COMMITTEE DECISIONS WITH COMPLEMENTARY VALUATION

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PREFATORY NOTE

The following monograph was first presented for publication in November, 1949. When, after eighteen months, it was found impossible to publish it in a journal, the authors sought to have it published as a separate monograph; and they have been enabled to do so by the generous financial assistance of the Carnegie Trust for Scottish Universities.

The only alterations made in the original paper have been a change in the title and the correction of a few defects of style.

June, 1951.

THE ASSUMPTIONS AND THE ARGUMENT

We consider always a committee of three persons which reaches its decisions by a simple majority; though the argument of Section III is extended to a committee of any number of members (§45).

In Section I we suppose that a number of motions have been put forward in a committee. When the number of motions is finite, being, say, *m* in number, the members' valuations of them can be expressed in terms of a discrete variable with *m* values. When the number of motions is taken to be infinite, the members' valuations can be expressed in terms of a continuous variable.

The committee procedure in use, we assume, is that encountered in practice, where every motion enters the voting process once, and continues to be put against other motions until it is either defeated by another motion or finally emerges as the decision of the committee.

For these circumstances we investigate the conditions that are necessary and sufficient for the existence of a *majority decision*, *i.e.*, a motion which would be able to get at least a simple majority over every other motion. When such a motion exists, it is bound to emerge as the decision of a committee that follows the procedure we assume.

With a discrete variable (number of motions finite), a construction is given that enables us to test whether or not a majority decision exists, and, where it exists, to pick it out.

With a continuous variable (number of motions infinite), the same conditions as before must be satisfied for the existence of a majority decision. These conditions may be expressed by reference to a certain three-dimensional solid, or by reference to a two-dimensional diagram. As before we show how to test whether a majority decision exists, and, where there is one, how to pick it out.

Section II of the paper employs the same assumptions as Section I, but a member's valuation of any motion is taken to depend on two aspects of the motion; and each aspect is specified by a separate variable. With a given value of one of the variables—one aspect given—his valuation of the motion will depend on the associated value of the other variable: *i.e.*, his valuations will be complementary in nature.

As before we seek to find the *majority decision* of a committee, where this majority decision is again that motion, if any, that would be able to get at least a simple majority over each of the others.

When the two variables by which the motions are specified are discrete, the majority decision can be picked out in accordance with the rules of Section I; and this case in effect degenerates into that in which valuations are expressible in terms of a single variable.

When the two variables are continuous, a member's preferences in regard to the motions can no longer be specified in terms of curves, but must be specified in terms of preference surfaces. For the most part we consider only the simplest type of preference surface, the " simple hump " which has a single peak and whose sections by vertical planes are single-peaked.

By consideration of the indifference contours on such humps, a rule is obtained which shows whether a majority decision exists, and which enables us to pick out the majority decision where there is one.

In Section III the members are assumed to value, not only motions put forward in the committee, but also " sets of circumstances " which they envisage. When all of the members envisage the same set of circumstances, it too, when represented by a symbol—*i.e.*, by the value of a variable—can be regarded as finding a definite level on the preference curve or preference surface of each of the members concerned.

As regards the procedure that it follows, we suppose now that the committee is reaching decisions on two topics alternately, first on the one topic, then on the other; and that then a second decision is taken on the first topic, a second decision on the second topic; then a third decision on the first topic and so on.

Each member's valuations in regard to these topics are assumed to be complementary and will be expressed as a function of two variables. His valuation of any arrangement on the one topic will depend on the arrangement that he expects in the other. Some definite assumption about members' expectations must, therefore, be introduced; and we suppose that when a decision is being taken on either topic, each member expects that the decision last arrived at on the other topic will remain unchanged.

With these assumptions we seek to discover, not majority decisions—for, with the committee procedure now in use, the committee decision arrived at need not correspond to a majority decision—but *positions of stable equilibrium*, *i.e.*, positions which, once arrived at through this committee procedure, will not be departed from.

By reference to the indifference contours, it is shown that these stable positions will be given by the points of intersection of two loci. Sometimes the stable position will be approached by an ascending or descending staircase, or by a converging spiral of decisions. Sometimes, for a given group of schedules, there will exist several positions that would give stability and one of them might be reached after a sufficient number of votes. Which one, if any, will be reached, will depend in general on the initial position from which the voting starts and on whether the initial vote is taken in regard to one variable or the other.

At other times the feature of the system is instability: each new vote leads to divergence from the preceding decision, without any tendency to find a point of rest. Or the voting may lead to some endless circle or some endless web of decisions.

The conditions for stability or instability, or for an endless circle or an endless web of votes are investigated, and the main classes of cases are distinguished.

Section IV shows that the results arrived at in the preceding Sections do not depend on the systems of measurement used. " Utility " or level of preference cannot be measured in terms of any definite unit. We show that if any given function represents the system of preferences of a member, this function, when subjected to an arbitrary monotonic increasing transformation, will still correspond to the same valuational facts; and it will provide the same results as the original function would have done.

Likewise the variable or variables to represent the motions can be subjected to arbitrary monotonic increasing transformations; and, under the transformation, the results derived by our theory, Sections I–III, will still remain valid.

SECTION I

INDEPENDENT VALUATION: PROCEDURE IN WHICH VOTING IS BETWEEN VALUES OF a

1. Let us consider a committee of three members A, B and C that reaches its decisions using a simple majority. An original motion is put forward, we suppose, and after that a further motion (or amendment); and a vote is taken to select one of these two motions. Then a further motion (amendment) is put forward and a vote is taken between the two motions now in the field; and so on until, after a time, no further motions are put forward and the committee has as its decision that one of all the motions put forward which, at the end, remains undefeated.

When m motions are put forward (m finite or infinite), we can denote them by $a_1, a_2 \ldots a_m$.

We assume that each of the motions finds a definite place in the scale of valuations of each member of the committee, in such a way that each member's scale of valuations is consistent within itself. Thus, if for one member, a_i is above a_j on his scale of preferences and a_j is above a_q, then a_i will also be above a_q. And if a_r and a_s are at the same level on his scale of preferences, and also a_s and a_t, then a_r and a_t will be at the same level.

When a finite number of motions, $a_1, a_2, \ldots a_m$, have been put forward, a member's scale of preferences can be represented by a vertical straight line on which the motions stand at the appropriate levels of preference, or by a two-dimensional figure in which the motions are represented by points on the horizontal axis while order of preference is taken in the vertical direction.[1]

The two-dimensional mode of representation still applies when m, the number of motions, becomes infinite. The points representing them on the horizontal axis can be taken infinitely close together so as to form a continuous straight line, and the motions become the values of a continuous variable,

[1] Cf. Black, "The Decisions of a Committee Using a Special Majority," *Econometrica*, July, 1948, §§ 1–5.

a say, measured along the horizontal axis. In the present paper we will assume that in these circumstances the points on a member's scale of preferences form a continuous curve such as that of Fig. 1.

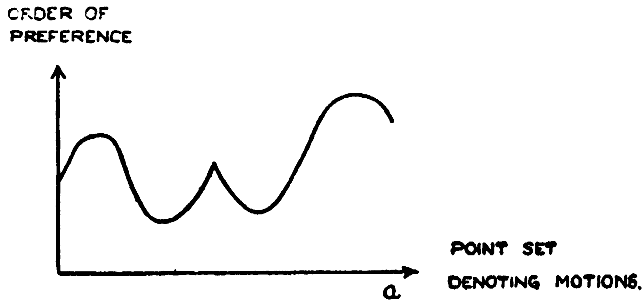


FIG. 1

We will denote the preference function of a member by $U(a)$; and we will assume that U is a bounded function of a in the given range over which a varies. A function U will be such that for any given value of a in the range, there corresponds one and only one value of U, though there may be more than one a at the same U-level.

2. The preference scales of the members of the committee, A, B and C respectively, we take to be given by the equations

$$h = U_A(a)$$
$$k = U_B(a)$$
$$l = U_C(a)$$

The first pair of these equations are the parametric equations of a curve in the h–k plane. Thus the preference scales of A and B can be represented by the single curve obtained by giving a its different values within the given range; or, when a has only a finite number of values, the scales of A and B will be represented by a set of points in the h–k plane.

Again the three equations shown are the parametric equations of a curve in h–k–l space, the curve being the locus of the point (h, k, l) as a varies in the range to be considered.

3. *Definition and Significance of a Majority Decision.*—By a majority decision we mean a motion that would be able to get a simple majority over every other.

Now where a majority decision, say the motion a_v, exists, it must give the decision arrived at by a committee that follows the procedure that we have assumed. For by hypothesis, a_v enters into the voting process at some stage; and when it does it will defeat any motion against which it is put. It must therefore emerge as the decision arrived at by the committee.

4. *The Conditions for a Majority Decision.*—Let us suppose that any given value of a, say $a = a_v$, corresponds to a majority decision. And let us denote the value of h, A's level of preference, at $a = a_v$, by h_v; while B's and C's levels of preferences, k and l respectively, are denoted by k_v and l_v for $a = a_v$.

Similarly let us denote the members' levels of valuation for another motion, a_1 say, by h_1, k_1 and l_1 respectively. Then if the member C values the motion a_1 above a_v, we have $l_1 > l_v$. In these circumstances if a_v is to be the majority decision, A and B must value a_v above a_1, *i.e.*, we must have $h_1 < h_v$ and $k_1 < k_v$.[1]

The same argument holds when we look, not to a particular motion a_1, but to *any* given motion (other than a_v): if C's valuation of the given motion is greater than that of a_v, *i.e.*, $l > l_v$, then for a_v to be a majority decision we must have A's and B's valuations of the given motion less than their valuations of a_v. That is, we must have $h < h_v$ and $k < k_v$. Thus we obtain our first condition for the existence of a majority decision:

$$\text{If } l > l_v, \text{ then } \left.\begin{matrix} h < h_v \\ k < k_v \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The following two conditions can be established in the same way.

If $a \neq a_v$ and $l = l_v$, then,

$$\text{either } \left.\begin{matrix} h < h_v \\ k < k_v \end{matrix}\right\} \text{ or } \left.\begin{matrix} h < h_v \\ k = k_v \end{matrix}\right\} \text{ or } \left.\begin{matrix} h = h_v \\ k < k_v \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

[1] There is a class of cases which, throughout the text, we have chosen to disregard. As an instance, in the vote between the motions a_i and a_j, A might be indifferent between the two, while B preferred a_i and C preferred a_j. So far the result would be indeterminate; but it would become determinate if either B or C were chairman and given the right of a casting vote.

It is easy enough to take into account possibilities of this kind when they crop up in particular examples (see §§ 43–4, *infra*), but it would embarrass the text to take them into account in the main exposition.

If $l < l_v$, then either $h < h_v$ with no restriction on k, or $k < k_v$ with no restriction on h, or $\left.\begin{array}{l} h = h_v \\ k = k_v \end{array}\right\}$ . . . . . (iii)

5. *Uniqueness Theorem.*—There cannot be more than one value a_v satisfying these conditions and there cannot, therefore, be more than one majority decision.

This can readily be established and is indeed obvious to commonsense.

6. We now wish to give another representation, in a single diagram, of *A*'s and *B*'s scales of preferences in relation to the motions put forward. We will take h, the order of preferences for the member *A*, to be measured along the directed straight line Wh, and k, the order of preference for *B*, along the directed straight line Wk, where Wh and Wk are at right angles, as in the accompanying figure. Valuation, we are supposing, is relative in its nature; and it will therefore be ordinal, not cardinal quantities that we mark along these two axes. For example let us take the motion a_1, say. It stands at a certain level of preference *in relation to* the other motions, on *A*'s and on *B*'s scale, being higher, let us suppose, than a_2 on *A*'s scale, and lower than a_2 on *B*'s scale. This relation can be represented by any two points a_1 and a_2 in the h–k plane, such that a_2 has a smaller h-co-ordinate than a_1 and a greater k-ordinate than a_1. If two motions a_3 and a_4, say, stand at the same level as each other on *A*'s scale and at the same level as each other on *B*'s scale, then the points in the h–k plane corresponding to a_3 and a_4 will be coincident.

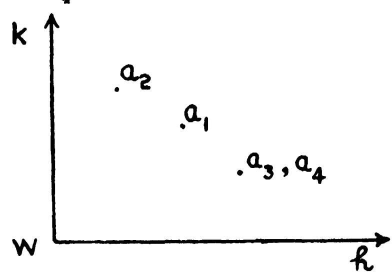


FIG. 2

Following these rules we can show all of *A*'s and of *B*'s preferences in relation to the motions put forward, in a single two-dimensional diagram.

7. Let us interpret in diagrammatical terms the conditions (i)–(iii) of § 4, that must be satisfied by any motion a_v that is to be a majority decision. In Fig. 3 we show the co-ordinates of a_v as (h_v, k_v).

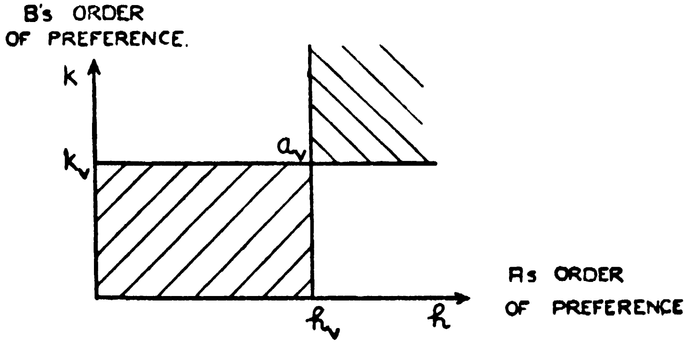


FIG. 3

Condition (i) of § 4 refers to any motion valued by the member C higher than he values a_v. It states that if $l > l_v$, then $h < h_v$ and $k < k_v$; that is, the point (h,k) in the h–k plane, which shows A's and B's valuations of this other motion, must lie inside the shaded area below and to the left of a_v, the top and right-hand boundaries of the area being excluded.

From conditions (ii), with $l = l_v$ the point (h,k) must lie either in the area or on its boundaries, but the point (h,k) may not be at a_v.

From conditions (iii), with $l < l_v$ the point (h,k) must lie either in or in the upper unshaded area, excluding its right-hand boundary, or in the lower unshaded area, excluding its upper boundary, or possibly at a_v.

Here we observe, as we may prove directly, that if a_v is to be a majority decision, in no case can there be any other point in the area above and to the right of it, or on the boundaries of this area, though there may be a point or points coincident with a_v at the lower left-hand corner of this area, provided C's valuation of any such point is less than his valuation of a_v.

To sum up, suppose we are given any motion whatever, say a_t, and know its position in the h–k plane. Then the necessary and sufficient conditions that a_i must satisfy in order to be a majority decision are as follows. All points of higher preference than a_i for the candidate C, must lie in the area defined by a_i, excluding its top and right-hand boundaries; all points that for C are at equal preference with a_i must occur in , or on its boundaries, the point a_i being excluded; and points of lower preference for C than a_i may be coincident with a_i, or may occur anywhere except in the prohibited area or on its boundaries.

8. *A Discrete Variable.* Let us suppose that we have discrete values for a, these values being a_1, a_2, ... a_6; and let the members' schedules be as in Fig. 4. The valuations of A and B are shown also in Fig. 5, by the set of points in the h–k plane. We are free to join these points in any manner. And we have still to represent the valuations of C. To do this let us join the points a_1, a_2 . . . . a_6 in the h–k plane, in accordance

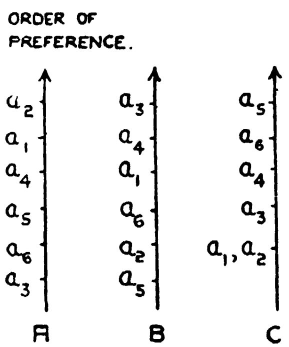


FIG. 4

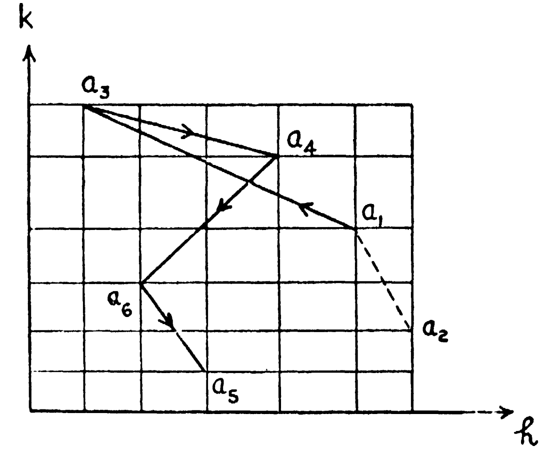


FIG. 5

with C's order of increasing preference for the points; or, if two or more motions stand at the same level on C's schedule of preferences, let us join them by a dotted line. Thus in the diagram we join a_1 and a_2 by a dotted line: then we join a_1 to a_3 by a directed line, a_3 to a_4, etc., the direction of the lines being indicated by the arrows. Instead of joining a_1 to a_3, we might, of course, have joined a_2 to a_3, since a_1 and a_2 stand at the same level on C's preference scale.

Thus when we are dealing with a discrete variable we are able to show the valuations of all three members in relation to the motions put forward, in a single diagram.

9. This construction, when taken along with the conditions stated in § 7, enables us to find the majority decision, where one exists. For any particular motion to be the majority decision there must be no points in the area equivalent to of Fig. 3, above and to the right of the motion. Thus in Fig. 5 the only possible points corresponding to a majority decision would be a_3, a_4, a_1 and a_2. Let us consider the point a_3. For this to be the majority decision, points that stand higher on *C*'s scale of preferences must lie in the area of Fig. 3; but this condition is not borne out and a_3 cannot be the majority decision. For the point a_4, however, there are no points in the area above and to the right of it, and the only points standing higher on *C*'s scale of preferences lie in the area below and to the left of it. The motion a_4, therefore, is the majority decision.

As regards the points a_1 and a_2, it may readily be seen that they violate the conditions of § 7, although it is now not necessary to examine them in detail, since by the uniqueness theorem there can only be one majority decision and this has been shown to be a_4.

As another example, the *data* of Fig. 6 are represented in the *h*-*k* plane in Fig. 7. For a_1, a_2, a_3, there are no points in

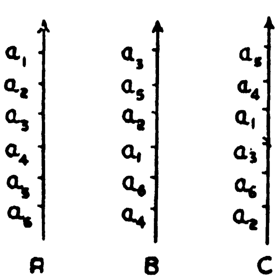


FIG. 6

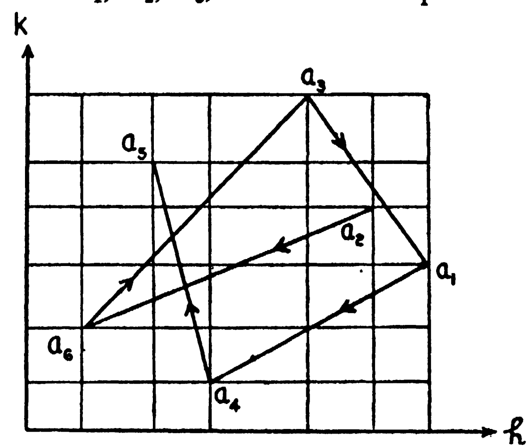


FIG. 7

the areas above and to the right of them; but for none of these points are the motions standing higher on *C*'s order of preference confined to the area  below and to the left of it, so that none of them gives a majority decision.

If, however, *A*'s and *B*'s schedules of preference remained as in Fig. 6, and if we were free to rearrange *C*'s preference in regard to the motions put forward, any of the points a_1, a_2, a_3 could be converted into a majority decision. The point a_1 would be the majority decision if a_2, a_3, a_5 were below a_1 on *C*'s schedule, with no restriction on the positions of a_4 and a_6 on *C*'s schedule. The point a_2 would be the majority decision if a_1, a_3, a_5 were below a_2 on *C*'s schedule, with no restriction on a_4 and a_6. The point a_3 would be the majority decision if a_1, a_2 were below a_3 on *C*'s schedule, with no restriction on a_4, a_5, a_6.

10. *A Continuous Variable.* When the motions are infinite in number and are represented by a continuous variable, the set of points in the *h–k* plane becomes a continuous curve, and *C*'s valuations can no longer be represented in the manner of § 7.

To test for the existence of a majority decision we may now consider parametric curves in the *h–k*, *k–l* and *l–h* planes. Then a_v will be the majority decision only if the curve in the *h–k* plane satisfies the conditions of § 7 and if the curves in the *k–l* and *h–k* planes satisfy the corresponding conditions.

11. It follows that for a_v to be the majority decision, the three-dimensional curve in *h–k–l* space must pass through the point $T_v = (h_v, k_v, l_v)$ and must lie in a very restricted region

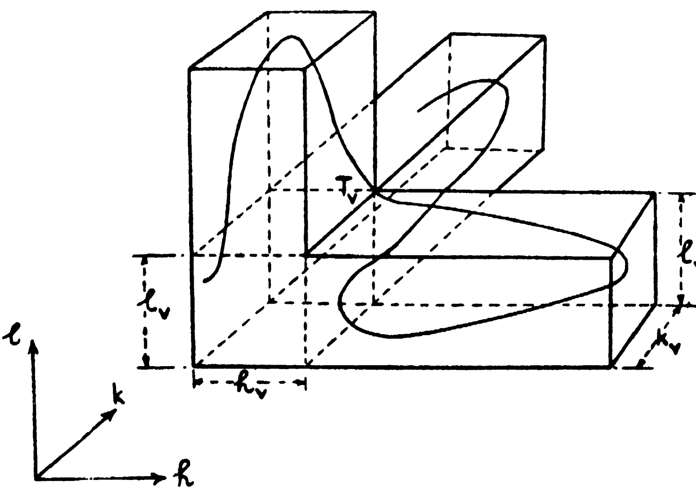


FIG. 8

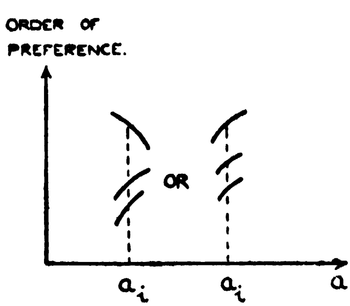
in the space. (See Fig. 8.) The curve must pass *once* through the point T_v, and must move inside the three arms of the solid to attain its maximum values of h, k, l.

Thus the geometrical condition for an equilibrium solution is that there shall be a point T_v of the curve in h–k–l space for which the above-shaped solid may be constructed to contain the *whole* curve.

This is a very restrictive condition, and the existence of a majority decision would seem to be exceptional. But, given any point T_v, it is possible to construct an infinity of curves defining possible valuations for the members A, B and C, that give T_v as the majority decision. It is only necessary to draw curves through T_v passing up the three arms of the solid as indicated in Fig. 8.

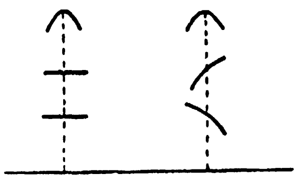
12. For given valuations of A, B and C, the detection of a majority decision will involve the examination of a curve in three dimensions, or alternatively, the simultaneous examination of the projections of this curve on the three co-ordinate planes. This procedure would be extremely laborious. Fortunately, in any given case in which a varies continuously, most a-values can be eliminated at once in testing for a majority decision, and detailed investigation is necessary only for a small number of a-values. Further, it is possible, as will be shown, to represent the relevant aspects of the valuations of all three members in a single diagram.

Let us consider a value $a = a_i$, not at either extremity of the a-range, at which $\frac{dh}{da}, \frac{dk}{da}, \frac{dl}{da}$ are all non-zero. It is at once evident that there will be values $a \pm \delta a$ rated higher than a_i by at least two of the three voters A, B, C; and a_i cannot be the majority decision.



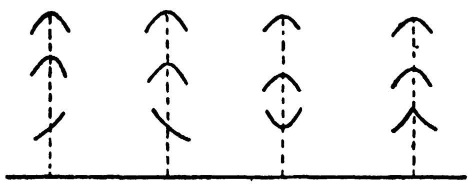
It follows that a_i cannot be a majority decision unless the preference level at $a = a_i$ is greater than for neighbouring values of a for at least one member; that is, *the majority decision,* *if it exists, occurs at a point of maximum for at least one member.*[1] This is a *necessary* (but not sufficient) condition for a majority decision. The first step in searching for a majority decision is therefore to note the values of a, say, $a_1, a_2, \ldots a_q$ at which points of maxima occur in one or other of the three preference curves.

By considering the preference curves in the neighbourhoods of these maxima, a further elimination may be made. It is in fact necessary only to consider values of a at which:



(1) One of the U-curves has a maximum, the other two curves sloping in opposite directions, or both having finite horizontal segments.

Or (2), Two of the U-curves have maxima, with no restriction on the third curve.



The end points, say $a = a_1$, and $a = a_2$, must also be considered to see whether conditions (1) or (2) are satisfied at these points.

Thus for $a = a_i$ to be a majority decision it is necessary that it should satisfy condition (1) or condition (2), though these conditions alone do not ensure that a_i shall be the majority decision. To determine whether or not a_i is the majority decision, we might examine the level of preference of a_i, by comparison with the whole range of a-values, on the preference scales of the three members. Or we may determine whether a_i is a majority decision by reference to one of the parameter curves.

13. On this latter method a suitable procedure is as follows:

(a) Construct one of the parameter curves, say that in the h-k plane. In drawing this curve use a thick line to denote those ranges of a over which, in C's valuation, $\dfrac{dl}{da} > 0$, a thin

[1] There may be several such points of maxima on the preference curve of any member. Cf. Fig. 1.

line to denote those ranges of a over which $\frac{dl}{da} < 0$, and a dotted line to denote those finite ranges, if any, over which $\frac{dl}{da} = 0$.

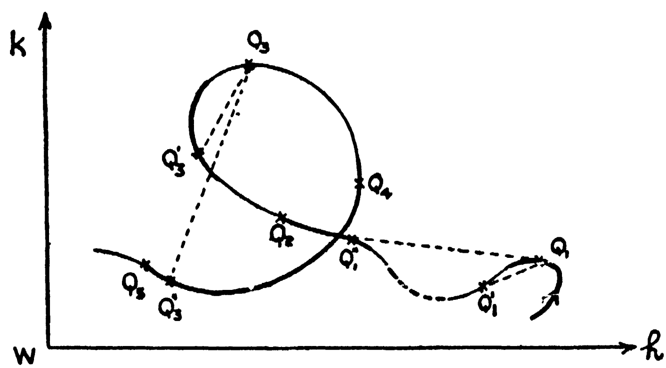
(b) Mark off on the curve the points Q_1, Q_2, ... corresponding to the values a_1, a_2, ... at which the condition (1) or the conditon (2) is satisfied. These points can be got either from the parameter curve itself (as explained below), or by inspection of the preference curves.

(c) Eliminate from consideration any of the points Q_1, Q_2, ... that has a portion of the curve in the prohibited region above and to the right of it, for it cannot define a majority decision.

And (d) for any point, say Q_1 corresponding to $a = a_1$, that remains, mark off on the curve those points Q'_1, Q''_1, ... corresponding to the values a'_1, a''_1, ... which have the same valuations as a_1 on C's preference scale.

We can now tell from the diagram which tracts of the curve stand at a higher or at the same level of preference on C's scale, as the point Q_1 under consideration. And we can therefore tell whether or not a_1, to which Q_1 corresponds, satisfies the conditions for being a majority decision.

Fig. 9, which is a parameter curve in the h-k plane, provides



In this diagram the arrow denotes the direction of a increasing

FIG. 9

an illustration. A majority decision must occur, if at all, when the curve in the h–k plane has either a local h-maximum or a local k-maximum, or else must occur at the end of a thickened portion of the curve where there is a local l-maximum. When each such point is examined to see whether it satisfies condition (1) or condition (2), we can narrow down the possible majority decisions to the values of a corresponding to Q_1, Q_2, Q_3, Q_4 and Q_5. For example, it is seen from the curve that Q_1 corresponds to a local maximum in B's preference curve, and the other two preference curves have slopes in opposite directions, since A's preference is decreasing and C's is increasing in the neighbourhood of a_1. But Q_1 cannot be a majority decision since the member C has points of higher preference on the curve to the left of Q_1'', and these are external to the region below and to the left of Q_1.

The points Q_2 and Q_5 can be eliminated as there are points of the curve in the corresponding prohibited areas, above and to the right of them.

The point Q_4 can be eliminated, since it is evident from the section $Q_4Q''_3Q_5$ that C prefers Q_3 to Q_4, and Q_3 is external to the region below and to the left of Q_4.

As regards Q_3 the only sections of the curve for which there are points preferred by C to Q_3, are Q'_3Q_3 and the section to the left of Q''_3. The point Q_3 thus satisfies all the necessary conditions, and a_3, the corresponding value of a, is the majority decision.

14. *Example.* Votes are taken on the variable a in the range $-0{\cdot}3 \leqslant a \leqslant 3{\cdot}0$, the preferences being defined by the following functions:—

For A, $h = 5{\cdot}5 + a(a-2)^2$

For B, $k = 5 - a(a-1)^2$	when	$-0{\cdot}3$	$\leqslant a \leqslant 0$
$k = 5 - 10a(a-1)^2$	when	0	$\leqslant a \leqslant 1$
$k = 5 - \frac{1}{2}(a^2-1)$	when	1	$\leqslant a \leqslant 3$
For C, $l = 4 + 40a^3 - 30a^2$	when	$-0{\cdot}3$	$\leqslant a \leqslant 0{\cdot}75$
$l = 4 + 2a^3 - 1{\cdot}5a^2$	when	$0{\cdot}75$	$\leqslant a \leqslant 1{\cdot}5$
$l = 4{\cdot}5(a-2)(a-3)+4$	when	$1{\cdot}5$	$\leqslant a \leqslant 2$
$l = 4$	when	2	$\leqslant a \leqslant 2{\cdot}5$
$l = 4 + (a-2{\cdot}5)^2$	when	$2{\cdot}5$	$\leqslant a \leqslant 3$

The three preference curves are as indicated in Fig. 10. The parameter curve in the *h–k* plane is shown in Fig. 11. By inspection of Fig. 10 it is seen that $a = 0$, $a = 1$, $a = 3$ satisfy the conditions (1) or (2) of § 12. These values of a must be tested as possible majority decisions, and the points corresponding to these three values of a, Q_1, Q_2, and Q_3 respectively, are therefore marked on the curve in Fig. 11.

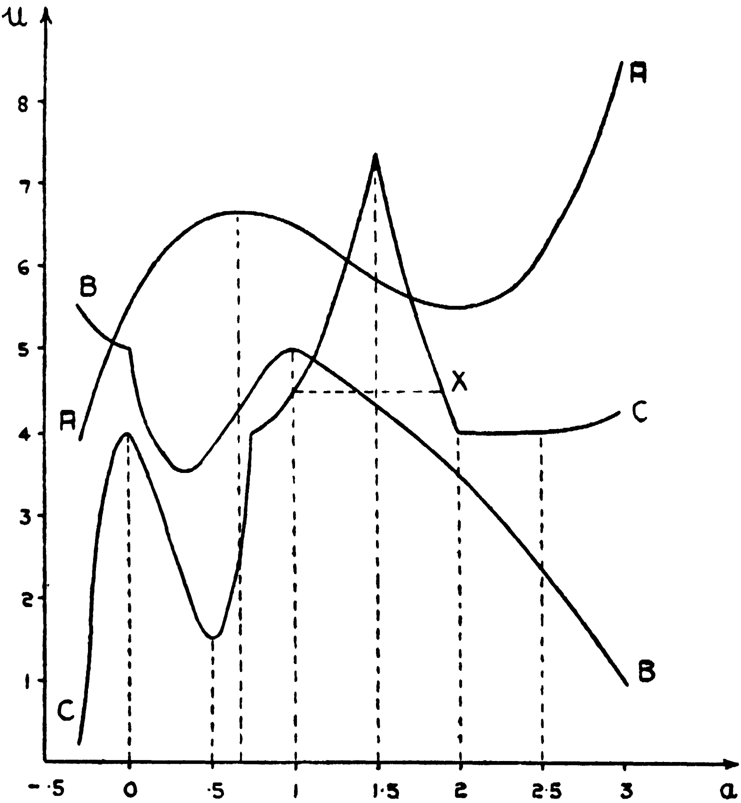
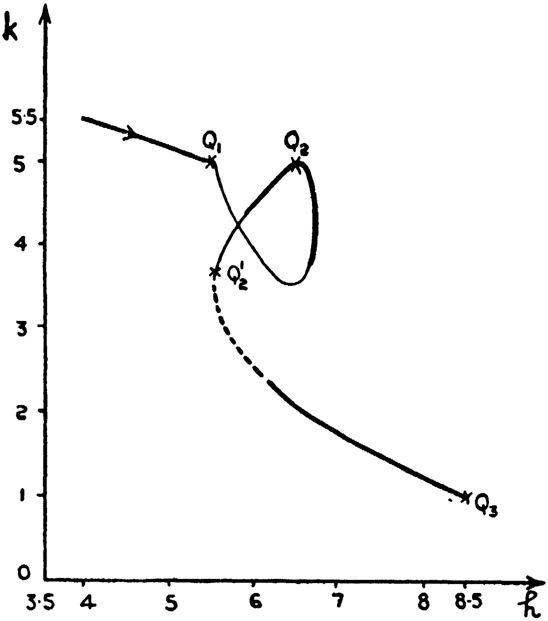


FIG. 10

The curve is external to the region below and to the left of Q_3; and some sections of the curve are preferred by the member C to the point Q_3, because in Fig. 10, C's curve is higher at some points than at the value $a = 3$. The point Q_3, therefore, cannot define a majority decision.

In the same way, the point Q_1 cannot define a majority decision, for the curve is external to the region below and to



In this diagram the arrow denotes the direction of a increasing

FIG. 11

the left of Q_1, and from Fig. 10 there are points external to this region corresponding to values of a preferred by C to the value $a = 0$ associated with the point Q_1.

In testing the remaining point Q_2, corresponding to the value $a = 1$, the point Q'_2 marked on the curve, corresponds to the point X on C's preference curve. The arc $Q_2Q'_2$ of the curve in the h–k plane corresponds to values of a preferred by C to the value $a = 1$, since C's preference is increasing at Q_2, continues to increase, and then decreases as Q'_2 is approached. It is evident that Q_2 corresponds to the majority decision, which is therefore $a = 1$.

(It will be noticed that the slope of the curve in the h–k plane is discontinuous at the point Q_1. This corresponds to the discontinuity in the slope of B's preference curve at $a = 0$).

SECTION II

Complementary Valuation: Procedure in which Voting is between Values (a, b)

15. We assume that the committee follows the same procedure as in Section I. (See § 1.) Motions are put forward and voted on. Any motion that is defeated by another in a vote is eliminated; and the committee's decision is that motion which emerges undefeated from the last vote.

But in this Section we assume that each motion either has two distinguishable aspects or consists of two separate parts. For instance one part of the motion might specify the total expenditure to be made on the Navy and Air Force, while the second part of the motion specified the amount to be spent on the Navy. Or one part of a motion might specify the total sum that a firm would pay towards superannuation in a particular period, while the other part of the motion specified the amounts to be distributed to the particular grades of workers.

In these circumstances, a motion would be defined by two separate characteristics. Let us denote one of these characteristics by the variable a and the other by the variable b.

The part a of a motion might be given; and the motion itself, and therefore the member's valuation of it, would not be determinate until the part b had also been specified. That is, a member's valuations in these circumstances will be a function of the two variables a and b. In the following theory the variables a and b will be taken to be either both discrete or both continuous.

As an example, if the values of a are $a_1, a_2, \ldots a_5$ and those of b are $b_1, b_2, \ldots b_7$, the voting will be between motions characterized as (a_1,b_1), $(a_1,b_2), \ldots (a_5,b_7)$. Putting this briefly we can say that voting takes place between the values of (a,b).

A member's preferences now depend on two variables. We will take his preference function to be of the form $U = U(a,b)$, where U is a bounded function of a and b. The function is assumed to be such that for any given a and given b, there is one and only one value of U.

We will denote the preference functions of A, B and C respectively, by

$$h = U_A(a,b),\ k = U_B(a,b) \text{ and } l = U_C(a,b).$$

16. *Definition and Significance of a Majority Decision.* By a majority decision we mean a motion characterized by a pair of values, (a_v, b_v) say, which is able to get a simple majority over every other motion.

It follows as before (§ 3), that if there is a majority decision it is bound to emerge as the decision of a committee that follows the procedure we assume.

17. *The Uniqueness Theorem.* It can be shown that at most there is only a single majority decision.

18. *Discrete Variables.* The case for a and b discrete may be treated in the same manner as that adopted previously. If the ranges are $a_1, a_2, \ldots a_6$, and $b_1, b_2, \ldots b_8$, we can fix b at any value, b_q say, and carry out the procedure defined in §§ 8–9. Repeating this 8 times we have 8 series of points which we can join up to form 8 curves in the h–k plane; and the majority decision, if there is one, can be found. Or it may be more convenient to fix a at a_s, and allow b to take on all its possible values. Six curves would then be obtained by making a take the values $a_1, a_2, \ldots a_6$. Alternatively the case can be treated as the case of a single discrete variable of which the values are (a_1,b_1), (a_1,b_2), $\ldots (a_6,b_7)$, (a_6,b_8). Precisely the same method as before will then apply.

19. *Continuous Variables.* When a and b are continuous variables, let the members' preference functions be such that they can be represented by simple humps with constantly down-sloping sides, *i.e.*, they have no subsidiary maxima but only one peak value.

The contour of such a surface in the a–b plane corresponding to a given level of preference, will be a closed curve or possibly the arc of a closed curve, *e.g.*, the arc XYZ in Fig. 12. Such a contour will really be an indifference curve, the member being indifferent in choice as between all points (a,b) that lie on the curve. Further, any such contour $U(a,b) = U_1$ say, will enclose all the points (a,b) for which $U > U_1$. Fig. 13

shows such a contour in the plane $U = U_1$. And the preference surface may be mapped on the *a–b* plane by a series of contour curves $U(a,b)$ = constant.

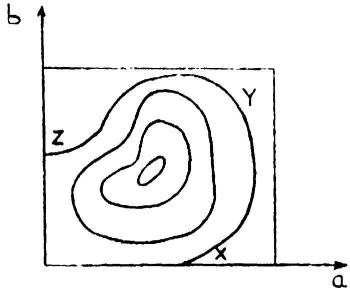


FIG. 12

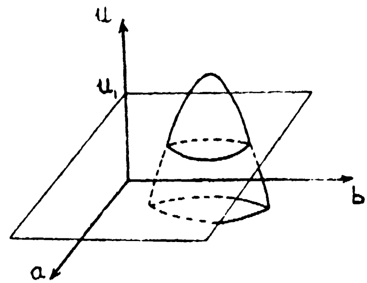


FIG. 13

20. *The Conditions for a Majority Decision.* The analytical conditions can be expressed, if we choose, after the manner of § 4.

21. We will investigate the geometrical conditions for the existence of a majority decision. Let us suppose that any point in the *a–b* plane, say *H*, is a majority decision; and let us consider the indifference contours through *H*. The contours of the three members have *H* as a common point.

Then if *H* is a majority decision no two of the contours can have any area in common. For suppose that two of them did have an area, say the area 1, in common. (Fig. 14.) Then

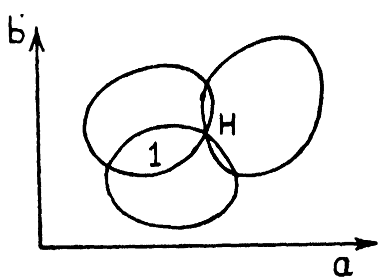


FIG. 14

any point in 1 would stand at a higher level of preference than *H* on the scales of the two members by whose contours it was enclosed; and it would be able to get a simple majority against *H* in a vote. That is, *H* would not be a majority decision. Thus any point in the *a–b* plane that is a majority decision, must lie on indifference contours of the three members, without these contours having any area in common. And this is a *necessary* condition for the existence of a majority decision.

Further it is a *sufficient* condition for such a point being a majority decision. For let us suppose that such contours through H are A', B' and C' respectively. (Fig. 15.) Then

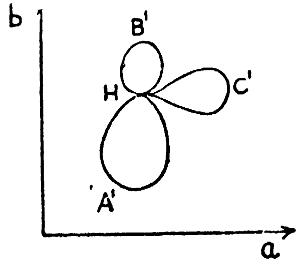


FIG. 15

in virtue of lying on A' and B', H can get a simple majority against any point lying outside these contours. Similarly it can get a simple majority against any point lying outside the contours B' and C', and against any point lying outside C' and A'. That is, H can get a simple majority against any other point in the plane and will represent a majority decision.

Now from the uniqueness theorem there is at most a single majority decision. Thus at most there will be a single point in the plane that satisfies the geometrical conditions of being a point common to three non-intersecting contours.

A point satisfying these geometrical conditions, it will be found, can arise only in one of three ways:—

(i) if at least one of the curves has a cusp[1] at H, as in Fig. 15;

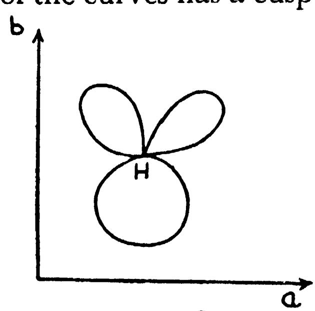


FIG. 16

(ii) if at least two of the curves have discontinuous tangents at H, as in Fig. 16; or

[1] A curve has a tangent at a cusp, but such a curve cannot be a convex curve.

(iii) if one of the indifference contours is a single point corresponding to the peak value of a member. Case (iii) is illustrated by Figs. 17 and 18, where *A*, *B* and *C* are the peak values of the members *A*, *B* and *C* respectively. The majority

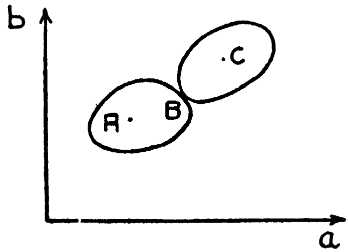


FIG. 17

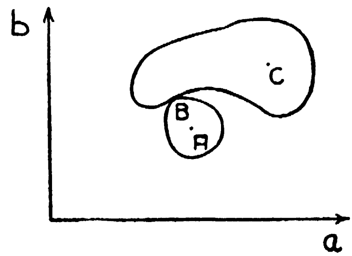


FIG. 18

decision in each of the diagrams is the peak value of the member *B*, and this result is independent of the nature of *B*'s contours.

If the contours are such that cases (i) and (ii) are excluded, the point *H* must correspond to the peak value of one member.

22. *Smooth Convex Contours.* Let it now be assumed that the contours are *convex* curves, *i.e.*, oval curves such that any straight line cuts any particular contour in not more than two points; and that the curves are smooth, so that at every point there is a unique tangent lying external to the area enclosed by the curve.[1]

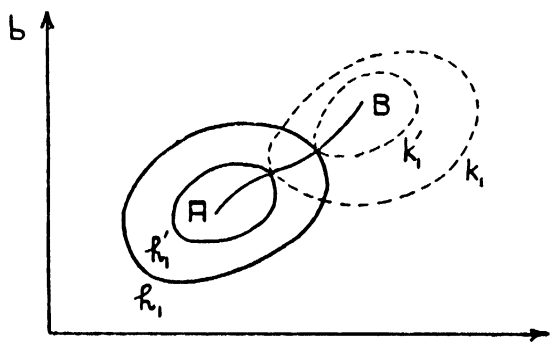


FIG. 19

Let the points *A* and *B* in Fig. 19 correspond to the optimum or peak values of the members *A* and *B* respectively; and let

[1] This excludes cases (i) and (ii) of the preceding paragraph.

$h = h_1$ and $k = k_1$ be two indifference contours that intersect, but do not enclose both A and B.

Now as h increases from h_1, the contour h will contract and there will be a value h'_1 of h, $(h'_1 > h_1)$, for which the contour $h = h'_1$ touches the curve $k = k_1$.

Similarly there will be a value k'_1 of k, $(k'_1 > k_1)$, for which the curve $k = k'_1$ touches the curve $h = h_1$. The locus of such points of contact will define a single curve in the a–b plane, joining A and B. (The curve may be extended beyond A and B by considering contours enclosing both A and B.) The curve of contact AB, or the corresponding curves BC, CA, got by reference to the contours of the other pairs of members, define the loci on which the majority decision, if any, must lie. There will be a majority decision at that point, if any, on one of these curves, at which the third member has his peak value.

Now let us take neighbouring points (a,b), $(a + \delta a, b + \delta b)$ which lie at the same level, $U =$ constant, on a preference surface. We have

$$\begin{aligned} U(a,b) &= U(a + \delta a, b + \delta b) \\ &= U + \frac{\partial U}{\partial a}\delta a + \frac{\partial U}{\partial b}\delta b \end{aligned}$$

and
$$\frac{db}{da} = -\frac{\partial U}{\partial a}\Big/\frac{\partial U}{\partial b}$$

Thus the slope of the h-contour through any point in the a–b plane is given by $-\dfrac{\partial h}{\partial a}\Big/\dfrac{\partial h}{\partial b}$, and the slope of the k-contour through the point is given by $-\dfrac{\partial k}{\partial a}\Big/\dfrac{\partial k}{\partial b}$.

The equation $\dfrac{\partial h}{\partial a}\Big/\dfrac{\partial h}{\partial b} = \dfrac{\partial k}{\partial a}\Big/\dfrac{\partial k}{\partial b}$ expresses the condition that the tangents to the two contours shall be in the same direction, *i.e.*, that the two contours shall touch at the point considered. This equation therefore defines the contact curve AB. Similarly the other curves of contact of the indifference contours are

$$\frac{\partial k}{\partial a}\Big/\frac{\partial k}{\partial b} = \frac{\partial l}{\partial a}\Big/\frac{\partial l}{\partial b} \quad \text{and} \quad \frac{\partial l}{\partial a}\Big/\frac{\partial l}{\partial b} = \frac{\partial h}{\partial a}\Big/\frac{\partial h}{\partial b}.$$

The condition for the existence of a majority decision, therefore, is that the peak value of the third member should satisfy one of these equations and also should lie between the other two peaks.

23. *Example.* Let the preference surfaces of the members A and B be simple humps with peaks (a_1,b_1) and (a_2,b_2), and let the contours of these surfaces be circles, then we have

$$h = (a - a_1)^2 + (b - b_1)^2$$
$$k = (a - a_2)^2 + (b - b_2)^2$$
$$\frac{\partial h}{\partial a} = 2(a - a_1), \qquad \frac{\partial h}{\partial b} = 2(b - b_1)$$
$$\frac{\partial k}{\partial a} = 2(a - a_2), \qquad \frac{\partial k}{\partial b} = 2(b - b_2)$$

The contact locus is $\dfrac{a - a_1}{b - b_1} = \dfrac{a - a_2}{b - b_2}$, *i.e.*, the straight line joining (a_1,b_1) to (a_2,b_2).

Then if C's peak, (a_3,b_3) say, lies on this line and is between (a_1,b_1) and (a_2,b_2), it will be the majority decision; and this will be true irrespective of the shape of C's contours. This result would, of course, already be obvious from § 21.

24. *General Contours.* If the contours are convex curves, but are no longer assumed to be smooth at every point, it is still possible to trace the three contact curves, and the majority decision, if any, will be one of the points that these three contact curves have in common. The conditions that the point must satisfy in order to be a majority decision, will be as set out above.

If the restriction that the contours are convex be removed,

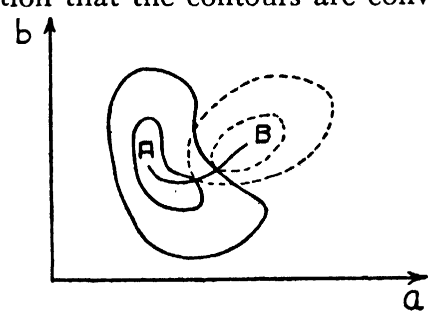


FIG. 20

there may still be a single curve joining any two peaks, *e.g.*, A and B in Fig. 20.

It is possible, however, that the two peaks A and B may be joined by more than one contact curve, or by branching curves.

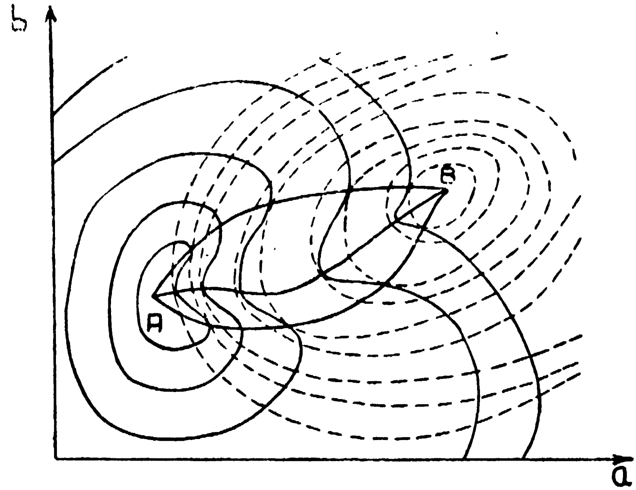


FIG. 21

In the case illustrated in Fig. 21, the contact locus between A and B splits up into three branches. Points on either of the two outer branches represent possible majority decisions, but no point on the central branch can be a majority decision, for the A and B contours through such a point will have areas 1 and 2 in common. See Fig. 22.

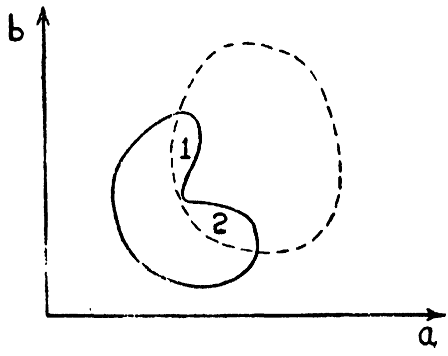


FIG. 22

In general, the majority decision, if there is one, must lie at one of various points on the contact locus, or on various arcs of the contact locus, or its branches, joining any pair of peaks. For example, in Fig. 23, the points H and K_1 both represent possible majority decisions, but the points H_1 and K do not. In the general case, therefore, it is necessary to give

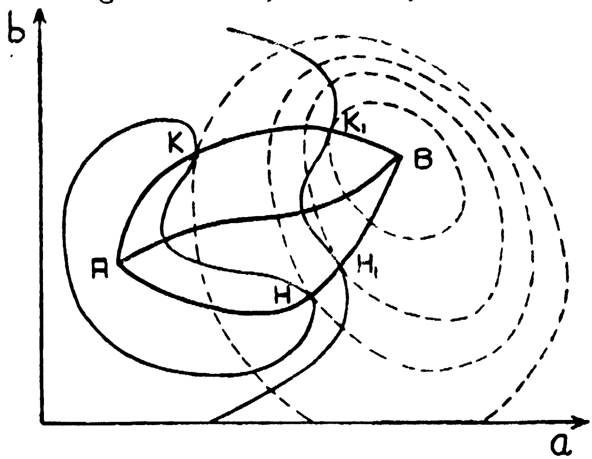


FIG. 23

detailed consideration to any point common to three contact curves *AB*, *BC*, *CA*, where *AB*, for example, is a branch of the contact locus joining *A* and *B*. By constructing the three indifference contours through the point, we can determine whether or not it is in fact the majority decision.

25. The above theory is also applicable to the case in which

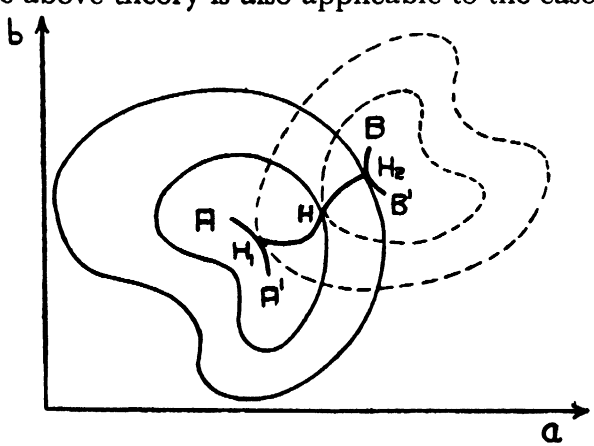


FIG. 24

the preference surfaces of one or more members have a horizontal ridge instead of a single point at the peak. For the contours illustrated in Fig. 24, there would be a single contact curve H_1HH_2 joining some point H_1 in *A*'s ridge *AA'* to some point H_2 in *B*'s ridge *BB'*. (The contact loci would consist of more than one curve beyond *AA'* and *BB'*.) A majority decision would exist if the curve H_1HH_2 contained *C*'s peak, or a point on *C*'s ridge.

If *C*'s peak or ridge does not contain a point on either of the ridges *AA'*, *BB'*, the majority decision, if any, must lie on the curve H_1HH_2, and this is true whatever the nature of *C*'s contours may be.

SECTION III

Complementary Valuation : Procedure in which Voting Selects Values of a and b Successively

26. Let us consider a committee that is reaching decisions on two different topics (or perhaps on separate and distinguishable aspects of the same topic). So far we are dealing with the same problem as in Section II. In the present Section, however, we wish to broaden our assumptions in relation to valuation and also to suppose that a different committee procedure is in use from that of the previous Section.

We will now suppose that what the committee members value are certain *sets of circumstances*—which, as we will explain, we will call " motions "—in relation to the two topics under consideration. On one topic these sets of circumstances (motions) which are envisaged by the members and valued by them will be denoted by a_1, a_2, ..., and on the other topic they will be denoted by b_1, b_2, ...

Some of these sets of circumstances may be formulated in words in the motions put forward in the committee. For instance if a_q were a motion put forward in relation to the relevant topic, it would be valued by all of the members. But we will also assume that some of the other values of a correspond to circumstances not formulated in words as motions, but that nevertheless these same circumstances are envisaged by all the members *as if* they had been formulated in words and put forward as motions. For instance a_r might be such a set of circumstances in relation to the first topic, that is envisaged and valued by all of the members *as if* it had been put forward as a motion. We might refer to a_r as being a *quasi-motion*—like a motion in every respect except that it has not been put forward as a definite proposal by a member of the committee, nor is it voted on. The variables a and b, therefore, now denote motions and quasi-motions on their respective topics. But for brevity of expression we will refer to all of the a's as being motions and all of the b's as being motions. And we will assume that on either topic the number of such motions may be either finite or infinite.

As regards the committee procedure in use, we assume that the committee reaches decisions on each of the two topics successively, taking a decision first in regard to a, say, then in regard to b. But the process may not stop there and a further decision may be taken in regard to a, then a further decision in regard to b, and so on indefinitely, or for as many motions as are put forward. And we wish to discover what the situation will be at any stage during the voting, and the situation after the voting has come to an end.

We will also assume that when a decision is being taken on one topic, each member of the committee expects the existing arrangement on the other topic to continue in operation. Thus if at any moment the existing arrangements on the two topics are those denoted by say, (a_p, b_q) and a vote is taken in relation to a, each member is assumed to expect that the value b_q of b will continue in operation. If the result of this is to select a value, a_r say, for a, in the next vote to determine b, each member will expect that the value a_r of a will persist. If this leads to the selection of a value b_s of b, in the next vote to determine a, each member will expect that the value b_s of b will persist. This may lead to the selection of a value a_t of a, and so on.

We will no longer take it that all motions (including quasi-motions) enter the voting process—the quasi-motions obviously do not. Instead we will assume that the system starts off from some initial position, say (a_0, b_0), and that each member values each of the motions (including each of the quasi-motions) over a given range of each variable. This range, in relation to the variable a, we denote by a_1a_2—all of the a-motions we take to lie within a_1a_2—and the range in relation to b we take to be b_1b_2. The initially-existent state of affairs (a_0, b_0) is taken to fall within the permissible ranges.

We suppose that each member puts forward in relation to the variable under consideration, the variable a say, as many values of a as he chooses, *including* that value of a within the given range a_1a_2 which, with the existing value of b, he considers preferable to any other. And we suppose that a vote is taken as between all the values put forward. A new value of a, let us suppose, is selected. With this new value of a in operation, a similar process is assumed to take place in relation to b. The members may put forward as many motions as they care, within the permissible range b_1b_2; and we suppose that these motions include what for each member, at the now-existing value of a, is the optimum b within b_1b_2. Some new value of b, which must, of course, lie inside the permissible range b_1b_2, will be selected. Then another vote is taken in relation to a, and so on.

27. What we seek to find in this Section is no longer a majority decision as in the preceding Sections. Instead we wish to examine for the existence of equilibrium positions. A point P is a position of *stable equilibrium* if, when the system starts from a point H, say, sufficiently near to P either in its a- or its b-co-ordinate, there will be convergence to P. A point P is a position of *unstable equilibrium* if, when the system starts off from the point H, say, sufficiently near to P either in its a- or its b-co-ordinate, there will be divergence from P.

We will investigate in particular, the conditions for the existence of one or more positions of stable equilibrium for a given group of preference schedules with any given starting point. When one or more such positions exist, we wish to discover which, if any of them, will be reached. We also wish to trace the path followed by the voting in moving to such a position of stability. Or, if no stable position exists, or if none is reached, we still wish to trace the path followed by the voting.

The members' valuations of the motions (including quasi-motions), will be assumed to be complementary. As before, we will take it that the variables a and b, in terms of which their valuations are expressed, are either both continuous or both discrete.

28. *Continuous Variables. Convex Indifference Contours.* A member's preference surface is of the form $U = U(a,b)$; and we assume that such a surface is single-peaked, with constantly down-sloping sides and with convex indifference contours.[1]

[1] The restriction to *convex* contours is not necessary for the present section. It would be sufficient if the contours were such that lines *parallel to one or other of the co-ordinate axes* intersected any particular contour in not more than two points.

These assumptions imply that any section $a =$ constant, or $b =$ constant, of a preference surface, is a curve with a single maximum. Within the range of a and b considered, the equation $\frac{\partial U}{\partial a} = 0$ has then at most a single real root for a given value of b, and $\frac{\partial U}{\partial b} = 0$ has at most a single real root for a given value of a.

The first of these equations will give the value of a for which, with the given b, U is a maximum, *i.e.* it will give the optimum or most-preferred a at that b. Similarly the second of these equations will give the member's optimum value of b at the given a.

We have shown earlier (§ 22) that the slope of the tangent at the point (a,b), on any given indifference contour, is

$$\frac{db}{da} = -\frac{\partial U}{\partial a}\Big/\frac{\partial U}{\partial b}$$

For the maximum value of a corresponding to any given b, we have $\frac{\partial U}{\partial a} = 0$, and therefore $\frac{db}{da} = 0$.

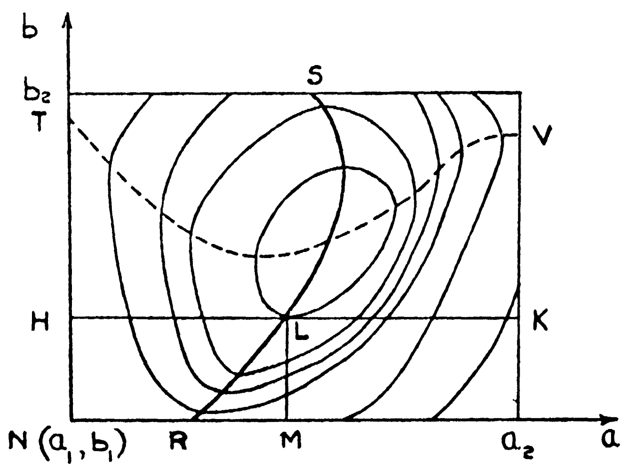


FIG. 25

In Fig. 25 the range of the variable a is from a_1 to a_2 and that of the variable b from b_1 to b_2. $\frac{\partial U}{\partial a} = 0$ where $\frac{db}{da} = 0$.

We can therefore trace $\frac{\partial U}{\partial a} = 0$ as the locus of points at which tangents to the contours are parallel to the a-axis, and in the diagram $\frac{\partial U}{\partial a} = 0$ is the curve RS.

To illustrate, let us select any value, say $b_1 + NH$, of b. Through H draw a straight line parallel to the a-axis. It touches only a single indifference contour, namely at the point L. And, consistently with the given value $b_1 + NH$ of b, this is the point of highest preference that the individual is able to reach. But L must lie on RS, the locus of points of contact of tangents parallel to the a-axis. Thus the curve RS, $(\frac{\partial U}{\partial a} = 0)$, gives the locus of optimum values of a for the individual member concerned corresponding to given values of b.

Similarly $\frac{\partial U}{\partial b} = 0$, the broken curve TV in the diagram, gives the locus of the optimum values of b for the member corresponding to given values of a. It can be traced as the locus of points at which tangents to the indifference contours are vertical.

(In our diagrams we will observe the convention that curves $\frac{\partial U}{\partial a} = 0$ are shown as continuous curves, and curves $\frac{\partial U}{\partial b} = 0$ are shown as broken curves.)

29. With preference surfaces of the type that we assume, there is one optimum value of a corresponding to any given b. There cannot, therefore, be more than one point of intersection between $\frac{\partial U}{\partial a} = 0$ and $b =$ constant ; nor can there be more than one point of intersection between $\frac{\partial U}{\partial b} = 0$ and $a =$ constant.

The member's peak value will be given by the point of intersection of the curves *RS* and *TV*; these curves have therefore not more than one point of intersection.

30. It may happen that the indifference contours do not possess tangents at every point, but in this case the peak loci *RS* and *TV* can still be traced. Our results do not require $U(a,b)$ to be a differentiable function of a and b at every point in the region of the a–b plane. For convenience, however, we continue to refer to these loci as $\frac{\partial U}{\partial a} = 0$ and $\frac{\partial U}{\partial b} = 0$ respectively.

We proceed to consider the various possible cases.

Slopes of $\frac{\partial U}{\partial a} = 0$ *and* $\frac{\partial U}{\partial b} = 0$ *of Same Sign.*

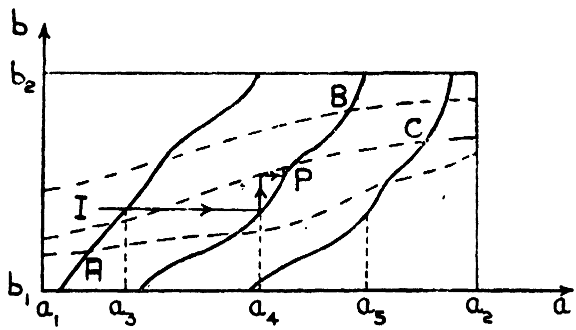


FIG. 26

31. With a committee of three members *A*, *B* and *C*, let us assume to begin with, that the curves $\frac{\partial U_A}{\partial a} = 0$, $\frac{\partial U_B}{\partial a} = 0$, $\frac{\partial U_C}{\partial a} = 0$ do not intersect, and that the curves $\frac{\partial U_A}{\partial b} = 0$, $\frac{\partial U_B}{\partial b} = 0$, $\frac{\partial U_C}{\partial b} = 0$ do not intersect. Let the curves be as indicated in Fig. 26, where their slopes are all positive and *A*, B and *C* the peak values for the three members.

Let us suppose that the initially-existing values of a and b are those shown at the point *I*, and that the committee first makes a decision in regard to *a*. With the given value of *b*, the optimum values of *a* for the three members are the intersections a_3, a_4, a_5 of a horizontal line—of which only part is shown—through *I* with the curves $\frac{\partial U_A}{\partial a} = 0$, $\frac{\partial U_B}{\partial a} = 0$, $\frac{\partial U_C}{\partial a} = 0$. By hypothesis, a_3, a_4 and a_5 are put forward as motions and the members' preference curves in relation to *a* are single-peaked with a_3, a_4, a_5 as their respective optima. It follows that a_4, corresponding to the median optimum, will be the value selected.[1] Thus the result of the first vote will be given by the point of intersection of the horizontal line through *I* with the midmost of the curves $\frac{\partial U}{\partial a} = 0$.

With $a = a_4$, if a vote is now taken in relation to the variable *b*, the value of *b* selected will be that given by the intersection of the vertical through a_4 with the midmost of the curves $\frac{\partial U}{\partial b} = 0$. If now, with this value of *b*, a vote is taken in relation to *a*, the value of *a* selected will be that given by the intersection of the horizontal through the now-existing value of *b* and the midmost curve $\frac{\partial U}{\partial a} = 0$, etc.

In fact if a large enough number of votes is taken on the two topics alternately, the decision of the committee will approach, by a converging staircase, the point *P*, which, once reached, will not be departed from. *P*, therefore, is a position of stable equilibrium. It can easily be verified by reference to this diagram, that whatever the position from which the system starts, and whether the initial vote is on *a* or *b*, the point *P* will be approached and will be reached after an infinite number of votes.[2]

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[1] Cf. Black, *op. cit.*, *Econometrica*, July, 1948, p. 250.

[2] In the case where the system starts with $a = a_p$ and the first vote is taken in relation to *b*, *P* will be reached in a single vote; or if the system starts from $b = b_p$ and the first vote is taken in relation to *a*, *P* will again be reached in a single vote.

A position of unstable equilibrium might similarly be reached (and maintained) as a result of the first vote.

The reasoning here depends only on the midmost of the curves $\frac{\partial U_A}{\partial a} = 0$, $\frac{\partial U_B}{\partial a} = 0$, $\frac{\partial U_C}{\partial a} = 0$, and on the midmost of the curves $\frac{\partial U_A}{\partial b} = 0$, $\frac{\partial U_B}{\partial b} = 0$, $\frac{\partial U_C}{\partial b} = 0$. In Fig. 26 these are the curves *PB* and *PC* respectively.

32. Sometimes, as in the diagram referred to, one of the curves $\frac{\partial U}{\partial a} = 0$ will be the midmost curve. At other times the midmost curve will consist of segments of the curves of the different members. This is illustrated in Fig. 27, where the midmost of the curves $\frac{\partial U}{\partial a} = 0$ is shown as a heavy continuous line.

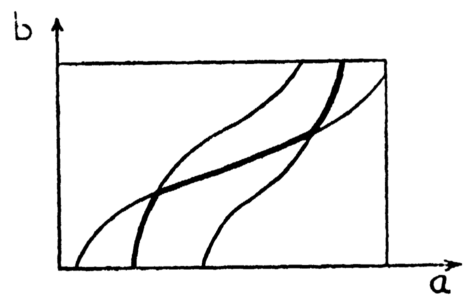


FIG. 27

Let us denote the midmost of the curves $\frac{\partial U}{\partial a} = 0$ by $\frac{\partial U_1}{\partial a} = 0$.

Similarly let us denote the midmost of the curves $\frac{\partial U}{\partial b} = 0$ by $\frac{\partial U_2}{\partial b} = 0$. As before this may be made up of segments of the three curves $\frac{\partial U_A}{\partial b} = 0$, $\frac{\partial U_B}{\partial b} = 0$, $\frac{\partial U_C}{\partial b} = 0$. See Fig. 28 where $\frac{\partial U_2}{\partial b} = 0$ is shown as the heavy broken line.

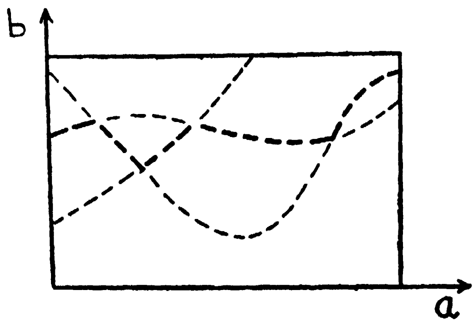


FIG. 28

The more general result, of which our finding of § 31 is a particular instance, can then be stated as follows:

Rule 1. If the curves $\frac{\partial U_1}{\partial a} = 0$ and $\frac{\partial U_2}{\partial b} = 0$ intersect in a single point P, and have slopes of the same sign, there will be convergence to P (or divergence from P) according as the magnitude of the slope of $\frac{\partial U_1}{\partial a} = 0$ exceeds (or is less than) that of $\frac{\partial U_2}{\partial b} = 0$ in the neighbourhood of P.

Example 1. In Fig. 29 the slopes of both curves are positive, that of $\frac{\partial U_1}{\partial a} = 0$ being the greater. $\frac{\partial U_1}{\partial a} = 0$ cuts $\frac{\partial U_2}{\partial b} = 0$ from below to above at P, a position of stable equilibrium.

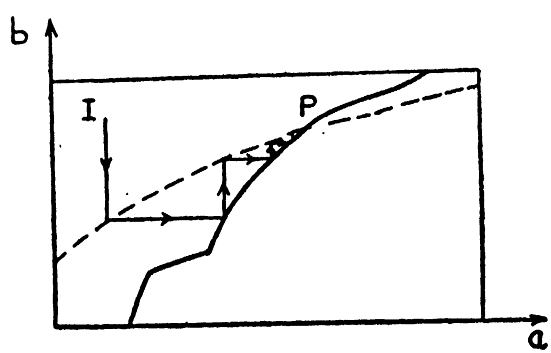


FIG. 29

The path followed by the voting will be shown by alternate horizontal and vertical lines and will rise or fall to P by a converging staircase.

Example 2. In Fig. 30 both curves are upward-sloping. $\frac{\partial U_2}{\partial b} = 0$, which has the steeper slope, cuts $\frac{\partial U_1}{\partial a} = 0$ from below to above in the single point P, which is a position of unstable equilibrium. In the figure the first vote is taken in relation to b, and each successive vote carries the decision further away from the point P until the origin of co-ordinates N is reached.

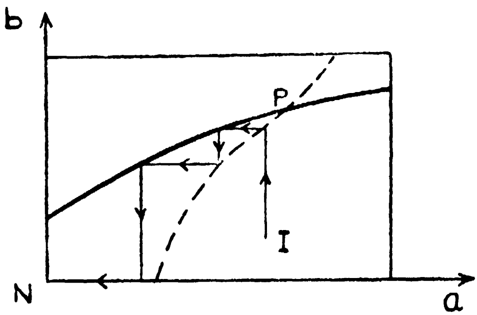


FIG. 30

Example 3. Both curves downward-sloping and $\frac{\partial U_2}{\partial b} = 0$ cuts $\frac{\partial U_1}{\partial a} = 0$ from below to above in the single point P. Then P will be a position of stable equilibrium. (No diagram drawn.)

Example 4. Both curves downward-sloping and $\frac{\partial U_2}{\partial b} = 0$ cuts $\frac{\partial U_1}{\partial a} = 0$ from above to below in the single point P. Then P will be a position of unstable equilibrium. (No diagram drawn.)

If the curves touch at P, there will be convergence on one side and divergence on the other side of *P*. Cf. Fig. 31.

Again, there may be no point of intersection of these curves within the prescribed ranges of the variables. For example, in

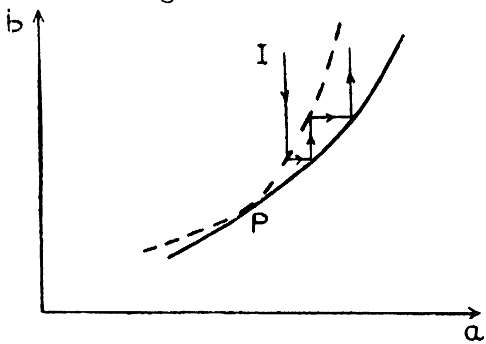


FIG. 31

Fig. 32, with the starting point *I* the position of stable equilibrium *Q* would be reached after 5 votes. Since to the left of A,

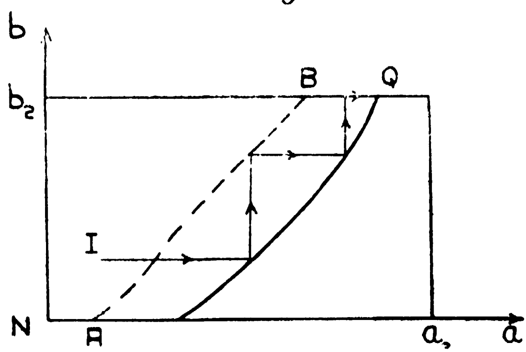


FIG. 32

$\frac{\partial U_2}{\partial b} = 0$ lies below $b = b_1$, and to the right of *B* lies above $b = b_2$, it is as if, for these ranges of the variable *a*, $\frac{\partial U_2}{\partial b} = 0$ were coincident with *NA* and *BQ*. It is therefore as if *Q* were the point of intersection of $\frac{\partial U_1}{\partial a} = 0$ and $\frac{\partial U_2}{\partial b} = 0$.

Slopes of $\frac{\partial U_1}{\partial a} = 0$ *and* $\frac{\partial U_2}{\partial b} = 0$ *of Opposite Sign.*

33. Next let us suppose that the curves $\frac{\partial U_1}{\partial a} = 0$, $\frac{\partial U_2}{\partial b} = 0$ have slopes of opposite sign and intersect within the region

considered. In this case successive votes will give a *spiral* that may, or may not, converge to P, the point of intersection of the two midmost curves.

Consider first the case when these curves are straight lines. It is evident from Fig. 33 (i) and (ii), that when the slope of $\frac{\partial U_1}{\partial a} = 0$ is positive, there will be a converging or diverging spiral according as the magnitude of the slope of $\frac{\partial U_2}{\partial b} = 0$ is less than or greater than that of $\frac{\partial U_1}{\partial a} = 0$. In the transition

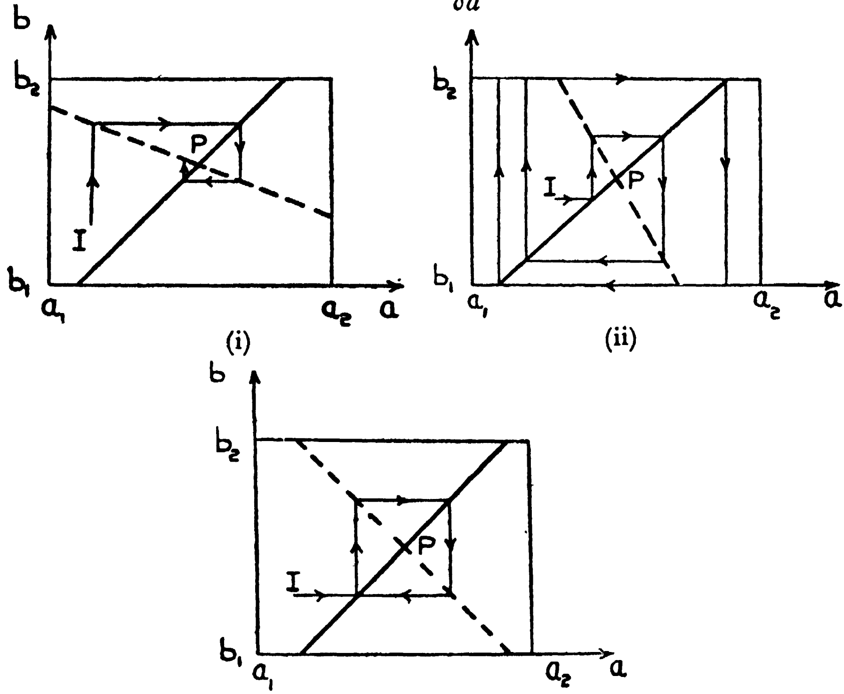


FIG. 33

case in which the two curves have slopes of equal magnitude, Fig. 33 (iii), there will be a repetitive cycle of votes.[1] It may

[1] The relevant literature in Economics is that relating to the "cobweb theorem." See pp. 368–74 of Paul A. Samuelson's essay in *A Survey of Contemporary Economics* edited by Howard S. Ellis, and Mordecai Ezekiel, "The Cobweb Theorem," *Quarterly Journal of Economics*, Vol. 52, 1937–38. Also in Mathematics, see E. T. Whittaker and G. Robinson, *The Calculus of Observations*, 2nd edition, pp. 81–2, on successive approximations.

be observed that neither case (ii) nor (iii) would arise if the section of the preference surfaces by the planes $a = a_1$, $a = a_2$, $b = b_1$, $b = b_2$ all had true peaks rising to a maximum and falling from it—*i.e.* if the line $\frac{\partial U_1}{\partial a} = 0$ intersected the lines $b = b_1$ and $b = b_2$ at points $a = a'$ such that $a_1 < a' < a_2$, and if the line $\frac{\partial U_2}{\partial b} = 0$ intersected the lines $a = a_1$ and $a = a_2$ at points $b = b'$ such that $b_1 < b' < b_2$.

The same results hold when the line $\frac{\partial U_1}{\partial a} = 0$ has a negative slope, and the line $\frac{\partial U_2}{\partial b} = 0$ has a positive slope.

34. When the curves $\frac{\partial U_1}{\partial a} = 0$ and $\frac{\partial U_2}{\partial b} = 0$ have opposite slopes but are not straight lines, any particular case may be investigated by the graphical method indicated. Looking for general rules, however, we can establish the following.

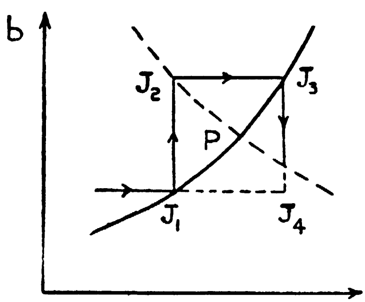


FIG. 34

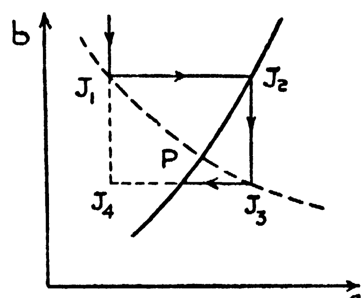


FIG. 35

Rule 2. Let J_1, J_2, J_3 be points on the curves denoting the results of three successive votes. Complete the rectangle $J_1J_2J_3J_4$. Then if the curve giving the result of the next vote emerges from the rectangle between J_3 and J_4 there will be either convergence or a perpetual spiral within the rectangle. If this curve emerges beyond J_4 there is bound to be divergence from the rectangle.

In Fig. 34 the first vote is taken in regard to a, and in Fig. 35 the first vote is taken in regard to b. In each case P is the point of intersection of $\frac{\partial U_1}{\partial a} = 0$ and $\frac{\partial U_2}{\partial b} = 0$. Then for convergence to P or a perpetual spiral to be possible, it is necessary that the curve giving the result of the next vote should leave the rectangle between J_3 and J_4; because if this condition is not satisfied, the voting web will pass outside the rectangle and will be unable to enter it at a later stage in the voting. See Fig. 36 in which two alternative shapes of the curve $\frac{\partial U_2}{\partial b} = 0$ are shown to the right of J_2.

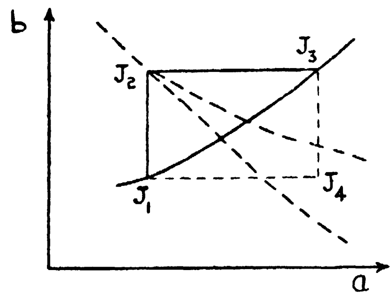


FIG. 36

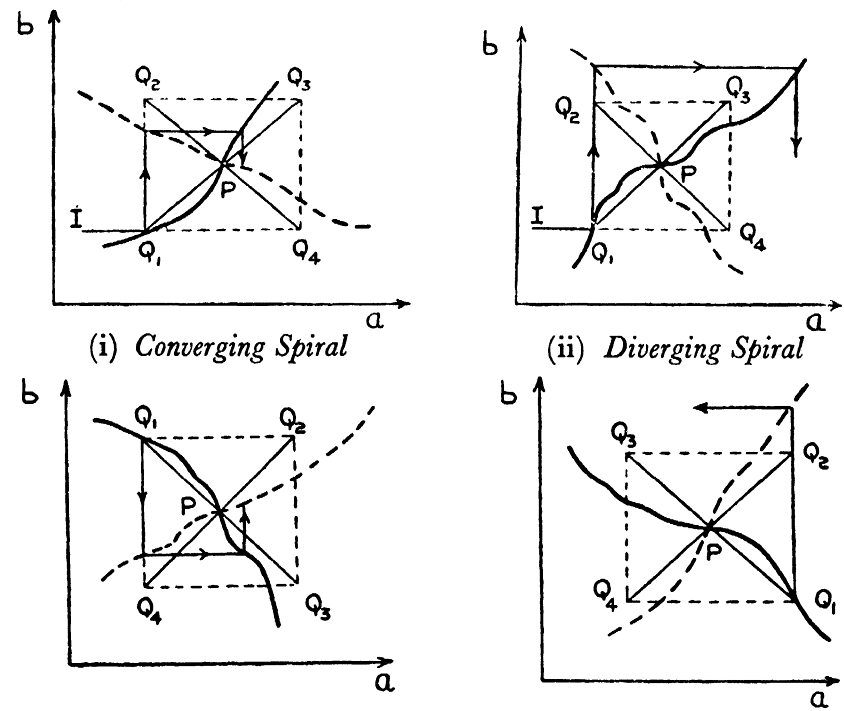
35. *Rule* 3. A necessary condition for convergence to a point P of intersection between the curves $\frac{\partial U_1}{\partial a} = 0$ and $\frac{\partial U_2}{\partial b} = 0$, when these curves have continuous and opposite slopes in the neighbourhood of P, is that the magnitude of the slope of the tangent to $\frac{\partial U_1}{\partial a} = 0$ at P must exceed that of $\frac{\partial U_2}{\partial b} = 0$ at P, except for the special case in which the sequence of votes involves a motion on b with $a = a_P$, or a motion on a with $b = b_P$. In such a special case P would be the point of stable or unstable equilibrium.

This follows from Rule 2: for if the conditions of convergence named in Rule 3 are not satisfied, the conditions of convergence of Rule 2 could be violated by taking the dimensions of the rectangle $J_1J_2J_3J_4$ sufficiently small.

36. Next we can establish certain conditions which are

sufficient, though not necessary, for convergence. Let the existing arrangement be represented by the point I, and let a vote taken in regard either to a or b, result in the new position Q_1. Join Q_1 to P and produce the line Q_1P to Q_3 so that $Q_1P = PQ_3$. Complete the rectangle $Q_1Q_2Q_3Q_4$.

Rule 4. Let the new position Q_1 result from a vote taken in regard to a. There will be a spiral converging to P if $\frac{\partial U_1}{\partial a} = 0$ lies above the diagonal Q_1Q_3 in the upper half of the rectangle and below it in the lower half, and if also the curve $\frac{\partial U_2}{\partial b} = 0$ lies below the diagonal Q_2Q_4 in the upper



(i) *Converging Spiral* (ii) *Diverging Spiral*

(iii) *Converging Spiral* (iv) *Diverging Spiral*

FIG. 37

half of the rectangle and above it in the lower half.

There will be corresponding results when the initial vote is taken in regard to b.

Where the first vote is taken in regard to a, convergence is illustrated in Fig. 37 (i) and (iii), and divergence in (ii) and (iv) of the same Figure.

The conditions stated are *sufficient* to ensure convergence to P, though not necessary, for in the cases shown in Figs. (i) and (iii) there would still be convergence if the two midmost curves were distorted in any manner provided that the points at the corners of the voting spirals remained unchanged.

37. It is, of course, possible to have a converging spiral for an initial vote on a, but a diverging spiral for an initial vote on b, and *vice versa*. Cf. Fig. 38.

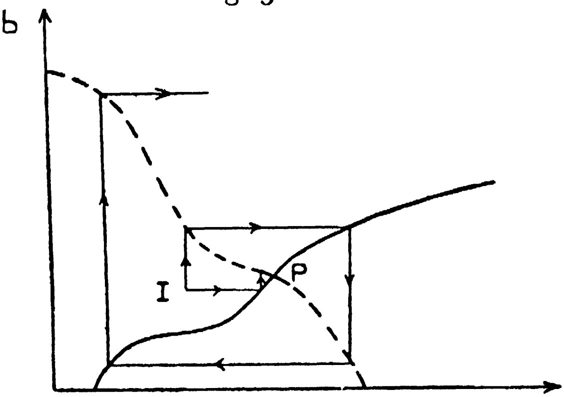


FIG. 38

38. A sub-case of the conditions in which the slopes of $\frac{\partial U_1}{\partial a} = 0$, $\frac{\partial U_2}{\partial b} = 0$ are of opposite sign, is where the curves

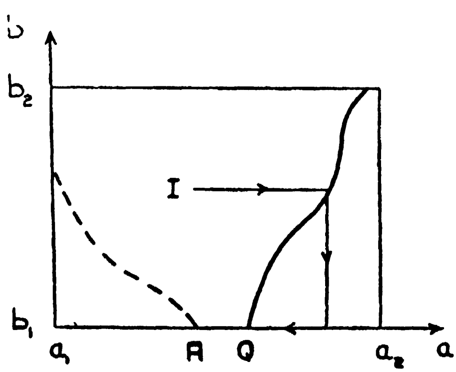


FIG. 39

have no point of intersection within the permissible ranges of the variables. This is illustrated in Fig. 39, where, to the right of the point A, the curve $\frac{\partial U_2}{\partial b} = 0$ is assumed to lie below the a-axis. The case works out as if, to the right of A, $\frac{\partial U_2}{\partial b} = 0$ were coincident with the a-axis, and Q is reached as a position of stable equilibrium after three votes.

When No Restriction is Placed on the Slopes of $\frac{\partial U_1}{\partial a} = 0$ *and* $\frac{\partial U_2}{\partial b} = 0$, *and when the Curves may have More than One Point in Common.*

39. In Fig. 40, with I as initial point, the small number of votes shown gives first convergence and then divergence.

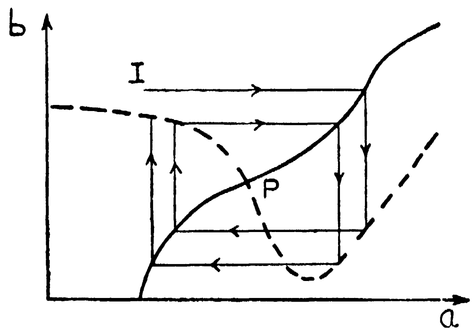


FIG. 40

40. If the two curves $\frac{\partial U_1}{\partial a} = 0$, $\frac{\partial U_2}{\partial b} = 0$ have always slopes of the same sign and intersect at several points, Rule 1 applies in the neighbourhood of each point of intersection. In Fig. 41,

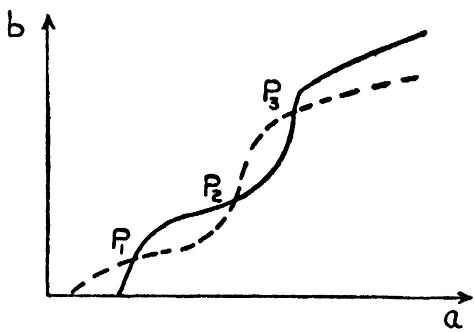


FIG. 41

for example, P_1 and P_3 are positions of stable equilibrium and P_2 is a position of unstable equilibrium. Whether it would be to P_1 or P_3 that, after a sufficient number of votes, the decision would converge, would depend on the initial position from which the system starts.

41. In the general case no restriction is placed on the shapes of the curves except that they satisfy the basic requirement that there is not more than one point of intersection between $\frac{\partial U_1}{\partial a} = 0$ and the line $b =$ constant, nor between $\frac{\partial U_2}{\partial b} = 0$ and the line $a =$ constant. For these conditions Rule 2 may be generalized as follows:

Rule 5. (i) Let J_1, J_2, J_3 be points on the curves denoting the results of three successive votes. Complete the rectangle $J_1J_2J_3J_4$. Then if the curves do not re-enter the rectangle once they have emerged from it, and if the curve giving the result of the next vote emerges from the rectangle between J_3 and J_4, there will be either convergence to one of the points of equilibrium $P_1, P_2 \ldots .$ enclosed by the rectangle, or a perpetual spiral within the rectangle. This holds without any other special restriction on the nature of the two midmost curves.

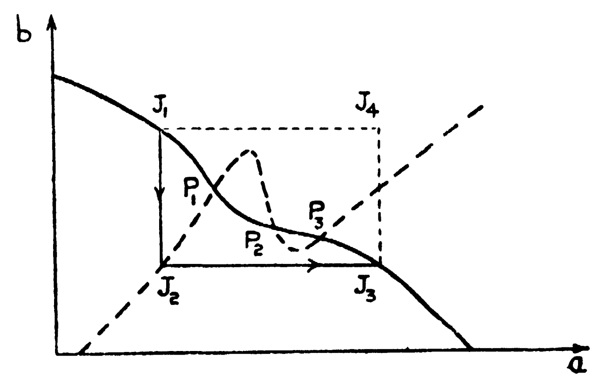


FIG. 42

Proof of part (i) of the rule is sufficiently obvious from Fig. 42 and from the discussion we have already given of Rule (2).

To state part (ii) of the rule, extend the sides of the rectangle $J_1J_2J_3J_4$ giving eight additional rectangular regions in the plane, and number these regions I....VIII, where the numbering follows the direction of the first three votes, and the region II has the points J_1 and J_2 at its corners. See Fig. 43.

Now let us examine the conditions under which there is bound to be divergence from the rectangle. Assuming that the curves do not re-enter the rectangle after they leave it, divergence can only occur when the curve giving the result of the fourth vote lies beyond J_4. In these circumstances there will be at least one part of one of the curves in three of the four corner regions I, III, V and VII, and the curve which emerges beyond J_4 may or may not enter the fourth corner region. It will not be permissible for a curve to enter certain regions in virtue of the basic requirement that $\frac{\partial U_1}{\partial a} = 0$ should define only one value of a for a given b, and $\frac{\partial U_2}{\partial b} = 0$ only one value of b for a given a.

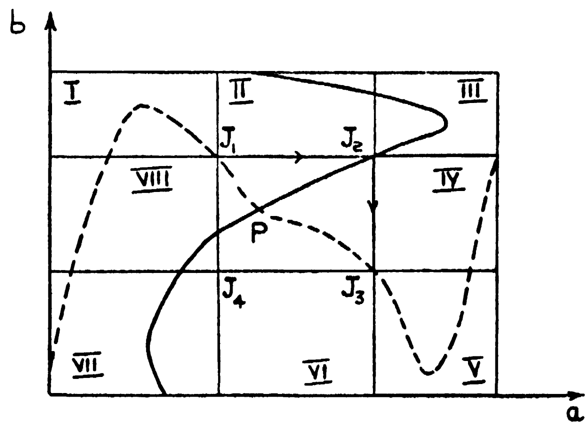


FIG. 43

The second part of *Rule* 5 can now be stated.

(ii) Let the curve giving the result of the fourth vote emerge from the rectangle into the region VIII, and let it move in this region and in the adjacent permissible corner region VII, in any manner. And let the curves that emerge at the corners J_1, J_2 and J_3 of the rectangle move in their respective corner regions in any manner. Then there is bound to be divergence provided the curves remain in the regions named, or provided that if $\frac{\partial U_1}{\partial a} = 0$ enters any of the regions other than those named, that is permissible to it, it does so only for values of b at which there is no curve $\frac{\partial U_2}{\partial b} = 0$, and provided that if $\frac{\partial U_2}{\partial b} = 0$ enters any of the regions other than those named, that is permissible to it, it does so only for values of a at which there is no curve $\frac{\partial U_1}{\partial a} = 0$.

It is impossible to move either horizontally or vertically from a point in one of the corner regions to the interior of the rectangle $J_1J_2J_3J_4$. The other conditions named ensure that we do not move horizontally on to a portion of $\frac{\partial U_1}{\partial a} = 0$ directly above or below $J_1J_2J_3J_4$; and that we do not move vertically on to a portion of $\frac{\partial U_2}{\partial b} = 0$ either directly to left or to right of $J_1J_2J_3J_4$. They therefore ensure divergence from the rectangle. The rule is exemplified in Fig. 43, in which the curves satisfy the conditions stated.

42. As another example we may give the following:

Example. If it is possible, as in Fig. 44, to find points K_1, K_2

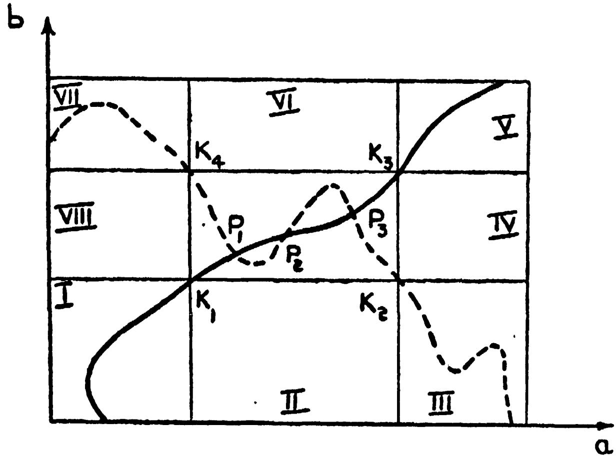


FIG. 44

on the midmost curve $\frac{\partial U_1}{\partial a} = 0$, and points K_2, K_4 on the midmost curve $\frac{\partial U_2}{\partial b} = 0$, such that $K_1K_2K_3K_4$ is a rectangle whose sides are parallel to the co-ordinate axes, and if, further, the curves are excluded from the regions II, IV, VI, VIII, the following general results hold without further restriction on the nature of the midmost curves:

(1) Convergence to a point of equilibrium within the rectangle is impossible, (i) for an initial vote taken in regard to *a* or to *b*, if the initial point lies in regions I, III, V or VII; (ii) for an initial vote taken in regard to *b*, if the initial point lies in region VIII or IV; and (iii) for an initial vote taken in regard to *a*, if the initial point lies in region II or VI.

(2) A vote on *a* from an initial point in VIII or IV, a vote on *b* from an initial point in II or VI, and a vote on either *a* or *b* from an initial point inside the rectangle, must lead either to an endless cycle confined within the rectangle, or to convergence to one of the points of equilibrium P_1, P_2. . . . enclosed by the rectangle.

Proof is sufficiently indicated by the figure.

43. *Discrete Variable.* Next let *a* and *b* be discrete, *a* having the values say $a_1, a_2, \ldots . a_5$ and *b* the values b_1, b_2, b_3, b_4. And we will suppose that the valuations of one or more of the members in relation to these values (motions), are complementary in a way that conforms to the requirements of § 28. If so, it must be possible to find an ordering of the *a*'s and *b*'s on the coordinate axes, such that given any value of *a*, each member's valuations of the *b*'s form a single-peaked curve; and, given any value of *b*, the member's valuations of the *a*'s form a single-peaked curve. This condition will be satisfied for example, if an ordering of the points on the axes can be found for which points of equal preference are represented as lying on convex contours.

Figs. 45, (A), (B) and (C) show the scales of preference of the members *A*, *B* and *C* respectively, in which the condition of having single-peaked curves is satisfied when the ordering of the motions on the *a*-axis is a_1, a_2, a_3, a_4, a_5 and on the *b*-axis is

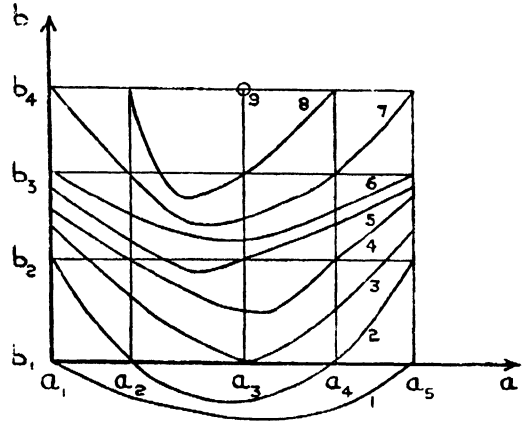


FIG. 45 (A)

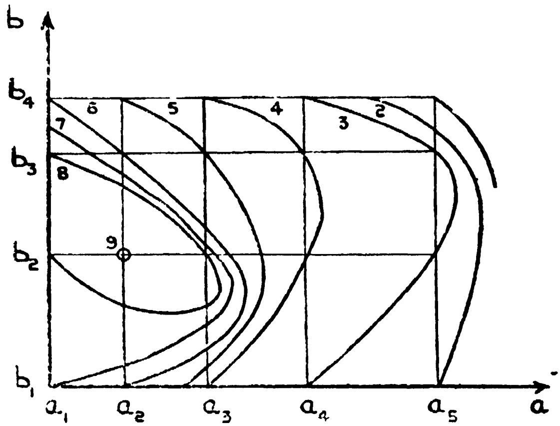


FIG. 45 (B)

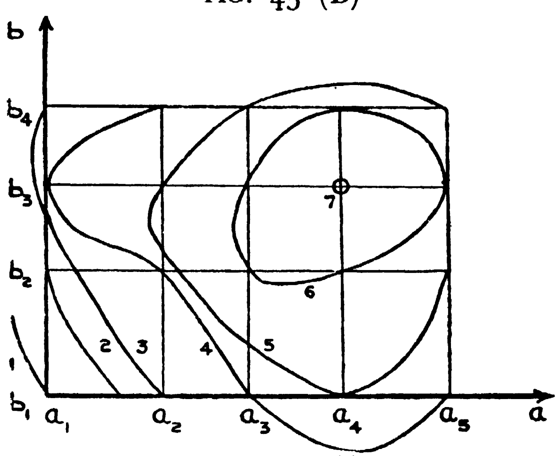
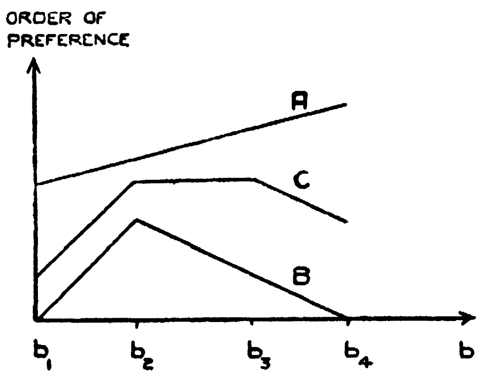


FIG. 45 (C)

b_1, b_2, b_3, b_4. The *a*- and *b*-axes in these diagrams are to be taken as being significant only at the points on them $a_1, \ldots . a_5$ and $b_1, \ldots . b_4$. The indifference contours, therefore, are significant only at the corresponding points. For convenience, however, in the diagrams we have shown the axes as continuous lines and the contours as continuous curves. And we have also named the contours 1, 2, 3, ... in ascending order of preference for the member concerned.

With the committee procedure supposed, let the initially existent values of *a* and *b* be given by (a_1, b_3), and let the first vote be taken in regard to *a*. Each member expects the value b_3 to persist. The optimum values of *a* for *A*, *B* and *C* respectively are a_3, a_1 and a_4; and a_3 is therefore the value selected.

In the next vote in relation to *b*, with a_3 as the expected value of *a*, the optimum values of *b* attainable by *A* and *B* respectively, become b_4 and b_2, while *C* has as his optima b_2 and b_3, being indifferent between them. In these circumstances the members' preference curves are as shown in Fig. 46,



Given $a = a_3$

FIG. 46

where the member's name is attached to the relevant curve, and it is seen that the result is indeterminate between b_2 and b_3. Let us suppose that *B* is chairman of the committee and has the right to a casting vote. Then b_2 becomes the value chosen.

With b_2 as the value of *b* in existence, the optimum values of a for the members A and B respectively, become a_3 and a_2; while C has optima a_3 and a_4, being indifferent between them; and a_3 is again the value selected. Thus with the starting point (a_1,b_3), (a_3,b_2) will be a position of stable equilibrium.

44. As a further example let us suppose that with members' scales of preferences as in Figs. 45, (A), (B) and (C), the initially existent values of the variables are given by the point (a_2, b_1), and the first vote is taken in relation to b. The process can be set out briefly thus.

Given a_2. Vote on b.

Member	A	B	C
Member's Optimum	b_4	b_2	b_3

and b_3 is selected.

Granted that B is chairman the result will work out as in the preceding example. Thus, as with the starting point (a_1,b_3), the starting point (a_2,b_1) gives (a_3,b_2) as a stable position.

Extension to a Committee with More than Three Members

45. The results of the present Section are valid when we have not 3, but any odd number 5, 7, . . . of members in the committee. We can examine the midmost of the curves $\frac{\partial U_1}{\partial a} = 0$ and the midmost of the curves $\frac{\partial U_2}{\partial b} = 0$. The same conditions as before in relation to these curves give rise to the same results.

When the number of members in the committee is even, 2, 4, . . . let us suppose that in the event of a tie in the voting the chairman has the right to a casting vote. Then if we regard each of the chairman's curves $\frac{\partial U}{\partial a} = 0$, $\frac{\partial U}{\partial b} = 0$ as consisting of two coincident curves, the above findings again apply.

SECTION IV

The Results are Independent of the System of Units

46. Committee decisions have been discussed in terms of the preference functions $U(a)$ and $U(a,b)$. Such functions represent the order of preference in which a committee member ranks the various motions under discussion. In any one particular representation, the mathematical technique implies that the preferences U are measurable, *e.g.* that the preference interval between $U(a_1,b_1)$ and $U(a_2,b_2)$ is equal to that between $U(a_3,b_3)$ and $U(a_4,b_4)$.

Now, psychologically, there is no sense in which two preference intervals can be said to be equal. All that we can assert about the preferences of the individual is that they are *ordered*. Thus if a given set of preferences is represented by the function U, any other function $U^* = F(U)$ will give an equally satisfactory representation, provided that the transformation $U^* = F(U)$ does not change the preference order. This will be the case if $F(U)$ is an arbitrary monotonic increasing[1]

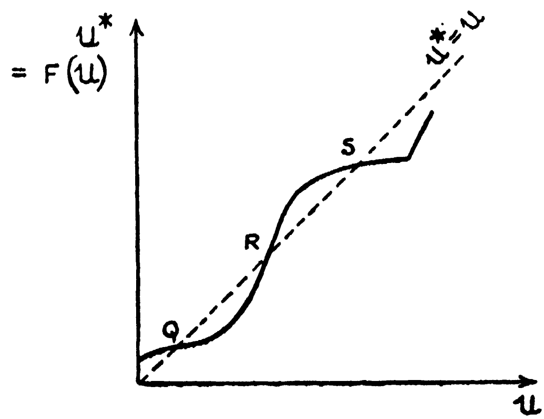


FIG. 47

function of U, *i.e.* if $F(U_1) > F(U_2)$ when $U_1 > U_2$, $F(U_1) = F(U_2)$ when $U_1 = U_2$, and $F(U_1) < F(U_2)$ when $U_1 < U_2$, for

[1] The graph of $F(U)$ is always upsloping, without any horizontal segments. Such a function is sometimes called "monotonic strictly increasing."

all U_1 and U_2. Such functions are single-valued functions: for any given value U_1 of U there is only one value U_1^* of $F(U)$, and for any given value U_1^* of $F(U)$ there is only one value of U_1 of U. The relation between U and $F(U)$ is represented graphically by a constantly up-sloping curve, which may or may not possess a tangent at every point. Regions in which $U^* < U$, *e.g.* the arc QR in Fig. 47, give compression of the preference surface U^* compared with U, while regions in which $U^* > U$, *e.g.* the arc RS, give expansion.

It remains to show that the results established in the previous sections will remain unchanged when the preference functions U_A, U_B, U_C are replaced by $F_A(U_A)$, $F_B(U_B)$, $F_C(U_C)$, *i.e.* that our results are invariant with respect to the transformations $U_A^* = F_A(U_A)$, $U_B^* = F_B(U_B)$, $U_C^* = F_C(U_C)$, where F_A, F_B, F_C denote monotonic increasing functions.

47. It is at once obvious that the results of Section I are unchanged by the transformation considered, for the results of that section are merely geometrical representations of the conditions in § 4, and these conditions are themselves invariant with respect to the transformation. *E.g.* the condition $\left.\begin{matrix} h<h_u \\ k<k_u \end{matrix}\right\}$ when $l>l_u$, implies $\left.\begin{matrix} h^*<h_u^* \\ k^*<k_u^* \end{matrix}\right\}$ when $l^*>l_u^*$, where $h^* = F_A(h)$, etc. It follows that the results deduced for the preference functions U_A, U_B, U_C will remain valid when these functions are replaced by U_A^*, U_B^*, U_C^*.

48. The results of Section II were derived from a study of the properties of the indifference contours $U(a,b)$ = constant. If the preferences are represented in terms of a new preference function $U^*(a,b)$, the indifference contours in the a–b plane are defined by the equation $U^*(a,b)$ = constant, *i.e.* $F\{U(a,b)\}$ = constant; this implies $U(a,b)$ = constant and we therefore have the same family of contours.

Geometrically, if we take any indifference contour $U(a,b)$ = constant = U_1, say, in the a–b plane, when the axis denoting level of preference is extended or contracted in the way permitted by our transformation, all of the points on this contour now correspond to a different height on the preference axis; but the contour at this new level is the same as the initial contour and the member is still indifferent as between points on it. Points on another contour $U(a,b)$ = constant = U_2, say, also remain together after the transformation. And that one of the two contours which initially stood at a higher level of preference will remain at a higher level of preference than the other after the extension or contraction in the preference axis has taken place. The results of Section II are therefore invariant with respect to the transformation considered.

49. The results of Section III were expressed in terms of the properties of curves derived from indifference contours, and are therefore also unchanged by the transformation $U^* = F(U)$.

50. Let us now assume that the variables a, b are ordered, but do not possess a natural measure, *i.e.* we assume that no significance is attached to equality of a-intervals or to the equality of b-intervals. The results established will remain valid if they can be shown to be invariant with respect to the transformation $a^* = \lambda(a)$, $b^* = \mu(b)$, where λ and μ denote arbitrary monotonic increasing functions. Now if U_1 be the preference level assigned to a_1 by a member, this will also be the preference level assigned to a_1^*, since a_1^* is merely a new label for the motion a_1. It follows that if $U \gtreqless U_1$ over a given range in a, the same preference relations will apply to the corresponding range in a^*. The conditions of § 4 will therefore continue to hold, and the results of Section I will remain valid, when expressed in terms of a^*.

51. The results of Sections II and III need to be discussed in greater detail. Let us first note some general results concerning the transformation $a \to a^*$, $b \to b^*$.

(i) There is a one-one correspondence between points in the a–b plane and points in the a^*–b^* plane, *i.e.* to any point (a,b) in the a–b plane, there corresponds one and only one point (a^*,b^*) in the a^*–b^* plane; and conversely, to any point (a^*, b^*) there corresponds one and only one point (a,b).

(ii) Straight lines parallel to the co-ordinate axes in the a–b plane transform to straight lines parallel to the axes in the a^*–b^* plane, and conversely; *e.g.* a = constant implies a^* = constant, and a^* = constant implies a = constant.

Straight lines not parallel to the co-ordinate axes, however, do *not* in general transform into straight lines.

(iii) The order of points on a curve is invariant under the transformation.

(iv) If a section of a curve has a positive (or negative) slope in one plane, the corresponding section of the transformed curve has a positive (or negative) slope in the other plane.

(v) If two curves $f_1(a,b) = c_1$, $f_2(a,b) = c_2$ intersect at a point (a_i,b_j) in the a–b plane, the transformed curves $f_1^*(a^*,b^*) = c_1^*$, $f_2^*(a^*,b^*) = c_2^*$ intersect at the corresponding point (a_i^*,b_j^*) in the a^*–b^* plane, and conversely. For if (a,b) is a point on a curve in the a–b plane, (a^*,b^*) is a point on the corresponding curve in the a^*–b^* plane, and therefore, if (a_i,b_j) is on the curve $f_1(a,b) = c_1$, (a_i^*, b_j^*) will be on the curve $f_1^*(a^*,b^*) = c_1^*$. Similarly (a_i^*,b_j^*) is on the second curve $f_2^*(a^*,b^*) = c_2^*$, *i.e.* the two curves have the point (a_i^*,b_j^*) in common. It follows that curves in the a^*–b^* plane intersect only at points corresponding to the intersection of corresponding curves in the a–b plane, and *vice versa*.

A number of important results follow:—

(*a*) Closed curves transform to closed curves.

(*b*) If (a_i,b_j) is a point inside (or outside) a closed curve in one plane, the corresponding point (a_i^*,b_j^*) is inside (or outside) the corresponding closed curve in the other plane.

(*c*) If in one plane a curve touches, but does not cross, a straight line parallel to one or other of the co-ordinate axes in one plane, the transformed curve will touch but not cross the corresponding line in the other plane.

52. For the purpose of Sections II and III it must be noted that it is possible that convex curves may not transform to convex curves, although if a curve is convex in one plane there will not be more than two points of intersection between the transformed curve and any straight line parallel to one or other of the co-ordinate axes in the second plane. Further, smooth contours will not necessarily transform into smooth contours. It may happen that a contour has continuous slope in the neighbourhood of a point (a_i,b_j) in the a–b plane, but that the transformed curve has a discontinuity in slope in the neighbourhood of the corresponding point (a_i^*,b_j^*) in the

a^*–b^* plane. This will happen at any point at which one or other of the functions $\frac{d}{da}\lambda(a)$, $\frac{d}{db}\,\mu(b)$ is discontinuous.

It is obvious, however, from the general properties of our transformation, that if two contours meet in a point Q in one plane, the transformed contours will meet in the corresponding point Q^* in the other plane. If there is a single contact locus between two sets of contours in one plane, there must be a single contact locus in the other plane.

If three contours meet in a point H, and no two of them have an area in common, in one plane, the transformed contours in the other plane will have no area in common, and will meet at the corresponding point H^*. If in one plane, the A, B and C contours are such that they can have a single point in common only if one of the contours collapses to a point, the transformed contours must also have this property. For any given set of contours, therefore, the results of Section II will be the same, whether we work in terms of the variables (a,b) or in terms of the variables (a^*,b^*).

53. The results of Section III were expressed in terms of the loci of points at which the indifference contours touched lines parallel to the co-ordinate axes. These loci were expressed as $\frac{\partial U}{\partial a} = 0$, $\frac{\partial U}{\partial b} = 0$. It is evident that the transformation of these loci will give the loci of points at which the indifference contours in the a^*–b^* plane touch lines parallel to the co-ordinate axes in that plane. The transformed loci will have properties corresponding to those of the original loci in the a–b plane; *e.g.* there will be not more than one intersection between the curve $\frac{\partial U}{\partial a^*} = 0$ and the line $b^* = b_j{}^*$; sections over which $\frac{\partial U_1}{\partial a^*} = 0$ has a positive slope will correspond to those over which $\frac{\partial U_1}{\partial a} = 0$ has a positive slope; and if the curves $\frac{\partial U_1}{\partial a} = 0$ and $\frac{\partial U_2}{\partial b} = 0$ intersect in a point P,

$\frac{\partial U_1}{\partial a^*} = 0$, $\frac{\partial U_2}{\partial b^*} = 0$ will intersect in the corresponding point P^*. Converging (or diverging) spirals, or staircases, will transform to converging (or diverging) spirals, or staircases, for horizontal and vertical lines transform to horizontal and vertical lines, and if Q_5 lies between Q_1 and P, Q_5^* must lie between Q_1^* and P^*. Cf. Figs. 48 and 49.

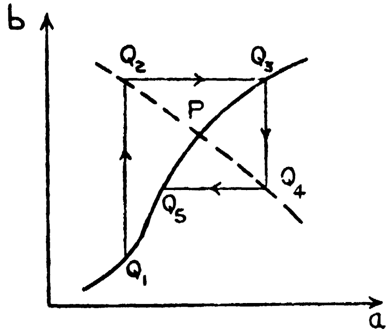


FIG. 48

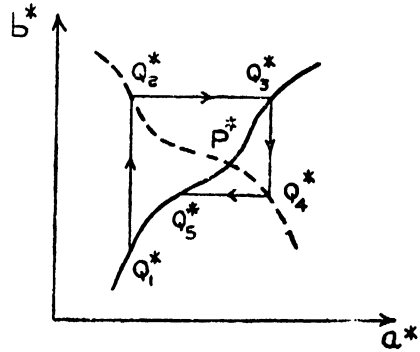


FIG. 49

The nature of the voting web is therefore invariant under the transformation. A converging, diverging or cyclic web in one plane implies a web of the same type in the other plane, so that our results are independent of the particular representation chosen for the variables a and b.

54. Special cases discussed in one plane, however, will not necessarily transform into the same special cases in the other plane. In § 33 the midmost curves were taken to be straight lines in the a–b plane. The corresponding curves in the a^*–b^* plane will also be straight lines only if the lines in the a–b plane are parallel to the co-ordinate axes. But convergence in one plane will imply convergence in the other.

If, in any particular case, conditions are satisfied in one plane which are sufficient, but not necessary, for convergence, the corresponding conditions in the other plane will not in general continue to be satisfied. In § 36 conditions sufficient for convergence to P were obtained by constructing a rectangle on Q_1PQ_3 as diagonal. The rectangle $Q_1Q_2Q_3Q_4$ will transform to a rectangle $Q_1^*Q_2^*Q_3^*Q_4^*$, but the diagonals Q_1Q_3,

Q_2Q_1 will not transform to the diagonals of the rectangle in the a^*–b^* plane; their transforms will be curves, and not straight lines. For this reason, the transform P^* of P will not in general be at the point of intersection of the diagonals of the rectangle $Q_1^*Q_2^*Q_3^*Q_4^*$. If the conditions are satisfied in the a–b plane, there will be convergence to P, and therefore there will be convergence to P^* in the a^*–b^* plane. In order to apply the test in the a^*–b^* plane it is necessary to construct a rectangle $Q_1^*Q_2^{*\prime}Q_3^{*\prime}Q_4^{*\prime}$, distinct from $Q_1^*Q_2^*Q_3^*Q_4^*$; but convergence to P^* does not imply that the conditions of § 36 with regard to the rectangle $Q_1^*Q_2^{*\prime}Q_3^{*\prime}Q_4^{*\prime}$ will be satisfied, since such conditions are not necessary for convergence. If, on the other hand, any *necessary* conditions for the convergence are satisfied in the one plane, they will also be satisfied in the other plane. Consider, for example, the conditions stated in § 34. In that paragraph we were concerned with the case in which the slope of the midmost curve $\frac{\partial U_1}{\partial a} = 0$ was always of one sign, while the slope of the other midmost curve $\frac{\partial U_2}{\partial b} = 0$ was always of the opposite sign. A necessary condition for convergence was obtained in terms of the point at which one of the two midmost curves crossed a side of a certain rectangle. It is obvious, from the general properties of monotonic increasing transformations, that in the a^*–b^* plane, the same relation will hold between the slopes of the two midmost curves, and that the condition stated is invariant under the transformation.