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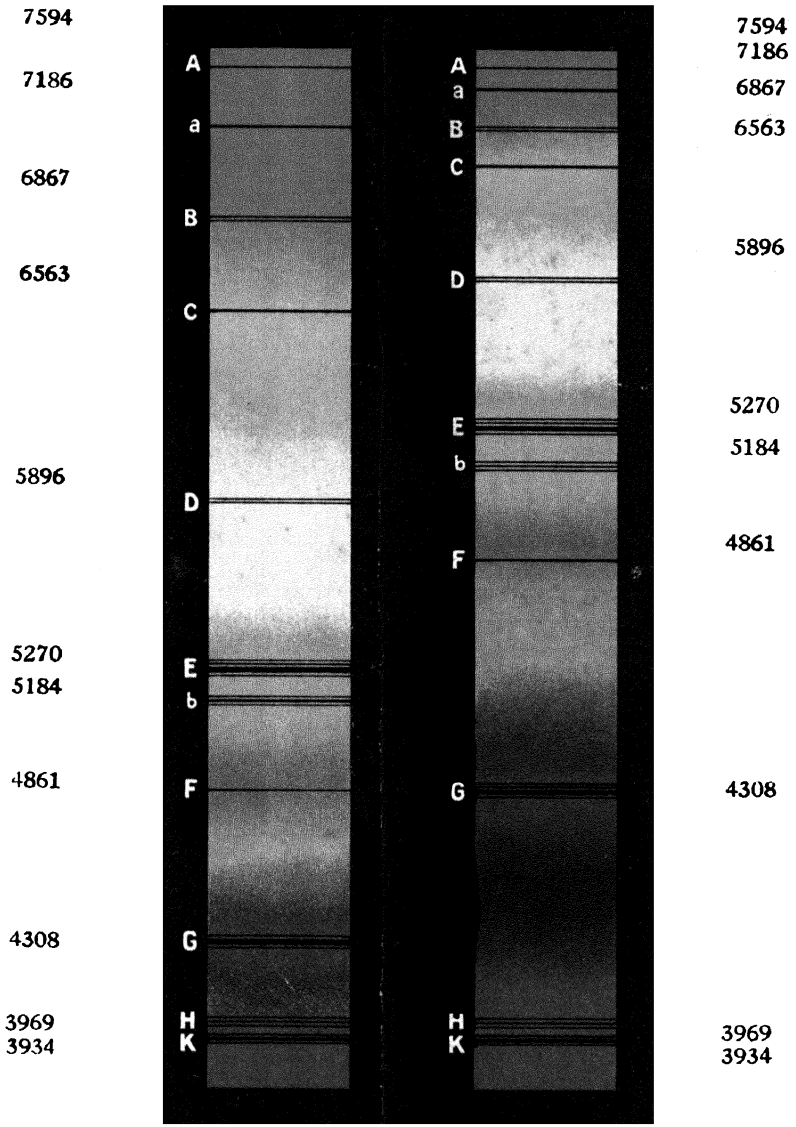
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AN INTRODUCTION TO PHYSICAL SCIENCE



Solar spectrum

(1) Formed by a prism.

(2) Formed by a diffraction grating.

(*See Chapter XXXIV*)

AN INTRODUCTION TO PHYSICAL SCIENCE

BY

CARL W. MILLER, PH.D.

Associate Professor of Physics in Brown University

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PREFACE

The subject matter of physics has increased so greatly during the last quarter of a century that it has become increasingly difficult for the college teacher to give an adequate survey of the field in the short space of a single year. Yet the average student finds time for only a single elementary course in this as in most other subjects. The possibility of compressing into this space the necessary fundamental training and at the same time giving the broader outlook over modern physics is not generally conceded, but it must be admitted that, if possible, it should be attempted. The contacts which physics makes with all other sciences and with the paraphernalia of everyday life make it more than ever before a *sine qua non* of a well-rounded education, and this primary function is not performed if the instruction must stop at the threshold of the newer developments.

The present book is an effort on the part of the author to lead the student by as natural steps as possible from the beginnings of scientific thought through to the surprising results of modern research. It has seemed best so far as possible to follow the historical development of the subject, but since it has required the unifying effect of recent discoveries to bring us to an appreciation of the close relationship between the various classical divisions of physics — mechanics, heat, light, and electricity — no attempt has been made to preserve the historical sequence where it did not appear to be the logical one.

Contrary to the opinion of many elementary students, the real difficulties of the subject are not mathematical. Nevertheless, it seems desirable, even at the expense of elegance, to avoid unnecessary mathematics, for the average student seldom approaches his elementary physics with enough facility in the use of such tools as the calculus for them to be of much service to him. Too often they confuse rather than clarify the main issues. This is the only excuse which the author is prepared to offer for the omission of all mathematics beyond that common in high school courses. An effort has been made, however, to make the task of applying the calculus an easy one for such instructors

as prefer to do so, and the step from the present treatment to that which is customary in advanced courses should not be difficult. This should be particularly true since wherever possible the modern point of view has been taken. Thus in the very first chapter the Galilean principle of relativity is emphasized as one of the fundamental laws of nature. This is used in Chapter IV to lead up to the laws governing planetary motion, in Chapter XXV for the study of wave propagation, and in Chapter XXXIII it is extended to the modern restricted theory of relativity. The treatment of image formation in Chapter XXVIII is similarly brought more nearly into accord with the methods used in advanced optics than is customary in an elementary text book. In this particular case the author ventures to hope that the treatment has gained in simplicity as well as in scope.

Although a part of the mathematics which frequently appears in elementary physics courses can be avoided, it must be confessed that it is the quantitative aspect of modern physics which makes it so convincing, and it is in order to exhibit this remarkable quantitative agreement of theory and experiment that such mathematics as that found in Chapters XIII, XXV, and XXXVI is introduced.

The goal of physical science is a fuller knowledge of the world in which we live, and the scientific method of approach to this goal is that of experiment, the formulation of hypotheses, and the checking of these hypotheses by still further experiment. The years which have passed since the beginning of the present century have been fruitful, not merely in the discovery of new facts and laws, but in the development of new tools for the laboratory. It is partly because of the certainty that these tools will be found increasingly useful in other sciences than physics that so much emphasis has been placed in the present book on experimental methods. Then, too, whether we wish it or not, experiment is destined in the future, as it has done in the past, to restrain us from too free speculation and to suggest new avenues of advance. This fundamentally practical viewpoint should be particularly well suited to the needs of students who contemplate future work in other sciences such as chemistry or biology.

It should be borne in mind that physics has been forced to pass through a succession of revolutions during the past quarter

of a century, which have shaken it to its very foundations. These revolutions have left the physicist in a chastened mood, in which he is prepared to admit the possibility or even the probability of future revolutions just as devastating to his present ideas. It would, therefore, be too optimistic to believe that the survey of the subject in the present book should not be found at fault as time goes by, a remark which applies with particular force to the last few chapters. These revolutions in physics have had a profound effect on the philosophical thought of the present generation, and it is particularly appropriate that a student should meet these new concepts in the practical and unromantic atmosphere of the physical laboratory which gave them birth.

The author is much indebted to Dr. Howard L. Andrews for his patience in reading the manuscript and for many valuable suggestions.

C. W. M.

PROVIDENCE, R. I.
February, 1932.

CONTENTS

CHAPTER PAGE

I. THE DEVELOPMENT OF THE SCIENTIFIC METHOD 1

1. The Beginnings of Astronomy. 2. The Problem of Falling Bodies. Galileo. 3. The Motion of a Body which has a Constant Acceleration. 4. The Principle of Galilean Relativity.

II. MASS AND FORCE 9

5. Newton's First Law of Motion. 6. Inertia and Mass. 7. The Measurement of Force. 8. Newton's Second Law of Motion. 9. Newton's Third Law of Motion.

III. COMPOSITION OF FORCES. EQUILIBRIUM 17

10. The Parallelogram of Forces. 11. The Triangle and Polygon of Force. 12. The Equilibrium of a Particle. 13. Components. 14. Friction.

IV. CURVILINEAR MOTION 24

15. Projectiles. 16. Planetary Motion. Centripetal Force. 17. Universal Gravitation. 18. Simple Harmonic Motion. The Pendulum.

V. MACHINES 34

19. The Lever. 20. The Law of Moments. 21. The Pulley System. 22. Work.

VI. ENERGY AND MOMENTUM 42

23. Kinetic Energy. 24. Potential Energy. 25. Conservation of Energy. 26. The Kinetic Energy of a Rotating Body. Moment of Inertia. 27. Impulse and Momentum. 28. Impact. Conservation of Momentum.

VII. FLUIDS 52

29. Viscosity. The Perfect Fluid. 30. Fluid Pressure. 31. Archimedes' Principle. 32. Atmospheric Pressure. The Barometer. 33. Fluids in Motion. Hydrodynamics.

VIII. HEAT 62

34. Temperature. 35. Quantity of Heat. Calorimetry. 36. Specific Heat. 37. Conversion of Mechanical Energy into Heat.

CHAPTER PAGE

IX. CHANGE OF STATE 69

38. Boyle's Law. 39. Liquefaction of a Gas. 40. The Continuity of State. Critical Temperature. 41. Latent Heat of Vaporization. 42. Fusion of a Solid.

X. TRANSFORMATION OF HEAT INTO MECHANICAL ENERGY.. 77

43. The Steam Engine. 44. The Carnot Engine. 45. The Second Law of Thermodynamics.

XI. ABSOLUTE TEMPERATURE AND ENTROPY 83

46. The Kelvin Scale of Temperature. 47. Absolute Zero. 48. The Law of Charles. 49. The Efficiency of Heat Engines. Carnot's Theorem. 50. Entropy. The Degradation of Energy.

XII. ATOMS AND ELEMENTS 92

51. Lavoisier and the Conservation of Mass. 52. John Dalton and the Law of Multiple Proportions. 53. Avogadro's Law. 54. The Ideal Gas Law. 55. The Periodic Table of the Elements.

XIII. THE KINETIC THEORY OF MATTER 102

56. The Pressure of a Gas. 57. The Interpretation of Temperature on the Kinetic Theory. 58. The Average Thermal Velocities of the Molecules. 59. Brownian Movements and Avogadro's Number. 60. Specific Heats of Gases. 61. Atomic Heats. The Law of Dulong and Petit. 62. The Kinetic Theory and the Second Law of Thermodynamics.

XIV. ELECTROSTATICS 115

63. Electrical Discoveries of the Eighteenth Century. 64. Distribution of Charge on a Conductor. Electrostatic Induction. 65. The Friction Machine. 66. Coulomb's Law.

XV. ELECTRIC POTENTIAL 122

67. Electric Potential. 68. Potential Due to a Point Charge. 69. Capacity. 70. Dielectric Constant. 71. The Voltaic Cell.

XVI. MAGNETISM 130

72. Fundamental Magnetic Phenomena. 73. The Earth's Magnetic Field. 74. The Magnetic Field Due to an Electric Current. 75. The Law of Biot and Savart. The Ampere. 76. The Tangent Galvanometer. 77. Rowland's Experiment. 78. Force Exerted by a Magnetic Field on a

CHAPTER PAGE

Wire Carrying a Current. 79. Suspended Coil Current-measuring Instruments.

XVII. The Electric Current and Energy 142

80. Joule's Law. 81. Electromotive Force. 82. Electric Resistance. 83. Fall in Potential along a Conductor. 84. Galvanic Cells. 85. Thermal Electromotive Force. The Seebeck Effect. 86. The Measurement of Resistance. The Wheatstone Bridge. 87. Ammeters and Voltmeters. 88. The Potentiometer.

XVIII. Electrolysis. The Atomic Nature of Electricity .. 154

89. Faraday's Laws of Electrolysis. 90. The Fundamental Charge on a Univalent Ion. 91. The Ratio of the Electromagnetic to the Electrostatic Unit of Charge. 92. The Atomic Nature of Electricity. Millikan's Experiment.

XIX. Cathode Rays. Electrons 163

93. The Electric Conduction of Gases. 94. Cathode Rays. 95. Velocity and Ratio of Charge to Mass of the Cathode Particles. 96. Relation between the Velocity of the Cathode Particle and the Voltage Applied to the Tube.

XX. X-rays and Positive Rays 172

97. Fundamental Properties of X-rays. 98. Thermionic Emission. The Edison Effect. 99. The Ionization of Gases. 100. Positive Rays. 101. e/m of Positive Rays. 102. Isotopes. 103. The Mass Spectrograph.

XXI. Radioactivity 181

104. Radium. 105. Alpha, Beta, and Gamma Rays. 106. The Nature and Properties of Beta Rays. 107. Alpha Particles. 108. The Rutherford Model of the Atom. 109. The Disintegration of Radioactive Elements. Radium Emanations. 110. The Radioactive Families of Elements.

XXII. Induced Electromotive Forces 194

111. E.M.F. Induced in a Wire by its Motion through a Magnetic Field. 112. E.M.F. due to Changing Magnetic Field. 113. Lenz's Law. 114. The Magnetic Properties of Iron. 115. The Electric Generator. 116. The Transformer. Self Inductance. 117. Power Transmission at High Voltage.

XXIII. Electric Oscillations 206

118. Damped Oscillations. 119. The Oscillatory Discharge of a Condenser. 120. Resonance. 121. Radio Communication. 122. The Electron Tube as a Detector. 123. The

CHAPTER PAGE

Electron Tube as an Amplifier. 124. The Electron Tube as a Generator of Oscillations. 125. The Mode of Propagation of Electric Oscillations from Transmitter to Receiver.

XXIV. SOUND 219

126. Noises and Musical Sounds. 127. Loudness and Pitch. 128. Quality. 129. The Telephone and the Phonograph. 130. The Oscillograph. 131. Harmonic Analysis and Synthesis. 132. Analysis by Electric Filters. 133. The Sounds of Speech. 134. Radio Telephony.

XXV. THE PROPAGATION OF SOUND 229

135. The Nature of Sound Waves. 136. The Velocity of Propagation of Sound. 137. The Propagation of a Wave along a Cord. 138. Longitudinal and Transverse Waves. 139. Reflection of Sound. Reverberation.

XXVI. PERIODIC WAVES. INTERFERENCE 238

140. Frequency, Velocity, and Wave-length. 141. The Interference of Two Sound Waves. Beats. 142. Interference Due to a Reflected Wave. Standing Waves. 143. Fundamental and Harmonics of a Vibrating Cord. 144. Standing Sound Waves. Kundt's Experiment. 145. Velocity of Sound in Solids. 146. The Resonance of Air Columns. Organ Pipes.

XXVII. THE VELOCITY OF LIGHT 248

147. The Nature of Light. 148. The Velocity of Propagation of Light. 149. Terrestrial Methods of Fizeau and Michelson. 150. The Velocity of Light in other Media. 151. The Velocity of Light of Different Colors.

XXVIII. THE FORMATION OF IMAGES 258

152. Images Formed by Mirrors. 153. Images Formed by Lenses. 154. Algebraic Method of Locating the Image Formed by a Single Refracting Surface. 155. The Image Formed by a Spherical Mirror. 156. Thin Lenses. 157. The Spectacle Lens.

XXIX. OPTICAL INSTRUMENTS 270

158. The Magnifying Glass. 159. The Telescope. 160. Total Reflection. The Prism Binocular. 161. The Microscope. 162. Correction of Objective for Chromatic and Spherical Aberration. 163. The Photographic Objective. Astigmatism.

XXX. INTERFERENCE AND DIFFRACTION 280

164. Diffraction. 165. The Interference of Light. Young's Pinhole Experiment. 166. The Lippmann Process

CHAPTER PAGE

of Color Photography. 167. The Effect of Diffraction on the Image Formed by a Lens. 168. The Resolving Power of a Telescope. 169. The Resolving Power of a Microscope. 170. The Diffraction Grating.

XXXI. POLARIZATION 293

171. Double Refraction and Polarization. 172. Double Refraction a Consequence of the Transverse Nature of Light. 173. The Nicol Prism. 174. Circularly and Elliptically Polarized Light. 175. Rotation of the Plane of Polarization.

XXXII. ELECTRIC WAVES 302

176. The Transition from an Elastic Solid to an Electromagnetic Ether. 177. The Generation and Detection of Electromagnetic Waves in the Laboratory. Heinrich Hertz. 178. Properties of Electric Waves Analogous to Light.

XXXIII. RELATIVITY 308

179. The Synchronization of Clocks in a Moving Laboratory. 180. The Michelson-Morley Experiment. 181. The Principle of Relativity. 182. The Lorentz Transformation. 183. The Four-dimensional Space-time of Minkowski. 184. The Composition of Velocities. 185. The Variation of Mass with Velocity and the Equivalence of Mass and Energy.

XXXIV. THE CONSTITUTION OF LIGHT. SPECTRA 322

186. The Continuous Spectrum from an Incandescent Solid. 187. Infra-red Light. 188. Line Spectra from Incandescent Gases. 189. Solar Spectra. Fraunhofer Lines. 190. Selective Absorption. The Color of Pigments. 191. Color Photography.

XXXV. SHORT WAVE-LENGTHS. X-RAYS 329

192. Ultra-violet Light. 193. The Nature of X-rays. 194. The X-ray Crystal Spectrometer. 195. X-ray Spectra. 196. The Relation between Wave-length and Atomic Number. 197. Gamma and Cosmic Rays.

XXXVI. THE ORIGIN OF SPECTRA. QUANTUM THEORY 338

198. The Distribution of Energy in a Continuous Spectrum. 199. The Photoelectric Effect. 200. The Bohr Theory of Line Spectra. 201. The Application of the Bohr Theory to X-ray Spectra. 202. The Compton Effect.

XXXVII. THE WAVE MECHANICS 352

203. Phase Waves. 204. de Broglie Waves and Quantum Theory. 205. The Experimental Discovery of de Broglie

CHAPTER PAGE

Waves. 206. The Electron as a Packet of Waves. 207. The Uncertainty Principle of Heisenberg.

XXXVIII. Cosmic Physics 362

208. Stellar Spectra. The Constitution of the Stars. 209. The Distance of the Stars. Spectroscopic Parallax. 210. The Nature and Dimensions of the Stars. 211. The Doppler Effect and Stellar Velocities. 212. The Origin of Cosmic Rays.

Supplementary Problems 374

Fundamental Physical Constants 386

Useful Relations Between Common Units 387

Table of Logarithms 388

Index 393

AN INTRODUCTION TO PHYSICAL SCIENCE

CHAPTER I

THE DEVELOPMENT OF THE SCIENTIFIC METHOD

As we look backward to the dawn of history, we see primitive man engaged in an oftentimes losing struggle against the forces of nature. All his energies seem to have been required to furnish food and clothing for himself and his family and to protect them from wild beasts and unfriendly neighbors. As gradually, however, his inventive genius devised ways and means of reducing the labor which was necessary for mere existence, his intellectual and aesthetic nature began to develop. He began to evince an interest in his surroundings entirely apart from the possibilities which they afforded for food and shelter. Carvings on the walls of caves which he occupied in Central Europe testify to the admiration with which he viewed the beauty and grace of the wild life about him. That admiration must early have embraced also the lakes and mountain peaks among which he lived and the starry sky which hung over him in the night watches.

1. The Beginnings of Astronomy. When that admiration ripened into curiosity and he began to speculate concerning the nature and origin of things, we do not know, but it must have been early indeed. Thus we find in the very oldest Egyptian and Mesopotamian civilizations a considerable and fairly well organized knowledge of astronomy. They had undoubtedly begun by noting the appearance and constant arrangement of the more striking groups of stars. To these they naturally enough gave names, and watched expectantly for their regular appearance and disappearance. Certain particularly bright objects in the heavens did not behave in such a simple way, and we can easily imagine the absorbing interest with which they watched and learned to predict the movement of these planets or wanderers, as they called them. By the sixth century B.C. this purely

observational astronomy had reached such a stage that Thales of Miletus was able to predict the solar eclipse of May 28, 585 B.C.

For the Greek mind the step from a practical knowledge of the stars to a philosophical attempt to explain them was a natural one. Thus we find the Pythagoreans late in the sixth century already speculating along these lines. They recognized the sphericity of the Earth, although on rather inadequate grounds, and Ecphantus of Syracuse explained the diurnal motion of the stars as due to the rotation of the Earth about its axis. The evidence, however, which the Pythagoreans were able to collect in favor of such a view was insufficient to attract much of a following. The more naïve suggestion of Eudoxus in the fourth century B.C. that Sun, Moon, planets, and stars were all rigidly attached to concentric spheres, which rotated about the Earth as a center, appealed to most of the ancients as the more reasonable. With this scheme of Eudoxus as a starting point, later astronomers such as Apollonius, Hipparchus, and Ptolemy were able to build up a system which was mathematically sound and was in accord with all the facts then known. This Ptolemaic System, as it was called, was accepted as correct up through the fifteenth century.

Early in the sixteenth century *Copernicus*, a Polish astronomer, after years of study and observation came to the conclusion that the motion of all the heavenly bodies could be better and more simply explained on two assumptions: (1) that the Earth is a sphere which revolves about its axis, thus giving rise to the daily changes in the positions of the Sun and the stars, and (2) that the Earth and the other planets revolve about the Sun as a center. The invention of the telescope shortly after the death of Copernicus opened the way for many new astronomical discoveries, all of which were found to be in accord with the Copernican Theory. Thus a study of the surface markings of Mars and Jupiter showed that these planets were undoubtedly rotating in just the way that Copernicus had assumed for the Earth. Accordingly, even before the time of Newton this theory had arrived at very general acceptance as the best means of coordinating the already large amount of astronomical knowledge. One of the first tasks of the present book will be to show how greatly the work of Newton added to the probability of the correctness of the Copernican viewpoint.

The picture of the Earth as one of a family of planets, each spinning about its own axis and at the same time sweeping in a great orbit about the Sun, was one of the first great contributions of science toward an appreciation of the world in which we live. Yet the spirit of those early investigators, and to a certain extent their methods of procedure, were identical with those which are being used successfully to-day to unlock the secrets of nature. Inspired simply by curiosity and a real thirst for accurate knowledge, the earlier workers devoted themselves tirelessly to accumulating by observation a body of fact which could later serve as a basis for a theory. Once such a theory has been formulated, it must be recognized as tentative, and must be carefully and honestly tested in the light of each new fact as it is discovered. The Copernican Theory has been unusually fortunate. It has met so successfully the test of vast quantities of astronomical data discovered during the last three hundred years that we can scarcely view it in any other light than that of an established fact. Most physical theories, however, are not so successful. Only too frequently a hypothesis which meets with great accuracy the demands of one group of phenomena, must be revised or abandoned altogether in the light of more complete knowledge of facts. Nevertheless, as the years go by, as theories are formed and changed in the light of new material, there are certain broad structural outlines of the universe in which we live which, like the Copernican Theory itself, seem to stand out more clearly and definitely.

There is one respect in which the modern worker has a distinct advantage over the ancient Greek. The Greeks never appreciated the value of experiment. In a study such as descriptive astronomy where experiment, as distinguished from mere observation, is impossible, they could make great strides, but they seem to have been wellnigh helpless in the face of a problem which required a simple practical test. Thus Aristotle confidently believed that a weight of two pounds would fall faster than a weight of one pound, and his belief was accepted without question for eighteen centuries. Even Archimedes, who stands out as the outstanding experimenter of classical antiquity, seemed to regard his experimental work in the nature of play, and of little importance in comparison with his pure mathematics.

2. The Problem of Falling Bodies. Galileo. As we look back to the time of Copernicus, we cannot wonder that so few of

his contemporaries took his suggestions seriously. The wonder is rather that he himself was ever able to work through to his correct conclusions. How could the Earth hurtle through space as he pictured it without our being conscious of the motion, without, in fact, snapping us off into the outer void? It seemed all too unreasonable. And yet, all that was required to make it the most natural and reasonable thing in the world was a little experimental mechanics interpreted by the genius of a Newton. It was Galileo (1564–1642) who performed these fundamental experiments in mechanics, and who thereby demonstrated for the first time the power of the experimental method. As a student in Pisa he distinguished himself by his unwillingness to accept facts simply on the authority of their authors. It is not surprising, therefore, that he took exception to the Aristotelian doctrine of falling bodies. Determined on an actual test, he dropped bodies of different weight and material from the leaning tower of Pisa, and observed that they fell with very nearly the same speed. The slight discrepancies which he noted, he attributed correctly enough to air resistance, and concluded that, except for air resistance, all bodies would fall in the same way. Shortly after the death of Galileo the invention of the suction pump made possible an interesting test of this conclusion. Into a long glass tube a guinea and a feather were introduced. The tube was then pumped free of air and inverted. As predicted by Galileo, guinea and feather fell side by side throughout the length of the tube.

Galileo was not content, however, with the mere qualitative observation that all bodies fall alike. He wished to know just how they fall. It was obvious that, as they dropped, they moved faster and faster, but just how did this increase in velocity take place? He was attacking here a problem which was a little too difficult for him, for he had no means of accurately measuring velocity or even short intervals of time, the water clock, which was the best he had, being entirely unsuited for such a purpose. Nevertheless, a shrewd observation that balls in rolling down an incline increase in velocity in much the same fashion, opened the way for further work. This slower motion he could measure, albeit roughly, and he soon satisfied himself that the increase in velocity was uniform. Each second the ball gained in speed exactly the same number of feet per second that it did in any other second, the precise amount of the gain, of course, depend-

ing on the slope of the incline. The increase in velocity each second is known as the *acceleration* of the body, and the statement that the speed of the body increases uniformly is equivalent to saying that it has a *constant* acceleration. By means of a theoretical argument which it is unnecessary to reproduce here, Galileo persuaded himself that a freely falling body, as well as a ball on an incline, obeyed this simple law. He had already shown experimentally that all bodies fall in the same way, i.e., with the same acceleration. At any one location on the Earth's surface this acceleration has a perfectly definite value, which is the same for all bodies; but it varies from 32.09 ft. per sec. each second at the equator to 32.25 ft. per sec. per sec. at the poles. At the latitude of New York it has a value approximately 32.16 ft. per sec. per sec. or 980.2 cm. per sec. per sec. The repetition of the word second in expressing an acceleration is awkward, and this quantity is usually written ft./sec.^{2}, and is read feet per second squared. The acceleration of a falling body is represented by the letter g. Thus $g = 32.16$ ft./sec.^{2} or 980.2 cm./sec.^{2} at New York.

3. The Motion of a Body Which Has a Constant Acceleration. The details of the motion of any body which possesses a constant acceleration, a, are easily calculated. Thus the amount by which its velocity increases in t seconds is at. Hence if we know its velocity v_0 at some initial instant, and wish to find its velocity v at an instant t seconds later, we need only add to the initial velocity v_0 the increase in velocity at. Or

$$v = v_0 + at \tag{1}$$

Since the velocity of the body is increasing uniformly, it will travel in this time a distance s which is the same as though it had been moving throughout with a velocity $(v_0 + v)/2$ midway between the initial and final velocities. This is called its average velocity during the time in question. Thus

$$s = \frac{v_0 + v}{2} t \tag{2}$$

By substituting in (2) the value of v given by (1), we get still another useful expression for s.

$$s = v_0 t + \tfrac{1}{2} a t^2 \tag{3}$$

If we eliminate t between (1) and (2), we get the additional relation

$$2as = v^2 - v_0^2 \tag{4}$$

These four relations between the quantities v_0, v, a, s, and t enable one to determine the remaining two when any three are known, and thus give the solution of any problem dealing with a body moving with a constant acceleration. Since the acceleration of a falling body is g, these four relations will hold for any falling body if a is replaced by g. Thus from (3) the distance fallen from rest by a freely falling body in time t is $s = \frac{1}{2}gt^2 =$ approximately $16t^2$ ft.

4. The Principle of Galilean Relativity. One other observation made by Galileo in connection with his experiments with the ball and incline was of fundamental importance. He noted that when a ball, after descending one plane, was forced to climb a second, it climbed very nearly up to its previous level. Galileo, who was not unfamiliar with friction, attributed the discrepancy to this cause, and concluded that, in the absence of friction, the ball would regain exactly its original height. His experiments convinced him that this would be true regardless of what might be the slope of the second incline. If, however, the second incline were horizontal, and the ball thus could not climb back to its original level, it would continue travelling along the horizontal plane indefinitely. This conclusion to which he was forced must have been a very startling one to Galileo. Hitherto it had been universally assumed that some sort of effort was required to maintain a body in motion. The results of his experiment seemed to indicate that, instead, a moving body would continue to move with undiminished speed unless something were done to stop it. The question is not, "Why does a body continue in motion," but, "Why does it stop?" The commonest answer to this latter question, as Galileo pointed out, is friction.

Imagine yourself seated in a moving railway coach before a smooth horizontal table with an ivory billiard ball in your hand. Place the ball gently on the table and observe it. I think you will be willing to believe that, if the train is moving steadily along with a uniform velocity and without bumps of any kind, the ball will remain motionless on the table. You would probably not be much surprised if you saw the ball behave in this prosaic fashion, and yet you would be in reality witnessing an

application of Galileo's law. The ball while in your hand was moving forward with the velocity of the train, and as the process of placing it on the table did nothing to disturb this forward motion, it continued so to move. This is only one very simple experiment in which a body in a uniformly moving laboratory (the train) behaves just as though the laboratory were at rest. If we tried any other mechanical experiment, such, for example, as dropping the ball, we should once more see it behave in every respect as though the train were at rest. The ball would apparently drop vertically downward with an acceleration of 32 ft./sec.^{2} In fact, no mechanical experiment could decide whether the laboratory in which it was performed was or was not in uniform motion at the time. This is the *Principle of Galilean Relativity*, which seems to have been recognized by both Galileo and Newton, although not, perhaps, stated in exactly these words. It is because of this fundamental law of nature that the Earth, the great laboratory in which we must perform all our experiments, can sweep through space on its journey about the Sun without our being conscious of any motion.

This same fundamental principle explains the observation of Galileo that a body falls with constant acceleration. During the first second of fall all bodies acquire a velocity of 32 ft./sec. Think of the body at this stage of its journey as surrounded by an imaginary laboratory which is moving downward uniformly with this same velocity, 32 ft./sec. With reference to this chamber the body is at rest. For an observer seated within, the body must acquire in one more second a velocity of 32 ft./sec. This would be viewed by one outside as simply an additional 32 ft./sec., and he would conclude that the body gained just as much in speed during the second second as it did during the first. He would accordingly decide that the body was moving downward with a constant acceleration.

PROBLEMS

1. A ball rolls the entire length of an inclined plane 10 ft. long in 10 sec. If it started from rest, what was its acceleration and its final velocity?

2. A stone, dropped over the perpendicular face of a cliff, is observed to strike the base 4 sec. later. Neglecting the resistance of the air, what is the height of the cliff?

3. A stone is thrown vertically upward with a velocity of 64 ft./sec. To what maximum height will it rise, and how long will it require to attain this height?

4. The brakes are applied to an automobile while it is travelling at the rate of 48 ft./sec. If 6 sec. are required to bring it to rest, how far does it travel in this time, and what is its acceleration?

5. If an automobile is accelerated from 5 miles an hour to 30 miles an hour in 10 sec., what is its acceleration?

CHAPTER II

MASS AND FORCE

In the very year, 1642, in which Galileo died in Italy, Sir Isaac Newton, who was destined to carry on his work, was born in England. Galileo had made it clear that a body, when left to itself, remained in its original state of rest or motion, but he did not come to a thorough understanding of the laws which govern changes in that motion. Simple as those laws seem to us to-day, it required a greater genius than that of Galileo to formulate them for the first time.

5. Newton's First Law of Motion. Newton's First Law of Motion was a statement of the results of Galileo's work. "A body at rest continues at rest, and a body in motion continues in motion in a straight line with constant velocity, unless acted on by an outside force." Perhaps the best definition of the word *force* which enters into this statement is contained in the law itself. A force is something which changes the motion of a body. If we see the velocity or the direction of motion of a body change, we immediately infer that there is a force acting on it. If, however, we are dissatisfied with such an abstract concept for force, we can get something a little more tangible from our own experience. Thus, if we wish to move a body which is at rest, we push it or pull it, and since pushes and pulls produce the desired change in motion, they must be forces. Forces can be applied, however, by so many different agencies and in so many different ways that it is convenient to have a single specific word, force, to apply in all cases.

The acceleration which a constant force imparts to a given body must be constant, if the Principle of Galilean Relativity is correct. The proof of this statement is exactly similar to that given in the last paragraph of the preceding chapter. Imagine the body starting from rest under the action of a constant force, which gives to it a velocity v in 1 sec. Now imagine yourself viewing its further motion from a laboratory, which is moving in the same direction with a constant velocity v. To you the body will appear initially at rest, but since it is acted on by the

same force as before, you must expect to see it acquire in one more second the apparent velocity v. To an observer at rest it must, therefore, appear to have gained an additional velocity v, exactly equal to its gain in the first second. In the same way it must gain a velocity v in each succeeding second. The acceleration of the body may, therefore, depend on the body itself and on the applied force, but not on its velocity.[1]

6. Inertia and Mass. Once it is recognized that a given force, when applied to a body, always produces the same acceleration, the question arises, " How large a force is required to produce a particular acceleration of the body? " Qualitatively we know from experience that the heavier the body, the greater must be the force, and the greater the desired acceleration, the greater must be the force. Before we can give a quantitative answer to the question, however, we must have some way of measuring forces, and some way of appraising the resistance which the body offers to acceleration. This resistance, which all bodies exhibit to a greater or less extent, is known as the *inertia* of the body, and its measure is known as the *mass* of the body. There are two common fundamental units of mass, the pound and the gram. A body is said to have a mass of 1 lb. if it has the same inertia as a standard pound in the Bureau of Standards at Washington, that is to say, if a given force imparts to it the same acceleration which it imparts to this standard. In the same way a body is said to have a mass of 1 gr., the fundamental metric unit of mass, if it has the same inertia as the standard gram. Once the unit of mass has been defined in this way, it is a simple matter to define multiples and fractions of it. Thus a duplicate could be made of the standard, and a mass of 2 lb. could be defined as one which has the same inertia as the standard and its duplicate when being accelerated together. Similarly a half pound could be defined by constructing two masses equal to each other, but which together have the same inertia as the standard. A set of such masses could be built up ranging all the way, let us say, from 0.001 lb. up to 100 lb., and the mass of any desired body within these limits could be determined by a direct comparison with the proper group of these " secondary " standards. A convenient way of making such a comparison

[1] This statement is not strictly correct, but for all velocities which can be obtained by purely mechanical means, the actual reduced effect of the force as velocity increases is experimentally imperceptible. See Chapter XXXIII.

will be described in the course of this chapter, but for the present it is merely necessary to see that we have here outlined a method of procedure for assigning to any body a number, so many pounds, which will be a measure of its mass. Having laid down this straightforward definition of mass, it remains for experiment to determine the way in which different masses are accelerated by the same force. When such an experiment is tried, it is found that nature is here behaving in the simplest manner possible. The 2-lb. mass receives just half the acceleration of the 1-lb. mass, and in general the acceleration is exactly inversely proportional to the mass. It is this experimental fact which forms the basis of Newton's *Second Law of Motion*. Before the law can be stated, however, we must agree on some method of measuring forces.

7. The Measurement of Force. The standard of force is best made to depend on the already accepted standard of mass, which can be done in either of two possible ways. The constant acceleration of a freely falling body indicates in the light of the present chapter that the falling body has acting on it, pulling it downward, a steady force. This force Newton viewed as a force of attraction existing between the Earth and all bodies near its surface. His researches led him to believe that it was only a particular manifestation of a very general law, to be discussed in Chapter IV, that all masses attract one another, the so-called *Law of Universal Gravitation*. What could be more natural, accordingly, than to define a force of 1 lb. as the force which the Earth exerts at its surface on a mass of 1 lb. This unit and the corresponding one in the metric system which is based on the kilogram or the gram has found a great deal of use, particularly in the engineering industry. It has, however, one serious disadvantage. The fact that the standard pound, in common with all other bodies, would suffer slightly different accelerations in falling at different points of the Earth's surface, indicates that the force which the Earth exerts on it varies somewhat from point to point. Although the difference is slight, not more than one-half of one per cent, it is great enough to make the pound or gram as a unit of force unsatisfactory for scientific purposes.

The other unit of force in common use lies at the basis of the so-called *Absolute System of Units*. In the metric system this unit of force is called the *dyne*, and is defined as the force

which would impart to a mass of 1 gr. an acceleration of 1 cm./sec.^{2} In the English system it is called the *poundal*, and is defined as the force which would give to 1 lb. an acceleration of 1 ft./sec.^{2} Not merely are these units free from the uncertainty inherent in the pound and the gram as units of force, but their use makes particularly simple, as we shall see, the statement of the fundamental law of motion. Once the unit of force has been defined, it is a simple matter to define multiples and fractions of it. Thus a spiral spring could be constructed which would exert a force of exactly 1 poundal when stretched to a certain length. If, now, a duplicate spring is made, the two together would necessarily exert a force of 2 poundals. By such a procedure a set of springs could be constructed similar to the set of masses already described; or better still, a single spring could be used, equipped with a pointer and a scale, which for each extension of the spring would indicate the force which it could exert. Such a contrivance is known as a *spring balance*, and is used extensively both inside and outside the laboratory for measuring force.

8. Newton's Second Law of Motion. Provided now with the means for measuring forces, we could carry out experiments to determine the accelerations produced by different forces on a given body. Such experiments are hardly necessary, however, as a moment's consideration will show. Imagine two cars, each having a mass of 1 lb., resting side by side as shown in Fig. 1. Imagine also two spiral springs, each in a position to exert a force of 1 poundal on the car to which it is attached. On releasing the two cars, they each exhibit momentarily, in accordance with our definition of poundal, an acceleration of 1 ft./sec.^{2} It will obviously make no difference in the result whether or not the two cars are connected together. The effect can be thought of as produced either by the application of a force of 1 poundal to each car individually or by the application of 2 poundals to the total mass of 2 lb. Now attach both springs to a single car, and observe the resulting acceleration. Since 2 poundals will

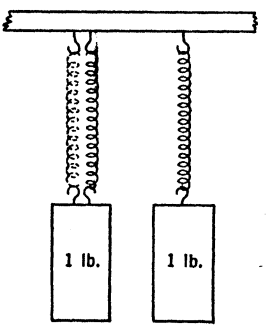


FIG. 1. Proportionality of Acceleration to Force.

impart to a mass of 2 lb. an acceleration of 1 ft./sec.^{2}, they must impart to one half the mass an acceleration twice as great or 2 ft./sec.^{2} This, in turn, is twice as great as that given the same mass by a force of 1 poundal. We accordingly infer that the acceleration given to a body is proportional to the applied force.

We are now in a position to calculate the acceleration which any given force will impart to any mass. For example, since 1 poundal gives to 1 lb. an acceleration of 1 ft./sec.^{2}, 8 poundals will give to a mass of 4 lb. an acceleration $\frac{8}{4} \times 1$ ft./sec.^{2} or 2 ft./sec.^{2} In general, the acceleration imparted by a force of F poundals to a mass of m pounds will be

$$a = \frac{F}{m}$$

Or

$$F = ma \tag{5}$$

The same formula will, of course, give the acceleration in centimeters per second per second which a force of F dynes will impart to a mass of m grams. This is Newton's Second Law of Motion.

It should be noted that F, which stands in this equation, must always be expressed in absolute units. It frequently happens, however, that in the data which must be used in the solution of a problem, the force is expressed in pounds or grams. A moment's consideration will show how to convert it into the corresponding absolute unit, the poundal or the dyne. Imagine a mass m, falling toward the surface of the Earth. In common with all other freely falling bodies, it will have an acceleration g, which means that, in accordance with Newton's Second Law, it must be acted on by a force $F = mg$. If m is its mass in pounds, it is acted on by a downward force of m pounds, and this must, accordingly, be equivalent to mg absolute units. Since g is 32 ft./sec.^{2} (approximately), in order to convert a force measured in pounds into poundals, it is only necessary to multiply it by this number. Similarly, to convert grams into dynes, it is only necessary to multiply by 980.

It is worth pausing for a moment to consider, in the light of the Second Law of Motion, the common acceleration possessed by all freely falling bodies. Since all bodies acquire the same acceleration g, the ratio of the gravitational force exerted on any body to its mass must always be g, and must, therefore, be the

same as the corresponding ratio for any other body. This, in turn, means that the gravitational attraction is always proportional to the mass of the body. The more one muses over this curious law of Nature, the more surprising it seems that two such dissimilar things as mass and gravitational attraction should be so intimately connected. Only within comparatively recent years has the Theory of Relativity enabled us to see a little further into the puzzle.

The fact that the gravitational force exerted by the Earth depends only on the mass of the body, furnishes a much more convenient means of comparing masses than that hitherto used in this chapter of comparing their accelerations under a standard force. Two masses are equal if the downward pull of the Earth on them is the same. The familiar equal-arm balance is, therefore, a satisfactory means of measuring mass, since the beam sets horizontally when the downward pull of gravity on the weights in one pan is exactly equal to the downward pull on the specimen being weighed in the other pan. The set of weights constitutes a set of standard masses such as was pictured earlier in the chapter, and with them any unknown object can be conveniently compared.

9. Newton's Third Law of Motion. Newton's Third Law of Motion states one more important property of forces. In its usual form it says that "Action and reaction are equal and opposite." Forces are very real tangible things, which are always exerted on real tangible bodies by other bodies. Whenever we have to deal with a force, it is always possible to state and it always should be stated on what body the force is exerted and by what body it is exerted. Once this fact is appreciated, the significance of the third law can be grasped. Whenever one body A exerts a force F on a second body B, B exerts an exactly equal and opposite force on A. If I push downward on the table with a force of 10 lb., the table is pushing upward on my hand with an exactly equal force. If I hold in my hand a bar of iron, and feel on it the tug of an electromagnet at the other side of the room, just so certainly do I by the bar exert an equal and opposite tug on the electromagnet itself. This reciprocal nature of forces should always be borne in mind, as with surprising frequency it suggests the method of approach to otherwise very puzzling problems.

In looking back over the material of the present chapter, one

can hardly fail to be impressed by the simplicity of the laws which have been discovered. The strict proportionality of acceleration to force, the inverse proportionality to mass, and the direct proportionality of gravitational attraction to mass could scarcely have been anticipated by even the most optimistic investigator. One of the effects of these discoveries of Newton was to bolster up a very common but unfounded belief that Nature is always simple. Nevertheless we must not accept such a doctrine too readily. At times Nature can be so prodigal in her complexity as to thwart effectively all efforts to untangle the skeins of her lore.

EXAMPLES

1. A mass of 10 lb., initially at rest, has applied to it a steady force of 3 lb. for 10 sec. How far does it move in this time?

The force of 3 lb. is equivalent to $3 \times 32 = 96$ poundals. In this example, therefore, $m = 10$ and $F = 96$.

Applying (5)	$96 = 10a$	
Or	$a = 9.6$ ft./sec.^{2}	
Furthermore	$v_0 = 0$	
and	$t = 10$	
Applying (3)	$s = \frac{1}{2}(9.6)(100) = 480$ ft.	*Ans.*

2. Two weights of 10 gr. and 8 gr. respectively are attached to the ends of a cord, which is then hung over the pulley P of Fig. 2. The weights are then

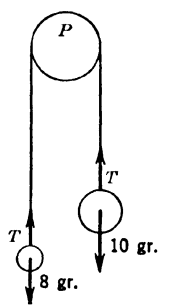


FIG. 2. Atwood's Machine.

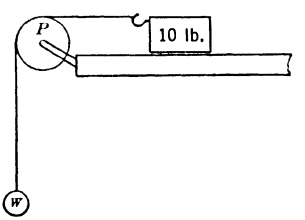


FIG. 3

released. With what acceleration do the weights move, and what is the tension in the cord?

The 10-gr. weight has acting on it two forces, which are represented in the figure by arrows. The first is the downward force of gravity of 10 gr. or $10 \times 980 = 9800$ dynes, and the second is the unknown upward pull of the cord, which is indicated by T in the figure. Since these two forces act in opposite directions, they are equivalent to a single force equal to their differ-

ence $(9800 - T)$. It is this single force to which they are equivalent, which must be used in the Second Law of Motion. Hence, applying (5),

$$9800 - T = 10a \qquad \text{(I)}$$

Similarly we can write down the equation of motion for the 8-gr. weight. Since it moves upward, drawn by the greater 10-gr. weight, the tension T must be greater than the downward pull of $8(980) = 7840$ dynes. Hence

$$T - 7840 = 8a \qquad \text{(II)}$$

Adding I and II,

$$1960 = 18a$$

Or

$$a = 108.8 \text{ cm./sec.}^2 \qquad \textit{Ans.}\ 1.$$

Substituting this value for a in I,

$$9800 - T = 10(108.8) = 1088$$

$$T = 8712 \text{ dynes} = 8.89 \text{ gr.} \qquad \textit{Ans.}\ 2.$$

PROBLEMS

1. A body, to which a force of 10 lb. is applied, is observed to increase in speed from 8 ft./sec. to 40 ft./sec. in 20 sec. What is the mass of the body?

2. A train weighing 100 tons is accelerated by a steady pull from the locomotive. If, starting from rest, it acquires a velocity of 60 ft./sec. in 2 min., what was the force in tons exerted by the locomotive? Neglect friction.

3. How far did the body in Problem 1 move during the 20 sec. that it was under observation? What was its average velocity?

4. A man weighing 150 lb. stands in an elevator which is being accelerated upward at the rate of 4 ft./sec.^{2} What is the force which his feet exert against the floor of the elevator? What force would they exert if the elevator were being accelerated downward at the same rate?

5. An airplane, striking a downward current of air, suffers for a short time a downward acceleration of 48 ft./sec.^{2} Calculate the downward force which the safety belt of a 150-lb. aviator exerts on him.

6. What is the least possible acceleration with which a 150-lb. man can slide down a rope which is only strong enough to sustain 100 lb.? What will be his velocity on reaching the ground, after descending thus a distance of 40 ft.?

7. If the 10-lb. weight of Example 2 consists of two individual 5-lb. weights connected by a cord, what is the tension in this cord?

8. A 10-lb. weight rests on a smooth horizontal table. A cord, attached to the weight, passes over the pulley P to the 8-lb. weight W, as shown in Fig. 3. How far will the 8-lb. weight fall in 2 sec.?

CHAPTER III

COMPOSITION OF FORCES. EQUILIBRIUM

In the last chapter we considered the motion of a body when acted on by a single force or at most by two forces which could be seen by inspection to be equivalent to a single force. It is, however, rather unusual to find a body which is influenced by only one force, and the study of the motion of most bodies must be preceded by a study of the combined effect of the various forces which act on them simultaneously. Such a study is particularly simple if the body under consideration happens to be a *particle*, that is, an object, which, for the purpose of a mechanical study of its motion, may be treated as a single point. Just what constitutes a particle may depend on the nature of the investigation being made. Thus, in a study of the orbital motion of the Earth about the Sun, the Earth may very well be considered a particle, but in a study of the mechanical causes which lie back of the precession of the equinoxes, it must be treated as an extended body. In general, if a determination of the motion of any one point in the object under consideration gives us all the information we require, such a body can be treated as a particle. A train moving along a track is, therefore, a particle, since every point of the train, neglecting the peculiar motion of the wheels and driving mechanism, moves like every other point. In dealing with a particle, since we are interested in only one point of the body, we can represent the entire body by this one point, and can imagine all the forces which act on it to be applied at this point. The complete proof that this is a legitimate method of procedure belongs to the field of rigid-body dynamics, and must be omitted in the present treatment.

If two forces are applied simultaneously to a particle, their combined effect will depend, obviously enough, both on their magnitude and on their direction. If they have the same direction, they will be equivalent to a single force oriented in exactly the same way, but having a magnitude equal to their sum; a fact of which we have already made use in the preceding chapter to define multiples and fractions of a standard force.

In the same way, if the two forces point in opposite directions, the force to which they are equivalent will be equal to their difference. If, however, they make any angle other than a straight angle with each other, a more difficult problem is presented, which has, nevertheless, a very simple graphical solution.

10. The Parallelogram of Forces. A quantity which, like a force, requires direction as well as magnitude for its complete description, is known as a *vector*. It is customary to represent such a quantity by an arrow, the direction of which indicates the direction of the vector, while its length indicates to a suitable scale the magnitude of the vector. That property, for which the barb of the arrow so conveniently stands, is called the *sense* of the vector, as distinguished from mere direction. In this technical way a segment of a line is thought of as having direction and magnitude, while an arrow, with its barb, has the added property of sense. A vector, such as force, has all three attributes.

Fig. 4 represents in this fashion two forces A and B acting on a particle P. The graphical solution of the problem in hand

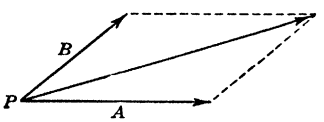


FIG. 4. Parallelogram of Force.

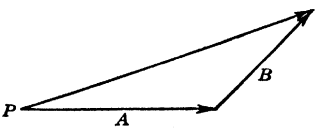


FIG. 5. Triangle of Force.

consists in completing the parallelogram which is determined by the two arrows, and drawing in that diagonal which has for one extremity the point P. This diagonal represents, then, both in direction and magnitude, the single force to which A and B are equivalent. It is called the *resultant* of A and B, and the entire diagram is known as the *parallelogram of force*. The accuracy of this construction can be proven experimentally in the laboratory, or by means of a theoretical argument based on the Second Law of Motion. It is thus possible for all mechanical purposes to substitute a single force for any two forces which are applied to a particle. This process can obviously be extended to the reduction of any number of forces to a single force. All that is necessary is to find the resultant of any two forces, then to compound this with a third force, and so on until all have been accounted for.

11. The Triangle and Polygon of Force. If the parallelogram is used to combine in this fashion very many forces, the figure soon becomes quite complicated, but a moment's consideration will show that a large number of the lines drawn are unnecessary. In Fig. 5, two out of the five lines of the parallelogram have been omitted, leaving only a triangle composed of the two forces themselves and their resultant. This triangle serves all the purposes served by the parallelogram, occupies less space, and lends itself much better to the combination of larger numbers of forces. In forming the *triangle of force*, as it is called, instead of placing the given forces tail to tail as in the parallelogram, the tail of one should be placed at the head of the other, and the resultant should be drawn from the tail of the first to the head of the latter. Without any confusion now the tail of a third force may be placed at the head of the resultant, and a new resultant obtained by drawing an arrow from the tail of the first resultant to the head of the third force. A glance at Fig. 6, in which this is done, shows that the intermediate resultant is entirely superfluous. The three given forces are drawn successively tail to head, and the overall resultant is drawn from the tail of the first force to the head of the last. This same process can clearly be extended to the composition of any number of forces. The result is known as the *polygon of forces*.

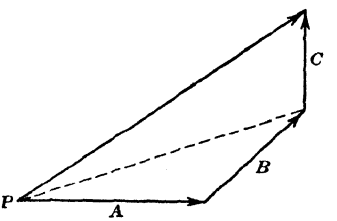


FIG. 6. Polygon of Force.

12. The Equilibrium of a Particle. The triangle and the polygon of force find some of their most valuable applications in the treatment of problems dealing with the equilibrium of a particle. A body is considered to be in equilibrium if it is at rest or is moving in a straight line with constant velocity, i.e., if, in the sense of the First Law of Motion, it is acted on by no force. There may, however, be a number of individual forces applied to such a body, if they so cancel one another that their resultant is zero. Since for all mechanical purposes the various forces are together equivalent to their resultant, the body must, in such a case, behave in every respect as though no forces whatever are acting on it. We have seen that the general method of finding the resultant of a group of forces is to draw the various

arrows which represent these forces tail to head, and finally to draw the resultant itself from the tail of the first force to the head of the last. If the resultant is zero, this last line must be of zero length, i.e., the head of the last force must exactly coincide with the tail of the first. Stated in another way, if all the forces acting on a body in equilibrium are drawn end to end, they must form a closed polygon. Conversely, if all the forces acting on a particle, when drawn end to end, form a closed polygon, the resultant of these forces is zero, and the particle must be in equilibrium. In particular, if only three forces act on a particle in equilibrium, they must form a closed triangle. This last is known as the *Law of the Triangle of Force*, and the more general theorem is known as the *Law of the Polygon Force*. By means of one or the other, any feasible problem dealing with the equilibrium of a particle can be solved.

EXAMPLE

1. The ring R in Fig. 7 is hung on a cord ARB, which is suspended at the points A and B. If the weight of the ring is 5 lb., and if the angle between the two segments of the cord is 90°, what is the tension in the cord?

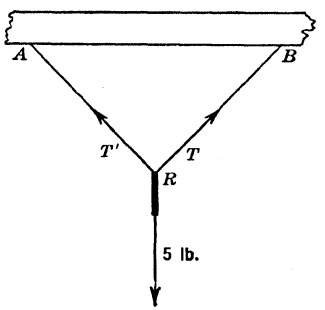


FIG. 7

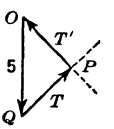


FIG. 7a

In attacking any such problem in the equilibrium of a particle, the various forces which act on it should be represented in the figure by arrows, which indicate, at least approximately, by their direction and length the direction and relative magnitude of the forces. Acting on the ring of Fig. 7 are three forces, the 5-lb. downward force of gravity, and the two upward tensions exerted by the cord, which it is the purpose of the problem to determine. These last two forces are labeled T and T' in the figure, although the very symmetry of the problem shows that they must be equal. The triangle of force is now formed in Fig. 7*a*, starting with the known 5-lb. force, which acts vertically downward. From its head a line should be drawn parallel to T and of indefinite length, since the magnitude of T is unknown. The arrow which is to

represent T' in this figure must close the triangle, and it must, accordingly, pass through O, the tail of the original 5-lb. force. We can, therefore, draw another line of indefinite length from this point parallel to T'. The intersection of this line, P, with the line QP representing T is the third vertex of the triangle. We can now add the appropriate barbs to the arrows QP and OP. These lines must represent by their lengths the magnitudes of the two tensions T and T'. Since in Fig. 7 the angle between T and T' is 90°, the angle which each makes with the horizontal and also with the vertical is 45°. Hence in Fig. 7*a* the angles QOP and OQP must be 45° angles, and both QP and OP must be $(1/\sqrt{2})(OQ)$. Hence both T and T' must be equal to $1/\sqrt{2}$ times the force which OQ represents or $1/\sqrt{2} \times 5$ lb. $\doteq$ 3.53 lb. *Ans.*

13. Components. We have seen that it is always possible to replace two or more forces which are applied to a particle by a single force. The converse process in which a single force is replaced by two other forces is much more useful than might, perhaps, be anticipated. Such a procedure is known as the

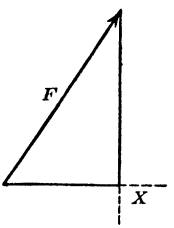


FIG. 8. Resolution of Force into Components.

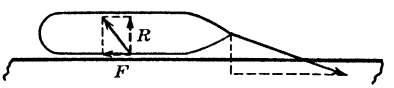


FIG. 9. Canal Boat.

resolution of a force into its *components*. There is nothing to prevent resolving a force into more than two components, and the directions of the components may be more or less arbitrarily chosen. In practice, however, in a two-dimensional problem it is seldom desirable to use more than two components, and it is usually best to split the force into components which are perpendicular to each other. The direction of one of the components may be chosen at will; that of the second is specified as perpendicular to the first. In Fig. 8, for example, the force F is resolved, as is frequently useful, into a vertical and a horizontal component. To this end a vertical line of indefinite length is drawn from the head of F, and a horizontal line, also of indefinite length, is drawn from its tail. The segments of these lines included between the force F and their own intersection X are the required components. That they are together equivalent to the original force is seen from the fact that, if they are placed

tail to head, the force F will just suffice to fill the gap which must be occupied by their resultant.

The most common circumstance under which it is desirable to resolve a force into its components occurs when a body is limited in its possible motion by a restraining surface. Thus in the classical example of a canal boat, which is dragged by a tow-rope from the bank, the boat is compelled by the retaining wall to move in the direction of the canal, whereas the force exerted by the rope is applied at an angle (Fig. 9). If, now, this force is resolved into components along the canal and perpendicular to it, the perpendicular component is always exactly balanced by the outward thrust of the retaining wall, and contributes nothing to the motion of the boat.

14. Friction. The outward thrust of the wall, however, is not confined to balancing the perpendicular component of the force exerted by the rope. The force which such a surface exerts on a body sliding past has, in general, a component tangent to the surface, in such a direction as to retard the motion. This tangential component of the force exerted by a surface is known as *friction*. It is subject to a few very simple laws, and is of the utmost importance in the economy of Nature. Thus it always acts in such a direction as to oppose relative motion of the surfaces between which it is exerted. It does not depend over wide ranges either on the velocity with which the surfaces slide past one another or on the area of those surfaces which are in contact. To a first approximation it depends only on the character of the surfaces and the perpendicular component of the force which they exert on each other. This perpendicular component of the force exerted by a surface is known as its *normal reaction*, and the tangential component or friction, when the surfaces are in motion with reference to each other, is strictly proportional to it. This simple proportionality makes possible the calculation of the frictional force in the greater number of cases which arise in practice. To this end the *coefficient of friction* is defined as the frictional force per unit of normal reaction, or simply as the ratio of the friction to the normal reaction. To determine the friction, accordingly, it is only necessary to multiply the normal reaction by the coefficient of friction. If F stands for friction, R for the normal reaction, and μ for the coefficient of friction,

$$F = \mu R \tag{6}$$

In the case of a body (Fig. 10) which is being dragged along a horizontal surface by means of a horizontal force, the upward thrust of the surface, i.e., its normal reaction, has only the weight, W, of the body to balance, and is accordingly equal to it. The frictional force, therefore, which must be overcome in dragging the body along is simply $F = \mu W$.

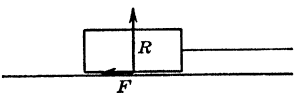


FIG. 10. Friction.

The coefficient of friction depends simply on the character of the surfaces. It is generally less than 2, approaching something in the vicinity of 2 for very rough surfaces, and approaching 0 for surfaces such as wet ice, which are notably slippery.

PROBLEMS

1. What is the resultant of two 10-lb. forces, (*a*) if they have the same direction, (*b*) if they have opposite directions, (*c*) if they make an angle of 90° with each other?

2. A picture is suspended from the wall by a single cord, attached to the corners of the frame, and hung over a pin. If the angle formed by the cord at the pin is 90°, and if the weight of the picture is 10 lb., what is the tension in the cord?

3. A 10-lb. weight rests on a rough inclined plane, which makes an angle of 30° with the horizontal. Calculate the magnitude of the friction and the normal reaction. Make use of the fact that the shorter arm of a right triangle having acute angles of 30° and 60° respectively is one-half the hypotenuse. If the weight is just on the point of slipping down the plane, i.e., if the maximum possible friction is required to hold the weight in equilibrium, calculate the coefficient of friction.

4. Prove that if the weight in the last problem is on the point of slipping when the angle which the plane makes with the horizontal is A, the coefficient of friction is tan A.

5. If the tow-rope of a canal boat makes an angle of 30° with the direction in which the boat is travelling, what part of a 200-lb. pull is effective in dragging the boat forward?

CHAPTER IV

CURVILINEAR MOTION

Newton's First Law of Motion states that a body in motion will continue in motion in a straight line with constant velocity unless acted on by some force. We have already considered the change in speed of a body as evidence of an applied force, and have learned to calculate the magnitude of the force from the observed acceleration. It remains now to consider the causes which may lead to the deviation of a body from rectilinear motion.

15. Projectiles. The guiding thread in this investigation is the Principle of Galilean Relativity which was discussed in the first chapter. According to this principle, every mechanical experiment will result in the same way in a laboratory which is in uniform motion as in one which is at rest. Let us apply this law to the study of a projectile which is thrown out horizontally in the Earth's gravitational field. It is started, let us say, with an initial horizontal velocity v_0. Imagine the projectile surrounded by a laboratory, which is also moving horizontally in the same direction with the uniform velocity v_0. Imagine yourself studying the motion of the projectile from the seclusion of this laboratory. Since projectile and laboratory are initially travelling with the same velocity, the projectile will appear to you initially at rest. You should expect, therefore, to see it fall vertically downward with an acceleration g, and in this you would not be disappointed. Its entire motion will, accordingly, consist of a uniform horizontal velocity v_0 equal to its initial velocity, plus an ordinary fall under the action of gravity. Its horizontal and vertical motion can be treated entirely independently. The horizontal distance which it has traversed in a given length of time can be calculated from the constant horizontal velocity, and the distance which it falls vertically downward can be calculated from the ordinary law of fall.

In exactly the same way rather more complicated problems in projectiles in which the body is launched at an angle with the horizon may be treated. Just as a force can be resolved into components, so can a velocity. Imagine a ball hurled

outward at an angle of 45° with the horizontal and having an initial velocity v_0. If the velocity v_0 is represented both in direction and magnitude by an arrow (Fig. 11), the horizontal and vertical components of this velocity may be drawn exactly as with a force. The significance of these components is fairly obvious. If, for example, a laboratory surrounding the ball is imagined travelling horizontally with a uniform velocity equal to the horizontal component of v_0, the ball to an inmate of the laboratory would appear to have no horizontal motion. It would appear to start vertically upward with a velocity equal to the vertical component of v_0, would rise to a maximum height, and then fall vertically downward, exactly the same as a ball thrown vertically upward from rest. Its total motion will, therefore, consist of a constant horizontal velocity, plus a vertical motion under gravity, which starts with an upward velocity equal to the vertical component of the initial velocity.

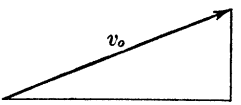


FIG. 11. Resolution of Velocity into Components.

Consider, for example, the problem of finding the range of such a projectile, which is thrown out with an initial velocity of 100 ft./sec. Since it is projected at an angle of 45° with the horizontal, both vertical and horizontal components will be $100/\sqrt{2}$ ft./sec. It will continue on its course until it has dropped to its original level. If we denote the vertical distance traversed by h, then the end of its journey will be characterized by the fact that $h = 0$. Applying, therefore, Formula 3 of Chapter I to the vertical motion,

$$h = 0 = \frac{100}{\sqrt{2}}t - \tfrac{1}{2}\,32t^2$$

This equation has two solutions, the first, $t = 0$, referring obviously to the starting instant, while the other,

$$t = \frac{(100)(2)}{(32)\sqrt{2}} = 4.41 \text{ sec.}$$

refers to the instant the projectile returns to its initial level, the ground. The horizontal distance, s, traversed in this time may be calculated from the constant horizontal component of the initial velocity, $100/\sqrt{2}$. Hence

$$s = \frac{100}{\sqrt{2}}\,(4.41) = 311.8 \text{ ft.}$$

The solutions which have been given of the projectile problems are, of course, strictly correct only in the absence of a resisting atmosphere. For slow velocities such as that of a baseball, the retarding effect of the air, although appreciable, does not entirely vitiate the conclusions which would be drawn from such an analysis. For very high velocities, however, such as those of rifle bullets, the air resistance becomes a dominating factor, and no even approximate solution can be obtained without taking it into consideration.

16. Planetary Motion. Centripetal Force. Another problem in curvilinear motion of great importance deals with the motion of a body in a circular or elliptic path, such as that of a planet about the Sun. We shall consider here only the simpler case of strictly circular motion, but the principles which emerge in such an investigation can be extended with a little more analysis to the general problem. In fact, Newton in his original treatise on mechanics, the "Principia," solved the whole problem of planetary motion in addition to laying down all the fundamental principles.

Imagine a body of mass m describing with uniform velocity v the circular path of Fig. 12. As it reaches the point P at the top of the circle, imagine yourself observing it once more from the seclusion of a laboratory which is also moving from left to right with the velocity v. To you it will appear to be at rest, and yet not quite at rest, for, owing to its circular motion it will fall, slowly at first, and then more rapidly, with reference to the strictly horizontal motion of your laboratory. It will have, therefore, a downward acceleration, which can easily be calculated. In Fig. 12, P' represents a point near P, which is occupied by the body an instant t later. The diameter through P, the lines joining P' with the extremities of the diameter, A and P, and the perpendicular $P'C$ are all drawn. The triangle CPP' is similar to the triangle $CP'A$ since the corresponding sides are mutually perpendicular. Hence

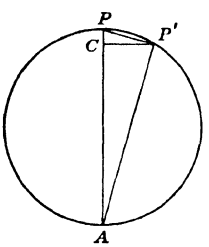


FIG. 12. Central Acceleration.

$$\frac{CP}{CP'} = \frac{CP'}{CA}$$

Or

$$CP = \frac{\overline{CP'}^2}{CA} \tag{7}$$

If t is taken as a very short interval, and if P' is, therefore, very close to P, CA is very nearly $2R$, where R is the radius of the circle. CP is the downward drop h noted by the observer in the moving laboratory, while CP' is the distance vt traversed in a horizontal direction by both the body and the observer. Hence

$$h = \frac{v^2t^2}{2R} \tag{7'}$$

If, however, the body were dropping from rest with a constant downward acceleration a, the distance fallen in time t would be

$$h = \tfrac{1}{2}at^2 \tag{8}$$

Equating (7') and (8) and cancelling $t^2/2$ from both sides of the equation,

$$a = \frac{v^2}{R} \tag{9}$$

It will be noted that the errors introduced by all the approximations which have been made in this calculation become increasingly negligible as the point P' is taken closer and closer to P. Hence in the immediate locality of P the body is undergoing a downward acceleration of precisely v^2/R. There is, however, nothing unique about the top of the circular path, and we must conclude that at all points in its excursion the body is being accelerated toward the center of the circle with the acceleration v^2/R.

The only thing which we have found which can produce an acceleration is a force, and the presence of such an acceleration directed toward the center of the circular path is proof of the existence of a force always pulling the body in the same direction. The magnitude of this *centripetal* force, as it is called, can be calculated from the second law of motion.

$$F = ma = \frac{mv^2}{R} \tag{10}$$

17. Universal Gravitation. It was the necessity for such a force to produce the observed acceleration which led Newton to the concept of a gravitational force of attraction between the Sun and each one of its planets, between the Earth and the Moon, in fact, between any two ponderable bodies. At this stage in his work the famous incident of the falling apple is said to have

occurred. Perhaps the very force which caused the apple to fall, which causes all bodies to fall to the Earth, might be the source of the Moon's acceleration in its orbit about the Earth. It was a simple matter to calculate the Moon's acceleration from its mean distance of 238,800 miles and the length of the lunar month, 27.32 days. Thus its velocity in feet per second is

$$v = \frac{2\pi(238{,}800)(5280)}{(27.32)(24)(3600)} = 3355 \text{ ft./sec.}$$

Its central acceleration is then

$$a = \frac{v^2}{R} = \frac{(3355)^2}{(238{,}800)(5280)} = 0.008928 \text{ ft./sec.}^2$$

This is, of course, much less than the value, 32 ft./sec.^{2}, which all bodies have at the surface of the Earth, but it was only to be anticipated that the force of gravity, if it existed, would fall off with the distance. The distance of the Moon from the center of the Earth is 60.26 times that of a body at its surface. If, therefore, the gravitational force, and with it the resultant acceleration, falls off as the square of the distance, the acceleration, 32.25 ft./sec.^{2}, at the surface of the Earth would be reduced to $32.25/(60.26)^2$ or 0.008882 ft./sec.^{2} at the Moon. The check between this and the observed acceleration of the Moon is so good as to leave little doubt as to the identity of the two forces considered and the correctness of the inverse square law. The discrepancy is to be attributed to the actual deviation of the Moon from the assumed circular path.

Assured of the inverse square law by this and other similar astronomical observations, and recognizing the fact already noted in Chapter II of the proportionality of gravitational attraction to mass, Newton was able to formulate the general law of gravitation as follows. The force of mutual attraction between two masses, m_1 and m_2, at a distance R from each other is

$$F = G\frac{m_1 m_2}{R^2} \tag{11}$$

G is known as the gravitational constant, and is clearly equal to the force of attraction between two unit masses when located at unit distance from each other. Thus, if we are using the absolute metric system, it is equal to the force in dynes exerted by a mass of 1 gr. on an equal mass at a distance of 1 cm. This

force is so excessively small that it defied all attempts to detect it in the laboratory for over a century after the time of Newton.

In 1798, however, Henry Cavendish succeeded in measuring it with considerable accuracy by means of a sensitive torsion balance. A light rod, to the ends of which were attached two small spheres, was suspended by a fine silver wire attached to its midpoint. When two heavy spheres weighing 158 kg. each were brought into the position shown in Fig. 13, the suspended rod was observed to be slightly twisted by the attraction of the heavy balls for the lighter ones. A tiny mirror was attached to the center of the rod, and the amount of twist was measured by

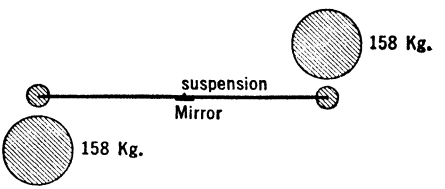


FIG. 13. Cavendish's Experiment.

observing with a telescope the apparent displacement of the image of a distant scale reflected in the mirror, exactly as the deflection of a wall galvanometer is measured at the present day. From a knowledge of the force required to produce such a twist, and the known separation of the attracting masses, G could be determined with the aid of Formula 11. Expressed in the usual absolute metric units, it is 6.66×10^{-8}, thus indicating an attraction of something over six hundred-millionths of a dyne between two masses of 1 gr. each at a distance of 1 cm.

With a knowledge of G, Formula 11 permits at once a calculation of the mass and the average density of the Earth. As already indicated in Chapter II, the force exerted by the Earth on 1 gr. is approximately 980 dynes. The distance of the surface of the Earth from its center, which must be used for R in Formula 11 if we are to calculate the attraction of the Earth for an object at its surface, is 6357 km. or 635,700,000 cm. Hence, substituting these values in (11) and letting M = the mass of the Earth,

$$980 = 6.66 \times 10^{-8} \frac{1(M)}{(6.357 \times 10^{8})^2}$$

or

$$M = 5.947 \times 10^{27} \text{ gr.}$$

The average density of the Earth can now be obtained by dividing this mass by its volume. The result, 5.5 times that of water, is nearly twice the average density of the surface of the Earth, and must be explained by a much greater abundance of heavier elements, such as iron, at the center.

18. Simple Harmonic Motion. The Pendulum. If a circular orbit such as that of the Moon about the Earth is viewed from a distant point in the plane of the orbit, it will seem to be a straight line, and the revolving body will appear to oscillate back and forth about the midpoint of this line. It was thus that Galileo saw the satellites of Jupiter when he first turned his telescope on this, the largest of the planets. The apparent motion is simply the component of the actual circular motion in a direction at right angles to the line of sight. It may be most readily visualized as the *projection* of the actual moving body on a diameter of the circle. Thus in Fig. 14 the apparent motion is that of the point P, obtained by dropping a perpendicular from the moving mass m on the diameter AB. It is clear that when m is at A or B the velocity v of m is perpendicular to AB, and the velocity of P is 0. When m is at C or D, on the other hand, the point P will lie at the center of the circle, and will be travelling with a velocity equal to v. P, therefore, oscillates back and forth between A and B. The time of one complete oscillation from A to B and back to A is called the *period* of the motion. This sort of oscillation is technically known as *simple harmonic motion.*

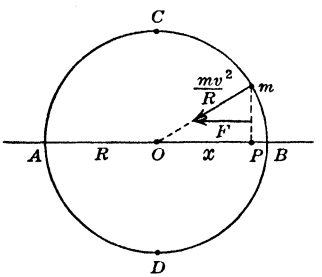


FIG. 14. Simple Harmonic Motion.

It suggests at once the familiar swing of a pendulum. That this suggestion is no mere analogy, and that the motion of the pendulum is actually simple harmonic, is readily shown if we use the fact that the fundamental law of motion, $F = ma$, is applicable to any component of an actual resultant motion. Thus the motion of P must be that which a mass m would exhibit if it were subjected to a force F equal at every instant to the component in the AB direction of the actual centripetal force. In Fig. 14 the displacement of P from O, the center of the circle, is represented by the letter x. By similar triangles

$$\frac{x}{R} = \frac{F}{\frac{mv^2}{R}}$$

Or
$$F = \left(\frac{mv^2}{R^2}\right)x \qquad (12)$$

The force F is, therefore, always proportional to x, and is always directed back toward O. It is this type of force which produces simple harmonic motion. Imagine, for example, a mass m, started from rest at B, but subjected to a force such as F back toward O. It will be gradually accelerated from right to left until, on arrival at O, the force F has dropped to zero, and it has acquired a velocity v. Due to its inertia, it will continue on even against the retarding force of F, which now acts in the opposite direction. This motion will continue until at A the velocity has been reduced to zero. The entire process will now be repeated over and over again.

Fig. 15 represents a pendulum bob of mass m suspended from the point O by a cord of length l. The only forces acting on the bob are the downward force of gravity mg and the upward tension T on the cord. Of these only the force of gravity mg has a component F in the direction of the arc along which the bob must move. It must, therefore, be this force which is responsible for the observed oscillations. By similar triangles

$$\frac{F}{mg} = \frac{x}{l}$$

Or
$$F = \left(\frac{mg}{l}\right)x \qquad (13)$$

Here also the force F is proportional to the displacement x, and is always directed approximately back toward the equilibrium position of the bob. The conditions necessary for the production of approximately simple harmonic motion are, therefore, satisfied.

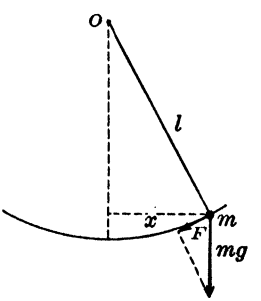


FIG. 15. The Simple Pendulum.

The most striking property of the pendulum is the constancy of its period. The fact that its time of swing depends to a very good approximation only on the length of the cord and not on the mass of the bob or its amplitude of oscillation was first dis-

covered by Galileo from observation of the great candelabra in the Cathedral of Pisa. When at dusk the caretaker drew it back to light the lamps, and then released it, the great scientist was thrilled by its rhythmic motion. He timed the swing by the throb of his own pulse, and then once more after its motion had decayed to a small fraction of its original amplitude. The period was the same, and a new law had been discovered.

The time of oscillation T of a mass m when subjected to a given restoring force can be readily calculated from (12). This time is equal to that required for the mass in Fig. 14 to describe its circular orbit.

That is,
$$T = \frac{2\pi R}{v}$$

or
$$\frac{R}{v} = \frac{T}{2\pi}$$

Substituting this expression for R/v in (12), and solving for F/x, the real measure of the restoring force,

$$\frac{F}{x} = \frac{4\pi^2 m}{T^2}$$

or
$$T = 2\pi \sqrt{\frac{m}{\frac{F}{x}}} \tag{14}$$

Applying this general relation to the pendulum, it is first necessary to calculate F/x from (13).

$$\frac{F}{x} = \frac{mg}{l}$$

Substituting in (14)

$$T = 2\pi \sqrt{\frac{l}{g}} \tag{15}$$

PROBLEMS

1. A stone is hurled horizontally outward from the brow of a vertical cliff with a velocity of 100 ft./sec. If the height of the cliff is 64 ft., what will be its time of fall, and how far from the base of the cliff will it strike? Neglect air resistance.

2. If the stone of the previous problem is thrown with the same initial velocity, but upward at an angle of 45° to the horizontal, what will be its time of flight, and how far from the base will it fall?

3. If the stone of the previous problems is thrown downward at an angle of 45°, how far from the base will it fall?

4. A weight of 10 lb. is whirled at the end of a wire 5 ft. long and in such a way that it makes 2 complete revolutions per second. What is its central acceleration, and what is the tension in the wire?

5. A car in a common circus exhibition loops the loop about a circle having a 15-ft. radius. How fast must it be travelling at the top of the loop in order not to fall from the track? *Hint:* The centripetal force necessary to make it describe the circular path is furnished in part by the weight of the car and in part by the inward thrust of the track. If the weight of the car is greater than the centripetal force required, it will describe a smaller circle with disastrous consequences.

6. The average radius of the orbit of the Earth about the Sun is 92,830,000 miles (1.493×10^{13} cm.), and the diameter of the Sun is 864,000 miles (1.39×10^{11} cm.). Calculate (*a*) the central acceleration of the Earth in its orbit, (*b*) the mass of the Sun, and (*c*) the average density of the Sun, i.e., its mass per cubic centimeter.

7. What is the length of a pendulum which describes a full swing in exactly 1 sec.?

8. What would be the period of a pendulum having a length equal to the radius of the Earth?

9. A mass of 10 gr. hangs at the end of a spiral spring. A downward force of 5 gr. displaces this mass from its equilibrium position a distance of 1 cm. If it is given a small downward displacement of this sort and is then released, what will be its period of oscillation?

CHAPTER V

MACHINES

From the colossal scale of the phenomena pictured in the last chapter the science of mechanics leads us now to some of the most practical of its applications to everyday life. Man has always been a bearer of burdens, but the progress of civilization very accurately parallels the progress he has made in devising mechanical means to aid him in his tasks. The civilization of all ages is appraised among other things by the extent, the permanency, and the general quality of the construction which it has undertaken. Permanent construction presupposes, in general, massive construction, and in building the huge structures which have remained as his monuments to future ages, man had to learn to raise and handle heavy loads. The great blocks of the pyramids were almost certainly put in place by the use of inclined planes, up which they were dragged by the exertion of hordes of slaves.

19. The Lever. It was Archimedes of Syracuse, in the third century B.C., who discovered the quantitative laws governing the use of the lever for raising heavy loads, although the device itself must have been known for many centuries previously. The laws here involved are of such a simple nature, and are so important for an understanding of similar devices, that it seems well to consider them at this point.

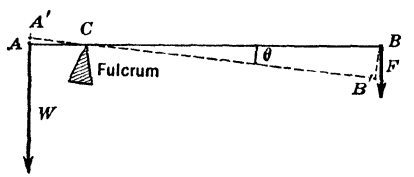


FIG. 16. The Lever.

The lever in its simplest form consists of a bar AB (Fig. 16), which is pivoted at the point C on some convenient support. This support is known as the *fulcrum* of the lever. If the lever is to be of any value in raising heavy loads, the fulcrum must be placed well toward the end at which the load is applied. If the load W is placed at A, and if the downward force F is exerted at B, it was shown by Archimedes that the force F required

to raise the load can be smaller than W in the ratio of AC to CB, or

$$\frac{W}{F} = \frac{CB}{AC} \tag{16}$$

Such a device, which makes it possible to exert a large force by means of a smaller one, is known as a *machine.* The ratio of the force W applied by the machine to the force F exerted on it is known as its *mechanical advantage.* It is clear from (16) that there is no theoretical limit to the mechanical advantage which can be secured by means of the lever. Practically, however, the shortness of the distance AC from the load to the fulcrum is limited by the nature of the task to be performed, and CB cannot be made too great owing to the elasticity of the material of the lever. These practical considerations, which limit the effectiveness of the lever, do not, however, affect unfavorably the performance of certain other types of machines, and there seems no doubt that machines can be devised for any particular task which will have any desired mechanical advantage.

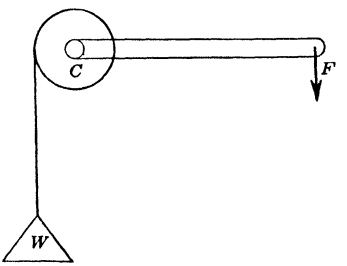


FIG. 17. The Windlass.

The windlass (Fig. 17) is clearly a modification of the lever. The bearing in which the axle rests serves as the fulcrum, the cable wrapped about the drum serves as a very convenient means of applying the load, and the crank arm is equally convenient for exerting the force on the machine. The limited motion, which the simple lever can impart to its load, is here extended to a distance equal to the whole length of the cable. The necessity at times of pressing upward on the crank arm reduces, however, the force which can be applied to the machine, and hence the load which can be raised with a given mechanical advantage.

The friction, which inevitably exists in the bearings of a machine, results always in an actual mechanical advantage somewhat under the theoretical one. The ratio of the actual to the theoretical mechanical advantage is known as the *efficiency* of the machine.

20. The Law of Moments. The law (16) which expresses the relation between the load W and the applied force F, i.e., the mechanical advantage of the lever, can, like the law of the polygon of forces, be tested experimentally in the laboratory, or it can be proven theoretically by an argument based on the fundamental laws of motion. Either method of approach would show that it is only a special case of the *Law of Moments*, a very general and important theorem, which governs the equilibrium of a rigid body. The expression (16) can be written

$$W \times AC = F \times CB \tag{17}$$

AC is known as the *lever arm* of the force W, or simply the *arm* of W, and CB is the *arm* of F. The product of the force and its lever arm is known as the *moment* of the force, and (17) expresses the fact that the moment exerted by the applied force is equal to that exerted by the load. In Fig. 16 the force F tends to rotate the lever in a clockwise sense, while W tends to rotate it in an anticlockwise sense. Taking into consideration the fact that there might be other forces applied to the body, the Law of Moments states that, if a body is in equilibrium, the sum of the moments which tend to rotate it in a clockwise sense is equal to the sum of those which tend to rotate it in an anticlockwise sense. In the case of the lever already treated, CB and AC are, from the very conditions of the problem, perpendicular to F and W, respectively. Whether or not this is so, the moment arm of any force must always be taken as the perpendicular distance from the point about which the moment is calculated to the line of action of the force.

21. The Pulley System. Another type of machine which has found wide application is the pulley system. It is capable of a great variety of modifications, depending on the purpose for which it is intended, but the following example will illustrate the method of determining its mechanical advantage. Fig. 18 shows a four-rope pulley system, with the load W applied to a hook on the lower block. If we neglect the effect of the friction which enters at the bearing of each wheel, i.e., if we treat the device as ideal, the tension T throughout the entire length of the rope will be the same, and will be equal to the applied force F. In the figure four segments of the rope are observed to be pulling upward on the lower block, and must, therefore, just balance the downward pull of the load W. Hence, neglecting

also the weight of the block itself, $W = 4F$, or the mechanical advantage is 4.

22. Work. Although there is no real limit to the mechanical advantage which can be obtained with a machine, this mechanical advantage is not achieved without some sacrifice. Thus, in the pulley system just considered, for every foot that the load is raised, the four supporting cords must each be shortened by 1 ft., and to this end 4 ft. of cord must be taken in by the operator. Expressed in a different way, the hand of the operator must move through 4 ft. for every foot through which the load is moved. If a ten-rope pulley with a mechanical advantage of 10 is used, the hand of the operator must move through ten times the distance the load is raised. The labor performed by him clearly depends, not merely on the force which he exerts, but on the distance through which this force is applied. Mechanically, therefore, the *work* he does is defined as the product of the force exerted and the distance through which his hand moves. If s is this distance, i.e., the length of rope which he must take in, the work which he does is Fs. Similarly, if h is the height through which the load is raised, the work done by the machine is Wh. Since, however, $W = 4F$ and $h = s/4$,

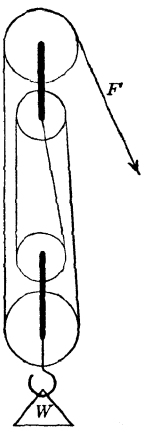


FIG. 18. Four-rope Pulley System.

$$Wh = Fs$$

The work done by the machine is equal to that done by the operator. On account of friction in the pulley bearings, the actual mechanical advantage is somewhat less than 4, the ratio of the actual to the theoretical value being the efficiency of the device. Or

$$W = (\text{Eff.})4F$$

Since the presence of friction can have no effect on the relation between h and s,

$$Wh = (\text{Eff.})Fs \tag{18}$$

Or the ratio of the work actually accomplished to that performed by the operator is the efficiency of the machine. This, instead

of the ratio of the actual to the theoretical mechanical advantage, is the usual definition of machine efficiency.

The law that the work done by the operator of a machine is equal to that done by the machine is readily shown to hold for the lever, and, in fact, for any ideal machine in which the effects of friction are neglected. In Fig. 16 the angle through which the lever is rotated about the fulcrum is designated by the Greek letter θ. For most mechanical purposes it is convenient to measure angles in *radians* rather than in degrees. The magnitude of an angle in radians is defined in terms of any circle which may be described about the vertex of the angle as the ratio of the length of the arc which the angle subtends to the radius of the circle. Since the length of an entire circumference is 2π times the radius, there must, accordingly, be 2π radians in the 360° surrounding a point, or each radian must be equivalent to $360°/2\pi$ or 57.3°. In these units the angle θ in Fig. 16 can be expressed as BB'/CB or its equal AA'/AC.

The work done by the operator is $F \cdot BB'$, and that accomplished by the machine is $W \cdot AA'$. Introducing the angle θ in these expressions for work, and letting $(\text{Work})_0$ represent the work done by the operator and $(\text{Work})_M$ that done by the machine,

$$(\text{Work})_0 = F \cdot BB' = F \cdot CB \cdot \theta$$

and

$$(\text{Work})_M = W \cdot AA' = W \cdot AC \cdot \theta$$

If we denote the moment of the force exerted by the operator, $F \cdot CB$, by L_0 and that exerted by the load, $W \cdot AC$, by L_M, it is clear that

$$(\text{Work})_0 = L_0\theta \tag{19}$$

and

$$(\text{Work})_M = L_M\theta \tag{19'}$$

Since, in accordance with the Law of Moments, the two moments are equal, the same must be true of the work done by the operator and that done by the machine.

Under all circumstances the work done by a force on a rotating body is the product of the moment which it exerts about the axis of rotation and the angle in radians through which the body is turned. It should be noted that this general relation $L\theta$ for the work done on a rotating body is strictly analogous to the expression Fs for the work done by a force F on a body which is displaced through a linear distance s.

The definition of work requires a slight modification if the direction of the applied force does not coincide with the direction in which the point of application moves. Fig. 19 illustrates this point for the simple lever. Owing perhaps to limitations in space, the force exerted by the operator is imagined as acting in the direction of the arrow F. The line of action of this force is extended in the dotted line BP. Its moment arm is now the perpendicular CP from C to this line of action. By the law of moments,

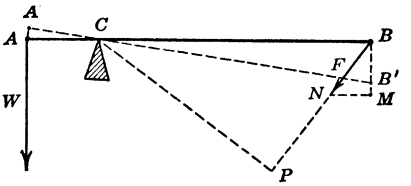


FIG. 19. Force Oblique to Direction of Motion.

$$F \cdot CP = W \cdot AC$$

As before, the work done by the machine is $W \cdot AA'$, which we have shown to be equal to $L_M\theta$. The work done by the operator must, however, now be defined as the *component* of F in the direction of motion, BM, multiplied by BB'. Or

$$(\text{Work})_0 = BM \cdot BB' = BM \cdot CB \cdot \theta$$

The triangle MNB is similar to triangle PBC. Hence

$$\frac{BM}{F} = \frac{CP}{CB}$$

Or $$BM \cdot CB = F \cdot CP = L_0$$

Hence $$(\text{Work})_0 = L_0\theta$$

The new definition for work has been such as to maintain the correctness of the expression $L\theta$ for the work done on a rotating body; and, furthermore, since $L_0 = L_M$, it has also been such as to maintain the original law that the work done by the operator is the same as that done by the machine.

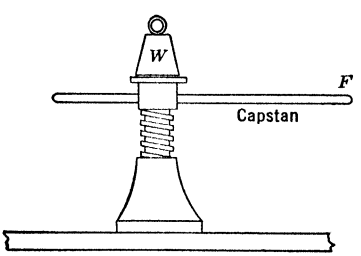


FIG. 20. Jackscrew.

Once persuaded of the correctness of this principle, it can be used to determine the mechanical advantage of machines which are otherwise rather difficult to treat. A good example of this is the jackscrew, Fig. 20. Let the screw be turned by means of

a capstan of length l, and let the pitch of the screw be h. In one complete turn of the screw the load will be raised a distance h, and the work accomplished will be Wh. Meanwhile the point of application of the force exerted by the operator will have moved a distance $2\pi l$ equal to the circumference of the circle in which the hand of the operator moved. If he has taken care always to exert his force at right angles to the capstan, i.e., in the direction of motion, the work he does is $2\pi lF$. Setting this equal to the work of the machine,

$$2\pi lF = Wh$$

Or the mechanical advantage, the ratio of load to applied force, is

$$\text{M.A.} = \frac{W}{F} = \frac{2\pi l}{h}$$

We have seen that work is always calculated by multiplying a force by a distance. The commoner units in which this is expressed are the *foot-pound*, in which the force is measured in pounds and the distance in feet, and the *erg*, in which the force is measured in dynes and the distance in centimeters.

EXAMPLE

1. A uniform beam is being carried by two men, one of whom is at one end of the beam and the other one-third of the length of the beam from the opposite end. What proportion of the total weight is carried by each? (Fig. 21.)

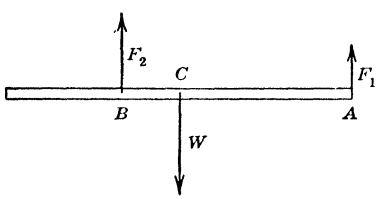


FIG. 21

It is possible to show that the weight of any extended body can be thought of, for all mechanical purposes, as a single force acting at some one point of the body. This point is known as its *center of gravity*. In any symmetrical body, such as the beam in the present problem, the center of gravity is located at its geometric center. Fig. 21 shows, accordingly, the beam with the force of gravity W acting downward at its midpoint and the two upward forces F_1 and F_2 applied at A and B respectively as required by the problem. A simple application of the law of moments yields the required solution. It remains only to choose the point about which to apply the law. The moment arm of any force which passes through the center of moments is, of course, zero, and the solution is accordingly simplified if the point of application of some one of the forces is chosen as the center of moments. If, for example, we choose the point A for this purpose, and denote the length of the beam by l, the moment of W about this point is $\frac{1}{2}lW$ while that of F_2 is $\frac{2}{3}lF_2$. This latter

moment tends to rotate the beam in a clockwise sense; W tends to rotate it in an anticlockwise sense. Hence in accordance with the Law of Moments,

$$\tfrac{1}{2}lW = \tfrac{2}{3}lF_2$$

Or

$$F_2 = \tfrac{3}{4}W$$

Since the two upward forces F_1 and F_2 must together be equal to the downward force W,

$$F_1 = \tfrac{1}{4}W$$

PROBLEMS

1. What force must be exerted with a lever 5 ft. long in order to raise a 200-lb. weight if the fulcrum is located 1 ft. from the load?

2. Solve Example 1 by choosing the point C as the center of moments.

3. At what point should the second man of Example 1 be placed in order that he might carry not three-quarters but two-thirds of the load?

4. If the diameter of the drum of a windlass is 6 in., and the length of the crankarm is 2 ft., what is its mechanical advantage? What force is required to raise a 200-lb. load?

5. Draw pulley systems which shall have mechanical advantages of 2, 3, and 5 respectively.

6. How long must be the capstan of an ideal jackscrew having a pitch of $\frac{1}{2}$ in. if it is to raise a load of 300 lb. by the application of a 4-lb. force? What must be the length of the capstan if the efficiency of the screw is 15 per cent?

CHAPTER VI

ENERGY AND MOMENTUM

23. Kinetic Energy. It is a familiar fact that a moving body is capable of performing work simply as a result of its motion. An automobile can climb some distance on a hill by virtue of its previous velocity, and the moving flywheel continues to drive the motor between the intermittent thrusts of the pistons. The capacity which a body has for doing work is known as *energy*, and the energy which it possesses by virtue of its motion is known as *kinetic energy*. The kinetic energy of a body depends only on its mass and velocity, and it is not difficult to obtain a quantitative expression for it.

Fig. 22 represents schematically an automobile, which is engaged in raising the load m_1 by means of a rope and the pulley P. If, while moving forward and raising the weight at a uniform velocity v, the motor is suddenly stopped, the impetus of the car will continue to carry it forward, and will raise the weight an additional height h. The work required for this in absolute units is m_1gh, and if the rolling friction of the car and that in the bearing of P is negligible, the combined kinetic energy of car and load will have been devoted to this purpose, and must, therefore, have been equal to m_1gh. In order to obtain an expression for their initial kinetic energy in terms of mass and velocity, it is only necessary to calculate h by the methods of Chapter II and substitute it in m_1gh. The retarding force acting on car and load is simply the downward force of gravity on m_1 or m_1g. Substituting this in Formula 5 of Chapter II, and letting m_2 be the mass of the car alone,

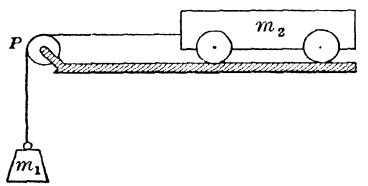


FIG. 22

$$m_1g = -(m_1 + m_2)a \qquad (20)$$

After travelling a distance h, the car comes to rest, and by Formula 4, Chapter I

$$2ah = -v^2$$

Or
$$a = -\frac{v^2}{2h} \tag{21}$$

Substituting the value of a from (21) in (20),

$$m_1 g = (m_1 + m_2)\frac{v^2}{2h}$$

Or
$$h = \frac{(m_1 + m_2)}{m_1 g} \cdot \frac{v^2}{2}$$

Substituting this value of h in m_1gh, which we have seen to be a measure of the initial kinetic energy of car and load,

$$m_1 gh = (m_1 + m_2)\frac{v^2}{2}$$

If we let m represent the total moving mass, and if we represent the kinetic energy by the abbreviation K.E.,

$$\text{K.E.} = \frac{mv^2}{2} \tag{22}$$

The kinetic energy of a body is equal to one-half the product of its mass and the square of its velocity.

The fact that a moving body is capable of performing a certain amount of work indicates that, if the principles discussed in the last chapter are correct, it should require exactly this same amount of work to give it its kinetic energy. That this is actually the case is readily seen if we imagine the body accelerated from rest by a force F. According to Newton's Law, $F = ma$. If it has acquired the velocity v after travelling a distance s, the work done by the applied force can be calculated by multiplying this equation through by s. Or

$$\text{Work done} = Fs = mas$$

Since the initial velocity of the body was zero, Equation 4 of Chapter I shows that $as = v^2/2$. Hence

$$\text{Work done} = \frac{mv^2}{2}$$

24. Potential Energy. It is always possible when a mass moves from a higher to a lower level to utilize the work done on it by gravity to do work on other bodies. Thus the cable which raises an elevator passes over a pulley to the counterweight, and

the work done by gravity on the counterweight, as it falls, furnishes a large part of the work necessary to raise the elevator. Here again the counterweight in its elevated position has a capacity for doing work. Such capacity as it possesses by virtue of its position is known as *potential energy*. Water at the top of a fall has potential energy, and this can be utilized by suitable machinery to do any desired type of work.

The gravitational potential energy of a body is measured by the work which gravity can be required to do on it. This is equal to the force which gravity exerts times the total distance through which the body can drop. Thus the potential energy of m pounds of water at the top of a waterfall of height h feet is mh foot-pounds or mgh foot-poundals. This is the energy which can be harvested from the drop of the m pounds at that one particular fall. If there are other falls below, the potential energy available at each will be mg times its height, and it would be perfectly correct to say that the total potential energy of the mass m is equal to mg times the total drop to sea level.

25. Conservation of Energy. In general, the potential energy of a body can be readily converted into kinetic energy. It is merely necessary to let it fall under the action of gravity. If a mass m falls through a height h with acceleration g, its final velocity will be related to h by the formula

$$2gh = v^2$$

Its kinetic energy will be, therefore,

$$\text{K.E.} = \frac{mv^2}{2} = mgh$$

The potential energy has been converted *in toto* into kinetic energy. Conversely, a body starting upward with a kinetic energy $mv^2/2$ will climb to such a height that its potential energy is exactly equal to its initial kinetic energy.

Such considerations as these, together with the facts about work which emerged in the last chapter, have led to the gradual recognition of energy as a very fundamental thing which Nature conserves. It may exist in various forms such as kinetic and potential energy, but when energy in one form disappears, it always appears in some other form. It may be converted from one form to another, but as a whole it is neither lost nor created. This law of the *Conservation of Energy* was first recognized about

the middle of the nineteenth century, and stands out as the most fruitful scientific generalization of that entire period.

Energy is probably the most important requirement of our modern industrial civilization. It turns the wheels of our mills and factories. It heats and lights our homes. It transports us from place to place. As time goes on, we are using energy at a rapidly increasing rate, and where the present trend will end, it is difficult to say. This energy is obtained in part from water-power and in part from fuels such as coal and petroleum. The laws which govern the ability of such fuels to provide energy are discussed in the tenth and eleventh chapters.

The rate at which energy is provided by a given source, or the rate at which it is being spent, is known as *power*. It is commonly measured in terms of the *horse-power*, which is defined as 550 ft.-lb./sec. or 33,000 ft.-lb./min.

26. The Kinetic Energy of a Rotating Body. Moment of Inertia. In one case already cited, that of an engine flywheel, kinetic energy is possessed by reason of a pure rotation. The

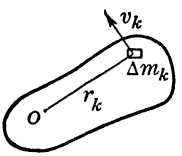


FIG. 23. Moment of Inertia.

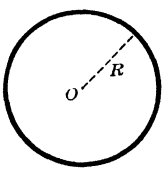


FIG. 23a. The Hoop.

expression $\frac{1}{2}mv^2$ is, of course, incapable of calculating directly this form of kinetic energy, but it can nevertheless be used to obtain a new formula for this purpose. Fig. 23 represents a body which is supposed to be rotating about an axis perpendicular to the paper at O. The body can be thought of as made up of a large number of tiny masses, Δm_1, Δm_2, $\cdots$ Δm_n, of which only one Δm_k is represented in the figure. This typical mass is at the instant pictured in the diagram moving with the velocity v_k, and possesses in consequence a definite kinetic energy $\frac{1}{2}mv_k^2$. The total kinetic energy of the rotating body, which we will represent in the usual way by K.E., should consist of the sum of the kinetic energies of the constituent masses, i.e.,

$$\text{K.E.} = \tfrac{1}{2}\Delta m_1 v_1^2 + \tfrac{1}{2}\Delta m_2 v_2^2 + \cdots + \tfrac{1}{2}\Delta m_n v_n^2$$

All these masses, however, possess one thing in common. They

are rotating about O through exactly the same angle θ in the same time t. The angle in radians through which a body rotates per second is known as its *angular velocity*, and is represented almost universally by the Greek letter ω. If we denote by s_k the distance moved by Δm_k in time t, its linear velocity is $v_k = s_k/t$, and its angular velocity $\omega = \theta/t$. If we represent the distance of Δm_k from O by r_k, and make use of the way in which the radian is defined,

$$s_k = r_k\theta$$

Hence

$$v_k = \frac{s_k}{t} = \frac{r_k\theta}{t} = r_k\omega$$

Substituting the corresponding expression in each term of the formula for the kinetic energy of the rotating body, and taking out the common factor $\frac{1}{2}\omega^2$,

$$\text{K.E.} = \tfrac{1}{2}\omega^2(\Delta m_1 r_1^2 + \Delta m_2 r_2^2 + \cdots + \Delta m_n r_n^2)$$

The quantity enclosed within the parentheses is known as the *moment of inertia* of the body, and is represented by the letter I.

Thus

$$\text{K.E.} = \tfrac{1}{2}I\omega^2 \tag{23}$$

where

$$I = (\Delta m_1 r_1^2 + \Delta m_2 r_2^2 + \cdots + \Delta m_n r_n^2) \tag{24}$$

In general, the calculation of such a moment of inertia is a rather intricate task, which requires the resources of the integral calculus; but in one simple case, that of a ring or hoop (Fig. 23*a*), the distance of all the little masses from the axis of rotation is the same, viz., the radius R of the hoop, and this may be factored out from the moment of inertia formula. Thus

$$I = R^2(\Delta m_1 + \Delta m_2 + \cdots + \Delta m_n)$$

Since, however, the sum of all the constituent masses is equal to the total mass M of the body,

$$I = MR^2 \text{ (hoop)}$$

As a matter of fact, the moment of inertia of any body contains the total mass as one factor and with it the square of one or more of its linear dimensions. A few typical moments of inertia are the following:

MOMENTS OF INERTIA

A uniform cylinder about its axis of symmetry	$\frac{1}{2}MR^2$
A sphere about a diameter	$\frac{2}{5}MR^2$
A uniform thin rod about an axis through its center and perpendicular to its length	$\frac{1}{12}Ml^2$

It should be noted that the expression (23) for the kinetic energy of a rotating body is quite analogous to the expression (22) for that of a body in rectilinear motion. The moment of inertia I takes the place of the mass m, and the angular velocity ω takes the place of the linear velocity v. This fact is typical of the thoroughgoing parallelism which exists between the mechanics of rotation and of translatory motion, a parallelism which we have already pointed out in the expressions for work.

27. Impulse and Momentum. A concept similar to that of work and energy, but of more restricted application, is that of *impulse* and *momentum*. Just as the work done by a force is the product of the force and the distance through which its point of application moves, so the impulse resulting from a force is the product of the force and the time during which it acts. If, for example, a force F is applied to a mass m, the resulting acceleration is given by the Second Law of Motion.

$$F = ma \tag{25}$$

If we multiply both sides of (25) by t, the time during which the force is applied, the left-hand side becomes the impulse exerted by the force.

$$Ft = mat \tag{26}$$

From Formula 1 of Chapter I, $at = v - v_0$, and substituting this in (26)

$$Ft = mv - mv_0 \tag{27}$$

The product mv is known as the *momentum* of the mass m when its velocity is v. (27) states accordingly that the difference between the initial momentum mv_0 and the final momentum mv is the impulse Ft. Or, the impulse exerted by a force is equal to the resulting change in momentum.

28. Impact. Conservation of Momentum. The importance of the concept of momentum is due to Newton's Third Law of Motion, the law of action and reaction. If one body exerts a force F on a second body for a time t, and thereby gives to it

an impulse Ft, the second body must exert an equal and opposite impulse on the first body. The change in momentum of one is, therefore, exactly equal to the corresponding change in the other, but is in the opposite direction.

For example, if a billiard ball A while travelling with a velocity v_a strikes a stationary ball B in the manner shown in Fig. 24, the reduction in momentum of the moving ball will be exactly equal to the gain in momentum of the stationary ball. Stated in another way, the sum of the momenta of the two balls will be unaffected by the collision. If, as is very nearly the case with ivory balls, there is a negligible loss of energy in the impact, we can calculate very simply the resulting motion of the two balls.

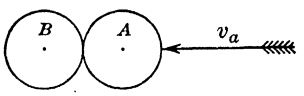


FIG. 24. Direct Impact of Two Spheres.

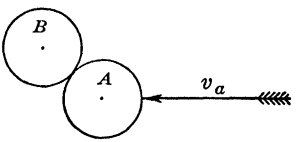


FIG. 24a. Oblique Impact of Two Spheres.

Let m be the mass of each ball, v_a' the velocity of A after collision, and v_b' that of B. Since B was initially at rest, both its initial momentum and its kinetic energy were zero. Since no change in total energy resulted,

$$\tfrac{1}{2}mv_a^2 = \tfrac{1}{2}mv_a'^2 + \tfrac{1}{2}mv_b'^2$$

Or

$$v_a^2 = v_a'^2 + v_b'^2 \tag{28}$$

Since there was no change in total momentum,

$$mv_a = mv_a' + mv_b'$$

Or

$$v_a = v_a' + v_b' \tag{29}$$

Squaring this last equation, and substituting the resulting value of v_a^2 in (28),

$$v_a'^2 + 2v_a'v_b' + v_b'^2 = v_a'^2 + v_b'^2$$

Or

$$v_a'v_b' = 0 \tag{30}$$

Either $v_a' = 0$, in which case from (29)

$$v_b' = v_a$$

Or $v_b' = 0$, and

$$v_a' = v_a$$

The first of these two alternatives must be the actual physical solution, since the latter, although mathematically possible, would be equivalent to no impact at all.

The case here treated, in which no energy is lost at impact, occurs when the bodies are *perfectly elastic*. The other extreme of zero elasticity, in which the two bodies remain in contact after collision, can be handled with equal ease, and can be observed experimentally when two balls of putty or wet plaster collide. In such a case, in place of the relation (28), which results from the conservation of energy, we would have the relation

$$v_a' = v_b'$$

In Fig. 24 the two balls were pictured as making contact when the direction of the line joining their centers was the same as the direction of motion of A. If, however, this line of centers was oblique to v_a, as shown in Fig. 24*a*, the resulting velocities would have components perpendicular to the initial velocity of A. It is easily shown that in this case the sum of the components of the momenta of A and B, both in the v_a direction and perpendicular to it, will be unaffected by the collision. Momentum, unlike kinetic energy, is a vector quantity, all the components of which are conserved in an impact.

The conservation of momentum, which has been observed in the problem of two colliding balls, is really of much more general application. Whenever two bodies are subject only to mutual forces, i.e., only to forces which they exert on each other, any resulting changes in their motion will be such as to maintain unaffected their original combined momentum. The expulsion of a projectile from a gun is a case in point. The combined momentum of gun and projectile must remain unaffected by the detonation of the cartridge. If the bullet of mass m attains a velocity v, the gun of mass M must suffer a recoil velocity V such that

$$MV = mv$$

The propulsion of a rocket is another example of this principle. Owing to the combustion of its charge, a mass of gas is discharged from the rocket with a high velocity, thus giving the rocket itself a recoil velocity in the opposite direction. It is obvious that this method of propulsion, unlike that due to an airplane propeller, is not dependent on the surrounding air, and would be applicable to the region above the Earth's atmosphere.

The Principle of the Conservation of Momentum can be extended from two to any number of bodies which are subject only to mutual forces. Thus the members of our solar system, if we neglect the probably small effect of distant fixed stars, must move in such a way as to maintain unchanged each component of their combined momentum.

EXAMPLES

1. Neglecting friction, what power is required to drive a 3000-lb. automobile up a 10 per cent grade at a speed of 30 miles an hour? A 10 per cent grade is one for which the rise is 1 ft. in every 10 ft. of travel along the road.

The car will travel forward 30 miles in each hour or $\frac{1}{2}$ mile in each minute, i.e., 5280/2 = 2640 ft./min. It is accordingly lifted in this time 2640/10 = 264 ft., a process which requires (264)(3000) = 792,000 ft.-lb. Since 1 horsepower is equivalent to 33,000 ft.-lb./min., the required power is 792,000/33,000 = 24 H.P.

2. A body slides down a smooth inclined plane of length l and height h (Fig. 25). If it starts from rest, what velocity does it attain at the base?

In accordance with the Law of the Conservation of Energy, the work done on the body by gravity is equal to the kinetic energy which it acquires. The

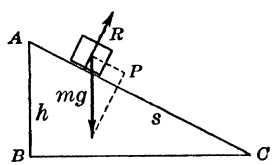


FIG. 25. Block on Incline.

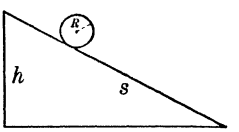


FIG. 25a. Ball on Incline.

component P of the gravitational force mg in the direction of motion can be calculated by a comparison of the triangle of which mg and P constitute the hypotenuse and one arm respectively with the triangle ABC. Since these two triangles are similar,

$$\frac{h}{s} = \frac{P}{mg}$$

Or

$$P = \frac{h}{s} mg$$

The work done by gravity Ps is, therefore, mgh, which, in turn, is equal to that which gravity would have done on the body if it had fallen vertically from A to B. Equating this work to the final kinetic energy of the body,

$$mgh = \tfrac{1}{2}mv^2$$

Or

$$v = \sqrt{2gh} \tag{31}$$

The final velocity is, therefore, equal to that which the body would have attained if it had dropped through the same vertical distance h.

3. A ball rolls without slipping down the incline of Example 2 (Fig. 25a). Calculate its final velocity.

The ball, unlike the sliding body of the previous problem, acquires kinetic energy of rotation as well as of translation. This, by Formula 23, is $(\text{K.E.})_R = \frac{1}{2}I\omega^2$, while its kinetic energy of translation by Formula 22 is $(\text{K.E.})_T = \frac{1}{2}mv^2$. Equating the sum of these kinetic energies to the loss which the ball suffers in potential energy,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

Since the motion is supposed to take place without slipping, the ball will travel a distance equal to its circumference $2\pi R$ while it is rotating through a full 2π radians. Hence, if we start to measure both s and θ at the same instant, $s = R\theta$, and correspondingly $v = R\omega$, or $\omega = v/R$. Substituting this value for ω in the expression for the total kinetic energy, and introducing at the same time the formula for the moment of inertia of a sphere, $I = \frac{2}{5}mR^2$,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\cdot\tfrac{2}{5}mR^2\cdot\frac{v^2}{R^2}$$

Or $$gh = \tfrac{7}{10}v^2$$

Hence $$v = \sqrt{\tfrac{10}{7}gh}$$

PROBLEMS

1. Calculate the kinetic energy of a body weighing 10 gr. which has fallen freely from a height of 5 meters (*a*) in gram centimeters, (*b*) in ergs.

2. Calculate the final velocity of the body of Example 2 when h is 5 ft. and s is 10 ft., (*a*) if the plane is smooth, (*b*) if the coefficient of friction between the body and the plane is $\frac{1}{10}$.

3. An automobile is travelling with a velocity of 30 miles/hour when the brakes are locked. If it succeeds in stopping in 88 ft., what is the coefficient of friction between its wheels and the ground?

4. If the automobile of the preceding problem had a mass of 3000 lb., what was its momentum before applying the brakes? If the retarding force due to the brakes was 1000 lb., how many seconds would be required to bring the machine to rest?

5. If 10,000 cu. ft. of water pass over a 75-ft. fall each minute, how much power can be derived from the fall if properly harnessed? The weight of a cubic foot of water is 62.5 lb.

6. If a 15-gr. bullet is expelled from a gun weighing 10 kg. with a velocity of 400 meters/second, what is the velocity of recoil of the gun? Calculate the kinetic energy of both gun and bullet at the instant the projectile leaves the barrel.

7. The projectile of the preceding problem is fired into a block of wood suspended by a long cord. If the mass of the block is 2 kg., to what height h above its initial level will the block be driven?

8. Calculate the velocity at the foot of the incline of Example 2 of (*a*) a solid cylinder, if it rolls without slipping, and (*b*) of a hoop if it also rolls without slipping. Which will arrive first at the base? Calculate the time of each if the distance s is 5 ft. and h is 2 ft.

CHAPTER VII

FLUIDS

If we look about us and attempt to classify all the materials with which we have daily contact, the most natural groups into which to divide them are the familiar ones of solids, liquids, and gases. The last two groups are distinguished from solids by the ease with which they accommodate themselves to the shape of any containing vessel. They have very similar properties, and it is frequently convenient to group them together under the common name of *fluids*. The facility with which they fit themselves to the shape of a container is due to the ability of one section of a fluid to slide past an adjacent section. Place a solid body on an inclined plane, and if the slope of the plane is not too steep, it will be held in position by friction. If a fluid is poured on an incline, however, it flows down to the base.

29. Viscosity. The Perfect Fluid. This behavior of a fluid is not due to an absence of friction, for a careful study would show that the fluid immediately in contact with the solid surface adheres to it quite closely. The flow is due to a sliding of the upper layers of the fluid past the lower layers. Even here there is a phenomenon, *viscosity*, which is closely analogous to friction. If a notably viscous fluid such as a heavy lubricating oil or molasses is poured on the incline, it progresses downward very slowly, obviously retarded by something in the nature of frictional forces between its various layers. The peculiarity of a fluid, however, is that, given time, it always will reach the lowest possible level. These viscous forces may retard, but unlike the friction which one solid exerts on another, they can never prevent altogether the downward motion.

It has already been pointed out that the frictional force between solids depends to a first approximation only on the nature of the surfaces in contact and the normal reaction between them. It does not, for example, depend on the velocity with which one body moves past another. A solid body, which starts of its own accord to slide down an incline, proceeds on its way with a constant acceleration. Friction balances a part of the down-

ward force of gravity, but the resultant force stays nearly constant as the speed of the body increases. If a viscous fluid is poured on the slope, the situation is quite different. It very quickly attains a certain maximum velocity, and then continues to flow on with a constant speed. The only explanation is that the retarding viscous forces increase with the velocity of shear of one layer past another until a speed is reached at which they just balance the force of gravity. When that speed is attained, every part of the fluid is in equilibrium, and moves with a uniform velocity. In just the same way a raindrop, retarded in its fall by the viscous forces of the atmosphere, soon acquires a maximum velocity, and then proceeds uniformly on its way.

The fact that a fluid, regardless of the magnitude of its viscosity, will always reach eventually the lowest possible level, is proof that these viscous forces not merely get smaller and smaller as the speed of shear decreases, but actually vanish at zero velocity. A *perfect fluid*, on the other hand, is one for which the viscous forces are always zero. It is, of course, an ideal material which does not exist in practice, but a large number of fluids are near enough to perfect fluids so that many of their properties can be very well predicted without taking account of viscosity, and even the most viscous fluids can be so treated when they are at rest.

30. Fluid Pressure. Consider, for example, a tank filled with fluid of density d, i.e., one such that each cubic centimeter has a mass of d grams. C_1 and C_2 (Fig. 26) represent two little cylinders attached to the walls of the tank at heights h_1 and h_2 above the base respectively. In each cylinder there fits a piston of area A. The *pressure* which a fluid exerts on any surface in contact with it is the force which it exerts per unit area. Thus if p_1 is the pressure on the piston in C_1 and p_2 is that on the piston in C_2, the corresponding forces must be p_1A and p_2A respectively. Now imagine the piston in C_1 pressed in from the outside very slowly, while at the same time and at the same rate the piston in C_2 is withdrawn. Let l be the distance each piston is moved. Since each has moved exactly the same amount, the level of the fluid in the tank has been unaffected, and the only change which

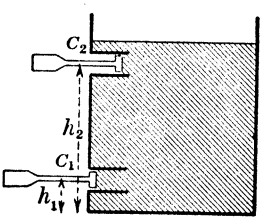


FIG. 26

has occurred has been the disappearance of an amount of fluid from C_1 and the appearance of exactly the same amount in C_2. If the pistons are moved very slowly indeed, the work done against viscous forces can be made negligibly small, and the work done by the operator in pushing in the piston of C_1 has been spent in only two ways: (1) in pushing out the piston of C_2 and (2) in lifting a volume of water equal to that displaced by the piston of C_1 from the level h_1 to the level h_2. If the pressure p_1 is expressed in dynes per square centimeters, the work done by the operator is

$$Fl = p_1 Al \text{ ergs}$$

That accomplished in pushing out the piston of C_2 is similarly $p_2 Al$ ergs. The volume displaced by the piston is Al cubic centimeters, its mass is Ald grams, and the work done in raising it a height $(h_2 - h_1)$ is $Ald(h_2 - h_1)$ gram-centimeters or $Aldg(h_2 - h_1)$ ergs. Hence, if the work done by the operator is equated to that accomplished,

$$p_1 Al = p_2 Al + Aldg(h_2 - h_1)$$

Dividing through by Al,

$$p_1 - p_2 = dg(h_2 - h_1) \tag{32}$$

The difference in pressure at any two points in the fluid is equal to dg times the difference in levels. Another, and perhaps more illuminating, way of expressing this result is to say that the pressure at the lower level is greater than that at the upper level by the weight of a column of the fluid of unit cross section and of height $(h_2 - h_1)$.

It should be noted that the orientation of neither cylinder enters into the proof, and we must conclude that at any one point of the fluid the pressure is the same in all directions. It is because of the simplicity of these two laws that the concept of pressure plays such an important rôle in all considerations dealing with fluids.

At any given level the pressure is the same throughout the fluid. This ability of a fluid to transmit undiminished a pressure from one point to another is frequently used in the construction of hydraulic machinery. Thus heavy bodies can be raised simply by using the very moderate pressure obtainable from our city water mains. The water enters the cylinder C (Fig. 27) by a

small pipe, in which is the valve V. In the cylinder is a large piston P of area A. When the valve is opened, the pressure p from the water mains is transmitted to all parts of the cylinder, and the total force which the water exerts against the piston is pA. If the pressure p is 30 lb./in.^{2}, a cylinder of area $\frac{2000}{3}$ sq. in. or $66\frac{2}{3}$ in.^{2} will raise a load of a ton.

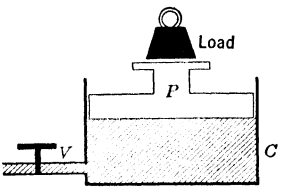


FIG. 27. Hydraulic Lift.

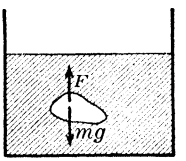


FIG. 28. Archimedes' Principle.

31. Archimedes' Principle. Another law of considerable importance was discovered by Archimedes, and is called in his honor Archimedes' Principle. It states that the lifting effect of a fluid on a body immersed in it is equal to the weight of the fluid displaced. A simple proof can be given here also by the principle of work and energy. Let d be the density of the fluid and m the mass of the body immersed in it (Fig. 28). Now imagine the body raised a distance h vertically upward. If F is the lifting force exerted on it by the fluid, the net downward force against which the work is done is $mg - F$, and the work required to raise the body through a distance h is $(mg - F)h$. Meanwhile, however, the space originally occupied by the body has been filled by the fluid, while the body has displaced an exactly equal amount in its present location. It is to all intents and purposes as though that quantity of fluid has been lowered through a distance h. Let W be the weight of the displaced fluid and V its volume, i.e., the volume of the immersed body. Then $W = dV$, and its loss in potential energy is dVh gram-centimeters or $dVhg$ ergs. The increase in potential energy of the body itself is mgh ergs. The work done in raising the body must be equal to the net increase in potential energy. Hence

$$(mg - F)h = mgh - dVgh$$

Or

$$F = dVg \text{ dynes}$$

Hence the lifting force is equal to the weight of the displaced fluid.

1596

This law furnishes one of the simplest means of measuring the density of a solid or a liquid. If the solid is weighed in air and then immersed in a fluid of density d, its loss in weight must be dVg dynes or dV grams. The gram was originally chosen to be the mass of 1 cc. of water at its maximum density. At room temperature this is so nearly the case that to a very good approximation its density in the metric system is 1 gr./cc. Hence the loss in weight of the specimen on immersing it in water is equal to the volume. Dividing the mass of the body by its volume found in this way, we get its density. In a similar way the density of an unknown fluid can be found. Any body, immersed successively in this fluid and in water, will lose in weight dV and V grams respectively. The ratio of its loss in weight in the unknown fluid to that which it loses in water is, accordingly, the required density.

32. Atmospheric Pressure. The Barometer. As already pointed out, the considerations of this chapter apply not merely to liquids but also to gases. The Earth's atmosphere exerts at sea level a uniform pressure in all directions, which, although depending somewhat on weather conditions, averages about 1033.3 gr./cm.^{2} or 14.7 lb./in.^{2} This was first proven by Torricelli, a pupil of Galileo, who filled a glass tube with mercury (Fig. 29) and inverted it in a vessel also filled with mercury. The experiment is easily performed by filling the tube to the brim and placing a finger tightly over the opening. After inverting the tube and dipping its end into the larger vessel with the finger still in place, the finger can be carefully withdrawn, thus permitting the mercury in the tube to come into contact with that in the dish. The level of the mercury in the tube will now drop until it is approximately 76 cm. above that of the mercury in the open dish. Since there is now one continuous fluid in tube and dish, we can apply the law that the pressure at one point of a fluid differs from that at any other point h centimeters higher by the weight of a column of fluid of unit cross section and of height h. This law indicates that the pressure on the upper surface of the mercury in the open vessel, i.e., the atmospheric pressure, is greater than that of the vacuum above the mercury column in the tube by the weight of a column of mercury

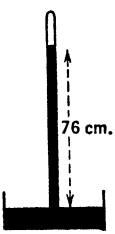


FIG. 29. Barometer.

of unit cross section and about 76 cm. high. Perhaps the commonest way in which such a pressure is expressed is to say that it is equal to that of a column of mercury 76 cm. high, or simply that it is a pressure of 76 cm. of mercury. If, however, the pressure is required in any other units, it can be readily calculated. Thus, since the density of mercury is 13.6 gr./cm.^{2}, the weight of a unit column of mercury h centimeters high is $13.6 \times h$ grams or $(13.6)(980) \times h$ dynes. A device such as this which is capable of measuring the pressure of the atmosphere is known as a *barometer*. It has proven of great value in the study of meteorological conditions and in estimating the elevation of various points of the Earth's surface above sea level. Many physical and chemical processes are very sensitive to changes in atmospheric pressure, and the barometer is, accordingly, an essential piece of laboratory apparatus.

In case it is desired to measure the pressure in a vessel which is closed to the outside atmosphere, either one of two types of

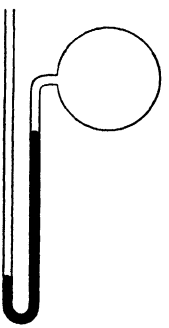


FIG. 30*a*. Open Manometer.

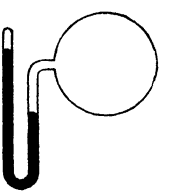


FIG. 30*b*. Closed Manometer.

manometer (Figs. 30*a* and 30*b*) may be used. In the open manometer (Fig. 30*a*) one end of a U-shaped tube partially filled with mercury is open to the outside atmosphere, and the other end is open to the vessel in which the pressure is desired. The difference in level between the mercury in the two arms of the tube measures the difference between the pressure in the vessel and atmospheric pressure. In order to determine the absolute pressure, an auxiliary barometer of ordinary type must be used to measure the pressure of the atmosphere itself.

The closed manometer (Fig. 30*b*) is a U-tube of which one end is sealed while the other is open to the vessel being studied.

If such a manometer has been properly constructed so that there is no air above the mercury in the closed column, the difference in level of the mercury in the two columns will measure precisely the pressure in the vessel.

33. Fluids in Motion. Hydrodynamics. Fig. 31 represents a vertical pipe through which a fluid of density d is being forced upward. Consider, in particular, a small volume of the fluid contained between two planes which are perpendicular to the direction of flow and a distance Δh apart. As it moves upward through the pipe, work must be done against the downward pressure on the top surface of the little volume, and work will at the same time be done on it by the pressure from below. Let p_1 be the pressure at the lower level, marked h_1 in the figure, and p_2 that at the upper level, which is marked h_2. The upper surface of the volume under consideration will move under ever decreasing pressure from a_1 to a_2, while the lower surface moves from b_1 to b_2. Both surfaces will be affected successively by the entire range of pressures between a_1 and b_2. Hence the work done on the upper surface of the little volume by the pressures within this interval will be exactly compensated by that which is done by the same pressures on the lower surface. The upper surface, however, travels a distance Δh further than the lower surface into the region where the pressure p_2 prevails, and if we may neglect the compressibility of the fluid, the lower surface travels the same distance Δh further under the action of the pressure p_1. If we denote the area of the cross section of the pipe by A, the force exerted by the pressure p_1 on the lower surface of the little volume is p_1A, and the work it does while this surface is moving from b_1 to a_1 is $p_1A\ \Delta h$. Similarly the work done by the pressure p_2 on the top surface while it is moving from b_2 to a_2 is $p_2A\ \Delta h$. $A\ \Delta h$ is the volume of fluid under consideration, and may be denoted by ΔV. Hence the net work, W, done on it during the entire journey will be

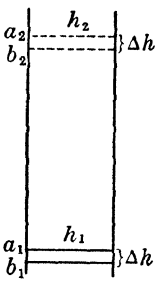


FIG. 31

$$W = \Delta V(p_1 - p_2)$$

Since this work is done on the volume ΔV, the work done on unit volume (1 cc. for example) would be this total work divided by ΔV or $p_1 - p_2$.

Every unit volume of fluid, in passing from a region where the pressure is p_1 into a region where the pressure is p_2, has an amount of work $p_1 - p_2$ done on it, and we may think of it as possessing everywhere an amount of potential energy equal to its pressure. $p_1 - p_2$ represents, accordingly, a quantity of *pressure potential energy*, which has in the present instance been converted into gravitational potential energy. This is true since the gravitational potential energy of the unit volume at height h_1 was dgh_1, and at h_2 it was dgh_2. The gain in gravitational potential energy was, therefore, $dgh_2 - dgh_1$. But by (32) this is precisely the decrease in pressure potential energy $p_1 - p_2$. We should, indeed, have expected this result, since we know that every particle of fluid in the pipe is in equilibrium, and it should, therefore, be possible to move any quantity of it from h_1 to h_2 without the expenditure of any energy. One form of potential energy is merely converted into another.

Although the concept of pressure potential energy is not of great importance in hydrostatics, it becomes of the greatest significance when the motion of fluids is considered. Each unit volume of fluid of density d at height h will have a gravitational potential energy dgh, a pressure potential energy p, and in general also a kinetic energy $dv^2/2$, if v is its velocity. Its total energy will, therefore, be

$$E = dgh + p + \frac{dv^2}{2} \tag{33}$$

If it is a perfect fluid so that no work is done by friction on the unit volume as it moves onward with the current, the conservation of energy requires that no change shall take place in this total energy. If the speed of flow increases, it must do so at the expense of some of the potential energy which is represented by the first two terms, while, if, on the other hand, the fluid passes into a region of increased pressure, the greater potential energy must be derived either from the gravitational or from the kinetic energy term. The essential facts represented by this law were discovered by Daniel Bernoulli about 1738, and it is known as *Bernoulli's Theorem*. Two examples will be given of its application.

If a fluid is allowed to gush from a hole near the base of a tank, it was discovered by Torricelli, a pupil of Galileo, that its velocity of efflux is equal to $\sqrt{2gh}$, where h is the height of the top surface

of the fluid above the aperture. This will be recognized as precisely the velocity which a body acquires in falling freely through a distance h. The pressure of the fluid both at the aperture and at its upper surface is that of the external atmosphere. Neglecting the weight of a column of air of height h, the pressure, and therefore the pressure potential energy, is the same at the two points. The decrease in gravitational potential energy of each cubic centimeter of fluid is dgh, and this must be equal to the kinetic energy which it has acquired.

$$dgh = \frac{dv^2}{2}$$

That is,

$$v = \sqrt{2gh}$$

The second example deals with the flow of a fluid through a pipe containing a constriction (Fig. 32). The total quantity of fluid which passes through the constriction per second must

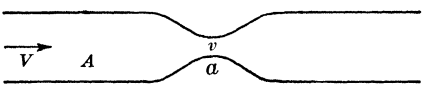


FIG. 32

be the same as that which passes through the open part of the pipe. Let the cross section of this wider portion be A, that of the constriction be a, and let V and v be the corresponding velocities of flow. The volume flowing past a given point in the open part of the tube per second will be VA, while that flowing through the constriction in the same time is va. Hence

$$VA = va$$

Or

$$v = V\frac{A}{a} \tag{34}$$

If the pipe is horizontal as indicated in the figure, no changes occur in the gravitational potential energy, and the gain in velocity in the constriction must take place at the expense of loss of pressure. We are accordingly forced to conclude that the pressure in the constriction is less than in the open part of the tube. If we denote the pressure in the constriction by p and that outside by P, the conservation of energy requires that

$$P + \frac{dV^2}{2} = p + \frac{dv^2}{2}$$

Or,

$$P - p = \frac{d}{2}(v^2 - V^2)$$

Substituting the value of v in terms of V from (34),

$$P - p = \frac{dV^2}{2}\left(\frac{A^2}{a^2} - 1\right) \tag{35}$$

These pressures can be readily measured by attaching manometer tubes to the pipe at and on either side of the constriction. From a reading of these manometers and a knowledge of the ratio of A to a, the velocity of flow of the fluid can be calculated. This method is commonly used in the Venturi meter for measuring the rate of flow of water through pipes, and it is also used to measure the velocity of airplanes with reference to the surrounding air.

PROBLEMS

1. If the density of sea water is 1.025 gr./cm.^{3}, or 64 lb./ft.^{3} what is the pressure at a depth of 1 mile below the surface of the ocean (*a*) in pounds per square inch, (*b*) in centimeters of mercury, and (*c*) in atmospheres?

2. The density of the air at a temperature of 21° Centigrade and at a pressure of 76 cm. of mercury is 0.0012 gr./cm.^{3} Under these conditions how much lower does the barometer stand on the top of a building 100 meters high than at the base? Neglect the reduction in density of the air with altitude.

3. A hydraulic press consists of a fluid chamber containing two pistons, one large one of diameter D for raising the load, and a small one of diameter d for exerting the force. What is the mechanical advantage of the device? If D is 12 in. and d is $\frac{1}{2}$ in., what force will be required to raise a load of 1 ton?

4. What is the density of a body which weighs 120 gr. in air and only 100 gr. when immersed in water? If the same specimen weighs 105 gr. when immersed in an unknown fluid, what is the density of the fluid?

5. What is the lifting effect of the atmosphere under the conditions of Problem 2 on an airship which contains 30,000 cubic meters of hydrogen? The density of hydrogen is approximately $\frac{1}{15}$ that of air.

6. How many kilograms of water will be discharged per minute from a hole 10 cm.^{2} in cross section in the base of a dam if it is located 30 meters below the upper surface of the water? Assume that the water is a perfect fluid.

7. If a constriction in a pipe through which water is flowing with a velocity of 2 meters/sec. (in the constriction) has a cross section only $\frac{1}{10}$ that of the open portion of the pipe, what will be the difference in pressure between the water in the constriction and that outside? Express the answer in centimeters of mercury.

CHAPTER VIII

HEAT

In the chapter dealing with machines it was pointed out that the work accomplished was equal to that done by the operator only in the absence of friction between the parts of the machine. If friction is present, as it always is in practice, energy is apparently lost in operating a machine, a fact which seems to be in contradiction to the Law of the Conservation of Energy. It has been known for many centuries, however, that, when one body is rubbed vigorously on another, a noticeable heating of both bodies takes place. Indeed, it is a universal fact that when work is done against friction, heat appears, and it would seem a very natural thing from our present standpoint to conclude that heat is a new form of energy into which mechanical energy has been converted. Substantially this opinion was held by Robert Boyle, a contemporary of Newton, as a result of his observation that heat always appeared when mechanical motion (kinetic energy) disappeared. Being unfamiliar with our present concept of energy, he could merely state that heat seemed to him to be a form of " motion."

Opposed to this viewpoint, however, was the belief almost universally held by the scientists of the eighteenth century that heat was a real substance. The reasons for such a belief are not far to seek. If a hot body is brought into contact with a cold body, it becomes gradually cooler, while the cold body becomes warmer, a process which continues until neither one is hotter than the other. It is just as though some material fluid passed from the hotter to the colder body. To be sure, no increase in mass is observed as a body is heated, but this may only mean that the heat material has too low a density to be detected by our rather crude experimental methods. Aside from this puzzling failure of a heated body to gain in weight, nearly all the known facts about heat can be very well handled from this standpoint, and as late as 1856 the article on heat in the Encyclopædia Britannica treated it as a material substance.

34. Temperature. Before any quantitative study of heat could be made, it was necessary to develop some method of measuring the hotness of a body, i.e., its *temperature.* Most ordinary thermometers depend for their action on the fact that in general substances expand when heated. The dimensions of a body can, therefore, be used to measure temperature, when once a proper basis has been agreed on. To Galileo belongs the credit of having constructed the first thermometer along these lines. It consisted of a glass bulb, to which was attached a long glass tube. Both tube and bulb were filled with air, and the open end of the tube was dipped into a vessel of water (Fig. 33). At the outset the bulb was slightly warmed so that, on cooling, the reduced pressure of the air would draw some of the water up into the tube. The elevation of the top surface of the water in the tube measured the volume of the air, and, therefore, indirectly its temperature. The disadvantage of this device lay in the fact that the position of the upper level of the water in the tube depended, not merely on the temperature, but also on the barometric pressure at the surface of the water in the vessel. The development of the thermometer from the time of Galileo was, however, rapid. Instead of air, alcohol or mercury was used as the material which, by its expansion, measured the temperature, and the open end of the tube was sealed to prevent evaporation or contamination of the expanding substance.

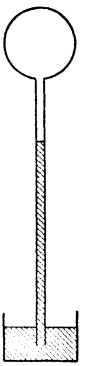


FIG. 33. Galileo's Thermometer.

The choice of a suitable scale required more time, and even at the present day there are in common use the Centigrade or Celsius, the Fahrenheit, and the Réaumur scales. Of these the Centigrade scale is almost universally used for scientific purposes, and will be employed exclusively in the present book. It makes use of the fact that pure water always boils or freezes under a standard barometric pressure at exactly the same temperature. 76 cm. of mercury, approximating as it does to average conditions at sea level, is chosen as the standard pressure, and the zero of the Centigrade scale is chosen as the freezing point of water at this pressure. Similarly the 100° temperature of the Centigrade scale is chosen as the boiling point of pure water under the same pressure conditions. If a thermometer tube of uniform bore is

used, the intermediate temperatures can be defined by dividing the space on this tube between the 0° and the 100° marks into 100 equal parts. Such a scale depends, however, to a considerable extent on the nature of the expanding material used. An extreme example of a very bad substance to use for this purpose would be water. It is a familiar fact that water expands on freezing, a circumstance which is responsible for the breakage of glass containers when ice is formed in them. It is not so commonly realized that this expansion starts at a temperature of 4° above 0° C. At this point water has, accordingly, a minimum volume, and if it were used in a thermometer tube, a position of its upper surface slightly above this minimum point might indicate a temperature either above or below 4° C.

Partly because of the limited range within which liquids such as alcohol or mercury can be used, between their freezing and boiling points, and partly for reasons which will appear in a later chapter, it has been customary to use a gas as the standard substance for defining these intermediate divisions. A large number of gases yield substantially the same result when used for this purpose, and serve to define very satisfactorily the size of the degree of temperature. The fact that two bodies in contact with each other, by transfer of heat from the hotter to the colder, eventually reach the same temperature, makes it possible to use a thermometer to measure the temperature of any material with which it can be brought into intimate contact.

35. Quantity of Heat. Calorimetry. Whether heat is viewed as a form of energy or as a material substance, it should be possible to measure the amount which is transferred from one body to another by the effect which it has on either body. To this end, water is chosen as the standard substance, and the common unit in which heat is measured, the *gram-calorie*, is defined as the quantity of heat which is required to raise the temperature of 1 gr. of water 1° C. Experiment shows that this quantity depends somewhat on the temperature of the water, and for higher precision it is usually defined as the heat required to raise the temperature of 1 gr. of water from 15° to 16° C. For any except the most precise work, however, particularly if the temperature range in a given experiment does not extend too far above or below room temperature, the simpler definition is adequate. The corresponding unit in the English system of measurement, the British Thermal Unit or B.T.U., is defined as the heat required

to raise the temperature of 1 lb. of water 1° F. One B.T.U. is equivalent to 252 gram-calories.

36. Specific Heat. The specific heat of any substance is defined as the number of calories (or the fraction of a calorie) which are required to raise 1 gr. of the material in question 1° C. Experience soon shows that practically all other substances require less heat for this purpose than water, and the specific heats of most materials are substantially less than unity. In the accompanying table the specific heats of a few substances are given

SPECIFIC HEATS

Aluminium	0.217	Ice	0.53
Copper	0.0912	Ethyl alcohol	0.548
Iron	0.113	Petroleum	0.511
Lead	0.0305	Air	0.2374
Silver	0.0562	Hydrogen (gas)	3.409
Glass (crown)	0.161	Hydrogen (liquid)	6.0

Once the specific heats of the various common materials with which we are likely to deal have been measured, it is easily shown that heat, like energy, is always conserved. For experiments of the sort to be described, a copper vessel or *calorimeter* is ordinarily used, as copper accommodates itself more rapidly to the temperature of the material with which it is in contact than most other substances. With such a copper container, it can always be assumed that the vessel shares the temperature of the enclosed substance.

Let us suppose that such a calorimeter is filled with water, and that into this water is transferred some solid material such as a lead cylinder, which is initially at a higher temperature than that of the water. The lead will be cooled by contact with the water, while the calorimeter and water will be correspondingly raised in temperature.

Let m_1 = the mass of the calorimeter
m_2 = the mass of the water
m_3 = the mass of the lead cylinder
s_1 = the specific heat of copper
s_2 = the specific heat of lead
t_1 = the initial temperature of water and calorimeter
t_2 = the initial temperature of the lead cylinder
t_3 = the final temperature of all.

If no heat is gained or lost in the process, the heat given up by the lead specimen must equal that received by the water and the calorimeter. Or

$$m_3 s_2(t_2 - t_3) = (m_1 s_1 + m_2)(t_3 - t_1)$$

On the left-hand side of this equation s_2 is the fraction of a calorie which 1 gr. of lead gives up when it is lowered 1° in temperature. $m_3 s_2$ is, therefore, the number of calories given up by the m_3 grams of lead per degree, and $m_3 s_2(t_2 - t_3)$ is the total number of calories given up during the entire drop from the initial to the final temperature. In the same way on the right-hand side $(m_1 s_1 + m_2)$ is the number of calories received by the water and the calorimeter when their temperature is raised 1°, while $(m_1 s_1 + m_2)(t_3 - t_1)$ is the total heat received during the entire rise from t_1 to t_3.

Such experiments as this can be used either to measure the specific heat of a substance, or, if the specific heats of all the materials involved are known, to test the completeness of the conservation of heat. The result of such tests leaves no doubt that the heat given over to a body to raise its temperature can always be recovered *in toto* when the temperature of the body is lowered to its original value, or that it can be handed on from one body to another by successive processes without loss at any point.

37. Conversion of Mechanical Energy into Heat. These experiments which deal with the transfer of heat from one body to another can, however, be explained with equal facility either by the hypothesis that heat is energy or that heat is a material. Only when we turn to a study of the heat which appears when mechanical energy disappears, do we find substantial evidence in favor of the energy standpoint. It is to Count Rumford that we owe the first striking experiment of this sort. In 1798 while engaged in boring cannon at Munich he was surprised by the great amount of heat developed in the process. Using a blunt borer and surrounding the cannon by a wooden box containing water, he was actually able to raise the water to the boiling point.

It was James Prescott Joule, however, about the middle of the nineteenth century, who, by a brilliant series of quantitative experiments, first succeeded in persuading the entire scientific world of the correctness of the energy viewpoint. These experiments were all concerned with a quantitative comparison of

the heat generated by frictional processes and the mechanical energy which was lost. He succeeded in showing that the heat which appears is always strictly proportional to the mechanical energy which has disappeared, and that, regardless of the nature of the process involved, exactly the same amount of heat is always gained from a given amount of mechanical energy.

The most successful of the experiments which he performed to test this conclusion is illustrated in Fig. 34. A paddle wheel A was so mounted that it could be set in rotation in a calorimeter C by means of two weights W_1 and W_2, suspended by cords wrapped around the drum of the paddle wheel. The calorimeter was filled with water, and the fall of the weights was

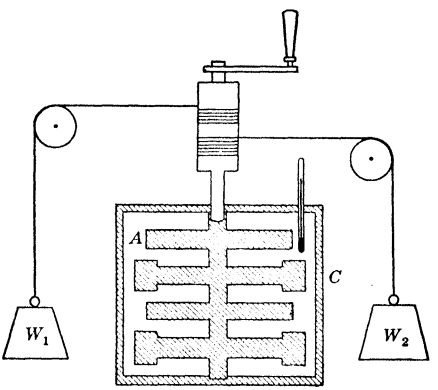


FIG. 34. The Conversion of Mechanical Energy into Heat.

retarded only by the viscous forces encountered by the paddle wheel in its rotation. The work done by gravity on W_1 and W_2 as they fell, i.e., the decrease in their potential energy, was thereby converted into heat in the water and calorimeter. In the experiment as performed by Joule, the weights were allowed to fall as far as possible, the cords were again wrapped up on the drum, and the process successively repeated until he had secured a convenient rise in temperature of the calorimeter. The results of his best measurements, corrected for slight differences between his thermometers and ours, indicate that 778 ft.-lb., when converted into heat, will produce 1 B.T.U. Stated in another way, the work done by gravity on a mass of 1 lb. falling through 778 ft., when converted into heat, is sufficient to raise the temperature of 1 lb. of water 1° F. Similarly in the absolute metric system

4.183×10^7 ergs (41,830,000 ergs) are equivalent to 1 gram-calorie.

In addition to the experiment described, Joule measured the heat obtained from mechanical work in various other ways. He rotated a small electromagnet in water between the poles of another magnet, thereby obtaining the heating effect indirectly through the electric currents induced in the coils of the magnet. He measured the heat which resulted from compressing air in a cylinder, and from forcing water through capillary tubes, in all cases obtaining substantially the same result for the equivalence of mechanical and heat energy.

PROBLEMS

1. If the density of air is 0.0012 gr./cm.^{3}, how many calories are required to raise the temperature of the air in a room 10 meters square and 3 meters high from 15° C. to 25° C.?

2. A quantity of lead shot weighing 100 gr. is heated to a temperature of 100° C., and is then transferred to a copper calorimeter. If the mass of the calorimeter is 100 gr., and if it contains 200 gr. of water at 20° C., to what final temperature will the mixture come?

3. The heat of combustion of coal averages about 14,000 B.T.U./lb. or 7800 cal./gr. How much coal if efficiently used will be required to raise the temperature of a cubic meter of water from 20° to boiling (100°)?

4. The brakes of an automobile weighing 3000 lb. and travelling 30 miles an hour are applied so gradually that the tires do not skid. How much heat is developed in the brake bands?

5. If the heat of combustion of gasoline is 20,250 B.T.U./lb., and if the automobile of the previous problem operates with an efficiency of 15 per cent, how much gasoline is wasted in coming to a stop from 30 miles an hour?

6. The height of the Horseshoe Falls at Niagara is 155 ft. How many British Thermal Units can be obtained from the drop of each cubic foot of water over these falls? If this energy were used to heat the water itself, through how many degrees Fahrenheit would its temperature be raised? The flow of water in the Niagara River averages about 200,000 ft.^{3}/sec. What is the maximum horse-power which could be derived from these falls?

CHAPTER IX

CHANGE OF STATE

38. Boyle's Law. It is a familiar fact that, when the volume of a gas is reduced, its pressure is correspondingly increased. The quantitative law which governs this behavior was first discovered by Robert Boyle, and states that the pressure remains throughout such a process inversely proportional to the volume. Thus if p_0 and v_0 are the initial pressure and volume of the gas respectively, and if p and v are its final pressure and volume,

$$\frac{p}{p_0} = \frac{v_0}{v}$$

Or

$$pv = p_0 v_0$$

The product of the pressure and the volume of a gas remains, therefore, constant throughout any compression or expansion. This relationship may be exhibited graphically by plotting the pressure p vertically and the volume v horizontally (Fig. 35). As the pressure is increased, the volume continually decreases, but in such a fashion that it never reaches zero. Accordingly, as the point which represents the pressure and volume of the gas in the graph rises higher and higher with increasing p, it constantly approaches the vertical p axis, but never quite reaches it. In a similar way, as v increases, the curve constantly approaches the v axis. This curve is known as a *hyperbola*, and the two axes which it ever approaches without reaching are called *assymptotes*.

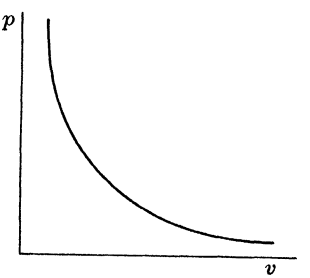


FIG. 35. Boyle's Law.

Boyle himself probably realized that his law was only an approximate one. In most cases the volume decreases a little too rapidly until very high pressures are reached, and if the observations are pushed to still more extreme pressures, it de-

creases less rapidly than would be anticipated from the simple law. The *ideal gas* is defined as one which would obey Boyle's Law throughout. No real gases are ideal, but the so-called permanent gases such as nitrogen, oxygen, hydrogen, and helium, exhibit such small discrepancies over a considerable pressure range, that Boyle's Law may be considered for them an excellent approximation.

39. Liquefaction of a Gas. Certain other gases even at very ordinary temperatures and pressures show few signs of accordance with Boyle's Law. The nature of these wide divergences were little understood until 1869 when Andrews carried out a classical investigation on carbon dioxide. The gas was compressed in a strong glass tube by a column of mercury, and the variation of pressure with volume was plotted for a number of temperatures between 13.1° and 48.1°. These curves, throughout each one of which the temperature was maintained constant, are called *isothermals*, and are shown in Fig. 36. At 48.1°, except for a slightly more rapid decrease in volume with pressure than is exhibited by the permanent gases, the trend of the curve is not substantially different from that to be anticipated from Boyle's Law. At 35.5° a kink has appeared at about 80 atmospheres pressure, which at 31.1° has developed into a horizontal point of inflection. Below this temperature the isothermal curves all have a horizontal section, along which the volume decreases uniformly without any increase in pressure. The explanation of this curious behavior was obvious from an examination of the glass tube containing the gas. Starting at the low-pressure end of an isothermal, and gradually increasing the pressure, Andrews noted that at just the instant the straight part of the curve was reached, liquid began to form in the bottom of the tube. As the volume was further reduced, more and more liquid appeared, until, at the sharp bend which occurs at the left-hand end of the straight section, the gas had completely disappeared. The rapid upward trend of the curve beyond this point corresponded

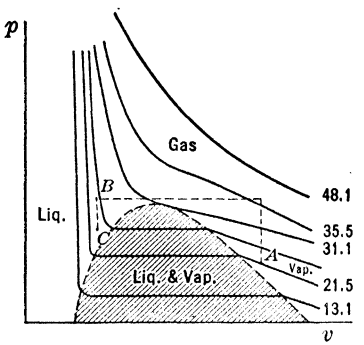


FIG. 36. Isothermals of Carbon Dioxide.

to the low compressibility of the liquid carbon dioxide. The pressure at which the transition from gas to liquid takes place, represented in the graph by the height of the horizontal section of the curve above the v axis, is known as the *saturated vapor pressure* of the substance. Its value depends on the temperature, increasing as the temperature is raised, until, at 31.3° where the straight portion of the curve vanishes, it is 73 atmospheres.

40. The Continuity of State. Critical Temperature. This temperature of 31.1° is known as the *critical temperature* of carbon dioxide. Below this point a simple increase in pressure will produce liquefaction, whereas above it no amount of pressure will convert gas into liquid. The shaded area in the figure, within which the process of condensation takes place, is known as the *steam dome.* A mass of carbon dioxide, the pressure and volume of which are such as to be represented by a point to the right of this region and below the critical 31.1° isothermal, is technically called a *vapor*, while a mass represented by a point to the left of the dome and below the same isothermal is called a liquid.

Although such a nomenclature is convenient, it does not represent any real distinction in the nature of the substances themselves. Indeed, the most striking fact brought out by the investigations of Andrews was that of the continuity of the liquid and the gaseous states. Start, for example, at A with a temperature of 21.5°, and instead of compressing the gas isothermally, maintain the volume constant, and raise the pressure along the dotted line to 80 atmospheres. This can, of course, be done by heating the gas in a closed vessel. Now, by cooling it at constant pressure, once more along the dotted line, the volume may be reduced to B, and again by cooling at constant volume, the point C may be reached. We have gained a point on the liquid side of the steam dome without at any stage cutting into it, and without, therefore, observing at any instant a mixture of liquid and vapor. In fact, the transition from gas to liquid has been a continuous one. All the properties of the gas — density, viscosity, and even refractive index — have changed continuously into the corresponding values in the liquid state. No point has been passed at which it could be said in more than a very arbitrary way that up to that point the substance was a gas and beyond it a liquid.

These facts, discovered by Andrews for carbon dioxide, are typical of gases in general. Since that time the so-called permanent gases have all been liquefied, despite the fact that the more refractory ones, hydrogen and helium, long resisted liquefaction because of their low critical temperatures.

41. Latent Heat of Vaporization. In Fig. 37 a mass of pure liquid is contained in the cylinder C beneath the piston P. If the piston is slowly withdrawn, the liquid will vaporize at a rate which will maintain the vapor at the pressure characteristic of the substance at the temperature in question. Since the vapor exerts a force on the piston as it moves outward, it must do work on it, and if the Law of the Conservation of Energy is to hold, this work must be done at the expense of heat in the liquid. The liquid must be expected to cool unless heat is supplied to it from outside the cylinder. The heat which must be supplied to a liquid during the process of vaporization to maintain its temperature constant is, however, much in excess of that necessary to account for the work which the vapor does in expanding. The total number of calories required to vaporize 1 gr. of the liquid at constant temperature is known as its *latent heat of vaporization*, or simply its heat of vaporization. At the critical temperature, at the peak of the steam dome, there is no physical distinction between vapor and liquid, and the heat required to transform one into the other must be zero. As the temperature is dropped below the critical point, the heat of vaporization constantly increases. In stating the heat of vaporization of a fluid, it is, therefore, necessary to specify the temperature or pressure to which the value applies. Since most vaporizations are carried out at atmospheric pressure, latent heats are most commonly measured and stated for this pressure. The table on page 73 gives the critical temperature and the heat of vaporization of a few of the commoner fluids.

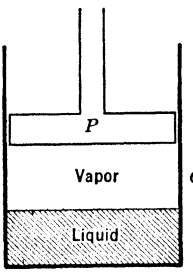


FIG. 37

If the Law of the Conservation of Energy is to hold, the heat which is required to vaporize a given mass of liquid must be recoverable *in toto* when the vapor is again condensed, and this is always found to be the case. If L is the latent heat of vaporization of a substance, it is also the number of calories which are liberated when each gram of vapor is condensed into liquid.

	Critical Temperature	Heat of Vaporization
Water	365 °C.	538.7 cal./gr.
Carbon dioxide	31.1	57.48*
Sulphur dioxide	155.4	95
Oxygen	−118	50.97
Nitrogen	−146	47.65
Hydrogen	−234.5	
Helium	−268	
Ammonia	130	310*

As the temperature of a liquid open to the atmosphere is raised to such a point that its vapor pressure is equal to that of the surrounding atmosphere, boiling takes place. Bubbles of vapor form in its interior, and, unhindered by a superior pressure from the outside, expand and force their way to the surface. The temperature which corresponds to a particular vapor pressure is, therefore, the boiling point of the liquid when subjected to that same outside pressure. The line OA in Fig. 38 shows the way in which the vapor pressure p of water varies with the temperature t. It may also be interpreted as showing the way in which the boiling point of water t varies with the external pressure p.

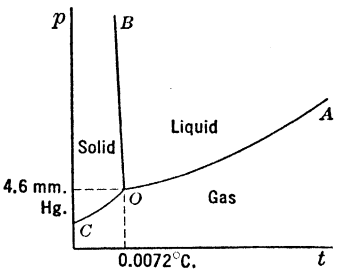


Fig. 38. Triple Point Diagram of Water.

42. Fusion of a Solid. In addition to the liquid and vapor states, most substances are capable of existing in a solid state. The temperature at which the transition from liquid to solid or *vice versa* takes place is known as the freezing point or the fusion point. Like the boiling point, it depends on the external pressure, and the curve OB in Fig. 38 exhibits graphically the dependence of the freezing point of water on the pressure. Just as in the case of vaporization, the transformation of a solid into a liquid is accompanied by an absorption of heat, while the converse process involves an evolution of heat. The number of calories required to melt 1 gr. of a solid substance is known as the *latent heat of fusion* of the material in question. The following table gives the heats of fusion of a few solids.

* The starred values of the heat of vaporization are for 0° C. The remaining values are for the temperature of normal boiling at atmospheric pressure.

HEATS OF FUSION

	Melting Point	Heat of Fusion
Ice	0° C.	79.24 cal./gr.
Iron, gray cast		23
Lead	327	5.86
Tin	232	14.0
Copper	1083	42.0
Platinum	1755	27.2
Paraffin	52.4	35.1
Sulphur	115	9.37
Mercury	−39	2.82

The line *OC* of Fig. 38 represents an exactly similar transition between solid and gas. This process, known as *sublimation*, is much less familiar to us than the corresponding processes of melting and vaporization. It accounts for the disappearance of ice and snow in the winter without the intermediate stage of melting; the formation of hoar frost is simply an example of the inverse phenomenon. It is for this reason that the line *OC* is usually referred to as the *frost line.* The three lines *OA*, *OB*, and *OC*, divide the entire plane into three sections representing solid, liquid, and gas respectively.

The point at which these three curves meet is known as the *triple point.* The only pressure and temperature at which all three states of a pure substance can exist side by side in equilibrium is the pressure and temperature of this triple point. If, for example, a specimen of ice is placed in a vessel of warm water, it will melt, thereby extracting from the water the necessary heat of fusion. This process will continue until the temperature of the water has been reduced to that of a point on the melting-point curve opposite the external pressure which prevails above the surface of the water. If this experiment is performed in an open vessel, the pressure in question is that of the atmosphere. If, however, it is performed under a bell jar, from which the air has been extracted, the pressure is due entirely to the water vapor, and must be the saturated vapor pressure appropriate to the temperature of the water. The point, therefore, which represents this pressure and temperature, must lie on the vapor-pressure curve as well as on the freezing-point curve, and must accordingly be the triple point.

This experiment can be performed in a very striking way by placing a small amount of water in a watch glass under a bell jar,

and extracting the air by a suction pump. Owing to the cooling which results from the necessary evaporation of the water, it will reach a point on the vapor-pressure curve at a somewhat lower temperature than that initially possessed by the water. If, furthermore, the water vapor is rapidly removed by the suction pump, or better with the help of concentrated sulphuric acid in a vessel beneath the watch glass, the evaporation and consequent cooling will continue until the temperature of the triple point is reached. The additional heat required for further vaporization can now be furnished only by the solidification of a portion of the water and the consequent release of its heat of fusion. The water will be observed to boil and freeze simultaneously.

If a liquid is brought temporarily into a region in which the temperature is lower than its freezing point, it will gradually give up its heat and constantly drop in temperature until this freezing point is reached. Still giving up its heat, its temperature will now remain constant until each gram of the liquid has lost a number of calories equal to its heat of fusion. It will then become solid. In Fig. 39 is shown a typical cooling curve, in which the temperature of the material is plotted against the time. The horizontal portion of the curve represents the process of transformation during which the heat of fusion is leaving the liquid.

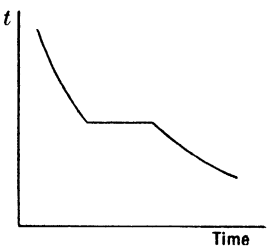


FIG. 39. Cooling Curve.

The existence of such latent heats is of the utmost importance in the economy of nature. They serve to retard in a very salutary way both the processes of freezing and vaporization. Were it not for the latent heat of fusion, bodies of water and all plant life would solidify the instant the freezing point was reached, and such sudden transformations would result in extraordinary hardship both to agriculture and industry. Similarly the sudden change of water into vapor at the boiling point would make well-nigh impossible the usual cooking processes.

PROBLEMS

1. How many calories are required to convert 100 gr. of ice at $-10°$ C. into steam at 100° C.?

2. Twenty grams of steam are passed into a copper calorimeter containing

1000 gr. of water at 25° C. If the mass of the calorimeter is 200 gr., to what final temperature will the mixture come?

3. A mass of copper weighing 100 gr. is heated to a temperature of 100° C. and is then placed on a block of ice at 0° C. How much ice will be melted?

4. The cooling effect in most small electric refrigerators is produced by the evaporation of liquid sulphur dioxide. How many grams of sulphur dioxide must be liquefied by the compressor and then evaporated to freeze a kilogram of ice cubes? Assume an initial temperature of the water to be frozen of 20° C.

5. The energy of a lead bullet in striking a steel target is largely converted into heat. If all the heat produced were used in warming the bullet, what must have been its velocity in order to be melted by the impact? Assume a temperature of 100° C. on striking the target.

6. From what altitude must a block of ice have fallen (in a vacuum) in order to be melted on striking the ground? Assume all its energy to be used in the melting process.

CHAPTER X

TRANSFORMATION OF HEAT INTO MECHANICAL ENERGY

Since heat is a form of energy into which mechanical energy can be converted *in toto*, simply by the agency of friction, it would seem natural that the inverse process, the transformation of heat into mechanical energy, should be possible. Nevertheless no such universal agent as friction has ever been discovered, or is ever likely to be discovered, for this inverse process. Man has, indeed, learned to get mechanical energy from heat, and thus from the fuels which produce heat, but only by the long and arduous procedure of experiment and invention.

43. The Steam Engine. The steam engine, the first practical device for this purpose, was developed during the eighteenth century. The atmospheric steam engine of Newcomen, built in Dartmouth, England, in 1705 was the first economical and convenient mechanism of this sort. It consisted of a piston moving in a cylinder, and connected by an overhead beam to the mechanism to be driven. Steam was led into the cylinder by a hand-operated cock, and held the piston up against the external pressure of the atmosphere. The cock was now closed, and the steam condensed by pouring cold water over the outer surface of the cylinder. The atmosphere then pushed the piston in, and the process was repeated. It is said that a boy, who was left to open and close the cock periodically, devised a system of catches and strings to do it mechanically, and thus showed the way for the present system of valve operation. A flywheel was added to the mechanism to produce steadier operation in 1736. Toward the latter part of the century James Watt tremendously increased its efficiency by introducing a separate condenser, thus dispensing with the cooling of the entire cylinder at each stroke.

Although the device itself was fairly well perfected, little progress had been made toward an understanding of the fundamental physical laws which underlay its operation until Sadi Carnot, the French engineer, in 1824 first grasped these basic facts. Much earlier than this the steam engine had undoubtedly

been recognized as a very inefficient device. At each stroke of the piston the steam in the cylinder had to be led out into a condenser, where it was liquefied; and no useful purpose was served by the heat of vaporization thus liberated. It was simply passed on to the water jacket and lost. All attempts to avoid this waste having failed, it remained for Carnot to point out the cause of this failure in a hitherto unsuspected law of Nature.

44. The Carnot Engine. The study of the conversion of heat into mechanical energy is known as *thermodynamics*. The Law of the Conservation of Energy is commonly known as the *First Law of Thermodynamics*, and the new law discovered by Carnot is called the *Second Law of Thermodynamics*. Its nature and operation can best be studied by turning our attention to a simple form of heat engine devised by Carnot and still known as a Carnot engine.

Imagine a cylinder in which a piston operates, filled for the sake of simplicity with an ideal gas. Imagine the sides of the cylinder so well insulated that no heat can be gained or lost through them, while the base is to be thought of as a good heat conductor. The cylinder is to be placed successively (Fig. 40) over a heat reservoir A at temperature t_1, an insulating stand B, and a second heat reservoir C at temperature t_2. These heat reservoirs may be thought of as vessels filled with such large quantities of water at the temperature in question that they are capable of giving up or receiving substantial amounts of heat without being materially changed in temperature. The cylinder is first placed with its base in contact with reservoir A, and the operation of the engine is supposed to begin when gas and cylinder alike have come to the temperature t_1. The piston is now moved slowly outward. The pressure of the gas against the piston does work on it, and if the Law of the Conservation of Energy is to hold, this work must be derived from the heat of the gas. If it were not for the contact between the cylinder

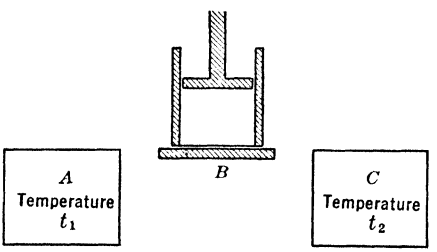


FIG. 40. Carnot Engine.

and the heat reservoir, the gas would, therefore, be cooled by this process. The outward motion of the piston, however, proceeds slowly enough so that the heat received from the reservoir maintains the temperature of the gas constant. The expansion is, therefore, isothermal, and is represented in Fig. 41 by the section of a hyperbola MN. The amount of heat received from A during this process is Q_1.

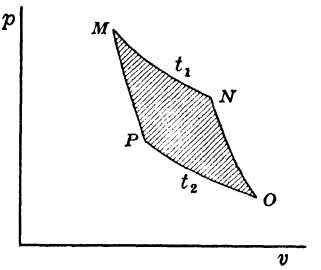


FIG. 41. Carnot Cycle.

The cylinder is now placed on the insulating stand B, and a further expansion of the gas is permitted, this time, however, without gain or loss of any heat. The gas, now doing work against the piston without any compensation for the loss of energy, will cool, and in cooling, its pressure will drop more rapidly than in the isothermal expansion. Such a process, in which no transfer of heat takes place, is called *adiabatic*. This expansion is continued until the temperature has dropped to t_2. It is represented in the figure by the curve NO.

The cylinder is now placed on the heat reservoir C, and the piston is slowly pushed in until a point P has been reached, from which a further adiabatic compression PM will restore the gas to its original state at M. During the first of these two compressions, while the cylinder is in contact with the reservoir C, the heating which would normally result from the work of compression is avoided by transferring the excess heat through the base of the cylinder to the reservoir. The process is in this way performed isothermally at temperature t_2. The heat transferred to C is Q_2. The cylinder is now placed on the insulating stand B, and a final adiabatic compression PM brings the gas back to its starting point.

Since the gas returns to its initial state, it is said to have passed through a *cycle* of operations. The execution of this cycle has resulted in the extraction of an amount of heat Q_1 from the reservoir A and the delivery of Q_2 to the reservoir C. In addition to this, the gas has done a net amount of work W against the piston, which can be calculated in the following way. In Fig. 42, MN represents the isothermal expansion of the gas at temperature t_1. Let u represent the first small increase in volume

of the gas, which results when the piston moves outward a short distance l. Let p_1 be the average pressure during this motion. The total force exerted by the gas on the piston is p_1A if A is the area of the piston. The work done will be p_1Al. But $Al = u$. Hence the work done $= p_1u$, or approximately the area under the section MM' of the curve MN. In the same fashion the work done by the gas in traversing any section of the curve is, at least approximately, the area beneath that section, and the total work done on the whole isothermal expansion is the total area under MN.

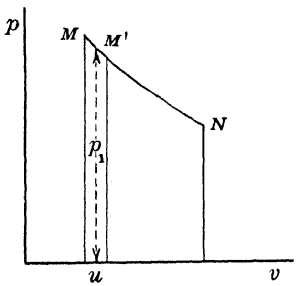


FIG. 42. Work of Expansion.

In just the same way, referring again to Fig. 41, the work done on the adiabatic expansion NO is equal to the area under this curve. The total work done by the gas in pushing out the piston is, therefore, the total area under the broken curve MNO. Similarly the work done by the operator in compressing the gas is equal to the area under the curves OP and PM. The difference between these two areas or the net work performed by the engine in each cycle is readily seen to be the closed area $MNOP$ between these two pairs of curves. Because of its importance as indicating the output of the engine, this area is shaded in Fig. 41.

45. The Second Law of Thermodynamics. Although the Carnot engine has never found practical application as a source of power, all actual engines share in its essential features. Thus all have a *working substance* corresponding to the ideal gas of the Carnot engine, which is carried through a cycle of operations (water and water vapor in the steam engine). All extract a quantity of heat from a heat reservoir or source, convert some of it into work, and then all, without exception, deposit the rest, and that a goodly portion, in another heat reservoir or *refrigerator*.

The necessity for passing on heat to the refrigerator is well seen in the Carnot engine. When the point O has been reached and the gas is to be compressed, it would retrace the adiabatic ON unless opportunity were given for the escape of heat. This, in turn, would mean that the area which measures the work delivered per cycle would be cut to zero. The escape of heat to the refrigerator is just as essential for getting work out of a heat

engine as the taking of heat from the source. It is this fact which Carnot recognized for the first time, and which, expressed in one way or another, lies at the basis of the Second Law of Thermodynamics. It can, perhaps, be best stated here in the following form. " It is impossible to construct an engine which will work in a complete cycle, and produce no effect except the raising of a weight and the cooling of a heat reservoir."

The Carnot engine is in accord with the law as thus stated, since, in addition to performing the work W, which could be used to raise a weight, and extracting heat from the source A, it warms the refrigerator C. There is another way of stating the law, which is of interest in itself even though it leads to the same conclusions. It is to the effect that " It is impossible by any self-acting cyclic process to transfer heat from a cooler to a warmer body." If, for example, such a transfer of heat were possible, it could be used to carry back to the source the heat deposited in the refrigerator by the Carnot engine, and the Carnot engine thus aided would constitute a cyclic device, which would produce no other effect than to cool the source and raise a weight. Such a mechanism, if possible, would be vastly profitable, as it would make possible the utilization of the energy contained in huge natural heat reservoirs such as the ocean. If some process were possible for transferring heat from a cooler to a warmer body without the expenditure of any energy, this process could be used to provide and maintain a refrigerator at a temperature considerably below that of the ocean. A heat engine could then be operated between the ocean as a source and this artificial refrigerator, and could thus convert the heat of the ocean into the desired mechanical energy.

The Second Law of Thermodynamics is thus intimately related to the tendency of heat to flow from a warmer to a cooler body. It constitutes, like the Law of the Conservation of Energy, a principle for which no satisfactory proof has ever been given, but which is, nevertheless, entirely in accord with human experience. Since its full significance dawned on Sadi Carnot over a century has passed. During this time the steam engine has risen to its present position as a wellnigh indispensable agency for supplying the power which is used in our modern civilization. The gasoline engine has been invented, perfected, and applied to the great variety of purposes which are familiar to everyone in this age of petroleum. Nevertheless, no violation

of the Second Law of Thermodynamics has ever been detected, and it looks as though all future engineering developments would have to be carried out with a view to the strict limitations which this law imposes.

CHAPTER XI

ABSOLUTE TEMPERATURE AND ENTROPY

Through any one point M on the pressure-volume diagram of a gas (Fig. 43) one and only one adiabatic can be drawn. This curve MP represents the way in which the pressure varies with the volume of the gas if, starting at M, the piston is moved in or out without permitting any gain or loss of heat. Let NO represent a second adiabatic, which passes through a neighboring point N. If, starting from any point of MP or its extension, we permit the gas to expand, not adiabatically but isothermally, the rate at which p falls with increasing v will be somewhat less rapid, and the isothermal curve thus determined will eventually intersect the second adiabatic NO. Any complete cycle which is made up of two such isothermals and the included sections of the two adiabatic curves is a Carnot cycle. If the gas describes this cycle in the same direction in which the hands of a clock move, an amount of work will be delivered by the engine per cycle which is represented by the area enclosed. Hence, in accordance with the Law of the Conservation of Energy, the heat taken in along the top isothermal is greater than that given up on the under isothermal by precisely this area.

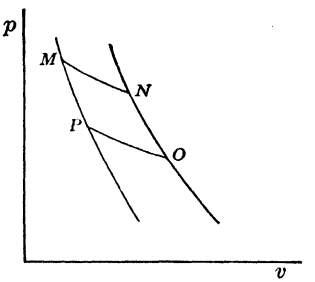


FIG. 43

46. The Kelvin Scale of Temperature. Considering, therefore, the whole family of isothermals which may cut across from MP to NO, we see that the higher the temperature at which a particular isothermal is described, the greater is the amount of heat Q which it extracts from its heat reservoir. Since for two given adiabatics this quantity of heat absorbed depends only on the temperature, it can be used as a measure for temperature in just the same way that the expansion of a liquid or gas is used for this purpose.

Consider, for example, the way in which we have defined temperature in terms of a mercury in glass thermometer. Let l_0 be the length of the column of mercury when the thermometer is dipped in a mixture of ice and water, and l_{100} the length when dipped in boiling water. Since this increase in length corresponds to 100°, the increase in length per degree is

$$\frac{l_{100} - l_0}{100}$$

and if l_t is the length for some intermediate temperature t, that temperature would clearly be defined as

$$t = \frac{l_t - l_0}{\frac{l_{100} - l_0}{100}} \tag{35}$$

Similarly, if we wish to define temperature in terms of the expansion of a gas such as hydrogen at constant pressure, we should merely need to replace the l's by v's denoting the volumes of the gas, and

$$t = \frac{v_t - v_0}{\frac{v_{100} - v_0}{100}} \tag{36}$$

Exactly the same sort of definition would be possible in terms of the heat absorbed by a gas in isothermal expansion between two adiabatics. Thus

$$t = \frac{Q_t - Q_0}{\frac{Q_{100} - Q_0}{100}} \tag{37}$$

where Q_t is the heat absorbed at temperature t, and Q_{100} and Q_0 are the amounts absorbed at the boiling and freezing points of water respectively. It can be shown theoretically by somewhat more elaborate methods that, if an ideal gas is used in (36), the temperature as so defined will agree exactly with that defined by (37). There is, however, one advantage of this last definition. Although for the sake of simplicity we have thought of a Carnot engine as running on an ideal gas, this is really not essential, as will be shown in the present chapter, and the definition can be made entirely independent of the particular working substance employed. This definition of temperature was first proposed by Sir William Thomson, Lord Kelvin, and is called in

his honor the Kelvin scale. It is now universally accepted as the standard for all temperature measurements.

47. Absolute Zero. The definition (37) may be written

$$t = \frac{Q_t}{\frac{Q_{100} - Q_0}{100}} - \frac{Q_0}{\frac{Q_{100} - Q_0}{100}} \qquad (37a)$$

The quantity $\frac{Q_0}{\frac{Q_{100} - Q_0}{100}}$ is obviously a constant independent of the particular temperature t with which we are concerned. Experiment proves it to have the value 273.09; that is to say, the heat absorbed between the two adiabatics at the temperature of melting ice is 273.09 times as great as the difference in the quantities absorbed along any two isothermals 1° apart. If we take the zero of our temperature scale 273.09° below the melting point of ice, and represent the temperature from this new origin by a capital T,

$$T = t + 273.09 = \frac{Q_t}{\frac{Q_{100} - Q_0}{100}} \qquad (38)$$

The new zero is known as *absolute zero*, and the temperature T measured from that point is known as the *absolute temperature*. From (38), absolute zero is characterized by the fact that the heat absorbed from a reservoir at that temperature by an isothermally expanding gas, Q_t, is zero. This can be true only if the pressure of the gas at that temperature is also zero, for otherwise the expanding gas will do work and violate the Law of the Conservation of Energy. Furthermore, the volume of an ideal gas will also approach zero if the temperature is reduced at constant pressure. As the temperature is lowered, the volume must be decreased in order to maintain the pressure constant. Nevertheless, no reduction of the volume will be able to combat the tendency of the pressure to approach zero at absolute zero, and if the effort is made, it is clear that the volume itself will approach zero.

48. The Law of Charles. From (38), $t = T - 273.09$. Since the definition of t in terms of the ideal gas (36) agrees with the Kelvin scale,

$$\frac{v_t - v_0}{\frac{v_{100} - v_0}{100}} = T - 273.09$$

and since $v_t = 0$ when $T = 0$

$$\frac{v_0}{\frac{v_{100} - v_0}{100}} = 273.09$$

and

$$\frac{v_t}{\frac{v_{100} - v_0}{100}} = T$$

The volume of an ideal gas is, therefore, proportional to the absolute temperature, and if T_1 and T_2 are the absolute temperatures corresponding to volumes v_1 and v_2

$$\frac{T_1}{T_2} = \frac{v_1}{v_2} \tag{39}$$

This law, which governs the variation of the volume of a gas with temperature when the pressure is maintained constant, is known in honor of its discoverer as the *Law of Charles.* Although strictly true only for ideal gases, it approximates very closely to the behavior of many gases at pressures and temperatures which are far enough removed from their liquefaction points.

The critical temperature of helium, the most difficult of all gases to liquefy, is 5° abs. It was first liquefied by Kamerlingh Onnes in Leyden in 1908 with a temperature of 4.3° abs. In 1923, by the evaporation of helium, he reached a temperature of 0.82° abs., and in 1926 liquid helium was actually solidified by a combination of low temperature and high pressure.

49. The Efficiency of Heat Engines. Carnot's Theorem. The fraction of the heat extracted from the source which an engine is able to convert into work is known as the *efficiency* of the engine. To Carnot we owe a theorem of the utmost importance concerning this efficiency. It states that "No engine working between the temperatures T_1 and T_2 can have a greater efficiency than a reversible engine working between the same two temperatures." By a *reversible* engine is meant one which can be driven backwards in such a way as to reverse every individual process connected with its operation. Thus, in particular, it must return to the source in each cycle exactly the same amount of heat, Q_1, which it takes from the source when being driven in the usual direction. It must extract from the refrigerator the amount of heat, Q_2, which it would normally deliver to it, and the

work which is required per cycle to drive it backwards must be exactly equal to its normal output. No real engines are strictly reversible, but if a Carnot engine runs slowly enough so that on the isothermal processes the gas in the cylinder differs in temperature from the heat reservoirs with which it is in contact by an infinitely small amount, these conditions are practically fulfilled. It is this practical reversibility of the Carnot cycle, together with its simplicity, which makes it of so much theoretical importance.

The proof of Carnot's Theorem is most readily given by assuming that it is untrue, and then showing that such an assumption leads to a contradiction of the Second Law of Thermodynamics. In Fig. 44, A and B represent a source of heat and a refrigerator at temperatures T_1 and T_2 respectively. R is a reversible engine which, when running between this source and refrigerator, extracts Q_1 from the source, delivers Q_2 to the refrigerator, and performs an amount of work W at each cycle. I is any other engine which extracts the same amount of heat Q_1 from the source and delivers an amount of work W'. We assume, in contradiction to Carnot's Theorem, that the efficiency of I is greater than that of R, i.e.,

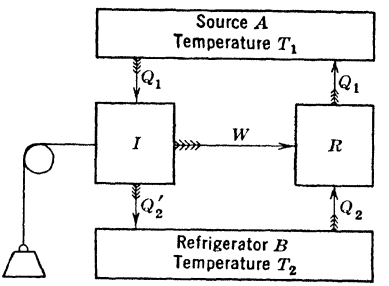


FIG. 44. Carnot's Theorem.

$$\frac{W'}{Q_1} > \frac{W}{Q_1}$$

Or

$$W' > W$$

Since R is reversible, it can be run backwards by a part of W' and thus restore Q_1 in its entirety to the source. Since $W' > W$, there will be left over some work, which can be made to raise a weight. As no change has been made in the heat content of the source, the energy required for raising the weight must have come from the refrigerator. We have, therefore, a cyclic device which produces no effects except the cooling of the refrigerator and the raising of a weight. This is in contradiction to the Second Law of Thermodynamics, and we must conclude that

our hypothesis that the efficiency of I is greater than that of R is incorrect. The theorem is, therefore, proven.

An immediate corollary to this theorem is to the effect that "All reversible engines working between the same two temperatures have the same efficiency." For, replace I by another reversible engine, R'. The same proof would show that the efficiency of R' cannot be greater than that of R, or that one reversible engine cannot have a greater efficiency than another.

A first result of these remarkable theorems is to establish a practical maximum for the efficiency of real heat engines. For, according to the Law of the Conservation of Energy, the work done by any engine per cycle is

$$W = Q_1 - Q_2$$

and its efficiency is

$$\text{Eff.} = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} \tag{40}$$

From (38) it follows directly that the heat absorbed in an isothermal expansion between two adiabatics is proportional to the absolute temperature. Hence

$$Q_1 = (\text{Constant}) \times T_1 \quad \text{and} \quad Q_2 = (\text{Constant}) \times T_2$$

Substituting these values in (40),

$$\text{Eff.} = \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1} \tag{41}$$

This must be at least as great as the efficiency of any real engine working between these same temperatures.

The feasible refrigerator temperatures are rather strictly limited in practice. Thus the condenser of a steam engine can never have a temperature below the freezing point of water and seldom below 10° C. or 283° abs. Even if the pressure of the steam in the boiler is 200 lb./in.^{2}, its temperature, and therefore that of the source, would only be 195° C. or 468° abs. The maximum efficiency of the engine, even at this high pressure, would then be only (468 − 283)/468 or about 40 per cent.

A second conclusion to be drawn from Carnot's Theorem applies to our definition of temperature. We have thought of the Carnot engine as running on an ideal gas. Suppose, however, that it is run on any other substance. Choose adiabatics such that the heat Q_1 extracted from the source along an isothermal

at temperature T_1 is the same as that extracted at the same temperature with an ideal gas as the working substance. Then, since the efficiency of this engine is the same as that of the ideal gas engine, the work delivered per cycle between T_1 and any other temperature T_2 will be the same. Hence also the heat Q_2 given up to the refrigerator will be the same as with the ideal gas. Thus the temperature defined by (38) with an arbitrary working substance will be the same as that defined with an ideal gas. The Kelvin definition of temperature is absolutely independent of any one particular material.

50. Entropy. The Degradation of Energy. From definition (38) it follows that if Q_1 and Q_2 are the quantities of heat extracted at T_1 and T_2 respectively,

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2} \tag{42}$$

The ratio of Q to T is of considerable importance both practical and theoretical. Q_1/T_1 is known as the *entropy* which is transferred from the source to the engine per cycle, and Q_2/T_2 is the entropy transferred from the engine to the refrigerator. The equation (42) states that, if we view the source and the refrigerator as a single system, there has been no net gain or loss of entropy during the cycle. Under all circumstances, since the working substance in the engine goes through a complete cycle, all its properties, including its entropy, at the end of the cycle must be the same as at the beginning. We conclude that the working of a reversible engine is associated with a strict conservation of entropy. If, however, an engine has been less efficient than a reversible engine, too large a fraction of the heat taken from the source will be handed over to the refrigerator, and Q_2/T_2 will be greater than Q_1/T_1. The gain in entropy by the refrigerator will more than compensate for the loss by the source. There has been a net increase in the entropy of the system as a whole.

Whenever an amount of heat Q flows from a warmer to a cooler body without running any heat engine whatever, the warmer body loses an amount of entropy Q/T_1, while the cooler body gains an amount Q/T_2. Since $T_1 > T_2$, $Q/T_2 > Q/T_1$, and once more we have had a net increase in entropy. Thus any failure to take full advantage of the transfer of heat from a warm to a cooler body is always accompanied by an increase in entropy. In fact, the increase in entropy is really a measure of such a

failure. Similarly, when mechanical energy is converted into an amount of heat Q by friction, the bodies which are thereby heated receive an amount of entropy Q/T, corresponding again to a net increase in entropy.

We have been led to recognize two fundamentally different types of energy, mechanical energy and heat. It is mechanical energy which we require for the great majority of purposes in our modern civilization. This, the more desirable type, can be readily converted into heat, with a simultaneous increase of entropy. Once existing as heat, it can be restored only in part to its original mechanical form. No energy has been lost, but it has become less *available* for practical purposes. Similarly, whenever heat flows from a warmer to a cooler body without passing through a reversible engine, an opportunity to secure mechanical energy has been forfeited, and the availability of that energy has been reduced. Now, friction is one of the commonest phenomena of nature, and the building of reversible engines is hardly a process which nature by herself would carry out. We must conclude that natural processes tend to increase entropy and thereby to decrease the availability of the total energy in the universe. This *Law of the Degradation of Energy*, as it is called, is by its sheer pessimism in startling contrast to the optimism of the Law of the Conservation of Energy. It points forward with inexorable certainty to a time in the far distant future when all fuels on the Earth will have been burned, all possible chemical reactions will have taken place, and all heat reservoirs equalized in temperature; when, in fact, the last possible erg of mechanical energy will have been extracted and dissipated, and life shall have ceased to exist.

PROBLEMS

1. A cubic meter of air at atmospheric pressure is compressed in a cylinder until it occupies a volume of only 1000 cc. Assuming that it behaves like an ideal gas and that its final temperature is the same as its initial temperature, what is its final pressure?

2. To what temperature must a mass of air at 20° C. be raised in order to double its volume if pressure is maintained constant?

3. A ½-horse-power motor is required to operate a certain air compressor. The heat produced in the gas is dissipated by means of a water jacket, through which a stream of water flows at the rate of a cubic foot in 10 minutes. How much is the temperature of the water raised?

4. A certain reversible engine delivers 5 horse-power. If the temperature of its source is 120° C. and that of its refrigerator is 10° C., how many B.T.U.'s

per second does it extract from the source, and how many does it deliver to the refrigerator?

5. If a reversible engine is run backwards to cool a refrigerator, how much energy is required to freeze 1 lb. of ice cubes? Assume the temperature of the room in which the refrigerator is operating to be 25° C. and the temperature of the cooling chamber of the refrigerator to be −10° C.

CHAPTER XII

ATOMS AND ELEMENTS

In order better to appreciate some of the laws which have been discussed in the last few chapters, it is necessary to turn to a consideration of the atomic theory of matter. The idea that all materials consist of an aggregation of large numbers of tiny particles, which themselves are incapable of subdivision, is not a new one. Certain of the Greek philosophers, notably Democritus, were led to such a viewpoint by metaphysical considerations which to-day would carry little or no weight. It was Robert Boyle, in the seventeenth century, who again proposed an atomic hypothesis. Throughout the middle ages the alchemists had been pursuing their search for some means of transmuting the baser elements into gold. Unsuccessful as they had been in their major purpose, the tireless experimentation which had been directed toward this end had contributed much to a knowledge of the various forms in which matter can and does exist. It had shown how in many cases two or more substances can be combined to produce a new material, which may have entirely different properties from any of the original ingredients. Only a maze of doubt existed, however, in the seventeenth century as to how much could be expected from such transformations. Was all matter made up of a very few fundamental substances, or was the number relatively large? Although the definite answer to this question required centuries of experimentation, Boyle at least visualized the problem in a way which could scarcely be improved on at the present time. To him the purpose of the study of chemistry was not to transmute baser metals into gold, nor even to discover new cures for disease, as had been held by many during the fifteenth and sixteenth centuries, but rather the simple investigation of the composition of substances. He defined an *element* as a substance which is incapable of decomposition into simpler substances, and a *compound* as a substance which is made up of two or more elements. What materials were really elements and what were compounds could be decided only after long and laborious ex-

perimentation. He pictured this combination of elements into compounds as an actual physical union between two or more of the tiny corpuscles which make up the mass of the constituent elements. The corpuscle of the compound thus formed could not be subdivided without losing thereby its own peculiar properties and taking on the properties of the elements from which it was made. To-day we call the fundamental corpuscle of the element an *atom* and that of a compound a *molecule*. The molecule is made up of two or more atoms, which are tied together by bonds which are even now rather imperfectly understood.

51. Lavoisier and the Conservation of Mass. Unfortunately the ideas of Boyle were very little recognized during the following century, and although much new chemical knowledge was accumulated, little progress was made in correlating this work into a real science of elements and compounds. The next really important advances were made by Anton Laurent Lavoisier (1743–1794), who, because of the great significance of his work, rightly deserves the title of the founder of modern chemistry. The key to his great success lay in his emphasis on the strictly quantitative measurement of chemical processes. He showed that, whenever two or more substances unite to form a compound, the mass of the new material formed is always equal to that of the original ingredients. This law that no mass is ever gained or lost in a chemical reaction is known as the *Law of the Conservation of Mass*. It has been tested repeatedly during the last century with all the refinements of modern measurement, and would seem to be true up to the limits of accuracy of the most sensitive chemical balances.

The quantitative methods advocated by Lavoisier were well calculated to throw light on the problem which had not yet been solved as to whether or not chemical compounds were always of exactly the same composition. Lavoisier was well satisfied by his researches that this was the case, but his opinion was not universally accepted for a number of years after his untimely death in the French revolution.

52. John Dalton and the Law of Multiple Proportions. In 1808, John Dalton in England published the first part of his "New System of Chemical Philosophy," in which he restated the atomic hypothesis, and for the first time advanced substantial evidence in its favor. One of the outstanding facts which up to this time had prevented a general acceptance of the constancy

of chemical composition was the unquestioned existence of compounds such as carbon dioxide and carbon monoxide, both of which contain only carbon and oxygen but in different proportions. Dalton pointed out in his law of multiple proportions that, when such was the case, the proportions of the two constituents bore a simple integral relation, which could be readily interpreted on the atomic hypothesis. Consider, for example, the very simple case of water and hydrogen peroxide. Both consist of hydrogen and oxygen only. In water the ratio by weight of hydrogen to oxygen is 1 to 8; in hydrogen peroxide it is 1 to 16. If, now, it is assumed that the atom of oxygen weighs 16 times as much as the atom of hydrogen, and that the molecule of hydrogen peroxide consists of 1 atom of hydrogen to 1 of oxygen, while that of water contains 2 atoms of hydrogen to 1 of oxygen, exactly the observed ratios would be accounted for. Hydrogen, as the lightest of the known elements, was chosen by Dalton as the standard, and the weights of all other elements were to be calculated in terms of it. This ratio of the weight of an atom of an element to that of hydrogen is known as its *atomic weight.* Because of the large number of elements with which oxygen forms compounds, it has been found rather more convenient in practice to take oxygen as the standard. Since its weight is very nearly 16 times that of hydrogen, it has been given the arbitrary value of exactly 16, and the atomic weights of other elements are calculated on this basis. The accurate value, for example, of the atomic weight of hydrogen in terms of oxygen as 16 is 1.008.

The sum of the atomic weights of the various atoms which make up the molecule of a compound is known as its *molecular weight.* Thus if a molecule of water consists of 2 atoms of hydrogen and 1 of oxygen, its molecular weight would be 2 + 16 or 18. The determination of the exact number of atoms of each constituent in a molecule cannot, however, be determined without ambiguity from chemical evidence alone. Thus, for example, if water consisted of 4 atoms of hydrogen and 2 of oxygen, the ratio by weight of oxygen to hydrogen would still be 8, and the same would be true of any integral multiple of 2 atoms of hydrogen and 1 of oxygen. This possibility has been ruled out by chemists as an unnecessary complication, since all known compounds can be just as well accounted for on the basis of the simpler formula for water.

53. Avogadro's Law. An observation on gases, originally made by Gay-Lussac and interpreted by Avogadro in 1811, proved particularly useful in making decisions regarding chemical formulas. Gay-Lussac noted that, if a new gas were formed by the union of two or more other gases, the volume occupied by the new gas was very simply related to the original volumes of the constituents. Thus 2 volumes of hydrogen united with 1 volume of oxygen to form 2 of steam. If the hydrogen molecule consists of 2 atoms of hydrogen, and that of oxygen also consists of 2 atoms of oxygen, as we believe to-day, 2 molecules of hydrogen would clearly unite with 1 of oxygen to form 2 molecules of water. Making use of the usual chemical notation, in which an atom of hydrogen is represented by the symbol H, a molecule of hydrogen by H_2, an atom of oxygen by O, and a molecule of water by H_2O, this combination of hydrogen and oxygen would be represented by the following equation.

$$2H_2 + O_2 = 2H_2O$$

Since on the atomic theory 2 molecules of hydrogen unite with 1 molecule of oxygen to form 2 molecules of steam, and since in practice 2 volumes of hydrogen unite with 1 volume of oxygen to form 2 volumes of steam, we must conclude that in the same volume each one of these gases contains the same number of molecules. It was Avogadro who first recognized this interpretation on the atomic theory of the observation of Gay-Lussac. The general conclusion, known as Avogadro's Law, may be stated as follows: "Under the same conditions of temperature and pressure, equal volumes of all gases contain equal numbers of molecules."

If Avogadro's Law is correct, the mass of any volume of a gas must be proportional to the molecular weight of the gas in question. Thus 2 gr. of hydrogen would contain the same number of molecules and occupy the same volume as 32 gr. of oxygen. Two grams of hydrogen, 32 gr. of oxygen, or M gr. of a substance of which M is the molecular weight, is known as a *gram-molecule* or simply a *mol* of that substance. A gram-molecule of one gas should, therefore, occupy the same volume as that of any other gas, and this volume under standard conditions of temperature and pressure (0° C. and 76 cm. of mercury) is found to be 22.4 liters. Whenever the formula of a gas is known and the observation is made at a temperature far enough above

the critical point so that the gas behaves approximately as an ideal gas, this law is found to hold experimentally. If the formula of a gas is not known with certainty, a measure of its density, and in particular of the mass occupying 22.4 liters, will determine approximately its molecular weight, and thus decide among several possibilities for its chemical formula. Consider, for example, the case of water and hydrogen peroxide. Assuming that the formula of water is H_2O, and that its molecular weight is accordingly 18, 18 gr. of water vapor constitute a gram-molecule, and at a suitably high temperature will occupy a volume, which, when reduced to 0° C. and one atmosphere by the laws of ideal gases, will yield 22.4 liters. If the formula of hydrogen peroxide is HO, its molecular weight would be 17, while if it were H_2O_2, also a possibility so far as the ratio of H to O by weight is concerned, its molecular weight would be 34. This last is found to be the correct value when its density under standard conditions of temperature and pressure is determined.

54. The Ideal Gas Law. We are now in a position to obtain a single ideal gas law which will combine the laws of Boyle, Charles, and Avogadro. Imagine a mass m of a gas of molecular weight M contained in a cylinder initially at a pressure of one atmosphere and a temperature of 0° C. The volume which it occupies will be a fraction m/M of 22.4 liters. Or

$$v_0 = \frac{m}{M}(22.4)$$

First compress the gas isothermally, and let p be the final pressure in atmospheres. Since Boyle's Law applies to this process, it may be used for the calculation of the final volume v_1

$$\frac{v_1}{v_0} = \frac{1}{p}$$

Or

$$v_1 = \frac{1}{p}\frac{m}{M}(22.4)$$

Now heat the gas at constant pressure, a process to which we can apply the Law of Charles. If T is the final absolute temperature, and v the resulting volume,

$$\frac{v}{v_1} = \frac{T}{273}$$

Or

$$v = \frac{T}{273}\frac{1}{p}\frac{m}{M}(22.4)$$

Or
$$pv = \left(\frac{22.4}{273}\right)\frac{m}{M}T \tag{43}$$

The quantity 22.4/273 is a constant of nature entirely independent of the particular problem on which we may be engaged. We can, therefore, replace it by a single letter R.

$$pv = \frac{m}{M}RT \tag{44}$$

In the preceding discussion we have thought of the pressure as measured in atmospheres and the volume in liters. If, however, we wish to measure either of these quantities in other units, the Formula 44 would still apply if we suitably modified the constant R. Thus, for example, if we wish to measure v in cubic centimeters while still maintaining the pressure in atmospheres, it is only necessary to use for R $(22.4/273) \times 1000$.

EXAMPLE

What pressure would 10 gr. of carbon dioxide exert on the walls of a 5-liter container if it were heated to a temperature of 100° C.?

We will first make use of the fact that a gram-molecule of any gas under standard conditions occupies 22.4 liters. The formula for carbon dioxide is CO_2, and since the atomic weight of carbon is 12, the molecular weight of CO_2 is $12 + 32 = 44$. Suppose we choose to calculate the required pressure in centimeters of mercury. In this unit the standard pressure of one atmosphere is 76. Substituting the appropriate values in (44),

$$(76)\ (22.4) = \frac{44}{44}R(273) \tag{45}$$

We will now write down Equation 44 again with the new conditions inserted. p, the unknown quantity, will be introduced directly on the left-hand side. Thus

$$p(5) = \frac{10}{44}R(373) \tag{46}$$

Dividing 46 by 45 to eliminate R, and solving for p,

$$p = \frac{10}{44}\,\frac{373}{273}\,\frac{(76)\ (22.4)}{5} = 105.7 \text{ cm. of mercury.}$$

55. The Periodic Table of the Elements. With the atomic theory as a guide and Avogadro's Law as a great aid in establishing formulas, the next half century was a period of unprecedented advance in chemistry. Gradually the real elements began to be definitely distinguished from compounds until by 1860 enough empirical knowledge was at hand to form the basis for a rational

arrangement. Certain of the elements had markedly similar properties, and the work of Lothar Meyer in Germany, Newlands in England, and Mendelyeev in Russia between 1860 and 1870 showed the possibility of so arranging the elements, roughly in the order of their atomic weights, as to exhibit these similarities in a very graphic way. The outgrowth of these attempts at a rational arrangement of the elements is our present *periodic table.*

All the elements are arranged in eight columns and seven rows. The first row contains only the two lightest elements, hydrogen and helium, in the first and last columns respectively. Beginning with lithium, the third lightest element, the second row is filled by the next heavier elements in the order of their atomic weight. To the left of the symbol is the *atomic number*, the numerical order in which the element appears in the table, a quantity which has in recent years attained a considerable theoretical significance. The third row follows exactly as the second, while the fourth, fifth, and sixth are double rows. In each case the elements in the first half of the double rows are displaced to the left-hand side of their squares; those in the second half of the row are displaced to the right. An exception is made in the first half of each of the double rows in the eighth column, where the appropriate space is occupied, not by a single element, but by a triad of closely related elements. Between the third and the fourth columns of the sixth row belongs the group of fourteen *rare earths*, which are arranged by themselves in two rows beneath the main table. Each space in the seventh row is occupied again by a single element, displaced to the left of its square, and the last two columns here are entirely unoccupied.

With this arrangement, chemically related elements lie under one another. Thus, with the exception of hydrogen, the elements to the left of the first column, known as the alkali metals, are all strikingly similar. The three elements to the right of this column in the double rows, copper, silver, and gold, are the so-called noble metals, characterized by their resistance to corrosion. Similarly the elements to the left of the second column are similar in their properties, and are known as the alkaline earths. The elements to the right of the last column are all chemically inert or "noble" gases. The complete study of these elements comprises the science of chemistry, and is far beyond the scope of the present book. Suffice it to say that the total number of the elements is 92. Two, numbers 85 and 87,

	I	II	III	IV	V	VI	VII	VIII
1	1H 1.008							2 He 4.00
2	3 Li 6.94	4 Be 9.02	5 B 10.82	6 C 12.00	7 N 14.008	8 O 16.000	9 F 19.00	10 Ne 20.2
3	11 Na 23.00	12 Mg 24.32	13 Al 26.96	14 Si 28.06	15 P 31.02	16 S 32.07	17 Cl 35.46	18 A 39.91
4	19 K 39.10 29 Cu 63.57	20 Ca 40.07 30 Zn 65.37	21 Sc 45.1 31 Ga 69.72	22 Ti 47.9 32 Ge 72.4	23 V 51.0 33 As 74.96	24 Cr 52.0 34 Se 79.2	25 Mn 54.93 35 Br 79.92	26 Fe 27 Co 28 Ni 55.84 58.97 58.68 36 Kr 82.9
5	37 Rb 85.45 47 Ag 107.88	38 Sr 87.63 48 Cd 112.4	39 Y 89.0 49 In 114.8	40 Zr 91.0 50 Sn 118.7	41 Nb 93.1 51 Sb 121.8	42 Mo 96.0 52 Te 127.5	43 Ms 53 I 126.92	44 Ru 45 Rh 46 Pd 101.7 102.9 106.7 54 X 130.2
6	55 Cs 132.8 79 Au 197.2	56 Ba 137.4 80 Hg 200.6	57 La 138.9 81 Tl 204.4	72 Hf 178.6 82 Pb 207.2	73 Ta 181.5 83 Bi 209.0	74 W 184.0 84 Po 210.0	75 Re 85	76 Os 77 Ir 78 Pt 190.9 193.1 195.2 86 Em 222.0
7	87	88 Ra 226.0	89 Ac 226	90 Th 232.1	91 Pa 230	92 U 238.2		

58 Ce 140.2	59 Pr 140.9	60 Nd 144.3	61 Il	62 Sm 150.4	63 Eu 152.0	64 Gd 157.3
65 Tb 159.2	66 Dy 162.5	67 Ho 163.5	68 Er 167.7	69 Tu 169.4	70 Yb 173.5	71 Cp 175.0

had until the year 1931 resisted all attempts to detect them in the laboratory, but the discovery of both was announced during this year. They are called eka-iodine and eka-caesium respectively on account of the positions which they occupy in the table beneath iodine and caesium, and the consequent similarity which they would be expected to exhibit to these elements. It seems highly unlikely that any further elements now remain to be discovered.

LIST OF THE CHEMICAL ELEMENTS WITH THEIR SYMBOLS AND ATOMIC NUMBERS

1 Hydrogen	H	32 Germanium	Ge	63 Europium	Eu
2 Helium	He	33 Arsenic	As	64 Gadolinium	Gd
3 Lithium	Li	34 Selenium	Se	65 Terbium	Tb
4 Beryllium	Be	35 Bromine	Br	66 Dysprosium	Dy
5 Boron	B	36 Krypton	Kr	67 Holmium	Ho
6 Carbon	C	37 Rubidium	Rb	68 Erbium	Er
7 Nitrogen	N	38 Strontium	Sr	69 Thulium	Tu
8 Oxygen	O	39 Yttrium	Y	70 Ytterbium	Yb
9 Fluorine	F	40 Zirconium	Zr	71 Cassiopeium	Cp
10 Neon	Ne	41 Niobium	Nb	72 Hafnium	Hf
11 Sodium	Na	42 Molybdenum	Mo	73 Tantalum	Ta
12 Magnesium	Mg	43 Masurium	Ma	74 Tungsten	W
13 Aluminium	Al	44 Ruthenium	Ru	75 Rhenium	Re
14 Silicon	Si	45 Rhodium	Rh	76 Osmium	Os
15 Phosphorus	P	46 Palladium	Pd	77 Iridium	Ir
16 Sulphur	S	47 Silver	Ag	78 Platinum	Pt
17 Chlorine	Cl	48 Cadmium	Cd	79 Gold	Au
18 Argon	A	49 Indium	In	80 Mercury	Hg
19 Potassium	K	50 Tin	Sn	81 Thallium	Tl
20 Calcium	Ca	51 Antimony	Sb	82 Lead	Pb
21 Scandium	Sc	52 Tellurium	Te	83 Bismuth	Bi
22 Titanium	Ti	53 Iodine	I	84 Polonium	Po
23 Vanadium	V	54 Xenon	Xe	85 Eka-iodine	
24 Chromium	Cr	55 Caesium	Cs	86 Ra-emanation	Em
25 Manganese	Mn	56 Barium	Ba	87 Eka-caesium	
26 Iron	Fe	57 Lanthanum	La	88 Radium	Ra
27 Cobalt	Co	58 Cerium	Ce	89 Actinium	Ac
28 Nickel	Ni	59 Praseodymium	Pr	90 Thorium	Th
29 Copper	Cu	60 Neodymium	Nd	91 Protoactinium	Pa
30 Zinc	Zn	61 Illinium	Il	92 Uranium	U
31 Gallium	Ga	62 Samarium	Sm		

PROBLEMS

1. What volume would be occupied by 1 gr. of (*a*) nitrogen, N_2, and (*b*) neon, Ne, at 100° C. and one atmosphere of pressure?

2. Calculate the mass of air contained in a certain automobile tire when it is pumped to a pressure of 40 lb./in.^{2} The volume of the tire is the same as that of a cylinder 30 in. long and 4 in. in diameter. The temperature is 25° C. Assume the molecular weight of air to be midway between that of oxygen (O_2) and nitrogen (N_2).

3. How much is the lifting effect of a balloon containing 1,000,000 cu. ft. of gas reduced by the substitution of helium for hydrogen?

4. A certain gas contains 6 parts of carbon to 1 of hydrogen by weight. Its density under standard conditions of temperature and pressure is 0.00125 gr./cm.^{3} What is its chemical formula?

CHAPTER XIII

THE KINETIC THEORY OF MATTER

Although the formation of compounds from elements is in accord with laws which suggest an atomic structure of matter, it must be admitted that such chemical evidence falls far short of proving the actual existence of atoms. Probably the closest approach to such a proof which we have at present is provided by the remarkable experimental check with the predictions of the *kinetic theory of gases*. It was Robert Boyle who first suggested that the pressure which a gas exerts on the walls of its container might be due to the impact of the molecules themselves, which he visualized in rapid motion, constantly colliding with the bounding walls and with one another. We owe the further development of this theory to Daniel Bernoulli in the eighteenth century and Boltzmann and Maxwell during the latter part of the nineteenth.

56. The Pressure of a Gas. Although the kinetic theory requires for its complete expression a rather elaborate mathematical treatment, some of its more interesting results can be obtained by a few very simple calculations. Consider, for example, the pressure exerted by a gas on the walls of the containing vessel. For simplicity let us think of the container as a rectangular box (Fig. 45), having edges of lengths a, b, and c. Consider, in particular, the force exerted by the molecules on the side of the box perpendicular to a. The number and frequency of these impacts is so great that they exert an apparently steady pressure. The molecules are to be thought of as hard, perfectly elastic spheres, impinging on a smooth and perfectly elastic wall. If the surface of the wall is smooth, it can exert no force parallel to itself, and the components of the velocity of the molecule parallel to both b and c must, therefore, be unchanged by the impact. The assumption of perfect elasticity

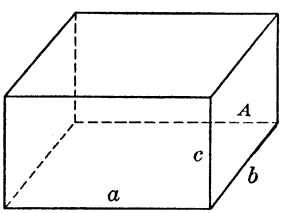


FIG. 45

means that the kinetic energy of the molecules is unchanged by impact. This can be true only if the resultant velocity is unchanged, and since two of the three components of this resultant velocity are unaffected, the third component parallel to a must have the same numerical value after collision as before. The molecule is, however, prevented by the wall from pursuing its original course, and we must conclude that it is reflected backwards in such a fashion that the magnitude of its velocity parallel to a is unaffected. The situation can be readily visualized in terms of a ball striking the cushion of a billiard table.

Let u, v, and w represent the components of the velocity of the molecule parallel to a, b, and c respectively before impact, and let u', v', and w' be the corresponding components after impact. Then

$$u' = -u \qquad v' = v \qquad w' = w$$

If μ is the mass of the molecule, the component of its momentum in the a direction before striking the wall was μu. This was reduced by the impact momentarily to zero, and was then brought to $-\mu u$, equal to its initial value in magnitude, but having the opposite direction. The force exerted on the molecule by the wall has, therefore, changed its momentum by an amount $2\mu u$. Since u, the component of its velocity in the a direction, is unchanged in magnitude by the successive impacts which it makes with the opposite walls of the container, it will travel a distance u in the a direction in 1 sec., and will make u/a impacts in all or $u/2a$ impacts with the single wall A. The force exerted on it by A will, therefore, change its momentum by an amount $(2\mu u^2)/2a$ or $\mu u^2/a$ in each second. If there are altogether n such molecules shuttling back and forth in the box, having velocities $u_1, u_2, \cdots u_n$ respectively, the total change produced per second in their momentum will be

$$\frac{\mu}{a}(u_1^2 + u_2^2 + \cdots + u_n^2)$$

and this must be equal to the impulse imparted to them in this time by the wall A. If we denote the average force which the wall exerts on these molecules by F, the impulse imparted to them in time t would be Ft, and in 1 sec. it would be F. This, in accordance with the law of action and reaction, is also the average force exerted on the wall by the impinging molecules.

The pressure exerted by the gas on the container is, therefore,

$$p = \frac{F}{bc} = \frac{\mu}{abc}(u_1^2 + u_2^2 + \cdots + u_n^2)$$

Or, since abc is equal to the volume V of the container,

$$p = \frac{\mu}{V}(u_1^2 + u_2^2 + \cdots + u_n^2)$$

If we denote by $\overline{u^2}$ the average value of these squared velocities,

$$p = \frac{n\mu}{V}\overline{u^2} \tag{47}$$

It remains to investigate the relation which u bears to the resultant velocity W of the molecule. In Fig. 46 a rectangular box is drawn with W as the diagonal (MP). The edges of the box are u, v, and w respectively. By the Pythagorean Theorem,

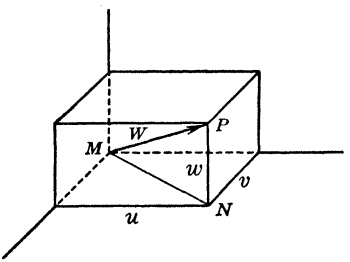


FIG. 46. Velocity Components.

$$\overline{MN}^2 = u^2 + v^2$$

But $$W^2 = \overline{MP}^2 = \overline{MN}^2 + w^2$$
$$= u^2 + v^2 + w^2$$

If there is no motion of the gas as a whole, a velocity in one particular direction cannot be considered more likely than that in any other direction. Or

$$\overline{u^2} = \overline{v^2} = \overline{w^2}$$

Hence $$\overline{W^2} = 3\overline{u^2} \quad \text{or} \quad \overline{u^2} = \tfrac{1}{3}\overline{W^2} \tag{48}$$

The validity of this assumption of *molecular chaos*, as it is called, depends on the largeness of the number of molecules in the box. If there are only a few molecules present, there may be instants at which (48) is very far from true. If, however, the number of molecules is huge, as we have every reason to believe, the chances are overwhelmingly opposed to any substantial violation of this relation between $\overline{u^2}$ and $\overline{W^2}$. Substituting it in (47), multiplying through by V, and replacing the product of n and μ by m, the total mass of the gas in the box,

$$pV = \tfrac{1}{3}m\overline{W^2} \tag{49}$$

57. The Interpretation of Temperature on the Kinetic Theory. The full significance of (49) is seen by comparing it with the

empirical gas law (44) of the last chapter.

$$pV = \frac{m}{M} RT \tag{44}$$

(49) and (44) are in agreement provided

$$\frac{1}{3}\overline{W^2} = \frac{1}{M} RT$$

or if

$$\frac{1}{2}\overline{MW^2} = \frac{3}{2} RT \tag{50}$$

that is to say, if the kinetic energy of a gram-molecule of the gas is equal to $\frac{3}{2}RT$. For any given temperature this kinetic energy of a gram-molecule should, therefore, be the same for all gases. Since a gram-molecule of all gases contains the same number, N, of individual molecules, the average kinetic energy of these molecules $(\overline{MW^2})/2N$ must depend only on the temperature, and must be proportional to it. In particular, at absolute zero this average kinetic energy, and with it the kinetic energy of every individual molecule, must be zero.

It should be remembered that the whole concept of temperature is based on the fact that two bodies in intimate contact with each other tend, by a flow of heat from the warmer to the cooler, to equalize in temperature. Thus, if two vessels, each containing a mass of gas, are brought into contact, an interchange of heat takes place until they arrive at the same final temperature. The process of this interchange is easy to visualize. The gas molecules, in striking a wall of the warmer container, set the molecules of the wall into motion, probably a vibration about their natural positions of equilibrium. The higher the temperature of the gas, the more intense are these impacts, and the more violent are the vibrations produced in the walls of the vessel. The physical sensation of warmth which we experience in touching the wall is due fundamentally to this molecular motion. The motion in the wall of one container is passed on to the wall of the adjoining cooler vessel, and when a gas molecule in this other vessel strikes the wall, it is very likely to impinge on a molecule already in violent motion, and thus to be reflected back with a greater than its original kinetic energy. In this way the average kinetic energy of the molecules in the cooler vessel is gradually raised until it is identical with that of the warmer one. The temperatures have been equalized.

Consider also the case of a mixture of two gases of molecular weights M and M' respectively. The temperature should be measured by the average kinetic energy of either one of the two types of molecules, and if a definite, unambiguous result is to be obtained, the two average kinetic energies, $\overline{MW^2}/2$ and $\overline{M'W'^2}/2$, should be the same. The greater mass of one of the components should be exactly compensated for by the smaller average velocity. It was first proven by Maxwell on purely dynamical grounds that in such a mixture the mere process of collision between the molecules will tend over a comparatively short period practically to equalize these average kinetic energies. It should be emphasized here that it is only these average kinetic energies which are thus simply related to the temperature. The velocities of the individual molecules may and do vary widely about the average value.

58. The Average Thermal Velocities of the Molecules. Equation 50 gives a means of calculating the average velocity, or more accurately the *root mean square* velocity $\sqrt{\overline{W^2}}$ of a molecule at any chosen temperature T. The value of R in absolute metric units can be calculated from (44) by making use of the fact that a gram-molecule under standard conditions of temperature and pressure occupies 22.4 liters or 22,400 cc. The pressure of one atmosphere in dynes per square centimeter is

$$(76)(13.6)(980) = 1{,}013{,}000$$

Hence $$(1{,}013{,}000)(22{,}400) = R(273)$$

Or $$R = 8.3 \times 10^7 \text{ ergs per degree Centigrade}$$

Thus, if we wish to calculate the average velocity of a nitrogen molecule at room temperature (293° abs.), we should substitute in (50) $R = 8.3 \times 10^7$, $T = 293$, and $M = 28$. Or

$$\sqrt{\overline{W^2}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3(8.3 \times 10^7)(293)}{28}} = 51{,}040 \text{ cm./sec.}$$
$$= 510 \text{ meters/sec.}$$

A remarkable verification of this sort of calculation was provided in 1920 by O. Stern in Germany. In 1916 the development of the mercury vapor diffusion pump made possible for the first time the attainment of exceedingly high vacua, so high, in fact, that a molecule might travel in its thermal motion entirely across the containing vessel without colliding with another

molecule on the way. Making use of this possibility, Stern measured directly the velocity of molecules of silver evaporated from a hot silver filament. The filament, heated, as in an ordinary incandescent lamp, by an electric current, was placed at the center O of a cylindrical vessel, which could be rotated about the filament as an axis (Fig. 47). An inner cylinder C with a slit S was placed between the periphery of the vessel and the filament. A glass plate P was mounted opposite S on the wall of the cylinder. Silver atoms, set in violent vibration by the high temperature of the filament, would occasionally fly off from its surface. The vessel was so well evacuated that a considerable proportion of these atoms which started initially in the direction of the slit S would pass through, and reach P without suffering any collisions on the way. When silver atoms strike a glass plate in this way, they adhere quite firmly to it, and if such a bombardment is continued long enough, they will build up a visible deposit. It was shown by Stern, furthermore, that much fewer molecules than are necessary for immediate visibility can be added to and rendered visible by a process very similar to photographic development. Briefly his experiment consisted in rotating the vessel, and with it the inner cylinder, at a high rate of speed while the filament was lighted. While the silver molecules were travelling from S to P, the plate would advance slightly in its rotation so that the molecules would build up their deposit at O_1 a little to one side of O' on the line passing through O and S. The vessel was then rotated in the opposite direction, thereby producing a deposit at O_2 on the other side of O'. If W is the velocity of the silver molecule, R the distance OO' from filament to plate, and d the distance SO' from slit to plate, the time required by the molecule in moving from S to O' would be

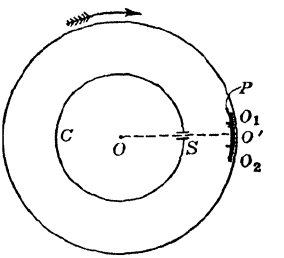


FIG. 47. Measurement of Molecular Velocity.

$$t = \frac{d}{W} \tag{51}$$

If n is the number of revolutions per second made by the chamber, P will move during this time t a distance

$$O'O_1 = \tfrac{1}{2}O_2O_1 = 2\pi Rnt \tag{52}$$

Substituting the value of t from (51) in (52) and solving for W,

$$W = \frac{4\pi nRd}{O_1O_2} \tag{53}$$

The deposits at O_1 and O_2 will not be sharp lines, since there will be present among the molecules leaving the filament a wide range of velocities, centering, however, about an average value, and getting scarcer and scarcer at velocities more and more remote from this value. We should expect to get, exactly as observed by Stern, two rather diffuse lines. The distance between the centers of these lines yielded from (53) a velocity W, which checked remarkably well with that calculated from (50) for a silver molecule at the observed filament temperature.

59. Brownian Movements and Avogadro's Number. Perhaps even more convincing evidence of the thermal motion of molecules is provided by the *Brownian movements*, first observed by the botanist Robert Brown in 1827. If tiny particles of smoke, floating in the air, or any fine particles such as those of India ink suspended in a liquid are examined by a microscope, they are observed to be in constant motion. The cause of this motion, which for many years remained a puzzle, is rather obvious on the basis of the kinetic theory. We have seen that the constant bombardment of a large surface by huge numbers of molecules results in a practically steady pressure. The smaller this surface is, however, the fewer would be the molecules striking it in a short interval of time, and the less continuous would be the pressure. When we reach the size of smoke particles or those of India ink, which may be so small as to be barely visible under the microscope, we might well expect to find, in place of a uniform pressure on all sides, a more or less irregular force, now greater on one side of the particle and now greater on the other, depending on the number and velocity of the impinging molecules. Such irregular forces would result in the observed motions.

In 1905 Einstein carried out a theoretical study of the behavior to be anticipated from such tiny particles, and Perrin made a series of experimental measurements to test the theory. In every respect the observations were such as to confirm the kinetic-theory hypothesis as to their origin. These measurements of Perrin not merely furnished qualitative support for the theory. They provided an experimental method for estimating the number of molecules per unit volume of the fluid in which the

particles were suspended. This was done most conveniently by a measurement of the vertical distribution of the particles.

Earlier in the present chapter it was pointed out that, if two different gases were mixed in the same container, the average kinetic energy for each type of molecule would be the same. Each gas would behave in this respect as though the other gas were not present. Similarly, many other properties of the individual gas would be unaffected. In particular, a detailed study of the kinetic theory shows that each gas will distribute itself vertically in the gravitational field entirely independently of the presence of other gases. An interesting example of this is provided by the Earth's atmosphere, which is a mixture of nitrogen, oxygen, and a number of rarer gases such as helium. If d is the average density of the atmosphere between a level h_1 above the sea and a higher level h_2, the hydrostatic law of pressure developed in Chapter VII states that the pressure p_1 at that level is related to that at the higher level by the formula

$$p_1 - p_2 = dg(h_2 - h_1) \tag{54}$$

This law should also hold for the partial pressures exerted by the individual gases of a mixture. Thus, for example, the decrease in pressure with altitude of a light gas such as helium should be much less rapid than that of a heavier gas such as oxygen. We should expect, therefore, the percentage of helium to be much greater in the higher levels of the Earth's atmosphere than in the lower, an expectation which is well substantiated by observed phenomena.

Returning to the Brownian-movement problem, if the tiny particles under observation are very carefully prepared so that they are of uniform size, they may be thought of as molecules of a very heavy gas, mixed, in the case of smoke particles, with the lighter air molecules. Owing to their relatively great weight, the variation in their numbers with elevation should be quite rapid. The variation in the pressure which they exert can be calculated from (54). The density d, which enters into this formula, can be determined from the gas law (44).

$$d = \frac{m}{V} = \frac{pM}{RT}$$

But the molecular weight M is equal to the number of molecules

in a gram-molecule, N, times the mass of one molecule μ. Hence

$$d = \frac{pN\mu}{RT} \tag{55}$$

If we substitute this value for d in (54), and divide the resulting equation through by p to determine the proportional increase in pressure $(p_1 - p_2)/p$,

$$\frac{p_1 - p_2}{p} = \frac{\mu g(h_2 - h_1)}{RT} N \tag{56}$$

Since by (47) the pressures are proportional to the numbers of molecules present in a given space, we can replace $(p_1 - p_2)/p$ by $(n_1 - n_2)/n$, the ratio of the increase in numbers between h_2 and h_1 to the average number. Or

$$N = \frac{RT}{\mu g(h_2 - h_1)} \frac{n_1 - n_2}{n} \tag{57}$$

Thus, by a study of the distribution of particles at different levels, it is possible to calculate Avogadro's number N. The result obtained by Perrin in 1908 was 6.8×10^{23}. It was checked by entirely independent observations on the average velocity of the particles and by a study of their rate of diffusion. Errors which entered into Perrin's original work have since been corrected, and we now believe this important number to be approximately

$$N = 6.06 \times 10^{23}$$

One is immediately impressed by the hugeness of the number and the resulting minuteness of the molecules themselves. Thus the hydrogen molecule must have a mass μ equal to its molecular weight 2 divided by 6.06×10^{23} or 3.3×10^{-24} gr. The density of hydrogen in the liquid form, in which we believe the molecules to be practically contiguous, is 0.07 gr./cm.^{3} Dividing this by 3.3×10^{-24} yields 2.1×10^{22} as the number of molecules in a cubic centimeter of liquid hydrogen. The reciprocal of this number should give the volume occupied by a molecule, and the cube root of the reciprocal, 3.6×10^{-8} should give the order of magnitude of the diameter of a molecule. Since a gram-molecule of an ideal gas under standard conditions of temperature and pressure occupies 22,400 cc., the number of molecules per cubic centimeter, which by Avogadro's Law should be the same for all gases, is $6.06 \times 10^{23}/22{,}400 = 2.70 \times 10^{19}$.

60. Specific Heats of Gases. Of scarcely less importance than the Brownian movements, as corroboration of the kinetic theory, are the remarkable experimental checks with the predictions of the theory in regard to specific heats. From Equation 50 we see that, in order to raise the temperature of a gas, energy must be delivered to it to increase the kinetic energy of the molecules. If, for example, we wish to raise a gram-molecule 1° C. from T to $T + 1$, an amount of energy would be required which by (50) is

$$(\tfrac{1}{2}\overline{MW^2})_{T+1} - (\tfrac{1}{2}\overline{MW^2})_T = \tfrac{3}{2}R(T + 1) - \tfrac{3}{2}RT = \tfrac{3}{2}R \quad (58)$$

This quantity is known as the *molecular heat* of the substance. Since 1 gram-molecule of substance contains M grams, the molecular heat is seen to be M times the specific heat. According to the kinetic theory it should be a constant for all gases and equal to $\frac{3}{2}R$. This is precisely in accord with experiment if

(*a*) the gas is heated in a container which prevents any simultaneous change in volume, and

(*b*) if the gas is monatomic, i.e., if the molecule consists of a single atom. (*Examples:* He, Hg, Ar, Ne, Kr, Xe.)

The simple result must be restricted to the case of heating at constant volume, since a gas does work in expanding and therefore requires additional energy. In case pressure rather than volume is maintained constant, this additional work is easily calculated. In Chapter X it was shown that a gas in expanding does an amount of work equal to the product of the pressure and the increase in volume. The gas law (44), when applied to a gram-molecule, for which $m = M$, reduces to

$$pV = RT \quad (59)$$

If the temperature is raised 1° to $T + 1$, and V is allowed to increase to V' in such a fashion as to maintain p constant,

$$pV' = R(T + 1) \quad (60)$$

Subtracting (59) from (60),

$$p(V' - V) = R \quad (61)$$

Or the additional energy required if a gram-molecule of gas is to be raised 1° at constant pressure is R. We distinguish these two simple conditions of constant volume and constant pressure, under which a gas can be heated, by designating the molecular

heat in the first case as c_v, the molecular heat at constant volume, and in the second case by c_p, the molecular heat at constant pressure. Clearly

$$c_v = \tfrac{3}{2}R \quad \text{and} \quad c_p = \tfrac{3}{2}R + R = \tfrac{5}{2}R \tag{62}$$

Hence also

$$\frac{c_p}{c_v} = \frac{5}{3} \tag{63}$$

Here again this is the experimentally observed ratio for all monatomic gases.

The reason that our result does not apply to gases of a more complicated structure lies in the fundamental assumptions at the beginning of this chapter. It will be remembered that we neglected the possibility of collisions with other molecules, as our typical molecule shuttled back and forth from one side of the container to the other. This neglect led to no error in the case of monatomic molecules, which can be visualized as tiny, perfectly elastic spheres. If, for example, two such molecules, while travelling at the same speed in opposite directions, enter into a head-on collision, they will rebound with exactly their original speed, and the rôles of the two molecules will be merely interchanged. In the case of collisions between molecules having different speeds, there will be a new distribution of momentum between them, but the loss in momentum of one molecule and its correspondingly reduced contribution to the pressure against a particular wall will be exactly compensated for by the increased momentum of the other molecule.

The difference between monatomic molecules and those of a more complicated structure lies in the possibility that such collisions may transform the translational energy of one molecule into rotational energy of the other. When such an event occurs, a net loss has been suffered in the pressure which these molecules can exert on the walls of the vessel. To be sure, the opposite process is also possible. A rotating molecule may in a collision give up some of its rotational energy to another molecule in the form of a greater translational velocity. But the fact remains that, when such a gas is heated, the energy of rotation as well as the energy of translation of its molecules must be increased. For diatomic molecules such as O_2 and N_2 a more complete study of this problem shows that not $\frac{3}{2}R$ but $\frac{5}{2}R$ should be taken as the molecular heat at constant volume, and for still more complicated structures $\frac{3}{2}R$ should be used. This leads to values of

c_p/c_v which are $\frac{5}{8}$, $\frac{7}{8}$, and $\frac{4}{8}$ respectively. These ratios, in turn, are very well substantiated experimentally.

61. Atomic Heats. The Law of Dulong and Petit. It is interesting that the kinetic picture of matter can be extended with some success from gases to solids. Of course, in a solid the molecules can neither shuttle back and forth as they do in a gas, nor do they in general rotate. They can, however, execute vibrations about their positions of equilibrium, and the detailed studies of Maxwell and Boltzmann led to the conclusion that such molecules would be expected to absorb, on being heated, the same amount of kinetic energy per gram-molecule for a rise of 1° C. as a monatomic gas. In addition, however, such vibrating atoms would be capable of storing up potential energy, for it is easily shown that the average potential energy of any body executing simple harmonic motion is the same as its average kinetic energy. In order, therefore, to raise the temperature of a gram atom of a solid element through 1° C. an amount of energy equivalent to $\frac{3}{2}R$ must be allocated to molecular kinetic energy and an equal amount, $\frac{3}{2}R$, to molecular potential energy. The total amount, $\frac{6}{2}R = 3R$, agrees fairly well with the *atomic heat*, i.e., the product of the specific heat and the atomic weight, of a considerable number of elements.

The R which must be used here can be calculated from the form of the gas law which applies to a gram-molecule, $pv = RT$. Using absolute units, i.e., dynes/cm.^{2} for p, cm.^{3} for v, and °C. for T, R would be obtained in ergs/°C. per gram-molecule. In order to obtain atomic heats in the usual units, cal./°C. per gram-atom, this result must be divided by $J = 4.18 \times 10^7$ ergs/cal. Or

$$3R = 3\frac{pv}{TJ} = 3\frac{(76)(13.6)(980)(22{,}400)}{(273)(4.18 \times 10^7)} = 5.96$$

The following table shows a few atomic heats, experimentally determined in the range between −188° C. and +18° C.

ATOMIC HEATS

Aluminium	4.5	Copper	5.1
Antimony	5.7	Nickel	5.1
Lead	6.1	Sulphur	4.1
Iron	4.8	Selenium	5.2
Cobalt	4.9	Zinc	5.5

The tendency of a considerable number of solids to exhibit atomic heats of about this magnitude was first discovered experimentally by Dulong and Petit in 1818, and has been known as the *Law of Dulong and Petit.* From what we know at present of the very complicated way in which the molecules of a solid may group themselves, and the complexity of the possible motions which they may execute, it is not at all surprising that no greater regularity is observed.

62. The Kinetic Theory and the Second Law of Thermodynamics. In the last chapter we saw that heat was a form of energy fundamentally different from mechanical energy, and in general less available for practical purposes. In the present chapter we have come to recognize heat as kinetic energy of the molecules. If the present viewpoint is correct and heat energy is also mechanical, what is the cause of the basic distinction which is evidenced by the Second Law of Thermodynamics? The answer is not far to seek. Thermal energy is an *incoordinated* hither-and-yon motion of the individual molecules, which to be made available for mechanical purposes must be transformed into a coordinated unidirectional motion. It is this function which the heat engine performs, and to which the Second Law of Thermodynamics sets such a relentless limit. Thermal energy is chaotic, and the conversion of heat into mechanical energy is the recovery of order from chaos. The inexorable increase in entropy, on the other hand, and the corresponding degradation of energy, give evidence of the disorganizing tendency of most natural processes.

PROBLEMS

1. Calculate the gas constant R in the usual units, ergs/°C. per gram-molecule. Calculate its value if assessed, not on a gram-molecule, but on an individual molecule. This last quantity is known as the Boltzmann constant, and is represented by the letter k. How can the average kinetic energy of a monatomic gas molecule be obtained from k?

2. Calculate the root mean square velocity of a molecule of oxygen, (*a*) at 20° C. and (*b*) at 5000° C.

3. Calculate the root mean square velocity of a spherical smoke particle which is composed of carbon (density 0.5), is 5×10^{-5} cm. in radius, and is floating in air at a temperature of 20° C.

4. Calculate the amount of heat required to raise the temperature of a room 10 meters long, 10 meters wide, and 3 meters high from 10° C. to 20° C. (*a*) if the room is so sealed as to prevent the escape of air, and (*b*) if the room is sufficiently open to permit the barometric pressure to remain constant.

CHAPTER XIV

ELECTROSTATICS

We have thus far recognized only two types of forces: those which are exerted by the direct contact of one body on another, and the gravitational forces which are transmitted over immense distances without any apparent connecting medium. These are the forces which enter most directly into our daily experience, but even the ancient Greeks were acquainted with a few samples of those which we to-day call *electric* forces. Thales of Miletus knew that, if amber was rubbed, it would attract to itself light particles of all sorts of materials. As a result of this, the oldest experimentally produced electric phenomenon, our modern word *electricity* is derived from *electron*, the Greek word for amber. Here again we encounter a case, like that of gravitation, in which a force is exerted by one body on another without any visible intervening medium. The ancients were immensely puzzled by such forces, and although to-day after centuries of familiarity we take them for granted, we have arrived at little more than an empirical knowledge of their nature.

63. Electrical Discoveries of the Eighteenth Century. Despite the fact that electric phenomena were known and some experimental work was done in the sixteenth and seventeenth centuries, the first substantial progress toward our modern electric age came during the eighteenth century. One of the most important steps in this advance was the development in the friction machine of a device for producing more intense phenomena. Otto von Guericke in the seventeenth century had shown that a rotating sphere of sulphur could be strongly electrified by holding against it the dry palm of the hand. Stephen Gray in England early in the eighteenth century showed that the entire human body could be electrified if it were suspended by silk cords while the hand was held against such a rotating sphere. He discovered that certain substances, notably the metals, would readily conduct away the electric charge, whereas others such as silk or glass would not. The former materials are called *conductors*, the latter *insulators*. He showed that a charged conductor could be in-

sulated and thus made to hold its charge if placed on a cake of resin.

Slightly later, in France, Dufay made use of this possibility to demonstrate that all bodies including conductors could be electrified by friction. The difficulty in the past had been that the charge leaked away from the conductor as fast as it was produced. Dufay and later Ebenezer Kinnersley of Philadelphia discovered independently that there are two kinds of electricity, which we to-day call positive and negative respectively. If, for example, a hard rubber rod is stroked with a woolen cloth, it becomes charged. If the rod is now touched to a pith ball as it hangs suspended by a silk cord, a portion of its charge is passed on to the ball. If a second ball is electrified in the same fashion, the two will repel each other (Fig. 48). Since both balls received their charge from the same source, we must infer that like charges repel one another. If, on the other hand, a porcelain or glass rod is used in place of the rubber, it also will be electrified, and may be made to pass on some of its charge to a pith ball. If this ball is now brought into the vicinity of one of those electrified by the hard rubber, it will be attracted rather than repelled. A different sort of electrification has taken place, and we conclude that unlike charges attract one another.

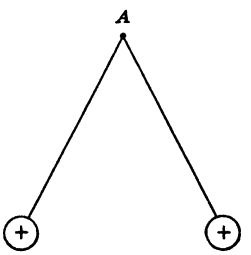


FIG. 48. Repulsion of Pith Balls.

A further study shows that, whenever one body is brought into intimate contact with another of different material, one becomes positively charged and the other negatively charged. In fact, all substances can be arranged consecutively in such a fashion that any material which is brought into contact with one higher up in the list will become negatively charged, whereas if brought into contact with one lower down, it will become positively charged. A short list of this sort is as follows:

fur
wool
glass
silk
hard rubber
sealing wax
resin
sulphur

Dufay thought of the charging process as the addition of an electric fluid to the electrified body. In order to account for the two types of charge he postulated the existence of two kinds of electric fluid, which he called vitreous and resinous respectively. Benjamin Franklin, on the other hand, suggested that a single fluid would account for the observed phenomena if matter exerted an attraction on it, and if the fluid particles repelled one another. He thought of matter as normally permeated by the electric fluid. When an excess was present, the body was positively charged, whereas if it contained less than the usual amount, it was negatively charged. Such a theory was well calculated to explain all electrostatic phenomena, and approximates very closely to our modern views.

64. Distribution of Charge on a Conductor. Electrostatic Induction. In a conductor the electricity is free to move about, whereas in an insulator it is pretty definitely localized to the

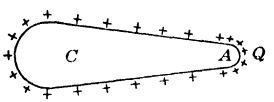


FIG. 49. Distribution of Charge on Conductor.

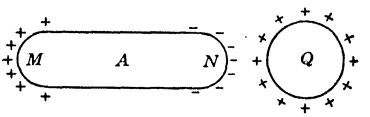


FIG. 50. Electrostatic Induction.

point at which it is deposited. As a result of the freedom with which it moves about on a conductor and the mutual repulsion of the particles of the electric fluid, any charge placed at one point tends to spread itself over the entire surface. Although the repulsive force which one particle exerts on another prevents any penetration of electricity into the metal, the way in which it distributes itself over the surface depends on the shape of the conductor. Fig. 49 shows how their mutual repulsion tends to crowd electric charges into a region where the curvature of the surface is great. It should be remembered that, since the charges on a conductor are free to move about, they will never be at rest unless they are in equilibrium under the action of the forces exerted on them by all the charges in the vicinity. A charge Q near the point A of the conductor C is so located that most of the other charges tend to thrust it still further toward A. Only those which lie beyond it will oppose this tendency. Accordingly, if Q is to be in equilibrium, the density with which the

charges are packed together near A must be decidedly greater than at more remote points.

A related phenomenon which is easily understood with the same hypothesis as to the nature of electricity is that of *electrostatic induction*. If a positive charge Q (Fig. 50) is brought into the vicinity of the conductor A, it will tend to repel the positive electric fluid in A to the far side M, thus leaving the near side N negatively charged. Such charges are known as *induced charges*. Their distribution can best be studied with the aid of the gold leaf *electroscope*, which was invented by Abraham Bennet in 1786. It consists of two strips of gold foil (Fig. 51) suspended from a metal rod R. A box C equipped with glass windows for observing the leaves protects them from air currents. The rod R passes through an insulating plug P in the top of this box, while two plates, joined by conducting leads to the walls, are so placed that they will be touched by the tips of the leaves if they diverge too widely. Usually a plate or ball B is attached to the top of the rod R where it projects from the box.

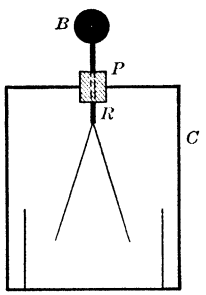


FIG. 51. Electroscope.

If now a positively charged rod is brought into the vicinity of B, a negative charge will be induced on B and a positive charge on the gold leaves. The strips of foil, being charged with the same kind of electricity, repel each other, and diverge until their mutual repulsion is balanced by the downward pull of gravity. The extent to which the leaves diverge is a measure of their charge. If the charged rod is now removed, the induced charges will disappear, and the leaves will collapse. If, however, the finger is placed on B while the charged rod is still held in the vicinity, the induced positive charge will be driven through the body of the operator to the ground, and the leaves will collapse, but the surface of B closest to the charged rod will still possess its induced negative charge. If the finger is now removed from the electroscope, this induced negative charge on B will be definitely isolated. When the rod is removed from the vicinity, it will distribute itself over the leaves, and they will again diverge, this time with a negative charge.

The electroscope, thus charged, is particularly convenient for testing the sign of charged bodies. Thus, if a positive charge

is brought near B, it will tend to induce a positive charge on the leaves, thereby reducing the already existing negative charge, and causing them to collapse. Conversely, if a negative charge is brought up, the leaves will still further diverge. If an insulated conducting disk or *proof plane*, as it is called, is touched to M of Fig. 50, it will receive part of the positive charge located at that point, and if brought near B, it will produce a partial collapse of the gold leaves. If touched to N, however, exactly the opposite effect will take place, thus proving the presence of both types of induced charge.

65. The Friction Machine. The existence of such induced charges and the tendency of electricity to concentrate on the sharper points of a conductor were utilized in the eighteenth century in the development of the friction machine. In its final

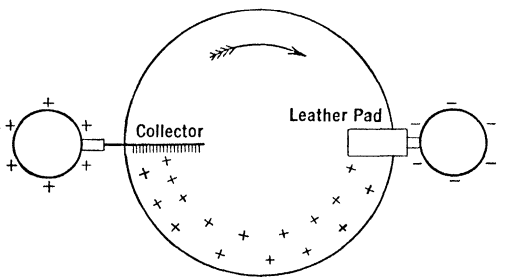


FIG. 52. Friction Machine.

form this extremely important device consisted of a glass disk, which could be rotated against a leather pad (Fig. 52). Opposite the leather pad was a metal rod, to which metallic points were attached like the teeth of a comb. As the disk, positively charged by contact with the leather, passed this comb-like collector, intense negative charges were induced in the sharp points, so intense, in fact, that they were dragged over to the glass by the attraction of the inducing charges. Thus the comb and the connecting ball were left positively charged, and the charge on the glass was neutralized.

The similarity of lightning to the electrostatic phenomena produced in the laboratory suggested to Benjamin Franklin the idea that it also might be an electric discharge. If the thunder clouds were for some reason electrically charged, and if the lightning bolt was simply the discharge to an induced charge on the Earth, might not the same device of sharp points be used

to collect the charge from the clouds? His suggested experiment of mounting a sharply pointed rod on an insulating stand at some high elevation, and thus attempting to extract some of this atmospheric electricity, was first tried with complete success in France. His own later successful experiment with a kite is too well known to require repetition. Once the electrical nature of thunder storms was demonstrated, his suggestion regarding the use of lightning rods with sharp points for the protection of buildings was a logical step.

66. Coulomb's Law. Although much unpublished work was done by Cavendish in England before 1773, the first reported quantitative experiments were carried out by Charles Augustin Coulomb in France between 1785 and 1789. Using a torsion balance very similar to that employed by Cavendish in 1798 to measure the gravitational attraction of ponderable bodies, he studied the forces of attraction and repulsion between electric charges. His apparatus is shown in Fig. 53. A small conducting sphere A was mounted at the end of an insulating rod R. This, in turn, was suspended by a fine silver wire from the torsion head Q. If the sphere A is positively charged, and is brought into the vicinity of a second positively charged sphere B, the repulsion between them will tend to twist the rod R about the silver suspension. If, however, Q is turned in the opposite direction by the proper amount, any actual rotation of the rod can be prevented, and from the amount through which Q is turned the force between A and B can be estimated. A series of measurements was made with different distances separating the two spheres, and it was discovered that the force between the charges varied inversely as the square of the distance, exactly as with gravitational forces. In order to test the effect of varying charges, the apparatus was so arranged that a third conducting sphere C exactly the same size as B could be brought in contact with B. If C was originally uncharged, the initial charge on B would distribute itself equally between the two spheres. If C was now removed, the force exerted by B on A with exactly one-

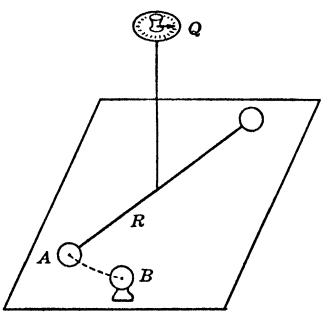


FIG. 53. Measurement of Electrostatic Repulsion.

half of the original charge could be measured. Such experiments showed that the force was strictly proportional to the charge. Since no units for measuring charge were in use up to his time, Coulomb very naturally defined a unit charge as one which would exert a unit force (1 dyne) on an equal charge at a distance of 1 cm. With this unit the two laws of force between charges could be combined into the simple law

$$F = \frac{e_1 e_2}{r^2}$$

where e_1 and e_2 are the respective charges, r is their distance apart in centimeters, and F is the force which they exert on each other in dynes. It differs from the law of gravitational attraction (Formula 11, Chapter IV) only in the absence of the constant G, which is rendered unnecessary by the choice of units.

PROBLEMS

1. Calculate the force of attraction between a positive charge of 5 electrostatic units and a negative charge of 20 electrostatic units if they are 20 cm. apart.

2. Two pith balls each weighing $\frac{1}{10}$ gr. are suspended from a common point A (Fig. 48) by means of two light silk cords each 10 cm. in length. What positive charge must each bear in order that their mutual repulsion shall hold them 10 cm. apart?

CHAPTER XV

ELECTRIC POTENTIAL

In general, when an electric charge is brought into the vicinity of other charges, it experiences a force. Any such region within which a body is subject everywhere to a definite force is known as a *field of force*. As a charge moves about in an electric field, work may be done on it by the field at the expense of its electrical potential energy, in just the same way that a body moving to a lower level in the Earth's gravitational field has work done on it at the expense of its gravitational potential energy. Conversely, outside work may be required to move such a charge to a point of greater potential energy. If, however, after any such journey it returns to its starting point, the net amount of work required to describe the entire closed path must have been zero, for, if the charge itself and all other charges in the field occupy exactly their initial positions, it is difficult to see where any balance of energy could have been derived or deposited. The charge has described a complete cycle, and the entire electric field is exactly as it was at the start of the cycle.

67. Electric Potential. If, in order to describe a part of such a cycle, from A to B, for example, in Fig. 54, an amount of external work W has been required, the charge in returning to A will recover this amount of work from the field regardless of the particular path by which it makes the return journey from B to A. Since A and B are any two arbitrary points, we must conclude that the work done in carrying a unit charge from one point to another in an electrostatic field does not depend on the path chosen for the journey but only on the location of the initial and final points. Since this work is due to the difference between the potential energy of the unit charge when it is located at A and B respectively, it is natural to call it the *difference in potential* between A and B. If work is done by the field in carrying the charge from A to B, A is said to be at a higher po-



FIG. 54

tential than B; if work has been required from an external force, B is at the higher potential.

In a conductor the charges are free to move about, and hence any static charge which it may possess must be so distributed that every portion of it is in equilibrium. The resultant force exerted on it by the rest of the charge is zero. A small charge, therefore, can be moved from one point of the conductor to any other point without the expenditure of any work. In other words, a conductor on which all charges are at rest possesses everywhere the same potential. For many purposes the Earth itself may be considered a very fair conductor, and it is frequently useful to define the potential of the Earth as zero. With such a definition of zero potential, the difference in potential between the Earth and any other body is taken as the potential of that body. For certain problems, however, it is useful to imagine the charges in which we are interested as isolated from all other charges or conductors, and in such cases it is customary to define zero potential as that of a point infinitely distant.

68. Potential Due to a Point Charge. Perhaps the simplest as well as one of the most interesting special cases which we

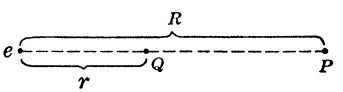


FIG. 55

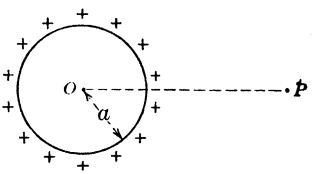


FIG. 56. Electric Field near Charged Sphere.

might consider is that of the potential in the vicinity of an isolated point charge. In Fig. 55 a unit positive charge, originally located at P, R centimeters from a point charge e, is moved to Q at a distance of r centimeters from e. The *intensity of the electric field* at any point is defined as the force in dynes which the field at that point exerts on a unit charge. It is usually represented by the letter E. At P this intensity, calculated by Coulomb's Law, is e/R^2; at Q it is e/r^2. The calculation of the work required for this change of position from P to Q is complicated by the fact that the force is constantly increasing during the motion, but if we assume that the average force is equal to the geometric mean of its initial and final values, as can be

proven by a more rigorous analysis, this average force will be

$$F = \sqrt{\frac{e}{r^2} \times \frac{e}{R^2}} = \frac{e}{rR}$$

The work done will be

$$F(R - r) = e\frac{R - r}{rR} = e\left(\frac{1}{r} - \frac{1}{R}\right)$$

If we denote the potential at P by V_p and that at Q by V_q,

$$V_q - V_p = e\left(\frac{1}{r} - \frac{1}{R}\right)$$

If $R = \infty$, $V_p = 0$ by definition, and hence

$$V_q = \frac{e}{r}$$

The importance of this conclusion is considerably increased by an interesting property of spherical conductors. Owing to the complete symmetry of a sphere, any charge on a spherical conductor will spread itself uniformly over the entire surface. If we wished to estimate the effect of such a spherically distributed charge at a point P outside (Fig. 56), we should observe that some portions of the charge are nearer to P than the center of the sphere, while other portions are at a greater distance. It can be shown, however, that the resultant effect is exactly as though the entire charge were concentrated at the center of the sphere. Thus, if e is the total charge on the sphere, the intensity of the electric field at P is e/R^2 if R is the distance from O to P. Similarly the potential at P is e/R. A case of particular interest is that of a point on the surface of the sphere itself. If a is its radius, the potential at the surface should be e/a. It is, therefore, proportional to the charge and inversely proportional to the radius. Within the sphere it can be shown that the intensity of the electric field is everywhere zero, and the potential, therefore, constant and equal to that at the surface.

69. Capacity. If V is the potential of the sphere,

$$V = \frac{e}{a}$$

Or

$$e = aV \tag{64}$$

The charge which is required to raise a body to unit potential is known as the *capacity* of the body. If we set $V = 1$ in (64),

$e = a$, and the capacity of the sphere must be numerically equal to its radius.

Since the potential at any point outside the sphere is e/R, it must be constant over any spherical surface which is concentric with the conductor itself. Such a surface over which the potential is constant is known as an *equipotential surface.* If a spherical conducting shell of radius b is placed about the charged sphere (Fig. 57), it will occupy one of these equipotential surfaces. Although there will be an induced charge of opposite sign on its inside surface, the effect of such a spherically distributed charge is zero at points within, and it will in no way affect either the electric field or the potential at points between it and the original sphere. Hence the potential of its inner surface must be e/b. The difference in potential between sphere and shell is, therefore,

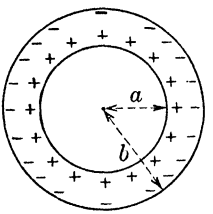


FIG. 57. Spherical Condenser.

$$V_a - V_b = \frac{e}{a} - \frac{e}{b} = e\left(\frac{b-a}{ab}\right)$$

Or

$$e = \frac{ab}{b-a}(V_a - V_b) \tag{65}$$

Such a device, in which two oppositely charged surfaces face each other, is known as a condenser. The reason for such a term is readily seen. If the collector of a friction machine is attached to a conducting sphere and the machine is set in operation, the sphere will increase in charge until it comes to the same potential as the glass plate. No amount of continued operation will further increase either charge or potential. If a larger quantity of electricity is to be collected, a larger sphere must be used (Formula 64), or better, a device with two opposing conducting surfaces such as the sphere surrounded by the spherical shell. From (65) it is seen that, if the separation of the two surfaces $(b - a)$ is reduced, the charge e is increased even with the same potential difference $(V_a - V_b)$. Large quantities of electricity can thus be collected and "condensed" on the opposing surfaces by relatively small differences of potential.

It should be noted from (65) that the charge on the spherical condenser is proportional to the difference in potential between these opposing surfaces, a fact which can be shown to be true

for condensers of any shape. As with the single sphere, the capacity of the device is defined as the charge which it carries for unit difference of potential between its surfaces. If we denote this capacity by C, and the difference in potential by the single letter V,

$$e = CV \tag{66}$$

From (65) it is clear that the capacity of the spherical condenser is

$$C = \frac{ab}{b - a} \tag{67}$$

If the separation of the surfaces, $d = b - a$, is small compared to the radius of the sphere, this capacity is approximately a^2/d. Since the total area of the sphere is $4\pi a^2$, the capacity per unit area of the spherical condenser is $\frac{1}{4\pi a^2} \cdot \frac{a^2}{d} = \frac{1}{4\pi d}$. Any small portion of the surface of a very large sphere is approximately plane, and we conclude that the capacity of two plane parallel conductors at a distance d apart is approximately $\frac{1}{4\pi d}$ per unit area or $\frac{A}{4\pi d}$ for an area A.

70. Dielectric Constant. Coulomb's Law of force between two charges, as it has thus far been stated, applies only to the case in which the attracting charges are separated by a vacuum. The discrepancy if air is the separating medium is very slight indeed, but if any liquids or solids are substituted, the force is always reduced by a factor ϵ, which is characteristic of the substance. Thus

$$F = \frac{e_1 e_2}{\epsilon r^2} \tag{68}$$

The quantity ϵ is known as the *specific inductive capacity* or the *dielectric constant* of the material. If a substance of dielectric constant ϵ is placed between the plates of a condenser, a simple repetition of the previous work with the quantity ϵ included will show that the capacity of the condenser is increased in proportion to ϵ. For the parallel plate condenser this yields

$$C = \frac{A\epsilon}{4\pi d} \tag{69}$$

Below is a table of the dielectric constants of some of the commoner materials

DIELECTRIC CONSTANTS

Crown glass	6.96
Mica	5.66–5.97
Paraffin	2.1
Cast sulphur	4.22
Air	1.00059
Water	81.07
Ethyl alcohol	25.8
Petroleum	2.13

The first condenser was accidentally discovered in 1745 when Ewald Georg von Kleist in Pomerania attempted to charge a bottle from a friction machine. He attached the collector to a small nail in the bottle while holding the bottle in one hand. This hand formed one plate of the condenser, and the nail, acting as the other plate, became so intensely charged that he received a stunning shock when he touched it with the other hand. The same discovery was made independently the next year in Leyden, a circumstance which gave the device its name of *Leyden jar*. It soon took the form of a glass jar coated inside and out with tin foil. The spectacular character of the discharge from one or more of these jars did much to stimulate interest in electric phenomena during the eighteenth century.

71. The Voltaic Cell. During the last quarter of that century a series of epoch-making investigations were being carried out in Italy by Galvani and Volta, which culminated in the discovery by Volta in 1800 of a chemical means for obtaining steady electric potentials. He showed that if a strip of copper and one of zinc are dipped into a cup of dilute sulphuric acid, a steady difference of potential is established between them. This potential difference is very much smaller than those which can be obtained by frictional methods, so small in fact that, if the two plates are connected directly to the stem and casing of an electroscope, no divergence of the leaves will be observed. If, however, as Volta showed, the stem of the electroscope is equipped with a flat plate P (Fig. 58), and if a second plate P', provided with an insulating handle H, is insulated from

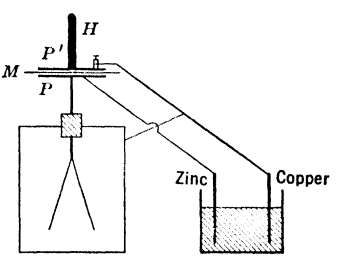


FIG. 58. Condensing Electroscope.

the first by a sheet of mica M, the difference in potential between the copper and zinc can be immediately detected. Connect, for example, the copper plate simultaneously to P' and the electroscope case, and touch the under plate momentarily with a wire attached to the zinc strip. If there is a difference in potential between the strips, the condenser formed by P and P' will now be charged to this potential. The greater part of the charge on the under plate will be bound by the attraction of the charge on P' to the top surface of P. If the top plate is now lifted upward by means of the insulating handle, the capacity of the condenser will be greatly reduced without altering in any way the charge. The difference in potential between the plates, and therefore between the gold leaves and the electroscope casing, must correspondingly increase. This, in turn, requires an increased charge on the gold leaves, which is obtained from a redistribution of the charge on P. This increased charge results in a divergence of the leaves, thus giving evidence of the initial potential difference between the copper and zinc plates. If the sign of the charge is now tested by bringing up a charged rubber rod, the leaves will still further diverge, proving them to be already negatively charged. If the experiment is reversed, and the copper rather than the zinc plate is brought into contact with the gold leaves, the charge will be found to be positive.

Volta showed that the potential difference obtainable could be increased by building up a series of such cells, connecting the copper plate of one cell to the zinc plate of the next, and so on. In this way batteries of cells could be constructed with sufficient potential to produce measurable divergence in an electroscope without recourse to the auxiliary condenser.

Perhaps the most significant feature of Volta's experiment was the persistence with which the necessary charge was supplied. As often as the condenser was discharged and the zinc and copper plates were again connected to it, an equal charge was once more furnished. Here was a source of electricity infinitely more reliable and suitable for experimental work than the old friction machine.

The most important results which were to come from this discovery, however, were to be in the field of *current electricity*. Not merely would the cell repeatedly charge a condenser when it was discharged, but if the zinc and copper electrodes were themselves joined by a conductor, there took place a steady flow of

electricity from one to the other. Such a flow is known as an electric current. The first evidence of its existence was furnished only six weeks after Volta described his experiments in a letter to the president of the Royal Society in London. William Nicholson and Sir Arthur Carlisle constructed a battery in accordance with Volta's directions, and joined its terminals by a conductor which included a vessel filled with acidulated water (Fig. 59). As the electric current passed through this vessel, it demonstrated its existence by effecting for the first time the decomposition of water into its constituents, hydrogen and oxygen. The oxygen appeared as bubbles of gas at the electrode B connected to the copper or positive pole of the battery, while the hydrogen appeared in a similar way at the negative pole A.

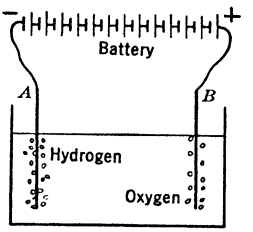


FIG. 59. Electrolysis of Water.

PROBLEMS

1. What is the potential at a distance of 10 cm. from a positive charge of 100 E.S.U.?

2. How much work is required to carry a charge of 10 units from a point 10 cm. distant from the 100 E.S.U. charge of the last problem

(*a*) to a point 1 cm. distant?

(*b*) to a point 10^{-8} cm. distant?

3. Using the dimensions given in Section 17, calculate the capacity of the Earth. What must be the area of a plate condenser in order that it may have this same capacity? Assume the plates to be separated by mica $\frac{1}{100}$ mm. thick.

4. If the Earth is charged to a potential of 1 E.S.U., what electric force will it exert on a unit electrostatic charge at its surface?

5. The hydrogen atom contains a negative charge of 4.77×10^{-10} E.S.U. at a distance of $.53 \times 10^{-8}$ cm. from a positive charge of the same magnitude. How much energy would be required to separate them to an infinite distance apart?

6. Apply the same considerations and the data of Section 17 to calculate the energy required to carry a mass of one gram from the surface of the Earth to a point infinitely distant. Calculate the number of calories required. From a comparison with familiar heats of combustion, what do you infer as to the possibility of leaving the Earth by a rocket ship operating on present fuels?

CHAPTER XVI

MAGNETISM

Magnetite, an iron ore which occurs in Asia Minor, possesses in its native state the property of attracting to itself small bits of iron. Such pieces of ore are called lodestones, and were a source of much interest to the ancients. Not merely do lodestones themselves exert an attraction on particles of iron, but pieces of iron which have been rubbed in contact with them acquire the same property. Just when it was discovered that a needle, magnetized in this way, would orient itself in an approximately north and south direction is unknown, but this fact was certainly known and utilized by the Chinese in the eleventh century.

72. Fundamental Magnetic Phenomena. In 1600, William Gilbert in England published a notable book, "De Magnete," in which the fundamentals of magnetism were for the first time clearly stated. In general, a magnet, whether a piece of a natural lodestone or an artificially magnetized iron bar, possesses two opposite poles. If it is so suspended that it is free to turn about a vertical axis, one of these poles, called the north-seeking or simply the north pole, is attracted toward the north, while the other or south pole is urged southward. If a second magnet is brought into the vicinity of the first, its north-seeking pole is found to repel that of the suspended magnet, whereas its south-seeking pole attracts it. Like poles repel, and unlike poles attract.

The reason for the tendency of a compass needle to orient itself in a northerly direction was a source of much speculation in the middle ages, but it is to Gilbert that we owe the suggestion, now definitely proven, that the Earth itself behaves like a gigantic lodestone with one pole not far from the northern extremity of its axis of rotation and the other near its south pole.

Although in many ways the phenomena of magnetism suggest the related phenomena of electrostatics, they differ in a number of important respects. Unlike positive or negative electric charges, individual north or south poles never exist by them-

selves. They always occur in pairs, and any attempt to isolate a single pole by cutting a magnet in two results in the formation of two magnets, each having both a north and a south pole. The phenomena of electrostatics are universal, and can be observed in all sorts of materials, but substantial magnetic effects are to be detected only in the three related elements, iron, cobalt, and nickel. Electrostatic phenomena are transient. The charges leak away whenever even a very imperfect conductor joins the charged body to the Earth, whereas a good permanent magnet may be handled at will over a period of many years without seriously impairing its magnetization.

In other respects there is a very striking similarity between electric and magnetic phenomena. In both cases the forces are exerted by one body on another without any intervening medium, and there are two types of magnetic poles which obey the same laws of attraction and repulsion as electric charges. It was shown by Coulomb, by exactly the same methods which he used in studying electric charges, that the force exerted by one pole on another is inversely proportional to the square of the distance, and although there was no such simple way available for halving a magnetic pole as there was for an electric charge, he could at least show that two equal poles exerted just twice the force of one alone. It was natural, therefore, to define a unit magnetic pole as one which, when placed at a distance of 1 cm. from an equal pole, would repel it with a force of 1 dyne, and to state the law of attraction or repulsion in the form.

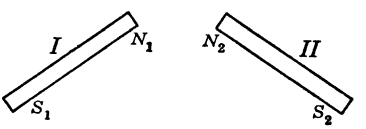


FIG. 60. Repulsion of Bar Magnets.

$$F = \frac{m_1 m_2}{r^2} \tag{70}$$

where m_1 and m_2 are the pole strengths of the two repelling poles. There are two difficulties in the practical application of this formula, which are not encountered in electrostatics. It is never possible to deal with the force exerted by just one north pole on another (Fig. 60) because of the necessary presence of the two south poles. The resultant force exerted by the magnet I on the magnet II is made up of the force exerted by N_1 on N_2,

that exerted by S_1 on S_2, that exerted by N_1 on S_2, and that exerted by S_1 on N_2. Furthermore, the location of the poles of the magnet cannot be inferred with any accuracy simply by inspection. Although in a good magnet they are located not far from the extremities of the bar, they are not exactly at the ends, and their position can be determined only by rather careful experiment.

73. The Earth's Magnetic Field. As part compensation for these difficulties, we are provided in the Earth's magnetic field at any one location with a very uniform field of force, for which we have no analogue in electrostatics. Just as in electrostatics, the intensity of this or any other magnetic field is defined as the force in dynes which it exerts on a unit magnetic pole. It is usually represented by the letter H. The unit of field intensity thus defined is known as the *gauss* in honor of the great German mathematician and physicist. A knowledge of the intensity of the Earth's field is of so much importance that the methods by which it is obtained merit a brief consideration.

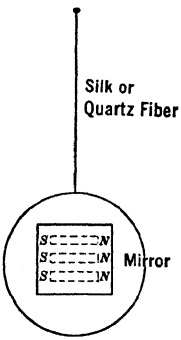


FIG. 61. Magnetometer.

The instrument which is used for all fundamental magnetic measurements is known as the *magnetometer*. It consists of a tiny mirror (Fig. 61) which is suspended by a fine silk or quartz fiber. To the back of the mirror are cemented several short pieces of magnetized steel.

As these little magnets tend to orient themselves in a north and south direction, the instrument should be so mounted that it will face east or west.

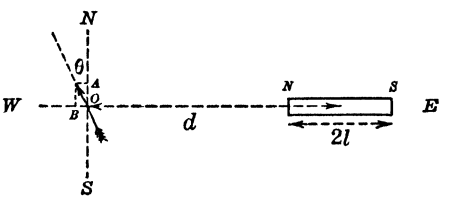


FIG. 62. Magnetometer in Field of Bar Magnet.

If a bar magnet NS (Fig. 62) is placed directly east of the magnetometer needle with its axis in the EW direction, and its north pole facing the instrument, the north pole of the needle will be repelled and the south pole attracted as shown in the figure. The needle will come to rest in a position of equilibrium under the action of the Earth's field H_e and that of the magnet H_m. If m

is the pole strength of the magnet, l the half distance between the poles, and d the distance from its center to the magnetometer needle, the field H_m can be calculated as follows. It is the resultant of the field due to the north pole of the magnet, $m/(d-l)^2$, and that due to the south pole $m/(d+l)^2$. Since these forces have opposite directions, the resultant field will be

$$H_m = \frac{m}{(d-l)^2} - \frac{m}{(d+l)^2} = \frac{4mld}{(d^2-l^2)^2}$$

The product of the length of the magnet $2l$ and its pole strength is of considerable importance. It is known as the *moment* of the magnet, and is represented by the letter M. In terms of this quantity

$$H_m = \frac{2Md}{(d^2-l^2)^2} \tag{71}$$

If m' is the pole strength of the magnetometer needle, the force exerted on each of its poles by H_m is $m'H_m$. The moment exerted by each of these forces on the needle is $m'H_m(OA)$, and the total moment is $2m'H_m(OA)$. Similarly the moment exerted by the Earth's field will be $2m'H_e(OB)$. Since the needle is in equilibrium under these two moments,

$$2m'H_m(OA) = 2m'H_e(OB)$$

Or

$$\frac{H_m}{H_e} = \frac{OB}{OA} = \tan\theta \tag{72}$$

if we denote by θ the angle through which the needle has been deflected from the NS direction.

Substituting the value of H_m from (71) in (72),

$$\frac{M}{H_e} = \frac{(d^2-l^2)^2}{2d}\tan\theta \tag{73}$$

The angle θ is obtained by noting with the help of a telescope and scale the angle through which the magnetometer mirror is turned. The value of M can be measured, at least theoretically, by determining the force which one magnet exerts on another, but in practice it is much more convenient to suspend it in a stirrup (Fig. 63) and observe its oscillations in the Earth's field. The greater the value of M, or the more intense the Earth's field at the point of the experiment, the greater is the rapidity with which the oscillations take place. The period of

this oscillation can be readily calculated by the fundamentals of simple harmonic motion developed in Chapter IV. It was shown there (Formula 14) that the time T of one complete oscillation of a mass m subject to a restoring force per centimeter of displacement F/x is

$$T = 2\pi \sqrt{\frac{m}{\frac{F}{x}}} \tag{14}$$

We are here concerned with a case of rotational motion in which the mass m in this formula should be replaced by the moment of

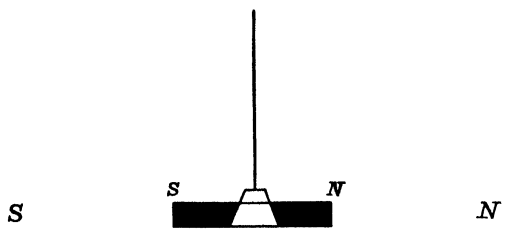


FIG. 63. Suspended Magnet.

inertia I of the magnet, the force F by the restoring moment L exerted by the Earth's field, and x by the angle θ through which the magnet is displaced from its equilibrium position.

Hence
$$T = 2\pi \sqrt{\frac{I}{\frac{L}{\theta}}} \tag{14'}$$

From Fig. 64 it is clear that the moment exerted by the Earth's field on each pole is $H_e m(OB)$, which for small angles of displacement is approximately equal to $H_e m(l\theta)$. The total restoring moment is, therefore, $L = 2H_e ml\theta = H_e M\theta$. L/θ is, accordingly, simply $H_e M$.

Hence
$$T = 2\pi \sqrt{\frac{I}{H_e M}} \tag{74}$$

and

$$MH_e = \frac{4\pi^2 I}{T^2} \tag{75}$$

Since both the product and the ratio of M and H_e are known, either can be immediately determined. The value of H_e obtained in this way is, of course, only the horizontal component

of the total field intensity. Its value at New York is approximately 0.17 gauss, but it varies from time to time, owing to magnetic storms and to more or less regular changes which are as yet little understood.

The north magnetic pole of the Earth is located in the region of Hudson Bay at about 78° latitude and 100° west longitude; the south magnetic pole is nearly opposite in the south polar region. Directly over either magnetic pole a needle which is free to turn about a horizontal axis will point vertically down-

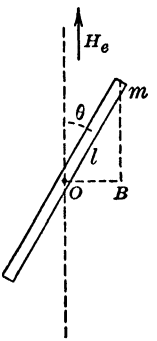


FIG. 64. Moment Exerted by Earth's Field.

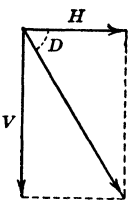


FIG. 65. Angle of Dip.

ward, and only at a point on the Earth's surface midway between the two poles will it set itself in a horizontal position. Throughout the greater part of the northern hemisphere the north-seeking pole will dip below the horizontal by an amount which is known as the angle of dip. At New York this angle is approximately 73°. Since the needle will set itself in the direction of the resultant field, it is clear from Fig. 65 that the ratio of the vertical to the horizontal component of the Earth's field will yield the tangent of the angle of dip.

$$\frac{V}{H} = \tan D \tag{76}$$

A measurement of the horizontal component, combined with the angle of dip, will, accordingly, yield both the vertical component and the resultant field.

74. The Magnetic Field Due to an Electric Current. The many points of similarity between electrostatic and magnetic phenomena undoubtedly suggested some relation between the

two even during the eighteenth century. Nevertheless it remained for Hans Christian Oersted in 1819 to demonstrate experimentally that such a relation really existed. In his experiments with galvanic cells of the sort described by Volta he had repeatedly tried to deflect a compass needle by bringing it into the vicinity of a wire carrying a current. In every instance, however, he had pointed the wire directly toward the needle without effect. It was merely by chance on one occasion that he placed the wire parallel to the needle and immediately observed a tendency on the part of the needle to set itself at right angles to the wire. A magnetic field was indeed produced by the wire, but it was *perpendicular* to the direction of the current.

Among those who were stimulated to further investigation by this discovery was André Marie Ampère in Paris. He

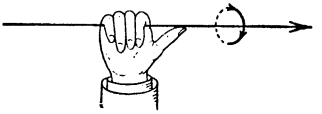


FIG. 66. Ampère's Rule.

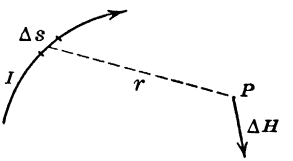


FIG. 67. Law of Biot and Savart.

showed that the magnetic field encircles the wire, tending at every point to urge a north pole in the direction of the fingers of the right hand if the wire is so grasped (Fig. 66) that the current flows in the direction of the thumb.

75. The Law of Biot and Savart. The Ampere. At about the time that Ampère was working in Paris, Biot and Savart succeeded in showing that the total magnetic field due to a complete electric circuit can be thought of as the sum of the effects due to all the little elementary pieces into which the circuit can be cut (Fig. 67). Any one such element of a circuit Δs produces at an arbitrary point P in the vicinity a field which is perpendicular both to Δs itself and to the line joining Δs with P. Its magnitude is inversely proportional to the square of the distance between Δs and P and directly proportional to the length of the element Δs. Although in most cases the calculation of the field due to an entire circuit involves a rather difficult summation of the effects of the individual elements, there is one particularly simple arrangement, which can be easily handled.

It had been shown by Ampère that, if a wire is coiled into a

spiral, the field at the center of the coil assumes very substantial proportions. Consider, for example, the case of a single turn of wire in the form of a circle (Fig. 68). The magnetic field at P due to Δs will, by Ampère's rule, be such as to urge a north pole into the paper. This will likewise be true for any other little element of the coil, and hence all the elements will combine to produce an intense field at P. If we denote the field at P due to Δs by ΔH, and apply the observation of Biot and Savart,

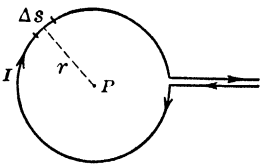


FIG. 68. Magnetic Field Due to Circular Coil.

$$\Delta H \propto \frac{\Delta s}{r^2}$$

Summing up the effects of all the elements, and noting that the sum of the lengths of the elements is the circumference of the circle, while the radius r is the same for all elements,

$$H \propto \frac{2\pi r}{r^2} = \frac{2\pi}{r}$$

If two turns of wire are wrapped about the same circle, a simple observation shows that the magnetic field is doubled. Two coincident wires carrying the same current should, however, be equivalent to a single wire carrying twice the current. We must conclude, therefore, that the field is proportional to the current. Or, if we denote the current by I,

$$H \propto \frac{2\pi I}{r} \tag{77}$$

Since no agreement had yet been reached as to the units in which the current should be measured, it was natural to choose a unit in which the proportionality sign in (77) could be replaced by an equality sign.

$$H = \frac{2\pi I}{r} \tag{78}$$

For this purpose the unit current was defined as one which, flowing about a single circular coil of unit radius (1 cm.), will exert a force of 2π dynes on a unit pole at the center. This is the *electromagnetic unit of current.* It is frequently called the *absamp.* In this system the unit charge is defined as the quan-

tity of electricity which flows by any point in the circuit in 1 sec. when unit current is flowing. The total quantity which flows past in t seconds is, therefore, It.

The *ampere*, which lies at the basis of the so-called *practical* system of units, is defined as $\frac{1}{10}$ of an absamp, and the *coulomb* the unit of charge in this system, is defined as the quantity which passes a point in the circuit each second when one ampere is flowing.

76. The Tangent Galvanometer. Nearly all current-measuring instruments depend for their action on the magnetic field which is set up by the current being measured. The simplest of all such instruments, and one which has been much used in the past for the calibration of other types, is the *tangent galvanometer*. It consists of a vertical coil of wire (Fig. 69), which is set with its plane in the magnetic meridian (north and south). At the center of the coil is mounted a short compass needle, the orientation of which can be observed against a circular scale or by a mirror and telescope as in the magnetometer. When no current is flowing through the coil, the needle is held in the north and south direction by the Earth's field H. When a current of I absamps flows, it produces a magnetic field in the east-west direction of magnitude $2\pi nI/r$ if n is the number of turns on the coil and r is its radius. The ratio of this field to the Earth's field is by (72) the tangent of the angle θ through which the needle is deflected. Hence

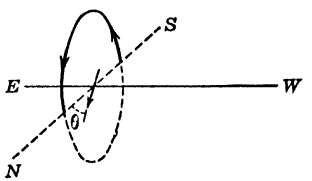


FIG. 69. Tangent Galvanometer.

$$\frac{\frac{2\pi nI}{r}}{H_e} = \tan\theta$$

Or

$$I = \frac{H_e r \tan\theta}{2\pi n}$$

If the Earth's field has been correctly measured, this furnishes a means of measuring directly a current I in absamps. The current in amperes is obtained by multiplying this result by 10.

77. Rowland's Experiment. An experiment of the greatest significance in this connection was performed by Professor Henry

Augustus Rowland of Johns Hopkins University in 1889. The purpose of the experiment was to determine whether or not a moving electrostatic charge was capable of producing the magnetic effects which are exhibited by an electric current, whether, in fact, it might or might not be legitimate to think of current electricity as moving electrostatic charges. The difficulty in performing the experiment lay in the fact, which will be discussed more in detail in Chapter XVIII, that the charges which are encountered in electrostatic experiments are very minute in comparison to those which move in electric currents.

Rowland mounted a hard rubber disk in such a fashion that it could be spun about its axis at a high rate of speed. He also provided means for placing on it a heavy electrostatic charge. If an electric current is simply a moving electrostatic charge, and if conversely a moving electrostatic charge constitutes a current, the rotating charged disk should be equivalent to a coil of wire carrying a current, and should produce along its axis a magnetic field identical with that produced by the coil of a tangent galvanometer. The experiment consisted, therefore, in searching for this magnetic field with a sensitive magnetometer needle suspended close to the disk and on its axis. Despite the technical difficulties involved, the experiment was a success, and left no doubt as to the fundamental identity of electric currents and moving static charges.

78. Force Exerted by a Magnetic Field on a Wire Carrying a Current. In Fig. 68 the coil exerts a force of $2\pi I/r$ dynes on a unit pole at P. According to the law of action and reaction it must itself experience an equal and opposite force from the magnetic pole. Since the force on a north pole located at P is toward the paper, the force which such a unit pole exerts on the coil must be directly out from the paper, and must have a magnitude of $2\pi I/r$ dynes. The force on each centimeter of the wire is obtained by dividing this total force by the length of the coil $2\pi r$, a calculation which yields I/r^2 dynes per unit length. This result can be stated in a little better form by introducing in place of r^2 the intensity of the magnetic field in which the coil finds itself. Every point of the coil is at a distance r from the unit pole, and is, therefore, in a field of intensity $H = 1/r^2$. The force acting on unit length of the coil, I/r^2, is accordingly HI. Throughout its length this field is perpendicular to the wire, and we conclude that a wire carrying a current of I absamps at

right angles to a magnetic field of intensity H is acted on by a force of HI dynes per unit length or HIl dynes for a total length l. It should be noted that this force is perpendicular both to the current and to the magnetic field. Its sense can always be derived from Ampère's rule by remembering that in Fig. 68 it is the reaction of the force which the coil exerts on the pole at P. Perhaps the most convenient of the various rules which have been given for remembering this direction is the following. If the forefinger of the right hand points in the direction of the current and the second finger in the direction of the magnetic field (i.e., in the direction in which a north pole is urged), the thumb will point in the direction of the required force.

79. Suspended Coil Current-Measuring Instruments. If a bar magnet is bent (Fig. 70) somewhat in the form of a horseshoe so that the north and south poles face each other, a fairly uniform and intense magnetic field is provided between them. If a coil of wire is now suspended in this field and a current is passed through it in the direction indicated in the figure, every wire on the left-hand side of the coil will be carrying the current downward while those on the right-hand side will be carrying it upward. On each left-hand wire there will be a force out from the paper, while each right-hand wire will be pushed toward the paper. All these forces will combine to produce a rotation of the coil, and the magnitude of the turning moment exerted by the magnetic field will be a measure of the current.

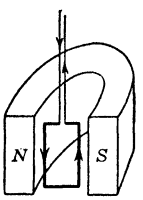


FIG. 70. Moving Coil Galvanometer.

In the D'Arsonval type of wall galvanometer this principle is used to produce a very sensitive current-measuring instrument. A light coil with many turns of fine wire is suspended by a thin phosphor bronze strip between the poles of a strong permanent magnet. When a small current is passed through the coil, the magnetic field rotates the coil until its torque is just balanced by the twist of the suspension. A small mirror is attached to the coil, and the amount of rotation is observed by a telescope and scale. With such an instrument it is possible to detect currents as small as 10^{-10} ampere.

The same general principle is utilized in nearly all portable current-measuring instruments such as ammeters and milliammeters. The coil is mounted on jewel bearings between the

poles of a magnet, and is provided with a pointer which moves over a circular scale as the coil rotates.

PROBLEMS

1. What is the force in dynes which a north pole of 10 E.M.U. exerts on a second north pole of 50 E.M.U. at a distance of 10 cm.?

2. Two magnets, each 10 cm. long and having poles of 20 E.M.U. intensity, rest side by side and parallel at a distance of 10 cm. apart. Calculate the total force which one exerts on the other.

3. One of the magnets of the preceding problem is placed with its axis east and west and with its center 50 cm. from a magnetometer needle. If the strength of the horizontal component of the Earth's field is 0.17, through what angle will the needle be deflected?

4. The magnetometer needle of the last problem is placed at the center of a vertical coil of wire with its plane in the magnetic meridian (N and S). If the coil consists of 10 turns, each with a radius of 15 cm., what current in amperes will be required to deflect the needle through 45°? What is the strength of the magnetic field in gausses at the center of the coil when a current of 1 ampere is flowing through the coil?

5. A straight wire 1 meter long is carrying a current of 10 amperes. What force is exerted on it by the Earth's magnetic field

(*a*) if it is vertical,

(*b*) if it is horizontal and lies in a north and south direction?

Assume an angle of dip of 73° and a horizontal component of 0.17 gauss.

6. A square coil hangs in the region between two magnetic poles where the field strength is 100 gausses as shown in Fig. 65. If the length of one side of the square is 1.5 cm., and if there are 100 turns in the coil, calculate the turning moment which a current of 10^{-6} ampere will exert on the coil.

CHAPTER XVII

THE ELECTRIC CURRENT AND ENERGY

Two phenomena have thus far been described in which an electric current may be made to furnish energy. At the close of Chapter XV it was pointed out that, if a current is passed through slightly acidulated water, the water is separated into its fundamental constituents, hydrogen and oxygen. Since the reunion of hydrogen and oxygen, i.e., the combustion of hydrogen, is productive of a large amount of heat, their separation must be accomplished at the expense of energy furnished by the current. Similarly, the force which a magnetic field exerts on a wire carrying a current may be used to do work on the wire. In fact, the electric motor depends for its action on the utilization of this force. Still a third way in which a current can deliver energy is familiar to all in the heating of wires through which a current passes. A current is, therefore, capable of furnishing chemical, mechanical, and thermal energy.

80. Joule's Law. A very simple experiment suffices to show that the heat produced by an electric current is proportional to the time t during which the current flows and to the square of the current I. This last law, first stated by Joule, is known as *Joule's Law*. If we denote the heat produced by H,

$$H \propto I^2 t$$

Or

$$H = I^2 R t \tag{79}$$

where R is a factor of proportionality.

81. Electromotive Force. It is not difficult in general to locate the source from which this energy comes. If, for example, the current flows from a galvanic cell, zinc from the electrode passes into solution. A chemical reaction is taking place in which zinc and sulphuric acid are replaced by hydrogen and zinc sulphate.

$$Zn + H_2SO_4 = H_2 + ZnSO_4$$

This is a chemical process which would normally yield a definite

amount of heat, substantially more than is observed when it takes place in the galvanic cell to produce a current. Energy is, therefore, furnished by the chemical reaction. It is transmitted into the wire by the electric current, and there appears as heat. The amount of zinc which goes into solution is proportional to the total quantity of electricity which flows, i.e., to It, where I is the current and t is the time during which it flows. The reason for this proportionality will be discussed in the next chapter. The amount of energy derived from the chemical process for each unit charge which passes through the cell is known as the *electromotive force* of the cell, and is represented by the letter E. The total energy derived thus during the passage of a current I for the time t is, therefore, EIt. This, in turn, by the Law of the Conservation of Energy should be equal to the energy which appears as heat in the wire. Thus

$$EIt = I^2Rt$$

Or

$$E = IR \tag{80}$$

The current should, therefore, be proportional to the electromotive force. This law was discovered experimentally by George Simon Ohm in 1826, and is known as *Ohm's Law.*

It seemed well to choose a unit in which to measure electromotive force approximately the size of the original galvanic cell. This unit is called the *volt*, and is defined as an electromotive force such that 10^7 ergs of energy are furnished by the cell for every coulomb passing through it. This unit of energy, 10^7 ergs, is called the *joule.* Since a calorie is equivalent to 4.18×10^7 ergs, it is equivalent to 4.18 joules. The corresponding unit of power, the joule per second, is called the *watt*; 746 watts = 1 horse-power.

82. Electric Resistance. From (80) a better idea can be gained of the significance of the proportionality constant R. With a cell of given electromotive force E as a source, the current I will be inversely proportional to R. R measures, therefore, the opposition of the circuit to the passage of the current, and is called the *electric resistance.* The unit in which R is measured in the practical system is called the *ohm*, and might be defined in accordance with (80) as the resistance through which an electromotive force of 1 volt will drive a current of 1 ampere. It was shown by Ohm that the resistance of any uniform conductor such as a copper wire was proportional to the length of

the conductor and inversely proportional to the area of its cross section.

$$R = \frac{\rho l}{A} \tag{81}$$

The constant of proportionality ρ is known as the *resistivity* of the material from which the conductor is made. Although ρ for all conductors is enormously less than for non-conductors, it varies very greatly from one metal to another. In the following table a few typical resistivities are given.

Substance	Resistivity	Temperature
Silver	1.5–1.7×10^{-6} ohm cm.	0° C.
Copper	1.54×10^{-6}	"
Aluminium	2.6–3 "	"
Iron	9.7–12 "	"
German silver	30 "	"
Constantin	49 "	"
Gas carbon	5000 "	"
Dry wood	5–10×10^{8}	
Distilled water	0.5×10^{6}	18° C.
Ethyl alcohol	0.3×10^{6}	15
Glass	9×10^{13}	20
Paraffin	3–3.0×10^{16}	
Sulphur, prismatic	70–390×10^{13}	

It should be noted that the fundamental electrical units in the practical system have been so chosen that Ohm's Law (80) gives directly the relation between current, electromotive force, and resistance in amperes, volts, and ohms respectively, while Joule's Law (79) gives the corresponding relation between the heat produced in joules, the current in amperes, and the resistance in ohms. Similarly the power spent in a resistance in watts is I^2R, and the power delivered by an electromotive force E in watts is IE.

83. Fall in Potential along a Conductor. If a current I passes through a resistance R, the heat liberated in the time t is I^2Rt. The number of coulombs flowing past any point of the wire in the same time is It. Since this same quantity of electricity enters at one end of the wire and leaves at the other end during the same period, the current I amounts to a transfer of It coulombs from one end of the wire to the other in t seconds. The heat produced by the transfer of 1 coulomb is, therefore, $I^2Rt/It = IR$ joules. The energy from which this heat was

derived existed in the form of potential energy in the electricity when it entered the wire, and it would accordingly require an expenditure of energy equivalent to IR joules to carry each coulomb as it emerged from the wire back to its starting point. By definition, however, the difference in potential between the extremities of the wire is precisely this, the work required to carry a unit charge from one end to the other. We accordingly conclude that the difference in potential V between the ends of the wire in joules per coulomb, i.e., in volts, is IR.

$$V = IR \tag{82}$$

The fact that both electromotive force and potential difference are expressible in the same units, volts, leads often to a confusion of the two concepts. They are, however, far from being interchangeable terms. Consider, for example, a galvanic cell which is delivering a current I to a resistance R (Fig. 71). The electromotive force E measures the chemical energy which is handed over to every coulomb passing through the cell, and is independent of the rate at which it flows through. A part of this energy, under favorable conditions the greater part, is spent in heating the resistance R. But under all circumstances a portion will be spent in heating the electrolyte between the plates of the cell. This latter part can be taken into account by attributing to the cell an *internal resistance* r. The drop in potential across the external resistance R and therefore between the terminals of the cell is IR, while that taking place in the internal resistance is Ir. The sum of these two potential drops is the potential energy which the cell must deliver to each coulomb, i.e., the electromotive force E. Thus

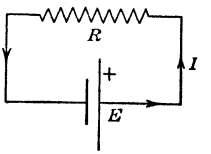


FIG. 71

$$E = IR + Ir$$

If we denote by V the drop in potential IR between the terminals of the cell,

$$V = E - Ir \tag{83}$$

If the current drawn from the cell I is zero, this drop in potential is equal to the electromotive force. Just so soon, however, as the cell begins to deliver current, the drop in potential between its terminals begins to fall below its electromotive force, and the two cease to be identical. Electromotive forces can be pro-

vided in various ways, but in all cases they are accompanied by the equivalent of the internal resistance of the cell, and the potential difference which they provide always falls as the current increases. The power supply of a modern radio receiver is an example of a different type of voltage source in which there is a marked decrease in voltage as the current drawn by the receiving set is increased. Such a source is not designed to deliver a very great current, and the few milliamperes taken by a fairly good voltmeter have in most cases an appreciable effect on the voltage.

84. Galvanic Cells. A good galvanic cell should furnish a steady, reliable electromotive force, and should have a low internal resistance. The original voltaic cell was, however, defective in both respects, since the necessary accumulation of hydrogen on the copper anode tended to obstruct the passage of the current, and at the same time affected the voltage. This difficulty, known as *polarization*, has been overcome to a large extent in modern cells by replacing the copper electrode by carbon impregnated with manganese dioxide or a bichromate, which rapidly removes the hydrogen. The corrosive acid is also replaced by an ammonium chloride solution. In the so-called dry cells the solution is held between the electrodes in some porous material such as blotting paper. Such cells are capable of delivering momentary currents up to 30 amperes or more, but are of little value for furnishing steady currents greater than a quarter of an ampere.

For larger currents, storage batteries with lead plates and sulphuric acid as electrolyte can be used. When fully charged, the positive electrode is heavily coated with lead oxide, PbO_2, while the negative electrode is pure lead. As the cell delivers current, the hydrogen arriving at the anode reduces the PbO_2 to PbO, which in turn is converted by the sulphuric acid to lead sulphate, $PbSO_4$. Lead sulphate is also formed at the negative electrode. As with the original galvanic cell, these chemical reactions furnish the energy required for the electric current, and possess the advantage of being completely reversed when a current is passed through the cell in the opposite direction. Accordingly, after the cell has been delivering current for some time, it is possible to restore both electrodes to their original condition by passing a current through the cell from the positive to the negative pole.

85. Thermal Electromotive Force. The Seebeck Effect. In addition to the chemical methods for producing an electromotive force, there are purely thermal methods by which heat is directly converted into electrical energy. Thus if two copper wires are connected to the ends of an iron wire (Fig. 72), and if the two junctions are maintained at different temperatures, an electromotive force will exist between them. If the free ends of the copper wires are now connected to a galvanometer, a steady current will be indicated just so long as the difference in temperature between the junctions is maintained. The current so obtained could be used to excite an electromagnet and raise weights, or it could be used directly to run a motor. In either case, there would be an indirect conversion of heat into mechanical energy, and the device would constitute a heat engine. While the current is flowing, the hotter junction is extracting heat from the heat source in which it is located, and the colder junction, in accordance with the laws of thermodynamics, is delivering heat to its surroundings. The process as a whole must, of course, be subject to Carnot's efficiency theorem just as any other heat engine. This method of producing an electric current was discovered by Johann Seebeck in 1821, and is called for him the *Seebeck effect.* It was utilized by Ohm in his experiments in 1826, as he found that the electromotive force so produced was much more reliable than that which he could obtain from a voltaic cell. The electromotive force, although not strictly proportional to the difference in temperature between the junctions, depends only on this difference and the materials of which the junctions are made. The Seebeck effect has accordingly found a very wide application in the measurement of temperatures, particularly those which are too high or too low for a mercury thermometer. By building up a so-called *thermopile* from a great many tiny junctions in series, it is possible to produce an instrument which is exceedingly sensitive to minute changes in temperature. Thermopiles have been used with great success in measuring the heat from individual stars and for many very delicate measurements in the laboratory.

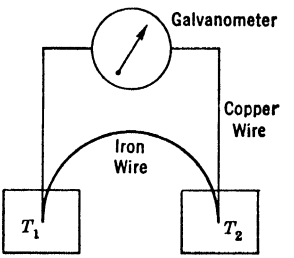


FIG. 72. Thermocouple.

86. The Measurement of Resistance. The Wheatstone Bridge. The earlier methods of measuring resistance were dependent directly on equation (80), and required a known and constant source of potential. In 1833, S. Hunter Christie in England suggested a device, later described by the more famous Sir Charles Wheatstone, which has since been known as the *Wheatstone bridge.* It consists of four wires, a, b, r, and x, arranged in the form of a quadrilateral (Fig. 73). a, b, and r have known resistances, while x is the unknown to be measured. The leads from a galvanometer G are connected between M and N, and a galvanic cell is placed between P and Q. When the keys K_1 and K_2 are pressed, a current flows from the cell, splitting at P into two parts, one part passing through a and the other through r. The resistances a, b, and r are so adjusted that no current flows through G. Under these circumstances, according to (82), M and N must be at the same potential, and the drop in potential between P and M must equal that between P and N. Or, if i_1 is the current through a and i_2 that through r,

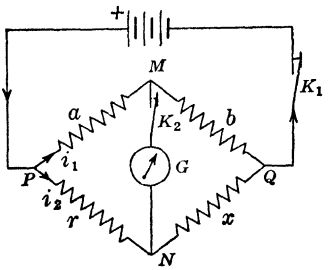


FIG. 73. Wheatstone Bridge.

$$i_1 a = i_2 r \tag{84}$$

Similarly

$$i_1 b = i_2 x \tag{85}$$

Dividing (85) by (84)

$$\frac{b}{a} = \frac{x}{r}$$

Or

$$x = \frac{b}{a} r \tag{86}$$

The method is in no way dependent on the steadiness of the cell which furnishes the current.

There are two common forms in which the bridge is used. In the simpler type the resistances a and b are the two segments of a german silver wire, the leads to the galvanometer being connected to M by a sliding contact. If the wire is uniform, the ratio of b to a is the same as the ratio of the lengths of the corresponding segments of the slide wire, and the unknown resistance x is obtained by multiplying the known resistance r

by this ratio. In the so-called box form of bridge a and b take the form of carefully constructed coils, which are usually provided with resistance ratios of 0.001 to 1, 0.01 to 1, 0.1 to 1, 1 to 1, 10 to 1, 100 to 1, and 1000 to 1. r consists of a bank of resistances which can be adjusted to an accuracy of one part in ten thousand by turning contact buttons. Such a box is capable of measuring resistances to an accuracy of better than 0.1 per cent.

87. Ammeters and Voltmeters. The *ammeter* or current-measuring instrument consists in general of a coil suspended in a magnetic field. Its sensitiveness depends on the number of turns in the coil, the strength of the magnetic field, and the stiffness of the suspension. It is common practice to use a fairly sensitive instrument and to reduce this sensitiveness when necessary by a *shunt*, i. e., a suitable low resistance in parallel with the coil. Fig. 74*a* shows such an instrument being used to measure the current flowing from A to B. Let I be the total current to be measured, i_1 and i_2 the currents through instrument and shunt respectively, r the resistance of the instrument, and S that of the shunt. The drop in potential through the instrument is $i_1 r$ and that through the shunt $i_2 S$. Since these drops in potential are equal,

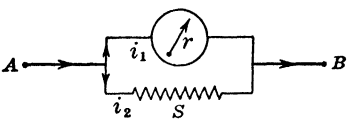


FIG. 74*a*. Ammeter with Shunt.

$$i_1 r = i_2 S$$

Or
$$i_2 = \frac{r}{S} i_1 \tag{87}$$

But
$$i_1 + i_2 = I \tag{88}$$

Or substituting (87) in (88)

$$i_1 + i_1 \frac{r}{S} = i_1 \left(1 + \frac{r}{S}\right) = I \tag{89}$$

If, for example, the shunt is intended to reduce the sensitiveness of the instrument to one-tenth of its original value,

$$i_1 = \frac{1}{10} I$$

Substituting this value for i_1 in (89),

$$\frac{1}{10} I \left(1 + \frac{r}{S}\right) = I$$

Or $$1 + \frac{r}{S} = 10$$

$$S = \frac{1}{9}r$$

If ρ is the overall resistance of the instrument including the shunt, the drop in potential across it is $I\rho$. If the ammeter is being used to measure the current delivered by a battery to some piece of apparatus, the difference in potential applied to the apparatus will be less than that at the terminals of the battery by this amount. In order to minimize the loss in potential which the use of an ammeter entails, the resistance ρ of the instrument should be kept as low as possible, and it is for this reason that a shunt rather than a series resistance is used to reduce its sensitiveness.

An instrument of exactly the same type can be used to measure potential difference. Since the drop in potential across an ammeter of resistance R carrying a current I is IR, it is merely necessary to know the resistance of the instrument in order to calculate the voltage between its terminals. The scale of the instrument may, of course, be calibrated to indicate this potential in volts. Fig. 74*b* shows such a voltmeter being used to measure the potential applied by the battery B to the instrument A. If the source of the current B has an internal resistance, the potential supplied will decrease as the current increases. In order, therefore, to have as little effect as possible on the potential, the voltmeter should have the highest possible resistance. It is for this reason that any necessary reduction in the sensitiveness of a voltmeter is accomplished by a series rather than a shunting resistance. If, for example, it is desired to reduce its sensitiveness to one-tenth of the original value, a series resistance R is used such that the drop in potential resulting from the current I is ten times as great as that across the instrument alone. Thus

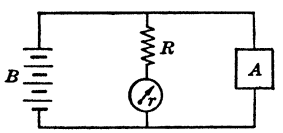


FIG. 74*b*. Use of Voltmeter.

$$(r + R)I = 10rI$$

Or $$r + R = 10r$$

Or $$R = 9r$$

88. The Potentiometer. If a voltmeter is used to measure the electromotive force of a cell, its reading will necessarily be somewhat in error, since even the small current taken by a good voltmeter will lower the terminal voltage somewhat below the electromotive force.

It is sometimes desirable to measure the drop in potential between two points in a circuit without drawing any current for the measuring instrument and thereby disturbing the circuit under observation. This can be done either by an *electrostatic voltmeter*, which operates on a principle similar to that of an electroscope, or by a *potentiometer*. In the potentiometer a galvanic cell passes a current through the resistance r between A and B of Fig. 75. An auxiliary circuit containing the voltage to be measured, V, a galvanometer G, and a key K is connected between A and C. In some simple potentiometers the resistance r is a slide wire similar to that in a Wheatstone bridge, and the position of C is altered by moving a contact along the wire. In more elaborate instruments r consists of sets of coils so arranged that the proportion of the total resistance between A and C can be varied by turning contact buttons. In any case, the position of C is so chosen that no current passes through the galvanometer G and therefore through the auxiliary circuit. If i is the current through r, and if R is the resistance of that portion of r between A and C, the drop in potential between A and C is iR. Since no current is flowing through the auxiliary circuit between A and C, the only drop in potential which will be encountered in passing from A to C through V will be V itself. Hence

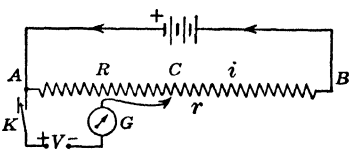


FIG. 75. Potentiometer.

$$V = iR \tag{90}$$

If a standard cell of known voltage E_0 is now substituted for the potential V which is being measured, and if R' is the new resistance between A and C when no current flows through the galvanometer,

$$E_0 = iR' \tag{91}$$

Dividing (90) by (91) and solving for V,

$$V = \frac{R}{R'} E_0$$

If the potential V is that across the terminals of a galvanic cell, it is strictly equal to the electromotive force of the cell, since it was measured with no current flowing.

The value of the potentiometer as an instrument for comparing potentials has made the development of reliable standard cells a matter of vital importance. Fortunately a number of such cells have been devised, among which the Weston cell, with a remarkably constant electromotive force of 1.0184 volts at 20° C., is the most commonly used.

PROBLEMS

1. A coil of insulated wire is immersed in a beaker containing 100 gr. of water. If the resistance of the coil is 20 ohms, and if a current of 2 amperes is passed through it, how long will be required to raise the temperature of the water from 20° C. to boiling? Neglect the heat capacity of the calorimeter.

2. Calculate (*a*) the power in watts spent in the coil of the preceding problem, (*b*) the total number of joules furnished by the current, (*c*) the difference in potential between the terminals of the coil.

3. Calculate (*a*) the current required to operate a 500-watt incandescent lamp on a 110-volt line, and (*b*) the resistance of such a lamp.

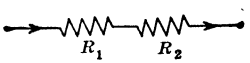


FIG. 76*a*. Resistances in Series.

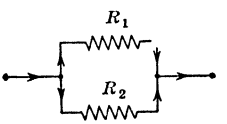


FIG. 76*b*. Resistances in Parallel.

4. If a difference of potential V is impressed across two resistances R_1 and R_2 in series, Fig. 76*a*, show that the current is $V/(R_1 + R_2)$ and that a single resistance $R_1 + R_2$ is equivalent to the two resistances when so connected.

5. If a difference of potential V is impressed across two resistances R_1 and R_2 in parallel, Fig. 76*b*, show that the total current is $V(R_1 + R_2)/R_1R_2$, and that, if R is a resistance equivalent to the two in parallel,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

6. What is the resistance of a copper wire $\frac{1}{10}$ mm. in diameter and 10 meters long? What would be the resistance of a wire of the same dimensions made of aluminium?

7. A certain circuit consists of a resistance of 5 ohms in series with two cells, one of which has an E.M.F. of 1 volt and an internal resistance of 2 ohms, and the other has an E.M.F. of 2 volts and a resistance of 1 ohm. Calculate (*a*) the current through this circuit, (*b*) the drop in potential across the resistance, (*c*) the drop in potential across each cell. Assume the positive pole of one cell to be connected to the negative pole of the other cell.

8. Repeat the preceding problem for the case that the positive poles of the two cells are joined to each other.

9. A galvanometer has a resistance of 100 ohms and suffers a deflection of 1 cm. when a current of 10^{-7} ampere passes through it. What shunting resistance is necessary to convert it into a direct-reading ammeter such that 1 ampere will produce a deflection of 10 cm.? Assume that the deflection is proportional to the current.

10. What series resistance would be required to convert the galvanometer of the preceding problem into a direct-reading voltmeter, for which a 10-cm. deflection would indicate a potential of 1 volt?

CHAPTER XVIII

ELECTROLYSIS. THE ATOMIC NATURE OF ELECTRICITY

In Chapter XV it was pointed out that one of the first results achieved by the galvanic cell was the electrolysis of water. If two platinum electrodes are dipped into a vessel filled with acidulated water, and are then connected to the terminals of a battery, a current flows through the water, the existence of which is evidenced by bubbles of gas which appear at both electrodes. If the containing vessel is of the shape shown in Fig. 77, and if the platinum electrodes A and B are sealed into the base of the two columns M and N, the gas formed at each electrode will be collected in the column above it. After collection it can be drawn off by the stop cocks P and Q and analyzed. If this is done, it will be found that the gas evolved from the *cathode* or negative electrode is hydrogen, whereas that from the *anode* or positive electrode is oxygen. Furthermore, a careful measurement would show that the volume of the hydrogen at the same pressure is always exactly twice that of the oxygen.

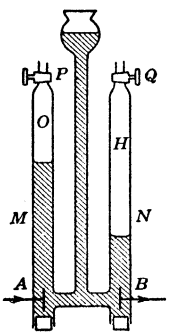


FIG. 77. Apparatus for Electrolysis of Water.

89. Faraday's Laws of Electrolysis. The chemical formula of water is H_2O, that of hydrogen is H_2, and of oxygen is O_2. There are, therefore, in 2 molecules of water, 2 molecules of hydrogen and 1 of oxygen. According to Avogadro's Law, equal numbers of molecules should, under the same conditions of temperature and pressure, occupy equal volumes. Hence the appearance of twice as much hydrogen as oxygen in electrolysis is consistent with the view that they are the products of water which has been decomposed by the electric current. These facts were recognized by Avogadro in 1811, and undoubtedly helped lead him to his law of gases. The purely electrical laws governing electrolysis could not, however, be appreciated until

the work of Oersted (1819), Ampère, and Ohm (1826) had provided a practicable method of measuring electric currents.

Between 1831 and 1834, Michael Faraday, the greatest experimenter of the nineteenth century, carried out a series of investigations in this field, in which he discovered all the fundamental laws which have since formed the basis of *electrochemistry*. He showed in the first place that the amount of gas evolved is proportional to the total quantity of electricity, It, which flows through the electrolyte. It does not depend on the strength of the current by itself, on the size of the electrodes, or on the concentration of the acid. The same law is found to apply, not merely to the electrolysis of water, but also to that of all other solutions. It is well known, for example, that, when a current is passed through a solution of copper sulphate, copper is deposited on the cathode. Faraday showed that the weight of copper deposited is strictly proportional to It, and does not depend on any other factors.

He next tried the experiment of comparing the amounts of different elements which are deposited in electrolysis by the same quantity of electricity, and was able to show that, if the same current is passed through a number of different electrolytes, acidulated water, copper sulphate, silver nitrate, etc., for the same length of time, the amounts deposited are proportional to the *combining weights* of the elements in question. The combining weight of any element which can unite chemically with hydrogen is defined as the mass of material which, in case of such union, combines with 1 gr. of hydrogen. Thus the combining weight of oxygen in water (H_2O) is 8, since 1 gram-atom of oxygen, 16 gr., unites with 2 gram-atoms of hydrogen, 2 gr. The number of atoms of hydrogen which unite with 1 atom of a given element is known as the *valence* of the given element. Thus the valence of oxygen is 2, and the combining weight of oxygen is seen to be the ratio of its atomic weight to its valence.

There are, of course, many elements which do not unite with hydrogen, but if such a material combines with another substance which, in turn, combines with hydrogen, the valence and the combining weight of the element in question can still be logically defined. Thus an atom such as calcium, which combines in calcium oxide, CaO, with 1 atom of oxygen, should have the same valence as oxygen, 2. The atomic weight of calcium is 40.07. Hence 40.07 gr. of calcium combines with 16 gr. of oxygen, and

since 16 gr. of oxygen is capable of combining with 2 gr. of hydrogen, it is natural to define the combining weight of calcium as 40.07/2. The combining weight, defined in this way, is once more the ratio of the atomic weight to the valence.

It frequently happens that an element will have one valence in one compound and a different valence in another. Thus iron can form two chlorides, ferrous chloride, $FeCl_2$, in which its valence is 2 (since the valence of chlorine is 1), and ferric chloride, $FeCl_3$, in which its valence is 3. In ferrous chloride the combining weight of iron is its atomic weight 55.84 divided by 2; in ferric chloride it is 55.84 divided by 3. It is these combining weights to which the masses of material deposited in electrolysis are proportional.

Elements having a valence of 1 are known as *univalent*, and for all such the combining weight is equal to the atomic weight. Thus 107.88 gr. of silver, a univalent element, would be deposited by the same quantity of electricity which deposits 1 gr. of hydrogen. Since, however, 107.88 gr. of silver and 1 gr. of hydrogen each constitute a gram-atom of these respective materials, and since, as was shown in Chapter XIII, a gram-atom of every substance contains the same number of atoms (6.06×10^{23}), it follows that the number of atoms of all univalent elements deposited by the same quantity of electricity is the same. This fact at once suggests that each univalent atom as it moves through the solution carries with it the same charge as every other univalent atom, and that this charge is deposited with it on the cathode.

Our picture, therefore, of the conduction of electricity through an electrolyte is that of charged atoms or *ions*, as they are called, moving along in the solution and carrying with them the charges which are eventually handed over to the electrodes. Since the combining weight of a *bivalent* atom (valence 2) is one-half that of a univalent atom, just half the number of gram-atoms are deposited by the same quantity of electricity, which in turn means half the number of atoms. Each ion must, therefore, carry just twice the charge of the univalent ion. Similarly, the trivalent ion carries a threefold charge.

90. The Fundamental Charge on a Univalent Ion. The actual charge in coulombs carried by a univalent ion can be determined in a single experiment by weighing the mass of silver deposited by a measured number of coulombs. Thus, if I

amperes flow through a solution of silver nitrate for t seconds and deposit m grams of silver, the number of coulombs required to deposit 1 gr. would be It/m, and that required to deposit a gram-atom would be ItA/m if A is the atomic weight. This has been found by careful experiment to be 96,500 coulombs. Because of the significance of this quantity of electricity, it has been called the *faraday* in honor of Michael Faraday. Thus 1 faraday = 96,500 coulombs. It is easily seen that the quantity of electricity Q expressed in faradays which is required to deposit a mass m of an element of atomic weight A and valence n is

$$Q = \frac{mn}{A} \tag{92}$$

Since there are 6.06×10^{23} atoms in a gram-atom, the charge on each univalent ion is $96{,}500/6.06 \times 10^{23} = 1.59 \times 10^{-19}$ coulomb. This represents, accordingly, the smallest charge which is encountered in electrolysis, and there was some speculation even during the latter part of the nineteenth century as to whether it might not be the ultimate atom of electric charge, the smallest charge which was capable of independent existence. These speculations were strikingly corroborated by a group of entirely different experiments which were performed at just about the beginning of the present century. Historically the first of these was carried out by J. J. Thomson in England in 1898, but the later experiment of Robert A. Millikan in Chicago in 1909 was so convincing and at the same time capable of such high precision that it will be described here.

91. The Ratio of the Electromagnetic to the Electrostatic Unit of Charge. In order to appreciate the factors which are involved in Millikan's experiment, an understanding is necessary of the relative magnitude of the electrostatic unit of charge, which is used to measure the size of electrostatic charges, and the coulomb, which is used in dealing with current electricity. The determination of the ratio of these two units is one of the most fundamentally important in the whole field of electricity and magnetism. The best direct measurement of it was made in 1907 by Rosa and Dorsey at the Bureau of Standards in Washington. One of their methods consisted in charging a spherical condenser and then discharging it through a carefully calibrated galvanometer. By means of an automatic switching device the condenser was repeatedly charged and discharged so rapidly that

the galvanometer gave only a steady deflection indicative of the average current flowing through it. This average current $\bar{I}$ consisted of the quantity of electricity q with which the condenser was each time charged, multiplied by the number of times which it was charged and discharged per second. Or $\bar{I} = nq$. q was determined in electrostatic units by multiplying the capacity of the condenser by the potential in electrostatic units to which it was charged. The capacity of the condenser, in turn, was obtained from its geometric dimensions and Formula 67 of Chapter XV. On the other hand, q in coulombs was obtained by dividing the $\bar{I}$ of the galvanometer reading by n. The result of the experiment showed that the coulomb was excessively large in comparison with the electrostatic unit, being, in fact, about 3×10^9 times as great. The ten times greater electromagnetic unit was, accordingly, 3×10^{10} times as large as the electrostatic unit.

The first conclusion to be drawn from this result is that, despite the relatively high potentials (several thousand volts) which are available in simple electrostatic experiments, the quantity of electricity involved is usually very small compared to that which is moving in even very moderate electric currents. On the other hand, it is usually very large compared to the charge on an individual ion. Consider, for example, the charge which a potential of 100 volts would place on a sphere of radius 1 cm. The capacity of such a sphere, as pointed out in Chapter XV, is equal to its radius or 1 E.S.U. Two points are at a difference of potential of 1 volt if the work required to transfer a coulomb from one point to the other is 1 joule, while the electrostatic unit of potential is such that 1 erg is required for the transfer of 1 E.S.U. of charge. Since the coulomb is equal to 3×10^9 E.S.U., and the joule is 10^7 ergs, the E.S.U. unit of potential $= 3 \times 10^9/10^7 = 300$ volts. 100 volts is therefore equivalent to $\frac{1}{3}$ E.S.U., and the charge on the sphere in E.S.U. would be

$$q = 1 \cdot \tfrac{1}{3} \text{ E.S.U.} = \frac{\frac{1}{3}}{3 \times 10^9} = \tfrac{1}{9} \times 10^{-9} \text{ coulomb}$$

This would still be $(\frac{1}{9} \times 10^{-9})/(1.59 \times 10^{-19}) = 7 \times 10^8$ times the charge on a univalent ion. If, as has been suggested, the smallest possible electric charge is that on a univalent ion, there would still be several hundred million such charges on the sphere, and it would be altogether hopeless to try to prove that its charge

was an exact multiple of the fundamental charge. If, however, the radius of the sphere could be reduced to microscopic dimensions, it would seem to be possible to place on it charges of the same order of magnitude as that on a single ion, and if only exact multiples of such a charge existed, it should be possible to demonstrate this fact experimentally.

92. The Atomic Nature of Electricity. Millikan's Experiment. Millikan, in his experiments, used for the charged spheres minute drops of oil, which were formed by spraying from an atomizer. Such of these drops as came in intimate contact with the sprayer were thereby automatically charged, and required no additional

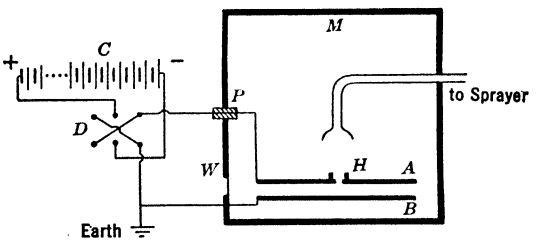


FIG. 78. Millikan's Oil Drop Experiment.

charging process. The apparatus for detecting and measuring this charge is shown in Fig. 78. Two parallel brass plates, A and B, were mounted in a horizontal position in the metal box M. The lower plate was connected to the box and to the Earth. A storage battery C of several thousand volts was so placed that, by means of the commutator D, either pole could be connected through the insulating plug P to the upper plate, while the other pole was connected to the Earth and thus to the lower plate. The charged oil drops were led from the sprayer into that region of the box above the plates. A small hole H was drilled through the upper plate, so that, as the drops slowly fell under the action of gravity, a few would find their way into the space between the two plates. This region was illuminated through the window W, and was viewed by a microscope perpendicular to the plane of the paper. Any drop of oil entering the field of the microscope would be brightly illuminated from the side, and would appear like a bright star against the dark background of the back of the box.

If the drop is uncharged or if the storage battery is disconnected, such a drop will be observed to move gradually across

the field of view, because of the downward pull of gravity. The rate of fall will be slow owing to the retarding viscous force of the atmosphere in which it moves, and, like the raindrop of Chapter VIII, it will fall with a uniform velocity. Since the downward pull of gravity is proportional to the mass of the drop and, therefore, to the cube of its radius, while the retarding viscous forces are proportional to the area of its cross section, that is, to the square of its radius, the gravitational force decreases more rapidly than the retarding force as the size of the drop is reduced. The smaller the drop, therefore, the slower will be its rate of fall. It was shown mathematically by Stokes that a sphere dropping through a viscous medium moves with a velocity v such that

$$v = \frac{2}{9}\frac{\rho a^2 g}{\eta} \tag{93}$$

where ρ is the density of the drop, g is the acceleration of gravity, a the radius of the drop, and η is a constant known as the *coefficient of viscosity* of the medium, which measures the amount of its viscosity. Knowing the material of which the drop is made and therefore its density, it is possible to calculate from the observed velocity of fall, v, the radius of the drop. With the extremely minute drops which were used in Millikan's experiment, this proved to be the best method of measuring their size. It was found by him that Stokes's Law did not hold in all strictness for such minute drops, but he was able to modify it in such a way as to get a working method for making this necessary measurement.

If, now, the positive and negative poles of the battery are connected respectively to the plates A and B, there will be established a difference of potential and a consequent electric field in the region between A and B. If the separation of the plates is small compared to their diameter, as was the case in Millikan's experiment, this resulting electric field is quite accurately uniform except in the vicinity of the edges. If the difference in potential between the two plates is V volts, i.e., $V/300$ E.S.U., the work necessary to carry a unit electrostatic charge from one plate to the other would be $V/300$ ergs or $Vq/300$ ergs for a charge of q electrostatic units. Let E be the strength of field between the plates, which, as has been pointed out, is practically uniform. The force which would be experienced between the plates by a unit charge would be E, and that experienced by a

charge q would be Eq. If d is the distance apart of the plates, the work done in carrying the charge q from one plate to the other would be Eqd.

Hence $$Eqd = \frac{Vq}{300}$$

Or $$E = \frac{V}{300d} \tag{94}$$

If the oil drop has a negative charge, and if the top plate A is connected to the positive pole of the battery, the drop will experience an upward electric force, which, by proper choice of V, may be made exactly to balance the downward pull of gravity. Under such circumstances the downward fall of the drop would be arrested, and it would remain indefinitely at exactly the same point in the field of view. For such a value of V, the downward force of gravity mg would be exactly equal to the upward electric force on the drop Eq.

$$mg = Eq \tag{95}$$

m could be determined from the rate of fall without the electric field and the known density of the oil, and E could be calculated from (94). Formula (95) furnishes, therefore, a simple and direct means of measuring the charge on the drop.

The experiment, when tried, showed conclusively that the charge on the drop was always an integral multiple of the charge on the univalent ion of electrolysis. In no case was a charge less than that occurring in electrolysis discovered, and in general it was found to be between three and twenty times as great. It would be difficult to conceive of a more convincing proof of the atomic nature of electricity. The value of the fundamental charge, e, at which Millikan arrived as the result of a long series of experiments was 4.774×10^{-10} E.S.U. or 1.592×10^{-19} coulomb. His estimated probable error in this value was 0.1 per cent, which is certainly better than the best determinations of Avogadro's number. Since the value of the faraday F is known to better than this accuracy, the value of N which is to-day accepted as that most probably correct is obtained by dividing F by Millikan's value of e.

$$N = \frac{F}{e} \tag{96}$$

The result so obtained, 6.06×10^{23}, is, within the errors of ex-

periment, equal to the best independent determinations of N, such as that described in Chapter XIII.

PROBLEMS

1. How many grams of copper would be deposited from a solution of copper sulphate $CuSO_4$ by a current of 2 amperes in 1 hour? The valence of copper in this compound is 2.

2. How many cubic centimeters each of hydrogen and oxygen would be produced by the electrolysis of 1 gr. of water if the operation is carried out at 20° C.? How long would it take a current of 1 ampere to accomplish this separation?

3. If the heat of combustion of hydrogen is 33,900 calories per gram, what is the minimum potential difference between the electrodes of an electrolysis apparatus which will separate the water into its constituents? Assume all the electrical energy spent in the electrolyte to go toward this separation.

4. What is the potential of a drop of mercury 0.001 cm. in radius if it contains one fundamental unit of charge?

5. If the density of a certain oil is 0.8, how large must be a droplet composed of this oil if the electric field produced between two parallel plates 1 cm. apart by a difference of potential of 2000 volts is just sufficient to balance the downward pull of gravity on the drop? Assume that the drop contains three fundamental units of charge.

CHAPTER XIX

CATHODE RAYS. ELECTRONS

The last two chapters have dealt with the conduction of electricity through metals and solutions respectively. The present chapter will deal with a few of the extremely important conclusions which have been deduced from the behavior of gases as electric conductors.

93. The Electric Conduction of Gases. Under ordinary circumstances, gases are among the best insulators which we have. Nevertheless, if the difference in potential between two electrodes is gradually increased, a voltage is eventually reached at which any gas between the electrodes loses its insulating property and, to the accompaniment of a noisy spark or arc, permits the passage of relatively large currents. The magnitude of the breakdown voltage depends on a number of factors, such as the size, shape, and separation of the plates, and the nature and pressure of the gas.

In general, as the pressure of the gas is reduced below that of the atmosphere, the electric discharge takes place more readily and becomes less and less noisy. These changes can be most easily observed by sealing two electrodes into the ends of a tube (Fig. 79), from which the air can be gradually extracted by the pump P. If the electrodes are connected to the terminals of an induction coil, which furnishes for this purpose a very convenient high potential, no discharge will in general take place until the pressure has been reduced to a centimeter or so of mercury. When the discharge first appears, it will be in the form of a thin and noisy spark. As the pressure is further reduced, however, the current increases and the spark gradually broadens out until at a pressure of perhaps a millimeter it wellnigh fills the tube with a glow, the color of which depends on the nature of the gas. The brilliant red of such a discharge through neon is familiar to everyone from

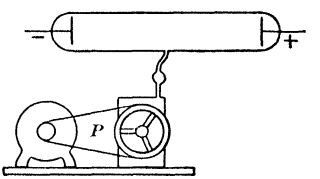


FIG. 79. Gas Discharge Tube.

its extensive use for advertizing purposes. If the reduction in pressure is continued, a point will be reached beyond which the discharge passes again with greater difficulty. Meanwhile the appearance of the tube will pass through a series of rather characteristic changes, sometimes exhibiting most beautiful striations. At a pressure of a few hundredths of a millimeter the glow will have practically disappeared, and an entirely new phenomenon will set in.

94. Cathode Rays. A beam of bluish light will be observed streaming out into the tube from the cathode. Wherever this beam strikes the wall of the tube it glows with a characteristic green *fluorescence*. These streamers, probably first observed by Hittorf in 1869, were called *cathode rays* by Goldstein in 1876. For nearly a quarter of a century after their discovery a lively controversy took place as to their nature, certain observers, notably Sir William Crookes in England, holding the belief that they consisted of streams of matter in some hitherto unknown form, while others considered them to be some sort of ether disturbance similar to light. During this period, however, the evidence in favor of material particles gradually accumulated, and this viewpoint was finally rendered almost unassailable by the epoch-making experiment of Sir J. J. Thomson in 1897.

The fundamental properties of the rays, all of which are consistent with such a hypothesis, are the following. Any substantial obstacle in their path stops them completely and casts a sharp shadow on the fluorescing walls of the tube (Fig. 80). The rays, though stopped by heavy obstacles, are capable of passing through thin sheets of metal such as aluminium or gold foil. They heat to incandescence light metallic objects against which they strike in the tube, and are capable of exerting pressure against objects in their path, being able, in fact, to set in rotation small paddle wheels of mica on the vanes of which they impinge.

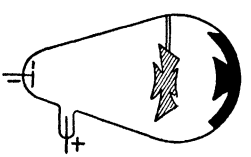


FIG. 80. Cathode Ray Shadow.

They were found to be deflected by a magnetic field, the direction of deflection being at right angles both to the path of the rays and to the magnetic field. One is at once reminded of the curious force which a magnetic field exerts on a wire carrying a current, since this also is perpendicular to the current and to the magnetic field. The sense in which the cathode rays are urged,

however, is that in which an electric conductor would be thrust if it were carrying a current in a direction opposite to that taken by the rays. It should be remembered, nevertheless, that the direction of flow of an electric current was rather arbitrarily defined as that in which positive electricity moves, and that in the case of Rowland's famous experiment, in which the hard rubber disk possessed a negative charge, the direction of the magnetic field was such as would have been caused by an electric current flowing in a direction opposite to that in which the disk revolved. A moving negative charge produces, therefore, a magnetic field opposite to that produced by a current flowing in the direction of motion of the charge; and, as a consequence, the force exerted on such a charge by a magnetic field must be opposite to that exerted on a conductor carrying a current in the same direction. The magnetic deflection of the cathode rays indicated, therefore, that if, as seemed likely, they consisted of moving particles, these particles carried negative charges.

It remained for Perrin in 1895 to show very directly that such was the case. His method consisted in passing a stream of cathode rays, generated at the cathode *A* in the side tube *T* (Fig. 81), through the slit *S* and into the large glass bulb *U*. Travelling on in a straight line, they would produce a fluorescent spot on the wall of the bulb at *B*. Well to one side of the direct path of the rays there was mounted in the bulb a brass cylinder *C* with a small aperture *D* facing the side tube *T*. The cylinder was connected by the wire *W* to the leaves of an electroscope, or better to a more sensitive *electrometer*. In order to protect the cylinder *C* from the intense electrostatic fields inside the bulb, a second cylinder *G* was mounted around *C* and connected to the Earth. As soon as the cathode rays began to enter the bulb a small negative charge was always indicated by the electroscope, but if the stream of rays was deflected by a magnetic field along the path *P* so that it entered the aperture *D*, an enormously greater negative charge was received. The small initial charge was apparently due to incomplete shielding of the inner cylinder; the large charge which appeared when the stream entered directly into the chamber was obviously the negative charge carried by

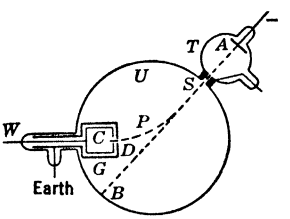


FIG. 81. Perrin's Apparatus.

the rays themselves. The shielded cylinder which was used for collecting the charge has found a wide range of usefulness in similar experiments, and is known as a *Faraday cylinder.*

95. Velocity and Ratio of Charge to Mass of the Cathode Particles. Two years after this experiment was performed by Perrin, in 1897, Sir J. J. Thomson carried out another experiment in England, which had such far-reaching consequences that it might almost be said to have marked the beginning of modern physics. If the cathode rays were really streams of negatively charged particles, it should obviously be possible to deflect them by an electric as well as by a magnetic field. Although a number of attempts had been made to show this effect, there were technical difficulties which were first overcome by J. J. Thomson at this time. His apparatus, built for the purpose of measuring quantitatively both the magnetic and electric deflection, was as follows. In the tube of Fig. 82 a narrow pencil of cathode rays, generated at the cathode A, and limited by the narrow slits D and D', entered the bulb U, and when undeflected, produced a bright fluorescent spot on the tube at B. As the rays excite much brighter fluorescence when impinging on barium platinocyanide or powdered willemite, the end of the tube in this region was covered with a coating of one or the other of these two materials. C and F were two parallel metal plates between which the cathode rays passed, and which, when connected to the poles of a storage battery, could furnish the requisite field for their electrostatic deflection. Two electromagnets, one on each side of the tube at G, indicated in the figure by a single circular coil to represent one turn of the exciting current, were mounted in such a way as to produce a magnetic field perpendicular both to the plane of the paper and to the direction of the electrostatic field. If the top plate C was connected to the positive pole of the storage battery while the lower plate F was attached to the negative pole, any negatively charged particles such as constituted the cathode rays would be attracted upward, and the spot at which they struck the fluorescent screen would rise from

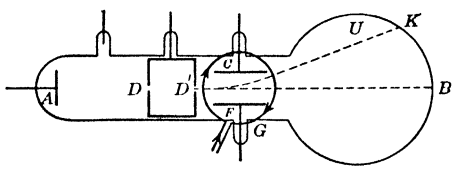


FIG. 82. Thomson's Apparatus for Measuring e/m.

B to K. If the current flowed through the coils of the magnet in the direction indicated by the arrow in the figure, the magnetic field would tend to thrust a positive current moving in the direction of the rays upward, or would tend to pull downward a stream of negative charges. The possibility was therefore open of balancing the upward pull of the electric field by the downward pull of the magnetic field. If e is the charge on one individual particle and E the intensity of the electric field, the force exerted by the electric field is Ee.

It was shown in Chapter XVI that the force in dynes exerted on 1 cm. of wire carrying a current I by a magnetic field of strength H at right angles to the wire is

$$F = HI$$

According to our present viewpoint, this is a force which the magnetic field exerts on charges flowing through the wire. It should be possible, therefore, for us to calculate the force which the field H would exert on a charge e moving with a velocity v, for the current I consists of I units of electric charge flowing past a given point in the wire per second. If each of these charges is moving with a velocity v, all the charges in v centimeters of wire will move past such a point in 1 sec. There are, therefore, I units of charge in v centimeters of wire or I/v units in 1 cm. Since the force on 1 cm. is HI, the force on each unit charge is $HI/(I/v) = Hv$, and the force F in dynes on a charge e would be

$$F = Hev \tag{97}$$

It is, therefore, proportional to the strength of the magnetic field, to the size of the charge, and to the velocity with which the charge is moving. This force is always perpendicular both to H and to the direction of motion of the charge. As fast as the direction of the motion changes, the direction of the force changes, yet always in such a fashion as to remain at right angles to the path of the charge. The only previous example of such a force, which we have encountered, occurred in Chapter IV in the discussion of the motion of a body in a circle. It was there proven that if a body of mass m is moving in a circle of radius R with uniform velocity v, it is acted on by a force of magnitude mv^2/R directed toward the center of the circle. This force was called a centripetal force. A force, however, which is

directed toward the center of a circle is always at right angles to the circumference of the circle, i.e., to the path of the body, and it can be proven conversely that, if a body of mass m travelling with velocity v has exerted on it a force F always perpendicular to its path, it will move in a circle of radius R such that

$$F = \frac{mv^2}{R} \tag{98}$$

This is exactly the sort of force which a moving charge experiences in a magnetic field, and we should accordingly expect the cathode rays to be deflected by the magnetic field into a circular path. This was found actually to be the case. Just so long as the path of the rays lies in a uniform field, it is accurately circular. The relation between the radius of the circle and the other factors involved, e the charge on the particle, m its mass, and v its velocity, can be obtained by equating the two values of F from (97) and (98).

$$Hev = \frac{mv^2}{R} \tag{99}$$

Cancelling out one v and solving for the ratio of e to m,

$$\frac{e}{m} = \frac{v}{HR} \tag{100}$$

The velocity of motion of the particles, v, was very simply determined by so adjusting the intensity of the magnetic field that it just balanced the electric field. When such an adjustment had been effected, the upward pull of the electric field, Ee, was exactly equal to the downward pull of the magnetic field, Hev.

$$Ee = Hev \tag{101}$$

Solving for v,

$$v = \frac{E}{H} \tag{102}$$

The velocity in centimeters per second is, therefore, given by the ratio of the electric field (in electromagnetic units) to the magnetic field (also in electromagnetic units).

Two plates are at unit difference of potential in the electromagnetic system, as in the electrostatic system, if the work required in transferring a unit charge from one plate to the other is 1 erg. Since the electromagnetic unit of charge is 3×10^{10}

greater than the electrostatic unit, the force on it at any point between the plates must be 3×10^{10} times that on the electrostatic unit, and if the total work done on it is to be only 1 erg, the difference in potential between the plates must be only $1/3 \times 10^{10}$ times as great as the electrostatic unit of potential difference. Since the volt is $\frac{1}{300}$ of the electrostatic unit, it must be 10^8 times as great as the electromagnetic unit. Thus, if we know the difference in potential in volts between the electrostatic deflecting plates, the difference in potential in E.M.U. is obtained by multiplying the voltage by 10^8, and the strength of the electric field E is found by dividing this potential difference by the distance apart of the plates in centimeters, exactly as was done in the oil-drop experiment of the last chapter. In Chapter XVI one very accurate method of measuring a magnetic field was described, and a means of calculating the field at the center of a coil carrying a current was also given. A rather more convenient method of measuring H will be described in a later chapter.

The value of v obtained in this experiment depended on the voltage between the cathode and the anode of the tube generating the cathode rays, but was of the order of magnitude of 10^9 cm./sec. or ten million meters a second. Such colossal velocities had never before been encountered, and had they not been checked in a remarkable way by many succeeding experiments, they might have raised serious doubt as to the validity of the methods used.

After determining v in this manner, the magnetic field alone was permitted to deflect the rays. From the magnitude of the deflection as observed on the fluorescent screen and a knowledge of the geometry of the tube, the radius R of the circular path in which the rays were bent could be calculated. Knowing v from the preceding experiment, H, and R, the value of e/m could be obtained from (100). This turned out to be independent[1] of the voltage at which the tube was operated, and was found to yield under all circumstances exactly the same value, viz., 1.769×10^7 E.M.U./gr.

If we make the very likely assumption that the charge carried by the particle is the smallest possible one, the fundamental charge encountered both in electrolysis and in the Millikan oil-

[1] Except for very high velocities where the effect of relativity was important.

drop experiment, then $e = 1.592 \times 10^{-20}$ E.M.U., and

$$m = \frac{e}{e/m} = \frac{1.592 \times 10^{-20}}{1.769 \times 10^{7}} = 9.0 \times 10^{-28} \text{ gr.}$$

The mass of the hydrogen atom, on the other hand, is

$$M = \frac{1.008}{6.06 \times 10^{23}} = 1.663 \times 10^{-24} \text{ gr.}$$

The cathode particle would, accordingly, appear to have a mass of only about $\frac{1}{1848}$ that of the hydrogen atom. This particle, which was first encountered in J. J. Thomson's famous experiment as the carrier of the individual charge in the cathode-ray stream, but which has constantly recurred in a wide variety of subsequent experiments, has received the name of *electron.*

96. Relation between the Velocity of the Cathode Particle and the Voltage Applied to the Tube. Even in Thomson's original experiment there existed a possibility of checking the validity of his methods in a very interesting and satisfactory manner. It has already been pointed out that the velocity of the particle depended on the voltage applied between cathode and anode of the generating tube. If we assume that the electrons leave the cathode with a relatively low velocity and acquire their final speed as a result of being dragged down the tube by the electric field, the total work done on each unit of charge in this process would be V, the difference in potential between the electrodes. The work done on each electron would be Ve, and if it travelled the entire length of the tube without dissipating any of the energy so obtained by encounters with molecules on the way, this entire energy would be stored up as kinetic energy. The magnitude of this kinetic energy in terms of the mass and velocity of the electron is $\frac{1}{2}mv^2$.

Thus $$Ve = \tfrac{1}{2}mv^2 \qquad (103)$$

Or $$v = \sqrt{2V\frac{e}{m}} \qquad (104)$$

If $V = 10{,}000$ volts or $(10{,}000)(10^8) = 10^{12}$ E.M.U.,

$v = \sqrt{2(10^{12})(1.769 \times 10^{7})} =$ approx. 6×10^{9} cm./sec.

Such a calculation gives in every case quite accurately the value of v obtained by the deflection experiment, and furnishes an extraordinarily convincing check on the correctness of the methods.

PROBLEMS

1. Calculate the force in dynes which is exerted on an electron between two parallel plates if the difference in potential between the plates is 1000 volts, and their separation is 1 cm.

2. What force in dynes is exerted on an electron by the Earth's magnetic field if it is travelling with a velocity of 10^8 cm./sec. at right angles to the resultant direction of the field? Assume the field intensity to be 0.54 gauss.

3. What is the radius of the circular path in which the electron of the previous problem moves?

4. What voltage applied between cathode and anode of a cathode-ray tube is required to produce an electron velocity of 10^8 cm./sec.?

5. Making use of the fact, discussed in Chapter IV, that the motion of an electron along the axis of the tube of Fig. 77 is independent of its motion at right angles to the axis, calculate the amount by which a " 300-volt electron " is deflected by a pair of deflecting plates 1 cm. apart while it is travelling a distance of 10 cm. between these plates. Assume the plates to be connected to the terminals of a 6-volt storage battery, and assume the initial direction of motion of the electron to be parallel to the plates.

CHAPTER XX

X-RAYS AND POSITIVE RAYS

In 1895, Wilhelm Konrad Roentgen, while experimenting with a cathode-ray tube, observed that a paper screen covered with barium platinocyanide was excited to fluorescence even though it was outside and at some distance from the discharge tube. Since it was well known that the cathode rays were completely stopped by relatively thin sheets of glass, it was obvious that the fluorescence was due to some other type of ray. Further investigation showed that these new rays, to which he gave the name *X-rays*, were produced by the impact of cathode-ray particles on metallic fixtures inside the tube. Taking advantage of the fact that the cathode rays emerge always at right angles to the face of the cathode, it was found possible to focus from a concave cathode considerable streams of such rays on a heavy metal block, interposed for the purpose. The X-rays then emerged with considerable intensity from the face of the block. Because of its position opposite the cathode, it was called the *anticathode* or *target*. The anode, which played a subordinate part in the generation of the new rays, was withdrawn into a side tube (Fig. 83).

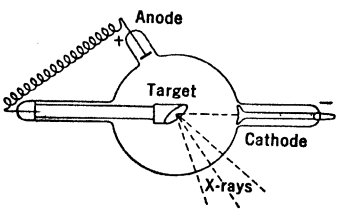


FIG. 83. X-ray Tube.

97. Fundamental Properties of X-rays. The outstanding feature of these X- or Roentgen rays was their soon demonstrated ability to pass through considerable thicknesses of such materials as paper, wood, and aluminium, which are opaque to ordinary light. Since they penetrated flesh more readily than bone, they were soon utilized by the medical profession in the study of fractures and various diseased conditions. The fact that, like light, they affected a photographic plate made it possible to produce permanent shadow pictures of fractured bones, a great boon to physicians. The nature of these Roentgen rays remained

for many years a complete mystery, thus thoroughly justifying the title of X-rays given them by their discoverer. It was early shown, however, that unlike cathode rays they were not deflected by electric or magnetic fields. If they were particles, they were either uncharged or were travelling at such huge velocities that their deflection could not be detected.

98. Thermionic Emission. The Edison Effect. One great practical difficulty with the older type of X-ray tube was the necessity for maintaining in it exactly the proper vacuum. Unless the gas pressure was correct within rather narrow limits, the cathode rays, and thus also the X-rays, were not produced. The remedy for this difficulty lay in the proper understanding of an observation made by Thomas Edison in 1883. While engaged in the development of the incandescent lamp, he sealed into an evacuated bulb a filament F and an electrode A (Fig. 84). By connecting the positive pole of a battery to A and the negative pole to F he was able, when the filament was heated, to pass a considerable current through the bulb. If, however, the negative pole was attached to A and the positive pole to F, no current resulted. Such an observation, extremely puzzling in the 'eighties, was entirely cleared up by the discovery of the electron. If electrons were spontaneously emitted by the glowing filament, they would be attracted to the positively charged electrode and would so carry a current through the vacuum. If, however, the electrode was negatively charged by contact with the negative pole of the battery, the electrons would only be driven back to the filament. The proof that this is really the correct explanation of the phenomenon consisted in substituting a filament for the cathode in a cathode-ray tube, and pumping the tube to a vacuum so high that, in the absence of the hot filament, no discharge could take place. Under these conditions, with relatively low potentials between filament and anode, a copious stream of electrons was obtained, which exhibited all the properties of cathode rays, exciting fluorescence in a barium platinocyanide screen, and being subject to deflection by electric and magnetic fields. The value of e/m obtained for such rays turned out to be the same as for the cathode particles. Since the discovery of this property of

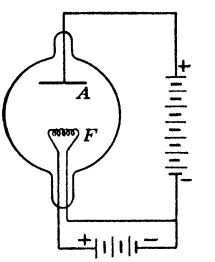


FIG. 84. The Edison Effect.

hot filaments they have been very extensively used as a source of electrons. The Coolidge X-ray tube, for example, is one in which the cathode is a hot filament. Since sufficient electrons are supplied by the filament, the tube can be pumped to the highest possible vacuum, and will remain in good condition as long as the filament lasts.

The fact that electrons are emitted from a hot filament regardless of the material of which it is made, has led among other things to the belief that electrons occur in all kinds of matter, and that they are, in fact, a constituent part of the atoms of all elements. Much of the evidence for such a viewpoint will be presented in the course of this and later chapters.

99. The Ionization of Gases. In addition to affecting a photographic plate and exciting fluorescence in sensitive materials, X-rays were found to have the property of rendering gases conducting even when they were spanned by very low voltages. The passage of electricity through a gas must be accompanied by the deposition of negative charges at the anode, positive charges at the cathode, or both. That these charges are transported through the gas by charged molecules or *gaseous ions* would seem likely by analogy with electrolytic conduction. Whereas in solutions the charged ions are always present, the gas molecules are apparently electrically neutral and incapable of transporting an electric current unless they are disrupted by excessively high potentials or are excited by some agent such as X-rays. The function of the X-rays is apparently to tear one or more electrons from the neutral gas molecule, thereby leaving it positively charged. The freed electron may act for a short time independently as a carrier of the electric current, but it doubtless unites very soon with a neutral atom to form a negative ion. There are, therefore, present in the ionized gas both positive and negative ions and presumably some free electrons. Owing, however, to the attraction between positive and negative charges, the ions in collision very rapidly neutralize one another when the ionizing source is removed, and the gas regains its insulating properties.

In carrying out his oil-drop experiment, Millikan frequently exposed the region between the horizontal plates to X-rays, thereby ionizing the gas through which the drops were falling. Whenever a gaseous ion thus produced came in contact with an oil drop, it would adhere to it, and would thereby alter the charge on the drop by the quantity of electricity which it carried. If,

for example, the drop was in equilibrium under the electric field and gravity, it would suddenly start to rise or fall according as the captured charge was positive or negative. From the observed velocity after the capture, the charge could be calculated, and under all circumstances the gain or loss was found to be equal to the fundamental charge or an exact multiple of it, thus furnishing very strong evidence that the charge on a gaseous ion is always an exact multiple of the charge on the electron.

Until these fundamental facts about gaseous ionization were learned, the behavior of gases when exposed to high voltages had been most puzzling. Why an almost perfect insulator should suddenly become a good conductor and allow the passage of huge currents was a complete mystery. Although the last word has not yet been said in the solution of this problem, the very high conductivity of the gas once the discharge has well started is certainly due to the ability of an ion to ionize neutral molecules with which it collides. Under the action of the electric field in the discharge tube it gradually increases in speed, and if between collisions it acquires sufficient velocity, it is likely to knock one or more electrons from such molecules, thereby increasing the number of carriers for the electric current. Once such a process is well under way, it might be expected to proceed with accelerating rapidity.

100. Positive Rays. Perhaps the most convincing evidence of the accuracy of this picture is furnished by the phenomenon of *positive rays*. In the ordinary cathode-ray tube there is a considerable amount of residual gas, which, ionized by the impact of electrons and other ions, plays an important part in conducting electricity through the tube. The negative ions travel toward the anode, and the positive ions toward the cathode. As the positive ions approach the cathode, they are accelerated by the electric field in the tube, and reach it with a velocity which depends on the distance which they have travelled since they were formed or since they suffered their last collision. If a hole is drilled through the cathode normal to its surface, the momentum of the positive particle will frequently be sufficient to carry it clear through the hole and on into the region behind. A tube blown into the shape shown in Fig. 85, with the cathode in the middle, provides

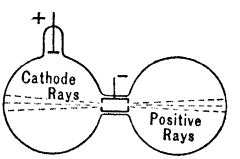


FIG. 85. Positive Ray Tube.

a considerable space behind the cathode where these rays can continue on until stopped by collision with a gas molecule or with the wall of the vessel. If the tube contains helium, the path of the positive ray is indicated by a red glow in sharp contrast to the bluish glow produced by the cathode rays in the other side of the tube. These positive rays were first observed by Goldstein in 1886, and were called by him *canal rays* because of the long passage or *canal* through which they traversed the cathode.

101. e/m of Positive Rays. The problem of measuring the velocity and the ratio of the charge to the mass of these particles turned out to be much more difficult than that of measuring the corresponding quantities in the cathode-ray stream. They proved more intense ionizers of the gas through which they passed than the cathode rays, and rapidly added a collection of new ions formed in the region behind the cathode. This difficulty was overcome by J. J. Thomson in 1910 by separating the positive-ray portion of the tube from that in which the cathode rays were generated except for a very long and fine hole through the cathode. By means of a fast-working pump he was then able to keep the bulb in which he was studying the positive rays free enough from gas to prevent any substantial production of new ions.

Since, however, there was present among the positive-ray particles a wide range of velocities, the simple method of measurement used for the cathode particles was still not applicable. The procedure taken by J. J. Thomson consisted in deflecting the positive rays up or down by a magnetic field and simultaneously deflecting them horizontally by an electric field. If all the particles in the ray were identical and moving with the same velocity, both electric and magnetic fields would deflect all by exactly the same amount, and all would strike the fluorescent screen at exactly the same point, for example P in Fig. 86. If, on the other hand, the particles were all identical but travelling with different velocities, those having the higher velocities would be deflected less by each field than those having lower velocities. It can easily be shown that

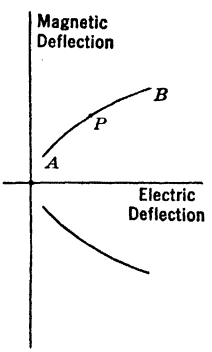


FIG. 86. Positive Ray Parabola.

under these circumstances the particles would be spread out on the screen along the arc of a parabola AB. The shape and dimensions of the parabola would depend only on the ratio e/m of the charge to the mass of the particle. From measurements made on the screen and a knowledge of the intensity of the fields, this ratio could be calculated. The impact of the positive rays was found to affect a photographic plate, and in practice the measurements were made at leisure on a plate which had been substituted for the fluorescent screen.

If there were present in the tube not one but several gases, or if, as frequently happened, not all the particles carried the same charge, several parabolas would appear on the photographic plate, each one corresponding to a particular ratio of e to m. By reversing the direction of the magnetic field during the exposure, a symmetrical group of such parabolic arcs could be produced below the horizontal axis, which aided considerably in the necessary measurements.

It usually happens that the photographs are fairly complicated. Thus on an exposure with atmospheric nitrogen in the tube there were found parabolas corresponding to nitrogen molecules (N_2) with one charge, nitrogen atoms (N) with both one and two charges, oxygen, argon, and neon atoms with single charges, CO_2 and C (from stop cock grease) with one charge, and mercury (from the pump) with one, two, and three charges. Nevertheless, despite the complexity of the photographs, it was usually possible adequately to explain all the parabolas by some such simple assumptions as to the nature and charge of the particles. There were, however, certain notable exceptions. Thus Thomson showed that the neon parabola which corresponded to a mass of about twenty times that of the hydrogen atom was always accompanied by a second fainter parabola indicating a mass twenty-two times that of hydrogen. This suggested at once that neon, the chemically measured atomic weight of which was 20.2, really consisted of a mixture of two elements having atomic weights of 20 and 22 respectively.

102. Isotopes. It had been suggested as early as 1815 by William Prout, an English physician, that perhaps all the heavier elements were constructed from hydrogen atoms. The fact that a goodly number of the lighter elements such as carbon (12), nitrogen (14), and oxygen (16) had atomic weights that were approximately exact multiples of that of hydrogen seemed to

substantiate this view. Nevertheless, as the atomic weights were measured with greater and greater accuracy, it was soon apparent that such a hypothesis as applied to the great majority of chemical elements was untenable. Although the accuracy with which the masses of positive-ray ions could be measured was still inadequate to decide whether the weight of each was or was not an exact multiple of that of hydrogen, the result of the experiment on neon suggested the possibility that wherever a

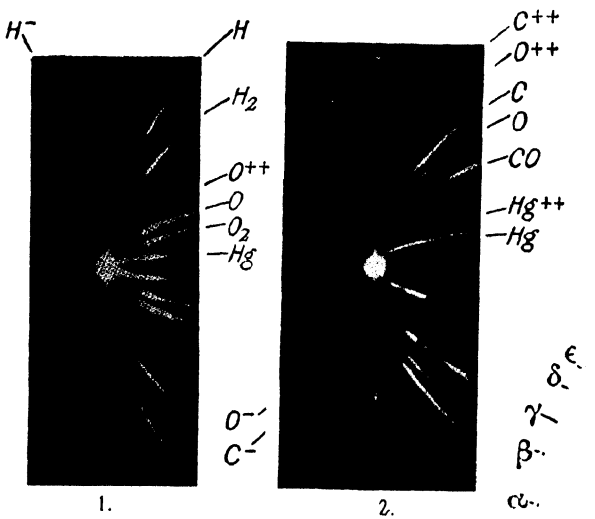


PLATE I. Typical Positive Ray Parabolas From "Isotopes" by F. W. Aston. Courtesy of Edward Arnold & Co.

chemical atomic weight was not an integer, the element might consist of two or more elements having different, but still integral, atomic weights. Such elements, which have identical chemical properties but different atomic weights, have been called *isotopes*.

It was shown in Chapter XIII that in a vessel containing a mixture of several different gases the heavier constituents tend to diffuse toward the lower parts of the vessel. Shortly after his positive-ray measurements on neon J. J. Thomson attempted to separate the heavier component of neon from the lighter by such a diffusion method, and his success was sufficient to corroborate in a striking fashion the complex nature of the gas.

103. The Mass Spectrograph. The great difficulty with the original method of J. J. Thomson for measuring positive-ray

masses lay in the fact that the various molecules of one given constituent in the tube were spread out on the photographic plate along the arc of a parabola. Where the number of such molecules was not great, as in the case of the rarer constituent of neon, the resulting trace on the plate was very weak or imperceptible. In 1919, F. W. Aston designed a new apparatus in which the rays were first deflected electrically and then magnetically in such a way that all ions of a given kind were brought to a focus at one particular point on the photographic plate. Molecules having different atomic weights or different charges, however, were deflected by different amounts, thus finally arriving at different points on the plate. The collection of parabolas which resulted from a complex gas in Thomson's method was thus replaced by a succession of tiny spots on the photographic plate in Aston's apparatus, and because all ions of a given sort were focussed to a single small spot on the plate, much smaller amounts of a given gas were required to insure its detection. The similarity of such photographs to those which had been obtained for many years by the *spectrograph* in studying the constitution of light[1] gave to the new instrument its name of the *mass spectrograph.*

The increased sensitivity of the method made it possible to apply it successfully to a wide range of gases which were known to have non-integral atomic weights, and in all cases they were found to be mixtures of isotopes, the weights of which were at least very nearly integers. In 1922, Dempster in Chicago extended the method in such a way as to make it applicable to certain solids as well as gases. The following table shows the results obtained with a number of the chemical elements. Following the name of the element is the chemically obtained atomic weight, which is now recognized as no more than a weighted average of the weights of the isotopes. In the next column are listed the atomic weights of the definitely proven isotopes.

Silver	107.880	107, 109
Boron	10.82	11, 10
Cadmium	112.41	110, 111, 112, 113, 114, 116
Chlorine	35.458	35, 37
Iron	55.84	56, 54
Potassium	39.095	39, 41

[1] Chapter XXXIV.

Krypton	82.9	84, 86, 82, 83, 80, 78
Magnesium.....	24.32	24, 25, 26
Neon.....	20.2	20, 22
Nickel.....	58.69	58, 60
Tin.....	118.70	120, 118, 116, 124, 119, 117, 122, 121
Zinc.....	65.38	64, 66, 68, 70

PROBLEMS

1. Calculate the velocity (*a*) of an electron, (*b*) of a singly charged hydrogen ion, and (*c*) of a singly charged argon ion, after they have " fallen " through a difference of potential of 1000 volts.

2. Calculate the radius of the circular path which (*a*) the electron, and (*b*) the hydrogen ion, of the preceding problem would describe in the Earth's magnetic field at a point where the magnitude of the field is 0.54 gauss.

CHAPTER XXI

RADIOACTIVITY

In 1896, only one year after the discovery of X-rays by Roentgen, Henri Becquerel in Paris discovered that uranium salts possessed the property of emitting spontaneously quite similar radiations. Thus a small amount of any uranium preparation, when placed in the vicinity of a photographic plate, produced exactly the same effect on it as light or X-rays. Like X-rays also these new radiations were found capable of passing through materials opaque to light and of ionizing a gas. The electroscope furnished an extraordinarily sensitive and convenient means of detecting and measuring the ionization produced. Thus if the gold leaves of an electroscope were connected to one of two parallel plates while the casing was connected to the other, any gaseous ions formed between the plates would permit the passage of a current and the discharge of the electroscope. The rate of discharge could be timed with a watch, and used as a measure of the intensity of ionization.

104. Radium. In 1897, Mme. and Pierre Curie, inspired by the discovery of Becquerel, undertook an investigation of various uranium minerals by the electroscope method. They soon found that certain of these minerals, notably pitchblende, a uranium ore from St. Joachimthal in Bohemia, exhibited greater activity in the way of ionization than would be anticipated from their uranium content. Separating the ore into its chemical constituents, and examining these with the electroscope, they ascertained that the greater part of the activity was possessed by the bismuth and barium also contained in the pitchblende. Since bismuth and barium compounds from other sources did not show this property, it was obvious that it was due to small amounts of some other substances which behaved chemically very much the same as these elements. In July, 1898, after long and laborious treatment of these active compounds, they were able to announce that the activity in the bismuth was due to an extraordinarily active new element which they called *polonium* in honor of Mme. Curie's

native country of Poland, and in December of the same year they reported the discovery of still another new element, *radium*, which was responsible for the activity of the barium compounds. Both of these new elements were found to occur in extremely minute quantities, so minute, in fact, that they would probably never have been discovered were it not for their ability to ionize gases and the extraordinary sensitivity of the electroscope in detecting such a process. They occupy spaces in the periodic table directly under bismuth and barium respectively, and are accordingly so similar chemically to these elements that their separation can be accomplished only by very laborious crystallization methods. Of the two, radium has proven to be much the more important.

The discovery of radium with an activity 100,000 times greater than uranium aroused at once questions regarding the nature and significance of these radiations, which might otherwise have been long in capturing the popular imagination. Thus, for example, the radiation from a radium salt was soon shown to be able to maintain the temperature of a solution in which it was dissolved somewhat above that of its surroundings. A simple calculation showed that enough heat was generated each hour from radium to raise the temperature of an equal mass of water from freezing to boiling. Yet despite this liberation of energy, the mass or activity of the radium was not apparently diminished. Was there here a contradiction to the Law of the Conservation of Energy?

The experimental activity which was stimulated by such questions during the next decade soon disposed of the possibility that the principle of the conservation of energy was at stake, but it resulted in a number of entirely new ideas which were almost as devastating to the scientific concepts of the nineteenth century. It would be both tedious and confusing to attempt to follow historically the growth of these new ideas. Instead, the facts which were learned regarding the radiations themselves will first be described, and then the conclusions to which they have led concerning radium and related substances.

105. Alpha, Beta, and Gamma Rays. Investigation soon showed that the radiations were much more complicated than was originally supposed, consisting, in fact, of three distinct types of rays, which were called for convenience α-, β-, and γ-rays. The α-rays were all absorbed by a few centimeters of air or less

than $\frac{1}{10}$ mm. of aluminium foil, the β-rays were capable of traversing several meters of air or very substantial thicknesses of aluminium, while the γ-rays, most penetrating of all, could be detected after traversing several centimeters of lead. The α-rays were deflected by electric and magnetic fields in the direction to be anticipated if they were positively charged particles, the β-rays in that to be expected if they carried negative charges, while the γ-rays could not be deflected at all. The γ-rays were in this respect and in their penetrating power suggestive of X-rays, and all subsequent experiments tended to emphasize this similarity. Although most γ-rays are even more penetrating than our artificially produced X-rays, this would seem to be due to limitations in our X-ray apparatus, since increasing the voltage across the X-ray tube, and thereby increasing the velocity of the impinging cathode particles, increases the penetrating power of the X-radiation. Recent experiments with higher voltages on the X-ray tubes have very largely bridged the gap in penetrating power between these and naturally produced γ-rays. Certain important and very significant properties of X-rays were discovered during the second decade of the present century, and when these are discussed in Chapter XXXV, more will be said regarding γ-rays.

106. The Nature and Properties of β-Rays. Just as γ-rays are suggestive of X-rays, so the fundamental properties of the β-rays suggest cathode rays. Like them they carry negative charges, and though capable of passing through small thicknesses of light metals such as aluminium, they would certainly not be classed as very penetrating radiations. The experiment of determining the ratio of their charge to mass by electric and magnetic deflection was, as with positive rays, complicated by the presence of a wide range of velocities. In order to make measurements of this sort, Kaufmann in 1902 used a method similar to that of J. J. Thomson with positive rays. The traces of the rays on the photographic plate, unlike those obtained by Thomson, were not exactly parabolas. As was pointed out in the last chapter, the faster rays were less deflected both by electric and magnetic fields than the slower ones. Those which suffered the least deflection of all were, therefore, the fastest rays, and the record which they made on the photographic plate provided a means of calculating the velocity of the fastest of the positive particles. In the experiment of Kaufmann with β-rays the

minimum deflection was so slight as to indicate that the β-rays with the maximum velocity were travelling at a rate but little less than that of light (186,000 miles/sec.). The deviation of the trace on the plate from parabolic shape was such as to indicate that the mass of the particle increased with the velocity. Later, similar experiments by Bucherer showed that for the highest speeds it was over five times that at low velocities.

In 1905, Albert Einstein proposed his well-known theory of *relativity*, which showed among other things that such an increase in mass with velocity was to be anticipated, and that it should first become appreciable for velocities greater than one-tenth that of light. The precise nature of these expected variations will be discussed in Chapter XXXIII. Although the behavior of very high-speed β-particles was of the greatest theoretical importance as furnishing one of the early corroborations of the theory of relativity, the lower-speed particles behaved in a very prosaic fashion indeed, yielding for e/m precisely the value obtained in all previous experiments on electrons. The β-rays were thus undoubtedly electrons, which were spontaneously thrown off from the radioactive materials.

107. Alpha Particles. The positively charged α-rays, although more easily absorbed than either β- or γ-rays, were found to be responsible for the major part of the fluorescence produced by radium in barium platinocyanide. It was discovered by Sir William Crookes in 1903 that this fluorescence, when examined under a magnifying glass, consisted, not of a continuous glow, but of a multitude of tiny, scintillating spots, which appeared for an instant and then disappeared. There seemed little doubt that each little scintillation was the result of the impact of a single α-particle, and further investigation corroborated this conclusion to a remarkable extent. A very simple and direct method was provided by these scintillations, first for counting the number of α-particles emitted from a given specimen of radium, and second for determining the charge carried by each particle. Thus, if a tiny specimen of radioactive material is thought of as located at the center of an imaginary sphere of radius R, the number of particles which pass across any area A on the surface of the sphere (Fig. 87) will bear to the total number of particles leaving the specimen the same ratio as the area A to the total area $4\pi R^2$ of the sphere. Accordingly, if ν scintillations appear in one minute on an area A of the fluo-

rescent screen, and if the distance between screen and specimen is R, the total number of particles leaving the specimen per minute is $n = (4\pi R^2/A)\nu$. If, instead of allowing the particles to fall on a fluorescent screen, they are permitted to enter a Faraday cylinder such as that described in Chapter XIX, the total charge q received in one minute can be measured. If the area of the aperture through which they enter the cylinder is A_1 and its distance from the specimen is R_1, the total charge carried by all the particles emitted would be $q(4\pi R_1^2/A_1)$. The charge on the individual particle is obtained by dividing this by n. This experiment, first performed by Regener, yielded a value for the positive charge on the α-particle exactly twice that on the electron.

Two other methods have been successfully used to signal the arrival of individual α-particles. In 1908, Rutherford and

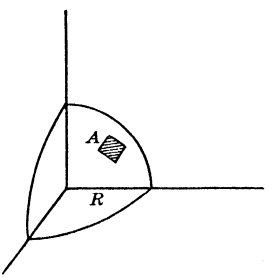


FIG. 87

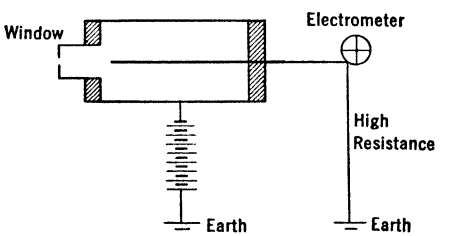


FIG. 88. Geiger Counter.

Geiger arranged a modified form of Faraday cylinder in which a rod terminating in a sharp point, mounted along the axis, was maintained at a potential of about 1500 volts negative with respect to the cylinder itself (Fig. 88). As an α-particle enters the chamber, the large number of ions produced permits a momentary current to flow between the point and the cylinder, resulting in a sudden deflection of the electrometer needle. Thus the arrival of every α-particle is announced by the electrometer. The method has been extended in recent years, the tiny ionization currents being amplified by apparatus similar to that used in radio reception. By such means the advent of an α-particle can be made audible to a large audience, or it can be recorded automatically by an electric counter.

In 1897, C. T. R. Wilson in England had shown that, in the absence of dust particles on which tiny drops could form, an

atmosphere might become very heavily supersaturated with water vapor before precipitation would take place. Not merely would particles of dust, by acting as nuclei, facilitate condensation, but gaseous ions also possessed that property. In 1911 he showed that, if an α-particle is shot into a region filled with supersaturated water vapor, the water would begin to condense on the ions thereby produced, and would mark the path of the particle by a string of tiny droplets. This experiment is very easily performed, and a great deal of work has been done by C. T. R. Wilson and others in the way of photographing such tracks. This experiment demonstrates very vividly the reality of these tiny invisible particles which are shot off from radioactive materials. The β-particles can also be made visible by the same device, but unlike the α-particles they are very frequently deflected by collision with molecules on the way, and describe very devious paths.

The velocity and the ratio of charge to mass of the α-particles was determined by electric and magnetic deflection as with cathode and β-rays. Their maximum velocity, unlike that of the β-rays, was somewhat less than one-tenth that of light, and their mass was accordingly affected to an inappreciable extent by the relativity variation of mass with velocity. The ratio of charge to mass turned out to be 4823 E.M.U./gr., regardless of the radioactive material from which the particles originated. It was pointed out in Chapter XVIII that the charge carried by a gram-molecule of any univalent ion in electrolysis was 96,500 coulombs or 9650 E.M.U. Since the mass of 1 gram-atom of hydrogen is 1 gr., the ratio of the charge of one atom to its mass is 9650. This, it will be observed, is almost exactly twice that obtained for the α-particle. Since the charge on the α-particle is twice that of a univalent ion, its mass must be four times that of the hydrogen atom and equal to that of helium.

That the α-particle not merely possesses the same mass as helium but is actually a positive helium ion requiring only the addition of two electrons to become a helium atom, was proven in 1903 by Ramsay and Soddy. The most sensitive way of detecting many elements, including helium, is by spectroscopic examination of the electric discharge through a gas containing some of the suspected element. This method, which will be discussed in considerable detail in Chapter XXXIV, is capable of yielding absolute evidence for or against the existence of extra-

ordinarily minute quantities of these substances. Ramsay and Soddy sealed a small amount of radioactive material into the thin-walled glass tube A (Fig. 89), which was in turn sealed into the larger tube B. The larger tube was provided with electrodes D and E, between which a discharge could be passed. The walls of A were so thin as to permit the passage of α-particles from A into B. When the tube was first sealed, no evidence of a helium spectrum could be detected in the discharge from D and E, but after a few hours it appeared, and continued to grow brighter as the time elapsed. The α-particles, after losing their huge velocity by collisions with gas molecules, were clearly picking up electrons in the tube B and becoming ordinary helium atoms.

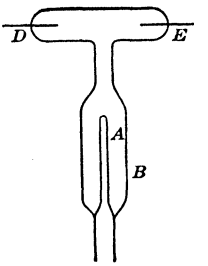


FIG. 89. Production of Helium from α-particles.

108. The Rutherford Model of the Atom. Conclusions of the utmost importance have been drawn from the behavior of α-particles in passing through matter. In the ray-track experiments of C. T. R. Wilson, nearly all the paths were accurately straight, and only in a few instances was there evidence that the particle had suffered a substantial deflection in its course by impact with a gas molecule. Similarly, in experiments carried out in 1912 by Rutherford on the passage of α-particles through thin sheets of metals, it was the exceptional particle which was deflected far from a rectilinear path. The only conclusion which can be drawn from such observations is that the particle itself is extraordinarily small, and that the part of the atom which is effective in deflecting it is also very small. If the particles on leaving the active material had anything but excessively small targets to hit, the impacts would occur more frequently. From such considerations Rutherford estimated that the diameters of the colliding particles were of the order of magnitude of 10^{-13} cm. The kinetic theory of gases, however, gives for the radius of an atom rather more than 10^{-8} cm. The most satisfactory way of reconciling these apparent discrepancies is to picture the atom with Rutherford as consisting of a positively charged *nucleus* surrounded by enough electrons to render the atom as a whole electrically neutral. On this viewpoint it is the nucleus alone which is effective in producing the major deflections of the α-particles. Since the electrons have almost negligible masses,

the nucleus must not only be extremely small, but it must contain nearly all the mass of the atom. By a quantitative study of the number of α-particles which are deflected through definite angles, Rutherford and Chadwick were able to measure with considerable accuracy the charge on the deflecting nucleus. This turned out to be, within the experimental errors, equal to the product of the fundamental charge on the electron and the atomic number of the element. A reference to the periodic table of the elements in Chapter XII will recall the meaning of this number. If the elements, arranged as they are in the table, for the most part in the order of their atomic weights, are numbered consecutively from H1 to U92, the number so assigned to each element is its atomic number. Throughout the early part of the table this number is approximately one-half the atomic weight, but such regularity soon vanishes as the lower rows are entered. The Rutherford atom is, therefore, pictured as a structure having an excessively minute nucleus which carries a positive charge of magnitude Ze, where Z is the atomic number and e is the charge on the electron. The nucleus is surrounded by a total of Z electrons, which give to the whole atom a radius of somewhere in the neighborhood of 10^{-8} cm. In order to explain the failure of the electrons to be drawn into the nucleus by the attraction of the positive charge, he assumed that they were in orbital motion about it in much the same way that the planets of our solar system are in orbital motion about the Sun. There is much evidence to substantiate such a hypothesis of orbital motion, some of which will be presented in Chapter XXXVI, but it does not rest on the solid experimental foundation on which our belief in the existence of a nucleus with its attendant electrons is built.

109. The Disintegration of Radioactive Elements. Radium Emanation. If we turn from this picture of the atom back to fundamental radioactive phenomena, we find ourselves in a position to appreciate certain of the epoch-making discoveries which were made in the first few years of the present century. In 1901, Mme. Curie and A. Debierne discovered that a gaseous material, itself radioactive, was constantly being produced by radium. This new gas was called *radium emanation*. The direct measurement of its density was a matter of extraordinary experimental difficulty owing to the minute quantities in which it could be obtained, but the task was eventually accomplished

by Ramsay and Gray, who obtained a value which corresponded to a molecular weight of approximately 223. The atomic weight of radium had been found by Mme. Curie to be 226. It would be difficult indeed to escape the conclusion that the radium emanation was a monatomic gas of atomic weight, not 223, but 222, which had been formed from radium by the ejection of an α-particle, i.e., a helium ion of atomic weight 4. Such a conclusion is borne out by a consideration of the charges on the nuclei. A loss of an α-particle from the nucleus of radium would reduce its total nuclear charge by 2, which should leave an element of atomic number 2 less than that of radium. Radium, it will be remembered, occupies a space at the bottom of the second column of the periodic table. A loss of one integer in atomic number would place it in the first row of the table; a loss of two would carry it over to the last column of the row above. All the elements in this last column, however, are monatomic noble gases, which checks completely with the known properties of radium emanation. The new gas was thus not merely a product, along with helium, of the disintegration of one element, radium, a hitherto unheard-of process. It was itself a new element, occupying a hitherto vacant place in the periodic table.

This disintegration hypothesis was first proposed by Rutherford and Soddy in 1902. In order to be acceptable, however, it had to check in some way with the observed fact that the mass of a given specimen of radium did not suffer any apparent diminution over a period of years. Since every atom of radium, on disintegrating, signals the event by the emission of an α-particle, it is merely necessary to count by the scintillation method the number of particles emitted in order to measure the rate at which disintegration takes place. Knowing the total number of atoms present (i.e., the fraction of a gram-atom $\times$ 6.06×10^{23}), the fractional loss in weight to be anticipated in a given length of time can be calculated. It is customary to estimate in this way the time which would be required for the mass of the element to be reduced to one-half of its original value. This time, known as the *half-value period*, turns out for radium to be 1580 years. The length of this period is sufficient explanation for the failure of early experiments to detect any decrease in the substance of radium as a result of its activity. The heating effect, which has been referred to, can be quantitatively accounted for as due to the stoppage of the known number of α-particles and

the consequent conversion of their known kinetic energy into heat.

110. The Radioactive Families of Elements. The recognition of radium emanation opened the way for the discovery of a long series of successive disintegration products. Thus it was soon found that if the emanation was kept in a thin-walled glass capsule, the pressure in the capsule would gradually fall, indicating a disappearance of the gas. Along with this loss of material there was a corresponding reduction in activity. That this was not due to escape of the gas was shown by the fact that, as the gas disappeared, the walls of the capsule itself became active, indicating the deposition on them of some active solid. It was eventually established that the emanation atom, by emitting an α-particle, became a new element, radium A, two places further down in the periodic table, and thus an isotope of polonium. The half-value period of radium emanation was found to be 3.825 days, while that of radium A is 3.05 minutes. The radium A, in turn, by the emission of an α-particle, becomes radium B, an isotope of lead with a half-value period of 26.8 minutes.

The radium B, though radioactive, was found to emit not α- but β-rays. The loss of a β-particle with its single unit of negative charge would be expected to have an inappreciable effect on the atomic weight of the element, but should increase the positive charge on the nucleus and therefore the atomic number of the element by one unit. As anticipated, the product of radium B or radium C, as it is called, was found to occupy a place in the next column to the right, the fifth, of the periodic table. It has a half-value period of 19.7 minutes, and emits both α- and β-rays. The emission of both types of rays meant two possible products, radium C_1 and C_2 respectively, but these in turn both produced first radium D, and then successively radium E, F, and G. With radium G, an isotope of lead, the series apparently ends, since radium G itself is no longer radioactive, and is, therefore, a permanent element. The disintegration of radium into radium G, if the above analysis is correct, is accompanied by the emission of five α-particles, which should correspond to a loss of twenty units of atomic weight, thus yielding for radium G an atomic weight of 226–20 or 206. The accepted atomic weight of lead was 207.19. That radium G is an isotope of lead and not identical with it was placed beyond question by T. W. Richards of Harvard University, who measured the

atomic weight of the lead obtained from a very pure uranium mineral, and found it to be 206.08, in excellent agreement with the value to be expected if it was obtained from the disintegration of radium. All these conclusions have been substantiated by positive-ray measurements recently made by Aston.

The known radioactivity of uranium, together with the fact that radium is always found in uranium minerals, suggested very early the possibility that radium itself might be a product of the disintegration of uranium. This genetic relationship between uranium and radium has since been definitely established, and the various intermediate stages in the process have been learned. Table I exhibits the entire series of changes between uranium

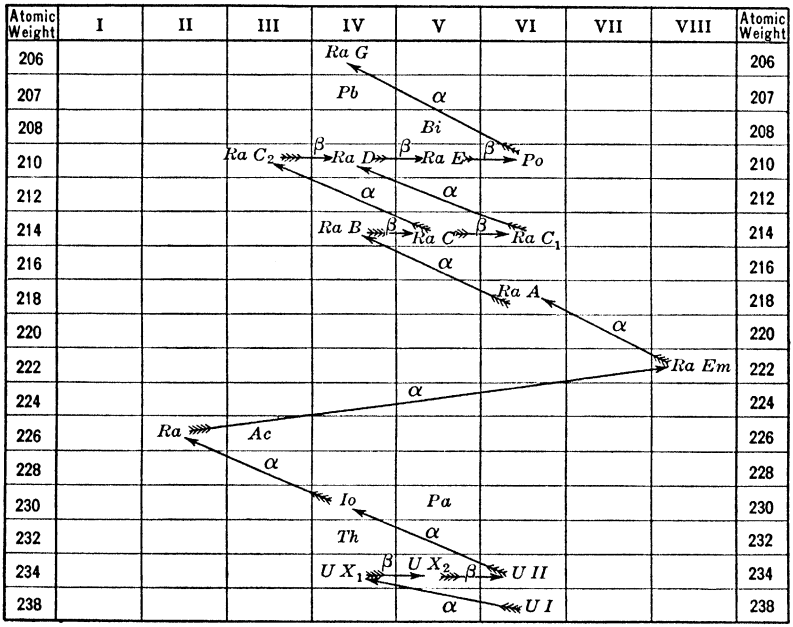


TABLE I. Radioactive Elements of the Radium Family.

and radium G. The table is so arranged that the atomic weight and the position in the periodic table of each product are clearly indicated. The arrows between successive elements indicate in each case the direction of the disintegration and the type of ray which accompanies the process.

Two other very similar radioactive series have been discovered, one starting with *thorium*, and the other starting probably with

an isotope of uranium and having as its key element *actinium*. Polonium, the first intensely active element discovered by Mme. Curie, is identical with radium F, and has a half-value period of 136.5 days.

Among the most startling facts which have been discovered in connection with radioactivity are the great differences which

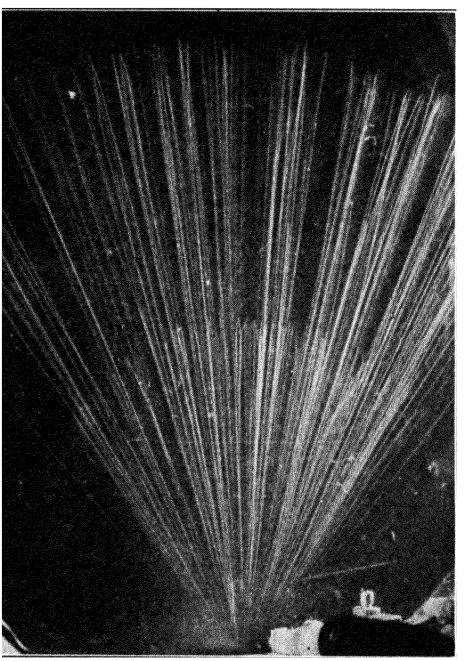


PLATE II. α-particle Tracks in Water Vapor.

Photograph by W. D. Harkins of the University of Chicago. The longer tracks are due to thorium C_1 and the shorter tracks to thorium C.

exist between the rates of disintegration of different elements. Thus the half-value period of uranium is 5×10^9 or five billion years, whereas that of thorium C_1 is 10^{-11} second. The distance an α-particle is capable of travelling through a gas has been shown to be closely related to the half-value period of the element from which it was ejected. The shorter the life of the element, the greater is the range of the particle, and it is indirectly by a measurement of these ranges that the shorter periods are estimated. Plate II shows very vividly the difference in range

between the short lived thorium C_1 and the comparatively long lived thorium C.

These rates of disintegration seem to be definite properties of the elements in question, and no way has ever been found of increasing or decreasing them by any measurable amount. Thus from any large collection of uranium atoms we are morally certain that at the end of 5,000,000,000 years one-half will have expelled each an α-particle and ceased to be uranium. If, however, the attention is fixed on one particular atom, we cannot speak with such definiteness. We can merely say that the chances are even that at the end of 5,000,000,000 years it will have disintegrated. It may expel its α-particle during the next second, or it may not expel it for 10^{20} years. Probability tells us much regarding the behavior of large groups which are subject to its laws, but very little regarding individuals. In radioactivity we seem to be face to face with physical phenomena which we can interpret only by such considerations.

CHAPTER XXII

INDUCED ELECTROMOTIVE FORCES

It was shown in Chapter XVII that an electromotive force can be produced either by chemical or by thermal means. In one case chemical energy and in the other case heat is converted into electrical energy. Of much greater practical importance than either of these methods, however, is the possibility, provided by one of the fundamental properties of a magnetic field, of obtaining an electric current from mechanical energy. The discovery of this possibility and its early development are due almost exclusively to the genius of Michael Faraday.

111. E.M.F. Induced in a Wire by its Motion through a Magnetic Field. According to our present viewpoint, the current in a wire consists of a flow of negative electrons taking place

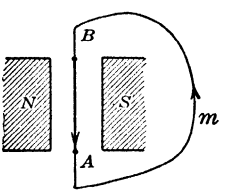


FIG. 90

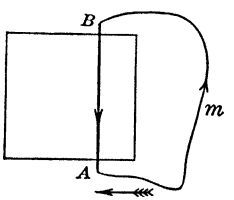


FIG. 91

E.M.F. Induced in Moving Wire.

in a direction opposite to that in which the current is conventionally thought of as moving. Whether a current is flowing or not, the electrons form an essential part of the conductor, being always present and ready to move when subjected to an electric field. There is, however, nothing in most of the phenomena of metallic conduction to indicate whether the observed effects are due to a flow of positive or negative electricity, and as a matter of convenience the conventional viewpoint of moving positive charges will be adopted in the present chapter.

Imagine, therefore, between the jaws of the electromagnet *NS* (Fig. 90) a wire *AB*, which is thought of as containing a quantity of positive electricity, ready to move in any direction

in which it may be urged by an electric field. Let H be the strength of the magnetic field. Suppose, now, that the wire is moving into the plane of the paper with a velocity v. Every unit charge of positive electricity in the wire must move with the same velocity, and must, in accordance with Formula 97 of Chapter XIX, be acted on by a force along the wire from B to A of magnitude

$$F = Hv \tag{105}$$

If A and B are now joined by another wire AmB, the downward force experienced by the electric charges between the jaws of the magnet will tend to drive a current around the circuit so formed. Every unit of charge taking part in this flow and passing from B to A will have done on it an amount of work equal to Fl or Hvl, if l is the length of the wire in the magnetic field. This work drives the charge on around the circuit in the same way that the work done on a charge passing through a galvanic cell drives it through the wire which joins the terminals of the cell. It constitutes, therefore, an electromotive force E such that

$$E = Hvl \tag{106}$$

It is easily shown that the current so produced is in accord with the Law of the Conservation of Energy. If, for example, the resulting current is I, the force which the magnetic field exerts on the wire in accordance with Section 78 is $F = HIl$, and it has a direction which, according to the three-finger rule, is out from the paper. In order, therefore, to push the wire into the paper with a velocity v for 1 sec., an amount of energy $Fv = HIlv$ is required. But this by (106) is equal to EI, precisely the work which the electromotive force E performs in the same time. There is, therefore, a direct conversion into electric energy of the work done in moving the wire through the magnetic field.

112. E.M.F. Due to Changing Magnetic Field. There is another way of viewing this same process, which can be best appreciated if the magnetic field is imagined perpendicular to the paper. Fig. 91 represents, therefore, the electromagnet as it would appear to one looking in the direction of the magnetic field. AB is once more the wire, which is moving from right to left with velocity v. It will be noted that, as the wire moves, more and more of the circuit AmB will be penetrated by the magnetic field. This rough observation can be made quantitative

by the introduction of a new concept, that of *magnetic flux.* The magnetic flux across any surface is defined as the intensity of the magnetic field perpendicular to the surface multiplied by its area. As the wire moves from right to left, the amount of flux through the circuit AmB will increase each second by an amount which is equal to the product of the field strength H and the area swept out by AB. This area is vl, and the increase in flux through the circuit is accordingly Hvl. This is recognized as precisely the electromotive force produced. We accordingly conclude that the electromotive force produced is equal to the change in magnetic flux through the circuit per second. Or better, if we represent the flux by Φ, and the change in flux in time Δt by $\Delta\Phi$,

$$E = \frac{\Delta\Phi}{\Delta t} \tag{107}$$

The advantage of this method of stating the law is seen if we modify somewhat the experiment, and think of the magnet as being moved from left to right with velocity v while the circuit AmB is held fast. If the field of the magnet is uniform, it is difficult to see how the electricity in the wire would recognize and respond to such a motion. Yet the flux through the circuit is being increased, and the electromotive force predicted by (107) is in fact observed.

Another way in which the flux through the circuit could be changed without moving either the magnet or the circuit itself would be by changing the current through the coils of the electromagnet. Experiment shows that here also an electromotive force of the magnitude predicted by (107) is produced. We cannot but conclude that (107) is a fundamental and accurate law, which can be relied on to predict the correct electromotive force regardless of the way in which the change in flux takes place.

113. Lenz's Law. Formula 107 says nothing with regard to the direction in which the current is set up. For the simple case of a wire moving across the magnetic field, the three-finger rule may once more be invoked to obtain this information. If the forefinger of the right hand represents the direction in which the electricity is moving, i.e., the direction in which the wire is pushed, and if the second finger again represents the direction of the magnetic field (from the north to the south pole), the thumb will give the direction in which the electricity is urged by the

magnetic field, i.e., the direction of the resulting electromotive force.

It was shown by Heinrich Friedrich Emil Lenz that the direction of the induced current is under all circumstances such as to oppose the change in the magnetic field which produces the current. Consider, for example, the case of a coil C (Fig. 92) and a bar magnet NS in its vicinity. The coil is to be thought of as lying in the plane of the paper while the magnet is perpendicular to the paper with its north pole downward. If the magnet is now suddenly raised vertically upward, the magnetic flux through the coil will decrease, and a current will be induced in it. The direction of the current is that indicated by the arrow in the figure, for such a current would result in a magnetic field at the center of the coil, which, in accordance with Ampère's rule, would pull a north magnetic pole vertically downward into the paper. This will accordingly hinder the upward motion of the magnet. The energy liberated by the current in the form of heat in the coil is, therefore, derived from the work of raising the magnet against this attracting force. Similarly, if the magnet is lowered toward the paper, a current is produced in the opposite direction, thus opposing the approach of the magnet.

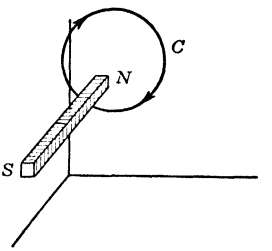


FIG. 92. E.M.F. Induced by Moving Bar Magnet.

114. The Magnetic Properties of Iron. The examples already given suffice to show that the changing magnetic flux of (107) may be furnished either by a coil carrying a current or by a permanent magnet, or by both. Thus far the only method which has been described for magnetizing a specimen of iron is that of contact with a natural lodestone. It is, however, a familiar fact that a piece of iron can be readily magnetized by the magnetic field from a coil. If the specimen is one of hard iron, some rapping is necessary while under the influence of the field to produce intense magnetization, but once produced, the magnetization is retained with great tenacity. If, on the other hand, the iron is soft, it is readily magnetized by simple exposure to the field, and just as readily loses the greater part of this magnetization when the field is removed. If a bar of such soft iron is placed inside a coil carrying a current, the magnetic field in the vicinity will be made up of two parts, that due to the coil

alone, and that due to the magnetization produced in the iron. With a good quality of soft iron this total field may be several hundred times that of the coil alone, and it is to this fact that the importance of iron in producing intense magnetic fields is due.

A typical electromagnet consisting of two heavy iron cores, each surrounded by many turns of wire, is shown in Fig. 93. For the purpose of Formula 107 the flux which emerges from the pole of an electromagnet must be considered as existing also within the iron itself. The total flux through the iron, therefore, is the sum of that due to the coil alone and that due to the magnetization of the iron. Just as the flux per unit area due to the coil is known as the intensity of the magnetic field H in which the iron is located, so the total flux per unit area is known

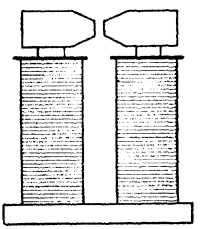


FIG. 93. Electromagnet.

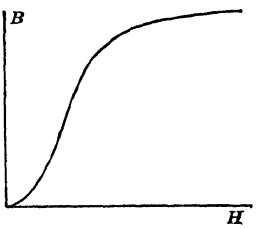


FIG. 94. Magnetization Curve.

as the *magnetic induction* B, and the Φ which is used in (107) must be taken as the product of B and the area of the region within the coil which is occupied by the iron. The ratio of B to H depends on the character of the iron and on the intensity of the field H. It is known as the *magnetic permeability* of the iron, and is represented by the letter μ. Fig. 94 is a typical graph depicting the variation of B with H. It should be noted that, whereas the increase in B is very nearly proportional to the increase in H throughout a considerable portion of the graph, it becomes much less rapid as the intensity of the field is increased, and finally gains very little from further additions to H. The iron is said to be magnetically *saturated.*

115. The Electric Generator. If a coil is so mounted (Fig. 95) that it can be rotated about an axis perpendicular to the magnetic field NS, a maximum magnetic flux will pass through it when vertical, whereas that passing through it in a horizontal position is zero. If one end of the coil is attached to the ring A and the other to B, and if two metal strips or *brushes* S_1 and

S_2 make contact with these rings, a momentary electromotive force will exist between S_1 and S_2 whenever the flux through the coil is changing. If the coil is rotated at a uniform rate, the flux will be changing most rapidly when the coil is horizontal, and the electromotive force at that instant will be a maximum. From that point the electromotive force will gradually decrease until, when the coil is vertical, it will have fallen to zero. As the rotation continues, the electromotive force will once more increase, but in the opposite direction, becoming again a maximum when the coil is horizontal. With a uniform field and a uniform speed of rotation, it can be shown that the electromotive force will vary with the time in exactly the same way that the sine

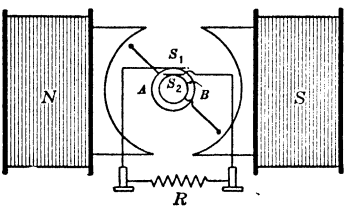


FIG. 95. Alternating Current Generator.

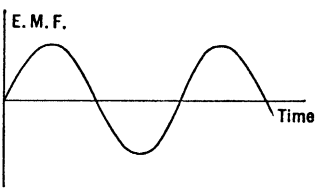


FIG. 96. E.M.F. from Alternating Current Generator.

or cosine of an angle varies with the angle. This is shown in Fig. 96. Such a curve is known as a *sine curve*, and an electromotive force which varies in this fashion is known as a *sinusoidal alternating* electromotive force. If a resistance is connected between S_1 and S_2, a current will flow through it, faithfully reproducing the fluctuations of the electromotive force. It is accordingly called a sinusoidal alternating current.

Such a rotating coil is a very simple form of alternating-current generator, and the generators which supply the huge quantities of electrical energy which are used in the world to-day are dependent for their action on the same fundamental principles. If a direct rather than an alternating current is required, the ends of the coil *CD* can be attached to the two segments of the *commutator A* of Fig. 97. Two brushes as before bear on opposite sides of the commutator, and at the instant the coil becomes vertical, they exchange segments, thereby maintaining continuously the same direction of electromotive force. With a single coil the electromotive force would then vary with the time as shown in Fig. 98, but by incorporating in the generator a

large number of coils in series, which reach their maximum voltages at different times, the irregularities can be very largely smoothed out, and a fairly steady current obtained.

The few disadvantages which an alternating current has in comparison with a direct current are more than compensated for by other alternating-current advantages, and direct currents

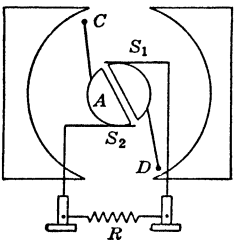


FIG. 97. Direct Current Generator.

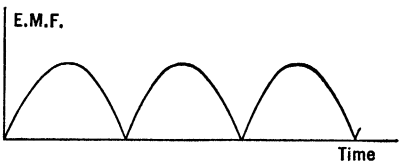


FIG. 98. E.M.F. from Direct Current Generator.

are being used less and less for industrial purposes. Thus an alternating current which describes 60 complete cycles (120 alternations) per second, as nearly all commercial currents do, is just as satisfactory for heating and lighting as a direct current. Modern alternating-current motors also are perhaps more satisfactory than direct-current motors, and only for certain special purposes such as electroplating are direct currents really required.

116. The Transformer. Self Inductance. The great advantage of an alternating current, on the other hand, is the ease with which its voltage can be raised or lowered by means of a *transformer*. If a number of turns of wire are wound about an iron core (Fig. 99), an alternating electromotive force applied to the ends of the coil will send through it an alternating current. Whenever the current increases in the coil, the magnetic field in the iron core must increase with it. An increasing magnetic field, however, produces an electromotive force which, in accordance with Lenz's Law, opposes the growth of the current. Similarly a decrease in the current is also opposed by an induced electromotive force from the same

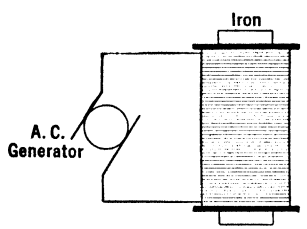


FIG. 99. Alternating Current through a Coil.

source. This qualitative observation can be made quantitative only by an understanding of the way in which alternating currents are measured. The magnitude of such a current could, of course, be adequately described in a variety of ways. The maximum value which the current attains at the peak of the cycle or the average value during a half cycle would both be reasonable measures to adopt. It is, however, more convenient to use the square root of the mean square (R.M.S.) current and the corresponding *root mean square* value of the electromotive force. It can be shown that the R.M.S. value of either current or electromotive force is equal to the maximum value divided by $\sqrt{2}$, and, when both quantities are measured in this way, the most important direct-current relations between current, voltage, and power are still valid. Thus if an R.M.S. voltage E drives an R.M.S. current I through a resistance R,

$$E = IR \quad \text{or} \quad I = \frac{E}{R} \tag{108}$$

The power in watts delivered by the electromotive force E to a resistive load is still

$$P = IE \tag{109}$$

and the power spent in heating the resistance R is

$$P = I^2R \tag{110}$$

According to Equation 107, the electromotive force produced in each turn of a coil C by a change in the magnetic flux passing through it is proportional to the rate at which the change is taking place. This would be proportional during every part of the cycle to the maximum flux at the peak of the cycle, and this, in turn, if the current is not large enough to saturate the iron, is approximately proportional to the current I through C and to the number of turns N of wire in the coil. The rate at which the flux is changing should also be proportional to the rapidity of alternation of the current, i.e., to its *frequency* n, the number of complete cycles per second. The total electromotive force produced in C is equal to that produced in each turn multiplied by N, and is, therefore, proportional to N^2 rather than to N alone. Hence, if we denote this opposing induced electromotive force by E_L,

$$E_L \propto N^2nI \tag{111}$$

It is customary to write this proportionality, as we have all similar relations, in the form of an equation

$$E_L = 2\pi nLI \tag{112}$$

where L is known as the *self-inductance* of the coil. By comparison of (111) with (112) it is seen that L must be approximately proportional to N^2, the square of the number of turns in the coil. In all alternating-current work it has been found convenient to have a 2π always associated with the frequency n, and it is for this reason that the self-inductance L has been so defined that a 2π has been left out as a factor in (112). The product $2\pi n$ is usually represented by the Greek letter ω. Thus (112) might be written

$$E_L = \omega LI \tag{112'}$$

If an alternating external electromotive force E is impressed on such a coil or *inductance*, as it is called, the R.M.S. current through it will be of such a magnitude that the opposing electromotive force E_L will just balance the impressed electromotive force E, i.e.,

$$E = \omega LI \quad \text{or} \quad I = \frac{E}{\omega L} \tag{113}$$

The practical unit of self-inductance L which yields the current I in amperes when the electromotive force is expressed in volts is called the *henry* in honor of the great American physicist and engineer of that name. ωL is known as the *impedance* of the coil.

In case the electromotive force E is applied to an inductance L and a resistance R in series (Fig. 100), the situation is complicated by the fact that the maximum value of the opposing induced electromotive force occurs at the instant the current is changing most rapidly, while the maximum value of the potential drop in the resistance occurs at the peak of the cycle. It can be shown that, when such factors are taken into consideration, the R.M.S. current in such a circuit can be obtained from the R.M.S. electromotive force by the following modification of Ohm's Law.

$$I = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \tag{114}$$

The impedance of the circuit is now the radical in the denominator. Although an inductance, like a resistance, impedes the

passage of a current, it does so without the dissipation of any energy. Energy, to be sure, is required to set up the magnetic field in the iron, but, if the core is made of a good quality of soft iron, this energy is almost completely recovered when the magnetization again drops to zero. The power spent in the circuit of Fig. 100 is, therefore, simply I^2R.

If two coils of wire, P and S, are wound about a core A (Fig. 101), and if an alternating electromotive force is impressed between the ends of P, the alternating magnetic flux produced in the core will induce an alternating electromotive force in S. If the ends of S are attached to a resistance R, an alternating current will be set up in the circuit so formed. Such a device is known as a *transformer*. The iron core serves to lead the mag-

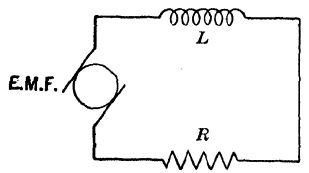


FIG. 100. Alternating Current through Inductance and Resistance.

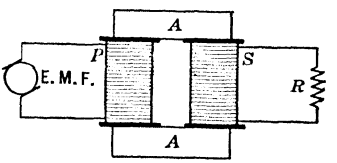


FIG. 101. Transformer.

netic flux from the primary coil through the turns of the secondary, a function which in a well-designed transformer it performs very nearly completely. The induced electromotive force in each turn of S is by (107) proportional to the rate of change of the flux through it, and this rate of change is proportional to the maximum value of the flux, which we may denote by Φ. The total induced electromotive force in S, E_S, is N_S times that in the individual turn, if N_S is the number of turns on the secondary, i.e.,

$$E_S = kN_S\Phi \tag{115}$$

where k is a constant of proportionality which depends simply on the electromotive force produced per turn per unit of magnetic flux. Similarly the induced electromotive force in the primary, which is equal to the impressed electromotive force E, is

$$E_P = kN_P\Phi \tag{116}$$

Hence, dividing (115) by (116),

$$\frac{E_S}{E_P} = \frac{N_S}{N_P} \tag{117}$$

It is, therefore, possible by varying the number of turns in primary and secondary to obtain a voltage in the secondary E_S, which shall bear any desired relation to that in the primary. Thus, if the voltage in the primary is 110, and the ratio of N_S to N_P is 20, the voltage of the secondary will be (110) (20) or 2200 volts. Similarly, if lower voltage is required in the secondary, it is merely necessary to make N_S less than N_P. The former type is a *step-up* transformer, and the latter is a *step-down* transformer.

117. Power Transmission at High Voltage. The extensive industrial use of transformers and alternating currents is due to the fact that power can be transmitted much more economically at high voltages than at low. Consider, for example, a transmission line which has a resistance of 1 ohm. If 10,000 watts is to be delivered along this line at a potential of 100 volts, it will require a current of 10,000/100 = 100 amperes. The power which will be dissipated in heat in the line (I^2R) will be $(100)^2 \times 1 = 10{,}000$ watts. For every 10,000 watts delivered, 10,000 watts would be wasted in the line. The efficiency of transmission would be only 50 per cent. If, however, the same power was transmitted at 10,000 volts, only 1 ampere would be required. The power wasted in the line would be only $(1)^2 \times 1 = 1$ watt, and the efficiency of transmission would be 99.99 per cent. It is customary, therefore, after generating the power, to raise the voltage by means of a transformer to 2200 volts, 10,000 volts, or even higher, depending on the distance to which it must be transmitted, and then to step down the voltage again to a convenient value in the immediate locality where it is to be used.

In recent years the construction of many high-tension lines has made possible, not merely the efficient utilization of water power over considerable distances, but it has enabled many small power houses to combine their resources in such a way as to safeguard themselves against any unusual demands for power.

PROBLEMS

1. What electromotive force is induced in a car axle 2 meters long when the car is travelling due south with a velocity of 30 miles an hour at a point where the vertical component of the Earth's magnetic field is 0.5 gauss? State the answer both in electromagnetic units and in volts.

2. A single circular turn of wire 1 meter in diameter lies in a horizontal position at a point where the vertical component of the Earth's field is 0.5.

What average electromotive force is induced in this coil if it is suddenly drawn out into two contiguous straight wires? Assume the operation to require $\frac{1}{10}$ sec.

3. What average current would flow through the coil of the preceding problem if its resistance was 1 ohm? Calculate the total quantity of electricity in coulombs which would flow through the coil during the process of straightening out the loop.

4. A coil of wire consisting of 10 turns, each 50 cm. in diameter, lies horizontally in the Earth's magnetic field. Assuming the intensity of the field to be the same as in the last problem, and the resistance of the coil to be 10 ohms, calculate the total quantity of electricity which would flow through the coil during the process of turning it about a horizontal axis through an angle of 180°.

5. An *exploring coil* consisting of 100 turns of fine wire, each having a diameter of 2 cm., is used to measure the intensity of the magnetic field between the jaws of a large electromagnet. The terminals of the coil are attached to a *ballistic galvanometer*, which by its deflection measures the total quantity of electricity which flows through it. The combined resistance of coil and galvanometer is 100 ohms. If, when the coil is suddenly snatched from between the jaws of the magnet, the galvanometer indicates the passage of 10^{-3} coulomb, what is the intensity of the magnetic field?

6. If the resistance of a certain line is 2 ohms, at what voltage must 10,000 watts be transmitted in order that the efficiency of transmission shall be 90 per cent?

CHAPTER XXIII

ELECTRIC OSCILLATIONS

118. Damped Oscillations. In Chapter IV a brief section was devoted to oscillatory motion, and the time of swing of the simple pendulum was calculated. If the motion of the pendulum took place in a vacuum, and if no energy were dissipated by friction with the air or in bending the cord, the bob would rise after each full swing exactly to its original elevation, and the motion would continue indefinitely without any reduction in amplitude. In

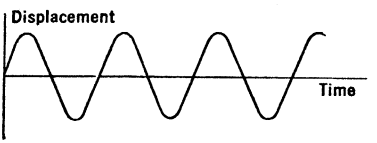


FIG. 102. Graph of Pendulum Displacement against Time.

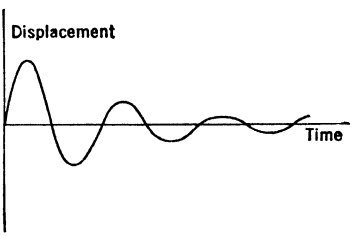


FIG. 103. Damped Oscillations.

practice, however, a certain amount of energy is converted by friction into heat on each swing, and the amplitude of oscillation becomes less and less until the motion ceases altogether. Such a process is known as *damping*, and oscillations which are reduced in this way are known as *damped* oscillations.

If the amplitude of swing of the pendulum is not too great, and if the damping is slight, the motion can be very simply represented in graphical fashion. Thus, if the displacement x of the bob is plotted vertically against the time t which has elapsed since any one passage of the equilibrium position, the graph will have the form of the familiar sine curve shown in Fig. 102. The motion is sinusoidal or simple harmonic. If damping is appreciable, as it usually is in practice, the amplitude of each oscillation is less than that of the preceding oscillation, and the graph takes the form shown in Fig. 103. Although damping occurs in most actual oscillations which are not artificially sus-

tained, approximately simple harmonic motion is quite frequently encountered.

Many materials are elastic in the sense that, if they are deformed in any way, they tend to recover their original shape. Thus a stretched steel wire, if not drawn out beyond its elastic limit, will regain exactly its original length when the elongating force is removed. It was pointed out by Robert Hooke, a contemporary of Newton, that the elastic restoring force in such cases is very nearly proportional to the displacement. Accordingly, if a heavy weight is suspended from a long steel wire, and is displaced slightly up or down from its equilibrium position, it will describe approximately simple harmonic motion under the action of the elastic restoring forces. The oscillatory motions which result from elastic forces are very common, being, indeed, responsible for the sound emitted by all musical instruments. In general, the motion is quite complicated, and cannot be even approximately viewed as that of a particle. The graph of displacement against time under such circumstances, although exhibiting a regular repetition of the same shape, is more complicated than the simple sine curve. Such oscillations, though not simple harmonic, are called *periodic* because of the constant recurrence of the same fundamental form. They will be considered more in detail in the next chapter.

119. The Oscillatory Discharge of a Condenser. In 1842 the American physicist, Joseph Henry, attempted to magnetize a steel needle by passing the discharge from a Leyden jar through a coil surrounding the needle. The experiment was successful, thereby demonstrating that moving electrostatic charges are capable of producing magnetic effects, but on repeating it a number of times, he found that the direction in which the needle was magnetized could not be predicted. Although the positive coating of the jar was always connected to the same end of the coil, sometimes one end of the needle became the north pole and sometimes the other. He drew from this observation the only possible conclusion, viz., that the discharge current was oscillatory. A similar observation had undoubtedly been made by Felix Savary at an earlier date, and Helmholz made the discovery independently in 1847. In 1853, Lord Kelvin showed theoretically that an oscillatory discharge was to be expected. As the discharging current from the Leyden jar passes through the coil, it must be accompanied by a magnetic field within the

coil. The setting up of this field as the discharge starts produces a back electromotive force, which tends to retard the growth of the current. Similarly, when the jar has lost its entire charge, the stoppage of the current through the coil must be accompanied by the disappearance of the magnetic field, which, in turn, results in an electromotive force which opposes any reduction in the current. Every change in the magnitude of the current through the coil is, therefore, opposed by the electromotive force which results from the changing magnetic field. The coil gives to the current a property analogous to the inertia of a mechanical system. Just as the inertia of a pendulum bob keeps it in motion after the restoring force is reduced to zero, so the inductance of the coil maintains the current for a time after the potential difference between the condenser plates is reduced to zero. This continuance of the current can result only in charging the condenser in the opposite direction, thus building up an opposing potential drop, which eventually stops the current flow. If no energy has been dissipated in heat in the coil, the potential energy of the charged condenser, and therefore its charge, must now be the same as before the discharge started. Since its terminals are still connected by the coil, it will again start to discharge, this time in the opposite direction, and the whole process will be repeated. Plotting the current through the coil against the time, we get here also a simple sine curve if the energy loss in the coil is negligible, and a damped sine curve in cases where the energy loss is appreciable.

It was shown mathematically by Lord Kelvin that the number of oscillations per second of the discharge current or the *frequency*, as it is called, should be

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \tag{118}$$

Or, setting as in the last chapter $2\pi n = \omega$,

$$\omega = \frac{1}{\sqrt{LC}} \tag{118'}$$

The unit of capacity in the practical system is the *farad*, and is the capacity of a condenser on which a charge of 1 coulomb can be placed with a resulting potential difference of 1 volt. Since the coulomb is 3×10^9 E.S.U. of charge, while the volt is only 1/300 of the electrostatic unit of potential, the farad must

be $3 \times 10^9 \times 300 = 9 \times 10^{11}$ times the size of the electrostatic unit of capacity. The electrostatic unit, the capacity of a sphere 1 cm. in radius, although small, is not excessively small, and it accordingly follows that the farad is, practically speaking, an impossibly huge capacity. It is customary to use in its place the *microfarad* (μf.), 1/1,000,000 of the farad, when large capacities are involved, and the *micromicrofarad* ($\mu\mu$f.), 1/1,000,000 of the microfarad, when a comparatively small unit is desired. The micromicrofarad is seen to be just $\frac{9}{10}$ of the electrostatic unit.

The large paper condensers which are extensively used in the power supply units of radio receivers range from 1 to 4 μf., and the variable tuning condensers in such a receiver are between 100 and 500 $\mu\mu$f. The large *choke* coils in the power supply filter have inductances around 30 henries, and the tuning coils of the receiver are between 100 and 300 microhenries. Applying (118) to calculate the frequency of discharge of a 4-μf. condenser through a 30-henry inductance,

$$n = \frac{1}{2\pi} \frac{1}{\sqrt{(30)(4)10^{-6}}} = 14.5$$

Applying it to the discharge of a 100-$\mu\mu$f. condenser through a 100-microhenry coil,

$$n = \frac{1}{2\pi} \frac{1}{\sqrt{(100)(100)10^{-6}10^{-12}}} = 1{,}590{,}000$$

Thus the frequency of condenser discharge which can be obtained from the ordinary equipment of a radio receiver ranges from only a few cycles per second up to at least several million, and with specially designed apparatus it is not difficult to secure a frequency of 10,000,000,000 cycles.

The high frequencies possible with such electric oscillations are important because of the distance over which they can be detected. In the last chapter it was pointed out that the electromotive force which results in a coil from a periodically changing magnetic flux is proportional to the frequency at which the alternations in the flux take place. Let P and S in Fig. 104 represent two coils such as are used in a radio receiver for tuning purposes. When the high-frequency alternating current from a discharging condenser is passed through P, some of the resulting magnetic flux will link the second

coil S even though it may be at a considerable distance away. The root mean square electromotive force induced in S will be proportional, of course, to the root mean square value of this flux, but it will also be proportional to the frequency of its alternation, and the increase in electromotive force resulting from the high frequency may more than compensate for the minuteness of the flux which arrives from the distant primary. It is, of course, a well-known fact that the distance over which such signals can be detected does not increase indefinitely as the frequency is raised. The reason for this apparent contradiction is the fact that an electromotive force is induced, not merely in the receiving coil, but also in any conductors, such as the surface of the Earth itself, which may be exposed to the magnetic flux. The induced currents which result in the Earth's surface are also proportional to the frequency, and are always such as to reduce or nullify the original disturbance. Nevertheless, despite this conflicting effect, there is unqualified gain in the intensity of the received signals up to a frequency of several hundred thousand or a million cycles.

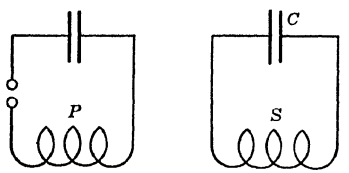


FIG. 104. Magnetically Coupled Circuits.

120. Resonance. Another phenomenon, that of resonance, is of importance in making possible the reception of such signals from a distance. If a slight tap is given to a pendulum bob as it hangs in its equilibrium position, it will be displaced slightly, and set into oscillation over a small arc. If, after a full swing, it is once more tapped as it passes the same point, the amplitude of swing will be increased; and if the tap is repeated over and over again, the pendulum will oscillate through a greater and greater range until the energy which is lost in each cycle by friction just equals that received from the tap. The only requirement is that the tap shall occur at exactly the frequency at which the pendulum tends naturally to swing. Similarly, if a condenser C is connected between the ends of the coil S in Fig. 104, such that the natural frequency of its discharge through S is n, an oscillating current of considerable intensity can be set up by a very small electromotive force of this same frequency n. This phenomenon, known as *resonance*, takes the simple form here described only if the proportion of the total flux from P

which passes through S is not too great. If the coils are too tightly coupled to each other, the reaction of the current in S on P leads to a much more complicated situation. If the electromotive force induced in S is E, and has a frequency $n = \omega/2\pi$, the current produced in S is

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{119}$$

where L is the inductance of the coil and C is the capacity of the condenser. If C is variable, and is gradually increased from 0 to a comparatively large value, the current, calculated from (119) for various capacities, can be plotted in a graph against C. Fig. 105 shows a typical example of such a *resonance curve*. The maximum value of the current is seen from (119) to occur for a value of C such that $\omega L - 1/\omega C = 0$, i.e., such that $\omega = \sqrt{1/LC}$. This value of ω is seen to be that of the natural discharge of C through L, and it is for this value that resonance occurs. The magnitude of the maximum current is seen by (119) to be $I = E/R$, in accordance with Ohm's Law. By constructing a coil of very low resistance, it is possible to obtain large currents from comparatively small electromotive forces.

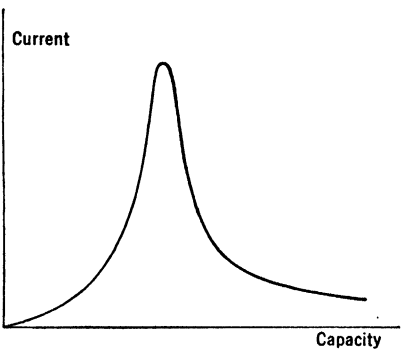


FIG. 105. Resonance Curve.

121. Radio Communication. The first to take advantage of the opportunity which high frequency and the phenomenon of resonance provided for detecting electric oscillations at a distance was Heinrich Hertz. In 1888 he devised apparatus both for the production and the reception of these oscillations over a distance of many meters. As his experiments were of the greatest significance in establishing the method by which the disturbance at one locality is transmitted to a distant point, they will be discussed in detail in Chapter XXXII. The application of this phenomenon to practical wireless communication was, on the other hand, the work of Guglielmo Marconi.

There are two ways by which a high-frequency electric oscillation, generated in one place and detected in another, can carry a message. The *modulation* of such an oscillation so that it carries with it the precise sounds of speech or music will be discussed in the next chapter, but the simpler and older method of *code* transmission will be considered here. The oscillations can be produced by charging and discharging the transmitting condenser many times a second by means of a transformer and a spark gap. At each discharge a group of damped oscillations is produced, and the succession of many such discharges in each second produces a *train* of damped oscillations (Fig. 106). The

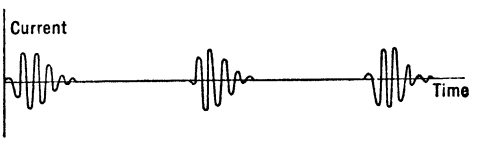


FIG. 106. Train of Damped Oscillations.

primary circuit of the transformer is closed by means of a key, and the length of such a train of oscillations can be made long or short at will, thus producing the familiar dots and dashes of the International code. It must be remembered that the oscillations which are thus set up in the receiving apparatus are of such a high frequency that they would not be detected by any mechani-

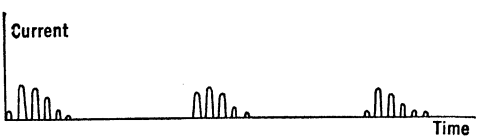


FIG. 107. Rectified Train of Damped Oscillations.

cal receiving instrument such as a galvanometer or the diaphragm of a telephone receiver. If, however, the high-frequency current is made to pass through a device, known as a *detector*, which permits the current to flow in one direction only, the graph of current against time will have the form shown in Fig. 107. There will result a one-way current throughout every series of damped oscillations, and if this current is passed through a telephone receiver, the diaphragm will be pulled toward the magnets once for each group of oscillations. A buzz similar to that which would be produced by the current in the primary

of the transmitter is heard, and will last just so long as the transmitter key is depressed.

In the early days of wireless communication a number of detectors were discovered, which had this property of passing current in one direction only. Perhaps the most satisfactory of all these early devices was the point of contact between a fine wire and the surface of a natural crystal. The current was found with many such crystals to flow more readily from the crystal to the wire point than in the reverse direction. In other words, the electrons pass quite readily from the tip of the wire into the crystal, but only with difficulty in the opposite direction. Fig. 108 shows the way in which such a crystal and telephone receiver can be connected to the tuned circuit in which the signal is received.

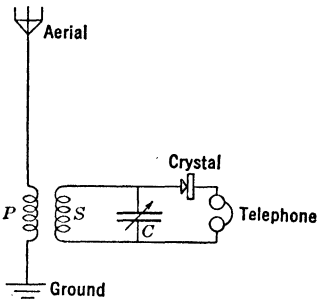


FIG. 108. Radio Receiver.

122. The Electron Tube as a Detector. It has already been pointed out that a vacuum tube containing a hot filament and a cold *plate* possesses this same property of unilateral conductivity. Its use as a detector was first suggested by J. A. Fleming in England, but it first became of great importance for radio reception when Lee de Forest in 1906 inserted a third electrode or *grid* between filament and plate. This device, which to-day is universally used for transmission as well as reception, owes its importance to the peculiar way in which the current from plate to filament depends on the voltage between grid and filament. Fig. 109*a* shows the current through one of the common receiving tubes plotted vertically against the voltage between plate and filament; Fig. 109*b* shows the plate current plotted against the potential between grid and filament. In the former curve the grid is maintained throughout at the potential of the filament; in the latter a positive potential of about 90 volts is kept on the plate. Such curves are known as *characteristics* of the tube being studied. It will be seen that as the plate potential in Fig. 109*a* starts at zero and gradually increases, the current through the tube increases, at first slowly, then more rapidly, and finally once again more slowly. With very high values of the plate potential, the gain in plate current from considerable additions

to the plate potential is practically zero. The slow initial growth of the current is due to the retarding effect of the swarms of electrons which normally surround the hot filament, while the failure of the current to increase in the upper part of the curve is due to the fact that a potential e_{p_1} is sufficient to carry over prac-

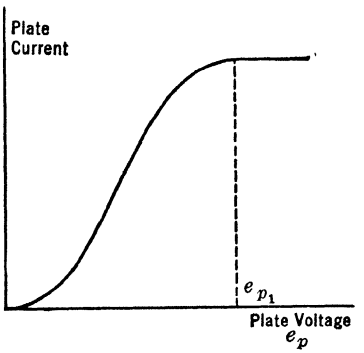


FIG. 109*a*. Plate Current — Plate Voltage Characteristic.

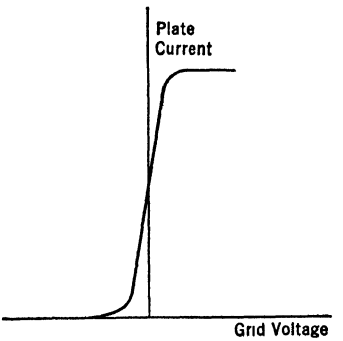


FIG. 109*b*. Plate Current — Grid Voltage Characteristic.

tically all the electrons emitted by the filament. A further increase in e_p must, therefore, be ineffective in producing additional current. It will be noted that the curve in Fig. 109*b* is of exactly the same character, but the slope of its middle portion is considerably greater. An increase of 1 volt in the potential of the grid for the particular tube in question increases the plate current by the same amount as an 8-volt increase in the plate potential. The *amplification factor* of the tube is 8.

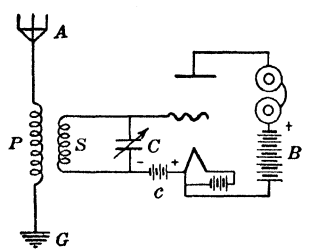


FIG. 110. Circuit Using an Electron Tube as Detector.

Such a tube is capable of performing two important functions in a radio receiver. It can act as a detector of the high-frequency oscillations, or it can *amplify* them. Fig. 110 shows a typical arrangement by means of which the oscillations can be impressed on the tube. The high-frequency signals are usually passed through a coil P, which is connected between the antenna A and the ground G. Some of the magnetic flux from this coil passes through the coil S, which, shunted by the condenser C, is connected between grid and filament of the tube. There may or may not be a battery, c, be-

tween the coil and the filament. In the simplest case the plate of the tube would be connected to one terminal of a telephone receiver, while the other terminal is connected to the positive pole of the 90-volt battery B. The condenser C is so adjusted that the circuit made up of S and C is in resonance to the incoming signal, thereby producing the largest possible current through S and the greatest possible alternating potential difference between grid and filament. If the battery c is omitted and the plate voltage is so low that, with grid and filament at the same potential, the plate current is still well down on the lower bend of the characteristic curve, the tube is in a suitable condition to operate as a detector. When the alternating voltage due to the signal is impressed on S, the potential of the grid will oscillate, first becoming positive with reference to the filament, and then negative. With the current initially on the lower bend of the characteristic, a negative voltage on the grid can produce only a relatively slight decrease in current, while a positive voltage will greatly increase it. The effect of the signal, therefore, is to produce a considerably increased average current through the tube, which will last just so long as the oscillations continue. If it is a train of damped oscillations which is being received, the telephone diaphragm will be drawn closer to the magnets for every group in the train, thus reproducing the sound of the transmitter.

123. The Electron Tube as an Amplifier. A somewhat different arrangement is required if the tube is to be used as an amplifier. A larger plate voltage is desirable, and the grid is so adjusted by means of the battery c that the plate current is well up on the straight part of the characteristic. To obtain a large amplification, several tubes may be used in series, the primary of the coupling transformer occupying the place of the telephone receiver in the simple detector circuit. As the signal arrives, the potential of the grid oscillates above and below its original value, but as the original current through the tube was on the straight part of the characteristic, the increase in current during one part of the cycle is no greater than the decrease at the other half of the cycle. There is, therefore, no average change in the plate current. The fluctuations in the grid potential, however, result in up and down fluctuations of the plate current, which in a properly designed amplifier faithfully reproduce the potential changes on the grid. It can be shown that, if the impedance of the

primary of the transformer in the plate circuit of the tube is sufficiently large, the R.M.S. drop in potential across this primary is greater than that impressed between grid and filament by very nearly the amplification factor, μ, of the tube. If, furthermore, the secondary of the transformer has n times as many turns as the primary, the R.M.S. voltage applied to the next tube is again multiplied by n. The overall amplification of tube and transformer is, therefore, under favorable circumstances nearly μn. If μ is 8 and n is 5, as it is in many common transformers which are used for this purpose, the total amplification of tube and transformer is nearly 40, and two such stages would give an amplification of nearly 40^2 or 1600. By proper use of the amplifying possibilities of such three-electrode tubes it is possible to increase the intensity of incoming signals by an amount which is limited only by the noises due to thermal agitations in the tubes! In general the distance from which signals can be received is limited, not by the weakness of the signals themselves, but by the masking effect of atmospheric disturbances.

Although the most familiar applications of amplifying tubes have been to radio reception, they have found a wide range of usefulness in many other fields of work. Long-distance telephony has been made possible, the way has been opened to the development of medical instruments of inestimable value such as the electrocardiograph, and new applications are being found on every hand. Direct currents as well as alternating currents can be amplified, and only recently such amplifiers have been so perfected as to make them by far the most sensitive current-measuring instruments which have thus far been devised.

124. The Electron Tube as a Generator of Oscillations. Still a third important function which the vacuum tube performs is that of generating oscillations. Fig. 111 shows one of many simple arrangements by which this can be done. C and L are a capacity and an inductance respectively. They are connected in such a way that if the condenser is charged, it will immediately discharge through L at a frequency $\frac{1}{2\pi}\sqrt{\frac{1}{LC}}$. An inductance L_g, so placed that it will be threaded by any magnetic flux through L, is connected between the grid and filament of the tube. If any momentary charge is now given to C, and it starts to discharge through L, the resulting oscillations will be picked

up by L_g and impressed on the grid of the tube. This in turn will result in an additional current through L of exactly the same frequency as the original current and so timed as to increase the amplitude of the initial disturbance. The tube in this way maintains the oscillatory discharge of C through L in much the same way that the spring and escapement mechanism of a clock maintain the pendulum in oscillation. The requisite energy is furnished by the battery B. The arrangement as a whole may be viewed as a device for converting the steady current of the B battery into an alternating current of any desired frequency. If proper precautions are taken in designing the apparatus, the current may be made quite accurately sinusoidal.

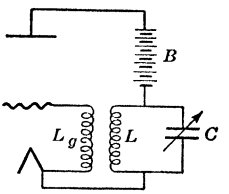


FIG. 111. The Electron Tube as a Generator.

125. The Mode of Propagation of Electric Oscillations from Transmitter to Receiver. The present chapter has been devoted almost exclusively to the generation and reception of electric oscillations. The mystery as to how the signal travels through apparently empty space from sender to receiver has been scarcely touched. It has, indeed, been stated that the magnetic field associated with a current in the sending circuit serves to couple it to the receiving apparatus. That this is only a partial truth was shown in 1873 by James Clerk Maxwell. In the course of a mathematical study of the electric and magnetic field in the vicinity of an electric current, he was forced to the conclusion that there is a fluctuating electric field associated with every fluctuating magnetic field. This electric field which accompanies the magnetic field from a transmitting antenna is even more important in the usual method of reception than the magnetic field, producing, as it does, the difference in potential between an antenna and the ground, which is responsible for the signals in the receiving apparatus. From his mathematical study Maxwell concluded that any disturbance in an electric or a magnetic field, such as that due to the oscillation of a discharging condenser, would send out *waves* from the source with a velocity in centimeters per second equal to the ratio of the electromagnetic to the electrostatic unit of charge, i.e., 3×10^{10} cm./sec. He recognized that this was precisely the known velocity of propagation of light, and suggested, accordingly, that light itself

might consist of just such electric and magnetic or *electromagnetic* waves. The proof of the correctness of this bold hypothesis must be postponed until the fundamental properties of wave propagation have been discussed in the next few chapters. It should be pointed out, however, that the extraordinary fruitfulness of Maxwell's work in the field of electricity and magnetism entitle him to a position among the very greatest of the scientific geniuses of all time. It was Maxwell's work which served as the direct inspiration to Hertz in his discovery of these electromagnetic waves in the laboratory, and which thus led to the development of radio communication in our own day. Of scarcely less importance in its influence on later scientific developments was the mathematical structure which he built, and of which the prediction of electromagnetic waves was a mere by-product. This must still be viewed as one of the most colossal steps which the human mind has yet taken in its effort to learn more about the physical world.

PROBLEMS

1. Calculate the frequency of oscillation of a condenser of (*a*) 1 μf. when it is discharged through a 1-henry coil, and (*b*) of 1 $\mu\mu$f. when discharged through 1 microhenry.

2. What current will flow through a 30-henry choke coil when a 110-volt, 60-cycle, alternating E.M.F. is impressed across it? Assume the resistance of the choke to be 25 ohms. What capacity would be necessary to resonate the 30-henry inductance to the 60-cycle E.M.F., and what would be the current at resonance? How much power would be expended in the coil, (*a*) without the condenser, and (*b*) with the necessary capacity to secure resonance? Neglect the resistance of the condenser itself.

CHAPTER XXIV

SOUND

Of the various means by which the individual is made conscious of his surroundings and by means of which he attempts to form opinions as to the nature of those surroundings, light and sound are the most important. It has long been known that the sensation of sound is due to a motion of the ear drum caused by changes in the air pressure against it. Thus an explosion produces a momentary compression of the atmosphere, which is transmitted to the ear drums of those who hear it.

126. Noises and Musical Sounds. If we attempt to classify the sounds with which we are familiar, the first major distinction which we would draw is between a noise and a musical sound. A noise is a transient disturbance of the atmospheric pressure or a group of unrelated transient disturbances; a musical sound is of longer duration, and has been shown to consist always of periodic changes. It should be emphasized, however, that, in defining a musical sound as such a periodic disturbance, we are including in this classification a great many sounds which are not ordinarily thought of as music. Thus very unmusical speech, if intelligible, consists of a succession of musical sounds, and the vibration of blocks of hard wood, when struck together, are also periodic, at least over a short interval of time.

127. Loudness and Pitch. We recognize three outstanding characteristics of all musical sounds: *loudness*, *pitch*, and *quality*. The loudness of a sound is quite obviously determined by the range or amplitude of the fluctuations in air pressure. The ear drum is an extremely sensitive organ, and the changes of pressure and the accompanying motion of the air molecules which it can detect are excessively minute. Thus for even the loudest sounds the motion of the individual molecules is not more than 2 or 3 mm., and displacements as small as 10^{-7} millimeter are readily detected.

The pitch of a sound is very easily shown to be determined by the frequency with which the periodic changes in pressure

succeed one another. Thus if air from a small nozzle (Fig. 112) is blown against a rotating disk in which holes have been drilled at regular intervals along the circumference of a circle, the successive puffs of air which emerge periodically through the holes produce the effect of a musical note. The faster the disk is rotated, the higher is the pitch of the note; and for any given speed, a direct measurement of the rate of rotation and a knowledge of the number of the holes provide a means of calculating the frequency. By a comparison of other sounds with that from this *siren disk*, their frequency can be determined with an accuracy limited only by the sensitiveness of the ear. Middle C has, for example, a frequency of 256, the lowest note on the piano 27.2, and the highest note 4138.4. The lowest note on most pipe organs is 16, but a few have a still lower pipe with a frequency of 8. The question as to whether this last is really a musical note is simply the question whether the ear can detect the individual throbs, coming as they do only 8 times per second, or whether they come so rapidly as to give only a resultant sensation. This question may very well be answered differently by different individuals, but most people will fail to get any musical sensation from such a low note. If we pass to the high-frequency end of the musical domain, we encounter once more a wide difference in individuals. Some people can hear sounds with a frequency as high as 30,000, others can hear nothing above 4000 or 5000, and the average person has an upper limit of somewhere around 20,000. Frequencies above 5000 or 6000 are of little practical importance either for music or speech. Those between 16 and 20,000 are known as *audio* frequencies to distinguish them from the high or *radio* frequencies discussed in the last chapter.

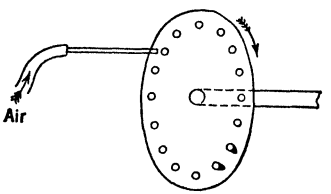


FIG. 112. The Siren.

128. Quality. Most musical sounds are produced by the vibration of elastic bodies, and are materially modified by the resonant motion of masses of air or other material in the vicinity. The peculiar quality of the sound from a particular instrument is due in part to the vibrator, the string in a piano or the reed in a reed organ pipe, but in a much larger measure to the resonator, the sounding board in the piano or the air column in the organ pipe. The fundamental physical distinction between two

sounds of the same pitch which come from different sources was first discovered by Helmholz in 1863, but only within comparatively recent years have experimental methods been developed for the precise investigation of these differences. A quite simple consideration serves, however, to limit very considerably the nature of the process which must account for the wealth of variety present in the sounds which come to us. Since the sensation of sound is due entirely to fluctuations in the pressure of the air against the ear drum, a single graph of pressure against time must contain all the factors which go to make up a complete description of any given sound. It is because of this fundamental simplicity of even the most complicated sounds that the *telephone* and the *phonograph* are possible.

129. The Telephone and the Phonograph. The telephone consists of two parts, the *transmitter* and the *receiver*. The receiving apparatus was invented in almost its present form by Alexander Graham Bell in 1876; the present transmitter is a development of the carbon transmitter devised by Thomas Edison in 1877. This latter instrument depends for its action

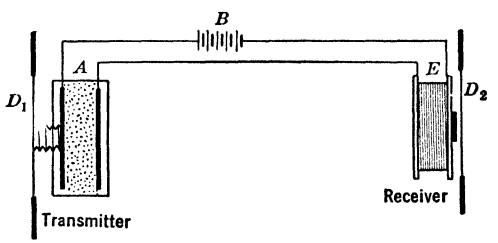


FIG. 113. The Telephone.

on the fact that pressure tends to lower the resistance of a loose contact such as that between carbon particles. In the modern *microphone* a small box of carbon particles A (Fig. 113) forms part of an electric circuit containing a battery B. A metal disk or diaphragm D_1, supported on its circumference, presses at its center against the carbon particles. The increase in pressure associated with a sound thrusts the diaphragm inward, tightening the contact between the particles, and thereby increasing the current in the circuit. If properly designed, the fluctuations in current thus produced will quite faithfully reproduce the fluctuations in pressure against the diaphragm. This fluctuating current now passes through the Bell receiver, which consists

of an electromagnet E (Fig. 113) with a second iron diaphragm D_2 quite close to the iron core. As the current through the coils of the magnet varies, the pull which the magnet exerts on the diaphragm varies, and this accordingly vibrates in synchronism with the transmitter diaphragm.

In the phonograph, invented by Edison in 1877, the fluctuations of pressure in the sound are reproduced, not in an electric current, but in the horizontal oscillations of a spiral groove cut in the record disk. The phonograph needle, moving through this groove, transmits these oscillations to the diaphragm of the *reproducer*. In the manufacture of a modern record the sound to be reproduced is converted by a microphone into a fluctuating electric current. This current, after amplification by a vacuum-tube amplifier, operates the machine which cuts the record. The perfection with which the most complicated sounds are thus reproduced is final proof that every element in the sound may be recorded in the single wavy line of the groove.

130. The Oscillograph. The microphone and the phonograph record have been very successfully used in recent investigations of the characteristics of sounds. Once a record when played has

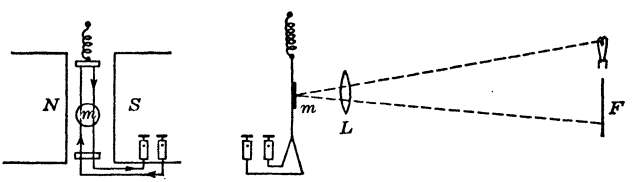


FIG. 114. The Duddell Oscillograph.

shown that it possesses all the important characteristics of the sound to be studied, the reproducer needle may be replaced by a needle making contact with the carbon granules of a microphone. In this way an electric current is obtained, which reproduces the oscillations in the groove of the record. This electric current can now be studied in either one of two ways, both of which have been used by the Bell Telephone Laboratories in recent investigations. In the first method the fluctuations in the current are permanently recorded on photographic film by an *oscillograph*. Two types of this valuable instrument are in common use. In the first or mechanical type, known as the Duddell oscillograph, a loop of fine wire is hung under slight tension between the poles of an electromagnet (Fig. 114). Cemented to this loop is a tiny

mirror m, by means of which, with the help of the lens L, the light from a fine filament is reflected and focussed on the film F. The current to be studied flows down one wire of the loop and up the other so that its effect in the magnetic field is to rotate the mirror through an angle which is approximately proportional to its intensity. As the current fluctuates, the spot of light moves back and forth across the film in synchronism with it. If, now, the film is drawn up or down by clockwork at a uniform rate, a curve will be photographically recorded on the film which will represent the variation of the current with the time. This type of oscillograph is satisfactory if the changes in current which it is desired to record do not take place so rapidly that the mirror and loop, owing to their inertia, are unable accurately to follow them.

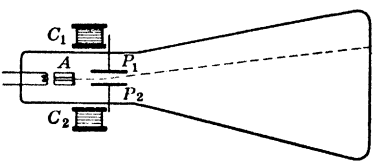


FIG. 115. Cathode-ray Oscillograph.

For the study of very rapid fluctuations, the cathode-ray oscillograph is used. This consists (Fig. 115) of a cathode-ray tube in which a fine pencil of rays is produced by passing them through a narrow hole in the cylindrical anode A. If the cathode particles are now subjected to an electric field between the parallel plates P_1 and P_2 or to the magnetic field from the coils C_1 and C_2, they will be deflected, and, owing to their small mass, the deflection will follow quite accurately exceedingly rapid alternations of the field. If the rays fall on a uniformly moving film, as in the other type of oscillograph, the variation of the field with the time here also will be directly recorded

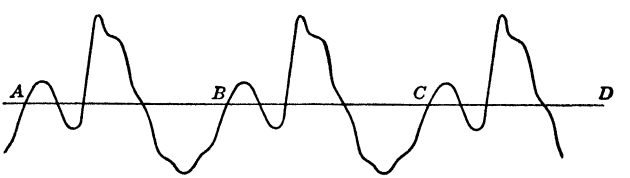


FIG. 116. Oscillograph Record of Organ Pipe. From "The Science of Musical Sounds," by D. C. Miller.

131. Harmonic Analysis and Synthesis. Fig. 116 shows, as an example of such a curve, that obtained from the sounding of an organ pipe. It will be observed that, although bearing little

resemblance to a sine curve, it is definitely periodic. The portion of the curve between A and B, for example, is accurately repeated between B and C and between C and D. This periodicity is characteristic of all musical sounds. It was shown by the French mathematician Joseph Fourier that any such periodic curve is equivalent to the sum of an infinite number of simple sine curves. The oscillations of the first of these sine curves recur with the same frequency n as the fundamental pattern of the periodic curve to be represented, and the frequency of the other curves are $2n$, $3n$, $4n$, etc., respectively. The component of the lowest frequency n is known as the *fundamental* component, and the others are known as *overtones* or *harmonics*. In general, the amplitudes of the harmonics decrease with increasing frequency, so that the sum of a limited number of such sine curves is quite adequate to reproduce very well all the irregularities of the required curve. Fig. 117 shows the fundamental and the first nine overtones which are necessary to represent the organ-pipe curve of Fig. 116. After careful measurement of any given curve, it is possible to calculate the amplitude of each harmonic which is necessary for its representation. Machines have, however, been devised which automatically record these amplitudes if the stylus of the machine is made to describe the contour of the required curve. This process of splitting up a given periodic curve into its various harmonics is known as *harmonic analysis*.

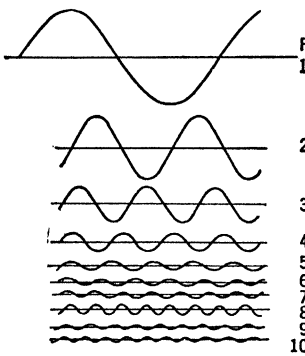


FIG. 117. Analysis of Organ Pipe Record. From "The Science of Musical Sounds," by D. C. Miller.

Perhaps the most convincing proof of the correctness of such an analysis is furnished by the converse process. After the various components of a sound such as that from an organ pipe have been determined, it is possible by sounding pure sinusoidal notes of the frequency and amplitude indicated by the analysis to simulate with astonishing fidelity the sound of the original pipe. This process is known as *harmonic synthesis*.

132. Analysis by Electric Filters. Another and somewhat less laborious method of analyzing sounds has been developed in recent years by the Bell Telephone Laboratories. Suppose, for example, it is desired to study the characteristics of the vowel *a* as sounded in *father*. A phonograph record is made, in which the word *father* or some other containing the same sound occurs. By a simple mechanical device the microphone needle is made to pass repeatedly over the section of the record occupied by the desired vowel sound. An electric current is thus obtained which, although containing all the fluctuations of the required sound, can be maintained as long as desired. Instead of recording this current oscillographically and subjecting the resulting curve to harmonic analysis, the current is analyzed directly by means of electric *filters*. In the last chapter it was pointed out that if an alternating potential of frequency n is applied to an inductance L and a capacity C in series, a maximum current is obtained if

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

The sharpness of the resonance at this frequency, i.e., the narrowness of the range of frequencies for which substantial resonance occurs, is determined by the resistance also present in the circuit. With such a simple circuit and with the lowest practicable resistance, the range of frequencies which give rise to substantial currents is rather wide, but somewhat more complicated arrangements of inductances and capacities can be devised which will respond almost uniformly to any desired range of frequencies, between 200 and 250 cycles, for example, and will block very nearly completely the passage of currents above or below this range. Such a device is known as an electric filter or more precisely as a *band pass* filter. By passing the complex current to be studied successively through a series of such filters which cover consecutively the entire range of audio frequencies, the precise frequency constitution of the sound under investigation can be very simply determined. This process is strictly analogous to the familiar one of determining with a radio receiver what particular stations may at any one time be broadcasting. Many different frequencies of varying amplitude are present simultaneously in the electromagnetic field surrounding the antenna. Starting at the low-frequency end of the broadcast

band, by varying the tuning condenser the receiving circuit is successively tuned to a large number of narrow frequency bands. If a station happens to be broadcasting on one of these little bands, for example in the vicinity of 1000 kilocycles, the signal will be received with a certain definite intensity when the receiver is tuned to this frequency. After all the broadcasting frequencies have been investigated in this way, an accurate knowledge will have been gained of the constitution of the complex electromagnetic disturbance around the antenna. Just as on almost any evening in a large city the electromagnetic field is extraordinarily complicated, so most sounds are found by the analogous process to be quite complex.

133. The Sounds of Speech. Among the most interesting studies which have been made in this way are those dealing with the sounds of speech. The vowels, for example, all involve a vibration of the vocal cord, but they are differentiated from one another by the way in which the mouth is formed. The cavities of the mouth and throat constitute the resonators which are a part of all musical instruments. In the vibrations of the vocal cord there are present a fundamental and all its harmonics, but the oral cavities are in resonance with and emphasize certain particular harmonics. It was first shown by Helmholz, and his conclusions have been borne out by recent experiments, that we distinguish one vowel from another by the absolute frequency of the overtone or overtones which are emphasized. The vowel *a* of father is characterized in this way by a frequency of about 1000 cycles. If, for example, it is sounded by a male voice at the pitch of middle C, 256 cycles, there will be present in the sound the frequencies 256, 512, 768, 1024, 1280, etc., but the oral cavity will be formed in such a way as to resonate to the frequency 1024, and a very large proportion of the total energy of the sound will be contained in this one frequency, i.e., in the fourth harmonic. A female voice, on the other hand, with a fundamental of frequency 512, would throw the greater part of the energy into the second harmonic at 1024. Certain of the vowel sounds are characterized in this way by one particular predominating frequency, others are characterized by two, and still others have been shown by recent studies to have a third region of resonance. Fig. 118 shows the predominating frequencies in a considerable number of the sounds of speech. Each sound is so plotted as to show by its distance above the horizontal axis its relative inten-

sity as compared to the other sounds represented. Certain consonants such as *s* and *h* are characterized by very high frequencies, and are, therefore, rather difficult to reproduce with a

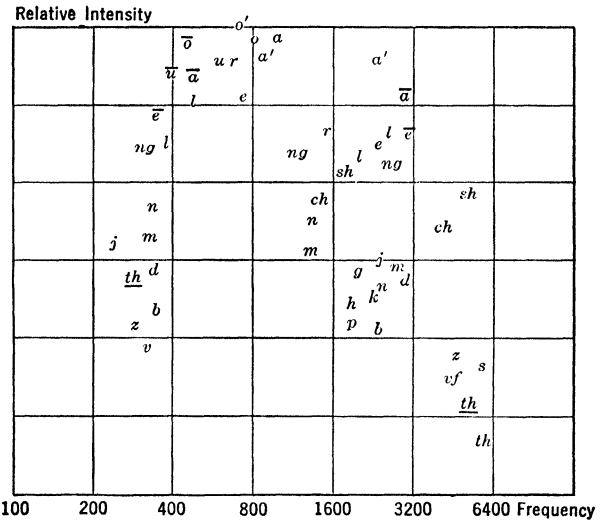


FIG. 118. Frequency and Intensity Associated with Sounds of Speech.
From "Speech and Hearing," by Harvey Fletcher. Courtesy of D. Van Nostrand Company, Inc.

telephone or phonograph diaphragm. As a result, these sounds were entirely absent in the earlier types of such apparatus.

134. Radio Telephony. With this understanding of the nature of sound, it is not difficult to see how a radio frequency electric

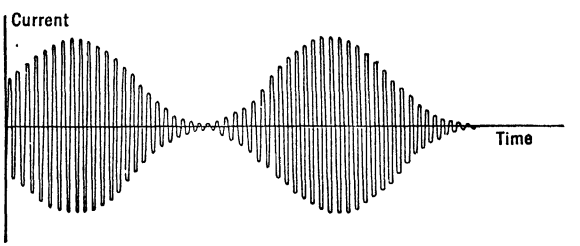


FIG. 119. Modulated Radio Frequency.

oscillation can carry with it all the characteristics of speech or music. It is merely necessary for the amplitude of the high frequency to vary with the time exactly as the audio frequency fluctuations of the sound to be reproduced. Fig. 119 shows a

high-frequency oscillation " modulated " in this way by a simple sinusoidal audio oscillation of frequency n. Fig. 120 shows the same modulated oscillation after it has passed through the detector of the receiving set. It will be observed that the average current through the detector increases and decreases periodically n times a second. These audio frequency changes actuate the diaphragm of the telephone receiver, and transmit to the ear the desired note of frequency n. There is obviously plenty of room for distorting the form of the original sound both in the process of

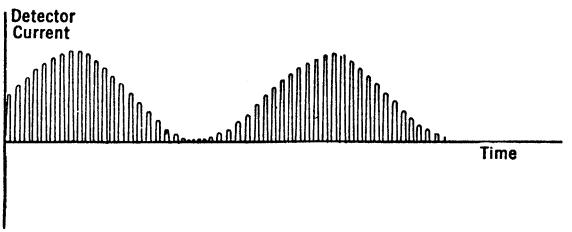


FIG. 120. Modulated Radio Frequency after Passage through Detector.

modulating the original oscillation in the transmitter and in detecting and amplifying it in the receiver. When suitable precautions are taken, however, it is possible to carry out the entire process with surprising fidelity.

CHAPTER XXV

THE PROPAGATION OF SOUND

In the last two chapters the nature of an electromagnetic or a sound disturbance has been described without entering upon a discussion of the way in which they are propagated from one place to another. It has been known for many years that sound is transmitted through the air in the form of a wave. That air is essential for the transmission of sound is readily shown by operating an electric bell in a chamber from which the air is slowly being pumped. The sound presently becomes fainter, and if the bell is so mounted that no vibrations are transmitted through the base or walls of the chamber, it becomes eventually inaudible.

135. The Nature of Sound Waves. The waves with which we are most familiar in everyday life are those which are formed on the surface of a body of water. Everyone has, for example, observed the circular wave which spreads out from the source of the disturbance when a stone is dropped into a smooth pond. In exactly the same way a sudden compression of the air, such as that produced by an explosion, travels uniformly outward; not, to be sure, on a surface in two dimensions as in the case of the water wave, but in three dimensions. Just as a water wave, if undisturbed, moves with the same velocity in all directions and hence takes a circular form, so the sound wave spreads out in the form of a sphere. It will be shown in Chapter XXVII that the passage of light through the atmosphere depends on the density of the air, and everyone has observed the curious distortion of the images of familiar objects when observed through the heated air above a radiator. The air immediately above the radiator is less dense than the cooler air in the vicinity, and the light rays are accordingly bent as they pass through. As the air is in constant motion, the images seen through such an inhomogeneous region are constantly being distorted into fantastic shapes. The changes in pressure associated with a sound wave also produce just such distortion, but because they travel

with such rapidity, they cannot be observed with the eye. Very striking instantaneous photographs have been taken of such waves, however, and were used by Professor Wallace C. Sabine of Harvard in studying the acoustics of auditoriums.

136. The Velocity of Propagation of Sound. The fact that sound does not travel instantaneously from one point to another must have been recognized in very ancient times, but the first actual measurement of its rate of propagation was made by Père Mersenne in Paris about the time that Galileo was working in Italy. He used the obvious method of noting the time which elapsed between the flash of a gun and the arrival of the sound. At 0° C. the velocity thus obtained is 1088 ft./sec. or 331.7 meter/sec. Newton attempted in his " Principia " a theoretical calculation of the velocity of sound, but arrived at too low a value owing to his neglect of the heat generated by the compression in the sound wave. A rather simple derivation can be given, which takes, however, full account of these subsidiary phenomena.

If a compression takes place at a point P, the air in the vicinity of this point will tend to expand outward into the surrounding region. This expansion will have two effects, the reduction of the pressure in the immediate neighborhood of P and the compression of the surrounding air. It is in this way that the disturbance at P is transmitted outward. As the distance from the source increases, any small section of the spherical wave approximates more and more closely to a plane surface. A strictly plane wave, though never actually occurring in practice, is easier to visualize than the spherical wave, and is more convenient for calculation. Fig. 121 represents the instantaneous pressure at and in the vicinity of such a compressional wave. In the plane A the pressure has the maximum value P_m, while on either side it gradually falls off until at B and C the pressure is substantially equal to the normal atmospheric pressure P_0. This compression at

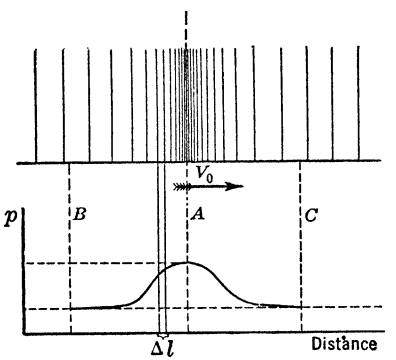


FIG. 121. Wave of Compression.

A must be thought of as travelling from right to left with the velocity V_0 which we wish to calculate.

The whole phenomenon is immensely simplified if, taking advantage of the principle of Galilean relativity, we imagine ourselves viewing the process from a laboratory which is moving uniformly to the left with precisely the velocity V_0. To us the air will appear to be streaming to the right with the velocity V_0, and the compression will remain stationary before us to be inspected at our leisure. At first sight the absence of any tendency on the part of the compression to expand would seem strange to us. The explanation of this behavior would have to be sought in the motion of the air passing through the compression. Consider a little slab of air of width Δl at the left of the region of compression, but parallel to the plane of the wave. If ρ_0 is the density of the air (at atmospheric pressure P_0), the mass of 1 sq. cm. of the slab would be $\rho_0 \Delta l$. As the air is moving from left to right with a velocity V_0, its kinetic energy is $\frac{1}{2}\rho_0 \Delta l V_0^2$. As it enters the region of compression, and passes to the peak at A, its pressure potential energy increases, and it suffers a corresponding reduction in kinetic energy. We can, it turns out, calculate independently the resulting decrease in velocity and the increase in pressure potential energy. Since the air flows in such a way that every point in the compression remains at the same pressure, the mass of air passing the peak of the compression per second must be equal to that passing the point B per second. That passing B in a stream 1 cm.^{2} in cross section in each second is $V_0\rho_0$, while that crossing A in the same time is $V\rho$, if V is the velocity and ρ the density at that point. Hence

$$V_0\rho_0 = V\rho$$

Or

$$V = \frac{\rho_0}{\rho} V_0 \tag{120}$$

From Section 33 the increase in potential energy per unit volume of a fluid when it moves from a region where the pressure is p_0 to one where the pressure is p is $(p - p_0)$. The gain in potential energy, therefore, of the little slab of thickness Δl is $(p - p_0)\Delta l$. Equating this to the loss in kinetic energy,

$$\frac{\rho_0 \Delta l}{2}(V_0^2 - V^2) = (p - p_0)\Delta l$$

Substituting the value of V in terms of V_0 from (120), and

cancelling Δl from the two sides of the equation,

$$\frac{\rho_0 V_0^2}{2} \frac{\rho^2 - \rho_0^2}{\rho^2} = p - p_0$$

$\rho^2 - \rho_0^2$ can be factored into $(\rho - \rho_0)(\rho + \rho_0)$, and since the change in pressure or density associated with the sound wave is small, $\rho + \rho_0$ can be replaced by $2\rho_0$ and ρ^2 by ρ_0^2. Hence

$$V_0^2(\rho - \rho_0) = p - p_0$$

Letting $\rho - \rho_0 = \Delta\rho$, the increase in density associated with the compression, and $p - p_0 = \Delta p$, the corresponding increase in pressure, and solving for V_0

$$V_0 = \sqrt{\frac{\Delta p}{\Delta \rho}} \tag{121}$$

This is, accordingly, the only velocity with which sound can be propagated, which is consistent with the principle of Galilean relativity.

The alternate compression and expansion of the air traversed by the sound wave takes place so rapidly that there is no time for the heat generated to be transmitted away from the immediate vicinity of the compression. The process is, therefore, adiabatic, and the increase in pressure resulting from the compression is greater than would be the case if the temperature remained constant, i.e., if Boyle's Law were obeyed. It can be shown that, under these circumstances, instead of pv remaining constant, pv^k is constant, where k is the ratio of the specific heat at constant pressure to that at constant volume. It was precisely at this point that Newton made his mistake, and assumed the validity of Boyle's Law. This error was first corrected by Laplace over a century later.

Using, accordingly, the correct relation between p and v, and noting that the density ρ is inversely proportional to v,

$$\frac{p}{\rho^k} = \frac{p_0}{\rho_0^k} \tag{122}$$

Letting $\rho = \rho_0 + \Delta\rho$ and $p = p_0 + \Delta p$, and making use of the binomial theorem,

$$p_0 + \Delta p = \frac{p_0}{\rho_0^k}(\rho_0 + \Delta\rho)^k = \frac{p_0}{\rho_0^k}(\rho_0^k + k\Delta\rho\,\rho_0^{k-1} + \cdots) \tag{123}$$

The third term and all succeeding terms in the binomial expansion can be neglected, since each contains a square or a higher power of the small quantity $\Delta\rho$. Simplifying (123),

$$p_0 + \Delta p = p_0 + \frac{kp_0\Delta\rho}{\rho_0}$$

Or
$$\frac{\Delta p}{\Delta\rho} = \frac{kp_0}{\rho_0} \tag{124}$$

Hence from (121)
$$V_0 = \sqrt{\frac{kp_0}{\rho_0}} \tag{125}$$

The dependence of this velocity on the pressure is only apparent, however, as can be shown by an application of the general gas law, $pv = mRT/M$. Since the density ρ is equal to m/v, this law can be written

$$\frac{p}{\rho} = \frac{RT}{M}$$

Substituting this value in (125)

$$V_0 = \sqrt{\frac{kRT}{M}} \tag{126}$$

The velocity of sound is, therefore, proportional to the square root of the absolute temperature. The consequent increase in velocity per degree Centigrade at 0° is about 60 cm./sec.

It should be noted that the velocity is proportional to the square root of k and inversely proportional to the square root of the molecular weight. As a result of the presence of k in the velocity formula, any measurement of the velocity of sound in a given gas can be used to calculate the ratio of the two specific heats of the same gas. A very simple method is, therefore, provided for the measurement of this constant, which plays such an important rôle in the kinetic theory.

137. The Propagation of a Wave along a Cord. A somewhat different type of wave, which in some respects is even simpler than a sound wave, is that which can be set up on a cord or wire. Nearly everyone has snapped the end of a long rope, and has watched the resulting wave travel along its entire length. The velocity with which such a wave advances can be calculated in much the same way as that of the sound wave. Consider, for example, a cord which is snapped at its right-hand extremity. The wave will travel from right to left with a certain velocity V. If this process is viewed from a laboratory which is moving from

right to left with exactly this same velocity V, the wave will appear to stand still, but the cord will seem to be moving through it from left to right with the velocity V. As a little element of the cord Δl (Fig. 122) passes through the wave, it is subjected to an acceleration, not in the direction of its motion, but, owing to its curvilinear path, at right angles to it. The source of this central acceleration can be seen if the forces acting on the little element are drawn as in Fig. 122. These forces consist of the two tensions t_1 and t_2, exerted on it by the cord on either side. Owing to the curvature of the path travelled by the element, these two tensions are inclined slightly to each other, and have a resultant T, the magnitude of which can be calculated by the law of the triangle of forces. In Fig. 123 the arrow representing

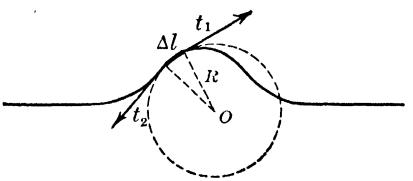


FIG. 122. Wave on Cord.

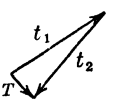


FIG. 123

the force t_2 is placed with its tail at the head of that representing t_1. The resultant T is then the third arrow drawn from the tail of t_1 to the head of t_2. The small section of the wave, on which Δl lies, approximates very closely to a circle, the center of which is located at O. In Fig. 122 the lines from O to the ends of Δl are drawn. Owing to the fact that any tangent to a circle is perpendicular to the radius drawn to its point of contact, these two lines are perpendicular respectively to t_1 and t_2, and the angle between t_1 and t_2 in Fig. 123 is, accordingly, equal to that subtended by Δl at O. Hence, by the familiar property of similar triangles,

$$\frac{\Delta l}{R} = \frac{T}{t} \quad \text{or} \quad T = \frac{t\,\Delta l}{R}$$

where t is the common magnitude of t_1 and t_2. The centripetal force T must, however, as was shown in Chapter IV, be equal to mV^2/R. The mass of the element Δl is $\rho\,\Delta l$, if ρ is its mass per unit length. Hence

$$\frac{t\,\Delta l}{R} = \frac{\rho\,\Delta l\,V^2}{R}$$

Or, cancelling R and Δl from each side of the equation, and solving for V,

$$V = \sqrt{\frac{t}{\rho}} \tag{127}$$

Once again this is the only possible velocity consistent with Galilean relativity, with which a wave can travel along a cord. It depends only on the tension of the cord and its linear density. The expression (127) is very similar to the relation (121) for the velocity of sound in a gas.

138. Longitudinal and Transverse Waves. There is one fundamental difference between this wave and the sound wave, however. Whereas in the sound wave the motion of the individual gas particle is always in the direction in which the wave is travelling (or opposite to it), the particles of the cord are displaced in a direction at right angles to that in which the wave is being propagated. Such a wave is called a *transverse* wave, whereas sound is an example of a *longitudinal* wave. Even though we were unable to see the wave on the cord, a very simple experiment would prove to us its transverse character. Imagine the cord (Fig. 124) passing through a slit. If the slit is so held that it lies in the plane of the wave, it will in no wise interfere with its progress. If, however, it is rotated about an axis parallel to the direction of propagation until it is at right angles to this plane, it will prevent effectively the passage of the wave Sound, in common with all longitudinal waves, has no *preferred direction* in a plane perpendicular to its direction of propagation, and it is inconceivable that the rotation of any sort of obstacle about this axis could have any effect on its progress.

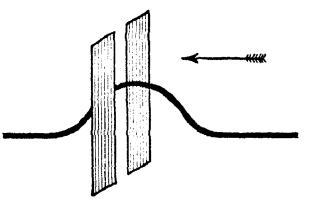


FIG. 124. Transverse Wave Passing through Slit.

139. Reflection of Sound. Reverberation. Although sound is usually thought of as being propagated in air, the same sort of longitudinal waves can be set up in liquids and solids. Thus, when a sound wave encounters a solid barrier, a certain proportion of its energy will be devoted to transmitting a similar wave into the material of the barrier. In general, the energy so spent is more rapidly dissipated or absorbed by internal friction in

such a solid than is the case in air, but the amount of this dissipation depends to a large extent on the material of which the barrier is composed. In addition to the transmitted wave, a portion of the energy of the incident wave is reflected in much the same way that light is reflected from a mirror. It is this reflected wave which gives rise to the familiar *echo*, the time which elapses between the generation of the sound and the arrival of the echo being simply that necessary for the disturbance to travel to the barrier and back.

In an enclosed space such as an auditorium any given wave is reflected many times from different parts of the walls, floor, and ceiling, and only after a substantial interval of time will the resulting waves drop to an intensity so small as to be inaudible. The persistence of sound due to multiple reflection is known as *reverberation*. A certain amount of reverberation is desirable as it adds to the intensity of the sound which reaches the ear of the listener. If, however, the time required for a given sound to fall in intensity to one-millionth of its original value is greater than 2.5 seconds, the resulting confusion is such as to make the auditorium unfit for orchestral music, and an even smaller amount of reverberation is desirable for speech. Professor Sabine made a thorough study of this problem, and found that the reverberation time was proportional to the volume of the auditorium and inversely proportional to what he called its absorbing power. Certain substances such as hair felt are particularly effective absorbers, destroying at each reflection something like three-quarters of the total energy incident on it. Plaster or brick, on the other hand, absorb only 2 to 4 per cent of the incident energy. It is possible, therefore, by judicious choice of materials, so to design any auditorium that it shall have the correct reverberation time for the purpose for which it is built. Each person in the audience is, of course, responsible for a definite amount of absorption, and it is inevitable that the reverberation time will be greater for a small audience than when the hall is filled to capacity.

It should be remarked that the development of amplifying systems for large auditoriums has greatly extended the possibility of making speeches and proceedings audible to large numbers of people. Although these can compensate to any desired extent for the defect of insufficient reverberation, they are, nevertheless, unable to remove the damaging effects of excessive

reverberation. This can still only be satisfactorily treated by the introduction of suitable amounts of absorbing material.

PROBLEMS

1. Calculate the velocity of propagation of sound in (*a*) nitrogen, and (*b*) helium, at a temperature of 20° C.

2. Calculate the velocity of sound in air at (*a*) 20° C. and (*b*) −268° C.

3. The density of a certain gas at 0° C. and 1 atmosphere is 0.00364 gr./cm.^{3} The velocity of sound in this gas at 0° C. is 215.3 meters/sec. Calculate (*a*) its molecular weight, (*b*) the ratio of its two specific heats, (*c*) its atomic weight. Identify the gas by comparison with the table of atomic weights in Chapter XII.

4. Calculate the velocity of propagation of a wave along a steel piano wire 0.1 mm. in diameter and stretched to a tension of 1 kg. The density of steel is 7.8 gr./cm.^{3}

CHAPTER XXVI

PERIODIC WAVES. INTERFERENCE

140. Frequency, Velocity, and Wave-length. As was pointed out in Chapter XXIV, a large proportion of the sources of sound produce periodic disturbances, resulting in a succession of alternate compressions and rarefactions. At a particular instant over any spherical surface surrounding the source of the sound the pressure is constant, but as one passes from the source outward along a radius of such a sphere, successive compressions and rarefactions are encountered. The distance between two adjacent maxima of compression, or, as we might say, between two adjacent wave crests, is the *wave-length* of the sound, and is represented by the symbol λ. The number of wave crests which pass any given point per second is the *frequency* n of the sound. If both the source and the observer are at rest, this is equal to the number of compressions sent out from the source per second. Since the distance between two successive wave crests is λ, and the number of crests passing per second is n, the velocity with which each crest advances must be

$$V = n\lambda \tag{128}$$

This fundamental relation between frequency, wave-length, and velocity is of basic importance in all discussions of periodic wave motion.

141. The Interference of Two Sound Waves. Beats. A group of alternate compressions and rarefactions, such as that produced by the sounding of a musical instrument, is called a *train* of waves. The presence in such a train of rarefactions as well as compressions, of wave hollows as well as wave crests, results in the fact that two superposed wave trains may interfere with each other in such a way as to produce a smaller effect than either wave by itself. In fact, if two sinusoidal disturbances of exactly the same frequency and amplitude are so initiated that the maximum compression of one coincides with the maximum rarefaction of the other, the two sounds will exactly annul each other. Two waves of the same frequency, for which the crests

coincide, are said to be in *phase* and to reënforce each other, whereas two waves for which the crest of one coincides with the hollow of the other are said to be just out of phase and to *interfere* with each other.

This interference can be very strikingly exhibited by two sources of sound, such as two tuning forks, which differ slightly in frequency. At a particular instant the crests of the two waves coincide at the ear of the listener, but as the crests of one wave succeed each other a little more rapidly than those of the other, they soon fall out of phase. If the frequency of one fork is n_1 and of the other is n_2, the time elapsing between the arrival of successive crests of the first wave is $1/n_1$, while that between crests of the second wave is $1/n_2$. If $n_1 > n_2$, each crest of the second wave lags behind the corresponding crest of the first wave more than its predecessor, the loss between two adjacent crests being $1/n_2 - 1/n_1 = (n_1 - n_2)/n_1n_2$ seconds. When the sum of these losses is equal to a whole cycle of the first fork, $1/n_1$, a full wave-length will have been lost, and the two sounds will once more be in phase. The number of wave crests from the second fork which will have passed the ear of the listener before the two waves are thus once more in a position to reënforce each other is

$$\frac{\frac{1}{n_1}}{\frac{n_1 - n_2}{n_1n_2}} = \frac{n_2}{n_1 - n_2}$$

Since there are n_2 crests per second, the number of times per second that the two forks will be in phase is

$$\frac{n_2}{\frac{n_2}{n_1 - n_2}} = n_1 - n_2$$

Each time the two waves are in phase the listener will note an increase in the intensity of the sound, while midway between these instants, when the crest of one wave coincides with the hollow of the other, there will be a sharp reduction in the intensity. If these successive changes in intensity occur infrequently enough, they will be detected as *beats* by the listener. These beats are of much value in tuning a musical instrument, as it is possible to observe them when they are many seconds apart, and thereby to adjust the pitch of the instrument to an accuracy of a small fraction of a cycle.

the returning wave crest is reflected from the fork at just the instant that the fork sends out a new crest, i.e., if the length of the cord is an exact multiple of $\lambda/2$, the new and reflected waves will reënforce each other. This process can be viewed in a little different manner as follows. A single wave, sent out by the fork, is reflected first from the pulley and then from the fork, and continues to move up and down the cord until it is finally damped out by frictional losses. This damping is prevented, however, if each time the wave is reflected from the fork it receives a new increment of energy. Not merely will the damping be eliminated, but the amplitude of the wave will be increased at each reflection from the fork until the energy received each time is just equal to that lost in the journey up and down the string. Under these circumstances, when the time required for the wave to travel up and down the cord is an integral multiple of the period of the fork, the string is in resonance to the fork, and a very substantial amplitude of vibration results.

Instead of adjusting the length of the cord for resonance, the same result can be achieved by varying the tension and thereby the velocity with which the wave travels. The maximum tension and therefore the maximum velocity which will produce resonance is that for which the wave travels once up and down the cord during one vibration of the fork. One wave just returns to the fork in time to be reënforced by the next succeeding wave. If the frequency of the fork is n, if its period consequently is $1/n$, and if l is the length of the cord, the required velocity is

$$V = \frac{2l}{\frac{1}{n}} = 2nl$$

The corresponding tension t is, in accordance with Formula 127 of Chapter XXV

$$t = \rho(2nl)^2 = 4n^2l^2\rho$$

With this particular tension and velocity, the cord as a whole will be set into intense vibration with a point of maximum vibration or *loop*, as it is called, midway between the two ends. If the velocity is one-half as great, i.e., nl, the first wave will return to the fork in time to be reënforced by the second succeeding wave. On the return journey it will have interfered with the wave immediately following it at the middle of the string. Although resonance will here again take place, the midpoint of the

142. Interference Due to a Reflected Wave. Standing Waves. Another and even more interesting way in which interference can take place is encountered when a periodic wave is reflected. If one end of a rope is attached to a hook and the other end is snapped, it will be observed that the wave not merely travels toward the hook, but also returns back along the cord. It was *reflected* at the hook, and in such a way that, if it travelled toward the hook on the upper side of the cord, it returned on the lower side. If, instead of one snap, a succession of snaps is given the cord at regular intervals, the wave returning on the under side of the cord will meet a wave travelling toward the hook on the upper side of the cord, and the two will interfere.

This experiment can be very conveniently performed by attaching one end of a light silk cord to the prong of an electrically driven tuning fork F (Fig. 125), while a weight W is attached to the other end of the cord and hung over the pulley P. Waves sent down the cord by the vibrating fork are reflected at the pulley, and interfere with one or more of those which left the fork at later instants. The distance between successive wave crests as they travel down the cord is the wave-length λ. When, therefore, any wave has travelled back from the pulley a distance $\lambda/2$, it must meet the next following wave crest, which still has a distance $\lambda/2$ to travel in order to reach the pulley. At that instant the second succeeding crest is a distance $3\lambda/2$ back from the pulley, but it also will be met by the returning wave when it has travelled an additional distance $\lambda/2$, and is, therefore, at a distance λ from the pulley. At each point where two waves interfere in this way the cord will remain permanently at rest. Such a point is called a *node*, and the foregoing discussion shows that such nodes will occur at a distance of one-half wave-length from the pulley and at each succeeding half wave-length.

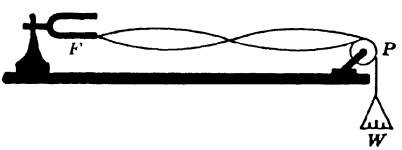


FIG. 125. Melde's Experiment.

With an apparatus such as that here described the phenomenon is complicated by the fact that, in addition to the new waves sent along the cord, the returning waves are again reflected at the fork, and unless precautions are taken to provide the proper length of cord, these multiply-reflected waves tend to obscure the whole phenomenon. If, however, the length of cord is so chosen that

string will be a node, and there will be a loop one-quarter of the length of the cord from each end. The cord will appear to vibrate in two segments as is shown in Fig. 125. If the tension is still further reduced, points will be reached at which the cord will vibrate in three, four, five, and still larger numbers of segments. This beautiful experiment, first described by F. Melde, is known as Melde's experiment.

143. Fundamental and Harmonics of a Vibrating Cord. The time required for a wave to travel back and forth along a string is the lowest period to which the cord will resonate, and is, therefore, its fundamental period of vibration. Since the velocity V of propagation of such a wave is $\sqrt{t/\rho}$, the time T required for such a journey is given by the relation

$$T = \frac{2l}{V} = 2l\sqrt{\frac{\rho}{t}}$$

The frequency of vibration n is

$$n = \frac{1}{T} = \frac{1}{2l}\sqrt{\frac{t}{\rho}} \tag{129}$$

When a string is plucked or bowed, the character of its vibration is very complicated, and consists of a superposition of this fundamental vibration and the various overtones produced by vibrating in segments. If it vibrates in two segments, the length of each segment is half that of the whole string, and in accordance with (129) the frequency of the first overtone would be twice that of the fundamental. The next possible overtone, produced by a vibration in three segments, would be three times the fundamental, and one is readily convinced that in this way all possible integral multiples of the fundamental frequency could be obtained. The intensity of the various overtones present in a complex vibration of the string depends on the way in which the vibration is started, plucking and bowing, for example, producing entirely different distributions of energy in the various harmonics, thus accounting for the difference in the quality of the sounds produced.

144. Standing Sound Waves. Kundt's Experiment. An experiment precisely analogous to that of Melde can be performed with sound waves. A column of air in a long horizontal glass tube takes the place of the string, and a vibrating telephone diaphragm can be used in place of the tuning fork. If the end of the

tube opposite the diaphragm is closed by a plug, thus preventing any motion of the air particles at that point, the analogy between the two experiments becomes complete. Although the motion of the air particles, unlike that of the cord, is invisible, it can be very simply brought into evidence by sprinkling a little lycopodeum powder on the bottom of the tube. As loops and nodes are set up in the air column, the lycopodeum powder is shaken from regions of intense motion of the air particles (loops) into region of calm (nodes). The closed end of the tube must necessarily be a node. One-half wave-length from this point a reflected compression meets the next following compression, and although each tends to thrust the air particles before it, the opposing action of the two compressions results in a point of no motion, i.e., a node. As in Melde's experiment, a node will occur not merely at one-half wave-length from the closed end, but at two half wave-lengths, three half wave-lengths, and at each succeeding half wave-length. The intensity of motion at the loops will be insufficient to shake out satisfactorily the lycopodeum powder unless the length of the tube is so chosen that it resonates to the frequency of the diaphragm. This length must be determined experimentally by adjusting the position of the plug. Once in adjustment, the wave-length of the sound can be determined with considerable accuracy by measuring the distance between successive piles of lycopodeum powder. For any one gas at a given temperature the velocity of propagation is definitely determined, and cannot be changed as was the case with the velocity of propagation of a wave on a cord. If the distance between two successive dust piles is l, $\lambda = 2l$, and $V = 2nl$. If the tube is first filled with air, and if l_a is the measured distance between nodes,

$$V_a = 2nl_a \tag{130}$$

If, now, the air is replaced by a second gas, for which the velocity of propagation V_x is required, and the distance between nodes is observed to be l_x,

$$V_x = 2nl_x \tag{131}$$

Dividing (131) by (130)

$$\frac{V_x}{V_a} = \frac{l_x}{l_a} \tag{132}$$

The velocities of propagation are directly proportional to the

distances between piles, and if the velocity of sound in air is known, that in the other gas can be determined even without a knowledge of the frequency n.

145. Velocity of Sound in Solids. A simple extension of this experiment makes possible also the measurement of the velocity of sound in a solid such as glass or steel. Imagine, for example, a rod of the material to be investigated clamped firmly to the table at its midpoint C (Fig. 126), and supplied with a light disk or cork at the end A to take the place of the telephone diaphragm. If the end B is now stroked with a resined cloth, it is alternately pulled out and released as the resin grips it momentarily. Each time it is released a wave of compression is sent down the rod, travelling with the velocity characteristic of an elastic disturbance in such a medium. It is reflected successively at A and B, and travels a number of times up and down the rod until it is damped

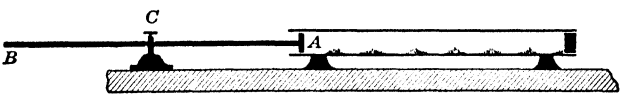


FIG. 126. Kundt's Experiment.

out by friction or is stopped by a new grip of the resined cloth. The rod thus executes damped longitudinal vibrations of such a sort that the midpoint, at which it is clamped, is a node and the two free ends are loops. The total length of the rod is, therefore, one-half wave-length. If L is its length, V_s the velocity with which the elastic waves move through it, and n_s the resulting frequency of the damped vibrations,

$$V_s = 2n_sL \tag{133}$$

If l_a is the distance between the dust piles which are set up in the glass tube, the expression (130) once more applies. Dividing (133) by (130),

$$\frac{V_s}{V_a} = \frac{L}{l_a} \tag{134}$$

The velocity of propagation of the elastic waves in the solid bears, therefore, to the velocity of sound in air the same relation as the length of the rod to the distance between dust piles in the tube. This one experiment, known as Kundt's experiment, furnishes, accordingly, a very direct means for measuring the velocity of

sound not merely in gases but also in solids. The following table gives the velocity in a few different materials.

Substance	Temperature °C.	Velocity Meters per Second
Air	0	331.7
CO_2	0	258
Water (liquid)	19	1461
Alcohol	20.5	1213
Iron	20	5130
Brass		3500
Glass		5000
Vulcanized rubber	0	54

146. The Resonance of Air Columns. Organ Pipes. The resonance of an air column to a particular frequency or frequencies, which plays such an important part in the establishment of intense loops and nodes in Kundt's experiment, may be very beautifully shown by holding a vibrating tuning fork at the open end of a tube (Fig. 127). If the other end of the tube is closed by a movable piston, it will be found that for certain particular lengths of the air column resonance will take place with such intensity that it can be heard at a considerable distance. The closed end of the tube must be a node, while the open end, where the air is set in motion by the vibrating fork, is a loop. The shortest tube, therefore, which will resonate to the fork is one having a length equal to the distance from a node to a loop, i.e., one-quarter of a wave-length (Fig. 127*a*). If, however, the length of the tube is three-quarters of a wave-length, there can still be a node at the closed end, a loop at the open end, and an additional node one-third of the way down the tube as shown in Fig. 127*b*. Similarly resonance will take place for a tube of length $5\lambda/4$, $7\lambda/4$, and in general $(2n + 1)\lambda/4$, if n is any integer (Figs. 127*c* to 127*d*). More complete analysis shows that the diameter of the tube has an influence on the length at which the tube resonates, and that,

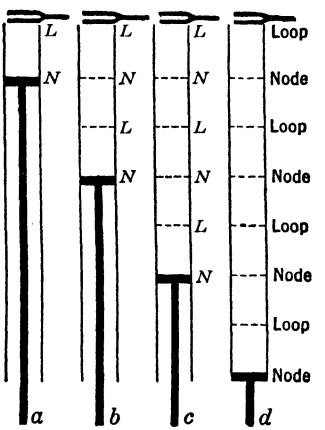


Fig. 127. Resonating Air Columns.

in fact, the effective length of the tube to be used in the above relations is the actual length plus one-quarter of its diameter.

In the organ pipe a blast of air at the mouth of the pipe sets the air column within the pipe into vibration at all the frequencies to which it can resonate. The situation is, indeed, very closely analogous to the motion produced in a stretched string by bowing or plucking. Whereas, however, both ends of the string are fixed and must be nodes, the mouth of the pipe is always a loop, and the other end may be either a loop or a node, depending on whether it is open or closed. In the closed pipe this upper end is a node, while the lower end is a loop, and the lowest possible frequency to which it can resonate is that for which the length of the pipe is one-quarter wave-length. If we denote this, the fundamental frequency, by n_0 and the corresponding wave-length by λ_0,

$$= \frac{V_a}{\lambda_0} = \frac{V_a}{4l}$$

The next higher frequency to which the pipe can resonate, i.e., its first overtone, is that for which there is one intermediate node. The length of the pipe is, therefore, three-fourths of the wave-length λ_1 of this frequency n_1. Or

$$n_1 = \frac{V_a}{\lambda_1} = \frac{3V_a}{4l}$$

This is seen to be three times the fundamental frequency. Unlike the vibrating string, therefore, the second harmonic (making use of the customary notation by which the fundamental is called the first harmonic) is missing. It is easily seen by an extension of the same reasoning that every *even* harmonic will be absent in the same way as the second harmonic.

With an open organ pipe, on the other hand, the upper end must be a loop as well as the mouth. The length is accordingly one-half the fundamental wave-length λ_0, and if, as before, n_0 is the fundamental frequency,

$$n_0 = \frac{V_a}{\lambda_0} = \frac{V_a}{2l}$$

The next higher frequency to which the pipe will resonate is that which provides an additional loop at its midpoint. The length of the pipe is, therefore, exactly equal to the wave-length of

this frequency n_1. Or

$$n_1 = \frac{V_a}{\lambda_1} = \frac{V_a}{l}$$

This first overtone is twice the frequency of the fundamental, as was the case in the vibrating string, and a continuation of the same argument would show here also the presence of all harmonics. The sound of the closed pipe is, therefore, distinguished from that of the open pipe of the same fundamental frequency by its greater *purity*.

It should be pointed out in closing the present chapter that the phenomenon of interference, which has been here discussed, and in particular the formation of standing waves in the vicinity of a reflecting surface, is a fundamental attribute of periodic wave motion. No one has ever been able to show how any other sort of disturbance could produce such effects, and whenever we encounter such stationary wave systems, we are accustomed to accept them as evidence that the phenomenon under investigation is periodic in character, is in the nature of a wave, and is being propagated with a finite velocity.

PROBLEMS

1. Calculate the wave-lengths in air at 0° C. of sounds having the frequencies 60, 256, 5000, and 50,000.

2. Two piano wires are of such a weight that, when exactly 1 meter long and stretched to a certain tension, their fundamental frequency of vibration is 500. How many beats per second would be detected between them if one was shortened by 5 mm.?

3. What are the fundamental frequency and the first three overtones of an open organ pipe 2 meters long? Assume a temperature of 20° C. How much would the frequency of the fundamental be altered by a rise of 10° in temperature?

4. What would be the frequency of the organ pipe of the last problem if it were filled with helium?

5. If the tube used in Kundt's experiment is filled with air at 0° C., and if the distance apart of dust piles is 10 cm., at what frequency is the tube excited? If a brass rod, clamped in the middle, is used for producing the vibrations, what must be the length of the rod?

CHAPTER XXVII

THE VELOCITY OF LIGHT

147. The Nature of Light. Of all the senses which make the individual aware of the material world about him, that of sight is the most valuable, and light, that mysterious connecting link between our visual consciousness and external things, has always aroused the curiosity of thoughtful men. It requires only a little casual thought to persuade one that the source of light is always some glowing or luminous body, for our optical nerves are made aware of the existence of non-luminous bodies only by the indirect process of reflection. When, for example, an incandescent lamp is turned on at night, we are made immediately conscious of all the varied objects in the room by the light which they reflect; but if the lamp is turned off, our optical senses, at least, cease to be of any use to us. Just as some vibrating body is the source of every sound, so a luminous body is always the source of light; and just as the vibrations which the sound produces in our ear drums make us conscious of it, so, perhaps, some similar process in the eye signals to us the arrival of light.

We cannot, however, press the analogy with sound too far, for in the case of sound we are able to observe and measure the changes of pressure in the air through which the sound moves, and even to take instantaneous photographs of the wave front as it advances; while the simplest sort of experiment convinces one that the air in which sound is propagated plays an almost infinitesimal part in the propagation of light. If, for example, the highest possible vacuum is obtained in a chamber through which light is passing, the effect on the passage of the light is almost negligible. Air, which is so essential to the passage of sound, plays a surprisingly subordinate rôle in the propagation of light. If light is really a wave disturbance similar to sound, it must be propagated in some medium which, although permeating all transparent materials, still remains, very little affected, in vessels which have been evacuated to the best of our ability. It must be a medium which is indetectible, not merely by our physical senses, but by our most sensitive instruments as well. Neverthe-

less, as we shall see, the scientists of the nineteenth century were led to believe in the existence of such a mysterious medium, and called it the *luminiferous* or light-bearing *ether.*

With such a difficulty in the way of any rational wave theory of light, it is natural that some other alternative should be sought, and no less a genius than Newton was in the seventeenth century the advocate of a *corpuscular* theory. Why might not light consist of streams of tiny invisible particles, shot out from the luminous source, and finally coming to rest in the eye of the observer? To us of the present day, who are familiar with the invisible α- and β-particles which are shot off from radioactive materials, such a hypothesis should appear even more plausible than to the scientists of that period. And so it came about that in the seventeenth century there was an open disagreement as to the nature of light, the followers of Newton favoring a corpuscular hypothesis, while Huygens and his coworkers in Holland were even then persuaded of the wave character of the phenomenon.

In addition to the problem as to the means by which light is propagated from source to observer, there remains, on the one hand, the problem as to the way in which the disturbance is produced in the source, and, on the other hand, the way in which it is detected by the observer. In the case of sound, both of these questions have been quite adequately answered, with the exception, of course, of the final stage in which the vibration of the ear drum tells its story to the brain of the listener. In the case of radio signals we also have a very satisfactory picture of the way in which they are initiated by alternating currents in the transmitting antenna, and also of the way in which the electric and magnetic fields of the disturbance set up similar currents in the receiving antenna. Some of the most significant of modern scientific developments have come about as a result of attempts made by physicists in recent years to gain a corresponding insight into the production and reception of light. The conclusions to which such attempts have led will be discussed in later chapters.

148. The Velocity of Propagation of Light. Whether light is a corpuscular or a wave disturbance, it should most certainly be propagated with a finite velocity. Galileo made the first recorded attempt to measure this velocity by an extension of the method which had been successful with sound. Two observers were stationed within sight of each other on hill tops, each carry-

ing a lantern. The first observer at some prearranged time was to uncover his lantern, and the second observer, on seeing the signal, was to expose his own as nearly instantaneously as possible. It was hoped that the time elapsing between the first signal and the arrival of the response would be a measure of the time required for light to travel back and forth over the distance between the two observers. Admittedly it would need to be corrected for the time required for both observers to respond to the signal when they saw it, and Galileo attempted to make this correction by a series of experiments performed at close range. When, however, the correction was applied, it was found that, within the limits of accuracy of the experiment, the light had been transmitted instantaneously.

The first evidence of other than an infinite velocity of propagation was obtained by the eminent Danish astronomer Roemer. In the course of a study of the eclipses which take place when one of the satellites of Jupiter passes into the shadow of the planet itself, he noted that the time elapsing between successive eclipses of the same satellite was not always the same, being, in fact, appreciably greater than average when the brightness of the planet was decreasing, i.e., when the Earth was moving away from it. Although eleven years were still to elapse before the publication of Newton's "Principia," there was already a certain amount of confidence in the regularity with which celestial phenomena take place, and an outstanding discrepancy such as that discovered in the motion of Jupiter's moons called for explanation. It occurred to Roemer that if light were propagated with a finite velocity, it might account for the difficulty. Thus, if one eclipse was observed when the Earth was at the point *A* of its orbit (Fig. 128), and the next eclipse of the same satellite was observed when the Earth in its motion had passed on to *B*, a part of the intervening time must have been taken for the light to pass from *A* to *B*, and the observed time between eclipses would be greater than the actual time by just this amount. Since during half the year the Earth is receding from Jupiter and

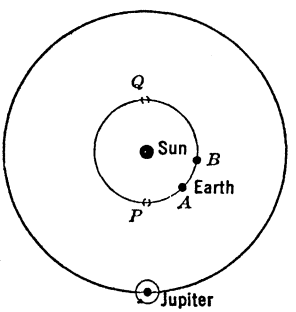


FIG. 128. Roemer's Measurement of the Velocity of Light.

during the other half is approaching it,[1] the average of all the intervals observed between eclipses for an entire year should be very close to the correct time of revolution of the satellite. The difference between this average time and that elapsing between the eclipse observed at A and that at B must be the time required for the light to travel from A to B. The sum of all these discrepancies occurring while the Earth is travelling from P to Q must be the time required for light to travel across the Earth's orbit, a distance which was, at least approximately, known to Roemer. He made an estimate of 22 minutes for this total time, which, although not very close to the correct value of 16 minutes

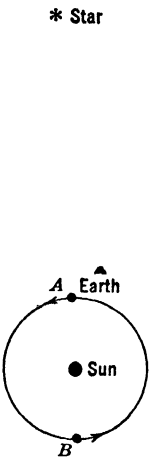


FIG. 129. Aberration of the Stars.

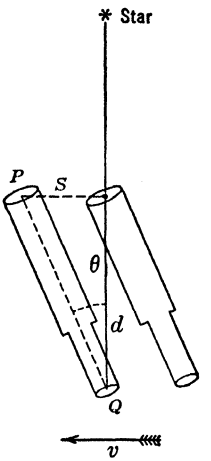


FIG. 129a

and 36 seconds, still enabled him to calculate the velocity of light to the right order of magnitude. The hugeness of this velocity, approximately 186,000 miles a second, accounted for the failure of previous terrestrial attempts to measure it.

Despite the brilliancy of Roemer's achievement, his conclusions were not generally accepted until over a half century later (1728) they were corroborated by the English astronomer Bradley by a method, which, although still astronomical, was based on entirely different considerations. In attempting to determine the distance of some of the stars by the straightforward trigonometric method of observing their apparent position in the heavens when

[1] Owing to its long period of revolution about the Sun, Jupiter has changed its position in the sky comparatively little during this time.

viewed from opposite sides of the Earth's orbit, he noted that all the stars appeared to be in slightly different positions when viewed at right angles to the motion of the Earth from the two opposite points A and B of Fig. 129. This phenomenon, known as *aberration*, was exceedingly puzzling to him until he suddenly realized that it also could be explained by a finite velocity of light.

If, for example (Fig. 129*a*), his telescope moved with the Earth a distance s from right to left while light from the star was travelling the distance d down the telescope tube, the light corpuscle, if we think of it as such for convenience, would appear to travel along the dotted line PQ, and would appear to arrive from the direction in which PQ points, i.e., at an angle θ in advance of its true direction. From the figure

$$\frac{s}{d} = \tan \theta$$

If t is the time required for such a motion, v the velocity of the Earth (and telescope), and c that of light, $s = vt$, and $d = ct$.

Or
$$\frac{s}{d} = \frac{v}{c} = \tan \theta \tag{135}$$

At the point A, therefore, a distant star should appear in a direction θ to the left of its true direction, while at B it should appear to the right by the same amount. The difference in the apparent direction of the star at the two opposite points of the Earth's orbit could, therefore, be used to determine the ratio of the Earth's velocity to that of light, and thus indirectly the velocity of light itself. The value which Bradley obtained, though more accurate than that of Roemer, agreed sufficiently well to furnish convincing evidence of the correctness of both explanations.

149. Terrestrial Methods of Fizeau and Michelson. The difficulty of measuring such a huge velocity terrestrially acted as a sufficient deterrent to such attempts for over a hundred years. In 1849, however, the French scientist Fizeau succeeded in making such a measurement. His apparatus consisted essentially of a toothed wheel, which could be rotated in front of a beam of light in such a fashion that the teeth periodically intercepted it. A mirror, located on a hill 8633 meters distant, reflected the beam of light back to the toothed wheel. If the velocity of light were

infinite, it would return instantaneously to pass through the same gap between the teeth through which it emerged. If, however, the light required an amount of time equal to that occupied by the next tooth in rotating in to close the gap, the reflected beam would be effectually stopped at the wheel. If the wheel was then rotated more rapidly, the returning beam should be visible from behind the wheel when the next following gap arrived in time to replace the first. The value found by Fizeau for the velocity of light checked within the experimental errors with those of Roemer and Bradley.

Later measurements were made by Foucault in France in 1862 and by Michelson in the United States in 1880, using a revolving mirror in place of the toothed wheel. In 1926 the increased importance of the velocity of light, owing to intervening discoveries which will be discussed in later chapters, justified a new attempt at measuring it with greater accuracy. This was, accordingly, undertaken by Michelson at that time, and cannot yet be said to be completed (1931). The result so far obtained, 299,796 km./sec., cannot be in error by more than 4 km./sec. The revolving mirror used was an octagonal glass prism, silvered on all eight faces. It was so mounted that, when at rest in the position shown in Fig. 130, light from the source S was reflected from the surface 1 to the distant mirror M. After the return journey it was caught by the face 2, and directed into the observing telescope T. If, however, the mirror is rotating slowly, the orientation of the face 2 will have changed during the journey of the light to M and back so that the returning beam is reflected to the right of its position when the prism is at rest. If the image of S, as seen from T, coincides initially with the cross hair of the telescope, it will move slowly to the right as the speed of rotation is increased until it passes altogether out of the field of view. A speed will, however, eventually be reached at which the face 3 will have taken the place of 2 when the light returns from M, and

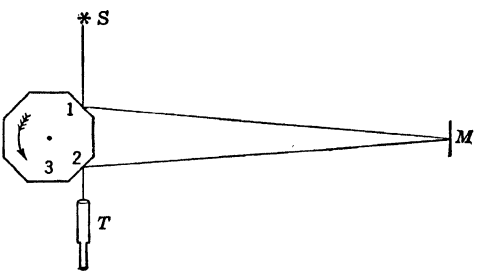


FIG. 130. Foucault-Michelson Apparatus for Measuring the Velocity of Light.

the image will once more appear in the telescope. When it coincides again with the cross hair, the speed of rotation must be such that the prism has made one-eighth of a complete revolution during the journey of the light signal. The velocity of propagation of light can, therefore, be directly calculated from the measured rate of rotation of the prism and the distance of the fixed mirror M. In Michelson's 1926 experiment the rotating mirror was located on Mount Wilson in California and the fixed mirror was 22 miles away on Mount San Antonio.

150. The Velocity of Light in Other Media. The velocity as measured by astronomical methods was naturally that through the vacuum of interstellar space, while the terrestrial experiments were carried out in the air. The substantial agreement between the results showed that air played little if any part in the propagation of light. That denser media such as water or glass have a much more substantial effect on the passage of light is obvious from the phenomenon of *refraction*. It is a familiar fact that the apparent direction of an object in a pool of water is not always the true direction. If, for example, some body, located at P of Fig. 131 beneath the surface of the water, is observed from Q, it will appear to lie in the direction QR. The light, as it passes out of the water at O, is bent away from the normal OM to the surface of the water. Conversely, it is easily shown that light, on entering a denser medium such as water from a rarer medium such as air, is bent toward the normal to the bounding surface. Curiously enough, this phenomenon can be equally well explained on a wave and on a corpuscular theory. On the corpuscular theory the bending of the ray as it leaves the water is attributed to the greater attraction of the heavy water molecules for the corpuscle. On account of this attraction, however, the normal component of the velocity of the particle should be reduced on leaving the denser medium, while the component parallel to the surface should be unaltered. There should, accordingly, be a net reduction in its velocity, and the speed of propagation in air should be less than that in water. The wave theory, on the other hand, requires a greater velocity in the rarer than in the denser medium, and gives, in

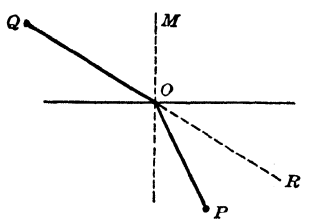


FIG. 131. Refraction.

fact, a very simple means of calculating the ratio of these velocities from the amount of bending observed.

In order to show this, we will make use of a plane wave, which, as pointed out in an earlier chapter, is simply the limiting case of a spherical wave as it gets larger and larger. Since the direction of propagation of any small section of a spherical wave is directly away from the source of the disturbance, and, therefore, in the direction of the radius drawn to the section in question, the direction of propagation of the plane wave must be normal to its surface. In Fig. 131*a*, *OS* is the intersection with the plane of the paper of the plane wave which is about to enter the denser medium *W* from the rarer medium *A*. *OP* is the surface separating *W* from *A*. *OI* is drawn perpendicular to *OS* at *O*, and indicates, therefore, the direction in which the wave was travelling before entering *W*. As the section of the wave at *O* first enters the denser medium, it is retarded with reference to the section at *S*, and travels a distance *OR* while the disturbance at *S* is travelling the greater distance *SP*. If V_0 is the velocity of propagation in *A* while *V* is the velocity of propagation in *W*, and if a time *t* is required for the wave front to take up its new position *RP*,

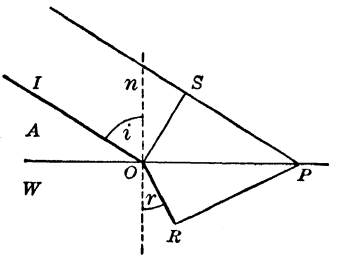


FIG. 131*a*. Significance of Refractive Index.

$$OR = Vt \tag{136}$$

$$SP = V_0 t \tag{137}$$

Or, dividing (137) by (136)

$$\frac{SP}{OR} = \frac{V_0}{V} \tag{138}$$

The angle between the original or *incident* direction of propagation of the wave and the normal to the bounding surface is called the *angle of incidence*, and is represented by the letter *i*. The corresponding angle between the direction of propagation in *W* and the same normal, or the *angle of refraction* as it is called, is *r*. It is readily seen that the angle *SOP* is equal to *i* since *OI* is perpendicular to *OS* and *On* is perpendicular to *OP*. Hence

$$\frac{SP}{OP} = \sin i \tag{139}$$

and similarly $$\frac{OR}{OP} = \sin r \tag{140}$$

Dividing (139) by (140) $$\frac{SP}{OR} = \frac{\sin i}{\sin r} \tag{141}$$

If the medium A is a vacuum or even, for all practical purposes, air, the ratio $\sin i/\sin r$, which measures the amount of bending of the light, is known as the *index of refraction* of the medium W, and is represented by the Greek letter μ. Comparing (141) with (138), and replacing V_0 by c, the velocity of light in a vacuum,

$$\frac{c}{V} = \frac{\sin i}{\sin r} = \mu \tag{142}$$

The index of refraction of any medium W is, accordingly, equal to the ratio of the velocity of light in a vacuum (or air) to that in W. Since light, in passing from air to a denser medium, is always bent toward the normal, i is always greater than r, and the velocity in air is greater than that in the denser medium. According to the proof which has just been given, the ratio of sin i to sin r should always be equal to that of c to V. It should, therefore, be the same for all values of the angle of incidence i. That this is so, was shown experimentally by Willebrord Snell early in the seventeenth century, and the law is known as *Snell's Law*.

By either of the terrestrial methods of measuring the velocity of light which have been described, the velocity can be measured in water as well as air, though, of course, over much shorter distances. This measurement, first made by Fizeau in 1850, shows that the velocity in water is approximately one-fourth less than in air, and accordingly favors the wave as against the corpuscular theory of light.

151. The Velocity of Light of Different Colors. If light from the sun is incident on the face AB of a triangular glass prism, as shown in Fig. 132, it will be bent at O in the direction OP, and on emerging from the prism at P will once more be bent, this time in the direction PQ. If the light is here received on a sheet of white paper, a succession of colors will be observed ranging from red through orange, yellow, green, and blue, to violet. The correct explanation for this phenomenon, known since very ancient times, was first given by Newton. He showed quite conclusively that the original white light was composed of all these colors.

The index of refraction of the glass, however, is different for each color, the amount of refraction being greater for violet light and least for red. This phenomenon, known as *dispersion*, indicates that the velocity of light in glass, and indeed in all optically dense media, is different for different colors. There can be no question, however, that all colors travel with the same velocity through a vacuum such as that of interstellar space. If, for example, red light travelled to us from the Sun with a greater velocity than violet light, the reappearance of the Sun after eclipse should first be signalled to us by the red light, and the edge of the Sun should first appear red.

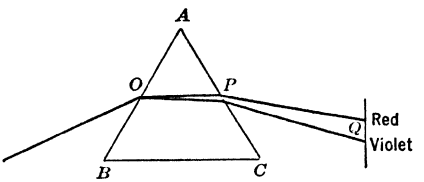


FIG. 132. The Dispersion of Light.

PROBLEMS

1. How long is required for the light of the Sun to traverse the 93,000,000 miles between it and the Earth? What is the equivalent in miles of the common astronomical unit, the light year, i.e., the distance travelled by light in one year?

2. Calculate 2θ, the difference in the apparent direction of a distant star as viewed from opposite points on the Earth's orbit. In calculating the velocity of the Earth, assume that it traverses the circumference of a circle of radius 93,000,000 miles in 365 days.

3. Referring to Fig. 130, calculate the rate at which the revolving mirror must be spun in order for face 3 to take the place of face 2 while the light beam is travelling the 44 miles to and from Mount San Antonio.

4. If the index of refraction of a certain specimen of crown glass is 1.52, and if the angle i at which a ray of light is incident on it is 45°, calculate the angle of refraction r. What is the velocity of propagation of light in this type of glass?

CHAPTER XXVIII

THE FORMATION OF IMAGES

Important as light is in making the individual aware of his surroundings, the method by which it accomplishes this is rather indirect. Images of external objects are formed on the retina, and complicated nerve systems carry to the brain the story which these images tell. Although we understand very little in detail about this last stage in the process of vision, much has been learned in the last three centuries about the formation of images by systems similar to that in the eye. Such images can be produced either by reflection or refraction, and can, up to a certain point, be equally well treated on a wave or a corpuscular hypothesis. Although there are certain details which can only be adequately explained by a wave theory, the process of image formation can, perhaps, be more simply presented if we think of light as being propagated in narrow beams or rays, a concept which would be a very natural one on a corpuscular theory. According to this viewpoint, rays of light travel out from the source in all directions, each ray pursuing an accurately rectilinear course until it comes to the boundary of a different medium such as glass. When such a boundary is reached, it is usually split into two parts, one of which is reflected and the other refracted. Snell's Law, which governs the direction of the refracted beam, has already been discussed. It states that the ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant, known as the index of refraction of the new medium. The law which gives the direction of the reflected beam is still simpler, being, in fact, that the angle which the reflected beam makes with the normal to the bounding surface is exactly equal to the angle which the incident beam makes. It is the same law which governs the reflection of a billiard ball from a perfectly elastic cushion, and is thus readily understandable on the corpuscular theory. It can also be derived with somewhat more difficulty from the wave theory.

152. Images Formed by Mirrors. The simplest case of image formation is that in a plane mirror. A ray, leaving the point P of the arrow in Fig. 133 and travelling along PN perpendicular to the mirror M, will be reflected at N, and will travel back along its own incident path to P. To an eye located at P it will appear to come from behind the mirror in the direction NP'. On the other hand, a ray which follows the dotted path from P to O, the foot of the perpendicular from the midpoint of the arrow to the mirror, will be reflected back along the dotted line to Q, in such a way that the angle of incidence POA is equal to the angle of reflection AOQ. To an eye located at Q, therefore, the ray would appear to come from behind the mirror in the direction OP'. In a similar way it can be shown that all rays which leave P and are reflected from the mirror will return as though from the direction of P'. At P' there will accordingly appear an image of the point P, and similarly the various other points of the arrow PQ will be imaged along the dotted line $P'Q'$. Since the triangle PNO is equal to the triangle $P'NO$, having one side NO in common and all angles equal, the distance of P' behind the mirror is exactly equal to that of P in front of the mirror, and since the triangle POQ is equal to $P'OQ'$, the size of the image is equal to that of the object.

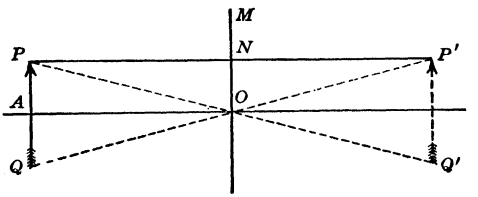


FIG. 133. Virtual Image in Plane Mirror.

There is one other case in which a very perfect image is formed by reflection. It is a fundamental property of the parabola that rays originating at its focus are all reflected straight out parallel to the axis, as shown in Fig. 134. This fact is utilized in the parabolic reflector of a searchlight to produce a narrow and intense beam of light. Conversely, if parallel rays of light such as those from a distant star are incident on such a mirror, they will all be reflected to the focus. If a small card is placed as shown in the dotted line of Fig. 134, it will intercept only a very small proportion of the incident rays, but will receive on its inner surface all the reflected rays. These will form at that point an image of the star. Such an image, which can actually be caught on a screen, is known as a *real* image; that formed behind the

plane mirror, being only the apparent source from which the reflected rays diverge, is called a *virtual* image.

If a spherical mirror is used in place of the parabolic mirror, the outer rays will be reflected a little closer to the mirror itself than those nearer the axis OM, but if only a small area of the mirror near M is utilized, the spherical surface will approximate so closely to the parabolic that an excellent image will still be obtained. The failure of a large spherical surface to form a sharp image is known as *spherical aberration.* It will be shown later in the present chapter that the point F, which is known as the

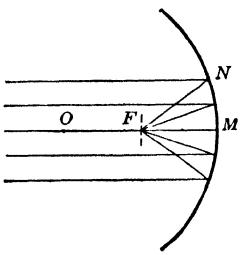


FIG. 134. Image of Star in Parabolic Mirror.

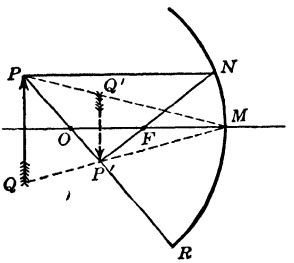


FIG. 135. Real Image in Concave Mirror.

principal focus of the mirror, is half way between the surface of the mirror and its center of curvature O.

A spherical mirror is capable of forming an image not merely of a distant object, but of nearer objects as well. In Fig. 135, for example, the ray PN parallel to the axis OM will be reflected in the direction NF, intersecting at P' the ray which passes through O and is reflected from the mirror at R. If only a small area of the mirror about M is used, all other rays from P, including the dotted ray which is incident on the mirror at M, will be reflected approximately to P', and will there form an image of the tip of the arrow. In similar fashion each point on the line PQ will be imaged at the appropriate point on $P'Q'$, thus forming a complete image of the arrow. If the distance of the center of the arrow from M is p, and that of the image $P'Q'$ is q, and if the length of the object PQ is l_0 and that of the image $P'Q'$ is l_i, it is obvious from similar triangles that

$$\frac{l_0}{l_i} = \frac{p}{q} \tag{143}$$

Fig. 135 illustrates a very generally useful geometrical construction for locating the image of any object point in such a spherical mirror. To find the image of P, for example, it is only necessary to draw PN parallel to the axis, and from N the line NF through F. The intersection of this line with that from P through O is the desired image point. The same construction can be used, as shown in Fig. 136, to locate the image of an object which lies inside the principal focus F. In this case, as with the plane mirror, the image is virtual and is located behind the mirror. Fig. 137 shows the corresponding construction for a

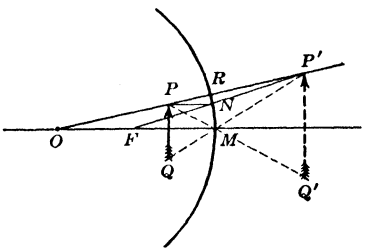


FIG. 136. Virtual Image in Concave Mirror.

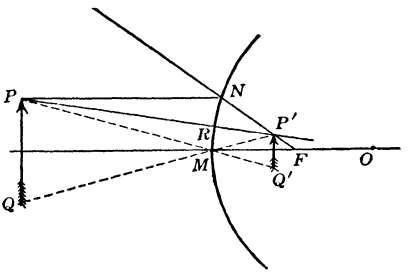


FIG. 137. Image Formed by Convex Mirror.

convex mirror. As the diagram indicates, the image here must always be virtual.

153. Images Formed by Lenses. Images can also be formed by refraction at one or more spherical surfaces. If, for example, a bundle of parallel rays, such as that shown in Fig. 134, falls on the spherical surface separating air from glass, these rays will all be refracted approximately toward (or away from) a single point, the accuracy of the concentration in one point depending, as with the spherical mirror, on the area of the refracting surface which is utilized. If the refracted beam now meets a second refracting surface on passing again from glass to air, as shown in Fig. 138, it will again be refracted, and will converge toward the point F, which is known as the principal focus of the *lens*. Any ray passing through the center of a thin lens such as that pictured in Fig. 138 will have the refraction at the first surface practically annulled by that at the second surface, and will emerge parallel to and very nearly collinear with the incident ray. A construction for locating the image of P formed by such a thin lens is, therefore, provided, which is very similar to that for finding the image in a

spherical mirror. It lies, viz., at the intersection of the ray PM through the center of the lens and the ray from N through F. If both surfaces of the lens are convex as shown in Fig. 138, a beam of rays parallel to the axis will be converged toward a point F on the opposite side of the lens; if both surfaces are concave as shown in Fig. 139, they will diverge as though from the point F on the same side of the lens. The first is known as a *convergent* lens, and the second as a *divergent* lens. Fig. 139 shows the way in which a divergent lens forms a virtual image on the same side of the lens as the object. Lenses can, of course,

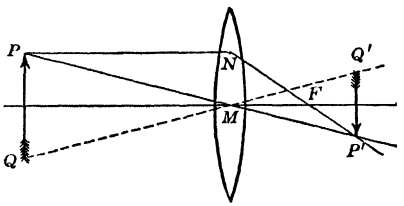


FIG. 138. Real Image Formed by Convergent Lens.

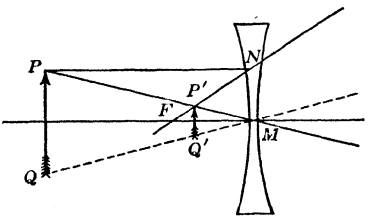


FIG. 139. Virtual Image Formed by Divergent Lens.

be formed, which have one side convex and the other concave. If the center of such a lens is thicker than the edge, it will be convergent; if it is thinner than the edge, it will be divergent.

154. Algebraic Method of Locating the Image Formed by a Single Refracting Surface. In addition to these geometric methods for locating the images formed by lenses and mirrors, it is possible to obtain comparatively simple formulas which express the relation between the image distance and the object distance. Perhaps the most generally useful of these formulas is one which expresses this relation for a single refracting or reflecting surface. Once this formula has been obtained, the corresponding relation for a thin lens can be immediately derived by applying it successively to the two refracting surfaces. In deriving the first general formula, we will imagine a single surface of separation LO between two media of refractive indices such that the velocity of light in the first medium is v_1 and that in the second medium is v_2. We will assume, as indicated by the arrow in Fig. 140, that the light is travelling from left to right. The arc MO represents a spherical wave, which, starting from the object P, has just reached the refracting surface at O. If the velocity v_2 in the new medium is less than that in the old, the

center of the wave will now be retarded with reference to the edge, and it will move forward the distance ON while the edge is travelling ML, where

$$\frac{ML}{ON} = \frac{v_1}{v_2} \tag{144}$$

The wave front will in this way be distorted into the new approximately spherical surface LN, which is converging toward Q to form there a real image of P. Let

the object distance $PO = p$
the image distance $OQ = q$
the radius of curvature OS of the surface $= R$
the distance $MX = d$

The distances ML and ON can be calculated in terms of p, q, R, and d by making use of the geometric relation expressed by Formula 7 of Chapter IV. The distances Ox, Oy, and Ny are

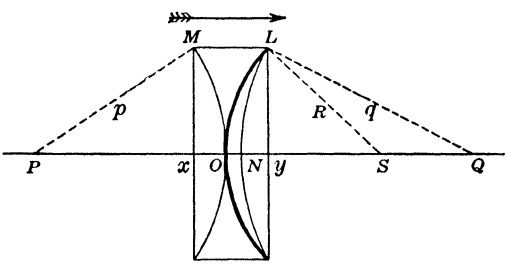


FIG. 140. Formation of Image by Spherical Refracting Surface.

known as the *sagittas* of the arcs MO, LO, and LN respectively. It should be noted that these distances, which we will represent by the letters h_1, h_2, and h_3, correspond to the distance CP of Formula 7. In order to simplify the application of the final formula to all possible positions of image and object, it is convenient to attach a positive algebraic sign to all distances to the right of O and a negative sign to all distances to the left. It should be borne in mind, however, that this sign convention is applicable only if the motion of the light wave through the refracting system is from left to right. The figure should, therefore, always be drawn in such a way that this is the case. With this convention the distances p and h_1, represented in Fig. 140, are negative, and q, R, h_2, and h_3, are all positive. If, however,

the refracting surface had its center of curvature to the left of O, R would be negative, and any image lying to the left of O would have a negative image distance q. With this notation and choice of signs

$$ML = h_2 - h_1$$

and

$$ON = h_2 - h_3$$

Hence from (144) $$h_2 - h_1 = \frac{v_1}{v_2}(h_2 - h_3) \tag{145}$$

Applying (7) to h_1, h_2, and h_3 respectively,

$$h_1 = \frac{d^2}{2p} \qquad h_2 = \frac{d^2}{2R} \qquad h_3 = \frac{d^2}{2q}$$

Substituting these values for the h's in (145)

$$\frac{d^2}{2R} - \frac{d^2}{2p} = \frac{v_1}{v_2}\left(\frac{d^2}{2R} - \frac{d^2}{2q}\right)$$

Or, cancelling the d's, and setting $v_1/v_2 = n$,

$$\frac{n}{q} = \frac{1}{p} + \frac{n-1}{R} \tag{146}$$

An interesting case in which light passes through only one refracting surface is that in which it comes from an object beneath the surface of a body of water. Since n is the ratio of the velocity of light in the first medium to that in the second, it is in this case equal to the ratio of the velocity in water to that in air, i.e., the reciprocal $1/\mu$ of the index of refraction of water. Since the surface of the water is plane, $R = \infty$, and $1/R = 0$. Hence from (146)

$$q = \frac{1}{\mu}p \tag{147}$$

Since p is initially negative, q must also be negative, and the image formed by the refracting surface is on the same side of the surface as the object. It lies, however, nearer the surface in the ratio of $1/\mu$ to 1, which for water is approximately $\frac{3}{4}$.

155. The Image Formed by a Spherical Mirror. Fig. 141 represents a spherical wave MO which is reflected from the spherical mirror LO. It is seen to differ from Fig. 140 only in the location of the wave LN, which now lies to the left of the surface LO instead of to the right. Since the velocity of propa-

gation is the same after reflection as before, $ML = NO$, and the relation (145) is replaced by

$$h_2 - h_1 = h_3 - h_2 = -(h_2 - h_3) \tag{148}$$

This is seen to differ from the corresponding formula of Section 154 only in the substitution of -1 for v_1/v_2 or n, and since all succeeding steps in the calculation are identical with those in Section 154, the final formula (146) can be adapted to the spherical mirror by substituting -1 for n.

Or
$$\frac{1}{q} = -\frac{1}{p} + \frac{2}{R} \tag{149}$$

For a concave mirror, the center of curvature is to the left of O, and R is accordingly negative; for a convex mirror it is positive. Since p is negative, the two terms on the right-hand side of (149) have, for the concave mirror, opposite signs, and the sum is negative if $1/p < 2/R$, i.e., if $p > R/2$. Accordingly the image distance is negative, and a real image is formed to the left of the mirror, if the object lies at a greater distance from the mirror than $R/2$. This conclusion checks with the geometric construction of Fig. 135. If, on the other hand, $p < R/2$, the right-hand side of (149) is positive, q is positive, and a virtual image is formed to the right of O, i.e., behind the mirror. A similar argument shows that the image formed by a convex mirror is always behind the mirror and virtual. The distance from the mirror at which the image of an infinitely distant object is formed is known as the focal length of the mirror. From (149) if $p = \infty$, $1/p = 0$, and $q = R/2$. The focal length of either convex or concave mirror is, therefore, one-half its radius of curvature.

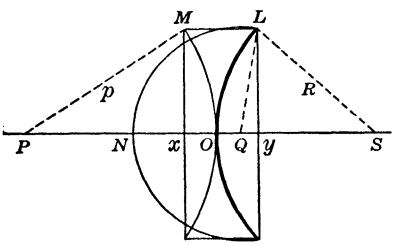


FIG. 141. Formation of Image by Spherical Reflecting Surface.

156. Thin Lenses. In order to locate the image formed by a lens, Formula 146 can be applied successively to its two surfaces. The image formed by the first surface at distance q_1 may be treated as the object for the second surface, and if the lens is thin, the new object distance p_2 may be set equal to q_1. The ratio of the velocity of light in the first medium, air, to that in

the second medium, glass, is the index of refraction of the glass, μ. Hence $n = \mu$, and

$$\frac{\mu}{q_1} = \frac{1}{p_1} + \frac{\mu - 1}{R_1} \tag{150}$$

where R_1 is positive if the first surface is convex and negative if it is concave. Hence

$$\frac{1}{p_2} = \frac{1}{q_1} = \frac{1}{\mu p_1} + \frac{1 - \frac{1}{\mu}}{R_1}$$

The value of n which must be used in (146) for the second surface of the lens is the ratio of the velocity in glass to that in air, or $1/\mu$. Hence

$$\frac{1}{\mu q_2} = \left[\frac{1}{\mu p_1} + \frac{1 - \frac{1}{\mu}}{R_1}\right] + \frac{\frac{1}{\mu} - 1}{R_2}$$

Or

$$\frac{1}{q_2} = \frac{1}{p_1} + (\mu - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

Denoting the distance from object to lens by p and the distance from lens to image by q,

$$\frac{1}{q} = \frac{1}{p} + (\mu - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{151}$$

The distance from a lens at which the image of an infinitely distant object is formed is called the *focal length* of the lens, and is denoted by f. If $p = \infty$, $1/p = 0$, and $q = f$. Hence

$$\frac{1}{f} = (\mu - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{152}$$

For a double convex lens R_1 is positive and R_2 is negative, thus yielding in all cases a positive sign for f. For a double concave lens R_1 is negative and R_2 is positive, a combination which always yields a negative sign for f. The focal length of a convergent lens is, therefore, positive, and that of a divergent lens negative. Formula 151 can be simplified by introducing $1/f$ in place of the second term on the right-hand side.

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{f} \tag{153}$$

157. The Spectacle Lens. As an application of this formula, consider the action of a divergent spectacle lens. In normal vision the lens L of the eye (Fig. 142) forms an image of the object PQ on the retina R. Since the distance of a particular object from the eye cannot in general be chosen to suit the convenience of the observer, it is necessary to adjust either the image distance q or the focal length f of the lens. In most cameras the focussing is done by varying the distance of the plate from the lens, but in the eye it is accomplished by changing the shape of the lens itself and thereby its focal length. It frequently happens, however, that the eye loses its power of accommodation over the range which is necessary in everyday life, and may, for example, be unable to focus sharply the image of a distant object. The rays of light coming from such a distant object are practically parallel, and the rays from a nearer object are diverging as they approach the lens of the eye. The difficulty in nearsightedness clearly lies in the inability of the eye to prevent the parallel rays from converging too rapidly and coming to a focus inside the retina. The function of the diverging spectacle lens is, therefore, to cause the parallel rays to diverge and thereby to make it possible for the eye to bring them together accurately at the retina. Stated in another way, the spectacle lens forms a virtual image $P'Q'$ of the object PQ at a point nearer the eye, and the eye, which is unable to accommodate itself to the distant object, is still able to focus accurately on the nearer image.

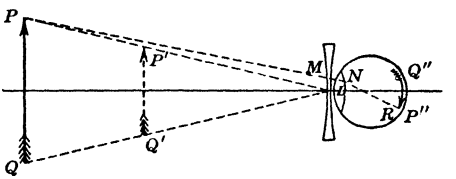


FIG. 142. Divergent Spectacle Lens.

EXAMPLES

1. The distance from the lens of the eye to the retina is 25 mm. A diverging spectacle lens with a focal length of 50 cm. is placed 1 cm. in front of the eye. What must be the focal length of the eye lens in order to focus accurately on an object 1 meter distant?

Consider first the image formed by the spectacle lens. The object distance p_1 is -100 cm. $f_1 = -50$. Hence, applying (153),

$$\frac{1}{q_1} = -\frac{1}{100} - \frac{1}{50} = -\frac{3}{100}$$

Or $$q_1 = -33\tfrac{1}{3} \text{ cm.}$$

The image formed by the spectacle lens is, therefore, to the left of the lens and $33\frac{1}{3}$ cm. distant. Its distance from the eye lens is $33\frac{1}{3} + 1 = 34\frac{1}{3}$ cm. Hence if we denote the object distance for the eye lens by p_2,

$$p_2 = -34\tfrac{1}{3}, \quad \frac{1}{p_2} = -\frac{3}{103}, \quad q_2 = 2.5, \quad \frac{1}{q_2} = \frac{2}{5}$$

Hence
$$\frac{2}{5} = -\frac{3}{103} + \frac{1}{f}$$

Or
$$\frac{1}{f} = \frac{2}{5} + \frac{3}{103} = \frac{221}{515}$$

$$f = \frac{515}{221} = 2.33 \text{ cm.}$$

This problem illustrates very well the possibility of applying Formula 153 successively to two or even more lenses. If a converging spectacle lens were used in place of the diverging lens, the rays from a distant object, after passing through it, would be converging toward some point behind the eye. The lens of the eye would then attempt, perhaps successfully, still further to converge these rays to the retina. Although the image toward which the spectacle lens was converging the rays before they encountered the eye lens has no real existence, it may still be treated for the purpose of Formula 153 as the object for the eye lens. Since it lies to the right of the eye, the object distance in this case will be positive.

2. A certain camera lens has a focal length of 5 in. An additional converging lens of focal length 10 in. can be mounted immediately in front of this lens to serve as a *portrait attachment.* Neglecting the finite thickness of the camera lens and the separation between the two lenses, calculate the position of the image formed of an object 20 in. away.

$$p_1 = -20, \quad f_1 = 10$$

Applying (153)
$$\frac{1}{q_1} = -\frac{1}{20} + \frac{1}{10} = \frac{1}{20}$$

Hence
$$q_1 = 20 \text{ in.}$$

The portrait lens alone would, therefore, form a real image of the object 20 in. behind the lens. This image, although having no real existence, may be taken as the object for the camera lens.

Hence
$$p_2 = 20, \quad f_2 = 5$$

$$\frac{1}{q_2} = \frac{1}{20} + \frac{1}{5} = \frac{5}{20} = \frac{1}{4}$$

$$q_2 = 4$$

The image will be formed 4 in. behind the combination.

PROBLEMS

1. Locate geometrically the image which a concave mirror having a focal length 3 in. forms of an object 2 in. distant. Locate the image formed by a convex mirror of the same radius of curvature.

2. Locate geometrically the image formed by a convergent lens having a focal length 3 in. of an object 2 in. distant. Is this image real or virtual?

3. Using the formulas developed in the present chapter, calculate the position of the images in the first two problems.

4. A goldfish is swimming in a spherical bowl of radius 6 in. at a distance of 3 in. from the spherical surface. What is its apparent location? Neglect the effect of the glass of the bowl, and assume the index of refraction of water to be $\frac{4}{3}$.

5. A single camera lens is of the meniscus type, i.e., one surface is concave and the other is convex. If the convex surface has a radius of curvature of 2 in., the concave surface a radius of 3 in., and if the index of refraction of the glass is 1.5, what is the focal length of the lens?

6. If two watch glasses, each having a radius of curvature of 4 in., are cemented together in such a way that their convex surfaces are outward, thus leaving a lens-shaped space between them, and if they are immersed in a vessel of water, what will be the effective focal length of the air space between them? Neglect the refractive effect of the glass itself.

7. How will the focal length of a 5-in. camera lens be affected by placing a 15-in. diverging lens immediately in front of it? Treat both lenses as thin lenses.

8. What type of spectacle lens must be used by a man who can see clearly distant objects, but has difficulty with nearer objects? What focal length would be required in order for him to see an object 12 in. away as readily as a distant object?

9. A goldfish is swimming in a basin formed by a spherical mirror of radius 12 in. If it is 2 in. above the mirror and if the water is 6 in. deep, where is his apparent image in the mirror? *Hint:* Find first the image of the fish in the concave mirror, and then find the image formed of this image by the surface of the water.

10. A certain thick lens is formed by two convex surfaces, one of radius 2 in. and the other of radius 3 in., with a maximum thickness of 1 in. The index of refraction of the glass is 1.5. Calculate the position of the image which it forms of a distant object. *Hint:* Apply Formula 146 to the two surfaces successively.

CHAPTER XXIX

OPTICAL INSTRUMENTS

The ability of a thin lens to focus the rays of the Sun was known to the ancient Greeks as early as the time of Aristophanes (424 B.C.), and it is not likely that they would have experimented with burning-glasses without noticing the magnifying effect of such a lens.

158. The Magnifying Glass. Valuable as this property is for certain kinds of work, it is easy to show that the amount of magnification which can be obtained in this way is rather limited. Consider, for example, the jeweller's eye lens, which is worn very close to the eye. Its effectiveness depends on the fact that it forms a virtual image, $P'Q'$ in Fig. 143, of a very near object PQ at a convenient distance from the eye. If the focal length of the lens is 2 in. and the object is located $1\frac{3}{4}$ in. in front of it, according to Formula 153 of the preceding chapter

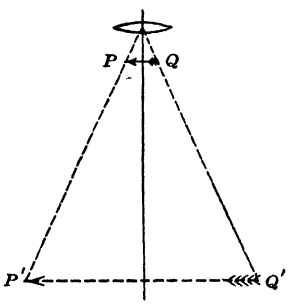


FIG. 143. Magnifying Glass.

$$\frac{1}{q} = -\frac{4}{7} + \frac{1}{2} = -\frac{1}{14}$$

Or $$q = -14 \text{ in.}$$

The image is virtual, and is located at a distance of 14 in. on the same side of the lens as the object. The angle which the image subtends at the eye is, indeed, approximately the same as that which the object subtends, but without the aid of the lens it would be necessary for convenient observation to hold the object at a distance of at least 12 in. from the eye. Held, as it is, at approximately 2 in., it appears about six times as large as it would without the lens. The advantage of the device lies in the possibility which it provides of holding the object closer to the eye. As the lens must always be held between object and eye, it is clear that this distance cannot be decreased indefinitely, and the possible magnification is rather limited.

159. The Telescope. It was not until the period of the renaissance that the possibility was discovered of combining lenses in such a way as to obtain very great magnifications, not merely of near objects, but of distant objects as well. The invention of the telescope and the compound microscope occurred almost

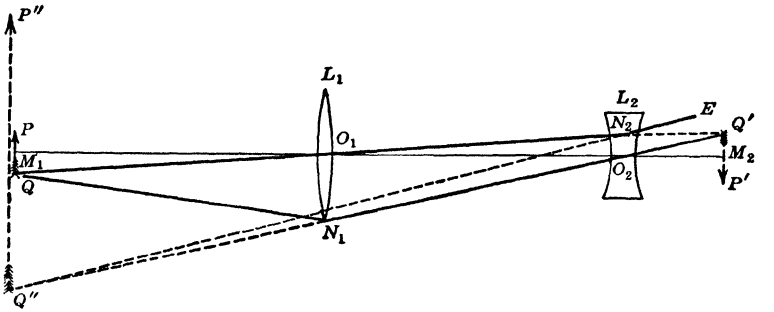


FIG. 144. Galilean Telescope.

simultaneously. The time was indeed ripe for these discoveries, and there still exists much doubt as to priority in the construction of both. The present evidence favors Hans Lippershey in Holland as the inventor of the telescope in 1608 and Zacharias

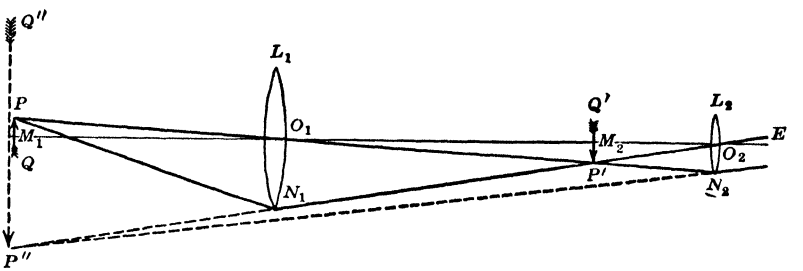


FIG. 145. Astronomical Telescope.

Joannides as the inventor of the microscope at some time prior to 1610. Both instruments originally consisted of a convergent *objective* lens and a divergent *eyepiece*. The substitution of a convergent eyepiece for the divergent one was due to the great astronomer Kepler. Fig. 144 shows the telescope with the divergent eyepiece, and Fig. 145 with the convergent eyepiece. The former type was used by Galileo for the discovery of a wealth of new astronomical facts, and is called in his honor the Galilean telescope. It is still used extensively in opera and field glasses.

The divergent lens L_2 is interposed between the objective lens L_1 and the image $P'Q'$ which it alone would form. The rays QO_1Q' and QN_1Q', which before encountering the lens L_2 are converging toward Q', are by L_2 diverged so that they appear to an eye at E to originate at Q''. A virtual and erect image is thus formed at $P''Q''$. In practice, of course, the object on which such an instrument is focussed is always at a great distance from the telescope, and it is represented in the immediate vicinity of the objective merely for convenience in the figure. Hence the distance $O_1M_2 = f_1$ if f_1 is the focal length of the objective. If, now, the eyepiece is to form an image of $P'Q'$ at an infinite distance, it must be located at a distance O_2M_2 equal to its focal length f_2. The angle subtended by that portion of the object between M_1 and Q at the telescope is M_1O_1Q or its equal $Q'O_1M_2$. The angle subtended by the corresponding part of the image M_1Q'' is M_1O_2Q'' or its equal $Q'O_2M_2$. But since all the angles are small,

$$\frac{Q'O_2M_2}{Q'O_1M_2} = \frac{O_1M_2}{O_2M_2} = \frac{f_1}{f_2}$$

The *magnification* of the telescope, i.e., the ratio of the angle subtended by the image to that subtended by the object, is, therefore, simply the ratio of the focal length of the objective to that of the eyepiece. Exactly the same proof can be used to show that the magnification of the telescope in Fig. 145 is also the ratio of the focal length of the objective to that of the eyepiece. It should be noted that the image formed by the telescope with the converging eyepiece is inverted, whereas that formed by the Galilean telescope is erect. This inversion of the image is no disadvantage for astronomical purposes, and the telescope in this form is usually called the astronomical telescope. It can be adapted for terrestrial purposes by introducing an additional lens or system of lenses between the image $Q'P'$ and the eyepiece.

It should also be noted that for the same magnification the astronomical telescope must be longer than the Galilean telescope by twice the focal length of the eyepiece. This advantage of the Galilean telescope in compactness is, however, compensated for by a substantially smaller field of view. If the telescope of Fig. 145 is held at arm's length, a bright disk will be observed a short distance behind the eyepiece between it and the eye. This is the image of the objective formed by the eyepiece,

and is known as the *exit pupil* of the telescope. All rays emerging from the telescope must pass through this circular area. It is, indeed, a sort of window through which an eye from behind looks out through the telescope. Ordinarily the eye is held between this point and the eyepiece so that one is not sensible of any restriction which it imposes on the angle of vision. In fact, with the astronomical telescope, the angle of vision is determined by the angle over which the eyepiece can form a distinct image, and this, in turn, is determined by the care and skill with which the eyepiece is designed. In practice a diaphragm is placed in the plane of $P'Q'$ to screen out those parts of the image which are too far from the axis to be adequately imaged by the eyepiece. In the case of the Galilean telescope, however, the exit pupil, i.e., the image of the objective formed by the eyepiece, lies between the eyepiece itself and the objective and the observer is always sensible of the material restriction which it imposes on his angle of vision. It should be noted that the diameter of this window is proportional to the diameter of the objective, and the field of vision accordingly increases with the size of the objective. It is for this reason that the Galilean type of telescope has, in general, a larger objective than the astronomical type. Along with this larger objective it has a greater *light-gathering* capacity, which is advantageous when the object under observation is poorly illuminated.

160. Total Reflection. The Prism Binocular. The greater field of the astronomical telescope, an erect image, and compactness equal to that of the Galilean telescope, are all secured by the use of reversing prisms in the modern prism binocular. Fig. 146 shows a single prism in the form of an isosceles right triangle. A ray PN_1 normal to the first surface of the prism will be reflected successively at N_1 and N_2, and will emerge from the prism parallel to its original direction. In general a ray PN_1 incident on a surface of separation between two media will be split into a reflected and a refracted component. The refracted ray, in passing from a denser to a rarer medium, will be bent away from the normal to the surface, and the angle r which it makes with

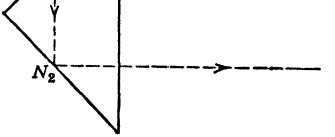


FIG. 146. Total Reflection.

the normal can be calculated by Snell's Law. If the sine of this angle turns out to be equal to or greater than 1, there can be no refracted component, and the reflected component will contain the entire energy of the incident beam. *Total reflection* takes place. Since the incident ray in the right-angle prism makes an angle of 45° with the normal, its sine is $1/\sqrt{2}$, and total reflection will take place if the index of refraction of the glass is greater than that given by the relation

$$\mu = \frac{1}{\frac{1}{\sqrt{2}}} = 1.414$$

A right-angle prism made of any glass whatever will, accordingly, exhibit total reflection. In the prism binocular each optical system consists of an objective O (Fig. 147), one horizontal right-angle prism P_1, a vertical prism P_2, and an eyepiece E. It is easily shown that, as the beam of light enters the objective and traces the path M_1ABCDM_2, the two prisms invert the image and interchange right and left to produce finally an erect and

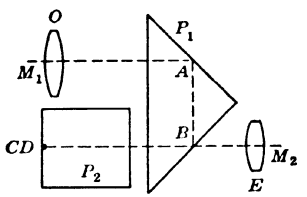


FIG. 147. Prism Binocular.

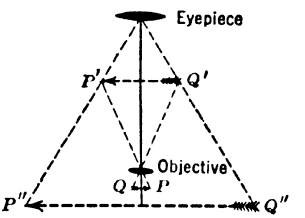


FIG. 148. The Compound Microscope.

correctly oriented image. Such an instrument, if well made, is necessarily expensive, as the faces of the prism must be ground accurately plane.

161. The Microscope. The compound microscope (Fig. 148) is similar in construction to the astronomical telescope. In place of the long-focus objective of the telescope, it has a very short-focus lens, which forms an enlarged real image $P'Q'$ of the object PQ just inside the principal focus of the eyepiece. A still further enlarged virtual image $P''Q''$ is then formed by the eyepiece at a convenient distance from the eye for careful observation.

162. Correction of Objective for Chromatic and Spherical Aberration. If a single lens is used as the objective of a telescope, the sharpness of the resulting image is seriously impaired by

spherical aberration, and all the early telescopic observations were made under this handicap. In addition to this difficulty, the fact that the index of refraction of glass is different for different colors gives to such an objective a different focus for each color. This defect is known as *chromatic aberration*. If the telescope is sharply focussed for yellow light, the violet focus will be nearer the lens, and the red focus at a greater distance from it, resulting in a colored ring surrounding the yellow image. In general, the dispersion of a refracting material, i.e., the amount by which the index of refraction for red light differs from that for violet light, is greater the greater the index of refraction itself. If the dispersion always increased strictly hand in hand with increasing refractivity, it would be impossible to eliminate chromatic aberration by a combination of lenses made from different kinds of glass. Newton persuaded himself by a rather too meager series of experiments that this was the case, and gave up hope of correcting an objective for these color defects. It was nearly a century later that John Dolland in London discovered that the excess in the dispersion of flint glass over that of crown was greater than its greater refractive index would lead one to expect, and that it was accordingly possible to combine a diverging flint-glass lens with a converging crown-glass lens in such a way as to leave a substantial positive focal length, and at the same time to balance the dispersion of the flint against that of the crown. In order to correct for chromatic aberration it was only necessary for the two lenses to possess proper focal lengths, and a wide variety of shapes was still possible for each lens. It was found, however, that by properly choosing the shapes of the two components, it was possible simultaneously to correct for spherical aberration as well. A typical *achromatic lens*, formed in this way, is shown in Fig. 149. It consists of a flint-glass lens with one face very nearly plane and the other concave and of the same radius of curvature as one face of the double convex crown-glass lens. It may either rest in contact with the crown-glass surface of the same curvature or be cemented to it with Canada balsam.

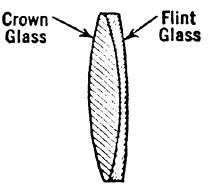


FIG. 149. Achromatic Lens.

In most of the very large modern telescopes the defects of spherical and chromatic aberration are obviated by using a concave parabolic mirror instead of a lens as objective. Ade-

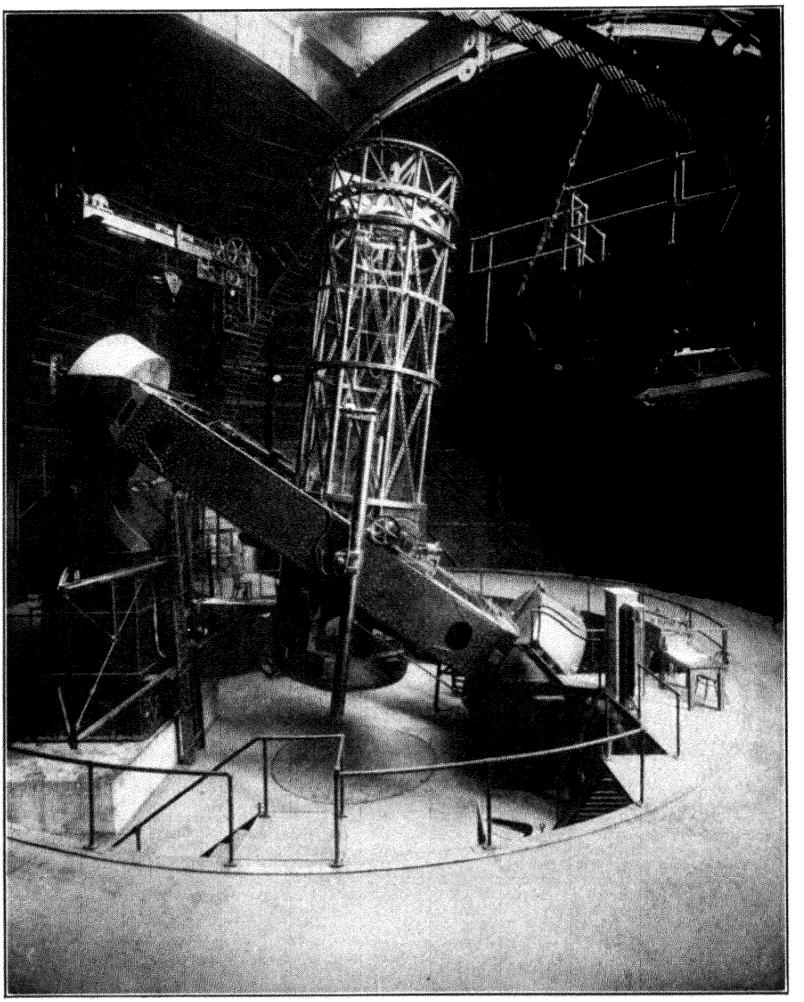


PLATE III. 100 in. Mt. Wilson Reflecting Telescope.

quate correction of a lens for spherical aberration can only be accomplished in the simple manner which has here been considered if the ratio of focal length to diameter is greater than 15. The total absence of spherical aberration in a parabolic mirror, however, makes possible much shorter telescopes, thus greatly reducing the cost, and increasing the intensity of the light which is available for photographic purposes. Another great advantage of the reflecting telescope is the fact that, since the light does not penetrate into the interior of the mirror, it can be made of glass which is much inferior to that required for a lens. Plate III shows the great 100 in. Mt. Wilson reflector, the largest in the world.

163. The Photographic Objective. Astigmatism. Although the problem of a satisfactory telescope objective was thus very simply solved by the latter part of the eighteenth century, the discovery of photography during the second quarter of the nineteenth century presented a much more difficult optical problem. Whereas the angle of vision required in a telescope or microscope is only a few degrees, it is desirable in a camera to form an image which may embrace 40° or 50°. Furthermore, it is desired to form an undistorted image of a plane surface such as a printed

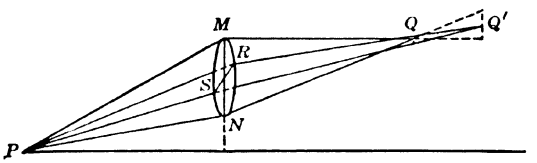


FIG. 150. The Cause of Astigmatism.

page accurately on the plane surface of the photographic plate. A single lens will under all circumstances form a distorted image, representing a square as somewhat pincushion or barrel shaped depending on whether the diaphragm, which limits the area of the lens to be utilized, is placed in front of or behind it. If, however, two identical lenses are used with the diaphragm between, this distortion is eliminated. The so-called *rapid rectilinear* lens consists of two identical achromatic lenses mounted in this way with reference to the diaphragm.

The outstanding residual difficulty and one which proved to be most difficult to correct is known as *astigmatism*. It is due to the fact that a diagonal bundle of rays MN which is refracted by points on the lens lying in the plane of the paper (Fig. 150) is focussed to an

entirely different point Q from that Q' to which the perpendicular bundle RS is directed. In general, neither of these foci will lie in the plane of the photographic plate, so that the fault of astigmatism is very intimately associated with the so-called *curvature of field* of the lens. These difficulties could be corrected only by the use of new kinds of glass with substantially different optical properties. The development of these new glasses was accomplished during the last two decades of the nineteenth century at the Zeiss works in Jena under the leadership of Professor E. Abbe. The perfection which has been achieved in photographic objectives can be appreciated only when it is realized that the fault of astigmatism, like most other optical defects, increases rapidly as the area of the lens which is utilized is increased. On the other hand, the *speed* of the lens, the amount of light which passes through it during the brief interval of a photographic exposure, is proportional to this area. Up to late in the last decade of the nineteenth century no lens capable of forming a sharp image over the angle required for landscape or portrait work existed with a ratio of diameter to focal length of more than 1 : 8, i.e., with a maximum aperture greater than $f/8$. Since then, with the opticians matching their skill against the insatiable demands of the motion-picture industry for more light, the speed of lenses has been steadily increased until at present apertures of $f/1.5$ and even $f/0.9$ are available. In this last case, the diameter of the lens is actually $1\frac{1}{8}$ as great as the distance from lens to plate. It is difficult to conceive of any more thrilling technical development than that which has been achieved with glass during the first quarter of the present century.

PROBLEMS

1. Calculate the position of the final image formed by a Galilean telescope of an infinitely distant object if the focal length of the objective is 6 in., that of the eyepiece is 2 in., and their distance apart is $3\frac{1}{2}$ in.

2. The indices of refraction for ordinary crown glass and a medium flint glass for a certain wave-length in the blue and a second wave-length in the orange are as follows:

	Crown	Flint
Blue	1.528	1.649
Orange	1.518	1.628

Calculate the focal length of a lens for each of these wave-lengths, if it is formed by combining a double convex crown lens with a plano-concave flint

lens. Assume the radii of all surfaces, except, of course, the plane surface of the flint lens, to be 4 in.

3. A reading glass having a focal length of 10 in. is held 9 in. from the object being examined and 12 in. from the eye. Calculate the location of the image and the apparent magnification.

4. Calculate the size and position of the exit pupil of a Galilean telescope if the focal length of the objective is 6 in., that of the eyepiece is 1 in., and the diameter of the objective is $1\frac{1}{2}$ in. What is the approximate angle of view?

5. What is the ratio of the size of the image to the size of the object in a compound microscope if the objective has a focal length of 3 mm., and if the eyepiece has a focal length of 15 mm.? Assume the image formed by the objective to be 20 cm. above the objective, and the final image to be in the plane of the object. Treat both lenses as thin lenses.

CHAPTER XXX

INTERFERENCE AND DIFFRACTION

In the last two chapters the process of image formation has been treated on the assumption that light travels out from a luminous source in rays, which are accurately rectilinear except at reflecting or refracting surfaces. If any group of these rays originates in a point source, and then, as a result of reflection or refraction, once more converges to a point, this point constitutes an image of the source. On such a purely geometric basis there would be no limit to the magnification possible with a telescope or a microscope. It would be merely a question of the skill of the optician in eliminating the geometrical defects which were discussed in the last chapter. If, however, light is a periodic wave disturbance similar to sound, the requirements for image formation would necessarily be more rigorous. We can still use the ray picture, but we must remember that the ray represents merely the direction of propagation of the wave or the normal to the wave front.

The essential complication which the wave picture introduces is that of phase. Not merely must each ray in the bundle which produces the image pass through the image point, but the wave front which one ray represents must arrive in phase with those which the other rays of the bundle represent. It would appear at first sight that this condition is very well satisfied in the process of image formation by a mirror or lens. As shown in the derivation of the lens formula in Chapter XXVIII, that portion of a wave crest which travels from source to image via the edge of the lens compensates for its longer journey outside the glass by a shorter path through the lens itself, and arrives at Q at the same time as the portion of the wave crest which travelled along the axis of the lens. All portions of the wave arrive, therefore, in phase.

164. Diffraction. The weakness of this measure of comfort lies in the well-known ability of waves to spread out. If, for example, a partition P containing a hole O (Fig. 151) divides a tray of water into two parts, and if a wave approaches O from

one side of the partition, the disturbance thus produced will serve as the source of a new circular wave spreading out from O on the opposite side of the partition. This spreading property of a wave, known as *diffraction*, is familiar enough in the case of sound. A sound entering an open door from some source outside is not restricted to regions from which the source itself can be seen, but spreads out in all directions. The fact that light does not apparently spread out in this way would at first seem conclusive evidence against its wave nature, and it was, as a matter of fact, the apparent rectilinear propagation of light which led Newton with considerable hesitation to take his stand against the wave theory. There are, however, two possible explanations of the sharp shadow cast by an opaque object. It may be, as Newton thought, that no light penetrates the region of shadow, or it may be that a great deal of light enters this region, but with crests and hollows so confused and so annulling one another that there is no resultant effect. This last, it turns out, is the correct explanation, and it remained for Thomas Young in England in 1802 and Augustin Jean Fresnel later but independently in France to devise a series of beautiful experiments to demonstrate this.

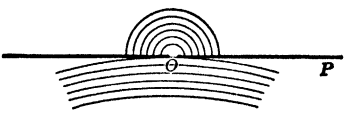


FIG. 151. Diffraction.

165. The Interference of Light. Young's Pinhole Experiment. In Young's original experiment two tiny holes A and B of Fig. 152 were drilled very close together in a thin metal partition,

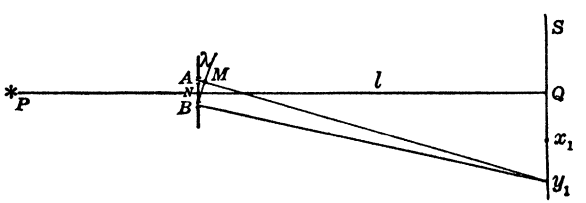


FIG. 152. Young's Pinhole Experiment.

which shades the screen S from the distant source of light P. On the corpuscular theory of light, particles from the source will pass through both these holes and impinge on the screen S at the points where the lines PA and PB intersect it. We should accordingly expect to find two bright spots on the screen at exactly

the geometric projection of the source through the two apertures. On the wave theory, however, each little hole will be the source of a new ripple, and at points on the screen where these two ripples arrive in phase there will be light, whereas at points where the crest from one aperture encounters a hollow from the other aperture there should be darkness. If the two holes are equidistant from the source, the crest of a given wave will arrive at them simultaneously, and will give rise simultaneously to crests in the little circular ripples behind the screen. At Q, the foot of the perpendicular dropped from a point midway between A and B, where the distance to the two holes is the same, the two crests will arrive simultaneously, and will produce a bright spot. At a point x_1 to one side of Q, however, such that the distance to A is one-half wave-length greater than that to B, the crest from A will arrive only in time to find a hollow from B, and the two waves will interfere to form a dark spot. Still further along at y_1, where the distance from A is a whole wave-length greater than that from B, a crest from A will encounter the next succeeding crest from B, and will reënforce it to produce a bright spot. Similarly at every point where the path difference of the two rays is an even number of half wave-lengths, there will be a bright spot, while at points where the path difference is an odd number of half wave-lengths there will be a dark spot. There should appear, therefore, on either side of the central bright spot a series of alternately dark and light bands. When Young tried the experiment, these predictions of the wave theory were brilliantly substantiated.

It is a simple matter to calculate the distance of any one particular band from the center in terms of the wave-length of light. Let, for example, the distance between the apertures be d, the perpendicular distance from them to the screen be l, and the distance from the center to the first light band y. Draw the line BM so that $y_1B = y_1M$. Then since y_1 is the first maximum of brightness to the left of Q, $MA = \lambda$. Owing to the minuteness of the separation between A and B in comparison to l, the angle BMA is very nearly a right angle, and the triangle ABM is similar to the triangle y_1QN, having the corresponding sides mutually perpendicular.

Hence
$$\frac{\lambda}{d} = \frac{y}{l}$$

and $$y = \frac{\lambda l}{d} \quad \text{or} \quad \lambda = \frac{yd}{l}$$

From a knowledge of the wave-length λ, the position of the bands can be predicted, or if the wave-length is unknown, it can be calculated from the measured distance y. If white light is used for the source P, the bands are colored and somewhat diffuse, indicating that the wave-length is different for different colors. If, however, monochromatic light is used, such as would be obtained by utilizing a narrow range of the spectrum formed by a prism, the maxima and minima of the bands are clear and distinct, and permit of fairly accurate measurement. The wave-length thus obtained for the extreme red end of the spectrum is a little less than 8×10^{-5} cm., and for the violet end about 4×10^{-5} cm., in each case having the order of magnitude of a few ten thousandths of a millimeter. The experiment, therefore, not merely demonstrated the wave nature of light, but showed that its wave-length was excessively short, a very fortunate circumstance, as we shall see, for the uses to which light is put.

The interference of two beams of light is a much commoner phenomenon than might at first be supposed, and some of these examples of interference presented outstanding problems even in the time of Newton. Thus Newton himself discovered that light reflected from a convex lens in contact with a plane glass surface shows a series of colored rings surrounding the point of contact between lens and glass. This phenomenon, known as Newton's rings, is due to the interference of light reflected from the under surface of the lens with that reflected from the upper surface of the glass plate. The colors are due to the difference in wave-length of the different colors. Very similar in origin are the colors to be observed on thin films of oil and on soap bubbles.

166. The Lippmann Process of Color Photography. Of still greater interest is the possibility of setting up standing waves similar to those encountered in Kundt's and Melde's experiments. These waves were first demonstrated by Wiener in Leipzig, but were somewhat later utilized by Lippmann in Paris as the basis for an extraordinarily interesting process of color photography. If a photographic plate with a very thin sensitive film is placed in contact with a mercury surface (Fig. 153) in such a way that the film side is next to the mercury, and if light is incident on the glass side of the plate, it will pass through glass

and film, and will be reflected at the mercury surface. The returning wave interferes in the sensitive film with the incident wave in exactly the same way that it does in Melde's experiment, and loops and nodes of light intensity result in the emulsion. The loops produce a blackening of the silver grains in the emulsion, and as in Melde's experiment, the blackened layers are one-half wave-length apart. If a real image of some object or landscape is formed by a camera lens on such a plate, the separation of the darkened layers in any part of the picture will be determined by the color which falls on it. If, now, white light is allowed to fall normally on a plate which has been made in this way, a portion of the light will be reflected from each layer. The difference in the length of path traversed by light reflected from one layer and that reflected from the next succeeding layer will be twice their distance apart or λ. For the particular color which was responsible for the image in the first place, the light reflected from all these successive layers will be in phase, and will produce a strong resultant reflection, while for other colors destructive interference will take place. The picture will, therefore, appear in its original natural colors. The technique of the process is difficult, and although it reproduces pure colors better than any other, it is not so successful with the mixture of many spectrum colors which are so often encountered in nature.

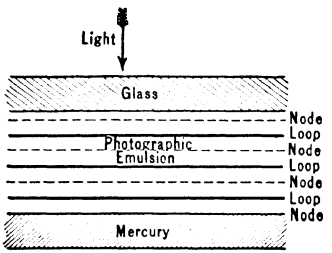


FIG. 153. Lippmann Process of Color Photography.

167. The Effect of Diffraction on the Image Formed by a Lens. In order to come to an understanding of the effect which the wave nature of light has on the formation of images, consider first the case of a converging lens (Fig. 154) before which is placed a slit AB perpendicular to the plane of the paper. Imagine that this slit is illuminated by a plane wave, or what amounts to the same thing, by a bundle of parallel rays from an infinitely distant source. This bundle will, on the simple geometric theory of the preceding chapters, be converged to the principal focus of the lens F. The only effect of the slit AB would be to limit the size of the bundle admitted and hence the brightness of the image. In a similar way any diagonal bundle of parallel rays will be

brought to a focus at some other point in the focal plane of the lens, indicated by F_1, and because of the peculiar properties of the lens the members of each bundle will arrive at their focus in phase. Our present problem is to decide in what way diffraction will affect these conclusions.

We are prepared to admit that, if an opaque screen with a tiny aperture is placed at any point in the path of the beam, the narrow horizontal ray which passes through this aperture would be diffracted into a bundle of rays radiating outward in all directions. That this diffraction takes place at each point in a wave front regardless of whether the rest of the wave is screened or not, was first recognized by the Dutch physicist Huygens, and is

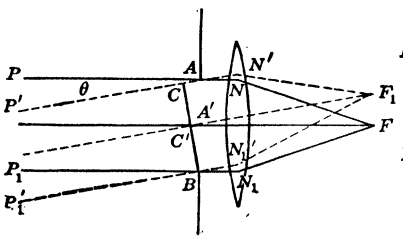


FIG. 154

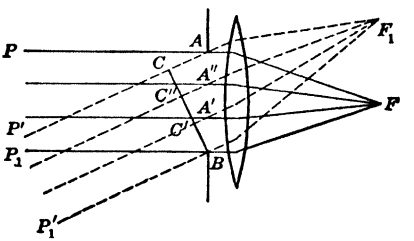


FIG. 155

Effect of Diffraction on Image Formed by Telescope Objective.

known as Huygen's Principle. The fact that the great majority of these diffracted rays produce no apparent effect on a screen lying in the focal plane through F is due to their interference with one another. In accordance with Huygen's Principle, the ray PA will at A be diffracted into a bundle of rays, which will include the ray ANF predicted by the geometric theory, but also the dotted ray $AN'F_1$. Similarly the ray P_1B and all intervening parallel rays will give rise to dotted rays, which, converging toward F_1, trace precisely the paths which the members of the bundle parallel to $P'A$ follow beyond the slit. Whether or not they produce a bright spot at F_1 depends on how completely they interfere with one another. In order to answer this question, it is necessary to consider the phase relations of these rays at and in the vicinity of the slit.

The diffracted rays are all in phase when they leave the plane of the slit, but in order to be in phase when they arrive at F_1, their crests should have coincided along the line BC, since BC is perpendicular to the rays which we know converge in proper phase

toward F_1. If, therefore, the distance between the intersection of any dotted ray with BC and its intersection with AB is a single half wave-length or any odd number of half wave-lengths, it will be out of phase with BF_1, whereas if this difference is an even number of half wave-lengths, it will be in phase with it. Consider, for example, the case that $AC = \lambda$. Then $A'C'$, the corresponding distance for a ray midway between A and B, will be $\lambda/2$. This ray will, accordingly, just cancel the effect of the ray BF_1. In similar fashion a ray just above B will be cancelled by a ray just above A', and pair by pair the rays between A' and B will cancel those between A and A' to produce absolute darkness at F_1. If, however, the distance AC is $3\lambda/2$, the aperture AB may be divided into three instead of two sections, the rays in the lower section being cancelled by those in the middle section, but those in the upper section being left (Fig. 155). These will combine to produce a substantial amount of light at F_1. Similarly there will be a light spot at the focus of any group of diffracted rays for which AC is an odd number of half wave-lengths. As a result of diffraction, therefore, the single group of rays parallel to PA will produce, not merely a bright spot at F as would be expected from the geometric theory, but a series of bright spots on either side of F. For the formation of the first of these diffraction spots, only about $\frac{1}{3}$ of the slit contributes, resulting in an intensity only about $\frac{1}{3}$ that of the central spot. For the next bright spot, only $\frac{1}{5}$ of the slit contributes ($CA = \frac{5}{2}\lambda$), and its intensity is only $\frac{1}{5}$ that of the central image. The diffraction images fall off, therefore, very rapidly in intensity, and only the first few are noticeable except under exceptionally favorable conditions. Their importance lies in the fact that the intensity of the central bright image falls off gradually to the first dark spot, and then increases gradually to the first subordinate maximum. Thus, instead of obtaining a sharp-point image of a distant star, we get a diffuse spot accompanied by other much weaker spots on either side. If a second star is so located that its geometric image would fall on the first dark spot in the diffraction pattern of the first star, the diffuse central image of the second star would so blend with that of the first that they could not be distinguished as two separate images. The telescope would be unable to *resolve* them.

These conclusions at which we have arrived on the assumption of a slit-shaped aperture before the lens are very little affected

if the slit is replaced by the usual circular aperture. Because of the symmetry of the circle, the diffraction spots on either side of the central image are expanded into diffraction rings, but the diffuseness of the central spot remains, and sets a limit to the *resolving power* of the apparatus.

168. The Resolving Power of a Telescope. This resolving power is defined precisely as the angle subtended at the telescope by two distant objects which can barely be distinguished as separate objects. This can be very easily calculated from Fig. 154. If the point F_1 is the first dark spot, $AC = \lambda$. The direction of an object which will form a central image at F_1 is along the dotted line AP' at an angle θ with AP. It is, therefore, precisely this angle θ which measures the resolving power of the apparatus. But since CB is perpendicular to AP', and AB is perpendicular to AP, the angle $ABC = \theta$. Or, if we let $d = AB$, the width of the slit,

$$\sin \theta = \frac{\lambda}{d}$$

In practice, of course, θ is a very small angle, for which the angle itself, measured in radians, is practically identical with the sine. Hence

$$\theta = \frac{\lambda}{d}$$

The greater the width of the slit, the smaller will be the angle between two distinguishable objects, and the greater will be the resolving power.

Returning to the practically more important case of a circular aperture, we can think of this as a slit of varying width; and, as a matter of fact, the position of the first dark ring may be calculated as with the slit by using a width equal to 0.8 times the diameter of the circle. If we set D equal to this diameter, therefore, the resolving power with the circular aperture will be

$$\theta = \frac{\lambda}{0.8D}$$

A limit to the resolving power of any telescope or even of the eye itself is set by the diameter of the aperture. The pupil of the eye varies in diameter with the light intensity, but taking a fair average of 0.3 cm. and the wave-length 6×10^{-5} of yellow light,

$$\theta = \frac{6 \times 10^{-5}}{(0.8)(0.3)} = 0.00025$$

Thus, with such an eye functioning in an optically perfect manner, it should be just able to distinguish two objects $\frac{1}{4}$ cm. apart at a distance of 10 meters. One of the most important requirements in a telescope for astronomical work is a large aperture. In fact, in order to increase the detail with which observations can be made, it is only necessary to increase the size of the objective. Of course, this cannot reduce the difficulty which results from the fact that all our telescopes are immersed in the great sea of our own atmosphere, and it is doubtful if the building of larger telescopes will enable us to see any better the details on other planets. The advantage to be hoped for is rather in their light-gathering power or speed, which, it was pointed out in the last chapter, is proportional to the area of the objective or the square of its diameter.

169. The Resolving Power of a Microscope. Whereas the resolving power of a telescope can be increased indefinitely by increasing the diameter of its objective, it is easily shown that no such possibility exists for the microscope. In securing the higher magnifications with a compound microscope, very short-focus objectives must be used. In Fig. 156 such an objective is shown very close to the two objects P and P_1 which it is hoped to resolve. Let a be the distance from P to P_1. Two rays from P, PNF and PN_1F, are shown, which converge toward its geometrical image F. Similarly the corresponding dotted rays indicate the way in which the geometric image of P_1 is formed. The rays PN_1 and PN, however, give rise to bundles of diffracted rays at N and N_1, among which are the dotted rays NF_1 and N_1F_1, which trace the same paths as the rays originating in P_1. These diffracted rays will, as with the telescope, produce diffuseness in the images at F and F_1 and series of concentric rings. If the images F and F_1 are too close together, they will accordingly be indistinguishable. If P and P_1 are so illuminated that simultaneous wave crests move from them toward the lens, and if a is one-half wave-length, the diffracted ray PNF_1, having had approximately one-half wave-length less to travel than the ray P_1NF_1, will arrive one-half period ahead of it. Similarly the diffracted ray PN_1F_1 will arrive one-half period be-

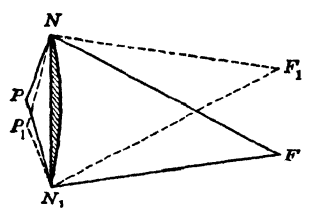


FIG. 156. Effect of Diffraction on Microscope Image.

hind the ray $P_1N_1F_1$. It will accordingly arrive a full period later than the diffracted ray NF_1 and a half period later than a ray from P which is diffracted at the midpoint of the lens. Just as in the case of the telescope lens, therefore, the rays from the lower half of the lens will exactly cancel pair by pair those from the upper half of the lens to produce darkness at F_1. For a separation of objects of one-half wave-length, therefore, the geometrical image of P_1 will fall approximately on the first dark diffraction ring of the image of P, and will be indistinguishable from it. It is accordingly impossible with any microscope to resolve objects which are appreciably closer together than a wave-length of light, and no increase in the aperture of the objective can help this situation.

The only improvement possible is accomplished by using a shorter wave-length, either by utilizing the extreme violet end of the spectrum and making the observations photographically or by inserting an oil of high refractive index between the object and the microscope lens. Since the velocity of propagation of light in this oil is less than in air, and since the frequency of the light disturbance cannot be altered by passing through such a medium, the wave-length must be shortened in accordance with the fundamental relation $V = \lambda n$. The increase in the effectiveness of the microscope resulting from either or both these expedients is not very great, and we are forced to recognize an absolute limit beyond which the microscope can never hope to go.

170. The Diffraction Grating. Of scarcely less importance than the telescope and the microscope in the wealth of information which it has gathered from Nature's treasure house, is the *diffraction grating*. This consists in its simplest form of a plane plate of glass, on which are scratched a large number of parallel lines. Such a grating will pass light freely through the clear spaces between the scratches, and may be thought of as a series of slit-shaped apertures very close together. As ordinarily used, it is placed in the path of a beam of light furnished by a *collimator*. This last consists of a telescope objective C (Fig. 157) having at its principal focus a narrow slit

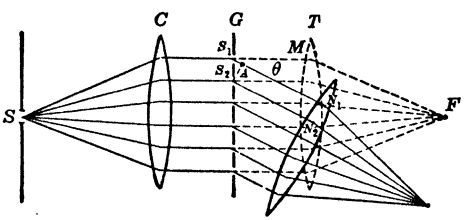


FIG. 157. Diffraction Grating.

perpendicular to the plane of the paper. From each point of this slit a bundle of rays travels to C, where it is so refracted that all the rays of the bundle emerge parallel to one another. In particular, a bundle of rays which originates on the axis of the collimator will emerge parallel to this axis. If this beam of parallel rays is now incident normally on the grating G, each ray in passing through one of the grating slits will give rise to a bundle of diffracted rays. Beyond the grating is a telescope T containing a cross hair at its principal focus. Any beam of parallel rays, incident on the telescope objective, will be focussed to the plane of the cross hair, but only when these rays arrive in phase, will they result there in a bright spot.

Among the many diffracted rays, those which emerge from the grating perpendicular to its plane will be brought to a focus on the extension of the axis of the collimator. The dotted picture of the telescope in Fig. 157 shows the position in which it must be placed in order to receive these rays. In this position the beam which originated on the axis of the collimator will be brought to a focus on the axis of the telescope, while those which originate at other points of the slit will be brought to a focus in the same plane, but in such a way as to form there a complete image of the slit.

If, however, the telescope occupies the position shown in the full-line drawing, such that the distance traversed by the diffracted ray S_1N_1 from one slit is just one wave-length greater than that S_2N_2 traversed by the parallel ray from the next slit, these diffracted rays will arrive at the focal plane of the telescope in phase, to produce once more a bright spot. If monochromatic light of wave-length λ is incident on the slit of the collimator, there will be formed at this point another image of the slit. The angle through which the telescope must be moved from the position A to the new position B in order for the cross hair of the telescope to lie on the new slit image is readily calculated. If we denote the angle MS_1N_1 by θ, $\theta = AS_2S_1$, since S_1N_1 is perpendicular to S_2A, and MS_1 is perpendicular to S_1S_2. Hence $\sin \theta = \lambda/d$, if d is the distance between successive apertures of the grating.

If, instead of monochromatic light, a mixture of several different colors is present in the incident light, an image of the slit will be formed for each wave-length, and if all colors are present as in white light, these images will blend continuously

into one another to form a continuous spectrum. Additional images of the slit will be formed for each color at angles for which S_1A is any integral multiple of λ. Or

$$\sin \theta = \frac{n\lambda}{d} \tag{154}$$

If $n = 1$, the case which has already been discussed, the spectrum is known as the *first-order* spectrum, for $n = 2$, it is the second-order, and for $n = n$ it is the nth-order spectrum. From a knowledge of d, the separation of successive lines of the grating, and from θ, it is possible to calculate the wave-length of the light producing any particular image. The sharpness of the image and the consequent accuracy with which θ can be measured is dependent simply on the aperture of the telescope (or that of the grating if it happens to be smaller than the telescope), the width of this somewhat diffuse image being, as in the case of a telescope star image, inversely proportional to the diameter of the aperture. If two wave-lengths which are very close together are present in the incident light, they will produce two distinguishable slit images if they are sufficiently separated in the focal plane of the telescope, and if the images themselves are sharp enough. It is easily seen from Formula 154 that the separation of the images is very nearly inversely proportional to the distance d between grating lines, or, what amounts to the same thing, it is directly proportional to the number of lines per centimeter. The sharpness of the image, however, is proportional to the length of the grating, or at least to the length which is utilized. What we may call, therefore, the resolving power of the grating, its ability to distinguish wave-lengths which are very close together, is proportional to the product of the number of lines per centimeter and the length of the grating, i.e., to the total number of its lines.

It was Thomas Young who first outlined the theory of the grating and showed its value for measuring wave-length. Professor Henry A. Rowland at Johns Hopkins University became interested in the problem of manufacturing such gratings during the early 'eighties of the last century, and, by the construction of a remarkable dividing engine for the purpose, he was enabled to make gratings of almost unbelievable accuracy and fineness, ruling as many as 1700 lines to the millimeter.

PROBLEMS

1. If the thickness of the emulsion of a plate used for the Lippmann process of color photography is 0.01 mm., how many layers of reduced silver grains will there be at a point where the wave-length of the incident light is 6×10^{-5} cm.?

2. Calculate the distance of the first subordinate bright spot in the Young pinhole experiment if the distance apart of the apertures is 0.1 mm., the distance to the screen is 1 meter, and the wave-length is 5.89×10^{-5} cm.

3. Calculate the resolving power of (*a*) a binocular objective having a diameter of 3 cm., and (*b*) the Mount Wilson telescope with a 100-in. aperture. What is the ratio of the light received by the Mount Wilson telescope to that received by (*a*) the eye, and (*b*) the binocular objective of the first part of the problem.

4. Calculate the angle at which the first-order image of the sodium line of wave-length 5.890×10^{-5} cm. is formed by a grating with 5000 lines to the centimeter. What will be the angular separation of the images of the two adjacent sodium lines at 5.890 and 5.896 in the first- and third-order spectra respectively?

CHAPTER XXXI

POLARIZATION

In Chapter XXV, examples were given of two types of wave motion, longitudinal and transverse. With the wave nature of light definitely indicated by interference experiments, it is natural to inquire further as to the character of the wave.

171. Double Refraction and Polarization. The first evidence of a disturbance more complicated than that involved in sound propagation was the discovery in 1669 by Erasmus Bartholinus in Copenhagen of *double refraction* in a crystal of Iceland spar. All objects when viewed through such a crystal in a direction oblique to the optic axis are seen double. Despite many efforts to gain an insight into the cause of this phenomenon, no progress was made until early in the nineteenth century when Etienne Louis Malus, a French engineer, made by chance an observation which led to the correct conclusions. The French Institute had proposed as a prize question a mathematical theory of double refraction, and Malus, having decided to enter the competition, was examining a specimen of Iceland spar. He happened to look through the crystal at an image of the Sun reflected from a window of the Luxembourg Palace, and was surprised to note that as he rotated the crystal about the direction of the incident light, first one image of the Sun and then the other disappeared. He soon persuaded himself that the phenomenon was due to the fact that the light reflected from the window was fundamentally different from ordinary sunlight. Further investigation showed that no double refracting crystal was necessary to detect this changed character of the reflected light. If, for example, light reflected

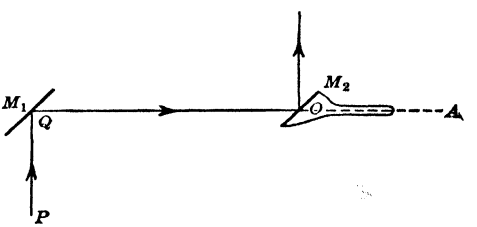


FIG. 158. Polarization of Light by Reflection.

from one mirror M_1 as shown in Fig. 158 was again reflected from a second mirror M_2, the intensity of the resulting beam depended on the orientation of the mirror M_2. If, in particular, it was so oriented that the resultant beam emerged perpendicular to the plane of the paper, its intensity was a minimum, and if the ray fell on the two mirrors at one particular angle of incidence, no light whatever was reflected. This angle is known as Brewster's angle in honor of the British investigator who did much of the early work in this field of study. A mere rotation of the mirror M_2 about an axis OA in the direction of the incident light causes, therefore, the reflected beam to alternate in intensity over a wide range.

The early explanations of this phenomenon were based on a corpuscular theory of light, and the initial failure of the wave theory to account for such a curious effect weighed heavily against it in the early years of the nineteenth century. Once more it was Young and Fresnel who independently suggested that the phenomenon merely demonstrated the transverse nature of the light disturbance. For, if the vibration which constitutes the light wave was perpendicular to the direction of propagation, it was to be anticipated that the mirror M_1 (Fig. 158) would reflect to a quite different extent vibrations in the plane of and perpendicular to the paper respectively. If the initial ray PQ consisted, therefore, of light vibrating in all directions perpendicular to PQ, and if M_1 reflected most copiously the vibrations perpendicular to the paper, the ray QO would contain a larger proportion of such vibrations. If the mirror M_2 were oriented as shown in the figure, it would be in a position to reflect best precisely these same vibrations, but if it were turned through 90° in such a way as to direct the reflected beam upward from the paper, reflection would be much less complete. The fact that for a given angle of incidence the intensity of the reflected beam is actually zero indicates that for this particular angle the mirror fails completely to reflect one of these two types of vibration. The reflected light is completely *polarized*, i.e., it is vibrating in only one particular direction. The plane passing through the direction of propagation and the direction of vibration is known as the *plane of polarization*. This experiment and all other purely optical experiments give no indication as to whether the actual direction of vibration in the ray QO is in the plane of the paper or perpendicular to it.

172. Double Refraction a Consequence of the Transverse Nature of Light. Once the transverse nature of the light disturbance was appreciated, it was possible to develop a satisfactory picture of the phenomenon of double refraction. The crystals which exhibit this phenomenon are all *anisotropic*, i.e., certain of their properties are different along different directions in the crystal. Thus, for example, the elastic constants, which measure the ease with which a material can be compressed or stretched, have different values when the deformation is produced in the direction of the *crystallographic axis* or at right angles to it. Such a crystal has, in fact, many properties similar to those of a block of wood, the axis of the crystal corresponding to the direction of the grain of the wood. The velocity of propagation of an elastic wave depends on the magnitude of these elastic constants, and is, therefore, different when the deformation associated with the wave takes place in the direction of the

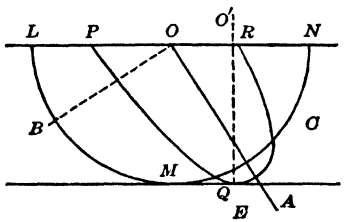


FIG. 159. Double Refraction of Light.

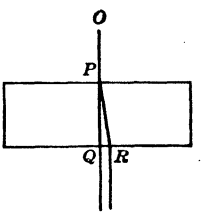


FIG. 160

axis or at right angles to it. It is not surprising, therefore, that the velocity of propagation of a light wave through a crystal should be different in different directions. This concept is, in fact, all that is required for an understanding of double refraction.

Consider, for example, the particularly simple case of a light wave which starts at the point *O* on the boundary of the crystal *C* of Fig. 159. The direction of the crystallographic axis is indicated by the line *OA*. If we picture the light as a transverse disturbance, we limit the direction of vibration to a plane perpendicular to the direction of propagation. The light spreads into the crystal from *O* in all directions, but the vibration must everywhere be at right angles to the direction in which the wave is progressing. Whatever may be this direction of vibration, we can resolve it into two components, one perpendicular to the paper and the other in the plane of the paper. The vibration

perpendicular to the paper will always be at right angles to the optic axis regardless of the direction from O in which it is travelling. It will, therefore, advance with the same velocity in all directions, and the trace of the wave front in the plane of the paper will be a circle LMN. The vibration in the plane of the paper, on the other hand, will be perpendicular to the axis OA where the wave is advancing in the direction OA, but it will be along the axis where the wave is travelling in the direction OB. Its velocity of propagation will accordingly be different in these two directions. In the case of Iceland spar the velocity is greater in the direction of the crystallographic axis, and the shape of the resulting wave PQR is elliptical with the major axis in the direction OA. The circular wave, due to a vibration at right angles to the optic axis, is known as the *ordinary* wave; the elliptical wave is called the *extraordinary* wave. Only a small portion of the elliptical wave about Q will enter the pupil of an eye at E. This small portion is approximately spherical with its center on the line $O'E$, and the disturbance will accordingly appear at E to come from this direction. The ordinary wave, on the other hand, will appear to come from O, and since both waves enter the eye of the observer, he will see two images of the source, in one of which, however, the light is polarized in the plane of the paper and in the other at right angles to the plane of the paper.

173. The Nicol Prism. A more complete analysis leads to similar conclusions when the source of the light is no longer thought of as in contact with the crystal. Using the terminology of light rays instead of light waves, a ray of unpolarized light, which, starting from O (Fig. 160), is normally incident on the crystal at P, will be split into two rays, an ordinary ray PQ, which advances exactly as it would in an isotropic medium, and the extraordinary ray PR. The rays emerge parallel to each other, the distance between them QR depending on the thickness of the crystal. The fact that they travel in different directions through the crystal makes possible a convenient method for producing completely polarized light. Fig. 161*a* shows the shape of a natural crystal of Iceland spar with the optic axis indicated by a dotted line. The natural cleavage planes are such that a rhomb can be split out as indicated by the lighter lines within the crystal. When this is done, the acute angle (Fig. 161*b*) between the two faces is 71°. The opposite faces are now ground off as shown in Fig. 161*c* to an angle of 68°. If the

crystal is now cut through diagonally along the line *MN*, and the two parts are cemented together with Canada balsam, a prism is produced, which in honor of its inventor is called a *Nicol prism.* A ray of light impinging on the upper surface of the prism is split into an ordinary ray *PQ* and an extraordinary ray *PR.* The ordinary ray with its greater refractive index is more intensely refracted on entering the crystal, and encounters the layer of Canada balsam at such an angle that it suffers total reflection. The extraordinary ray, on the other hand, passes through the other half of the prism. The result is a beam of completely polarized light.

If such a beam is incident on a second nicol, it will pass through with very little further loss of intensity if the second prism is oriented in the same way as the first, but it will be completely

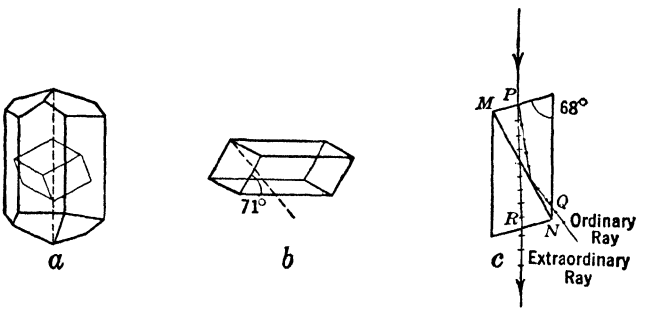


FIG. 161. Construction of Nicol Prism.

blocked if the second prism is now rotated through an angle of 90°. The first prism of such a combination is known as the *polarizer*, and the additional prism, because of its ability to distinguish thus between polarized and unpolarized light, is called the *analyzer.* Two prisms oriented in such a way that the light passed by the first is blocked by the second are called *crossed nicols.* If the second prism is so turned that it is neither at right angles to nor parallel to the first, it automatically resolves the incident vibration into two components, one of which it passes and the other of which it blocks, in exactly the same way that the first prism picks out vibrations in one particular plane from the unpolarized light.

174. Circularly and Elliptically Polarized Light. If plane polarized light from a nicol is allowed to fall on a natural crystal, let us say Iceland spar, in a direction at right angles to the optic

axis, the component of the incident vibration in the direction of the optic axis will be propagated through the crystal with a different velocity from that at right angles to the axis. The incident polarized light will, therefore, be split into two plane polarized components travelling with different velocities through the crystal. Fig. 162 shows how a single vibration can be resolved in this way into two vibrations at right angles to each other. The initial motion can be represented by the oscillation of the point P back and forth along the line PQ. Its displacement from O at any one instant can be resolved into a horizontal displacement OP_1 and a vertical displacement OP_2. As P oscillates between P and Q, P_1 oscillates between P_1 and Q_1 and

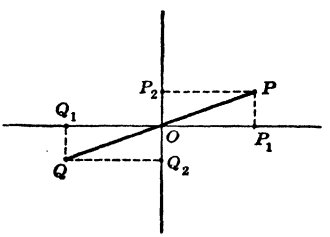


FIG. 162. Resolution of Vibration into Components.

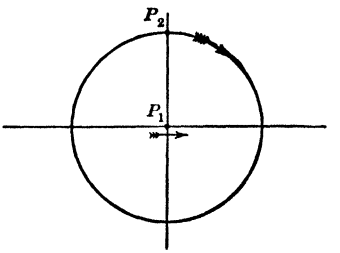


FIG. 162*a*

P_2 between P_2 and Q_2. These latter two vibrations are the component vibrations into which the initial one is resolved. It should be noted that the maximum displacements of P_1 and P_2 occur at the same instant as the maximum displacement of P. The two component vibrations are in phase. If, however, the vertical component is propagated through the crystal with a greater velocity than the horizontal component, P_1 will lag more and more behind P_2 the further we pass into the crystal, until a point is eventually reached at which this lag amounts to a quarter cycle, and P_1 will be at the midpoint of its path when the displacement of P_2 is a maximum. Fig. 162*a* shows the way in which the two components will combine at this particular point. Since the displacement of P_1 at that instant is 0, the point P will lie on the vertical axis at P_2. An instant later, however, the point P_2 will have started downward, and the point P_1 will have moved to the right. The resultant point P will accordingly move outward into the upper right-hand quadrant. When P_2 reaches O, P_1 will lie at its maximum displace-

ment to the right, and P will be located on the horizontal axis. It is easily shown that, if the two component oscillations are sinusoidal and equal in magnitude, the point P will describe a circle about O as a center, but if they are unequal, the path of P will still be an ellipse. Such a disturbance is said to be *circularly polarized.*

If the light impinging on the surface of the crystal is plane polarized, the difference in phase between its two components, initially zero, will increase as it penetrates deeper and deeper into the crystal until at a particular distance the angle which measures this phase difference is 90°, and the light is circularly polarized. As it penetrates still deeper, the phase angle continues to increase until at 180° it is once more plane polarized. At intermediate points it is said to be *elliptically* polarized. The thickness of crystal which is required, however, to produce a given phase difference depends on the wave-length of the light, and if the crystal is of such a thickness that the emerging light is plane polarized for one color of light, it will still be elliptically polarized for all others. Accordingly, if such light is analyzed by passing it through a nicol prism so oriented as to block the plane polarized light, the emerging component of the elliptical vibration will appear colored.

Among the many applications of this phenomenon is the common method of testing optical glass for strain. Glass, although ideally isotropic, when subjected to strain of any kind becomes doubly refracting with the optic axis in the direction of the strain. If polarized light from a nicol prism is passed through such a strained specimen of glass in a direction perpendicular to the strain, it will undergo progressive changes into elliptically and circularly polarized light, and when examined by a second nicol, will exhibit characteristic coloring. Since only by the most careful annealing is the formation of strain in optical glass to be avoided, this means of rapidly testing specimens is of great value.

175. Rotation of the Plane of Polarization. In case the horizontal component of the light represented in Fig. 162*a* travels more rapidly than the vertical component, the resultant point P will describe its circle in an anticlockwise or *left-handed* sense, whereas if the vertical component advances with the greater speed, the circle will be described in a clockwise sense. The former type of circular polarization is called *left-handed*; the latter is known as *right-handed.* Although a beam of polar-

ized light travelling at right angles to the optic axis passes through successive stages of plane, elliptical, and circular polarization in this fashion, a beam moving in the direction of the optic axis retains the character of polarization which it possessed on entering the crystal. Strange as it may seem, however, although vibrations in all directions perpendicular to the optic axis are propagated with the same velocity, there exist certain crystals, such as quartz, which propagate right and left circularly polarized light in the direction of the optic axis with different velocities. There are also a number of liquids, such as sugar solutions, which, although isotropic, possess the same property. This phenomenon is due to a basic right- or left-handedness of the crystal form when viewed along the crystallographic axis in the case of the solid crystal, and to a similar right- or left-handedness of the molecular form in the case of a solution.

The importance of this observation lies in the fact that a beam of plane polarized light is equivalent to a right circularly polarized beam superposed on a left circularly polarized beam of the same frequency and intensity. This is readily seen from Fig. 163. The left circularly polarized disturbance is represented by the point P_1 describing the circle P_1AB in an anticlockwise sense; the right circularly polarized disturbance is represented by P_2 describing the circle in the opposite sense. If P_1 passes P_2 at a point on the horizontal axis, and if they describe the circle in the same time, their vertical components will always cancel each other, and the resultant motion will be plane polarized along the horizontal axis. Since two such equal and opposite circular motions are equivalent to a plane oscillation, it must be conversely true that any plane oscillation can be thought of as the resultant of two circular motions, and if the plane disturbance is incident on a crystalline medium through which its two circular components travel with different velocities, they will be automatically separated, and will travel along independently. If the motion represented by P_1 advances through the crystal more rapidly than

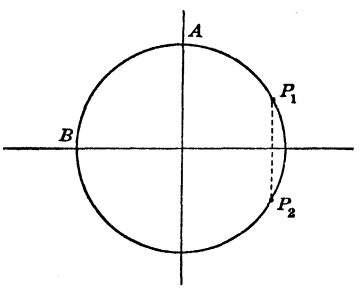


FIG. 163. Resolution of Plane Polarized Light into Two Circularly Polarized Beams.

that represented by P_2, it will farther along in the crystal reach the horizontal axis sooner than P_2, and will pass P_2 at some point above the axis. Since the resultant plane of polarization is determined by this point at which P_1 and P_2 pass, it is clear that at this point in the crystal the plane of polarization no longer lies in the direction of the horizontal axis but makes with it a small positive angle. The crystal has effected an anticlockwise rotation of the plane of polarization. The amount of rotation must clearly be proportional to the distance which the disturbance has travelled in the rotating medium. This fact is easily checked experimentally.

The rotation of the plane of polarization by a sugar solution is regularly used to estimate the purity of ordinary commercial sugars. The instrument designed for this purpose, which in its simplest form consists of a polarizing and an analyzing nicol prism with space between for a tube containing the sugar to be tested, is known as a *saccharimeter*.

CHAPTER XXXII

ELECTRIC WAVES

In the preceding chapter the disturbance which characterizes a light wave has been rather loosely designated as a vibration. The fact that light is capable of travelling through space which is to all appearances empty, makes the reality of any such actual vibration more than dubious. Nevertheless the workers of the nineteenth century were so impressed by the wavelike properties of light that they readily assumed the existence of a *luminiferous ether* which pervaded all space whether empty or not, and in which the light vibrations were pictured as actually taking place. The apparent absence of a longitudinal or compressional wave such as that of sound was explained by attributing to this ether complete incompressibility. The actually observed transverse wave was thought of as an elastic *shearing* of the medium, and the huge distances through which light passes through interstellar space without appreciable diminution were associated with a theoretical perfect elasticity of the ether.

176. The Transition from an Elastic Solid to an Electromagnetic Ether. The introduction of the electromagnetic theory of light by Maxwell changed this picture to a certain extent without removing the intuitive necessity for an ether. From serving as the vehicle for propagating an elastic disturbance it came to be viewed as the seat of the electric and magnetic forces which from the newer standpoint constituted the light wave. Consider first the picture which the electromagnetic theory gives us of the wave sent out from a vertical radio antenna (Fig. 164). An electric current oscillates up and down in this antenna at a frequency of perhaps 1,000,000 cycles per second. If a steady current were flowing upward in it, we know that at all points in the vicinity there would be an encircling magnetic field. At the point P, for example, the direction of this field would be that of the tangent to the circle MN at P. This direction is horizontal, i.e., perpendicular to the vertical antenna and perpendicular also to the shortest line PO drawn from P to the antenna. A natural question to ask is whether, if the steady current were suddenly

established in the antenna, the appropriate magnetic field would instantaneously appear at all distances away, or if it would travel outward as a sound wave moves out from the source of a sound disturbance. The mathematical theory of Maxwell unqualifiedly predicts the latter of these two alternatives, and sets the velocity of propagation of the wave as equal to the ratio of the electromagnetic to the electrostatic unit of charge. This ratio, as was pointed out in Chapter XXIII, is experimentally equal to the known velocity of light.

If, in place of the suddenly established steady current in the antenna, the current is allowed to vary in an approximately sinusoidal fashion, as it does in practice, we should have a maximum magnetic field passing outward at a certain instant, succeeded by a field which varies sinusoidally with the current. At any one point in space such as at P the maxima of magnetic field will succeed one another at a rate equal to that of the exciting oscillations in the antenna. Any one particular maximum or wave crest, as it might be called, will travel outward with the velocity of light. In order to complete the picture it is only necessary to imagine in addition to the magnetic field an electric field at right angles both to the magnetic field and to the direction of propagation. At the point P, for example, the magnetic field will be horizontal and tangent to MN, whereas the electric field will be vertical, i.e., parallel to the antenna. The theory further predicts that such waves will be reflected at a conducting metallic surface, and that the reflected wave will interfere with the incident wave to produce loops and nodes exactly as sound and light waves are known to do.

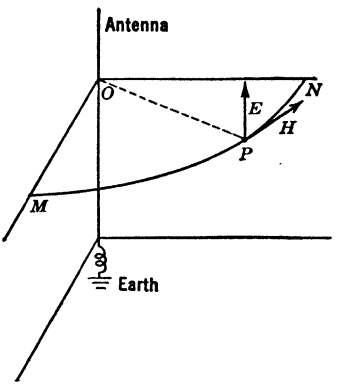


FIG. 164. Electromagnetic Field about an Antenna.

It is to be noted that these are transverse waves such as the phenomenon of polarization has shown light to be. Both the electric and the magnetic fields are at right angles to the direction of propagation. A plane drawn through either of these two fields and the direction in which the disturbance is advancing

could be thought of as the plane of polarization. As we shall see, most of the experimental and physiological effects of a light wave are due to the electric field, and it would seem more natural for it to determine the plane of polarization. In contrast to the disturbance from an oscillating antenna, ordinary light is unpolarized, i.e., electric and magnetic fields occur in all directions at right angles to the direction of propagation. The broadcasting stations which transmit to us the light waves are myriads of atoms in the luminous body, and although the disturbance sent out from each may very well be polar, the resultant wave should clearly be unpolarized.

177. The Generation and Detection of Electromagnetic Waves in the Laboratory. Heinrich Hertz. Despite the fact that the electromagnetic theory was developed in detail by

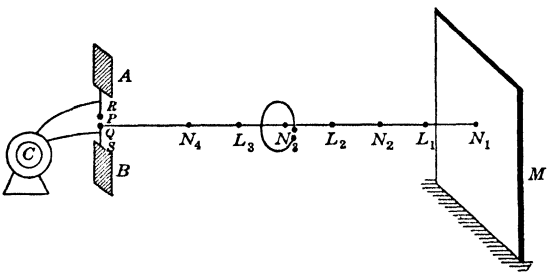


FIG. 165. Interference of Electromagnetic Waves.

Maxwell as early as 1865, nearly a quarter of a century was to pass before any experimental evidence was obtained for the reality of these electric waves. It was Heinrich Hertz who first generated and detected such waves in the laboratory. His original generating apparatus (Fig. 165) consisted of two metal plates, A and B, each 40 cm. square, to which were attached rods, R and S, 30 cm. long, terminated in little spheres, P and Q. These spheres were connected by wires to the terminals of an induction coil C. Such an induction coil consists of an iron core, about which is wound a primary coil containing a few turns of heavy wire and a secondary coil with many turns of finer wire. A buzzer, placed in the primary circuit, periodically interrupts the primary current. The sudden resulting reduction in the magnetic field through the core produces a momentary high voltage in the secondary. The two plates A and B constitute a condenser, which, because of the wide separation of the plates,

has a rather small capacity. The momentary high voltage, established between these plates by the induction coil, charges the condenser, but the width of the gap between the spheres P and Q is so adjusted to the potential supplied by the coil that the condenser, once charged, begins to discharge by a spark across the gap. Owing to the inductance of the rods R and S, the discharge is oscillatory with a frequency determined by the shape and dimensions of the apparatus. This frequency for the given dimensions was about 32,000,000 oscillations per second.

From such an oscillator vertically polarized waves should travel outward in accordance with the electromagnetic theory. In order to detect these waves, Hertz used a simple loop of wire (Fig. 166) terminating in a spark gap G. The inductance and capacity of such a loop 35 cm. in radius were such as to make it resonate to the oscillations from the generator. If the loop was placed with its plane perpendicular to the line drawn from loop to generator as shown in Fig. 165, the electric field, if sufficiently intense, would produce sparking at the gap. In order to prove the wave nature of the disturbance, it was only necessary to set up a metallic reflector M and explore with the loop the region in front of the reflector. When this was done by Hertz, he found a series of loops and nodes of electric intensity exactly as predicted by the theory. At the reflector itself, N_1, the absence of sparking in the loop indicated a node, and additional nodes were found at N_2, N_3, etc., at distances of every half wave-length between M and the oscillator. Midway between consecutive nodes he discovered loops L_1, L_2, etc., at which sparking was a maximum. For the apparatus described, the distance between successive loops or nodes, i.e., the half wave-length, was found to be 4.8 meters. Hertz attempted to calculate the frequency of the oscillator from its shape and dimensions, and so to determine the velocity with which the waves were propagated from the familiar relation, $V = \lambda n$. Although he was unable at the time to obtain very precise results in this way, the velocity obtained was about the order of magnitude of the known velocity of light, and later experiments have shown this relationship to be exact.

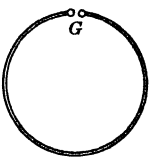


FIG. 166.
Hertz's Detector.

178. Properties of Electric Waves Analogous to Light. These first experiments of Hertz were performed in 1888. The next

few years were spent by Hertz himself and many others in efforts to obtain more intense waves. From analogy with optical experiments it seemed desirable to concentrate large beams of these electric waves on the detector by means of concave mirrors. The initial difficulty lay in the great length of the waves first used and the consequent size of the mirrors required. Gradually, however, the size of the oscillator and the resulting wave was reduced until the feasibility of such reflectors was demonstrated beyond a doubt. Hertz himself, using as oscillator two simple rods with a total length of 26 cm. (Fig. 167) mounted along the axis of a cylindrical mirror M_1, obtained waves only 66 cm. in length, which he detected by a similar pair of rods placed at the focus of a second mirror M_2. Such a receiver signalled the arrival of waves by sparks leaping across the tiny gap G.

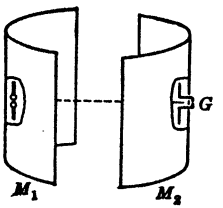


FIG. 167. Parabolic Reflectors for Short Electromagnetic Waves.

With this oscillator he was able to reproduce many of the familiar experiments of optics. By rotating the receiving mirror about the line joining it to the oscillator, he was able to prove the polar nature of the wave (Fig. 168). When the axes of the two mirrors were parallel, a lively sparking took place at the receiver, but when they were turned at right angles to each other, no waves were detected. This was even more vividly shown by interposing between oscillator and detector a grid of parallel wires (Fig. 169). When the wires were parallel to the reflector axes, no sparking took place, the entire energy of the wave being absorbed or reflected from the grid, but when rotated to a position at right angles to the reflectors, the grid had little effect in stopping the waves. Its behavior was quite analogous

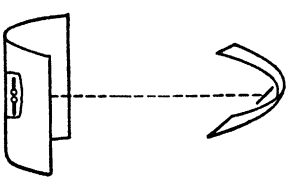


FIG. 168

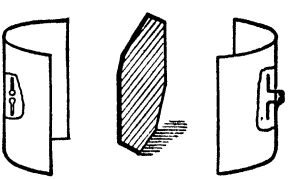


FIG. 169

Proof of Polar Character of Waves.

to that of a nicol prism in the path of a beam of plane polarized light. Finally he found that he could refract these electric waves by passing them through a huge prism of pitch, which he made by casting the pitch in a wooden mould.

In 1897, Rhigi at the University of Bologna used two spheres only 8 mm. in diameter in place of the rods of the Hertzian oscillator, and by immersing them in oil he was able to get intense enough oscillations to perform many interesting experiments. The waves obtained with this apparatus were only 2.6 cm. in length. They were reflected by small coins, suffered refraction and total internal reflection in prisms of sulphur and paraffin, and were doubly refracted by blocks of fir wood. Lebedew by similar methods obtained waves only 6 mm. long, with which the double refraction of natural crystals could be shown.

Within recent years there has been a revival of interest in the attempt to secure still shorter waves. The method used has been in general to set up oscillations among a large number of tiny spherical particles imbedded in a heavy oil. The use of many different particles together with continuous stirring of the oil has eliminated to a large extent the difficulty formerly experienced in the carbonizing of the spheres. Among the overtones sent out by such oscillating systems Glagolewa-Arladjewa in Russia reported in 1924 the detection of waves as short as 0.04 mm.

Such experiments have very naturally given rise to an almost unbounded confidence in the fundamental correctness of the electromagnetic theory of light. Seldom has man had the temerity to build such a colossal theoretical structure as Maxwell did in the early 'sixties, and certainly no similar structure has ever been more convincingly supported by experimental evidence such as that accumulated by Hertz and his successors. The new discoveries of the first quarter of the twentieth century have shown, however, that, though perhaps correct as far as it goes, the theory is inadequate. Light has properties which seem to find no place in such a picture, and it is to a description of these strange properties that the major portion of the next few chapters is devoted.

CHAPTER XXXIII

RELATIVITY

In order to make clear the wave characteristics of electromagnetic radiation, we have probably overemphasized the similarity between light and sound. Whereas sound is propagated in a medium, the air, with the chemical composition and physical properties of which we are familiar, the ether in which light waves are propagated is much more mysterious. Unlike sound waves, these have never been photographed, and even the calculation by Maxwell of their velocity involved no precise knowledge of the nature of the medium in which they move. Indeed, the only real evidence which we have for the existence of this ether is the phenomenon of interference and the wave nature of radiation which it suggests.

179. The Synchronization of Clocks in a Moving Laboratory. Toward the close of the nineteenth century it seemed likely that the rapid development of experimental technique would make possible some more tangible grasp on the ether. In order better to appreciate this possibility, let us consider for a moment the apparently simple problem of comparing two watches on board a moving freight train. One member of the crew at the forward end of the train has agreed in advance to notify his comrade at the rear when his watch indicates precisely the stroke of noon. At exactly the proper instant he calls back to his friend, who confidently sets his own watch by the signal. Has he obtained in this way the correct time? You answer that he has approximately, but that a small error has been introduced by the time required for the sound of his friend's voice to travel the length of the train. Assuming, for example, that the train is 1088 ft. long, it would take the sound approximately 1 sec. to travel this distance, and the man at the rear of the train would have set his watch 1 sec. late. There is, however, still another error introduced by the motion of the train. The sound is travelling in air, and the forward motion of the train, by carrying the listener forward during the passage of the sound, causes him to hear it a little sooner than the estimated second. If the train is travelling

60 miles an hour, i.e., 88 ft./sec., and if t is the time which elapses between the transmission of the signal and its reception, it is clear that the distance which the sound has travelled during this time, $1088t$, is less than the total length of the train by the distance, $88t$, which the man at the rear has advanced. Or

$$1088t = 1088 - 88t$$

Hence

$$t = \frac{1088}{1176} = 0.925 \text{ sec.}$$

It should be noted that, in order to synchronize the two watches accurately, a knowledge of the velocity of sound in still air and of the velocity of the train is required. If the two men were in box cars which hid from them the land through which they were moving, and if they had no other means of estimating the speed of the train, an accurate synchronization of their watches would be impossible.

It might be objected that this is a silly way to synchronize watches, that the signal should be sent by a motion of the arm, i.e., by light and not by the slow sound wave. This is undoubtedly true for the practical purpose of setting a watch on a moving train, but it would solve none of the difficulties if watches were to be synchronized over much greater distances, or if a much higher degree of precision was required. The common method of sending out radio signals from Washington to all parts of the country, for example, involves an error of an appreciable fraction of a second, the time required for the signal to travel from transmitter to receiver with the velocity of light. Furthermore, if the Earth is travelling through the ether as the train was moving through the air, the same sort of correction would have to be made for this motion. Here, however, we encounter difficulty, for the velocity of the Earth through the ether is unknown. True, a point on the surface of the Earth, owing to the rotation of the Earth about its axis, is travelling 25,000 miles a day or about $\frac{1}{3}$ mile a second, and owing to its revolution about the Sun it is moving about 18 miles a second. We know, however, that the Sun is moving with reference to other fixed stars, just how fast we cannot say, and we have no basis for an estimate of the rate at which these stars are moving. In the absence of such information, it is accordingly impossible to say whether two clocks in New York and Paris, for example, are in synchronism or not. The likely error is, of course, only a fraction of a second,

but if some day we should attempt such a synchronism between the Earth and a distant planet, the uncertainty might be considerable.

Let us suppose, however, that the two members of the train crew had in some way previously synchronized their watches so that they had confidence in their accuracy. It should be theoretically possible to use the time required for the passage of the sound signal as a measure of the speed at which the train is moving. Let V_0 be the known velocity of sound in still air and V its apparent velocity on the train, obtained by dividing the length of the train by the time required for the signal to traverse this distance. If the sound travels a distance s_a through the air in time t, and if the train advances the distance s_t in the same time, the apparent velocity of the sound will be

$$V = \frac{s_a \mp s_t}{t}$$

the $-$ or $+$ sign depending on whether the sound is travelling in the direction in which the train is moving or in the opposite direction. Since $s_a/t = V_0$ and $s_t/t = v$, the velocity of the train,

$$V = V_0 \mp v$$

Hence $$v = V - V_0 \quad \text{or} \quad V_0 - V \tag{155}$$

180. The Michelson-Morley Experiment. It is substantially this experiment which scientific workers late in the nineteenth century proposed to use to measure the rate at which the Earth is moving with reference to the ether. The difficulty of the experiment lay in the minuteness of the amount by which even the great velocity of the Earth about the Sun might be expected to affect the measured velocity of light. From (155) the anticipated percentage effect is seen to be equal to the ratio of the velocity of the Earth, perhaps 18 miles per second, to the velocity of light, 186,000 miles a second, or 1/100 of 1 per cent. Yet even this minute amount should have been readily detected by an extraordinarily sensitive instrument, the interferometer, invented by Michelson, and used by him and Morley in 1887 for this purpose.

The Michelson interferometer (Fig. 170) consists of a fixed mirror A and a mirror B which is so mounted on a screw that it can be moved forward or backward in the direction BO. A

third glass plate O is slightly silvered, and is so mounted at an angle of 45° that a beam of light from the source S is partly transmitted in the direction of B and partly reflected toward A. After reflection at these two mirrors both beams return again to O, where the reflected portion of the beam from B and the transmitted portion from A pass on into the telescope T. If the paths traversed by these two beams is the same, or if they differ by an integral number of whole wave-lengths, they will reinforce each other in the focal plane of the telescope, but if they differ by an odd number of half wave-lengths, they will interfere to produce a null effect. In practice, if the instrument is in correct adjustment, and if the two paths are such as to produce reënforcement, the very center of the field of the telescope will be bright, but the bright spot will be surrounded by alternate light and dark rings. As the mirror B is moved slowly in or out, the central bright spot changes to black, and the light and dark rings have the appearance of expanding or contracting. When the mirror has moved a distance of one-half wave-length, thus increasing the total path of the light beam by one wave-length, the central spot will again be light. By counting the number of changes from black to white and back to black again, the number of half wave-lengths through which the mirror is moved can be accurately determined. Such an interferometer constitutes, therefore, an extraordinarily sensitive instrument for measuring length. The number of wave-lengths by which a metal rod expands when heated can, for example, be measured by mounting the mirror B on one end of such a rod, and counting the number of changes from light to dark in the center of the telescope field.

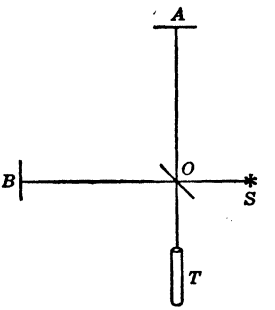


FIG. 170. Michelson Interferometer.

In order to apply this instrument to the measurement of the rate at which the Earth is moving through the ether, it was only necessary to mount it in such a way that it could be turned as a whole through an angle of 90°. It was initially oriented with OB in the supposed direction of motion of the Earth, and carefully adjusted. A rotation of 90° now brought OA into this direction and OB at right angles to it. The experiment consisted in an attempt to detect a change in the central spot of the inter-

ference pattern and tne accompanying change in position of the surrounding fringes. That such a change would be anticipated, is readily shown.

Let us suppose that the distance from O to A is equal to that from O to B. Let this distance be D. If the instrument were at rest, each beam of light would have to travel a distance $2D$ before returning to O. Since, however, the apparatus is moving in the direction OB with the velocity v, the distance travelled by the two beams is not equal. While the beam OB is travelling from O to B, the mirror B retreats a distance vT_1 if T_1 is the time required. If d_1 = the actual distance travelled by the beam,

$$d_1 = D + vT_1$$

If c is the velocity of light, $T_1 = d_1/c$

Hence $$d_1 = D + d_1\frac{v}{c}$$

Solving for d_1, $$d_1 = \frac{cD}{c - v}$$

Similarly if d_2 is the distance travelled by the beam in returning to O,

$$d_2 = \frac{cD}{c + v}$$

The total distance traversed is, therefore,

$$d = cD\left[\frac{1}{c - v} + \frac{1}{c + v}\right] = \frac{2c^2D}{c^2 - v^2} = \frac{2D}{1 - \frac{v^2}{c^2}} = 2D\left[1 + \frac{v^2}{c^2}\right]^{1} \quad (156)$$

While the beam OA is travelling from O to A, the mirror advances a distance vT_2 in a direction at right angles to OA. Here T_2 is the time required for the journey. The distance travelled is

$$d_1' = \sqrt{D^2 + v^2T_2^2}$$

But as before $$T_2 = \frac{d_1'}{c}$$

Or $$d_1' = \sqrt{D^2 + \frac{v^2}{c^2}d_1'^2}$$

[1] This last result, which is only approximate, may be obtained most simply by the long division of numerator by denominator, neglecting powers of v/c greater than the second.

Solving for $d_1'^2$, $$d_1'^2 = \frac{D^2}{1 - \frac{v^2}{c^2}}$$

and $$d_1' = \frac{D}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The return journey back to O must in this case be identical in length with the outward journey. Hence

$$d' = \frac{2D}{\sqrt{1 - \frac{v^2}{c^2}}} = 2D\left(1 + \frac{v^2}{2c^2}\right)^1 \qquad (157)$$

The difference in path between the two beams is

$$d - d' = 2D\left(1 + \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right) = D\frac{v^2}{c^2} \qquad (158)$$

The ratio of v^2 to c^2 is excessively small, but a velocity v only one-tenth the probable value of 18 miles per second should have been readily detected by a shift of the fringes when the Michelson interferometer was rotated. Actually no such shift was detected.

A number of explanations for the failure of the experiment have been suggested. The most obvious is that the Earth drags the ether with it exactly as a closed freight car carries its own air with it. It is difficult to reconcile such a behavior with the other known properties of the ether, but this viewpoint has received some support even so recently as 1921 when D. C. Miller in California obtained experimental results which he believed favored such an explanation. In 1895 G. F. Fitzgerald and H. A. Lorentz independently suggested the possibility that moving matter might contract in the direction of motion and thus obscure any actual motion with reference to the ether. By a comparison of (156) and (157) it is seen that the path traversed by light in the OB direction is to that traversed in the OA direction as 1 is to $\sqrt{1 - v^2/c^2}$. If, therefore, the distance D between O and B on the interferometer was reduced in this same ratio, the two paths would become equal, and no shift in the fringes would result. This *Fitzgerald contraction*, which on the surface seems so improbable, received a great deal of support from the investigations of Lorentz on the mechanics of moving electric charges.

[1] *Ibid.*, see page 312.

181. The Principle of Relativity. Nevertheless the most far-reaching suggestion, and the one which is now generally believed to give the correct explanation of the difficulty, was proposed by Albert Einstein in 1905. We have already pointed out that the laws of mechanics are such that any mechanical experiment, performed in a laboratory which is moving with a constant velocity, will come out exactly the same if it is performed in a stationary laboratory. The result of the Michelson-Morley experiment suggested that perhaps not merely mechanical experiments but all other physical experiments might be subject to the same relativity principle. This, in essence, was Einstein's great contribution. When, however, we attempt to apply it to the propagation of light, we encounter some rather astonishing results. Nevertheless, whenever an experimental test has been possible, such conclusions have been corroborated in a striking fashion.

182. The Lorentz Transformation. In order to understand these conclusions of the relativity hypothesis, imagine once more a railroad train passing by a station platform with velocity v (Fig. 171). The point O represents the location of the station agent on the platform, O' that of a brakeman on the train, OX a line on the station platform parallel to the direction of motion of the train, and $O'X'$ a similar line on the train. Let us imagine that O' passes O at exactly noon, and that at just this instant the station agent sends out a radio time signal, which, as we have seen, will travel outward with the velocity of light c. Every station employee and every trainman sets his watch by this signal, but each is careful to correct it for the time that it required to reach him. If each has done this correctly, the time of arrival of the signal at a station employee located at A, a distance x from O, will be

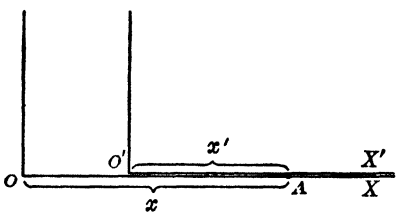


FIG. 171

$$t = \frac{x}{c} \tag{159}$$

and the time of its arrival at a trainman opposite A will be

$$t' = \frac{x'}{c} \tag{160}$$

where x' is the distance of this trainman from O'. The distance x would naturally be measured by laying down a meter stick a certain number of times between O and A, and the corresponding distance x' would be measured by a similar process on board the train. The principle of relativity unqualifiedly requires that the velocity of light c as measured on the moving train shall be the same as that on the station platform, and since the train has moved forward during the time required for the radio wave to advance to A, it would seem obvious that t' cannot be equal to t. That is, the watch of the trainman, set by the time signal, and duly corrected for the time which it required to reach him, cannot check with that of the station employee opposite him. Actually, however, the problem is not quite so simple, for we must not overlook the possibility, suggested by Fitzgerald, that the moving meter stick might be shorter than the stationary one, and might have to be laid down too many times between O' and A.

The whole problem reduces to that of finding the relations between the times t and t' and the distances x and x' which are consistent with the relativity principle. In order to calculate these relations, let us assume that the length of the moving stick is, to an observer on the station platform, reduced by a factor a, i.e., its apparent length to such an observer is not 1 meter but a meters. To him, therefore, the relation between x and x' is easily calculated as follows

$$x = ax' + vt \tag{161}$$

If, as the relativity principle postulates, any physical experiment is destined to come out the same in a moving laboratory as if it were performed in a stationary laboratory, it should be forever impossible to determine whether a body is moving or not. Our assumption, therefore, that the station is at rest was unjustified, and, in fact, it might be that, owing to the motion of the Earth as a whole, the train was actually at rest and the station moving past it with a velocity v. To the trainman, therefore, not his own meter stick, but that of the station employee would be reduced by the factor a, and he would conclude the following to be the correct relation between x and x'.

$$x' = ax - vt' \tag{162}$$

If the principle of relativity is valid, both (161) and (162) must

be true. Substituting the value of t from (159) in (161) and that of t' from (160) in (162),

$$x = ax' + \frac{vx}{c} \quad \text{or} \quad x\left(1 - \frac{v}{c}\right) = ax' \tag{163}$$

and

$$x' = ax - \frac{vx'}{c} \quad \text{or} \quad x'\left(1 + \frac{v}{c}\right) = ax \tag{164}$$

Multiplying (163) by (164)

$$xx'\left(1 - \frac{v^2}{c^2}\right) = a^2x'x$$

Cancelling x and x' and solving for a

$$a = \sqrt{1 - \frac{v^2}{c^2}} \tag{165}$$

This is seen to be precisely the value of the Fitzgerald contraction ratio, which is required to explain the Michelson-Morley result. Solving (161) and (162) for x' and x respectively,

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{166 and 166'}$$

Substituting x from (166') into (161)

$$x' = a^2x' + avt - vt'$$

Solving for t, and noting that $1 - a^2 = v^2/c^2$

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{167}$$

Similarly

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{167'}$$

(166), (166'), (167), and (167') were originally obtained by H. A. Lorentz in his study of moving electric charges, and constitute the essential part of what is known as the *Lorentz Transformation*.

Our suspicion that the watches of trainman and station employee might not agree is seen to be justified. If the station

agent at $x = 0$ compares his own watch with those which pass him on the train, he observes a time t' (obtained from Formula 167′ by setting $x = 0$) such that $t' = t/\sqrt{1 - v^2/c^2}$. The time in the moving system, thus obtained, appears to go faster than that in the fixed system. Nevertheless, a particular moving clock (at $x' = 0$, for example) will show a time t' which by (167) is less than the time t shown by a clock opposite it on the station platform. For, setting $x' = 0$ in (167), $t' = t\sqrt{1 - v^2/c^2}$. To an observer on the station platform this moving clock will appear slow, a fact which is the basis for the usual statement that a clock moving with reference to an observer appears to him to run slowly.

183. The Four-dimensional Space-time of Minkowski. If the time which elapses between two events, such as the coincidence of O' and O and the arrival of the radio wave at A, is measured by clocks on the train and on the station platform, the result will not in general be the same. In fact, two events which seem to the station agent to occur simultaneously may not seem simultaneous at all to a trainman. Related to this temporal distance between two events, is the spacial distance between two objects. Owing to the Fitzgerald contraction, which we have seen is required by the relativity principle, this distance also will appear to depend on the state of motion of the body.

It was Minkowski who showed how this complicated relation can be most simply represented. Since time and space are apparently inextricably mixed, it was a natural step to add to the three dimensions of space a fourth dimension which should depend on the time, and to represent any event by a point in this four-dimensional manifold. The location at which the event occurred would then be determined by the three spacial coordinates of this *world point*, as it is called, while the time at which it took place is represented by the fourth coordinate. If, instead of using the time itself as the last coordinate, the quantity $\sqrt{-1}\ ct$ is used, the distance apart in this four-dimensional manifold of any two events is found to be no longer dependent on the velocity of the laboratory from which the events are observed, and the complicated formulas of the Lorentz transformation can be geometrically interpreted as a simple rotation of the coordinate axes. It is because of this and a number of other remarkable simplifications, which follow from the same method of representation, that the world in which we live has in recent years

come to be thought of as in reality a four-dimensional rather than a three-dimensional manifold.

184. The Composition of Velocities. Another interesting consequence of the relativity hypothesis can be obtained from the relations (166) to (167). Consider, for example, a man who is walking along the train in the direction of its motion with a velocity u'. What will be his apparent velocity with reference to the station platform? On the basis of the older mechanics we should have answered without question $u' + v$. Suppose, however, that at the time t_1' he is at a distance x_1' from O', and that at t_2' he is at x_2'. His velocity as seen from the train would be $x_2' - x_1'/t_2' - t_1'$. As viewed from the station platform his velocity would be $x_2 - x_1/t_2 - t_1$, a quantity which we can calculate by the formulas of the Lorentz transformation. Thus

$$x_1 = \frac{x_1' + vt_1'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad x_2 = \frac{x_2' + vt_2'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Subtracting the first of these expressions from the second,

$$x_2 - x_1 = \frac{(x_2' - x_1') + v(t_2' - t_1')}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Similarly

$$t_2 - t_1 = \frac{(t_2' - t_1') + \frac{v}{c^2}(x_2' - x_1')}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{(x_2' - x_1') + v(t_2' - t_1')}{(t_2' - t_1') + \frac{v}{c^2}(x_2' - x_1')}$$

Or, dividing numerator and denominator by $(t_2' - t_1')$, and substituting u' for $x_2' - x_1'/t_2' - t_1'$ and u for $x_2 - x_1/t_2 - t_1$,

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \tag{168}$$

This is the Einstein formula for the addition of velocities. It is seen to be always less than the intuitive value $u' + v$, and less in a curious way. Consider, for example, the velocity of a light beam. This beam is travelling along the train with a velocity c.

What will be its velocity as observed from the platform? Setting $u' = c$, we get

$$u = \frac{c + v}{1 + \frac{cv}{c^2}} = c$$

The velocity of the beam as seen from the platform is also c, and it is clear that any body with a velocity on the train of less than c will also appear to a person on the platform to have a velocity less than c. The full significance of this observation is realized if, starting with a body having the velocity v, we attempt to accelerate it to a velocity greater than c. The process of acceleration consists in combining with the initial velocity v an additional velocity u'. (168) shows that such a combination can never result in a velocity greater than c. The speed of light is, therefore, an upper limit, which the velocity of no body can surpass. This conclusion is in accord with all physical observations which have thus far been made. β-rays have been observed, for example, with velocities within 1 per cent of the velocity of light, but none have been found to exceed it.

185. The Variation of Mass with Velocity and the Equivalence of Mass and Energy. The impossibility of accelerating a body to a velocity greater than that of light calls for a significant revision of the old Newtonian mechanics. If, according to the classical mechanics, a constant force was applied to a body, it would increase in speed by the same amount per second indefinitely, and would eventually exceed c. In order to be in accord with the principle of relativity, it must be increasingly hard to accelerate a body as it approaches this velocity. Since the measure of the difficulty of producing such an acceleration is the mass of the body, this mass must increase with the velocity. If we let m be this variable mass, and if we denote by m_0 the mass of the same body as seen by an observer moving with it (the so-called *rest mass*) the detailed calculation shows that

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{169}$$

If $v = 0$, $m = m_0$, as it should, but if $v = c$, $m = \infty$, and no force however great can accelerate it further. The correctness of this

formula has been checked by measurements of the ratio of e to m for β-particles.

As the velocity of a body increases, its energy increases, and here also the relativity principle leads to a new expression.

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{170}$$

This expression for energy, unlike the corresponding formula in the classical mechanics, does not vanish when $v = 0$, but reduces to m_0c^2. A body at rest is, therefore, supposed to contain an amount of energy proportional to its rest mass, and for even small masses, this amount, owing to the magnitude of c^2, is huge. This law, known as the *equivalence of mass and energy*, is probably the most important single conclusion which has been drawn from the relativity principle. It has found repeated application in the study of a wide variety of problems, and is generally recognized as one of the most important of the physical laws with which we are to-day familiar. The expression (170) lends itself readily to expansion by the binomial theorem. Thus

$$\begin{aligned} E &= m_0c^2\left[1 + \frac{v^2}{2c^2} + \frac{3}{16}\frac{v^4}{c^4} + \ldots\right] \\ &= m_0c^2 + \frac{m_0v^2}{2} + \frac{3}{16}\frac{m_0v^4}{c^2} + \ldots \end{aligned} \tag{171}$$

The second term in this expansion is the familiar kinetic energy of the classical mechanics. No purely mechanical experiments would detect the constant term m_0c^2, and the succeeding terms which contain c^2 or higher powers of c in the denominator are so small as to be impossible of detection by methods at present available.

If we accept the fundamental postulate of relativity, that no physical experiment can decide whether a body is at rest or in uniform motion, we must give up hope of obtaining any tangible grip on the ether. The traditional ether was a vast motionless sea in which all bodies moved, and which served as a mutual bond between them. It was a stationary *frame of reference* to which the motion of all bodies could be referred. If no such frame of reference exists, or what amounts to the same thing, if we can never detect it by experiment, it becomes nonsense to retain it in our physical theories. Such, at any rate, is the stand of the more

thoughtful of our present-day workers, and it cannot be denied that this viewpoint is rapidly becoming the accepted one.

PROBLEMS

1. Calculate the amount by which the diameter of the Earth in its direction of motion is apparently reduced by the Fitzgerald contraction. Calculate similarly the proportional reduction in the diameter of an electron which is moving with a velocity $0.99c$.

2. One of the most remote of the spiral nebulae is receding from the Earth with a velocity of 2500 miles per second, and is 140,000,000 light years distant. Calculate the difference in time between a clock on the nebula and one attached to a frame of reference associated with the Earth, but located at the same distance.

3. How much greater is the mass of the electron of Problem 1 than its rest mass?

4. Calculate the energy in calories associated with a mass of 1 gr. At 14,000 B.T.U./lb. how many tons of coal would be required to furnish by combustion the same amount of energy?

5. Assuming that the heat obtained from radium is due entirely to the kinetic energy of the α-particles which it emits, and that each atom of radium in disintegrating to radium lead expels 5 α-particles with an average velocity one-tenth that of light, calculate the radioactive energy in calories present in each gram of radium.

CHAPTER XXXIV

THE CONSTITUTION OF LIGHT. SPECTRA

It was pointed out in Chapter XXX that the wave-length of light ranged from 4×10^{-5} cm. in the violet to 8×10^{-5} cm. in the red. The corresponding frequencies range from $3 \times 10^{10}/4 \times 10^{-5} = 7.5 \times 10^{14}$ to $3 \times 10^{10}/8 \times 10^{-5} = 3.8 \times 10^{14}$. Just as sounds having frequencies between 16 and 20,000 are audible to the normal ear, so light within this higher range of frequencies is visible to the normal eye. If we push the analogy between light and sound a little further, and inquire as to the frequency or wave-length constitution of the light which comes to our eyes, it is easily shown that, like sound, it is very complex. Whereas the sound from even a large orchestra has its origin in a comparatively small number of sources, and the number of overtones from each source which is included in the audible range is rather limited, any light beam originates in myriads of more or less independent atoms. At first sight we should accordingly anticipate even greater complexity than is present in sound waves, and the first and most natural observations would seem to justify this expectation.

186. The Continuous Spectrum from an Incandescent Solid. Thus if light from an incandescent solid such as the filament of an incandescent lamp is allowed to fall on the slit of a spectrometer (Chapter XXX), each wave-length present in the incident light will be brought to a focus by the telescope in a particular direction from the grating or prism. No matter how fine the slit of the spectrometer is made, the light from the incandescent solid is spread out in this way into a perfectly continuous spectrum. All colors within a certain range are present, and the individual atoms are apparently radiating their energy without any vestige of unanimity. To be sure, not all colors are emitted to the same extent, as can be readily shown by mounting a narrow thermopile at the cross hairs of the telescope, and thus measuring the varying heat energy which falls on it as the telescope is rotated to receive different parts of the spectrum.

187. Infra-red Light. With the usual incandescent lamp, the amount of energy thus received increases as the red end of the spectrum is approached, and continues to increase even after the thermopile has passed the region of any apparent light. It is clear that beyond the visible wave-lengths there must be radiations which carry heat energy capable of raising the temperature of the thermopile. This long wave-length invisible region of the spectrum, known as the *infra-red*, was first discovered by Sir William Herschel in 1800. Such radiation has been shown to exist in particular sources over the entire region which separates visible light from the short electric waves discussed in Chapter XXXII, i.e., from 8×10^{-5} to 0.034 cm. The hugeness of this region, which until the dawn of the nineteenth century was entirely unsuspected, is best appreciated by utilizing the idea of octaves so familiar in music. An octave may be defined as the interval between two notes, one of which has a frequency twice that of the other. There are thus a little over seven octaves on the piano keyboard. There are eight octaves in the infra-red, sixteen in the region of very short electric waves between 0.03 cm. and the 20-meter amateur band, and eight octaves more between this and the 5000-meter commercial band. In contrast to this great range of thirty-two octaves, the visible portion of the spectrum comprises only a single octave.

The region in the spectrum of a hot solid which contains the largest amount of energy depends on the temperature of the body. The hotter it is, the shorter the wave-length of the energy maximum, a law which is qualitatively familiar to anyone who has watched a metal gradually heated through the dull red and the bright red to the white heat stage. The way in which the position of this maximum depends on the temperature has been carefully studied, and furnishes the basis for one of the most reliable methods of measuring high temperatures.

188. Line Spectra from Incandescent Gases. If the light from an incandescent gas such as that from a mercury arc is examined by the spectrometer, the spectrum to be observed is entirely different. In place of the continuous succession of all colors, only certain very definite wave-lengths are present. Each color forms its own image of the collimator slit in the focal plane of the telescope, and the spectrum as a whole is simply a group of such fine parallel lines. Such *line spectra* were first observed by Thomas Melvill, a Scotsman, in 1752, but it was

over a century later in 1859 that Kirchhoff and Bunsen showed that the wave-lengths present are absolutely characteristic of the elements producing them. Thus sodium, one of the commoner elements, always reveals its presence by two conspicuous and closely adjacent lines in the yellow at 5.890×10^{-5} and 5.896×10^{-5} cm. respectively.

Certain substances are so readily volatilized that their characteristic line spectra are readily obtained by holding them in the flame of a Bunsen burner, whereas other substances, including most metals, must form one of the poles of an electric arc in order to be sufficiently vaporized. Even if some

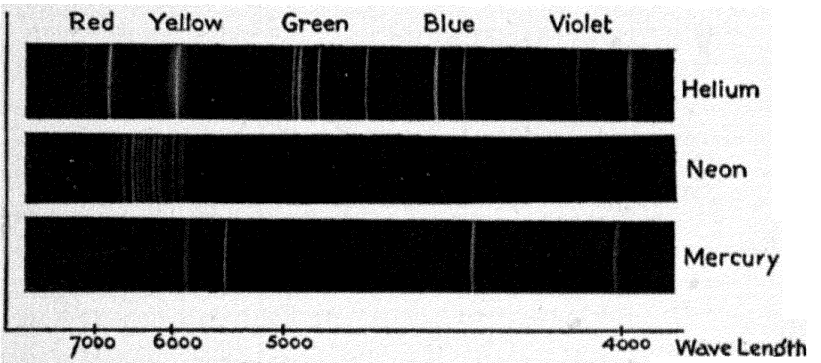


PLATE IV. Line Spectra.

lines can be obtained in a flame, many others can usually be obtained by arc excitation. Nevertheless, although a given spectrum may be quite complicated, and may depend very intimately on the method of excitation, it is quite characteristic of the element producing it, and can never be exactly duplicated by any other. As a rule, the elements of higher atomic weight have more complex spectra, but this, like other atomic properties, depends on the position of the element in the periodic table. Members of the iron, cobalt, and nickel group, for example, are characterized by extraordinarily complicated spectra containing thousands of lines. The frequencies of the various lines in the spectrum of a single element bear no simple harmonic relation to one another, and until recently it has been a completely unsolved puzzle how a structure as simple as the individual atom seems to be can produce such an extraordinary variety of wave-lengths. Thus hydrogen, the simplest of all atoms, gives rise to some scores of different spectrum lines. Since the work of Kayser

and Runge late in the nineteenth century it has been known that the spectrum lines of many elements can be arranged in a number of groups or series, each of which can be described by a relatively simple mathematical formula. It remained for Bohr in 1913 and succeeding workers to throw light on the significance of these series relations. The discussion of this highly interesting work must, however, be postponed until Chapter XXXVI.

189. Solar Spectra. Fraunhofer Lines. A quite different type of spectrum is obtained from sunlight. When viewed casually or with a broad slit, it appears similar to the continuous spectrum from an incandescent lamp, but if the slit is made narrower, a large number of dark lines are observed traversing the continuous spectrum. These lines were first observed by Wollaston, a London physician, early in the nineteenth century, but it was Joseph Fraunhofer, an optician of Munich, who, having discovered them independently, reported the results of his careful observations in 1814, and for whom the lines are now called *Fraunhofer lines.* Kirchhoff was the first to solve the riddle as to their origin by showing in the laboratory that a flame, which normally emits a particular wave-length, may absorb light of the same wave-length from a light beam containing it. All colors of light originating in the Sun's disk or *photosphere* must pass through an outer gaseous envelope, which absorbs the particular frequencies which the gases there present are able to emit. This outer layer of gas, because of its peculiar property, is known as the *reversing layer.* The dark lines observed in the solar spectrum give, therefore, a knowledge of the elements present in this region of the Sun. A large proportion of the elements existing on the Earth have been thus proven to exist in the Sun, and the spectrum of one, helium, was discovered in sunlight before it was found among the rarer gases of the Earth's atmosphere.

The more prominent of the Fraunhofer lines have been designated by letters, and they serve as very useful standards for calibrating spectroscopes. The following table gives the wave-lengths of some of these lines and the symbol of the element which is responsible for each. The wave-lengths are expressed in terms of the Ångström unit, which is defined as 10^{-8} cm.

FRAUNHOFER LINES

Line	Source	Wave-length
A	O	7594
B	O	6867
C	H	6563
D_1	Na	5896
D_2	Na	5890
E	Fe	5270
F	H	4861
G	Fe, Ca	4308
H	Ca	3969
K	Ca	3934

The position of these lines is indicated on the solar spectrum shown in the frontispiece.

190. Selective Absorption. The Color of Pigments. Certain liquids and solids as well as gases are capable of absorbing particular sections from the spectrum of white light. In fact, the colors of most objects are due to this sort of absorption. Light impinging on such a body penetrates a short distance, is scattered in various directions, and eventually comes out once more to the surface. A green material appears green because the other parts of the spectrum have been absorbed from the scattered light during this process.

It is clear from the foregoing paragraphs that the light which informs us of the color of objects may be very complex indeed. If the object is viewed in sunlight, an analysis would show a continuous spectrum traversed by the dark Fraunhofer lines and also by certain broader bands or areas, which are the source of its peculiar color. If viewed in the light of a mercury arc, only the sharp lines of the mercury spectrum would be present, but of these, certain ones might be partially or even quite completely absorbed by the pigment in the body. In the analogous case of complex musical sounds, a trained ear is capable of detecting and of estimating the relative intensity of practically all its constituents. The eye, however, is no such sensitive instrument. If three so-called primary colors such as red, yellow, and blue are suitably chosen from the visible spectrum, they can be combined in such a way as to simulate accurately any spectral combination which may fall on the eye. It would appear, as was first pointed out by Thomas Young, that our sensation of color is due to the

effect of three separate mechanisms in the eye which are capable of responding independently to these three different regions of the spectrum.

191. Color Photography. This curious physiological fact has made it possible to develop highly satisfactory methods of color photography. In all methods now used the object is photographed through a screen of some sort, and the resulting picture is viewed through the same screen or a similar one. This procedure, suggested by Maxwell, was first successfully carried out by Dr. John Joly of Dublin in 1897. On a glass plate alternate red, green, and blue lines were ruled, and the photographic plate was exposed beneath this screen. Light which passed through the red dye lost all except the red part of its composition, and only that coming from an object containing some red, after passing through this part of the screen, would darken the emulsion. Similarly any green light from the object would penetrate the green dye and affect the emulsion directly beneath it. If a positive transparency is now made from such a negative, and is viewed by transmitted light through the screen used in taking the original, it will appear red where red light fell on it from the object, green in green areas of the picture, and blue in blue areas. Furthermore, because of the familiar ability of such photographic emulsions to register all gradations of light intensity, and because of the ability of the three primary colors in proper proportion to simulate all intervening colors, these will also be quite faithfully reproduced.

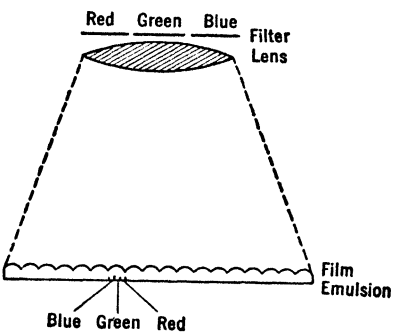


FIG. 172. Kodacolor Process.

Of the modern methods of color photography, the Finlay process follows precisely the Joly scheme, but with a much improved screen; in the Lumiere and Agfa processes the screen is incorporated in the sensitive plate itself. The Kodacolor method is an ingenious adaptation of the Joly type of screen to motion-picture requirements. It would, of course, be impracticable to rule the hundreds of feet of motion picture film required for even a single short exposure, but it turned out to be a not im-

possible task to mould or press the back of the film into tiny cylindrical corrugations or lenses (Fig. 172). This is done, and a special filter, consisting of three vertical segments, red, green, and blue, respectively, is placed during exposure before the camera lens. Light from the lens passes through the corrugated back of the film before reaching the emulsion, and the curvature of the corrugations is so designed that they focus on the emulsion images of the parallel red, green, and blue bands of the filter. There is thus formed optically on the film at the time of exposure the necessary system of parallel colored lines similar to those of the Joly process. The resulting motion-picture film is intended for projection, and it is merely necessary to project it through a filter exactly like that used in taking the picture in order to obtain all the natural colors.

CHAPTER XXXV

SHORT WAVE-LENGTHS. X-RAYS

192. Ultra-violet Light. Just as beyond the red end of the spectrum there exists a region, the infra-red, which, though invisible to the eye, reveals its presence by the effect which it has on a thermopile, so also beyond the violet end of the spectrum there are invisible radiations known as the ultra-violet. These rays were discovered by Johann Wilhelm Ritter and Wollaston early in the nineteenth century. Whereas infra-red rays are most readily detected by their heating effect on a thermopile, the heat energy associated with ultra-violet light is even less than that in the visible region. There are, however, certain materials which have the property, known as *fluorescence*, of absorbing light of one frequency and emitting simultaneously light of another, in general lower, frequency. Such substances, among which may be cited barium platinocyanide, fluorspar, uranium glass, sulphate of quinine, and aqueous solutions of fluorescein and eosin, absorb readily the energy in the invisible ultra-violet radiations, and reëmit the energy as a greenish or bluish glow. They provide, therefore, a simple means of detecting these rays.

The property of *phosphorescence*, possessed by a variety of materials such as calcium sulphate, differs from fluorescence in the fact that the emitted glow persists, sometimes for many minutes, after the removal of the exciting rays. Although phosphorescence may be produced by visible light, it is particularly intense with ultra-violet light.

It is a familiar fact that photographic plates are more sensitive to the violet than to the red end of the spectrum, ordinary or *non-orthochromatic* plates being practically unaffected by light beyond the blue. This sensitivity persists out into the ultra-violet, and provides the most convenient method of studying this region. Thus, if this section of the spectrum of the mercury arc is photographed, it is found to contain a wealth of sharp lines. Such investigations, however, encountered unexpected difficulties, as it was soon discovered that ultra-violet light was increas-

ingly absorbed by glass and even by air as the wave-length decreased. Quartz and later fluorite were found to be more transparent in this region than glass, and spectrometer lenses were accordingly ground from these materials. Still better, concave reflection gratings were developed, which removed the necessity for any lenses whatever in the spectrometer, and finally such spectrometers were completely evacuated to eliminate the residual absorption of the atmosphere. By such methods spectrum lines were discovered by Theodore Lyman in 1906 as far down as 500 A (Ångström units) or 500×10^{-8} cm., and Millikan in 1920 reached 136 Å. If we define for the present the ultra-violet region as that between 4000 Å and 136 Å, it will still be seen to comprise nearly five octaves or five times the extent of the visible spectrum.

In addition to their action on fluorescent material and on the photographic plate, ultra-violet radiations have a marked bacteriocidal and a consequent physiological effect. Rays shorter than a certain critical wave-length tend to destroy all living organisms, and it is fortunate indeed that these short waves are absorbed from the light of the Sun by the atmosphere. The use of such rays, however, in limited doses from the quartz mercury vapor arc and from sunlight at high altitudes has been increasingly successful in the treatment of many skin and blood diseases.

193. The Nature of X-Rays. In Chapter XX the method of producing X-rays was described, and the problem as to their nature was formulated. No sure solution to this problem was reached, however, until the work of Max v. Laue in 1912. That they were charged particles such as α- and β-rays was rendered unlikely by the absence of any observable electric or magnetic deflection, but the corresponding absence of any certain interference phenomenon made their wave nature equally doubtful. In 1900, Haga and Wind had noted an apparent spreading out of a beam of X-rays on passing through a fine wedge-shaped slit, such as might be attributed to diffraction. A repetition of this experiment by Walter and Pohl in 1911 showed, according to measurements by Sommerfeld, an amount of spreading so small that it could be explained on the basis of diffraction only by assuming a wave-length of not more than 6×10^{-9} cm. It occurred to v. Laue that, if X-rays really consisted of such short waves, it should be possible to use the regularly spaced molecules of a

natural crystal as a diffraction grating to produce from them substantial interference phenomena. The separation of adjacent lines in a diffraction grating of 3000 lines per centimeter is 3.3×10^{-4} cm., or a little over five times the wave-length of sodium light. Rock salt, NaCl, is one of a number of common substances which exist in cubical crystals, and which by their isotropy indicate that the atoms are regularly spaced at the corners of little elementary cubes in the crystalline structure. From the known number of molecules in a gram-molecule and

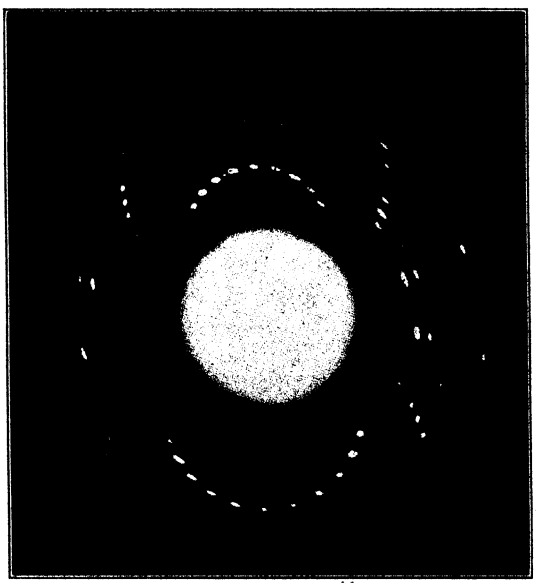


PLATE V. X-ray Diffraction Pattern from a Nickel Sulphate Crystal.

the known density of the crystalline material, it is a simple matter to calculate the distance apart of adjacent molecules in such a crystal. For rock salt this calculation yields 2.81×10^{-8} cm., or less than five times the estimated wave-length for X-rays. At the suggestion of v. Laue, Friedrich and Knipping in 1912 tried the experiment of passing a narrow pencil of X-rays through such a crystal and then allowing it to fall on a photographic plate. Without the intervention of the crystal such a pencil would produce only a single dark spot on the plate, but with the crystal interposed, a regular pattern of dark spots surrounding the dark central one was obtained (Plate V). The additional spots were

clearly the result of diffraction, occurring at points where the waves, scattered by individual atoms of the crystal, arrived in phase.

194. The X-ray Crystal Spectrometer. The geometry of such three-dimensional diffraction gratings is rather complicated, and although v. Laue was able mathematically to explain the observed spots on the basis of the probable arrangement of the molecules in the crystal, it remained for Sir William H. Bragg and his son W. L. Bragg in England to develop a somewhat simpler method of performing the same sort of experiment. Instead of passing the pencil of X-rays through the crystal, the Braggs reflected it from a face of the crystal. They visualized the crystal as made up of successive layers of atoms, and thought of the reflection as taking place at these various layers. Actually each atom served as the scattering center for a sheaf of diffracted rays, which were sent out in all directions, just as, in the chapter on diffraction, we thought of each ray incident on an aperture as giving rise to a sheaf of rays of light spreading out in all possible directions. Only in certain directions, however, could these rays reëforce one another to produce substantial effects. In Fig. 173 the horizontal lines represent the surface layer of atoms and the next underlying layers. If a pencil of X-rays PM strikes the top layer of atoms at M, an atom located at that point will scatter a small fraction of the incident beam in all directions about M. Among these directions will be that, MQ, which would be taken by a beam of light optically reflected at M. By far the greater part of the pencil will, however, continue on through N and S, a small additional amount being scattered at each of these points. As with the energy scattered at M, a certain portion will travel in the direction NO normally taken by reflected light. This latter ray will be in phase with that scattered along MQ if and only if the distance MNO which it traverses before emerging from the crystal is greater than the distance MQ by an integral number of wave-lengths. Since, as is apparent from the figure, OQ is drawn perpendicular to MQ, and

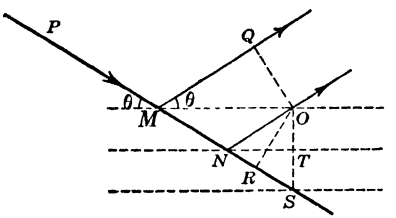


FIG. 173. Diffraction of X-rays from Crystal.

OR is perpendicular to MS, $MQ = MR$. But $NO = NS$ since the triangle NOT is equal to the triangle NST. Hence the path traversed by the ray through the crystal = MNS, and this is greater than the path MQ by RS or $OS \sin \theta$. Hence, if we let d = the distance between successive planes, $OS = 2d$, and the ray MNO will reënforce MQ if and only if

$$2d \sin \theta = n\lambda \qquad (172)$$

where n is any integer. It should be noted that if the X-rays diffracted in the direction θ from one layer differ in this way from those diffracted at the next succeeding layer by an integral number of wave-lengths, they will also differ by an integral number of wave-lengths from those diffracted at all succeeding layers. Thus, even though only a small portion of the incident beam is scattered in this direction at each layer, the combined contributions of all the layers reached by the beam may be very considerable. In such a direction the crystal will behave very much like an optical mirror, and may exhibit very intense reflection.

From a measurement of the angle θ at which such intense reflection takes place, and from Formula 172, the wave-length λ can be calculated if d is known, or d can be found if λ is known. A method was provided, therefore, for studying both the constitution of a beam of X-rays and the structure of crystals. Several quite different methods have been used for making such measurements. The Braggs allowed a fine pencil of X-rays from a narrow slit S (Fig. 174) to fall on the crystal at a small angle θ, and placed a photographic plate or film F in such a position as to receive the reflected beam. Only, however, if the condition of Formula 172 was satisfied, would any substantial reflection take place. It was necessary, therefore, to rotate the crystal slowly about a vertical axis, thereby changing the angle θ. Whenever in this process θ became such as to fulfil the conditions of (172), reflection took place with an intensity proportional to the amount of radiation of that particular wave-length present

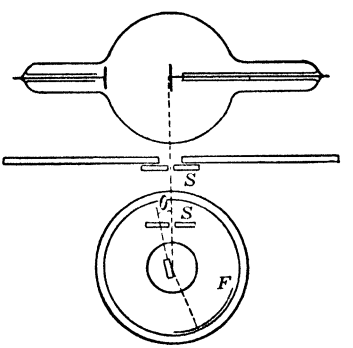


FIG. 174. X-ray Spectrometer.

in the beam, and left its impress on the film. In the course of rotating the crystal in this way through a number of degrees, all wave-lengths in the incident X-rays would be at some time intensely reflected, and would be permanently recorded photographically. The dark areas on the film would then constitute a complete spectrum of the radiation.

In place of the photographic film as detector of the rays, ionization chambers have found considerable use. Such a chamber with a narrow slit-shaped aperture will indicate by the deflection of an electrometer when substantial amounts of reflected rays are being received. By turning simultaneously both crystal and ionization chamber, the whole region of wave-lengths can be explored. This method is more laborious than the photographic one, but it is capable of somewhat greater precision. In cases when large crystals are not available, it was shown by P. Debye and Scherrer that a mass of small or powdered crystals can be used. Among such a collection of tiny crystals there will be individual crystals having all conceivable orientations, and which, therefore, are in a position to reflect each and every constituent of the incident radiation. If a film is placed about the container (Fig. 175), it will have recorded on it all these wave-lengths.

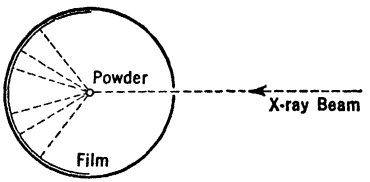


FIG. 175. Use of Powdered Crystals for X-ray Diffraction.

195. X-ray Spectra. The spectra obtained in this way, although noticeably simpler than most optical spectra, are still comparatively complex. They contain, in general, an area within which all wave-lengths are present, which constitutes, therefore, a continuous spectrum analogous to that of light from an incandescent solid. In addition, there are always present a number of sharp lines. The region covered by the continuous portion of the spectrum depends on the voltage applied to the X-ray tube, but the position of the sharp lines depends only on the material of which the target or anticathode is made. Just as optical line spectra are characteristic of the elements producing them, so also these X-ray line spectra earn in this way the title of *characteristic* spectra. Unlike optical spectra, however, the general arrangement of these lines is the same for large numbers of elements. Early in this study two different series of

lines were recognized, the so-called K and L series, and more recently an M and an N series have been added in the region of comparatively long and easily absorbed wave-lengths. The K series consists of four lines, the K_α, $K_{\alpha'}$, K_β, and K_γ. The K_α and $K_{\alpha'}$ are so close together that they can be separated only by apparatus of very high resolving power, and the K_γ is relatively weak and so close to K_β that it was not distinguished from it in the early work. The K series has the appearance, therefore, roughly of two lines, the stronger and longer wave-length $K_{\alpha\alpha'}$ and the weaker $K_{\beta\gamma}$.

196. The Relation between Wave-length and Atomic Number. The first study of the spectra from different elements was made in 1913, by H. G. J. Mosely, who constructed an X-ray tube in such a way that a target, composed successively of all the elements in the periodic table between calcium and copper (except scandium), could be interposed in the path of the cathode rays. The resulting spectra were studied by the photographic method, and were found to consist in each case of the two K lines. The wave-lengths of these lines, however, decreased progressively as the atomic weight of the element increased. If the square root of the frequency of either line was plotted in a graph against the atomic number (Fig. 176), a smooth and very nearly straight line was obtained. The fact that atomic numbers and not atomic weights had to be used in this sort of graph constituted the first evidence for their greater fundamental significance. The order in which cobalt and nickel had been previously placed in the periodic table was found to be wrong, and nickel with its slightly lower atomic weight was found to require the higher atomic number. Further work extended the range of such measurements to a much larger number of elements, and to the L and M series, where precisely similar laws were found to hold. In

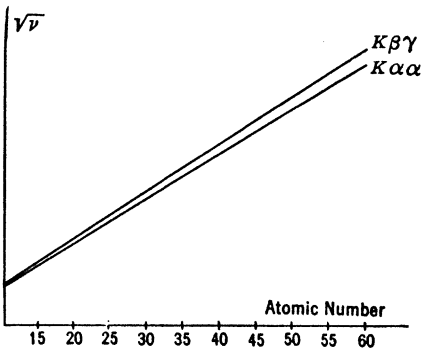


FIG. 176. Mosely Diagram of Frequency against Atomic Number.

Fig. 177, due to Siegbahn, the discoverer of the M series, the wave-lengths of the more conspicuous lines of all these series are plotted opposite the atomic number of the element producing them. The longer wave-lengths are, in general, much more readily absorbed than the shorter ones, a fact which accounts for the absence in this graph of K lines below atomic number 11 (Na), of L lines below 35 (Br), and of M lines except in the very heaviest elements. The shorter the wave-length, as we shall see in the next chapter, the higher the voltage which must be applied to the tube in order to excite it. It is the magnitude of the

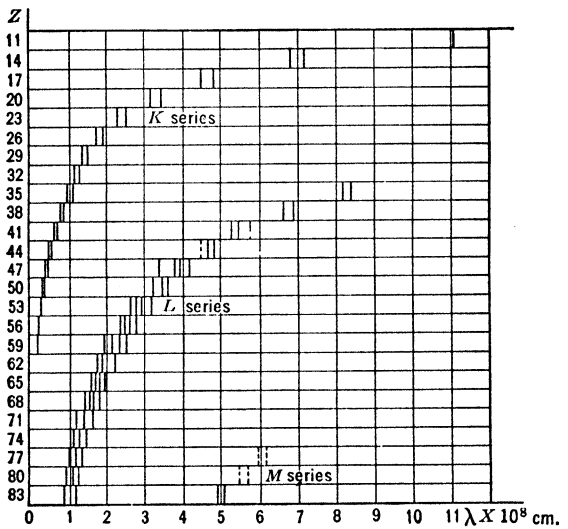


FIG. 177. X-ray Spectra of Elements between Sodium and Bismuth.

required voltage which breaks off the K series at atomic number 59 (Pr) and not its actual absence in heavier elements. Although in this table no lines are shown of longer wave-length than between 11 and 12 Å, Siegbahn has actually reached 17.66 Å. The region of X-rays which has already been studied comprises, therefore, between five and six octaves, and is separated from the short ultra-violet wave-lengths of Millikan by at most three octaves. There is no reason to expect that this short intervening region contains anything startlingly new. It remains only for a little refinement in technique to bridge the entire gap.

197. Gamma and Cosmic Rays. The similarity of γ- to X-rays was pointed out in Chapter XXI. Like X-rays they have

been subjected to crystal analysis, and have been found to possess a definite line spectrum. In wave-length they occupy the region between the very shortest X-rays and about one-twentieth of this wave-length, an additional four or five octaves. Corresponding to their short wave-length, γ-rays have a very high penetrating power, being able to pass through a good many millimeters of lead.

The last to be discovered in this great family of electromagnetic radiations if, indeed, it belongs to the family, are the so-called *cosmic* rays. Ever since the invention of the electroscope the inability of even the best insulators to prevent such an instrument from losing its charge has been a source of trouble to investigators. In 1903, Rutherford and McLennan found that the rate of this discharge was reduced if the electroscope was surrounded by a thick-walled metal box. Clearly some sort of very penetrating radiation was concerned with the discharge. Experiments with balloons, however, showed that the rate of discharge at high altitudes did not decrease as would be expected if the difficulty were due to radioactive materials on the Earth's surface. In 1911 and 1913 respectively Victor F. Hess and Werner Kolhorster in new balloon experiments at altitudes up to 5.6 miles found that, instead of decreasing, the rate of discharge increased very substantially with altitude. In 1922, Millikan and Bowen, first with balloons and then by sinking their electroscopes deep in the water of snow-fed mountain lakes in California, established beyond a doubt that highly penetrating radiations from outside the Earth's atmosphere were causing the discharge. These radiations, after suffering an amount of absorption in the atmosphere equivalent to 23 ft. of water, could still be detected at a depth of 45 ft. below the surface of the lake. This total of 68 ft. of water was equivalent to 6 ft. of lead, since the absorbing effect of any material on such rays as these is proportional to its density. Careful study of the intensity of the rays at various depths indicated that they were probably complex and very likely possessed a line spectrum. According to Millikan's estimate, they extend over a spectral region three octaves wide, all well below the shortest γ-rays. That they originate somewhere in interstellar space, and therefore deserve their title of cosmic rays, can scarcely be doubted. The problem as to their origin will be touched on in the last chapter.

CHAPTER XXXVI

THE ORIGIN OF SPECTRA. QUANTUM THEORY

It has been pointed out that during the latter part of the nineteenth century the electromagnetic nature of light was proven beyond any reasonable doubt. By reducing the size of the Hertzian oscillator, it had been found possible to obtain wave-lengths very nearly if not quite as short as the longest infra-red radiations. On the other hand, Kirchhoff and Bunsen had shown that the line spectra emitted from glowing materials were strictly characteristic of the emitting elements. What could, therefore, be more natural than to guess that the atom itself was a sort of miniature broadcasting station, each particular element being so designed as to send out its own unique message. True, the message was confusedly complex, but might we not some day learn to read it, and gain thereby an insight into the construction of the tiny transmitter?

198. The Distribution of Energy in a Continuous Spectrum. The continuous spectrum of an incandescent solid seemed to offer the most promising field for such an investigation, for here all substances behaved very nearly alike. To be sure, very dark or black substances absorbed light more completely than lighter materials, and as Benjamin Franklin had shown, the better a body absorbed, the more freely would it emit when heated. The perfect absorber or *black body*, as it is technically called, was, therefore, the perfect emitter, and a great deal of effort was expended in the closing years of the nineteenth century to learn the mechanics of black-body radiation. The problem was briefly to explain the distribution of energy in such a spectrum at one particular temperature, and then to learn the way in which this distribution varied with changing temperature. It has already been pointed out that the maximum amount of energy is emitted at shorter and shorter wave-lengths as the temperature of the incandescent body is raised. On either side of this maximum, however, the energy falls off in the peculiar but highly characteristic fashion shown in Fig. 178. What was the reason for this shape of curve? The problem was a very irritating one

because all the early and very plausible attempts at a theory led to completely erroneous results, assigning in all cases too much energy to the violet end of the spectrum. Finally in 1900 a theory was proposed by Max Planck, which yielded the correct sort of curve. It made the strange assumption that light, instead of being emitted by the atom continuously, was sent out in discreet impulses or *quanta*, as he called them. The size of the individual quanta were supposed to increase in direct proportion to the frequency of the light. Or, if E represents the energy in one quantum,

$$E = h\nu \qquad (173)$$

where h was a new constant of proportionality, which has since been called *Planck's constant.* The detailed derivation of the black-body law was a somewhat intricate statistical task, but it is easily realized that the atom, with a certain amount of energy at its disposal, could not afford to emit too many of the large quanta at the violet end of the spectral region. The magnitude of the constant h could be calculated from measurements made on any observed distribution curve, a process which yielded

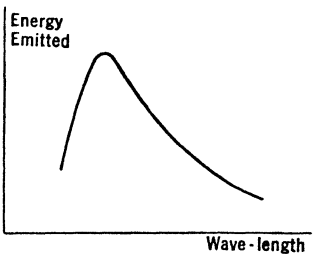


FIG. 178. Energy Distribution in Black-body Radiation.

$$h = 6.5 \times 10^{-27}$$

This idea of light quanta, which originated solely to meet the exigencies of a theory of black-body radiation, has since recurred so persistently that we can no longer doubt that it represents some very real attribute of radiation.

199. The Photoelectric Effect. In Chapter XX the ability of X-rays to produce ionization in gases was described. They are able, however, to remove electrons not merely from gas molecules but also from solids on which they impinge. These electrons are liberated with considerable velocity, and are known as secondary electrons to distinguish them from the cathode rays which produced the X-rays. This property, which radiation has, of freeing electrons from solid materials, was first observed with ultra-violet light. In 1887, Hertz noted that an electric spark discharge took place more readily between two electrodes if ultra-violet light fell on the cathode. As with Edison's similar

observation of the unilateral conductivity of a vacuum tube, made four years earlier, the correct explanation of this phenomenon had to wait for the discovery of the part which the electron plays in electric conduction. Just as a metal, when heated to incandescence, was found to emit electrons copiously, so even a cold metal was discovered to send out electrons when it was exposed to ultra-violet light or X-rays. Indeed, electrons are emitted from the alkali metals (lithium, sodium, potassium, etc.) when they are exposed to visible light.

This phenomenon and the peculiar sensitivity of the alkali metals has been utilized in the *photoelectric cell* for photometry, and has made possible the recent development of *television*. The photoelectric cell in its usual form consists of an evacuated bulb (Fig. 179) on one side of which a coating of potassium is deposited. In the middle of the tube is a small *plate*, which is connected to the positive pole of a battery of perhaps one hundred volts, while the layer of potassium, by means of an electrode sealed into the tube, is connected to the negative pole. Whenever light falls on the potassium surface, the emitted electrons are drawn to the positively charged plate, and constitute an electric current through the tube. This may be measured with a sensitive galvanometer or, after vacuum-tube amplification, it can be used to operate relays and thus control considerable amounts of power.

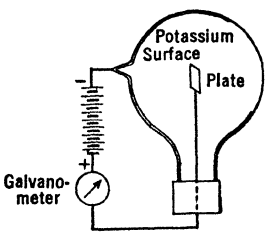


FIG. 179. Photoelectric Cell.

In addition to these practical applications, the greater part of the interest in the *photoelectric effect*, as it is called, has centered on the velocity with which the electrons are emitted. The reason for this interest was the discovery of the puzzling way in which this velocity depends on the intensity and the wave-length of the light. Although the velocity of the individual electrons cannot, of course, be measured, it is a simple matter to measure the velocity of the fastest of them. If, for example, the potassium surface is placed in front of a Faraday cylinder (Fig. 180), and if

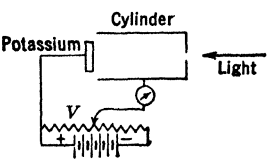


FIG. 180. Apparatus for Studying the Velocity of Emission of Photoelectrons.

the cylinder is placed at a negative potential with reference to the potassium, this opposing negative potential will tend to drive the photoelectrons back to the emitting surface. It should be remembered that by definition the difference in potential between two points is the work done in carrying a unit charge from one to the other. If V is, therefore, the difference in potential between the potassium surface and the cylinder, the work done in retarding the electron is Ve. If its initial energy $\frac{1}{2}mv^2$ just suffices to carry it to the cylinder,

$$\tfrac{1}{2}mv^2 = Ve \tag{174}$$

In performing such an experiment, the opposing potential V, starting from zero, is gradually increased, and the current through the cell is measured at regular intervals. Even with very low voltages the slower electrons are driven back to the emitting surface, but as the voltage is increased, the current through the cell decreases in the manner shown in Fig. 181. The fact that at a quite critical voltage the current is reduced to zero, indicates that there exists a definite maximum velocity among the electrons emitted. This maximum velocity can be calculated by (174) from the measured critical voltage.

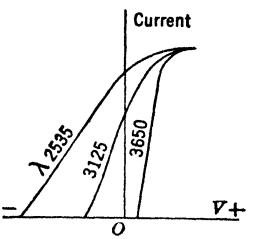


FIG. 181. Photoelectric Current Plotted against the Potential Difference between Emitting Surface and Collecting Cylinder.

The first accurate measurements of this sort were made by Millikan in 1916. He found, surprisingly enough, that this maximum velocity was independent of the intensity of the light. There was, of course, nothing in the photoelectric effect itself which could be disconcerting to the electromagnetic interpretation of the wave theory of light. What could be more natural than for an oscillating electric field to set into sympathetic oscillation an electron in a potassium atom, and if the disturbing field was sufficiently intense, even to tear it free from its moorings. We would expect, however, that such a process would liberate electrons with increasing velocity as the intensity of the electric field increased, and above all things we would expect the light to be entirely ineffective if its intensity were less than a certain critical minimum. Actually no such minimum has been observed. True, the number of electrons decreases with

decreasing light intensity, but however weak it may be, some electrons are drawn out with the maximum velocity indicated in Fig. 181.

Although the velocity of the photoelectrons does not depend on the light intensity, Fig. 181 shows that it does depend on the wave-length or the frequency of the light. As Einstein had already pointed out, this is exactly what would be expected if light travelled in quanta of a size $h\nu$ dependent on the frequency, for, if the entire energy of a light quantum was spent in liberating the electron, its resulting kinetic energy should be less than $h\nu$ by precisely the energy required to free the electron from the metal. If we set this energy equal to P,

$$h\nu = \tfrac{1}{2}mv^2 + P \tag{175}$$

This relation, first suggested by Einstein, seems to fit the photoelectric effect with surprising accuracy. In the case of metals, the electrons thus liberated are probably the "free" electrons which are responsible for the electrical conductivity. The entire energy P is, therefore, that which is required to remove the electron from the surface of the metal, and has been found to check as it should with the known contact potential difference between the surface of the metal and a point just outside the surface. According to (174), $\frac{1}{2}mv^2 = Ve$, and substituting this in (175), $h\nu = Ve + P$, a relation which Millikan found to hold accurately between the frequency of the incident light and the voltage required to drive the fastest electrons back to the emitting surface. This retarding voltage is plotted in Fig. 182 against the frequency of the incident light for sodium. From the slope of the straight line thus obtained, h could be directly calculated, and yielded an average of 6.57×10^{-27} for all the metals investigated. This, in turn, checked very well with Planck's value from the radiation curve. The reason that only the alkali metals respond to visible radia-

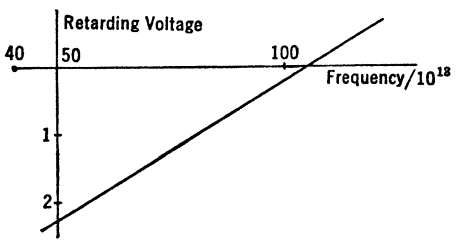


FIG. 182. Relation between the Energy of the Fastest Electrons and the Frequency of the Incident Light.

tions lies in the fact that only with them is P itself smaller than $h\nu$ in this region.

The application of (175) to X-rays is particularly impressive. Here $h\nu$ is so large that P may be neglected without introducing a very large percentage error. The energy $h\nu$ in the X-ray quantum or *photon*, as it is called, is directly converted at impact with the metal into the kinetic energy of the liberated electron. The process is almost the opposite of that by which the X-ray was originally produced. There an electron with a kinetic energy $\frac{1}{2}mv_0^2$, on impact with the target, gave rise to an X-ray quantum with energy $h\nu$. If the Law of the Conservation of Energy is to hold, $h\nu$ should not be greater than $\frac{1}{2}mv_0^2$, and we actually find this to be the case. The highest possible frequency which can be produced by a cathode ray of velocity v_0 is that for which

$$h\nu = \tfrac{1}{2}mv_0^2$$

The continuous X-ray spectrum is broken off sharply on the short wave-length end at precisely this point, and no line of the characteristic spectrum will appear of shorter wave-length. If, however, a characteristic line should appear at exactly the frequency ν for which $h\nu = \frac{1}{2}mv_0^2$, the entire energy of the cathode-ray electron will be converted into the X-ray of this frequency. This X-ray photon now moves away from the target, and no matter how far away it travels, it retains its original frequency and its consequent ability to free a photoelectron with the kinetic energy $\frac{1}{2}mv^2 = h\nu$, that is, with the same total energy which it originally received from the cathode-ray particle. If the X-ray were a spherically expanding wave, only a very small part of its surface could impinge on a particular atom of the metal, and it is very difficult to see how it could thus transfer its entire energy to one small locality. If, however, the X-ray were a corpuscle such as Newton visualized in his concept of light, such a process would be both comprehensible and natural. The photoelectric effect gives to the quantum of Planck the attributes of a corpuscle, and strange as it may seem, this picture of a light quantum or photon seems best fitted to explain a considerable variety of such phenomena.

200. The Bohr Theory of Line Spectra. In 1913, Niels Bohr, the Danish physicist, discovered how to apply the new quantum theory to the problem of the origin of line spectra. Only one additional hypothesis was necessary, and this was provided with

rare insight by Bohr. It is known as the hypothesis of *stationary states*, and assumes that an atom can exist in a number of different states, in each of which it possesses a definite amount of energy. Under certain conditions it can change from one state to another. If it changes from a state with a smaller amount of energy to one with a larger amount, it must receive the necessary difference in energy from some outside source, perhaps from an impinging electron, perhaps from a photon. If, on the other hand, it changes to a state with a smaller amount of energy, it must lose the difference, and this it does by emitting a quantum of radiation. If the energy of its initial state is E_1 and that of its final state is E_2, the frequency ν of the emitted radiation must be given by the relation

$$E_1 - E_2 = h\nu \tag{176}$$

This hypothesis of stationary states, which Bohr invented to meet the requirements of a theory of line spectra, has since been checked most remarkably by independent experiments on the ionization of gases. When a stream of electrons of controllable velocity is shot into a body of gas, the velocity of its individual particles will be little affected by the collisions which they make with the gas molecules if their velocity is less than a certain critical amount. Apparently the electrons rebound from the heavy atoms with very slight loss of energy. If, however, a certain critical velocity is exceeded, the electrons, after passing through the gas, are found to have lost a substantial part of their energy. We might suppose that this critical velocity is that required to tear an electron free from the atom, in other words to ionize it, and this was the interpretation originally given to the phenomenon. As, however, more careful studies of this sort were made, notably by Franck and Hertz in Germany, it was found that for many gases there were, not one, but several such critical velocities. It is customary to represent such a critical velocity by the accelerating voltage which is required to produce it in the electron. In the case of helium, no substantial loss of velocity is noted in the bombarding electrons until the voltage used to accelerate them is raised to 19.75 volts. An additional loss is sustained at 20.55 volts, and a third at 24.5 volts. The last of these is believed to be the actual *ionization potential*, i.e., the voltage required to give an electron speed enough to tear one of the helium electrons free. The other two represent the energy

required to carry the helium atom from its *normal* stationary state into two different states of higher energy content In general, when an atom is lifted in this way to a higher energy level, it begins to radiate one or more spectrum lines, a process which on the Bohr theory indicates a return of the atom to its original normal state. It is for this reason that the lower critical potentials are known as *radiation* potentials.

Bohr did more, however, than enunciate the hypothesis of stationary states and their application to the production of line spectra. He found a simple rule which enabled him to calculate the energy in all the stationary states of the simplest atom, that of hydrogen. This rule can best be understood, as Bohr originally understood it, in terms of the Rutherford atomic model. In the case of the hydrogen atom, this consisted of a single positive nucleus or proton, about which a single electron described a planetary orbit. If the orbit is circular, and Bohr originally confined himself to circular orbits, both kinetic and potential energies of the electron are easily calculated. Thus, if e is the numerical value of the charge on the electron, E the charge on the nucleus, and r the radius of the orbit, the potential energy, as shown in Chapter XV, would be

$$V = -\frac{eE}{r} \tag{177}$$

The minus sign which appears in this and subsequent expressions is due to the fact that zero potential energy has been defined in accordance with Chapter XV as that possessed by the electron at an infinite distance from the nucleus. Since electron and nucleus attract each other, it is clear that their potential energy for such an infinite separation is greater than when they are closer. Hence the potential energy for these smaller separations must be negative. The kinetic energy of the electron is

$$\text{K.E.} = \frac{mv^2}{2}$$

The magnitude of the attractive force between electron and nucleus is

$$F = \frac{eE}{r^2}$$

and according to the fundamental law of centripetal force,

$$F = \frac{eE}{r^2} = \frac{mv^2}{r}$$

Thus also $$\text{K.E.} = \frac{mv^2}{2} = \frac{eE}{2r}$$

The total energy $$\text{E} = V + \text{K.E.} = -\frac{eE}{r} + \frac{eE}{2r}$$

Or $$\text{E} = -\frac{eE}{2r} = -\frac{mv^2}{2} \qquad (178)$$

One of the fundamental laws discovered by Kepler which are applicable to any planetary orbit, whether circular or not, is that which states that the area swept out per second by the line joining the Sun and the planet is the same at all parts of the orbit. For a circular orbit this area swept out per second is readily calculated. The total area of a circle is πr^2, and since its circumference is $2\pi r$, the area of a sector subtended by an arc of unit length is $\pi r^2/2\pi r = r/2$. The area of a sector subtended by an arc of length s is, therefore, $rs/2$. The distance travelled per second by an electron in its orbit is v, and hence the area swept out is $rv/2$. The product mv is the momentum of the electron, and mvr is called quite naturally the *moment of momentum* or less accurately the *angular momentum*. This moment of momentum, like the area swept out per second by the line joining Sun and planet, is constant throughout any orbit, and has always occupied a prominent place in the literature of celestial mechanics. The rule which Bohr found to be obeyed by the electron in its stationary states is simply that this moment of momentum is always an integral multiple of $h/2\pi$. Or

$$mvr = \frac{nh}{2\pi} \qquad (179)$$

where n is an integer. Rewriting (178) in slightly different form,

$$mv^2r = eE \qquad (178')$$

In (179) and (178′) we are provided with two simple equations for determining v and r, which when solved yield

$$v = \frac{2\pi eE}{nh} \quad \text{and} \quad r = \frac{n^2h^2}{4\pi^2meE} \qquad (180) \text{ and } (180')$$

Substituting this value for v in (178),

$$\text{E} = -\frac{2\pi^2me^2E^2}{n^2h^2} \qquad (181)$$

As a result of what may be called the quantum restriction on the moment of momentum, only certain orbits are possible for the electron. The radii of these orbits according to (180′) are

$$a_0 = \frac{h^2}{4\pi^2 meE},\ 4a_0,\ 9a_0,\ 16a_0,\ 25a_0,\ \ldots\ n^2a_0$$

In the case of the hydrogen atom, $E = e$, somewhat simplifying these expressions. All the quantities necessary for the calculation of these radii, h, m, and e, are known, and the smallest radius, a_0, which we have good reason to identify with the normal stationary state of the atom, turns out to be 0.532×10^{-8}, in good agreement with other estimates of atomic dimensions. The electron is supposed to remain normally in this orbit, and when driven out by the impact of an electron or photon, it very soon returns, with the consequent emission of radiation. If it falls from the orbit determined by setting $n = n_1$ into the orbit for which $n = n_2$, the frequency emitted is obtained by setting

$$h\nu = \mathrm{E}_1 - \mathrm{E}_2$$

$$\text{Or} \qquad h\nu = -\frac{2\pi^2 me^2E^2}{h^2}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) = \frac{2\pi^2 me^2E^2}{h^2}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right)$$

$$\text{Or} \qquad \nu = \frac{2\pi^2 me^2E^2}{h^3}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \qquad (182)$$

If the electron falls directly into its normal orbit, $n_2 = 1$. By setting $n_1 = 2$, 3, 4, etc., the frequencies emitted when the electron falls thus from the second, third, and fourth orbits can be calculated. They check accurately with the frequencies of a series of lines discovered by Lyman in the ultra-violet, and are called the Lyman series. If the electron falls into the second orbit, $n_2 = 2$, and by setting $n_1 = 3$, 4, 5, etc., the frequencies of the Balmer series in the visible spectrum are obtained with equal accuracy. Similarly if $n_2 = 3$, and if $n_1 = 4$, 5, 6, etc., we get the Paschen series in the infra-red, and thus, with the exception of a few lines in the extreme infra-red obtained by setting $n_2 = 4$, all the lines in the spectrum of the hydrogen atom are obtained.

201. The Application of the Bohr Theory to X-ray Spectra. The application of (182) is not restricted to the hydrogen spectrum. If E is set equal to the charge on the nucleus of a heavier element, i.e., if $E = Ze$, X-ray frequencies emitted by the element in question are obtained. This is taken on the Bohr theory as

evidence that X-rays are produced by an exactly similar process taking place at the heart of the heavier atoms. In each case an inner electron is describing an orbit about the charged nucleus just as the corresponding electron describes its orbit about the hydrogen nucleus. The possible orbits are limited by the same restriction on the angular momentum, and exactly as with the hydrogen atom, radiation takes place when the electron falls from an outer into an inner orbit. The higher frequency of this radiation is due to the larger value of E and the resulting greater energy. If $n_2 = 1$, the lines of the K series are obtained, whereas if $n_2 = 2$, 3, and 4 respectively, we get the lines of the L, M, and N series. If we set $R = 2\pi^2 me^4/ch^3$, the so-called *Rydberg constant*, largely used in spectroscopy,

$$\nu = cRZ^2\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \qquad (182')$$

and if we consider the K_α line for which $n_2 = 1$ and $n_1 = 2$,

$$\nu_{K_\alpha} = \tfrac{3}{4}\, cRZ^2$$

The explanation for Mosely's straight-line graph between $\sqrt{\nu}$ and Z is obvious.

The years immediately following Bohr's brilliant discovery were devoted to many extensions and refinements of the theory. In the initial work, Bohr had treated the heavy nucleus as a fixed point, unmoved by the attraction of the planetary electron. When account was taken of the slight resulting motion of the nucleus, a small but experimentally substantiated correction was made in the original calculation. Elliptical orbits had to be considered, and the high speed of the electron at the perihelion required a relativity correction which, in turn, led to the prediction of a " fine structure," a splitting up of the individual spectrum lines into multiple lines, a phenomenon which was also checked experimentally. The behavior of the spectrum lines when the source was subjected to intense electric and magnetic fields, and the extension of the theory to other elements, opened up a vast field for research. On the whole the theory proved its worth in a very striking fashion, but it was soon apparent that it was not telling the whole story. The confidence with which the Rutherford picture was hailed as the correct representation of the atom has gradually waned, particularly since all the results achieved by using this model have in recent years been

obtained by quite different methods. These newer methods, which will be described very briefly in the next chapter, have also been successful in the solution of problems on which the Bohr theory threw no light at all or a quite false light. They do not give, however, any satisfactory picture of the atom to replace the tiny solar system of the original theory, and it is questionable if we will ever get a better picture. Thus, although we no longer believe it to be entirely accurate, we can still quite legitimately continue to use the Rutherford atom as a tangible guide to our thoughts, and in most cases it will not lead us astray. Nature is often a very good imitator. She constructs an insect to look like the twig of a tree, and she constructs an atom to simulate in many respects a planetary system.

202. The Compton Effect. Perhaps the most striking experiment in which light appears to have the attributes of a corpuscle is that first performed by A. H. Compton in 1922. In this experiment a beam of monochromatic X-rays is allowed to fall on some light element such as carbon, and the scattered radiation is subjected to crystal analysis. It is found to contain not merely the frequency present in the incident X-rays but an additional frequency produced by the scattering process. The mere presence of a new frequency would be most puzzling on the electromagnetic wave theory. Thus in the analogous case of a radio wave, no amount of reflection or scattering by receiving aerials will produce in the wave any frequency not present in the original radiation. Still more remarkable, however, Compton was able to calculate the frequency of the new spectrum line by assuming that it was produced by the collision of a photon of the incident radiation with a free electron of the scattering material. This calculation is particularly interesting as it shows how completely every attribute of radiation except its corpuscular property can be overlooked when dealing with an experiment of this character.

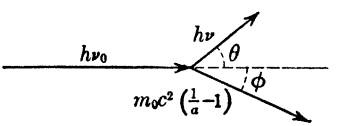


FIG. 183. Impact between Photon and Electron.

Fig. 183 represents a photon of frequency ν_0 and a consequent energy $h\nu_0$ colliding with an electron. Photon and electron are imagined at collision to behave exactly as two perfectly elastic billiard balls, the direction and magnitude of their sub-

sequent motion being completely determined by the Laws of the Conservation of Energy and Momentum. The photon is represented as continuing on at an angle θ to its original direction, and the electron at an angle ϕ. Since some of the original energy of the photon has been passed over to the electron, its energy $h\nu$ after collision is less than that $h\nu_0$ before collision, its frequency is correspondingly less, and its wave-length greater. This is Compton's explanation of the observed shift in wave-length.

According to the relativity principle, the total energy of an electron is $mc^2 = \dfrac{m_0c^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \dfrac{m_0c^2}{a}$.

Its kinetic energy would, then, be the difference between this total energy and its rest energy m_0c^2 or $m_0c^2(1/a - 1)$. The conservation of energy, as applied to electron and proton, would then require that

$$h\nu + m_0c^2\left(\frac{1}{a} - 1\right) = h\nu_0 \tag{183}$$

The mass of a particle moving with velocity v has been shown to be m_0/a if m_0 is its rest mass. Its momentum is, accordingly, m_0v/a. The ratio of its energy to its momentum is, therefore,

$$\frac{\dfrac{m_0c^2}{a}}{\dfrac{m_0v}{a}} = \frac{c^2}{v}$$

This, in turn, in the case of a photon travelling with the velocity of light, reduces simply to c. Thus the momentum of a photon of energy $h\nu$ is $h\nu/c$. The initial momentum $h\nu_0/c$ of the photon is transferred in part to the electron, but in such a way that the total momentum in the original direction is unchanged, and the sum of the resulting momenta at right angles to this direction is zero. The first of these requirements yields

$$\frac{h\nu}{c}\cos\theta + \frac{m_0v}{a}\cos\phi = \frac{h\nu_0}{c} \tag{184}$$

and the second requirement yields

$$\frac{h\nu}{c}\sin\theta = \frac{m_0v}{a}\sin\phi \tag{185}$$

Solving (184) for $m_0v/a \cos \phi$ and squaring,

$$\frac{m_0^2v^2}{a^2} \cos^2 \phi = \frac{h^2}{c^2}(\nu_0^2 - 2\nu\nu_0 \cos \theta + \nu^2 \cos^2 \theta) \tag{186}$$

Squaring (185)

$$\frac{m_0^2v^2}{a^2} \sin^2 \phi = \frac{h^2}{c^2} \nu^2 \sin^2 \theta \tag{187}$$

Adding (186) and (187), and remembering that $\sin^2 \theta + \cos^2 \theta = 1$,

$$\frac{m_0^2v^2}{a^2} = \frac{h^2}{c^2}(\nu_0^2 + \nu^2 - 2\nu\nu_0 \cos \theta) \tag{188}$$

Since $a^2 = 1 - \frac{v^2}{c^2}$, $v^2 = c^2(1 - a^2)$ and $\frac{m_0^2v^2}{a^2} = \frac{m_0^2c^2}{a^2} - m_0^2c^2$ (189)

From (183) $\frac{m_0^2c^2}{a^2} = m_0^2c^2 + 2m_0h(\nu_0 - \nu) + \frac{h^2}{c^2}(\nu_0 - \nu)^2$ (190)

Hence, from (189) and (190)

$$\frac{m_0^2v^2}{a^2} = 2m_0h(\nu_0 - \nu) + \frac{h^2}{c^2}(\nu_0 - \nu)^2$$

Substituting in (188) and solving for $\nu_0 - \nu$,

$$\nu_0 - \nu = \frac{h}{m_0c^2} \nu_0\nu(1 - \cos \theta) \tag{191}$$

Or, since $c = \lambda\nu$,

$$\lambda - \lambda_0 = \frac{h}{m_0c}(1 - \cos \theta) \tag{192}$$

It is to be observed that the difference between the new and the original wave-length is zero for radiation scattered in the direction of the incident X-rays ($\theta = 0$). It becomes, however, progressively greater as θ is increased, until at $\theta = 180°$ it is $2h/m_0c$. This characteristic of the phenomenon as well as the absolute magnitude of the shift has been repeatedly checked experimentally. Furthermore, a study by the C. T. R. Wilson ray-track method has been successful in identifying the trails produced by electrons which have shared in this way the energy of the incident photons.

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CHAPTER XXXVII

THE WAVE MECHANICS

203. Phase Waves. One of the most surprising consequences of the relativity principle is the fact that events which appear simultaneous in one laboratory or *frame of reference* do not appear simultaneous when viewed from a laboratory which is moving with reference to the first. The time has now come to investigate a little more closely this difference in the viewpoint of two such laboratories. Consider, for example, a number of clocks, each with a hand which makes one revolution per second, scattered at regular intervals (Fig. 184) on the station platform along Ox and on the train along $O'x'$. Those arranged along the line $O'x'$ are moving to the right with a velocity v with reference to those along Ox. Each group of clocks is supposed to have been synchronized as described in Chapter XXXIII by light or radio signals, and we suppose in addition that at the instant that O' passed O the hands of the clocks at O and O' pointed vertically upward. Will the other clocks on the train be in synchronism with those opposite them on the station platform? Formula 167′ of Chapter XXXIII, which gives the relation between the time t in the fixed system and the corresponding time t' in the moving system, shows that this is not the case. Thus even at the instant $t = 0$ when O' is passing O, although $t' = 0$ at $x = 0$ it is not zero at any other point. In particular, at $x = 1$

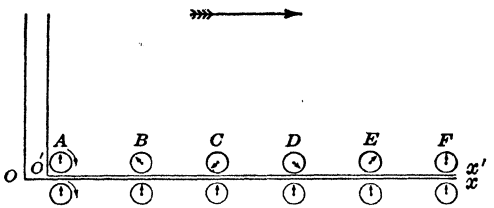


FIG. 184. Comparison between "Stationary" and Moving Clocks.

$$t' = \frac{-\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and at } x = 2, \quad t' = \frac{-\frac{2v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The discrepancy between the clocks on the train and on the platform is greater the more remote they are from O. This is exhibited graphically in Fig. 184. As the figure shows, the only clock on the train which shows the same time as that opposite it on the station platform is that at $O' = 0$. All the clocks on the train to the right of O' are approaching, but have not yet reached the vertical position.

As time goes on, clock B will be the next to reach this position, and will be succeeded consecutively by clocks C, D, E, F, etc. Since the arrival of the hand of a particular clock at this vertical position indicates the time $t' = 0$ at its particular location, we can easily calculate the distance x of a clock which gives this reading at a time t. Thus, setting $t' = 0$ in (167′), we see that

$$t = (v/c^2)x \quad \text{or} \quad x = (c^2/v)t$$

If each clock could flash a red signal light as its hand reaches the vertical position, and if in some superhuman way a stationary observer could see these flashes at the exact instant of their occurence, he would see a wave of red flashes travelling from left to right with a velocity $u = c^2/v$, obtained by dividing the distance of a light flash x by the time t at which it occurred. Since v, in accordance with the principle of relativity, is always less than c, the velocity c^2/v is always greater than the velocity of light. There is, however, no conflict with the fundamental conclusions of the relativity principle, since these waves, if they exist, could never by any conceivable means be made to transmit a signal.

If we turn our attention once more to the point $x = 0$, a clock opposite this point on the moving train will again point vertically upward at $t' = 1$, i.e., by Formula 167′ at the time $t = \sqrt{1 - v^2/c^2}$. At this instant the next clock on the train which seems from the platform to point upward is at a distance

$$x = ut = \frac{c^2}{v}t = \frac{c^2}{v}\sqrt{1 - \frac{v^2}{c^2}}$$

In fact, such points will occur regularly along the track at intervals of $(c^2/v)\sqrt{1 - v^2/c^2}$ from each other. If the hands of the train clocks make their revolutions, not in one second, but in time T, the distance λ between such points will clearly be

$$\lambda = T\frac{c^2}{v}\sqrt{1 - \frac{v^2}{c^2}} \tag{193}$$

It should not be forgotten that, from the standpoint of the train, these clocks are in synchronism, and reach the vertical position simultaneously. Any event, therefore, such as this arrival of the hands of the train clocks at the vertical position, which occurs simultaneously throughout the train, will appear from the station to be a localized event travelling as a wave crest from left to right with a velocity c^2/v. Similarly any periodic process which takes place everywhere on the train simultaneously, such as the repeated arrival of the hands of the clocks in this position, will appear to an observer on the station platform as a periodic wave with a velocity of propagation c^2/v and a wave-length $(Tc^2/v)\sqrt{1 - v^2/c^2}$.

204. de Broglie Waves and Quantum Theory. It is not at all certain that we should expect to find any such waves in practice, since they are contingent on the simultaneous occurrence of a periodic phenomenon over a considerable region of a moving body, and it is not at all certain that such processes take place in nature. It was Louis de Broglie who showed first in 1924 that the postulation of such waves associated with a moving electron led to a deeper insight into quantum phenomena. That the electron in linear motion should exhibit any periodicity, was at that time a new concept, but the relation which the quantum theory seems to require between energy and frequency left no doubt as to what frequency, if any, was to be associated with it. According to the relativity principle, the energy of the electron, as viewed from the moving electron itself, should be m_0c^2, and the frequency of the periodic disturbance in the electron, as viewed from the electron, should accordingly be given by the relation

$$m_0c^2 = h\nu \tag{194}$$

Or

$$T = \frac{1}{\nu} = \frac{h}{m_0c^2}$$

By (193) the wave-length should accordingly be

$$\lambda = \frac{h}{m_0c^2}\frac{c^2}{v}\sqrt{1 - \frac{v^2}{c^2}} = \frac{h}{m_0v}\sqrt{1 - \frac{v^2}{c^2}}$$

This wave-length takes a still simpler form if we replace the rest-mass of the electron by its mass m when viewed by an observer from the " station platform," that is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence $$\lambda = \frac{h}{mv} \tag{195}$$

One of the most unsatisfactory aspects of the quantum theory as outlined in Chapter XXXVI was the absence of any theoretical foundation for the relation (179), which specified the possible orbits of the revolving electron in the hydrogen atom. If we multiply both sides of this relation by $2\pi/mv$, it becomes

$$2\pi r = n\frac{h}{mv} \tag{179'}$$

In this form it merely states that the circumference of the electron orbit must be an integral number of de Broglie wave-lengths. This is the sort of relation which governs the possible vibration frequencies of a stretched string, and would seem to give for the first time some measure of basis for the mysterious quantum relation. This idea of de Broglie was expanded by Schroedinger into a mathematical wave theory of the electron, in which the rather vague derivation of the fundamental quantum condition of de Broglie was replaced by one which was mathematically rigorous and compelling. It resulted, in fact, directly from properties of partial differential equations, which had been familiar to mathematicians for many years.

205. The Experimental Discovery of de Broglie Waves. It is interesting that the theory of these phase waves, or *matter waves* as they are sometimes called from their association with one of the fundamental constituents of matter, the electron, was developed without any experimental evidence for their actual existence. In this respect they resembled the electromagnetic waves of Maxwell, but, unlike these, the experimental proof of their existence did not have to wait for a quarter of a century. The discovery of such electron waves in the laboratory occurred in 1927 in an extraordinary series of experiments by Davison and Germer of the Bell Telephone Laboratories in New York. A stream of electrons from a hot filament was directed against the face of a single crystal of nickel, and the returning electrons were studied. It would scarcely be anticipated that the reflected electrons would obey any such simple laws as a billiard ball in its

rebound from a smooth cushion. The fact that the crystal face was made up of regularly spaced individual atoms would be expected to preclude this. In fact, it was hoped in the early conduct of the experiments to get from them information regarding the shape of the nickel atoms, or at least regarding the character of the electric field in their vicinity. Actually the electrons were found to be reflected in great intensity only in certain definite directions, exactly as X-rays are reflected from similar crystal surfaces. The phenomenon was, in fact, so strictly analogous to X-ray reflection that, following the same methods of calculation used for X-rays, the wave-length associated with the incident electrons could be computed, and was found to check surprisingly well with that of the de Broglie waves, h/mv.

That such results could be expected, is due to the fact that the velocity v of an electron which has been accelerated through 150 volts yields a wave-length of 10^{-8} cm., i.e., about the same as that of an average X-ray. The spacing of the natural crystal grating was, accordingly, a convenient one for the study of these new waves. The actual wave-lengths calculated by Davison and Germer from these experiments were from 10 to 30 per cent shorter than the de Broglie waves predicted from the known electron velocities, but the discrepancy was quite naturally attributed to an increase in the velocity of the electron as it approached the crystal surface owing to the electric field in the vicinity of the nickel atoms.

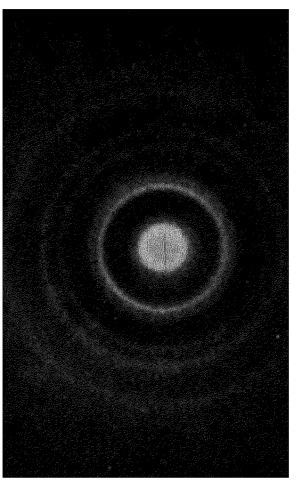


PLATE VI. Electron Diffraction by Gold Film.

Courtesy of Professor G. P. Thomson of the Imperial College of Science and Technology, London.

In 1928, G. P. Thomson in England, a son of J. J. Thomson, performed a similar experiment by shooting streams of electrons through thin sheets of metal. Since these metallic sheets consisted of aggregates of small crystals with all possible orientations, the experiment was analogous to the X-ray method used by Debye and Scherrer for studying powdered crystals. Just as in the Debye-Scherrer experiment, a photographic plate

behind the metal sheet showed, not merely a dark spot due to electrons which passed directly through the sheet, but a succession of concentric rings, each corresponding to the reflection of electrons from a particular crystal face (Plate VI). Knowing from X-ray data the arrangement and spacing of the individual molecules of the crystal, it was possible to calculate here also the length of the electron waves. In one respect the experiment was more satisfactory than that of Davison and Germer. Thomson used much faster electrons, which had been accelerated through 20,000 to 30,000 volts, and owing to their great initial energy, the change in wave-length produced by the attraction of the metal was negligible. As a result, the observed experimental wave-lengths checked quite accurately with the expected de Broglie waves.

Mention should here also be made of some interesting experiments which have been under way for a number of years under the direction of Debye in an effort to study the size and form of individual gas molecules by X-ray diffraction. The starting point in these experiments was the idea that gas molecules, like the individual crystals in an aggregate of powdered crystals, should reflect, or rather scatter, incident X-rays in particular directions. The carrying out of the experiment, and the proper interpretation of the results, have been matters of great difficulty, but in certain cases, notably that of carbon tetrachloride and related molecules, it has been possible in this way to determine with extraordinary accuracy the separation of the individual ions of the molecule. These measurements have now been made with electrons in place of X-rays, and dimensions have been obtained which check completely with those resulting from X-ray experiments.

206. The Electron as a Packet of Waves. In all these phenomena the corpuscular nature of the electron has played no part. It has behaved in every respect like a train of waves of wave-length h/mv. We are thus confronted by the fact that not merely radiation, but also electrons, have a dual aspect, in certain experiments behaving like corpuscles, whereas in other experiments they behave like systems of waves. Since the electron is one of the constituent parts of all matter, and since we can scarcely doubt that similar though shorter waves are associated with the proton, the other building stone of the elements, it would seem likely that matter as a whole as well as radiation possesses this dual nature.

If, however, we attempt to view the electron simply as a system of waves, we can scarcely fail to be troubled by the vagueness which its spacial boundaries take on in this light. For a 150-volt electron, as already pointed out, the wave-length has the same order of magnitude as the dimensions of the atom in which the electron is supposed to reside, and for the wave-length to have any meaning, the electron should consist of at least a number of waves. As a matter of fact, the mathematical properties of the Schroedinger theory seem to require that the wave system associated with an electron shall extend over very large areas, if not, indeed, throughout all space. The electron has apparently lost its local or atomic character. This difficulty was recognized by de Broglie, and was met by him in a very interesting and ingenious way.

If we imagine, not a single wave train of wave-length λ_1, but two superposed trains of wave-lengths λ_1 and λ_2, a crest of one of these trains may happen at a particular point to coincide with a crest of the other (Fig. 185). If the wave-lengths λ_1 and λ_2 are different, the next adjoining crests will not coincide, and the intensity of the resultant wave will decrease on either side of this point of reënforcement, eventually reaching a minimum, and then once more increasing to a second maximum or point of reënforcement. If the waves are advancing in the same direction and with the same velocity, the point at which the waves reënforce each other will also advance with this same speed. In fact, the phenomenon which we are describing is nothing more nor less than that which we have already met in the production of beats by two superposed sounds which have the same velocity of propagation but slightly different wave-lengths. If, however, two de Broglie waves of slightly different wave-lengths are superposed in this fashion, the situation is not quite the same, for their velocities of propagation are different. Thus, if we denote the velocity of the electrons which they represent by v_1 and v_2, the wave-lengths are h/m_1v_1 and h/m_2v_2, while the corresponding

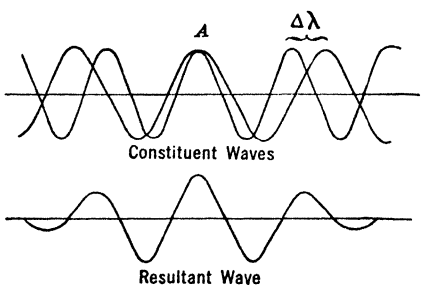


FIG. 185. Formation of Wave Packet by Two Superposed Waves.

velocities of the de Broglie waves are c^2/v_1 and c^2/v_2 respectively. The masses m_1 and m_2 would be the same in the two cases if it were not for the relativity variation of mass with velocity, and in many practical cases their difference is not great. If we neglected this small discrepancy, both wave-lengths and wave velocities would be inversely proportional to the electron velocities. The velocities of propagation would, accordingly, be proportional to the wave-lengths, and this, in turn, would mean that any point at which the two waves are in phase would remain always fixed in position, and would not advance at all with the individual waves. If, however, the difference between m_1 and m_2 due to relativity is taken into account, it was shown by de Broglie that such a point of phase coincidence would travel forward with a velocity which approaches v_1 as the difference between v_1 and v_2 is taken smaller and smaller.

He pictured, therefore, associated with an electron, not a single wave train, but a *group* of wave trains containing a range of wave-lengths. The location of the electron was thought of as the region where the crests of this group of waves happened to coincide. Such a *packet* of wave crests would then be expected to advance at about the velocity v of the electron with which the waves are associated. This picture, although attractive in many respects, is disconcerting in others. If, for example, we ask what the velocity of the wave packet or " electron " is, we are unable to answer with precision. The mechanical velocity v associated with each individual wave of the packet is connected to λ by the relation $\lambda = h/mv$, but since the wave-lengths of the members of the group are different, their velocities, v, are different, and the velocity of the " electron " is left somewhat indeterminate. At first sight it might seem that this difficulty could be minimized by taking the range of wave-length present in the packet very small or even infinitely small. If we do this, however, we encounter new difficulty in the size of the region within which the waves of the group reënforce one another to form the " electron." Consider, for example, two waves which differ in wave-length by a small amount $\Delta\lambda$. As shown in Fig. 185, the crest of one may coincide with the other at a particular point A, and owing to the difference in wave-length $\Delta\lambda$, the coincidence of phase will be less complete at the next adjacent wave crest, and will be progressively less at succeeding crests. The broadness of the region within which the two waves will be substantially in phase is

determined by the size of $\Delta\lambda$, and will become progressively greater as $\Delta\lambda$ is taken smaller and smaller. Thus a reduction in $\Delta\lambda$, which would seem to be required in order to specify more accurately the *velocity* of the "electron," is precisely the step which tends to make more uncertain its *location.*

207. The Uncertainty Principle of Heisenberg. This picture of the electron as a packet of de Broglie waves would appear to be defective, since it can give the exact position of the electron only at the expense of uncertainty as to its velocity, or conversely it can give its exact velocity only at the expense of uncertainty as to its position. During the past few years a number of attempts have been made to formulate a new quantum mechanics which should retain all the advantages of the Schroedinger wave mechanics and at the same time avoid its difficulties. Although some of these attempts, notably that of Dirac in England, have been astonishingly successful in accounting for a wide variety of physical phenomena, the same difficulty in the way of simultaneous definition of position and velocity or position and momentum has appeared in every case. In 1927, Heisenberg suggested that, rather than a defect in the new mechanics, this difficulty might correspond to a fundamental attribute of nature itself. At any rate, as he pointed out, we are unable experimentally to measure, with high precision, one of these two properties of a body, without making some sacrifice in the accuracy with which we can measure the other.

Suppose, for example, that we wish to locate accurately the position of a tiny body such as an electron. The microscope would seem to be the proper instrument for such a task, but the accuracy with which it can designate the position of any object is limited by diffraction to approximately a single wave-length of the light used. In order to locate the object with greater precision, light of shorter wave-length would be required, and it is not inconceivable that some day efforts should be made to design a γ-ray microscope for such a purpose. The use, however, of shorter wave-lengths would entail photons with higher energy and higher momentum, and this greater momentum would be passed on to the electron under observation by the Compton effect. The very process of measuring accurately the position of the electron would, therefore, alter its momentum. The extent of the interdependence of position and momentum is easily calculated in this case. The momentum of the photon,

as shown in the paragraph dealing with the Compton effect, is $h\nu/c$ or simply h/λ. A simple numerical calculation shows that the change in wave-length of the scattered radiation due to impact with an electron is a small fraction of the total wave-length. If, therefore, the maximum scattering angle of 180° is considered, the change in momentum of the photon will be approximately twice its initial momentum, namely, $2h/\lambda$, and this will also be the amount by which the momentum of the electron is changed. This will represent, therefore, the maximum uncertainty in the momentum of the electron introduced by the experiment to determine its position. It is an interesting and apparently significant fact that the product of the uncertainty of the position measurement, λ, and that in the momentum, $2h/\lambda$, is of the order of magnitude of h.

The persistence with which these uncertainties occur led Heisenberg to suggest as a new physical principle that the simultaneous measurement of position and momentum is subject to an indetermination such that the product of the uncertainties is of the order of magnitude of h. This *principle of indetermination* has been more disturbing to the older physical concepts than even the principle of relativity. For, although relativity forced us to abandon the concepts of time and space as independent physical entities, it was found possible to regain everything essential which had been lost by picturing all processes as taking place in four-dimensional space-time. The independent existence of position and velocity, however, would seem to lie at the very basis of our scientific thought. Our ability, for example, to predict the future behavior of a physical system would seem to depend on our knowledge of the present state of the system, i.e., of the present position and velocity of all its constituent parts. The belief has been freely expressed in some quarters that this new principle undermines the very concept of strict causality in physical phenomena.

CHAPTER XXXVIII

COSMIC PHYSICS

It is a fair statement that the most epoch-making of the developments in physics during the last two or three decades have had to do with the nature of matter and, in particular, with the fundamental unit of matter, the atom. In thinking of modern physics, therefore, we are accustomed to think of the smallest things about which we have any knowledge, a submicroscopic universe, if you will. Nevertheless, the knowledge which has been gained about atoms and radiation has contributed greatly to our understanding of the universe as a whole. Wellnigh all of this information about the vast spaces outside our own planet has come to us through light. So long as the only known property of light was its rectilinear mode of propagation, the information which it could bring was very limited. By trigonometric observations the dimensions of our solar system could be determined and the distances of a very few stars. Most of these, however, were so far away that no difference in their apparent direction from opposite sides of the Earth's orbit or *parallax*, as it is called, could be detected. It could be said with certainty only that most of the stars are so remote that more than 20 years are required for light to traverse the distance between them and us.

208. Stellar Spectra. The Constitution of the Stars. The discovery of line spectra, the Fraunhofer lines in the spectrum of sunlight, and the relation between such spectra and the emitting or absorbing element, opened up a direct means for studying the chemical constitution of the Sun and the other stars. The first stellar spectra were photographed about 1871 by Professor Henry Draper of New York University. In order to produce such photographs it is necessary only to replace the telescope eyepiece by a photographic plate, and to interpose before the objective, or between plate and objective, a thin prism. This prism serves to spread out the point image of the star into the desired complete spectrum. The distance of the star renders the rays of light incident on the telescope parallel, thus making

the use of a collimator unnecessary, and the minuteness of the star image makes it possible to dispense with the collimator slit. Such a star spectrum, like that of the Sun, consists in general of a continuous spectrum traversed by dark lines. As was first pointed out by Kirchhoff, the dark lines are almost certainly due to absorption of light from the incandescent material of the star by gases on its periphery. In the light of our present knowledge of spectra, they give direct evidence of the existence of elements familiar to us on the Earth in both the Sun and the other stars. Thus fifty elements have been definitely recognized in the Fraunhofer lines of the Sun, and seven others are probable.

The underlying continuous spectrum serves, on the other hand, as a means of estimating the temperature of the star. As has already been pointed out, the wave-length at which the maximum energy is emitted from an incandescent body is a known function of the temperature, and the application of this knowledge to stellar spectra has shown that the surface temperatures of the stars vary from about 3000° abs. in the redder type to 10,000 or 13,000 in the so-called blue stars. Our own Sun has a temperature of about 6000°. Of course, such measurements are incapable of giving any information as to the temperatures which may exist in the interior of the stars, but theoretical considerations indicate that these may be as high as several million degrees.

A relation was very early recognized between the line spectra present and the temperature so determined. Thus the hotter stars were found to exhibit almost exclusively the lines of hydrogen and helium. They were called for this reason hydrogen or helium stars. Did this mean that only hydrogen and helium existed in them, and if so, why did the constitution of the star depend so intimately on its temperature? It was the Bohr theory which in the second decade of the present century first answered this question with complete satisfaction. The lines in the visible spectrum are produced in all cases by changes which take place in the *state* (or orbit) of an electron in the outskirts of the atom. Owing to the process of ionization these outer electrons may be removed, and when such has been the case, the atom can no longer emit or absorb these visible radiations. Such ionization can be produced by the impact of electrons, ions, or other molecules. As the temperature of the gas rises, the average energy of the individual particles increases, and with it the severity of the impacts between them. Unless the

energy of the impinging molecules is at least equivalent to that of an electron which has fallen through the potential required to ionize the atom, the impact will be perfectly elastic, and no ionization can take place; but if it exceeds this amount, ionization is probable. The explanation, therefore, for the absence of any spectrum except that of hydrogen and helium is simply that the heavier elements have lost in this way all their radiating electrons. This theory is corroborated by the discovery in the laboratory that the ionization potentials of He and H are the highest of any of the elements, and that they would accordingly be the last to cease radiating as the temperature of the star increased. In the case of the heavier elements, the innermost electrons are responsible for the X-ray spectra, whereas the intermediate electrons are capable with suitable excitation of emitting ultra-violet radiation. That the high temperature of the Sun and the stars is adequate to excite some of these intermediate electrons can scarcely be doubted, but the radiation which they would thus emit would certainly be absorbed in the upper regions of our own atmosphere, thus producing intense ionization.

That such a layer of ions really exists in the upper atmosphere is placed beyond a doubt by the way in which radio waves are refracted to follow the curvature of the Earth, and by the phenomenon of the *aurora borealis*. This is almost certainly an electric discharge taking place in the ionized region 50 to 100 miles above the surface of the Earth. Because its existence was first postulated independently by Kennelly in this country and Heaviside in England in order to explain electric wave propagation, this ionized layer is known as the Kennelly-Heaviside layer.

209. The Distance of the Stars. Spectroscopic Parallax. The knowledge derived from stellar spectra is not restricted to information concerning the constitution of the stars. Experiment in the laboratory leaves no doubt that the extent of ionization in a gas due to molecular impact is a function not merely of temperature but of pressure as well, and the pressure of an absorbing gas in the periphery of a star should quite obviously depend on the intensity of the gravitational field in which it finds itself. This, in turn, should depend on the mass and the size of the star. At lower pressures, i.e., in weaker gravitational fields, the separated electrons will come less frequently within

the attractive power of positive ions, and the formation of new ions will be compensated to a smaller extent by the recombination of those already existing. The line spectrum of a star, therefore, in combination with the information which its continuous spectrum gives regarding temperature, should be capable of giving us information concerning its size. Although these methods, developed by Adams and Kohlschütter at the Mount



PLATE VII. Spiral Nebula.
Courtesy of the Mt. Wilson Observatory.

Wilson observatory in California, are not yet capable of giving absolute quantitative information concerning size, pressure, mass, etc., it has been definitely shown that stars which have the same types of both continuous and line spectra have approximately the same intrinsic brightness, i.e., they would appear equally bright if they were located at the same distance from the Earth. Such a conclusion has been arrived at, of course, by a study of those stars which are so near that their distance can be measured trigonometrically by determining their parallax. Where the distance of one star of a particular type has been thus trigonometrically measured, the distance of any other star of the same

spectral type can now be determined by a simple comparison of their apparent brightness.

Although other corroborative methods have been developed, it is mainly by such evidence as this that we have been learning in recent years the extent and distribution of the stars in the stellar universe. Practically all the stars which are visible to the naked eye lie in a huge system shaped somewhat like a watch, perhaps 50,000 light years in its greatest diameter and 10,000 to 12,000 light years in thickness. It is this watchlike shape of our system of stars which makes their apparent density so much greater in the direction of the Milky Way, for there we are looking out edgewise in the direction of the remotest stars of the group. Just within or just outside this immediate system there are some ninety *globular clusters* consisting of perhaps 100,000 stars each, located at distances which range up to more than 100,000 light years. Still more remote are the spiral nebulae, which seem in many respects to resemble our own local system of stars. Some thousands of these objects are already known, and their distances vary from the 1,000,000 light years of the Andromeda nebula to at least 140,000,000 light years. Because of their resemblance to our own stellar system and their remoteness, they have been very appropriately termed *island universes*.

210. The Nature and Dimensions of the Stars. What are these stars which, to the number of several billions, fill the night sky? That they are of the same general character as our own Sun can scarcely be doubted. The diameter of the Sun is about 864,000 miles, and its average density is 1.4, i.e., a little greater than that of water. Its outer layers are certainly gaseous, and considering the high temperatures which exist in its interior, this must also be gaseous. The source of the energy which it has been radiating since the remotest geological epochs is still a puzzle. That the intensity of this radiation a million years ago was not greatly different from that which we are at present receiving, is proven by the similarity of the vegetation which thrived at that period, and which has been discovered in fossils. Yet, if the heat of the Sun were the result of a process of chemical combustion or even of radioactive disintegration, it should have been entirely expended in such a long period of time. The idea was first suggested by Eddington in 1924, and has since been quite generally accepted, that the actual matter of the Sun is

being converted into radiation somewhere in the great crucible of its interior. The energy mc^2 of an atom of mass m is somehow or other changed into a photon of radiant energy $h\nu$. Then, perhaps by successive impacts with electrons as it makes its way to the surface, its frequency is reduced by the Compton effect to that which we receive in the visible and infra-red region.

Until recently the Sun was the only star the dimensions of which could be directly measured. That the other stars were similar in physical dimensions as well as in spectra and temperature, could only be guessed. That they appeared even to the most powerful telescope as tiny points of light, rather than as disks similar to the Sun, was due to their remoteness. Assuming as correct the distance of any given star as obtained by trigonometric or spectroscopic measurement, it is possible to calculate

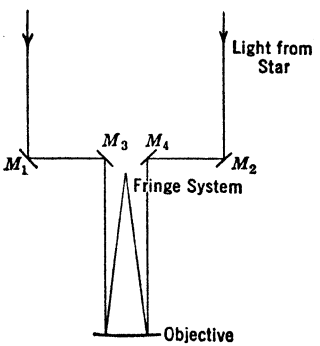


FIG. 186. Diagram of Interferometer for Measuring Stellar Diameters.

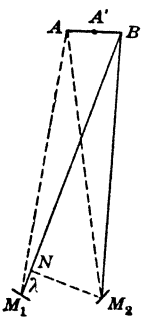


FIG. 187

the angle which our Sun would subtend if it were substituted for the star in question, and in all cases this angle is less than the resolving power of even the Mount Wilson telescope. Nevertheless, in the case of some of the nearer stars, this angle is still large enough to be detected and measured by an interferometer devised by Michelson. In the form in which it has been applied to the Mount Wilson telescope, it consists of two mirrors, M_1 and M_2 of Fig. 186, placed at the ends of a 20-ft. beam mounted at the end of the telescope. The mirrors M_3 and M_4 serve to reflect the light from M_1 and M_2 to the great objective. The theory of the device is intimately related to that of the Young pinhole experiment.

The mirrors M_1 and M_2 behave like two gigantic pinholes which would be expected to form a series of interference fringes at the focal plane of the telescope. In the discussion of Young's experiment we assumed that the light originated at a point source, and that a given wave crest arrived simultaneously at the two apertures. If, however, a difference of phase existed between the light which emerged at a particular instant from the two holes, the fringe system would still be produced, but would

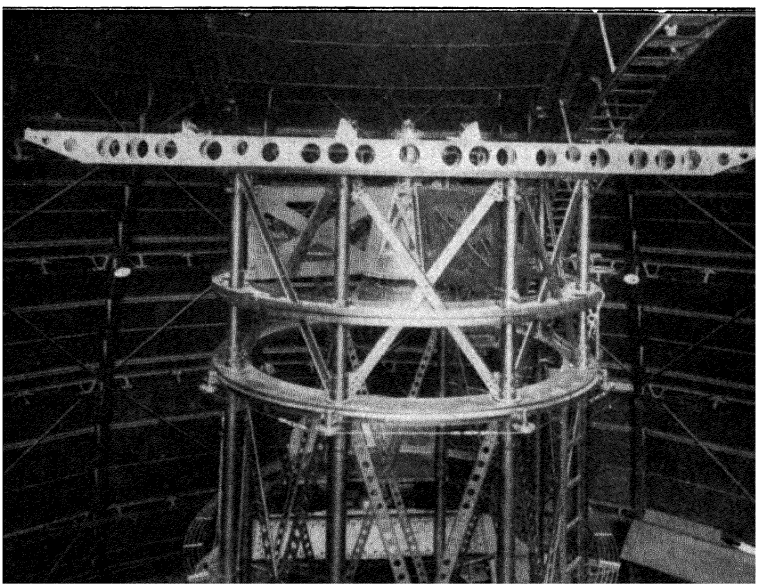


PLATE VIII. Michelson Interferometer Beam attached to 100 in. Mt. Wilson Reflector.

Courtesy of the Mt. Wilson Observatory.

clearly be displaced from its normal position on the screen. If, as in the interferometer designed by Michelson for astronomical purposes, the source is actually a disk of diameter AB (Fig. 187), the fringe system produced by the light from B will be displaced slightly with reference to that produced by A. Assume, for example, that $AM_1 = AM_2$ and that the difference between BM_1 and BM_2 is one wave-length λ. Then the difference between $A'M_1$ and $A'M_2$ will be $\lambda/2$ if A' is midway between A and B. At a point on the focal plane of the telescope where the light

from A via M_1 reënforces that via M_2, the light from A' via M_1 will destructively interfere with that via M_2, and more generally the entire fringe system from A will be blotted out by that from A'. Similarly the fringe system originating from every point between A and A' will be cancelled by that from a corresponding point between A' and B. With this particular distance between M_1 and M_2 no fringe system will, accordingly, be visible. In practice the mirrors M_1 and M_2 are gradually separated until this point is reached at which the fringes vanish. The angle θ subtended by AB at the telescope is then equal to M_1M_2N. Or $\sin\theta = \lambda/(M_1M_2)$. Here, as in the analogous problem of diffraction from a circular aperture, we have thought of the source as a slit of width AB. Here also we may think of the actual circular source as a slit of variable width, and can obtain the angle subtended by the true diameter by dividing the $\lambda/(M_1M_2)$ already found by 0.8.

The first star to be measured in this way, Betelgeuse in the constellation of Orion, was found to have a diameter 250 times that of the Sun; Antares, the largest thus far measured, is 460 times the size of the Sun. The smallest stars known are probably still a few tenths of the diameter of the Sun, and the ratio of the largest to the smallest is, accordingly, perhaps one or two thousand diameters, a surprisingly narrow spread if we take into account the entire gamut of linear dimensions from that of an atom to that of the universe as a whole. It is instructive to note that, in size, the human body is about midway between the atom and an average star.

211. The Doppler Effect and Stellar Velocities. There is still another entirely different type of evidence which can be gleaned from the spectra of the stars. It is a familiar fact that the pitch of a locomotive whistle is higher when it is approaching and lower when it is receding. This is due to the fact that the velocity v with which a given wave crest advances is independent of the motion of the source of the sound, and that, accordingly, if this source advances a distance V in the same direction during the same time (one second), a number of waves n will be crowded into the space $v - V$ instead of the usual space v. The wavelength is altered from v/n to $v - V/n$, and the number of crests which arrive at the ear of a stationary observer per second is increased from n to $[v/(v - V)]n$. Similarly, if the source of the sound is receding with velocity v, the frequency with which the

waves arrive at the ear is reduced to $[v/(v + V)]n$. This phenomenon is known as the *Doppler effect*, and applies equally well to light waves which originate in a moving source. Owing to the huge velocity of light, the velocity of any source V is small in comparison with the light velocity c. Hence a very good approximation to the altered frequency can be obtained by replacing

$$\frac{c}{c \mp V} n = \frac{n}{1 \mp \frac{V}{c}} \text{ by } n\left(1 \pm \frac{V}{c}\right)$$

The proportionate increase or decrease in the frequency is, accordingly, the ratio of the source velocity to that of light. The thermal velocity of the molecules in an incandescent gas results in the apparent emission of a narrow range of frequencies instead of the single frequency predicted by the quantum theory, and this is, in fact, one of a number of causes which lead to a slight broadening of spectrum lines.

Of still greater interest, however, is the information which it gives concerning the motion of stars in the line of sight. By a direct comparison of a given spectrum line emitted by a star with the same line as generated in the laboratory, it can be determined whether the star is approaching or receding from the Earth, and, if so, with what velocity. A surprisingly large proportion of the stars have been found to be *binary*, i.e., they consist of two stars describing orbits about each other. A study of the Doppler effect observed in the spectra of such stars shows which of the two is receding and which is approaching at a particular instant in its orbital motion, and a star can be recognized as such a binary even when its two components cannot be telescopically resolved, if its spectrum shows normally single lines split up in this way into doublets.

In recent years the results of the general theory of relativity have led to much speculation concerning the size of the universe. This general theory seems to indicate that the universe as a whole is finite, and permits some sort of estimate of its size. It does not, however, answer with any certainty the question as to whether this size is or is not constant. An expanding universe would mean that the members of it, the stars, are on the average receding from one another. This is, in fact, exactly the picture which a study by means of the Doppler effect gives of the motion

of the more remote spiral nebulae. All are receding, and the more remote they are, the greater is their velocity of recession. The more distant ones are apparently moving outward with a speed of several thousand miles per second.

The Doppler effect is not the only phenomenon which produces an apparent shift of spectrum lines from their normal position. Another cause of great interest is the gravitational field in which the light is generated. We have come to recognize that a light quantum or photon possesses a momentum h/λ and a corresponding mass $h/\lambda c$. If the photon is emitted by a star, it will be subject for a time to the star's gravitational attraction. It will accordingly have its momentum reduced and its wave-length correspondingly increased. All spectrum lines which originate in the intense gravitational field of a massive star should, therefore, be displaced somewhat toward the red end of the spectrum, and the amount of the *red shift*, as it is called, should give a measure of the mass of the star. This phenomenon was first predicted by Einstein on the basis of the general theory of relativity, and its existence has been definitely established by astronomical observations. The most interesting measurements of this sort have been those made on a few *white dwarfs*, as they are called. These stars, of which the Companion of Sirius is the best known, would appear from measurements of their surface brightness and total radiation to possess extraordinarily high densities, and their spectrum lines were found to be shifted toward the red by an amount which would require a density from 30,000 to 60,000 times that of water. Such great concentrations of matter would seem to require rather complete ionization of the atoms composing it and fairly close packing of the nuclei.

212. The Origin of Cosmic Rays. No account of the information which is being accumulated by modern physical methods regarding the vast spaces of the universe would be complete without returning once more to the newly discovered cosmic rays. That they originate not in the stars themselves, but in the spaces between them, is likely, because of the uniformity with which they seem to come from all directions. In Chapter XXXV, reference was made to their great penetrating power, and the conclusion was there stated that, if they belonged to the great family of electromagnetic radiation, they must be of extremely high frequency, well out beyond the γ-rays of radium. Although the question has not yet been finally settled whether they are

such high-frequency photons, high-speed electrons, or even hypothetical uncharged particles to which the name *neutron* has been given, they must in any case carry vastly more energy than has to date been detected in photons or electrons which originate in terrestrial processes. It is a fascinating pastime to speculate as to the probable source of this energy. Sir James Jeans inclines to the belief that the cosmic ray is a photon which has its origin in the coalescence of a proton and an electron. If such a union took place, the electric charges would be expected to cancel each other, and according to Jeans's theory, their combined mass m with its attendant energy mc^2 is transformed into a photon of energy $h\nu$. This at least would yield a frequency not far from those which are probably present in the cosmic rays.

Another alternative has been suggested by Millikan. If the atomic weights of the elements as obtained by positive-ray measurements, or even the chemically determined weights of elements which have been found to possess no isotopes, are carefully scrutinized, it will be observed that they are not precisely multiples of the atomic weight of hydrogen, i.e., of the weight of a proton. The discrepancy has been attributed to a reduction in the potential energy of the constituent electrons and protons when they are combined in the nucleus of one of the heavier elements. This reduction in the potential energy would, in the light of the equivalence of mass and energy, correspond to a reduction in mass. If, however, energy is to be conserved, this difference in energy must be radiated at the time of the formation of the atom, and it is precisely this radiation which Millikan believes to constitute cosmic rays. From his viewpoint these mysterious rays are the signals which come to us to announce the building up of heavier and more complicated elements from lighter ones. Whether, however, the cosmic ray is really announcing the birth or the disappearance of matter, it seems destined eventually to open up an entirely new volume in our knowledge of the universe about us.

The theory of Millikan is rendered more plausible by recent experiments performed in Berlin by W. Bothe and H. Becker. It was shown in 1919 by Rutherford that the impact of an α-particle on a light element such as nitrogen and aluminium will sometimes separate a proton from the nucleus. This proton is driven off in certain cases with a kinetic energy greater than that of the incident α-particle, the difference in energy being ap-

parently derived from that ordinarily locked in the atom. In 1931 Bothe and Becker, while bombarding beryllium atoms in this way, obtained evidence of the emission of a penetrating radiation similar to cosmic-rays, which they interpreted as the radiation accompanying the formation of carbon from beryllium. Here also a greater amount of energy is present in the new radiation than was possessed by the α-particle, and energy originally locked in the atom has been transformed into a different and perhaps more available form. These experiments constitute the first cases in which atomic energy has been artificially extracted, and it is not impossible that they may be the first steps toward an understanding of the means by which this vast storehouse of energy may be tapped for useful purposes.

SUPPLEMENTARY PROBLEMS

1. If Hiawatha could shoot 10 arrows in 6 seconds, and if, as the poet relates, "the tenth had left his bow-string ere the first to Earth had fallen," what initial velocity must he have imparted to them, and to what height did they rise? Neglect air resistance.

2. If air resistance could be neglected, how long would be required for a bullet, shot vertically downward from a balloon at an elevation of 5000 ft., to reach the ground? Assume an initial velocity of 600 ft./sec.

3. In an attempt to calibrate a camera shutter, a steel ball was photographed as it dropped in front of a centimeter scale. On the film the ball appeared as a streak extending for 130 cm. downward from a point 360 cm. below the point of release. What was the time of exposure?

4. An automobile weighing 3200 lb. is travelling with a velocity of 30 miles an hour when the brakes are locked. If it comes to rest after sliding a distance of 50 ft., what is the frictional force exerted on it? What is the coefficient of friction between the tires and the road?

5. A stone is thrown along a smooth icy surface. If the coefficient of friction between the stone and the ice is 0.1, and it comes to rest after sliding 300 ft., what was the initial velocity of the stone?

6. What is the shortest possible time in which a man can do the 100-yd. dash if the coefficient of friction between his feet and the track is $\frac{1}{3}$?

7. A locomotive exerts a constant pull of 2000 lb. on a train of 50 tons. If we neglect the frictional force, how long will it take the train to acquire a velocity of 30 miles an hour? How long will it take if there is an opposing frictional force of 1000 lb.? How far will the train have gone in this last case?

8. What would a 100-gr. mass weigh in an elevator which is moving upward with an acceleration of 2 meters/sec.: (*a*) if it were weighed by a spring balance, (*b*) if it were weighed by a chemical balance?

9. A monkey of mass m clings to a rope which passes over a pulley to another mass m. The monkey starts to climb the rope. Describe the ensuing motion: (*a*) if there is no friction in the pulley, (*b*) if there is appreciable friction.

10. Two pulleys A and B are attached to the under side of a horizontal beam. A load of 100 gr. is hung by a hook to a third pulley, which is suspended between A and B by a cord which passes over both. If a load of 40 gr. is attached to one end of this cord and a load of 60 gr. to the other end, how will the system move? Neglect the weight of the pulleys and friction. Assume that the cords supporting the third pulley are vertical.

11. A mass of 20 lb. is suspended by a cord from a vertical wall. It is pulled outward by a horizontal force until the suspending cord makes an angle of 30° with the vertical. What is the tension in this cord, and what is the horizontal force?

12. A 30-ft. ladder rests against a smooth vertical wall. The distance of its base from the wall is 3 ft. If the weight of the ladder is 40 lb., and if it is uniformly distributed, calculate the force which it exerts against the wall.

13. A crane consists of an upright mast, to the base of which is attached a beam. The other end of the beam is connected to a pulley on the upper end of the mast by a cable. Mast, beam, and cable form an equilateral triangle. If a load of 100 lb. is hung from the end of the beam, what is the tension in the cable?

14. If the beam of the crane in Problem 13 is hung horizontally in such a way as to form with mast and cable an isosceles right triangle, what is the tension in the cable?

15. What is the tension in a 30-ft. tight rope if a 150-lb. man depresses it only 1 ft. when standing at the center?

16. A 100-gr. weight rests on a plane which is inclined at an angle of 45° with the horizontal. What force parallel to the plane will hold the weight in equilibrium if the coefficient of friction between plane and weight is $\frac{1}{3}$?

17. What acceleration would be experienced by the weight of Problem 16 if it were allowed to slide freely down the plane?

18. A 150-lb. man lies in a hammock. If one of the supporting ropes makes an angle of 30° with the horizontal and the other an angle of 60°, what is the tension in each?

19. An airplane while travelling at a speed of 200 miles an hour describes a horizontal curve of 2000-ft. radius. If the operator weighs 150 lb., what is the direction and the magnitude of the resultant force acting on him?

20. If two 1-gr. masses were situated at a distance apart of 1 meter, and if they were subject only to their mutual attraction, approximately how long would be required for their separation to be reduced by 1 cm.? Assume a constant acceleration equal to that possessed by them initially.

21. What force of attraction does the Sun exert on every gram at the surface of the Earth?

22. A certain roof makes an angle of 45° with the horizontal. If the coefficient of friction between a mass of ice and this roof is 0.1, if the height of the eaves above the ground is 40 ft., and if the mass of ice becomes dislodged at a point 10 ft. above the eaves, how far from the house will it strike the ground?

23. A ski jumper gains speed by sliding down an incline 300 ft. long at a pitch of 30°. If the coefficient of friction between snow and skis is 0.1, what velocity will he attain? If he takes off horizontally, and if the slope below continues at an angle of 30°, what will be the length of his jump?

24. An automobile while travelling on a level road is about to negotiate a curve of radius 100 ft. If the coefficient of friction between road and car is $\frac{1}{3}$, what is the maximum speed at which the curve can be safely made without skidding?

25. A 16-lb. hammer is whirled on the end of a wire in a circle of radius 6 ft. It is released by the athlete with a velocity of 60 ft./sec., and at an initial angle of 45° with the horizontal. (*a*) What force must the athlete exert on the wire at the instant he releases the hammer? (*b*) Neglecting the resistance of the atmosphere, calculate the horizontal distance which the hammer will travel and its time of flight.

26. Calculate the central acceleration of a body at the equator due to the Earth's rotation. How much less will a 1-gr. mass weigh at the equator than at the pole owing to this acceleration?

27. Show that if a hole were drilled completely through the center of the Earth, and if a body were dropped into this hole from the Earth's surface, it would execute simple harmonic motion. Assume that the body will be attracted only by material which lies nearer to the center of the Earth than itself, and that the density of the Earth is uniform. Calculate the time required for one such oscillation.

28. A simple pendulum 1 meter long oscillates through a range of 20 cm. Calculate the velocity of the bob as it passes the equilibrium position.

29. In a chain hoist the load is raised by pulling downward on a heavy chain. It is observed that, when the chain is pulled down a distance of 4 ft., the load is raised only 1 in. What is the theoretical mechanical advantage? If its efficiency is 15%, what force would be required to raise a load of 1 ton?

30. A painter's sling is suspended by means of a rope which is hung over a pulley, and is attached at both ends to the sling. The painter is able to raise himself by pulling downward on one of these ropes and thus shortening the distance between himself and the pulley. If his own weight and that of the sling total 160 lb., what force must he exert for this purpose? If the pulley is hung by a hook to the gutter of the house being painted, how great a strain is brought to bear on it? What would be this strain if the painter were being raised by a comrade from the ground?

31. Design a modification of the pulley system used in the previous problem, which shall reduce the required force.

32. An automobile jack is operated by pushing downward on a long lever. The distance which the end of this lever is depressed in a full stroke is 1 ft. If 40 strokes are required to lift the load a distance of 6 in., what is the theoretical mechanical advantage of the device? If the efficiency is 15%, what force is required to raise a load of 1000 lb.?

33. What is the kinetic energy of a 3-ton flywheel when making 180 R.P.M. if the average diameter of its rim is 12 ft.? Assume the mass to be all at the rim. How many horse-power would be required to bring it up to speed in 1 minute?

34. A water wheel operates with a fall of 25 ft. If the velocity of the water in the exhaust is 6 ft./sec., what is the efficiency of the wheel?

35. What steady forward force must a horse exert while travelling 4 miles an hour if he is to work at the rate of 1 horse-power?

36. An automobile weighing 3000 lb. and travelling 50 miles an hour collides with the rear of an automobile weighing 2500 lb. and travelling in the same direction with a velocity of 25 miles an hour. What will be their common velocity after collision?

37. If in Fig. 22 of Chapter VI the load m_1 is initially resting on some horizontal support, and if there is sufficient slack in the rope so that the automobile can attain a speed of 30 miles an hour before the cord tightens, calculate the common velocity of car and load at the instant they begin to move together. The mass of the car is 2500 lb. and that of the load is 500 lb. How high will the load rise?

38. A wheel and axle is set in rotation by a cord wrapped about the axle and a weight of 1 lb. attached to the end of the cord. The wheel can be treated as a solid cylinder of mass 3 lb. and diameter 10 in. How long is required for the 1-lb. weight to fall 3 ft., if the diameter of the axle is 1 in.?

39. The total length of the cord in the preceding problem is 3 ft., and after it has all unwound, it begins once more to wind up on the axle. If by this process the weight is raised 2 ft. above its lowest point, what frictional force is exerted by the bearing in which the axle rotates?

40. A fire hose 3 in. in diameter throws a stream of water with a velocity of 64 ft./sec. If this strikes the vertical wall of a building, what force does it exert? Neglect the velocity with which the water rebounds from the wall, and calculate the momentum which the wall imparts per second.

41. A machine gun discharges one hundred 50-gr. bullets per minute with a velocity of 400 meters/sec. What is the average force of recoil?

42. The density of a fluid may be found by means of a device known as balancing columns. A Y-shaped tube is inverted with one tube in a beaker of water and the other tube in a vessel containing the unknown fluid. When a little air is sucked from the tube above these fluids, the water climbs to a height of 30 cm. above the level outside, while the unknown fluid climbs but 25 cm. What is the density of the unknown fluid, and why? What is the pressure of the air remaining in the tube if the barometric pressure is 76 cm.?

43. A rectangular coal barge has vertical sides. If a depth of 3 ft. is submerged in sea water of density 65.0 lb./ft.^{3}, what depth will be submerged in fresh water of density 62.4 lb./ft.^{3}?

44. Oil of density 0.7 is poured into one branch of a U-tube which contains enough mercury to keep the bend full. When the column of oil is 39 cm. high, how much higher will it stand than the mercury in the other branch?

45. A lead sinker is hung beneath a 100-gr. block of wood. When both block and sinker are immersed in a vessel of water, their weight is 50 gr., whereas when the block is above the water and only the sinker is immersed, their weight is 175 gr. What is the density of the block? If the density of lead is 11.3 gr./cm.^{3}, what is the mass of the sinker?

46. The level of the kerosene in a certain 5-gal. can is 30 cm. above the faucet. Calculate (*a*) the pressure in excess of atmospheric pressure at the faucet when it is closed, and (*b*) the velocity with which the kerosene flows out when the faucet is opened. The density of kerosene is 0.8 gr./cm.^{3} Assume that the kerosene is a perfect fluid.

47. Water is flowing through a pipe which contains a constriction 1 cm.^{2} in area. The pipe outside the constriction has a cross section of 4 cm.^{2} A U-tube partially filled with mercury is hung below the pipe with one branch open to the constriction and the other attached to the open part of the tube. If the difference in levels of the mercury in the two sides of the tube is 10 cm., what is the velocity of flow of the water in the pipe?

48. If 5 horse-power is delivered to the propeller of a boat, how many B.T.U.'s per hour goes into the water, (*a*) if the boat is tied, (*b*) if it is moving through the water with a constant velocity?

49. The heat of combustion of alcohol is 7183 cal./gr. How many grams of alcohol will be required to convert 1 liter of water at 20° C. into steam at 100° C.?

50. The heat of combustion of dynamite is 1300 cal./gr. How much dynamite will be required to blow a 1-ton stone 2 ft. into the air if all the energy is used for this purpose?

51. A certain ocean liner develops 20,000 H.P. If the heat of combustion

of coal is 14,000 B.T.U./lb., and if the efficiency of the engines is 25%, how many tons would be required for a 9-day run?

52. A vessel of negligible mass contains a mixture of water and ice, the mixture weighing 1000 gr.; 125 gr. of steam at 100° C. are passed into the mixture and the final temperature is found to be 25° C. How much ice was in the original mixture?

53. One thousand grams of melted lead at a temperature of 340° C. are dropped into a vessel containing 1000 gr. of water at 20° C. Neglecting the heat capacity of the vessel, calculate the final temperature of the water. The specific heat of the lead may be taken as the same in the liquid and the solid state.

54. One hundred grams of lead shot at 100° C. and 50 gr. of ice at 0° C. are dropped into a calorimeter weighing 100 gr. and containing 500 gr. of water at 20° C. What happens? Assume the specific heat of the calorimeter to be 0.1.

55. How much gas must be burned under a 2-lb. aluminium kettle containing 10 lb. of water at 50° F. to boil away half the water? Assume $\frac{3}{4}$ of the heat to be effective. The heat of combustion of gas is about 540 B.T.U./ft.^{3}

56. An old rough rule for a gasoline engine is to the effect that 1 pt./hr. will yield 1 horse-power. If this is the case, what is the efficiency of the engine? The heat of combustion of gasoline is 20,250 B.T.U./lb. The density of gasoline is 0.7. Assume 1 pt. of water to weigh 1 lb.

57. A Carnot engine operates between temperatures of 600° C. and 200° C. and delivers 10 H.P. What is the efficiency of this engine? The exhaust is used to heat water. If 50 lb. of water flows through the heater per minute, how much will its temperature be raised?

58. What would be the maximum possible efficiency of an engine which used mercury as its working substance and which operated between the boiling point of mercury, 357° C., as a source and 0° C. as a refrigerator? What would be the maximum number of foot-pounds of work which such an engine could obtain from a ton of coal?

59. A tank contains gas at a pressure of 3000 lb./in.^{2} and a temperature of 20° C. If 5000 lb./in.^{2} is the maximum pressure which it will stand, at what temperature will it explode?

60. The specific heat at constant pressure of carbon dioxide is 0.2025 cal./gr. °C. How does this compare with the value which you would expect from theoretical considerations? To what do you attribute the discrepancy?

61. How much liquid oxygen would have to be sealed into a closed bomb with 1 gr. of carbon in order to ensure complete combustion to CO_2 on ignition?

62. The density of water at 100° C. is 0.958 gr./cm.^{3}, whereas that of steam at the same temperature is 0.000589 gr./cm.^{3} If it takes 539 calories to convert each gram of water into steam, what fraction is retained in the form of heat by the steam, and what fraction is spent in expansion?

63. An air bubble at the bottom of a pond 6 meters deep has a volume of 0.01 cc. Find the volume just as it reaches the surface, the barometer standing at 76 cm.

64. Ten grams each of hydrogen, oxygen, and nitrogen are placed together in a vessel of 1 cubic meter volume. What is their combined pressure at 0° C.?

65. Calculate (*a*) the molecular heat at constant volume of nitrogen, (*b*) its specific heat at constant volume, and (*c*) its specific heat at constant pressure.

66. Show that if two condensers of capacity C_1 and C_2 respectively are connected in series, they are equivalent to a single condenser of capacity $1/\frac{1}{C_1}+\frac{1}{C_2}$.

67. If a condenser of 10 E.S.U. is connected in series with one of 20 E.S.U., and if a potential of 2 E.S.U. is applied to them when so connected, what will be the magnitude of the charge which they take?

68. Express the data and the answer in Problem 67 in practical units.

69. Calculate the capacity of a spherical condenser in which the radius of the inner sphere is 4 cm. and its separation from the spherical shell is 1 cm. (*a*) if the two spheres are separated by air, and (*b*) if they are separated by paraffin.

70. What force would be exerted on a unit charge if it were located midway between the sphere and the shell in Problem 69, and if the potential difference between sphere and shell was 2 E.S.U.?

71. What initial velocity would have to be imparted to a projectile to enable it to escape from the Earth's attraction, i.e., to travel outward to an infinite distance?

72. A certain bar magnet may be thought of as a rod of length 10 cm. and mass 50 gr. If the distance between the poles is 8 cm., and the pole strength is 100 E.M.U., what will be its period of oscillation if suspended in the Earth's field at a point where the horizontal intensity is 0.17 gauss?

73. A magnetized bar of cobalt steel 5 cm. long and weighing 50 gr. is supported from beneath by the repulsion of a similar magnet parallel to it and 1 cm. distant. Assuming the distance between the poles to be 4 cm., what is the pole strength?

74. A tangent galvanometer contains 20 turns of wire each 10 cm. in radius. If it is set up at a point where the intensity of the Earth's field is 0.17 gauss, through what angle would the needle be deflected by a current of 0.25 ampere?

75. A group of two hundred 50-watt lamps are supplied by a generator through a line of 0.1 ohm resistance. If the lamp terminals are maintained at a difference of potential of 100 volts, calculate: (*a*) the total current taken by the lamps, (*b*) the drop in potential in the line, (*c*) the power supplied the lamps and the power wasted in the line, (*d*) the horse-power of the generator.

76. One hundred grams of water at 20° C. are contained in a copper vessel. If a 75-watt heater is connected to the 110-volt mains for 1 hour, what is the final state of the water? Neglect the heat capacity of the vessel. The specific heat of steam is 0.5 cal./gr. °C.

77. A generator of E.M.F. 110 volts is used to charge a 12-volt storage battery. If the rate of charge is to be 10 amperes, and the internal resistance of the battery is 0.1 ohm, what must be the resistance of the rheostat used in series with the battery? What power is spent (*a*) in charging the battery, (*b*) in heating the electrolyte, and (*c*) in heating the rheostat?

78. Compare the cost of electricity for heating at \$0.10 a kilowatt-hour and gas at \$1.00 per 1000 ft.^{3} The heat of combustion of gas is 540 B.T.U./ft.^{3}

79. If coal costs \$10.00 a ton and provides 14,000 B.T.U./lb., what is the

minimum price per kilowatt-hour at which electrical energy can be supplied? Assume the power to be generated by steam engines working between 10° C. and 190° C.

80. A certain galvanometer, having a resistance of 1000 ohms, is in series with a resistance of 9000 ohms. Both are shunted by a resistance of 10,000/9999 of an ohm, and a current of 1 ampere passes through the galvanometer and shunt together. Find (*a*) the current through the galvanometer, and (*b*) the drop in potential across it.

81. The resistance of a certain direct-reading milliammeter is 10 ohms. What shunt will convert it into a direct-reading ammeter?

82. A voltmeter, the maximum reading of which is 3 volts, has a resistance of 600 ohms. What resistance in series with this meter will convert it into an instrument which will read up to 150 volts?

83. A current of 10 amperes flowed through a silver electrolytic cell for 2 hours and deposited 80 gr. of silver. What is the atomic weight of silver? How many ions took part in the transfer?

84. The electrochemical equivalent of an element is the mass which is deposited in electrolysis by 1 coulomb. Calculate the electrochemical equivalent of Cu in $CuSO_4$, Cu in CuCl, Fe in $FeCl_3$, and O in H_2O.

85. The voltage across an electrolytic cell containing acidulated water is 2.7. Find the energy in kilowatt-hours required to liberate 1 cubic meter of oxygen under standard conditions of temperature and pressure.

86. A certain spherical condenser has an inner sphere of radius 30 cm. and an outer shell of radius 31 cm. What is its capacity? If it is charged to a potential of 3000 volts and discharged 20 times a second through a sensitive galvanometer, what average current will flow?

87. If the coefficient of viscosity of air at room temperature is 184.4×10^{-6}, calculate the rate of fall under gravity of a mercury droplet 10^{-4} cm. in diameter.

88. If the droplet of Problem 87 is uncharged and is falling through an electric field of 3000 volts/cm., calculate the change in its velocity if it captures a single fundamental charge of such a sign as to increase its rate of fall. Make use of the fact that its rate of fall is proportional to the total force acting on it.

89. An oscillograph tube designed by the Western Electric Co. contains a hot filament from which a stream of electrons is emitted. This stream of electrons is accelerated by a potential of 300 volts. What velocity do the electrons acquire? If the tube is so set up that the electrons move at right angles to the Earth's magnetic field ($H = 0.6$), what is the radius of the circle in which they are deflected?

90. An electron passes between two metal plates 3 cm. apart, which are brought to a difference of potential of 3000 volts. If a magnetic field of 100 gausses in such a direction as to compensate for the deflecting action of the electric field exactly balances the electric field, what is the velocity of the electron?

91. How many volts would be required to give to a He nucleus the energy of an α-particle with a velocity $\frac{1}{10}$ that of light?

92. What would be the difference in the velocities of the two isotopes of chlorine if, when possessed each of a unit positive charge, they were accelerated through a difference of potential of 1000 volts? What would be the difference in energy of the two isotopes under the same circumstances?

93. The initial velocity of an α-particle from radium is 1.61×10^9 cm./sec. Calculate its energy.

94. If the half-value period of radium is 1580 years, calculate the average number of α-particles emitted from a gram of radium per hour during this period. How many calories would be released in this way in an hour?

95. A D.C. motor is identical in construction with a D.C. generator, and while the armature is in rotation it generates a back E.M.F. equal to that which it would produce as a generator when rotating with the same speed. The energy spent in driving a current through the armature against this back E.M.F. is converted by the motor into mechanical work. If the back E.M.F. is 105 volts and the armature current is 10 amperes, what is the horse-power delivered by the motor?

96. If the resistance of the armature of a certain 110-volt motor is 1 ohm, and if the motor delivers 2 H.P., what current flows through the armature? How much power is spent in heating the armature? What is the back E.M.F.?

97. What steady turning moment must be applied to a 5-H.P. generator if it spins at the rate of 1200 R.P.M.?

98. A square ring of copper is constructed from a copper rod 5 mm. in diameter. The length of each side of the square is 10 cm. What is the electrical resistance of the ring? If it is thrust between the pole pieces of an electromagnet into a region where the field strength is 10,000 gausses, with a velocity of 100 cm./sec., what current is induced in it?

99. What force was required to push the ring of Problem 98 into the magnetic field?

100. What current from a 110-volt, 60-cycle line will flow through a circuit containing an inductance of 10 henries, a resistance of 200 ohms, and a condenser of 1 microfarad? If such a circuit is set into oscillation, what is its frequency?

101. An air condenser is used in an oscillating circuit to generate oscillations at the rate of 1,000,000 per sec. How much would this frequency be altered by extracting the air from between the condenser plates?

102. An electron tube which is widely used in radio receiving sets has an amplification factor of 8. When the grid is connected directly to the filament, and the plate is maintained 40 volts positive with reference to the filament, a current of 2 milliamperes flows through the tube. If the plate potential is raised to 76 volts, what negative voltage between grid and filament will suffice to maintain the current at 2 milliamperes?

103. The grid of a vacuum tube is frequently separated from the filament by a "grid condenser." When such is the case, a high resistance, known as a "grid leak," is always necessary to shunt the condenser in order to provide a way of escape for electrons which flow to the grid. If the resistance of this leak is 1,000,000 ohms, and if the flow of electrons along it from grid to filament amounts to a current of 4 microamperes, what is the difference in potential between grid and filament?

104. A radio-frequency choke coil is widely used to allow the passage of D.C. and to prevent the passage of radio frequencies. If such a coil is a pure inductance of 300 microhenries, calculate the current which will be passed through it by a R.M.S. potential of 500 volts at (*a*) 1,000,000 cycles, and (*b*) 10,000,000 cycles. To what wave-lengths are these frequencies equivalent?

105. If the coil of Problem 104 offers a capacitative path for the radio-frequency current to flow between its coils, which is equivalent to a shunting condenser of 20 $\mu\mu f$, what current will pass in this way under the conditions of Problem 104?

106. How many electrons flow per second from filament to plate of an electron tube when the plate current is 5 milliamperes? If the plate potential is 90 volts, with what velocity do the electrons arrive?

107. How many calories are spent per hour in heating the plate of a certain power tube which passes a current of 125 milliamperes at a potential of 1000 volts?

108. An open organ pipe 3 ft. long emits a tone, the fundamental of which has a frequency of 200 vibrations per second. What is the velocity of propagation of sound in the gas in the pipe? What are the frequencies of the first and second overtones?

109. A note of high pitch is being sounded in front of a vertical wall and at a distance of a few feet from it. On moving a sensitive flame about between the source of the sound and the wall, a series of points are found at which the flame is not affected. Explain this. If the distance apart of two such adjacent points is 5 cm., what is the pitch of the sound? Assume the temperature of the room to be 20° C.

110. Assuming the force required to break a steel wire to be 2.4×10^{10} dynes/cm.^{2} and to be proportional to the area of its cross section, calculate the maximum velocity of propagation of a transverse wave on such a wire, and show that it is independent of the size of the wire. The density of steel is 7.8 gr./cm.^{3}

111. Calculate the maximum fundamental frequency of transverse vibrations in a steel wire 1 meter long.

112. If a 50-cm. silk cord having a mass of 0.01 gr./cm. is set in vibration by a 100-cycle tuning fork, what tension is necessary to make it vibrate (*a*) as a whole, and (*b*) in three segments?

113. If a Kundt's tube, when filled with air, is so excited that the dust piles are 10 cm. apart, what will be the separation of the piles if the tube is filled with (*a*) helium, and (*b*) carbon monoxide (CO) at the same temperature? Assume the molecular weight of air to be 30.

114. A lens is made of glass of index of refraction 1.5. It is 1.5 cm. thick at the center and of zero thickness at the edge. Its diameter is 10 cm. Calculate its focal length and show that to a good approximation this does not depend on the individual curvatures of the two surfaces.

115. A 7-in. photographic lens is used to enlarge a negative which is $4\frac{1}{4}$ in. long. At what distance from the lens must the negative be placed in order for the enlargement to be 10 in. long?

116. If the lamp and film of Fig. 114 are located at a distance of 1 meter from the oscillograph mirror, what must be the focal length of the lens L? Assume that the film is so close to the lamp that the light may be thought of as perpendicularly incident on the mirror.

117. An object is placed 75 cm. in front of a concave mirror which has a radius of curvature of 25 cm. Where will the image be formed? Where will the image be formed if a converging lens of 10-cm. focal length is placed between the mirror and the object and 5 cm. from the latter?

118. A divergent lens of 10-cm. focal length is placed 40 cm. behind a convergent lens of 50-cm. focal length. Calculate the position of the image which this combination of lenses forms of a distant object. Will this image be real or virtual?

119. A microscope has an objective lens of 2-cm. focal length and a converging eyepiece lens of 2.2-cm. focal length. The two lenses are placed 12 cm. apart. If an object is 2.5 cm. in front of the objective, where is the image formed by the eyepiece? Is it real or virtual, erect or inverted?

120. Calculate the diameter of the first two dark rings in the diffraction image of a star formed by a camera lens of focal length f if its aperture is (*a*) $f/16$ and (*b*) $f/45$. Assume a wave-length of 5×10^{-5} cm.

121. At what aperture of a motion picture lens of 2.5-cm. focal length is the resolving power approximately the same as for a normal eye?

122. What is the diameter of the first light ring in the "Newton's rings" pattern formed by a plane glass surface and a lens having a radius of curvature of 50 cm.? Assume a wave-length of 5.89×10^{-5} cm.

123. Two rectangular glass plates are in contact at one edge and are separated at the opposite edge by a hair 0.001 cm. thick. How many dark interference bands can be observed by light of wave-length 5.89×10^{-5} cm. reflected from these plates?

124. The index of refraction of the ordinary ray in Iceland spar is 1.658, whereas that of the extraordinary ray is 1.486. The index of refraction of Canada balsam is 1.530. Calculate the critical angle of total reflection, in one case when the beam is about to enter the Canada balsam film of a Nicol prism, and in the other case when it is about to leave it. Explain from these calculations how the Nicol prism construction is possible.

125. Through what thickness of quartz perpendicular to the optic axis must light, initially plane polarized, pass in order to be circularly polarized? The index of refraction of quartz for sodium light is 1.544 for the ordinary ray and 1.553 for the extraordinary ray.

126. At what angle of incidence must an X-ray having a wave-length of 0.5 Ångström units be incident on the face of a rock salt crystal to be intensely reflected? The distance between the planes in the crystal is 2.81×10^{-8} cm. Make the calculation for both the first and the second order spectra.

127. If the wave-length of the K_α line of nickel is 1.655×10^{-8} cm., calculate the approximate location of the same line in the spectrum of silver.

128. If the K_α radiation from silver ($\lambda = 0.562 \times 10^{-8}$ cm.) is incident on the face of a certain crystal at a glancing angle of 8.82°, a second order spectrum is obtained. What is the grating space of the crystal used?

129. What is the minimum wave-length which can be excited by the application of 30,000 volts to an X-ray tube?

130. Calculate the velocity of a photo-electron which is liberated by an X-ray having a wave-length of 3 Ångström units. What voltage would be required to stop this electron?

131. Calculate the velocity of the electron in the smallest Bohr orbit.

132. Calculate and plot to scale the wave-lengths of the first five lines in the Balmer series of hydrogen. What is the wave-length of the "head" of the series, i.e., the wave-length toward which the shorter waves of the series converge?

133. What energy in ergs is required to ionize the hydrogen atom, i.e., to remove the electron from its normal orbit? What energy in calories would be required in this way to ionize a gram of hydrogen?

134. What would need to be the wave-length of a quantum of radiation if it is to have the same energy as an α-particle with a velocity $\frac{1}{10}$ that of light?

135. What would be the velocity of a proton having the same kinetic energy as the α-particle of the last problem? Use for this calculation the relativity formula for kinetic energy (page 350).

136. What must be the velocity of an electron if it is to possess the same energy as the α-particle of Problem 135?

137. Calculate the maximum difference between the wave-length of the "modified" line produced by the Compton effect and that of the incident X-ray. Why should this phenomenon be more readily detected in the X-ray than in the visible region of the spectrum?

138. What is the maximum velocity which an X-ray having a wave-length of 2 Ångström units can impart to a stationary electron by the Compton effect? What would be the maximum velocity which visible light of wave-length 6×10^{-5} cm. could impart to an electron in this way?

139. Calculate the length of the de Broglie wave associated with an electron which is moving with a velocity of 10^9 cm./sec. What wave-length would be associated with a proton having the same velocity?

140. What de Broglie wave-length would be associated with a bullet having a mass of 10 gr. and travelling with a velocity of 300 meters/sec.?

141. At what angle should a beam of electrons which has been accelerated through 500 volts be incident on a crystal with a grating space of 2.71×10^{-8} cm. in order to be intensely reflected?

142. If the angle subtended by Betelgeuse is 0.046″, what must have been the separation of the two mirrors M_1 and M_2 of Fig. 186 in order to cause the fringe system to vanish? Assume light of wave-length 4×10^{-5} cm.

143. The brightest of all the stars, Sirius, appears to be 0.76″ of arc nearer to a certain very distant star when viewed from one side of the Earth's orbit than when it is viewed from the opposite side. Calculate the distance of Sirius (*a*) in miles, and (*b*) in light years.

144. If the whistle of a certain locomotive has a frequency of 100 when at rest, what will be its apparent frequency (*a*) when it is approaching at a speed of 40 miles an hour, and (*b*) when it is receding at the same speed?

145. A man at rest makes 4 strokes of a hammer per second. Another man, approaching at the rate of 25 ft./sec., hears 41 strokes in 10 sec. What is the velocity of sound?

146. If the K line of the solar spectrum can be observed in the light from a spiral nebula which is receding with a velocity of 2000 miles/sec., how much will this line appear to be shifted toward the red?

147. The solar constant, i.e., the energy which falls on each square centimeter of the Earth's surface per minute, is 1.92 calories. What is the total energy emitted per second by the Sun? How long would the Sun last at this rate if it were composed entirely of carbon? The heat of combustion of carbon is 14,000 B.T.U./lb.

148. If the heat from the Sun could be concentrated by a concave mirror

10 ft. in diameter and used to run an engine, how many horse-power could be obtained with an efficiency of 30%?

149. Using the relativity expression for energy, calculate the amount of energy which would be liberated if a gram of hydrogen could be converted into helium.

150. Using the method suggested for Problem 6, page 129, calculate the gravitational potential energy of a photon of wave-length 5861 Ångström units (due to H) at the surface of the Companion of Sirius. Assume the density of this star to be 30,000 gr./cm.^{3} and its diameter to be 13,000 miles. How much would the wave-length of this line be shifted toward the red by the gravitational attraction of the star for the escaping photon?

FUNDAMENTAL PHYSICAL CONSTANTS

SYMBOL	QUANTITY	VALUE
c	Velocity of Light	2.9986×10^{10} cm./sec.
G	Gravitation Constant	6.66×10^{-8} cm.^{3} gr.$^{-1}$ sec.$^{-2}$
e	Electronic Charge	4.774×10^{-10} E.S.U.
		1.592×10^{-20} E.M.U.
e/m_0	Electronic Ratio	5.305×10^{17} E.S.U./gr.
		1.769×10^{7} E.M.U./gr.
F	Faraday	9.6500×10^{4} coulombs
v_0	Vol. of 1 gr. mol. at 0° C. and 1 atm.	22.4115 liters/gr. mol.
h	Planck's Constant	6.554×10^{-27} erg sec.
T_0	Ice Point, abs.	273.1° C.
R	Gas Constant	8.315×10^{7} ergs/°C. gr. mol.
N_0	Avogadro's Number	6.061×10^{23}/ gr. mol.
M_H	Mass of Hydrogen Atom	1.663×10^{-24} gr.
m_0	Electronic Mass	8.999×10^{-28} gr.
r_1	Radius of 1st Bohr orbit of hydrogen	0.5305×10^{-8} cm.

GREEK LETTERS USED AS SYMBOLS

α	alpha	μ	mu
β	beta	ν	nu
γ	gamma	ξ	xi
Δ, δ	delta	π	pi
ϵ	epsilon	ρ	rho
ζ	zeta	Σ, σ	sigma
η	eta	τ	tau
θ	theta	ϕ	phi
κ	kappa	ψ	psi
λ	lambda	ω	omega

USEFUL RELATIONS BETWEEN COMMON UNITS

Length

1 in. = 2.540 cm.
1 ft. = 30.48 cm.
1 micron (μ) = 0.001 mm. = 10^{-4} cm.
1 millionth micron ($\mu\mu$) = 10^{-10} cm.
1 Ångström unit = 10^{-8} cm.

Volume

1 liter = 1000 cc. = 61.024 in.^{3} = 1.05671 qt.

Mass

1 lb. = 453.59 gr.
1 kg. = 2.2046 lb.

Angles

1 circumference = 360° = 2π radians.
1 radian = 57.2958°.

Density

1 gr./cm.^{3} = 62.4 lb./ft.^{3}

Pressure

1 atmosphere = 76 cm. of mercury = 14.697 lb./in.^{2}
= 1013200 dynes/cm.^{2}

Work or Energy

1 ft.-lb. = 1.356×10^7 ergs.
1 joule = 10^7 ergs.
1 gr. cal. = 4.183×10^7 ergs.
1 B.T.U. = 777.8 ft.-lb. = 252.2 gr. cal.

Power

1 H.P. = 33,000 ft. lb./min. = 550 ft. lb./sec. = 746 watts.
1 watt = 1 joule/sec.

Electrical Units

1 ampere = 10^{-1} absamps = 3×10^9 E.S.U.
1 volt = 10^8 E.M.U. = $\frac{1}{3} \times 10^{-2}$ E.S.U.
1 coulomb = 10^{-1} E.M.U. = 3×10^9 E.S.U.
1 ohm = 10^9 E.M.U. = $\frac{1}{9} \times 10^{-11}$ E.S.U.
1 farad = 10^{-9} E.M.U. = 9×10^{11} E.S.U.
1 henry = 10^9 E.M.U. = $\frac{1}{9} \times 10^{-11}$ E.S.U.

Natural Numbers	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Natural Numbers	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9026	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Logarithms	0	1	2	3	4	5	6	7	8	9	Proportional Parts 1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Logarithms	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

INDEX

A

Abbe, E., 278
Aberration, chromatic, 275
  of light, 251
  spherical, 260, 275
Absolute temperature, 85
  units, 11
  zero, 85
Absorption of cosmic rays, 337
  of light, 325, 326, 344
  of sound, 235
Acceleration, 5, 7, 12
  central, 27
  of gravity, 5, 13
  of Moon, 28
Actinium, 192
Action and Reaction, 14, 47, 139
Adams, 365
Adiabatic, 79
Alchemists, 92
α-particles, 184
  disintegration by, 372
  ray tracks, 186
  relation to helium, 186
Alternating current, 199
Ammeter, 140, 149
Ampère, André Marie, 136
Ampere, 136, 138
Ampère's Rule, 136
Amplification factor, 214
Amplifier, 215
Andrews, 70
Angle, radian measure of, 38
  of incidence and refraction, 255
Ångström unit, 325
Angular momentum, 346
  velocity, 46
Anode, 154
Antares, 369
Anticathode, 172
Apollonius, 2
Archimedes, 3, 34, 55
Archimedes' Principle, 55
Aristophanes, 270
Aristotle, 3
Astigmatism, 277
Aston, F. W., 179
Astronomy, 1, 362
Atom, 93
  building, 372
  of electricity, 156
Atomic energy, 372
  heats, 113
  nature of electricity, 159
  number, 98, 188
    and X-ray spectra, 335
  weight, 94
Auditoriums, acoustics of, 230, 236
Aurora borealis, 364
Avogadro, 95, 154
Avogadro's Law, 95, 154
  number, 110, 156, 161

B

Balance, equal arm, 14
Balmer series, 347
Barometer, 56
Bartholinus, Erasmus, 293
Battery of voltaic cells, 128
Beats, 238, 358
Becker, H., 372
Becquerel, Henri, 181
Bell, Alexander Graham, 221
Bennet, Abraham, 118
Bernoulli, Daniel, 59, 102
Bernoulli's theorem, 59, 231
β-particles, 183, 319
Betelgeuse, 369
Binocular, 274
Biot and *Savart*, Law of, 136
Black body, 338
  radiation, 338
Bohr, Niels, 325, 343
  theory of line spectra, 343
    application to hydrogen, 347
    application to stellar spectra, 363
    application to X-rays, 347
Boiling point, 63, 73
Boltzmann, 102
Bothe, W., 372
Bowen, 337
Boyle, Robert, 62, 69, 92, 102
Boyle's Law, 69, 232
Bradley, 251

Bragg, William H., and *W. L.*, 332
Brewster, 294
Brewster's angle, 294
British Thermal Unit, B.T.U., 64
Brown Robert, 108
Brownian movements, 108
Bunsen, 324
  burner, 324
Burning glass, 270

C

Calorie, 64
Calorimeter, 65
Calorimetry, 64
Capacity, 124
  of parallel plate condenser, 126
  of sphere, 125
  of spherical condenser, 125
Carbon dioxide, liquefaction of, 70
Carlisle, Sir Arthur, 129
Carnot, Sadi, 77
  cycle, 79, 83
  engine, 78
Carnot's theorem, 86
Cathode, 154
  rays, 164
    electric deflection of, 166
    magnetic deflection of, 164, 166
    oscillograph, 223
Causality, 361
Cavendish, Henry, 29, 120
Cavendish's experiment, 29
Cell, galvanic, 146
  standard, 152
  storage, 146
Celsius scale of temperature, 63
Center of gravity, 40
Centigrade scale of temperature, 63
Centripetal force, 26, 30, 167
Chadwick, 188
Charge, electric, 121
  fundamental unit of, 161
  on α-particle, 185
  on atomic nucleus, 188
  on cathode particle, 169
Charles, Law of, 86
Chemical symbols, 95
Chemistry, 92
Christie, S. Hunter, 148
Cobalt, magnetic property of, 131
  position in periodic table, 335
Code, international, 212
Coefficient of friction, 22
  of viscosity, 160
Coil, 136
  choke, 209
  inductance of, 202
  induction, 304
  suspended coil instruments, 140
  turning moment in magnetic field, 140
Collimator, 289
Color, nature of, 256
  of pigments, 326
  photography, 327
  primary, 326
  Young's theory of color vision, 326
Combining weight, 155
Commutator, 199
Companion of Sirius, 371
Compass, 130
Components of force, 21
  of velocity, 25
    in three dimensions, 104
Composition of forces, 17
  of velocity, 25
    relativistic, 318
Compound, 92
Compression, of gas, isothermal and adiabatic, 79
  propagation as wave, 230, 238
Compton, A. H., 349
  effect, 349, 360, 367
Condenser, electric, 125
    oscillatory discharge of, 208
  of steam engine, 77
Conduction of electricity through gases, 163
Conductor, 115
Constants, fundamental physical, 386
Constriction in pipe, 60
Cooling curve, 75
Copernicus, 2
Copernican theory, 2
Cord, fundamental and harmonics of, 242
  standing waves on, 240
  velocity of wave on, 233
Corpuscular theory of light, 249, 254, 258, 281, 343
Cosmic rays, 337
  origin of, 371
Coulomb, Charles Augustin, 120, 131
Coulomb's Law, 120, 131
Coulomb, 138
Coupled circuits, 210
Crookes, Sir William, 164, 184
Crystal, anisotropic nature, 295
  axis of, 295

Crystal, spacing of molecules in, 331
spectrometer, 332
structure of, 333
Curie, Mme. and *Pierre*, 181, 188, 189
Current, alternating, 199
electric, 128
energy of, 142
induced, 195
magnetic field due to, 135
Curvilinear motion, 24
Cycle, 79, 200

D

Dalton, John, 93
Davison, 355
de Broglie, Louis, 354
waves, 354
Debierne, A., 188
Debye, P., 334, 357
de Forest, Lee, 213
Democritus, 92
Dempster, 179
Density of Earth, 30
of fluid, 53
of gas, 96
Detector, 212
crystal, 213
electron tube, 213
Hertz's, 305
Dielectric constant, 126
Diffraction, effect on image formed by lens, 284
grating, 289
of electrons, 356
of X-rays, 330
Diffusion, pump, 106
separation of isotopes by, 178
Dirac, 360
Discharge, of condenser, 207
through gases, 163
Disintegration of elements by α-particles, 372
of radioactive elements, 188
Dispersion, 257, 275
Distortion of photographic image, 277
Distribution of brownian particles, 109
of energy in the spectrum, 338
of stars in the universe, 366
Dolland, John, 275
Doppler effect, 369
broadening of spectrum lines, 370
motion of stars, 370
Dorsey, 157
Double refraction, 293
Dufay, 116
Dulong and *Petit*, Law of, 113
Dyne, 11, 13

E

Earth, 2, 27, 29
magnetic field of, 135
dip, 135
horizontal and vertical components, 135
Echo, 236
Eclipse, 2
of Jupiter's satellites, 250
Ecphantus, 2
Eddington, 366
Edison, Thomas, 173, 221, 222
effect, 173
Efficiency of heat engines, 86, 88
of machines, 35, 37
Einstein, Albert, 108, 314, 342, 371
Elastic, perfectly, 49, 102
solid theory of light, 302
Electric charge, 115, 121
distribution of, 117
conduction through gases, 163
current, 128
energy of, 142
force experienced in magnetic field, 139
magnetic field due to, 135
field, 122
filters, 225
generator, A. C. and D. C., 199
potential, 122
of conductor, 123
waves, 302
Electricity, 115
positive and negative, 116
Electrochemistry, 155
Electrolysis, 154
Faraday's laws of, 154
of water, 129, 154
Electromagnet, 68, 198
Electromagnetic units, (E.M.U.), charge, 138
ratio to E.S.U., 158
current, absamp, 137
gauss, 132
pole strength, 131
potential, 169
Electromagnetic waves, 302
generation and detection of, 304
polarization of, 306
reflection of, 306
refraction of, 307

Electromotive force, due to changing magnetic field, 195
three-finger rule, 196
induced in moving wire, 194
of galvanic cell, 142
thermal, 147
Electron, 115, 170
as a wave packet, 357
coalescence with proton, 372
diffraction of, 356
emission from hot solids, 173
part of atom, 187
tube, 213
as detector and amplifier, 213
as generator of oscillations, 216
Electroscope, 118, 181
condensing, 127
Electrostatic units, (E.S.U.), capacity, 124, 126
charge, 121
ratio to E.M.U., 158
field strength, 160
potential, 158
Electrostatics, 115
induction, 117
Elements, 92
list with symbols, 100
periodic table, 99
radioactive, 190
Energy, 42
atomic, 373
availability of, 90
conservation of, 44, 78, 90
degradation of, 90
equivalence with mass, 320
kinetic, 42, 320, 350
of gas molecules, 111
of rotation, 46
relativistic, 350
of electric current, 142
potential, 43
pressure, 59
Engine, Carnot, 78
reversible, 86
steam, 77
Entropy, 89
Equilibrium, 19
Equipotential surface, 125
Erg, 40
Ether, luminiferous, 249, 302, 320
electromagnetic, 302
Eudoxus, 2
Exit pupil, 273
Expansion, isothermal and adiabatic, 79
Expansion, work of, 80, 111
Eye, 267
color sensitiveness, 326
resolving power, 288
Eyepiece, 271

F

Fahrenheit, scale of temperature, 63
Falling bodies, 3
Farad, 208
Faraday, Michael, 155, 194
cylinder, 166, 185
Faraday, 157
Field, electromagnetic, 217
magnetic, 133, 217
of force, 122
due to conducting sphere, 124
due to point charge, 123
of telescope, 273
Fitzgerald, G. F., 313
contraction, 313
Fizeau, 252, 256
Fleming, J. A., 213
Fluid, 52
in motion, 58
perfect, 53
pressure, 53
Fluorescence, 329
induced by α-particles, 184
induced by cathode rays, 164
Flux, magnetic, 196, 203, 210
Focus, principle, of mirrors and lenses, 260, 261
Foot-pound, 40
Force, 9
Foucault, 253
Fourier, Joseph, 224
Franck, 344
Franklin, Benjamin, 117, 119, 338
Fraunhofer, Joseph, 325
lines, 325, frontispiece
origin of, 325
Freezing point, 73
table, 74
Frequency, audio, 220
of alternating current, 201
of de Broglie waves, 354
of wave disturbance, 238
radio, 208
Fresnel, Augustin Jean, 281
Friction, 6, 22
coefficient of, 22
Friedrich, 331
Frost line, 74

Fulcrum, 34
Fundamental, component of sound, 224
of organ pipe, 246
Fusion, 73
heat of, table, 74

G

Galileo, 3, 9, 30, 32, 56, 59, 63, 249, 271
Galvanic cells, 146
standard, 152
storage, 146
Galvanometer, D'Arsonval, 140
tangent, 138
γ-rays, 182, 336
Gas, constant, R, 97, 106
electric conduction of, 163
ideal, 70, 78
kinetic theory of, 102
law, 96
liquefaction of, 70
pressure of, 102, 109
Gauss, unit of magnetic field, 132
Gay-Lussac, 95
Geiger, counter, 185
Generator, electric, 198
of high frequencies, 216
Germer, 355
Gilbert, William, 130
Glagolewa-Arladjewa, 307
Glass, development at Jena, 278
double refraction from strain, 299
flint and crown, 275
Globular clusters, 366
Goldstein, 164, 176
Gram, 11
molecule, 95
Gravitation, universal, 11, 27
constant of, 28
Gray, J. A., 189
Gray, Stephen, 115
Greek letters, used as symbols, 386
Guericke, Otto von, 115
Guinea and feather experiment, 4
Gun, recoil of, 49

H

Haga, 330
Half-value period of radium, 189
Harmonic, 224
analysis of sound, 223
of organ pipes, 246
simple harmonic motion, 30, 206
Heat, 62
from electric current, 142
from radium, 182
of fusion, 73, 74
of vaporization, table, 73
reservoir, 78
specific, table, 65
Heaviside, Oliver, 364
Heisenberg, 360
Helium, in Sun, 325
ionization and radiation potentials, 344
liquefaction of, 86
product of radioactive disintegration, 186
Helmholz, 207, 221, 226
Henry, Joseph, 202, 207
unit of inductance, 202
Herschel, Sir William, 323
Hertz, G., 344
Hertz, Heinrich, 211, 218, 304, 339
Hess, Victor F., 337
Hipparchus, 2
Hittorf, 164
Hooke, Robert, 207
Horse-power, 45
Huygens, 249, 285
Hydraulic lift, 54
Hydrodynamics, 58
Hydrogen, atomic weight, 94
Bohr theory of line spectrum, 347
in stars, 363
mass and dimensions of molecule, 110
produced by electrolysis, 129, 154

I

Ice, 74
Iceland spar, 293
Image, affected by diffraction, 284
formed by lens, 261
formed by mirror, 259
single refracting surface, 262
size of, 260
Impact, 47
Impedance, 202
Impulse, 47
Inclined plane, 6, 34, 50
Indetermination principle, 361
Index of refraction, 256
Inductance, 202
part in oscillatory discharge, 208
Induction coil, 304
electromagnetic, 195
electrostatic, 117

Inertia, 10
of electric current, 208
moment of, 45, 47
Infra-red light, 323
Insulator, 115
Interference of electromagnetic waves, 305
of light, 280
of sound, 238
Interferometer, 310, 367
Ionization of gases, 174
by collision, 175
by radioactive material, 181
by X-rays, 174
in upper atmosphere, 364
Ions, 156, 174
Iron, magnetic properties of, 131, 197
Isothermal, 70
Isotopes, 177, 179, 190

J

Jackscrew, 39
Jeans, Sir James, 372
Joannides, Zacharias, 271
Joly, John, 327
Joule, James Prescott, 66, 142
Joule's Law, 142
unit of energy, 143
Jupiter, 30, 250

K

Kaufmann, 183
Kayser, 324
Kelvin, Lord, 84, 207
scale of temperature, 84, 88
Kennelly, 364
-Heaviside layer, 364
Kepler, 271, 346
Kinetic theory, 102
and Second Law of Thermodynamics, 114
Kinnersley, Ebenezer, 116
Kirchhoff, 324, 325
Kleist, Ewald Georg von, 127
Knipping, 331
Kodacolor process, 327
Kolhorster, Werner, 337
Kohlschütter, 365
Kundt's experiment, 242

L

Laplace, 232
Latent heat, 72, 73
v. Laue, Max, 330
Lavoisier, Anton Laurent, 93
Lead, disintegration product of radium, 190
Lebedew, 307
Lens, 261
achromatic, 275
aperture of, 278
convergent and divergent, 262
focal length of, 266
formula, 266
spectacle, 267
speed of, 278
successive lenses, 268
Lenz, Heinrich Friedrich Emil, 197
Lenz's Law, 197
Lever, 34
arm, 36
Leyden jar, 127
Light, constitution of, 322
infra-red, 323
nature of, 217, 248
ultra-violet, 329
velocity of, 217, 249
in denser media, 255
of different colors, 256
visible, 322
wave-length, 283
Lightning, 119
Lippershey, Hans, 271
Lippmann, 283
process of color photography, 283
Liquid, 52, 70
Lodestone, 130
Logarithms, table of, 388
Lorentz, H. A., 313
transformation, 314
Lyman, Theodore, 330

M

Machine, 34
Magnet, 130
Magnetic field, 132
due to current, 135
Earth's, 132
flux, 196, 210
induction, 198
permeability, 198
pole, 130
Magnetism, 130
Magnetite, 130
Magnetometer, 132
Magnification of jeweler's eye lens, 270
of telescope, 272
Malus, Etienne Louis, 293

Manometer, 57
Marconi, Guglielmo, 211
Mass, 10, 28
  conservation of, 93
  equivalent to energy, 320
  of Earth, 29
  of electron, 170
  of gas molecules, 103, 110
  of hydrogen atom, 170
  spectrograph, 178
  variation with velocity, 319
Maxwell, James Clerk, 102, 106, 217, 303, 307, 327
McLennan, 337
Mechanical advantage, 35
Mechanical equivalent of heat, 67
Melde, F., 242
Melde's experiment, 240
Melting point, table, 74
Melvill, Thomas, 323
Mendelyeev, 98
Mercury, 56, 57, 63, 106
Mersenne, Père, 230
Meyer, Lothar, 98
Michelson, 252, 310, 368
  -*Morley* experiment, 310
  interferometer, 310, 368
Microfarad, 209
Microphone, 221
Microscope, compound, 274
  resolving power, 288
  γ-ray, 360
Miller, D. C., 313
Millikan, Robert A., 157, 330, 337, 341, 372
Millikan's oil drop experiment, 159, 174
Minkowski, 317
Mirror, parabolic, 259
  plane, 259
  revolving, 253
  spherical, concave and convex, 260
Modulation, 212, 227
Mol, 95
Molecular chaos, 104, 114
  heat, 111
  weight, 94, 233
Molecule, 93
  mass of, 110
  spacing of atoms in, 357
  velocity of, 106
Moments, of a force, 36
  of inertia, 45, 47
  law of. 36
Momentum, 47
  angular, 346
  conservation of, 47, 349
  of gas molecules, 103
  of light photons, 350
  related to position, 360
Moon, 2, 27
Morley, 310
Mosely, H. G. J., 335
  relation between frequency and atomic number, 335, 348
Motion, curvilinear, 24
  laws of, 6, 9, 11, 14
  simple harmonic, 30
Motor, 142
Mount Wilson observatory, 277, 365
Multiple proportions, law of, 94

N

Neon, isotopes of, 177
Newcomen, 77
Newlands, 98
Newton, Sir Isaac, 9, 26, 230, 249, 256, 275, 281, 283
Newton's rings, 283
Nicholson, William, 129
Nickel, magnetic properties of, 131
  position in periodic table, 335
Nicol, prism, 297
Nodes, 240
Noise, 219
Nucleus, 187
  charge on, 188

O

Objective, 271
  photographic, 277
Oersted, Hans Christian, 136
Ohm, George, Simon, 143
Ohm's Law, 143, 201, 211
  unit of resistance, 143
Onnes, Kamerlingh, 86
Optical instruments, 270
Organ pipes, 246
Orthochromatic plates, 329
Oscillations, elastic, 207
  electric, 207
  of pendulum, 30, 206
Oscillograph, cathode ray, 223
  Duddell, 222
Overtones, 224
Oxygen, atomic weight, 94
  produced by electrolysis, 129, 154

P

Parabolas, positive ray, 177
Parallax, 362
  spectroscopic, 364
Parallelogram of forces, 18
Particle, 17
Paschen series, 347
Pendulum, 30
Period, of oscillatory motion, 30
    of magnet in Earth's field, 134
    of pendulum, 30
  half-value period of radioactive substances, 189
Periodic disturbance, 207
  table of the elements, 97, 99, 190
  wave, 238
Perrin, 108, 165
Petit, 113
Phase, of wave disturbance, 239, 280
  waves, 352
Phonograph, 222
Phosphorescence, 329
Photoelectric cell, 340
  effect, 339
Photoelectrons, velocity of, 341
Photon, 343, 349
  energy and momentum of, 350
Photosphere, 325
Physical constants, table of, 386
Pitch, 219
Pitchblende, 181
Planck, Max, 339
Planck's constant, 339
Planets, 1, 2, 26
Pohl, 330
Polarization, circular and elliptic, 297
  of electromagnetic waves, 306
  of galvanic cell, 146
  of light, 293
  plane of, 294
    rotation of, 299
Pole, magnetic, 130
  Earth's, 130
  north and south, 130
Polonium, 181
Polygon of forces, 19
Positive rays, 175
Potential, difference of, 122
  electric, 122
    across a galvanic cell, 145
    along a conductor, 144
  ionization and radiation, 344
  of charged sphere, 124
  of conductor, 123
Potential of point charge, 123
  of the Earth, 123
  zero, 123
Potentiometer, 151
Pound, 11
Poundal, 12, 13
Power, 45
Practical system of units, ampere, 138
  coulomb, 138
  farad, 208
  henry, 202
  joule, 143
  ohm, 143
  volt, 143
  watt, 143
Pressure, fluid, 53
  of atmosphere, 56
  of gas, 69, 102, 109
  potential energy, 59
Prism, reversing, 273
  triangular for producing spectra, 257
Projectiles, 24
Proton, 345
  coalescence with electron, 372
Prout, William, 177
Pulley system, 36
Ptolemy, 2
Ptolemaic system, 2
Pythagoras, 2

Q

Quality of sound, 220
Quanta, 339
Quantum theory, 338
Quartz, 300, 330

R

Radian, 38
Radioactivity, 181
Radio communication, 211, 227
  receiver, 213
Radium, 181
  emanation, 188
Ramsay, Sir William, 186, 189
Range, of α-particles, 192
  of projectile, 25
Ratio of charge to mass,
    of α-particle, 186
    of β-particle, 184
    of electron, 169, 173
    of hydrogen ion, 186
    of positive rays, 176
  of specific heats of a gas, 112, 232, 233

Réaumur, 63
Red shift, 371
Reflection of electromagnetic waves, 306
of light, 258
total, 273
of sound, 235
of X-rays, 332
Refraction, 254
double, 293
of electromagnetic waves, 307
Refractive index, 71, 256
Refrigerator, 80
Regener, 185
Relativity, galilean, 6, 24, 26, 231, 233
general, 14, 370
special, 308, 314, 348, 350, 352, 354, 361, 367
Reservoir of heat, 78
Resistance, electric, 143
in series and parallel, 152
internal, 145
measurement of, 148
Resistivity, table, 144
Resolution of force into components, 21
of velocity into components, 25
of vibration into circular components, 300
of vibration into components, 298
Resolving power, 287
of diffraction grating, 291
of eye, 288
of microscope, 288
of telescope, 287
Resonance of air columns, 245
of electric circuits, 210
of vibrating cord, 241
Resultant, 18
Reverberation, 235
Reversible engine, 86
Reversing layer, 325
Rhigi, 307
Richards, T. W., 190
Ritter, Johann Wilhelm, 329
Rocket propulsion, 49
Rock-salt, crystal spacing of, 331
Roemer, 250
Roentgen, Wilhelm Konrad, 172
Root mean square, current and E.M.F., 201
velocity of molecules, 106
Rosa, 157
Rotation, energy of, 46
of gas molecules, 112
Rotation, of plane of polarization, 299
work of, 38
Rowland, Henry Augustus, 138, 291
Rumford, Count, 66
Runge, 325
Rutherford, Sir Ernest, 185, 187, 189, 337, 372
model of atom, 187, 345, 348
Rydberg constant, 348

S

Sabine, Wallace C., 230, 236
Saccharimeter, 301
Saturated vapor pressure, 71
Savart, 136
Savary, Felix, 207
Scherrer, 334
Schroedinger, 355
Seebeck, Johann, 147
effect, 147
Series of spectra, 324
Shunt, 149
Siegbahn, 336
Siren, 220
Snell, Willebrord, 256
Snell's Law, 256
Soddy, 186, 189
Sodium spectrum, 324
Sommerfeld, 330
Sound, 219
analysis and synthesis of, 223, 225
pitch, 219
propagation of, 229
quality of, 220
velocity of, 230
Space-time, 317
Spectra, Bohr theory of, 343
continuous, 322
distribution of energy in, 338
line, 323
origin of, 338
solar, 325, frontispiece
stellar, 362
Spectrometer, 330
X-ray, 332
Speech, 226
Spiral nebulae, 366
Spring balance, 12
Stars, 2
binary, 370
constitution of, 362
dimensions of, 367, 369
distance of, 364

Stars, spectra of, 362
temperature of, 363
velocities of, 370
State, change of, 69
continuity of, 71
stationary, 344
Steam dome, 71
Stern, O., 106
Stokes, 160
Stokes's Law, 160
Sublimation, 74
Sun, 2, 27, 325, 363, 366
source of energy, 366
Sunlight, spectrum of, 325, frontispiece
Synchronization of clocks, 308, 316

T

Target, of X-ray tube, 172
Telephone, 221
Telescope, astronomical, 271
galilean, 271
reflecting, 275
resolving power, 287
Television, 340
Temperature, 63
critical, 71
table, 73
interpretation of on kinetic theory, 104
Kelvin scale, 84
of stars, 363
Thales, 2, 115
Thermionic emission, 173
Thermodynamics, 78
first and second laws of, 78, 81, 114
Thermometer, 63
Thermopile, 147
Thomson, G. P., 356
Thomson, J. J., 157, 164, 176, 178
Thomson, Sir William, Lord Kelvin, 84, 207
Thorium, 191
Time, in a moving system, 316, 352
of swing of a pendulum, 31
related to space, 317
Torricelli, 56, 59
Torsion balance, 120
Transformer, 203
Transmission of power at high voltage, 204
of signals by code, 212
Triangle of forces, 19
Triple point, 74

U

Ultra-violet light, 329
Uncertainty principle, 360
Units, useful relations between, 387
Univalent ion, charge on, 156
Universal gravitation, 27
Universe, expanding, 370
island, 366
Uranium, radioactivity of, 181

V

Vacuum pump, 4, 106
tube, 213
Valence, 155
Vapor, 71
pressure, 71
Vaporization, 72
heat of, table, 73
Vector, 18
Velocity, 5
angular, 46
components of, 25, 104
related to frequency and wavelength, 238
relativistic addition of, 318
Venturi meter, 61
Viscosity, 52
Volt, 143
Voltmeter, 149, 151
Volta, 127
Voltaic cell, 127
Vowels, 226

W

Walter, 330
Water, chemical composition, 94
density of, 56
expansion on freezing, 64
minimum volume, 56, 64
Watt, James, 77
unit of power, 143
Waves, de Broglie, 354
effect of waves on optical image, 280, 284
electromagnetic, 217, 304
longitudinal and transverse, 235, 293
matter, 355
ordinary and extraordinary, 296
phase, 352
sound, 229
standing, 240, 242, 283, 305

Wave-length,
of cosmic rays, 337
of de Broglie waves, 355
of electromagnetic waves, 305, 307, 323
of light, 283, 322, 323, 330
of X-rays, 336
relation to atomic number, 335
Wave theory of light, 248, 255, 281, 343
Weight, 14
combining, 155
Weston cell, 152
Wheatstone, Sir Charles, 148
bridge, 148
Wiener, 283
Wilson, C. T. R., 185
Wind, 330
Windlass, 35
Wireless communication, 211
Wollaston, 325, 329
Work, 37
of expanding gas, 80, 111
of heat engine, 80
Working substance, 80
World point, 317

X

X-rays, 172
diffraction of, 330
nature of, 330
photoelectric effect with, 343
spectrum of, 334

Y

Young, Thomas, 281, 291, 326
Young's pinhole experiment, 281, 368

Z

Zeiss works, Jena, 278