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ESSENTIALS OF TRIGONOMETRY WITH APPLICATIONS

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PREFACE

The distinguishing features of this book are (1) a concise presentation of the fundamental part of plane trigonometry up to the solution of triangles; (2) the use of five-place logarithms in the solution of triangles and in other applications, with due allowance for the accuracy of data; (3) a chapter on spherical trigonometry; (4) a chapter on applications, which has been written to give essential preparation to those who may enter upon military or naval service. This text, therefore, should not only meet the needs of students for whom trigonometry is mainly useful as an introduction to further courses in mathematics, but it should be especially valuable for those who are interested in applications.

The first five chapters are taken from the authors' text *A Brief Course in Trigonometry* with minor revision. In the next three chapters part of the material has been adapted from the same text and from the authors' *Trigonometry, Plane and Spherical*. Most of the last chapter is new.

Use of Coördinates. As in their other texts, the authors have here made use of coördinates, almost from the start. In preparatory courses in mathematics, students have become familiar with rectangular coördinates. The briefest and best introduction to trigonometry, we believe, is based on the utilization of this knowledge, together with the use of polar coördinates. This makes it relatively simple to start with the general angle and to give proofs that apply to all cases. It also lays a firm foundation for the study of analytic geometry and calculus.

Applications. Throughout the first eight chapters both text and exercises present many applications of trigonometry to problems of engineering and of other sciences. The ninth chapter is devoted to the use of trigonometry in surveying, in problems connected with maps and with artillery, and in navigation. Under the last head will be found sections on plane sailing, parallel sailing, dead reckoning, great circle sailing, and celestial navigation. In connection with the use of artillery such terms as the mil are explained, and problems of aiming are briefly treated.

Exercises and Answers. An unusually large number of exercises have been included. Those in Chapter IX should be especially useful as illustrations of the applications of trigonometry which are not found in most textbooks on the subject.

Answers are given to odd-numbered problems where giving an answer would not spoil the exercise. Where the proper number of significant figures is a part of the exercise, attention is paid to this point in the answer. Except where the contrary is indicated, the five-place tables given in this book were used in computing answers for the last four chapters.

Four- and Five-Place Tables. Both four-place and five-place tables are available. The tables have been set in large, clear type and are noteworthy for their legibility.

Number of Lessons. A brief course in the essentials of plane trigonometry is contained in the first seven chapters and may be covered in from 30 to 35 lessons. By omission of starred sections and by a reduction in the number of exercises which are assigned, the course may be shortened by 5 lessons. By inclusion of Chapter VIII on Spherical Trigonometry, Chapter IX on Applications, and the starred sections, a course of 50 to 60 lessons is available.

D. R. CURTISS
E. J. MOULTON

CONTENTS

CHAPTER I

TRIGONOMETRIC FUNCTIONS

SECTION	PAGE
1. The general angle in trigonometry	1
2. Measurement of angles	2
3. Coördinates. Standard position of an angle. Triangle of reference	4
4. Quadrants	6
5. The six trigonometric functions	6
6. Signs of the functions	8
7. Trigonometric tables obtained by measurements	9
8. Functions of 45° and related angles	12
9. Functions of 30° , 60° , and related angles	13
10. Functions of quadrantal angles	14
11. Problems in which a function is given	16
★12. Projection on coördinate axes	17
★13. Vectors. Components. Resultants	18

CHAPTER II

RIGHT TRIANGLES

14. Relations between the sides and acute angles of a right triangle . .	20
15. Relations between functions of the acute angles of a right triangle	20
16. Solution of right triangles	21
17. Tables of values of functions	21
18. Interpolation	22
19. The table of squares of numbers. Square roots	24
20. Significant figures	26
21. Typical solutions of right triangles	27
22. Geometrical applications	28
23. Heights and distances. Bearings	30

CHAPTER III

REDUCTION FORMULAS. LINE VALUES. GRAPHS

24. Functions of a general angle expressed as functions of the corresponding acute angle	35
--	----

SECTION	PAGE
25. Functions of $180^\circ - \theta$	37
26. Functions of $n \cdot 180^\circ \pm \theta$	39
27. Functions of $n \cdot 90^\circ \pm \theta$, where n is odd	40
28. Line values	41
29. Properties of the sine function	43
30. Graph of the sine function in rectangular coördinates	44
31. Properties and graph of the cosine	45
32. Properties and graphs of the tangent and secant	46

CHAPTER IV

TRIGONOMETRIC IDENTITIES

33. Fundamental identities involving one angle	48
34. Other identities involving one angle	50
35. Addition formulas	52
36. Addition formulas for sine and cosine functions	53
37. Formulas for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$	55
38. Formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$	58
39. Functions of 2α	59
40. Functions of $\frac{\alpha}{2}$	60
41. Formulas for sums and differences of two sines or two cosines	64

CHAPTER V

RADIAN MEASURE. INVERSE FUNCTIONS.
TRIGONOMETRIC EQUATIONS

42. Radian measure	68
43. Inverse functions and their principal values	71
44. Trigonometric equations	75

CHAPTER VI

LOGARITHMS. FIVE-PLACE TABLES

45. The definition of a logarithm	78
46. The table of five-place logarithms of numbers	78
47. Logarithms of positive numbers	80
48. Laws of exponents and of logarithms	82
49. Computation with the use of logarithms to base 10	83
★50. Cologarithms	84
51. Five-place logarithms of trigonometric functions	85

CHAPTER VII

SOLUTION OF TRIANGLES. FIVE-PLACE LOGARITHMS

SECTION	PAGE
52. Right triangles	89
53. The problem of solving an oblique triangle	91
54. The law of sines	92
55. Check formulas involving all six parts	93
56. Case I. Given two angles and one side	94
57. Case II. Given two sides and an angle opposite one of them	95
58. The law of tangents	98
59. Case III. Given two sides and the included angle.	99
60. The law of cosines	100
61. The half-angle formulas	101
62. Case IV. Given three sides	103
63. Two formulas for the area of a triangle	104

CHAPTER VIII

SPHERICAL TRIGONOMETRY

64. Spherical triangles	109
65. Polar triangles	110
66. Formulas for right triangles	111
67. Napier's rules	111
68. Proofs of the formulas for right triangles	112
69. Rules regarding the relative magnitude of parts.	115
70. Solution of right spherical triangles	116
71. Isosceles triangles. Quadrantal triangles.	119
72. The law of sines	120
73. The laws of cosines	121
74. Formulas for half-angles and half-sides	122
75. Napier's analogies	124
76. Solution of oblique spherical triangles	126
77. Case I. Given three sides	127
78. Case II. Given three angles	128
79. Case III. Given two sides and the included angle.	129
80. Case IV. Given two angles and the included side.	130
81. Case V. Given two sides and an angle opposite one of them (ambiguous case)	131
82. Case VI. Given two angles and a side opposite one of them (ambiguous case)	133
83. Area of a spherical triangle.	134

CHAPTER IX

APPLICATIONS

SECTION	PAGE
84. Introduction	136
85. Plane surveying	136
86. Use of maps in artillery problems	140
87. Measurement of angles. The mil	142
88. Plane sailing	146
89. Parallel sailing	147
90. Middle latitude sailing	148
91. Dead reckoning	150
92. Great circle sailing	153
93. Positions on the celestial sphere	156
94. The astronomical triangle, and problems connected with its solution	158
ANSWERS TO ODD-NUMBERED PROBLEMS	161

ESSENTIALS OF TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS

In trigonometry we are concerned with relations involving angles, which are best expressed by means of six *trigonometric functions*. Trigonometry may be described as a study of these functions and their uses.

1. The general angle in trigonometry. An *angle* as defined in plane geometry is a figure, such as BAC in Figure 1, formed by two rays, AB and AC , with common end-point at A . The word *ray* here denotes a portion of a straight line extending indefinitely in one direction from a point A . The rays AB and AC are said to be the *sides* of the angle BAC .

In trigonometry we add to this definition by considering an angle BAC as *generated by rotating a ray* from the position AB until it coincides with AC . We then call AB the **initial side** of the angle, and AC the **terminal side**.

Rotation may be in either direction, and the amount of rotation

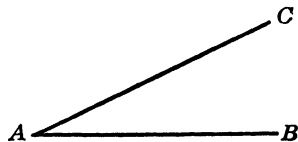


FIG. 1

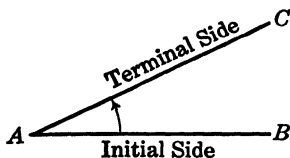


FIG. 2

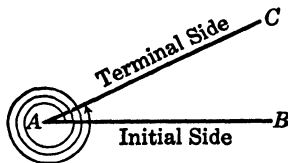


FIG. 3

may include one or more complete revolutions. In each of the Figures 2, 3, 4, 5 the curved arrow indicates the direction and amount

of rotation for the angle BAC as thus defined. Figure 3, for example, indicates an angle of more than three complete revolutions, while the angle in Figure 5 has more than two, in the opposite direction.

In order to designate the *direction* of rotation, we call an angle **positive** if it is generated by **counterclockwise rotation** (that is, rotation in the direction opposite to that in which the hands of a clock move; see the angles in Figures 2 and 3) and **negative** if it is generated by **clockwise rotation**, as in Figures 4 and 5.

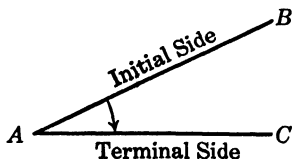


FIG. 4

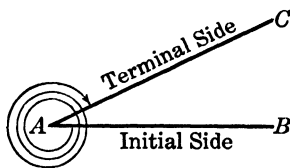


FIG. 5

2. Measurement of angles. A familiar system for the measurement of angles employs *degrees*, *minutes*, and *seconds*. An angle of one degree is the ninetieth part of a right angle, while a sixtieth of a degree is one minute, and a sixtieth of a minute is one second. Thus, if we employ the usual notation,

$$1 \text{ right angle} = 90^\circ, \quad 1^\circ = 60', \quad 1' = 60''.$$

In designating the measure of an angle we prefix a minus sign if the angle is negative, but if the angle is positive we either prefix a plus sign, or no sign at all. Thus in Figure 6 the measure of the first angle is 90° , of the second angle is 585° , and of the third is -225° .

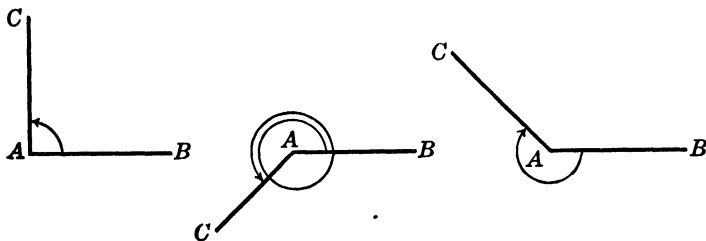


FIG. 6

In another system of measurement a right angle is divided into 100 equal parts called *grades*, a grade into 100 *minutes*, and a minute into 100 *seconds*. Still another system, that of *radian* measure, is discussed in Chapter V of this book.

A useful instrument for drawing or measuring angles is the **protractor**, whose use is illustrated in Figure 7. Here the instrument

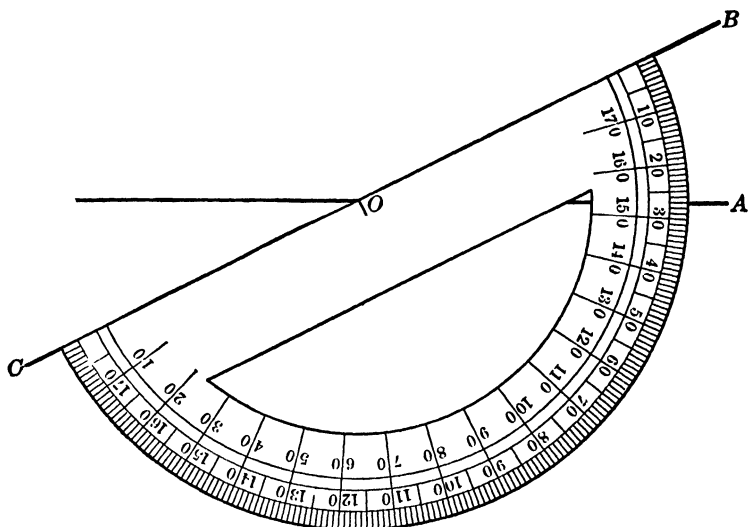


FIG. 7

is so placed as to measure the angle AOB ; we see that it is an angle of 27° . From the same figure we infer how to draw a ray OB making an angle of 27° with OA , or to draw OC making an angle of -153° with OA .

EXERCISES

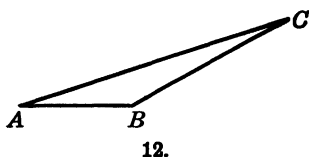
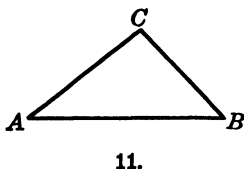
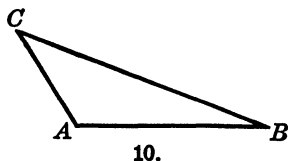
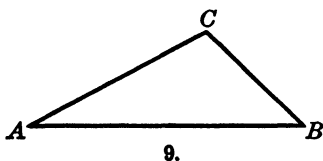
Draw the angles whose magnitudes are the following:

1. 270° , 135° , -225° , -630° , 2.5 right angles.
2. 180° , 225° , 1170° , -300° , 1.5 right angles.
3. -270° , 315° , 1215° , -855° , -3.5 right angles.
4. -90° , 240° , -315° , -1215° , -2.5 right angles.

With a protractor draw the angles given in one of the following sets:

5. 10° , 210° , -83° , 527° , -933° .
6. 70° , 160° , 318° , -482° , -872° .
7. 40° , 110° , 262° , -198° , -753° .
8. 80° , 200° , 478° , -413° , -1291° .

Estimate the measure in degrees of the angles A , B , C , of the following triangles, then measure them with a protractor:



13. Through an angle of how many degrees does the minute hand of a watch turn in 140 minutes? The second hand? The hour hand?

14. Through an angle of how many degrees does the minute hand of a watch turn in a week? The second hand? The hour hand?

15. An auto wheel whose circumference is 8.4 ft. rolls forward 14 ft. Through an angle of how many degrees does a spoke turn as viewed from the left-hand side of the car? As viewed from the right-hand side?

16. A bicycle wheel whose circumference is 7.5 ft. rolls forward 17.5 ft. Through an angle of how many degrees does a spoke turn as viewed from the left-hand side of the bicycle? As viewed from the right-hand side?

3. Coördinates. Standard position of an angle. Triangle of reference. The student is familiar with the use of coördinates in drawing graphs. Axes $X'OX$ and $Y'OY$ are drawn perpendicular to each other, intersecting at an origin O as in Figure 8. To locate a point

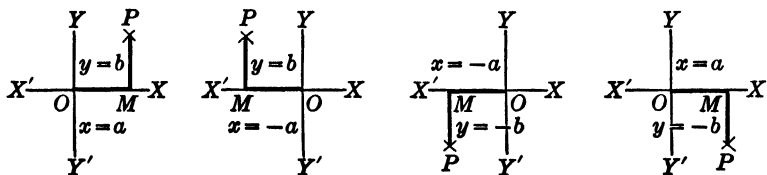


FIG. 8

P we drop a perpendicular PM to the $X'OX$ axis (more briefly, the x -axis) and, choosing a unit of length, measure OM and MP in terms

of this unit. If the length of $OM = a$ units, and the length of $MP = b$ units, then the *rectangular coördinates* x, y , of P with respect to the axes $X'OX, Y'OY$ are defined as follows:

If M is to the right of O , that is, on the ray OX , then

$$x = a;$$

if M is to the left of O , that is, on the ray OX' , then

$$x = -a.$$

If P is above the x -axis, then

$$y = b;$$

if P is below the x -axis, then

$$y = -b.$$

If P is on the x -axis, then

$$y = 0;$$

if P is on the y -axis, then

$$x = 0.$$

If the coördinates of a point P are $x = a$ and $y = b$, we refer to it as the point (a, b) .

In trigonometry the **standard position** of an angle is that in which its initial side is OX . On the terminal side take any point P (not O), and drop the perpendicular PM on the x -axis. If P is not on one of the coördinate axes, then OMP forms a triangle called a **triangle of reference for the angle**.

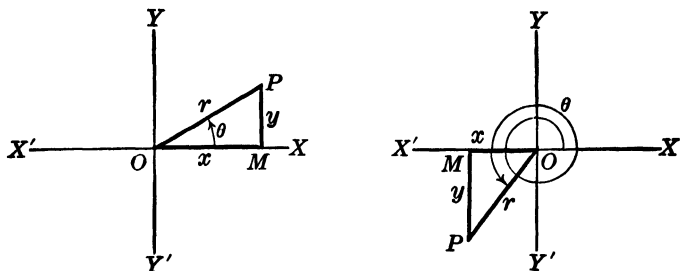


FIG. 9

If θ (Greek letter "theta") is a measure of the angle XOP , and if
length of $OP = r$ units,

then r and θ are called **polar coördinates** of the point P (see Figure 9).

4. Quadrants. The axes $X'OX$ and $Y'OY$ divide a plane into four parts called *quadrants*. If a point is in the first quadrant, both of its rectangular coördinates are positive; if it is in the second quadrant, its x -coördinate is negative, its y -coördinate positive; if it

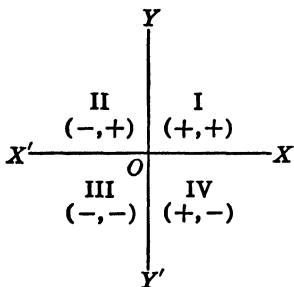


FIG. 10

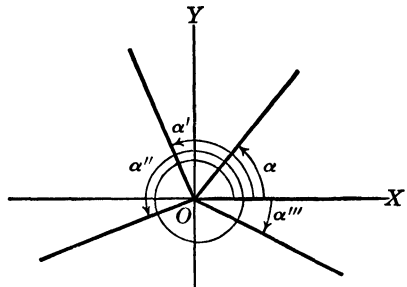


FIG. 11

is in the third quadrant, the coördinates are both negative; if it is in the fourth quadrant, its x -coördinate is positive, its y -coördinate negative.

An angle is said to **terminate in the first quadrant** when, if it is in standard position, its terminal side is in the first quadrant. Figure 11 illustrates such an angle, and also angles that terminate in each of the other quadrants.

An angle whose terminal side is OX , OY , OX' , or OY' when the initial side is OX is called a *quadrantal angle*.

Thus an angle of 0° is a quadrantal angle, as well as any positive or negative angle whose measure is a multiple of 90° . For such an angle there is, strictly speaking, no triangle of reference, but the rectangular coördinates (x, y) and the polar coördinates (r, θ) of a point P on the terminal side of the angle still have a definite meaning.

5. The six trigonometric functions.

Construct a triangle of reference, OMP , for any angle θ . In doing this the point P on the terminal side is taken at an arbitrary distance from O . If we take a second point P' we obtain a second triangle of reference, $OM'P'$, as in Figure 12.

The two triangles are similar, since their corresponding angles are equal. Hence the ratios of corresponding sides are equal.

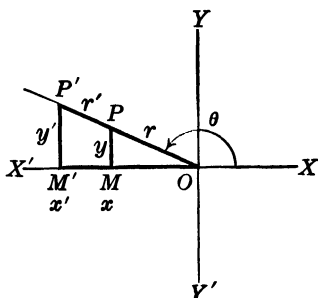


FIG. 12

Moreover, if the coördinates of P are (x, y) or (r, θ) , and of P' are (x', y') or (r', θ) , the corresponding coördinates have the same signs. Thus, in Figure 12 x and x' are negative, y and y' are positive, and r and r' are positive.

It follows that

$$\begin{aligned} \frac{y}{r} &= \frac{y'}{r'}, & \frac{x}{r} &= \frac{x'}{r'}, & \frac{y}{x} &= \frac{y'}{x'}, \\ \frac{r}{y} &= \frac{r'}{y'}, & \frac{r}{x} &= \frac{r'}{x'}, & \frac{x}{y} &= \frac{x'}{y'}. \end{aligned}$$

These six ratios, whose values thus depend upon θ but not on the position of P on the terminal side, are called the **trigonometric functions** (or ratios) of the angle θ . For convenience special names are given to them, as listed below, where an abbreviation for each is shown (the student may refer to Figures 9 and 13 for typical angles θ):

$$\begin{aligned} \text{sine of } \theta &= \sin \theta = \frac{y}{r}; & \text{cosecant of } \theta &= \csc \theta = \frac{r}{y}; \\ (1) \quad \text{cosine of } \theta &= \cos \theta = \frac{x}{r}; & \text{secant of } \theta &= \sec \theta = \frac{r}{x}; \\ \text{tangent of } \theta &= \tan \theta = \frac{y}{x}; & \text{cotangent of } \theta &= \cot \theta = \frac{x}{y}. \end{aligned}$$

These six definitions should be memorized, since they are of constant use in trigonometry.

In the case of quadrantal angles 0° , 90° , 180° , ..., there is no triangle of reference, but a point P on the terminal side has co-

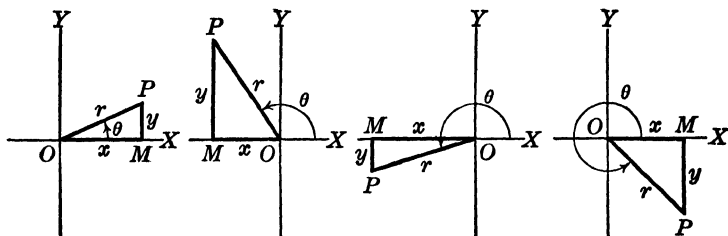


FIG. 13

ordinates, x , y , r , and the equations (1) are used for such angles also, defining values of the functions in all cases which do not involve division by zero. Where the definition would involve division by zero, the function is not defined.

From the definitions we observe that $\sin \theta$, for example, has one value for each angle θ . But there are many angles which have the

same sine; in fact, there are infinitely many (**co-terminal**) angles differing by multiples of 360° , for which the one triangle of reference may be used, and $\sin \theta$ would have the same value for all of these angles. The problem of finding all angles for which a given trigonometric function has a particular value will be studied later.

Besides the six trigonometric functions defined by (1), there are others sometimes encountered. For example:

$$\begin{aligned}\text{versed sine of } \theta &= \text{vers } \theta = 1 - \cos \theta, \\ \text{coversed sine of } \theta &= \text{covers } \theta = 1 - \sin \theta, \\ \text{haversine of } \theta &= \text{havers } \theta = \frac{1}{2} \text{ vers } \theta.\end{aligned}$$

6. Signs of the functions. The signs of x and y may be either positive or negative, depending on the quadrant in which θ terminates, but r is always positive. Hence the functions are positive for some angles and negative for others.

For example, by reference to Figure 13, we observe that when θ terminates in the second quadrant

$$\begin{aligned}\sin \theta &= \frac{y}{r} = \frac{+}{+} = +, \\ \cos \theta &= \frac{x}{r} = \frac{-}{+} = -, \\ \tan \theta &= \frac{y}{x} = \frac{+}{-} = -,\end{aligned}$$

and so on.

It is readily seen, by a similar discussion, that the signs of the functions for the various quadrants are those indicated in Figure 14.

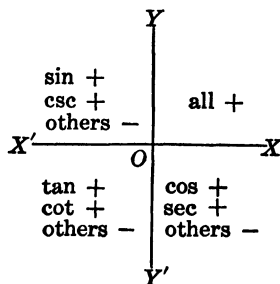


FIG. 14

EXERCISES

Locate points whose polar coordinates are given as follows:

1. $A(5, 30^\circ)$, $B(10, 135^\circ)$, $C(6, 300^\circ)$, $D(8, 270^\circ)$, $E(4, -110^\circ)$.
2. $A(6, 70^\circ)$, $B(4, 170^\circ)$, $C(4, 200^\circ)$, $D(8, 280^\circ)$, $E(6, -180^\circ)$.
3. $A(8, 50^\circ)$, $B(5, 145^\circ)$, $C(4, -150^\circ)$, $D(8, -400^\circ)$, $E(4, -270^\circ)$.
4. $A(7, 420^\circ)$, $B(5, 210^\circ)$, $C(4, -90^\circ)$, $D(5, 460^\circ)$, $E(8, -630^\circ)$.

Locate by means of a protractor and ruler points whose polar coordinates are given as follows; and write down approximate rectangular coordinates for each point with respect to the axes OX and OY :

5. $A(9, 38^\circ)$, $B(6, 249^\circ)$, $C(5, -81^\circ)$, $D(7, -585^\circ)$.
6. $A(8, 42^\circ)$, $B(6, 116^\circ)$, $C(12, 283^\circ)$, $D(10, -407^\circ)$.
7. $A(5, 91^\circ)$, $B(9, 171^\circ)$, $C(12, -140^\circ)$, $D(8, -758^\circ)$.
8. $A(6, 67^\circ)$, $B(8, -222^\circ)$, $C(5, 263^\circ)$, $D(4, 317^\circ)$.

Plot the points whose rectangular coördinates are given in one of the following sets. Find for each point by computation the value of r , $\sin \theta$, $\cos \theta$, and $\tan \theta$, where r , θ are polar coördinates of the point.

9. $A(4, 3)$, $B(6, -8)$, $C(-9, -12)$, $D(5, -12)$, $E(0, -3)$.
10. $A(3, 4)$, $B(-8, 6)$, $C(-12, -9)$, $D(12, -5)$, $E(0, 5)$.
11. $A(5, 12)$, $B(-12, 5)$, $C(-8, -6)$, $D(9, -12)$, $E(5, 0)$.
12. $A(20, 21)$, $B(-5, 12)$, $C(-12, -5)$, $D(21, -20)$, $E(-5, 0)$.

For each of the following points $r = 10$. Compute the rectangular coördinates of each point in one of the following sets where $\sin \theta$ and $\cos \theta$ have the values given, and locate each point in a suitable figure:

13. Point	A	B	C	D	E
$\sin \theta$	$\frac{4}{5}$	$-\frac{3}{5}$	$\frac{1}{3}$	$-\frac{7}{5}$	1
$\cos \theta$	$\frac{3}{5}$	$\frac{4}{5}$	$-\frac{2}{3}$	$-\frac{4}{5}$	0
14. Point	A	B	C	D	E
$\sin \theta$	$\frac{4}{5}$	$\frac{1}{3}$	$-\frac{5}{13}$	$-\frac{2}{3}$	-1
$\cos \theta$	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{1}{3}$	$\frac{2}{3}$	0
15. Point	A	B	C	D	E
$\sin \theta$	$\frac{5}{13}$	$-\frac{7}{5}$	$\frac{5}{13}$	$-\frac{4}{5}$	0
$\cos \theta$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{3}{5}$	-1
16. Point	A	B	C	D	E
$\sin \theta$	$\frac{1}{3}$	$\frac{4}{5}$	$-\frac{5}{13}$	$-\frac{2}{3}$	0
$\cos \theta$	$\frac{5}{13}$	$-\frac{3}{5}$	$\frac{1}{3}$	$\frac{2}{3}$	1

For the following sets of points, find for each point the six trigonometric functions of θ when the terminal line passes through the point designated:

17. $A(9, 12)$, $B(7, -24)$, $C(-21, 20)$, $D(-8, -15)$.
18. $A(4, 3)$, $B(20, -21)$, $C(-7, 24)$, $D(-6, -8)$.
19. $A(8, 15)$, $B(-4, 3)$, $C(21, -20)$, $D(-16, -30)$.
20. $A(20, 21)$, $B(-30, 16)$, $C(-8, -6)$, $D(3, -4)$.

Find in what quadrants θ may terminate:

21. If $\sin \theta = -\frac{3}{5}$.
22. If $\cos \theta = \frac{4}{5}$.
23. If $\cot \theta = -2$.
24. If $\tan \theta = -\frac{1}{2}$.
25. If $\sec \theta = -3$.
26. If $\csc \theta = 2$.
27. If $\sin \theta = \frac{1}{3}$.
28. If $\cos \theta = -\frac{1}{2}$.

Give the signs of the six trigonometric functions of each angle in the following exercises:

29. 78° , 213° , -310° , -601° , 1111° .
30. 13° , 471° , -500° , 1000° , -1500° .
31. 25° , 301° , -140° , -800° , 2000° .
32. 107° , 265° , -513° , 640° , 3000° .

7. Trigonometric tables obtained by measurements. In using the trigonometric functions it is convenient to have at hand a table of approximate values of the functions for a large number of angles.

We shall now show how such a table may be made by use of measurements.*

In Figure 15 a circle of radius 50 millimeters is graduated to 5° intervals; the cross-rulings are 2 millimeters apart. To find approximate values of the trigono-

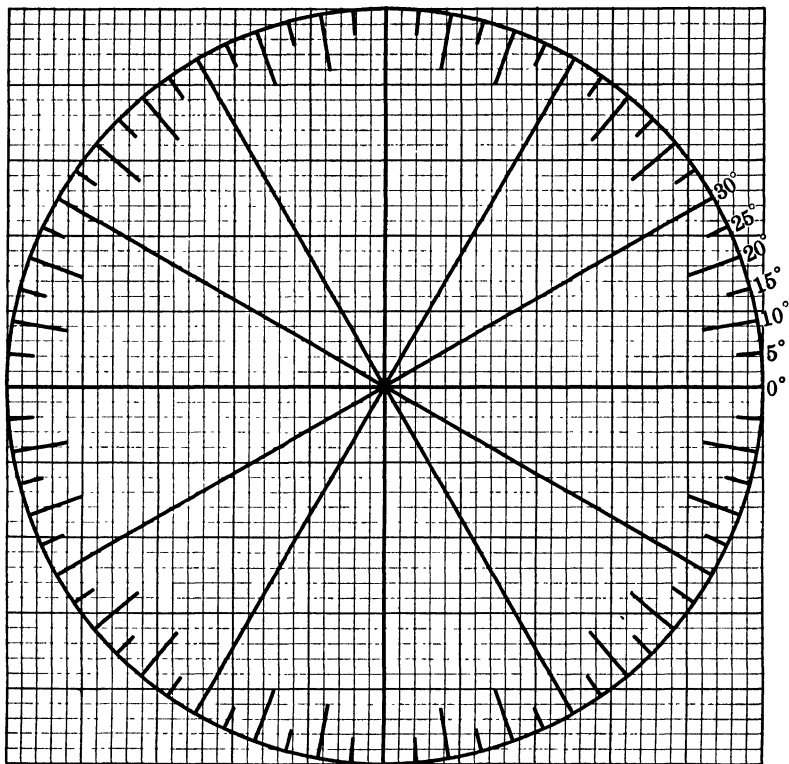


FIG. 15

metric functions of an angle whose measure is 20° , for instance, we read off for a point P at the 20° mark on the circle the approximate values of x , y , and r , correct to two significant figures, $x = 47$, $y = 17$, $r = 50$.

Hence, approximately,†

$$\begin{aligned}\sin 20^\circ &= \frac{17}{50} = .34; \quad \cos 20^\circ = \frac{47}{50} = .94; \quad \tan 20^\circ = \frac{17}{47} = .36; \\ \csc 20^\circ &= \frac{50}{17} = 2.9; \quad \sec 20^\circ = \frac{50}{47} = 1.1; \quad \cot 20^\circ = \frac{47}{17} = 2.8.\end{aligned}$$

* At the end of the book is a more extensive and accurate set of tables, calculated by other methods, which will be used in the next chapter. The methods of calculation are too difficult to explain in this book.

† By $\sin 20^\circ$ we mean "the sine of an angle whose measure is 20° "; this form of abbreviation is constantly used in trigonometry.

Similarly for 250° we have

$$x = -17, y = -47, r = 50;$$

and hence

$$\sin 250^\circ = \frac{-47}{50} = -.94; \cos 250^\circ = \frac{-17}{50} = -.34; \tan 250^\circ = \frac{-47}{-17} = 2.8;$$

and so on.

By the method indicated we may make a table of values such as that shown below.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
0°	.00	1.00	.00		1.00	
5°	.09	1.00	.09	11.4	1.00	11.5
10°	.17	.98	.18	5.67	1.02	5.76
15°	.26	.97	.27	3.73	1.04	3.86
20°	.34	.94	.36	2.75	1.06	2.92
25°	.42	.91	.47	2.14	1.10	2.37
30°	.50	.87	.58	1.73	1.15	2.00
35°	.57	.82	.70	1.43	1.22	1.74
40°	.64	.77	.84	1.19	1.31	1.56
45°	.71	.71	1.00	1.00	1.41	1.41
50°	.77	.64	1.19	.84	1.56	1.31
55°	.82	.57	1.43	.70	1.74	1.22
60°	.87	.50	1.73	.58	2.00	1.15
65°	.91	.42	2.14	.47	2.37	1.10
70°	.94	.34	2.75	.36	2.92	1.06
75°	.97	.26	3.73	.27	3.86	1.04
80°	.98	.17	5.67	.18	5.76	1.02
85°	1.00	.09	11.4	.09	11.5	1.00
90°	1.00	.00		.00		1.00
95°	1.00	-.09	-11.4	-.09	-11.5	1.00
100°	.98	-.17	-5.67	-.18	-5.76	1.02

By use of the table, approximate solutions of certain important problems are readily obtained, as illustrated in the following examples.

Example 1. If the rectangular coördinates of a point are (11, 60), what are its polar coördinates?

Solution. We have $x = 11, y = 60$. And since

$$r^2 = x^2 + y^2,$$

we find that $r = 61$. Also

$$\tan \theta = \frac{y}{x} = 5.45.$$

Since the point (11, 60) is in the first quadrant, the angle θ must terminate there. In the table we find that $\tan 75^\circ = 3.73, \tan 80^\circ = 5.67$. Hence, approximately, $\theta = 80^\circ$. The polar coördinates as thus determined are (61, 80°).

Example 2. If the polar coordinates of a point are $(70, 100^\circ)$, what are the rectangular coordinates?

Solution. Since $\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$, we have the often used formulas,

$$x = r \cos \theta, \quad y = r \sin \theta.$$

From the table, we have, approximately, $\cos 100^\circ = -.17$, $\sin 100^\circ = .98$. Since $r = 70$, we therefore have $x = 70 \times -.17 = -11.9$, $y = 70 \times .98 = 68.6$. Thus to two significant figures the rectangular coordinates are $(-12, 69)$.

EXERCISES

Extend the table on page 11 to 200° in the column indicated:

1. Sin. 2. Cos. 3. Tan. 4. Cot. 5. Sec. 6. Csc.

By use of the table on page 11 compute approximate polar coordinates (nearest 5° for θ) for points whose rectangular coordinates are given, or vice versa:

7. $A(5, 12)$. 8. $B(12, 5)$. 9. $C(-2, 11)$. 10. $D(6, 10)$.
11. $A(8, 25^\circ)$. 12. $B(25, 65^\circ)$. 13. $C(31, 95^\circ)$. 14. $D(40, 100^\circ)$.

8. Functions of 45° and related angles. There are special angles the *exact values* of whose trigonometric functions can be found from geometric properties of their triangles of reference.

Thus for an angle of 45° the triangle of reference (see Figure 16) is isosceles. We may take P on the terminal line so that $x = 1$, $y = 1$. Then, by a law of right triangles, $x^2 + y^2 = r^2$, so that $r = \sqrt{2}$. Hence

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\tan 45^\circ = 1, \quad \cot 45^\circ = 1,$$

$$\sec 45^\circ = \sqrt{2}, \quad \csc 45^\circ = \sqrt{2}.$$

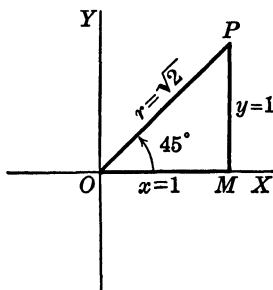


FIG. 16

We give below figures for angles of 135° , 225° , 315° . In each triangle of reference the acute angle MOP is an

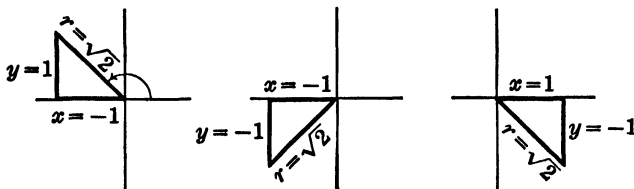


FIG. 17

angle of 45° , and the lengths of OM , MP , and OP are 1 , 1 , $\sqrt{2}$. The signs of x and y are as shown.

Thus the functions of 135° have the following values:

$$\sin 135^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\cos 135^\circ = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2},$$

$$\tan 135^\circ = -1,$$

$$\cot 135^\circ = -1,$$

$$\sec 135^\circ = -\sqrt{2},$$

$$\csc 135^\circ = \sqrt{2}.$$

Note that we could have obtained these values by taking the corresponding values for 45° and prefixing the minus sign where a function of an angle terminating in the second quadrant should be minus (see § 6).

We could similarly write down values for the functions of 225° , 315° , 405° , and so on.

9. Functions of 30° , 60° , and related angles. Let ABC , Figure 18, be an equilateral triangle each of whose sides is of length 2 units. Each angle is then an angle of 60° . If we drop a perpendicular from

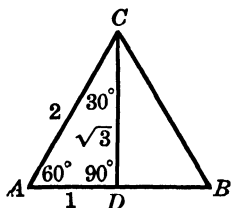


FIG. 18

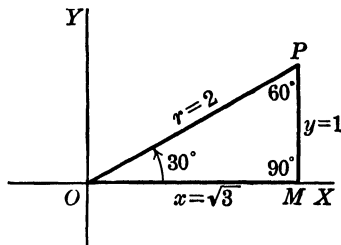


FIG. 19

C to AB it bisects the angle C and the base AB . Hence, in the right triangle ADC thus formed we have

$$A = 60^\circ, D = 90^\circ, C = 30^\circ,$$

$$AD = 1, AC = 2.$$

Since $\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2$, it follows that $DC = \sqrt{3}$.

A triangle of reference whose angles are 30° , 60° , 90° and whose sides are 1, $\sqrt{3}$, 2 is used in each of Figures 19 and 20.

We readily obtain from Figure 19

$$\sin 30^\circ = \frac{1}{2},$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2},$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

$$\cot 30^\circ = \sqrt{3},$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

$$\csc 30^\circ = 2.$$

From Figure 20 we have

$$\sin 60^\circ = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{1}{2},$$

$$\tan 60^\circ = \sqrt{3},$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

$$\sec 60^\circ = 2,$$

$$\csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3};$$

and

$$\sin 120^\circ = \frac{\sqrt{3}}{2},$$

$$\cos 120^\circ = -\frac{1}{2},$$

$$\tan 120^\circ = -\sqrt{3},$$

$$\cot 120^\circ = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3},$$

$$\sec 120^\circ = -2,$$

$$\csc 120^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

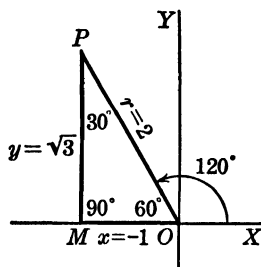
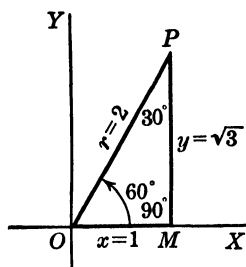


FIG. 20

In a similar manner we obtain exact values of functions of 150° , 210° , 240° , 300° , 330° , and angles differing from these by multiples of 360° .

10. Functions of quadrantal angles.

When $\theta = 0^\circ$, the point P is on OX . If we choose P so that $r = 1$, as in Figure 21, we shall have

$$x = 1, y = 0, r = 1.$$

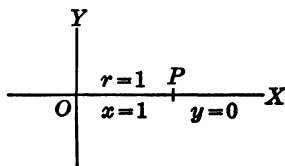


FIG. 21

Hence

$$\sin 0^\circ = \frac{0}{1} = 0,$$

$$\cos 0^\circ = \frac{1}{1} = 1,$$

$$\tan 0^\circ = \frac{0}{1} = 0,$$

$$\cot 0^\circ = \left(\frac{0}{1}\right) \text{ impossible},$$

$$\sec 0^\circ = \frac{1}{1} = 1,$$

$$\csc 0^\circ = \left(\frac{0}{1}\right) \text{ impossible}.$$

We have indicated that the ratios for $\cot \theta$ and $\csc \theta$ lead, in this case, to a division by 0, which is impossible; * these functions have no value for $\theta = 0^\circ$.

Similarly, from Figure 22, we have

$$\begin{array}{ll} \sin 90^\circ = 1, & \cos 90^\circ = 0, \\ \tan 90^\circ \text{ impossible,} & \cot 90^\circ = 0, \\ \sec 90^\circ \text{ impossible,} & \csc 90^\circ = 1; \end{array}$$

and

$$\begin{array}{ll} \sin 180^\circ = 0, & \cos 180^\circ = -1, \\ \tan 180^\circ = 0, & \cot 180^\circ \text{ impossible,} \\ \sec 180^\circ = -1, & \csc 180^\circ \text{ impossible.} \end{array}$$

The student should write down the values of the functions of $270^\circ, 360^\circ, 450^\circ, 540^\circ$, and other angles differing from these by multiples of 360° .

As a memorizing device it is interesting to note that the values of the sines of the angles $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ may be written as one half the square roots of the successive integers:

$$\sin 0^\circ = \frac{\sqrt{0}}{2}; \sin 30^\circ = \frac{\sqrt{1}}{2}; \sin 45^\circ = \frac{\sqrt{2}}{2}; \sin 60^\circ = \frac{\sqrt{3}}{2}; \sin 90^\circ = \frac{\sqrt{4}}{2};$$

and the cosines of these angles are found by taking these numbers in the reverse order. Values of all six functions of these angles are given in the following table:

Angle	Sin	Cos	Tan	Cot	Sec	Csc
0°	0	1	0		1	
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	1	0		0		1

* By definition of division as the inverse of multiplication, $\frac{1}{0}$, if it has a value, must be such a number that its product by the denominator 0 is the numerator 1. But there is no such number.

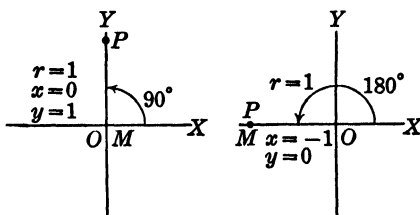


FIG. 22

EXERCISES

For the following sets find by measurement and calculation the values of the sine, cosine, and tangent of each angle, using Figure 15.

1. 70° , 80° , 90° , 100° , 260° .
2. 40° , 60° , 80° , 110° , 280° .
3. 120° , 240° , 250° , 270° , 280° .
4. 140° , 240° , 260° , 280° , 300° .

For the following sets find, by use of figures, approximate values of θ and θ' :

5. If $\sin \theta = \frac{3}{5}$, $\cos \theta = -\frac{4}{5}$. If $\cos \theta' = -1$.
6. If $\cos \theta = \frac{5}{13}$, $\tan \theta = -\frac{12}{5}$. If $\tan \theta' = 0$, $\cos \theta'$ positive.
7. If $\tan \theta = \frac{12}{5}$, $\sin \theta = -\frac{12}{13}$. If $\sin \theta' = -1$.
8. If $\cot \theta = -\frac{3}{4}$, $\cos \theta = -\frac{3}{5}$. If $\cot \theta' = 0$, $\sin \theta'$ negative.

For the following sets find from suitable figures the exact values of the six functions of each of the angles given:

- | | |
|---------------------------------|----------------------------------|
| 9. 315° , 210° . | 13. -60° , 270° . |
| 10. 300° , 135° . | 14. -90° , -30° . |
| 11. 150° , 225° . | 15. 180° , -150° . |
| 12. 240° , 330° . | 16. 540° , -45° . |

Prove the following statements:

17. $\cos 60^\circ \sin 330^\circ - \cos 30^\circ \sin 300^\circ = \frac{1}{2}$.
18. $\cos 30^\circ \cos 330^\circ + \sin 45^\circ \cos 225^\circ = \frac{1}{2}$.
19. $\sin 45^\circ \cos 300^\circ - \cos 60^\circ \sin 225^\circ = \frac{1}{2}\sqrt{2}$.
20. $\sin 30^\circ \cos 300^\circ - \sin 60^\circ \cos 210^\circ = 1$.

11. Problems in which a function is given.

Example 1. Given $\sin \theta = \frac{3}{5}$; construct possible angles θ and find values of the other functions of θ .

Solution. Our construction depends on finding a point (or points) P for which $\frac{y}{r} = \frac{3}{5}$. We can take $y = 3$, $r = 5$ (or we could take $y = 6$, $r = 10$). Since $x^2 + y^2 = r^2$, we see that $x^2 = r^2 - y^2$, so that in the present example, if $y = 3$, $r = 5$, we have $x^2 = 25 - 9 = 16$, and x may be either $+4$ or -4 . In Figure 23 we have drawn the two corresponding triangles of reference, and have indicated two angles θ_1 , θ_2 , for each of which the value of the sine is $\frac{3}{5}$. Of course, all co-terminal angles have the same sine.

From the figure we read off the values of the other functions of the angles. We have, for example,

$$\begin{aligned}\cos \theta_1 &= \frac{4}{5}, \tan \theta_1 = \frac{3}{4}, \\ \cos \theta_2 &= -\frac{4}{5}, \tan \theta_2 = -\frac{3}{4}.\end{aligned}$$

The student should write the values of the other functions.

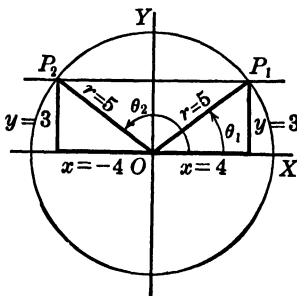


FIG. 23

Example 2. Given $\tan \theta = \frac{5}{12}$; construct possible angles θ and find values of the other functions of θ .

Solution. Since $\frac{y}{x} = \frac{5}{12}$, we may take $x = 12$, $y = 5$, or $x = -12$, $y = -5$. The relation $r^2 = x^2 + y^2$ gives $r^2 = 144 + 25 = 169$, so that $r = 13$ (we do not allow r to be negative). Figure 24 shows how to construct two angles θ_1 , θ_2 , and their triangles of reference. By using the data given in this figure, the values of the functions of θ_1 and θ_2 are found.

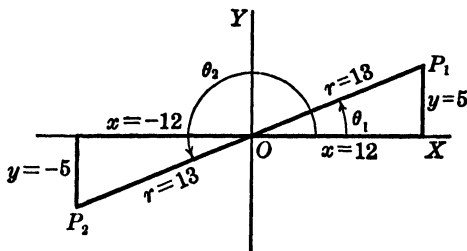


FIG. 24

EXERCISES

Find the values of the other functions of the angle θ , when it is given that:

- | | | |
|---|--|--|
| 1. $\csc \theta = \frac{5}{3}$. | 7. $\sec \theta = \frac{1}{\frac{1}{5}}$. | 13. $\cos \theta = -\frac{4}{5}$. |
| 2. $\sin \theta = -\frac{1}{\frac{7}{4}}$. | 8. $\tan \theta = \frac{1}{\frac{2}{3}}$. | 14. $\tan \theta = 2$. |
| 3. $\tan \theta = -\frac{2}{7}$. | 9. $\tan \theta = -\frac{3}{4}$. | 15. $\sin \theta = -\frac{3}{5}$. |
| 4. $\cos \theta = \frac{2}{3}$. | 10. $\sin \theta = -\frac{1}{3}$. | 16. $\cot \theta = -\frac{4}{3}$. |
| 5. $\sin \theta = \frac{1}{2}$. | 11. $\cot \theta = -1$. | 17. $\sec \theta = \frac{1}{\frac{1}{2}}$. |
| 6. $\cos \theta = -\frac{3}{5}$. | 12. $\sec \theta = -\frac{2}{\frac{3}{4}}$. | 18. $\csc \theta = -\frac{1}{\frac{1}{2}}$. |

Express all six of the trigonometric functions in terms of the following:

- | | | |
|---------------------|---------------------|---------------------|
| 19. $\sin \theta$. | 21. $\tan \theta$. | 23. $\sec \theta$. |
| 20. $\cos \theta$. | 22. $\cot \theta$. | 24. $\csc \theta$. |

★12. Projection on coördinate axes. Consider a directed line AB which makes an angle θ with the x -axis of a system of rectangular coördinates, and let CD be a segment of AB (Figure 25). On a directed line through O making the angle θ with the x -axis, take $OP = CD$. Then the projection $C'D'$ of CD on the x -axis equals OM , the projection of OP on the x -axis. This may be written

$$\text{Proj}_x CD = \text{Proj}_x OP = OM.$$

Since $\cos \theta = \frac{OM}{OP}$, we have $OM = OP \cos \theta$; hence

$$\begin{aligned} \text{Proj}_x CD &= OP \cos \theta, \\ \text{Proj}_x CD &= CD \cos \theta. \end{aligned}$$

(1)

Similarly

(2)

$$\text{Proj}_y CD = CD \sin \theta.$$

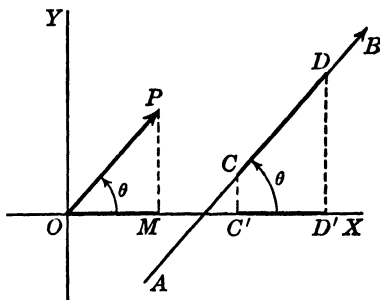


FIG. 25

The student should verify these formulas not only for the angle of Figure 25 but also for that of Figure 26 and other figures.

★13. Vectors. Components.

Resultants. A quantity which may be represented by a directed line segment CD is often called a vector quantity. Thus force, velocity, and acceleration are vector quantities. The projections of a vector quantity on the x - and y -axes are called components of the vector.

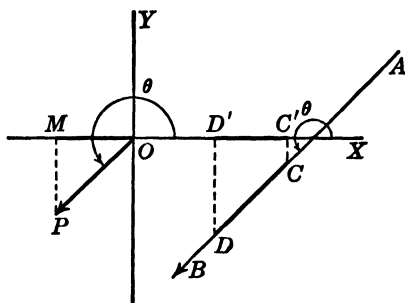


FIG. 26

If F is the magnitude of a force which makes an angle θ with the x -axis, and if F_x and F_y are the components of the force, then, by formulas (1) and (2) of § 12,

$$F_x = F \cos \theta, \quad F_y = F \sin \theta.$$

Similar formulas hold for velocity and acceleration.

If the components F_x and F_y are given, the vector F is called the resultant. It is seen that the magnitude of F is $\sqrt{F_x^2 + F_y^2}$. The direction of F is given by the angle θ , where $\tan \theta = \frac{F_y}{F_x}$.

If two forces, represented in magnitude and direction by AB and AC , act on a particle at A , they are equivalent to a single force, called the resultant force, acting on the particle. The magnitude and direc-

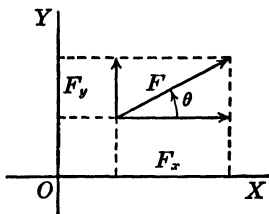


FIG. 27

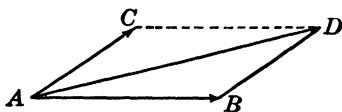


FIG. 28

tion of this resultant are represented by the diagonal AD of the parallelogram of which AB and AC are two sides. This principle is known as the Parallelogram Law of Forces. A similar law holds for velocities and accelerations.

Example. A force of 20 lb. acts at an angle of 40° with the horizontal. What two forces, one horizontal, the other vertical, would be equivalent?

Solution. In the vertical plane of the force, let the x -axis be horizontal, the y -axis vertical. Then

$$F_x = 20 \cos 40^\circ, \quad F_y = 20 \sin 40^\circ.$$

Using the table on page 11, we have

$$\cos 40^\circ = .77, \quad \sin 40^\circ = .64.$$

Hence

$$F_x = 15.4 \text{ lb.}, \quad F_y = 12.8 \text{ lb.}$$

These values are of course approximations.

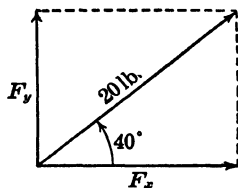


FIG. 29

EXERCISES

1. Draw a figure similar to those in § 5, making θ an angle terminating in the second quadrant, and verify formulas (1) and (2) of § 12. Note that the signs as well as the magnitudes are correct.

2. Proceed as in Exercise 1, making θ an angle terminating in the fourth quadrant.

3. If a boat is traveling NE with a speed of 15 mi. per hr., what is the component of its velocity in the eastward direction? In the northward direction?

4. If a boat is making 30 knots per hour on a course N 40° E, what are its components of velocity in the eastward and the northward directions respectively?

5. A swimmer in crossing a stream puts forth efforts which in still water would carry him directly across at 2 mi. per hr. If the current is 5 mi. per hr., what are the actual direction and speed of the swimmer?

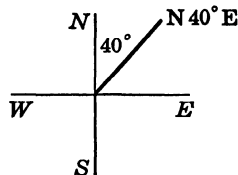
6. An airplane heads west, running so that in still air it would have a speed of 120 mi. per hr. There is a wind from the south blowing with a speed of 30 mi. per hr. What are the actual direction and speed of the airplane?

7. A force of 15 lb. acts vertically upward and another of 25 lb. acts horizontally on a particle. What are the magnitude and the direction of the single force equivalent to the two?

8. A force of 3 lb. acts horizontally, another of 4 lb. acts vertically on a particle. What are the magnitude and the direction of the resultant force?

9. A boat sails on a course S 20° W (compare Exercise 4) with a speed of 14 knots per hour. What are the westward and southward components of its velocity?

10. A surveyor runs a line 600 yd. N 10° W (compare Exercise 4) from A to B. How far west and how far north is B from A?



CHAPTER II

RIGHT TRIANGLES

14. Relations between the sides and acute angles of a right triangle.

Let ABC be a right triangle whose acute angles are at A and B . We designate the angles as A, B, C , and the sides as a, b, c (see Figure 30). It is convenient to call a the opposite side to angle A , and b the adjacent side; c is the hypotenuse.

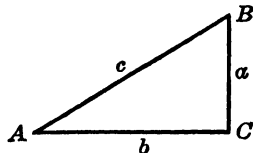


FIG. 30

If Figure 30 were placed suitably with respect to rectangular axes, the triangle shown would be a triangle of reference for the angle A , with $x = b, y = a, r = c$. The definitions of the functions of A give us the following relations between a, b, c , and A .*

$$\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}},$$

$$\cot A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}},$$

$$\cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}},$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}},$$

$$\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

There is a corresponding set of formulas for the angle B . These can be written by analogy with the above set if we recall that b is opposite to B , and a is adjacent.

$$\sin B = \frac{b}{c},$$

$$\cos B = \frac{a}{c},$$

$$\tan B = \frac{b}{a},$$

$$\csc B = \frac{c}{b},$$

$$\sec B = \frac{c}{a},$$

$$\cot B = \frac{a}{b}.$$

15. Relations between functions of the acute angles of a right triangle. By comparing the two sets of formulas of § 14, we obtain the following relations between functions of B and functions of A :

$$\sin B = \cos A,$$

$$\cos B = \sin A,$$

$$\tan B = \cot A,$$

$$\csc B = \sec A,$$

$$\sec B = \csc A,$$

$$\cot B = \tan A.$$

* The reader may note that we are using the same letter for a line segment, for its measure in some unit of length, and for the number of units in that measure; similarly for angles. The ambiguity need cause no difficulty, since it will be clear from the context which designation is meant.

A convenient way of remembering this last set of formulas is suggested by the prefix "co" in the names of some of the functions. We shall accordingly call cosine the cofunction of sine, and we shall also call sine the cofunction of cosine; similarly for the pair tangent, cotangent, and the pair secant, cosecant.

We may then summarize relations between A and B as follows:

Each function of one of the angles A , B is equal to the corresponding cofunction of the other angle.

Since $B = 90^\circ - A$, the angles A and B are said to be complementary (note that this word also begins with "co"), and the above rule can take the following form:

Each function of the complement of an acute angle is equal to the corresponding cofunction of the angle.

For example, 60° is the complement of 30° , hence $\cos 60^\circ = \sin 30^\circ$.

16. Solution of right triangles. Each formula of § 14 gives us an equation connecting three numbers, one of which is a value of a function of an acute angle, while the other two refer to sides of a right triangle.

If two sides are given, the value of a function of A is thereby determined, and by the use of tables we can find A . To find the third side we may use another of the equations, or we may use the equation $a^2 + b^2 = c^2$. Finally, we find B from the relation $B = 90^\circ - A$. When we have thus found the sides and angles which were not given, we are said to have **solved the triangle**.

If a side and an acute angle are given, we can also solve the triangle.

In solving a triangle we use tables of trigonometric functions such as are to be found at the end of this book. Their use is explained in the following section.

17. Tables of values of functions. On pages 4-8 of the "Logarithmic and Trigonometric Tables" at the end of this book, values of the six trigonometric functions are given for angles at intervals of $10'$ from 0° to 90° . The following illustrative examples will serve to explain how the table is used.

Example 1. To find $\sin 21^\circ 30'$, turn to page 6, go down the column headed "Degrees" to find $21^\circ 30'$, go across to the column headed "Sin," and find

$$\sin 21^\circ 30' = .3665,$$

the value being correct to four decimal places. It is understood that the value to five decimal places is between .36645 and .36655, the tabulated value being as close as may be given by four-place tables.

Example 2. Similarly we find $\sin 21^\circ 40' = .3692$, the first digit being carried down from above in the table.

Example 3. To find $\tan 63^\circ 20'$, we must go to the last column, with "Degrees" at the bottom of the page, to find $63^\circ 20'$ (on page 6). At the bottom find the column labeled "Tan," and opposite $63^\circ 20'$ find the value, $\tan 63^\circ 20' = 1.991$.

It is observed that angles in the first column range from 0° to 45° ; for functions of these angles find the name of the function at the top of the pages. Angles in the last column range from 45° to 90° ; for functions of these angles use the names at the bottom.

Example 4. To find A when $\cot A = 4.061$, we look in the "Cot" column for 4.061, and find, on page 5, that $A = 13^\circ 50'$.

Example 5. To find A when $\cos A = .4975$, we note that we must use a column with "Cos" at the bottom, on page 7, and read the angle in the right-hand column. We find that $\cos 60^\circ 10' = .4975$, and hence $A = 60^\circ 10'$.

EXERCISES

Complete the following equations, using four-place tables:

- | | | | |
|---------------------------|---|-----------------------|---|
| 1. $\sin 18^\circ 30' =$ | ; | $\cos 7^\circ 0' =$ | . |
| 2. $\tan 55^\circ 50' =$ | ; | $\cot 42^\circ 30' =$ | . |
| 3. $\sec 17^\circ 40' =$ | ; | $\csc 69^\circ 40' =$ | . |
| 4. $\sin 28^\circ 0' =$ | ; | $\cos 82^\circ 30' =$ | . |
| 5. $\tan 15^\circ 40' =$ | ; | $\cot 18^\circ 50' =$ | . |
| 6. $\sec 63^\circ 30' =$ | ; | $\csc 49^\circ 40' =$ | . |
| 7. $\sin 37^\circ 40' =$ | ; | $\cos 10^\circ 30' =$ | . |
| 8. $\tan 48^\circ 0' =$ | ; | $\cot 19^\circ 50' =$ | . |
| 9. $\sec 52^\circ 30' =$ | ; | $\csc 25^\circ 40' =$ | . |
| 10. $\sin 47^\circ 20' =$ | ; | $\cos 5^\circ 50' =$ | . |
| 11. $\tan 18^\circ 40' =$ | ; | $\cot 87^\circ 30' =$ | . |
| 12. $\sec 75^\circ 10' =$ | ; | $\csc 24^\circ 40' =$ | . |

Find the angle A in each of the following equations, using four-place tables:

13. $\sin A = .6361$, $\cos A = .9750$, $\tan A = 7.115$, $\cot A = .3249$.
 14. $\sin A = .3746$, $\cos A = .5225$, $\tan A = 8.144$, $\cot A = 3.305$.
 15. $\sin A = .5299$, $\cos A = .9990$, $\tan A = .1554$, $\cot A = .4314$.
 16. $\sin A = .9903$, $\cos A = .9781$, $\tan A = .2309$, $\cot A = 1.653$.

18. Interpolation. In finding the value of a function of an angle, such as $17^\circ 23'$, which is not given in the table but lies between two angles that appear, we use the method of interpolation, as illustrated in the following Examples 1 and 2. In Examples 3 and 4 the method is applied in finding the angle when the value of one of its functions is given.

Example 1. Find $\sin 17^\circ 23'$.

Solution. The given angle, $17^\circ 23'$, is three tenths of the way from $17^\circ 20'$ to $17^\circ 30'$. We assume that $\sin 17^\circ 23'$ is three tenths of the way from $\sin 17^\circ 20'$ to $\sin 17^\circ 30'$. The sine of $17^\circ 23'$ will then be obtained by taking $\frac{3}{10}$ of the amount by which $\sin 17^\circ 30'$ exceeds $\sin 17^\circ 20'$, and adding this correction to $\sin 17^\circ 20'$. Hence

$$\begin{aligned}\sin 17^\circ 23' &= \sin 17^\circ 20' + \frac{3}{10}(\sin 17^\circ 30' - \sin 17^\circ 20') \\ &= .2979 + \frac{3}{10}(.0028) = .2979 + .00084 \\ &= .2987 \text{ approximately.}\end{aligned}$$

Since the Tables give values to only four places, we give only four places in our value of $\sin 17^\circ 23'$. This amounts to calling the correction .0008 instead of .00084. We would have used .0008 for any correction greater than .00075 and less than .00085. It is customary to disregard the decimal point in the tabulated values and call the tabular difference 28 instead of .0028, and the correction 8 instead of .0008.

Another way to explain the preceding interpolation is to state that we have assumed that when an angle increases, its sine increases proportionally; or, in other words, that differences between angles are proportional to differences between their sines. For the example just solved the accompanying small table indicates these differences. We thus have

$$\frac{x}{28} = \frac{3}{10}.$$

Then $x = 8.4 = 8$ approximately; and

$$\sin 17^\circ 23' = .2979 + .0008 = .2987.$$

Angle	Sin
$10 \left[\begin{array}{c} 3 \left[\begin{array}{c} 17^\circ 20' \\ 17^\circ 23' \\ 17^\circ 30' \end{array} \right] x \end{array} \right] 28$	$\begin{array}{c} .2979 \\ \hline .3007 \end{array}$

The assumption just made that differences between angles are proportional to differences between the values of a function of those angles is not exactly true, but it gives rise to errors which are negligible when the differences involved are small.

Example 2. Find $\cot 17^\circ 15'$.

Solution. From the little table at the right we have

$$x = \frac{5}{10} \times 33 = 16.5.$$

Angle	Cot
$10 \left[\begin{array}{c} 5 \left[\begin{array}{c} 17^\circ 10' \\ 17^\circ 15' \\ 17^\circ 20' \end{array} \right] x \end{array} \right] 33$	$\begin{array}{c} 3.237 \\ \hline 3.204 \end{array}$

The correction x can be called either 16 or 17. Since the cotangent decreases when we go from $17^\circ 10'$ to $17^\circ 20'$, the correction, which should take us $\frac{5}{10}$ of the way from $\cot 17^\circ 10'$ to $\cot 17^\circ 20'$, must be subtracted from the former, giving either $3.237 - .016 = 3.221$ or $3.237 - .017 = 3.220$. In such cases we shall always use the result which ends in an even digit; thus, $\cot 17^\circ 15' = 3.220$.

Example 3. Given $\tan A = .4361$. Find A .

Solution. We find that the angle A lies between $23^\circ 30'$ and $23^\circ 40'$, as shown at the right. By the principle of proportional differences we have

$$\text{Hence } x = \frac{13}{10} \times 10 = \frac{130}{10} = 3.7.$$

$$A = 23^\circ 30' + 4' = 23^\circ 34'.$$

Angle	Tan
$10 \left[\begin{array}{c} x \left[\begin{array}{c} 23^\circ 30' \\ A \\ 23^\circ 40' \end{array} \right] 13 \end{array} \right] 35$	$\begin{array}{c} .4348 \\ .4361 \\ \hline .4383 \end{array}$

Example 4. Given $\cos A = .4100$. Find A .

Solution. Proceeding as before, we have

$$x = \frac{2}{3} \times 10 = 8.$$

Hence

$$A = 65^\circ 48'.$$

Angle		Cos
10	$x \left[\begin{array}{c} 65^\circ 40' \\ A \\ 65^\circ 50' \end{array} \right]$	$\left[\begin{array}{c} .4120 \\ .4100 \\ .4094 \end{array} \right]$
		20
		26

EXERCISES

By interpolation find the values of the following, using four-place tables:

- | | | | |
|--------------------------|-----------------------|---------------------------|-----------------------|
| 1. $\sin 41^\circ 51'$; | $\sin 48^\circ 32'$. | 7. $\cos 8^\circ 15'$; | $\sin 51^\circ 35'$. |
| 2. $\tan 26^\circ 16'$; | $\tan 57^\circ 6'$. | 8. $\sin 37^\circ 8'$; | $\cot 68^\circ 18'$. |
| 3. $\cos 11^\circ 37'$; | $\csc 52^\circ 21'$. | 9. $\tan 42^\circ 24'$; | $\sec 73^\circ 11'$. |
| 4. $\cot 21^\circ 17'$; | $\tan 80^\circ 25'$. | 10. $\sin 20^\circ 52'$; | $\cot 79^\circ 13'$. |
| 5. $\sec 15^\circ 21'$; | $\sec 63^\circ 17'$. | 11. $\cos 35^\circ 14'$; | $\sec 48^\circ 5'$. |
| 6. $\sin 35^\circ 12'$; | $\cos 70^\circ 12'$. | 12. $\tan 14^\circ 43'$; | $\cos 84^\circ 54'$. |

By interpolation find the values of A to the nearest minute:

- | | |
|------------------------|------------------------|
| 13. $\sin A = .2320$. | 21. $\cot A = 1.300$. |
| 14. $\cot A = 3.245$. | 22. $\tan A = .8446$. |
| 15. $\tan A = .1115$. | 23. $\cos A = .9940$. |
| 16. $\sin A = .5207$. | 24. $\cos A = .9105$. |
| 17. $\sin A = .8100$. | 25. $\cos A = .5105$. |
| 18. $\cos A = .4035$. | 26. $\tan A = .4050$. |
| 19. $\tan A = 1.127$. | 27. $\cot A = .6845$. |
| 20. $\cot A = .1540$. | 28. $\sin A = .9500$. |

19. The table of squares of numbers. Square roots. In solving right triangles we use such formulas as

$$a^2 = b^2 + c^2, \quad a = \sqrt{b^2 + c^2}.$$

Table I, at the end of this book, shortens the work of finding squares and square roots.

Example 1. Find $(3.54)^2$.

Solution. On page 2, go down the column headed N to 3.5, then across to the column headed 4. The entry is 12.53; hence $(3.54)^2 = 12.53$, approximately.

Example 2. Find $(3.543)^2$.

Solution. We interpolate, using a principle of proportional parts similar to that employed in § 18. The tabular difference, $(3.55)^2 - (3.54)^2$, is found to be .07. The correction is $\frac{3}{10} \times .07 = .02$; hence $(3.543)^2 = 12.53 + .02 = 12.55$, approximately.

Example 3. Find $(35.43)^2$ and $(.03543)^2$.

Solution. $(35.43)^2 = (10 \times 3.543)^2 = 100 \times (3.543)^2 = 1255$.

$$(.03543)^2 = \left(\frac{3.543}{100} \right)^2 = \frac{12.55}{10000} = 0.001255.$$

Example 3 illustrates the **rule**: If the number N whose square is to be found is not between 1 and 10, move the decimal point to the right (or left) k places so as to obtain a number N_1 between 1 and 10. Find N_1^2 from the Table. Then N^2 can be obtained from N_1^2 by moving the decimal point $2k$ places to the left (or right). Thus if $N = 35.43$, we have $N_1 = 3.543$, $k = 1$.

The square root of a number n in the interior of Table I is obtained by reading off the corresponding number N from Table I. The first two digits of N are at the left of the row in which n lies, and the third digit is at the top of the column where n is located.

Example 4. Find $\sqrt{1.800}$.*

Solution. By use of Table I we find $\sqrt{1.796} = 1.34$, $\sqrt{1.823} = 1.35$. By interpolation, we thus obtain

$$\sqrt{1.800} = 1.34 + \frac{4}{7} \times (.01) = 1.341, \text{ approximately.}$$

Example 5. Find $\sqrt{18.00}$.

Solution. We note that $\sqrt{18.00}$ is located between the entries 17.98 and 18.06 in the lower part of page 2. By interpolation, $\sqrt{18.00} = 4.242$, approximately (we chose the final digit as the even digit 2; see § 18, Example 2).

Example 6. Find $\sqrt{180.0}$ and $\sqrt{.001800}$.

Solution. The numbers under the radical sign are to be expressed as numbers between 1 and 100, multiplied or divided by 100 (or powers of 100). Thus

$$\begin{aligned}\sqrt{180.0} &= \sqrt{100 \times 1.800} = 10 \times \sqrt{1.800} = 13.41. \\ \sqrt{.001800} &= \sqrt{\frac{18.00}{(100)^2}} = \frac{1}{100} \times \sqrt{18.00} = .04242.\end{aligned}$$

Example 6 illustrates this **rule**: if the number n whose square root is to be found is not between 1 and 100, move the decimal point to the right (or left) an even number of places, say $2k$, so as to obtain a number n_1 between 1 and 100. Find $\sqrt{n_1}$ from the Table. Then $\sqrt{n}$ can be obtained from $\sqrt{n_1}$ by moving the decimal point k places to the left (or right).

EXERCISES

Find the squares of the following numbers by use of a Table of Squares:

- | | | |
|-----------|------------|--------------|
| 1. 3.67. | 5. 0.0882. | 9. 67.2. |
| 2. 28.4. | 6. 63100. | 10. 0.666. |
| 3. 319. | 7. 8.19. | 11. 87600. |
| 4. 0.218. | 8. 385. | 12. 0.00827. |

* We write 1.800 to indicate that we wish an answer to four significant figures (see § 20).

By interpolation find the squares of the following to four places by use of a Table of Squares:

13. 82.46.	17. 0.08218.	21. 6232.
14. 217.9.	18. 0.7526.	22. 7219.
15. 8227.	19. 876500.	23. 982.4.
16. 1.234.	20. 23.43.	24. 6.666.

Find the square roots of the following numbers to three places by use of a Table of Squares:

25. 9.797.	29. 1648.	33. 537300.
26. 19.98.	30. 2007.	34. 156800.
27. 47.89.	31. 0.9293.	35. 0.002401.
28. 85.38.	32. 0.4914.	36. 0.009409.

By interpolation find the square roots of the following numbers to four places by use of a Table of Squares:

37. 11.72.	41. 625.9.	45. 51870.
38. 21.98.	42. 7890.	46. 87550.
39. 68.91.	43. 2194.	47. 0.007750.
40. 22.08.	44. 9202.	48. 0.05842.

20. Significant figures. Tables I and II do not give exact values of squares or of trigonometric functions; they give the four-place numbers that are nearest to the true values. The latter are usually unending decimals. We should not expect that computations based on these tables will give better than four-place accuracy. We now make these ideas more precise.

Let us define the **significant figures** (or digits) of a number as its digits beginning with the first that is not zero and ending with the last that is not zero, unless some other convention is explicitly stated. An example of such a convention is to say that the number 32000 is given only to four significant figures; we are then counting only the first two of the final three zeros. We shall always count final zeros that are written as part of a number following a decimal point. Thus in .0100, the final two zeros are significant (the zero just after the decimal point is not significant).

In general, if we multiply or divide numbers each of which is given approximately, we should not retain more significant figures in the result than the smallest number of significant figures in the data. When we are using four-place tables, our results will be given to four significant figures unless data contain numbers of less than four significant figures. In this last case we may compute as if all data were

given to four significant figures, but indicate that our final result is reliable to fewer places.

Since we are to compute with measurements of angles as well as of line segments, we need a definition of significant figures in degree measure. By referring to the tables, one may note that in general the sine and cosine change their second significant figure by about 1 when the angle changes by 1° . Similar observations have led to this **general rule for angles**:

An angle measurement to the nearest degree is taken to correspond to a number of two significant figures; a measurement to the nearest $10'$ means three significant figures, one to the nearest $1'$ four significant figures, one to the nearest $5''$ five significant figures.

21. Typical solutions of right triangles. In the examples which follow, some of the data are given to less than four significant figures. Our computations are all carried out to four or more figures and the answers are first thus given, and then rounded off to the proper number of significant figures.

An answer is **checked** by substitution in a formula that has not been used in the solution and which involves at least two parts of the answer. This tests the accuracy of the computation; if a check fails, the answer is probably wrong.

Example 1. Given $A = 30^\circ 46'$, $c = 1.50$. Find B , a , and b .

Solution. The work may be conveniently arranged as follows:

Formulas

$$\frac{a}{c} = \sin A, \quad \frac{b}{c} = \cos A,$$

$$B = 90^\circ - A, \quad a = c \sin A, \quad b = c \cos A.$$

Computation

$90^\circ = 89^\circ 60'$	$\sin 30^\circ 46' = .5115$	$\cos 30^\circ 46' = .8593$
$A = 30^\circ 46'$	$c = 1.5$	$c = 1.5$
$B = 59^\circ 14'$	$\frac{25575}{5115}$	$\frac{42965}{8593}$
	$a = .76725$	$b = 1.28895$

Answer to three significant figures. $B = 59^\circ 10'$; $a = .767$; $b = 1.29$.

Check to four significant figures. $b = a \tan B$.

$$b = 1.289 \quad a \tan B = .7672 \times 1.679 = 1.288$$

The check is as close as should be expected.

Example 2. Given $b = 100$, $c = 232$ (to three significant figures). Find A , B , and a .

Solution.

	<i>Formulas</i>	
$\sec A = \frac{c}{b},$	$B = 90^\circ - A,$	$a = \sqrt{c^2 - b^2}.$
	<i>Computation</i>	
$\sec A = \frac{232}{100}$		$a = \sqrt{53820 - 10000}$
$= 2.320$		$= \sqrt{43820}$
$A = 64^\circ 28'$	$B = 90^\circ - 64^\circ 28'$	$a = 209.3$
	$= 25^\circ 32'$	

Answer to three significant figures. $A = 64^\circ 30'$; $B = 25^\circ 30'$; $a = 209$.

Check to four significant figures. $a = b \tan A$.

$$a = 209.3 \quad b \tan A = 100 \times 2.094 = 209.4$$

EXERCISES

In these exercises, two parts of a right triangle are given, $C = 90^\circ$, and the other three parts are to be found. Check all solutions to four significant figures.

In Exercises 1–12, assume that the data were correct to four significant figures, and give answers to four figures.

- | | |
|---------------------------------------|-------------------------------|
| 1. $A = 27^\circ 16'$, $c = 8.100$. | 7. $a = 1.66$, $b = 2.48$. |
| 2. $A = 18^\circ 34'$, $c = 44.00$. | 8. $a = 54.1$, $b = 81.2$. |
| 3. $B = 42^\circ 20'$, $c = 27.00$. | 9. $a = 11.2$, $b = 12.4$. |
| 4. $B = 35^\circ 15'$, $c = 12.00$. | 10. $a = 1.35$, $b = 1.27$. |
| 5. $A = 42^\circ 17'$, $b = 18.00$. | 11. $a = 4.02$, $c = 3.04$. |
| 6. $A = 50^\circ 42'$, $b = .4800$. | 12. $a = 46.3$, $c = 57.8$. |

In Exercises 13–24, compute to four or more significant figures, but round off answers to the appropriate number of figures.

- | | |
|--------------------------------------|---------------------------------|
| 13. $B = 80^\circ 20'$, $a = .27$. | 19. $b = 508$, $c = 817$. |
| 14. $B = 68^\circ 34'$, $a = 6.6$. | 20. $b = 25.4$, $c = 41.6$. |
| 15. $A = 62^\circ 54'$, $a = 9.2$. | 21. $a = 40.55$, $b = 35.62$. |
| 16. $A = 56^\circ 39'$, $a = .67$. | 22. $a = 5.725$, $b = 4.125$. |
| 17. $B = 48^\circ 45'$, $b = 83$. | 23. $a = 26.61$, $b = 15.07$. |
| 18. $B = 34^\circ 18'$, $b = .44$. | 24. $a = 4.027$, $b = 6.153$. |

22. Geometrical applications. *Isosceles triangles.* If ABC is an isosceles triangle, lettered as in Figure 31, with $a = b$, then the perpendicular dropped from C to AB divides the triangle ABC into two equal right triangles. This makes it possible to solve an isosceles triangle if a side and an angle are given, or if the three sides are given.

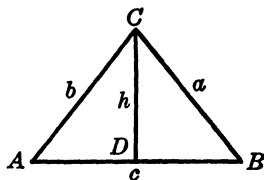


FIG. 31

Example. Solve the isosceles triangle, where the base is 21.25 ft. and the angle at the vertex is $37^{\circ} 26'$.

Solution. In the triangle ADC we have

$$D = 90^{\circ}, \quad AD = \frac{21.25}{2} = 10.62,$$

$$\angle ACD = \frac{37^{\circ} 26'}{2} = 18^{\circ} 43'.$$

Then

$$\begin{aligned} AC &= AD \csc \angle ACD \\ &= 10.62 \times 3.116 = 33.09 = BC, \\ A &= 90^{\circ} - \angle ACD = 71^{\circ} 17' = B. \end{aligned}$$

Regular polygons. Lines drawn from the center of a regular polygon to the vertices are equal in length, so that a triangle such as ABC in Figure 32 is isosceles. Here, if n is the number of sides of the polygon, we easily see that angle C is $\frac{360^{\circ}}{n}$, and angles A and B may then be computed. If, in addition, a side AB is given, or the radius CD of the inscribed circle (CD is also called the apothem), or the radius AC of the circumscribed circle, the other parts may be computed.

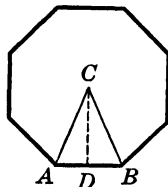


Fig. 32

Oblique triangles. In Chapter VII the solution of oblique triangles is discussed. A different, but usually more laborious method consists in computing with two auxiliary right triangles, formed by dropping a perpendicular from a vertex to an opposite side, as illustrated by Figure 33. Thus if b , c , and A were given, we could solve the right triangle ACD , thus obtaining AD and DC . Since $DB = c - AD$, we would now know two sides of the right triangle DCB and could solve for a and B , while $C = 180^{\circ} - (A + B)$.

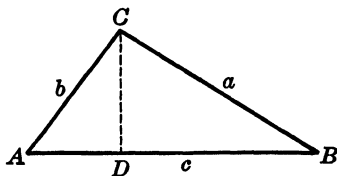


Fig. 33

EXERCISES

The data below are parts of isosceles triangles ABC , A and B being the equal base angles and C the vertex angle. Solve and check your solution.

1. $A = 46^{\circ} 22'$, $c = 1262$; find C , a , and b .
2. $C = 62^{\circ} 18'$, $c = 1072$; find A , a , and b .
3. $a = 1504$, $c = 1240$; find A , C , and b .
4. $A = 51^{\circ} 15'$, $a = 10.51$; find C , c , and b .
5. $A = 62^{\circ} 12'$, $a = 88.66$; find C , c , and b .
6. $C = 135^{\circ} 8'$, $b = 1.347$; find A , a , and c .

The following problems 7–10 refer to regular pentagons. Find the parts asked for by means of those given.

7. The side is 4.56 in.; find the radii of the inscribed and circumscribed circles.

8. The apothem is 8.44 in.; find the perimeter and the radius of the circumscribed circle.

9. The radius of the circumscribed circle is 6.63 in.; find the length of a side and the radius of the inscribed circle.

10. The perimeter is 25.50 in.; find the apothem and the radius of the circumscribed circle.

11. In a regular octagon the apothem is 1.76 in.; find the perimeter and the radius of the circumscribed circle.

12. For a regular decagon the radius of the inscribed circle is 2.72 in.; find its perimeter and the radius of the circumscribed circle.

An oblique triangle may sometimes be solved by solving right triangles formed by dropping a perpendicular from a vertex to the opposite side. Solve the following in this way, the angles being designated by capital letters, the opposite sides by corresponding small letters.

13. $A = 13^\circ 27'$, $b = 37.21$, $c = 47.26$; find B , C , and a .

14. $B = 46^\circ 16'$, $a = 81.22$, $c = 92.17$; find A , C , and b .

15. $C = 98^\circ 12'$, $a = 4.176$, $b = 2.873$; find B , A , and c .

16. $A = 59^\circ 21'$, $b = 0.1222$, $c = 1.242$; find B , C , and a .

23. Heights and distances. Bearings. The applications of trigonometry in all the sciences are so numerous and important that no attempt can be made to list them here. As examples, we shall consider a few problems in surveying and navigation.

Suppose a surveyor wishes to find the distance between two trees A and B on opposite sides of a stream. He can measure on one shore

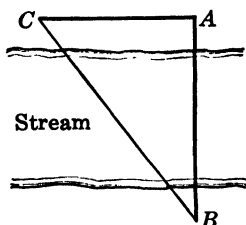


FIG. 34

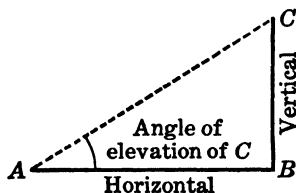


FIG. 35

along a line perpendicular to AB a convenient distance AC (Figure 34), measure the angle ACB , and find the required distance by solving the right triangle ACB .

Suppose he wishes to find the distance from a position A to a flag-pole BC of known height (Figure 35) without leaving the position A .

Assuming that A and B are in the same horizontal plane, and that BC is vertical, he may measure the angle BAC , which is called the **angle of elevation** of C for the observer at A , and solve the right triangle ABC for the required distance AB .

Suppose a navigator on board ship wishes to find how far he is from a certain rock R on shore at the water's edge. If he sights with the appropriate instrument from A and observes that the line AR (Figure 36) is depressed below the horizontal line AH by a certain amount, called the **angle of depression** of R as observed from A , and if he knows the height AB of his instrument above the water, he may solve the right triangle ABR and find the required distance. (We observe that the angle of depression of R for an observer at A equals the angle ARB , which is the angle of elevation of A for an observer at R .)

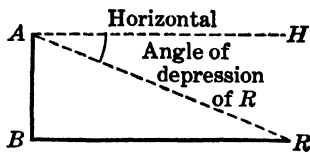


FIG. 36

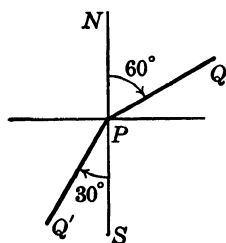


FIG. 37

In surveying and navigation it is a common practice to describe the direction of a line from a point P to a point Q by using the north-south line through P as a line of reference and indicating the angle not greater than 90° which PQ makes with the north or south line through P . The **bearing** of Q from P states first whether the north or south half line was used, the number of degrees in the angle from this ray to PQ , and whether Q is to the east or west of NS . In Fig-

ure 37, the bearing of Q from P is north 60° east, written in abbreviated form N 60° E. The bearing of Q' from P is S 30° W.

At the end of this section we shall give a number of exercises more or less like those we have just presented. It will be helpful for the student to adopt the following **method of procedure**:

(1) Read the problem carefully, then draw a figure to some convenient scale which will show those lines and angles which are given and those to be found.

(2) Draw auxiliary lines if necessary, and decide on the simplest plan for solving the problem.

(3) Write down all necessary formulas.

(4) Carry out the numerical calculations, retaining the appropriate number of significant figures in each answer.

(5) Check the results.

EXERCISES

Solve the following problems:

1. The angle of elevation of the top of a flagpole measured at a point on the same level as the base of the flagpole and 153 ft. from it is $22^{\circ} 30'$. How tall is the flagpole?
2. A vertical marker on a sundial is exactly 3 in. high. What is the elevation of the sun when the shadow cast is 5.8 in. long?
3. In a lighthouse, at 58.2 ft. above water level, the angle of depression of a small boat is $10^{\circ} 30'$. How far is the observer from the boat?
4. Two trees stand at points A and B , separated by a deep ravine. In order to find the distance between them a line AC , at right angles to AB , is measured. If AC is 102 ft. long and the angle ACB is found to be $42^{\circ} 20'$, how far apart are the trees?
5. A surveyor wishes to measure the distance between two points A and B at opposite ends of a lake. To do this he runs a line AC , 171 yd. long, and a line CB at right angles to AC , 205 yd. long. What is the distance AB ?
6. A ladder 15 ft. long is leaned against a wall. How high on the wall will it reach when the foot is 8.5 ft. from the base of the wall, if the ground is level? What angle will the ladder make with the ground?
7. Two towns are connected by a right-angled system of roads. To go from one town to the other one must go 8 miles east and 23 miles south. How far apart are the towns?
8. A guy wire of a telephone pole is 25 ft. long; one end is attached to the pole 17 ft. above the ground. Find the distance of the other end of the wire from the base of the pole, if this end is attached to a post at the level of the ground, the ground being level and the pole vertical.
9. The top of a ladder is placed at a window 21 ft. above the ground. What angle does the ladder make with the ground if the foot is 11 ft. from the house (we assume the ground to be level)? How long is the ladder if 1 ft. 6 in. projects above the point of contact with the window sill, in this position?
10. An observer at the same level as the bottom of a flagpole finds the angle of elevation of the top to be $27^{\circ} 20'$. If the flagpole is known to be 65.2 ft. tall, how far is the observer from the base of the pole? From the top?
11. A shaft is designed to descend 35 ft. for each 100 ft. measured along the shaft. What angle will it make with the horizontal? When the shaft is 152 ft. long, how far is its end below the starting level? How much horizontal distance has been traversed?
12. A right triangle with sides 4.32 and 2.41 inches long respectively is inscribed in a circle. What is the diameter of the circle? What are the acute angles of the triangle?
13. A and B stand on a level plain and sight at the same instant a balloon directly above a point on a line joining them. The angle of elevation of the balloon is $35^{\circ} 20'$ at A 's station and $53^{\circ} 40'$ at B 's station. If the two stations are exactly a mile apart, how high was the balloon at the time of observation?
14. From one point of observation, on a level with the base of a hill, the angle of elevation of the top is 44° . From a point 253 yards farther away on the same level surface the elevation of the top is 35° . How high is the hill?

15. A flagpole is mounted on the top of a wall. At a point level with the base of the wall and 25.5 feet away, the angle of elevation of the bottom of the pole is $37^\circ 40'$, and of the top is $54^\circ 30'$. What is the height of the pole?

16. A power plant has built a new smokestack. At a point 252 feet from the new stack and 176 feet from the old, the angle of elevation of the top of each is $38^\circ 30'$. If the point of observation is in the same horizontal plane as the base of each stack, how much taller is the new stack than the old one?

17. A ship sails $N 13^\circ 20' E$ for 26.3 miles, then turns 90° to the right and travels 8.20 miles. What is its position then with respect to the starting point?

18. An aeroplane flies $N 26^\circ 35' W$ for 137.2 miles, then $S 53^\circ 25' W$ for 62.4 miles. In what direction should it then fly to return to the starting point in a straight line, and how far must it go?

19. Two observers at A and B in a horizontal plane observe a captive balloon C . The points A , B , and C lie in a vertical plane, with C above a point between A and B . The distance AB is 1570 yards. At A the angle of elevation of C is $25^\circ 20'$, at B it is $34^\circ 30'$. How high is the balloon above the plane of the observers?

20. From a ship running on a course $N 5^\circ E$ along a shore the bearing of a rock is observed to be $N 32^\circ E$. When the ship has run 350 yards the bearing of the rock is $N 51^\circ E$. If the ship continues on its course, how close will it come to the rock?

21. The angle of elevation of the top of a spire from a point A in a horizontal plane is $22^\circ 23'$; from a point B which is 120 feet nearer it in the same plane the angle of elevation is $35^\circ 12'$. How high is the top of the spire above the plane?

22. A boat is traveling on a course $N 30^\circ E$. When it is at a point A a rock R is due north. After the boat has gone 500 feet to B , the bearing of the rock is $N 25^\circ W$. How close will the boat get to the rock if it continues on its course?

23. As a boat travels a certain course the bearing of a rock R when the boat is at A is α (alpha) to the left of the boat's course; when it is at B the bearing is β (beta) to the left; and the distance $AB = d$ is measured. Find how much farther the boat is to travel before it is at its closest proximity to the rock, and find how close it will then be.

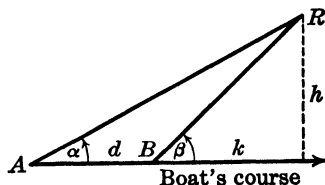


FIG. 38

$$\text{Ans.} \quad k = \frac{d \cot \beta}{\cot \alpha - \cot \beta};$$

$$h = \frac{d}{\cot \alpha - \cot \beta}.$$

Also find two angles α and β such that the answers will be

$$h = d; \quad k = 2d.$$

24. The planet Venus goes around the sun in an orbit which is practically circular, its distance from the sun being about 67×10^6 miles. The earth's orbit is also nearly circular, the distance from the earth to the sun being about 93×10^6 miles. What is the largest possible value of the angle SEV formed

by the line through the earth and sun with the line through the earth and Venus? Will Venus ever be seen in the east in the evening?

25. In latitude 41° on the earth's surface the sun gets to an altitude of $72^\circ 30'$ above the horizon when it is at its highest point for the year. What is the shortest possible shadow which may be cast on a horizontal plane by a vertical pole 75 feet high? And if a pole of unknown height casts a shadow which is at least 40 feet long in a horizontal plane, how tall is the pole?

CHAPTER III

REDUCTION FORMULAS. LINE VALUES. GRAPHS

24. Functions of a general angle expressed as functions of the corresponding acute angle. Trigonometric tables extend only from 0° to 90° . To find the value of a function of an angle θ' that is not between 0° and 90° we reduce the problem to one concerning an acute angle.

One such method of reduction uses the **corresponding acute angle θ** , which is the acute angle MOP of the triangle of reference for the

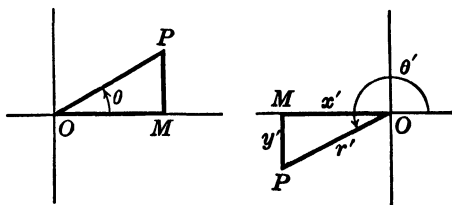


FIG. 39

angle θ' . By moving the triangle OMP into the first quadrant we could use it as a triangle of reference for θ , but its sides would then be considered positive, whereas, as in Figure 39, x' and y' may be negative. We thus obtain the **rule**:

Each function of an angle θ' is equal either to the same-named function of the corresponding acute angle θ , or to the negative of that function.

We observe that when θ' is given we find θ by taking the numerical difference between the numerical value of θ' and the nearest multiple of 180° .

Thus

$$\text{if } \theta' = 160^\circ, \text{ then } \theta = 180^\circ - 160^\circ = 20^\circ;$$

$$\text{if } \theta' = -570^\circ, \text{ then } \theta = 570^\circ - 540^\circ = 30^\circ.$$

We conclude that

$$\sin 160^\circ = \pm \sin 20^\circ, \quad \cos (-570^\circ) = \pm \cos 30^\circ.$$

It remains for us to determine whether to use the plus or the minus sign. Since functions of acute angles are positive, so that $\sin 20^\circ$ and $\cos 30^\circ$ are both positive, we can answer this question by finding whether $\sin 160^\circ$ and $\cos (-570^\circ)$ are positive or negative. Since 160° is an angle terminating in the second quadrant, its sine must be positive, and hence $\sin 160^\circ = + \sin 20^\circ$; since -570° terminates in the second quadrant, its cosine is negative, and hence $\cos (-570^\circ) = - \cos 30^\circ$.

We may now complete our rule as follows:

In the equation

function of $\theta' = \pm$ same-named function of θ

we take the + sign if θ' terminates in a quadrant where the function of θ' is positive, the - sign if the function of θ' is negative.

The method of reduction given above is not the only possible one, since a function of θ is equal to the cofunction of $90^\circ - \theta$, and this latter angle will also be acute. Thus

$$\begin{aligned}\sin 160^\circ &= \sin 20^\circ = \cos 70^\circ; \\ \cos (-570^\circ) &= -\cos 30^\circ = -\sin 60^\circ.\end{aligned}$$

It may be observed that $90^\circ - \theta$ may be computed by taking the numerical difference between the numerical value of θ' and the nearest odd multiple of 90° .

Example 1. Find the value of $\sin (-497^\circ)$.

Solution. The corresponding acute angle is $3 \times 180^\circ - 497^\circ = 43^\circ$; also -497° terminates in the third quadrant, where the sine is negative. Hence

$$\sin (-497^\circ) = -\sin 43^\circ = -.6820.$$

Also

$$\sin (-497^\circ) = \sin (-5 \times 90^\circ - 47^\circ) = -\cos 47^\circ = -.6820.$$

It may aid the student if he draws a suitable figure.

Example 2. Find an angle θ' terminating in the fourth quadrant, and such that $\cot \theta' = -.5000$.

Solution. The corresponding acute angle θ must be such that $\cot \theta = +.5000$. From the tables we find $\theta = 63^\circ 26'$. We can obtain a value of θ' by subtracting θ from such a multiple of 180° that θ' will terminate in the fourth quadrant. Thus one solution is

$$\theta' = 360^\circ - 63^\circ 26' = 296^\circ 34'.$$

We may add to θ' any positive or negative multiple of 360° and thus obtain additional solutions.

EXERCISES

With the aid of a suitable figure, express each of the following in terms of a function of an acute angle, and find its value by use of Tables:

- | | | |
|---------------------------|--------------------------------|---------------------------|
| 1. $\sin 119^\circ$. | 11. $\cos 162^\circ 10'$. | 21. $\tan 100^\circ$. |
| 2. $\sin 222^\circ$. | 12. $\cos 216^\circ 20'$. | 22. $\tan 200^\circ$. |
| 3. $\sin 303^\circ$. | 13. $\cos 312^\circ 40'$. | 23. $\tan 310^\circ$. |
| 4. $\sin 456^\circ$. | 14. $\cos 471^\circ 50'$. | 24. $\tan 490^\circ$. |
| 5. $\sin 682^\circ$. | 15. $\cos 569^\circ 20'$. | 25. $\tan (-80^\circ)$. |
| 6. $\sin (-38^\circ)$. | 16. $\cos (-27^\circ 20')$. | 26. $\tan (-195^\circ)$. |
| 7. $\sin (-138^\circ)$. | 17. $\cos (-127^\circ 30')$. | 27. $\cot 230^\circ$. |
| 8. $\sin (-216^\circ)$. | 18. $\cos (-210^\circ 10')$. | 28. $\cot (-70^\circ)$. |
| 9. $\csc 625^\circ$. | 19. $\sec 530^\circ 40'$. | 29. $\cot 850^\circ$. |
| 10. $\csc (-888^\circ)$. | 20. $\sec (-1500^\circ 10')$. | 30. $\cot (-311^\circ)$. |

REDUCTION FORMULAS. LINE VALUES. GRAPHS 37

Find an angle terminating in the quadrant indicated which satisfies the given equation:

- | | |
|------------------------------------|-----------------------------------|
| 31. II; $\sin \theta = 0.8624$. | 43. II; $\tan \theta = -2.345$. |
| 32. III; $\sin \theta = -0.2144$. | 44. III; $\tan \theta = 5.432$. |
| 33. IV; $\sin \theta = -0.8188$. | 45. II; $\cot \theta = -0.4321$. |
| 34. III; $\sin \theta = -0.1111$. | 46. III; $\cot \theta = 0.4321$. |
| 35. II; $\cos \theta = -0.1234$. | 47. IV; $\cot \theta = -0.4321$. |
| 36. III; $\cos \theta = -0.4321$. | 48. II; $\cot \theta = -4.321$. |
| 37. IV; $\cos \theta = 0.4321$. | 49. II; $\sec \theta = -1.234$. |
| 38. II; $\cos \theta = -0.4321$. | 50. III; $\sec \theta = -1.234$. |
| 39. II; $\tan \theta = -0.1234$. | 51. IV; $\sec \theta = 1.234$. |
| 40. III; $\tan \theta = 0.1234$. | 52. III; $\sec \theta = -4.321$. |
| 41. IV; $\tan \theta = -0.1234$. | 53. III; $\csc \theta = -4.321$. |
| 42. II; $\tan \theta = -1.234$. | 54. IV; $\csc \theta = -4.321$. |

Find all of the angles, if there are any, which satisfy the given equation:

- | | | |
|-------------------------------|-------------------------------|-------------------------------|
| 55. $\sin \theta = 0.2345$. | 63. $\tan \theta = 0.2345$. | 71. $\sec \theta = 0.2345$. |
| 56. $\sin \theta = -0.2345$. | 64. $\tan \theta = -0.2345$. | 72. $\sec \theta = -0.2345$. |
| 57. $\sin \theta = 2.345$. | 65. $\tan \theta = 2.345$. | 73. $\sec \theta = 2.345$. |
| 58. $\sin \theta = -2.345$. | 66. $\tan \theta = -2.345$. | 74. $\sec \theta = -2.345$. |
| 59. $\cos \theta = 0.2345$. | 67. $\cot \theta = 0.2345$. | 75. $\csc \theta = 0.2345$. |
| 60. $\cos \theta = -0.2345$. | 68. $\cot \theta = -0.2345$. | 76. $\csc \theta = -0.2345$. |
| 61. $\cos \theta = 2.345$. | 69. $\cot \theta = 2.345$. | 77. $\csc \theta = 2.345$. |
| 62. $\cos \theta = -2.345$. | 70. $\cot \theta = -2.345$. | 78. $\csc \theta = -2.345$. |

Find rectangular coördinates of the point P whose polar coördinates are as given:

- | | | |
|--------------------------|--------------------------|--------------------------|
| 79. $P(4, 119^\circ)$. | 81. $P(8, 303^\circ)$. | 83. $P(12, -38^\circ)$. |
| 80. $P(6, -222^\circ)$. | 82. $P(10, 456^\circ)$. | 84. $P(2, -138^\circ)$. |

Find polar coördinates of the point P whose rectangular coördinates are as given:

- | | | |
|--------------------|-------------------|--------------------|
| 85. $P(-7, 24)$. | 87. $P(7, -24)$. | 89. $P(-12, -5)$. |
| 86. $P(-7, -24)$. | 88. $P(-12, 5)$. | 90. $P(12, -5)$. |

Prove that if θ is a positive acute angle the following relations hold:

- | | |
|--|--|
| 91. $\sin(180^\circ - \theta) = \sin \theta$. | 97. $\sin(90^\circ + \theta) = \cos \theta$. |
| 92. $\sin(360^\circ - \theta) = -\sin \theta$. | 98. $\sin(270^\circ + \theta) = -\cos \theta$. |
| 93. $\sin(540^\circ - \theta) = \sin \theta$. | 99. $\sin(450^\circ + \theta) = \cos \theta$. |
| 94. $\sin(720^\circ - \theta) = -\sin \theta$. | 100. $\sin(630^\circ + \theta) = -\cos \theta$. |
| 95. $\sin(-180^\circ - \theta) = \sin \theta$. | 101. $\sin(-90^\circ + \theta) = -\cos \theta$. |
| 96. $\sin(-360^\circ - \theta) = -\sin \theta$. | 102. $\sin(-270^\circ + \theta) = \cos \theta$. |

25. Functions of $180^\circ - \theta$. If $\theta' = 180^\circ - \theta$, then the angle θ , if acute, must be the corresponding acute angle for θ' , and θ' must terminate in the second quadrant. Hence if θ is acute we obtain the following formulas from the rules of §24:

$$\begin{array}{ll} \sin(180^\circ - \theta) = \sin \theta, & \cos(180^\circ - \theta) = -\cos \theta, \\ \tan(180^\circ - \theta) = -\tan \theta, & \cot(180^\circ - \theta) = -\cot \theta, \\ \sec(180^\circ - \theta) = -\sec \theta, & \csc(180^\circ - \theta) = \csc \theta. \end{array}$$

It is an interesting property of many formulas in trigonometry that each of them holds, not merely for acute angles, but for *all permissible angles*; that is, for *angles such that the formula does not imply a division by zero*. The above formulas are a good example; they hold for all permissible values of θ , and not merely for values between 0° and 90° .

We shall prove the statement by comparing the triangle of reference for $180^\circ - \theta$ with that for θ , in the four cases which arise when θ terminates in the first quadrant, the second quadrant, the third quadrant, or the fourth quadrant.

Figure 40a shows the triangle of reference OMP for an angle θ terminating in the first quadrant, and the triangle of reference

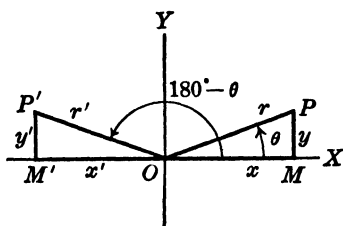


FIG. 40a

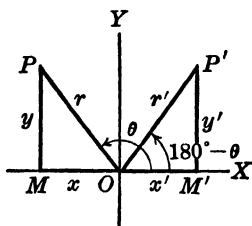


FIG. 40b

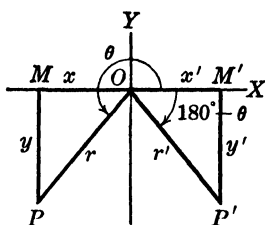


FIG. 40c

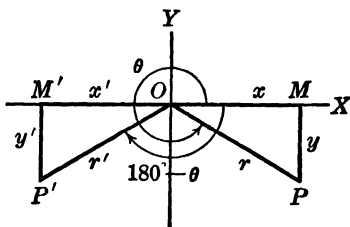


FIG. 40d

$OM'P'$ for $180^\circ - \theta$. The lengths of corresponding sides are equal. In Figure 40b the two triangles are drawn for an angle θ terminating in the second quadrant; in Figures 40c, 40d, the angle θ terminates in the third and fourth quadrants respectively. In each case the triangle OMP is geometrically equal to $OM'P'$; we insure this by taking OP' equal to OP , and we then observe that the angles at O are equal. The two right triangles have two sides and an acute angle of one equal to corresponding parts of the other, and hence the triangles are equal. It follows that $r' = r$; but we note that x' and x ,

while numerically equal, are of opposite sign in each of the four cases, while y' and y are of the same sign. That is, in all cases,

$$x' = -x, y' = y, r' = r.$$

We complete the proof of the formulas for three of the functions of $180^\circ - \theta$ as follows:

$$\sin (180^\circ - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin \theta,$$

$$\cos (180^\circ - \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\cos \theta,$$

$$\tan (180^\circ - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\tan \theta.$$

The student should write out the proof for the other three formulas.

Quadrantal angles θ do not possess triangles of reference, but even for them the x, y, r relations hold, and our proof is valid except when the angle and function are such that the function of that angle has no value ($\tan 90^\circ$, for example); that is, the proofs are valid for *all permissible* angles.

26. Functions of $n \cdot 180^\circ \pm \theta$. If θ is an acute angle, it is the corresponding acute angle for the angle $n \cdot 180^\circ \pm \theta$, where n is an integer, positive or negative or zero. Thus we have the **rule**:

If θ is an acute angle, any function of $\theta' = n \cdot 180^\circ \pm \theta$ is equal to plus or minus the same-named function of θ , according as the function of θ' is itself plus or minus.

We obtain particular formulas for particular values of n ; and it is true, as in the preceding section, that these formulas, including the $+$ or $-$ sign, hold not only when θ is acute, but also for all permissible values of θ . We reproduce a few of these formulas:

$$\sin (180^\circ + \theta) = -\sin \theta,$$

$$\cos (180^\circ + \theta) = -\cos \theta,$$

$$\tan (180^\circ + \theta) = \tan \theta,$$

$$\cot (180^\circ + \theta) = \cot \theta,$$

$$\sin (-\theta) = \sin (360^\circ - \theta) = -\sin \theta,$$

$$\cos (-\theta) = \cos (360^\circ - \theta) = \cos \theta,$$

$$\tan (-\theta) = \tan (360^\circ - \theta) = -\tan \theta,$$

$$\cot (-\theta) = \cot (360^\circ - \theta) = -\cot \theta.$$

Proofs should be supplied by the student, who should draw triangles of reference and compare them as was done in § 25.

27. Functions of $n \cdot 90^\circ \pm \theta$, where n is odd. If θ is an acute angle and n is odd, then the corresponding acute angle for $\theta' = n \cdot 90^\circ \pm \theta$ is $90^\circ - \theta$, and the rules of § 24 enable us to express a function of θ' as plus or minus the same-named function of $90^\circ - \theta$, and finally as plus or minus the cofunction of θ . We have, for example,

$$\begin{aligned}\sin(90^\circ - \theta) &= \cos \theta, \cos(90^\circ - \theta) = \sin \theta, \tan(90^\circ - \theta) = \cot \theta; \\ \sin(90^\circ + \theta) &= \cos \theta, \cos(90^\circ + \theta) = -\sin \theta, \tan(90^\circ + \theta) = -\cot \theta; \\ \sin(270^\circ - \theta) &= -\cos \theta, \cos(270^\circ - \theta) = -\sin \theta, \tan(270^\circ - \theta) = \cot \theta; \\ \sin(270^\circ + \theta) &= -\cos \theta, \cos(270^\circ + \theta) = \sin \theta, \tan(270^\circ + \theta) = -\cot \theta.\end{aligned}$$

These formulas all hold for all permissible values of θ . This may be proved by comparing related triangles of reference.

As an illustration we give in Figure 41 a drawing for a case where θ terminates in the second quadrant, so that $90^\circ - \theta$ terminates in the fourth. We note here that

$$\begin{aligned}x' &= y, y' = x, r' = r, \\ \sin(90^\circ - \theta) &= \frac{y'}{r'} = \frac{x}{r} = \cos \theta,\end{aligned}$$

and similarly for other functions of $90^\circ - \theta$.

We may summarize all of these formulas in the following rule:

If θ is an acute angle and n is an odd integer, a function of $\theta' = n \cdot 90^\circ \pm \theta$ is equal to plus or minus the corresponding cofunction of θ , according as the function of θ' itself is plus or minus. Each formula thus obtained holds not only when θ is acute, but for all permissible values of θ .

For example, $\sin(270^\circ + \theta) = -\cos \theta$ whether θ is acute or not, according to the rule just stated, since this formula is correct when θ is acute.

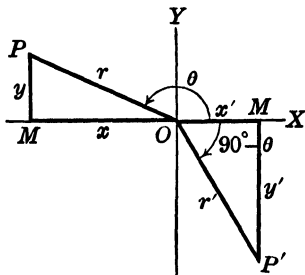


FIG. 41

EXERCISES

Use the general rules of §§ 26, 27 to express each of the following as a function of an acute angle in two ways:

- | | | |
|---------------------------|----------------------------|----------------------------|
| 1. $\sin 237^\circ$. | 9. $\tan 386^\circ$. | 17. $\sec 281^\circ$. |
| 2. $\sin 1237^\circ$. | 10. $\tan 1386^\circ$. | 18. $\sec (-281^\circ)$. |
| 3. $\sin (-237^\circ)$. | 11. $\tan (-386^\circ)$. | 19. $\sec 1281^\circ$. |
| 4. $\sin (-1237^\circ)$. | 12. $\tan (-1386^\circ)$. | 20. $\sec (-1281^\circ)$. |
| 5. $\cos 156^\circ$. | 13. $\cot 123^\circ$. | 21. $\csc 482^\circ$. |
| 6. $\cos 256^\circ$. | 14. $\cot (-123^\circ)$. | 22. $\csc (-482^\circ)$. |
| 7. $\cos 356^\circ$. | 15. $\cot 1123^\circ$. | 23. $\csc 1482^\circ$. |
| 8. $\cos 1356^\circ$. | 16. $\cot (-1123^\circ)$. | 24. $\csc (-1482^\circ)$. |

By the method used in § 25 (which requires drawing triangles of reference and comparing them) prove that if θ is an angle terminating in the second quadrant each of the following formulas holds:

- | | |
|---|---|
| 25. $\sin (180^\circ + \theta) = -\sin \theta.$ | 33. $\sin (270^\circ - \theta) = -\cos \theta.$ |
| 26. $\cos (180^\circ + \theta) = -\cos \theta.$ | 34. $\cos (270^\circ - \theta) = -\sin \theta.$ |
| 27. $\tan (180^\circ + \theta) = \tan \theta.$ | 35. $\tan (270^\circ - \theta) = \cot \theta.$ |
| 28. $\cot (180^\circ + \theta) = \cot \theta.$ | 36. $\cot (270^\circ - \theta) = \tan \theta.$ |
| 29. $\sin (-\theta) = -\sin \theta.$ | 37. $\sin (270^\circ + \theta) = -\cos \theta.$ |
| 30. $\cos (-\theta) = \cos \theta.$ | 38. $\cos (270^\circ + \theta) = \sin \theta.$ |
| 31. $\tan (-\theta) = -\tan \theta.$ | 39. $\tan (270^\circ + \theta) = -\cot \theta.$ |
| 32. $\cot (-\theta) = -\cot \theta.$ | 40. $\cot (270^\circ + \theta) = -\tan \theta.$ |

By drawing triangles of reference and comparing them, prove that if θ is an angle terminating in the third quadrant each of the following formulas holds:

- | | |
|---|---|
| 41. $\sin (180^\circ + \theta) = -\sin \theta.$ | 44. $\cot (90^\circ + \theta) = -\tan \theta.$ |
| 42. $\cos (180^\circ + \theta) = -\cos \theta.$ | 45. $\sec (270^\circ - \theta) = -\csc \theta.$ |
| 43. $\tan (180^\circ + \theta) = \tan \theta.$ | 46. $\csc (450^\circ + \theta) = \sec \theta.$ |

By drawing triangles of reference and comparing them, prove that if θ is an angle terminating in the fourth quadrant each of the following formulas holds:

- | | |
|--|---|
| 47. $\sin (-\theta) = -\sin \theta.$ | 50. $\cot (270^\circ - \theta) = \tan \theta.$ |
| 48. $\cos (90^\circ + \theta) = -\sin \theta.$ | 51. $\sec (270^\circ + \theta) = \csc \theta.$ |
| 49. $\tan (180^\circ + \theta) = \tan \theta.$ | 52. $\csc (180^\circ + \theta) = -\csc \theta.$ |

28. Line values. In the triangle of reference OMP for an angle θ , the length of OP is at our choice. If, for example, θ is an acute angle, and $OP = 1$, we have

$$\sin \theta = \frac{MP}{OP} = \frac{MP}{1} = MP,$$

$$\cos \theta = \frac{OM}{OP} = \frac{OM}{1} = OM.$$

These lengths, MP , OM , are called **line values** of $\sin \theta$, $\cos \theta$ respectively.*

The other four functions of an acute angle θ are also assigned line values in a figure such as Figure 43 on page 42. We first construct the auxiliary Figure 42 by drawing the coordinate axes OX , OY , locating on them the points A whose coordinates are $(1, 0)$ and B whose coordinates are $(0, 1)$, then drawing the circle of unit radius and center O , together with tangents at A and B . In Figure 43 we add to Figure 42 by placing the angle θ in standard position, drawing its triangle of reference OMP so that $OP = 1$, and prolonging OP until it meets at T the tangent line through A , and at T' the tan-

* It was once customary to use these as first definitions of the sine and cosine.

gent line through B . Since triangles OMP , OAT , and OBT' are similar, we have

$$\begin{aligned}
 \sin \theta &= MP; \\
 \cos \theta &= OM; \\
 \tan \theta &= \frac{MP}{OM} = \frac{AT}{OA} = \frac{AT}{1} = AT; \\
 \cot \theta &= \frac{OM}{MP} = \frac{BT'}{OB} = \frac{BT'}{1} = BT'; \\
 \sec \theta &= \frac{OP}{OM} = \frac{OT}{OA} = \frac{OT}{1} = OT; \\
 \csc \theta &= \frac{OP}{MP} = \frac{OT'}{OB} = \frac{OT'}{1} = OT'.
 \end{aligned}
 \tag{1}$$

We have thus found six line segments whose lengths are the line values of the six functions of an acute angle.

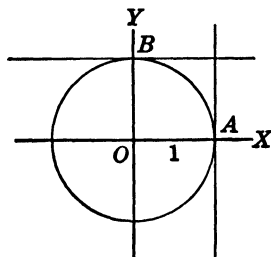


FIG. 42

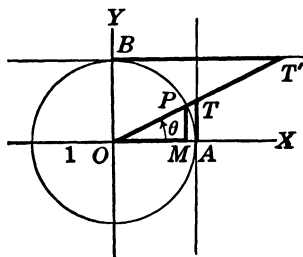


FIG. 43

How may these definitions be adapted so as to apply to angles that are not acute? We answer this question by drawing a figure according to the same verbal specifications as for an acute angle. We have done this in Figure 44 for an angle θ terminating in the third quadrant; that is, we have taken the fundamental Figure 42 and inserted the triangle of reference OMP with $OP = 1$, and have prolonged OP (in this case backwards) until it intersects at T and T' the tangent lines through A and B . It is clear that equations (1) now hold numerically, but they do not exhibit the minus signs that $\sin \theta$, $\cos \theta$, $\sec \theta$, $\csc \theta$ should have as functions of an angle terminating in the third quadrant. To remedy this we interpret MP , OM , AT , BT' , OT , OT' not

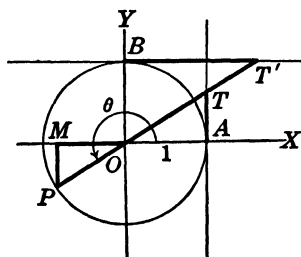


FIG. 44

merely as lengths, but as *directed lengths* according to the following rule:

By MP , OM , AT , BT' we now mean, respectively, the ordinate of P , the abscissa of P , the ordinate of T , and the abscissa of T' , while OT is to be positive if P is between O and T , otherwise OT is negative; for OT' the same rule holds as for OT .

With this rule as to signs, formulas (1) hold for any permissible angle θ . The student should draw a figure for an angle θ terminating in the second quadrant, and also one for an angle θ terminating in the fourth quadrant; he should then verify formulas (1) as above interpreted.

29. Properties of the sine function. The interpretation of $\sin \theta$ as the line value MP exhibits clearly the following properties of the sine function:

(1) $\sin \theta$ is never greater than 1 or less than -1 .

(2) As θ increases steadily from one quadrantal angle to the next greater one, $\sin \theta$ either increases steadily or decreases steadily.

Thus as θ increases from 0° to 90° , $MP = \sin \theta$ increases steadily from 0 to 1; as θ increases from 90° to 180° , MP decreases steadily from 1 to 0; as θ increases from 180° to 270° , MP decreases steadily from 0 to -1 ; as θ increases from 270° to 360° , MP increases steadily from -1 to 0.

(3) $\sin \theta$ has the period 360° . By this we mean that for every angle θ we have $\sin (\theta + 360^\circ) = \sin \theta$.

(4) $\sin (-\theta) = -\sin \theta$.

Two more properties can be directly deduced from (2) and its explanatory paragraph:

(5) If θ and θ' are two different angles terminating in the same quadrant, then $\sin \theta$ cannot be equal to $\sin \theta'$.

(6) If a is a number between -1 and $+1$, there are at most two angles θ between 0° and 360° for which $\sin \theta = a$.

Property (5) follows from (2) because, according to (2), if θ' is greater than θ , then $\sin \theta'$ is either greater than or less than, but not equal to, $\sin \theta$. To prove the statement of (6), note that the discussion of (2) shows that if a is positive, and we let θ increase steadily from 0° to 90° , $\sin \theta$ will increase steadily from 0 to 1 and hence will have the value a for at least one value of θ (but not two θ 's, by (5)). Similarly, for positive a there will be one θ between 90° and 180° for which $\sin \theta = a$; but there will be no θ between 180°

and 360° for which $\sin \theta = a$. The student should supply the proof for a negative. If $a = 0$, the equation $\sin \theta = a$ has two solutions, $\theta = 0^\circ$ and $\theta = 180^\circ$, that are not less than 0° and not as large as 360° . We must interpret the word "between" in (6) accordingly. With this understanding, we can make (6) more precise by saying that if $a = -1$, or if $a = +1$, there is just one value of θ between 0° and 360° for which $\sin \theta = a$, but for all other values of a between -1 and $+1$ there are just two values of θ between 0° and 360° for which $\sin \theta = a$.

30. Graph of the sine function in rectangular coördinates. The properties given in § 29 are visualized if we draw a graph of the sine function as given in the equation $y = \sin x$. In this equation x and y are no longer coördinates of a vertex P of a triangle of reference; instead, x now stands for the number of degrees in an angle and y is the positive or negative number that gives the value of $\sin x$.

We take axes OX , OY , as in Figure 45, and for a sufficient number of angles plot points (x, y) for which x is the number of degrees in

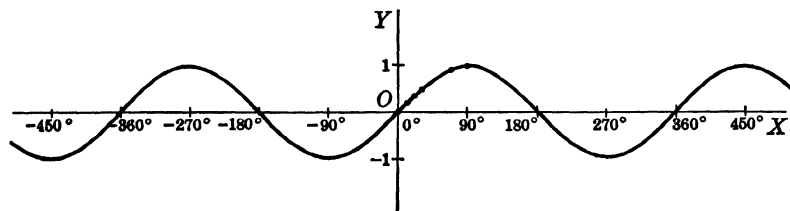


FIG. 45. $y = \sin x$

the angle, and y is the value of $\sin x$. For acute angles, we may use a table of trigonometric functions to determine y . When x is between 90° and 180° , we may make use of the reduction formula $\sin x = \sin (180^\circ - x)$ and obtain values for $180^\circ - x$ from the tables. In Figure 45, the black dots on the curve at O and near it to the right, indicate values of x, y that correspond to the following tabulation:

x	0°	10°	20°	30°	60°	90°
y	0	0.1736	0.3420	0.5000	0.8660	1.000

When enough points have been plotted we draw through them the smooth curve that forms the first right-hand arch in Figure 45, from

$x = 0^\circ$ to $x = 180^\circ$. Property (4) of § 29 enables us then to sketch the curve from $x = 0^\circ$ to $x = -180^\circ$; here an ordinate for $x = -\theta$ is of the same length as for $x = +\theta$, but is drawn downwards instead of upwards. Property (3) of § 29 then tells us that the remainder of the curve is a repetition of the part that we have now drawn; the part from 180° to $180^\circ + 360^\circ$, for example, reproduces the part from -180° to $+180^\circ$, since if x is the number of degrees in an angle θ , and x' the number in $360^\circ + \theta$, then $\sin x' = \sin x$.

The graph in Figure 45 gives a geometrical picture from which we readily infer the properties listed in § 29. The curve never rises above $+1$ or falls below -1 ; this statement implies property (1). It rises steadily from $x = -90^\circ$ to $x = 90^\circ$, falls steadily from $x = 90^\circ$ to $x = 270^\circ$, and so on; this implies property (2). Properties (3) and (4) are inherent in the construction of the curve. Property (5) is a statement of the nonequality of two ordinates in certain arcs of the curve, and (6) tells how many ordinates may be equal from $x = 0^\circ$ to $x = 360^\circ$.

31. Properties and graph of the cosine. The statements (1), (2), (3) of § 29 are true for $\cos \theta$ as well as for $\sin \theta$, but the explanatory

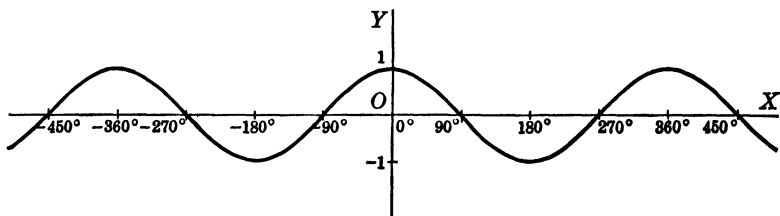


FIG. 46. $y = \cos x$

paragraph under (2) must be altered; $\cos \theta$ decreases as θ increases from 0° to 180° , and increases as θ increases from 180° to 360° . Statement (4) must be replaced by the following:

$$(4') \cos(-\theta) = \cos \theta.$$

Statements (5) and (6) hold for the cosine as well as for the sine.

In Figure 46 a graph of the equation

$$y = \cos x$$

has been drawn. The student should observe how it exhibits properties (1), (2), (3), (4'), (5), (6). An inspection of the figure shows that the cosine curve would coincide with the sine curve if it were moved

to the right so that $x = 0^\circ$ became $x = 90^\circ$. This is a consequence of the formula $\cos \theta = \sin (90^\circ + \theta)$.

32. Properties and graphs of the tangent and secant. Figures 47 and 48 represent graphs of the equations $y = \tan x$, $y = \sec x$, respectively. A study of these graphs, or a reference to the definitions by line values, shows that the statements (2), (3), (5) of § 29 hold for $\tan \theta$ and $\sec \theta$ also, but the discussion of (2) must be modified. It remains to see how (1), (4), (6), and the discussion of (2) are to be changed.

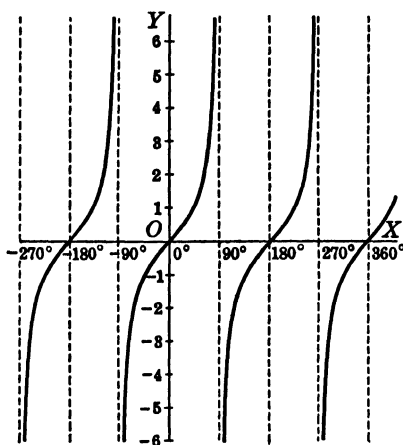


FIG. 47. $y = \tan x$

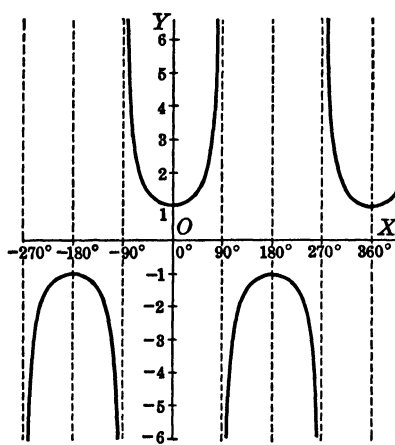


FIG. 48. $y = \sec x$

The function $\tan \theta$ is not limited; it can in particular be greater than $+1$ or less than -1 , but when, for example, θ is between -45° and $+45^\circ$, $\tan \theta$ is between -1 and 1 . The function $\sec \theta$ can take on any value greater than $+1$ or less than -1 , but its value cannot lie between -1 and $+1$. These statements replace (1).

The formulas analogous to (4) are

$$\tan (-\theta) = -\tan \theta, \sec (-\theta) = \sec \theta.$$

As for (6), the equation $\tan \theta = a$ has two and only two solutions for θ between 0° and 360° (including 0° but not 360°), and there is no restriction on a . The student should supply the corresponding statement for the equation $\sec \theta = a$.

REDUCTION FORMULAS. LINE VALUES. GRAPHS 47

The graph of $y = \tan x$ shows clearly that as x increases from 0° to 90° , $\tan x$ becomes steadily and unlimitedly larger. We say that $\tan x$ becomes infinite as x thus approaches 90° ; to write $\tan 90^\circ = \infty$ (the symbol for "infinity") must be regarded as an abbreviation of this statement, and not as an assertion that "infinity" is a number or that $\tan 90^\circ$ has a value. The graph, in fact, shows that $\tan x$ always increases with x , has no value at $x = 90^\circ, 270^\circ, \dots, -90^\circ, \dots$, but becomes infinite at those points.

The student should supply a similar discussion for $\sec x$.

EXERCISES

Draw a graph of each of the following equations and discuss its properties:

- | | | |
|--------------------|--------------------------------|-----------------------------|
| 1. $y = \cot x$. | 4. $y = \frac{1}{2} \sin 3x$. | 7. $y = \tan 3x$. |
| 2. $y = \csc x$. | 5. $y = \cos 2x$. | 8. $y = \sec \frac{x}{2}$. |
| 3. $y = \sin 2x$. | 6. $y = \frac{1}{2} \cos 3x$. | 9. $y = 2 + \sin x$. |

10. Draw a graph of $y = \sin x + \cos x$. Does the function $\sin x + \cos x$ have a period? If so, what is it?

11. Draw a graph of $y = \frac{1}{2} \sin x + \frac{3}{4} \sin 2x$. Does the function $\frac{1}{2} \sin x + \frac{3}{4} \sin 2x$ have a period? If so, what is it?

12. Draw a graph of $y = \sin x - \frac{1}{2} \sin 2x$. Does the function $\sin x - \frac{1}{2} \sin 2x$ have a period? If so, what is it?

13. Draw a graph of $y = \cos x - \frac{1}{2} \cos 2x$. Does the function $\cos x - \frac{1}{2} \cos 2x$ have a period? If so, what is it?

14. Draw a graph of $y = \sin^2 x$. Does the function $\sin^2 x$ have a period? If so, what is it?

In the same figure draw graphs of the two equations, for each of the following pairs:

- | | |
|---|--|
| 15. $y = \sin x, y = \sin(90^\circ + x)$. | 19. $y = 2 \cos x, y = \cos 2x$. |
| 16. $y = \cos x, y = \cos(180^\circ + x)$. | 20. $y = \cos x, y = \cos(90^\circ - x)$. |
| 17. $y = 2 \sin x, y = \sin 2x$. | 21. $y = \sin x, y = 1 + \sin x$. |
| 18. $y = \cos x, y = 2 \cos x$. | 22. $y = \cos x, y = \cos x - 1$. |

CHAPTER IV

TRIGONOMETRIC IDENTITIES

33. Fundamental identities involving one angle. By an identity in an angle θ we mean an equation involving θ which is true for all permissible values of θ . Here we change slightly a meaning previously given to the term *permissible values* by making it include all values except those for which a function in the identity is undefined, or a denominator is zero.

The reduction formulas of Chapter III are examples of trigonometric identities in one angle; they hold for all permissible values of θ .

From the definitions of the functions in § 5, formulas (1), we see at once that those written below in the same column are reciprocals of each other:

$$(1) \quad \begin{array}{lll} \csc \theta = \frac{1}{\sin \theta}, & \cot \theta = \frac{1}{\tan \theta}, & \sec \theta = \frac{1}{\cos \theta}, \\ \sin \theta = \frac{1}{\csc \theta}, & \tan \theta = \frac{1}{\cot \theta}, & \cos \theta = \frac{1}{\sec \theta}. \end{array}$$

The following two identities also follow directly from the definitions of the functions:

$$(2) \quad \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}.$$

We prove the first of these identities, for example, as follows:

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan \theta.$$

We obtain three more identities from the relation $x^2 + y^2 = r^2$, which holds because, in the triangle of reference, the sum of the squares of two sides is equal to the square of the hypotenuse; the sides are $\pm x$, $\pm y$, but their squares are always x^2 , y^2 . We write

these identities in a customary notation where $\sin^2 \theta$, for example, means $(\sin \theta)^2$:

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1, \\ (3) \quad 1 + \tan^2 \theta &= \sec^2 \theta, \\ 1 + \cot^2 \theta &= \csc^2 \theta. \end{aligned}$$

To prove any one of these three we start with

$$x^2 + y^2 = r^2$$

and divide by one of the three terms. Thus, dividing by r^2 , we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1,$$

$$(\sin \theta)^2 + (\cos \theta)^2 = 1,$$

which is another way of writing the first of formulas (3). Similarly, if we divide by x^2 we have

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2;$$

and if we divide by y^2 we have, with a change of order of terms,

$$1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2.$$

These last two formulas reduce to the last two of identities (3).

Formulas (1), (2), (3) are fundamental in trigonometry, and *must be memorized*.

Example 1. By means of identities (1), (2), and (3), find the values of the other functions of θ if $\tan \theta = -\frac{3}{4}$, and θ terminates in the second quadrant.

Solution. In examples of this sort it is necessary to use one of formulas (3); except for this, (1) and (2) are sufficient. Thus, first using the second of formulas (3), we have

$$\sec^2 \theta = 1 + \tan^2 \theta = 1 + \left(-\frac{3}{4}\right)^2 = \frac{25}{16};$$

$$\sec \theta = -\frac{5}{4} \text{ (negative since } \theta \text{ is in the second quadrant);}$$

$$\cos \theta = \frac{1}{\sec \theta} = \frac{-4}{5} \text{ (by the last of formulas (1));}$$

$$\sin \theta = \cos \theta \tan \theta = -\frac{4}{5} \times \left(-\frac{3}{4}\right) = \frac{3}{5} \text{ (by the first of (2));}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{-4}{3} \text{ (by a formula of (1));}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{5}{3} \text{ (by a formula of (1)).}$$

If the quadrant in which θ terminates had not been specified, we would have written $\sec \theta = \pm \frac{5}{4}$, and similarly for $\cos \theta$, $\sin \theta$, $\csc \theta$.

Example 2. Express $\csc \theta + \cos \theta + \tan^2 \theta$ in terms of $\sin \theta$ only.

Solution. We have from the fundamental identities (1), (2), (3),

$$\csc \theta = \frac{1}{\sin \theta},$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta},$$

$$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\sin^2 \theta}{1 - \sin^2 \theta}.$$

Hence

$$\csc \theta + \cos \theta + \tan^2 \theta = \frac{1}{\sin \theta} \pm \sqrt{1 - \sin^2 \theta} + \frac{\sin^2 \theta}{1 - \sin^2 \theta},$$

where the $\pm$ sign indicates that $\cos \theta$ is positive or negative according to the quadrant where θ terminates.

34. Other identities involving one angle. An unlimited number of other identities can be formed, and can be proved by means of the fundamental identities (1), (2), (3) of § 33. The following examples illustrate three methods of proof.

Example 1. Prove the identity

$$\frac{\cos \theta}{1 - \sin^2 \theta} = \sec \theta$$

by transforming the left side only.

Solution. By means of the fundamental formulas and elementary algebra we have

$$\frac{\cos \theta}{1 - \sin^2 \theta} = \frac{\cos \theta}{\cos^2 \theta} = \frac{1}{\cos \theta} = \sec \theta.$$

Example 2. Prove the identity

$$\cos \theta - \cot \theta = \cot \theta (\sin \theta - 1)$$

by reducing both sides to the same expression.

Solution. We arrange the work in parallel columns:

$$\begin{array}{l|l} \cos \theta - \cot \theta & \cot \theta (\sin \theta - 1) \\ \cos \theta - \frac{\cos \theta}{\sin \theta} & \frac{\cos \theta}{\sin \theta} (\sin \theta - 1) \\ & \cos \theta - \frac{\cos \theta}{\sin \theta} \end{array}$$

Example 3. Prove the identity of Example 2 by deducing from it, by algebraic and trigonometric transformations, a chain of identities such that the given one is true if the second is, the second is true if the third is, and so on until we obtain an identity which states that an expression is equal to itself.

Solution. Such a chain is

$$(A) \quad \cos \theta - \cot \theta = \cot \theta (\sin \theta - 1);$$

$$(B) \quad \cos \theta - \cot \theta = \cot \theta \sin \theta - \cot \theta;$$

$$(C) \quad \cos \theta = \frac{\cos \theta}{\sin \theta} \sin \theta;$$

$$(D) \quad \cos \theta = \cos \theta.$$

If we merely write down such a chain as constituting a proof of (A), we must understand that it is to be interpreted as saying that (A) is true if (B) is true, (B) is true if (C) is true, (C) is true if (D) is true.

In proving identities it is usually best to avoid using radicals which may have plus or minus signs. A method often effective is to reduce all other functions to sines and cosines, as was done in the preceding Examples 2, 3.

EXERCISES

By means of the fundamental identities (1), (2), (3) of § 33, find the values of all the functions of θ from the given conditions:

- | | | |
|------------------------------------|-----------------------------------|------------------------------------|
| 1. $\sin \theta = \frac{3}{5}$. | 5. $\tan \theta = \frac{1}{5}$. | 9. $\sec \theta = -\frac{4}{3}$. |
| 2. $\sin \theta = -\frac{5}{13}$. | 6. $\tan \theta = -\frac{4}{3}$. | 10. $\sec \theta = \frac{5}{4}$. |
| 3. $\cos \theta = -\frac{4}{5}$. | 7. $\cot \theta = -\frac{3}{4}$. | 11. $\csc \theta = \frac{1}{8}$. |
| 4. $\cos \theta = -\frac{1}{2}$. | 8. $\cot \theta = -\frac{1}{5}$. | 12. $\csc \theta = -\frac{4}{3}$. |

By means of the fundamental identities (1), (2), (3) of § 33, express the following in terms of $\sin \theta$ and $\cos \theta$ only:

- | | |
|---|---|
| 13. $\tan \theta + \sec \theta$. | 20. $\cot^2 \theta - \frac{1}{\csc^2 \theta}$. |
| 14. $\cot \theta + \csc \theta$. | 21. $\cos \theta \tan \theta + \sec \theta$. |
| 15. $\sin^2 \theta + \sec^2 \theta$. | 22. $\sin \theta \cot \theta + \csc \theta$. |
| 16. $\cos^2 \theta + \csc^2 \theta$. | 23. $\sin \theta \cos \theta \tan \theta \cot \theta$. |
| 17. $\tan^2 \theta + \sec^2 \theta$. | 24. $\tan \theta \cot \theta \sec \theta \csc \theta$. |
| 18. $\cot^2 \theta + \csc^2 \theta$. | |
| 19. $\tan^2 \theta + \frac{1}{\sec^2 \theta}$. | |

By means of the fundamental identities (1), (2), (3) of § 33, express the following in terms of $\sin \theta$ only:

- | | |
|---|---|
| 25. $\sec^2 \theta + \tan^2 \theta$. | 29. $\cos \theta \cot \theta + \csc \theta$. |
| 26. $2 \cos^2 \theta - 1$. | 30. $\sin \theta \tan \theta + \sec \theta$. |
| 27. $\sec \theta \tan \theta + \csc \theta$. | 31. $\sec^2 \theta \csc^2 \theta - \tan^2 \theta$. |
| 28. $\csc \theta \cot \theta - \sec \theta$. | 32. $\cot^2 \theta - \sin \theta \tan \theta$. |

By means of the fundamental identities (1), (2), (3) of § 33, prove each of the following identities by transforming the left side only:

- | | |
|---|---|
| 33. $\sin \theta \cot \theta = \cos \theta$. | 41. $\frac{\csc^2 \theta - \cot^2 \theta}{\sec^2 \theta} = \cos^2 \theta$. |
| 34. $\cos \theta \tan \theta = \sin \theta$. | 42. $\frac{\sin^2 \theta}{\cos^4 \theta + \cos^2 \theta \sin^2 \theta} = \tan^2 \theta$. |
| 35. $\sec \theta \cot \theta = \csc \theta$. | 43. $\cos \theta \csc \theta = \cot \theta$. |
| 36. $\csc \theta \tan \theta = \sec \theta$. | 44. $\sin \theta \sec \theta = \tan \theta$. |
| 37. $(1 + \tan^2 \theta) \sin^2 \theta = \tan^2 \theta$. | 45. $\tan \alpha + \cot \alpha = \sec \alpha \csc \alpha$. |
| 38. $(1 + \cot^2 \theta) \cos^2 \theta = \cot^2 \theta$. | 46. $\sin \alpha - \tan \alpha = \tan \alpha (\cos \alpha - 1)$. |
| 39. $\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \sec^2 \theta$. | 47. $\cot \alpha \cos \alpha = \csc \alpha - \sin \alpha$. |
| 40. $\frac{\sec^2 \theta - \tan^2 \theta}{\sin^2 \theta} = \csc^2 \theta$. | 48. $\sec^2 \alpha + \csc^2 \alpha = \sec^2 \alpha \csc^2 \alpha$. |

$$49. \frac{\sin^4 \alpha - \cos^4 \alpha}{\sin \alpha - \cos \alpha} = \sin \alpha + \cos \alpha.$$

$$51. \frac{\tan \alpha - 1}{\tan \alpha + 1} = \frac{1 - \cot \alpha}{1 + \cot \alpha}.$$

$$50. \frac{1 + \tan \alpha}{\sec \alpha} = \sin \alpha + \cos \alpha.$$

$$52. \frac{\sin \alpha - \cos \alpha}{\sin \alpha + \cos \alpha} = \frac{\tan \alpha - 1}{\tan \alpha + 1}.$$

By means of the fundamental identities of (1), (2), (3) of § 33, and a chain of identities, prove the following:

$$53. \sec \alpha - \cos \alpha = \sin \alpha \tan \alpha.$$

$$56. \cot^2 \alpha - \cos^2 \alpha = \cot^2 \alpha \cos^2 \alpha.$$

$$54. \tan \alpha + \cot \alpha = \sec \alpha \csc \alpha.$$

$$57. (\csc \alpha - \cot \alpha)^2 = \frac{1 - \cos \alpha}{1 + \cos \alpha}.$$

$$55. \sin^2 \alpha \sec^2 \alpha + 1 = \sec^2 \alpha.$$

$$58. \cot \alpha \cos \alpha = \sin \alpha + \csc \alpha (1 - 2 \sin^2 \alpha).$$

$$59. \left(\frac{\sec \alpha + \csc \alpha}{1 + \tan \alpha} \right)^2 = \frac{\tan \alpha + \cot \alpha}{\tan \alpha}.$$

$$60. \sec^4 \alpha - 2 \tan^2 \alpha = 1 + \tan^4 \alpha.$$

$$61. \frac{\cos^2 \alpha}{\tan^2 \alpha} + 1 = \tan^2 \alpha + \cot^2 \alpha - \sin^2 \alpha \tan^2 \alpha.$$

$$62. \sin \alpha (1 + \tan \alpha) - \sec \alpha = \csc \alpha - \cos \alpha (1 + \cot \alpha).$$

$$63. (\sin \alpha + \cos \alpha - 1)^2 = 2(\sin \alpha - 1)(\cos \alpha - 1).$$

$$64. \frac{(\tan \alpha - \cos \alpha \cot \alpha) \cot \alpha}{\csc \alpha} = \sin \alpha - \frac{\cos \alpha \cot \alpha}{\sec \alpha}.$$

35. Addition formulas. Important formulas involve functions of two angles, such as $\sin(\alpha + \beta)$ where α, β are the Greek letters "alpha," "beta." If one were careless he might assume that $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$, which is usually false; for example,

$$\begin{aligned} \sin(30^\circ + 60^\circ) &= \sin 90^\circ = 1, \\ \sin 30^\circ + \sin 60^\circ &= \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3} + 1}{2}, \end{aligned}$$

and hence $\sin(30^\circ + 60^\circ)$ does not equal $\sin 30^\circ + \sin 60^\circ$.

In sections which follow we shall prove the following formulas:

$$(1) \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta;$$

$$(2) \quad \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta;$$

$$(3) \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta;$$

$$(4) \quad \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta;$$

$$(5) \quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta};$$

$$(6) \quad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$

These are called the *addition formulas for the sine, cosine, and tangent functions*. They hold for all permissible values of the angles.

36. Addition formulas for sine and cosine functions. We shall now give two proofs of formulas (1) and (3) of § 35. Figures 49 and 50 illustrate the case in which α , β , and $\alpha + \beta$ are positive acute angles. The first proof to be given in this section would require modification if one or more of the angles were not acute, but the second proof applies without change of wording when the angles are not thus restricted.

First proof, for α , β , and $\alpha + \beta$ acute. Draw α in the standard position with respect to XY -axes, with origin at O . Then take X_1Y_1 -axes with the same origin such that OX_1 is the terminal side of α , and draw β in the standard position with respect to the X_1Y_1 -axes. From P , a point on the terminal side of β , drop perpendiculars PQ to OX_1 , and PM to OX ; then from Q drop a perpendicular QN to OX .

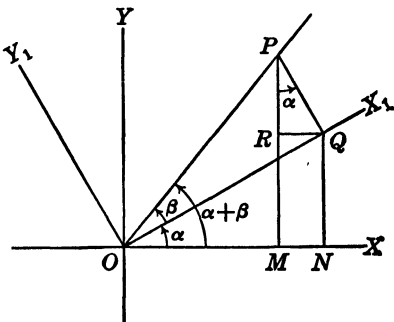


FIG. 49

The construction of Figure 49 is completed by dropping the perpendicular QR from the point Q to the line MP . We observe that angle RPQ will then be equal to α , since its sides RP and QP are respectively perpendicular to the sides OX , OX_1 of angle XOX_1 .

We are to prove that

$$(1) \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

If we use the triangle of reference OMP for $\alpha + \beta$, the triangle OQP for β , and the triangles ONQ and RPQ for α , formula (1) can be replaced by

$$\frac{MP}{OP} = \frac{NQ}{OQ} \cdot \frac{OQ}{OP} + \frac{RP}{QP} \cdot \frac{QP}{OP},$$

which is equivalent to

$$\frac{MP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP},$$

or

$$(2) \quad MP = NQ + RP.$$

Formula (2) is, however, obviously true, and we have thus proved formula (1).

Similarly, the formula

$$(3) \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

can be replaced by

$$\frac{OM}{OP} = \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{RQ}{QP} \cdot \frac{QP}{OP},$$

which is equivalent to

$$\frac{OM}{OP} = \frac{ON}{OP} - \frac{RQ}{OP},$$

or

$$(4) \quad OM = ON - RQ.$$

Since (4) is obviously true, formula (3) must hold.

Second proof, with no restrictions on the angles. In Figure 50 we have altered Figure 49 as follows: We have, for simplicity, omitted the letters M , N , R . We have drawn X_2Y_2 -axes, with Q as origin, parallel to the XY -axes. We have omitted the letter α for the angle RPQ and have added $90^\circ + \alpha$ for the angle X_2QP . We have indicated coördinates thus:

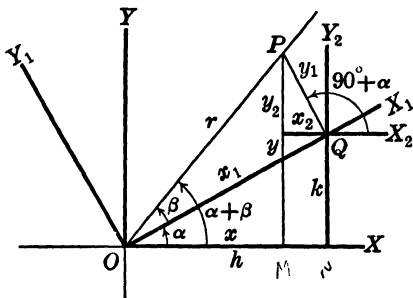


FIG. 50

The coördinates of P with respect to the XY -axes are (x, y) , to the X_1Y_1 -axes are (x_1, y_1) , to the X_2Y_2 -axes are (x_2, y_2) .

The coördinates of Q with respect to the XY -axes are (h, k) .

Finally, the length of OP is designated as r .

Following exactly the directions given above, we could construct a figure corresponding to Figure 50 for any angles α, β , whether positive or negative, without restriction as to their magnitude. In case y_1 were negative, however, angle X_2QP would be $270^\circ + \alpha$ (or a coterminal angle).

Formulas (1) and (3) of § 35 will now be proved by comparing coördinates for the three sets of axes and using the familiar formulas

$$x = r \cos \theta, \quad y = r \sin \theta$$

for the various sets of axes, with appropriate interpretation of x, y, r, θ .

Taking into account the signs of the quantities represented by the letters, we see that

$$(5) \quad y = k + y_2,$$

$$(6) \quad x = h + x_2 \quad (\text{in Figure 50, } x_2 \text{ is negative}).$$

In all cases we have *

$$y = r \sin (\alpha + \beta),$$

$$k = x_1 \sin \alpha = (r \cos \beta) \sin \alpha,$$

$$y_2 = y_1 \sin (90^\circ + \alpha) = (r \sin \beta) \cos \alpha.$$

When these expressions are substituted in (5), and the resulting equation is divided by r , we obtain

$$\sin (\alpha + \beta) = \cos \beta \sin \alpha + \sin \beta \cos \alpha,$$

which gives formula (1), § 35.

Similarly, we have

$$x = r \cos (\alpha + \beta),$$

$$h = x_1 \cos \alpha = (r \cos \beta) \cos \alpha,$$

$$x_2 = y_1 \cos (90^\circ + \alpha) = r \sin \beta (-\sin \alpha).$$

By substituting these expressions in (6) and dividing by r , we obtain

$$\cos (\alpha + \beta) = \cos \beta \cos \alpha - \sin \beta \sin \alpha,$$

which gives formula (3), § 35.

The wording of these proofs of formulas (1) and (3) as well as the directions for constructing Figure 50, apply to all angles α , β , whether positive or negative, without restriction as to their magnitude.

The student should draw the figure and verify the proof for cases where α , β , and $\alpha + \beta$ are not all positive acute angles.

The formulas should be verified when α and β , one or both, are quadrantal angles, by use of the rules of §§ 26 and 27.

37. Formulas for $\sin (\alpha - \beta)$ and $\cos (\alpha - \beta)$. In § 36 we have shown that the formulas (1) and (3) of § 35, for $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$, are true even when one of the angles is negative, or when both are negative. Hence we have

$$\sin (\alpha - \beta) = \sin (\alpha + (-\beta))$$

$$= \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta)$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta,$$

and we thus have proved formula (2) of § 35.

* The equation for y_2 holds even when y_1 is negative, for then

$$y_2 = -y_1 \sin (270^\circ + \alpha) = y_1 \sin (90^\circ + \alpha).$$

To prove formula (4) of § 35 we proceed similarly:

$$\begin{aligned}\cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) \\ &= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) \\ &= \cos \alpha \cos \beta + \sin \alpha \sin \beta.\end{aligned}$$

Example. Find the exact value of $\cos 15^\circ$ by use of an addition formula.

Solution. We have

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.\end{aligned}$$

EXERCISES

By use of the addition formulas, find the exact values of the following:

1. $\sin 15^\circ$. 2. $\sin 75^\circ$. 3. $\cos 75^\circ$. 4. $\sin(-15^\circ)$.

Apply the addition formulas to the following to obtain exact values, and then check by obtaining the exact values in another way:

- | | |
|----------------------------------|------------------------------------|
| 5. $\sin(60^\circ + 30^\circ)$. | 10. $\cos(60^\circ - 30^\circ)$. |
| 6. $\cos(60^\circ + 30^\circ)$. | 11. $\sin(90^\circ + 60^\circ)$. |
| 7. $\sin(45^\circ + 45^\circ)$. | 12. $\cos(90^\circ + 30^\circ)$. |
| 8. $\cos(45^\circ + 45^\circ)$. | 13. $\sin(270^\circ + 60^\circ)$. |
| 9. $\sin(60^\circ - 30^\circ)$. | 14. $\cos(270^\circ - 45^\circ)$. |

By use of addition formulas prove the following:

- | | |
|---|---|
| 15. $\sin(90^\circ + \theta) = \cos \theta$. | 20. $\cos(180^\circ + \theta) = -\cos \theta$. |
| 16. $\cos(90^\circ + \theta) = -\sin \theta$. | 21. $\sin(270^\circ - \theta) = -\cos \theta$. |
| 17. $\sin(180^\circ - \theta) = \sin \theta$. | 22. $\cos(270^\circ - \theta) = -\sin \theta$. |
| 18. $\cos(180^\circ - \theta) = -\cos \theta$. | 23. $\sin(270^\circ + \theta) = -\cos \theta$. |
| 19. $\sin(180^\circ + \theta) = -\sin \theta$. | 24. $\cos(270^\circ + \theta) = \sin \theta$. |

By use of Tables find approximate values of the following:

25. $\sin(47^\circ + 32^\circ) - (\sin 47^\circ \cos 32^\circ)$.
 26. $\cos(18^\circ + 28^\circ) - (\cos 18^\circ \cos 28^\circ)$.
 27. $\sin(25^\circ - 10^\circ) - (\sin 25^\circ \sin 10^\circ)$.
 28. $\cos(35^\circ - 10^\circ) - (\cos 35^\circ \cos 10^\circ)$.

By use of addition formulas find exact values of the following, assuming that α and β are angles terminating in the first quadrant:

29. $\sin(\alpha + \beta)$ if $\sin \alpha = \frac{3}{5}$, $\cos \beta = \frac{5}{13}$.
 30. $\cos(\alpha + \beta)$ if $\cos \alpha = \frac{3}{5}$, $\cos \beta = \frac{5}{13}$.
 31. $\sin(\alpha + \beta)$ if $\sin \alpha = \frac{1}{3}$, $\cos \beta = \frac{2}{3}$.
 32. $\cos(\alpha + \beta)$ if $\sin \alpha = \frac{1}{3}$, $\sin \beta = \frac{2}{3}$.
 33. $\sin(\alpha - \beta)$ if $\sin \alpha = \frac{1}{3}$, $\sin \beta = \frac{1}{3}$.
 34. $\cos(\alpha - \beta)$ if $\cos \alpha = \frac{2}{3}$, $\cos \beta = \frac{5}{13}$.
 35. $\sin(\alpha - \beta)$ if $\cos \alpha = \frac{1}{3}$, $\cos \beta = \frac{5}{13}$.
 36. $\cos(\alpha - \beta)$ if $\sin \alpha = \frac{1}{3}$, $\cos \beta = \frac{5}{13}$.
 37. $\sin(\alpha - \beta)$ if $\sin \alpha = \frac{1}{7}$, $\tan \beta = \frac{1}{3}$.
 38. $\cos(\alpha - \beta)$ if $\sin \alpha = \frac{1}{7}$, $\tan \beta = \frac{1}{3}$.

Use addition formulas to prove the following identities:

$$39. \sin (45^\circ + \theta) = \frac{\sin \theta + \cos \theta}{\sqrt{2}}.$$

$$40. \sin (45^\circ - \theta) = \frac{\cos \theta - \sin \theta}{\sqrt{2}}.$$

$$41. \sin (60^\circ + \theta) = \frac{\sqrt{3} \cos \theta + \sin \theta}{2}.$$

$$42. \sin (30^\circ - \theta) = \frac{\cos \theta - \sqrt{3} \sin \theta}{2}.$$

$$43. \cos (45^\circ + \theta) = \frac{\cos \theta - \sin \theta}{\sqrt{2}}.$$

$$44. \cos (60^\circ - \theta) = \frac{\cos \theta + \sqrt{3} \sin \theta}{2}.$$

$$45. \cos (30^\circ + \theta) = \frac{\sqrt{3} \cos \theta - \sin \theta}{2}.$$

$$46. \cos (60^\circ + \theta) = \frac{\cos \theta - \sqrt{3} \sin \theta}{2}.$$

$$47. \sin (A + B) \cos B - \cos (A + B) \sin B = \sin A.$$

Hint. Let $A + B = \alpha$, $B = \beta$.

$$48. \cos (A - B) \cos B - \sin (A - B) \sin B = \cos A.$$

$$49. \sin (x + y + z) = \sin x \cos y \cos z - \sin x \sin y \sin z + \cos x \sin y \cos z + \cos x \cos y \sin z.$$

$$50. \cos (x + y + z) = \cos x \cos y \cos z - \cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z.$$

$$51. \sin (x - y + z) = \sin x \cos y \cos z + \sin x \sin y \sin z - \cos x \sin y \cos z + \cos x \cos y \sin z.$$

$$52. \cos (x - y - z) = \cos x \cos y \cos z - \cos x \sin y \sin z + \sin x \sin y \cos z + \sin x \cos y \sin z.$$

53. Prove the formula for $\sin (\alpha + \beta)$ directly from a figure in which α and β are positive acute angles but $\alpha + \beta > 90^\circ$.

54. Prove the formula for $\sin (\alpha + \beta)$ directly from a figure in which α and β are positive angles, α terminating in the first quadrant, β in the second quadrant, and $\alpha + \beta$ in the third quadrant.

55. Prove the formula for $\cos (\alpha + \beta)$ directly from a figure in which α and β are positive angles terminating in the second quadrant and $\alpha + \beta$ an angle terminating in the third quadrant.

56. Prove the formula for $\cos (\alpha + \beta)$ directly from a figure in which α and β are positive angles, α terminating in the first quadrant, β in the fourth quadrant, and $\alpha + \beta$ in the first quadrant.

57. Prove the formula for $\sin (\alpha + \beta)$ directly from a figure in which α and β are positive angles, α terminating in the first quadrant, β in the third quadrant, and $\alpha + \beta$ in the fourth quadrant.

58. Prove the formula for $\sin (\alpha - \beta)$ directly from a figure in which α , β , $\alpha - \beta$ all terminate in the first quadrant.

59. Prove the formula for $\cos (\alpha - \beta)$ directly from the figure of Exercise 58.

38. Formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$. From formula (2) of § 33 and formulas (1) and (3) of § 35, we have

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}.$$

We can express the last fraction in terms of $\tan \alpha$ and $\tan \beta$ if we divide both numerator and denominator by $\cos \alpha \cos \beta$. We thus obtain

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\ &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}}. \end{aligned}$$

From this identity we at once derive formula (5) of § 35,

$$(1) \quad \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

If we treat in the same way the identity

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta},$$

we obtain formula (6) of § 35,

$$(2) \quad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$$

EXERCISES

Find exact values of the following by use of addition formulas:

1. $\tan 15^\circ$.

2. $\tan 75^\circ$.

3. $\tan 105^\circ$.

4. $\tan 195^\circ$.

By use of addition formulas prove the following identities:

5. $\tan(45^\circ + \theta) = \frac{1 + \tan \theta}{1 - \tan \theta}$.

7. $\tan(60^\circ - \theta) = \frac{\sqrt{3} - \tan \theta}{1 + \sqrt{3} \tan \theta}$.

6. $\tan(30^\circ + \theta) = \frac{1 + \sqrt{3} \tan \theta}{\sqrt{3} - \tan \theta}$.

8. $\tan(135^\circ - \theta) = \frac{-1 - \tan \theta}{1 - \tan \theta}$.

Find exact values of the following by using addition formulas, assuming that α and β are positive acute angles:

9. $\tan(\alpha + \beta)$ if $\sin \alpha = \frac{3}{5}$, $\sin \beta = \frac{4}{5}$.
10. $\tan(\alpha + \beta)$ if $\sin \alpha = \frac{4}{5}$, $\cos \beta = \frac{3}{5}$.
11. $\tan(\alpha - \beta)$ if $\tan \alpha = \frac{3}{4}$, $\sin \beta = \frac{1}{2}$.
12. $\tan(\alpha - \beta)$ if $\sec \alpha = \frac{5}{3}$, $\csc \beta = \frac{5}{4}$.

By use of addition formulas prove the identities:

13. $\frac{\tan(x+y) - \tan y}{1 + \tan(x+y)\tan y} = \tan x$. (Hint. Let $x+y = \alpha$, $y = \beta$.)
14. $\frac{\tan(x-y) + \tan y}{1 - \tan(x-y)\tan y} = \tan x$.
15. $\frac{\tan(x+y)}{\cot(x-y)} = \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$.
16. $\cot(\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}$.
17. $\cot(\alpha - \beta) = \frac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha}$.

39. Functions of 2α . By taking β equal to α in the formulas for the sine, cosine, and tangent of $\alpha + \beta$, we obtain formulas for the corresponding functions of 2α ; these are sometimes called double-angle formulas. For example, we have

$$\begin{aligned}\sin 2\alpha &= \sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha \\ &= 2 \sin \alpha \cos \alpha;\end{aligned}$$

and similarly

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha.$$

This last formula may be transformed as follows:

$$\cos 2\alpha = \cos^2 \alpha - (1 - \cos^2 \alpha) = 2 \cos^2 \alpha - 1;$$

also

$$\cos 2\alpha = (1 - \sin^2 \alpha) - \sin^2 \alpha = 1 - 2 \sin^2 \alpha.$$

We list below the formulas for $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, including the three we have just obtained for $\cos 2\alpha$:

- (1) $\sin 2\alpha = 2 \sin \alpha \cos \alpha;$
- (2a) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$
- (2b) $= 2 \cos^2 \alpha - 1$
- (2c) $= 1 - 2 \sin^2 \alpha;$
- (3) $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$

Example. Find the sine, cosine, and tangent of 120° by means of the double-angle formulas.

Solution. We have $\sin 120^\circ = \sin(2 \times 60^\circ) = 2 \sin 60^\circ \cos 60^\circ$

$$= 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}.$$

$$\cos 120^\circ = \cos^2 60^\circ - \sin^2 60^\circ = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}.$$

$$\tan 120^\circ = \frac{2 \tan 60^\circ}{1 - \tan^2 60^\circ} = \frac{2\sqrt{3}}{1 - 3} = -\sqrt{3}.$$

40. Functions of $\frac{\alpha}{2}$. Just as the algebraic identity

$$x(1 - x) = x - x^2$$

remains an identity if we substitute $\frac{x}{2}$ for x , obtaining

$$\frac{x}{2}\left(1 - \frac{x}{2}\right) = \frac{x}{2} - \left(\frac{x}{2}\right)^2,$$

so likewise in a trigonometric identity in an angle α , we may substitute $\frac{\alpha}{2}$ for α if we do it in every place where α occurs.* If we do this in formulas (1), (2b), and (2c) of § 39, we have

$$(1) \quad \sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2};$$

$$(2) \quad \cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1;$$

$$(3) \quad \cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}.$$

We obtain two **half-angle formulas** by solving (3) for $\sin \frac{\alpha}{2}$, and (2) for $\cos \frac{\alpha}{2}$. Thus, from (3) we have

$$2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha;$$

and hence

$$(4) \quad \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

Similarly

$$(5) \quad \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}.$$

We obtain a formula for $\tan \frac{\alpha}{2}$ by dividing (4) by (5):

$$(6a) \quad \tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}.$$

If α is between 0° and 180° (inclusive, but 180° is not permissible in (6a)), the $+$ sign is to be used in the above formulas; for other angles we decide whether to use the $+$ or $-$ sign by the rules for the signs of functions of $\frac{\alpha}{2}$ according to the quadrant in which $\frac{\alpha}{2}$ lies.

* We could, in fact, substitute for α any expression involving α provided we make the same substitution wherever α occurs.

Two additional formulas for $\tan \frac{\alpha}{2}$ which avoid the $\pm$ sign of (6a) are these:

$$(6b) \quad \tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha},$$

$$(6c) \quad \tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}.$$

We prove (6b) by reducing the right side to the left side with the aid of formulas (3) and (1) of the present section:

$$\begin{aligned} \frac{1 - \cos \alpha}{\sin \alpha} &= \frac{1 - \left(1 - 2 \sin^2 \frac{\alpha}{2}\right)}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \frac{2 \sin^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \\ &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \tan \frac{\alpha}{2}. \end{aligned}$$

The identity formed by equating the right side of (6b) to the right side of (6c) is easily proved.

Example 1. Find the sine, cosine, and tangent of $\alpha/2$ if α is an angle between 180° and 270° for which $\tan \alpha = 3$.

Solution. Here α terminates in the third quadrant and $\alpha/2$ in the second. Hence $\sin \alpha$ and $\cos \alpha$ are negative; $\sin (\alpha/2)$ is positive, $\cos (\alpha/2)$ and $\tan (\alpha/2)$ are negative. We have

$$\begin{aligned} \cos \alpha &= \frac{1}{\sec \alpha} = \frac{-1}{\sqrt{1 + \tan^2 \alpha}} = -\frac{1}{\sqrt{10}}, \\ \sin \alpha &= \cos \alpha \tan \alpha = -\frac{3}{\sqrt{10}}, \\ \sin \frac{\alpha}{2} &= +\sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{10}}\right)} \\ &= \sqrt{\frac{1}{2}\left(\frac{10 + \sqrt{10}}{10}\right)} = \sqrt{\frac{10 + \sqrt{10}}{20}}, \\ \cos \frac{\alpha}{2} &= -\sqrt{\frac{1 + \cos \alpha}{2}} = -\sqrt{\frac{10 - \sqrt{10}}{20}}, \\ \tan \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{\sin \alpha} = \frac{1 + \frac{1}{\sqrt{10}}}{-\frac{3}{\sqrt{10}}} = -\frac{\sqrt{10} + 1}{3}. \end{aligned}$$

Example 2. Prove the identity

$$\cos 2A - 4 \cos A + 3 = 8 \sin^4 \frac{A}{2}.$$

Solution. We reduce both sides to the same expression by means of formulas (2b) of § 39 and (4) of § 40.

$$\begin{array}{l} 2 \cos^2 A - 1 - 4 \cos A + 3 \\ 2 \cos^2 A - 4 \cos A + 2 \\ 2(1 - \cos A)^2 \end{array} \quad \left| \begin{array}{l} 8 \left(\pm \sqrt{\frac{1 - \cos A}{2}} \right)^4 \\ 8 \left(\frac{1 - \cos A}{2} \right)^2 \\ 2(1 - \cos A)^2 \end{array} \right.$$

EXERCISES

By use of the formulas of § 39 find exact values of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ in the following cases:

1. $\sin \alpha = \frac{3}{5}$, if $0^\circ < \alpha < 90^\circ$.
2. $\cos \alpha = -\frac{4}{5}$, if $90^\circ < \alpha < 180^\circ$.
3. $\tan \alpha = \frac{1}{2}$, if $180^\circ < \alpha < 270^\circ$.
4. $\sec \alpha = \frac{1}{2}$, if $270^\circ < \alpha < 360^\circ$.
5. $\cot \alpha = \frac{4}{3}$, if $360^\circ < \alpha < 450^\circ$.
6. $\csc \alpha = -\frac{1}{2}$, if $540^\circ < \alpha < 630^\circ$.
7. $\sin \alpha = -\frac{1}{2}$, if $-90^\circ < \alpha < 0^\circ$.
8. $\cos \alpha = -\frac{1}{2}$, if $-180^\circ < \alpha < -90^\circ$.

By use of the formulas of § 40 find exact values of $\sin \frac{\alpha}{2}$, $\cos \frac{\alpha}{2}$, and $\tan \frac{\alpha}{2}$ in the following cases:

9. $\sin \alpha = \frac{3}{5}$, if $0^\circ < \alpha < 90^\circ$.
10. $\cos \alpha = -\frac{4}{5}$, if $90^\circ < \alpha < 180^\circ$.
11. $\tan \alpha = \frac{1}{2}$, if $180^\circ < \alpha < 270^\circ$.
12. $\sec \alpha = \frac{1}{2}$, if $270^\circ < \alpha < 360^\circ$.
13. $\cot \alpha = \frac{4}{3}$, if $360^\circ < \alpha < 450^\circ$.
14. $\csc \alpha = -\frac{1}{2}$, if $540^\circ < \alpha < 630^\circ$.
15. $\sin \alpha = -\frac{1}{2}$, if $-90^\circ < \alpha < 0^\circ$.
16. $\cos \alpha = -\frac{1}{2}$, if $-180^\circ < \alpha < -90^\circ$.

By use of formulas of § 40, find exact values of the following:

17. $\sin 15^\circ$, $\cos 15^\circ$, $\tan 15^\circ$.
18. $\sin 22\frac{1}{2}^\circ$, $\cos 22\frac{1}{2}^\circ$, $\tan 22\frac{1}{2}^\circ$.
19. $\sin 75^\circ$, $\cos 75^\circ$, $\tan 75^\circ$.
20. $\sin 105^\circ$, $\cos 105^\circ$, $\tan 105^\circ$.

Prove the following identities:

21. $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$.

22. $1 + \cos 2\theta = 2 \cos^2 \theta$.

23. $\sin 2\alpha = \frac{2 \tan \alpha}{\sec^2 \alpha}$.

24. $\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}$.

25. $\tan A = \frac{1 - \cos 2A}{\sin 2A} = \frac{\sin 2A}{1 + \cos 2A}$.

26. $\tan \theta \sin 2\theta = 2 \sin^2 \theta$.
 27. $\cos^3 x - \sin^3 x = (\cos x - \sin x)(1 + \frac{1}{2} \sin 2x)$.
 28. $\cos^4 x - \sin^4 x = \cos 2x$.
 29. $\sec 2\alpha = 1 + \tan 2\alpha \tan \alpha$.
 30. $\sin \alpha + \cot \alpha \sin 2\alpha = 1 + \cos 2\alpha + \sin \alpha$.
 31. $\sec 2\alpha = \frac{1}{2 \cos^2 \alpha - 1} = \frac{\sec^2 \alpha}{2 - \sec^2 \alpha}$.
 32. $\frac{\cos 2\alpha}{1 + \sin 2\alpha} = \frac{\cos^2 \alpha - \sin^2 \alpha}{(\cos \alpha + \sin \alpha)^2} = \frac{1 - \tan \alpha}{1 + \tan \alpha}$.
 33. $2 \cot 2A = \cot A - \tan A$.
 34. $2 \cot \alpha = \cot \frac{\alpha}{2} - \tan \frac{\alpha}{2}$.
 35. $\sin 4A = 2 \sin 2A \cos 2A = 4 \sin A \cos^3 A - 4 \sin^3 A \cos A$.
 (Hint. Let $\alpha = 2A$.)
 36. $\cos 4A = 1 - 2 \sin^2 2A = 1 - 8 \sin^2 A \cos^2 A$.
 37. $\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$.
 38. $\cos 4A = \cos^4 A - 6 \sin^2 A \cos^2 A + \sin^4 A$.
 39. $\tan 4A = \frac{\sin 4A}{\cos 4A} = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$.
 40. $\sin 4A = 4 \sin A \cos A - 8 \sin^3 A \cos A$.
 41. $(1 + \cos 2\alpha)(1 + \tan \frac{\alpha}{2} \tan \alpha)^2 = 2$.
 42. $\sin 3\alpha = \sin (2\alpha + \alpha) = 3 \sin \alpha - 4 \sin^3 \alpha$.
 43. $\sin 3\alpha = 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha$.
 44. $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$.
 45. $\cos 3\alpha = \cos \alpha - 4 \cos \alpha \sin^2 \alpha$.
 46. $\tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$.
 47. $\sin 2x + \cos 2x + 1 = 2 \cos x (\sin x + \cos x)$.
 48. $\cos \theta - \sin \frac{\theta}{2} = \left(1 - 2 \sin \frac{\theta}{2}\right) \left(1 + \sin \frac{\theta}{2}\right)$.
 49. $\left(1 - \cos \frac{\theta}{2}\right) = \frac{\cos \frac{\theta}{2} - \cos \theta}{1 + 2 \cos \frac{\theta}{2}}$.
 50. $\sec \frac{x}{2} + \csc \frac{x}{2} = \frac{2 \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)}{\sin x}$.
 51. $\frac{1 - \tan \frac{\alpha}{2}}{1 + \tan \frac{\alpha}{2}} = \frac{\cos \alpha}{1 + \sin \alpha}$.
 52. $\sin \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$.

41. Formulas for sums and differences of two sines or two cosines.

From the formulas of § 36 we obtain, by addition and subtraction,

- (1) $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta,$
- (2) $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta,$
- (3) $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta,$
- (4) $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin \alpha \sin \beta.$

If we read these formulas from right to left they express products of a sine or cosine of one angle by the sine or cosine of another as equal to one half of sums or differences of sines or cosines.

For purposes of computation it is often more convenient to deal with products of functions than with their sums. The four formulas express sums as products, but a change of notation is advantageous. Let us make the substitutions

$$\alpha + \beta = A, \quad \alpha - \beta = B.$$

By adding and subtracting these equations we obtain

$$\begin{aligned} 2\alpha &= A + B, & 2\beta &= A - B, \\ \alpha &= \frac{A + B}{2}, & \beta &= \frac{A - B}{2}. \end{aligned}$$

When α and β are replaced by these values, formulas (1) to (4) become

- (5) $\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2},$
- (6) $\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2},$
- (7) $\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2},$
- (8) $\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}.$

A good way to memorize these identities is to put them in words. Thus formula (5) is equivalent to the statement: *The sum of the sines of two angles is equal to twice the sine of half the sum of the angles, multiplied by the cosine of half the difference.*

Example 1. Prove that $\sin 40^\circ + \sin 20^\circ = \sin 80^\circ$.

Solution. By formula (5)

$$\begin{aligned} \sin 40^\circ + \sin 20^\circ &= 2 \sin \frac{40^\circ + 20^\circ}{2} \cos \frac{40^\circ - 20^\circ}{2} \\ &= 2 \sin 30^\circ \cos 10^\circ = 2 \cdot \frac{1}{2} \cdot \cos 10^\circ \\ &= \cos 10^\circ = \sin 80^\circ. \end{aligned}$$

Example 2. Prove that $\frac{\sin A - \sin B}{\cos A - \cos B} = -\cot \frac{A+B}{2}$.

Solution. By formulas (6) and (7),

$$\begin{aligned}\frac{\sin A - \sin B}{\cos A - \cos B} &= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{-2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}} \\ &= \frac{-\cos \frac{A+B}{2}}{\sin \frac{A+B}{2}} = -\cot \frac{A+B}{2}.\end{aligned}$$

EXERCISES

By use of formulas (1), (2), (3), (4) of § 41, express the following products as sums or differences of sines or of cosines:

1. $2 \cos \theta \sin 3\theta$.
2. $2 \sin \theta \cos 8\theta$.
3. $2 \sin 2\alpha \cos 5\alpha$.
4. $2 \cos 3\alpha \cos 4\alpha$.
5. $2 \sin 3\alpha \sin 2\alpha$.
6. $2 \sin \alpha \sin 5\alpha$.
7. $\cos \alpha \cos 2\alpha$.
8. $\sin 5\alpha \sin 2\alpha$.
9. $\sin 5\alpha \cos 6\alpha$.
10. $\cos 3\alpha \sin 2\alpha$.

By use of formulas (5), (6), (7), (8) of § 41, prove the following identities:

11. $\sin x + \sin 3x = 2 \sin 2x \cos x$.
12. $\sin 5x - \sin 3x = 2 \cos 4x \sin x$.
13. $\cos 3x = \cos x - 2 \sin 2x \sin x$.
14. $\cos 8x + \cos 4x = 2 \cos 6x \cos 2x$.
15. $\sin 3\alpha = \sin \alpha + 2 \cos 2\alpha \sin \alpha$.
16. $\cos 7\alpha = 2 \cos 6\alpha \cos \alpha - \cos 5\alpha$.
17. $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}$.
18. $\frac{\sin A - \sin B}{\sin A + \sin B} = \tan \frac{A-B}{2} \cot \frac{A+B}{2}$.
19. $\frac{\cos 3\alpha - \cos \alpha}{\sin \alpha - \sin 3\alpha} = \tan 2\alpha$.
20. $\frac{\sin \alpha - \sin 2\alpha}{\cos 2\alpha - \cos \alpha} = \cot \frac{3}{2}\alpha$.
21. $\sin \alpha (\sin \alpha + \sin 3\alpha) = \cos \alpha (\cos \alpha - \cos 3\alpha)$.
22. $\sin x - \cos 2x - \sin 3x = -\cos 2x (2 \sin x + 1)$.
23. $\cos (45^\circ - \alpha) - \cos (45^\circ + \alpha) = \sqrt{2} \sin \alpha$.
24. $\sin (45^\circ + \alpha) + \sin (45^\circ - \alpha) = \sqrt{2} \cos \alpha$.
25. $\sin (60^\circ + \beta) - \sin (60^\circ - \beta) = \sin \beta$.
26. $\cos (60^\circ - \beta) + \cos (60^\circ + \beta) = \cos \beta$.

Express each sum or difference as a product:

27. $\sin 36^\circ + \sin 44^\circ$.
28. $\sin 54^\circ + \sin 26^\circ$.
29. $\sin 40^\circ - \sin 50^\circ$.
30. $\sin 70^\circ - \sin 10^\circ$.
31. $\cos 40^\circ + \cos 50^\circ$.
32. $\cos 30^\circ + \cos 70^\circ$.
33. $\cos 20^\circ - \cos 30^\circ$.
34. $\cos 50^\circ - \cos 20^\circ$.
35. $\sin 2\alpha + \sin 4\alpha$.
36. $\sin 4\alpha + \sin 6\alpha$.
37. $\sin 3\alpha - \sin \alpha$.
38. $\sin 2\alpha - \sin 5\alpha$.
39. $\cos \alpha + \cos 4\alpha$.
40. $\cos 3\alpha + \cos 2\alpha$.
41. $\cos 3\alpha - \cos \alpha$.
42. $\cos 2\alpha - \cos 3\alpha$.

Express as a product involving only tangents and cotangents:

$$43. \frac{\sin 20^\circ + \sin 30^\circ}{\cos 20^\circ + \cos 30^\circ}.$$

$$44. \frac{\sin 50^\circ + \sin 70^\circ}{\cos 50^\circ + \cos 70^\circ}.$$

$$45. \frac{\sin 40^\circ - \sin 20^\circ}{\sin 40^\circ + \sin 20^\circ}.$$

$$46. \frac{\sin 70^\circ - \sin 50^\circ}{\sin 70^\circ + \sin 50^\circ}.$$

$$47. \frac{\cos \alpha + \cos 3\alpha}{\sin \alpha + \sin 3\alpha}.$$

$$48. \frac{\cos 5\alpha + \cos \alpha}{\sin 5\alpha + \sin \alpha}.$$

$$49. \frac{\cos 2\alpha - \cos \alpha}{\cos 2\alpha + \cos \alpha}.$$

$$50. \frac{\cos 3\alpha - \cos \alpha}{\sin 3\alpha - \sin \alpha}.$$

MISCELLANEOUS EXERCISES

Prove the following identities:

$$1. (\sin \alpha + \cos \alpha)(\tan \alpha + \cot \alpha) = \sec \alpha + \csc \alpha.$$

$$2. 1 + \sec^4 \alpha = 2 \sec^2 \alpha + \tan^4 \alpha.$$

$$3. \frac{1}{1 + \cos^2 2A} = \frac{\sec^2 2A}{2 + \tan^2 2A}.$$

$$4. \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\cot \beta - \cot \alpha}{\cot \alpha \cot \beta + 1}.$$

$$5. \sin 2x \csc x = 2 \cos x.$$

$$6. \csc 2x \sin x = \frac{1}{2} \sec x.$$

$$7. \sin 2x \cot x = \cos 2x + 1.$$

$$8. \cot 2x \sin 4x = 1 + \cos 4x.$$

$$9. \sin 3x \cos x - \cos 3x \sin x = \sin 2x.$$

$$10. \cos x \cos 3x - \sin x \sin 3x = \cos 4x.$$

$$11. \frac{\sin 2x}{\sec x} - \frac{\cos 2x}{\csc x} = \sin x.$$

$$12. \frac{\cos 2x}{\sec x} - \frac{\sin 2x}{\csc x} = \cos 3x.$$

$$13. \sin 7x \cos 5x + \cos 7x \sin 5x = \sin 12x.$$

$$14. \cos 7x \cos 5x + \sin 7x \sin 5x = \cos 2x.$$

$$15. 2 \cot 4\alpha - \cot 2\alpha = -\tan 2\alpha.$$

$$16. \sin 2\alpha - \sin 4\alpha + \sin 6\alpha = 4 \sin \alpha \cos 2\alpha \cos 3\alpha.$$

$$17. 2 \sin 2A = \sin 4A + 8 \sin^3 A \cos A.$$

$$18. \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan \alpha \tan \beta}{1 + \tan \alpha \tan \beta}.$$

$$19. \cot \frac{\alpha}{2} - \sin \alpha = 2 \cot \alpha \cos^2 \frac{\alpha}{2}.$$

$$20. \tan \alpha = \csc 2\alpha - \cot 4\alpha - \csc 4\alpha.$$

$$21. \cos 4\alpha - 4 \sin 2\alpha = 3 - 2(\sin \alpha + \cos \alpha)^4.$$

$$22. \sin 3\alpha - \sin 2\alpha + \sin \alpha = 4 \cos \frac{3}{2}\alpha \cos \alpha \sin \frac{1}{2}\alpha.$$

$$23. \frac{1 - \cos \alpha + \sin \alpha}{1 + \cos \alpha + \sin \alpha} = \tan \frac{\alpha}{2}.$$

$$24. \cot \frac{\alpha}{2} - \tan \frac{\alpha}{2} = \cos \alpha \left(\cot \frac{\alpha}{2} + \tan \frac{\alpha}{2} \right).$$

When A, B, C are angles of a triangle we have $A + B + C = 180^\circ$. Prove that in this case the following four identities hold:

$$25. \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

$$26. \sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

$$27. \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$28. \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Other proofs than those given in § 36 for the addition formulas for sine and cosine are proposed in the following exercises. It is assumed that A and B are acute angles.

29. Let A and B be angles of a triangle ABC as shown in the adjoining figure, and R the radius of the circumscribed circle. Show that:

$$(1) c = a \cos B + b \cos A;$$

$$(2) a = 2R \sin A, b = 2R \sin B, c = 2R \sin C;$$

$$(3) \sin C = \sin (A + B).$$

From (1), (2) and (3), show that

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

30. For Figure 52, show that:

$$\text{Area } OPQ = \frac{1}{2}xy \sin A;$$

$$\text{Area } OQR = \frac{1}{2}zy \sin B;$$

$$\text{Area } OPR = \frac{1}{2}xz \sin (A + B);$$

and hence that

$$\begin{aligned} \sin (A + B) &= \frac{y}{z} \sin A + \frac{y}{x} \sin B \\ &= \sin A \cos B + \cos A \sin B. \end{aligned}$$

31. For Figure 52, show that:

$$\text{Area } ORS = \frac{1}{2}wz \cos (A + B);$$

$$\text{Area } OQS = \frac{1}{2}wy \cos A;$$

$$\text{Area } OQR = \frac{1}{2}yz \sin B.$$

Hence deduce that

$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

32. In Figure 53, α is an acute angle which is inscribed in a semicircle of radius a . Show that:

$$AP = 2a \cos \alpha;$$

$$MP = a \sin 2\alpha;$$

$$BP = 2a \sin \alpha;$$

and then derive the formulas

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha;$$

$$2 \cos^2 \alpha = 1 + \cos 2\alpha;$$

$$2 \sin^2 \alpha = 1 - \cos 2\alpha;$$

$$\tan \alpha = \frac{1 - \cos 2\alpha}{\sin 2\alpha} = \frac{\sin 2\alpha}{1 + \cos 2\alpha}.$$

In these formulas substitute $\alpha = \frac{x}{2}$, and derive the half-angle formulas.

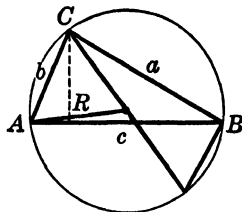


FIG. 51

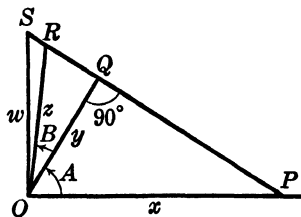


FIG. 52

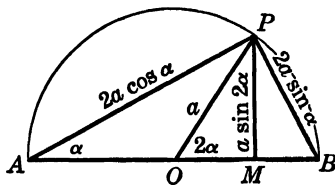


FIG. 53

CHAPTER V

RADIAN MEASURE. INVERSE FUNCTIONS. TRIGONOMETRIC EQUATIONS

42. Radian measure. So far we have been using the angle of 1° as our unit of measurement for angles. Another unit, especially useful in higher mathematics, is the *radian*.

An angle of **1 radian** is defined as *the angle at the center of a circle of radius r that subtends an arc of length r .*

From a theorem of geometry, changing r would not change the angle thus defined.

Another theorem of geometry states that measures of angles at the center of a circle are proportional to the arcs which they subtend. If we apply this to the angle θ and the angle of 1 radian shown in Figure 54, we have

$$\frac{\text{radian measure of } \theta}{1 \text{ radian}} = \frac{s}{r}$$

Hence

$$(1) \quad \text{radian measure of } \theta = \frac{s}{r} = \frac{\text{subtended arc}}{\text{radius}}.$$

We can compare measurements in radians and degrees by substituting for θ , in (1), an angle of 180° . Since such an angle subtends a semicircumference, we have $s = \pi r$, and (1) reduces to

$$\text{radian measurement of an angle of } 180^\circ = \frac{\pi r}{r} = \pi.$$

We write this, in abbreviated form,

$$(2) \quad 180^\circ = \pi \text{ radians.}$$

Recalling that $\pi = 3.141593$ approximately, we find from (2)

$$(3) \quad 1^\circ = \frac{\pi}{180} \text{ radians} = 0.0174533 \text{ radians};$$

$$(4) \quad 1 \text{ radian} = \frac{180^\circ}{\pi} = 57.29578^\circ = 57^\circ 17' 45''$$

to the indicated number of significant figures.

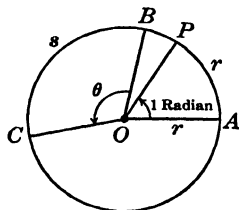


FIG. 54

Our Tables enable us to change the measurement of an angle in degrees and minutes to its measurement in radians and vice versa. The first and second columns of Table II serve this purpose for acute angles. Other angles are expressed as multiples of an angle of 90° plus or minus an acute angle, and the radian measure of these two terms is combined, if we are changing from degrees to radians. Similarly, for the reverse process.

Example 1. Express $110^\circ 23'$ in radians.

Solution. We shall first obtain the result as a fraction times π radians, a form that is often convenient; then as a decimal number of radians.

$$\begin{aligned} \text{(a) } 110^\circ 23' &= (110 + \frac{23}{60})^\circ = (\frac{6623}{60})^\circ \\ &= \frac{6623}{60} \times \frac{\pi}{180} = \frac{6623}{10800} \pi \text{ radians.} \\ \text{(b) } 110^\circ 23' &= 110 \times 0.0174533 + \frac{23}{60} \times 0.0174533 \\ &= 1.9199 + 0.0067 \\ &= 1.9266 \text{ radians (to four decimal places).} \end{aligned}$$

We could have obtained result (b) from Table II as follows:

$$110^\circ 23' = 90^\circ + 20^\circ 23'.$$

From the Table, $90^\circ = 1.5708$ radians. The radian value of $20^\circ 23'$ is found by interpolation to be $0.3549 + \frac{23}{60}(.0029) = 0.3558$. Hence

$$110^\circ 23' = (1.5708 + 0.3558) \text{ radians} = 1.9266 \text{ radians.}$$

Example 2. Express 5 radians in degrees and minutes.

$$\begin{aligned} \text{Solution. } 5 \text{ radians} &= 5 \times (57^\circ 17' 45'') \\ &= 286^\circ 29' \text{ to the nearest minute.} \end{aligned}$$

Example 3. Express $\frac{\pi}{6}$ radians in degrees.

$$\text{Solution. } \frac{\pi}{6} \text{ radians} = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ.$$

Example 4. If the radius of a circle is 10 ft., find in terms of π the length of arc subtended by an angle of 15° with vertex at the center of the circle.

Solution. According to formula (1), $s = r \times \text{number of radians in } \theta$; hence, for this Example,

$$s = 10 \times \frac{\pi}{180} \times 15 = \frac{5}{6} \pi.$$

Example 5. Find both the radian and the degree measure of an angle θ subtended at the center of a circle of radius 6 inches by an arc of length 10 inches.

Solution. According to formula (1),

$$\text{radian measure of } \theta = \frac{10}{6} = 1.6667.$$

To change this to degrees and minutes, note that $90^\circ = 1.5708$ radians, and write

$$1.6667 = 1.5708 + 0.0959;$$

then from Table II

$$\theta = 90^\circ + 5^\circ 30' = 95^\circ 30'.$$

EXERCISES

Give the radian measures in terms of π for the following angles:

1. 45° ; 60° ; 135° ; 300° .
2. $22\frac{1}{2}^\circ$; 75° ; 195° ; 540° .
3. 30° ; 225° ; -45° ; -270° .
4. 15° ; $67\frac{1}{2}^\circ$; 330° ; 465° .

Express in radians in decimal form, using Tables:

5. 35° ; 50° ; 60° ; 100° .
6. 30° ; 40° ; 80° ; 110° .
7. 20° ; 45° ; 70° ; 130° .
8. 10° ; 75° ; 90° ; 135° .
9. $15^\circ 27'$; $132^\circ 15'$.
10. $61^\circ 13'$; $111^\circ 11'$.
11. $43^\circ 12'$; $141^\circ 40'$.
12. $67^\circ 17'$; $188^\circ 17'$.

Without using Tables express the following angles, whose radian measure is given, in terms of degrees and minutes:

13. $\frac{\pi}{8}$; $\frac{2\pi}{3}$; $\frac{7\pi}{2}$; 2.
14. $\frac{\pi}{6}$; $\frac{3\pi}{4}$; $-\frac{4\pi}{5}$; 3.
15. $\frac{\pi}{5}$; $\frac{3\pi}{7}$; $\frac{7\pi}{3}$; 4.
16. $\frac{\pi}{9}$; $\frac{2\pi}{5}$; $-\frac{5\pi}{9}$; 5.

Using Tables express the following angles, whose radian measures are given, in terms of degrees and minutes:

17. 0.9427; 1.0378.
18. 0.8127; 1.0836.
19. 0.5432; 1.5432.
20. 0.7171; 1.7171.
21. 0.8623; 2.8883.
22. 0.7979; 3.6542.
23. 1.0114; 3.1104.
24. 0.9876; 2.6789.

In each of the following problems the radius of a circle and an arc on that circle are given; find the angle at the center subtended by the arc. Express the result first in radians, then in degrees and minutes, assuming the measurements of radius and arc exact.

25. Radius = 10 ft., arc = 15 ft.
26. Radius = 12 ft., arc = 18 ft.
27. Radius = 20 in., arc = 24 in.
28. Radius = 18 in., arc = 15 in.
29. Radius = 16 in., arc = 2 ft.
30. Radius = 8 in., arc = 4 ft.
31. Radius = 5 mi., arc = 11 mi.
32. Radius = π ft., arc = 2π ft.

In each of the following problems an angle at the center of a circle and the arc it intercepts are given; find the radius of the circle. (Assume that given measurements are exact.)

33. Angle = 2 radians, arc = 9 in.
34. Angle = 1.5 radians, arc = 12 in.
35. Angle = 3.333 radians, arc = 20 in.
36. Angle = 1.111 radians, arc = 27.9 in.
37. Angle = 1.4286 radians, arc = 14 in.
38. Angle = 1.25 radians, arc = 8 in.
39. Angle = 20° , arc = 6 in.
40. Angle = 30° , arc = 7 in.
41. Angle = $46^\circ 11'$, arc = 201 ft.
42. Angle = $58^\circ 11'$, arc = 103 in.
43. Angle = $152^\circ 13'$, arc = 25.5 in.
44. Angle = $307^\circ 10'$, arc = 68.8 ft.
45. Angle = 300° , arc = 70.7 ft.
46. Angle = 9° , arc = 15.708 ft.

43. Inverse functions and their principal values. The statement that $\sin \theta$ is equal to a is equivalent to saying that θ is an angle whose sine is a , or, more briefly, that θ is the *inverse sine of a* . Two notations are in use to express this relation,

$$\theta = \arcsin a, \text{ and } \theta = \sin^{-1} a.$$

When the latter notation is used, we must be careful not to interpret $\sin^{-1} a$ as the -1 power of $\sin a$.

The three equations

$$\sin \theta = a, \quad \theta = \arcsin a, \quad \theta = \sin^{-1} a$$

mean exactly the same thing.

We define similarly the other inverse functions $\arccos a$, $\arctan a$, and so on; the alternative notation is $\cos^{-1} a$, $\tan^{-1} a$, and so on.

The inverse functions are *many-valued*. For example, we have

$$\arcsin \frac{1}{2} = 30^\circ, 150^\circ, 390^\circ, \dots$$

In order that $\theta = \arcsin a$ have a value, it is necessary that a be not greater than 1 or less than -1 , since $a = \sin \theta$. If a has a value so chosen, $\arcsin a$ has infinitely many values, of which *one and only one* lies between -90° and $+90^\circ$ (inclusive). This is called the **principal value** of $\arcsin a$, and is written $\text{Arcsin } a$ (with the initial letter A capitalized); it may also be written $\text{Sin}^{-1} a$ (with capital S). For $\text{Arcsin } a$ we will say that the **range** of this principal value and the **interval of definition** are as follows:

$$-90^\circ \leq \text{Arcsin } a \leq 90^\circ, \quad -1 \leq a \leq 1.$$

For principal values of all six inverse functions, the **ranges and intervals of definition** are given in the following table:

$-90^\circ \leq \text{Arcsin } a \leq 90^\circ,$	$-1 \leq a \leq 1;$
$0^\circ \leq \text{Arccos } a \leq 180^\circ,$	$1 \geq a \geq -1;$
$-90^\circ < \text{Arctan } a < 90^\circ,$	$a \text{ may have any value;}$
$0^\circ < \text{Arccot } a < 180^\circ,$	$a \text{ may have any value;}$
$0^\circ \leq \text{Arcsec } a \leq 180^\circ,$	$a \geq 1 \text{ or } a \leq -1;$
$-90^\circ \leq \text{Arccsc } a \leq 90^\circ,$	$a \leq -1 \text{ or } a \geq 1.$

In each case, when a is in the interval of definition, the inverse function has one and only one value, the principal value, in the range indicated.

The problem of finding further values of $\theta = \arcsin a$ is that of finding solutions θ of the equation $\sin \theta = a$ other than the principal value, θ_1 . We easily solve this by observing that there is always a

secondary solution $180^\circ - \theta_1$, since $\sin(180^\circ - \theta_1) = \sin \theta_1 = a$ (the principal and the secondary solution here coincide if $\theta_1 = 90^\circ$). The two solutions, θ_1 and $180^\circ - \theta_1$, both lie in the range $-90^\circ \leq \theta < 270^\circ$, and are the only ones in that range. *All other values of $\theta = \arcsin a$ are obtained by adding positive or negative multiples of 360° to the principal and secondary values of θ .*

For the six inverse functions, if θ_1 is a principal value, secondary values are as follows:

$$\begin{aligned} 180^\circ - \theta_1 &\text{ for } \arcsin \theta \text{ and } \operatorname{arccsc} \theta; \\ -\theta_1 &\text{ for } \arccos \theta \text{ and } \operatorname{arcsec} \theta; \\ 180^\circ + \theta_1 &\text{ for } \arctan \theta \text{ and } \operatorname{arccot} \theta. \end{aligned}$$

In each case the principal and secondary values are the only ones on a range of 360° , and all other values are obtained from the principal and secondary values by adding positive or negative multiples of 360° .

Example 1. Find all values of $\arccos\left(\frac{\sqrt{3}}{2}\right)$, giving results in degrees.

Solution. The principal value is

$$\operatorname{Arccos}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ,$$

and the secondary value is -30° . Hence

$$\arccos\left(\frac{\sqrt{3}}{2}\right) = 30^\circ \pm n 360^\circ, -30^\circ \pm n 360^\circ.$$

Example 2. Find all values of $\arctan(-3.000)$, giving results in radians.

Solution. Let $\theta' = \arctan(-3.000)$; then we are to solve the equation

$$\tan \theta' = -3.000.$$

As a first step we obtain from the Tables the corresponding acute angle θ , for which $\tan \theta = 3.000$; in radians, $\theta = 1.2490$. A value of θ' will therefore be -1.2490 radians, and this will be the principal value, since it lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ radians (equivalent to -90° and $+90^\circ$). Thus for our Example the principal value is

$$\operatorname{Arctan}(-3.000) = -1.2490,$$

the secondary value is $-1.2490 + \pi$, and the general solution of our problem is

$$\arctan(-3.000) = -1.2490 \pm 2n\pi, -1.2490 + \pi \pm 2n\pi.$$

Example 3. Find the values of $\tan(\arcsin \frac{1}{2})$.

Solution. We could, with the aid of the Tables, find all angles which are values of $\arcsin(\frac{1}{2})$ and then use the Tables and reduction formulas to find the value of the tangent of each of these angles. It is simpler, however, to observe

that if we write $\alpha = \arcsin(\frac{1}{2})$, we are to find $\tan \alpha$, given $\sin \alpha = \frac{1}{2}$. We have solved problems of this sort on pages 16, 17, and 49. The solution is

$$\tan \alpha = \tan \arcsin(\frac{1}{2}) = \pm \frac{1}{\sqrt{3}},$$

the positive sign to be taken for the principal value $\text{Arcsin}(\frac{1}{2})$ and coterminal angles $\pm n 360^\circ + \text{Arcsin}(\frac{1}{2})$, the negative sign for the secondary value and coterminal angles.

Example 4. Simplify the expressions $\sin \sin^{-1} x$, $\sin^{-1} \sin x$, and $\text{Sin}^{-1} \sin x$.

Solution. We are here using the alternative notation for inverse functions; for example, $\text{Sin}^{-1} a$ means the principal value $\text{Arcsin } a$.

Obviously, the phrase, "the sine of an angle whose sine is x " can have but one meaning,

$$\sin \sin^{-1} x = x.$$

On the other hand, $\sin^{-1} \sin x$ (an angle whose sine is equal to $\sin x$) has many values. It is clear that x is an angle whose sine is equal to $\sin x$, but it is also true (in radian notation) that $\pi - x$ is an angle whose sine is equal to $\sin x$. All other values are given by coterminal angles, hence

$$\sin^{-1} \sin x = x \pm 2n\pi, \quad \pi - x \pm 2n\pi.$$

Finally, $\text{Sin}^{-1} \sin x$ is equal to x if x is between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; but if x is not between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, then $\text{Sin}^{-1} \sin x$ is an angle in this interval that is coterminal either with x , or else with $\pi - x$.

★*Example 5.* Prove that if x and y are positive, and each is less than 1, then

$$\text{Arctan } x + \text{Arctan } y = \text{Arctan } \frac{x+y}{1-xy}.$$

Solution. This is an example of so-called identities in terms of inverse functions, which are true only with restrictions on the variables, as x and y here.

In this problem the two sides of the identity are acute angles and our identity will be true if

$$\begin{aligned} \tan [\text{Arctan } x + \text{Arctan } y] &= \tan \left[\text{Arctan } \frac{x+y}{1-xy} \right], \\ \frac{\tan (\text{Arctan } x) + \tan (\text{Arctan } y)}{1 - \tan (\text{Arctan } x) \tan (\text{Arctan } y)} &= \tan \left[\text{Arctan } \frac{x+y}{1-xy} \right]. \end{aligned}$$

The last equation reduces to

$$\frac{x+y}{1-xy} = \frac{x+y}{1-xy},$$

and the original identity is thus proved.

Remark. If x and y are positive, then $\text{Arctan } x$ and $\text{Arctan } y$ lie between 0° and 90° , as noted above. If, however, $1 - xy$ is negative, then $\text{Arctan } \frac{x+y}{1-xy}$ lies between -90° and 0° . Hence the stated formula is not valid in this case. But since the tangent of the left member equals the tangent of the right member, the two members differ by a multiple of 180° . It follows that when x is positive, y is positive, and $1 - xy$ is negative,

$$\text{Arctan } x + \text{Arctan } y = \text{Arctan } \frac{x+y}{1-xy} + 180^\circ.$$

EXERCISES

Express the following angles in degrees and minutes:

- | | | |
|---|--|------------------------------------|
| 1. $\text{Arcsin} \left(\frac{\sqrt{2}}{2} \right)$. | 11. $\text{Arccos } 0.8988$. | 22. $\text{Arctan} (-1)$. |
| 2. $\text{Arccos} \left(-\frac{\sqrt{2}}{2} \right)$. | 12. $\text{Arccos } 0.4067$. | 23. $\text{Arctan} (-0.4245)$. |
| 3. $\text{Arcsin } 0.4848$. | 13. $\text{Arcsin} \left(\frac{\sqrt{3}}{2} \right)$. | 24. $\text{Arctan} (-2.300)$. |
| 4. $\text{Arcsin } 0.8949$. | 14. $\text{Arccos} \left(-\frac{\sqrt{3}}{2} \right)$. | 25. $\text{Arccot } \sqrt{3}$. |
| 5. $\text{Arcsin} \left(-\frac{1}{2} \right)$. | 15. $\text{Arccos} (-0.4067)$. | 26. $\text{Arccot} (-\sqrt{3})$. |
| 6. $\text{Arcsin} \left(-\frac{\sqrt{3}}{2} \right)$. | 16. $\text{Arccos} (-0.8141)$. | 27. $\text{Arccot } 1.483$. |
| 7. $\text{Arcsin} (-0.8572)$. | 17. $\text{Arctan } 1$. | 28. $\text{Arccot} (-2.006)$. |
| 8. $\text{Arcsin} (-0.9205)$. | 18. $\text{Arctan } \sqrt{3}$. | 29. $\text{Arcsec } 1.206$. |
| 9. $\text{Arccos} \left(\frac{1}{2} \right)$. | 19. $\text{Arctan } 0.6873$. | 30. $\text{Arcsec } \frac{1}{2}$. |
| 10. $\text{Arccos} \left(\frac{\sqrt{3}}{2} \right)$. | 20. $\text{Arctan } 1.492$. | 31. $\text{Arcsec } \sqrt{2}$. |
| | 21. $\text{Arctan} \left(-\frac{\sqrt{3}}{3} \right)$. | 32. $\text{Arcsec} (-2)$. |
| | | 33. $\text{Arcsec } 1$. |
| | | 34. $\text{Arcsec} (-2)$. |
| | | 35. $\text{Arcsec } 1.729$. |
| | | 36. $\text{Arcsec} (-2.215)$. |

Express the following angles in radians, by use of tables:

- | | | |
|-------------------------------|---------------------------------|--------------------------------|
| 37. $\text{Arcsin } 0.0669$. | 40. $\text{Arctan } 343.8$. | 43. $\text{Arcsec } 68.76$. |
| 38. $\text{Arcsin } 0.9890$. | 41. $\text{Arccos } 0.9986$. | 44. $\text{Arcsec} (-1.010)$. |
| 39. $\text{Arctan } 0.0670$. | 42. $\text{Arccos} (-0.9986)$. | 45. $\text{Arcsec} (-68.76)$. |

Express in radians, and also in degrees and minutes, the following:

- | | | |
|-------------------------------|--|--|
| 46. $\text{arctan } 1$. | 50. $\text{arccot } 4.011$. | 54. $\text{arccsc } 3$. |
| 47. $\text{arccos} (-1)$. | 51. $\text{arccsc} (-6.392)$. | 55. $\text{arctan} \left(\frac{2}{3} \right)$. |
| 48. $\text{arcsec} (-2)$. | 52. $\text{arcsec} (-\sqrt{2})$. | 56. $\text{arccot} (-1)$. |
| 49. $\text{arcsin } 0.9613$. | 53. $\text{arcsin} \left(\frac{2}{3} \right)$. | 57. $\text{arccos} \left(\frac{1}{3} \right)$. |

Find values of the following:

- | | | |
|--|---|--|
| 58. $\sin \text{Arctan} \left(\frac{3}{4} \right)$. | 63. $\cos \text{arccos} \left(\frac{3}{5} \right)$. | 68. $\text{arctan } \tan 15^\circ$. |
| 59. $\tan \text{Arcsin} \left(\frac{2}{5} \right)$. | 64. $\sin \text{arccos} (-1)$. | 69. $\text{Arctan } \cot 220^\circ$. |
| 60. $\cos \text{Arcsin} \left(\frac{1}{3} \right)$. | 65. $\cos \text{arcsin } \frac{1}{2}\sqrt{2}$. | 70. $\text{Arcsin } \cos (-100^\circ)$. |
| 61. $\sin \text{Arcsin} \left(\frac{3}{5} \right)$. | 66. $\tan \text{arccos} \left(-\frac{3}{5} \right)$. | 71. $\text{Arccos } \cos 550^\circ$. |
| 62. $\cot \text{arctan} \left(\frac{2}{3} \right)$. | 67. $\text{arcsin } \sin 20^\circ$. | 72. $\text{Arccot } \tan 100^\circ$. |
| 73. $\sin (\text{Arcsin} \left(\frac{3}{5} \right) + \text{Arctan} \left(\frac{3}{4} \right))$. | 76. $\sin (2 \text{Arccos} \left(\frac{3}{5} \right))$. | |
| 74. $\cos (\text{Arccos} \left(\frac{1}{2} \right) + \text{Arctan} \left(\frac{1}{2} \right))$. | 77. $\cos (2 \text{Arctan} \left(\frac{5}{12} \right))$. | |
| 75. $\tan (\text{Arctan} \left(\frac{1}{2} \right) + \text{Arctan} \left(\frac{3}{4} \right))$. | 78. $\sin (\text{Arctan} \left(\frac{8}{15} \right) + 2 \text{Arcsin} \left(\frac{3}{5} \right))$. | |

Prove the following:

- ★79. $\text{Arctan} \left(\frac{1}{2} \right) + \text{Arctan} \left(\frac{1}{3} \right) = \frac{\pi}{4}$.
- ★80. $\text{Arcsin} \left(\frac{3}{5} \right) + \text{Arcsin} \left(\frac{4}{5} \right) = \text{Arccos} \left(-\frac{1}{5} \right)$.
- ★81. $\text{Arccos } x - \text{Arccos } y = \text{Arccos} (xy \pm \sqrt{1-x^2} \sqrt{1-y^2})$. (State restrictions which should be placed on x and y .)

44. Trigonometric equations. In § 43 we have discussed the solution of a very simple type of equations in trigonometric functions; the equation $\sin \theta = a$, in which θ is the unknown, has the infinitely many solutions $\theta = \arcsin a$. In the following Examples we show how to solve equations of other types.

Example 1. Find in degrees all solutions of the equation $3 \tan^2 \theta - 1 = 0$.

Solution. We have

$$\begin{aligned}\tan^2 \theta &= \frac{1}{3}, \\ \tan \theta &= \frac{1}{\sqrt{3}}, \quad \text{or} \quad \tan \theta = -\frac{1}{\sqrt{3}}.\end{aligned}$$

Hence all solutions are given by

$$\theta = \arctan \frac{1}{\sqrt{3}}, \quad \theta = \arctan \left(-\frac{1}{\sqrt{3}} \right).$$

The principal value of $\arctan (1/\sqrt{3})$ is 30° , and that of $\arctan (-1/\sqrt{3})$ is -30° . In degrees, the above solutions are therefore

$$\begin{aligned}\theta &= 30^\circ \pm n 360^\circ, \quad \theta = -30^\circ \pm n 360^\circ, \\ &210^\circ \pm n 360^\circ, \quad 150^\circ \pm n 360^\circ.\end{aligned}$$

Example 2. Find in radians all solutions of $2 \cos 3x = 1$ that lie between 0 and π .

Solution. We have $\cos 3x = \frac{1}{2}$, for which the general solution is

$$3x = \frac{\pi}{3} \pm 2n\pi, \quad -\frac{\pi}{3} \pm 2n\pi.$$

The values of x that lie between 0 and π are

$$x = \frac{\pi}{9}, \quad \frac{1}{3} \left(-\frac{\pi}{3} + 2\pi \right), \quad \frac{1}{3} \left(+\frac{\pi}{3} + 2\pi \right),$$

or

$$x = \frac{\pi}{9}, \quad \frac{5\pi}{9}, \quad \frac{7\pi}{9}.$$

Example 3. Find in radians all solutions of $\sin 2x - \cos x = 0$ that lie between 0 and 2π .

Solution. We have

$$\begin{aligned}\sin 2x - \cos x &= 0, \\ 2 \sin x \cos x - \cos x &= 0, \\ \cos x (2 \sin x - 1) &= 0,\end{aligned}$$

hence the solutions are found by solving the two equations (not simultaneous)

$$\cos x = 0; \quad \sin x = \frac{1}{2}.$$

The solutions are

$$x = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

Example 4. Find in degrees all solutions of $\sin x + \sin 3x = 0$ such that $0 \leq x \leq 180^\circ$.

Solution. We have, by § 41,

$$\sin x + \sin 3x = 2 \sin 2x \cos x,$$

hence solutions of our equation are found by solving the two equations

$$\sin 2x = 0; \quad \cos x = 0.$$

Hence we have

$$x = 0^\circ, 90^\circ, 180^\circ.$$

Example 5. Find in degrees all solutions of $\cos \theta + 2 \sin \theta = 2$ such that $-90^\circ \leq \theta \leq 90^\circ$.

Solution. We reduce this to a quadratic equation in $\sin \theta$ as follows:

$$\cos \theta + 2 \sin \theta = 2,$$

$$\cos \theta = 2 - 2 \sin \theta,$$

$$\cos^2 \theta = (2 - 2 \sin \theta)^2,$$

$$(1 - \sin^2 \theta) = 4 - 8 \sin \theta + 4 \sin^2 \theta,$$

$$5 \sin^2 \theta - 8 \sin \theta + 3 = 0,$$

$$(\sin \theta - 1)(5 \sin \theta - 3) = 0.$$

Hence all solutions of our equation are found by solving the two equations

$$\sin \theta = 1; \quad \sin \theta = \frac{3}{5};$$

and the solutions of these equations in the range proposed are

$$\theta = 90^\circ; \quad \theta = \text{Arcsin}(\frac{3}{5}) = 36^\circ 52'.$$

These are the only possible solutions of our problem, but we might doubt whether they are *actual* solutions, since at one stage we squared both sides of the equation $\cos \theta = 2 - 2 \sin \theta$, and it does not always follow that all solutions of a squared equation are solutions of the equation itself. We should finish by substituting our *possible* solutions. In the present Example this is easily done, and we find that both the above values of θ are *actual* solutions.

It is interesting to note that if $x = \cos \theta$, $y = \sin \theta$, we have $x^2 + y^2 = 1$, and our problem reduces to this: Solve algebraically for y the simultaneous equations

$$x^2 + y^2 = 1, \quad x + 2y = 2;$$

if the solutions are y_1, y_2 , solve the equations

$$\sin \theta = y_1, \quad \sin \theta = y_2.$$

Many trigonometric equations are similarly related to systems of algebraic equations.

Another method is suggested by the observation that if, in an expression $a \cos \theta + b \sin \theta$, we replace a by $r \cos \alpha$ and b by $r \sin \alpha$, we have

$$a \cos \theta + b \sin \theta = r \cos \alpha \cos \theta + r \sin \alpha \sin \theta = r \cos(\alpha - \theta) = r \cos(\theta - \alpha).$$

Thus the equation $\cos \theta + 2 \sin \theta = 2$ can be replaced by

$$1 = r \cos \alpha, \quad 2 = r \sin \alpha, \quad r \cos(\theta - \alpha) = 2.$$

These equations are equivalent to the following:

$$r = \sqrt{5}, \quad \alpha = \arccos \frac{1}{\sqrt{5}}, \quad \theta = \arccos \frac{2}{r} + \alpha.$$

The last equation gives the solutions when we use appropriate values of the inverse functions.

EXERCISES

Find in degrees all solutions of the following equations:

- | | |
|---------------------------|----------------------------------|
| 1. $3 \cos \theta = 2.$ | 9. $\sin 2x = 2 \sin x.$ |
| 2. $4 \tan \theta = 5.$ | 10. $\cos 2x = 2 \cos x.$ |
| 3. $3 \sin \theta = 2.$ | 11. $\tan 2x = 2 \tan x.$ |
| 4. $4 \sec \theta = 5.$ | 12. $\sec 2x = 2 \sec x.$ |
| 5. $4 \sin^2 \theta = 3.$ | 13. $\sin 3x = \sin x.$ |
| 6. $2 \cos^2 \theta = 1.$ | 14. $\cos 3x = \cos x.$ |
| 7. $3 \cot^2 \theta = 1.$ | 15. $\sin^2 2x - \cos^2 2x = 0.$ |
| 8. $3 \sec^2 \theta = 4.$ | 16. $\cos^2 2x - \cos^2 x = 0.$ |

Find in radians all solutions of the following equations:

- | | |
|--------------------------|-------------------------------------|
| 17. $\tan^2 x = 2.$ | 21. $\sec^2 \theta = 2.$ |
| 18. $2 \sin^2 x = 1.$ | 22. $\csc \theta = 2 \sec \theta.$ |
| 19. $\sin x = 2 \cos x.$ | 23. $\sin 2\theta = 2 \cos \theta.$ |
| 20. $\cos x = 2 \sin x.$ | 24. $\cos 3\theta = \cos \theta.$ |

Find in degrees all solutions of the following equations such that $0 \leq x \leq 180^\circ$:

- | | |
|--------------------------------|---------------------------------|
| 25. $\sin x - \sin 2x = 0.$ | 29. $\sin 2x + 2 \cos^2 x = 1.$ |
| 26. $\cos 2x - \cos^2 x = 0.$ | 30. $\sin 5x - \sin x = 0.$ |
| 27. $\tan^2 x + \sec^2 x = 7.$ | 31. $\cos 5x + \cos 3x = 0.$ |
| 28. $2 \cos x + \sin x = 2.$ | 32. $3 \cos x + 4 \sin x = 5.$ |

Find in degrees and in radians all solutions, if there are any, of the following equations such that $0 \leq x \leq 360^\circ$:

- | | |
|--|--------------------------------------|
| 33. $\cos 2x - \cos x = 0.$ | 42. $8 \sin x + \cos x = 7.$ |
| 34. $\cos 4x - \cos 2x = 0.$ | 43. $4 \sin x - 7 \cos x = 1.$ |
| 35. $\tan^2 x + 2 = 3 \tan x.$ | 44. $\cos 3x = 4 \cos^3 x.$ |
| 36. $2 \cos^2 x + 3 = 5 \cos x.$ | 45. $6 \csc x - 6 \sin x = 5.$ |
| 37. $\cos 2x + \cos x + 1 = 0.$ | 46. $\cos^2 x - \sin^2 x = 1.$ |
| 38. $\sec x + 2 \cos x = 3.$ | 47. $\sec^2 x = \tan^2 x + 2.$ |
| 39. $2 \cos 2x - 4 \sin x + 1 = 0.$ | 48. $3 \csc^2 x - 2 \csc x = 1.$ |
| 40. $\sin 4x - \sin 2x = \cos 3x.$ | 49. $\cos^2 x - \cos x = 6.$ |
| 41. $\sin x + \sin 3x = \cos x + \cos 3x.$ | 50. $\tan^4 x + 3 \tan^2 x + 2 = 0.$ |

CHAPTER VI

LOGARITHMS. FIVE-PLACE TABLES

45. The definition of a logarithm. It is a consequence of the definitions of real numbers that every positive real number N can be expressed as a power of 10,

$$(1) \quad N = 10^x,$$

where x is a real number. For a given N there is only one x , and to increase N is to increase x .

Definition. If $N = 10^x$, the exponent x is called the *logarithm of N to the base 10*.

In symbols we write

$$\log_{10} N = x,$$

or, the base being taken for granted,

$$\log N = x.$$

Equation (1) can then be written

$$(2) \quad 10^{\log N} = N.$$

We can generalize the definition of a logarithm so as to use another base than 10. If a is any positive real number other than 1, every positive real number N can be expressed in one and only one way in the form

$$(3) \quad N = a^x.$$

We then say that x is the *logarithm of N to base a* , and write

$$(4) \quad \log_a N = x, \quad a^{\log_a N} = N.$$

46. The table of five-place logarithms of numbers. Logarithms are, in general, unending decimals and only approximate values can be given in tables. When we say that .30103 is the five-place logarithm of 2 to base 10, we mean that .30103 is the five-place number which is nearest to the true solution x of the equation

$$10^x = 2.*$$

* We recall that $10^{.30103}$ means 10 to the power .30103/100000, that is, the hundred-thousandth root of 10^{30103} . Needless to say, tables of logarithms are not actually computed by taking hundred-thousandth roots.

On pages 74–91 of the Tables at the end of this book, five-place logarithms to base 10 are given for numbers N ranging from 1.0000 to 9.9990. The first three digits of N are given in the left-hand column, under N , the fourth digit in line with N near the top of the page, and the fifth digit is zero. A decimal point is to be placed after the first digit of N and before the first digit of each logarithm. The last three digits of $\log N$ are to be found in the line which contains the first three digits of N and in the column headed by the fourth digit of N . We find the first two digits of $\log N$, if the last three digits of $\log N$ are not preceded by a star, by going back to the first column of logarithms and going up to the first logarithm given to five places; the first two digits of that logarithm are the same as the first two digits of the logarithm we are seeking. If the last three digits are preceded by a star, use for the first two digits those to be found in the first logarithm of the line below. Thus, for example, we read from page 76 of the Table

$$\log 2.3360 = .36847,$$

$$\log 2.3460 = .37033.$$

The following two Examples illustrate interpolation by the principle of proportional parts.

Example 1. Find $\log 2.5432$ to five places.

Solution. We have, from page 77 of the Table,

$$\begin{aligned} \log 2.5432 &= \log 2.5430 + \frac{1}{10} (\log 2.5440 - \log 2.5430) \\ &= .40535 + \frac{1}{10} \times (.40552 - .40535) \\ &= .40535 + \frac{1}{10} \times (.00017) \\ &= .40535 + .00003 \\ &= .40538. \end{aligned}$$

N		$\log N$
10	$2 \left[\begin{array}{c} 2.5430 \\ 2.5432 \\ 2.5440 \end{array} \right]$	$\left[\begin{array}{c} .40535 \\ ? \\ .40552 \end{array} \right] \begin{array}{c} x \\ 17 \end{array}$

At the right of page 77, under **Prop. Pts.** (Proportional Parts) for the black type entry **17**, it can be read off that $\frac{1}{10} \times 17 = 3.4$, from which we could have inferred that the correction .00003 was to be added to .40535.

Example 2. If $\log N = .54991$, find N to five places.

Solution. The logarithms in the Table that are nearest to .54991 are

$$\begin{aligned} \log 3.5470 &= .54986, \\ \log 3.5480 &= .54998. \end{aligned}$$

N		$\log N$
10	$x \left[\begin{array}{c} 3.5470 \\ ? \\ 3.5480 \end{array} \right]$	$\left[\begin{array}{c} .54986 \\ .54991 \\ .54998 \end{array} \right] \begin{array}{c} 5 \\ 12 \end{array}$

The given logarithm .54991 is $\frac{5}{12}$ of the way from $\log 3.5470$ to $\log 3.5480$; hence N is to be taken $\frac{5}{12}$ of the way from 3.5470 to 3.5480, so that

$$\begin{aligned} N &= 3.5470 + \frac{5}{12} (.0010) \\ &= 3.5470 + .0004 = 3.5474. \end{aligned}$$

In the column of Proportional Parts for the black number **12** we would have found that $\frac{4.8}{12} = .4$, from which we could see that the interpolation is .0004.

In the next section we shall explain how to find logarithms of numbers greater than 10 or less than 1.

47. Logarithms of positive numbers. We have now explained how to find the logarithm to base 10 of a number between 1 and 10. Logarithms of positive numbers without such restriction are obtained with the aid of the following fundamental property of numbers:

I. *Every positive number N can be expressed in the form*

$$(1) \quad N = 10^{c+m}$$

where c is 0 or a positive or negative integer, and m is a positive number less than 1 (or m may be 0).

We are here merely rewriting formula (1) of § 45, with $x = c + m$. To say that x has this form is merely to say that a given x always lies between two integers c and $c + 1$, that is, $c \leq x < c + 1$.

Another way of stating I is as follows:

II. *Every positive number N has a logarithm to base 10 of the form*

$$(2) \quad \log N = c + m,$$

where c , called the **characteristic**, is an integer (positive, negative, or zero) and m , called the **mantissa**, is positive (or zero) and less than 1.

To find the logarithm of a number N to base 10 we must determine its characteristic and its mantissa. The solution is clearer if we rewrite (1), by using a law of exponents (see the following § 48),

$$N = 10^c \times 10^m.$$

Since m is between 0 and 1, we see that 10^m is a number M between 1 and 10, so that

$$(3) \quad N = 10^c M, \quad 1 \leq M < 10.$$

$$(4) \quad M = 10^m.$$

We thus have the following rule for finding the characteristic and mantissa of $\log N$:

III. *Express the number N in form (3); then the characteristic of $\log N$ is c and the mantissa of $\log N$ is $\log M$, which can be found by using tables.*

Example 1. Find $\log 65.520$.

Solution. We have

$$65.520 = 10^1 \times 6.5520.$$

Hence

$$c = 1,$$

and from the Table

$$m = \log 6.5520 = .81637,$$

so that

$$\log 65.520 = 1 + .81637 = 1.81637.$$

Example 2. Find $\log 0.065520$.

Solution. We have

$$0.065520 = 10^{-2} \times 6.5520,$$

$$\log N = -2 + .81637.$$

Note that in Example 2 it would have been incorrect to write -2.81637 for $\log N$, since -2.81637 means $-2 - .81637$. We do not wish to change $-2 + .81637$ into -1.18363 , so we use the notation

$$\log N = 8.81637 - 10.$$

The above Examples illustrate the fact that, in formula (3), M has the same digits as N , but in M the decimal point is just after the first digit. If c is positive, multiplying M by 10^c merely moves the decimal point so that in the product, N , the first significant figure will be just c places to the left of *units' place*, the place just before the decimal point; if c is zero, the first significant figure of N is in *units' place*; if c is negative, the first significant figure of N is c places to the right of *units' place*. Note that in Example 2 we kept a zero in *units' place* to make sure that we should count correctly.

The converse of each statement in the preceding paragraph is also true.

We state the rule for the characteristic as follows:

IV. *To find the characteristic of $\log N$, first find how far it is from the first significant figure of N to the units' place. If units' place is*

k places to the right, the characteristic is k ;

k places to the left, the characteristic is $-k$.

The student should write the converse of IV which states where the *units' place* must be in N for a given characteristic of $\log N$.

EXERCISES

Find the characteristics for the following logarithms:

1. $\log 34.210$; $\log 58370$.

5. $\log 0.32723$; $\log 0.00723$.

2. $\log 54760$; $\log 3.4023$.

6. $\log 0.01525$; $\log 0.000742$.

3. $\log 325.47$; $\log 824300$.

7. $\log 0.00273$; $\log 0.000039$.

4. $\log 5.0732$; $\log 4217000$.

8. $\log 0.00071$; $\log 0.002423$.

Find the following logarithms, using five-place tables:

- | | |
|---|-------------------------|
| 9. $\log 3752.0$; $\log 2.8350$. | 19. $\log 371.39$. |
| 10. $\log 36.840$; $\log 49990$. | 20. $\log 289520$. |
| 11. $\log 218300$; $\log 0.01284$. | 21. $\log 48437$. |
| 12. $\log 6172000$; $\log 0.6496$. | 22. $\log 0.62148$. |
| 13. $\log 0.8984$; $\log 2982000$. | 23. $\log 0.0071782$. |
| 14. $\log 0.007263$; $\log 92,920,000$. | 24. $\log 5.0425$. |
| 15. $\log 43434$. | 25. $\log 0.63787$. |
| 16. $\log 5284.2$. | 26. $\log 0.031814$. |
| 17. $\log 6.7316$. | 27. $\log 0.00025678$. |
| 18. $\log 82.477$. | 28. $\log 0.079784$. |

Find to five significant figures the numbers N whose logarithms are given as follows:

- | | |
|-------------------------------|-------------------------------|
| 29. $\log N = 0.50664$. | 41. $\log N = 3.70030$. |
| 30. $\log N = 0.60438$. | 42. $\log N = 5.50042$. |
| 31. $\log N = 0.82000$. | 43. $\log N = 6.41143$. |
| 32. $\log N = 0.90020$. | 44. $\log N = 0.90029$. |
| 33. $\log N = 2.47712$. | 45. $\log N = 7.43100$. |
| 34. $\log N = 3.62180$. | 46. $\log N = 3.81020$. |
| 35. $\log N = -2 + .71383$. | 47. $\log N = 6.44444 - 10$. |
| 36. $\log N = -3 + .84634$. | 48. $\log N = 7.69920 - 10$. |
| 37. $\log N = 8.90428 - 10$. | 49. $\log N = 9.87358 - 10$. |
| 38. $\log N = 7.94507 - 10$. | 50. $\log N = 8.78996 - 10$. |
| 39. $\log N = 2.41670$. | 51. $\log N = 4.70067 - 10$. |
| 40. $\log N = 4.61940$. | 52. $\log N = 3.49131 - 10$. |

48. Laws of exponents and of logarithms. If a is a positive number, and m and n are real numbers, the following *laws of exponents* hold:

$$(I) \quad a^m \times a^n = a^{m+n},$$

$$(II) \quad \frac{a^m}{a^n} = a^{m-n},$$

$$(III) \quad (a^m)^n = a^{mn}.$$

In books on algebra these laws are discussed; in accordance with these formulas, we have

$$(IV) \quad a^0 = 1,$$

$$(V) \quad a^{-m} = \frac{1}{a^m}.$$

Now let us turn these laws of exponents into corresponding *laws of logarithms*. For this purpose we write

$$\begin{aligned} M &= a^m, & m &= \log_a M; \\ N &= a^n, & n &= \log_a N. \end{aligned}$$

Thus formula (I) gives

$$MN = a^{m+n},$$

and we have

$$\log_a(MN) = m + n = \log_a M + \log_a N.$$

Similarly, formula (II) gives

$$\log_a \frac{M}{N} = m - n = \log_a M - \log_a N,$$

and formula (III) gives

$$\log_a M^n = mn = n(\log_a M).$$

Thus our three laws of exponents have been expressed as the following **laws of logarithms** for all positive real numbers M , N , and positive base a ($a \neq 1$):

$$(1) \quad \log MN = \log M + \log N,$$

$$(2) \quad \log \frac{M}{N} = \log M - \log N,$$

$$(3) \quad \log M^n = n \log M.$$

Since

$$\sqrt[r]{M} = M^{\frac{1}{r}},$$

we may write, as a corollary of (3),

$$(3') \quad \log \sqrt[r]{M} = \frac{1}{r} \log M.$$

Formulas (IV) and (V) become

$$(4) \quad \log 1 = 0,$$

$$(5) \quad \log \frac{1}{M} = -\log M.$$

49. Computation with the use of logarithms to base 10. From law (1) of § 48 we see that we can compute a product by finding the logarithms of the factors M and N , adding them to get $\log MN$, and then finding the number MN from the Tables. Products of more than two factors are handled by an extension of law (1) which states that the logarithm of a product of two or more factors is the sum of the logarithms of the factors. For example,

$$\begin{aligned} \log MNP &= \log (MN)P = \log MN + \log P \\ &= \log M + \log N + \log P. \end{aligned}$$

Example 1. Find $N = 2.8255 \times .0074232$.

Solution.

$$\begin{array}{r} \log 2.8255 = 0.45110 \\ \log .0074232 = 7.87059 - 10 \\ \text{sum} = \log N = 8.32169 - 10 \\ N = 0.0020974. \end{array}$$

Example 2. Find $N = \frac{42.734}{5402.7}$.

Solution.

$$\begin{array}{r} \log 42.734 = 11.63077 - 10 \\ \log 5402.7 = 3.73261 \\ \text{difference} = \log N = 7.89816 - 10 \\ N = 0.0079097. \end{array}$$

Example 3. Find $x = \frac{M}{N} = \frac{0.38283 \times 0.048296}{0.062192 \times 8348.4}$.

Solution.

$$\begin{array}{r} \log 0.38283 = 9.58301 - 10 \\ \log 0.048296 = 8.68391 - 10 \\ \text{sum} = \log M = 18.26692 - 20 \\ \log N = 12.71533 - 10 \\ \text{difference} = \log x = 5.55159 - 10 \\ x = 0.000035612. \end{array} \quad \begin{array}{r} \log 0.062192 = 8.79373 - 10 \\ \log 8348.4 = 3.92160 \\ \text{sum} = \log N = 12.71533 - 10 \end{array}$$

Example 4. Find $\sqrt[3]{0.37285}$.

Solution.

$$\begin{array}{r} \log 0.37285 = 29.57154 - 30 \\ \log \sqrt[3]{0.37285} = \frac{1}{3} \log 0.37285 = 9.85718 - 10 \\ \sqrt[3]{0.37285} = 0.71975. \end{array}$$

Note that the characteristic of $\log 0.37285$ was taken as $29 - 30$ in order to facilitate division by 3 at the next step.

Negative numbers have no real logarithms. A computation involving them, where logarithmic tables are to be used, must be made to depend on one involving only positive numbers. Thus, to find the product of $-M$ and N , where M and N are positive, we compute MN by using logarithms, and prefix a minus sign to the result.

The student must remember that $\log(a + b)$ is *not* equal to $\log a + \log b$ (unless $a + b = ab$). The logarithms defined in this chapter are not useful in obtaining sums and differences of numbers.

★50. Cologarithms. In finding $\log\left(\frac{M}{N}\right)$ we subtracted $\log N$ from $\log M$. This is the same as adding $\log M$ and $(-\log N)$. When $-\log N$ is so written that the decimal part is positive (as we would

write $\log \left(\frac{1}{N} \right)$, which is equal to $-\log N$, it is called the *cologarithm* of N . Thus

$$\text{colog } N = \log \frac{1}{N} = -\log N,$$

and the law for division becomes

$$\log \frac{M}{N} = \log M + \text{colog } N.$$

In complicated computations, this device is often useful. We shall not employ it in this text.

EXERCISES

Use a five-place table of logarithms to find the number N to five significant figures in the following:

- | | |
|---------------------------------------|--|
| 1. $N = 32.34 \times 2.185$. | 11. $N = (3.1624)^3$. |
| 2. $N = 7.588 \times 87.62$. | 12. $N = (71.346)^2$. |
| 3. $N = 625.44 \times 3.8717$. | 13. $N = (2.2333)^{10}$. |
| 4. $N = 14923 \times 0.38761$. | 14. $N = (0.31724)^2 \times (4.1557)^3$. |
| 5. $N = 1066.1 \times 0.077244$. | 15. $N = \sqrt{0.4940}$. |
| 6. $N = 0.087319 \times 0.11146$. | 16. $N = \sqrt[3]{87.27}$. |
| 7. $N = \frac{28.746}{37.254}$. | 17. $N = \frac{-61.872 \times 23.750}{7279.3 \times 0.071637}$. |
| 8. $N = \frac{0.025183}{0.0035677}$. | 18. $N = \frac{\sqrt[3]{-6254} \times \sqrt{0.88888}}{-37.247}$. |
| 9. $N = \frac{0.25764}{1.3579}$. | 19. $N = \sqrt[3]{\frac{(3.5723)^2 \times (-1.1236)^3}{-0.37162 \times (88.714)^3}}$. |
| 10. $N = \frac{96.172}{0.042755}$. | 20. $N = \sqrt{\frac{37.611 \times 0.082727}{458.34 \times 0.071714}}$. |

51. Five-place logarithms of trigonometric functions. Table VI gives five-place logarithms of four trigonometric functions for angles from 0° to 90° . For angles less than 45° , we find the number of degrees at the top of the page, and the number of minutes in the left-hand column; for angles greater than 45° , the number of degrees is given at the bottom of the page, the number of minutes in the column next to the **Prop. Pts.** column, and the column designations to be used are those at the bottom of each column. After each logarithm as given in this Table we must supply " -10 "; for example,

$$\log \sin 18^\circ 0' = 9.48998 - 10.$$

We interpolate, usually to the nearest second,* by the principle of proportional parts as illustrated in the following Examples.

Example 1. Find $\log \sin 18^\circ 54' 25''$.

Solution. On page 45 of the Tables (the page headed 18°) we go down the first column to $54'$, and opposite this entry and the next, in the **L Sin** column, we find

	Angle	L Sin
$\log \sin 18^\circ 54' = 9.51043 - 10,$		
$\log \sin 18^\circ 55' = 9.51080 - 10.$		
	$60 \left[\begin{array}{c} 25 \left[\begin{array}{c c} 18^\circ 54' 0'' & 9.51043 \\ 18^\circ 54' 25'' & ? \\ 18^\circ 55' 0'' & 9.51080 \end{array} \right] x \end{array} \right] 37$	

The angle $18^\circ 54' 25''$ is $\frac{25}{60}$ of the way from $18^\circ 54'$ to $18^\circ 55'$, and we conclude that (approximately) its $\log \sin$ is $\frac{25}{60}$ of the way from $\log \sin 18^\circ 54'$ to $\log \sin 18^\circ 55'$. Thus

$$\begin{aligned} \log \sin 18^\circ 54' 25'' &= 9.51043 - 10 + \frac{25}{60} \times .00037 \\ &= 9.51043 - 10 + .00015 \\ &= 9.51058 - 10, \text{ to five decimal places.} \end{aligned}$$

By making use of other columns on page 45 we could compute the interpolation as follows:

From the column next to the right of the **L Sin** column we could have read off the difference between $\log \sin 18^\circ 54'$ and $\log \sin 18^\circ 55'$ (the tabular difference) as 37 in the last two of the five decimal places, and in the **Prop. Pts.** (Proportional Parts) column, under the black number entry **37** we are told how much an increase of an angle by the indicated number of seconds increases the last two of the five decimal places of the logarithm where the tabular difference is 37. Since to increase $18^\circ 54'$ by $25''$ is to increase it by $20'' +$ one tenth of $50''$, we read off from the **L Sin** column and from the **Prop. Pts.** column

$$\begin{aligned} \log \sin 18^\circ 54' 25'' &= 9.51043 - 10 + .000123 + .0000308 \\ &= 9.51058 - 10, \text{ to five places of decimals.} \end{aligned}$$

Example 2. Find $\log \tan 54^\circ 17' 42''$.

Solution. On page 62 of the Tables we find

$$\log \tan 54^\circ 17' = 10.14326 - 10.$$

The **cd** column to the left gives the tabular difference as 27, and in the **Prop. Pts.** column under entry 27 we find that to the last two places of $\log \tan 54^\circ 17'$ we must add, when the angle is increased by $42''$, the correction $18.0 + 0.9 = 19$ (to the nearest whole number). Thus

$$\begin{aligned} \log \tan 54^\circ 17' 42'' &= 10.14326 - 10 + .00019 \\ &= 10.14345 - 10. \end{aligned}$$

Example 3. Find $\log \cos 73^\circ 23' 36''$.

Solution. On page 43 of the Tables we find 73° at the bottom of the page.

* Interpolation to the nearest five seconds would be more consistent with the statements of page 27.

In the column at whose foot we find **L Cos** we use the entry opposite 23 in the *right-hand* column of minutes. We thus see that

$$\log \cos 73^\circ 23' = 9.45632 - 10.$$

We compute the correction for $36''$ by using **d** and **Prop. Pts.** columns as in preceding examples, but note that the correction is now to be *subtracted*, since **L Cos** decreases as we go from $73^\circ 23'$ to $73^\circ 24'$. Thus

$$\begin{aligned} \log \cos 73^\circ 23' 36'' &= 9.45632 - 10 - .00026 \\ &= 9.45606 - 10. \end{aligned}$$

Example 4. Find the acute angle A , given

$$\log \cot A = 8.97123 - 10.$$

Solution. By running through **L Cot** entries in the Table we find that those of about the right magnitude are on page 32, for the column with **L Cot** at the foot. We see that

$\log \cot 84^\circ 39' = 8.97150 - 10,$	A	L Cot
$\log \cot 84^\circ 40' = 8.97013 - 10.$	$60 \left[\begin{array}{c c} x \left[\begin{array}{c} 84^\circ 39' 0'' \\ ? \\ 84^\circ 40' 0'' \end{array} \right. & \begin{array}{c} 8.97150 \\ 8.97123 \\ 8.97013 \end{array} \end{array} \right] \begin{array}{c} 27 \\ 137 \end{array}$	

Since $8.97123 - 10$ is $\frac{27}{137}$ of the way from $8.97150 - 10$ to $8.97013 - 10$ we infer that the angle A should be $\frac{27}{137}$ of the way from $84^\circ 39'$ to $84^\circ 40'$, that is,

$$\begin{aligned} A &= 84^\circ 39' + \frac{27}{137} \times 60'' \\ &= 84^\circ 39' 12'', \text{ to the nearest second.} \end{aligned}$$

To find that $12''$ is $\frac{27}{137} \times 60''$ we could have used the **Prop. Pts.** table for 137. In the column under that entry, the number next less than 27 is 22.8, which corresponds to $10''$; the other 4.2 in 27 corresponds to about one tenth of $20''$; hence we infer that the total correction is $12''$.

Pages 21-24 of the Tables give special methods of interpolation for angles near 0° and 90° .

EXERCISES

Find by use of five-place tables the values of the following:

- | | |
|-----------------------------------|-------------------------------------|
| 1. $\log \sin 22^\circ 37' 36''.$ | 7. $\log \tan 71^\circ 19' 48''.$ |
| 2. $\log \cos 32^\circ 51' 12''.$ | 8. $\log \cot 79^\circ 17' 54''.$ |
| 3. $\log \tan 42^\circ 36' 20''.$ | 9. $\log \sin 85^\circ 25' 42''.$ |
| 4. $\log \cot 51^\circ 22' 40''.$ | 10. $\log \cos 87^\circ 21' 36''.$ |
| 5. $\log \sin 55^\circ 18' 35''.$ | 11. $\log \sin 113^\circ 12' 20''.$ |
| 6. $\log \cos 62^\circ 47' 25''.$ | 12. $\log \tan 209^\circ 5' 40''.$ |

By use of five-place tables find acute angles A satisfying the following equations:

- | | |
|-----------------------------------|-----------------------------------|
| 13. $\log \sin A = 9.55793 - 10.$ | 17. $\log \sin A = 9.90197 - 10.$ |
| 14. $\log \cos A = 9.95798 - 10.$ | 18. $\log \cos A = 9.76021 - 10.$ |
| 15. $\log \tan A = 9.52031 - 10.$ | 19. $\log \tan A = 0.14406.$ |
| 16. $\log \cot A = 0.46880.$ | 20. $\log \cot A = 8.82620 - 10.$ |

21. $\log \sin A = 9.80923 - 10.$

22. $\log \cos A = 9.86359 - 10.$

23. $\log \tan A = 9.11514 - 10.$

24. $\log \cot A = 0.75740.$

25. $\log \sin A = 9.99069 - 10.$

26. $\log \cos A = 9.54572 - 10.$

27. $\log \tan A = 0.27541.$

28. $\log \cot A = 9.57519 - 10.$

29. $\log \sec A = 0.04634.$

30. $\log \csc A = 0.32839.$

31. $\log \sec A = 1.15604.$

32. $\log \csc A = 0.14142.$

CHAPTER VII

SOLUTION OF TRIANGLES. FIVE-PLACE LOGARITHMS

52. Right triangles. In Chapter II we solved right triangles by making computations which involved multiplication and division. We shall now illustrate how the computations may be carried out by using logarithms.

It saves time, and tends to greater accuracy in computing, to outline the solution completely before referring to the Tables. The following steps are desirable:

- I. Draw a figure from the data and estimate the unknown parts of the triangle.
- II. Write down all of the formulas which will be used in the solution and in the check.
- III. Lay out the plan of computation, arranging for the placing of each number which must be written in a position which insures easy and accurate calculation. This must include the **labeling of each step so that a reader may follow without difficulty.**
- IV. Use the Tables to carry out the plan of Step III. If the check is not satisfactory, go over each step to locate errors. If the check is satisfactory, write down the computed values (of the parts which were unknown) at the end of your computation.

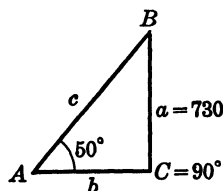
We note that in Step III different plans are possible. Completeness and neatness are important in avoiding errors and resultant re-computations.

In the following Examples, the work of Step III is indicated in bold type, and is done before any of the numbers are supplied.

Example 1. Solve the triangle, given $C = 90^\circ$, $A = 49^\circ 45' 20''$, $a = 734.40$.

Solution.

Step I.



Estimates

$B = 40^\circ$

$b = 600$

$c = 950$

FIG. 55 (1 cm. = 400)

Step II.

Formulas

$$B = 90^\circ - A$$

$$b = a \cot A$$

$$\log b = \log a + \log \cot A$$

$$c = \frac{a}{\sin A}$$

$$\log c = \log a - \log \sin A$$

$$\text{Check: } b^2 = c^2 - a^2 \quad 2 \log b = \log (c - a) + \log (c + a)$$

Steps III and IV.

Computation

$$90^\circ = 89^\circ 59' 60''$$

$$\log a = 2.86593$$

$$\log a = 12.86593 - 10$$

$$A = 49^\circ 45' 20''$$

$$(+)\log \cot A = 9.92757 - 10 \quad (-)\log \sin A = 9.88269 - 10$$

$$B = 40^\circ 14' 40''$$

$$\log b = 12.79350 - 10$$

$$\log c = 2.98324$$

$$b = 621.59$$

$$c = 962.14$$

Check

$$c = 962.14$$

$$\log (c - a) = 2.35744$$

$$a = 734.40$$

$$\log (c + a) = 3.22956$$

$$c - a = 227.74$$

$$\text{sum} = 5.58700$$

$$c + a = 1696.54$$

$$2 \log b = 5.58700$$

Satisfactory check

$$\text{Answers: } B = 40^\circ 14' 40'', \quad b = 621.59, \quad c = 962.14.$$

Example 2. Given $a = 0.23402$, $b = 0.31968$, $C = 90^\circ$. Find A , B , c .

Solution.

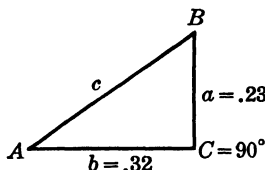


FIG. 56 (1 cm. = 0.15)

Estimates

$$A = 36^\circ$$

$$B = 54^\circ$$

$$c = 0.39$$

Formulas

$$B = 90^\circ - A$$

$$\tan A = \frac{a}{b}$$

$$\log \tan A = \log a - \log b$$

$$c = \frac{a}{\sin A}$$

$$\log c = \log a - \log \sin A$$

$$\text{Check: } b = c \sin B$$

$$\log b = \log c + \log \sin B$$

Computation

$$\log a = 19.36926 - 20$$

$$\log a = 19.36926 - 20$$

$$(-)\log b = 9.50471 - 10$$

$$(-)\log \sin A = 9.77137 - 10$$

$$\log \tan A = 9.86455 - 10$$

$$\log c = 9.59789 - 10$$

$$A = 36^\circ 12' 23''$$

$$c = 0.39618$$

$$B = 53^\circ 47' 37''$$

Check

$$\log c = 9.59789 - 10$$

$$(+)\log \sin B = 9.90682 - 10$$

$$\text{sum} = 19.50471 - 20$$

$$\log b = 9.50471 - 10$$

Satisfactory check

Answers (angles to nearest multiple of $5''$):

$$A = 36^\circ 12' 25'', \quad B = 53^\circ 47' 35'', \quad c = 0.39618.$$

Hereafter we shall give angles in answers to the nearest multiple of 5".

A rule generally observed in such computation is to use formulas involving the given data as far as possible, in preference to formulas which express a part of the solution in terms of another part of the solution. If we had followed this rule in Example 2, we should not have used the formula $c = \frac{a}{\sin A}$ which expresses the part c in terms of the solution for A ; for, if there had been a mistake in obtaining A , the solution c would then almost certainly be wrong. The formula $a^2 + b^2 = c^2$ would have been the one to use, according to the rule; this formula is not well adapted to logarithmic computation, however, and we have followed a method of solution here that uses only logarithmic tables.

In both Examples we have followed the rule that a check formula should be one not used in computing the solution, and should involve at least two parts of the solution.

EXERCISES

Make an outline similar to those of pages 89-90 for the logarithmic solution of the right triangles where the following parts are given ($C = 90^\circ$):

1. A and a .

2. B and c .

3. b and c .

Solve, by use of five-place logarithms, the right triangles whose given parts are as follows. Check the solutions. For each problem, prepare a complete outline, as illustrated on pages 89-90, before using the Tables. In all these exercises, $C = 90^\circ$. Write answers to the nearest multiple of 5".

4. $A = 39^\circ 30' 10''$, $a = 2670.5$.

12. $B = 18^\circ 38' 25''$, $c = 0.073462$.

5. $B = 61^\circ 30' 50''$, $b = 4320.4$.

13. $A = 72^\circ 14' 45''$, $c = 7.2975$.

6. $A = 43^\circ 23' 15''$, $a = 0.023570$.

14. $A = 84^\circ 22' 20''$, $b = 0.027325$.

7. $B = 19^\circ 27' 35''$, $b = 0.35244$.

15. $B = 79^\circ 44' 40''$, $c = 27.198$.

8. $a = 241.35$, $b = 415.05$.

16. $a = 1.7234$, $b = 3.2196$.

9. $c = 0.89236$, $b = 0.70254$.

17. $b = 271.42$, $c = 704.76$.

10. $a = 0.023413$, $c = 0.035672$.

18. $a = 0.17239$, $b = 0.092460$.

11. $a = 0.042347$, $c = 0.062309$.

19. $c = 27335$, $b = 20298$.

53. The problem of solving an oblique triangle. A triangle has six parts, its three sides a , b , c , and the three angles A , B , C respectively opposite those sides. In plane geometry we find that, if three parts of a triangle are given, the triangle can be constructed in each of the following Cases:

Case I. One side and two angles are given.

Case II. Two sides and an angle opposite one of them are given.

Case III. Two sides and the included angle are given.

Case IV. Three sides are given.

We shall see later that in Case II there are sometimes two triangles having given parts, but otherwise there is just one triangle.*

The problem of solving a triangle is that of finding by computation the unknown parts. One well-known formula which we may use is

$$A + B + C = 180^\circ.$$

Others will be derived in the following paragraphs as we prepare to solve in succession Cases I, II, III, IV.

54. The law of sines. In Figure 57 we have drawn three triangles ABC , in each of which h is the perpendicular distance from C to

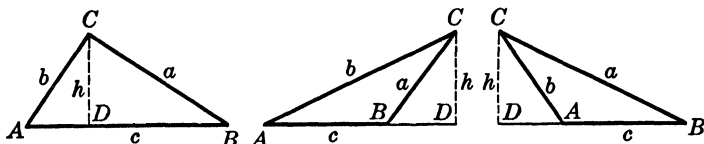


FIG. 57

AB (produced if necessary). These illustrate respectively the cases where angles A and B are both acute, A is acute and B obtuse, A is obtuse and B acute. In all three cases we have

$$\frac{h}{b} = \sin A, \quad \frac{h}{a} = \sin B.$$

It is readily verified that these formulas hold also when A or B is a right angle.

From the above equations it follows by division that

$$(1) \quad \frac{a}{b} = \frac{\sin A}{\sin B}.$$

By dropping a perpendicular from B to AC we could prove that

$$(2) \quad \frac{a}{c} = \frac{\sin A}{\sin C};$$

* It is possible, however, to give values to three of the letters a, b, c, A, B, C which would be impossible for parts of a triangle (for example, a value of a greater than the sum of b and c would be impossible for parts of a triangle since one side cannot exceed the sum of the other two).

the proof would be as before, with an interchange of letters. Similarly

$$(3) \quad \frac{b}{c} = \frac{\sin B}{\sin C}.$$

These three equations constitute the *law of sines*. They can be expressed as the one continued equation

$$(4) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

We shall use the law of sines to solve triangles in Cases I and II, but before doing so we shall derive check formulas which involve all six parts of a triangle and can be used in all the Cases.

55. Check formulas involving all six parts. From formulas (2) and (3) of the preceding section we obtain, by addition,

$$(1) \quad \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}.$$

The right-hand side may be transformed by means of the identities

$$\begin{aligned} (2) \quad \sin A + \sin B &= 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B), \quad (\text{by (5), p. 64}), \\ &= 2 \sin \frac{1}{2}(180^\circ - C) \cos \frac{1}{2}(A-B) \\ &= 2 \sin (90^\circ - \frac{1}{2}C) \cos \frac{1}{2}(A-B) \\ &= 2 \cos \frac{1}{2}C \cos \frac{1}{2}(A-B), \end{aligned}$$

$$(3) \quad \sin C = 2 \sin \frac{1}{2}C \cos \frac{1}{2}C, \quad (\text{by (1), p. 60}).$$

By substitution from (2) and (3), formula (1) becomes

$$(4) \quad \frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}.$$

Similarly

$$(5) \quad \frac{b+a}{c} = \frac{\cos \frac{1}{2}(B-A)}{\sin \frac{1}{2}C}.$$

All six parts appear in formulas (4) * and (5); hence these formulas are useful for checking solutions.

EXERCISES

Prove the following identities:

$$1. \quad \frac{a-b}{b} = \frac{2 \sin \frac{1}{2}C \sin \frac{1}{2}(A-B)}{\sin B}.$$

$$2. \quad \frac{b}{c} = \frac{\sin B}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C}.$$

* Formula (4) is sometimes called Newton's formula. The pair consisting of this formula and that of Exercise 3 of the following set are called Mollweide's equations.

$$3. \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}.$$

$$4. \frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin C}.$$

$$5. a = b \cos C + c \cos B.$$

Hint. $a : b : c = \sin A : \sin B : \sin C$; $A = 180^\circ - (B + C)$.

56. Case I. Given two angles and one side. The following Example illustrates the solution of a triangle in this Case. The Steps I, II, III, IV described in § 52 are again important. Step III is indicated below by boldface type. Details of the computation were carried out only after Step III was completed. It is recommended that the student use a similar plan for exercises which follow.

Example. Given $A = 32^\circ 24' 0''$, $C = 63^\circ 17' 0''$, $c = 32.345$. Find B , a , b .

Solution.

Estimates

$$B = 84^\circ \quad a = 19 \quad b = 36$$

Formulas

$$B = 180^\circ - (A + C)$$

$$a = \frac{c \sin A}{\sin C} \quad b = \frac{c \sin B}{\sin C}$$

$$\text{Check: } \frac{b+a}{c} = \frac{\cos \frac{1}{2}(B-A)}{\sin \frac{1}{2}C}$$

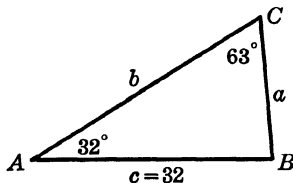


FIG. 58 (1 cm. = 10)

Computation

$A = 32^\circ 24'$	$\log c = 1.50980$	$\log c = 1.50980$
$C = 63^\circ 17'$	$(+) \log \sin A = 9.72902 - 10$	$(+) \log \sin B = 9.99786 - 10$
$A + C = 95^\circ 41'$	$\text{sum} = 11.23882 - 10$	$\text{sum} = 11.50766 - 10$
$B = 84^\circ 19'$	$(-) \log \sin C = 9.95097 - 10$	$(-) \log \sin C = 9.95097 - 10$
	$\log a = 1.28785$	$\log b = 1.55669$
	$a = 19.402$	$b = 36.032$

Check

$A = 32^\circ 24'$	$\log \cos \frac{1}{2}(B-A) = 9.95381 - 10$	$b+a = 55.434$
$B-A = 51^\circ 55'$	$(-) \log \sin \frac{1}{2}C = 9.71984 - 10$	$\log(b+a) = 1.74377$
$\frac{1}{2}(B-A) = 25^\circ 57' 30''$	$\text{diff.} = D = 0.23397$	$(-) \log c = 1.50980$
$\frac{1}{2}C = 31^\circ 38' 30''$		$\text{diff.} = D' = 0.23397$

Since $D = D'$, the work checks. We cannot always expect exact agreement.

Answers: $B = 84^\circ 19' 0''$, $a = 19.402$, $b = 36.032$.

EXERCISES

Solve and check the triangles whose given parts are as follows. Use five-place logarithms. Write answers to the nearest multiple of 5''.

- $A = 27^\circ 16' 10''$, $B = 74^\circ 29' 40''$, $a = 16000$.
- $A = 44^\circ 20' 12''$, $C = 55^\circ 42' 23''$, $c = 0.17285$.
- $B = 62^\circ 27' 35''$, $C = 42^\circ 54' 45''$, $a = 24.182$.
- $B = 29^\circ 42' 25''$, $C = 63^\circ 18' 50''$, $b = 70.796$.
- $A = 82^\circ 24' 36''$, $C = 60^\circ 5' 24''$, $a = 0.023478$.

6. $B = 79^\circ 33' 23''$, $C = 55^\circ 31' 12''$, $b = 0.072924$.
 7. $A = 66^\circ 15' 35''$, $B = 52^\circ 17' 45''$, $a = 192.41$.
 8. $A = 16^\circ 44' 50''$, $C = 82^\circ 3' 20''$, $b = 4.1292$.
 9. $A = 112^\circ 32' 48''$, $B = 33^\circ 41' 12''$, $b = 527.37$.
 10. $B = 45^\circ 37' 20''$, $C = 99^\circ 14' 20''$, $c = 129.62$.

11. An observer at C on a hillside measures the angles of depression of two points A and B in a horizontal plane below him. A and B are in the same direction from the observer and C , A , B are in the same vertical plane. The angle of depression of A is $36^\circ 28' 30''$, and that of B is $22^\circ 16' 0''$. If the distance from A to B is 4125.0 feet, find to five significant figures the distances of C from A and B .

12. For three points A , B , C , in a horizontal plane, the bearing of C from A is $N 34^\circ 27' 20'' E$, the bearing of C from B is $S 72^\circ 40' 40'' E$, and the bearing of B from A is $N 15^\circ 24' 30'' E$. The distance from A to B is 2450.5 yards. Find to five significant figures the distance from A to C .

57. Case II. Given two sides and an angle opposite one of them. The following six figures illustrate the possibilities when values are given to a , b , A as indicated:

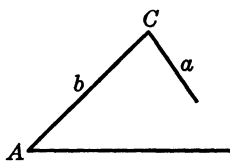


FIG. 59

$A < 90^\circ$, $a < b$
 $a < b \sin A$

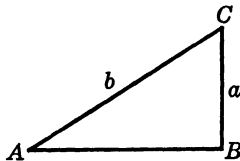


FIG. 60

$A < 90^\circ$, $a < b$
 $a = b \sin A$

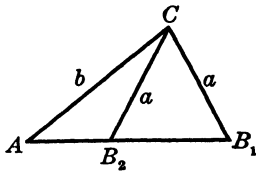


FIG. 61

$A < 90^\circ$, $a < b$
 $a > b \sin A$

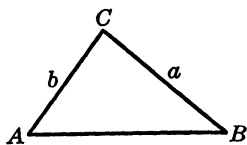


FIG. 62

$A < 90^\circ$, $a \geq b$

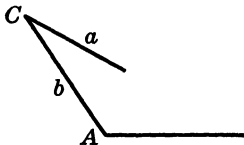


FIG. 63

$A \geq 90^\circ$, $a \leq b$

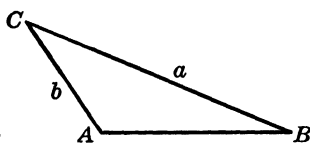


FIG. 64

$A \geq 90^\circ$, $a > b$

We see that solutions are determined as follows:

- No solution** when the data are as stated under Figures 59, 63.
- One solution only** when the data are as stated under Figures 60, 62, 64.
- Two solutions** when the data are as stated under Figure 61.

On account of the possibility (c), Case II is sometimes called the **ambiguous case**.

When the data correspond to Figure 63, the decision that there is no solution is immediate. If they correspond to Figure 62 or Figure 64 we solve as illustrated in Example 3 of the Examples which follow this paragraph. The subcases (where the data correspond to Figure 59 or Figure 61) are illustrated by Examples 1 and 2. Note that in these Examples we rely on the logarithmic computation to determine whether conclusion (a), (b), or (c) holds. In Figure 60, ABC is a right triangle.

Example 1. Given $A = 47^\circ 13' 50''$, $a = 0.20633$, $b = 0.70812$. Find B , C , c .

Solution.

Estimate

No solution

Formula

$$\sin B = \frac{b \sin A}{a}$$

(Only formula needed to show that there is no solution)

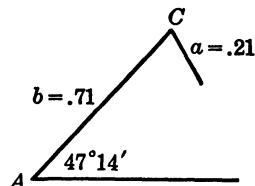


FIG. 65 (1 cm. = 0.25)

Computation

$$\begin{aligned} \log b &= 9.85010 - 10 \\ (+) \log \sin A &= 9.86575 - 10 \\ \text{sum} &= 19.71585 - 20 \\ (-) \log a &= 9.31456 - 10 \\ \log \sin B &= 10.40129 - 10 \\ \text{No such angle.} \end{aligned}$$

Example 2. Given $A = 47^\circ 13' 35''$, $a = 0.60631$, $b = 0.70815$. Find B , C , c .

Solution.

Estimates

Two solutions:

$B_1 = 60^\circ$	$C_1 = 73^\circ$	$c_1 = 0.8$
$B_2 = 120^\circ$	$C_2 = 12^\circ$	$c_2 = 0.17$

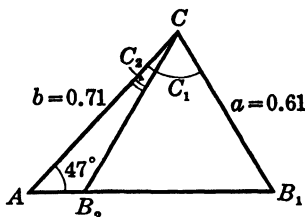


FIG. 66 (1 cm. = 0.25)

Formulas

$$\sin B = \frac{b \sin A}{a} \qquad C_1 = 180^\circ - (A + B_1) \qquad c_1 = \frac{a \sin C_1}{\sin A}$$

$$B_2 = 180^\circ - B_1 \qquad C_2 = 180^\circ - (A + B_2) \qquad c_2 = \frac{a \sin C_2}{\sin A}$$

$$\text{Check: } \frac{b+a}{c_1} = \frac{\cos \frac{1}{2}(B_1 - A)}{\sin \frac{1}{2}C_1} \qquad \frac{b+a}{c_2} = \frac{\cos \frac{1}{2}(B_2 - A)}{\sin \frac{1}{2}C_2}$$

Computation

$\begin{aligned} \log b &= 9.85013 - 10 \\ (+) \log \sin A &= \frac{9.86572 - 10}{19.71585 - 20} \\ (-) \log a &= \frac{9.78270 - 10}{9.93315 - 10} \\ \log \sin B &= \frac{9.93315 - 10}{9.93315 - 10} \\ B_1 &= 59^\circ 1' 8'' \\ (+) A &= 47^\circ 13' 35'' \\ A + B_1 &= 106^\circ 14' 43'' \\ C_1 &= 73^\circ 45' 17'' \\ \log a &= 9.78270 - 10 \\ (+) \log \sin C_1 &= \frac{9.98230 - 10}{19.76500 - 20} \\ (-) \log \sin A &= \frac{9.86572 - 10}{9.89928 - 10} \\ \log c_1 &= \frac{9.89928 - 10}{0.79302} \\ c_1 &= 0.79302 \end{aligned}$	$\begin{aligned} B_2 &= 120^\circ 58' 52'' \\ (+) A &= 47^\circ 13' 35'' \\ A + B_2 &= 168^\circ 12' 27'' \\ C_2 &= 11^\circ 47' 33'' \\ \log a &= 9.78270 - 10 \\ (+) \log \sin C_2 &= \frac{9.31041 - 10}{19.09311 - 20} \\ (-) \log \sin A &= \frac{9.86572 - 10}{9.22739 - 10} \\ \log c_2 &= \frac{9.22739 - 10}{0.16881} \\ c_2 &= 0.16881 \end{aligned}$
--	---

Check for triangle AB_1C_1

$\begin{aligned} B_1 - A &= 11^\circ 47' 33'' \\ \frac{1}{2}(B_1 - A) &= 5^\circ 53' 46'' \\ \frac{1}{2}C_1 &= 36^\circ 52' 38'' \\ \log \cos \frac{1}{2}(B_1 - A) &= 9.99769 - 10 \\ (-) \log \sin \frac{1}{2}C_1 &= \frac{9.77823 - 10}{0.21946} \\ \text{diff.} = D_1 &= \frac{0.21946}{0.21946} \end{aligned}$	$\begin{aligned} b &= 0.70815 \\ a &= 0.60631 \\ b + a &= 1.31446 \\ \log (b + a) &= 10.11875 - 10 \\ (-) \log c_1 &= \frac{9.89928 - 10}{0.21947} \\ \text{diff.} = D_1' &= \frac{0.21947}{0.21947} \end{aligned}$
--	---

Check for triangle AB_2C_2

$\begin{aligned} B_2 - A &= 73^\circ 45' 17'' \\ \frac{1}{2}(B_2 - A) &= 36^\circ 52' 38'' \\ \frac{1}{2}C_2 &= 5^\circ 53' 46'' \\ \log \cos \frac{1}{2}(B_2 - A) &= 9.90305 - 10 \\ (-) \log \sin \frac{1}{2}C_2 &= \frac{9.01168 - 10}{0.89137} \\ \text{diff.} = D_2 &= \frac{0.89137}{0.89137} \end{aligned}$	$\begin{aligned} \log (b + a) &= 10.11875 - 10 \\ (-) \log c_2 &= \frac{9.22739 - 10}{0.89136} \\ \text{diff.} = D_2' &= \frac{0.89136}{0.89136} \end{aligned}$
--	---

The checks are satisfactory.

Answers: $B_1 = 59^\circ 1' 10''$, $C_1 = 73^\circ 45' 15''$, $c_1 = 0.79302$;
 $B_2 = 120^\circ 58' 50''$, $C_2 = 11^\circ 47' 35''$, $c_2 = 0.16881$.

Example 3. Given $A = 132^\circ 47' 20''$, $a = 0.90635$, $b = 0.70810$. Find C .

Solution. Since $A > 90^\circ$ and $a > b$ there is but one solution (Figure 64). We shall see how this fact presents itself in the computation of C_1 and C_2 , following the plan of Example 2.

$\begin{aligned} \log b &= 9.85009 - 10 \\ (+) \log \sin A &= \frac{9.86561 - 10}{19.71570 - 20} \\ \text{sum} &= 19.71570 - 20 \\ (-) \log a &= \frac{9.95729 - 10}{9.75841 - 10} \\ \log \sin B &= \frac{9.75841 - 10}{9.75841 - 10} \end{aligned}$	$\begin{aligned} B_1 &= 34^\circ 59' 0'' \\ A &= 132^\circ 47' 20'' \\ A + B_1 &= 167^\circ 46' 20'' \\ C_1 &= 12^\circ 13' 40'' \end{aligned}$	$\begin{aligned} B_2 &= 145^\circ 1' 0'' \\ A &= 132^\circ 47' 20'' \\ A + B_2 &= 277^\circ 48' 20'' \\ C_2 &\text{ is impossible.} \end{aligned}$
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EXERCISES

Solve and check the triangles whose given parts are as follows. Use five-place tables. Write answers to the nearest multiple of 5".

1. $A = 42^\circ 37' 0''$, $a = 24.135$, $b = 20.270$.
2. $A = 78^\circ 23' 30''$, $a = 0.37190$, $b = 0.34215$.
3. $B = 95^\circ 41' 18''$, $a = 4594.3$, $b = 7234.4$.
4. $C = 107^\circ 55' 36''$, $a = 0.023566$, $c = 0.045978$.
5. $A = 40^\circ 5' 10''$, $a = 7003.2$, $c = 8104.7$.
6. $A = 42^\circ 11' 40''$, $a = 443.25$, $b = 502.76$.
7. $A = 54^\circ 13' 54''$, $a = 0.50079$, $b = 0.19950$.
8. $B = 24^\circ 25' 12''$, $b = 125.34$, $c = 267.85$.
9. $B = 73^\circ 32' 20''$, $a = 2.3196$, $b = 3.7214$.
10. $C = 62^\circ 18' 40''$, $b = 0.072381$, $c = 0.10495$.
11. $C = 65^\circ 23' 36''$, $b = 32.504$, $c = 30.333$.
12. $B = 53^\circ 17' 25''$, $b = 7231.4$, $c = 7952.7$.
13. $A = 109^\circ 20' 20''$, $a = 11.305$, $b = 17.402$.
14. $B = 98^\circ 15' 40''$, $b = 2725.7$, $c = 3010.3$.
15. $A = 121^\circ 27' 24''$, $a = 0.49236$, $c = 0.32584$.
16. $B = 97^\circ 44' 48''$, $b = 6.2375$, $c = 4.2191$.

17. Three points A, B, C in a horizontal plane are so situated that the bearing of B from A is $N 17^\circ 30' 0'' E$ and the bearing of C from A is $S 24^\circ 20' 20'' E$. If the length of AB is 3210.5 yards and that of BC is 4715.0 yards, find the bearing of B from C .

58. The law of tangents. In solving triangles under Case III we shall need a formula derived from the law of sines

$$(1) \quad \frac{a}{b} = \frac{\sin A}{\sin B}.$$

If we subtract 1 from each side of (1) we obtain

$$(2) \quad \frac{a - b}{b} = \frac{\sin A - \sin B}{\sin B},$$

and if we add 1 to each side of (1) we have

$$(3) \quad \frac{a + b}{b} = \frac{\sin A + \sin B}{\sin B}.$$

If we divide the sides of equation (2) by the corresponding sides of (3) we obtain

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

From formulas (6) and (5) of § 41 (p. 64), it follows that

$$\frac{a - b}{a + b} = \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)},$$

and hence

$$(4) \quad \frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

Formula (4) is known as the **law of tangents**. Since a and b are any two sides of the triangle, we may state the law of tangents thus:

In any triangle the difference of any two sides is to their sum as the tangent of one half the difference of the opposite angles is to the tangent of one half their sum.

There are thus five other formulas similar to (4); for example,

$$\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}, \quad \frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}.$$

59. Case III. Given two sides and the included angle. The method of solving triangles in Case III by using the law of tangents is illustrated in the following Example.

Example. Given $b = 0.60733$, $c = 0.49188$, $A = 43^\circ 12' 30''$. Find a , B , C .

Solution.

	<i>Estimates</i>	
$a = 0.42$	$B = 83^\circ$	$C = 54^\circ$

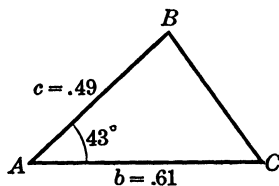


FIG. 67 (1 cm. = 0.2)

Formulas

$$B + C = 180^\circ - A$$

$$\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \tan \frac{1}{2}(B+C)$$

$$B = \frac{1}{2}(B+C) + \frac{1}{2}(B-C)$$

$$C = \frac{1}{2}(B+C) - \frac{1}{2}(B-C)$$

$$a = \frac{b \sin A}{\sin B}$$

$$\text{Check: } \frac{b+a}{c} = \frac{\cos \frac{1}{2}(B-A)}{\sin \frac{1}{2}C}$$

Computation

$$\begin{aligned} b &= 0.60733 \\ c &= 0.49188 \\ b-c &= 0.11545 \\ b+c &= 1.09921 \\ A &= 43^\circ 12' 30'' \\ B+C &= 136^\circ 47' 30'' \\ \frac{1}{2}(B+C) &= 68^\circ 23' 45'' \\ \frac{1}{2}(B-C) &= 14^\circ 51' 15'' \\ B &= 83^\circ 15' 0'' \\ C &= 53^\circ 32' 30'' \end{aligned}$$

$$\begin{aligned} \log(b-c) &= 9.06240 - 10 \\ (-) \log(b+c) &= 0.04108 \\ \text{diff.} &= 9.02132 - 10 \\ (+) \log \tan \frac{1}{2}(B+C) &= 10.40229 - 10 \\ \log \tan \frac{1}{2}(B-C) &= 9.42361 - 10 \\ \frac{1}{2}(B-C) &= 14^\circ 51' 15'' \\ \log b &= 9.78342 - 10 \\ (+) \log \sin A &= 9.83547 - 10 \\ \text{sum} &= 19.61889 - 20 \\ (-) \log \sin B &= 9.99698 - 10 \\ \log a &= 9.62191 - 10 \\ a &= 0.41871 \end{aligned}$$

Check

$b = 0.60733$	$B = 83^\circ 15' 0''$	$C = 53^\circ 32' 30''$
$(+) a = 0.41871$	$(-) A = 43^\circ 12' 30''$	$\frac{1}{2}C = 26^\circ 46' 15''$
$b + a = 1.02604$	$B - A = 40^\circ 2' 30''$	
$\log(b + a) = 10.01117 - 10$	$\frac{1}{2}(B - A) = 20^\circ 1' 15''$	
$(-) \log c = 9.69186 - 10$	$\log \cos \frac{1}{2}(B - A) = 9.97293 - 10$	
$\text{diff.} = D = 0.31931$	$(-) \log \sin \frac{1}{2}C = 9.65362 - 10$	
	$\text{diff.} = D' = 0.31931$	

We note that $D = D'$.*Answers:* $a = 0.41871$, $B = 83^\circ 15' 0''$, $C = 53^\circ 32' 30''$.**EXERCISES**

Solve and check the triangles whose given parts are as follows. Use five-place tables. Write answers to the nearest multiple of 5".

- $a = 32.705$, $b = 54.803$, $C = 41^\circ 56' 20''$.
- $a = 0.45235$, $b = 0.67413$, $C = 52^\circ 42' 40''$.
- $b = 1.7294$, $c = 2.7148$, $A = 108^\circ 29' 30''$.
- $b = 78542$, $c = 30314$, $A = 141^\circ 34' 50''$.
- $a = 5.0237$, $c = 4.7923$, $B = 29^\circ 23' 48''$.
- $a = 60235$, $c = 45921$, $B = 72^\circ 34' 24''$.
- $b = 0.071462$, $c = 0.042314$, $A = 123^\circ 24' 15''$.
- $b = 72.234$, $c = 23.275$, $A = 164^\circ 5' 20''$.
- $a = 7.0252$, $b = 2.3414$, $C = 75^\circ 28' 40''$.
- $a = 782.36$, $b = 924.27$, $C = 21^\circ 27' 50''$.
- Two sides of a parallelogram are 11.055 feet long, the other two sides are 13.267 feet long, and one interior angle is $72^\circ 15' 30''$. Find the lengths of the diagonals.
- A surveyor runs a line $N 35^\circ 30' 30'' E$ from A to B , the length of AB being 1246.5 feet. From B he runs a line $S 25^\circ 14' 0'' E$ to C , and measures BC as 1729.6 feet long. How long is AC ?

60. The law of cosines. The law of cosines is given by the formulas

- $a^2 = b^2 + c^2 - 2bc \cos A$,
- $b^2 = a^2 + c^2 - 2ac \cos B$,
- $c^2 = a^2 + b^2 - 2ab \cos C$.

We shall prove (1); an interchange of lettering in this proof would establish (2) and (3).

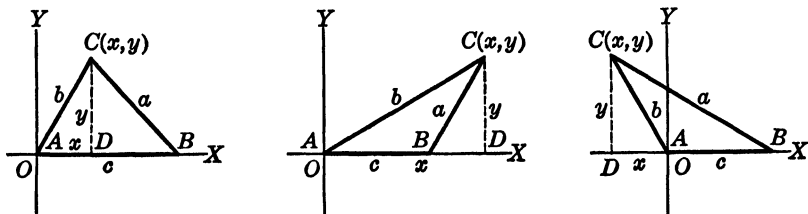


FIG. 68

We draw coördinate axes so that the origin is at A , and AB is on the positive x -axis. Three cases are illustrated in Figure 68. In all cases, if (x, y) are the coördinates of C , we have

$$\begin{aligned}a^2 &= y^2 + \overline{DB}^2, \\y^2 &= b^2 - x^2,\end{aligned}$$

and hence

$$(4) \quad a^2 = b^2 - x^2 + \overline{DB}^2.$$

The length of DB is either $c - x$ or $x - c$, so that

$$\overline{DB}^2 = c^2 - 2cx + x^2.$$

Thus equation (4) becomes

$$(5) \quad a^2 = b^2 + c^2 - 2cx.$$

Since in every case

$$x = b \cos A,$$

formula (1) follows.

Note that, if A is a right angle, (1) becomes $a^2 = b^2 + c^2$.

When the three sides a, b, c of a triangle are given (Case IV), formulas (1), (2), and (3) enable us to compute the angles. However, these formulas are poorly adapted for logarithmic computation; in the next section we derive other formulas better suited for this purpose.

61. The half-angle formulas. Formula (1) of § 60 may be written

$$(1) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Hence we have

$$\begin{aligned}1 - \cos A &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\&= \frac{a^2 - (b - c)^2}{2bc} \\&= \frac{(a + b - c)(a - b + c)}{2bc}.\end{aligned}$$

Similarly

$$1 + \cos A = \frac{(b + c - a)(b + c + a)}{2bc}.$$

If we substitute these expressions in the formula ((6a), p. 60)

$$\tan \frac{1}{2}A = \sqrt{\frac{1 - \cos A}{1 + \cos A}},$$

we obtain

$$(2) \quad \tan \frac{1}{2}A = \sqrt{\frac{(a-b+c)(a+b-c)}{(b+c-a)(a+b+c)}}.$$

Let $2s$ be the perimeter of the triangle; then

$$\begin{aligned} a+b+c &= 2s, & a-b+c &= 2s-2b, \\ b+c-a &= 2s-2a, & a+b-c &= 2s-2c. \end{aligned}$$

Formula (2) can now be written

$$\begin{aligned} \tan \frac{1}{2}A &= \sqrt{\frac{2(s-b) \cdot 2(s-c)}{2(s-a) \cdot 2s}} \\ &= \sqrt{\frac{(s-b)(s-c)}{(s-a)s}} \\ &= \frac{1}{s-a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \end{aligned}$$

or

$$(3) \quad \tan \frac{1}{2}A = \frac{r}{s-a},$$

where

$$(4) \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad s = \frac{a+b+c}{2}.$$

Similarly

$$(5) \quad \tan \frac{1}{2}B = \frac{r}{s-b}, \quad \tan \frac{1}{2}C = \frac{r}{s-c}.$$

We now show that r is the radius of the inscribed circle of the triangle ABC . In Figure 69, the center of this circle, O , is shown as the intersection of the bisectors of the angles. We have

$$\tan \frac{1}{2}A = \frac{OF}{AF}.$$

Since $AF = AE$, $CE = CD$, and $BF = BD$, the perimeter will be

$$\begin{aligned} 2s &= 2AF + 2CD + 2BD \\ &= 2AF + 2(CD + BD) \\ &= 2AF + 2a. \end{aligned}$$

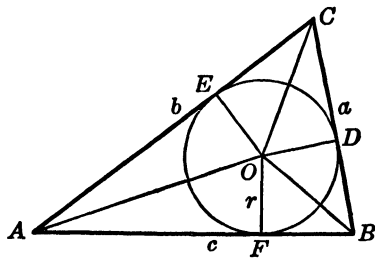


FIG. 69

Hence $AF = s - a$, and therefore

$$OF = AF \tan \frac{1}{2}A = (s-a) \frac{r}{s-a} = r.$$

62. Case IV. Given three sides. We carry out the logarithmic computation with the aid of formulas (3), (4), and (5) of § 61. We check the final results by the formula

$$A + B + C = 180^\circ.$$

A check on a part of the computation comes from the relation

$$(s - a) + (s - b) + (s - c) = 3s - (a + b + c) = s.$$

Example. Given $a = 250.32$, $b = 316.06$, $c = 287.42$. Find A , B , C .

Solution.

$$\begin{array}{lll} \text{Estimates} \\ A = 49^\circ & B = 72^\circ & C = 59^\circ \end{array}$$

Formulas

$$2s = a + b + c \quad r^2 = \frac{(s - a)(s - b)(s - c)}{s}$$

$$\tan \frac{1}{2}A = \frac{r}{s - a} \quad \tan \frac{1}{2}B = \frac{r}{s - b}$$

$$\tan \frac{1}{2}C = \frac{r}{s - c}$$

$$\text{Check: } (s - a) + (s - b) + (s - c) = s \quad A + B + C = 180^\circ$$

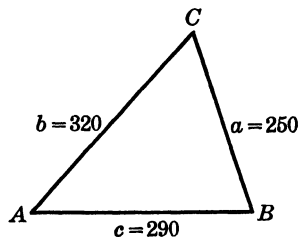


FIG. 70 (1 cm. = 100)

Computation

$$\begin{array}{lll} a = 250.32 & s - a = 176.58 & \log(s - a) = 2.24694 \\ b = 316.06 & s - b = 110.84 & (+) \log(s - b) = 2.04470 \\ c = 287.42 & s - c = 139.48 & (+) \log(s - c) = 2.14451 \\ 2s = 853.80 & \text{sum} = 426.90 & \text{sum} = 6.43615 \\ s = 426.90 & (\text{check}) & (-) \log s = 2.63033 \\ & & \log r^2 = 2 \log r = 3.80582 \end{array}$$

$$\begin{array}{l} \log r = 11.90291 - 10 \\ (-) \log(s - a) = 2.24694 \\ \log \tan \frac{1}{2}A = 9.65597 - 10 \\ \frac{1}{2}A = 24^\circ 21' 51'' \end{array}$$

$$\begin{array}{l} \log r = 11.90291 - 10 \\ (-) \log(s - b) = 2.04470 \\ \log \tan \frac{1}{2}B = 9.85821 - 10 \\ \frac{1}{2}B = 35^\circ 48' 31'' \end{array}$$

$$\begin{array}{l} \log r = 11.90291 - 10 \\ (-) \log(s - c) = 2.14451 \\ \log \tan \frac{1}{2}C = 9.75840 - 10 \\ \frac{1}{2}C = 29^\circ 49' 36'' \end{array}$$

$$A = 48^\circ 43' 42''$$

$$B = 71^\circ 37' 2''$$

$$C = 59^\circ 39' 12''$$

$$\text{Check: sum} = 179^\circ 59' 56''$$

Answers: $A = 48^\circ 43' 40''$, $B = 71^\circ 37' 0''$, $C = 59^\circ 39' 10''$.

EXERCISES

Solve and check the triangles whose given parts are as follows. Use five-place tables. Write answers to the nearest multiple of 5".

1. $a = 144.82$, $b = 132.65$, $c = 163.87$.

2. $a = 0.15790$, $b = 0.14462$, $c = 0.08213$.

3. $a = 9.2407$, $b = 15.065$, $c = 12.847$.

- | | | |
|---------------------|-----------------|-----------------|
| 4. $a = 31194,$ | $b = 36758,$ | $c = 43825.$ |
| 5. $a = 96.551,$ | $b = 56.859,$ | $c = 131.32.$ |
| 6. $a = 360.45,$ | $b = 300.20,$ | $c = 250.77.$ |
| 7. $a = 0.17042,$ | $b = 0.23105,$ | $c = 0.10097.$ |
| 8. $a = 3007.4,$ | $b = 1629.9,$ | $c = 1842.0.$ |
| 9. $a = 79.126,$ | $b = 62.767,$ | $c = 70.923.$ |
| 10. $a = 871.42,$ | $b = 1007.4,$ | $c = 1241.6.$ |
| 11. $a = 0.043256,$ | $b = 0.061232,$ | $c = 0.078944.$ |
| 12. $a = 0.079413,$ | $b = 0.091477,$ | $c = 0.065231.$ |

For the triangles whose given sides are as follows find the three angles by using the law of cosines (§ 60) and the Table of Four-Place Values of Functions and Radians (Tables, pages 4-8). Give solutions to as many significant figures as are used for the given sides.

- | | | |
|------------------|--------------|--------------|
| 13. $a = 20,$ | $b = 30,$ | $c = 40.$ |
| 14. $a = 50,$ | $b = 20,$ | $c = 60.$ |
| 15. $a = 6.20,$ | $b = 7.00,$ | $c = 9.10.$ |
| 16. $a = 34.00,$ | $b = 65.00,$ | $c = 76.00.$ |

63. Two formulas for the area of a triangle. Let S be the area of triangle ABC (Fig. 71). Then, since

$$S = \frac{hc}{2}, \quad h = b \sin A,$$

we have

$$(1) \quad S = \frac{1}{2}bc \sin A.$$

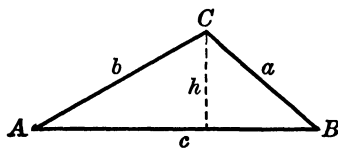


FIG. 71

This formula gives the area in terms of two sides and the included angle.

Another formula, often proved in texts on plane geometry, expresses S in terms of three sides as follows:

$$(2) \quad S = \sqrt{s(s-a)(s-b)(s-c)},$$

where

$$s = \frac{1}{2}(a + b + c).$$

Formula (2) is easily derived if we observe, from Figure 69, that S is the sum of the areas of three triangles, each of which has altitude r , and whose respective bases are a, b, c . Hence

$$S = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}r(a + b + c) = rs.$$

If we now substitute the value of r given by formula (4) of § 61, we have

$$S = s \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{s(s-a)(s-b)(s-c)}.$$

EXERCISES

1. Prove the formula

$$S = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin C \sin A}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}.$$

2. Prove that, if R is the radius of the circumscribed circle for triangle ABC , then

$$\frac{a}{2R} = \sin A;$$

hence

$$2R = \frac{a}{\sin A}.$$

3. Prove that, if S is the area of triangle ABC , and R is the radius of the circumscribed circle,

$$S = \frac{abc}{4R}.$$

Find the areas, to five significant figures, of the triangles which have the following given parts. Use the formula of Exercise 1 for triangles where one side and two angles are given.

- | | | |
|--------------------|----------------------------|----------------------------|
| 4. $a = 4.5625$, | $b = 3.1974$, | $c = 3.7286$. |
| 5. $a = 1729.3$, | $b = 4512.7$, | $c = 3621.9$. |
| 6. $b = 0.21230$, | $c = 0.40396$, | $A = 72^\circ 24' 5''$. |
| 7. $a = 70.827$, | $c = 90.265$, | $B = 42^\circ 32' 36''$. |
| 8. $b = 342.50$, | $A = 29^\circ 14' 10''$, | $C = 109^\circ 21' 45''$. |
| 9. $c = 47.231$, | $B = 120^\circ 20' 20''$, | $C = 40^\circ 25' 30''$. |
| 10. $a = 75.040$, | $b = 90.727$, | $B = 115^\circ 20' 0''$. |
| 11. $b = 6.3754$, | $c = 4.2378$, | $B = 97^\circ 44' 20''$. |

MISCELLANEOUS EXERCISES

Solve and check solutions for the following triangles, using five-place logarithms and writing answers to five significant figures; or show that no triangle can have the given parts:

- | | | |
|--------------------------|----------------------------|---------------------------|
| 1. $a = 327.04$, | $b = 431.01$, | $C = 27^\circ 30' 40''$. |
| 2. $b = 3.6312$, | $c = 6.3327$, | $A = 77^\circ 20' 35''$. |
| 3. $b = 1.7325$, | $A = 90^\circ$, | $B = 60^\circ$. |
| 4. $c = 245.03$, | $B = 90^\circ$, | $A = 30^\circ$. |
| 5. $a = 607.4$, | $c = 408.5$, | $C = 41^\circ 13'$. |
| 6. $b = 0.4035$, | $c = 0.376$, | $C = 58^\circ 23' 20''$. |
| 7. $b = 3.7125$, | $c = 7.6604$, | $B = 29^\circ 51' 5''$. |
| 8. $a = 724.02$, | $c = 2912.0$, | $A = 15^\circ 20'$. |
| 9. $b = 0.2714$, | $B = 7^\circ 40' 15''$, | $A = 74^\circ 20' 45''$. |
| 10. $c = 0.08535$, | $B = 112^\circ 40' 35''$, | $C = 42^\circ 10' 10''$. |
| 11. $a = 476$, | $B = 79^\circ 10'$, | $C = 112^\circ 30'$. |
| 12. $b = 4.0259$, | $A = 88^\circ 16' 10''$, | $B = 95^\circ 42' 20''$. |
| 13. $a = 315.25$, | $b = 415.75$, | $c = 615.12$. |
| 14. $a = 0.62451$, | $b = 0.31783$, | $c = 0.49116$. |
| 15. $A = 74^\circ 25'$, | $B = 62^\circ 13'$, | $C = 51^\circ 14'$. |

16. $A = 25^\circ 10'$,	$B = 75^\circ 16'$,	$C = 75^\circ 16'$.
17. $a = 82435$,	$c = 91257$,	$C = 123^\circ 27' 15''$.
18. $c = 4243.2$,	$b = 6012.4$,	$B = 99^\circ 14' 20''$.
19. $a = 237.01$,	$b = 373.05$,	$c = 625.18$.
20. $a = 4.0234$,	$b = 7.9165$,	$c = 3.80021$.
21. $a = 0.62455$,	$b = 0.31783$,	$c = 0.49112$.
22. $a = 0.71202$,	$b = 0.41526$,	$c = 0.42510$.
23. $a = 71.773$,	$c = 68.442$,	$B = 120^\circ 40' 20''$.
24. $b = 28.414$,	$c = 40.510$,	$A = 112^\circ 16' 25''$.
25. $a = 4.0727$,	$B = 134^\circ 25'$,	$C = 22^\circ 17'$.
26. $c = 7.0909$,	$B = 114^\circ 26'$,	$C = 31^\circ 5'$.

Find to five significant figures the areas of the triangles having given parts as follows:

27. $b = 324.03$,	$c = 215.60$,	$A = 129^\circ 15' 20''$.
28. $b = 8.0845$,	$c = 7.1727$,	$B = 106^\circ 25' 10''$.
29. $a = 16.241$,	$b = 13.212$,	$c = 27.427$.
30. $A = 18^\circ 20' 20''$,	$B = 110^\circ 45' 25''$,	$c = 36.202$.
31. $B = 75^\circ 35' 25''$,	$C = 62^\circ 12' 40''$,	$b = 0.88250$.
32. $a = 0.08888$,	$b = 0.09999$,	$c = 0.06666$.

Find the radii of the inscribed and of the circumscribed circles of the triangles having given parts as follows:

33. $a = 315.12$,	$b = 415.04$,	$c = 615.12$.
34. $a = 33.340$,	$b = 29.816$,	$c = 45.728$.
35. $b = 0.27150$,	$A = 74^\circ 20' 10''$,	$B = 7^\circ 40' 40''$.
36. $c = 0.085320$,	$B = 112^\circ 40' 0''$,	$C = 42^\circ 10' 55''$.
37. $b = 32.315$,	$c = 42.449$,	$B = 47^\circ 8' 15''$.
38. $a = 124.56$,	$c = 74.812$,	$A = 122^\circ 24' 35''$.
39. $a = 36.214$,	$c = 62.195$,	$B = 44^\circ 37' 0''$.
40. $b = 41972$,	$c = 67537$,	$A = 83^\circ 51' 20''$.

In the following problems, use five-place logarithms to get answers to five significant figures. Then write the answers to the number of significant figures appropriate for the implied accuracy of the data.

41. The radius of a circle is 3.15 in. Find the angle subtended at the center by a chord 1.74 in. long.

42. In a circle of radius 3.21 in., an angle of $29^\circ 30'$ is drawn with vertex at the center. What is the length of the chord for the intercepted arc?

43. In a circle a chord of length 3.4 in. subtends an angle of $31^\circ 10'$ at the center. Find the radius of the circle.

44. Two angles of a triangle are $71^\circ 10'$ and $65^\circ 30'$; the radius of the inscribed circle is 5.17 in. Find the lengths of the sides of the triangle.

45. The sides of a triangle are in the proportion 3 : 4 : 5; the area of the triangle is 108 sq. in. Find the radius of the inscribed circle.

46. An army officer observes the angle of elevation of an airplane to be $67^\circ 50'$, its distance to be 2197 yards. How high is it above the horizontal plane of the observer?

47. In tunneling under a river, a tunnel AB was first made at an angle of depression of $12^\circ 30'$, then a horizontal tunnel BC 610 ft. long, then a tunnel CD rising at an inclination of $12^\circ 30'$, the points A and D lying in a horizontal plane. If the maximum depth of the tunnel is 55 ft., how long is the tunnel and how far apart are A and D ? Assume that A, B, C, D lie in a vertical plane.

48. In going down in a mine, a man goes 576 ft. from A to B at a dip of $29^\circ 17'$, then 481 ft. from B to C at a dip of $68^\circ 29'$, and finally 359 ft. from C to D at a dip of $12^\circ 25'$. How far is D below A ? If the descent was always toward the south, how far south of A is D ? Assume that A, B, C, D lie in a vertical plane.

49. The latitude of the Dearborn Observatory is $42^\circ 3' 27.2''$ N. Assuming that the earth is a sphere of radius 3959 miles, find the length of the circle of latitude through this observatory. How far does the observatory move in a minute due to the earth's rotation on its axis once in 24 hours?

50. A sailor travels from A to B going 13.25 miles N 28° E, and then from B to C going 12.55 miles N 82° E, then from C to D going 14.35 miles S 15° E. He wishes then to go from D to A by the shortest route. What is the distance and the direction?

51. A flagpole 35 ft. tall stands on top of a corner of a building 54 ft. high. What angle does the flagpole subtend at a point lying in the horizontal plane at the foot of the building at a distance of 92 ft. from the building?

52. From a boat a tree on shore has a bearing of N $41^\circ 30'$ E. The boat travels due north for 10 minutes at 9 mi. per hour; then the bearing of the tree is N $72^\circ 40'$ E. How far east of the boat is the tree?

53. At a distance of 117 ft. from the foot of a building 63 ft. high a flagpole on top of the front wall of the building subtends an angle of $23^\circ 30'$. How tall is the flagpole?

54. Two observers A and B are 1.834 miles apart, B being due north from A . At a certain instant they observe an airplane in the northern sky; for A the angle of elevation is $67^\circ 31'$, and for B it is $82^\circ 16'$. How high is the airplane above the plane of the observers A and B ?

55. What is the radius of the largest gas tank that could be placed on a triangular lot whose sides are 84.027 ft., 77.526 ft., and 102.473 ft. long respectively?

56. Two circles whose radii are 3.8224 in. and 2.1916 in. long intersect at an angle of $18^\circ 35' 25''$. Find the distance between their centers and the length of their common chord.

57. Two chords from a point A on a circle are of length 3.4115 in. and 2.1897 in., and the angle between them is $96^\circ 10' 35''$. Find the radius of the circle.

58. The angles of a triangle are $36^\circ 20' 20''$, $79^\circ 30' 40''$, and $64^\circ 10' 0''$; the radius of the circumscribed circle is 2.2534 in. long. Find the lengths of the sides and the area of the triangle.

59. The hands of a clock are 2.250 ft. and 1.725 ft. long respectively. How far apart are their tips when the time is 2:35?

60. Two sides of a triangle are 187.3 and 218.4, and the angle between them is $151^\circ 18'$. Find the lengths of the segments into which the opposite side is divided by the bisector of this angle.

61. Two forces of 36.27 lb. and 18.83 lb. act at an angle of $32^\circ 30'$ with each other. What is the resultant force in magnitude and direction?

62. A boat sailed 2317 yards N 30° E, from A to B , then turned to the right

and sailed 1680 yards from B to C , then to the right again and sailed 1380 yards from C to A , its starting point. What was the bearing of BC and of CA ?

63. The distance from the sun, S , to the planet Venus, V , is 67,000,000 miles; and from S to the earth, E , is 92,830,000 miles. At a certain time an observer measured the angle SEV and found it to be $41^\circ 35'$. How far was Venus from the earth at that time?

64. The distance from the sun, S , to the earth, E , is 92,830,000 miles; and from the sun to the planet Mars, M , is 141,500,000 miles. An astronomer observes at a certain time that the angle SEM is $74^\circ 18'$. How far is Mars from the earth at that time?

65. A submarine is sailing $N 48^\circ 20' E$ at the rate of 21 miles per hour from a point A . A chaser is sailing $N 31^\circ 30' E$ at the rate of 32 miles per hour from a point B . The bearing of A from B is $N 38^\circ 25' W$ and the distance AB is 9.35 miles. How far apart will they be after 18 minutes and what will then be the bearing of the submarine from the chaser?

66. To find the height of a mountain peak, P , above a horizontal plane, two observers A and B , who were 2184 yards apart, made measurements as follows: At A the angle of elevation of P was $4^\circ 18'$, the bearing of P was $N 32^\circ 14' E$, and the bearing of B was $S 82^\circ 10' E$. At B the bearing of P was $N 22^\circ 12' E$. How high was the mountain?

67. Two buoys, A and B , on a lake are known to be 1210 yards apart, and one is due north of the other. An observer on a hilltop due north of the buoys observes with a range finder that the distance to A is 3240 yards and that the distance to B is 4350 yards. What is the elevation of the observer, to the nearest ten yards?

68. Two astronomers in the same longitude observe the zenith distance of the center of the moon when it crosses their meridian. Their difference of latitude is $92^\circ 14' 12''$; the observed zenith distances are: $A = 44^\circ 54' 21''$ and $B = 48^\circ 42' 57''$. Taking the earth's radius to be 3959 mi., find the distance from earth to moon.

CHAPTER VIII

SPHERICAL TRIGONOMETRY

64. Spherical triangles. A **great circle** on the surface of a sphere has certain properties that correspond to those of a straight line in a plane. A great circle, it will be recalled, is the intersection of the sphere with a plane through the center of the sphere. The **poles** of a great circle are the points in which a line through the center, perpendicular to the plane of the great circle, cuts the sphere.

Two great circles intersect in two points that are diametrically opposite. They divide the spherical surface into four parts, each of which is called a **lune**.

The angle CAB between two arcs AC and AB of great circles is, by definition, the angle $T'AT$ between the tangents to these arcs at A (Fig. 72). This angle also measures the corresponding dihedral angle between the planes of the great circles.

A **spherical triangle** ABC is a figure on the spherical surface bounded by three arcs of great circles. The planes of these great circles intersect at O , the center of the sphere. The dihedral angles of the trihedral angle $O-ABC$ (Fig. 72) are equal to the corresponding angles of the spherical triangle ABC , while the face angles AOB , BOC , COA measure the sides AB , BC , CA .

Consequently every theorem regarding spherical triangles corresponds to a theorem concerning trihedral angles.

It has been noted that the angle AOB measures the side AB of the spherical triangle ABC . The length of AB may be obtained by writing formula (1) of § 42 (p. 68) in the form $s = r\theta$ (θ measured in radians) when the angle AOB and the radius OA are known. It is customary, instead of giving the actual length of a side AB of a spherical triangle, to state the number of degrees, n , in angle AOB , and to write $AB = n^\circ$.

If a spherical triangle has a side or an angle greater than 180° , the

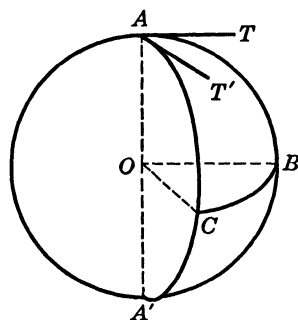


FIG. 72

great circles on which its sides lie will bound another spherical triangle whose sides and angles are all less than 180° . The sides and angles of the former triangle are easily determined when those of the latter are known. In case a side or an angle is equal to 180° the triangle degenerates into a lune. Hence *we shall hercafter consider only spherical triangles each of whose angles and sides is less than 180° .*

For spherical triangles thus defined, the following two propositions are proved in books on solid geometry:

The sum of the sides is less than 360° .

The sum of the angles is greater than 180° (and, of course, less than 540° since each angle is less than 180°).

65. Polar triangles. The vertices A, B, C of a spherical triangle are the poles of three great circles which bound various other spherical

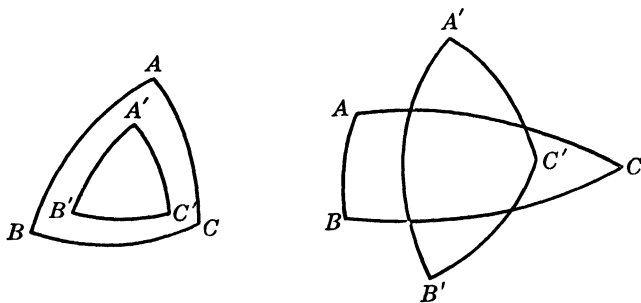


FIG. 73

triangles. Of these there is one, $A'B'C'$, where A is a pole of the circle on which $B'C'$ lies, or more briefly A is a pole of $B'C'$, B is a pole of $C'A'$, C is a pole of $A'B'$, such that A and A' are on the same side of BC , B and B' are on the same side of CA , and C and C' are on the same side of AB . This triangle $A'B'C'$ is the **polar triangle** of ABC (Fig. 73).

It is proved in solid geometry that if $A'B'C'$ is the polar triangle of ABC , then ABC is the polar triangle of $A'B'C'$. Further, if the sides opposite angles A, B, C of the triangle ABC are designated a, b, c , and those opposite angles A', B', C' in triangle $A'B'C'$ are designated a', b', c' , respectively, then the following important relations are true:

$$\begin{aligned} A &= 180^\circ - a', & B &= 180^\circ - b', & C &= 180^\circ - c', \\ a &= 180^\circ - A', & b &= 180^\circ - B', & c &= 180^\circ - C'. \end{aligned}$$

Each angle of a spherical triangle is the supplement of the corresponding side of the polar triangle; and each side is the supplement of the corresponding angle of the polar triangle.

EXERCISES

1. On a sphere of radius 2 ft. an arc AB 5 ft. long is drawn on a great circle. What is the angular measure of the arc?

2. Under the general definition of a spherical triangle, we may have one whose sides are $a = 120^\circ$, $b = 120^\circ$, $c = 300^\circ$. What are the lengths a_1 , b_1 , c_1 of the sides of the smallest spherical triangle bounded by arcs of the same great circles?

3. Under the general definition of a spherical triangle we may have one whose angles are $A = 90^\circ$, $B = 90^\circ$, $C = 270^\circ$. How does its area compare with that of the whole spherical surface?

4. A man travels 791.8 miles on the earth's surface due north from a point on the equator. What would his latitude be at the end of the journey if the earth were a sphere of radius 3959 miles?

RIGHT SPHERICAL TRIANGLES

66. Formulas for right triangles. In Chapter II we obtained certain relations between the sides and angles of a plane right triangle ABC with angle C equal to 90° . We now use a similar notation for a spherical right triangle ABC , where side a is opposite to angle A , b is the side opposite to angle B , and c is the side opposite to the right angle C . Since $A + B$ is greater than 90° for the spherical right triangle, instead of being equal to 90° as in the plane, the situation here is more complicated. We shall now list ten formulas, each of which contains a different three of the five parts a , b , c , A , B . The following section gives a convenient memorizing device, and § 68 contains proofs.

- | | |
|--------------------------------|---------------------------------|
| (1) $\sin a = \sin c \sin A$, | (2) $\sin a = \tan b \cot B$, |
| (3) $\sin b = \sin c \sin B$, | (4) $\sin b = \tan a \cot A$, |
| (5) $\cos A = \cos a \sin B$, | (6) $\cos A = \tan b \cot c$, |
| (7) $\cos B = \cos b \sin A$, | (8) $\cos B = \tan a \cot c$, |
| (9) $\cos c = \cos a \cos b$, | (10) $\cos c = \cot A \cot B$. |

67. Napier's rules. A device which makes it unnecessary to memorize the ten preceding formulas is due to Napier, who also invented logarithms. In Napier's two **rules of circular parts** for a spherical triangle ABC with right angle at C , we call a , b , $90^\circ - A$, $90^\circ - c$, $90^\circ - B$ the **circular parts**. The last three, which are complements of A , c , B , we designate A_c , c_c , B_c . If the five circular parts are marked on the triangle, or in the same order around a circle, as

in Figure 74, each part has two others **opposite** to it and two **adjacent** to it. Napier's rules are these:

1. The sine of any circular part is equal to the product of the cosines of the opposite parts. (The "cos-opp" rule.)

2. The sine of any circular part is equal to the product of the tangents of the adjacent parts. (The "tan-adj" rule.)

Of the ten formulas of the preceding section, those in the first column can be written down by using the "cos-opp" rule, those in

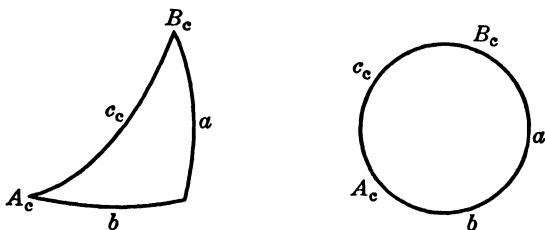


FIG. 74

the second column by using the "tan-adj" rule. These rules do not prove the formulas; they are merely a substitute for memorizing them.

To illustrate the use of the rules, let us obtain a formula which expresses a function of A in terms of functions of b and c . The corresponding circular parts are A_c and the adjacent parts b and c . Rule 2 for these parts gives $\sin A_c = \tan b \tan c$, and hence

$$\begin{aligned}\sin(90^\circ - A) &= \tan b \tan(90^\circ - c), \\ \cos A &= \tan b \cot c \quad (\text{formula (6), § 66}).\end{aligned}$$

68. Proofs of the formulas for right triangles. In Figures 75a and 75b the triangle ABC has a right angle at C , and OA , OB , OC are radii of the sphere on which ABC lies. The point P may be taken anywhere between O and B on OB ; the plane PRQ is perpendicular to the line OA and intersects the planes OAB , OBC , OAC in lines RP , QP , RQ respectively.

We shall now prove that each of the four faces of the pyramid $OPQR$ is a right triangle, the right angles being ORQ , ORP , OQP , PQR . We recall that the plane PRQ is perpendicular to line OA at point R , hence all lines in that plane that pass through R are perpendicular to OA , and angles ORQ and ORP are right angles. Further, the plane PQR is perpendicular to the plane OAC since it is perpendicular to the line OA of OAC , and the plane OBC is also

perpendicular to the plane OAC since C is a right angle. Hence planes PQR and OBC have their line of intersection PQ perpendicular to the plane OAC , and therefore to lines OC and QR ; in other words, angles OQP and PQR are right angles. We have thus shown that

$$(1) \quad \angle ORQ = \angle ORP = \angle OQP = \angle PQR = 90^\circ.$$

If the sides of triangle ABC and the angles they subtend at O are measured in degrees (or any other common unit), and if we ab-

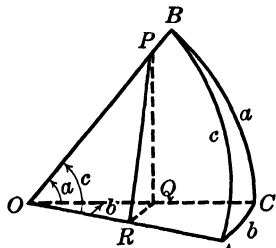


FIG. 75a

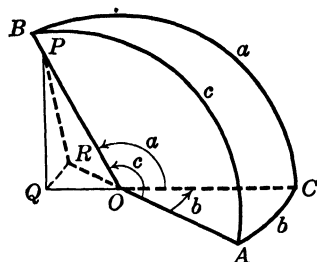


FIG. 75b

breviate by writing the symbol for an angle or side instead of the measure of that angle or side, we have

$$(2) \quad \angle COB = a, \quad \angle AOC = b, \quad \angle AOB = c.$$

Since RP and RQ are perpendicular to OA , the angle QRP measures a dihedral angle between the planes OAB and OAC ; hence it is equal either to angle A , as in Figure 75a, or to $180^\circ - A$, as in Figure 75b.

Let us first consider the case of triangles for which $a < 90^\circ$, $b < 90^\circ$, as in Figure 75a. Here Q must fall between O and C , and R between O and A . We have

$$(3) \quad \angle QRP = A, \quad \angle QOP = a, \quad \angle ROQ = b, \quad \angle ROP = c.$$

Hence, in view of equations (1),

$$(4) \quad \begin{aligned} \sin A &= \frac{QP}{RP} = \frac{QP}{OP} \div \frac{RP}{OP} = \frac{\sin a}{\sin c}, \\ \cos A &= \frac{RQ}{RP} = \frac{RQ}{OR} \div \frac{RP}{OR} = \frac{\tan b}{\tan c}, \\ \tan A &= \frac{QP}{RQ} = \frac{QP}{OQ} \div \frac{RQ}{OQ} = \frac{\tan a}{\sin b}, \\ \cos c &= \frac{OR}{OP} = \frac{OR}{OQ} \times \frac{OQ}{OP} = \cos b \cos a. \end{aligned}$$

These four formulas are equivalent to (1), (6), (4), and (9) of page 111.

If, instead of drawing a plane through a point of OB perpendicular to OA , we had drawn one through a point of OA perpendicular to OB , we should have obtained formulas similar to the above, but in which the pair A, a is interchanged with the pair B, b . Thus from the first three equations (4) of this section we would obtain three equivalent to (3), (8), (2) of page 111.

It remains to prove formulas (5), (7), (10) of page 111. If we multiply the left sides of (3), (9), (6) of that set and equate the result to the product of the right sides, we have

$$\begin{aligned}\sin b \cos c \cos A &= \sin c \sin B \cos a \cos b \tan b \cot c \\ &= \sin b \cos c \cos a \sin B,\end{aligned}$$

from which (5) follows ($\cos c$ is not zero when $a < 90^\circ$, $b < 90^\circ$). We obtain (7) from (5) by interchanging A, a with B, b . Finally, we derive (10) by multiplying the left sides of (2), (4), (9) and equating the result to the product of the right sides.

Figure 75b illustrates the case where $a > 90^\circ$, $b < 90^\circ$. Here, in equations (3) and (4) of the present section, we must replace A, a, c by $180^\circ - A, 180^\circ - a, 180^\circ - c$, but the last four of the resulting equations are equivalent to (4); hence the ten formulas of § 66 still hold. Similar remarks apply to cases where $b > 90^\circ$.

It remains only to consider cases where a or b is 90° . Here the angle opposite a 90° side is a right angle, and the triangle ABC is **birectangular** or **trirectangular**. The odd-numbered formulas of § 66 are still true; others may be meaningless because $\tan 90^\circ$ is impossible.

The formulas of § 66 hold in all cases where the functions in them have a meaning.

EXERCISES

1. In a birectangular spherical triangle, $B = 90^\circ$, $C = 90^\circ$, $A \neq 90^\circ$. Which of the formulas of § 66 are valid? Prove by means of these formulas that $b = c = 90^\circ$, and $a = A$.
2. Prove by means of the formulas of § 66 that in a trirectangular spherical triangle the sides are each equal to 90° . Prove the converse statement.
3. In a spherical triangle, $b = 90^\circ$, $C = 90^\circ$. Prove from the formulas of § 66 that the triangle is birectangular or trirectangular.
4. In a spherical triangle, $c = 90^\circ$, $C = 90^\circ$. What conclusion follows from each formula of § 66 that is not meaningless?
5. Prove formulas (4) of § 68 for the case $a > 90^\circ$, $b < 90^\circ$, illustrated in Figure 75b.
6. Draw a figure and prove formulas (4) of § 68 for the case $a > 90^\circ$, $b > 90^\circ$.
7. By using the relations of § 65 between the parts of a spherical right triangle

ABC and its polar triangle $A'B'C'$, show that $c' = 90^\circ$, and obtain from the formulas of § 66 ten formulas for a spherical triangle $A'B'C'$ in which $c' = 90^\circ$.

8. In a right spherical triangle on the earth's surface, $A = 30^\circ$, and the length of c is 100 miles. If the earth's radius, to two significant figures, is 4000 miles, find the length of a , in miles, from formula (1) of § 66 and compare it with the length obtained by treating the triangle as a plane triangle. Make a similar comparison when $c = 1000$ miles.

9. By use of Napier's rules, obtain formula (7) of § 66.

10. By use of Napier's rules, obtain formula (10) of § 66.

69. Rules regarding the relative magnitude of parts. In solving right spherical triangles certain propositions are convenient for the purpose of checking results. We quote first from solid geometry two theorems that are true for all spherical triangles, whether right-angled or not:

1. *The sum of any two sides of a spherical triangle is greater than the third side.*

2. *If two angles of a spherical triangle are equal, the opposite sides are equal, and conversely. If two angles are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle. The converse is also true.*

In addition we have the two so-called **rules for species**, their name arising from the fact that they are concerned with the question of whether sides or angles are of the same or different *species*, that is, whether they terminate in the same quadrant or in different quadrants. We shall first state these rules, then prove them.

3. *In a right spherical triangle ABC with right angle at C , the angle A is of the same species as side a (both terminate in the same quadrant); and B is of the same species as b .*

4. *In a right spherical triangle ABC with right angle at C , if $c < 90^\circ$, then a and b are of the same species; if $c > 90^\circ$, then a and b are of different species.*

The proof of rule 3 follows from formulas (5) and (7) of § 66. Thus by formula (5), since $\sin B$ is positive, $\cos A$ and $\cos a$ cannot be of opposite sign, hence A and a cannot terminate in different quadrants. Rule 4 follows from formula (9) of § 66; for if $c < 90^\circ$, then $\cos c$ is positive and $\cos a$ and $\cos b$ are of like sign, so that a and b are of the same species; if $c > 90^\circ$, then $\cos c$ is negative, $\cos a$ and $\cos b$ are of unlike sign, and a and b are of different species.

70. Solution of right spherical triangles. If values for any two parts other than the right angle are given, we use equations (1)–(10) of § 66 to compute the three unknown parts. There may be **no solution** (for instance, when a rule of § 69 is violated), **just one set of solutions**, or **just two sets of solutions** (this can occur only when the given parts are a and A , or b and B). To each possible set of solutions of the equations an actual triangle corresponds, and the only possible triangles with the given parts correspond to solutions of the equations.

The student should prepare a complete outline of the computation before turning to the Tables.

To check solutions we use a formula involving the three parts that were not given.

In each of the following Examples, we shall, by use of Napier's rules, find the three formulas needed, each of which contains one unknown and two known parts. If a formula contains a part greater than 90° , we shall modify the formula so that it applies to the supplement of that part. This is done because our tables do not extend beyond 90° and because we must avoid minus quantities when using logarithms (negative numbers have no real logarithms). If θ is a part, a convenient notation for its supplement is θ_s ;

$$\theta_s = 180^\circ - \theta$$

(just as, with Napier's rules, we used θ_s for the complement of θ). From the formulas of page 39, we have

$$\begin{aligned} \sin \theta_s &= \sin \theta, & \cos \theta_s &= -\cos \theta, & \tan \theta_s &= -\tan \theta, \\ & & \cot \theta_s &= -\cot \theta. \end{aligned}$$

As an example, suppose given parts are b and A , both greater than 90° , so that we are to solve for a , c , B by using formulas (4), (6), and (7) of § 66, so modified as to contain b_s and A_s and to avoid minus signs. Formula (4) is

$$\sin b = \tan a \cot A,$$

and, in terms of b_s , A_s , this becomes

$$\sin b_s = \tan a (-\cot A_s).$$

To avoid the minus sign, we substitute $-\tan a_s$ for $\tan a$; this gives us

$$\sin b_s = \tan a_s \cot A_s.$$

Similarly, we obtain from (6) and (7)

$$\begin{aligned}\cos A_e &= \tan b_e \cot c, \\ \cos B_e &= \cos b_e \sin A_e.\end{aligned}$$

A useful rule is the following:

In any of the formulas of § 66 we can without change of sign replace the sine of a part by the sine of the supplement of that part; and we can replace by their supplements any two parts for each of which the cosine, tangent, or cotangent appears in the formula.

In the following examples, the -10 that should follow logarithms of functions will be omitted, for brevity.

Example 1. Given $c = 72^\circ 10' 45''$, $B = 30^\circ 43' 0''$. Find a , b , A .

Solution.

Formulas

$$\begin{array}{lll}\sin B_e = \tan a \tan c_e & \sin b = \cos c_e \cos B_e & \sin c_e = \tan A_e \tan B_e \\ \cos B = \tan a \cot c & \sin b = \sin c \sin B & \cos c = \cot A \cot B \\ \tan a = \tan c \cos B & & \cot A = \cos c \tan B\end{array}$$

$$\begin{aligned}\text{Check: } \sin b &= \tan a \tan A_e \\ \sin b &= \tan a \cot A\end{aligned}$$

Computation

$$\begin{array}{lll}\log \tan c = 10.49286 & \log \sin c = 9.97865 & \log \cos c = 9.48578 \\ (+) \log \cos B = \underline{9.93435} & (+) \log \sin B = \underline{9.70824} & (+) \log \tan B = \underline{9.77390} \\ \log \tan a = \underline{10.42721} & \log \sin b = \underline{9.68689} & \log \cot A = \underline{9.25968} \\ a = 69^\circ 29' 52'' & b = 29^\circ 5' 47'' & A = 79^\circ 41' 39''\end{array}$$

Check

$$\begin{aligned}\log \tan a &= 10.42721 \\ (+) \log \cot A &= \underline{9.25968} \\ \log \sin b &= 9.68689 & \text{sum} = \underline{9.68689}\end{aligned}$$

Answers (to the nearest multiple of $5''$):

$$a = 69^\circ 29' 50'', \quad b = 29^\circ 5' 45'', \quad A = 79^\circ 41' 40''.$$

Here the equation for $\log \sin b$ would also admit the additional solution $b = 180^\circ - 29^\circ 5' 47''$, but this must be rejected, since it is not of the same species as B (Rule 3, § 69).

The check we have used tests only the correctness of the logarithms involved. We might have used the following modified form of our check formula:

$$\sin b \cot a \tan A = 1.$$

Example 2. Given $a = 156^\circ 3' 0''$, $A = 154^\circ 44' 20''$ (an example of an "ambiguous case" where there are two sets of solutions). Find b , c , B .

Solution.

Formulas

$$\begin{array}{lll} \sin b = \tan a \tan A_c & \sin a = \cos c_c \cos A_c & \sin A_c = \cos a \cos B_c \\ \sin b = \tan a \cot A & \sin a = \sin c \sin A & \cos A = \cos a \sin B \\ \left. \begin{array}{l} \sin b_j \\ \sin b_s \end{array} \right\} = \tan a_c \cot A_s & \left. \begin{array}{l} \sin c \\ \sin c_s \end{array} \right\} = \frac{\sin a_s}{\sin A_s} & \left. \begin{array}{l} \sin B \\ \sin B_s \end{array} \right\} = \frac{\cos A_s}{\cos a_s} \end{array}$$

$$\text{Check: } \sin b = \cos c_c \cos B_c$$

$$\sin b = \sin c \sin B$$

Computation

$$a_s = 23^\circ 57' 0'' \quad A_s = 25^\circ 15' 40''$$

$$\begin{array}{lll} \log \tan a_s = 9.64756 & \log \sin a_s = 9.60846 & \log \cos A_s = 9.95635 \\ (+) \log \cot A_s = 10.32618 & (-) \log \sin A_s = 9.63017 & (-) \log \cos a_s = 9.96090 \\ \log \sin b = 9.97374 & \log \sin c = 9.97829 & \log \sin B = 9.99545 \end{array}$$

$$\begin{array}{lll} b = 70^\circ 16' 30'' & c = 72^\circ 2' 0'' & B = 81^\circ 43' 0'' \\ b_s = 109^\circ 43' 30'' & c_s = 107^\circ 58' 0'' & B_s = 98^\circ 17' 0'' \end{array}$$

The rules for species allow two sets of solutions. We shall not check here.

$$\text{Answers: } b_1 = 70^\circ 16' 30'', \quad c_1 = 107^\circ 58' 0'', \quad B_1 = 81^\circ 43' 0'';$$

$$b_2 = 109^\circ 43' 30'', \quad c_2 = 72^\circ 2' 0'', \quad B_2 = 98^\circ 17' 0''.$$

EXERCISES

Obtain by using Napier's rules formulas for the solution of the right spherical triangles of which the following are given parts, using supplements of parts greater than 90° ; write a complete outline for logarithmic computation, but do not solve:

- | | | | |
|----------------------|-------------------|-----------------------|-------------------|
| 1. $a = 40^\circ$, | $b = 25^\circ$. | 6. $b = 70^\circ$, | $A = 135^\circ$. |
| 2. $a = 40^\circ$, | $c = 120^\circ$. | 7. $b = 75^\circ$, | $B = 85^\circ$. |
| 3. $b = 70^\circ$, | $c = 106^\circ$. | 8. $c = 80^\circ$, | $A = 110^\circ$. |
| 4. $a = 160^\circ$, | $A = 150^\circ$. | 9. $c = 120^\circ$, | $B = 135^\circ$. |
| 5. $a = 100^\circ$, | $B = 50^\circ$. | 10. $A = 100^\circ$, | $B = 100^\circ$. |

Solve the right spherical triangles of which the following parts are given. Check each set of answers. Find both sets of answers in case there are two. Write answers to the nearest multiple of $5''$.

- | | | | |
|--------------------------------|----------------------------|--------------------------------|----------------------------|
| 11. $c = 56^\circ 55' 0''$, | $A = 18^\circ 43' 30''$. | 21. $b = 14^\circ 59' 35''$, | $A = 75^\circ 21' 55''$. |
| 12. $c = 102^\circ 2' 30''$, | $B = 65^\circ 51' 0''$. | 22. $a = 102^\circ 35' 25''$, | $B = 112^\circ 13' 50''$. |
| 13. $a = 35^\circ 40' 15''$, | $A = 35^\circ 40' 15''$. | 23. $b = 47^\circ 18' 45''$, | $c = 95^\circ 48' 30''$. |
| 14. $b = 127^\circ 17' 45''$, | $B = 123^\circ 40' 15''$. | 24. $a = 48^\circ 54' 55''$, | $c = 50^\circ 2' 55''$. |
| 15. $A = 143^\circ 50' 20''$, | $B = 73^\circ 7' 40''$. | 25. $a = 29^\circ 11' 0''$, | $b = 121^\circ 23' 35''$. |
| 16. $a = 134^\circ 31' 40''$, | $b = 40^\circ 42' 20''$. | 26. $A = 72^\circ 49' 50''$, | $B = 125^\circ 13' 25''$. |
| 17. $a = 43^\circ 18' 30''$, | $c = 47^\circ 34' 30''$. | 27. $b = 77^\circ 21' 50''$, | $B = 83^\circ 56' 40''$. |
| 18. $b = 44^\circ 29' 0''$, | $c = 109^\circ 52' 30''$. | 28. $a = 140^\circ 16' 5''$, | $A = 101^\circ 59' 20''$. |
| 19. $b = 113^\circ 22' 15''$, | $A = 19^\circ 29' 0''$. | 29. $c = 67^\circ 2' 50''$, | $B = 110^\circ 39' 40''$. |
| 20. $a = 130^\circ 28' 30''$, | $B = 165^\circ 1' 0''$. | 30. $c = 36^\circ 21' 35''$, | $A = 58^\circ 59' 40''$. |

71. Isosceles triangles. Quadrantal triangles. Rule 2 of § 69 states that the base angles of an isosceles triangle are equal. The great circle joining the vertex to the mid-point of the base bisects the angle at the vertex and divides the isosceles triangle into two symmetrical right triangles. The solution of isosceles triangles can therefore be made to depend upon that of right triangles. The given parts may be any two of the following four: base, base angle, side, angle at vertex.

A **quadrantal triangle** is one of which a side is equal to 90° . If this side is lettered c then the polar triangle will be right-angled at C' , since $C' = 180^\circ - c = c_s$, according to a relation given in § 65 (p. 110). If two parts other than c are given, their supplements are parts of the polar triangle. When we have solved the latter triangle, the supplements of sides or angles found will be angles or sides of the original triangle. Thus if c, a, B are given parts, with $c = 90^\circ$, we obtain for the polar triangle

$$C' = 90^\circ, \quad A' = a_s, \quad b' = B_s.$$

We should then solve for c', a', B' , and finally obtain C, A, b from the relations

$$C = c_s', \quad A = a_s', \quad b = B_s'.$$

EXERCISES

Solve the oblique spherical triangles of which the following are given parts. Note that Exercises 4, 8, 11, 15 have two sets of solutions. Write answers to the nearest multiple of 5".

1. $a = b = 47^\circ 32' 0''$, $c = 52^\circ 50' 0''$.
2. $a = b = 69^\circ 54' 0''$, $C = 54^\circ 24' 0''$.
3. $c = 86^\circ 22' 30''$, $A = B = 167^\circ 22' 30''$.
4. $c = 153^\circ 32' 30''$, $C = 166^\circ 30' 30''$, $a = b$.
5. $c = 144^\circ 43' 20''$, $A = B = 28^\circ 49' 55''$.
6. $a = b = 117^\circ 3' 0''$, $A = 106^\circ 34' 10''$.
7. $A = B = 100^\circ 10' 55''$, $C = 60^\circ 46' 10''$.
8. $c = 44^\circ 30' 20''$, $C = 54^\circ 57' 15''$, $a = b$.
9. $c = 90^\circ$, $a = 53^\circ 7' 0''$, $b = 132^\circ 26' 0''$.
10. $c = 90^\circ$, $a = 40^\circ 56' 0''$, $C = 116^\circ 13' 0''$.
11. $c = 90^\circ$, $b = 96^\circ 45' 30''$, $B = 103^\circ 14' 30''$.
12. $c = 90^\circ$, $a = 142^\circ 4' 40''$, $B = 22^\circ 48' 20''$.
13. $c = 90^\circ$, $a = 149^\circ 8' 45''$, $b = 108^\circ 24' 0''$.
14. $c = 90^\circ$, $A = 53^\circ 14' 5''$, $C = 125^\circ 47' 50''$.
15. $c = 90^\circ$, $a = 96^\circ 3' 20''$, $A = 102^\circ 38' 20''$.
16. $c = 90^\circ$, $A = 48^\circ 39' 15''$, $B = 120^\circ 31' 35''$.

OBLIQUE SPHERICAL TRIANGLES

72. The law of sines. There are laws of sines and of cosines for spherical triangles; the reader should compare their form and their derivation with those of the corresponding laws for plane triangles (see § 54, pp. 92-93, and § 60, pp. 100-101).

The **law of sines** may be stated as follows:

In every spherical triangle the sines of the sides are proportional to the sines of the corresponding opposite angles. This may be written

$$(1) \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

The proof is analogous to that of § 54. We here draw the great circle arc CD , whose angular measure is h , perpendicular to AB .

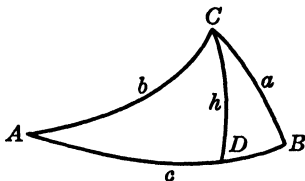


FIG. 76a

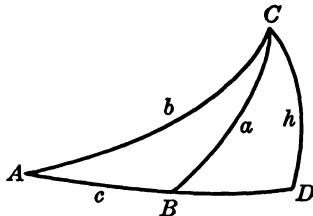


FIG. 76b

Figure 76a illustrates the case where D is between A and B , Figure 76b a case where D is outside AB .

In both cases, formula (1) of § 66 applied to the right triangles ADC and BDC gives *

$$(2) \quad \sin h = \sin b \sin A, \quad \sin h = \sin a \sin B.$$

Hence

$$\sin a \sin B = \sin b \sin A,$$

or

$$(3) \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

If D coincides with A or B , the triangle ABC is a right triangle, and formula (3) here corresponds to either formula (1) or (3) of § 66.

We can show similarly the equality of the last two expressions in (1), and thus complete the proof of that formula.

* For Figure 76b, the angle at B in BDC is not B , but B_1 . However, since $\sin B_1 = \sin B$, the second of formulas (2) is still true.

73. The laws of cosines. The first law of cosines is given by the formulas

$$\begin{aligned}(1) \quad & \cos a = \cos b \cos c + \sin b \sin c \cos A, \\(2) \quad & \cos b = \cos c \cos a + \sin c \sin a \cos B, \\(3) \quad & \cos c = \cos a \cos b + \sin a \sin b \cos C.\end{aligned}$$

The second law of cosines is given by the formulas

$$\begin{aligned}(4) \quad & \cos A = -\cos B \cos C + \sin B \sin C \cos a, \\(5) \quad & \cos B = -\cos C \cos A + \sin C \sin A \cos b, \\(6) \quad & \cos C = -\cos A \cos B + \sin A \sin B \cos c.\end{aligned}$$

To prove formula (1) we again make use of Figures 76a, 76b, and begin by applying formula (9) of § 66 to the two right triangles BDC and ADC . This gives

$$\begin{aligned}(7) \quad & \cos a = \cos h \cos DB, \\(8) \quad & \cos b = \cos h \cos AD.\end{aligned}$$

In Figure 76a, DB is $c - AD$; and in Figure 76b, DB is $AD - c$. Since $\cos \theta = \cos (-\theta)$, we have in both cases

$$(9) \quad \cos DB = \cos (c - AD).$$

If we solve (8) for $\cos h$ and substitute the result in (7), we have

$$\cos a = \frac{\cos b}{\cos AD} \cos DB;$$

and if we use (9) this becomes

$$\cos a = \cos b \frac{\cos (c - AD)}{\cos AD} = \cos b \frac{\cos c \cos AD + \sin c \sin AD}{\cos AD},$$

hence

$$(10) \quad \cos a = \cos b \cos c + \cos b \sin c \tan AD.$$

Formula (6) of § 66, applied to the right triangle ADC , gives

$$\cos A = \tan AD \cot b$$

or

$$\tan AD = \tan b \cos A;$$

hence for the last term in (10) we have

$$\cos b \sin c \tan AD = \cos b \sin c \tan b \cos A = \sin b \sin c \cos A,$$

and with this substitution (10) reduces to (1).

The proof of formulas (2) and (3) is similar. The proofs of the three formulas must be modified if ABC is a right triangle or if $AD = 90^\circ$, but the formulas remain true in all cases.

Formula (4) is proved by reference to the polar triangle $A'B'C'$ and use of the relations

$$a' = 180^\circ - A = A_s, \quad b' = B_s, \quad c' = C_s, \quad A' = a_s.$$

Thus we have

$$\begin{aligned}\cos a' &= \cos b' \cos c' + \sin b' \sin c' \cos A', \\ \cos A_s &= \cos B_s \cos C_s + \sin B_s \sin C_s \cos a_s, \\ -\cos A &= (-\cos B)(-\cos C) + \sin B \sin C (-\cos a), \\ \cos A &= -\cos B \cos C + \sin B \sin C \cos a.\end{aligned}$$

The proofs for formulas (5) and (6) are similar.

EXERCISES

1. Show that if havers θ has the meaning given it on page 8, the relation

$$\text{hav} a = \text{hav} (b - c) + \sin b \sin c \text{ hav} A$$

is equivalent to formula (1) of § 73.

2. Show that if havers θ has the meaning given it on page 8, the relation

$$\text{hav} A_s = \text{hav} (B + C) - \sin B \sin C \text{ hav} a$$

is equivalent to formula (4) of § 73.

Solve the following problems without using logarithms:

3. Given $a = 40^\circ$, $A = 30^\circ$, $c = 25^\circ$, find C .
4. Given $b = 50^\circ$, $B = 60^\circ$, $a = 45^\circ$, find A .
5. Given $A = 30^\circ$, $C = 130^\circ$, $c = 70^\circ$, find a .
6. Given $B = 135^\circ$, $C = 65^\circ$, $b = 150^\circ$, find c .
7. Given $b = 45^\circ$, $c = 55^\circ$, $A = 50^\circ$, find a .
8. Given $a = 120^\circ$, $b = 70^\circ$, $C = 40^\circ$, find c .
9. Given $B = 85^\circ$, $C = 65^\circ$, $a = 60^\circ$, find A .
10. Given $A = 125^\circ$, $C = 110^\circ$, $b = 120^\circ$, find B .
11. Given $a = 45^\circ$, $b = 55^\circ$, $C = 55^\circ$, find c , A , B .
12. Given $A = 60^\circ$, $B = 95^\circ$, $c = 130^\circ$, find C , a , b .

74. Formulas for half-angles and half-sides. For the half-angle formulas of spherical trigonometry we shall use notations similar to those of § 61 for plane triangles. Thus we shall write

$$(1) \quad 2s = a + b + c,$$

$$(2) \quad r = \sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}}.$$

Then the half-angle formulas are

$$(3) \quad \tan \frac{A}{2} = \frac{r}{\sin(s-a)}, \quad \tan \frac{B}{2} = \frac{r}{\sin(s-b)}, \quad \tan \frac{C}{2} = \frac{r}{\sin(s-c)}.$$

It will suffice to prove the first of these formulas. We begin by solving formula (1) of the preceding section for $\cos A$ as follows:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

We shall substitute this value of $\cos A$ in formula (6a) of § 40, using the positive sign before the radical, since $A < 180^\circ$. Thus we have

$$(4) \quad \tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}},$$

where

$$\begin{aligned} 1 - \cos A &= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c} \\ &= \frac{\cos (b - c) - \cos a}{\sin b \sin c}. \quad (\text{by formula (4), p. 52}). \end{aligned}$$

Formula (8) of page 64 gives

$$\begin{aligned} \cos (b - c) - \cos a &= -2 \sin \frac{b - c + a}{2} \sin \frac{b - c - a}{2} \\ &= 2 \sin \frac{a + b - c}{2} \sin \frac{a - b + c}{2}. \end{aligned}$$

And from formula (1) of this section we have

$$\frac{a + b - c}{2} = s - c, \quad \frac{a - b + c}{2} = s - b.$$

Hence we may write

$$(5) \quad 1 - \cos A = \frac{2 \sin (s - c) \sin (s - b)}{\sin b \sin c}.$$

Similarly

$$(6) \quad 1 + \cos A = \frac{2 \sin s \sin (s - a)}{\sin b \sin c}.$$

From (4), (5), and (6) we have

$$\begin{aligned} (7) \quad \tan \frac{A}{2} &= \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}} \\ &= \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s \sin^2 (s - a)}}, \end{aligned}$$

which reduces to the first of formulas (3).

In the half-side formulas we use the notations

$$(8) \quad 2S = A + B + C,$$

$$(9) \quad R = \sqrt{\frac{\cos(S-A) \cos(S-B) \cos(S-C)}{-\cos S}}.$$

The half-side formulas are

$$(10) \quad \cot \frac{a}{2} = \frac{R}{\cos(S-A)}, \quad \cot \frac{b}{2} = \frac{R}{\cos(S-B)}, \quad \cot \frac{c}{2} = \frac{R}{\cos(S-C)}.$$

We prove the first of formulas (10), for example, by applying the first of formulas (3) to the polar triangle $A'B'C'$, which gives us the formula

$$(11) \quad \tan \frac{A'}{2} = \frac{r'}{\sin(s' - a')}.$$

We must now write this in terms of the sides and angles of triangle ABC .

We have

$$(12) \quad \begin{aligned} s' &= \frac{1}{2}(a' + b' + c') = \frac{1}{2}(A_s + B_s + C_s) \\ &= \frac{1}{2}[540^\circ - (A + B + C)] = 270^\circ - S. \end{aligned}$$

Similarly

$$(13) \quad \begin{aligned} s' - a' &= 90^\circ - (S - A), & s' - b' &= 90^\circ - (S - B), \\ s' - c' &= 90^\circ - (S - C). \end{aligned}$$

By definition,

$$(14) \quad r' = \sqrt{\frac{\sin(s' - a') \sin(s' - b') \sin(s' - c')}{\sin s'}};$$

formulas (12) and (13) reduce (14) to the form

$$(15) \quad r' = R.$$

We complete the reduction of (11) to the first of formulas (10) by noting that

$$\sin(s' - a') = \sin[90^\circ - (S - A)] = \cos(S - A),$$

$$\tan \frac{A'}{2} = \tan \frac{a_s}{2} = \tan\left(90^\circ - \frac{a}{2}\right) = \cot \frac{a}{2}.$$

75. Napier's analogies.* In this section we shall obtain formulas involving five of the six parts of a spherical triangle. The following four **Napier's analogies** are typical; others may be obtained from them by interchange of letters.

* *Analogy* is here used in an obsolete sense, meaning *proportion*.

$$\begin{aligned}
 (1) \quad & \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c}. \\
 (2) \quad & \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c}. \\
 (3) \quad & \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(A - B)}{\cot \frac{1}{2}C}. \\
 (4) \quad & \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(A + B)}{\cot \frac{1}{2}C}.
 \end{aligned}$$

We shall prove formula (1) as a consequence of the first two of formulas (3) of § 74. From these we have

$$\begin{aligned}
 \frac{\tan \frac{1}{2}A - \tan \frac{1}{2}B}{\tan \frac{1}{2}A + \tan \frac{1}{2}B} &= \frac{\frac{r}{\sin(s-a)} - \frac{r}{\sin(s-b)}}{\frac{r}{\sin(s-a)} + \frac{r}{\sin(s-b)}}, \\
 \frac{\frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} - \frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B}}{\frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} + \frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B}} &= \frac{\sin(s-b) - \sin(s-a)}{\sin(s-b) + \sin(s-a)}, \\
 \frac{\sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B} &= \frac{2 \cos \frac{1}{2}(2s - a - b) \sin \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(2s - a - b) \cos \frac{1}{2}(a - b)}, \\
 \frac{\sin(\frac{1}{2}A - \frac{1}{2}B)}{\sin(\frac{1}{2}A + \frac{1}{2}B)} &= \frac{\cos \frac{1}{2}c \sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c \cos \frac{1}{2}(a - b)}, \\
 \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} &= \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c}.
 \end{aligned}$$

We obtain the second of Napier's analogies by transforming the first. This is done by means of an auxiliary formula as follows:

From the law of sines (§ 72) we easily deduce the relation

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\sin a - \sin b}{\sin a + \sin b}.$$

To both sides of this equation we apply formulas of page 64, thus obtaining

$$\begin{aligned}
 \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)} &= \frac{2 \cos \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)} \\
 &= \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}(a + b)}.
 \end{aligned}$$

From this we deduce the desired auxiliary formula

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\cos \frac{1}{2}(A - B) \tan \frac{1}{2}(a - b)}{\cos \frac{1}{2}(A + B) \tan \frac{1}{2}(a + b)}.$$

If we replace the left member of (1) by the right member of this last equation, the result is easily reduced to formula (2).

We obtain formulas (3) and (4) by applying (1) and (2) to the polar triangle $A'B'C'$. Thus formula (1) for $A'B'C'$ is

$$\frac{\sin \frac{1}{2}(A' - B')}{\sin \frac{1}{2}(A' + B')} = \frac{\tan \frac{1}{2}(a' - b')}{\tan \frac{1}{2}c'},$$

from which we deduce

$$\begin{aligned} \frac{\sin \frac{1}{2}(a_s - b_s)}{\sin \frac{1}{2}(a_s + b_s)} &= \frac{\tan \frac{1}{2}(A_s - B_s)}{\tan \frac{1}{2}C_s}, \\ \frac{\sin \frac{1}{2}(b - a)}{\sin \frac{1}{2}(a + b)} &= \frac{\tan \frac{1}{2}(B - A)}{\cot \frac{1}{2}C}. \end{aligned}$$

This may be at once reduced to formula (3), since

$$\begin{aligned} \sin \frac{1}{2}(b - a) &= -\sin \frac{1}{2}(a - b), \\ \tan \frac{1}{2}(B - A) &= -\tan \frac{1}{2}(A - B). \end{aligned}$$

Formula (4) may be obtained in a similar manner.

76. Solution of oblique spherical triangles. Cases may be listed according to the given parts as follows:

- I. Given three sides.
- II. Given three angles.
- III. Given two sides and the included angle.
- IV. Given two angles and the included side.
- V. Given two sides and an angle opposite one of them.
- VI. Given two angles and a side opposite one of them.

In the following sections we shall explain the solution under each case by logarithmic computation. For Cases I and II we shall use the half-angle and half-side formulas (§ 74); for Cases III and IV, Napier's analogies (§ 75); for Cases V and VI, the law of sines (§ 72) and Napier's analogies.

Here, as in solving right triangles, we omit the number -10 that should follow certain logarithms. We keep results to the nearest second in computations, but give answers to the nearest multiple of five seconds.

The student should prepare a complete outline of the computation before using the Tables.

77. Case I. Given three sides. We here use the half-angle formulas of § 74, and check by the law of sines, or one of Napier's analogies that involves all three angles. We replace $\sin \theta$ by $\sin \theta$, when θ is greater than 90° .

Example. Given $a = 125^\circ 10' 0''$, $b = 110^\circ 22' 15''$, $c = 30^\circ 48' 45''$. Find A , B , C .

Solution.

Formulas

$$2s = a + b + c$$

$$\tan \frac{A}{2} = \frac{r}{\sin(s-a)} \quad \tan \frac{B}{2} = \frac{r}{\sin(s-b)} \quad \tan \frac{C}{2} = \frac{r}{\sin(s-c)}$$

$$r^2 = \frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}$$

$$\text{Check: } \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Computation

$a = 125^\circ 10' 0''$	$s - a = 8^\circ 0' 30''$	$\log \sin(s-a) = 9.14400$
$b = 110^\circ 22' 15''$	$s - b = 22^\circ 48' 15''$	$(+) \log \sin(s-b) = 9.58837$
$c = 30^\circ 48' 45''$	$s - c = 102^\circ 21' 45''$	$(+) \log \sin(s-c)_s = 9.98981$
$2s = 266^\circ 21' 0''$	$\text{sum} = 133^\circ 10' 30''$	$\text{sum} = 28.72218$
$s = 133^\circ 10' 30''$	$(\text{Check: sum} = s)$	$(-) \log \sin s_s = 9.86289$
$s_s = 46^\circ 49' 30''$	$(s-c)_s = 77^\circ 38' 15''$	$\log r^2 = 2 \log r = 18.85929$

$$\log r = 9.42964$$

$$(-) \log \sin(s-a) = 9.14401$$

$$\log \tan \frac{A}{2} = 10.28563$$

$$\frac{A}{2} = 62^\circ 36' 48''$$

$$A = 125^\circ 13' 36''$$

$$\log r = 9.42964$$

$$(-) \log \sin(s-b) = 9.58836$$

$$\log \tan \frac{B}{2} = 9.84128$$

$$\frac{B}{2} = 34^\circ 45' 20''$$

$$B = 69^\circ 30' 40''$$

$$\log r = 9.42964$$

$$(-) \log \sin(s-c)_s = 9.98981$$

$$\log \tan \frac{C}{2} = 9.43983$$

$$\frac{C}{2} = 15^\circ 23' 35''$$

$$C = 30^\circ 47' 10''$$

Check

$\log \sin A_s = 9.91216$	$\log \sin B = 9.97162$	$\log \sin C = 9.70913$
$(-) \log \sin a_s = 9.91248$	$(-) \log \sin b_s = 9.97195$	$(-) \log \sin c = 9.70947$
9.99968	9.99967	9.99966

Answers: $A = 125^\circ 13' 35''$, $B = 69^\circ 30' 40''$, $C = 30^\circ 47' 10''$.

78. Case II. Given three angles. We here use the half-side formulas of § 74 and check by the law of sines or one of Napier's analogies that involves all three sides. It can be shown that S , $S - A$, $S - B$, $S - C$ are all between -90° and $+90^\circ$; hence we write

$$R^2 = \frac{\cos(S - A) \cos(S - B) \cos(S - C)}{\cos S},$$

and in using this formula remember that $\cos(-\theta) = \cos \theta$.

Example. Given $A = 144^\circ 20' 20''$, $B = 47^\circ 24' 35''$, $C = 56^\circ 35' 45''$. Find a , b , c .

Solution.

Formulas

$$2S = A + B + C$$

$$R^2 = \frac{\cos(S - A) \cos(S - B) \cos(S - C)}{\cos S}$$

$$\cot \frac{a}{2} = \frac{R}{\cos(S - A)}$$

$$\cot \frac{b}{2} = \frac{R}{\cos(S - B)}$$

$$\cot \frac{c}{2} = \frac{R}{\cos(S - C)}$$

$$\text{Check: } \frac{\cos \frac{1}{2}(A - B)}{\cos [\frac{1}{2}(A + B)]} = \frac{\tan [\frac{1}{2}(a + b)]}{\tan \frac{1}{2}c}$$

Computation

$A = 144^\circ 20' 20''$	$S - A = -20^\circ 10' 0''$	$\log \cos(S - A) = 9.97252$
$B = 47^\circ 24' 35''$	$S - B = 76^\circ 45' 45''$	$(+) \log \cos(S - B) = 9.35982$
$C = 56^\circ 35' 45''$	$S - C = 67^\circ 34' 35''$	$(+) \log \cos(S - C) = 9.58144$
$2S = 248^\circ 20' 40''$	$\text{sum} = 124^\circ 10' 20''$	$\text{sum} = 28.91378$
$S = 124^\circ 10' 20''$	$(\text{Check: } \text{sum} = S)$	$(-) \log \cos S = 9.74949$
$S_a = 55^\circ 49' 40''$		$\log R^2 = 2 \log R = 19.16429$

$$\log R = 9.58214$$

$$(-) \log \cos(S - A) = 9.97252$$

$$\log \cot \frac{1}{2}a = 9.60962$$

$$\frac{1}{2}a = 67^\circ 51' 8''$$

$$a = 135^\circ 42' 16''$$

$$\log R = 9.58214$$

$$(-) \log \cos(S - B) = 9.35982$$

$$\log \cot \frac{1}{2}b = 10.22232$$

$$\frac{1}{2}b = 30^\circ 56' 11''$$

$$b = 61^\circ 52' 22''$$

$$\log R = 9.58214$$

$$(-) \log \cos(S - C) = 9.58144$$

$$\log \cot \frac{1}{2}c = 10.00070$$

$$\frac{1}{2}c = 44^\circ 57' 14''$$

$$c = 89^\circ 54' 28''$$

Check

$$A - B = 96^\circ 55' 45''$$

$$\frac{1}{2}(A - B) = 48^\circ 27' 52''$$

$$A + B = 191^\circ 44' 55''$$

$$\frac{1}{2}(A + B) = 95^\circ 52' 28''$$

$$[\frac{1}{2}(A + B)]_c = 84^\circ 7' 32''$$

$$a + b = 197^\circ 34' 38''$$

$$\frac{1}{2}(a + b) = 98^\circ 47' 19''$$

$$[\frac{1}{2}(a + b)]_c = 81^\circ 12' 41''$$

$$\frac{1}{2}c = 44^\circ 57' 14''$$

$$\begin{aligned}\log \cos \frac{1}{2}(A - B) &= 9.82157 \\ (-) \log \cos [\frac{1}{2}(A + B)]_s &= 9.01008 \\ \hline &0.81149\end{aligned}$$

$$\begin{aligned}\log \tan [\frac{1}{2}(a + b)]_s &= 10.81078 \\ (-) \log \tan \frac{1}{2}c &= 9.99930 \\ \hline &0.81148\end{aligned}$$

This gives a better check than the law of sines would yield, since c is almost 90° .

Answers: $a = 135^\circ 42' 15''$, $b = 61^\circ 52' 20''$, $c = 89^\circ 54' 30''$.

EXERCISES

Solve the spherical triangles of which the following are given parts. Write answers to the nearest multiple of $5''$.

1. $a = 108^\circ 0' 30''$, $b = 96^\circ 42' 0''$, $c = 40^\circ 34' 10''$.
2. $a = 150^\circ 20' 40''$, $b = 43^\circ 3' 20''$, $c = 129^\circ 8' 20''$.
3. $a = 116^\circ 0' 15''$, $b = 57^\circ 57' 50''$, $c = 137^\circ 20' 45''$.
4. $a = 153^\circ 38' 45''$, $b = 40^\circ 0' 20''$, $c = 118^\circ 20' 15''$.
5. $a = 47^\circ 42' 0''$, $b = 50^\circ 1' 40''$, $c = 63^\circ 15' 10''$.
6. $a = 82^\circ 3' 5''$, $b = 82^\circ 1' 30''$, $c = 70^\circ 14' 10''$.
7. $a = 64^\circ 23' 15''$, $b = 100^\circ 49' 30''$, $c = 99^\circ 40' 50''$.
8. $a = 55^\circ 42' 10''$, $b = 150^\circ 57' 5''$, $c = 134^\circ 15' 55''$.
9. $A = 122^\circ 40' 10''$, $B = 31^\circ 20' 20''$, $C = 37^\circ 30' 0''$.
10. $A = 145^\circ 46' 30''$, $B = 125^\circ 12' 0''$, $C = 108^\circ 12' 30''$.
11. $A = 22^\circ 20' 20''$, $B = 47^\circ 21' 10''$, $C = 146^\circ 40' 10''$.
12. $A = 25^\circ 10' 15''$, $B = 36^\circ 42' 45''$, $C = 160^\circ 32' 20''$.
13. $A = 98^\circ 52' 40''$, $B = 60^\circ 44' 25''$, $C = 27^\circ 52' 55''$.
14. $A = 116^\circ 55' 25''$, $B = 40^\circ 10' 30''$, $C = 54^\circ 6' 20''$.
15. $A = 25^\circ 51' 0''$, $B = 155^\circ 14' 25''$, $C = 11^\circ 34' 10''$.
16. $A = 70^\circ 20' 40''$, $B = 110^\circ 10' 0''$, $C = 133^\circ 18' 0''$.

79. Case III. Given two sides and the included angle. If the given parts are a , b , C , we use Napier's analogies (3) and (4) of § 75 to determine $\frac{1}{2}(A - B)$ and $\frac{1}{2}(A + B)$ respectively, and from these values we find A and B . The side c is then found by using either (1) or (2) of § 75. We check by the law of sines, or by a Napier's analogy which we did not use in our solution. We choose our formulas so that each expression in them is positive. Parts greater than 90° are replaced by their supplements.

Example. Given $a = 65^\circ 40' 0''$, $c = 124^\circ 7' 20''$, $B = 159^\circ 44' 40''$. Find A , C , b .

Solution.

Formulas

$$\tan \frac{1}{2}(C - A) = \frac{\sin \frac{1}{2}(c - a)}{\sin [\frac{1}{2}(c + a)]_s} \cot \frac{1}{2}B \quad \tan [\frac{1}{2}(C + A)]_s = \frac{\cos \frac{1}{2}(c - a)}{\cos [\frac{1}{2}(c + a)]_s} \cot \frac{1}{2}B$$

$$\tan \frac{1}{2}b = \frac{\sin [\frac{1}{2}(C + A)]_s}{\sin \frac{1}{2}(C - A)} \tan \frac{1}{2}(c - a)$$

$$\text{Check: } \frac{\cos \frac{1}{2}(B - A)}{\cos [\frac{1}{2}(B + A)]_s} = \frac{\tan [\frac{1}{2}(b + a)]_s}{\tan \frac{1}{2}c}$$

Computation

$c = 124^\circ 7' 20''$ $a = 65^\circ 40' 0''$ $c - a = 58^\circ 27' 20''$ $c + a = 189^\circ 47' 20''$ $\frac{1}{2}(c - a) = 29^\circ 13' 40''$ $\frac{1}{2}(c + a) = 94^\circ 53' 40''$ $[\frac{1}{2}(c + a)]_s = 85^\circ 6' 20''$ $\frac{1}{2}B = 79^\circ 52' 20''$ $\frac{1}{2}(C - A) = 5^\circ 0' 11''$ $\frac{1}{2}(C + A) = 118^\circ 41' 36''$ $C = 123^\circ 41' 47''$ $A = 113^\circ 41' 25''$	$\log \sin \frac{1}{2}(c - a) = 9.68867$ $(+) \log \cot \frac{1}{2}B = 9.25195$ $\text{sum} = 18.94062$ $(-) \log \sin [\frac{1}{2}(c + a)]_s = 9.99841$ $\log \tan \frac{1}{2}(C - A) = 8.94221$ $\frac{1}{2}(C - A) = 5^\circ 0' 11''$ $\log \cos \frac{1}{2}(c - a) = 9.94085$ $(+) \log \cot \frac{1}{2}B = 9.25195$ $\text{sum} = 9.19280$ $(-) \log \cos [\frac{1}{2}(c + a)]_s = 8.93105$ $\log \tan [\frac{1}{2}(C + A)]_s = 10.26175$ $[\frac{1}{2}(C + A)]_s = 61^\circ 18' 24''$ $\frac{1}{2}(C + A) = 118^\circ 41' 36''$
--	---

We omit the computation for b . The result is $b = 159^\circ 50' 56''$.

Check

$B = 159^\circ 44' 40''$ $A = 113^\circ 41' 25''$ $B - A = 46^\circ 3' 15''$ $B + A = 273^\circ 26' 5''$ $\frac{1}{2}(B - A) = 23^\circ 1' 38''$ $\frac{1}{2}(B + A) = 136^\circ 43' 2''$ $[\frac{1}{2}(B + A)]_s = 43^\circ 16' 58''$ $\log \cos \frac{1}{2}(B - A) = 9.96394$ $(-) \log \cos [\frac{1}{2}(B + A)]_s = 9.86211$ $\text{diff.} = 10.10183$	$b = 159^\circ 50' 56''$ $a = 65^\circ 40' 0''$ $b + a = 225^\circ 30' 56''$ $\frac{1}{2}(b + a) = 112^\circ 45' 28''$ $[\frac{1}{2}(b + a)]_s = 67^\circ 14' 32''$ $\frac{1}{2}c = 62^\circ 3' 40''$ $\log \tan [\frac{1}{2}(b + a)]_s = 10.37727$ $(-) \log \tan \frac{1}{2}c = 10.27545$ $\text{diff.} = 10.10182$
---	---

Answers: $A = 113^\circ 41' 25''$, $C = 123^\circ 41' 45''$, $b = 159^\circ 50' 55''$.

80. Case IV. Given two angles and the included side. The procedure corresponds to that in Case III. We use Napier's analogies to find the other two sides and the other angle. For the check we may use a Napier's analogy or the law of sines.

EXERCISES

Solve the spherical triangles of which the following are given parts. Write answers to the nearest multiple of 5''.

1. $a = 99^\circ 40' 0''$, $b = 64^\circ 20' 0''$, $C = 96^\circ 10' 0''$.
2. $a = 60^\circ 6' 0''$, $b = 100^\circ 8' 0''$, $C = 31^\circ 10' 0''$.
3. $a = 138^\circ 20' 30''$, $c = 117^\circ 15' 45''$, $B = 22^\circ 20' 15''$.
4. $b = 60^\circ 6' 20''$, $c = 100^\circ 5' 40''$, $A = 31^\circ 11' 0''$.
5. $a = 56^\circ 19' 40''$, $b = 20^\circ 16' 40''$, $C = 114^\circ 20' 20''$.
6. $a = 53^\circ 49' 25''$, $b = 97^\circ 44' 20''$, $C = 102^\circ 14' 10''$.
7. $a = 146^\circ 37' 15''$, $c = 135^\circ 49' 20''$, $B = 105^\circ 8' 10''$.
8. $b = 102^\circ 16' 10''$, $c = 130^\circ 46' 0''$, $A = 136^\circ 19' 10''$.

9. $A = 122^\circ 20' 0''$, $B = 37^\circ 40' 0''$, $c = 34^\circ 3' 0''$.
 10. $A = 58^\circ 18' 0''$, $B = 76^\circ 22' 0''$, $c = 103^\circ 4' 0''$.
 11. $A = 152^\circ 14' 30''$, $C = 122^\circ 41' 15''$, $b = 48^\circ 20' 45''$.
 12. $B = 92^\circ 38' 20''$, $C = 124^\circ 54' 40''$, $a = 61^\circ 20' 0''$.
 13. $A = 80^\circ 20' 10''$, $B = 54^\circ 8' 30''$, $c = 61^\circ 47' 25''$.
 14. $A = 59^\circ 4' 30''$, $B = 94^\circ 23' 10''$, $c = 129^\circ 11' 40''$.
 15. $A = 127^\circ 22' 0''$, $C = 107^\circ 33' 20''$, $b = 125^\circ 41' 45''$.
 16. $B = 133^\circ 28' 35''$, $C = 169^\circ 38' 10''$, $a = 85^\circ 50' 20''$.

81. Case V. Given two sides and an angle opposite one of them (ambiguous case). Here, as in the corresponding case for plane triangles (p. 95), there may be no solution, one solution, or two solutions.

If the given parts are a , b , A , we may use the law of sines to compute B :

$$(1) \quad \sin B = \frac{\sin b}{\sin a} \sin A.$$

If this gives $\log \sin B$ a positive value, there is *no solution*. But if $\log \sin B$ has an admissible value, and B_1 is the corresponding acute angle given by the Tables, then

$$B_2 = 180^\circ - B_1 = (B_1),$$

may be also a solution. We now use the theorem that the greater side is opposite the greater angle to determine the values of B that can be used.

If $a > b$, and $A > B_2$ (and therefore $A > B_1$) there are two solutions, one with parts a , b , A , B_1 , the other with parts a , b , A , B_2 ; if $a > b$, and $B_1 < A \leq B_2$, there is a solution corresponding to B_1 , but not one corresponding to B_2 ; and if $a > b$, and $A \leq B_1$, there is no solution.

If $a = b$ the triangle is isosceles, and in this case there is but one solution.

The student should discuss for himself the case $a < b$, using the principle that an angle $B = B_1$ or $B = B_2$ gives a solution when and only when $A < B$.

With a , b , A , and the admissible value or values of B , we calculate c and C by Napier's analogies, and check by another Napier's analogy or by the law of sines.

In preparing an outline of the work, one should first proceed as far as the determination of B , and make out the remainder of the outline only after the number of solutions has been found.

Example 1. Given $a = 30^\circ 0' 0''$, $b = 60^\circ 0' 0''$, $A = 75^\circ 0' 0''$. Find B , C , c .

Solution.

Formula

$$\sin B = \frac{\sin b}{\sin a} \sin A \quad (\text{Only formula needed.})$$

Computation

$$\begin{aligned} \log \sin b &= 9.93753 \\ (+) \log \sin A &= 9.98494 \\ \text{sum} &= 19.92247 \\ (-) \log \sin a &= 9.69897 \\ \log \sin B &= 10.22350, \text{ impossible.} \end{aligned}$$

Answer: No triangle exists with the given parts.

Example 2. Show that there is but one triangle for which $b = 60^\circ$, $c = 150^\circ$, $B = 45^\circ$.

We find, by computing with the law of sines,

$$\log \sin C = 9.61093, \quad C_1 = 24^\circ 5' 43'', \quad C_2 = [C_1]_s = 155^\circ 54' 17''.$$

We here have $c > b$, hence $C > B$. Thus C_2 is admissible, but C_1 is not.

Example 3. Given $a = 70^\circ 30' 20''$, $c = 125^\circ 15' 35''$, $C = 145^\circ 0' 0''$. Find A , B , b .

Solution.

Formulas

$$\begin{aligned} \sin A &= \frac{\sin a}{\sin c} \sin C, & \tan \frac{1}{2}b &= \frac{\sin \frac{1}{2}(C+A)}{\sin \frac{1}{2}(C-A)} \tan \frac{1}{2}(c-a) \\ \cot \frac{1}{2}B &= \frac{\sin \frac{1}{2}(c+a)}{\sin \frac{1}{2}(c-a)} \tan \frac{1}{2}(C-A) & \text{Check: } \frac{\sin A_1}{\sin a} &= \frac{\sin B_1}{\sin b_1} = \frac{\sin [B_2]_s}{\sin [b_2]_s}. \end{aligned}$$

Computation

$$\begin{aligned} \log \sin a &= 9.97436 \\ (+) \log \sin C &= 9.75859 \\ \text{sum} &= 19.73295 \\ (-) \log \sin c &= 9.91198 \\ \log \sin A &= 9.82097 \end{aligned}$$

$$A_1 = 41^\circ 27' 56''$$

$$A_2 = [A_1]_s = 138^\circ 32' 4''$$

$$c = 125^\circ 15' 35''$$

$$a = 70^\circ 30' 20''$$

$$c - a = 54^\circ 45' 15''$$

$$c + a = 195^\circ 45' 55''$$

$$\frac{1}{2}(c - a) = 27^\circ 22' 38''$$

$$\frac{1}{2}(c + a) = 97^\circ 52' 58''$$

$$[\frac{1}{2}(c + a)]_s = 82^\circ 7' 2''$$

$$C = 145^\circ 0' 0''$$

$$C = 145^\circ 0' 0''$$

$$A_1 = 41^\circ 27' 56''$$

$$A_2 = 138^\circ 32' 4''$$

$$C + A_1 = 186^\circ 27' 56''$$

$$C + A_2 = 283^\circ 32' 4''$$

$$\frac{1}{2}(C + A_1) = 93^\circ 13' 58''$$

$$\frac{1}{2}(C + A_2) = 141^\circ 46' 2''$$

$$[\frac{1}{2}(C + A_1)]_s = 86^\circ 46' 2''$$

$$[\frac{1}{2}(C + A_2)]_s = 38^\circ 13' 58''$$

$$C - A_1 = 103^\circ 32' 4''$$

$$C - A_2 = 6^\circ 27' 56''$$

$$\frac{1}{2}(C - A_1) = 51^\circ 46' 2''$$

$$\frac{1}{2}(C - A_2) = 3^\circ 13' 58''$$

$\log \sin [\frac{1}{2}(C + A_1)]_s = 9.99931$	$\log \sin [\frac{1}{2}(C + A_2)]_s = 9.79159$
$(+) \log \tan \frac{1}{2}(c - a) = 9.71420$	$(+) \log \tan \frac{1}{2}(c - a) = 9.71420$
sum = 19.71351	sum = 19.50579
$(-) \log \sin \frac{1}{2}(C - A_1) = 9.89514$	$(-) \log \sin \frac{1}{2}(C - A_2) = 8.75123$
$\log \tan \frac{1}{2}b_1 = 9.81837$	$\log \tan \frac{1}{2}b_2 = 10.75456$
$\frac{1}{2}b_1 = 33^\circ 21' 13''$	$\frac{1}{2}b_2 = 80^\circ 1' 11''$
$b_1 = 66^\circ 42' 26''$	$b_2 = 160^\circ 2' 22''$
$\log \sin [\frac{1}{2}(c + a)]_s = 9.99588$	
$(-) \log \sin \frac{1}{2}(c - a) = 9.66261$	
diff. = 0.33327	0.33327
$(+) \log \tan \frac{1}{2}(C - A_1) = 10.10356$	$(+) \log \tan \frac{1}{2}(C - A_2) = 8.75192$
$\log \cot \frac{1}{2}B_1 = 10.43683$	$\log \cot \frac{1}{2}B_2 = 9.08519$
$\frac{1}{2}B_1 = 20^\circ 5' 22''$	$\frac{1}{2}B_2 = 83^\circ 3' 46''$
$B_1 = 40^\circ 10' 44''$	$B_2 = 166^\circ 7' 32''$

Check

$\log \sin A_1 = 9.82097$	$\log \sin B_1 = 9.80968$	$\log \sin [B_2]_s = 9.37984$
$(-) \log \sin a = 9.97436$	$(-) \log \sin b_1 = 9.96308$	$(-) \log \sin [b_2]_s = 9.53323$
9.84661	9.84660	9.84661

Answers: $A_1 = 41^\circ 27' 55''$, $B_1 = 40^\circ 10' 45''$, $b_1 = 66^\circ 42' 25''$;
 $A_2 = 138^\circ 32' 5''$, $B_2 = 166^\circ 7' 30''$, $b_2 = 160^\circ 2' 20''$.

82. Case VI. Given two angles and a side opposite one of them (ambiguous case). The procedure corresponds to that in Case V. If the given parts are A , B , a , we use the law of sines to compute b . We allow for two solutions, $b = b_1$ and $b = [b_1]_s = b_2$, in case $\log \sin b$ has a value less than zero (if $\log \sin b > 0$ there is no solution). We retain both b_1 and b_2 if they obey the rule that a is greater than, equal to, or less than b , according as A is greater than, equal to, or less than B . If only one of the pair b_1 , b_2 obeys this rule, we retain that one; if neither obeys the rule, there is no solution. With the parts retained we compute c and C , and check, as in Case V.

EXERCISES

Find the number of solutions for the following given parts:

- $a = 110^\circ$, $b = 80^\circ$, $A = 95^\circ$.
- $a = 40^\circ$, $A = 100^\circ$, $B = 130^\circ$.
- $b = 100^\circ$, $c = 120^\circ$, $C = 130^\circ$.
- $B = 130^\circ$, $C = 140^\circ$, $b = 125^\circ$.
- $c = 105^\circ$, $A = 110^\circ$, $C = 100^\circ$.
- $a = 114^\circ$, $c = 47^\circ$, $A = 100^\circ$.

Solve the spherical triangles of which the following are given parts. If there are two solutions, find both. Write answers to the nearest multiple of $5''$.

- $a = 32^\circ 59' 30''$, $b = 47^\circ 45' 10''$, $A = 43^\circ 50' 40''$.
- $a = 47^\circ 29' 20''$, $b = 50^\circ 6' 20''$, $A = 34^\circ 29' 30''$.
- $b = 138^\circ 20' 15''$, $c = 117^\circ 9' 30''$, $C = 47^\circ 21' 45''$.
- $b = 107^\circ 33' 20''$, $c = 52^\circ 38' 0''$, $C = 55^\circ 47' 40''$.

11. $a = 96^\circ 34' 40''$, $c = 55^\circ 0' 20''$, $C = 42^\circ 30' 30''$.
 12. $a = 121^\circ 28' 10''$, $c = 154^\circ 46' 50''$, $A = 39^\circ 45' 10''$.
 13. $A = 111^\circ 42' 20''$, $B = 77^\circ 5' 0''$, $a = 108^\circ 30' 30''$.
 14. $A = 105^\circ 6' 40''$, $B = 58^\circ 40' 15''$, $b = 61^\circ 33' 5''$.
 15. $B = 31^\circ 26' 15''$, $C = 130^\circ 36' 35''$, $c = 71^\circ 15' 20''$.
 16. $B = 30^\circ 36' 55''$, $C = 95^\circ 50' 0''$, $c = 30^\circ 17' 35''$.
 17. $A = 133^\circ 18' 20''$, $C = 70^\circ 16' 10''$, $a = 155^\circ 5' 40''$.
 18. $A = 111^\circ 51' 45''$, $C = 126^\circ 18' 40''$, $c = 120^\circ 22' 40''$.

83. Area of a spherical triangle. We have seen that the sum of the angles of a spherical triangle ABC is greater than 180° . Following the usage of books on solid geometry we write

$$E = A + B + C - 180^\circ,$$

and call E the **spherical excess**. It is shown in such books that the area of the spherical triangle ABC is equal to the area of a lune whose angle is $\frac{E}{2}$; that the area of a lune is to the area of the entire spherical surface as the angle of the lune is to 360° ; and that the surface area of a sphere of radius R is equal to $4\pi R^2$. If we put these statements together we have

$$\frac{\text{Area of } \triangle ABC}{4\pi R^2} = \frac{\frac{1}{2}E}{360^\circ},$$

$$(1) \quad \text{Area of } \triangle ABC = \frac{\pi R^2 E}{180^\circ}.$$

It is here understood that E is given in degrees. Since the number of radians in E degrees is $\frac{\pi E}{180^\circ}$, formula (1) states that the area of $\triangle ABC$ is equal to the product of R^2 and the radian measure of E .

If the three angles of a spherical triangle are given, we compute its area directly from (1). When three other parts are given we may solve for the angles and then use (1). In more extended works on spherical trigonometry other formulas for area are developed; we give one here, called **L'Huilier's Theorem**, which enables us to compute E when the three sides are known:

$$(2) \quad \tan \frac{E}{4} = \sqrt{\tan \frac{1}{2}s \tan \frac{1}{2}(s-a) \tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)}.$$

The proof is omitted.

EXERCISES

Find the area of each triangle for which the following parts are given, when R has exactly the length indicated. Give answers to four significant figures for Exercises 1, 3, 5, 7, 9, 11, and to five significant figures for Exercises 2, 4, 6, 8, 10.

1. $A = B = C = 70^\circ 0'$; $R = 1$ ft.
2. $A = 50^\circ 14' 25''$, $B = 63^\circ 20' 40''$, $C = 134^\circ 37' 10''$; $R = 3959$ mi.
3. $a = 36^\circ 27'$, $c = 54^\circ 20'$, $C = 90^\circ$ (exactly); $R = 10$ in.
4. $b = 32^\circ 9' 15''$, $A = 50^\circ 19' 20''$, $C = 90^\circ$ (exactly); $R = 100$ ft.
5. $A = 143^\circ 5'$, $C = 120^\circ 54'$, $c = 90^\circ$ (exactly); $R = 50$ ft.
6. $a = 51^\circ 59' 55''$, $b = 121^\circ 33' 0''$, $c = 90^\circ$ (exactly); $R = 3959$ mi.
7. $a = 154^\circ 45'$, $c = 34^\circ 9'$, $B = 108^\circ 40'$; $R = 10$ ft.
8. $B = 104^\circ 49' 50''$, $C = 84^\circ 51' 40''$, $a = 96^\circ 53' 10''$; $R = 3959$ mi.
9. $a = 76^\circ 36'$, $b = 57^\circ 8'$, $c = 110^\circ 26'$; $R = 1000$ ft.
10. $a = 54^\circ 36' 50''$, $b = 147^\circ 37' 0''$, $c = 126^\circ 50' 50''$; $R = 47281$ ft.
11. Find the number of square miles in the triangle whose vertices are Baltimore (lat. $39^\circ 17' N$, long. $76^\circ 37' W$), New York (lat. $40^\circ 43' N$, long. $74^\circ 0' W$), and Chicago (lat. $41^\circ 53' N$, long. $87^\circ 38' W$), assuming the earth to be a sphere of radius 3959 mi.

CHAPTER IX

APPLICATIONS

84. Introduction. In this chapter we present briefly a few important applications of trigonometry. First we shall present problems which use *plane* trigonometry, (1) in plane surveying, (2) in artillery, and (3) in navigation (of the water or of the air); later we shall give applications of *spherical* trigonometry to problems of navigation on the earth regarded as a sphere.

In our brief discussion we must omit all description of instruments of measurement, of methods of handling them, and of corrections to be made to observations. For these important topics, we must refer to special books on surveying, artillery, and navigation.*

In computations we shall use five-place tables, and at the end of the work round off the answers to four places or less according to data.

PLANE SURVEYING

85. Plane surveying. In plane surveying we consider small portions of the earth's surface for which no important error is made if we assume that the earth is flat, i.e., that such a portion lies in a horizontal plane. A map of such a region is a drawing which pictures the relative distances and directions of points in the region. In the simplest map a coördinate system (or grid) is used, points being located by their coördinates.

A surveyor locates points with respect to some particular object or point, frequently a *bench mark*. Through this point *B* passes a

* E.g., Davis, R. E., and Foote, F. S., *Surveying: Theory and Practice*, McGraw-Hill, New York.

Breed, C. B., and Hosmer, G. L., *Principles and Practice of Surveying*, Wiley, New York.

Dutton, Capt. Benjamin, *Navigation and Nautical Astronomy*, U. S. Naval Institute, 1939.

Lyon, Thoburn C., *Practical Air Navigation*, U. S. Department of Commerce.

Bowditch, N., *American Practical Navigator*, U. S. Hydrographic Office, Navy Department, Washington, D. C. This contains extensive tables, which simplify calculations.

meridian line SN from south to north, and a line WE from west to east (Fig. 77). A point P is located by its distance CP north or south of WE , and its distance AP east or west of the meridian SN . In the surveyor's language, the **latitude** (or *difference in latitude*) of P is l units north or south, and the **departure** of P is p units east or west, where the length of CP is l units and of AP is p units (the unit usually being a foot, a yard, or a rod).

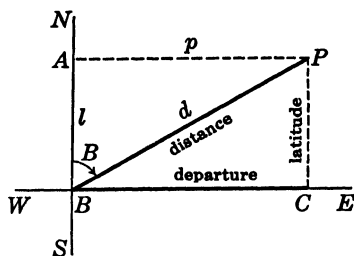


FIG. 77

The point P is also located by its **distance** BP , and the **bearing** of BP (which is the angle NBP if P is north of WE , and the angle SBP if P is south of WE), written as in Exercises 4, 9, 10, page 19. If the distance is d units and the bearing is B , then we have (Fig. 77)

$$(1) \quad l = d \cos B, \quad p = d \sin B,$$

and

$$(2) \quad \tan B = \frac{p}{l}, \quad d = \frac{p}{\sin B} = \frac{l}{\cos B}.$$

By use of (1) or (2) we may compute two of the four quantities l , p , B , d when the others are known.

When the line BP is measured it is called the **course**.

A surveyor frequently makes a series of measurements, going from B to P_1 to P_2 to P_3 , and so on. A problem is to locate the final point with respect to the starting point, as illustrated in the following Example.

Example. A surveyor's notebook shows the following measurements (shown in heavy type), calculations, and sketch.

Course	Distance	Bearing	Latitude		Departure	
			N	S	E	W
BP_1	200 yd.	N 15° E	193.19		51.76	
P_1P_2	300 yd.	S 80° E		52.09	295.44	
P_2P_3	200 yd.	S 5° W		199.24		17.43
Totals			193.19	251.33	347.20	17.43
Diff.				58.14	329.77	
BP_3	335 yd.	S 80° E				

Computation

$$\begin{aligned}
 l &= 58.14 \\
 p &= 329.77 \\
 \log p &= 2.51821 \\
 \log l &= 1.76448 \\
 \log \tan B &= 0.75373 \\
 B &= 80^\circ 0.1' \\
 \log p &= 2.51821 \\
 \log \sin B &= 9.99335 \\
 \log d &= 2.52486 \\
 d &= 334.85
 \end{aligned}$$

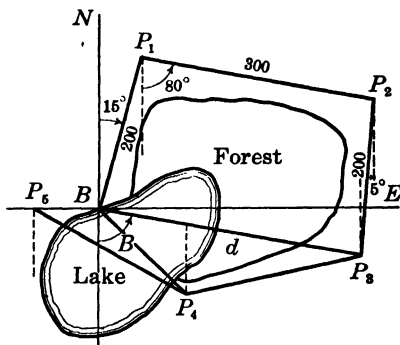


FIG. 78

In the preceding Example, the surveyor wished to know the length and bearing of BP_3 , and a lake and a forest caused the series of measurements. An alternative method of locating P_3 would have been to measure the bearing and distance for BP_5 (see Fig. 78), the bearings of BP_4 and P_5P_4 and the bearing and distance for P_4P_3 . By this method the distance BP_4 across the lake is found by solving the triangle BP_5P_4 . We then should have sufficient data to solve the triangle BP_4P_3 and thus find the distance and bearing of BP_3 .

To make a *map* of a region, we may first find latitude and departure for a number of points, and bearing and distance for a number of lines, and then make a drawing to scale. Angles on the drawing should be the same as corresponding angles in the field. Lengths on the drawing are reduced by a *constant scale* from those in the field.

EXERCISES

Use five-place tables in solving the following Exercises:

1. The bearing of BP is $N 38^\circ W$ and the distance is 324 yd. Find the latitude and departure of P from B .
2. The bearing of BP is $S 52^\circ E$ and the distance is 324 rods. Find the latitude and departure of P from B .

In a series of measurements a surveyor made the following tables. In each case find the latitude, departure, bearing, and distance for BP_3 . Draw a map to some selected scale (and give the scale).

3.	Course	Dist.	Bearing
	BP_1	150 yd.	$N 16^\circ W$
	P_1P_2	210 yd.	$S 72^\circ W$
	P_2P_3	320 yd.	$S 65^\circ E$

4.	Course	Dist.	Bearing
	BP_1	280 rd.	$N 35^\circ W$
	P_1P_2	302 rd.	$N 15^\circ E$
	P_2P_3	401 rd.	$S 27^\circ E$

5.	Course	Dist.	Bearing
	BP_1	328.1 ft.	N $3^\circ 25'$ E
	P_1P_2	218.7 ft.	N $88^\circ 12'$ E
	P_2P_3	301.6 ft.	S $4^\circ 5'$ W

6.	Course	Dist.	Bearing
	BP_1	288.4 yd.	S $2^\circ 11'$ E
	P_1P_2	129.3 yd.	S $59^\circ 4'$ W
	P_2P_3	307.2 yd.	S $6^\circ 34'$ E

7. A surveyor found for BP_1 the distance 248.1 yd., the bearing S $87^\circ 15'$ W. The bearing for BP_2 was S $27^\circ 14'$ E and for P_1P_2 was S $53^\circ 47'$ E. For P_2P_3 the distance was 587.7 yd. and the bearing N $76^\circ 14'$ E. Find the distance and bearing of BP_3 (see Fig. 79).

8. A surveyor found for BP_1 the distance 124.7 rd. and the bearing S $84^\circ 18'$ W. The bearing for BP_2 was S $23^\circ 13'$ E and for P_1P_2 was S $63^\circ 44'$ E. For P_2P_3 the distance was 310.6 rd. and the bearing N $82^\circ 4'$ E. Find the distance and bearing of BP_3 .

9. A surveyor made the following measurements: For BP_1 the distance was 220.1 yd., the bearing N $15^\circ 1'$ E. For P_1P_2 the distance was 329.9 yd., the bearing S $80^\circ 2'$ E. For P_2P_3 the distance was 219.8 yd., the bearing S $5^\circ 2'$ W. For P_3B' the distance was 368.5 yd., the bearing S $79^\circ 58'$ E. What was the calculated latitude and departure of B' from B ? If all measurements had been exact, B' and B would have coincided, since they were the same point of observation.

10. A surveyor made a traverse $ABCD A$ by making measurements shown in the adjoining table. By calculating the latitude and departure for each course find the resultant error in latitude and departure for the closed traverse.

Course	Distance	Bearing
AB	123.4 yd.	S $20^\circ 15'$ E
BC	129.2 yd.	N $71^\circ 15'$ W
CD	66.8 yd.	N $8^\circ 3'$ W
DA	90.4 yd.	N $85^\circ 1'$ E

According to a map, referred to a bench mark B several points (see Fig. 80) are located as shown in the following table. Using these data, solve Exercises 11 and 12.

	Diff. Lat.		Departure	
	N	S	E	W
C	427.31		631.75	
D	815.23		182.14	
E	128.72			581.56
F		132.68		622.28

11. Railroad surveyors at C can observe D but not E which lies over a ridge through which a tunnel is to be constructed. They calculate the angle DCE and determine the direction in which the tunnel is to be dug. Find this angle and the distance CE .

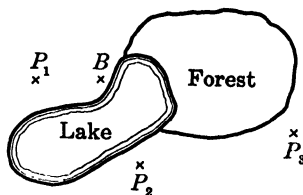


FIG. 79

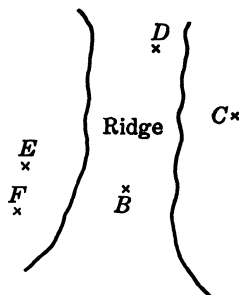


FIG. 80

12. Surveyors at E proceed similarly (see Ex. 11) and calculate the angle FEC , and determine the direction EC of the other end of the tunnel. Calculate angle FFC and the distance EC .

In the following Exercises the surveyors used observation points at different levels as stated. In making a map the points are projected on a horizontal plane. Find the required things and draw a map to scale (indicate the scale on the map) in Exercises 13 and 14.

13. In a mountainous country there were two points C and D which were situated so that an observer at one could not see the other, but both were visible from two points A and B which lay in a horizontal plane MM' . Surveyors at A observed that the bearing of AB was $S 6^\circ 20' E$, of AC was $N 62^\circ 34' E$, and of AD was $S 84^\circ 47' E$. They measured AB to be 1538 yd., and observed at B that the bearing of BC was $N 55^\circ 47' E$, of BD was $N 79^\circ 13' E$. The angle of elevation of BC was $5^\circ 11'$ and of BD was $4^\circ 28'$. Letting C' and D' be the points in the plane MM' directly under C and D , find the distance and bearing of $C'D'$ and the elevation of C and D above the plane MM' .

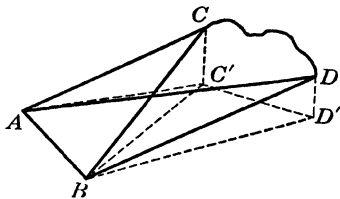


FIG. 81

14. Surveyors measured AB to be 235 yd. along a hillside whose inclination to the horizontal was $8^\circ 21'$. At A the angle of elevation of a point C on a distant hill was $6^\circ 13'$, the bearing of AB was $S 4^\circ 11' W$, of AC was $N 38^\circ 57' E$. At B the bearing of BC was $N 35^\circ 11' E$. What is the distance BC and how much higher is C than B ?

15. A line AB on a hillside is d yd. long, its inclination is α . The inclination of AC is β . The bearing of AB is $S \delta E$, of AC is $N \theta E$, and of BC is $N \phi E$. Find the heights h and k of B and C above the horizontal plane MM' through A , and find the horizontal distance from B to C .

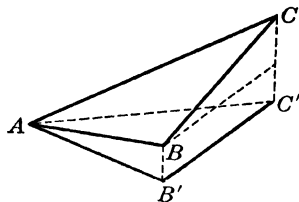


FIG. 82

16. Two points C and D are invisible from each other. At a point A the bearing of AC is $N \alpha E$ and of AD is $N \beta E$. At another point B which lies in a horizontal plane MM' with A , the bearing of BC is $N \alpha' E$ and of BD is $N \beta' E$. The angle of elevation of AC is θ , and the length of AB is d yd. Let C' and D' be points in the plane MM' which are directly below C and D . Find the distance $C'D'$ and the angle of elevation of CD .

ARTILLERY PROBLEMS

86. Use of maps. Maps are of great importance in military problems. Some of the principles of their construction have been briefly described in the preceding pages. By use of surveying instruments, which measure directions, angles, and distances, relative positions of points of interest are found and a drawing or map is made showing these points in their relative positions. And conversely when such a map is available we may determine from it directions, angles, and distances for use in the field.

Suppose that a map is at hand, and that a gun which is located at a mapped point G is to be fired at a mapped target T . The distance GT is scaled off from the map, and the angle of elevation of the gun is found from "firing tables" which give this angle for various distances under standard conditions. For very large guns and long ranges corrections are made for weather conditions and other factors which produce predictable results. If the point T is visible from G , the direction to point the axis of the gun may be determined by direct aiming, which is called "direct firing." If the target T is not visible from the gun G , "indirect firing" is used. One method is to use an aiming point A , which is mapped and which is visible from G . From the map the magnitude of the angle AGT is found with a suitable protractor, and then the gun is set so that its axis makes this angle with the observable direction GA .

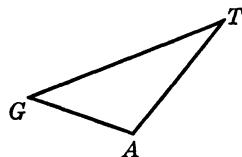


FIG. 83

More frequently, however, the map does not show G and T , and T is not visible from G . In this case, these points may be mapped by observing their directions from mapped points $A, B, C \dots$, and drawing lines on the maps through these points with these directions. Suppose three points A, B, C are used. Because of inaccuracies of observation and of drawing, these sets of lines will not exactly coincide at G or at T , but the set of intersecting lines will appear to locate

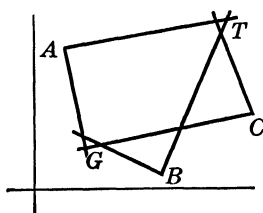


FIG. 84

G in a **triangle of position** and T in another such triangle (Fig. 84). Assuming that the points are at the centers of the triangles (perhaps roughly determined), the gunner may proceed with "indirect fire." In some cases more elaborate methods are used.

EXERCISES

1. By use of a range finder an officer at O observes that the distance to a gun G is 3127 yd., and that the distance to a target T is 2463 yd. The angle TOG is observed to be $88^\circ 21'$. The gunner wishes to know the angle OGT and the distance GT . What are they?

2. By use of a range finder an officer at O observes that the distance to a gun G is 1628 yd. and to a target T is 3181 yd. The angle TOG is observed to be $123^\circ 14'$. The gunner wishes to know the angle OGT and the distance GT . What are they?

The coördinates of points A, B, C, D on a map are as follows: $A(2.3, -1.5)$, $B(2.4, 6.6)$, $C(6.5, 3.4)$, $D(-2.1, 3.1)$. Use these points in solving Exercises 3–6.

3. Plot the points A, B, C on a sketch. From a gun emplacement G the following bearings are taken: For GA the bearing is $S 27^\circ E$; for GB the bearing is $N 27^\circ E$; for GC the bearing is $N 84^\circ E$. On the sketch draw GA, GB , and GC , and estimate the coördinates of G in the "triangle of position."

4. A target T is observed from A, B, C . The bearing of AT is $N 56^\circ E$, of BT is $S 80^\circ E$, and of CT is $N 72^\circ E$. On a sketch locate A, B , and C , draw AT, BT , and CT , and estimate the coördinates of T from the "triangle of position."

5. From a gun G the bearing of B is $N 38^\circ E$, of C is $S 86^\circ E$, and of D is $S 74^\circ W$. Make a sketch of B, C , and D , and estimate the coördinates of G from the "triangle of position."

6. The bearing of a target T from D is $S 82^\circ E$, from B is $S 61^\circ E$, and from A is $N 78^\circ E$. From a sketch estimate the coördinates of T in the "triangle of position."

7. Write down all of the formulas that would be used in a solution of the following problem. An observer at A (Fig. 85) finds angle α and distances a and b ; and an observer at B finds angle β and distance c . Find angle γ and distance d for the gunner at G who wishes to fire at target T .

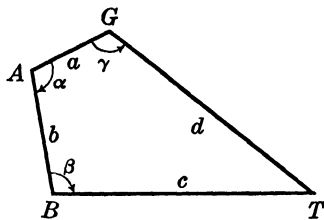


FIG. 85

8. From a gun G the target is hidden by a forest. An officer at A observes that $GA = 582$ yd., $AB = 629$ yd., and angle $GAB = 121^\circ 13'$. An officer at B observes that $BT = 1122$ yd., and angle $ABT = 98^\circ 15'$. Calculate the angle AGT and the range GT for the gunner.

87. Measurement of angles. The mil. In various branches of military service several systems of measuring angles are used. In giving bearings of a course or of an object from a given point, (a) in the navy angles are measured from the north clockwise from 0° to 360° ; (b) in the coast artillery angles are measured from the south clockwise from 0° to 360° ; and (c) in field artillery angles may be measured to the right or left of some observable directing point.

The angles are measured (a) in degrees and minutes, or (b) in degrees and hundredths of degrees, or (c) in mils, a unit which we shall now briefly discuss.

A **mil** is an angle which may be defined by the relation

$$1600 \text{ mils} = 90^\circ.$$

It follows that

$$\begin{aligned} 1 \text{ mil} &= 0.05625^\circ = 3.37500'; & 1 \text{ mil} &= 0.0009817 \text{ radians}; \\ 1 \text{ degree} &= 17.77778 \text{ mils}; & 1 \text{ radian} &= 1018.6 \text{ mils}; \\ 1 \text{ minute} &= 0.296296 \text{ mils}. \end{aligned}$$

The usefulness of the mil arises principally from the fact that a vertical yardstick (BD in Fig. 86) at a horizontal distance of a thousand yards from A subtends almost exactly an angle of one mil. More precisely, for an angle of A mils (Fig. 86),

$$(1) \quad \frac{BC}{AB} = \frac{A}{1000} \quad (\text{error about } 1.9\%),$$

and

$$\frac{BD}{AB} = \frac{A}{1000} \quad (\text{error less than } 2\% \text{ for angles less than } 340 \text{ mils, about } 19^\circ).$$

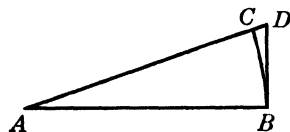


FIG. 86

The latter equation is equivalent to

$$(2) \quad \tan A = \frac{A}{1000} \quad (\text{error less than } 2\% \text{ when } A < 340).$$

The adjoining table shows that this approximation (2) is very close for small angles, being exact when $A = 0$ and also when A is about 235 mils (an angle of about $13^\circ 13'$); but for angles as large as 600 mils the error is more than 10%. For example, when $A = 400$,

$$\tan A = 0.41421,$$

$$\frac{A}{1000} = 0.40000,$$

and the error is 0.01421.

A mils	$\tan A$	Angle in degrees
0 mils	0.00000	0°
50 "	0.04913	$2^\circ 48.75'$
100 "	0.09849	$5^\circ 37.5'$
150 "	0.14834	$8^\circ 26.25'$
200 "	0.19891	$11^\circ 15'$
250 "	0.25048	$14^\circ 3.75'$
300 "	0.30335	$16^\circ 52.5'$
350 "	0.35780	$19^\circ 41.25'$
400 "	0.41421	$22^\circ 30'$
500 "	0.53451	$28^\circ 7.5'$
600 "	0.66818	$33^\circ 45'$

The following Examples illustrate how the mil is used.

Example 1. At a distance of d yd. from an observer at A , a line of unknown length x yd. subtends an angle of A mils, as shown in Figure 87. Find x .

Solution. From the approximation (2) we have

$$\frac{x}{d} = \frac{A}{1000}$$

and hence

$$(3) \quad x = \frac{Ad}{1000}.$$

This involves an error of less than 2% if A is less than 340 mils.

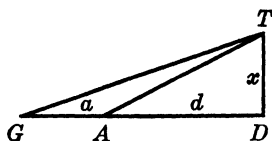


FIG. 87

Example 2. From a gun G (Fig. 87) a target T is invisible, but a directing point D is visible such that angle ADT is approximately a right angle. An observer at A on the line GD finds that $AG = a$ yd., $AD = d$ yd., and the angle $DAT = A$ mils. What is the angle DGT , which gives the deflection of the gun from the visible direction GD ?

Solution. If $DT = x$ we have, by (2),

$$\frac{x}{d} = \frac{A}{1000}, \quad 1000x = Ad,$$

and

$$\frac{x}{a+d} = \frac{G}{1000}, \quad 1000x = G(a+d).$$

Hence

$$(4) \quad G = \frac{Ad}{a+d}.$$

This involves an error due to the assumption that angle ADT is approximately a right angle. With a range finder, the observer at A can estimate the accuracy of the assumption; for practical purposes the approximation might suffice for giving the direction of a trial shot. If angle G is small the range GT is approximately $a + AT$.

Example 3. A gun G (Fig. 88) fires at a target T . An observer at B on the line GT observes that the shell S falls b mils to the right of BT , that $GB = b$ yd. and $BS = s$ yd. An observer at A , where AT is perpendicular to GT , notes that $AT = a$ yd. and that S is A mils to the right of T . What changes should the gunner make in setting his gun for another shot?

Solution. When S is close to T compared to distances b and s , we get a good approximation to angle TGS by use of (4), namely,

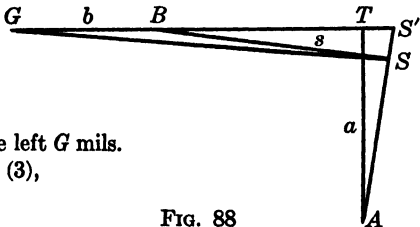
$$(5) \quad G = \frac{Bs}{b+s}$$

and the gun should be deflected to the left G mils.

If AS intersects GT at S' , then, by (3),

$$TS' = \frac{Aa}{1000}.$$

FIG. 88



The elevation of the gun should be decreased to shorten the range by this amount.

EXERCISES

Express the following angles in mils:

- | | |
|-------------------------------------|----------------------------------|
| 1. $10^\circ, 25^\circ, 45^\circ$. | 5. $7^\circ 19', 17^\circ 31'$. |
| 2. $8^\circ, 30^\circ, 55^\circ$. | 6. $5^\circ 23', 32^\circ 28'$. |
| 3. $4^\circ, 18^\circ, 35^\circ$. | 7. $2^\circ 35', 24^\circ 25'$. |
| 4. $2^\circ, 16^\circ, 40^\circ$. | 8. $6^\circ 28', 19^\circ 19'$. |

Express the following angles in degrees and minutes:

- | | |
|------------------------|------------------------|
| 9. 125 mils, 260 mils. | 11. 84 mils, 400 mils. |
| 10. 60 mils, 441 mils. | 12. 71 mils, 313 mils. |

Express the following angles in radians:

- | | |
|------------------------|------------------------|
| 13. 25 mils, 225 mils. | 15. 18 mils, 455 mils. |
| 14. 40 mils, 375 mils. | 16. 44 mils, 360 mils. |

Express the following angles in mils:

- | | |
|---------------------------------|---------------------------------|
| 17. 0.25 radians, 1.2 radians. | 19. 0.40 radians, 0.66 radians. |
| 18. 0.12 radians, 0.85 radians. | 20. 0.18 radians, 0.35 radians. |

Find a close approximation in mils for the following angles:

- | | |
|--------------------------|--------------------------|
| 21. $\tan^{-1} 0.0985$. | 25. $\tan^{-1} 0.0777$. |
| 22. $\tan^{-1} 0.4142$. | 26. $\tan^{-1} 0.1234$. |
| 23. $\tan^{-1} 0.3578$. | 27. $\tan^{-1} 0.2333$. |
| 24. $\tan^{-1} 0.6682$. | 28. $\tan^{-1} 0.3456$. |

Draw angles of the following magnitudes:

- | | |
|---------------|---------------|
| 29. 100 mils. | 33. 50 mils. |
| 30. 200 mils. | 34. 250 mils. |
| 31. 300 mils. | 35. 450 mils. |
| 32. 400 mils. | 36. 800 mils. |

In the following when answers are measures of angles express them in mils. Assume that in each of Exercises 37-42 the angle and the line concerned are parts of a right triangle.

37. At a distance of 1200 yd. a vertical stick 3 yd. long subtends an angle A . How large is A ?

38. How large is an angle that is subtended by a stick 7 ft. long at 350 yd.?

39. How tall is a building which subtends an angle of 60 mils at a distance of 175 yd.?

40. A target T is 120 yd. to the left of a directing point D as seen from a gun G . What is the angle DGT if $DG = 720$ yd.?

41. A target T appears 40 mils to the right of a directing point D which is 800 yd. from an observer at a gun G . How far is it from D to T ?

42. A chimney C which is 160 ft. tall subtends an angle of 50 mils at an observation post P . How far is it from P to C ?

43. From a gun G a target T is invisible, but a directing point D is visible such that angle GDT is approximately 90° (Fig. 89). From a point A to the rear of G and in line with GD , an observer finds that $AG = 237$ yd., $AD = 1872$ yd.,

and angle $GAT = 135$ mils. Find the angle of deflection of GT from GD , which enables the gunner to direct his gun on the target.

44. An officer at A observes that a target T and a point D are so situated that angle ADT is approximately a right angle, angle $DAT = 85$ mils, and AD is 2856 yd. A gun G is on the line AD and is 382 yd. from A as shown in Figure 89. Find the angle AGT , which may be used by the gunner in training his gun on the target T .



FIG. 89

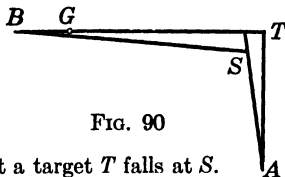


FIG. 90

45. A shell from a gun G (Fig. 90) fired at a target T falls at S . An observer at B observes that BS is 8 mils to the right of the line BGT , and that $BG = 152$ yd., $BT = 1892$ yd. An observer at A observes that AS is 21 mils to the left of AT and that $AT = 692$ yd. Assuming that AT is perpendicular to GT , what changes should be made in setting the gun for the next shot?

46. A target T is due east from a gun G . An observer A in an airplane at a height of 3000 yd. directly over T observes that a shell falls at S due southeast from T , and that the angle TAS is 16 mils. If GT is 1239 yd., what adjustments in bearing and range of the gun should be made for the next shot?

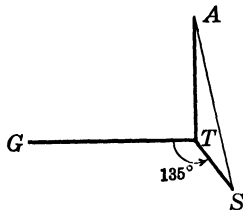


FIG. 91

NAVIGATION

88. Plane sailing. In navigation of the sea or air, it frequently suffices to consider the surface of the earth as a plane surface when only short distances are involved. This is the method of *plane sailing*.

We first define a few words which will be used. The student may note that some of these words have slightly different meanings in surveying and in various branches of navigation.

In plane sailing the line which runs north and south through a point on the earth is called the **meridian** of the point. The direction of a line is described by giving the angle it makes with a meridian. The **course** C is this angle, and it is measured from the north clockwise, through the east, south, and west, running from 0° to 360° .

In giving a ship's position with reference to some given place, five terms are employed — the **course**, the **distance** (d), the **difference of latitude** (DL), the **difference of longitude** (DLo), and the **departure** (dep). All of these terms except difference of longitude are shown in the plane triangle of Figure 92. Here TN is a segment of the meridian-

ian through T , and QN is the perpendicular to TN from the ship Q . The distances TN and NQ are the **difference of latitude** and the **departure**; the distance TQ is called the **distance**; and the angle C is the **course**.

We consider departures to the east as positive, and to the west as negative; and differences in latitude to the north as positive, and to the south as negative. We have the relations

$$(1) \quad \text{dep} = d \sin C;$$

$$(2) \quad DL = d \cos C;$$

$$(3) \quad \tan C = \frac{\text{dep}}{DL}.$$

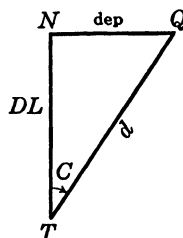


FIG. 92

Navigators use Traverse Tables which give values of $d \sin C$ and $d \cos C$ for integral values of d from 0 to 600 and angles at intervals of $1'$ from 0° to 90° . Without such tables at hand, we may compute these products by use of trigonometric tables.

89. Parallel sailing. Here we no longer treat parallels of latitude and meridians as straight lines, but consider them as circles on a sphere. If Q is a point on the earth's surface, its **parallel of latitude** is the circle cut from the sphere by a plane through Q parallel to the equatorial plane, and its **meridian** is the half great circle on the sphere that passes through Q and ends at the north and south poles.

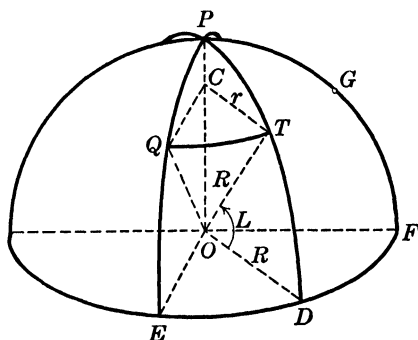


FIG. 93

In Figure 93, G is Greenwich, England, P is the north pole, O the center of the earth, FDE the equator, PQE a part of the meridian through Q , and PGF part of the meridian of Greenwich; then we define the latitude and longitude of Q as follows:

longitude of Q = angle GPQ

= angle FOE , measured to 180° E or 180° W;

latitude of Q = angle EOQ , measured to 90° N or 90° S.

Thus the position of Q in Figure 93 is described as follows:

Q , lat. $58^\circ 50' \text{ N}$, long. $110^\circ 20' \text{ W}$.

When a course is due east or due west, the path is along a parallel of latitude and we have parallel sailing. In this case the distance equals the departure, and $DL = 0$. The only problem is to determine the difference of longitude from the departure, or vice versa.

Referring again to Figure 93, we say that TQ represents the departure in parallel sailing from Q to T . The **difference of longitude** of T and Q is the angle DOE or TCQ , where TQ is an arc of a parallel of latitude with radius r and center at C . If R is the radius of the earth, we have $OD = OT = R$. The latitude L of T and Q is the angle DOT . We then have

$$\cos L = \frac{r}{R}, \quad \frac{TQ}{DE} = \frac{r}{R};$$

hence, since $TQ = \text{dep}$,

$$(1) \quad \text{dep} = DE \cos L,$$

$$(2) \quad DE = \frac{\text{dep}}{\cos L}.$$

A nautical mile is defined as the length of a $1'$ arc on a great circle of the earth. Hence a nautical mile on the equator corresponds to a difference of longitude of $1'$, and, if differences of longitude and departures are measured in nautical miles, we have

$$(3) \quad \text{dep} = DLo \cos L,$$

$$(4) \quad DLo = \frac{\text{dep}}{\cos L}.$$

90. Middle latitude sailing. In parallel sailing the latitude L is constant. In plane sailing the points T and Q have different latitudes, L_1 and L_2 . The formulas (3) and (4) of parallel sailing will give good approximations if for L we take the middle latitude $\frac{L_1 + L_2}{2}$. If we call this middle latitude L_m , we then have

$$(1) \quad \text{dep} = DLo \cos L_m,$$

$$(2) \quad DLo = \frac{\text{dep}}{\cos L_m}.$$

Thus, in general, to solve a problem in navigation (using only plane trigonometry) it is necessary to use both the plane sailing formulas of § 88 and the middle latitude sailing formula.

Example 1. A ship sails a course of $134^\circ 25'$ for a distance of 135 nautical miles. If the starting point T has a latitude of $38^\circ 15' \text{ N}$ and a longitude of $70^\circ 20' \text{ W}$, what are the latitude and longitude of the ship Q ?

Solution. Using plane sailing formulas, we have

$$\begin{aligned}\text{dep} &= 135 \sin 134^\circ 25', \\ DL &= 135 \cos 134^\circ 25' .\end{aligned}$$

Since $\sin 134^\circ 25' = \cos 44^\circ 25'$, and $\cos 134^\circ 25' = -\sin 44^\circ 25'$, we have

$$\begin{aligned}\log 135 &= 2.13033 \\ \log \cos 44^\circ 25' &= \underline{9.85386} \\ \log \text{dep} &= 1.98419 \\ \text{dep} &= 96.425 \\ \log 135 &= 2.13033 \\ \log \sin 44^\circ 25' &= \underline{9.84502} \\ \log (-DL) &= 1.97535 \\ DL &= -94.482\end{aligned}$$

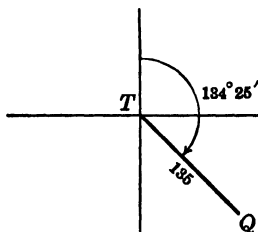


FIG. 94

Thus the departure is 96.425 nautical miles and the difference of latitude is -94.482 nautical miles or $-94.482'$ or $-1^\circ 34.5'$. Since DL is negative the latitude L of Q is less than that of T , and we find that the latitude of Q is $38^\circ 15' - 1^\circ 34.5' = 36^\circ 40.5'$.

The middle latitude L_m is

$$L_m = \frac{38^\circ 15' + 36^\circ 40.5'}{2} = 37^\circ 27.8'.$$

Hence by the formula of middle latitude sailing, we get

$$\begin{aligned}DLo &= \frac{96.425}{\cos 37^\circ 27.8'} \\ &= 121.48' \\ &= 2^\circ 1.5'.\end{aligned}\qquad \begin{aligned}\log 96.425 &= 1.98419 \\ \log \cos 37^\circ 27.8' &= \underline{9.89968} \\ \log DLo &= 2.08451\end{aligned}$$

Since the position of Q is in west longitude, and since the ship sails in an easterly direction, we subtract DLo from the longitude of T to get the longitude of Q . The result is $68^\circ 18.5'$ W longitude.

Hence the position of Q is lat. $36^\circ 40.5'$ N, and long. $68^\circ 18.5'$ W.

Example 2. If a ship is to sail from a point T whose longitude is $46^\circ 25'$ W and whose latitude is $44^\circ 14'$ N to a point Q whose longitude is $51^\circ 35'$ W and latitude is $45^\circ 10'$ N, what are the distance and the course?

Solution. We find the middle latitude $L_m = 44^\circ 42'$, and the difference of longitude $DLo = -5^\circ 10' = -310'$, using the negative sign since S is west of T (§ 88).

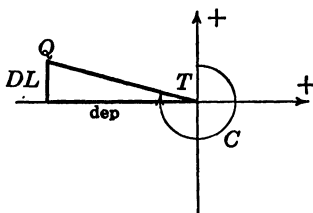


FIG. 95

By formula (1) above, we then have

$$\begin{aligned}\text{dep} &= -310 \cos 44^\circ 42' \\ &= -220.35;\end{aligned}$$

and by use of formula (3), § 88, since $DL = 56'$ or 56 nautical miles, we have

$$\tan C = \frac{\text{dep}}{DL},$$

from which we find, since the course is northwesterly,

$$C = 360^\circ - 75^\circ 44.4' = 284^\circ 15.6'.$$

Then by formula (1), § 88,

$$d = \frac{\text{dep}}{\sin C} = 227.35.$$

From T to Q the distance is 227.35 nautical miles and the course is $284^\circ 15.6'$.

91. Dead reckoning. Dead reckoning is the process of determining the position of a ship by use of the course and distance of its run from a position previously determined by other methods. The course is determined by use of a compass, and the distance by use of an instrument known as a log or by the number of revolutions of the ship's propeller. If there is a current affecting the ship's motion, this will not be observed on the compass or the log, and the velocity of the current must be added to the ship's observed velocity in determining the actual velocity relative to the ground, which is required in finding differences in latitude and in longitude.

The dead reckoning may be done by plotting on maps or by computation by the method of middle latitude sailing. In the following Examples the dead reckoning is done by computation.

Example 1. A ship starts at a point T whose latitude is $36^\circ 25' N$ and whose longitude is $74^\circ 36' W$. As determined by compass and log, it sails 21.2 nautical miles on a course of $132^\circ 45'$ in 10 hours. Assuming that there is a current of 6.2 knots (nautical miles per hour), its set being 65° , find the ship's position at the end of the ten hours.

Solution. We first find the velocity over the ocean floor by vector addition of the two velocities. In one hour the motion relative to the water is 21.2 nautical miles on course $132^\circ 45'$, and of the water relative to the floor is 6.2 nautical miles on course 65° . From Figure 96 we see how to find the resultant DL and departure for one hour:

$$\begin{aligned}DL &= 21.2 \cos 132^\circ 45' + 6.2 \cos 65^\circ = -11.771, \\ \text{dep} &= 21.2 \sin 132^\circ 45' + 6.2 \sin 65^\circ = 21.187.\end{aligned}$$

$$\begin{array}{l} \text{Computation} \\ \log 310 = 2.49136 \\ \log \cos 44^\circ 42' = 9.85175 \\ \log (-\text{dep}) = \overline{2.34311} \\ \log DL = 1.74819 \\ \log \tan C = \overline{0.59492} \end{array}$$

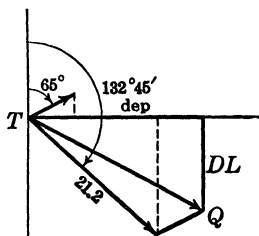


FIG. 96

Then for the resultant course and distance

$$\tan C = \frac{\text{dep}}{DL}, \quad C = 119^\circ 3.3', \quad d = \frac{\text{dep}}{\sin C} = 24.237.$$

The distance made good in ten hours is 242.37 nautical miles on course $119^\circ 3.3'$. In ten hours the $DL = -117.71'$ and the $\text{dep} = 211.87$ nautical miles. Hence the required latitude is $36^\circ 25' - 1^\circ 57.7' = 34^\circ 27.3'$. The middle latitude is found to be $L_m = 35^\circ 26.2'$. Hence by (2), § 90, and a brief calculation,

$$DL_o = \frac{\text{dep}}{\cos L_m} = 260.04' = 4^\circ 20.0',$$

to the nearest tenth of a minute, and therefore the required longitude is

$$74^\circ 36' - 4^\circ 20.0' = 70^\circ 16.0'.$$

Example 2. A ship is to sail from a point T whose latitude is $36^\circ 25' \text{ N}$ and longitude $74^\circ 36' \text{ W}$ to a point Q whose latitude is $35^\circ 42' \text{ N}$ and longitude is $73^\circ 12' \text{ W}$, its speed being 26.4 knots. There is a current of 4.6 knots on course 65° . What should be the compass course of the ship, and how long will it take to reach Q ?

Solution. Here we have $DL = -43'$, middle latitude $L_m = 36^\circ 3.5'$, and $DL_o = 84'$. Hence, by (1), § 90, omitting details of the calculation, we have

$$\text{dep} = DL_o \cos L_m = 67.908;$$

therefore, by (3) and (1), § 88,

$$\tan C = \frac{\text{dep}}{DL},$$

$$C = 122^\circ 20.5';$$

$$d = \frac{\text{dep}}{\sin C} = 80.377.$$

Let C' be the ship's compass course, TQ' the velocity relative to the water, and $Q'Q''$ the velocity of the current. Then angle $TQ''Q' = C - 65^\circ = 57^\circ 20.5'$, $TQ' = 26.4$, and $Q'Q'' = 4.6$.

Relettering the triangle $TQ''Q'$ as shown in Figure 97, we have

$$a = 4.6, \quad b = 26.4, \quad B = 57^\circ 20.5'.$$

By the law of sines, we get

$$\sin A = \frac{a \sin B}{b}, \quad A = 8^\circ 26.1'.$$

Then $c_1 = \frac{a \sin C_1}{\sin A} = 28.597$. Hence the ship's compass course is

$$C' = 122^\circ 20.5' + 8^\circ 26.1' = 130^\circ 47'.$$

The time required is $\frac{80.377}{28.597} = 2.81 \text{ hr.}$

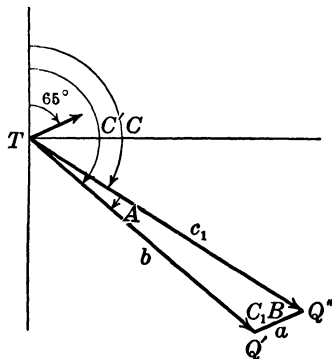


FIG. 97

EXERCISES

In each of Exercises 1-8 find the departure and the difference in latitude for the given distance and course (plane sailing):

- | | |
|---------------------------------------|---------------------------------------|
| 1. dist. = 125, $C = 57^\circ 28'$. | 5. dist. = 219, $C = 213^\circ 15'$. |
| 2. dist. = 201, $C = 182^\circ 13'$. | 6. dist. = 371, $C = 241^\circ 17'$. |
| 3. dist. = 178, $C = 151^\circ 52'$. | 7. dist. = 175, $C = 303^\circ 15'$. |
| 4. dist. = 82, $C = 265^\circ 13'$. | 8. dist. = 144, $C = 327^\circ 50'$. |

In each of Exercises 9-16 find the distance and course for the given departure and difference in latitude (plane sailing):

- | | |
|--------------------------------|---------------------------------|
| 9. dep = 73.7, $DL = 23.9$. | 13. dep = 64.4, $DL = -23.7$. |
| 10. dep = 86.8, $DL = -31.4$. | 14. dep = -29.9, $DL = -71.1$. |
| 11. dep = -70.7, $DL = 87.1$. | 15. dep = -72.3, $DL = -31.6$. |
| 12. dep = -81.1, $DL = 27.7$. | 16. dep = 67.7, $DL = -55.5$. |

17. A ship sailed from lat. $44^\circ 19' N$, long. $40^\circ 42' W$ due west for 121.4 nautical miles. What were then its latitude and longitude?

18. A ship sailed from lat. $13^\circ 12' N$, long. $145^\circ 42' W$ due east for 212.5 nautical miles. What were then its latitude and longitude?

19. A ship sailed from lat. $55^\circ 15' N$, long. $10^\circ 39' W$ to lat. $55^\circ 15' N$, long. $13^\circ 27' W$. How far did it go?

20. A ship sailed from lat. $63^\circ 12' S$, long. $2^\circ 5' E$ to lat. $63^\circ 12' S$, long. $4^\circ 15' W$. How far did it go?

21. A ship sailed from lat. $58^\circ 27' S$, long. $14^\circ 33' W$ for 147.2 nautical miles on course $78^\circ 15'$. What were then its latitude and longitude?

22. A ship sailed from lat. $23^\circ 16' N$, long. $5^\circ 10' W$ for 167.1 nautical miles on course $247^\circ 10'$. What were then its latitude and longitude?

23. A ship sailed from lat. $12^\circ 4' S$, long. $4^\circ 14' W$ for 155.4 nautical miles on course $151^\circ 25'$. What were then its latitude and longitude?

24. An airplane flew from lat. $38^\circ 40' N$, long. $74^\circ 13' W$ for 218.3 nautical miles on course $321^\circ 32'$. What were then its latitude and longitude?

25. A ship is to sail from lat. $17^\circ 32' N$, long. $15^\circ 12' W$ to lat. $19^\circ 19' N$, long. $13^\circ 7' W$. What are the distance and course?

26. A ship is to sail from lat. $14^\circ 15' S$, long. $120^\circ 15' W$ to lat. $16^\circ 6' S$, long. $122^\circ 6' W$. What are the distance and course?

27. A ship sailed a compass course $123^\circ 10'$ for 162.3 nautical miles, requiring 7.4 hr. There was a current of 3.7 knots with set $41^\circ 30'$. What were the difference of latitude and the departure over the ocean floor?

28. An airplane flew a compass course $62^\circ 30'$ at 97.3 mi./hr. for 2 hr. 15 min. There was a wind of velocity 35 mi./hr. with course $80^\circ 40'$. What difference of latitude and departure did it make with respect to the ground?

29. A ship sailed from lat. $37^\circ 43' N$, long. $66^\circ 18' W$ for 132.2 nautical miles on compass course $67^\circ 30'$, requiring 6 hours. There was a current of 7.2 knots with set 38° . What were then the latitude and longitude by dead reckoning?

30. An airplane is to go from A to B , traveling at 135 miles per hour with respect to the air. There is a wind of 47 miles per hour on course 125° . If the position of A is lat. $33^\circ 14' N$, long. $102^\circ 7' W$, and if the position of B is lat. $37^\circ 25' N$, long. $93^\circ 5' W$, what should be the compass course of the airplane, and how long will it take to fly from A to B ?

92. Great circle sailing. In the preceding sections, we have solved certain problems of navigation by means of plane trigonometry. We shall now apply spherical trigonometry to problems in navigation. While the earth is not exactly spherical, it is approximately so. In this section we shall take the earth to be a sphere whose radius is $R = 3959$ miles.

The line of shortest distance between two points on a sphere is the great circle on which the points lie, the distance being measured on the arc which is not greater than a semicircumference. In **great circle sailing** a ship (or airplane considered as traveling approximately on the earth's surface) has a great circle arc as its **track** from A to B .

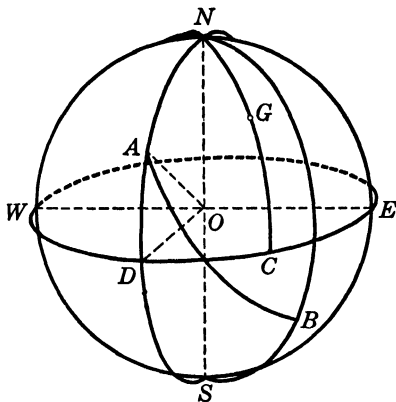


FIG. 98

Two principal problems of navigation are to determine the **distance** from A to B along the great circle arc AB and to determine the **direction** of that arc at any one of its points.

In § 89 we have defined the latitude and longitude of a point on the earth's surface, and the difference of longitude of two points. We now add two other definitions, referring to Figure 98.

Colatitude of A = angle NOA .

Bearing of A from B = the acute angle which the great circle BA makes with the meridian through B .

For bearings we use the notation of § 85. Thus, if A were west of B and angle NBA were 50° , the bearing of A from B would be $N 50^\circ W$; if B were east of A , and angle SAB were 75° , the bearing of B from A would be $S 75^\circ E$.

If the latitude and longitude of A and B are given, we can find the distance from A to B along the great circle passing through them, and the bearing of each point from the other. We obtain these by solving

the spherical triangle NAB , where

AN = colatitude of A ,

BN = colatitude of B ,

AB = great circle distance from A to B ,

(1) $\angle BNA$ = difference of longitude of A and B ,

$\angle NAB$ = bearing of B from A , or supplement of that bearing,

$\angle NBA$ = bearing of A from B , or supplement of that bearing.

When any three of these parts are given, we can solve for the other three.

In great circle navigation the term **course** is used to specify direction on a great circle track. Thus, if A is in the north hemisphere, the angle NAB taken as a positive angle between 0° and 180° is called the **course angle** of the track from A to B . The **initial course**, commonly designated by the symbol C_n , is the angle between 0° and 360° which gives the value of the angle NAB when measured clockwise from AN to AB .

To express distances on the earth's surface in linear units when they have been given in degrees, minutes, and seconds, we may use the **geographical** or **nautical** mile whose length is that of a great circle arc of $1'$, and which can be reduced to statute miles by means of the formulas:

$$1 \text{ nautical mile} = 1'.$$

$$1 \text{ nautical mile} = 6080 \text{ ft.}$$

$$(2) \quad 1 \text{ statute mile} = 5280 \text{ ft.}$$

$$1 \text{ nautical mile} = 1.1515 \text{ statute miles.}$$

$$1 \text{ statute mile} = 0.8684 \text{ nautical miles.}$$

To these formulas we add, for reference,

$$R = \text{earth's radius} = 3959 \text{ statute miles.}$$

Example. For the great circle track from New York, lat. $40^\circ 43' \text{ N}$, long. $74^\circ 0' \text{ W}$, to Liverpool, lat. $53^\circ 24' \text{ N}$, long. $3^\circ 4' \text{ W}$, find the distance, the bearing of each city from the other, the course angle, and the initial course.

Solution. If we reletter the ANB triangle as in Figure 99, with A at New York, B at Liverpool, and C at the north pole, we have

$$a = 90^\circ - 53^\circ 24' = 36^\circ 36',$$

$$b = 90^\circ - 40^\circ 43' = 49^\circ 17',$$

$$C = 74^\circ 0' - 3^\circ 4' = 70^\circ 56'.$$

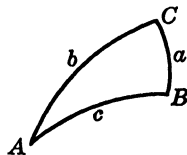


FIG. 99

The solution comes under Case III for spherical triangles (§ 79, p. 129). If we keep four significant figures in our answers, we obtain

$$A = 49^\circ 30', \quad B = 75^\circ 8', \quad c = 47^\circ 50'.$$

Hence

distance is $47^{\circ} 50'$, or 2870 nautical miles,
 bearing of Liverpool from New York is $N 49^{\circ} 30' E$,
 bearing of New York from Liverpool is $N 75^{\circ} 8' W$,
 course angle (New York to Liverpool) is $49^{\circ} 30'$,
 initial course C_n (New York to Liverpool) is $49^{\circ} 30'$.

We note that the initial course for the track from Liverpool to New York is $360^{\circ} - 75^{\circ} 8' = 284^{\circ} 52'$; also that, although New York is farther south than Liverpool, an air pilot would start in a northerly direction ($N 75^{\circ} 8' W$) in flying from Liverpool to New York by the shortest path.

EXERCISES

For the great circle track from the first to the second place specified in each of Exercises 1 to 6, find the initial course, the distance in nautical miles, and the bearing of each place from the other:

1. Boston, lat. $42^{\circ} 21' N$, long. $71^{\circ} 4' W$, and San Diego, lat. $32^{\circ} 43' N$, long. $117^{\circ} 10' W$.
2. New York, lat. $40^{\circ} 43' N$, long. $74^{\circ} 0' W$, and Cape Town, lat. $33^{\circ} 56' S$, long. $18^{\circ} 29' E$.
3. Valparaíso, lat. $33^{\circ} 2' S$, long. $71^{\circ} 39' W$, and Rio de Janeiro, lat. $22^{\circ} 54' S$, long. $43^{\circ} 10' W$.
4. Seattle, lat. $47^{\circ} 36' N$, long. $122^{\circ} 20' W$, and Manila, lat. $14^{\circ} 36' N$, long. $120^{\circ} 58' E$.
5. San Francisco, lat. $37^{\circ} 47' 30'' N$, long. $122^{\circ} 27' 50'' W$, and Sydney, lat. $33^{\circ} 51' 40'' S$, long. $151^{\circ} 12' 25'' E$.
6. Bombay, lat. $18^{\circ} 53' 45'' N$, long. $72^{\circ} 48' 55'' E$, and Greenwich, lat. $51^{\circ} 28' 40'' N$.

Find the shortest distance in statute miles between the places specified in Exercises 7, 8 (for latitudes and longitudes see Exercises 1 to 6):

7. New York and San Diego.
8. Bombay and Sydney.
9. A great circle track starts from New York (see Exercise 2) bearing $N 29^{\circ} 35' E$. At what angle will it cross the meridian of $0^{\circ} 30' W$ longitude, and how far will this point be from the north pole?
10. Determine the latitude and longitude of the point farthest north on the great circle specified in Exercise 9. *Hint.* If C is this point, A is at New York, and N at the north pole, then ACN is a right angle.
11. If ϕ is the latitude of A , and R is the earth's radius, then the parallel of latitude through A is a circle (but not a great circle) of radius $R \cos \phi$. What is the distance from Providence (lat. $41^{\circ} 50' N$, long. $71^{\circ} 24' W$) to Chicago (lat. $41^{\circ} 50' N$, long. $87^{\circ} 38' W$), (a) along their common parallel of latitude, (b) along the great circle through them?
12. From San Francisco (see Exercise 5) a ship has sailed just 2000 nautical miles on a great circle track that started due west. What is the ship's latitude and longitude?
13. Find the latitude and longitude of the point midway between Liverpool (lat. $53^{\circ} 24' N$, long. $3^{\circ} 4' W$) and San Francisco on the shortest air route.

93. Positions on the celestial sphere. In the following sections we shall consider the solution of certain problems of especial interest to navigators; for example, the problem of computing an observer's position, or the time of observation, from the position of the sun or the positions of stars.

We see the heavenly bodies as though they were situated on a vast spherical surface whose center is at the observer's eye. If the diameter of this sphere is taken so large that the length of the earth's radius is negligible by comparison, we call the center of the earth

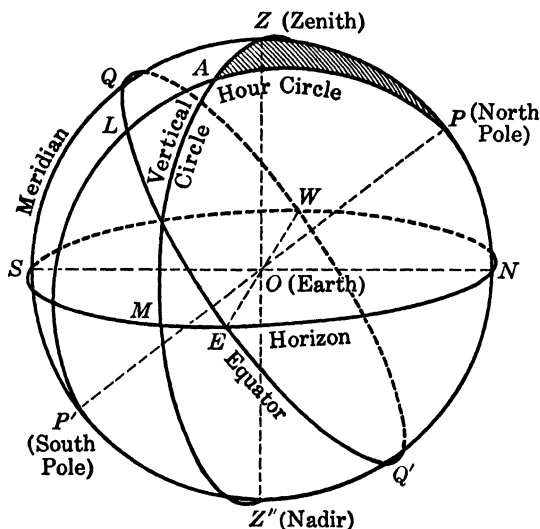


FIG. 100

the center of the **celestial sphere**. By the position of a heavenly body we mean its apparent position, that is, the position on the celestial sphere of its projection from the observer's eye.

We shall describe two coördinate systems on the celestial sphere comparable to the latitude-longitude system on the earth's surface. These are the **declination-hour angle** system, and the **altitude-azimuth** system.

The following are preliminary definitions, referring to Figure 100:

The **zenith**, Z , is the point vertically above the observer; the line OZ from the center of the earth to Z passes through the observer. The opposite point, Z' , is called the **nadir**.

The **horizon** of the observer is the great circle of which his zenith,

Z , is a pole; its plane is tangent to the earth at the observer's position. It is the circle $SWNE$ of Figure 100.

The **celestial poles**, P and P' , are the intersections of the earth's axis with the celestial sphere. In Figure 100, P is the **north pole** and P' the **south pole**.

The **celestial equator** is the great circle $QEQ'W$ of which P and P' are poles. It is the intersection of the earth's equatorial plane with the celestial sphere.

The **celestial meridian** of the observer is the great circle through Z and P . Its intersections with the horizon determine the north point, N , and the south point, S . If the observer faces north, the east point, E , is to his right, and the west point, W , to his left.

In the **declination-hour angle** system a position A on the celestial sphere is described as follows:

The **hour circle** of A is the great circle through P and A .

The **declination** of A is the distance of A (in degrees) from the celestial equator along the hour circle of A . In Figure 100, it is the measure of arc LA . It is analogous to latitude on the earth's surface. If A is 20° north of the equator its declination is given as $20^\circ N$; if it were 20° south its declination would be $20^\circ S$. The **codeclination** is 90° minus the declination (south declination taken as negative).

The **hour angle** of A is the angle ZPA between the celestial meridian and the hour circle of A . It is thus analogous to the longitude angle, but, instead of measuring it in degrees east or west of the celestial meridian, we use hours as a unit with the connecting formula

$$1 \text{ hour} = 15 \text{ degrees.}$$

This relation is suggested by the apparent revolution of the celestial sphere through 360° in 24 hours, and therefore through 15° in one hour. The hour angle is measured *westward* from the celestial meridian in hours, minutes, and seconds (time units) from 0 hr. to 24 hr.

For some purposes **right ascension** is used instead of hour angle; in the **declination-right ascension** system the zero meridian is the great circle joining the poles PP' and passing through the **vernal equinox**. The latter point is the intersection of the equator with the annual path of the sun in March.

In the **altitude-azimuth** system, the zenith takes the place of the north pole, and the horizon that of the equator in the declination-hour angle system. The corresponding definitions are:

The **vertical circle** of A is the great circle through Z and A .

The **altitude** of A is the distance of A (in degrees) above the horizon.

It is measured along the vertical circle of A ; in Figure 100 it is the measure of arc MA . Altitude is positive for positions above the horizon, and negative for positions below the horizon. The **coaltitude** is 90° minus the altitude.

The **azimuth** of A is the angle PZA between the celestial meridian and the vertical circle of A . It is measured from arc ZPN east or west up to 180° , and is designated as so many degrees, minutes, and seconds E or W.

Owing to the apparent daily revolution of the celestial sphere, the point Z is not a fixed point on the sphere over any interval of time; also at a given instant the zenith point is different for observers in different places. Thus hour angle, altitude, and azimuth are *instantaneous* and *local*, while declination is not.* As we shall see in the next section, these contrasting qualities enable us to solve problems of time and position when both sets of coördinates of a heavenly body are given.

94. The astronomical triangle, and problems connected with its solution. The triangle AZP is called the **astronomical triangle** corresponding to A . Various problems of time and position can be referred to the problem of solving the astronomical triangle.

The part ZP , which is independent of the position of A , is measured by the angle ZOP ; this angle is the colatitude, on the earth's surface, of the observer, since the direction of $P'OP$ is that of the earth's axis and OZ passes through the earth's center and the observer. Four of the other parts of triangle AZP are expressible in terms of the coördinates of A in the two systems. Thus we have the relations

$$\begin{aligned} AZ &= 90^\circ - MA = \text{coaltitude of } A, \\ AP &= 90^\circ - LA = \text{codeclination of } A, \\ ZP &= 90^\circ - QZ = \text{codeclination of the zenith}, \\ &= 90^\circ - NP = \text{coaltitude of the pole}, \\ &= \text{colatitude of the observer}, \\ \angle ZPA &= \text{hour angle of } A, \text{ if this hour angle is less than } 12 \text{ hr.,} \\ &= 24 \text{ hr.} - \text{hour angle of } A, \text{ if this hour angle is greater} \\ &\quad \text{than } 12 \text{ hr.,} \\ \angle PZA &= \text{azimuth of } A. \end{aligned}$$

* We assume here that the polar axis of the earth has an unvarying direction and that the celestial sphere is so large that its intersections with the axial line appear as the same point for all positions of the earth in its orbit.

In these equations we are to take MA as negative if A is below the horizon, and LA as negative if A is south of the equator.

We will now discuss some problems solvable in terms of the astronomical triangle.

(a) *To find the local solar time for an observer at a moment when the sun's declination is known and its altitude is measured. The observer's latitude is supposed to be known.*

Here A is the sun's position. The observer's colatitude gives ZP and the given data determine AP and AZ . We can therefore solve the triangle AZP , since three sides are known, and in particular we can find angle ZPA in degrees. We reduce this to hours, minutes, and seconds, and the result will be the time before noon, or after noon, as the case may be.

(b) *To find the time of sunrise or sunset.*

This is a special case of the preceding problem. The coaltitude of the sun is 90° at sunrise or sunset. The triangle APZ is quadrantal and is to be solved as in § 71 (p. 119).

(c) *To determine the observer's latitude after finding the altitude and hour angle of a heavenly body whose declination is known.*

This problem is at once reducible to that of finding ZP , given AZ , AP , and angle ZPA .

(d) *To find the azimuth of a heavenly body, given its declination and hour angle and the observer's latitude.*

This is the problem of determining angle PZA , given ZP , AP , and angle ZPA . The result is used to find compass corrections.

EXERCISES

In these Exercises, carry out the computation as though the data were exact:

1. An observer in latitude 30° N takes a morning observation and finds the sun's altitude to be 35° , its declination being 15° S. Find the time of observation.
2. At the U. S. Naval Observatory in Washington (lat. $38^\circ 55' 14''$ N), the altitude of the sun in the west was observed to be $60^\circ 20' 10''$ when its declination was $20^\circ 31' 30''$ N. Find the time of observation.
3. Find the time of sunrise at New York (lat. $40^\circ 43'$ N) when the sun's declination is $18^\circ 25'$ N.
4. At San Francisco (lat. $37^\circ 47' 30''$ N) what is the time of sunset when the sun's declination is $8^\circ 45'$ S?
5. Find the length of the longest day of the year (sun's declination $23^\circ 30'$ N) in Chicago (lat. $41^\circ 50'$ N). What angle does the sun's path make with the horizon at sunset?
6. Find the length of the shortest day of the year (sun's declination $23^\circ 30'$ S) in Edinburgh (lat. $55^\circ 57'$ N). What is the bearing of the sun at sunset?

7. On the longest day of the year in Chicago (see Exercise 5) at what time of day is the sun due west of the observer?

8. The compass bearing of a star is $N 47^{\circ} E$. The observer's latitude is $28^{\circ} 26' 55'' N$, the hour angle of the star is 16 hr. 59 min. 30 sec., and its declination is $37^{\circ} 54' 5'' N$. What is the error of the compass?

9. At 3:35 P.M., local solar time, the sun's altitude is $27^{\circ} 23'$, and its declination is $8^{\circ} 27' N$. Find the observer's latitude.

ANSWERS TO ODD-NUMBERED PROBLEMS

Answers have been omitted in cases where little would be left for the student to do if the answer were given. Where approximate results were required, four-place tables were used in Chapters I-V. In Chapters VI-IX, five-place tables were used. Answers obtained by using other tables may be slightly different from those given. Occasionally a problem may be solved in several ways, and answers when approximate may differ slightly according to the method used.

While the authors have attempted to check the answers with great care, they can hardly hope that all errors have been avoided. They will appreciate notice of errors which are discovered.

CHAPTER I

Art. 2, Pages 3-4

13. -840° ; $-50,400^\circ$; -70° . 15. 600° ; -600° .

Art. 6, Pages 8-9

9.

	r	$\sin \theta$	$\cos \theta$	$\tan \theta$
A	5	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$
B	10	$-\frac{4}{5}$	$\frac{3}{5}$	$-\frac{3}{4}$
C	15	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{4}{3}$
D	13	$-\frac{12}{13}$	$\frac{5}{13}$	$-\frac{12}{5}$
E	3	-1	0	—

11.

	r	$\sin \theta$	$\cos \theta$	$\tan \theta$
A	13	$\frac{12}{13}$	$\frac{5}{13}$	$\frac{12}{5}$
B	13	$\frac{5}{13}$	$-\frac{12}{13}$	$-\frac{5}{12}$
C	10	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{4}$
D	15	$-\frac{4}{5}$	$\frac{3}{5}$	$-\frac{4}{3}$
E	5	0	1	0

13. A (8, 6); B (8, -6); C ($-\frac{5}{13}$, $\frac{12}{13}$); D ($-\frac{4}{5}$, $-\frac{12}{5}$); E (0, 10).

15. A ($\frac{12}{13}$, $\frac{5}{13}$); B ($\frac{4}{5}$, $-\frac{12}{5}$); C ($-\frac{12}{13}$, $\frac{5}{13}$); D (-6, -8); E (-10, 0).

17.

	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
A	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{5}{4}$
B	$-\frac{24}{25}$	$\frac{7}{25}$	$-\frac{24}{7}$	$-\frac{7}{24}$	$\frac{25}{7}$	$-\frac{25}{24}$
C	$\frac{20}{29}$	$-\frac{21}{29}$	$-\frac{20}{21}$	$-\frac{21}{20}$	$-\frac{29}{21}$	$\frac{29}{20}$
D	$-\frac{15}{17}$	$-\frac{8}{17}$	$\frac{15}{8}$	$\frac{8}{15}$	$-\frac{17}{8}$	$-\frac{17}{15}$

19.

	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
A	$\frac{15}{17}$	$\frac{8}{17}$	$\frac{15}{8}$	$\frac{8}{15}$	$\frac{17}{8}$	$\frac{17}{15}$
B	$\frac{8}{17}$	$-\frac{15}{17}$	$-\frac{8}{15}$	$-\frac{15}{8}$	$-\frac{17}{8}$	$\frac{17}{8}$
C	$-\frac{20}{29}$	$\frac{21}{29}$	$-\frac{20}{21}$	$-\frac{21}{20}$	$\frac{29}{21}$	$-\frac{29}{20}$
D	$-\frac{15}{17}$	$-\frac{8}{17}$	$\frac{15}{8}$	$\frac{8}{15}$	$-\frac{17}{8}$	$-\frac{17}{15}$

21. III; IV.

23. II; IV.

25. II; III.

27. I; II.

29.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
78°	+	+	+	+	+	+
213°	-	-	+	+	-	-
-310°	etc.	(to be	completed by the	student)		
-601°						
1111°						

31.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
25°	+	+	etc.			
301°	-	+	etc.			
-140°	etc.	(to be	completed by the	student)		
-800°						
2000°						

Art. 7, Page 12

7. $r = 13$, $\theta = 65^\circ$.

9. $r = 11.2$, $\theta = 100^\circ$.

11. $x = 7.28$, $y = 3.36$.

13. $x = -2.79$, $y = 31.0$.

Art. 10, Page 16

1.

θ	70°	80°	90°	100°	260°
$\sin \theta$	.94	.98	1.00	.98	-.98
$\cos \theta$	.34	.17	0	-.17	-.17
$\tan \theta$	2.8	5.8	—	-5.8	5.8

3.

θ	120°	240°	250°	270°	280°
$\sin \theta$	.86	-.86	-.94	-1	-.98
$\cos \theta$	-.50	-.50	-.34	0	.17
$\tan \theta$	-1.72	1.72	2.8	—	-5.8

5. 143° ; 180° .

7. 247° ; 270° .

9.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2

11.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$

13.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
-60°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
270°	-1	0	—	0	—	-1

15.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
180°	0	-1	0	—	-1	—
-150°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2

Art. 11, Page 17

Exercise	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
1.	$\frac{5}{13}$	$\pm \frac{12}{13}$	$\pm \frac{5}{12}$	$\pm \frac{12}{5}$	$\pm \frac{13}{5}$	$\pm \frac{13}{5}$
3.	$\pm \frac{24}{25}$	$\pm \frac{7}{25}$	$\pm \frac{24}{7}$	$-\frac{7}{24}$	$\pm \frac{25}{7}$	$\pm \frac{25}{24}$
5.	$\pm \frac{3}{5}\sqrt{3}$	$\pm \frac{4}{5}$	$\pm \frac{3}{4}\sqrt{3}$	$\pm \frac{4}{3}\sqrt{3}$	$\pm \frac{5}{3}\sqrt{3}$	$\pm \frac{5}{2}$
7.	$\pm \frac{8}{17}$	$\pm \frac{15}{17}$	$\pm \frac{8}{15}$	$\pm \frac{15}{8}$	$\pm \frac{17}{8}$	$\pm \frac{17}{8}$
9.	$\pm \frac{3}{5}$	$\pm \frac{4}{5}$	$\pm \frac{3}{4}$	$-\frac{4}{3}$	$\pm \frac{5}{3}$	$\pm \frac{5}{3}$
11.	$\pm \frac{1}{2}\sqrt{2}$	$\pm \frac{1}{2}\sqrt{2}$	-1	$\pm \sqrt{2}$	$\pm \sqrt{2}$	$\pm \sqrt{2}$
13.	$\pm \frac{3}{5}$	$\pm \frac{4}{5}$	$\pm \frac{3}{4}$	$\pm \frac{4}{3}$	$-\frac{5}{4}$	$\pm \frac{5}{3}$
15.	$\pm \frac{3}{5}$	$\pm \frac{4}{5}$	$\pm \frac{3}{4}$	$\pm \frac{4}{3}$	$\pm \frac{5}{4}$	$-\frac{5}{3}$
17.	$\pm \frac{5}{13}$	$\pm \frac{12}{13}$	$\pm \frac{5}{12}$	$\pm \frac{12}{5}$	$\pm \frac{13}{5}$	$\pm \frac{13}{5}$

$$19. \cos \theta = \pm \sqrt{1 - \sin^2 \theta}; \quad \tan \theta = \frac{\pm \sin \theta}{\sqrt{1 - \sin^2 \theta}}; \quad \cot \theta = \frac{\sqrt{1 - \sin^2 \theta}}{\pm \sin \theta};$$

$$\sec \theta = \frac{\pm 1}{\sqrt{1 - \sin^2 \theta}}; \quad \csc \theta = \frac{1}{\sin \theta}.$$

$$21. \sin \theta = \frac{\pm \tan \theta}{\sqrt{1 + \tan^2 \theta}}; \quad \cos \theta = \frac{\pm 1}{\sqrt{1 + \tan^2 \theta}}; \quad \cot \theta = \frac{1}{\tan \theta};$$

$$\sec \theta = \pm \sqrt{1 + \tan^2 \theta}; \quad \csc \theta = \frac{\sqrt{1 + \tan^2 \theta}}{\pm \tan \theta}.$$

$$23. \sin \theta = \frac{\sqrt{\sec^2 \theta - 1}}{\pm \sec \theta}; \quad \cos \theta = \frac{1}{\sec \theta}; \quad \tan \theta = \pm \sqrt{\sec^2 \theta - 1};$$

$$\cot \theta = \frac{\pm 1}{\sqrt{\sec^2 \theta - 1}}; \quad \csc \theta = \frac{\pm \sec \theta}{\sqrt{\sec^2 \theta - 1}}.$$

Art. 13, Page 19

3. 10.6 mi. per hr.; 10.6 mi. per hr.

5. 68° downstream from a line straight across, at 5.4 mi. per hr.

7. 29.2 lb., at 31° angle with horizontal.

9. Westward 4.8 knots per hour; southward 13.2 knots per hour.

CHAPTER II

Art. 17, Page 22

- | | | |
|---|--|-------------------|
| 1. .3173; .9925. | 3. 1.049; 1.066. | 5. .2805; 2.932. |
| 7. .6111; .9833. | 9. 1.643; 2.309. | 11. .3378; .0437. |
| 13. $39^\circ 30'$; $12^\circ 50'$; $82^\circ 0'$; $72^\circ 0'$. | 15. $32^\circ 0'$; $2^\circ 30'$; $8^\circ 50'$; $66^\circ 40'$. | |

Art. 18, Page 24

- | | | |
|----------------------|---------------------|----------------------|
| 1. 0.6672; 0.7494. | 3. 0.9795; 1.263. | 5. 1.037; 2.224. |
| 7. 0.9896; 0.7835. | 9. 0.9131; 3.456. | 11. 0.8168; 1.496. |
| 13. $13^\circ 25'$. | 15. $6^\circ 22'$. | 17. $54^\circ 6'$. |
| 21. $37^\circ 34'$. | 23. $6^\circ 17'$. | 25. $59^\circ 18'$. |
| | | 27. $55^\circ 37'$. |

Art. 19, Pages 25-26

- | | | | | |
|----------------------|-----------------|-----------------|---------------|--------------|
| 1. 13.47. | 3. 101,800. | 5. 0.007779. | 7. 67.08. | 9. 4516. |
| 11. 7,674,000,000. | 13. 6800. | 15. 67,680,000. | 17. 0.006754. | |
| 19. 768,200,000,000. | 21. 38,840,000. | 23. 965,100. | 25. 3.13. | |
| 27. 6.92. | 29. 40.6. | 31. 0.964. | 33. 733. | 35. 0.0490. |
| 39. 8.301. | 41. 25.02. | 43. 46.84. | 45. 227.8. | 47. 0.08803. |

Art. 21, Page 28

1. $a = 3.711$; $b = 7.200$; $B = 62^\circ 44'$.
3. $a = 19.96$; $b = 18.18$; $A = 47^\circ 40'$.
5. $a = 16.37$; $c = 24.34$; $B = 47^\circ 43'$.
7. $A = 33^\circ 48'$; $B = 56^\circ 12'$; $c = 2.984$.
9. $A = 42^\circ 5'$; $B = 47^\circ 55'$; $c = 16.71$.
11. Impossible.
13. $A = 9^\circ 40'$; $b = 1.585$; $c = 1.608$. ($A = 10^\circ$; $b = 1.6$; $c = 1.6$.)
15. $B = 27^\circ 6'$; $b = 4.708$; $c = 10.33$. ($B = 27^\circ$; $b = 4.7$; $c = 10.3$.)
17. $A = 41^\circ 15'$; $a = 72.79$; $c = 110.4$. ($A = 41^\circ$; $a = 73$; $c = 110$.)
19. $A = 51^\circ 33'$; $B = 38^\circ 27'$; $a = 639.9$. ($A = 51^\circ 30'$; $B = 38^\circ 30'$; $a = 640$.)
21. $A = 48^\circ 42'$; $B = 41^\circ 18'$; $c = 53.97$.
23. $A = 60^\circ 29'$; $B = 29^\circ 31'$; $c = 30.58$.

Art. 22, Pages 29-30

- | | |
|--|---|
| 1. $C = 87^\circ 16'$; $a = b = 914.3$. | 3. $A = 65^\circ 39'$; $C = 48^\circ 42'$; $b = 1504$. |
| 5. $C = 55^\circ 36'$; $c = 82.70$; $b = 88.66$. | |
| 7. Inscribed radius = 3.137 in. (or 3.14 in. to indicated accuracy); circumscribed radius = 3.878 in. (or 3.88 in.). | |
| 9. Side = 7.794 in. (or 7.79 in. to indicated accuracy); inscribed radius = 5.364 in. (or 5.36 in.). | |

11. Perimeter = 11.66 in. (or 11.7 in. to indicated accuracy); circumscribed radius = 1.904 in. (or 1.90 in.).
 13. $B = 38^\circ 1'$; $C = 128^\circ 32'$; $a = 14.05$.
 15. $A = 49^\circ 59'$; $B = 31^\circ 49'$; $c = 5.394$.

Art. 23, Pages 32-34

- | | | | |
|------------------------------|---------------|--|-------------|
| 1. 63.4 ft. | 3. 319 ft. | 5. 267 yd. | 7. 24.4 mi. |
| 9. $62^\circ 20'$; 25.2 ft. | | 11. $20^\circ 30'$; 53.2 ft.; 142.4 ft. | |
| 13. 2460 ft. | 15. 16.1 ft. | 17. N $30^\circ 40'$ E; 27.6 mi. | |
| 19. 440 yd. | 21. 118.7 ft. | 25. 23.6 ft.; at least 126.9 ft. | |

CHAPTER III

Art. 24, Pages 36-37

- | | | | |
|-----------------------|----------------------|-----------------------|-----------------------|
| 1. .8746. | 3. — .8387. | 5. — .6157. | 7. — .6691. |
| 9. — 1.004. | 11. — .9520. | 13. .6777. | 15. — .8718. |
| 17. — .6088. | 19. — 1.013. | 21. — 5.671. | 23. — 1.192. |
| 25. — 5.671. | 27. .8391. | 29. — .8391. | 31. $120^\circ 25'$. |
| 33. $305^\circ 2'$. | 35. $97^\circ 5'$. | 37. $295^\circ 36'$. | 39. $172^\circ 58'$. |
| 41. $352^\circ 58'$. | 43. $113^\circ 6'$. | 45. $113^\circ 22'$. | 47. $293^\circ 22'$. |
| 49. $144^\circ 7'$. | 51. $324^\circ 7'$. | 53. $193^\circ 23'$. | |
55. $13^\circ 34' + n 360^\circ$; $166^\circ 26' + n 360^\circ$.
 57. None. 59. $\pm 76^\circ 26' + n 360^\circ$.
 61. None. 63. $13^\circ 12' + n 180^\circ$. 65. $66^\circ 54' + n 180^\circ$.
 67. $76^\circ 48' + n 180^\circ$. 69. $23^\circ 6' + n 180^\circ$. 71. None.
 73. $\pm 64^\circ 45' + n 360^\circ$. 75. None.
 77. $25^\circ 15' + n 360^\circ$; $154^\circ 45' + n 360^\circ$. 79. $x = -1.9392$, $y = 3.4984$.
 81. $x = 4.3568$, $y = -6.6696$. 83. $x = 9.4560$, $y = -7.3884$.
 85. $r = 25$, $\theta = 106^\circ 15'$. 87. $r = 25$, $\theta = 286^\circ 15'$.
 89. $r = 13$, $\theta = 202^\circ 13'$.

Art. 27, Pages 40-41

- | | |
|---|---|
| 1. $-\sin 57^\circ$; $-\cos 33^\circ$. | 3. $\sin 57^\circ$; $\cos 33^\circ$. |
| 5. $-\cos 24^\circ$; $-\sin 66^\circ$. | 7. $\cos 4^\circ$; $\sin 86^\circ$. |
| 9. $\tan 26^\circ$; $\cot 64^\circ$. | 11. $-\tan 26^\circ$; $-\cot 64^\circ$. |
| 13. $-\tan 33^\circ$; $-\cot 57^\circ$. | 15. $\cot 43^\circ$; $\tan 47^\circ$. |
| 17. $\csc 11^\circ$; $\sec 79^\circ$. | 19. $-\sec 21^\circ$; $-\csc 69^\circ$. |
| 21. $\sec 32^\circ$; $\csc 58^\circ$. | 23. $\csc 42^\circ$; $\sec 48^\circ$. |

Art. 32, Page 47

11. Period = 360° . 13. Period = 360° .

CHAPTER IV

Art. 34, Pages 51-52

13. $\frac{\sin \theta + 1}{\cos \theta}$.

17. $\frac{\sin^2 \theta + 1}{\cos^2 \theta}$.

21. $\sin \theta + \frac{1}{\cos \theta}$.

25. $\frac{1 + \sin^2 \theta}{1 - \sin^2 \theta}$.

29. $\frac{2 - \sin^2 \theta}{\sin \theta}$.

15. $\sin^2 \theta + \frac{1}{\cos^2 \theta}$.

19. $\cos^2 \theta + \frac{\sin^2 \theta}{\cos^2 \theta}$.

23. $\sin \theta \cos \theta$.

27. $\frac{1}{\sin \theta (1 - \sin^2 \theta)}$.

31. $\frac{1 + \sin^2 \theta}{\sin^2 \theta}$.

Art. 37, Page 56

1. $\frac{\sqrt{6} - \sqrt{2}}{4}$.

3. $\frac{\sqrt{6} - \sqrt{2}}{4}$.

25. -0.2797 .

27. 0.0098 .

29. $\frac{3}{8}$.

31. $\frac{3}{8}$.

33. $\frac{3}{8}$.

35. $-\frac{3}{8}$.

37. $\frac{21}{221}$.

Art. 38, Pages 58-59

1. $\frac{\sqrt{3} - 1}{1 + \sqrt{3}}$ or $2 - \sqrt{3}$.

3. $-2 - \sqrt{3}$.

9. $\frac{7}{16}$.

11. $-\frac{3}{4}$.

Art. 40, Pages 62-63

1. $\frac{2}{5}; \frac{7}{5}; \frac{2}{5}$.

3. $\frac{1}{16}; -\frac{1}{16}; -\frac{1}{16}$.

5. $\frac{2}{5}; \frac{7}{5}; \frac{2}{5}$.

7. $-\frac{1}{16}; -\frac{1}{16}; \frac{1}{16}$.

9. $\frac{1}{\sqrt{10}}; \frac{3}{\sqrt{10}}; \frac{1}{3}$.

11. $\frac{3}{\sqrt{13}}; -\frac{2}{\sqrt{13}}; -\frac{3}{2}$.

13. $\frac{-1}{\sqrt{10}}; \frac{-3}{\sqrt{10}}; \frac{1}{3}$.

15. $\frac{-2}{\sqrt{13}}; \frac{3}{\sqrt{13}}; -\frac{2}{3}$.

17. $\frac{\sqrt{2 - \sqrt{3}}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}; \frac{\sqrt{2 + \sqrt{3}}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}; 2 - \sqrt{3}$.

19. $\frac{\sqrt{6} + \sqrt{2}}{4}; \frac{\sqrt{6} - \sqrt{2}}{4}; 2 + \sqrt{3}$.

Art. 41, Pages 65-66

1. $\sin 4\theta + \sin 2\theta$.

3. $\sin 7\alpha - \sin 3\alpha$.

5. $\cos \alpha - \cos 5\alpha$.

7. $\frac{1}{2} \cos 3\alpha + \frac{1}{2} \cos \alpha$.

9. $\frac{1}{2} \sin 11\alpha - \frac{1}{2} \sin \alpha$.

27. $2 \sin 40^\circ \cos 4^\circ$.

29. $-2 \cos 45^\circ \sin 5^\circ$.

31. $2 \cos 45^\circ \cos 5^\circ$.

33. $2 \sin 25^\circ \sin 5^\circ$.

35. $2 \sin 3\alpha \cos \alpha$.

37. $2 \cos 2\alpha \sin \alpha$.

39. $2 \cos \frac{5\alpha}{2} \cos \frac{3\alpha}{2}$.

41. $-2 \sin 2\alpha \sin \alpha$.

43. $1 \cdot \tan 25^\circ$.

45. $\cot 30^\circ \tan 10^\circ$.

47. $1 \cdot \cot 2\alpha$.

49. $-\tan \frac{3\alpha}{2} \tan \frac{\alpha}{2}$.

CHAPTER V

Art. 42, Page 70

1. $\frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{3}$. 3. $\frac{\pi}{6}, \frac{5\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{2}$.
 5. 0.6109; 0.8727; 1.0472; 2.0594. 7. 0.3491; 0.7854; 1.2217; 2.2690.
 9. 0.2696; 2.3072. 11. 0.7540; 2.4376. 13. $22^\circ 30'$; 120° ; 630° ; $114^\circ 35.5'$.
 15. 36° ; $77^\circ 8.6'$; 420° ; $229^\circ 11'$. 17. $54^\circ 1'$; $59^\circ 28'$. 19. $31^\circ 7'$; $88^\circ 25'$.
 21. $49^\circ 24'$; $165^\circ 49'$. 23. $57^\circ 57'$; $172^\circ 32'$.
 25. 1.5 radians; $85^\circ 57'$. 27. 1.2 radians; $68^\circ 45'$.
 29. 1.5 radians; $85^\circ 57'$. 31. 2.2 radians; $126^\circ 3'$.
 33. 4.5 in. 35. 6.000 in. 37. 9.800 in. 39. 17.19 in.
 41. 249.4 ft. 43. 9.579 in. 45. 13.31 ft.

Art. 43, Page 74

1. 45° . 3. 29° . 5. -30° . 7. -59° . 9. 60° .
 11. 26° . 13. 60° . 15. 114° . 17. 45° . 19. $34^\circ 30'$.
 21. -30° . 23. -23° . 25. 30° . 27. 34° . 29. 34° .
 31. 45° . 33. 90° . 35. $35^\circ 20'$. 37. 0.0669 radians.
 39. 0.0669 radians. 41. 0.0524 radians. 43. 1.5563 radians.
 45. -0.0145 radians. 47. $\pi \pm 2n\pi$ radians; $180^\circ \pm n360^\circ$.
 49. $1.2915 \pm 2n\pi$ radians, $1.8501 \pm 2n\pi$ radians; $74^\circ \pm n360^\circ$, $106^\circ \pm n360^\circ$.
 51. $-0.1571 \pm 2n\pi$ radians, $-2.9845 \pm 2n\pi$ radians; $-9^\circ \pm n360^\circ$, $-171^\circ \pm n360^\circ$.
 53. $0.6431 \pm 2n\pi$ radians, $2.4985 \pm 2n\pi$ radians; $36^\circ 52' \pm n360^\circ$, $143^\circ 8' \pm n360^\circ$.
 55. $0.8760 \pm n\pi$ radians; $50^\circ 11' \pm n180^\circ$.
 57. $1.2310 \pm 2n\pi$ radians, $-1.2310 \pm 2n\pi$ radians; $70^\circ 32' \pm n360^\circ$, $-70^\circ 32' \pm n360^\circ$.
 59. $\frac{4}{3}$. 61. $\frac{2}{3}$. 63. $\frac{2}{3}$. 65. $\pm \frac{\sqrt{2}}{2}$.
 67. $20^\circ \pm n360^\circ$, $160^\circ \pm n360^\circ$. 69. 50° . 71. 170° .
 73. 1. 75. 2. 77. $\frac{1+\sqrt{3}}{2}$.

Art. 44, Page 77

1. $48^\circ 11' \pm n360^\circ$, $-48^\circ 11' \pm n360^\circ$.
 3. $41^\circ 49' \pm n360^\circ$, $138^\circ 11' \pm n360^\circ$.
 5. $60^\circ \pm n180^\circ$, $120^\circ \pm n180^\circ$. 7. $60^\circ \pm n180^\circ$, $120^\circ \pm n180^\circ$.
 9. $0^\circ \pm n180^\circ$. 11. $0^\circ \pm n180^\circ$.
 13. $0^\circ \pm n180^\circ$, $45^\circ \pm n90^\circ$. 15. $22^\circ 30' \pm n45^\circ$.

17. $0.9552 \pm n\pi$, $-0.9552 \pm n\pi$.
 21. $\frac{\pi}{4} \pm \frac{n\pi}{2}$.
 27. 60° , 120° .
 31. $22^\circ 30'$, $67^\circ 30'$, 90° , $112^\circ 30'$, $157^\circ 30'$.
 33. 0° , 120° , 240° , 360° ; 0 , $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, 2π .
 35. 45° , $63^\circ 26'$, 225° , $243^\circ 26'$; $\frac{\pi}{4}$, 1.1071 , $\frac{5\pi}{4}$, 4.2487 .
 37. 90° , 120° , 240° , 270° ; $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{3\pi}{2}$.
 39. 30° , 150° ; $\frac{\pi}{6}$, $\frac{5\pi}{6}$.
 41. $22^\circ 30'$, 90° , $112^\circ 30'$, $202^\circ 30'$, 270° , $292^\circ 30'$; $\frac{\pi}{8}$, $\frac{\pi}{2}$, $\frac{5\pi}{8}$, $\frac{9\pi}{8}$, $\frac{3\pi}{2}$, $\frac{13\pi}{8}$.
 43. $67^\circ 23'$, $232^\circ 53'$; 1.1761 , 4.0646 .
 45. $41^\circ 49'$, $138^\circ 11'$; 0.7298 , 2.4118 .
 47. None. 49. None.

CHAPTER VI

Art. 47, Pages 81-82

- | | | |
|----------------------------|-------------------|-------------------------------|
| 1. 1; 4. | 5. -1 ; -3 . | 9. 3.57426; 0.45255. |
| 11. 5.33905; 8.10857 - 10. | | 13. 9.95347 - 10; 6.47451. |
| 15. 4.63783. | 17. 0.82812. | 19. 2.56983. 21. 4.68517. |
| 23. 7.85601 - 10. | 25. 9.80473 - 10. | 27. 6.40957 - 10. 29. 3.2110. |
| 31. 6.6070. | 33. 300.00. | 35. 0.051740. 37. 0.080220. |
| 39. 261.04. | 41. 5015.3. | 43. 2578900. 45. 26978000. |
| 47. 0.00027825. | 49. 0.74745. | 51. 0.0000050196. |

Art. 50, Page 85

- | | | | | |
|-------------|-------------|--------------|-----------------|---------------|
| 1. 70.663. | 3. 2421.6. | 5. 82.350. | 7. 0.77163. | 9. 0.18973. |
| 11. 31.627. | 13. 3086.7. | 15. 0.70285. | 17. -2.8179 . | 19. 0.041168. |

Art. 51, Pages 87-88

- | | | |
|--------------------------|---------------------------|---------------------------|
| 1. 9.58515 - 10. | 3. 9.96366 - 10. | 5. 9.91500 - 10. |
| 7. 10.47122 - 10. | 9. 9.99862 - 10. | 11. 9.96336 - 10. |
| 13. $21^\circ 11' 0''$. | 15. $18^\circ 20' 0''$. | 17. $52^\circ 56' 0''$. |
| 21. $40^\circ 7' 44''$. | 23. $7^\circ 25' 38''$. | 25. $78^\circ 10' 40''$. |
| 29. $26^\circ 0' 0''$. | 31. $85^\circ 59' 47''$. | 27. $62^\circ 3' 33''$. |

CHAPTER VII

Art. 52, Page 91

5. $A = 28^\circ 29' 10''$, $a = 2344.4$, $c = 4915.4$.
7. $A = 70^\circ 32' 25''$, $a = 0.99750$, $c = 1.0579$.
9. $A = 38^\circ 4' 5''$, $B = 51^\circ 55' 55''$, $a = 0.55022$.
11. $A = 42^\circ 48' 50''$, $B = 47^\circ 11' 10''$, $b = 0.045708$.
13. $B = 17^\circ 45' 15''$, $a = 6.9500$, $b = 2.2252$.
15. $A = 10^\circ 15' 20''$, $a = 4.8423$, $b = 26.763$.
17. $A = 67^\circ 20' 55''$, $B = 22^\circ 39' 5''$, $a = 650.40$.
19. $A = 42^\circ 2' 55''$, $B = 47^\circ 57' 5''$, $a = 18308$.

Art. 56, Pages 94-95

1. $C = 78^\circ 14' 10''$, $b = 33650$, $c = 34188$.
3. $A = 74^\circ 37' 40''$, $b = 22.238$, $c = 17.076$.
5. $B = 37^\circ 30' 0''$, $b = 0.014419$, $c = 0.020530$.
7. $C = 61^\circ 26' 40''$, $b = 166.31$, $c = 184.63$.
9. $C = 33^\circ 46' 0''$, $a = 878.16$, $c = 528.49$.
11. $AC = 6368.1$ ft., $BC = 9990.8$ ft.

Art. 57, Page 98

1. $B = 34^\circ 39' 25''$, $C = 102^\circ 43' 35''$, $c = 34.770$.
3. $A = 39^\circ 11' 40''$, $C = 45^\circ 7' 5''$, $c = 5151.3$.
5. $B_1 = 91^\circ 44' 5''$, $C_1 = 48^\circ 10' 45''$, $b_1 = 10,870$.
 $B_2 = 8^\circ 5' 35''$, $C_2 = 131^\circ 49' 15''$, $b_2 = 1531.0$.
7. $B = 18^\circ 51' 30''$, $C = 106^\circ 54' 35''$, $c = 0.59051$.
9. $A = 36^\circ 42' 30''$, $C = 69^\circ 45' 10''$, $c = 3.6408$.
11. $A_1 = 37^\circ 37' 55''$, $B_1 = 76^\circ 58' 30''$, $a_1 = 20.370$.
 $A_2 = 11^\circ 34' 55''$, $B_2 = 103^\circ 1' 30''$, $a_2 = 6.6980$.
13. No solution.
15. $B = 24^\circ 10' 25''$, $C = 34^\circ 22' 15''$, $b = 0.23635$.
17. N $2^\circ 40' 30''$ E.

Art. 59, Page 100

1. $A = 35^\circ 39' 0''$, $B = 102^\circ 24' 40''$, $c = 37.504$.
3. $B = 26^\circ 41' 0''$, $C = 44^\circ 49' 30''$, $a = 3.6523$.
5. $A = 80^\circ 26' 15''$, $C = 70^\circ 10' 0''$, $b = 2.5006$.
7. $B = 36^\circ 9' 5''$, $C = 20^\circ 26' 40''$, $a = 0.10113$.
9. $A = 85^\circ 7' 40''$, $B = 19^\circ 23' 40''$, $c = 6.8254$.
11. 14.421 ft.; 19.688 ft.

Art. 62, Pages 103-104

1. $A = 57^\circ 19' 0''$, $B = 50^\circ 26' 20''$, $C = 72^\circ 14' 50''$.
3. $A = 37^\circ 37' 5''$, $B = 84^\circ 19' 30''$, $C = 58^\circ 3' 30''$.
5. $A = 41^\circ 39' 55''$, $B = 23^\circ 2' 50''$, $C = 115^\circ 17' 15''$.
7. $A = 42^\circ 15' 10''$, $B = 114^\circ 16' 15''$, $C = 23^\circ 28' 35''$.
9. $A = 72^\circ 17' 15''$, $B = 49^\circ 4' 55''$, $C = 58^\circ 37' 55''$.
11. $A = 32^\circ 58' 30''$, $B = 50^\circ 23' 40''$, $C = 96^\circ 37' 55''$.
13. $A = 30^\circ$, $B = 50^\circ$, $C = 100^\circ$ (to the nearest 10°).
15. $A = 43^\circ$, $B = 50^\circ$, $C = 87^\circ$ (to the nearest degree).

Art. 63, Page 105

- | | | | |
|---------------|------------|------------|-------------|
| 5. 2,945,600. | 7. 2161.4. | 9. 489.09. | 11. 8.8740. |
|---------------|------------|------------|-------------|

Miscellaneous, Pages 105-108

1. $A = 46^\circ 59' 0''$, $B = 105^\circ 30' 20''$, $c = 206.61$.
3. $C = 30^\circ$, $a = 2.0005$, $c = 1.0003$.
5. $A_1 = 78^\circ 26' 20''$, $B_1 = 60^\circ 20' 40''$, $b_1 = 538.76$.
 $A_2 = 101^\circ 33' 40''$, $B_2 = 37^\circ 13' 20''$, $b_2 = 375.02$.
9. $C = 97^\circ 59' 0''$, $a = 1.9578$, $c = 2.0135$.
11. No triangle.
13. $A = 27^\circ 56' 40''$, $B = 38^\circ 10' 10''$, $C = 113^\circ 53' 10''$.
15. No triangle.
17. $A = 48^\circ 54' 45''$, $B = 7^\circ 38' 0''$, $b = 14529$.
19. No triangle.
21. $A = 98^\circ 49' 0''$, $B = 30^\circ 11' 30''$, $C = 50^\circ 59' 30''$.
23. $A = 30^\circ 26' 20''$, $C = 28^\circ 53' 20''$, $b = 121.85$.
25. $A = 23^\circ 18' 0''$, $b = 7.3543$, $c = 3.9042$.
27. $S = 27048$.
29. $S = 73.157$.
31. $S = 0.23890$.
33. $r = 88.744$, $R = 336.93$.
35. $r = 0.12410$, $R = 1.0161$.
37. $r_1 = 10.412$, $R_1 = 22.043$.
 $r_2 = 6.6063$, $R_2 = 22.043$.
39. $r = 11.075$, $R = 31.622$.
41. $32^\circ 4' 0''$. (To three significant figures, $32^\circ 0'$.)
43. $r = 6.3281$ in. (To two significant figures, $r = 6.3$ in.)
45. $r = 3\sqrt{2}$ in. (By five-place logarithms, $r = 4.2427$ in.; to three significant figures, $r = 4.24$ in.)
47. Length = 1118.22 ft., $AD = 1106.16$ ft. (To three significant figures, length = 1120 ft., $AD = 1110$ ft.)
49. 18469 mi., 12.826 mi. (To four significant figures, 18470 mi., 12.83 mi.)
51. $13^\circ 38' 20''$. (To two significant figures, 14° .)
53. 85.69 ft. (To two significant figures, 86 ft.)
55. 24.188 ft.
57. 2.1361 in.
59. 3.6445 ft. (To three significant figures, 3.64 ft.)

61. Angle made with greater force = $10^{\circ} 58' 45''$, magnitude = 53.123 lb. (To four significant figures, $10^{\circ} 59'$, 53.12 lb.)
63. 95,758,000 mi., or 43,113,000 mi. (To four significant figures, 95,760,000 mi., or 43,110,000 mi.)
65. 6.9663 mi., N $61^{\circ} 27' 25''$ W. (To two significant figures, 7.0 mi., N 61° W.)
67. To the required accuracy, 1490 yd.

CHAPTER VIII

Art. 65, Page 111

1. $143^{\circ} 14'$ (to the nearest minute). 3. As 3 to 8.

Art. 70, Page 118

11. $a = 15^{\circ} 36' 15''$, $b = 55^{\circ} 28' 35''$, $B = 79^{\circ} 31' 0''$.
13. $b = 90^{\circ}$, $c = 90^{\circ}$, $B = 90^{\circ}$.
15. $a = 147^{\circ} 31' 50''$, $b = 60^{\circ} 32' 5''$, $c = 114^{\circ} 31' 5''$.
17. $b = 22^{\circ} 0' 50''$, $A = 68^{\circ} 19' 0''$, $B = 30^{\circ} 31' 0''$.
19. $a = 17^{\circ} 59' 30''$, $c = 112^{\circ} 9' 55''$, $B = 97^{\circ} 36' 10''$.
21. $a = 44^{\circ} 44' 0''$, $c = 46^{\circ} 40' 5''$, $B = 20^{\circ} 49' 50''$.
23. $a = 98^{\circ} 35' 5''$, $A = 96^{\circ} 19' 55''$, $B = 47^{\circ} 38' 5''$.
25. $c = 117^{\circ} 3' 5''$, $A = 33^{\circ} 11' 45''$, $B = 106^{\circ} 34' 15''$.
27. $a_1 = 28^{\circ} 14' 30''$, $c_1 = 78^{\circ} 53' 20''$, $A_1 = 28^{\circ} 49' 55''$.
 $a_2 = 151^{\circ} 45' 30''$, $c_2 = 101^{\circ} 6' 40''$, $A_2 = 151^{\circ} 10' 5''$.
29. $a = 140^{\circ} 12' 0''$, $b = 120^{\circ} 30' 10''$, $A = 135^{\circ} 57' 40''$.

Art. 71, Page 119

1. $A = B = 62^{\circ} 57' 20''$, $C = 74^{\circ} 11' 10''$.
3. $a = b = 136^{\circ} 6' 45''$, $C = 161^{\circ} 39' 35''$.
5. $a = b = 74^{\circ} 26' 5''$, $C = 163^{\circ} 11' 40''$.
7. $a = b = 107^{\circ} 50' 15''$, $c = 57^{\circ} 34' 0''$.
9. $A = 35^{\circ} 35' 20''$, $B = 147^{\circ} 31' 10''$, $C = 46^{\circ} 41' 15''$.
11. $a_1 = 30^{\circ} 54' 50''$, $A_1 = 30^{\circ} 14' 20''$, $C_1 = 78^{\circ} 35' 20''$.
 $a_2 = 149^{\circ} 5' 10''$, $A_2 = 149^{\circ} 45' 40''$, $C_2 = 101^{\circ} 24' 40''$.
13. $A = 154^{\circ} 47' 10''$, $B = 127^{\circ} 59' 10''$, $C = 123^{\circ} 50' 20''$.
15. $b_1 = 28^{\circ} 49' 30''$, $B_1 = 28^{\circ} 14' 5''$, $C_1 = 78^{\circ} 53' 20''$.
 $b_2 = 151^{\circ} 10' 30''$, $B_2 = 151^{\circ} 45' 55''$, $C_2 = 101^{\circ} 6' 40''$.

Art. 73, Page 122

3. $C = 19^{\circ} 11'$. 5. $a = 37^{\circ} 50'$. 7. $a = 38^{\circ} 55'$.
9. $A = 65^{\circ} 30'$. 11. $c = 42^{\circ} 27'$, $A = 59^{\circ} 8'$, $B = 84^{\circ} 0'$.

Art. 78, Page 129

1. $A = 109^\circ 57' 45''$, $B = 78^\circ 59' 10''$, $C = 39^\circ 59' 55''$.
3. $A = 94^\circ 49' 25''$, $B = 70^\circ 1' 40''$, $C = 131^\circ 18' 15''$.
5. $A = 55^\circ 52' 40''$, $B = 59^\circ 4' 0''$, $C = 88^\circ 12' 35''$.
7. $A = 65^\circ 33' 5''$, $B = 97^\circ 26' 30''$, $C = 95^\circ 38' 5''$.
9. $a = 64^\circ 11' 50''$, $b = 33^\circ 47' 45''$, $c = 40^\circ 37' 15''$.
11. $a = 27^\circ 22' 20''$, $b = 117^\circ 9' 35''$, $c = 138^\circ 20' 40''$.
13. $a = 47^\circ 6' 15''$, $b = 40^\circ 18' 20''$, $c = 20^\circ 17' 20''$.
15. $a = 82^\circ 56' 45''$, $b = 107^\circ 35' 10''$, $c = 27^\circ 9' 40''$.

Art. 80, Pages 130-131

1. $A = 96^\circ 8' 35''$, $B = 65^\circ 22' 35''$, $c = 99^\circ 40' 50''$.
3. $A = 146^\circ 33' 25''$, $C = 47^\circ 28' 40''$, $b = 27^\circ 17' 0''$.
5. $A = 55^\circ 52' 25''$, $B = 20^\circ 9' 55''$, $c = 66^\circ 20' 50''$.
7. $A = 142^\circ 12' 45''$, $C = 129^\circ 5' 30''$, $b = 60^\circ 4' 55''$.
9. $a = 64^\circ 20' 5''$, $b = 40^\circ 40' 50''$, $C = 31^\circ 39' 40''$.
11. $a = 159^\circ 6' 50''$, $c = 139^\circ 53' 30''$, $B = 102^\circ 33' 20''$.
13. $a = 64^\circ 47' 15''$, $b = 48^\circ 3' 25''$, $C = 73^\circ 46' 50''$.
15. $a = 124^\circ 12' 30''$, $c = 82^\circ 47' 40''$, $B = 128^\circ 41' 50''$.

Art. 82, Pages 133-134

1. No solution.
3. One solution.
5. One solution.
7. $B_1 = 70^\circ 20' 30''$, $C_1 = 80^\circ 13' 0''$, $c_1 = 50^\circ 46' 10''$.
 $B_2 = 109^\circ 39' 30''$, $C_2 = 34^\circ 4' 40''$, $c_2 = 26^\circ 7' 55''$.
9. $A = 22^\circ 20' 25''$, $B = 146^\circ 39' 30''$, $a = 27^\circ 22' 10''$.
11. $A_1 = 55^\circ 1' 25''$, $B_1 = 146^\circ 39' 0''$, $b_1 = 138^\circ 12' 5''$.
 $A_2 = 124^\circ 58' 35''$, $B_2 = 45^\circ 20' 20''$, $b_2 = 59^\circ 34' 45''$.
13. $b_1 = 84^\circ 10' 0''$, $c_1 = 71^\circ 42' 35''$, $C_1 = 68^\circ 28' 40''$.
 $b_2 = 95^\circ 50' 0''$, $c_2 = 40^\circ 49' 30''$, $C_2 = 39^\circ 49' 55''$.
15. $b = 40^\circ 35' 10''$, $a = 39^\circ 10' 5''$, $A = 30^\circ 25' 10''$.
17. $c_1 = 33^\circ 0' 20''$, $b_1 = 147^\circ 4' 40''$, $B_1 = 110^\circ 5' 10''$.
 $c_2 = 146^\circ 59' 40''$, $b_2 = 15^\circ 6' 20''$, $B_2 = 26^\circ 45' 55''$.

Art. 83, Page 135

1. 0.5236 sq. ft.
3. 26.15 sq. in.
5. 9335 sq. ft.
7. 232.6 sq. ft.
9. 873,000 sq. ft.
11. 43,310 sq. mi.

CHAPTER IX

Art. 85, Pages 138-140

NOTE. Five-place tables were used in obtaining the answers for this chapter, except in Art. 92 and 94. These answers are rounded off to the appropriate number of places and given in parentheses.

1. 255.32 yd.; 199.48 yd. (255 yd.; 199 yd.)
3. 55.943 yd.; 48.955 yd.; S 41° 11' 20" E; 74.338 yd. (56 yd.; 49 yd.; S 41° E; 74 yd.)
5. 33.550 ft.; 216.68 ft.; N 81° 11' 55" E; 219.25 ft. (33.55 ft.; 216.7 ft.; N 81° 12' E; 219.2 ft.)
7. 750.18 yd.; S 76° 51' 45" E. (750.2 yd.; S 76° 52' E)
9. 0.734 yd.; - 0.20 yd. 11. 51° 23' 10"; 1303.7. (51° 23'; 1304)
13. 7455.8 yd., S 38° 49' 40" E; 1102.0 yd., 427.0 yd. (7456 yd., S 38° 50' E; 1102 yd., 427 yd.)
15. $h = d \sin \alpha$, $k = \frac{d \cos \alpha \sin (\delta + \phi) \tan \beta}{\sin (\theta - \phi)}$; $\frac{d \cos \alpha \sin (\delta + \theta)}{\sin (\theta - \phi)}$.

Art. 86, Page 142

1. 38° 52' 3"; 3923.3 yd. (38° 52'; 3923 yd.) 3. (0.3, 2.6). 5. (0.4, 4.3).

Art. 87, Pages 145-146

Answers which follow when expressed in mils are given to the nearest mil, and when expressed in degrees and minutes are given to the nearest minute.

1. 178 mils; 444 mils; 800 mils. 3. 71 mils; 320 mils; 622 mils.
5. 130 mils; 311 mils. 7. 46 mils; 434 mils.
9. 7° 2'; 14° 38'. 11. 4° 44'; 22° 30'.
13. 0.025 radians; 0.221 radians. 15. 0.018 radians; 0.447 radians.
17. 255 mils; 1222 mils. 19. 407 mils; 672 mils.
21. 100 mils. 23. 350 mils. 25. 79 mils. 27. 233 mils.
37. 2 (or 3) mils. 39. 10.5 yd. 41. 32 yd. 43. 155 mils.
45. Increase range 1 yd., deflect gun 9 mils to left.

Art. 91, Page 152

1. 105.4 east; 67.2 north. 3. 83.9 east; 157.0 south.
5. 120.1 west; 183.1 south. 7. 146.4 west; 96.0 north.

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|--|---|
| 9. 77.5; $72^{\circ} 2'$. | 11. 112.2; $320^{\circ} 56'$. |
| 13. 68.6; $110^{\circ} 12'$. | 15. 78.9; $246^{\circ} 24'$. |
| 17. $44^{\circ} 19' N$; $43^{\circ} 32' W$. | 19. 95.8 mi. |
| 21. $57^{\circ} 57' S$; $10^{\circ} 0' W$. | 23. $14^{\circ} 20' S$; $2^{\circ} 58' W$. |
| 25. 165.3 mi.; $48^{\circ} 31'$. | 27. 68.3 mi. south; 154.0 mi. east. |
| 29. $39^{\circ} 7.6' N$; $63^{\circ} 8' W$. | |

Art. 92, Page 155

Four-place tables were used in computing the answers for this article, except in Exercise 5.

- 1.** Distance = 2239 nautical miles. Bearing of San Diego from Boston, $N 89^{\circ} 23' W$. Bearing of Boston from San Diego, $N 61^{\circ} 27' E$.
- 3.** Distance = 1621 nautical miles. Bearing of Valparaiso from Rio de Janeiro, $S 61^{\circ} 41' W$. Bearing of Rio de Janeiro from Valparaiso, $N 75^{\circ} 19' E$.
- 5.** Distance = 6445.5 nautical miles. Bearing of Sydney from San Francisco, $S 60^{\circ} 17' 20'' W$. Bearing of San Francisco from Sydney, $N 55^{\circ} 44' 44'' E$.
- 7.** Distance = 2428 statute miles.
- 9.** At an angle of $86^{\circ} 28'$, 1321 nautical miles from the north pole.
- 11.** (a) 835.8 statute miles; (b) 834.6 statute miles.
- 13.** Lat. $63^{\circ} 6' N$, long. $76^{\circ} 11' W$.

Art. 94, Pages 159-160

Four-place tables were used in obtaining the answers for this article.

- | | |
|---|--|
| 1. 9 hr. 49 min. A.M. | 3. 4 hr. 54 min. A.M. |
| 5. 15 hr. 3 min. Angle between sun's path and horizon = $43^{\circ} 21'$. | |
| 7. 4 hr. 4 min. P.M. | 9. $54^{\circ} 26' N$ or $26^{\circ} 10' S$. |

