of the perpendicular from C to AB, prove that $\overline{AC}^2 - \overline{BC}^2 = \overline{AD}^2 - \overline{BD}^2$.

9. The hypotenuse of a right triangle is 41 ft. One leg is 40 ft. Find the length of the other leg.

10. The median of a trapezoid is 8 in., and one of the bases is 12 in. How long is the other base? If the area is $\frac{1}{2}$ ft^2, what is the altitude?

11. In a right triangle, the altitude from the right angle is 14 in. From the foot of the altitude to a vertex is 8 in. How long is the hypotenuse?

Review Test

1. What is the area of a parallelogram with sides 10 and 8 and an included angle of 30°?

2. What is the area of a rhombus with side 10 if an angle is 45°?

3. Find the area of an isosceles right triangle whose legs are 20.

4. One of the bases of a trapezoid is 15, and its height is 6. Find the other base if the area is 75.

5. Two similar triangles, T and T', have two corresponding sides 6 and 15, respectively. What is the ratio of the area of T to the area of T'?

6. The length of a diagonal of a square is 15. Find the area.

7. Two similar polygons have areas of 250 and 160. If the length of a side of the first is 6, what is the length of the corresponding side of the second?

8. Two triangles are similar, and the lengths of the sides of the first triangle are 3, 5, and 6. If the length of the greatest side of the other triangle is 15, compute the lengths of the other sides.

9. If the legs of a right triangle are in the ratio of 5 to 4, find the ratio of the segments formed on the hypotenuse by the altitude to the hypotenuse.

10. If one leg of a right triangle has length 25, and the altitude to the hypotenuse is 7, what is the area of the triangle?

CHAPTER 11

CIRCLES AND REGULAR POLYGONS

11–1 INTRODUCTION

In this chapter we define what we mean by the *circumference* (or *length*) of a *circle* and then prove that the circumference C of a circle of radius r is given by the familiar formula, $C = 2\pi r$. This involves giving a definition of the number π. To do these things, we must learn a few facts about limits, because the study of circumference is made by approximating to the circle by polygons and considering polygons with a very large number of sides. It is therefore necessary, first, to prove some theorems about regular polygons.

11–2 CIRCLES AND REGULAR POLYGONS

Our study of the circle depends upon knowing about regular polygons inscribed in circles. We therefore study these matters first.

Definition 11–1. A regular polygon is *inscribed* in a circle if all the vertices of the polygon are on the circle. We also say that the circle *circumscribes the polygon.*

Exercise Group 11–1

1. With ruler and compass, inscribe a square in a given circle.

2. Using Exercise 1, show how to construct regular inscribed polygons of 8 sides; 16 sides; 32 sides.

3. With ruler and compass, inscribe a regular hexagon (6-sided polygon) in a given circle.

4. Using Exercise 3, show how to construct regular polygons of 3 sides; 12 sides; 24 sides; 48 sides.

Theorem 11–1. *There is exactly one circle circumscribing a given triangle.*

Proof. In Fig. 11–1, suppose that the triangle is $\triangle ABC$, and choose any two of the sides, say AB and BC. Let m and n be the perpendicular

bisectors of these sides. The lines m and n must intersect at a point O, for if m and n did not intersect, they would be parallel, CB would be perpendicular to m, and the points A, B, and C would be collinear. The point O is equidistant from A and B, and also equidistant from B and C. (Why?) Hence $\overline{OA} = \overline{OB} = \overline{OC}$, and the circle with center O and radius $\overline{OA}$ passes through A, B, and C. This proves that there is at least one circle through A, B, and C. To see that there can be only one, we observe that the center of such a circle must be equidistant from A, B, and C, and hence must lie on the perpendicular bisectors of AB and BC. These lines, m and n, intersect in only one point. This completes the proof.

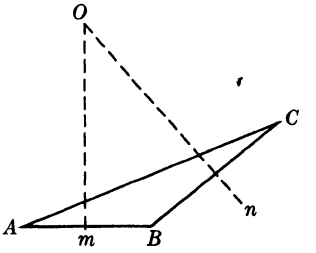


FIGURE 11–1

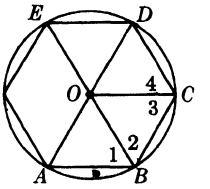


FIGURE 11–2

Theorem 11–2. *There is exactly one circle that circumscribes a regular polygon.*

Proof. In Fig. 11–2, suppose the polygon has successive vertices A, B, C, D,.... We show that the circle circumscribing $\triangle ABC$ actually circumscribes the polygon. Let O be the center of the circle circumscribing $\triangle ABC$, and consider the segments OA, OB, OC, OD, OE,.... By definition of circumscribing circle, $\overline{OA} = \overline{OB} = \overline{OC} = r$. We show that $\overline{OD} = r$. The same method can be used to prove that $\overline{OE} = \cdots = r$. Hence we need only show that $\overline{OD} = r$.

STATEMENTS	REASONS
1. $\angle 2 \cong \angle 3$.	1. Why?
2. $\angle ABC \cong \angle BCD$, and $AB \cong BC$.	2. The polygon is regular.
3. $\angle 1 \cong \angle 4$.	3. Addition and subtraction of angles postulate.
4. $\triangle OAB \cong \triangle ODC$.	4. S.A.S.
5. $OD \cong OA$.	5. Statement 4.
6. $\overline{OD} = r$.	6. Statement 5.

Exercise 11–2

Suppose that in Theorem 11–1 we had considered the perpendicular bisectors of AB and AC instead of AB and BC. Would the same point O have been found? What does the theorem say about the perpendicular bisectors of the sides of a triangle?

11–3 TANGENTS TO CIRCLES

To talk about circles *inscribed* in regular polygons, we need to know about tangents to circles.

Theorem 11–3. *If OP is a radius of a circle with center O, and a line l passes through P and is perpendicular to OP, then l has exactly one point in common with the circle.*

Proof. In Fig. 11–3, let A be any point on l different from P. Then $\triangle OPA$ is a right triangle, and $\overline{OA} > \overline{OP}$. (Why?) Hence A is outside the circle, and l intersects the circle only at the point P.

Definition 11–2. A line l is *tangent* to a circle if it has exactly one point in common with the circle; the point is called *the point of tangency.*

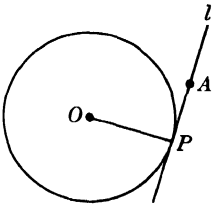


FIGURE 11–3

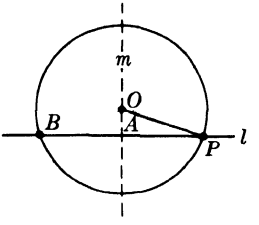


FIGURE 11–4

Theorem 11–4 Converse to Theorem 11–3. *A line tangent to a circle at a point P is perpendicular to the radius containing P.*

Proof. We will prove the contrapositive, that is, if the line is not perpendicular to the radius at P, then it is not tangent to the circle. Consider a circle (Fig. 11–4) with center O and a point P on the circle. Let l be a line through P, such that l is not perpendicular to OP.

STATEMENTS	REASONS
1. Let m be the line $\perp l$ passing through O. Let A be the intersection of m and l.	1. Why is there such a line?
2. Let B be the point on l on the opposite side of A from P, and such that $\overline{AB} = \overline{AP}$.	2. Why does such a point exist?
3. $\overline{OB} = \overline{OP}$.	3. Why? Several different arguments may be given.
4. B is on the circle and also on l.	4. Statements 2 and 3 and definition of a circle.
5. l intersects the circle twice, and hence l is not tangent to the circle. This completes the proof.	5. Statement 4 and definition of tangent.

Remark 1. In the above proof, the point A had to be inside the circle because $\overline{OA} < \overline{OP}$, by the Pythagorean theorem. The Pythagorean theorem shows that the perpendicular distance from a point to a line is less than the distance to any other point on the line.

Remark 2. An argument similar to the proof of Theorem 11–4, and using the Pythagorean theorem, can be made to show that a line can intersect a circle at most twice. Hence, a line will either be tangent to a circle, or intersect the circle in exactly two points, or fail to intersect the circle.

Exercise Group 11–3

1. Draw a circle, and construct (RC) the line tangent to the circle at a point of the circle.

*2. From a point outside a circle, construct a line tangent to the circle.

3. Prove that if A is a point not on a line l, and B and C are distinct points on l, with $AB \perp l$, then $\overline{AB} < \overline{AC}$.

Definition 11–3. A circle is said to be *inscribed* in a polygon, if it is tangent to each of the sides of the polygon. We also say that the polygon *circumscribes* the circle.

Theorem 11–5. *There is a circle inscribed in every regular polygon.*

STATEMENTS	REASONS
1. Let O be the center of the circle circumscribing the polygon (Fig. 11–5).	1. Theorem 11–2.
2. Let AB be a side of the polygon, and C the midpoint of AB.	2. AB has one, and only one, midpoint.
3. $OC \perp AB$.	3. Why?
4. The circle with center O and radius OC is tangent to AB at C.	4. Theorem 11–4.
5. Now let BD be a side of the polygon adjacent to AB, and E the midpoint of BD. Then $OE \cong OC$.	5. Why? This takes several steps.
6. Hence the circle with radius OC is also tangent to BD at E.	6. Statement 5 and the fact that $OE \perp BD$.

7. The above argument can be continued around the polygon to complete the proof.

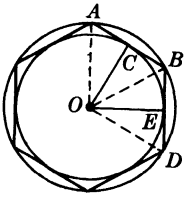


FIGURE 11–5

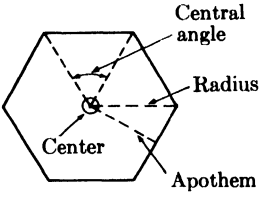


FIGURE 11–6

Exercise 11–4

Show that congruent chords in a circle are equidistant from the center of the circle.

Definition 11–4. The *center* of a regular polygon is the center of the inscribed or circumscribed circle. The *apothem* of a regular polygon is the length of the radius of the inscribed circle (that is, a number). The *radius* of a regular polygon is the radius of the circumscribed circle. A *central angle* of a regular polygon is an angle, with vertex at the center, whose sides contain adjacent vertices of the polygon. See Fig. 11–6.

Exercise Group 11–5

1. Inscribe a regular hexagon in a circle. Draw the circle inscribing the hexagon.

2. Using your experience in measuring angles in degrees (we will study this topic in the next chapter), how many degrees in a central angle of a regular pentagon? hexagon? n-gon?

3. What is the sum of the degree measures of the angles of a regular pentagon? hexagon? n-gon?

4. If an exterior angle of a regular polygon is as shown in Fig. 11–7, how many degrees in it if the polygon has 5 sides? 6 sides? n sides?

5. Prove that a central angle of a regular polygon is the supplement of any angle of the polygon.

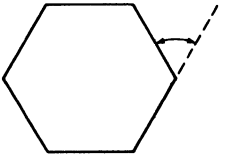


FIGURE 11–7

6. Show that an exterior angle of a regular polygon is congruent to a central angle.

7. In Definition 11–4, how do we know that the center of the inscribed polygon and the center of the circumscribed polygon are identical?

11–4 REGULAR POLYGONS AND SIMILARITY

All our proofs about area and circumference of circles depend upon the following theorem.

Theorem 11–6. *Two regular polygons are similar if, and only if, they have the same number of sides.*

Preliminary remark. Recall that two polygons are similar if there is a correspondence between their angles and sides, such that corresponding angles are congruent and corresponding sides are proportional. Thus, if the polygons are similar, they necessarily have the same number of sides, and the "only if" part of the theorem is obvious. Therefore, we have only to prove that if they have the same number of sides, they must be similar.

Proof. Suppose that two regular polygons have consecutive vertices $A, B, C, D, E, \ldots$ and $A', B', C', D', E', \ldots$, respectively, as in Fig. 11–8. Suppose also that the polygons are not similar. Since they are regular,

$$\frac{\overline{AB}}{\overline{A'B'}} = \frac{\overline{BC}}{\overline{B'C'}} = \frac{\overline{CD}}{\overline{C'D'}} = \frac{\overline{DE}}{\overline{D'E'}} = \cdots.$$

It must be, then, that $\angle A \ncong \angle A'$, $\angle B \ncong \angle B'$, etc. If $\angle A \ncong \angle A'$, then $\angle A > \angle A'$ or $\angle A < \angle A'$. Suppose that $\angle A < \angle A'$. Since the polygons are regular, $\angle B < \angle B'$, $\angle C < \angle C'$, etc. The exterior angle at A is greater than the exterior angle at A'. (Why?) This is true also for the exterior angles at B and B', C and C', etc. The sum of n exterior angles in polygon $ABCDE\ldots$ is two straight angles. Clearly then, the sum of n exterior angles in polygon $A'B'C'D'E'\ldots$ is not two straight angles. This contradiction completes the proof.

Remark. If we had studied measurement of angles, we could more easily have proved that $\angle AOB \cong \angle A'O'B'$, because the measure of these angles depends only upon the number of sides n. In degrees, the measure of $\angle AOB$ would be equal to $360/n$.

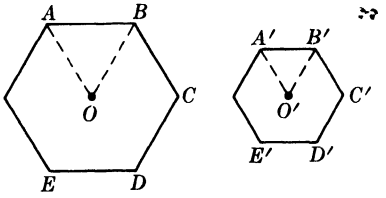


FIGURE 11–8

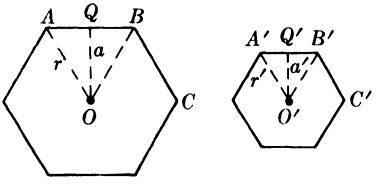


FIGURE 11–9

Theorem 11–7. *The perimeters of regular polygons, with the same number of sides, are proportional to their radii (or apothems).*

Proof. Suppose that two polygons (of n sides) are labeled as in Fig. 11–9, and have perimeters p and p'. We must show that $p/p' = r/r' = a/a'$.

STATEMENTS	REASONS
1. $\triangle AOQ \sim \triangle A'O'Q'$.	1. Why?
2. $r/r' = a/a' = \overline{AQ}/\overline{A'Q'}$.	2. Statement 1.
3. $p' = n \cdot \overline{A'B'} = 2n \cdot \overline{A'Q'}$, and $p = n \cdot \overline{AB} = 2n \cdot \overline{AQ}$.	3. Definition of perimeter and the fact that Q and Q' are midpoints.
4. $p/p' = \overline{AQ}/\overline{A'Q'}$.	4. Statement 3 and algebra.
5. $p/p' = r/r' = a/a'$.	5. Statements 2 and 4.

Theorem 11–8. *The area of a regular polygon is equal to one-half the product of apothem and perimeter; area* $= \frac{1}{2}a \cdot p$.

The proof is left as an exercise.

Theorem 11–9. *The areas of two regular polygons, with the same number of sides, are proportional to the squares of their radii (or the squares of their apothems, or the squares of their sides).*

The proof is left as an exercise.

Exercise Group 11–6

1. Two regular heptagons (7-gons) have sides 3 in. and 4 in., respectively. What is the ratio of their perimeters? of their areas?

2. A regular pentagon has twice the area of another regular pentagon. What is the ratio of their perimeters?

3. Find the radius and apothem of an equilateral triangle of side 2 in.

4. Two regular polygons of the same number of sides have perimeters of 28 and 42 in. The radius of the first polygon is x in. What is the radius of the second?

5. The apothems of two regular hexagons are 3 in. and 5 in. What is the ratio of their areas?

6. The area of a regular 17-gon is 16/9 times the area of a second 17-gon. What is the ratio of their perimeters? radii?

7. If a regular hexagon and an equilateral triangle are inscribed in a circle, the side of the hexagon is twice the apothem of the triangle. Show this.

8. Find the areas of the squares inscribed in and circumscribed about a circle of radius 1. Also find the perimeters of the squares.

9. Repeat Exercise 8, using hexagons instead of squares.

11–5 LIMITS

To investigate circumference and area of circles, it is necessary for us to discuss *limits*. Precise definitions relating to limits are difficult to grasp, and a careful study is first made in college calculus courses. For our purposes, only a little information about limits is required. All our statements about limits are statements about *numbers*. Therefore you may regard these statements as giving information about numbers that we will *assume* or postulate.

Definition 11–5. Suppose that for each positive integer n, there is associated in some way, a number x_n. The numbers $x_1, x_2, \ldots, x_n, \ldots$ then form an infinite sequence. The number x_n is called the nth term of the sequence. We shall indicate the sequence by $\{x_n\}$.

For example, let the nth term of a sequence be given by $x_n = 2n$. The positive integer 1 gives the first term $2 \cdot 1$. The positive integer 2 gives the second term $2 \cdot 2$. The third term is $2 \cdot 3$, and so on. The sequence given by $x_n = 2n$, then, would look like this:

Positive integers	→	1	2	3	4 . . .	n . . .
		↕	↕	↕	↕	↕
Terms of the sequence	→	2	4	6	8 . . .	$2n$ . . . .

Exercise Group 11–7

1. Write the first six terms of the sequence whose nth term is given by $x_n = 1/n$; by $x_n = \frac{1}{2}n$; by $x_n = \frac{1}{5}n$.

2. Write the 100th term for each of the sequences in Exercise 1. When n is very large, what number are the terms close to?

3. Write the first 10 terms of the sequence whose nth term is $x_n = (n + 1)/n$. Write the 100th term; the 1000th. What number are these terms close to when n is extremely large?

4. A sequence is given by $x_n = 2/(n + 3)$. Write the first six terms; the 100th term; the millionth term. Is there a number that the terms of the sequence are close to for large values of n?

5. Consider a regular polygon of n sides inscribed in a circle. Let p_n be the perimeter of the polygon. Does this define a sequence?

6. A sequence is given by $x_n = 2/\sqrt{n}$. Write the first six terms. Is there a number that the terms of the sequence are close to for large values of n?

7. Imagine $\sqrt{2}$ written as an infinite decimal, $\sqrt{2} = 1.4142\ldots$. Define a sequence x_n as follows: x_n = the decimal obtained by cutting off the decimal expansion of $\sqrt{2}$ after the nth decimal place. Then $x_1 = 1.4$, $x_2 = 1.41$, etc. What is x_4? x_5? To what number is x_n close when n is large?

We can now define the *limit* of a sequence, an idea that should be clear from the above exercises. We shall be content with expressing this idea in an intuitive way, but it should be mentioned that a more complicated definition is necessary in order to *prove* theorems about limits. We *assume* that the theorems we need concerning limits are true.

Definition 11–6. The number L is called the *limit* of the sequence $\{x_n\}$ if, for large values of n, the term x_n is necessarily arbitrarily close to L. We then say that L is *the limit of the sequence of terms* x_n and write "$L = \lim \{x_n\}$."

Exercise Group 11–8

In each of the following problems, the nth term of a sequence is given. You are to decide by inspection what the limit of the sequence is. Note that a sequence does not have to possess a limit.

1. $x_n = 1/n$.
2. $x_n = 1/10n$.
3. $x_n = (n + 1)/n$.
4. $x_n = \sqrt{n}$.
5. $x_n = 5$.
6. $x_n = 2n/(n + 1)$.
7. $x_n = n + 2$.
*8. $x_n = (2n + 100)/(4n + 50)$.
*9. $x_n = (n + 1)/n^2$.
*10. $x_n = (n^2 + 5n + 6)/[2n(n + 2)]$.

When we discuss the circumference and area of circles, we will need the following two theorems on limits. We assume that these theorems are true, and thus treat them as postulates. But they are not postulates about our *geometry;* they are postulates about the *number system.*

Theorem 11–10. *Consider two sequences with nth terms* x_n *and* y_n, *respectively, and suppose that* $\lim \{x_n\} = X$ *and* $\lim \{y_n\} = Y$. *Then the sequence whose nth term is* $x_n y_n$ *has a limit, and* $\lim \{x_n y_n\} = XY$.

Theorem 11–11. *With the same notation as in Theorem* 11–10, *and the additional hypothesis that* y_n *and* Y *are not zero, then the sequence whose nth term is* x_n/y_n *has a limit, and* $\lim \{x_n/y_n\} = X/Y$.

Exercise Group 11–9

1. Verify Theorem 11–10 if $x_n = (2n + 1)/n$ and $y_n = n/(n + 1)$.
2. If $\lim \{x_n\} = 2$, what is $\lim \{(x_n)^2\}$?
3. If $\lim \{x_n\} = 3$ and $\lim \{y_n\} = 4$, what is $\lim \{x_n/y_n\}$? $\lim \{x_n y_n\}$? $\lim \{y_n/x_n\}$?
4. If x_n is "very close" to 5 and y_n is "very close" to 7, what is $x_n y_n$ close to? x_n/y_n? y_n/x_n?

11–6 CIRCUMFERENCE

From your everyday experience, you already have an idea of the perimeter or circumference of a circle. For example, if you were asked to find the circumference of a wheel, you might wrap a string around the wheel, then pull it out straight, and measure its length. But such a practical method is not adequate for geometry, since we cannot describe this process in our geometry. It is necessary that we first *define* circumference. The definition must satisfy our intuition about what we *feel* the circumference should be, and should be one that can readily be *used* to prove theorems.

Observe that we had no difficulty in defining the perimeter of a polygon, because the polygon consisted of a finite number of segments, each of which had a length. A circle contains no segment of a line, so we cannot define its perimeter so simply. To arrive at our definition, we need the following theorem, the truth of which we assume.

Theorem 11–12. *For a given circle, let* p_n *be the perimeter of a regular polygon of n sides inscribed in the circle. The numbers* p_n $(n = 3, 4, 5, \ldots)$ *form a sequence that has a limit.*

Remarks. The proof of this theorem involves two steps. First, it is shown that the sequence is increasing, that is, each term is less than the one that follows it. Then it is shown that the sequence is *bounded,* that

is, the terms do not become indefinitely large. It then follows, as a fundamental property of real numbers, that such a sequence has a limit.

It is natural to ask how we know that regular polygons with n sides exist for all positive integers $n \geqq 3$. We do know that there are regular polygons with the number of sides equal to 2^2, 2^3, 2^4, etc., for we can inscribe a square in the circle, and, by bisecting the central angles, we get a regular octagon, and then a regular 16-gon, and so on, as in Fig. 11–10.

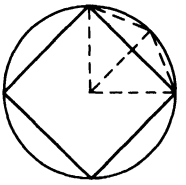


FIGURE 11–10

Therefore, the perimeters of these special regular polygons form a sequence $p_4, p_8, p_{16}, p_{32}, \ldots$, and by Theorem 11–12 and the definition of limit, this sequence has a limit.

We can now give a definition of circumference. Observe that in order to *define* the circumference of a circle, we are *necessarily forced* to use limits.

Definition 11–7. The *circumference* C of a circle is the limit of the sequence of perimeters p_n of regular inscribed polygons. $C = \lim \{p_n\}$.

Exercise Group 11–10

1. For a given circle, show that the perimeter of an inscribed square is less than that of an inscribed octagon.

2. Find the perimeter of a square inscribed in a circle of radius 1. Hence show that the circumference of the circle is more than $4\sqrt{2}$.

3. By inscribing a regular hexagon in a circle of radius 1, show that the circumference of the circle is more than 6.

The fundamental result on the circumferences of circles is the following:

Theorem 11–13. *The circumferences of two circles are proportional to their radii (or to their diameters). That is, if the circles have circumferences C and C', radii r and r', and diameters d and d', then $C/C' = r/r' = d/d'$.*

Proof.

STATEMENTS	REASONS
1. $C/C' = \lim \{p_n\}/\lim \{p'_n\}$, where p_n, p'_n, are the perimeters of the regular inscribed polygons of n sides in the circles.	1. Definition 11–7.

2. $C/C' = \lim \{p_n/p'_n\}$.	2. Theorem 11–11 on limits.
3. $p_n/p'_n = r/r' = d/d'$.	3. Theorem 11–7.
4. $\lim \{p_n/p'_n\} = \lim \{r/r'\}$.	4. $p_n/p'_n = r/r'$ for all n.
5. $\lim \{r/r'\} = r/r'$.	5. r and r' are the same for all n.
6. $\lim \{p_n/p'_n\} = r/r' = d/d'$.	6. Statements 3, 4, and 5.
7. $C/C' = r/r' = d/d'$.	7. Statements 2 and 6.

Remark. Observe that this theorem about circumferences is basically a theorem about similar polygons.

Theorem 11–14. *If C is the circumference of a circle, and d is its diameter, the ratio C/d is the same for all circles.*

Proof. This theorem is really just a corollary to Theorem 11–13. But it is so important that it is given a place by itself. The proof is as follows. From Theorem 11–13, we have $C/C' = d/d'$. This proportion implies the proportion $C/d = C'/d'$. This last equality says that C/d is the same number for any two circles. This completes the proof.

Definition 11–8. The number C/d, which is the same for all circles, will be denoted by the Greek letter π (pi).

Corollary 11–14–1. $C = \pi d = 2\pi r$.

Remark. The definition of π does not immediately give its numerical value. So a problem remains—namely that of calculating π to any desired degree of accuracy. We examine later a bit of the history of this number and methods for calculating its value. At the moment, and for purposes of the exercises, we mention two approximate values commonly used for π, namely 22/7 and 3.1416.

Exercise Group 11–11

1. Find, approximately, the circumference of a circle whose radius is 7 in.; 2.131 in.; 49 ft.

2. Find, approximately, the radius of a circle whose circumference is 66 ft; 12.125 in.; 100.00 ft.

3. Give an exact expression for the area of a square whose perimeter is equal to the circumference of a circle of radius 1.

4. Suppose that a circular steel band surrounds the earth at the equator (radius = 4000 mi). If the band is made longer by inserting a piece 1 ft long, about how far will the band be from its previous position?

5. A tire on a car has a diameter of 26 in. If it revolves at 10 revolutions per second, what is the approximate speed of the car in miles per hour?

6. About how many revolutions per minute does a car wheel make (diameter = 28 in.) when the car is traveling at 60 mph?

7. The radius of one circle is three times the radius of a second. How do their circumferences compare?

11-7 COMPUTATION OF PI

There are many ways of computing π, some much more convenient than others. The obvious way is to compute the perimeters p_n of regular polygons with a large number of sides. Then π would be given approximately by p_n/d. This is a rather tedious process, and we use it here to get a very rough estimate of π.

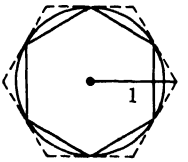


FIGURE 11-11

Consider a regular inscribed hexagon in a circle of radius 1, as shown in Fig. 11-11. The perimeter of the hexagon is 6. Increasing the number of sides will increase the perimeter. Hence $C > 6$, so that $\pi = C/d > 6/2 = 3$, and $\pi > 3$.

If we consider a regular circumscribed hexagon, we find that its perimeter is $4\sqrt{3}$, and since C is less than this perimeter,

$$\pi = \frac{C}{d} < \frac{4\sqrt{3}}{2} = 2\sqrt{3} < 3.465,$$

and $\pi < 3.465$.

In this way, we have obtained a very rough value for π. It lies somewhere between 3 and 3.5. By these same methods, Archimedes (3rd century B.C.) showed that π is between 3 1/7 and 3 10/71. The Babylonians used 3 as the value of π, and there is a verse in the Bible that implies that the writer believed $\pi = 3$.

By far the most convenient and rapid methods for calculating π are based on formulas derived in calculus.* We cannot take up these formulas here, but it is interesting to know that in the 17th century a Scot by the name of Shanks,† using one of these formulas, computed π to 707 decimal places. His reasons for doing this are at best obscure. More recently, a group working on one of the large electronic computers set up a program for the calculation of π on the computer and obtained, in

* A formula from calculus that can be used to calculate π is

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \cdots).$$

† One of your authors makes no claim to be a descendant of this misguided computer.

a few hours, π to more than 2000 decimal places! This demonstrates the extreme speed of modern computing machines. The corresponding job by hand would take hundreds of years. The correct value of π to 14 decimal places is 3.14159265358979.

The number π is an irrational number. A proof of this is not easy, and was given first by the German mathematician Johann Lambert in 1766.

Exercise Group 11–12

1. By considering inscribed and circumscribed squares, show that $2\sqrt{2} < \pi < 4$.

*2. By considering an inscribed regular octagon, show that $\pi > 3.06$.

*3. Find the perimeter of a regular 12-gon in a circle of radius 1. Hence find an approximate value for π. Can you continue to find the perimeter of a regular 24-gon?

4. What does the irrationality of π imply about its decimal representation?

11–8 AREA OF CIRCLES

In studying area we first obtained the area of a rectangle by counting squares. Areas of triangles may be obtained by forming parallelograms and observing that the triangle is half the area of the parallelograms. Areas of polygons were obtained by cutting up the polygons into triangles. To define what we mean by the area of a circle, it is necessary to use again the idea of limit, just as it was necessary to use limits in order to define circumference. We all have a fairly clear idea of what we *feel* the area of a circle should be, and we use this impression to define area. Since the area of the circle should be close to the area of the inscribed polygon with a large number of sides, we first need to know that the limit of the areas of inscribed polygons exists. This fact is explicitly stated as a theorem below, which we accept without proof.

Theorem 11–15. *Given a circle C, let S_n denote the area of an inscribed regular polygon of n sides; then* $\lim \{S_n\}$ *exists. In other words, there is a number S, such that the term S_n is arbitrarily close to S for large n.*

Remark 1. We know, from Section 11–6, that regular polygons exist with the number of sides equal to powers of 2. In using Theorem 11–15, we can consider only polygons that we know to exist.

Remark. 2. Theorem 11–15 is a theorem about numbers. It has little geometric content, so we need not feel that it is another geometric postulate. If we consider only polygons with 4, 8, 16, 32,... sides, it is easy to show that each area in our sequence S_4, S_8, S_{16}, S_{32},... is less than the one that follows. On the other hand, we can see that each of these polygons has an area less than the area of a circumscribed square. This

indicates that our sequence is not only increasing, but is "bounded above." It is a theorem about numbers that such a sequence has a limit.

Definition 11–9. The area S of a circle is the limit of the areas of inscribed regular polygons. That is, $S = \lim \{S_n\}$.

We can now prove the main theorem about the area of a circle. But because the definition of area uses the idea of limit, we must use some theorems about limits. In particular, we need to know the following fact about the apothems of the inscribed polygons.

Theorem 11–16. *Let a_n be the apothem of an inscribed regular polygon of n sides in a circle of radius r, then* $\lim \{a_n\} = r$.

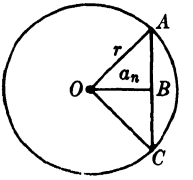


FIGURE 11–12

Proof. In Fig. 11–12, $\overline{AC}$ is the length of a side of the polygon, and is a small number when the number of sides of the polygon is large. By the Pythagorean theorem, $a_n = \sqrt{r^2 - \overline{AB}^2}$. Thus, when n is very large, $\overline{AB}^2$ is very small, and $r^2 - \overline{AB}^2$ is very near r^2. Hence a_n will be near r. This is exactly what we mean when we say that $\lim \{a_n\} = r$.

Theorem 11–17. *The area of a circle is equal to one-half the circumference times the radius.* $S = \frac{1}{2}$ (*circumference*) $\times$ (*radius*).

Proof.

STATEMENTS	REASONS
1. $S = \lim \{S_n\}$, where S_n is the area of the regular inscribed polygon of n sides.	1. Definition.
2. $S_n = \frac{1}{2}p_n \cdot a_n$, where p_n is the perimeter, and a_n is the apothem, of the polygon.	2. Theorem 11–8.
3. $\lim \{S_n\} = \frac{1}{2}(\lim \{p_n\}) \cdot (\lim \{a_n\})$.	3. Statement 2 and Theorem 11–10 on limits.
4. $\lim \{p_n\} = C$ = circumference.	4. Definition of circumference.
5. $\lim \{a_n\} = r$ = radius.	5. Theorem 11–16.
6. $\lim \{S_n\} = \frac{1}{2}C \cdot r$.	6. Statements 3, 4, and 5.
7. $S = \frac{1}{2}C \cdot r$.	7. Statements 1 and 6.

Corollary 11–17–1. *The area of a circle of radius r is πr^2.*

Corollary 11–17–2. *The areas of two circles are proportional to the squares of their radii (or diameters).*

The proofs are left to the student.

Exercise Group 11–13

1. Two circles have radii of 5 in. and 2.5 in. Find their areas. What is the ratio of their areas?

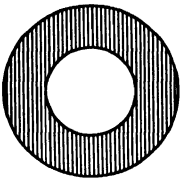


FIGURE 11–13

2. An annular ring (Fig. 11–13) has inner radius 2 ft and outer radius 4 ft. Find its area.

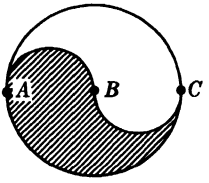


FIGURE 11–14

3. Figure 11–14 is composed of a circle and semicircles. If $\overline{AB} = \overline{BC} = 2$, B is the center of the circle, and AC is a diameter, find the area of the shaded portion.

4. Show that the area of the inscribed circle is half the area of the circumscribed circle of the square (Fig. 11–15).

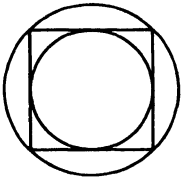


FIGURE 11–15

5. The area of a circle is 154 in^2. Find its radius. (Use $\pi = 22/7$.)

6. The area of a circle is 100 ft^2. Find its radius.

7. Semicircles are constructed on the sides of a right triangle, as in Fig. 11–16. Show that the area of the largest is the sum of the areas of the other two.

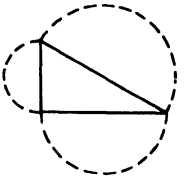


FIGURE 11–16

8. A circle has its circumference equal to its area. What is its radius?

9. A rectangular barn is 30 ft by 40 ft. A goat is tethered at one corner by a rope 40 ft long. Over what area can he graze? Over what area can he graze if the rope is 70 ft long?

10. The circumference of a circle is 50 ft. What is its area?

11. The area of a circle is 288 in^2. What is the area if the unit of length is a foot?

12. On Mars the standard unit of length is the *pfutth*, which is 0.88 as long as our foot. A Martian states that the area of a circle is 16 square *pfutths*. What is the area in square feet?

*13. Show that the area of a circle is the limit of the areas of circumscribed regular polygons.

14. The area of a circle is 49/4 times the area of another circle. What is the ratio of their radii?

REVIEW OF CHAPTER 11

The following definitions were given in this chapter. Be sure that you know them.

inscribed polygon (circle)
circumscribed polygon (circle)
tangent
center, radius, apothem of regular polygons
sequence, limit
circumference
π
area of a circle

The following is a list of the more important theorems of the chapter. Give complete statements of these theorems. Then try to give a proof for each. Do not refer to the text until you need help.

1. A regular polygon has a circumscribed circle.
2. A line tangent to a circle is perpendicular to a radius.
3. Converse of 2 (almost).
4. Regular polygons are similar if and only if they have the same number of sides.
5. The perimeters of similar polygons are proportional to their radii.
6. The area of a regular polygon is one half the product of apothem and perimeter.
7. The circumferences of two circles are proportional to their radii.
8. The area of a circle is equal to one half the circumference times the radius.

Miscellaneous Questions

1. Why is it important in this chapter to know that regular polygons of the same number of sides are similar?
2. Why is it important that circumference is proportional to the radius? (Tell why this enables us to define π.) This theorem depends on what theorems on similar polygons?
3. Tell why we must *define* circumference and area for circles.
4. In proving that area $= \pi r^2$, what theorems about polygons did we need? what theorems on limits?

CHAPTER 12

MEASUREMENT OF ANGLES AND ARCS

12–1 INTRODUCTION

In the first chapter of this book you used a protractor to measure representations of angles. Finally, at this late stage, we prove that the process of angle measurement can be made precise in our geometry. In order to discuss angle measurement, it is convenient first to define the measure of an arc of a circle. This definition is quite like the definition of circumference and really only makes precise the intuitive idea of *length of arc* that you already have. We needed limits to discuss circumference, and of course we also need limits in order to discuss arc length.

Once we know what is meant by arc length, it will be relatively easy to discuss measurement of angles. This discussion will go quite like Chapter 5 on measurement of length of segments.

One reason that a precise description of measurement of angles in degrees is difficult is that describing a unit angle is troublesome. (Is there an angle of one degree?) Furthermore, how do we know that it is possible to divide a degree into 60 congruent angles (minutes) and a minute into successively smaller pieces? This chapter shows how to resolve these difficulties. Incidentally, it should become quite clear that what we choose as the *unit angle* is quite arbitrary, and it is important for you to understand how measurements differ when different units are used. The idea that ties all this together is that of *length of circular arc.*

12–2 CIRCULAR ARCS

We have already described the interior and exterior of an angle (Chapter 3), and we use Definition 3–10 to describe an arc of a circle.

Definition 12–1. If $\angle AOB$ is a central angle of a circle, *the arc* $\overset{\frown}{AB}$ *of the circle* subtended by $\angle AOB$ is defined to be the set of points of the circle which are *interior* to $\angle AOB$ or are on the angle. The set of points

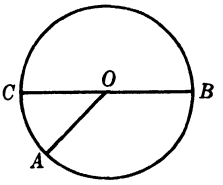


FIGURE 12–1

of the circle which are *exterior* to or on the angle is also called *an arc of the circle* (for example, the arc $\overset{\frown}{ABC}$ in Fig. 12–1). If the central angle is a straight angle (like $\angle COB$ in Fig. 12–1), then it has no interior, but the set of points of the circle which are on one side of CB or on CB is called a *semicircular arc* and is said to be subtended by $\angle COB$. Finally, when we speak simply of an *arc of a circle*, we mean an arc subtended by a central angle or the arc exterior to some central angle.

Exercise Group 12–1

1. In Fig. 12–2, show that if $\angle AOB > \angle COB$, with C in the interior of $\angle AOB$, then every point of the arc $\overset{\frown}{CB}$ is a point of the arc $\overset{\frown}{AB}$.

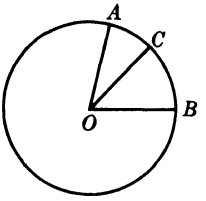


FIGURE 12–2

2. If the arc $\overset{\frown}{AB}$ is subtended by the central angle AOB, define the *midpoint of the arc*, and show that such a point exists.

3. Use the result of Exercise 2 to prove that there are infinitely many points on the arc.

4. In Fig. 12–3, the arc $\overset{\frown}{EF}$ is subtended by $\angle EOF$. Consider the segment EF and any point P on EF. Show that there is, on the ray OP, a point Q, which is on the arc $\overset{\frown}{EF}$. Show that if $P \neq P'$, then $Q \neq Q'$. How many points are there on the arc $\overset{\frown}{EF}$?

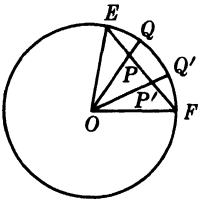
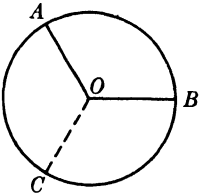


FIGURE 12–3



Midpoint

FIGURE 12–4

5. Referring to Fig. 12–4, suppose that an arc is exterior to a central angle AOB. Show how to define the *midpoint of the arc*. Why is $\angle AOC \cong \angle BOC$?

12–3 ARC LENGTH

In defining the length of a segment, we first had to choose a unit of length. So also in defining arc length, it will be understood that we have already chosen a segment of unit length. We shall also suppose that we have defined the midpoint of an arc as in Exercise Group 12–1.

We describe how to construct a regular inscribed polygonal path in an arc, and indeed, polygonal paths of 2, 4, 8, 16,... sides.

In Fig. 12–5, let AB be a circular arc, and let C be the midpoint of the arc. (See Exercise Group 12–1.) The segments AC and CB form a regular inscribed polygonal path from A to B, with two sides. There are now formed two congruent angles, $\angle AOC$ and $\angle COB$. Each of these angles subtends an arc. Let D and E be the midpoints of these arcs. We can now consider the regular inscribed polygonal path of four sides, $ADCEB$. We then consider each of the arcs $\overset{\frown}{AD}$, $\overset{\frown}{DC}$, $\overset{\frown}{CE}$, and $\overset{\frown}{EB}$ and their midpoints F, G, H, and I. Thus we obtain a regular inscribed path $AFDGCHEIB$ from A to B, having eight sides. Continuing in this manner, for each positive integer we obtain a regular inscribed polygonal path from A to B. That is to say, first we have a path of two sides, next a path of four sides, then a path of eight sides, and so on.

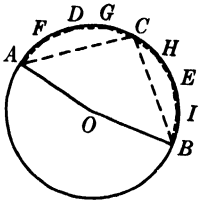


FIGURE 12–5

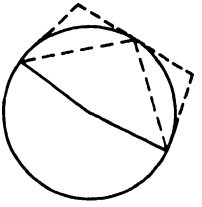


FIGURE 12–6

The following facts are clear from Fig. 12–6, and a complete proof is not difficult to give, although several steps are needed. We shall use these facts without proof. The *length* of the circumscribed path in Fig. 12–6 will be denoted by L. The length of a regular inscribed path of 2, 4, 8, 16,..., sides will be denoted by l_n, where n is used to denote the number of segments in the path.

(a) *For every* n, $l_n < L$; that is to say, the length of each regular inscribed path is less than the length of any circumscribed path.

(b) The length of each regular inscribed path of 2, 4, 8, 16,... sides is less than the one that follows, that is, $l_n < l_{2n}$.

(c) From (a) and (b) and properties of real numbers, it follows that $\lim \{l_n\}$ exists.

We are now able to define length of arc.

Definition 12–2. Given a circular arc, let l_n be the length of a regular inscribed polygonal path of 2, 4, 8, 16, 32,..., sides inscribed in the arc. The *length l of the arc* is defined to be the following limit: $l = \lim \{l_n\}$.

Exercise Group 12–2

1. A regular hexagon is inscribed in a circle of radius 1, as in Fig. 12–7. Show that the length of the arc ACB is greater than 2.

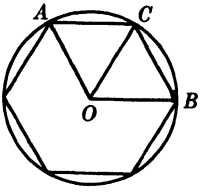


FIGURE 12–7

2. In a circle of radius 2, what is the length of arc subtended by a right angle? by a straight angle?

3. In Fig. 12–8, the central angles AOB and BOC are congruent, and $\angle AOC$ is a right angle. What is the length of the arc $\overset{\frown}{AB}$ if $\overline{AO} = 5$?

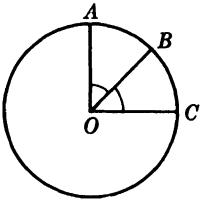


FIGURE 12–8

*4. Prove that the length of any arc is a positive number which is greater than the length of the line segment determined by the endpoints of the arc.

Theorem 12–1. *If, in a circle, two central angles are congruent, they subtend arcs of equal length.*

Proof. Because the angles are congruent, the inscribed polygons at any stage are also congruent. Thus the lengths of the corresponding inscribed polygonal paths are the same. Hence the arc lengths are the same.

Remark. In the same way it follows that the arcs exterior to the central angles also have the same length.

Having defined arc length, we now would like to prove some more theorems about arc lengths. Indeed, it would be possible to proceed at once to this task, but if we were to do so, our proofs would be quite troublesome. In constructing later proofs it will be most convenient for us to use a theorem (12–2 below) which tells us that in approximating to the circle by inscribed polygons, we need *not* restrict ourselves to regular polygons. The proof of this "helping" theorem is rather hard to make and involves greater knowledge of limits than is presupposed here. Nevertheless, the geometric significance of the theorem should be clear, and it should appeal to your intuition. We shall use this theorem without proof, so that actually we treat it like a postulate.

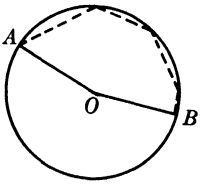


FIGURE 12–9

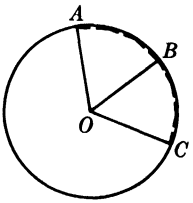


FIGURE 12–10

Theorem 12–2. *Let* $\{P_n\}$ *be any sequence of polygonal paths inscribed in the arc* $\overset{\frown}{AB}$ *of a circle (Fig.* 12–9*), and let* $\{l_n\}$ *be the sequence of lengths of these polygonal paths; let* s_n *be the length of the longest side of the path* P_n. *If* $\lim \{s_n\} = 0$, *then* $\lim \{l_n\} =$ *length* $\overset{\frown}{AB} = l$.

Remark. Evidently, when the length of the longest side of P_n is small, the number of sides, n, must be large. We can now use this theorem to prove some facts about arc length.

Theorem 12–3. *If* $\overset{\frown}{AB}$ *and* $\overset{\frown}{BC}$ *(Fig.* 12–10*) are adjacent arcs (that is, if they have only point B in common), then length* $\overset{\frown}{ABC}$ = *length* $\overset{\frown}{AB}$ + *length* $\overset{\frown}{BC}$.

Proof. The length of $\overset{\frown}{ABC}$ is, according to Theorem 12–2, the limit of lengths of inscribed polygonal paths from A to C. Let us always choose these paths so that B is one of the vertices of the path. The path from A to C can then be separated into a path from A to B and a path from B to C. Let l'_n be the length of the path from A to B, and l''_n be the length of the path from B to C. Then, if l_n is the length of the path from A to C, we have $l_n = l'_n + l''_n$.

Now we consider sequences of paths $\{P'_n\}$ and $\{P''_n\}$, with the sequences of lengths of longest sides approaching zero. Thus,

$$\left.\begin{aligned} \lim \{l_n\} &= \text{length } \overset{\frown}{ABC}, \\ \lim \{l'_n\} &= \text{length } \overset{\frown}{AB}, \\ \lim \{l''_n\} &= \text{length } \overset{\frown}{BC}. \end{aligned}\right\} \tag{12–1}$$

But from a theorem on limits, we have

$$\lim \{l'_n + l''_n\} = \lim \{l'_n\} + \lim \{l''_n\}. \tag{12–2}$$

From Eqs. (12–1) and (12–2), we see that

$$\text{length } \overset{\frown}{ABC} = \text{length } \overset{\frown}{AB} + \text{length } \overset{\frown}{BC}. \tag{12–3}$$

This completes the proof. The theorem might be stated briefly as "the length of the 'sum' is equal to the sum of the lengths."

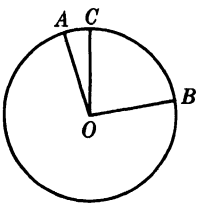


FIGURE 12–11

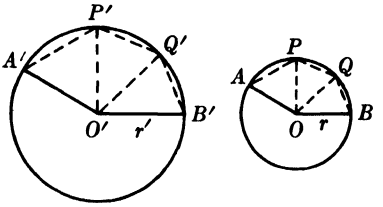


FIGURE 12–12

Theorem 12–4. *If, in a circle, one central angle is greater than another central angle, the greater angle subtends the longer arc.*

Proof. Since congruent angles subtend arcs of equal length, we may suppose that the central angles are as in Fig. 12–11, and that $\angle AOB > \angle COB$. The remainder of the proof (using Theorem 12–3) is left to the student.

Exercise Group 12–3

1. Apply Theorem 12–3 to the arcs subtended by a straight angle and two adjacent right angles.

2. State the form Theorem 12–3 would have for three adjacent arcs; for four adjacent arcs; for n adjacent arcs.

3. Two adjacent arcs in a circle of radius r have lengths $\pi r/3$ and $2\pi r/3$. What angle subtends the two arcs together?

4. Two arcs on a circle of radius r have lengths r and $\pi r/3$. Which is subtended by the greater central angle?

Theorem 12–5 Converse to Theorem 12–4. *If one arc of a circle is longer than a second arc of the circle, the longer arc is subtended by the greater central angle.*

The proof is left to the student.

Theorem 12–6. *Two central angles of a circle are congruent if and only if they subtend arcs of equal length.*

The proof is left to the student.

12–4 ARCS ON DIFFERENT CIRCLES

So far we have compared arcs on the same circle. We now compare the lengths of arcs on different circles. Before studying Theorem 12–7, try to discover the relation between the lengths of two arcs in different circles if they are subtended by congruent angles.

Theorem 12–7. *If $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ are arcs on two circles of radii r and r', respectively, and are subtended by congruent angles, then the arc lengths*

are proportional to the radii (*Fig.* 12–12). *That is, if* $\angle AOB \cong \angle A'O'B'$, *then*

$$\frac{\text{length } \overset{\frown}{AB}}{\text{length } \overset{\frown}{A'B'}} = \frac{r}{r'}, \quad \textit{or} \quad \frac{\text{length } \overset{\frown}{AB}}{r} = \frac{\text{length } \overset{\frown}{A'B'}}{r'}.$$

Proof. The proof depends upon three things: (1) the definition of arc length, (2) a theorem on limits, and (3) properties of similar triangles.

Suppose that $APQB$ is a polygonal path inscribed in the arc AB. Then there is a corresponding polygonal path $A'P'Q'B'$ inscribed in the arc $A'B'$ such that corresponding central angles are congruent, that is, $\angle AOP \cong \angle A'O'P'$, $\angle POQ \cong \angle P'O'Q'$, etc. (Why?) Hence, there is a one-to-one correspondence between polygonal paths inscribed in the two arcs. Let L_n be the length of a polygonal path of n sides inscribed in $\overset{\frown}{AB}$, and L'_n be the length of the corresponding polygonal path inscribed in $\overset{\frown}{A'B'}$. Because the corresponding central angles in the two polygons are congruent, for example $\angle AOP \cong \angle A'O'P'$, the triangles formed are similar. (Why?) Since $\overline{AP}/r = \overline{A'P'}/r'$, etc., we have $L_n/r = L'_n/r'$ and $L_n/L'_n = r/r'$.

From the definition of length of arc, length $\overset{\frown}{AB} = \lim \{L_n\}$ and length $\overset{\frown}{A'B'} = \lim \{L'_n\}$. From a theorem on limits,

$$\frac{\text{length } \overset{\frown}{AB}}{\text{length } \overset{\frown}{A'B'}} = \frac{\lim \{L_n\}}{\lim \{L'_n\}} = \lim \left\{\frac{L_n}{L'_n}\right\} = \frac{r}{r'}.$$

This completes the proof.

Exercise Group 12–4

1. A central angle B subtends an arc of length 5 in. on a circle of radius 4 in. A central angle $B' \cong \angle B$ subtends an arc of length 10 in. on another circle. What is the radius of the second circle?

2. A central angle subtends an arc of 2 in. on a circle of radius 5 in. What would be the length of an arc subtended by this same angle on a circle with the same center but with a radius of 2 ft?

3. The radii of two circles have a ratio of 3 to 5. If a central angle subtends an arc of length 2.454 on the first circle, what length arc would a central angle that is congruent to the first angle subtend on the second circle?

12–5 MORE ABOUT ARC LENGTH

In the preceding sections we learned that every arc has associated with it a number called its length. We shall not consider an exhaustive proof of the converse theorem: *For every positive number, less than the circumference, there is an arc having this number as its length.* The complete proof

of this theorem requires a rather difficult argument using limits, so we will limit our proof to certain special numbers. It will then be clear that the theorem is true.

In proving the theorem, we confine ourselves to arcs of length not more than one quarter of the circumference, that is to arcs which are subtended by acute angles or right angles.

Theorem 12–8. *Suppose that L is a positive number less than or equal to one fourth of the circumference of a circle of radius r ($L \leqq \pi r/2$).* Then there is an arc of the circle whose length is L.*

"Proof." If $L = \pi r/2$, then the required arc is subtended by a right angle. If $L = \frac{1}{2} \cdot \pi r/2$, then bisecting a central right angle will give two arcs $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, as in Fig. 12–13, of length L. If $L = \frac{3}{4} \cdot \pi r/2$, then the arc AD of Fig. 12–13 has length L, where OD bisects $\angle BOC$. Similarly, if OE bisects $\angle DOC$, then length $\overset{\frown}{AE} = \frac{5}{8} \cdot \pi r/2$.

Thus we see that if L is a certain special fractional part of $\pi r/2$, then there is an arc of length L. (Try to describe these fractions.) The fractions for which arcs are easy to construct are 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16,.... In other words, the arc is easy to construct if its length is given simply in terms of halves, quarters, eighths, etc., of a quarter of the circumference.

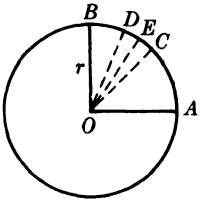


FIGURE 12–13

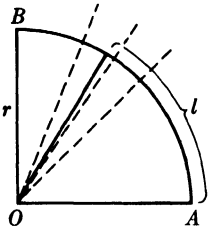


FIGURE 12–14

You are familiar with measuring lengths with a ruler whose scale is divided into halves, quarters, eighths, sixteenths, etc. Hence, if the number L is not exactly expressible in halves, quarters, eighths, etc., we can still construct an arc whose length is very close to L.

In Fig. 12–14, L appears to be a little greater than $\frac{5}{8} \cdot \pi r/2$. It is plausible therefore, that for every $L < \pi r/2$, there is exactly one arc AP, as in the figure, with length $\overset{\frown}{AP} = L$.

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* Since the circumference of a circle is $2\pi r$, it follows that one fourth of the circumference is $\frac{1}{4} \times 2\pi r$, or $\pi r/2$.

Exercise Group 12–5

1. In this section we gave a list of fractions whose associated arcs are easy to construct. The list is computed according to a definite rule. Continue the list to 20 more terms.

2. On a circle of radius r, construct arcs of length $7/8 \cdot \pi r/2$; $9/16 \cdot \pi r/2$; $7/4 \cdot \pi r/2$.

3. Suppose that there is an arc of length 3 on a circle of radius 2 in. What part (as a decimal) of $\pi r/2$ is 3? Show how to approximate this decimal by a sum of halves, quarters, eighths, etc. Use the approximation 3.14 for π, and then rework using 3.1416.

4. An arc of length r on a circle of radius r is what part of the circumference? of one quarter of the circumference?

5. On a circle of radius 32 in. is an arc of length 22 in. What part of a right angle does it subtend? Use $\pi = 22/7$. Give a construction approximating this arc.

12–6 MEASUREMENT OF ANGLES

In the previous section we have in effect been measuring angles by measuring arcs. In fact we have expressed the arc length as a fraction of $\pi r/2$ that is as a fraction of one quarter of the circumference. In doing this we have measured the corresponding central angle in terms of the right angle as a unit of angle. In this section we shall define the measure of an angle in terms of other units.

In deciding how to measure angles, we must first define the unit. This involves choosing some angle whose measure is 1. The choice is quite arbitrary, but there are some restrictions based on convenience. If the unit angle is small, the measures of common angles will be large numbers; whereas if the unit angle is large, the measures of common angles will be small numbers. It turns out that the familiar degree is a convenient size unit for ordinary measurement, because the numbers involved are a "good" size. Obviously, other unit angles could be selected. A natural one would be obtained by dividing the circle into 100 arcs of equal length, or perhaps the right angle into 100 congruent angles.

How does it happen that we use that rather strange unit, the degree? We measure angles in degrees because the ancient Sumerians* (2000 B.C.

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* The Sumerians had a highly developed civilization in Mesopotamia, in the valleys of the Tigris and Euphrates rivers, more than 5000 years ago. Roughly contemporary with ancient Egypt, the Sumerian culture discovered much about arithmetic, geometry, and astronomy. We inherit many things from these gifted people. Their method of counting was based on 60 rather than our base of 10. Thus we divide the hour into 60 minutes, and the minute into 60 seconds. They also invented place value in number notation, although they had no "zero." Many of our words go back to Sumerian roots. For more Sumerian history, see *A History of Science*, by George Sarton, Harvard University Press, 1952, or your encyclopedia.

and earlier) did. They divided the circle into 360 "equal parts," perhaps because they thought at one time that the year was 360 days long!

Definition 12–3. On a circle of radius r, an arc of length $1/360 \times 2\pi r$ is subtended by a central angle of one degree (1°).

Remark 1. The definition of degree is independent of the radius r, by Theorem 12–7.

Remark 2. Any angle congruent to an angle of 1° is also an angle of 1°, by Theorem 12–6.

Exercise Group 12–6

1. Write out the proof of Remark 1.
2. Write out the proof of Remark 2.
3. Define "minute" and "second."
4. Define a new unit of angle, which we will call the *centile*, by dividing one quarter of the circumference into 100 equal arcs. Using this unit, what is the measure of a 30° angle? a 45° angle? a straight angle?
5. Divide the circumference of a circle into two arcs of equal length. What angle is subtended by each arc? Each arc is what part of the circumference? What is the measure in degrees of the central angle that is subtended by each arc?
6. Divide the circumference of a circle into four arcs of equal length. What kind of an angle is subtended by each of these arcs? Each arc is what part of the circumference? What is the measure in degrees of each central angle?
7. Consider an arc whose length is one sixth of the circumference. What is the measure in degrees of the central angle that is subtended by this arc?
8. Consider an arc whose length is one sixth of the length of a semicircular arc. What is the measure in degrees of the central angle that is subtended by this arc?
9. If a semicircular arc has length 8 in., and a second arc has length 2 in., what part is the length of the second arc of the first? What is the measure in degrees of the central angle that is subtended by the second arc?
10. On a circle of radius 4, what is the length of an arc subtended by an angle of 1°? of 2°? of 30°?

Definition 12–4. If, on a circle of radius r, a central angle subtends an arc of length L, then the measure of the angle in degrees is $L/\pi r \times 180$.

Remark 1. This definition agrees with Definition 12–3 for angles of 1°.

Remark 2. By Theorem 12–7, the measure of an angle in degrees is independent of the radius.

Remark 3. If M is the measure of the central angle in degrees, we have from Definition 12–4 that $L = M\pi r/180$.

Exercise Group 12–7

1. If $\angle AOB$ and $\angle BOC$ are adjacent angles of 1° each, show that the measure of $\angle AOC$ is 2°.

2. Show that a straight angle has measure 180°.

3. On a circle of radius 6, find the number of degrees in a central angle that subtends an arc of length 3; 6; π; $2\pi/3$.

4. On a circle of radius 9, find the length of an arc subtended by an angle of 30°; 180°; 60°; $(180/\pi)°$.

5. Show that bisecting a right angle gives an angle whose measure is 45°.

6. Prove Remark 3 under Definition 12–4.

12–7 COUNTING DEGREES AND MEASURING ANGLES

Measurement of angles is quite similar to the counting process which led to the idea of length of a segment. In this section we give a description of this process and find that it gives the same measure of an angle as is given by Definition 12–4. Detailed proofs will be omitted. The following theorem shows how counting applies.

Theorem 12–9. *Suppose that $\angle AOB$ is an acute angle, and we consider successive arcs subtending central angles of 1° on the circle from B to A so that the point A is a point of the $(n + 1)$st arc* (Fig. 12–15). *Then the measure of $\angle AOB$ in degrees is either equal to n, or $(n + 1)$, or is between n and $(n + 1)$; $n \leqq$ measure of $\angle AOB \leqq n + 1$.*

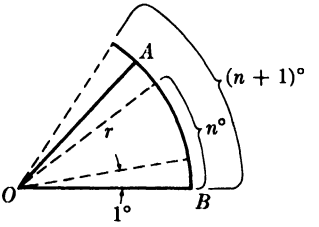


FIGURE 12–15

Proof. The length of the arc of $n°$ is $n \cdot \pi r/180$; the length of the arc of $(n + 1)°$ is $[(n + 1)\pi r]/180$. The length L of the arc $\overset{\frown}{AB}$ is then between these two numbers, by Theorem 12–5.

$$n \cdot \frac{\pi r}{180} \leqq L \leqq \frac{(n + 1)\pi r}{180}. \tag{12–4}$$

By Definition 12–4, the measure of $\angle AOB$ in degrees is $L \cdot 180/\pi r$. From Eq. (12–4), we see that

$$n \leqq L \cdot \frac{180}{\pi r} \leqq (n + 1),$$

and this is what we wished to prove.

Remark. The measure to the nearest minute could be obtained by counting in a similar way.

Consider an angle $A'O'B'$ (Fig. 12–16) that is congruent to $\angle AOB$ of Theorem 12–9. Consider also successive arcs subtending central angles of 1° on arc $B'A'$, as described for arc BA. If the point A is on the $(n + 1)$st arc, then the point A' is on the $(n + 1)$st arc. This can be shown using the addition and subtraction of angles postulate and the definition of "less than" and "greater than" for angles. This proves the following theorem:

Theorem 12–10. *If two angles are congruent, they have equal measures.*

The converse of Theorem 12–10 can be proved, using the fact that $\angle O > \angle O'$, or $\angle O \cong \angle O'$, or $\angle O < \angle O'$.

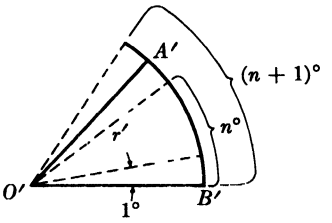


FIGURE 12-16

Exercise Group 12–8

Give your answers in terms of π.

1. If the radius of a circle is 14, what is the length of the arc subtended by an angle of 30°?

2. On a circle of radius 5 in. is an arc of length 6 in. What is the measure in degrees of the central angle subtending the arc? What if the radius is 20 in.? 6 in.?

3. A central angle of a circle subtends an arc of length equal to the radius. How many degrees in the angle?

4. An arc of length 8 in. on a circle of radius 6 in. subtends a central angle of how many degrees? an arc of length 4 in.? an arc of length 6 in.?

5. On a circle of radius π, there is an arc of length $\pi^2/4$. How many degrees in the central angle?

6. An arc of length $\pi r/2$ is subtended on a circle of radius r by what central angle?

7. Find the length of arc on a circle of radius r subtended by an angle of 1°.

We have been measuring arcs by their lengths. It is convenient to measure them by the measure in degrees of the central angles they subtend.

Definition 12–5. The measure in degrees of an arc subtended by a central angle is equal to the measure of the central angle. The measure in de-

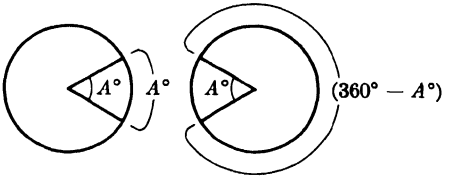


FIGURE 12–17

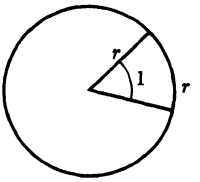


FIGURE 12–18

grees of an arc exterior to a central angle is equal to 360 minus the measure of the central angle. (See Fig. 12–17.)

Exercise Group 12–9

1. An arc of length π in. on a circle of radius 3 in. is of how many degrees?
2. An arc of length 4 in. on a circle of radius 6 in. has how many degrees?
3. On a circle of radius 2 in., how long is an arc of 120°? 240°?
4. How many degrees has an arc of length 15 in. on a circle of radius 3?

12–8 RADIAN MEASURE

Besides the familiar unit of degree for measuring angles, there is another one which is of great importance in advanced mathematics. The reasons for using it cannot be given here, but the definition is quite simple, and you will meet it again if you study trigonometry. The unit is called the *radian.*

Definition 12–6. An angle of one radian is one which subtends an arc of length r on a circle of radius r (Fig. 12–18).

Definition 12–7. The measure of an angle in radians is equal to L/r, where L is the length of the arc subtended on a circle of radius r.

Remark 1. Definition 12–7 agrees with Definition 12–6 for angles of 1 rad.

Remark 2. The measure in radians does not depend on the radius, by Theorem 12–7.

Exercise Group 12–10

1. On a circle of radius 2 is an arc of length 4. How many radians are there in the central angle?
2. How many radians are there in a right angle? in a straight angle?
3. On a circle of radius 2, find the length of arc subtended by a central angle of 30°. How many radians are there in this angle?
4. How many degrees are there in one radian?

5. How many radians are there in one degree?

6. From Exercises 4 and 5, derive the formulas for changing measure in degrees to measure in radians, and vice versa.

7. What are the measures in radians of the following angles? 30°; 45°; 60°; 90°; 120°; 135°; 150°; 180°.

8. The following numbers are measures of angles in radians. What are their measures in degrees? $\pi/4$; $5\pi/4$; 3; 2; 1.5; $\pi/3$; $\pi/6$; $\pi/2$.

9. Which is the larger unit, radian or degree? For which unit will the measure of an angle give the larger number?

10. Consider a central angle of 1 radian on a circle of radius r. What is the length of the arc subtended by this angle? an angle of 2 radians? an angle of 3 radians? an angle of n radians?

Consequence of Definition 12–7. If R is the measure in radians of a central angle in a circle of radius r, then the angle subtends an arc of length $r \cdot R$.

12–9 OTHER UNITS

Units other than degrees and radians could be chosen. One which has some use in artillery is the *mil*. It is approximately the angle which subtends an arc of 1 yd at a distance of 1000 yd. Apart from its military use, it is of little importance.

12–10 INSCRIBED ANGLES

Now that we have a measure for angles, it is easy to prove a theorem about the measure of angles *inscribed* in a circle. We recall the definition.

An angle is said to be *inscribed* in a circle if its vertex and one point on each side of the angle, other than the vertex, lie on the circle. The arc of the circle interior to the angle is said to be subtended by the angle. (See Fig. 12–19.)

Theorem 12–11. *The measure of an inscribed angle in degrees is equal to one half of the degree measure of the arc it subtends.*

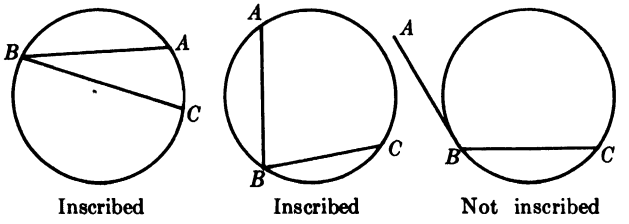


FIGURE 12–19

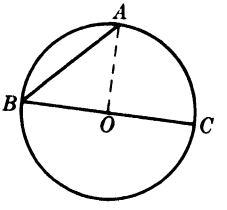


FIGURE 12–20

First, we prove a special case of this theorem, which will enable us to prove the theorem easily.

Lemma 12–11–1. *If $\angle ABC$ is an inscribed angle and BC is a diameter of the circle, then measure $\angle B = \frac{1}{2}$ measure of $\overset{\frown}{AC}$.* (See Fig. 12–20.)

Proof.

STATEMENTS	REASONS
1. $\angle OAB \cong \angle B$.	1. Why?
2. Measure $\angle OAB$ + measure $\angle OBA$ = measure $\angle COA$.	2. Why?
3. 2 (measure $\angle B$) = measure $\angle COA$.	3. Why?
4. Measure $\angle COA$ = measure $\overset{\frown}{AC}$.	4. Why?
5. Measure $\angle B = \frac{1}{2}$ (measure $\overset{\frown}{AC}$).	5. Why?

Proof of the theorem. There are two cases to consider to complete the proof of the theorem. They are: *Case* 1. The center of the circle is interior to $\angle ABC$ (Fig. 12–21). *Case* 2. The center of the circle is exterior to $\angle ABC$ (Fig. 12–22).

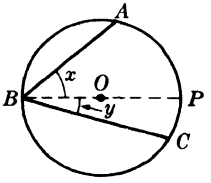


FIGURE 12–21

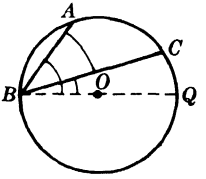


FIGURE 12–22

In Case 1 the diameter through B divides $\overset{\frown}{AC}$ into arcs $\overset{\frown}{AP}$ and $\overset{\frown}{PC}$, and $\angle ABC$ into angles x and y, as shown. By Lemma 12–11–1, measure $\angle x = \frac{1}{2}$ (measure $\overset{\frown}{AP}$), and measure $\angle y = \frac{1}{2}$ (measure $\overset{\frown}{PC}$). Hence measure $\angle ABC$ = measure $\angle x$ + measure $\angle y = \frac{1}{2}$ (measure $\overset{\frown}{AP}$) + $\frac{1}{2}$ (measure $\overset{\frown}{PC}$) = $\frac{1}{2}$ (measure $\overset{\frown}{AC}$).

The proof of Case 2 is left to the student.

Exercise Group 12–11

1. If $\angle ABC$ is inscribed in a circle and measure $\overset{\frown}{AC} = 80°$, how many degrees in $\angle B$?

2. An angle of 100° is inscribed in a circle. How many degrees are there in the arc it subtends?

3. Prove that an angle "inscribed in a semicircle" is a right angle (Fig. 12–23).

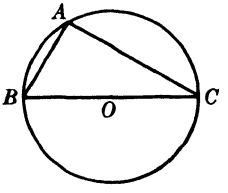


FIGURE 12–23

4. How large can an angle be that is inscribed in a circle?

5. Prove that two inscribed angles in one circle which subtend arcs of equal measure are congruent.

6. Prove that a parallelogram inscribed in a circle must be a rectangle. More generally, prove that if a quadrilateral is inscribed in a circle, then the opposite angles are supplementary.

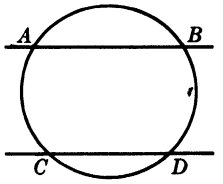


FIGURE 12–24

7. In Fig. 12–24 lines AB and CD are parallel. Prove that measure $\overset{\frown}{AC}$ = measure $\overset{\frown}{BD}$.

8. A trapezoid is inscribed in a circle. Prove that the base angles are congruent. [*Hint:* Use the result of Exercise 7.]

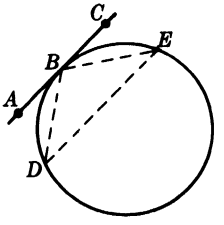


FIGURE 12–25

*9. In Fig. 12–25, AC is tangent to the circle. Prove that measure $\angle ABD = \frac{1}{2}$ measure $\overset{\frown}{BD}$. [*Hint:* Draw $DE \parallel AC$. Show that arcs $\overset{\frown}{BD}$ and $\overset{\frown}{BE}$ have the same measure. Then use $\angle ABD \cong \angle BDE \cong \angle BED$.]

*10. Prove, for Fig. 12–26, that measure $\angle A = \frac{1}{2}$ (measure $\overset{\frown}{CD}$ − measure $\overset{\frown}{BE}$). State this theorem in words.

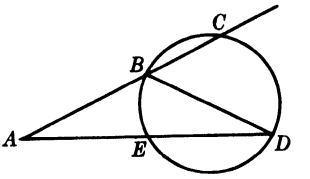


FIGURE 12–26

*11. State and prove theorems similar to that of Exercise 10 for cases in which either AC or AD, or both, are tangent to the circle.

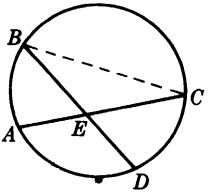


FIGURE 12–27

*12. Prove, for Fig. 12–27, that measure $\angle AEB = \frac{1}{2}$ (measure $\overset{\frown}{AB}$ + measure $\overset{\frown}{DC}$). [*Hint:* Draw BC.] State this theorem in words.

13. Using Fig. 12–27, prove that $\triangle AED \sim \triangle BEC$. Show that $\overline{AE} \cdot \overline{EC} = \overline{BE} \cdot \overline{ED}$. State this last property as a theorem.

*14. Using Fig. 12–26, prove that $\triangle AEC \sim \triangle ABD$. Hence prove that $\overline{AB} \cdot \overline{AC} = \overline{AE} \cdot \overline{AD}$. State this last property as a theorem.

15. Two chords of a circle intersect as in Fig. 12–28. If $\overline{AO} = 12$, $\overline{OB} = 4$, and $\overline{CO} = 8$, find $\overline{OD}$. [*Hint:* Use Exercise 13.]

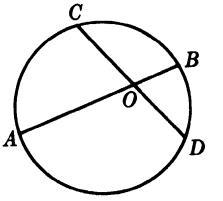


FIGURE 12–28

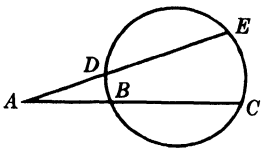


FIGURE 12–29

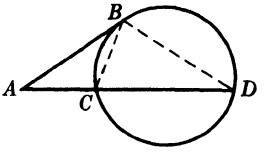


FIGURE 12–30

*16. From point A outside a circle, secants are drawn as in Fig. 12–29. If $\overline{AD} = 6$, $\overline{DE} = 10$, and $\overline{AB} = 8$, find $\overline{BC}$.

*17. In Fig. 12–30, AB is tangent to the circle at B. Prove that $\triangle ACB \sim \triangle ABD$. You will need the result of Exercise 9 for this. Prove also that $\overline{AB}^2 = \overline{AD} \cdot \overline{AC}$.

18. If, in Fig. 12–30, $\overline{AC} = 8$, and $\overline{CD} = 10$, determine $\overline{AB}$.

REVIEW OF CHAPTER 12

In this chapter definitions have been given of the following terms; be sure that you can state them in your own words: arc, subtended arc, inscribed polygonal path, arc length, degree, measure of angle in degrees, measure of arc in degrees, radian, measure of angle in radians, inscribed angle.

The following is an outline of the important definitions and theorems of the chapter. Study the outline so that you understand clearly how each step leads to the next.

Theorem: The limit of the lengths of a sequence of regular inscribed polygons exists.

Definition: Arc length.

Theorem: Can use inscribed polygons other than regular ones to define arc length.

Theorem: Length of "sum" of two arcs equals sum of lengths.

Theorem: Greater angle subtends longer arc.

Theorem: Angles congruent if and only if arc lengths equal.

Theorem: Arc length is proportional to radius.

Theorem: For every positive number less than the circumference, there is an arc having length equal to the number.

Definition: Measure of angles in degrees.

Definition: Measure of angles in radians.

Theorems: Angle measurement and counting.

Theorem: Measure of inscribed angle is one half measure of subtended arc.

Important Remark

The purpose of the chapter has been to define measurement of angles. To make the definition, we first defined *arc length*, and then *angle measurement* in terms of an arc length.

Review Exercises

1. From the definition of arc length, show that the length of arc subtended by a right angle (Fig. 12–31) is greater than $\sqrt{2}\,r$. Hence, conclude that $\pi/2 > \sqrt{2}$.

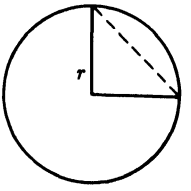


FIGURE 12–31

2. Angles of 15° subtend arcs of lengths 10 and 12 on two circles. What is the ratio of the radii of the circles? Find the radii.

3. Theorem 12–7 states that arc length is proportional to the radius. What theorem about triangles is used in the proof of Theorem 12–7?

4. State briefly what is involved in proving that the length of the "sum" of two adjacent arcs is equal to the sum of their lengths.

5. Compute the length of an arc in a circle of radius 10 if it is subtended by an angle of (a) 40 degrees, (b) 2 radians, (c) $\pi/4$ radians, (d) $\frac{1}{2}$ degree.

*6. A sequence of points A_1, A_2, A_3, . . . is marked on a circle so that each pair of consecutive points determines a central angle of 1 rad. The direction from A_1 to A_2 to A_3, etc., is counterclockwise. Compute the approximate length of the arc from A_{51} to A_1 in a counterclockwise sense if the radius of the circle is 10. Use $\pi = 3.1416$.

CHAPTER 13

LOCI AND SETS

13–1 TERMINOLOGY

In this chapter we use the term *locus** (plural, *loci*), which we inherit from the days when Latin was the universal language of science. (Even in the first part of the 19th century, most mathematics was published in Latin.) As will be clear from the definition given below, a locus is just a set of points. The language may be met in later work, so we devote this short chapter to it. We also collect a number of geometric constructions in this chapter.

Definition 13–1. A locus of points is the set of all points, and only those points, which satisfy some given condition.

We may now use the language, "The locus of points P such that...." This is just another way of saying, "The set of all points P such that..."

Examples:

(a) The *locus* of points at a given distance r from a given point is obviously a circle.

(b) The *locus* of points at a given distance d from a given line l is a pair of lines parallel to l, one on each side of l (Fig. 13–1).

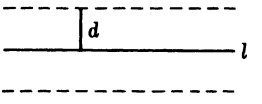


FIGURE 13–1

Example (b) requires a bit of proof, which is omitted. We will discuss such proofs in the next section. This locus could also be described as the *set* of all points at distance d from l.

* *Locus* is a Latin word meaning, literally, *place* in the sense that we use the word *location*.

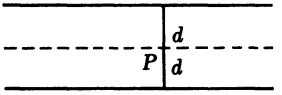


FIGURE 13–2

(c) The *locus* of points equidistant from two parallel lines is a line between the given lines.

An equivalent way of describing (c) would be to say: the *set* of all points equidistant from two parallel lines (Fig. 13–2). Again, the fact that locus (c) is a line requires proof.

Exercise Group 13–1

In the following exercises, state what you believe the locus to be. Also describe this locus as a *set*.

1. What is the locus of points P such that $\overline{PP}_0 = 4$, where P_0 is a given point?

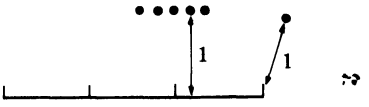


FIGURE 13–3

2. What is the locus of all points at a distance 1 from a line segment three units long (Fig. 13–3)? (By the distance from a point to a line segment, we mean the shortest distance from the point to any point of the segment.)

3. What is the locus of all points at a distance four units from the sides of a unit square? a square 12 units on a side? eight units?

4. A circle of radius five units is given. What is the locus of all points at a distance two units from the given circle?

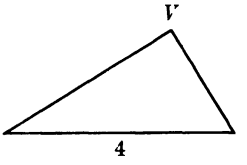


FIGURE 13–4

5. A triangle has a fixed base four units long (Fig. 13–4). What is the locus of all possible positions of the vertex V if the area is four units? if the area is 1 units?

13–2 SOME LOCUS THEOREMS

In many locus problems the set of points can be characterized in more than one way, or we can say that two locus statements may describe the same set of points. When this is the case, there is the following to be shown: (1) Each point of the first locus is also a point of the second locus. (The first set is a *subset* of the second set.) (2) Each point of the second locus is also a point of the first locus. (The second set is a subset of the first set.) If each is a subset of the other, the two sets are the same.

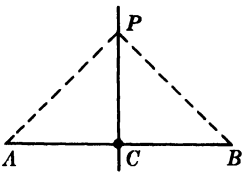


FIGURE 13–5

Theorem 13–1. *The locus of points P equidistant from two given points, A and B, is the set of points on the perpendicular bisector of the segment AB* (Fig. 13–5).

To make the proof we must show two things: (1) that if a point P is equidistant from A and B, then it is on the perpendicular bisector, and (2) that if a point is on the perpendicular bisector, then it is equidistant from A and B. These should be easy for you to prove now; in fact, both (1) and (2) were proven previously as theorems or exercises.

A brief proof will be sketched here, although your teacher may require you to write out the details. To prove (1), let C denote the midpoint of AB. Then if $P \neq C$, $\triangle PAC \cong \triangle PBC$. (Why?) Thus $\angle PCA \cong \angle PCB$, and P is on the perpendicular bisector. (How do we argue if $P = C$?) To prove (2), suppose that P is on the perpendicular bisector and $P \neq C$. Then $\triangle PCA \cong \triangle PCB$ (why?) and $PA \cong PB$. (How do we argue if $P = C$?)

A similar theorem concerns angle bisectors.

Theorem 13–2. *The locus of points interior to an angle and equidistant from the sides of the angle is the set of points on the bisector of the angle, excluding the vertex* (Fig. 13–6).

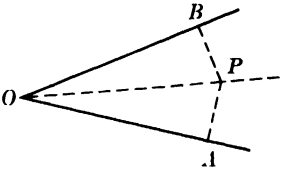


FIGURE 13–6

Proof sketch. Suppose that P is interior to the angle and equidistant from the sides. Let A and B be the feet of perpendiculars from P to the sides of the angle. (How do you know that perpendiculars from P meet the sides of the angle?) Then $\triangle POA$ and $\triangle POB$ are right triangles with two congruent sides and are congruent. Hence $\angle POA \cong \angle POB$, and

OP is the angle bisector. Conversely, if P is on the bisector, then $P \neq O$, PA and PB are perpendicular to the sides of the angle, and the right triangles POA and POB are congruent. Hence $\overline{PA} = \overline{PB}$.

Theorem 13–3. *Let AB be a given segment. The set of all points P such that $\triangle APB$ is right, with AB the hypotenuse, is the circle on AB as diameter, excluding points A and B* (Fig. 13–7).

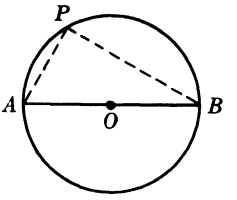


FIGURE 13–7

Let O be the midpoint of AB. It is necessary to show that if P is a point such that $AP \perp PB$, then P is on the circle whose center is O and which contains A and B.

Conversely, if P is on the circle, and different from A and B, it is necessary to show that $AP \perp PB$. But this follows from theorems on the measure of inscribed angles. (There are other proofs, too.) The details are left to the student.

Exercise Group 13–2

1. What is the set of all points that are centers of circles tangent to both sides of an angle?

2. What is the locus of all points exterior to a given circle and at a distance d from the circle?

3. A barn has dimensions 30 × 50 ft. A cow is tethered to a corner of the barn by a rope 60 ft long. What is the locus of points where the cow may graze?

*4. In Fig. 13–8, points A and B are fixed. What is the locus of all points P, on a given side of the line AB, such that $\angle APB$ is 50°?

5. What is the locus of points equidistant from three noncollinear points?

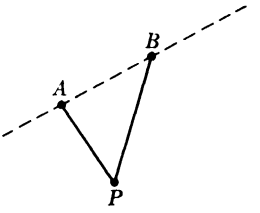


FIGURE 13–8

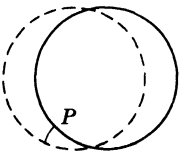


FIGURE 13–9

6. As in Fig. 13–9, sketch the path of a point on the rim of a wheel rolling on a line. The curve obtained is called a *cycloid.*

*7. With the same wheel as in Exercise 6, sketch the locus if P is on a spoke of the wheel; on an extension of a spoke.

8. A small circle of radius $R/4$ rolls on the inside of a circle of radius R (Fig. 13–10). What is the path of a point on the small circle? (The curve is called a *hypocycloid.*)

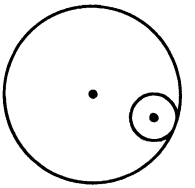


FIGURE 13–10

*9. What are the various possibilities that arise from Exercise 8 if the radius of the small circle is xR? Investigate the locus if $x = \frac{1}{2}$; $x = \frac{1}{3}$; x is a rational number; x is an irrational number.

13–3 UNIONS AND INTERSECTIONS

Some loci are just unions or intersections of other simpler loci. To be precise, we make a definition.

Definition 13–2. Given two sets of points S_1 and S_2, the union of the two sets is the set of all points of S_1 and S_2. The *intersection* of S_1 and S_2 is the set of points that are common to S_1 and S_2.

Exercise Group 13–3

1. Describe an angle as a union of two sets.

2. Describe an angle and its interior as the intersection of two sets.

3. Describe the interior of a triangle as the intersection of three sets; as the intersection of two sets.

4. Describe a triangle as the union of three sets.

5. Describe a quadrilateral as the union of four sets. Describe the interior of a simple convex quadrilateral as the intersection of two sets; as the intersection of four sets.

Theorem 13–4. *The bisectors of the angles of a triangle intersect in one point (called the incenter), which is equidistant from the three sides* (Fig. 13–11).

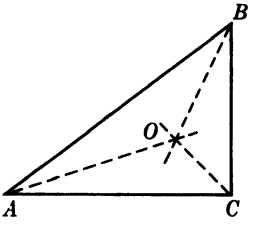


FIGURE 13–11

Proof sketch. The bisectors of angles A and B must intersect in one point O. (Why?) Then this point O is equidistant from AB and from BC. (Why?) Likewise, O is equidistant from AB and AC. Therefore, O is on the bisector of $\angle C$. (Why?) This proves the theorem.

The point O is called the *incenter* because it is the center of a circle *inscribed* in the triangle. One may also state the theorem by saying that *the bisectors of the angles are concurrent.*

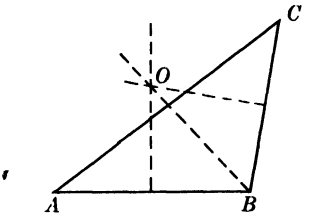


FIGURE 13–12

Theorem 13–5. *The perpendicular bisectors of the sides of a triangle intersect in one point (called the circumcenter), which is equidistant from the three vertices* (Fig. 13–12).

The proof is left to the student.

Exercise Group 13–4

1. Consider all lines that are parallel to the base of a triangle and that intersect the other sides in points C and D (Fig. 13–13). Show that the locus of midpoints of such segments CD is the median to the base (except for one point!).

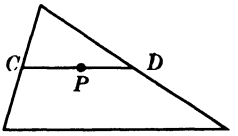


FIGURE 13–13

2. Show how to inscribe a circle in a triangle. State why your construction is correct. Why is there a unique inscribed circle?

3. Show how to circumscribe a circle about a triangle. State why there is a unique circumscribing circle.

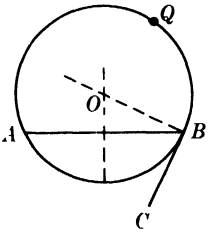


FIGURE 13–14

4. In Fig. 13–14, $\angle ABC$, neither right nor straight, is given. The point O is the intersection of the perpendicular bisector of AB and the line perpendicular to BC through B. Prove that there is a circle with center O containing A and B. If Q is a point of this circle and is on the same side of AB as O, prove that $\angle AQB \cong \angle ABC$.

Theorem 13–6. *The medians of a triangle intersect in one point (called the centroid), which is two thirds of the distance from each vertex to the midpoint of the opposite side.*

Outline of proof. (Details are left as an exercise.) (1) Any two medians, say CD and AE in Fig. 13–15, intersect. Call this intersection point O. (2) Choose F and G as the midpoints of OC and OA, respectively. (3) Show that D, E, F, and G are vertices of a parallelogram. (4) Thus $AG \cong GO \cong OE$, and $FC \cong FO \cong OD$. (5) The point O has, therefore, the required distance properties. Since there is but one such point on each median, the medians are concurrent.

The centroid is the "balance point" in the following sense. A sheet of metal of uniform thickness, the shape of the triangle, would balance if supported at O. Exercise 1 of Group 13–4 may help you to see why this is so.

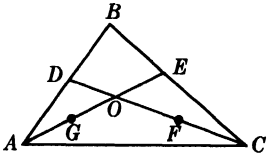


FIGURE 13–15

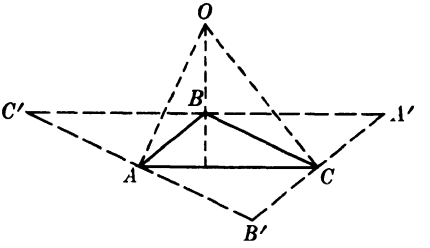


FIGURE 13–16

Theorem 13–7. *The altitudes of a triangle intersect in one point (called the orthocenter).*

Outline of proof. (Details are left as an exercise.) (1) Through each vertex, consider the line that is parallel to the opposite side of the given $\triangle ABC$. These lines will intersect to form $\triangle A'B'C'$, as in Fig. 13–16. (2) Show that $AC'BC$, $ABCB'$, and $ABA'C$ are parallelograms, and hence observe that $C'B \cong BA'$, $C'A \cong AB'$, and $B'C \cong CA'$. (3) Show that the circumcenter O of $\triangle A'B'C'$ is the point of intersection of the altitudes of $\triangle ABC$.

Exercise Group 13–5

1. Construct a triangle, given the lengths of its base, altitude, and one side.

2. Construct a triangle, given the base, the length of the altitude to this base, and one of the base angles.

*3. As in Fig. 13–17, construct a triangle, given the lengths of two sides and an angle other than the included angle. [*Hint:* Use Exercise 4 in Group 13–4.]

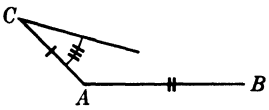


FIGURE 13–17

4. Construct a triangle, given the lengths of an altitude to one side and the other two sides.

5. Construct a triangle, given the midpoints of the three sides.

6. Construct a triangle, given the length of the altitude to the base and the two base angles.

7. Given a circle and a line segment AB, construct the locus of all points P such that tangents of length $\overline{AB}$ can be drawn to the circle from P.

8. Construct a right triangle with a given segment as hypotenuse and having a side of given length.

9. Construct a circle tangent to a given line at a given point and having a given radius.

10. Construct the locus of all points P such that there is a circle, with given radius and center at P, which is tangent to a given circle.

11. Construct all circles with a given radius which are tangent to two given circles of unequal radii.

12. Construct a line tangent to a given circle and perpendicular to a given line.

13. Given a circle and a line segment AB, construct the set of all points M such that M is the midpoint of a chord (of the circle) congruent to AB.

14. Given an arc of a circle, construct the center of the circle.

15. Construct an isosceles triangle, given the base, and the radius of the circumscribed circle.

16. Construct an isosceles triangle, given the base, and the radius of the inscribed circle.

17. Construct an isosceles triangle, given the base and the vertex angle.

18. Construct a triangle, given two sides and the length of the median to one of these sides.

*19. Let P be a given point not on a given circle whose center is O. Consider all segments PX such that X is on the circle. Show that the locus of midpoints M of these segments is a circle whose diameter is a segment on the line through P and O. Construct this circle.

*20. Given three segments, construct a triangle two of whose sides are congruent to two of these segments, and such that the median to the other side is congruent to the third segment.

REVIEW OF CHAPTER 13

In earlier chapters we were using the phrases *set of points* and *geometric figure* interchangeably. In this chapter we introduced a third term, *locus*, which has exactly the same meaning as the earlier expressions.

It is important that you understand what is involved in establishing that two different "rules" determine the same locus. A statement like

> *the set of points having property p is the same set as that having property q*

must be established by proving two things:

(1) Every point in the first set is also in the second. That is, every point having property p also has property q.

(2) Every point in the second set is also in the first. That is, every point having property q also has property p.

Review Exercises

1. If AB is a fixed segment, are the properties

p: $\angle AXB$ is acute

and

q: X is in the exterior of the circle on AB as diameter

equivalent to each other?

2. Restate Exercise 1 in conventional locus terminology.

3. If R is the set of points X such that, for a fixed segment AB, $\overline{AX} + \overline{XB} = AB$, and S is the set of points between A and B, prove that $R = S$. How else can you describe the locus of points X such that $\overline{AX} - \overline{XB} = \overline{AB}$? such that $\overline{BX} - \overline{XA} = \overline{AB}$?

4. Let AB and CD be two given segments. Describe in a second way the set of points X such that

$\angle AXB$ is obtuse and $\angle CXD$ is right;
$\angle AXB$ is obtuse or $\angle CXD$ is right;
$\angle AXB$ is acute and $\angle CXD$ is obtuse;
$\angle AXB$ is acute or $\angle CXD$ is obtuse.

CHAPTER 14

SPACE GEOMETRY*

14–1 INTRODUCTION

In this chapter we outline the geometry of space, that is, three-dimensional Euclidean space. Sometimes this study is called *solid* geometry. Just as plane geometry is an abstraction derived from our experience with flat surfaces, so solid geometry is an abstraction derived from our ordinary perception of objects in space.

In this chapter you will find many figures and many theorems and definitions, but very few proofs. Proofs are omitted because this chapter is intended to be only an introduction to the subject. Nevertheless, if you study the theorems in the order they occur, you will find that their arrangement is logical because they can be proved in that order. Furthermore, most of them have fairly simple proofs. To get an understanding of the *facts* of solid geometry, it will be sufficient to read over the theorems and be sure that you know what they say.

14–2 GEOMETRIC OBJECTS

As in plane geometry, we shall be concerned mostly with simple sets of points that you have already encountered, namely, planes, spheres,

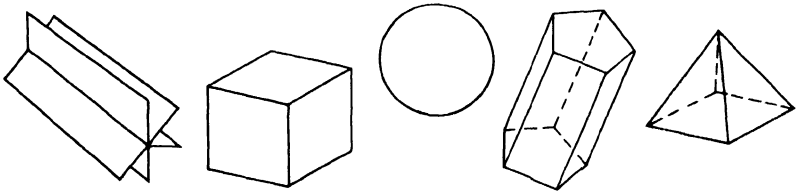


FIGURE 14–1

* This chapter is divided into three parts. For Part A (Section 14–3), Chapters 1 through 3 are prerequisites. For Part B (Sections 14–4 and 14–5), Chapters 4 through 8 are prerequisites. For Part C (Sections 14–6 through 14–9), Chapters 9 through 12 are prerequisites.

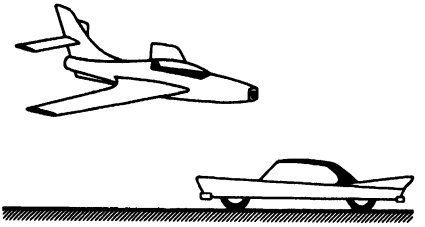


FIGURE 14–2

pyramids, cubes, prisms, etc., as in Fig. 14–1. More complicated sets of points, such as those in Fig. 14–2, will occur to you. As before, any set of points may be studied. The drawings in Fig. 14–2 represent sets of points that do not lie in a plane.

Exercise Group 14–1

1. Draw figures of several three-dimensional sets of points.
2. Sketch a polyhedron.
3. What might be meant by parallel planes? perpendicular planes?

A. LINES, PLANES, AND BETWEENNESS IN SPACE

14–3 THE POSTULATES

It is perhaps somewhat surprising that most of the essential difficulties in geometry already occur in the plane, so that extension to space geometry requires no fundamentally new concepts. As a matter of fact, very few more postulates are needed than for the plane, indeed only four more.

Postulate VII–1. There is a set of points called *space*. Certain subsets of these points are called planes. There are at least four points that are not in any one plane.

Postulate VII–2. There is exactly one plane containing any three non-collinear points (Fig. 14–3).

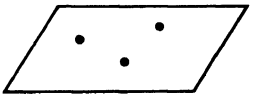


FIGURE 14–3

Postulate VII–3. Each plane satisfies all the postulates for plane geometry. In particular, the line joining two points of a plane lies in that plane

Furthermore, the postulates for congruence of segments and angles apply whether or not the segments and angles are in the same plane.

Postulate VII–4. Each plane separates space. By this we mean that the set of points of space not on the plane consists of two subsets, called the sides of the plane, with the following properties. If two points are in the same subset, then the segment joining them does not meet the plane, but if the two points are in different subsets, the segment joining them does intersect the plane.

Postulate VII–1 asserts that no plane fills up all of space, so that we have a geometry of more than two dimensions (Fig. 14–4).

Postulate VII–2 gives us a simple requirement for fixing a definite plane.

Postulate VII–3 states that our planes are as we want them to be from our study of plane geometry. In addition, this postulate relates congruent lines and angles in different planes.

Postulate VII–4 expresses the familiar fact that a wall between neighbors separates them.

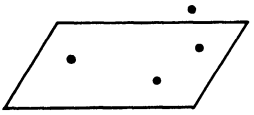


FIGURE 14–4

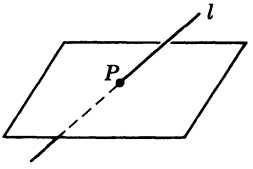


FIGURE 14–5

Exercise 14–2

Restate Postulate Group VII in your own words.

From these postulates and the first two postulate groups for plane geometry, we can prove a number of theorems concerning intersections of lines and planes.

Theorem 14–1. *If a line l does not lie in a plane π, and has at least one point in π, then l and π have exactly one point in common* (Fig. 14–5).

For the rest of the chapter, proofs will be left out, but we include a proof of Theorem 14–1 as a guide to start you out correctly.

Proof. There is, by assumption, at least one point P in both l and π. Suppose that there were a second point Q. Then, by Postulate VII–3, the line $l = PQ$ would lie in the plane π. But we have assumed that l is not in π. Hence P is the only point in both l and π.

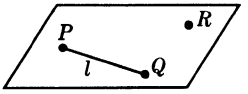


FIGURE 14–6

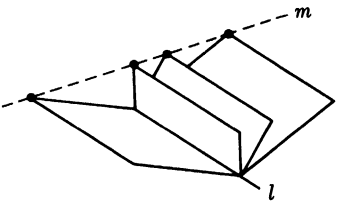


FIGURE 14–7

Theorem 14–2. *There is a unique plane containing a given line and a point not on the line* (Fig. 14–6).

Theorem 14–3. *There are infinitely many planes containing a given line.* Figure 14–7 may suggest a proof.

Theorem 14–4. *There is a unique plane containing two intersecting lines* (Fig. 14–8).

Theorem 14–5. *If two distinct planes have at least one point in common, they have precisely a line in common* (Fig. 14–9).

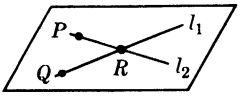


FIGURE 14–8

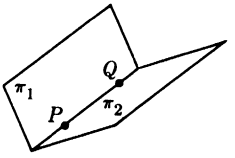


FIGURE 14–9

Definition 14–1. If two planes π_1 and π_2 intersect in a line l (Fig. 14–10), let π_1^+ and π_2^+ be half planes of π_1 and π_2, determined by l. The set of all points in π_1^+ and π_2^+ is called a *dihedral angle.* The line l is called the *edge,* and π_1^+ and π_2^+ are called the *faces,* of the dihedral angle.

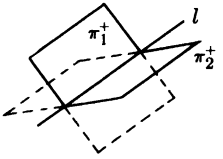


FIGURE 14–10

Definition 14–2. The interior of the dihedral angle with faces π_1^+ and π_2^+ is the set of all points of space on the same side of π_2 as π_1^+ and on the same side of π_1 as π_2^+.

Exercise Group 14–3

1. Give a precise definition for a tetrahedron (Fig. 14–11). The tetrahedron is usually considered to be just the surface. What do you define the interior to be?

2. Give a definition for a polyhedron (surface). See Fig. 14–12.

3. Give a precise definition for a half plane.

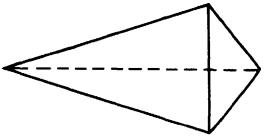


FIGURE 14–11

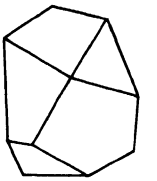


FIGURE 14–12

B. CONGRUENCE IN SPACE

14–4 PERPENDICULAR LINES AND PLANES

Before giving some theorems, we observe that all right angles are congruent regardless of whether they are in the same plane. We know this because our postulates for angle congruence state that, given an $\angle AOB$ in a plane π and a ray r' from O' in a plane π', there is exactly one ray r'' from O' (on a given side of the line of r' in the plane π') that forms with r' an angle congruent to $\angle AOB$. See Fig. 14-13.

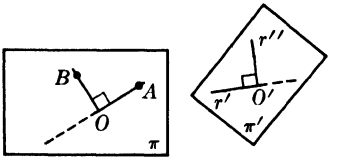


FIGURE 14–13

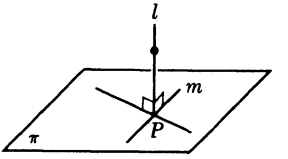


FIGURE 14–14

Definition 14–3. A line l is perpendicular to a plane π if l intersects π at a point P and l is perpendicular to every line in π that passes through P (Fig. 14–14).

Theorem 14–6. *If a line l is perpendicular to two intersecting lines, then it is perpendicular to the plane containing the lines* (Fig. 14–15).

[*Hint:* Choose A and A' such that $AP \cong A'P$. Choose B and C. Then prove that $\triangle ABC \cong \triangle A'BC$. Then prove that $\triangle ABQ \cong \triangle A'BQ$, so that $AQ \cong A'Q$. This will imply that line $m \perp l$.]

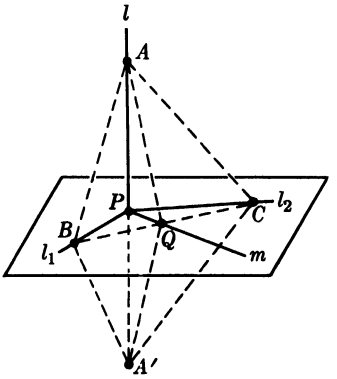


FIGURE 14–15

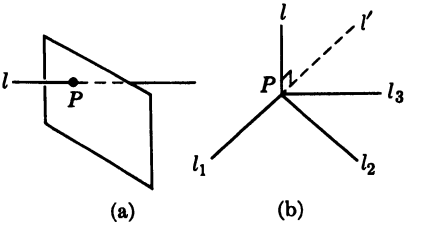


FIGURE 14–16

Theorem 14–7. *There is exactly one plane perpendicular to a line at a point of the line.* See Fig. 14–16(a).

[*Hint:* Use the previous theorem to show that there is at least one plane. To prove there is only one, it is sufficient to show that if l_1, l_2, and l_3 in Fig. 14–16(b) are distinct lines perpendicular to l at P, then they are in a single plane. Thus, the plane of l_2 and l_3 meets the plane of l and l_1 in a line $l' \perp l$. Since there is only one line in the plane of l and l_1 perpendicular to l at P, we have $l_1 = l'$. Hence l_1, l_2, and l_3 are coplanar.]

Theorem 14–8. *There is exactly one plane perpendicular to a line and containing a point not on the line* (Fig. 14–17).

Exercise Group 14–4

1. State two theorems, analogous to Theorems 14–7 and 14–8, concerning existence of lines perpendicular to planes.

2. Prove the theorems that you stated in Exercise 1.

Theorem 14–9. *If two lines are perpendicular to the same plane, they are parallel* (Fig. 14–18).

[*Hint:* First one must show that the two lines are in one plane.]

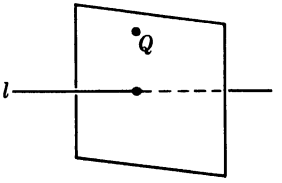


FIGURE 14–17

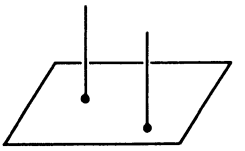


FIGURE 14–18

Definition 14–4. Two lines are called *skew* if they do not lie in one plane. Two planes are called *parallel* if they do not intersect.

Remarks. Two lines in one plane either intersect or are parallel. Two lines in parallel planes need not be parallel. See Fig. 14–19.

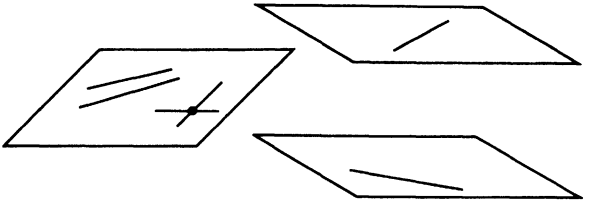


FIGURE 14–19

Theorem 14–10. *Through a point P not in a plane π, there is exactly one plane parallel to π* (Fig. 14–20).

[*Hint:* Choose intersecting lines l_1 and l_2 in π. Then P, with these lines, determines two planes π_1 and π_2, in which there are unique lines l_1' and l_2' parallel to l_1 and l_2, respectively, as in Fig. 14–21. These lines determine a plane parallel to π. The uniqueness is yet to be proved.]

Theorem 14–11. *Two planes perpendicular to the same line are parallel* (Fig. 14–22).

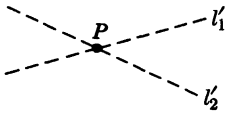


FIGURE 14–20

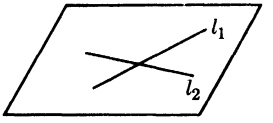


FIGURE 14–21

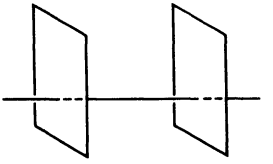


FIGURE 14–22

Theorem 14–12. *If two lines are cut by three parallel planes, the intercepted segments have lengths that are proportional. In Fig.* 14–23, *we would have* $\overline{AB}/\overline{A'B} = \overline{BC}/\overline{B'C'}$.

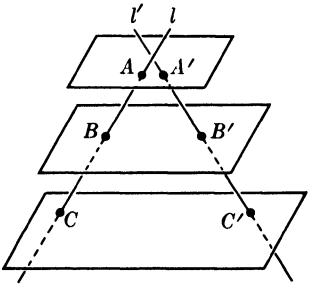


FIGURE 14–23

Exercise Group 14–5

Prove the following theorems.

1. Two planes parallel to one plane are parallel to each other.
2. If a line intersects one of two parallel planes, it intersects the other.
3. If a line is perpendicular to one of two parallel planes, it is perpendicular to the other.
4. If a plane intersects one of two parallel planes, it intersects the other.
5. If one of two parallel lines is perpendicular to a plane π, then the other line is perpendicular to π.

14–5 POLYHEDRA AND POLYHEDRAL ANGLES

Definition 14–5. Let $P_1P_2 \ldots P_n$ be a simple closed polygon in a plane π, and let V be a point not in π, as in Fig. 14–24. The set of points Q on rays from V through all the points P of the polygon is called a polyhedral angle. V is the *vertex*, and VP_1, VP_2, $\ldots$ VP_n are called *edges*. The planes VP_1P_2, VP_2P_3, $\ldots$ contain the *faces* of the angle. The angles $\angle P_1VP_2$, $\angle P_2VP_3$, $\ldots$ are called *face angles*. The polyhedral angle is called convex if the polygon $P_1 \ldots P_n$ is convex.

Exercise Group 14–6

1. Draw a trihedral angle and a tetrahedral angle. These have three and four faces, respectively.
2. Can a trihedral angle have three right-angled face angles?

Theorem 14–13. *The sum* of two face angles of a trihedral angle is greater than the third face angle* (Fig. 14–25).

[*Hint:* It suffices to prove the theorem in the case in which the third angle is the largest. Suppose that that angle is $\angle AVB$, as in Fig. 14–25.

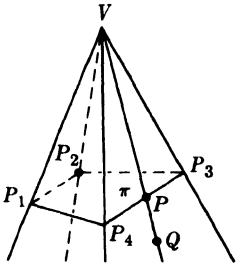


FIGURE 14–24

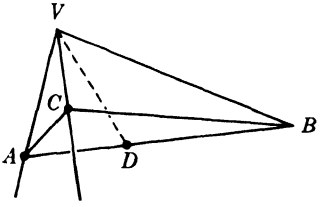


FIGURE 14–25

* The word *sum* is not defined, but the idea is clear.

Choose C such that $\angle CAV \cong \angle BAV$. Choose D on AB such that $AC \cong AD$. Now show that $VC \cong VD$ and $CB > DB$. Hence conclude that $\angle CVB > \angle DVB$.]

Theorem 14–14. *The sum of the face angles of a convex polyhedral angle is less than four right angles.*

Exercise 14–7

Illustrate the truth of Theorem 14–14 by constructing a polyhedral angle (Fig. 14–26) of paper and observing that when it is "flattened out" in a plane, it does not "encircle" a point.

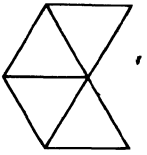


FIGURE 14–26

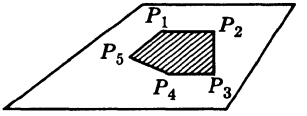


FIGURE 14–27

Definition 14–6. A *polygonal cell* (Fig. 14–27) is the sets of points on, or interior to, a simple plane polygon $P_1P_2 \ldots P_{n-1}$. The edges of the cell are the sides $P_1P_2, P_2P_3, \ldots$ of the polygon.

A *polyhedron* is the set of points on a finite number of polygonal cells joined together in the following way:

(1) Any two cells have either no points in common, exactly one vertex in common, or exactly one edge in common.

(2) Every edge is on precisely two cells.

Exercise 14–8

With Scotch tape and cardboard, construct a number of polyhedra.

Theorem 14–15. *A polyhedron separates space in the following sense. The points not on the polyhedron consist of two sets (called the inside and the outside) such that a polygonal path joining an inside point and an outside point must intersect the polyhedron, and two inside (or two outside) points can be joined by a polygonal path not intersecting the polyhedron.*

Exercise 14–9

Describe the interior of a tetrahedron.

Definition 14–7. A polyhedron is called *convex* if the segment joining any two points of the polyhedron lies either in the polyhedron or its interior. A polyhedron is *regular* if it is convex and all faces are congruent regular polygons.

Exercise 14–10

With Scotch tape and cardboard, construct regular polyhedra with faces that are triangles; squares; pentagons.

Theorem 14–16. *There are but five (except for size) regular polyhedra* (Fig. 14–28).

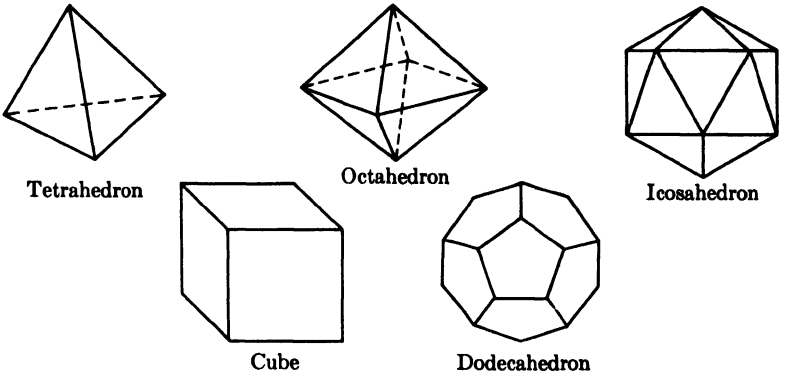


FIGURE 14–28

C. MEASUREMENT IN SPACE

14–6 ANGLES

Theorem 14–17. *Parallel planes intercept a dihedral angle in congruent angles* (Fig. 14–29).

Definition 14–8. The measure of a dihedral angle is equal to the measure of the angle intercepted by a plane perpendicular to the edge of the dihedral angle (Fig. 14–30).

Exercise 14–11

Restate Theorems 14–13 and 14–14, using degrees for measuring angles.

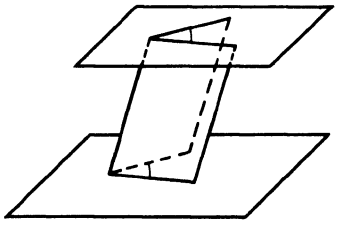


FIGURE 14–29

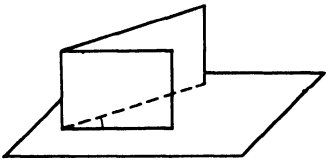


FIGURE 14–30

14–7 AREA

Definition 14–9. The area of a polyhedron is the sum of the areas of its faces.

Exercise Group 14–12

1. A *prism* is a polyhedron two of whose faces are congruent polygons in parallel planes, called bases, and whose remaining edges are all parallel to one another, as in Fig. 14–31. The lateral area is the total area of the faces less the area of the bases. Find a formula for the lateral area of a prism.

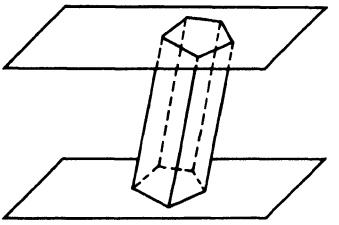


FIGURE 14–31

2. Using Fig. 14–32, define *pyramid* and *regular pyramid*. Find a formula for the lateral area of a regular pyramid.

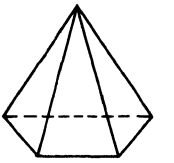


FIGURE 14–32

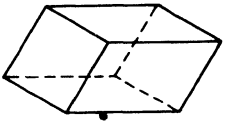


FIGURE 14–33

3. A *parallelepiped* is a special kind of prism. Give a definition that you think fits the drawing in Fig. 14–33. Prove that the opposite faces of a parallelepiped are congruent.

14–8 VOLUME

Just as area in the plane offers special difficulties, so does volume in space. We content ourselves with listing the theorems and sketching a few "proofs," obtained by drawing figures. Proofs that involve the notion of limit are not presented.

Theorem 14–18. *The volume of a rectangular parallelepiped (box) is equal to the product of the lengths of three concurrent edges. Volume = lwh.* See Fig. 14–34.

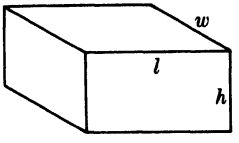


FIGURE 14–34

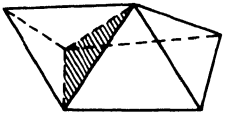


FIGURE 14–35

Theorem 14–19. *If a polyhedron is the union of two other polyhedra, with only a face and no interior points in common, then its volume is the sum of the volumes of the two polyhedra forming it* (Fig. 14–35).

Exercise 14–13

A parallelepiped is shown in Fig. 14–36. Show, by "chopping up" the parallelepiped to form a box, that the following theorem is true.

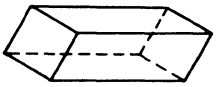


FIGURE 14–36

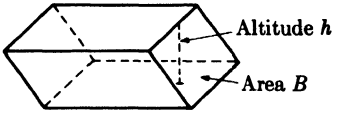


FIGURE 14–37

Theorem 14–20. *The volume of a parallelepiped is equal to the area of a base multiplied by the corresponding altitude* (Fig. 14–37).

The same formula holds for the volume of a prism. To see this, we first prove the theorem for prisms with triangular bases.

Exercise Group 14–14

1. By constructing a parallelepiped of which a triangular prism is half, show that the volume of the prism is equal to the area of a base times the altitude. See Fig. 14–38(a).

2. Show that the same result holds for an arbitrary prism. See Fig. 14–38(b).

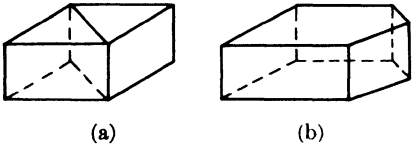


FIGURE 14–38

Theorem 14–21. *The volume of a prism is equal to the area of a base times the altitude.*

Volumes of pyramids are somewhat harder to discuss. The following theorem is to be taken without proof.

Theorem 14–22. *If B is the area of the base of a pyramid, then the volume* $= \frac{1}{3}Bh$, *where h is the altitude* (Fig. 14–39).

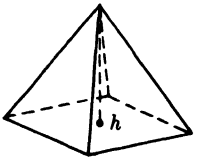


FIGURE 14–39

Exercise Group 14–15

1. Find the volume of a regular pyramid with a base 4 ft square and altitude equal to 4 ft.

2. Find the volume of a regular tetrahedron whose side has length a.

3. Show that if the formula for volume of a pyramid is correct for pyramids with triangular bases, then it is correct for all pyramids.

4. Define *similar polyhedra*, and prove that the volumes of similar polyhedra are proportional to the cubes of corresponding edges. Thus, $V/V' = l^3/l'^3$, as in Fig. 14–40.

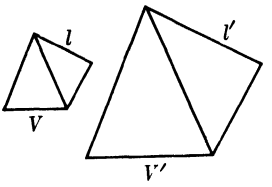


FIGURE 14–40

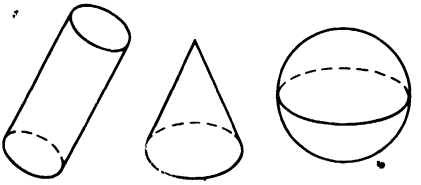


FIGURE 14–41

14–9 CYLINDERS, CONES, AND SPHERES

The familiar shapes in Fig. 14–41 are analogous to prisms, pyramids, and regular polyhedra.

The lateral surface area of cylinders and cones is obtained as a limit of the areas of prisms and pyramids.

Exercise Group 14–16

1. Give precise definitions for cylinders, cones, and spheres.

2. Find a formula for the lateral surface of a "regular" cone, called a *right circular cone* (Fig. 14–42).

The surface area of a sphere is harder to discuss. We merely state the result.

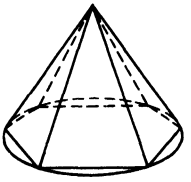


FIGURE 14–42

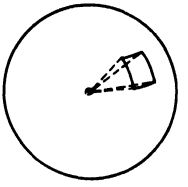


FIGURE 14–43

Theorem 14–23. *The area of the surface of a sphere of radius r is $4\pi r^2$.*

With the aid of Theorem 14–23, it is not too hard to arrive at the volume of a sphere. Try to prove the following theorem by considering small "pyramids," as shown in Fig. 14–43.

Theorem 14–24. *The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.*

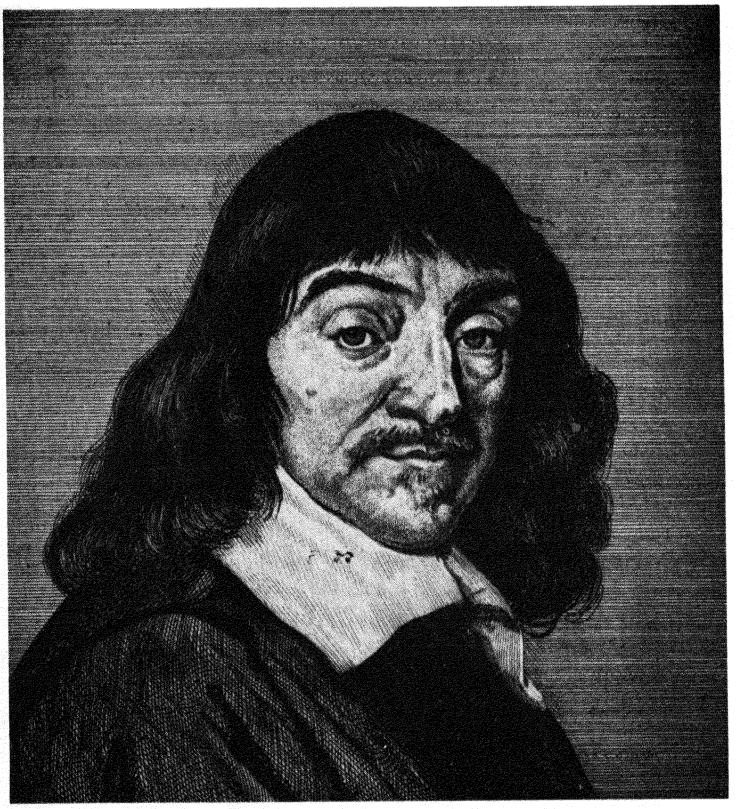
Exercise Group 14–17

1. Show that if the radius of a sphere is doubled, the volume is multiplied by a factor of 8.

2. Oranges 2 in. in diameter are 50¢ per dozen. Neglecting skin thickness, how much should one pay for oranges 3 in. in diameter?

3. Prove that if a plane intersects a sphere, then it intersects the sphere in either a circle or one point.

4. What is the formula for the lateral area of a cylinder?



René Descartes
(*1596–1650*)
Courtesy of *Scripta Mathematica*.

the date is generally set as 1637, when his *Geometry* was published. Descartes is also noted for his work in physics and in philosophy. In the latter, he aspired to write a philosophy that would be perfectly logical and free from objectionable assumptions. His efforts in philosophy include the book *Discourse on Method*, which expounds what is referred to today as Cartesian philosophy, and which has been a source of dispute ever since! But his *Geometry* is undisputed. Analytic geometry is today a part of a liberal education as well as a necessity for pure and applied science.

Some brief remarks about Descartes' life may be of interest. Born in 1596, he started school in a Jesuit institution at the age of eight. Never a strong, healthy boy, he was allowed by his school to stay in bed mornings to do his studies, and later in life he did much of his writing in bed. He left school in 1612, studied law, and even spent some time in the Dutch army as a gentleman volunteer, leaving in 1619. It appears that by 1620 most of his great ideas had been established in outline. In other words, he had thought them over and had a fairly clear picture of how they should go. Much of his life thereafter was spent in Holland. A devout man, he nevertheless had troubles with the church over his philosophical ideas. By the time his *Geometry* was published, he was a well-known figure. In 1649 he visited the court of Queen Christina of Sweden, after much urging, to teach her his system of philosophy. Assigned the outlandish hour of 5 a.m. three days a week for the instructions, he fell ill in February 1650 and died a few days later.

Geometry was his only book on mathematics and, in a sense, contained nothing totally new. What Descartes did was to show how a systematic use of coordinates helps to attack problems. He classified various kinds of curves and suggested further work. The book was suggestive and tremendously fruitful. We owe to him our convention of using the first letters of the alphabet (a, b, c) for knowns and the last letters (x, y, z) for unknowns.

It is not easy to give a concise definition of analytic geometry. In a sense, this entire chapter is such a definition, for a complete definition should include some of its uses. But for the moment, it may suffice to say that, in analytic geometry, geometric figures are studied by means of equations. Then, if we wish to know something concerning the geometric figure, it *may* be easy to learn this by examining an equation. The equations that are used involve numbers called coordinates, which serve to locate points. But why the word "analytic"? Analytic geometry does enable one to analyze geometric figures, but this is also done in regular Euclidean geometry. Perhaps a better description would be to call this new way of looking at geometry, plane *coordinate* geometry, or plane *Cartesian* geometry. Remember that it is the same plane we are studying all the time, no matter how we look at it.

Exercise Group 15–1

1. Read the history of mathematics during the 17th century.

2. Read a biography of Descartes in an encyclopedia or in *Men of Mathematics*, by E. T. Bell.

15–2 EXAMPLES

Although analytic geometry is distinguished by use of coordinates, the idea should already be familiar from some common situations. Some cities are laid out in squares, with the east-west and north-south roads called avenues and streets, respectively. The numbers of street and avenue fix any intersection. In the same way, you unconsciously use coordinates in describing a nearby town as, perhaps, 7 miles north and 5 miles west of "here."

Exercise Group 15–2

1. Try to think of other examples of coordinates, that is situations in which position is determined by a pair of numbers.

2. Discuss the need to have streets straight in order to "coordinatize" a city.

By far the most important example of coordinates is our method (Chapter 5) of putting coordinates on a line. We recall how this was done:

Given a line l and a unit of length, choose a point O on the line l, as in Fig. 15-1. Now choose* a point P_1 on l at a distance 1 from O. This choice determines which half of the line we will call the *positive axis*, or *positive direction*. For every positive real number x, there is exactly one point P_x of l on the same side of O as the point P_1 such that the length $\overline{OP}_x = x$. For every negative real number x', there is exactly one point $P_{x'}$ of l on the opposite side of O from P_1 such that the length $\overline{OP}_{x'} = -x' = |x'|$.

Each of the numbers x and x' is called the *coordinate* of the corresponding point of the line.

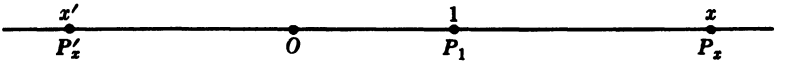


FIGURE 15–1

* There are two ways of choosing this point, namely on either side of O. Which side is chosen is not important.

Exercise Group 15–3

1. How are the points P_{x_1}, P_{x_2}, and O related if $x_1 < 0$ and $x_2 > 0$? if $0 < x_2 < x_1$? if $x_1 < x_2 < 0$?

2. If we agree that points with larger coordinates "lie to the right" of points with lesser coordinates, how are we thinking of the drawn line l and the chosen point P_1?

3. Two points on the line have coordinates -3 and 7. What is the distance between them? between 5 and 12? between x_1 and x_2?

Definition 15–1. We will refer to the above construction of coordinates as a coordinatization of the line l, with the point O as *origin of coordinates.*

15–3 THE EXISTENCE OF COORDINATES

The construction of plane coordinates is best carried out in a number of simple steps. We assume that a unit of length has been selected in the plane.

Definition 15–2. Choose a point O in the plane. Call this point the *origin of coordinates.* Choose any two perpendicular lines through O. Call these lines *coordinate axes.* Coordinatize each of the axes, with the point O as origin of coordinates. Select one of these lines and call it the *axis of abscissas.** The other axis is called the *axis of ordinates.*† See Fig. 15–2.

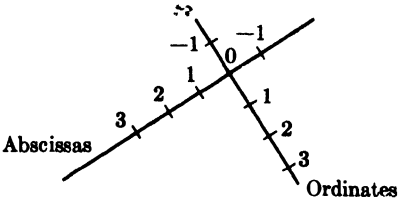


FIGURE 15–2

Exercise Group 15–4

1. Draw a figure, different from Fig. 15–2, illustrating Definition 15–2.

2. Compare your figure with your neighbor's. Did you both choose the "same" axis for abscissas? Turn your figure sideways. Do you have a different opinion about which is the first coordinate?

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* This is not a fortunately chosen word. It would be better to say "first coordinates." In most drawings in textbooks, the axis of abscissas is drawn horizontally as the reader views it. There is nothing sacred about this, we just agree to consider one coordinate before we do the other (something must be first) and call this first coordinate by the strange name *abscissa.*

† This word is not much better, but the literature of mathematics is filled with this word too, so make the best of it and learn it. Of course "second coordinate" is much better.

3. With the same drawing as in Exercise 1, reverse the coordinates on each of the axes. Is this a possible way of satisfying Definition 15–2? Suppose that we reversed the coordinates on only one of the axes.

Perhaps you will have observed a certain arbitrariness in the definition, in that we *choose* a point O and *choose* lines for axes. This is actually the way things operate in practice. In applying analytic geometry to problems in physics or engineering, it is usually necessary to *choose* coordinates. *There are no fixed coordinate systems inherent in nature.* Coordinate systems are convenient inventions that facilitate mathematical computations.

Having set up our axes, we can make certain statements concerning them. We give some of these in a theorem.

Theorem 15–1. *The coordinate axes separate the plane into four regions, called quadrants, such that if P and Q are points not on either axis and in the same quadrant, then the segment PQ does not meet either axis, whereas if P and Q are in different quadrants, then PQ meets at least one of the axes.* See Fig. 15–3.

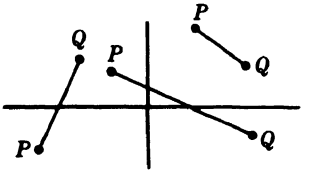

FIGURE 15–3

Proof. Each axis separates the plane. (Why?) Consider any point P not on any axis. The point P is on one side of the axis of abscissas and one side of the axis of ordinates. Let Q be any point on the same side of the axis of abscissas as P and on the same side of the axis of ordinates as P. Then PQ meets neither axis. (Why?) The set of all such points Q is a quadrant. If Q_1 and Q_2 are two points of that quadrant, then Q_1Q_2 does not intersect either axis. (Why?)

There are four ways of choosing the point P, according to which sides of the axes it is on. This gives rise to the four quadrants.

Exercise Group 15–5

1. Describe the four ways of choosing P to get the four quadrants.

2. Does the above proof require the axes to be perpendicular?

Definition 15–3. The half plane that contains the positive axis of abscissas will be called the *right half plane*, and similarly for *left half plane*.* The half plane that contains the positive axis of ordinates will be called the *upper half plane*, and similarly for *lower half plane*.

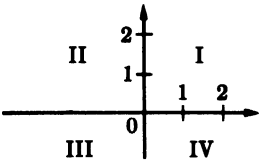

FIGURE 15–4

The quadrant that is in the right half plane and in the upper half plane will be called the *first quadrant;* the *second quadrant* lies in the upper and left half planes; the *third quadrant*, in the lower and left half planes; and the *fourth quadrant*, in the lower and right half planes. See Fig. 15–4.

Exercise Group 15–6

1. In Fig. 15–5, which is quadrant I? quadrant II? quadrant III? quadrant IV?

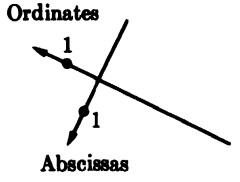

FIGURE 15–5

*2. Explain why Definition 15–3 does not require any previous knowledge of what the words "left" and "right" mean.

*3. Has there been any use made of the fact that the axes have been selected to be perpendicular? What changes would have to be made so far if the axes were not at right angles?

Definition 15–4. Let P be any point of the plane, and l_1 and l_2 the lines through P parallel, respectively, to the axis of ordinates and axis of abscissas. The line l_1 meets the first axis in a point A, with coordinate x. The line l_2 meets the second axis in a point B, with coordinate y. See Fig. 15–6. The two numbers x and y are called the *coordinates* of P. We write them in parentheses, (x, y), and refer to (x, y) as a *point*. The number x is the *abscissa* of P, and the number y is the *ordinate* of P.

Because the letters x and y are so frequently used for abscissa and ordinate, the axes are often called the *x-axis* and the *y-axis*. The plane then, when coordinatized in this way, is often called the *xy-plane*.†

* This agrees with the way axes are conventionally drawn in textbooks.

† Of course there is nothing really special about the letters x and y. Use of different letters, say u and v, would give us a uv-plane.

Given the coordinates of a point, if one marks that point in a figure, it is then said that the point has been *plotted.*

Remark. The point O has coordinates $(0, 0)$. (Why?)

If P does not lie on either axis, neither coordinate of P can be 0. If the first coordinate of P is 0, then P is on the axis of ordinates. The converse is also true.

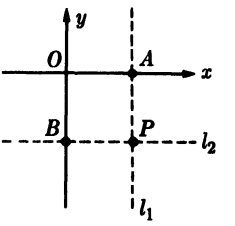

FIGURE 15–6

Exercise Group 15–7

1. Draw axes, and plot the points $(2, 4)$; $(-2, 4)$; $(-3, -\frac{1}{2})$; $(0, -5)$; $(-4, 0)$; and $(7\frac{1}{2}, -3)$.

2. A point P lies in the first quadrant. The perpendicular distance from the x-axis is $4\frac{1}{2}$ units; the distance from the y-axis is 3 units. What are the coordinates of P? What are the coordinates if distances are the same but P is in the second quadrant?

*3. Suppose that, using the same lines as axes, the unit of distance is chosen twice as large. What would be the effect upon the coordinates of points?

*4. Where are all the points in the plane with abscissa 0? abscissa negative? ordinate positive?

*5. Where are all the points in the plane whose ordinates are equal to their abscissas?

6. Draw the line that passes through the points $(-1, 5)$ and $(2, -1)$. Can you easily draw a line parallel to it, which passes through the origin?

*7. Carry out the construction of coordinates when the axes are not perpendicular.

8. Both coordinates of the point S are negative numbers. The points S and T are on the same side of the x-axis, but on opposite sides of the y-axis. What can you say about the coordinates of T?

9. The positive and negative x- and y-axes are rays from O. Describe the four quadrants as interiors of angles formed by these rays.

10. The abscissa of a point is 0. Where must it lie?

11. The ordinate of a point is 0. Where can it be?

12. What can you say about a point if the product of its coordinates is negative? 0? positive?

13. Plot five distinct points with first coordinate equal to 3. Describe the set of all points in the plane such that the first coordinate of every point of the set is 3.

14. Consider the set of all points in the plane such that the sum of the coordinates of each point is 10. Plot six such points, two in each of quadrants I, II, and IV. Why are there no such points in quadrant III?

15. Describe the set of all points (x, y) such that $x + 1 < 0$.

16. What is the perpendicular distance from $(3, -4)$ to the x-axis? to the y-axis? from $(-5, -7)$ to each axis?

17. How far is $(3, 4)$ from $(0, 0)$?

15–4 STRAIGHT LINES

Having coordinates, we naturally ask how lines may be described by coordinates. In exercises we have already examined a few lines.

For example, the set of all points (x, y) with abscissa 3 is the line shown in Fig. 15–7, and every point on the line parallel to the y-axis, at a distance 3 to the right of the y-axis, has its x-coordinate equal to 3. Hence, this line can be specified briefly by writing just $x = 3$. We say then that "$x = 3$" is the *equation* of this line.

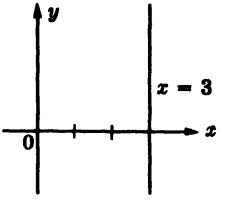

FIGURE 15–7

Conversely, when we see the equation $x = 3$, we ask ourselves, "What is the set of all points (x, y) for which $x = 3$?"

In the same way, all lines parallel to the y-axis have an equation $x = a$, where a is some real number, positive or negative. (If $a = 0$, what is the line?)

Likewise, lines parallel to the x-axis will have equations $y = b$.

Exercise Group 15–8

1. A line is parallel to the y-axis and passes through the point $(-3, 5)$. What is its equation? What if it is parallel to the x-axis?

2. What is the equation of the line that bisects the first and third quadrants? the second and fourth?

*3. Plot several points (x, y) whose coordinates satisfy the equation $x + y = 5$. Can you describe briefly the set of all points (x, y) whose coordinates satisfy this equation?

We will prove shortly that every straight line has an equation of a simple kind. But first we see that such simple equations turn out to be straight lines.

Example 1. Consider the set of all points (x, y) whose coordinates satisfy the equation $2x - 3y + 6 = 0$. We first find a few of these points, by choosing a value for x (or y) and then solving for y (or x). Thus, if $x = 0$, we have $0 - 3y + 6 = 0$, or $y = 2$. If $x = 3$, then $y = 4$. If $y = 0$, then $x = -3$.

x	y
0	2
3	4
−3	0

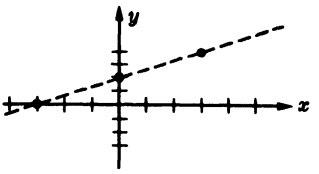

FIGURE 15–8

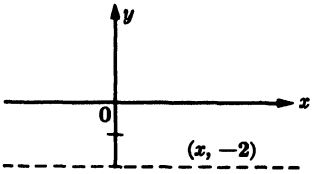

FIGURE 15–9

When these points are plotted, we see (by looking at Fig. 15–8) that they apparently lie on a line.

Find other points whose coordinates satisfy the equation, and plot them.

Example 2. Consider the set of all points (x, y) whose coordinates satisfy the equation $-4y - 8 = 0$. From the equation, we see that $y = -2$, regardless of what x is. The set of all points $(x, -2)$ is the line parallel to the x-axis and a distance 2 units below. See Fig. 15–9.

Exercise Group 15–9

Plot several points (x, y) whose coordinates satisfy the following equations:

1. $2x + 3y - 6 = 0$.
2. $x - 2y + 4 = 0$.
3. $3x - 9 = 0$.
4. $-x - y + 4 = 0$.
5. $2ax + 3ay - 6a = 0; a \neq 0$.
6. $-3x + 5y - 15 = 0$.
7. $y = 2x + 2$.
8. $x = 3y - 6$.

From the examples we observe that equations of the form $Ax + By + C = 0$, where A, B, and C are given numbers, are satisfied by the coordinates of points on lines. We now prove this theorem.

***Theorem 15–2.** *If A, B, and C are given numbers, with not both A and B zero, then all points (x, y) whose coordinates satisfy the equation $Ax + By + C = 0$ lie on a single line.*

Proof. We separate the proof into three cases.

Case 1. $A = 0$. Then $B \neq 0$ (why?) and the equation is $By + C = 0$, or $y = -C/B$. Then all points with this y-coordinate lie on the line parallel to the x-axis and passing through $(0, -C/B)$.

Case 2. $B = 0$. In this case, the points lie on a line parallel to the y-axis.

Case 3. $A \neq 0$ and $B \neq 0$. It suffices to show that any two points (x_1, y_1) and (x_2, y_2) whose coordinates satisfy the equation lie on one line with the point $(0, -C/B)$. This will be the case if and only if the right triangles indicated in Fig. 15–10 are similar, where

$$y_1 = \frac{-Ax_1 - C}{B}; \qquad y_2 = \frac{-Ax_2 - C}{B}. \tag{15–1}$$

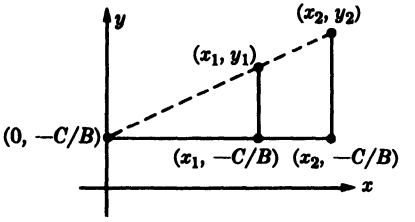

FIGURE 15–10

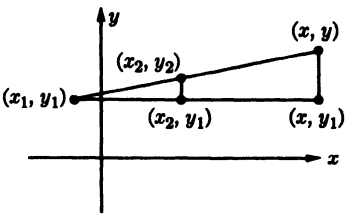

FIGURE 15–11

But these triangles are similar if and only if

$$\frac{y_2 - (-C/B)}{x_2} = \frac{y_1 - (-C/B)}{x_1}. \tag{15–2}$$

From Eq. (15–1) we see that

$$\frac{y_2 + C/B}{x_2} = \frac{-A}{B} = \frac{y_1 + C/B}{x_1}.$$

Hence we conclude that the triangles are similar and the points $[0, -(C/B)]$, (x_1, y_1), and (x_2, y_2) are collinear.

***Theorem 15–3.** *If l is a line, then there are numbers A, B, and C, with not both A and B zero, such that every point (x, y) on l has coordinates that satisfy the equation*

$$Ax + By + C = 0. \tag{15–3}$$

Proof. As in the previous theorem, we consider three cases.

Case 1. l is parallel to the x-axis. Then there is a point $(0, b)$ on l, and every point (x, y) on l satisfies $y = b$. This is of the form of Eq. (15–3), with $A = 0$, $B = 1$, and $C = -b$.

Case 2. l is parallel to the y-axis. This case is left to the student.

Case 3. l is parallel to neither axis. Choose (x_1, y_1) and (x_2, y_2), distinct points on l. Then $x_1 \neq x_2$ and $y_1 \neq y_2$. (Why?) Let (x, y) be any other point on l.

As shown in Fig. 15–11, the points (x_2, y_1) and (x, y_1) form, with (x_1, y_1), (x_2, y_2), and (x, y), two right triangles that are similar.

Because of similarity,†

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{15–4}$$

† Figure 15–11 is drawn so that $x_1 < x_2 < x$, and $y_1 < y_2 < y$, but all orders can occur for different points and different lines. In every case, however, Eq. (15–4) will be satisfied. Thus it makes no difference which point is considered as (x_1, y_1), so long as one is consistent in its use.

or

$$(y_2 - y_1)x - (x_2 - x_1)y + x_2y_1 - y_2x_1 = 0,$$

which is of the form of Eq. (15–3). This completes the proof.

Remark 1. An equation equivalent to Eq. (15–4) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{15–5}$$

This is a useful equation to remember because it enables one to write at once an equation satisfied by the coordinates of any point on a line through two given points.

Remark 2. Two different equations may correspond to the same line. Thus for example, $x = 2$ and $2x - 4 = 0$ represent the same line. Such equations are said to be *equivalent* equations. In arriving at equations for lines, we usually do not care which equivalent equation we obtain.

Definition 15–5. If $Ax + By + C = 0$ is an equation satisfied by the coordinates (x, y) of all points on a line l, we say that $Ax + By + C = 0$ is the equation of l. We also speak of the "*line* $Ax + By + C = 0$." Such an equation of the first degree in x and y will be called a *linear* equation in x and y.

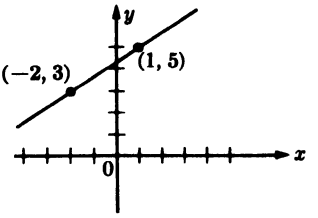

FIGURE 15–12

Example. Find the equation of the line through $(-2, 3)$ and $(1, 5)$ in Fig. 15–12, and draw the line. Using Eq. (15–5), we have

$$y - 3 = \frac{5 - 3}{1 - (-2)}[x - (-2)] \quad \text{or} \quad y = \tfrac{2}{3}x + \tfrac{13}{3}.$$

Exercise Group 15–10

1. Draw the lines

(a) $3x - 4y = 12$,
(b) $3x - 4y = 8$,
(c) $4x + 3y = 0$,
(d) $4x + 15 = 0$,
(e) $-2x - 3y + 5 = 0$,
(f) $y = \frac{-1}{2}x + 3$,
(g) $y = \frac{-1}{2}x$,
(h) $x = 3y + 3$,
(k) $5y - 12 = 0$,
(l) $2x + y = 7$.

2. Find the equations of the lines through the following pairs of points:
(a) $(-1, 1)$ and $(3, -1)$,
(b) $(2, 3)$ and $(-2, -3)$,
(c) $(a, 0)$ and $(0, b)$,
(d) $(-3, 0)$ and $(0, -5)$,
(e) $(-2, -4)$ and $(-5, -5)$,
(f) $(0, b)$ and $(1, m + b)$.

3. In Exercise 1, how many points did you plot to draw each line? Is there a reason for plotting more than two?

4. Find the coordinates of the point of intersection of the lines $3x - 5y = 7$ and $2x + 3y = -1$.

5. A line meets the y-axis in $(0, -3)$ and the x-axis in $(-4, 0)$. Where does this line intersect the line $y = 7x - 19$?

6. Show that the lines $2x + 3y = 0$ and $4x + 6y = 5$ are parallel.

15-5 DISTANCE

The distance between any two points of the plane is easy to calculate from the coordinates of the points. It is particularly easy if the line through the points is parallel to one of the axes. Thus, $\overline{AB} = 1 - (-3) = 4$, and $\overline{CD} = |x_2 - x_1| = |x_1 - x_2|$ in Fig. 15–13.

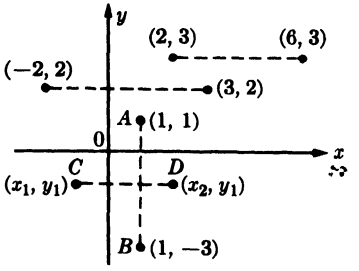

FIGURE 15–13

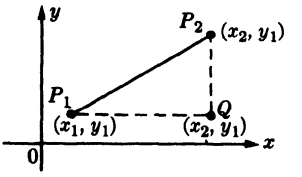

FIGURE 15–14

If the line through the points is not parallel to one of the axes, then with the point Q (x_2, y_1) a right triangle P_1P_2Q is formed, as in Fig. 15–14, and the Pythagorean theorem asserts that $\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{P_2Q}^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$. Taking square roots, we obtain $\overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Remark. In using this formula it makes no difference which point is considered to be (x_1, y_1) because $(x_1 - x_2)^2 = (x_2 - x_1)^2$ and $(y_1 - y_2)^2 = (y_2 - y_1)^2$.

Exercise Group 15–11

1. Show that the triangle with vertices $(2, 6)$, $(5, 7)$, and $(-8, -2)$ is a right triangle.

2. What kind of triangle is the one with vertices $(3, 2)$, $(-4, 4)$, and $(1, -5)$?

*3. Write an equation that states that the point (x, y) is equidistant from $(-3, 2)$ and $(2, 1)$. Show that this equation is equivalent to a linear equation. Hence conclude that the set of all points equidistant from $(-3, 2)$ and $(2, 1)$ is a straight line.

4. Show that $(-1, -18)$, $(-13, -2)$, $(11, -8)$, and $(-1, 8)$ are vertices of a parallelogram.

5. Show that if (x, y) is a point on a circle whose center is $(-2, 1)$ and radius is 3, then $x^2 + y^2 + 4x - 2y = 4$.

6. Show that the points $(2, 1)$, $(3, \frac{1}{2})$, and $(-2, 3)$ lie on a line. What geometric theorems do you use?

7. Show that the distance formula yields $|x_1 - x_2|$ if $y_1 = y_2$.

15-6 MIDPOINT

The midpoint of the segment P_1P_2 may be shown to have coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The proof is left to the student. If the line through the points is parallel to one of the axes, the proof is very simple. If $x_1 \neq x_2$ and $y_1 \neq y_2$, a right triangle may be formed as in Fig. 15–15, and parallels to the coordinate axes through the midpoint will bisect the sides in points whose coordinates are easily computed.

Example. Show in Fig. 15–16 that the midpoint formula actually gives a point equidistant from $(-1, 3)$ and $(1, -1)$.

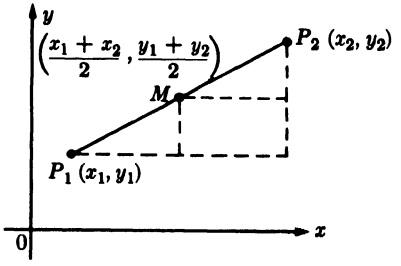

FIGURE 15–15

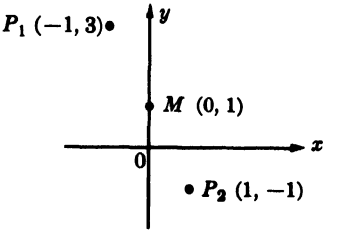

FIGURE 15–16

The point M is $\left(\frac{-1 + 1}{2}, \frac{3 + (-1)}{2}\right) = (0, 1)$. Then

$$\overline{P_1P_2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5},$$

$$\overline{P_1M} = \sqrt{1^2 + 2^2} = \sqrt{5},$$

and

$$\overline{MP_2} = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

Hence

$$\overline{P_1P_2} = 2\sqrt{5} = \sqrt{5} + \sqrt{5} = \overline{P_1M} + \overline{MP_2}.$$

Exercise Group 15–12

1. Find the midpoints of the sides, and the lengths of the medians, of the triangle whose vertices are (−4, 10), (12, 10), and (4, −2).

2. Find the coordinates of the points that divide the segment (−3, −7) (11, 17) into fourths.

*3. Find the points that trisect the segment (−3, −7) (6, 11).

*4. Derive a formula for the coordinates of the points that trisect the segment (x_1, y_1) (x_2, y_2).

*5. Show that the medians of the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) intersect in the point $[(x_1 + x_2 + x_3)/3, (y_1 + y_2 + y_3)/3]$.

6. The point (−3, 4) is the midpoint of a segment, one end of which is (5, 7). Find the other end.

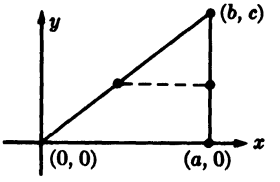

FIGURE 15–17

*7. In Fig. 15–17, show that the line joining the midpoints of the sides of a triangle is half as long as the third side, and parallel to it. [*Hint:* Choose your coordinate system so that one vertex is at the origin and another is on the positive x-axis.]

*8. Prove that the diagonals of a parallelogram bisect each other.

15–7 CURVES AND EQUATIONS

We have seen that certain sets of points can be described by equations. Thus a straight line has a linear or first degree equation, in the sense that the coordinates of every point on the line satisfy an equation of first degree in x and y of the form $Ax + By + C = 0$, and conversely. In the same way, other sets of points can be described by different equations. Perhaps the simplest example of this is furnished by circles.

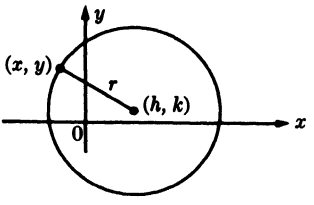

FIGURE 15–18

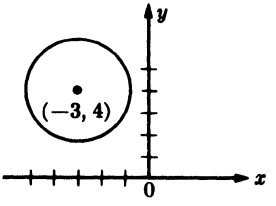

FIGURE 15–19

Suppose that (x, y) in Fig. 15–18 is a point on the circle whose center is (h, k) and whose radius is r, then

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or, equivalently,

$$(x - h)^2 + (y - k)^2 = r^2. \qquad (15\text{–}6)$$

Conversely, a point (x, y) whose coordinates satisfy Eq. (15–6) will be at a distance r from (h, k) and therefore lies on the circle.

We speak of Eq. (15–6) as "the" equation of a circle, although any equivalent equation would also be called "the" equation. For example, the equation of the circle whose center is at $(-2, 3)$, and whose radius is 4, is

$$(x + 2)^2 + (y - 3)^2 = 16$$

or

$$x^2 + y^2 + 4x - 6y = 3.$$

Given an equation in the form of Eq. (15–6), we can recognize it at once for the equation of a circle. Any equation equivalent to Eq. (15–6) is the equation of a circle. For example, consider the equation

$$x^2 + y^2 + 6x - 8y + 19 = 0.$$

Rewrite the equation as

$$(x^2 + 6x + ?) + (y^2 - 8y + ?) = -19 + ? + ?.$$

By completing the squares of the expressions in parentheses, we have

$$(x^2 + 6x + 9) + (y^2 - 8y + 16) = -19 + 9 + 16$$

or

$$(x + 3)^2 + (y - 4)^2 = 6,$$

which is the equation of a circle whose center is at $(-3, 4)$ and whose radius is $\sqrt{6}$. See Fig. 15–19.

Other curves (sets of points) will have other kinds of equations. The student is referred to books on analytic geometry for more detail.

Exercise Group 15–13

1. Write the equation of the circle whose center is (3, 5) and whose radius is 6.

2. Write the equation of the circle whose center is at $(-4, -3)$ and which (a) is tangent to the y-axis, (b) passes through the origin.

3. Find center and radius of the following circles:

(a) $x^2 + y^2 + 8x - 9 = 0$,
(b) $x^2 + y^2 - 6x - 10y = 2$,
(c) $x^2 + y^2 - 7 = 0$,
(d) $2x^2 + 2y^2 + 5x + 4y - 5 = 0$,
(e) $x^2 + y^2 - 6x - 8y + 25 = 0$.

*4. Find an equation satisfied by the coordinates of every point (x, y) whose distance from (2, 0) is equal to its distance from the line whose equation is $x + 2 = 0$.

*5. If the sum of the distances from (x, y) to $(-3, 0)$ and from (x, y) to (3, 0) is equal to 10, show that $16x^2 + 25y^2 = 400$.

CHAPTER 16

PHILOSOPHY OF MATHEMATICS

16–1 WHAT IS PHILOSOPHY?

The word "philosophy" is a combination of two Greek words and means literally *love of wisdom.* In the early Greek civilization, philosophy was the study of all branches of knowledge. As man's learning increased through the ages, certain disciplines grew until they split away from philosophy and became separate areas for study. We no longer think of medicine, law, mathematics, physics, chemistry, biology, economics, etc., as parts of philosophy, although philosophy was the father of all these sciences. That is, the first men who called attention to important concepts in mathematics, physics, chemistry, economics, political science, etc., were philosophers. It was philosophy that suggested how knowledge about these new areas should be collected and organized.

Philosophy has always concerned itself with the unknown—with those situations in which man is not sure of himself. Because of this, many of the conclusions that philosophers reach cannot be checked by experiment, as we check physical laws, and consequently many people have the mistaken belief that time devoted to philosophical thought is largely wasted.

It is impossible to describe briefly and clearly just what the subject matter of philosophy consists of today. Some of the branches into which philosophy is subdivided will be listed here, but we make no pretense of completeness.

Moral philosophy, or *ethics,* investigates the nature of right and wrong. It is concerned with distinguishing between good and bad conduct. It raises questions like, "Is an action to be judged good or bad according to its effect or its intent?" "What are *good* and *evil*?" "How shall we determine whether one action is better or worse than another?"

Religious philosophy is the study of concepts common to religions. On the other hand, *theology* is concerned with the logical structure of a single religion.

Political philosophy does not consider details of government, but rather the principles upon which various systems of government are constructed. "How much authority may the state exert over an individual?" "Are there fundamental human rights, which no state can usurp? If so, what are they?"

Metaphysics ("beyond" physics) is sometimes described as the study of final or ultimate meanings. Much that is nonsense has been written in this area, but it includes many deep and fascinating insights.

Epistemology can be described briefly as the theory of knowledge. It considers the general questions, "How do we learn?" and, "How shall we organize our knowledge?" As a matter of fact, the philosophy of *mathematics* is just a branch of epistemology; so you may think of this chapter as an introduction to a fragment of epistemology, which is itself a fragment of philosophy.

It is impossible to discuss the philosophy of mathematics without considering the philosophy of science, for mathematics is used as a tool in every science to organize and interpret scientific knowledge. We shall mention how mathematics thus often leads to the discovery of new scientific facts.

16–2 THE TASK OF SCIENCE

All information about what we call the *real* or *external* world is obtained from the five senses: sight, sound, taste, smell, and touch. Our sense organs, for example the eye, send messages to the brain. A newborn baby cannot do much with these messages, but as he grows older, he learns to interpret them. We are always amazed at the accomplishments of a person who is both blind and deaf (for example, Helen Keller), for it is difficult for us to grasp how such a person can have much understanding of the world. What we are inclined to forget is that all of us have learned how to interpret information from our sense organs so as to make useful responses. As you read this page, your eye receives impressions and sends messages to the brain. You have learned to interpret and respond to these messages.

Every scientific observation is, at first, a sense impression, that is a touch, smell, taste, sight, or sound, for some experimenter. It is the task of scientists to understand, explain, and predict happenings in the world of our sense impressions.

Some philosophers have suggested that because our only knowledge of the external world comes from sense perception, *we cannot even prove that the external world has any existence apart from our sense perception of it; that perhaps the world that we believe we see, touch, and smell exists only as things exist in a dream.*

The story is told that the great 17th-century philosopher Bishop Berkeley was so convinced by this line of thought that one day, when he was walking and meditating upon philosophical ideas, he encountered a large stone in his path and so much doubted the actual existence of the stone that he drew back his foot and gave it a vigorous kick. We have no record of the good bishop's thoughts as he hobbled home.

16–3 KINDS OF SCIENCE

The world is very complicated, and we have little understanding of it. To simplify the task of understanding our environment, certain topics that seem somewhat related are studied by themselves as separate sciences. For example, biology is the study of living organisms, and biology is itself separated into several subsciences, like zoology, the study of animal life, and botany, the study of plant life. To understand all aspects of biology, it is necessary to know some chemistry and physics. For this reason chemistry and physics are regarded as more basic, or more fundamental, than biology. Of the sciences concerned with the real world, the most fundamental is physics, which deals with the properties of matter.

All sciences become *quantitative* when they have progressed far enough. This means that measurements are made and numbers are introduced into the sciences. Those things which we measure, and to which we assign numbers, are called *quantities*. As soon as numbers are introduced into a science, mathematics becomes necessary for its study. As a matter of fact, one needs geometry in order to discuss the physics of motion of bodies, even before one makes measurements. Even psychology and sociology, two of the newer sciences that have only recently split away from philosophy, need statistics in order to organize and interpret the huge number of facts with which they deal. Thus all the sciences depend upon mathematics, so that it has been termed "the handmaiden of the sciences."*

Sometimes mathematics itself is classified as one of the sciences; and sometimes one hears the phrase "science *and* mathematics." In any event, call it what you may, mathematics is different from the other sciences in that it is *independent of experiments. Mathematics is a creation of the mind alone.*

16–4 IS PHYSICS TRUE?

We cannot give an acceptable definition of physics in a few words, but it will not take long to make the question of this section meaningful. You have studied a little physics previously, so you should know that

* E. T. BELL, *The Handmaiden of the Sciences.* Baltimore: The Williams and Wilkins Co., Baltimore, 1937.

physics is concerned with forces, motion, heat, sound, electricity, and much else. For example, in that part of physics called mechanics, it is learned that if at some instant we know the position and velocity of a ball, and if in addition we know the forces acting upon it, then we can *calculate* where it will be at any later time.

You need not concern yourself here with the exact method used to make this calculation. What is important is that *given certain information, we can predict what will happen.* Physics, and indeed every science, is just as "true" as its predictions are *reliable.* As soon as a physical theory fails to predict what will happen, then it becomes necessary to construct a new theory that will predict more accurately. In physics, the final test of "truth" is whether the predictions we get from our theory actually agree with our observations. As an example of this, Aristotle had written in his *Physics* that heavier bodies would fall more rapidly than lighter ones. This prediction is a logical consequence of Aristotle's physical *theories*, or *laws.* Every school child knows how Galileo *proved* these theories *false* by dropping balls of various sizes and weights from the leaning tower of Pisa. Every observer who was open-minded enough to watch the experiment (and there were indeed some who were not) became convinced that Aristotle's laws of motion were false and that it was necessary to replace them by new laws that would come closer to predicting the events that were actually observed.

The physical laws used by the physicist to make his predictions are actually *assumptions* and are expressed in terms of mathematical formulas. If the mathematical calculations employed do not lead to correct predictions, this does not mean that the mathematics is incorrect.* Instead it means that the physical laws do not agree well enough with reality, that the *physical assumptions* are at fault.

When the physicist defines physical concepts like force, weight, etc., and then gives physical laws that tell how these concepts are related, he is creating what we call a *mathematical model.* What he is saying is that *if* there are things in the real world that correspond to these concepts, and *if* these things are related to one another according to the rules called physical laws, *then* mathematics tells us what will happen in the real world.

If the predictions come out wrong, then the physicist must construct a better mathematical model. Often it happens that the physicist cannot solve the mathematical problems that arise within the mathematical model. Then he calls upon the mathematician for help. In this way physics supplies mathematics with interesting problems.

As an example of a simple mathematical model, consider our solar system. The physicist thinks of the sun and planets as points that are

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* Of course someone could make a computational mistake, but this is the fault of the computer, not of the mathematics he is using.

tracing out curves as they move through space. By using calculus and analytic geometry, these curves are described by mathematical equations that predict how the planets ought to move through the sky if the law of gravitation is valid.

It is most interesting that early in the 18th century, when astronomers calculated how the planets *should* move according to the law of gravitation, it was noticed that one of the planets, Uranus, did not follow the exact path predicted for it by the mathematical model. The astronomers had so much faith in the mathematical model that they became sure that there must be an "invisible" planet, which was pulling Uranus off its course with its gravitational attraction. The Englishman John C. Adams and the Frenchman LeVerrier set about attempting to solve the problem of determining the path of the "invisible" planet.* Working with the mathematical model, they computed where it *ought* to be to affect the course of Uranus as it did. When observers focused their telescopes upon that part of the heavens, they saw the planet Neptune precisely where the mathematical model predicted that it would be found. This is but one of the many dramatic illustrations of how an *abstract* mathematical model can lead us to the discovery of new facts about our universe.

16–5 IS MATHEMATICS TRUE?

If by this question we mean, "Does the real world behave like the mathematical models that we construct?" then we are asking a nonmathematical question. The *truth* of mathematics is independent of the real world. The *usefulness* of one kind of mathematics or another does depend upon the real world.

Pure mathematics is concerned with the logical consequences of postulate systems. In pure mathematics, postulate systems are studied primarily because they are interesting, and often without regard to any future use. In the history of mathematics, there are many postulate systems that have been studied for their own sakes and only much later found to be applicable to the external world. For example, the non-Euclidean geometry needed in the theory of relativity was first studied many years before any physical application of it was made.

Applied mathematics is concerned with those parts of pure mathematics that are of greatest use in the science of today. Scientific applications often suggest the constructions of new postulate systems. Mathematics and science thus serve as stimuli to each other.

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* A fascinating account of this work can be found in J. R. Newman's *The World of Mathematics*. Vol. 2, pp. 822–839.

We have not yet given a direct answer to the question, "Is mathematics true?" What is "truth," anyway? Each of us has some sort of feeling about what "truth" means. Truth, for the physicist, is *that which works*, and this attitude is characteristic of science. Truth, in practical affairs, usually refers to *whether some event actually took place*. Here, we often take the word of others whom we trust. Truth, in spiritual matters, depends upon *faith*. In general, agreement among people regarding spiritual beliefs is difficult to obtain, for there is no test by experience that we can all agree upon.

When we say that a system of mathematical postulates is "true," we mean only, *these postulates are free from contradiction*. That is, if it is impossible to prove from our postulates that a statement is true, and then also prove that the same statement is false, in this case and only in this case do we call our postulates true. Remember that we do not assert in mathematics that anything actually exists in the physical world. Rather, our theorems have the form: *If* a set of things has so-and-so properties, *then* it has such-and-such properties.

We have developed our geometry this year from statements that dealt with a set of points that we called our plane. A mathematician is naturally pleased to encounter in the physical world a set of objects that actually has the properties he has postulated for one of his mathematical systems, for then he can be sure that the set he finds will have other properties that he has established by studying his abstract system. A natural question to ask at the close of this year's study of geometry is whether there actually exists in the real world a set of things that satisfies the postulates of our geometry. Does the space we live in actually have the properties of the Euclidean space we study in geometry?

The answer will probably not satisfy you. The answer is, *we do not know. Furthermore, no mathematician ever expects to know.* Our geometric plane consists of *infinitely* many points. *It will never be possible to determine whether our universe is finite or infinite in extent.* Although we have no assurance that any part of the real world actually fits our geometric postulates, experience has shown us that much of our geometric theory is applicable to the world around us; and we can indeed use our geometry to make very accurate predictions.

16–6 FREEDOM FROM CONTRADICTION

In this section we discuss a very deep question. When we have a postulate system before us, how do we know that these postulates are *consistent?* How do we know that we shall never deduce from these postulates, by correct logical reasoning, two contradictory conclusions? Let us

give an example, first given by Bertrand Russell,* to illustrate what we mean.

Russell's barber. Let us assume that, in a certain small town, there is but one barber. We assume that this barber shaves everyone in the town who does not shave himself, and no one else.

We now ask who shaves the barber.

If the barber does not shave himself, *then* he is a person who does not shave himself, and therefore he is shaved by the barber—himself. Since this is impossible, we have proved that the statement *"The barber does not shave himself"* is false. Hence, the barber shaves himself. But, by our assumptions, *if* the barber shaves himself, *then* he is one of the men who is not shaved by the barber—again himself. This proves that the statement *"The barber shaves himself"* is false, and hence, *"The barber does not shave himself,"* is true. But now we have arrived at a contradiction, having shown from our postulates that the statement *"The barber does not shave himself" is both true and false.*

This example shows that in forming a set of postulates, we must be careful that our postulates are consistent; that is, that they do not lead to contradictory conclusions. When our postulate sets are very simple, it is sometimes possible to prove that no contradiction can ever be reached by logical reasoning. For more complicated postulate sets, this may not be possible.

David Hilbert spent much of the latter portion of his life trying to prove the consistency of certain interesting mathematical systems. At about the same time that he wrote his *Foundations of Geometry,* he showed that geometry is consistent if arithmetic is consistent. He accomplished this proof by using the ideas of Chapter 15. The details need not concern us here. All we need know is that our geometry is consistent if arithmetic is. We should remark that non-Euclidean geometries have also been proved to be consistent, providing that arithmetic is. Of course this has nothing to do with the question of whether the space we live in is Euclidean or non-Euclidean.

Now, how do we know that arithmetic is consistent? The answer may again seem disappointing, for the answer is, *we do not know!* Perhaps we never will know! There is no proof that we can never be led to an impossible conclusion in arithmetic. For example, we do *not* know that it is impossible to *prove,* by a valid chain of logical arguments, that $0 = 1$!

All mathematicians believe that arithmetic is consistent. This belief is based upon faith, not proof. Of course, arithmetic is useful. It works,

* An English philosopher who is interested in the foundations of mathematics. At about the turn of the century, he wrote *Principia Mathematica* with A. N. Whitehead. In the book they attempt to derive all mathematics from logic alone.

and helps us reach conclusions that turn out to be correct. But here is the position in which the scientist finds himself. If an inconsistency is discovered in arithmetic, the postulates for arithmetic will have to be changed. The possibility seems to be remote, so it does not worry working mathematicians too much.

16–7 MATHEMATICAL RESEARCH

Euclid's geometry is more than 2000 years old. The average person holds the erroneous opinion that all of mathematics was discovered long ago, and that today we study only old stuff. There are, however, many journals published throughout the world that are devoted primarily to *new* mathematics. In fact, during the past fifty years, more new mathematics has been invented* than in all previous history! Mathematical progress is stimulated by both intellectual curiosity and the needs of modern science. Because of the mathematical nature of modern science and engineering, mathematicians are in greater demand than ever before and command excellent salaries.

It is impossible to describe here the breadth and depth of current mathematical research. As yet, we have not undertaken enough elementary mathematics to appreciate such a description. However, in your study of plane geometry, you have made a fine beginning by studying a single mathematical system in some detail. The logical features of geometry are common to all mathematics, so that you have before you one example of a postulate system and some of its consequences.

16–8 MATHEMATICS AND LIFE

How may an understanding of the philosophy of mathematics affect you as an individual in your everyday life? Mathematicians are much like any other group. If you were to attend the International Congress of Mathematicians, which convenes once every four years, you would see a group of people differing widely in dress, speech, religious and political beliefs, and physical appearance. But you would, if you were observant enough, see many common characteristics. Everyone there would have a way to communicate with everyone else—not through English, French, German, or Russian, but through the universal symbolism of mathematics. You would soon realize that these folk, while actively interested in the real world, were still far more interested in the world of ideas.

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* Is mathematics *discovered* or *invented*? The first word suggests that mathematics is "sitting around waiting to be found." The second suggests that mathematics is created by the mind, that human activity is essential.

You would see that each one present had a tremendous respect for the personalities and intellectual capabilities of his neighbors.

The philosophy of mathematics speaks to those who will listen and assures them that even more important than wresting food, shelter, and material riches from the earth is the acquisition of knowledge and understanding.

SUGGESTED READING

From *The World of Mathematics*, by J. R. Newman.

1. PHILIP E. B. JOURDAIN, "The Nature of Mathematics." Vol. 1, pp. 4–71.
2. J. W. N. SULLIVAN, "Mathematics as an Art." Vol. 3, pp. 2015–2021.
3. RAYMOND L. WILDER, "The Axiomatic Method." Vol. 3, pp. 1647–1667.

APPENDIX

This appendix lists all the postulates for plane geometry, along with some of the definitions and theorems. In addition to being convenient for reference purposes, the appendix merits study, because reading it over and thinking about the order of the theorems will help you fix the structure of Euclidean geometry in your mind.

THE POSTULATES

There are three groups of postulates, and three postulates listed singly. They are:

Postulates of incidence (3 postulates)
Postulates of order or betweenness (5 postulates)
Postulates of congruence (7 postulates)
The postulate of Archimedes
The completeness postulate
The parallel postulate.

The Incidence Postulates

1. There are at least three points not all on a line.
2. For any two different points, there is exactly one line containing these points.
3. Every line contains at least two points.

The Betweenness Postulates

4. If B is between A and C, then A, B, and C are three different, or distinct, points on a line.
5. For every three points on a line, exactly one of them is between the other two.
6. Any four points on a line may be named A_1, A_2, A_3, A_4, so that the only betweenness relations are the same as the order of the subscript numbers. That is, the betweenness relations are: A_2 between A_1 and A_3; A_2 between A_1 and A_4; etc.
7. If A and B are two points, then there is at least one point C such that B is between A and C, and at least one point D such that D is between A and B.

8. Every line separates the plane. By this we mean that all points of the plane not on the line are divided into two sets, called the two sides of the line, having the following properties:

(a) If P and Q belong to one of these sets, then no point of l is between P and Q. We say then that P and Q are on the same side of l.

(b) If P and Q are in different sets, then there is a point on l which is between P and Q.

Linear Congruence Postulates

9. Given points A and B on a line l and a point A' on a line l', then on a given side of A' on l' there is exactly one point B' such that AB is congruent to $A'B'$. To indicate that AB is congruent to $A'B'$, we write "$AB \cong A'B'$."

In congruence the order of the points does not matter, that is, if $AB \cong A'B'$, then also $AB \cong B'A'$, $BA \cong A'B'$, and $BA \cong B'A'$.

10. Congruence also satisfies the following:

(a) $AB \cong AB$.

(b) If $AB \cong A'B'$, then $A'B' \cong AB$.

(c) If $AB \cong A'B'$ and $A'B' \cong A''B''$, then $AB \cong A''B''$.

11. (This postulate tells how to "add" and "subtract" segments.) Suppose that B is between A and C on a line l, and that B' is between A' and C' on a line l'.

(a) If $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.

(b) If $AC \cong A'C'$ and $BC \cong B'C'$, then $AB \cong A'B'$.

Archimedes' Postulate

12. If AB is any segment on a line l and CD any other segment, then there is a finite number n of points $A_1, A_2, \ldots, A_n$ on l such that the points $A, A_1, A_2, \ldots, A_n$ are arranged in this order, and all the segments $AA_1, A_1A_2, \ldots, A_{n-1}A_n$ are congruent to CD, and either $B = A_n$ or B lies between A_{n-1} and A_n.

The Completeness Postulate

13. For every line l and any given point A on l, and for any positive real number x, there is, on a given side of A, a point B such that $\overline{AB} = x$.

Congruence Postulates for Angles

14. If $\angle A$ is not a straight angle and if r' is a ray from A' on a line l', then on a given side of l' there is exactly one ray s' from A' such that the angle A', with sides r' and s', is congruent to angle A. We write this as "$\angle A' \cong \angle A$."

If $\angle A$ is a straight angle, then the straight angle at A' with one side r' is the only angle with one side r' congruent to angle A.

15. (a) $\angle A \cong \angle A$.

(b) If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.

(c) If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

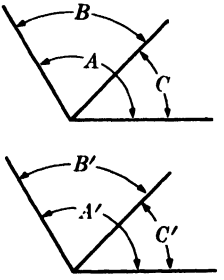
16. If in the figure any two of the pairs $\angle A$ and $\angle A'$, $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$ are congruent, then the third pair of angles are congruent. In other words:

(a) If $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$, then $\angle C \cong \angle C'$.

(b) If $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$, then $\angle A \cong \angle A'$.

(c) If $\angle C \cong \angle C'$ and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$.

The angle A may be a straight angle.



17. If for $\triangle ABC$ and $\triangle A'B'C'$, $AB \cong A'B'$, $AC \cong A'C'$, and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$.

The Parallel Postulate

18. Through a point not on a line there is no more than one line parallel to the given line.

SELECTED DEFINITIONS

Although every definition has a purpose, certain ones are more necessary than others, and most of these essential definitions are listed here. In some cases an abbreviated form of the definition is given when no misunderstanding can arise.

1. Each set of points in the plane is called a *geometric figure.*

2. If A and B are any two points on a line, then the set of points consisting of A and B and all the points between A and B will be called the *line segment* (or *segment*) determined by A and B. The segment will be indicated by "AB." The points A and B are called the *ends* of the segment.

3. If A, B, and C are three points not all on a line, the set of points of the segments AB, BC, AC is called a *triangle*. We call the points A, B, and C *vertices* and the segments AB, AC, BC *sides* of the triangle. To indicate "triangle ABC," we write "$\triangle ABC$."

4. If O is a point of a line l and A and B are two other points of l, then we say that A and B are on the *same side of* O if O is not between A and B. We say that A and B are on *opposite sides of* O if O is between A and B.

5. A *ray* from O, or a ray with end O, is a set of points consisting of O and all the points on one side of O of a line l through O.

6. The set of all points on two rays from a point O is called an *angle*. The point O is called the *vertex* of the angle, and the two rays are called the *sides* of the angle. If the two rays are on the same line, the angle is called a *straight angle*.

7. Consider an angle, not a straight angle, with vertex O and whose sides are the rays r_1 and r_2 on the lines l_1 and l_2, respectively. The *interior of the angle* is the set of all points on the same side of l_1 as r_2 and on the same side of l_2 as r_1. The *exterior of the angle* is the set of all points in the plane which are not points of the angle and are not in the interior of the angle.

8. Two angles are *adjacent* to each other if they have a common side and vertex and have no common interior points.

9. Let AB be any segment and $A'B'$ any other segment on a line l'. On the same side of A' as B', let B'' be the point such that $AB \cong A'B''$. If B' is between A' and B'', we say that $AB > A'B'$. If B'' is between A' and B', we say that $AB < A'B'$.

10. For the definition of *length of a segment* read Chapter 5, Section 4.

11. A *right* angle is one which is congruent to its supplement.

12. Two triangles are said to be *congruent* if they can be labeled $\triangle ABC$ and $\triangle A'B'C'$ such that $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. To indicate that "triangle ABC is congruent to triangle $A'B'C'$," we write "$\triangle ABC \cong \triangle A'B'C'$."

13. Let A and B be any angles, not straight angles. Let $\angle B'$ be an angle congruent to $\angle B$ such that one side of $\angle B'$ is the same as the first side of $\angle A$, and the other side of $\angle B'$ and the second side of $\angle A$ lie on the same side of the first side of $\angle A$. If the second side of $\angle B'$ is interior to $\angle A$, we say $\angle B < \angle A$. If the second side of $\angle B'$ is exterior to $\angle A$, we say $\angle B > \angle A$. If $\angle A$ is a straight angle and $\angle B$ is not a straight angle, we say $\angle A > \angle B$ and $\angle B < \angle A$.

14. Let $P_1, P_2, \ldots, P_n$ be any set of n points. The set of points on all the segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$ is called a *polygonal path*. The path is said to join, or connect, the endpoints P_1 and P_n. The points $P_1, \ldots, P_n$ are called the *vertices*. In particular, if $P_1 = P_n$, the polygonal path is called a *polygon* and designated as polygon $P_1P_2 \ldots P_{n-1}$.

A polygon is *convex* if for every two points on the polygon each point between them is either on the polygon or in the interior. A convex, equiangular, and equilateral polygon is called *regular*.

15. Two lines are *parallel* if they do not intersect.

16. If the ray OB is in the interior of $\angle AOC$ (or $\angle AOC$ is straight), then $\angle AOC$ is the *sum* of $\angle AOB$ and $\angle BOC$.

17. If AB and CD are segments, the ratio of AB to CD is the number $\overline{AB} \div \overline{CD}$. The statement that the segments AB and CD are proportional to the segments $A'B'$ and $C'D'$ means that $\overline{AB}/\overline{A'B'} = \overline{CD}/\overline{C'D'}$. We say that the sides of $\triangle ABC$ are proportional to the sides of $\triangle A'B'C'$ if $\overline{AB}/\overline{A'B'} = \overline{BC}/\overline{B'C'} = \overline{CA}/\overline{C'A'}$.

18. A *circle* is determined by a point O, called the center, and a positive number r. The circle is the set of all points A of the plane such that the distance $\overline{OA} = r$. If A is a point of the circle, then the segment OA is called a *radius* (plural, *radii*). If A and B are on the circle and the segment AB contains O, then AB is called a *diameter*.

19. If a circle has center O and radius r, the *interior of the circle* is the set of all points B such that $\overline{OB} < r$. The *exterior of the circle* is the set of all points C such that $\overline{OC} > r$. A *chord* is a segment determined by two points E and F on the circle. The line through E and F is a *secant*. A *central angle* of a circle is an angle whose vertex is at the center of the circle. An *inscribed angle* is an angle with vertex on the circle and a second point of the circle on each of the two sides of the angle.

20. Suppose that for each positive integer n there is associated in some way, a number x_n. The numbers $x_1, x_2, \ldots, x_n$ then form an *infinite sequence*. The number x_n is called the nth term of the sequence. We designate the sequence by the symbol $\{x_n\}$.

21. The number L is called the *limit of the sequence* x_n if, for large values of n, x_n is necessarily arbitrarily close to L. We then say that L is the limit of the sequence of terms x_n and write $L = \lim \{x_n\}$.

22. The *circumference* C *of a circle* is the limit of the perimeters p_n of regular inscribed polygons. That is, $C = \lim \{p_n\}$.

23. The number C/d, which is the same for all circles, will be denoted by the Greek letter π(pi).

24. The *area* S *of a circle* is the limit of the areas of inscribed regular polygons. That is, $S = \lim \{S_n\}$.

25. Given a circular arc, let l_n be the length of a regular polygonal path of 2^n sides inscribed in the arc. The *length* l *of the arc* is defined to be the following limit: $l = \text{limit } \{l_n\}$.

26. On a circle of radius r an arc of length $\pi r/180$ is subtended by a central angle of *one degree* (1°).

27. If on a circle of radius r a central angle subtends an arc of length L, then the *measure of the angle in degrees* is $(L/\pi r) \cdot 180°$.

SELECTED THEOREMS

The theorems selected below have been chosen to help you remember how the ideas have been developed and *in what order*. There is no attempt at completeness, and in some cases the statement is less precise than in the text; when this is the case, you should try to fill in the details or refer back to the text.

1. Two different lines intersect in at most one point.

2. Let O be any point of a line l. The point O separates the line into two sets, called the two sides of the point O on the line, having the following properties: (1) If P and Q belong to the same side of O, then either OPQ or OQP. (2) If P and Q belong to different sides of O, then POQ.

3. If P and Q are both in the interior of $\angle O$, then the segment PQ is in the interior of $\angle O$.

4. If rays r_1 and r_2 form $\angle O$, a ray r from O, other than r_1 or r_2 lies either completely in the interior or completely in the exterior of $\angle O$.

5. If P is in the interior of $\angle O$ and Q is in the exterior of $\angle O$, then the segment PQ contains a point of the angle.

6. If AB and $A'B'$ are any two segments, either $AB > A'B'$, or $AB \cong A'B'$, or $AB < A'B'$, and exactly one of these holds.

7. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.

8. If $AB \cong A'B'$, then $\overline{AB} \nsim \overline{A'B'}$.

9. If $\overline{AB} = \overline{A'B'}$, then $AB \cong A'B'$.

10. $AB > CD$ if and only if $\overline{AB} > \overline{CD}$.

11. The S.A.S. theorem for congruent triangles.

12. The A.S.A. theorem.

13. The S.S.S. theorem.

14. For any two angles $\angle A$ and $\angle B$ either $\angle A < \angle B$, or $\angle A \cong \angle B$, or $\angle A > \angle B$, and exactly one of these is true.

15. All right angles are congruent to one another.

16. An exterior angle of a triangle is greater than either opposite interior angle.

17. In $\triangle ABC$, $AB > AC$ if and only if $\angle C > \angle B$.

18. In $\triangle ABC$, $\overline{AB} + \overline{BC} > \overline{AC}$.

19. If l is any line and A any point not on l, then there exists at least one line on A parallel to l.

20. If parallel lines are cut by a transversal, the alternate interior angles are congruent.

21. The sum of the angles of a triangle is a straight angle.

22. If a transversal is cut by three parallel lines in congruent segments, the same is true for every transversal of these lines.

23. If for $\triangle ABC$ and $\triangle A'B'C'$, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$, then the triangles are similar.

24. If $\overline{AB}/\overline{A'B'} = \overline{AC}/\overline{A'C'} = \overline{BC}/\overline{B'C'}$, then $\triangle ABC$ is similar to $\triangle A'B'C'$.

25. In a right triangle the altitude on the hypotenuse forms two right triangles similar to the given triangle.

26. In $\triangle ABC$ with $\angle C$ a right angle, $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$.

27. The area of a triangle is one half the base times the altitude.

28. Suppose that circles with centers A and B each have radius $r > \frac{1}{2}\overline{AB}$. Then these circles intersect in two points on opposite sides of the line AB.

29. Exactly one circle is circumscribed about a regular polygon.

30. If regular polygons of n sides are inscribed in two circles, then the perimeters of the polygons are proportional to the radii of the circles.

31. For a given circle let p_n be the perimeter of a regular polygon of n sides inscribed in the circle. The numbers p_n, $n = 3, 4, 5, \ldots$, form a sequence which has a limit.

32. The ratio C/d (circumference to diameter) is the same for all circles.

33. If S_n is the area of a regular polygon of n sides inscribed in a circle, then limit $\{S_n\}$ exists.

34. The area of a circle is equal to one half the circumference times the radius.

35. If two central angles are congruent, they subtend arcs of equal length.

36. If $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are adjacent arcs (that is, if they have only point B in common), then length $\overset{\frown}{ABC}$ = length $\overset{\frown}{AB}$ + length $\overset{\frown}{BC}$.

37. The measure of an angle inscribed in a circle is one half the measure of the central angle which subtends the same arc.

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