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MODERN SPECTRAL THEORY

BY

HIDEGORÔ NAKANO

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PREFACE

Functional analysis, namely analysis in infinite-dimensional spaces, changed its feature by the introduction of the notion of semi-order, and spectral theory got an important part in it, as the integration theory did in the classical analysis.

Spectral theory was established first about Hermitean operators in Hilbert spaces. S.W.P. Steen attempted to modernize this spectral theory in Hilbert spaces, and constructed a spectral theory in semi-ordered rings. On the other hand, F. Riesz attempted the generalization of Lebesgue's decomposition in the integration theory, which was established already for totally additive measures, to finitely additive measures, and obtained a new spectral theory about functionals. This spectral theory was modernized by H. Freudenthal as one in semi-ordered linear spaces. Modernization of these two different spectral theories was attempted independently by S.W.P. Steen and H. Freudenthal and they established it at the same time in 1936. Furthermore we found that these two different spectral theories fell into the same contents by modernization.

The spectral theory in semi-ordered linear spaces constitutes the subject of the present volume. The modernized spectral Theory indicated just now will be called the first spectral theory, which is constructed by means of Stieltjes integral, considering every spectrum as a continuous one. On the contrary there is another type of spectral theory established by the author, which is constructed by means of Riemann integral in topological spaces, considering every spectrum as a point spectrum. This spectral theory will be called the second spectral theory in the text. The first spectral theory will occupy Chapter II.

However, after Chapter III, we shall be concerned only with the second spectral theory, which enables us to develop the theory further.

Totality of bounded continuous functions on a compact Hausdorff space constitutes obviously a semi-ordered ring. Characterization of this semi-ordered ring was considered by many mathematicians. This problem will be discussed precisely in Chapter VI, applying the second spectral theory.

Now I wish to say a few words about the structure of the book. It embodies the greater part of a course of lectures delivered by the author at Tokyo University during 1946-47. The reader needs only to be acquainted with elementary part of classical analysis. I attempted at first to get it published in Japanese, but did not succeed on account of the present circumstance of Japan. Then I have translated it into English, hoping it to be printed in U. S. A.. I have recognized however that it is also impossible in U. S. A.. Accordingly I have resolved to publish it by photographic process. I think this book will be a good introduction to the volume I of Tokyo Mathematical Book Series.

Tokyo, May 16, 1950

Hidegoro Nakano

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INTRODUCTION

1. Topological spaces

For a set A we define $a \in A$ or $A \ni a$ to mean that a is an element of A , and we shall write $a \notin A$ or $A \not\ni a$ if a is not contained in A . For two sets A and B we define $A \supset B$ or $B \subset A$ to mean that B is a subset of A , that is, A includes B .

We shall make use of the notation

$$\{x : C(x)\}$$

to denote the set consisting of all x satisfying the condition $C(x)$. We denote by $\{a_1, a_2, \dots\}$ the set composed of elements $a_1, a_2, \dots$, and the empty set will be denoted by O .

For a system of sets A_λ ($\lambda \in \Lambda$) we denote by $\sum_{\lambda \in \Lambda} A_\lambda$ the union of all A_λ , by $\prod_{\lambda \in \Lambda} A_\lambda$ their intersection, that is,

$$\prod_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\},$$

$$\sum_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Lambda\}.$$

For a sequence of sets A_ν ($\nu = 1, 2, \dots$) we may write

$$\sum_{\nu=1}^{\infty} A_\nu = A_1 + A_2 + \dots, \quad \prod_{\nu=1}^{\infty} A_\nu = A_1 A_2 \dots$$

A set R may be called a space, if we shall be concerned only with subsets of R , and then every subset of R may be called a point set, while every element of R may be called a point. Let R be a space. For every point set A in R we denote the complement of A by A' , that is,

$$A' = \{x : x \notin A\}.$$

Then we have obviously

$$\left(\sum_{\lambda \in \Lambda} A_\lambda\right)' = \prod_{\lambda \in \Lambda} A_\lambda', \quad \left(\prod_{\lambda \in \Lambda} A_\lambda\right)' = \sum_{\lambda \in \Lambda} A_\lambda'$$

for any system of point sets A_λ ($\lambda \in \Lambda$). For two point set

A and B , if $A \supset B$, then we denote AB' by $A - B$, that is,

$$A - B = \{x : x \in A, x \notin B\}.$$

A space R is said to be a topological space if open sets are

defined in R such that for any system of open sets A_λ ($\lambda \in \Lambda$) the union $\sum_{\lambda \in \Lambda} A_\lambda$ is open; for any two open sets A and B the intersection AB is open; and further the whole space R and the empty set O are open, i.e., if we denote by $\mathcal{T}$ the collection of all open sets, then $\mathcal{T}$ satisfies the topological conditions:

- 1) $A_\lambda \in \mathcal{T}$ ($\lambda \in \Lambda$) implies $\sum_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}$,
- 2) $A, B \in \mathcal{T}$ implies $AB \in \mathcal{T}$,
- 3) $R, O \in \mathcal{T}$.

A space without topology is called an abstract space. For an abstract space R , if a collection of point sets $\mathcal{T}$ in R satisfies the topological conditions, then it is evident by definition that we can introduce a topology uniquely into R such that $\mathcal{T}$ coincides with the totality of open sets by this topology.

For a topological space R , a collection of open sets $\mathcal{U}$ is said to be a neighbourhood system of R , if we have

$$A = \sum_{A \supset X \in \mathcal{U}} X$$

for every open set A . The totality of open sets is obviously a neighbourhood system. We see easily by definition that every neighbourhood system $\mathcal{U}$ satisfies the neighbourhood conditions:

- 1) $A, B \in \mathcal{U}$ implies $AB = \sum_{AB \supset X \in \mathcal{U}} X$,
- 2) $R = \sum_{X \in \mathcal{U}} X$ and $O \in \mathcal{U}$.

For an abstract space R , if a collection of point sets $\mathcal{U}$ satisfies the neighbourhood conditions, then we can introduce uniquely a topology into R such that $\mathcal{U}$ is a neighbourhood system of R by this topology. Because, if we denote by $\mathcal{T}$ the totality of point sets A such that

$$A = \sum_{A \supset X \in \mathcal{U}} X,$$

then $\mathcal{T}$ satisfies obviously the topological condition 1). We can conclude by the neighbourhood condition 1) that $\mathcal{T}$ satisfies the topological condition 2), and further by the neighbourhood condition 2) that $\mathcal{T}$ satisfies the topological condition 3).

Therefore we can introduce uniquely a topology into R such that $\mathcal{T}$ coincides with the totality of open sets by this topology, as remarked just above. For this topology it is evident by definition that $\mathcal{U}$ is a neighbourhood system of R . On the other hand, if we can introduce a topology into R such that $\mathcal{U}$ is a neighbourhood system of R , then $\mathcal{T}$ must coincide with the totality of open sets by this topology. Accordingly we have the uniqueness of such topology.

Let R be a topological space and let $\mathcal{T}$ be the totality of open sets in R . Every point set S in R may be considered itself as a space. If we put

$$\mathcal{T}^S = \{ SX : X \in \mathcal{T} \},$$

then we see easily that $\mathcal{T}^S$ satisfies the topological conditions. Therefore we can introduce uniquely a topology into S such that $\mathcal{T}^S$ coincides with the totality of open sets in S by this topology. This topology in S is called the relative topology. Thus every point set S may be considered itself as a topological space by the relative topology. Then we see easily from the construction of $\mathcal{T}^S$ that a point set $A \subset S$ is open in S by the relative topology if and only if there exists an open set $X \in \mathcal{T}$ for which we have $A = SX$.

For two topological spaces R and $\hat{R}$, if there exists a correspondence between points $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ such that this correspondence is one-to-one, that is, $a \neq b$ implies $a^{\hat{R}} \neq b^{\hat{R}}$, $R^{\hat{R}} = \hat{R}$; and $X^{\hat{R}}$ is open in $\hat{R}$ if and only if X is open in R , making use of the notation

$$X^{\hat{R}} = \{ x^{\hat{R}} : x \in X \},$$

then R is said to be homeomorphic to $\hat{R}$ by this correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$. For point sets $S \subset R$ and $\hat{S} \subset \hat{R}$ we also shall say that S is homeomorphic to $\hat{S}$ by a correspondence

$S \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, if as topological spaces by the relative topology, $\hat{S}$ is homeomorphic to $\hat{S}$ by this correspondence.

11. Topological notations

Let R be a topological space and let $\mathcal{T}$ be the totality of open sets in R . For an arbitrary point set A , the greatest open set included in A is called the opener of A and denoted by A° , that is,

$$(1) \quad A^\circ = \bigcup_{A \supset X \in \mathcal{T}} X,$$

since every union of open sets is open too, by the topological condition 1). Every point $a \in A^\circ$ is said to be an inner point of A . By definition (1) we see easily that a point $a \in R$ is an inner point of a point set A if and only if there exists an open set X for which we have $a \in X \subset A$. For every point set A , its opener A° is obviously open, and for every open set X we have $X^\circ = X$ by definition (1). Thus we may make X° to represent an arbitrary open set.

A point set A is said to be closed, if its complement A' is open, that is, if $A' \in \mathcal{T}$. With this definition we have that every intersection of closed sets is closed, since we have

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \left(\bigcup_{\lambda \in \Lambda} A'_\lambda \right)'$$

for any system of point sets A_λ ($\lambda \in \Lambda$). Therefore, for an arbitrary point set A we define its closure as the least closed set including A , denoting by A^- , that is,

$$(2) \quad A^- = \bigcap_{A \subset X, X' \in \mathcal{T}} X.$$

Every point $a \in A^-$ is called a contact point of A . We see easily by definition (2) that a point $a \in R$ is a contact point of a point set A if and only if we have $A \cap X \neq \emptyset$ for every open set $X \ni a$. For every point set A , its closure A^- is obviously closed, and for every closed set X we have $X^- = X$.

by definition (2). Thus we may make X^- to represent an arbitrary closed set.

For an arbitrary point set A we obtain immediately by definitions (1) and (2)

$$(3) \quad A^{\circ\prime} = A'^-, \quad A^{-\prime} = A'^{\circ}.$$

Since the whole space R and the empty set O are open by the topological condition 3), we have obviously

$$(4) \quad R^{\circ} = R, \quad O^{\circ} = O, \quad R^- = R, \quad O^- = O.$$

As A° is open, and A^- is closed, we have naturally

$$(5) \quad A^{\circ\circ} = A^{\circ}, \quad A^{--} = A^-.$$

We have further obviously by definitions (1) and (2)

$$(6) \quad A^{\circ} < A < A^-,$$

$$(7) \quad A \supset B \text{ implies } A^{\circ} \supset B^{\circ} \text{ and } A^- \supset B^-$$

For any point sets A and B we have $(AB)^{\circ} < A^{\circ}B^{\circ}$ by the formula (7). On the other hand, we have $AB \supset A^{\circ}B^{\circ}$ by the formula (6), and hence we obtain by the formula (7)

$$(AB)^{\circ} \supset (A^{\circ}B^{\circ})^{\circ} = A^{\circ}B^{\circ},$$

since $A^{\circ}B^{\circ}$ is open by the topological condition 2). Therefore we have $(AB)^{\circ} = A^{\circ}B^{\circ}$ for any point sets A and B . From this relation we can conclude $(A+B)^- = A^- + B^-$ by duality, that is,

$$\begin{aligned} (A+B)^- &= (A+B)^{\prime\prime\prime} = (A'B')^{\prime\prime} \\ &= (A'^{\circ} B'^{\circ})' = A'^{\circ\prime} + B'^{\circ\prime} = A^{-} + B^{-}. \end{aligned}$$

For every point sets A and B we have thus

$$(8) \quad (AB)^{\circ} = A^{\circ}B^{\circ}, \quad (A+B)^- = A^- + B^-.$$

As every union of open sets is open, every intersection of closed sets is closed, we have obviously

$$(9) \quad \left(\sum_{\lambda \in A} A_{\lambda} \right)^{\circ} = \sum_{\lambda \in A} A_{\lambda}^{\circ}, \quad \left(\prod_{\lambda \in A} A_{\lambda} \right)^- = \prod_{\lambda \in A} A_{\lambda}^-.$$

Since we have obviously $AB \subset A^-B^-$ by the formula (6), we obtain by the formulas (7) and (9)

$$(AB)^- \subset (A^-B^-)^- = A^-B^-.$$

As $A = AB \dot{+} AB' \subset AB \dot{+} B'$, we have by the formulas (7), (8), and (9)

$A^- \subset (AB \dot{+} B')^- = (AB)^- \dot{+} B'^- = (AB)^- \dot{+} B^{\circ'}$,
and hence $A^- B^{\circ} \subset (AB)^- B^{\circ} \subset (AB)^-$. Therefore we have for every point sets A and B

$$(10) \quad A^- B^{\circ} \subset (AB)^- \subset A^- B^-.$$

From this relation we can conclude by duality

$$(11) \quad A^{\circ} \dot{+} B^{\circ} \subset (A \dot{+} B)^{\circ} \subset A^{\circ} \dot{+} B^-.$$

For an arbitrary point set A we have

$$(12) \quad A^{-\circ-\circ} = A^{-\circ}, \quad A^{\circ--\circ} = A^{\circ-}.$$

Because, we have $A^{-\circ-\circ} \supset A^{-\circ}$ by the formula (6), and hence

$$A^{-\circ-\circ} \supset A^{-\circ\circ} = A^{-\circ}$$

by the formulas (7) and (5). On the other hand we have

$A^{-\circ} \subset A^-$ by the formula (6), and hence

$$A^{-\circ-\circ} \subset A^{-\circ\circ} = A^{-\circ}$$

by the formulas (7) and (5). Therefore we obtain the first relation. We can conclude the second from the first by duality.

A point set A is said to be regularly open, if $A^{-\circ} = A$. We see easily by the formula (12) that $A^{-\circ}$ is regularly open for every point set A . A point set A is said to be nowhere dense, if $A^{-\circ} = 0$. If both A and B are nowhere dense, then the union $A \dot{+} B$ is nowhere dense too, because we have by the formulas (5) and (11)

$(A \dot{+} B)^{-\circ} = (A^- \dot{+} B^-)^{\circ\circ} \subset (A^{-\circ} \dot{+} B^{-\circ})^{\circ} = B^{-\circ} = 0$
if $A^{-\circ} = 0$, $B^{-\circ} = 0$. A point set A is said to be dense, if $A^- = R$. For point sets $A \subset B$, A is said to be dense in B , if A is dense in the space B by the relative topology. A point set A is said to be a F_σ-set, if A is a union of countable closed sets, that is, there exists a sequence of closed sets A_{ν}^- ($\nu = 1, 2, \dots$) for which $A = \sum_{\nu=1}^{\infty} A_{\nu}^-$. An open set A is said to be G-open, if A is a F_σ-set.

For two point sets A and B we define $A \succ B$ to mean $A^\circ \supset B^-$. Then we see easily by the formula (6) that $A \succ B$ implies $A \supset B$, and that we have $A \succ A$ if and only if A is closed and open. Furthermore we have by the formulas (6) and (7) that $A \succ B \supset C$ implies $A \succ C$, and that $C \supset A \succ B$ implies $C \succ B$. We see easily by the formulas (10) and (11) that if $A_1 \succ B_1$ and $A_2 \succ B_2$, then we have

$$A_1 A_2 \succ B_1 B_2 \text{ and } A_1 + A_2 \succ B_1 + B_2.$$

A topological space R is said to be regular, if for every open set A° we have

$$A^\circ = \sum_{A^\circ \succ X^\circ} X^\circ,$$

that is, if to any point $a \in A^\circ$ there exists an open set X° for which we have $a \in X^\circ \subset A^\circ$.

A topological space R is said to be a Hausdorff space, if to any pair of different points a and b there exist open sets A° and B° such that $a \in A^\circ$, $b \in B^\circ$ and $A^\circ B^\circ = \emptyset$. In a Hausdorff space R , every point a is a closed set as a point set composed of a single point $\{a\}$, because to any point $x \in \{a\}'$ there exists an open set A_x° for which $\{a\}' \supset A_x^\circ \ni x$ and consequently the complement $\{a\}'$ is open.

For a point set S , a point set $A \subset S$ is open in S by the relative topology if and only if there exists an open set X° in R for which $A = S X^\circ$ as remarked already. Hence a point set $A \subset S$ is closed in S by the relative topology, if and only if there exists a closed set X^- in R for which $A = S X^-$. If $A \succ B$ in R , then we have by the formula (6)

$$AS \supset A^\circ S \supset B^- S \supset BS.$$

Since $B^- S$ is closed in S , $A^\circ S$ is open in S by the relative topology, we have $AS \succ BS$ in S by the relative topology. Therefore we have that $A \succ B$ in R implies $AS \succ BS$ in S by the relative topology. By this fact, we see easily that if

R is regular, then every point set S is a regular space by the relative topology. For a Hausdorff space, it is obvious by definition that every point set S is a Hausdorff space by the relative topology.

111. Compact sets

Let R be a topological space. A point set A is said to be compact, if to any system of open sets A_λ° ($\lambda \in \Lambda$) covering A , namely $\sum_{\lambda \in \Lambda} A_\lambda^\circ \supset A$, there exists a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$ for which $\sum_{i=1}^n A_{\lambda_i}^\circ \supset A$. We see easily by definition that for a neighbourhood system $\mathcal{U}$ of R , if to any system $U_\lambda \in \mathcal{U}$ ($\lambda \in \Lambda$) covering a point set A there exists a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$ for which $\sum_{i=1}^n U_{\lambda_i} \supset A$, then A is compact.

If a point set A is compact, then the intersection AB^- is compact for every closed set B^- Because, "if

$$AB^- \subset \sum_{\lambda \in \Lambda} A_\lambda^\circ,$$

then we have $A \subset AB^- \dot{+} B^{-'} \subset \sum_{\lambda \in \Lambda} A_\lambda^\circ \dot{+} B^{-'}$. Since $B^{-'}$ is open, there exists by assumption a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$ for which we have

$$A \subset \sum_{i=1}^n A_{\lambda_i}^\circ \dot{+} B^{-'},$$

and hence $AB^- \subset \sum_{i=1}^n A_{\lambda_i}^\circ$ Therefore AB^- is compact by definition.

For a system of closed sets A_λ^- ($\lambda \in \Lambda$), if $A \prod_{\lambda \in \Lambda} A_\lambda^- = 0$ for a compact set A , then there exists a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$ for which

$$A \prod_{i=1}^n A_{\lambda_i}^- = 0.$$

Because, if $A \prod_{\lambda \in \Lambda} A_\lambda^- = 0$, then we have obviously

$$A \subset (\prod_{\lambda \in \Lambda} A_\lambda^-)' = \sum_{\lambda \in \Lambda} A_\lambda^{-'}.$$

Since all $A_\lambda^{-'}$ ($\lambda \in \Lambda$) are open, there exists by assumption a

finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_x \in A$ for which

$$A \subset \sum_{\nu=1}^x A_{\lambda_\nu}^- = \left(\prod_{\nu=1}^x A_{\lambda_\nu}^- \right)',$$

that is, $A \prod_{\nu=1}^x A_{\lambda_\nu}^- = 0$.

We have obviously by definition that for a finite number of compact sets A_ν ($\nu = 1, 2, \dots, x$) the union $A_1 + A_2 + \dots + A_x$ is compact too.

Lemma 1. In a regular space R , for an open set B° , to any compact set $A \subset B^\circ$ there exists an open set X° for which we have $A \subset X^\circ \subset B^\circ$.

Proof. Since R is regular by assumption, we have

$$A \subset B^\circ = \sum_{B^\circ \supset X^\circ} X^\circ$$

by definition, and hence there exists by assumption a finite number of open sets X_ν° ($\nu = 1, 2, \dots, x$) such that

$$A \subset \sum_{\nu=1}^x X_\nu^\circ \quad \text{and} \quad X_\nu^\circ \subset B^\circ \quad (\nu = 1, 2, \dots, x).$$

For such open sets X_ν° ($\nu = 1, 2, \dots, x$), putting $X^\circ = \sum_{\nu=1}^x X_\nu^\circ$, we obtain $A \subset X^\circ \subset B^\circ$.

Lemma 2. In a regular space R , to any two sequences of compact sets A_ν and B_ν ($\nu = 1, 2, \dots$) such that

$$\left(\sum_{\nu=1}^\infty A_\nu \right)^- \left(\sum_{\nu=1}^\infty B_\nu \right)^- = \left(\sum_{\nu=1}^\infty A_\nu \right) \left(\sum_{\nu=1}^\infty B_\nu \right)^- = 0,$$

there exist open sets A° and B° such that

$$A^\circ \supset \sum_{\nu=1}^\infty A_\nu, \quad B^\circ \supset \sum_{\nu=1}^\infty B_\nu \quad \text{and} \quad A^\circ B^\circ = 0.$$

Proof. By virtue of Lemma 1 we can determine two sequences of open sets X_ν° and Y_ν° ($\nu = 1, 2, \dots$) consecutively such that

$$A_\nu \subset X_\nu^\circ \subset \left(\sum_{\nu=1}^\infty B_\nu \right)^- \setminus (Y_1^\circ + \dots + Y_{\nu-1}^\circ),$$

$$B_\nu \subset Y_\nu^\circ \subset \left(\sum_{\nu=1}^\infty A_\nu \right)^- \setminus (X_1^\circ + \dots + X_{\nu-1}^\circ + X_\nu^\circ).$$

For such open sets X_ν° and Y_ν° , if we put

$$A^\circ = \sum_{\nu=1}^\infty X_\nu^\circ, \quad B^\circ = \sum_{\nu=1}^\infty Y_\nu^\circ,$$

then A° and B° satisfy our requirement. In fact, it is evident that

$$A^\circ \supset \sum_{\nu=1}^\infty A_\nu \quad \text{and} \quad B^\circ \supset \sum_{\nu=1}^\infty B_\nu.$$

Since we have for every $\nu = 1, 2, \dots$

$$x_\nu^\circ (Y_1^\circ + \dots + Y_{\nu-1}^\circ) = 0,$$

$$Y_\nu^\circ (X_1^\circ + \dots + X_{\nu-1}^\circ + X_\nu^\circ) = 0,$$

we obtain $X_\nu^\circ Y_\mu^\circ = 0$ for every $\nu, \mu = 1, 2, \dots$, and hence

$$A^\circ B^\circ = 0.$$

Theorem 1. In a Hausdorff space R every compact set is closed.

Proof. Let A be a compact set. For a point $a \in A$, to every point $x \in A$ there exist open sets U_x° and V_x° such that $x \in U_x^\circ$, $a \in V_x^\circ$ and $U_x^\circ V_x^\circ = 0$, since R is a Hausdorff space by assumption. For such U_x° ($x \in A$) we have obviously

$$A \subset \sum_{x \in A} U_x^\circ,$$

and hence there exists by assumption a finite number of points

$x_1, x_2, \dots, x_k$ for which we have

$$A \subset \sum_{i=1}^k U_{x_i}^\circ.$$

For such points $x_1, x_2, \dots, x_k \in A$, putting $V^\circ = V_{x_1}^\circ V_{x_2}^\circ \dots V_{x_k}^\circ$, we obtain an open set V° for which we have $a \in V^\circ$ and $A V^\circ = 0$, since $U_{x_\nu}^\circ V^\circ = 0$ for every $\nu = 1, 2, \dots, k$. Therefore the complement A' is open by the topological condition 1), and hence A is closed by definition.

A topological space R is said to be compact, if R is compact as a point set. With this definition we have obviously that a point set A is compact, if and only if A is compact as a space by the relative topology.

A topological space R is said to be locally compact, if there exists a neighbourhood system consisting only of open sets whose closures are compact. Such a neighbourhood system is called a compact neighbourhood system. Compact spaces are obviously locally compact by definition.

Theorem 2. Every locally compact Hausdorff space is regular

Proof. Let $\mathcal{U}$ be a compact neighbourhood system. Since R is a Hausdorff space by assumption, we have

$$\{a\} = \bigcap_{a \in X \in \mathcal{U}} X^-$$

for every point $a \in R$. If $a \in A^\circ$ for an open set A° , then we have $\{a\} \cap A^{\circ'} = \emptyset$ and hence

$$\bigcap_{a \in X \in \mathcal{U}} X^- \cap A^{\circ'} = \emptyset.$$

Since $X^- \cap A^{\circ'}$ is closed and compact for every $X \in \mathcal{U}$, there exists a finite number of open sets $X_\nu \in \mathcal{U}$ ($\nu = 1, 2, \dots, \kappa$) such that

$$\bigcap_{\nu=1}^{\kappa} X_\nu^- \cap A^{\circ'} = \emptyset, \quad a \in X_\nu \quad (\nu = 1, 2, \dots, \kappa).$$

For such X_ν ($\nu = 1, 2, \dots, \kappa$), putting $Y^\circ = \bigcap_{\nu=1}^{\kappa} X_\nu$, we obtain an open set $Y^\circ \ni a$ for which we have by the formula (10)

$$Y^{\circ-} \subset \bigcap_{\nu=1}^{\kappa} X_\nu^- \subset A^\circ,$$

that is, $a \in Y^\circ \subset A^\circ$. Therefore the space R is regular by definition.

iv. Continuous functions

Let R be a topological space and let φ be a function on R , i.e., corresponding to every point $x \in R$ we have a real number or $\pm \infty$ as a function value $\varphi(x)$. For $\pm \infty$ we shall adopt the following convention: for every real number α we have

$$-\infty < \alpha < +\infty, \quad \frac{\alpha}{\pm \infty} = 0,$$

$$\alpha + (\pm \infty) = (\pm \infty) + \alpha = \pm \infty,$$

$$\alpha(\pm \infty) = (\pm \infty)\alpha = \pm \infty \quad \text{for } \alpha > 0,$$

$$-(\pm \infty) = \mp \infty, \quad |\pm \infty| = +\infty,$$

$$(\pm \infty) + (\pm \infty) = \pm \infty, \quad (\pm \infty)(\pm \infty) = +\infty$$

$$(\pm \infty)(\mp \infty) = -\infty.$$

A function φ on R is said to be upper semi-continuous at a point $a \in R$ if

$$\varphi(a) = \inf_{X^\circ \ni a} \left\{ \sup_{x \in X^\circ} \varphi(x) \right\},$$

that is, if to any real number $\lambda > \varphi(a)$ there exists an open set

$X^\circ \ni a$ such that $\varphi(x) < \lambda$ for every point $x \in X^\circ$.

If a function φ on R is upper semi-continuous at every point $a \in R$, then φ is said to be upper semi-continuous.

With this definition we have that in order that a function φ on R be upper semi-continuous, it is necessary and sufficient that for every real number λ the point set

$$\{x : \varphi(x) < \lambda\}$$

is open. In fact, if φ is upper semi-continuous, then to any point $a \in R$ such that $\varphi(a) < \lambda$, there exists by definition an open set $X^\circ \ni a$ such that $\varphi(x) < \lambda$ for every point $x \in X^\circ$, and hence every point of the point set $\{x : \varphi(x) < \lambda\}$ is its inner point, that is, this point set is open. Conversely, if the point set $\{x : \varphi(x) < \lambda\}$ is open for every real number λ , then for any real number $\lambda > \varphi(a)$, putting

$$X^\circ = \{x : \varphi(x) < \lambda\},$$

we have $X^\circ \ni a$ and $\varphi(x) < \lambda$ for every point $x \in X^\circ$, and hence φ is upper semi-continuous by definition.

From this fact we conclude immediately by duality that a function φ on R is upper semi-continuous if and only if the point set

$$\{x : \varphi(x) \geq \lambda\}$$

is closed for every real number λ .

For a point set S , every function φ on R may be considered as a function on S . A function φ defined in a point set S is said to be upper semi-continuous in S , if φ is upper semi-continuous as a function on the space S by the relative topology. With this definition we have obviously that if a function φ on R is upper semi-continuous, then φ also is upper semi-continuous in every point set S .

If a function φ is upper semi-continuous in a compact set S , then there exists a point $a \in S$ for which we have

$$\varphi(a) = \sup_{x \in S} \varphi(x).$$

Because, putting $\alpha = \sup_{x \in S} \varphi(x)$, to every real number $\lambda < \alpha$ there exists a closed set X_λ^- for which

$$X_\lambda^- S = \{x : \varphi(x) \geq \lambda\} S \neq \emptyset,$$

since φ is upper semi-continuous in S by assumption. For such closed sets X_λ^- ($\lambda < \alpha$) we have obviously

$$\prod_{\nu=1}^n X_{\lambda_\nu}^- S \neq \emptyset$$

for every finite number of real numbers $\lambda_\nu < \alpha$ ($\nu = 1, 2, \dots, n$), and consequently we have

$$\prod_{\lambda < \alpha} X_\lambda^- S \neq \emptyset,$$

since S is a compact set by assumption. For every point $a \in \prod_{\lambda < \alpha} X_\lambda^- S$ we have obviously $\varphi(a) \geq \alpha$, and hence

$$\varphi(a) = \sup_{x \in S} \varphi(x).$$

A function φ on R is said to be lower semi-continuous at a point $a \in R$, if

$$\varphi(a) = \sup_{X^\circ \ni a} \left\{ \inf_{x \in X^\circ} \varphi(x) \right\},$$

that is, if to every real number $\lambda < \varphi(a)$ there exists an open set $X^\circ \ni a$ such that $\varphi(x) > \lambda$ for every point $x \in X^\circ$. Then we see easily that a function φ on R is lower semi-continuous at a point $a \in R$ if and only if $-\varphi$ is upper semi-continuous at this point $a \in R$.

A function φ on R is said to be lower semi-continuous, if φ is lower semi-continuous at every point $a \in R$. For a point set S , a function φ defined in S is said to be lower semi-continuous in S , if φ is lower semi-continuous as a function on the space S by the relative topology. Considering $-\varphi$, we obtain similar properties for lower semi-continuous functions as we have obtained for upper semi-continuous functions. For instance, in order that a function φ on R be lower semi-continuous, it is necessary and sufficient that the point set

$$\{x : \varphi(x) > \lambda\}$$

be open for every real number λ ; or that the point set

$$\{x : \varphi(x) \leq \lambda\}$$

be closed for every real number λ .

A function φ on R is said to be continuous at a point $a \in R$, if φ is upper semi-continuous and lower semi-continuous at this point a . A function φ on R is said to be continuous, if φ is continuous at every point $a \in R$, that is, if φ is upper semi-continuous and lower semi-continuous at the same time. A function φ is said to be continuous in a point set S , if φ is upper and lower semi-continuous in S at the same time.

If a function φ on R is continuous in every open set S_λ° for $\lambda \in A$, then φ is continuous in the union $\sum_{\lambda \in A} S_\lambda^\circ$. Because, if a function φ is upper semi-continuous in every open set S_λ° ($\lambda \in A$), then the point set

$$\{x : \varphi(x) < \alpha\} S_\alpha^\circ$$

is open for every real number α , and hence the union

$$\{x : \varphi(x) < \alpha\} \sum_{\lambda \in A} S_\lambda^\circ$$

is open too for every real number α . Consequently φ is upper semi-continuous in the union $\sum_{\lambda \in A} S_\lambda^\circ$. We also can prove likewise that if a function φ is lower semi-continuous in every open set S_λ° ($\lambda \in A$), then φ is lower semi-continuous in the union

$$\sum_{\lambda \in A} S_\lambda^\circ.$$

A function φ is said to be finite in a point set S , if $-\infty < \varphi(x) < +\infty$ for every point $x \in S$. A function φ is said to be bounded in a point set S , if there exists a positive number α such that $|\varphi(x)| \leq \alpha$ for every point $x \in S$.

A function φ on R is said to be almost finite, if φ is finite in some open dense set. A function φ is said to be almost finite in a point set S , if φ is almost finite as a function on the space S by the relative topology.

If a continuous function φ is finite in a compact set S

then φ is bounded in S . Because, there exist two points a and $b \in S$ for which we have

$$\varphi(a) = \sup_{x \in S} \varphi(x), \quad -\varphi(b) = \sup_{x \in S} \{-\varphi(x)\},$$

as proved just above.

We see immediately by definition that if both functions φ and ψ on R are upper semi-continuous, then both

$$\text{Max} \{ \varphi(x), \psi(x) \} \quad \text{and} \quad \text{Min} \{ \varphi(x), \psi(x) \}$$

are upper semi-continuous functions on R . Therefore if both

φ and ψ are continuous, then both

$$\text{Max} \{ \varphi(x), \psi(x) \} \quad \text{and} \quad \text{Min} \{ \varphi(x), \psi(x) \}$$

are continuous again. Furthermore we see easily that if both

φ and ψ are continuous and finite, then both

$$\varphi(x) \psi(x) \quad \text{and} \quad \varphi(x) + \psi(x) \quad (x \in R)$$

are continuous, and that for every continuous function φ on R , all functions $|\varphi(x)|^{\frac{1}{\nu}}$ ($\nu = 1, 2, \dots$) are continuous too.

We define the oscillation of a function φ in a point set S as

$$\text{osc}_{x \in S} \varphi(x) = \sup_{x \in S} \varphi(x) - \inf_{x \in S} \varphi(x).$$

Then we see easily that a function φ on R is continuous if and

only if to any positive number ε there exists a system of open

sets A_λ° ($\lambda \in A$) such that

$$\text{osc}_{x \in A_\lambda^\circ} \varphi(x) < \varepsilon \quad (\lambda \in A) \quad \text{and} \quad \sum_{\lambda \in A} A_\lambda^\circ = R.$$

For a function φ defined in a point set S , if there exists a continuous function ψ over the whole space R such that

$$\varphi(x) = \psi(x) \quad \text{for every point } x \in S,$$

then such a function ψ is called a continuous extension of φ

over R . If a bounded function φ on a point set S has a

continuous extension over R , then φ has a continuous extension,

which is bounded in R too. Indeed, if $\alpha \leq \varphi(x) \leq \beta$ for

every $x \in S$ and if ψ is a continuous extension of φ over R ,

then putting

$$\psi_0(x) = \max \{ \min \{ \psi(x), \beta \}, \alpha \} \quad (x \in R),$$

we obtain a continuous function ψ_0 , which is obviously a continuous extension of φ and satisfies the condition

$$\alpha \leq \psi_0(x) \leq \beta \quad \text{for every point } x \in R.$$

Extension theorem. In a regular space R , if a closed set S is included in an open set A° whose closure $A^{\circ-}$ is compact, then every continuous function φ on S has a continuous extension ψ over R such that $\psi(x) = 0$ for $x \notin A^\circ$.

Proof. We shall consider first the case where R is regular and compact. Let φ be a continuous function on a closed set S . Since the totality of rational numbers is countable, we denote it by $\alpha_1, \alpha_2, \dots$. Then there exists a sequence of open sets X_ν° ($\nu = 1, 2, \dots$) such that

$$\begin{aligned} \{x : \varphi(x) < \alpha_\nu\} &\subset X_\nu^\circ, \\ X_\nu^{\circ-} \setminus \{x : \varphi(x) > \alpha_\nu\} &= O, \end{aligned}$$

and $X_\nu^\circ \subset X_\mu^\circ$ for $\alpha_\nu < \alpha_\mu$. Indeed, if we suppose that $X_1^\circ, \dots, X_{\nu-1}^\circ$ are determined in such a manner, then, since

$$\begin{aligned} \{x : \varphi(x) < \alpha_\nu\} &= \bigcup_{p=1}^{\infty} \{x : \varphi(x) \leq \alpha_\nu - \frac{1}{p}\}, \\ \{x : \varphi(x) > \alpha_\nu\} &= \bigcup_{p=1}^{\infty} \{x : \varphi(x) \geq \alpha_\nu + \frac{1}{p}\}, \\ \{x : \varphi(x) < \alpha_\nu\}^- &\subset \{x : \varphi(x) \leq \alpha_\nu\}, \\ \{x : \varphi(x) > \alpha_\nu\}^- &\subset \{x : \varphi(x) \geq \alpha_\nu\}, \end{aligned}$$

there exist by Lemma 2 open sets X_ν° and Y_ν° such that

$$\begin{aligned} \sum_{\alpha_\mu < \alpha_\nu, \mu < \nu} X_\mu^{\circ-} \cup \{x : \varphi(x) < \alpha_\nu\} &\subset X_\nu^\circ, \\ \sum_{\alpha_\mu > \alpha_\nu, \mu < \nu} X_\mu^{\circ-} \cup \{x : \varphi(x) > \alpha_\nu\} &\subset Y_\nu^\circ \\ \text{and } X_\nu^\circ \cap Y_\nu^\circ &= O. \end{aligned}$$

For such an open set X_ν° we have obviously by the first relation

$$\{x : \varphi(x) < \alpha_\nu\} \subset X_\nu^\circ, \quad X_\mu^\circ \subset X_\nu^\circ \text{ for } \alpha_\mu < \alpha_\nu, \mu < \nu,$$

and by the second relation

$$X_\nu^{\circ-} \setminus \{x : \varphi(x) > \alpha_\nu\} = O, \quad X_\mu^\circ \supset X_\nu^\circ \text{ for } \alpha_\mu > \alpha_\nu, \mu < \nu,$$

since $X_\nu^\circ \cap Y_\nu^\circ = O$ implies $X_\nu^{\circ-} \subset Y_\nu^{\circ-}$.

For such a sequence of open sets X_ν° ($\nu = 1, 2, \dots$), putting

$$\psi(x) = \begin{cases} \inf_{X_\nu^\circ \ni x} \alpha_\nu & \text{for } x \in \sum_{\nu=1}^{\infty} X_\nu^\circ, \\ +\infty & \text{for } x \notin \sum_{\nu=1}^{\infty} X_\nu^\circ, \end{cases}$$

we obtain a function ψ on R . This function ψ satisfies obviously

$$\{x : \psi(x) < \alpha_\nu\} \subset X_\nu^\circ \subset \{x : \psi(x) \leq \alpha_\nu\}$$

for every $\nu = 1, 2, \dots$. Accordingly we have for every real number ξ

$$\begin{aligned} \{x : \psi(x) < \xi\} &= \sum_{\alpha_\nu < \xi} \{x : \psi(x) < \alpha_\nu\} \\ &= \sum_{\alpha_\nu < \xi} \{x : \psi(x) \leq \alpha_\nu\} = \sum_{\alpha_\nu < \xi} X_\nu^\circ, \end{aligned}$$

and hence $\{x : \psi(x) < \xi\}$ is open for every real number ξ .

Consequently ψ is upper semi-continuous, as proved just above.

On the other hand, we have for every real number ξ

$$\begin{aligned} \{x : \psi(x) \leq \xi\} &= \prod_{\alpha_\nu > \xi} \{x : \psi(x) < \alpha_\nu\} \\ &= \prod_{\alpha_\nu > \xi} \{x : \psi(x) \leq \alpha_\nu\} = \prod_{\alpha_\nu > \xi} X_\nu^\circ = \prod_{\alpha_\nu > \xi} X_\nu^{\circ-}, \end{aligned}$$

since $\alpha_\nu > \alpha_\mu$ implies $X_\nu^{\circ-} \supset X_\mu^{\circ-} \supset X_\mu^\circ$. From this relation we conclude that the point set $\{x : \psi(x) \leq \xi\}$ is closed for every real number ξ , and hence ψ is lower semi-continuous. Therefore

ψ is a continuous function on R . Furthermore ψ is a continuous extension of φ over R . Because, if $\varphi(x) < \alpha_\nu$ for a point $x \in S$, then we have $x \in X_\nu^\circ$, and hence $\psi(x) \leq \alpha_\nu$ that is, $\varphi(x) < \alpha_\nu$ implies $\psi(x) \leq \alpha_\nu$. Consequently we have

$$\varphi(x) \geq \psi(x) \quad \text{for every point } x \in S.$$

On the other hand, if $\psi(x) < \alpha_\nu$ for a point $x \in S$, then we have $x \in X_\nu^\circ$, and hence $\varphi(x) \leq \alpha_\nu$, since

$$X_\nu^\circ \setminus \{x : \varphi(x) > \alpha_\nu\} = \emptyset.$$

Consequently we obtain likewise that $\psi(x) \geq \varphi(x)$ for every point $x \in S$. Therefore we conclude $\varphi(x) = \psi(x)$ for $x \in S$.

In a regular space R , if a closed set S is included in an open set A° and its closure $A^{\circ-}$ is compact, then S is obviously compact too and there exists by Lemma 1 an open set B°

such that $S \subset B^0 \subset A^0$. For a continuous function φ on S , putting

$$\varphi_0(x) = \begin{cases} \varphi(x) & \text{for } x \in S, \\ 0 & \text{for } x \in A^{0-} - B^{0-}, \end{cases}$$

we obtain a continuous function φ_0 on $S \cup A^{0-} - B^{0-}$, since φ_0 is continuous in $S \subset B^0$ as well as in $A^{0-} - B^{0-} \subset S'$, and further $B^0 \cup S' = R$. If we consider the space A^{0-} by the relative topology, then this space A^{0-} is regular and compact. Therefore φ_0 has a continuous extension ψ_0 over A^{0-} , as proved just now. For such a continuous extension ψ_0 , if we put

$$\psi(x) = \begin{cases} \psi_0(x) & \text{for } x \in A^{0-}, \\ 0 & \text{for } x \in A^0, \end{cases}$$

then ψ is a continuous function on R , because ψ is continuous in A^0 as well as in B^{0-} and $A^0 \cup B^{0-} = R$. Furthermore we have obviously $\psi(x) = \varphi(x)$ for every point $x \in S$. Therefore ψ is a continuous extension of φ over R , and

$$\psi(x) = 0 \quad \text{for } x \in B^0 \subset A^0.$$

v. Maximal theorem

As a method for infinite process we are permitted to make use of the following axiom due to Zermelo:

Choice axiom. For any space R there exists a correspondence between every point set $A \neq \emptyset$ and a point $a_A \in R$ such that

$$A \rightarrow a_A \in A.$$

By virtue of Choice axiom we will prove the following theorem due to Zorn, which will be applied often in this book instead of transfinite induction.

Maximal theorem. Let C be a condition for a finite number of points in a space S . If a point set A_0 satisfies the condition C , that is, if the condition $C(x_1, x_2, \dots, x_n)$

is satisfied for every finite number of points $x_1, x_2, \dots, x_n \in A_0$, then there exists a maximal point set A which include A_0 and satisfies the condition C , that is, there is no other than A which includes A and satisfies the condition C .

Proof. We can assume by Choice axiom that to every point set $X \neq \emptyset$ there is determined a point $x \in X$ corresponding to X . Let $\mathcal{K}_0$ be the totality of point sets which include A_0 and satisfy the condition C . For a point set $A \in \mathcal{K}_0$, if there exists a point $x \in A$ for which $\{A, x\} \in \mathcal{K}_0$, then we obtain a point a_A corresponding to the totality of such points x and we have

$$\{A, a_A\} \in \mathcal{K}_0, \quad a_A \in A.$$

We need only to prove that there exists a point set $A \in \mathcal{K}_0$ for which there is no such corresponding point a_A .

We suppose that corresponding to every point set $A \in \mathcal{K}_0$ there is determined such a point a_A . We shall consider subsets

$\mathcal{K} \subset \mathcal{K}_0$ which satisfy the conditions:

$$*) \quad \mathcal{K} \ni A_0,$$

$$**) \quad \mathcal{K} \ni A \text{ implies } \mathcal{K} \ni \{A, a_A\},$$

$$***) \quad \mathcal{K} \ni A_\lambda \ (\lambda \in \Lambda) \text{ implies } \mathcal{K} \ni \sum_{\lambda \in \Lambda} A_\lambda \text{ if } A_\lambda \ (\lambda \in \Lambda) \text{ are mutually comparable, that is, if to every two elements } \lambda_1, \lambda_2 \in \Lambda \text{ we have } A_{\lambda_1} \supset A_{\lambda_2} \text{ or } A_{\lambda_1} \subset A_{\lambda_2}.$$

$\mathcal{K}_0$ satisfies obviously these conditions. Let $\mathcal{K}_1$ be the intersection of all subsets $\mathcal{K} \subset \mathcal{K}_0$ satisfying these conditions. Then we see easily that $\mathcal{K}_1$ also satisfies these conditions, that is, $\mathcal{K}_1$ is the least subset of $\mathcal{K}_0$ satisfying these conditions.

Let $\mathcal{K}_2$ be the totality of point sets in $\mathcal{K}_1$ which are comparable with every point set in $\mathcal{K}_1$. Then we have obviously $\mathcal{K}_2 \ni A_0$ and we see easily that $\mathcal{K}_2$ satisfies the condition ***). Furthermore $\mathcal{K}_2$ satisfies the condition **). In fact, for any $A \in \mathcal{K}_2$, if we put

$\mathcal{K}_3 = \{A : A_2 \supset A \in \mathcal{K}_1\} \cup \{A : \{A_2, a_{A_2}\} \subset A \in \mathcal{K}_1\}$, then we have obviously $\mathcal{K}_3 \ni A_0$, and we see easily that $\mathcal{K}_3$ satisfies the condition ***). Furthermore $\mathcal{K}_3$ satisfies the condition **). Because, if $\mathcal{K}_3 \ni A$ but $\mathcal{K}_3 \not\ni \{A, a_A\}$, then, since A_2 is comparable with every point set of $\mathcal{K}_1$, and since $A \in \mathcal{K}_1$ implies $\{A, a_A\} \in \mathcal{K}_1$, we must have both

$$A \subset A_2 \quad \text{and} \quad \{A, a_A\} \supset A_2,$$

and consequently $A = A_2$ or $\{A, a_A\} = A_2$, contradicting the assumption $\{A, a_A\} \notin \mathcal{K}_3$. Therefore $\mathcal{K}_3$ satisfies the conditions *), **), ***), and hence we obtain $\mathcal{K}_3 = \mathcal{K}_1$, since $\mathcal{K}_1$ is the least subset of $\mathcal{K}_0$ satisfying the conditions *), **), ***). Consequently $\{A_2, a_{A_2}\}$ is comparable with every point set $A \in \mathcal{K}_1$, that is, $\{A_2, a_{A_2}\} \in \mathcal{K}_2$. Thus $\mathcal{K}_2$ also satisfies the conditions *), **), ***), and hence we obtain likewise $\mathcal{K}_2 = \mathcal{K}_1$. Accordingly $\mathcal{K}_1$ is a system of mutually comparable point sets, and hence putting

$$A_1 = \sum_{A \in \mathcal{K}_1} A,$$

we obtain $A_1 \in \mathcal{K}_1$ by the condition ***). For such A_1 we have $\{A_1, a_{A_1}\} \in \mathcal{K}_1$ by the condition **), and consequently we obtain $\{A_1, a_{A_1}\} \subset A_1$, contradicting the assumption $a_{A_1} \notin A_1$. Therefore there exists a point set $A \in \mathcal{K}_0$ for which there is no point $x \in A$ such that $\{A, x\} \in \mathcal{K}_0$. Such a point set A is obviously a maximal point set which includes A_0 and satisfies the condition C.

CHAPTER I
SEMI-ORDERED LINEAR SPACES

§1 Linear spaces

As an important idea in functional analysis we have linear spaces. A linear space is a commutative group with operators of all real numbers.

A space R is called a commutative group, if for every two elements a and $b \in R$ we have $a + b \in R$ such that

- 1) $a + b = b + a$,
- 2) $(a + b) + c = a + (b + c)$,
- 3) to any elements a and $b \in R$ there exists at least an element $c \in R$ for which

$$a = b + c.$$

Such an element $c \in R$ is uniquely determined. Because, if

$$a = b + c = b + c',$$

then there exist by the postulate 3) e and $d \in R$ such that

$$c = c' + d, \quad c' = a + e,$$

and we have by the postulates 1) and 2)

$$\begin{aligned} c' &= a + e = (b + c) + e = \{b + (c' + d)\} + e \\ &= \{(b + c') + d\} + e = (a + d) + e = (a + e) + d \\ &= c' + d = c. \end{aligned}$$

Therefore we denote such $c \in R$ by $a - b$ Then we have

$$a = b + (a - b)$$

for every elements a and $b \in R$. Especially

$$b = b + (b - b),$$

and hence we have by the postulates 1) and 2)

$$\begin{aligned} a &= \{b + (b - b)\} + (a - b) \\ &= \{b + (a - b)\} + (b - b) = a + (b - b). \end{aligned}$$

By the uniqueness of $a - a$ we conclude consequently

$$a - a = f - f$$

for every elements a and $f \in R$, that is, for every element $a \in R$ we obtain the same element $a - a$. This uniquely determined element $a - a$ is called the zero element of R and denoted by 0 .

For any element $a \in R$ we define $-a$ to mean $0 - a$. Then we have

$$-(-a) = a,$$

because $0 = a + (-a) = (-a) + \{-(-a)\}$. Since

$$f + \{a + (-f)\} = a + \{f + (-f)\} = a + 0 = a,$$

we have for every elements a and $f \in R$

$$a + (-f) = a - f.$$

We shall denote real numbers by Greek letters in the sequel. A commutative group R is called a linear space, if for every element $a \in R$ and for every real number α we have $\alpha a \in R$ such that

- 4) $\alpha(\beta a) = (\alpha\beta)a$,
- 5) $\alpha a + \beta a = (\alpha + \beta)a$,
- 6) $\alpha a + \alpha f = \alpha(a + f)$,
- 7) $1a = a$.

Since $0a + 0a = (0+0)a = 0a$ by the postulate 5), we have $0a = 0a - 0a \stackrel{1)}{=} 0$ for every element $a \in R$. Therefore we have $a + (-1)a = (1 + (-1))a = 0a = 0$ by the postulates 5) and 7), and hence

$$(-1)a = -a.$$

Furthermore we have for every elements a and $f \in R$

$$\alpha(a - f) = \alpha a - \alpha f,$$

because

$$\alpha(a - f) = \alpha(a + (-f)) = \alpha a + \alpha(-1)f = \alpha a - \alpha f.$$

§2 Semi-ordered linear spaces

Let R be a linear space. If for some elements a and $b \in R$ we have $a \geq b$ and

- 1) $a \geq a$ for every element $a \in R$,
- 2) $a \geq b$, $b \geq a$ implies $a = b$,
- 3) $a \geq b$, $b \geq c$ implies $a \geq c$,
- 4) $a \geq b$ implies $\alpha a \geq \alpha b$ for every positive number α
- 5) to every $a, b \in R$ there exists an element $c \in R$ such that $c \geq a$ and $c \geq b$,
- 6) $a \geq b$ implies $a + c \geq b + c$ for every element $c \in R$,

then R is said to be semi-ordered. We may also write $a \leq b$ instead of $b \geq a$.

Let R be a semi-ordered linear space. An element $a \in R$ is said to be positive, if $a \geq 0$. If we denote by R^+ the totality of positive elements, then we see easily that R^+ satisfies the order conditions:

- 1') $R^+ \ni 0$,
- 2') $R^+ \ni a$ and $\exists -a$ implies $a = 0$,
- 3') $R^+ \ni a$ and $\exists b$ implies $R^+ \ni a + b$,
- 4') $R^+ \ni a$ implies $R^+ \ni \alpha a$ for every positive number α
- 5') to every element $a \in R$ there exist b and $c \in R^+$ such that $a = b - c$.

Conversely we have:

Theorem 2.1. Let R be a linear space. If a subset R^+ of R satisfies the order conditions, then we can introduce uniquely a semi-order into R , such that R^+ coincides with the totality of positive elements.

Proof. If we define $a \geq b$ to mean $a - b \in R^+$, then we have obviously the postulate 6). Furthermore we obtain the postulates 1)-5) respectively by the order conditions 1')-5'). Therefore R is semi-ordered by definition, and R^+ coincides

with the totality of positive elements. Conversely for a semi-ordered linear space R , denoting by R^+ the totality of positive elements, we have by the postulate 6) that $a \geq b$ is equivalent to $a - b \in R^+$. Therefore such semi-order is uniquely determined.

Let R be a semi-ordered linear space in the sequel. A system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) is said to be upper bounded, if there exists an element $b \in R$ such that

$$a_\lambda \leq b \quad \text{for every } \lambda \in \Lambda,$$

and then such an element b is called an upper bound of a_λ ($\lambda \in \Lambda$). If a system of elements a_λ ($\lambda \in \Lambda$) has an upper bound $a \in R$ such that we have $a \leq b$ for every other upper bound b of a_λ ($\lambda \in \Lambda$), then we shall say that a_λ ($\lambda \in \Lambda$) has the least upper bound a and we shall write

$$a = \bigcup_{\lambda \in \Lambda} a_\lambda.$$

The uniqueness of the least upper bound is obviously by definition.

A system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) is said to be lower bounded, if there exists an element $b \in R$ such that

$$a_\lambda \geq b \quad \text{for every } \lambda \in \Lambda,$$

and then such an element b is called a lower bound of a_λ ($\lambda \in \Lambda$). If a system of elements a_λ ($\lambda \in \Lambda$) has a lower bound $a \in R$ such that we have $a \geq b$ for every other lower bound b of a_λ ($\lambda \in \Lambda$), then we shall say that a_λ ($\lambda \in \Lambda$) has the greatest lower bound a and we shall write

$$a = \bigcap_{\lambda \in \Lambda} a_\lambda.$$

The postulate 5) says that every two elements a and $b \in R$ are upper bounded. We see easily by the postulate 6) that

$$a \geq b \text{ is equivalent to } -b \geq -a.$$

By this fact we see easily that every two elements a and $b \in R$ are lower bounded, since $\{-a, -b\}$ is upper bounded. But

every two elements need not have least upper bound nor greatest lower bound.

Theorem 2.2. $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ implies $-a = \bigcap_{\lambda \in \Lambda} (-a_\lambda)$; and $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ implies $-a = \bigcup_{\lambda \in \Lambda} (-a_\lambda)$.

Proof. If $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, then we have $a \geq a_\lambda$ for every $\lambda \in \Lambda$, and hence $-a \leq -a_\lambda$ for every $\lambda \in \Lambda$. On the other hand, if $b \leq -a_\lambda$ for every $\lambda \in \Lambda$, then we have $-b \geq a_\lambda$ for all $\lambda \in \Lambda$, and consequently $-b \geq a$ by the definition of least upper bound, that is, $b \leq -a$. Therefore we have $-a = \bigcap_{\lambda \in \Lambda} (-a_\lambda)$ by definition. We also can prove likewise the other assertion.

By virtue of the postulate 4) we can prove likewise:

Theorem 2.3. For every positive number α , $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ implies

$$\alpha a = \bigcup_{\lambda \in \Lambda} \alpha a_\lambda ;$$

and $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ implies

$$\alpha a = \bigcap_{\lambda \in \Lambda} \alpha a_\lambda .$$

By means of the postulate 5) we can further prove likewise:

Theorem 2.4. For every element $b \in R$, $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ implies

$$a + b = \bigcup_{\lambda \in \Lambda} (a_\lambda + b) ;$$

and $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ implies

$$a + b = \bigcap_{\lambda \in \Lambda} (a_\lambda + b) .$$

Theorem 2.5. If $a_\lambda = \bigcup_{\gamma \in \Gamma} a_{\lambda, \gamma} (\lambda \in \Lambda)$, then $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ is equivalent to $a = \bigcup_{\lambda \in \Lambda, \gamma \in \Gamma} a_{\lambda, \gamma}$. If $a_\lambda = \bigcap_{\gamma \in \Gamma} a_{\lambda, \gamma} (\lambda \in \Lambda)$, then $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ is equivalent to $a = \bigcap_{\lambda \in \Lambda, \gamma \in \Gamma} a_{\lambda, \gamma}$.

Proof. If $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, $a_\lambda = \bigcup_{\gamma \in \Gamma} a_{\lambda, \gamma} (\lambda \in \Lambda)$, then we have obviously $a \geq a_{\lambda, \gamma}$ for every $\lambda \in \Lambda$ and $\gamma \in \Gamma$ by definition. On the other hand, if $b \geq a_{\lambda, \gamma}$ for every $\lambda \in \Lambda$ and $\gamma \in \Gamma$, then we have by definition $b \geq a_\lambda$ for every $\lambda \in \Lambda$,

and hence further $b \geq a$. Therefore we obtain $a = \bigcup_{\lambda \in \Lambda, r \in P} a_{\lambda, r}$ by definition. Conversely, if

$$a = \bigcup_{\lambda \in \Lambda, r \in P} a_{\lambda, r}, \quad a_{\lambda} = \bigcup_{r \in P} a_{\lambda, r} \quad (\lambda \in \Lambda),$$

then we have obviously $a \geq a_{\lambda}$ for every $\lambda \in \Lambda$. If $b \geq a_{\lambda}$ for every $\lambda \in \Lambda$, then we have $b \geq a_{\lambda, r}$ for every $\lambda \in \Lambda$ and $r \in P$, and hence $b \geq a$ by definition. Consequently we have $a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$. We also can prove likewise the other assertion.

Theorem 2.6. If $a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$, $b = \bigcup_{r \in P} b_r$, then we have

$$a + b = \bigcup_{\lambda \in \Lambda, r \in P} (a_{\lambda} + b_r).$$

If $a = \bigcap_{\lambda \in \Lambda} a_{\lambda}$, $b = \bigcap_{r \in P} b_r$, then we have

$$a + b = \bigcap_{\lambda \in \Lambda, r \in P} (a_{\lambda} + b_r).$$

Proof. If $a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$, $b = \bigcup_{r \in P} b_r$, then we have by

Theorem 2.4 and 2.5

$$a + b = \bigcup_{\lambda \in \Lambda} (a_{\lambda} + b) = \bigcup_{\lambda \in \Lambda} \left\{ \bigcup_{r \in P} (a_{\lambda} + b_r) \right\} = \bigcup_{\lambda \in \Lambda, r \in P} (a_{\lambda} + b_r).$$

We can prove likewise the other assertion.

If two elements a and $b \in R$ have the least upper bound $c \in R$, then we shall write $c = a \vee b$. The greatest lower bound of two elements a and $b \in R$, if it exists, will be denoted by $a \wedge b$. We see easily by definition that we have $a = a \vee b$ if and only if $a \geq b$; and that we have $a = a \wedge b$ if and only if $a \leq b$.

§3 Lattice ordered linear spaces

A semi-ordered linear space R is said to be lattice ordered, if to every element $a \in R$ there exists $a \vee 0$. This least upper bound $a \vee 0$ is called the positive part of an element a and denoted by a^+ .

Let R be a lattice ordered linear space in the sequel.

Theorem 3.1. To every pair of elements a and $b \in R$

there exist both $a \cup b$ and $a \cap b$, and we have

$$(a \cup b) + (a \cap b) = a + b.$$

Proof. by virtue of Theorem 2.4 we have

$$(a - b)^+ + b = \{(a - b) \cup 0\} + b = a \cup b,$$

that is, to every two elements a and b there exists the least upper bound $a \cup b$, and hence there exists also the greatest lower bound $a \cap b$ by Theorem 2.2, and furthermore we have

$$a \cap b = -\{(-a) \cup (-b)\}.$$

Since we have by Theorem 2.4

$$\{(-a) \cup (-b)\} + a + b = b \cup a,$$

we obtain hence $a + b = (a \cup b) + (a \cap b)$.

Theorem 3.2. $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ implies

$$a \cup b = \bigcup_{\lambda \in \Lambda} (a_\lambda \cup b)$$

for every element $b \in R$; and $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ implies

$$a \cap b = \bigcap_{\lambda \in \Lambda} (a_\lambda \cap b).$$

Proof. We suppose $a = \bigcup_{\lambda \in \Lambda} a_\lambda$. Then we have obviously

$$a \cup b \geq a_\lambda \cup b \quad \text{for every } \lambda \in \Lambda.$$

If $c \geq a_\lambda \cup b$ for all $\lambda \in \Lambda$, then we have naturally $c \geq a_\lambda$ for every $\lambda \in \Lambda$, and hence $c \geq a$ by definition. Consequently we have $c \geq a \cup b$. Therefore we obtain

$$a \cup b = \bigcup_{\lambda \in \Lambda} (a_\lambda \cup b)$$

by definition. We also can prove likewise the other assertion.

Theorem 3.3. For every element $b \in R$, $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ implies

$$a \cap b = \bigcup_{\lambda \in \Lambda} (a_\lambda \cap b);$$

and $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ implies

$$a \cup b = \bigcap_{\lambda \in \Lambda} (a_\lambda \cup b).$$

Proof. We suppose $a = \bigcup_{\lambda \in \Lambda} a_\lambda$. Then we have obviously

$$a \cap b \geq a_\lambda \cap b \quad \text{for every } \lambda \in \Lambda.$$

If $c \geq a_\lambda \cap b$ for all $\lambda \in \Lambda$, then we have by Theorem 3.1

$$c - b + (a_\lambda \cup b) \geq (a_\lambda \cap b) + (a_\lambda \cup b) - b = a_\lambda,$$

and hence $c - b + (a \vee b) \geq a_\lambda$ for every $\lambda \in \Lambda$. Therefore we have

$$c - b + (a \vee b) \geq a,$$

namely $c \geq a + b - (a \vee b) = a \wedge b$. Consequently we have by definition

$$a \wedge b = \bigcup_{\lambda \in \Lambda} (a_\lambda \wedge b).$$

We also can prove likewise the other assertion.

By means of Theorem 2.5 we can conclude easily from Theorems 3.2 and 3.3:

Theorem 3.4. If $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, $b = \bigcup_{r \in R} b_r$, then we have

$$a \vee b = \bigcup_{\lambda \in \Lambda, r \in R} (a_\lambda \vee b_r) \quad \text{and} \quad a \wedge b = \bigcup_{\lambda \in \Lambda, r \in R} (a_\lambda \wedge b_r).$$

If $a = \bigcap_{\lambda \in \Lambda} a_\lambda$, $b = \bigcap_{r \in R} b_r$, then we have

$$a \vee b = \bigcap_{\lambda \in \Lambda, r \in R} (a_\lambda \vee b_r) \quad \text{and} \quad a \wedge b = \bigcap_{\lambda \in \Lambda, r \in R} (a_\lambda \wedge b_r).$$

Theorem 3.5. For every element $a \in R$ we have

$$a \vee (-a) \geq 0 \quad \text{and} \quad a \wedge (-a) \leq 0.$$

Proof. Since $a \vee 0 \geq a$ and $a \vee 0 \geq 0$, we have obviously

$$2(a \vee 0) - a \geq 0.$$

On the other hand, we have by Theorem 2.4

$$2(a \vee 0) - a = \{(2a) \vee 0\} - a = a \vee (-a),$$

and hence $a \vee (-a) \geq 0$. Consequently we obtain further by Theorem 2.2

$$a \wedge (-a) = -\{(-a) \vee a\} \leq 0.$$

For every element $a \in R$, the positive part $(-a)^+$ of $-a$ is called the negative part of a and denoted by a^- . We have then obviously by definition

$$(-a)^- = a^+, \quad (-a)^+ = a^-.$$

Theorem 3.6. For every element $a \in R$ we have

$$a = a^+ - a^-, \quad a^+ \wedge a^- = 0.$$

Proof. By definition and by Theorem 2.2 we have

$$a^- = (-a) \vee 0 = -(a \wedge 0),$$

and hence we obtain by Theorem 3.1

$$a^+ - a^- = (a \vee 0) + (a \wedge 0) = a + 0 = a.$$

Furthermore we have by Theorems 3.3 and 3.5

$$a^+ \wedge a^- = (a \vee 0) \wedge \{(-a) \vee 0\} = \{a \wedge (-a)\} \vee 0 = 0.$$

We obtain immediately by definition:

Theorem 3.7. $a \geq b$ implies $a^+ \geq b^+$ and $a^- \leq b^-$.

Theorem 3.8. If $a = b - c$, $b \wedge c = 0$, then we have

$$b = a^+ \quad \text{and} \quad c = a^-.$$

Proof. Since $b \wedge c = 0$ by assumption, we have by Theorem 3.1

$$b + c = b \vee c.$$

We obtain consequently by Theorem 2.4

$$b = (b \vee c) - c = (b - c) \vee 0 = a \vee 0 = a^+,$$

and further similarly

$$c = (b \vee c) - b = 0 \vee (c - b) = 0 \vee (-a) = a^-.$$

By definition and by Theorem 2.3 we obtain at once:

Theorem 3.9. For every positive number α we have

$$(\alpha a)^+ = \alpha a^+, \quad (\alpha a)^- = \alpha a^-.$$

Theorem 3.10. For every elements a and $b \in R$ we have

$$(a + b)^+ \leq a^+ + b^+, \quad (a + b)^- \leq a^- + b^-.$$

Proof. Since $a \leq a^+$ and $b \leq b^+$ by definition, we have

$$a + b \leq a^+ + b^+,$$

and hence by Theorem 3.7

$$(a + b)^+ \leq (a^+ + b^+)^+ = a^+ + b^+.$$

Therefore we obtain furthermore

$$(a + b)^- = (-a - b)^+ \leq (-a)^+ + (-b)^+ = a^- + b^-.$$

By virtue of Theorems 2.2, 3.2, and 3.3 we see easily:

Theorem 3.11. If $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, then we have

$$a^+ = \bigcup_{\lambda \in \Lambda} a_\lambda^+ \quad \text{and} \quad a^- = \bigcap_{\lambda \in \Lambda} a_\lambda^-.$$

If $a = \bigcap_{\lambda \in \Lambda} a_\lambda$, then we have

$$a^+ = \bigcap_{\lambda \in \Lambda} a_\lambda^+ \quad \text{and} \quad a^- = \bigcup_{\lambda \in \Lambda} a_\lambda^-.$$

Theorem 3.12. For every elements a and $b \in R$ we have

$$a \vee b = (a - b)^+ + b,$$

$$a \wedge b = a - (a - b)^+.$$

Proof. We have obviously by definition and Theorem 2.4

$$(a - b)^+ + b = \{(a - b) \vee 0\} + b = a \vee b,$$

and further by Theorems 2.2 and 2.4

$$\begin{aligned} a - (a - b)^+ &= a - \{(a - b) \vee 0\} \\ &= a + \{(b - a) \wedge 0\} = b \wedge a. \end{aligned}$$

§4 Orthogonality

Let R be a lattice ordered linear space. For every element $a \in R$ we define the absolute $|a|$ of a by

$$|a| = a^+ + a^-.$$

Since both a^+ and a^- are positive, we have obviously

$$|a| \geq a^+ \quad \text{and} \quad |a| \geq a^-.$$

Therefore we have $|a| = 0$ if and only if $a = 0$.

We have by Theorem 2.6

$$a^+ + a^- = (a \vee 0) + \{(-a) \vee 0\} = a \vee (-a) \vee 0.$$

Since $a \vee (-a) \geq 0$ by Theorem 3.5, we obtain hence

$$(1) \quad |a| = a \vee (-a).$$

For every positive number α we have by Theorem 3.9

$$(\alpha a)^+ = \alpha a^+, \quad (\alpha a)^- = \alpha a^-,$$

and for every negative number α we have

$$(\alpha a)^+ = |\alpha| a^-, \quad (\alpha a)^- = |\alpha| a^+.$$

Therefore we obtain for every real number α

$$(2) \quad |\alpha a| = |\alpha| |a|.$$

We have obviously by Theorem 3.10

$$|a + b| \leq |a| + |b|.$$

Consequently we obtain by the formula (2)

$|a| \leq |a+b| + |-b| = |a+b| + |b|$,
that is, $|a| - |b| \leq |a+b|$. Consequently we also have

$$|b| - |a| \leq |a+b|,$$

and hence we obtain by the formula (1)

$$(3) \quad ||a| - |b|| \leq |a+b| \leq |a| + |b|.$$

By virtue of Theorems 2.2 and 2.6 we have

$$\begin{aligned} (a \vee b) - (a \wedge b) &= (a \vee b) + \{(-a) \vee (-b)\} \\ &= (a-b) \vee (b-a) \vee 0, \end{aligned}$$

and hence we obtain by the formula (1)

$$(4) \quad |a-b| = (a \vee b) - (a \wedge b).$$

Recalling Theorems 3.2 and 3.3, we have by this relation (4)

$$\begin{aligned} |(a \vee c) - (b \vee c)| &= \{(a \vee b) \vee c\} - \{(a \wedge b) \vee c\}, \\ |(a \wedge c) - (b \wedge c)| &= \{(a \vee b) \wedge c\} - \{(a \wedge b) \wedge c\}, \end{aligned}$$

and hence we obtain by Theorem 3.1 and by the formula (4)

$$(5) \quad |(a \vee c) - (b \vee c)| + |(a \wedge c) - (b \wedge c)| = |a-b|.$$

Consequently we have naturally

$$(6) \quad |(a \vee c) - (b \vee c)| \leq |a-b|, \quad |(a \wedge c) - (b \wedge c)| \leq |a-b|.$$

An element $a \in R$ is said to be orthogonal to an element $b \in R$ if $|a| \wedge |b| = 0$, and then we shall write

$$a \perp b.$$

With this definition it is evident that $a \perp b$ is equivalent to $b \perp a$.

We have obviously by definition:

Theorem 4.1. If $a \perp b$ and $|b| \geq |c|$, then we have $a \perp c$.

Theorem 4.2. $a \perp b$ implies $\alpha a \perp b$ for every real number α .

Proof. If $a \perp b$, then we have by Theorem 2.3

$$(|\alpha|+1)a \perp (|\alpha|+1)b.$$

Since $|\alpha a| \leq |(|\alpha|+1)a|$, $|b| \leq |(|\alpha|+1)b|$, we obtain $\alpha a \perp b$ by the previous theorem.

Theorem 4.3. If $a \perp c$ and $b \perp c$, then we have $a+b \perp c$.

Proof. Since $(|a| \vee |b|) \wedge |c| = (|a| \wedge |c|) \vee (|b| \wedge |c|)$ by

Theorem 3.3, if $a \perp c$ and $b \perp c$, then we have

$$|a| \vee |b| \perp c,$$

and hence $2(|a| \vee |b|) \perp c$ by the previous theorem. On the other hand we have by the formula (3)

$$|a + b| \leq |a| + |b| \leq 2(|a| \vee |b|).$$

Consequently we obtain $a + b \perp c$ by Theorem 4.1.

Theorem 4.4. For $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, or for $a = \bigcap_{\lambda \in \Lambda} a_\lambda$, if $a_\lambda \perp b$ for every $\lambda \in \Lambda$, then we have $a \perp b$.

Proof. We suppose that $a = \bigcup_{\lambda \in \Lambda} a_\lambda$. Then we have by Theorem 3.11

$$a^+ = \bigcup_{\lambda \in \Lambda} a_\lambda^+ \quad \text{and} \quad a^- = \bigcap_{\lambda \in \Lambda} a_\lambda^-.$$

If $a_\lambda \perp b$ for every $\lambda \in \Lambda$, then we have by Theorem 4.1

$$a^- \perp b \quad \text{and} \quad a_\lambda^+ \perp b \quad \text{for every } \lambda \in \Lambda.$$

From the latter we conclude by Theorem 3.3

$$a^+ \cap |b| = \bigcup_{\lambda \in \Lambda} (a_\lambda^+ \cap |b|) = 0.$$

Therefore we obtain $a = a^+ - a^- \perp b$ by Theorem 4.3. We

also can dispose likewise of the case: $a = \bigcap_{\lambda \in \Lambda} a_\lambda$.

Theorem 4.5. If $a \perp b$, then we have

$$(a + b)^+ = a^+ + b^+ = a^+ \vee b^+,$$

$$(a + b)^- = a^- + b^- = a^- \vee b^-,$$

$$|a + b| = |a| + |b| = |a| \vee |b|.$$

Proof. If $a \perp b$, then we have by Theorem 4.1

$$a^+ \perp b^+, \quad a^- \perp b^+, \quad a^+ \perp b^-, \quad a^- \perp b^-.$$

Since $a^+ \perp a^-$ and $b^+ \perp b^-$ by Theorem 3.6, we obtain hence by Theorem 4.3

$$a^+ + b^+ \perp a^- + b^-,$$

that is, $(a^+ + b^+) \cap (a^- + b^-) = 0$. On the other hand, we have by Theorem 3.6

$$a + b = (a^+ + b^+)^+ - (a^- + b^-),$$

and hence by Theorem 3.8

$$(a + b)^+ = a^+ + b^+, \quad (a + b)^- = a^- + b^-.$$

Consequently we obtain furthermore by definition

$$|a + b| = |a| + |b|.$$

Since $a^+ \wedge b^+ = 0$, we obtain by Theorem 3.1

$$a^+ + b^+ = a^+ \vee b^+,$$

and further similarly

$$a^- + b^- = a^- \vee b^-, \quad |a| + |b| = |a| \vee |b|.$$

A system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) is said to be an orthogonal system, if $a_\lambda \perp a_\mu$ for every pair of different elements λ and $\mu \in \Lambda$. An orthogonal system $a_\lambda \in R$ ($\lambda \in \Lambda$) is said to be complete, if $x \perp a_\lambda$ for all $\lambda \in \Lambda$ implies $x = 0$. In general, we may say that a system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) is complete, if $x \perp a_\lambda$ for all $\lambda \in \Lambda$ implies $x = 0$. In particular, an element $a \in R$ is said to be complete, if a is itself a complete system, that is, if $x \perp a$ implies $x = 0$.

By virtue of Maximal theorem we see easily:

Theorem 4.6. For every lattice ordered linear space R , there exists a complete orthogonal system of positive elements.

§5 Convergence

Let R be a semi-ordered linear space in the sequel. A sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be increasing and denoted by $a_\nu \uparrow_{\nu=1}^\infty$, if $a_1 \leq a_2 \leq \dots$. Similarly a sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be decreasing and denoted by $a_\nu \downarrow_{\nu=1}^\infty$, if $a_1 \geq a_2 \geq \dots$. If $a_\nu \uparrow_{\nu=1}^\infty$, and further $a = \bigvee_{\nu=1}^\infty a_\nu$, then we shall write $a_\nu \uparrow_{\nu=1}^\infty a$. Similarly if $a_\nu \downarrow_{\nu=1}^\infty$, and further $a = \bigwedge_{\nu=1}^\infty a_\nu$, then we shall write $a_\nu \downarrow_{\nu=1}^\infty a$. With this definition we have obviously:

Theorem 5.1. If $a_\nu \uparrow_{\nu=1}^\infty a$, then we have $a_{\mu_\nu} \uparrow_{\nu=1}^\infty a$ for any partial sequence a_{μ_ν} ($\nu = 1, 2, \dots$). If $a_\nu \downarrow_{\nu=1}^\infty a$, then we have $a_{\mu_\nu} \downarrow_{\nu=1}^\infty a$ for any partial sequence a_{μ_ν} ($\nu = 1, 2, \dots$).

By virtue of Theorem 2.2 we obtain immediately:

Theorem 5.2. $a_\nu \uparrow_{\nu=1}^\infty, a$ implies $-a_\nu \downarrow_{\nu=1}^\infty, -a$,
 $a_\nu \downarrow_{\nu=1}^\infty, a$ implies $-a_\nu \uparrow_{\nu=1}^\infty, -a$.

Theorem 5.3. If $a_\nu \uparrow_{\nu=1}^\infty, a$ and $b_\nu \uparrow_{\nu=1}^\infty, b$, then we have

$$\alpha a_\nu + \beta b_\nu \uparrow_{\nu=1}^\infty, \alpha a + \beta b$$

for every positive numbers α and β

Proof. It is obvious by Theorem 2.3 that $a_\nu \uparrow_{\nu=1}^\infty, a$ implies $\alpha a_\nu \uparrow_{\nu=1}^\infty, \alpha a$ for every positive number α . Thus we need only to prove that $a_\nu \uparrow_{\nu=1}^\infty, a$, $b_\nu \uparrow_{\nu=1}^\infty, b$ implies

$$a_\nu + b_\nu \uparrow_{\nu=1}^\infty, a + b.$$

We have obviously by assumption

$$a + b \geq a_\nu + b_\nu \quad \text{for every } \nu = 1, 2, \dots$$

If $x \geq a_\nu + b_\nu$ for all $\nu = 1, 2, \dots$, then we also have

$$x \geq a_\nu + b_\mu \quad \text{for every } \nu, \mu = 1, 2, \dots,$$

because $a_\nu \uparrow_{\nu=1}^\infty$ and $b_\nu \uparrow_{\nu=1}^\infty$ by assumption. As

$$a + b = \bigvee_{\nu, \mu} (a_\nu + b_\mu)$$

by Theorem 2.6, we obtain hence $x \geq a + b$. Therefore we have $a_\nu + b_\nu \uparrow_{\nu=1}^\infty, a + b$ by definition.

We also can prove likewise:

Theorem 5.4. If $a_\nu \downarrow_{\nu=1}^\infty, a$ and $b_\nu \downarrow_{\nu=1}^\infty, b$, then we have

$$\alpha a_\nu + \beta b_\nu \downarrow_{\nu=1}^\infty, \alpha a + \beta b$$

for every positive numbers α and β .

Let R be lattice ordered in the sequel. A sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be convergent to a limit $a \in R$, if there exist a decreasing sequence of elements $b_\nu \downarrow_{\nu=1}^\infty, 0$ and an increasing sequence of natural numbers μ_ν ($\nu = 1, 2, \dots$) such that

$$|a_{\mu_\nu} - a| \leq b_\nu \quad \text{for every } \mu \geq \mu_\nu, \nu = 1, 2, \dots,$$

and then we shall write

$$\lim_{\nu \rightarrow \infty} a_\nu = a.$$

Such a limit $a \in R$ is uniquely determined, if exists. Because, if there exist further $b \in R$, $R \ni b_\nu \downarrow_{\nu=1}^\infty, 0$ and μ_ν ($\nu = 1, 2, \dots$)

$$|a_\mu - b| \leq h_\nu \quad \text{for every } \mu \geq \mu_\nu, \nu = 1, 2, \dots,$$

then for every $\mu \geq \text{Max}\{\mu_\nu, \mu_\nu\}$ we have both

$$|a_\mu - a| \leq l_\nu \quad \text{and} \quad |a_\mu - b| \leq h_\nu,$$

and hence we obtain by the formula §4(3)

$$|a - b| \leq l_\nu + h_\nu \quad \text{for every } \nu = 1, 2, \dots$$

Since $l_\nu + h_\nu \downarrow_{\nu=1}^\infty 0$ by Theorem 5.4, we obtain consequently

$$|a - b| = 0, \text{ namely } a = b.$$

Theorem 5.5. If $a_\nu \uparrow_{\nu=1}^\infty a$, or if $a_\nu \downarrow_{\nu=1}^\infty a$, then
we have $\lim_{\nu \rightarrow \infty} a_\nu = a$.

Proof. From assumption we conclude by Theorem 2.4

$$a - a_\nu \downarrow_{\nu=1}^\infty 0 \quad \text{or} \quad a_\nu - a \downarrow_{\nu=1}^\infty 0$$

respectively. Putting $h_\nu = a - a_\nu$ or $h_\nu = a_\nu - a$ respectively, we have hence $h_\nu \downarrow_{\nu=1}^\infty 0$, and obviously

$$|a_\mu - a| \leq h_\nu \quad \text{for all } \mu \geq \nu, \nu = 1, 2, \dots$$

Theorem 5.6. If $a_\nu \uparrow_{\nu=1}^\infty$, and $\lim_{\nu \rightarrow \infty} a_\nu = a$, then we
have $a_\nu \uparrow_{\nu=1}^\infty a$. If $a_\nu \downarrow_{\nu=1}^\infty$, and $\lim_{\nu \rightarrow \infty} a_\nu = a$, then we
have $a_\nu \downarrow_{\nu=1}^\infty a$.

Proof. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, then there exist by definition $R \ni h_\nu \downarrow_{\nu=1}^\infty 0$ and μ_ν ($\nu = 1, 2, \dots$) such that

$$|a_\mu - a| \leq h_\nu \quad \text{for every } \mu \geq \mu_\nu, \nu = 1, 2, \dots,$$

and hence

$$a - h_\nu \leq a_\mu \leq a + h_\nu \quad \text{for every } \mu \geq \mu_\nu, \nu = 1, 2, \dots$$

In the case: $a_\nu \uparrow_{\nu=1}^\infty$, for any $\mu = 1, 2, \dots$ we have

$$a_\mu \leq a + h_\nu \quad \text{for every } \nu = 1, 2, \dots,$$

and hence $a_\mu \leq \bigcap_{\nu=1}^\infty (a + h_\nu) = a$ by Theorem 2.4. On the other

hand, if $x \geq a_\mu$ for every $\mu = 1, 2, \dots$, then we have

$$x \geq a - h_\nu \quad \text{for every } \nu = 1, 2, \dots,$$

and hence $x \geq \bigcup_{\nu=1}^\infty (a - h_\nu) = a$ by Theorems 2.2 and 2.4. There-

fore we obtain $a_\nu \uparrow_{\nu=1}^\infty a$. We also can dispose likewise of the other case.

By definition and by Theorem 5.1 we have obviously:

Theorem 5.7. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, then we have $\lim_{\nu \rightarrow \infty} a_{\mu_\nu} = a$ for any partial sequence a_{μ_ν} ($\nu = 1, 2, \dots$).

Theorem 5.8. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, $\lim_{\nu \rightarrow \infty} b_\nu = b$, then we have for every real numbers α and β

$$\lim_{\nu \rightarrow \infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b.$$

Proof. By assumption, there exist $R \ni l_\nu \downarrow_{\nu=1}^\infty 0$, $R \ni k_\nu \downarrow_{\nu=1}^\infty 0$, and μ_ν , ρ_ν ($\nu = 1, 2, \dots$) such that

$$\begin{aligned} |a_\mu - a| &\leq l_\nu & \text{for all } \mu \geq \mu_\nu, \nu = 1, 2, \dots, \\ |b_\rho - b| &\leq k_\nu & \text{for all } \rho \geq \rho_\nu, \nu = 1, 2, \dots. \end{aligned}$$

Putting $\kappa_\nu = \text{Max} \{ \mu_\nu, \rho_\nu \}$, we have then

$$|(\alpha a_\mu + \beta b_\rho) - (\alpha a + \beta b)| \leq |\alpha| l_\nu + |\beta| k_\nu$$

for every $\mu \geq \kappa_\nu$, $\nu = 1, 2, \dots$. Since $|\alpha| l_\nu + |\beta| k_\nu \downarrow_{\nu=1}^\infty 0$ by Theorem 5.4, we obtain hence by definition

$$\lim_{\nu \rightarrow \infty} (\alpha a_\nu + \beta b_\nu) = \alpha a + \beta b.$$

Furthermore we have by the formula §4(6)

$$\begin{aligned} |(a_\mu \cup b_\rho) - (a \cup b)| &\leq |(a_\mu \cup b_\rho) - (a \cup b_\rho)| + |(a \cup b_\rho) - (a \cup b)| \\ &\leq |a_\mu - a| + |b_\rho - b| \leq l_\nu + k_\nu \end{aligned}$$

for every $\mu = \kappa_\nu$, $\nu = 1, 2, \dots$, and similarly

$$|(a_\mu \cap b_\rho) - (a \cap b)| \leq l_\nu + k_\nu.$$

Therefore we also obtain:

Theorem 5.9. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, $\lim_{\nu \rightarrow \infty} b_\nu = b$, then we have

$$\begin{aligned} \lim_{\nu \rightarrow \infty} (a_\nu \cup b_\nu) &= a \cup b, \\ \lim_{\nu \rightarrow \infty} (a_\nu \cap b_\nu) &= a \cap b. \end{aligned}$$

As a special case, putting $b_\nu = 0$ ($\nu = 1, 2, \dots$) we have that $\lim_{\nu \rightarrow \infty} a_\nu = a$ implies $\lim_{\nu \rightarrow \infty} a_\nu^+ = a^+$. Therefore we obtain by Theorem 5.8.

Theorem 5.10. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, then we have

$$\lim_{\nu \rightarrow \infty} a_\nu^+ = a^+, \quad \lim_{\nu \rightarrow \infty} a_\nu^- = a^-, \quad \lim_{\nu \rightarrow \infty} |a_\nu| = |a|.$$

Theorem 5.11. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, $\lim_{\nu \rightarrow \infty} b_\nu = b$ and $a_\nu \leq b_\nu$ except for a finite number of ν , then we have $a \leq b$.

Proof. From assumption we conclude by Theorem 5.9

$$\lim_{\nu \rightarrow \infty} (a_\nu \vee b_\nu) = a \vee b.$$

Since $a_\nu \vee b_\nu = b_\nu$ except for a finite number of ν by assumption, we obtain by Theorem 5.7

$$b = \lim_{\nu \rightarrow \infty} b_\nu = a \vee b,$$

and hence $a \leq b$.

Theorem 5.12. If $\lim_{\nu \rightarrow \infty} a_\nu = \lim_{\nu \rightarrow \infty} b_\nu = c$ and $a_\nu \leq c_\nu \leq b_\nu$ except for a finite number of ν , then we have $\lim_{\nu \rightarrow \infty} c_\nu = c$.

Proof. Except for a finite number of ν , we have

$$c_\nu - c \leq b_\nu - c, \quad c - c_\nu \leq c - a_\nu$$

by assumption, and hence by the formula §4(1)

$$|c_\nu - c| \leq |b_\nu - c| + |c - a_\nu|.$$

From this relation we conclude similarly as in Proof of Theorem 5.8 that $\lim_{\nu \rightarrow \infty} c_\nu = c$.

We have obviously by definition:

Theorem 5.13. We have $\lim_{\nu \rightarrow \infty} a_\nu = a$, if and only if

$$\lim_{\nu \rightarrow \infty} |a_\nu - a| = 0.$$

For a sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$), if

$$\lim_{\nu \rightarrow \infty} (a_1 + a_2 + \dots + a_\nu) = a,$$

then we shall say that a series $\sum_{\nu=1}^{\infty} a_\nu$ is convergent with sum a and write

$$a = \sum_{\nu=1}^{\infty} a_\nu \quad \text{or} \quad a = a_1 + a_2 + \dots$$

We have then obviously by Theorem 5.8:

Theorem 5.14. If $a = \sum_{\nu=1}^{\infty} a_\nu$, $b = \sum_{\nu=1}^{\infty} b_\nu$, then we have

$$\alpha a + \beta b = \sum_{\nu=1}^{\infty} (\alpha a_\nu + \beta b_\nu)$$

for every real numbers α and β .

We obtain immediately by Theorems 4.5, 5.6, and 5.9:

Theorem 5.15. If $a = \sum_{\nu=1}^{\infty} a_\nu$ for an orthogonal sequence

$a_\nu \in R$ ($\nu = 1, 2, \dots$), then we have

$$a^+ = \sum_{\nu=1}^{\infty} a_\nu^+ = \sum_{\nu=1}^{\infty} a_\nu^+,$$

$$a^- = \sum_{\nu=1}^{\infty} a_{\nu}^- = \bigcup_{\nu=1}^{\infty} a_{\nu}^-,$$

$$|a| = \sum_{\nu=1}^{\infty} |a_{\nu}| = \bigcup_{\nu=1}^{\infty} |a_{\nu}|.$$

§6 Continuous semi-ordered linear spaces

A semi-ordered linear space R is said to be archimedean, if we have $\frac{1}{\nu}a \downarrow_{\nu=1}^{\infty} 0$ for every positive element $a \in R$. If R is archimedean, then we see easily that we have $\varepsilon_{\nu}a \downarrow_{\nu=1}^{\infty} 0$ for every positive $a \in R$ and for every sequence of numbers $\varepsilon_{\nu} \downarrow_{\nu=1}^{\infty} 0$.

Let a lattice ordered linear space R be archimedean. A sequence of elements $a_{\nu} \in R$ ($\nu = 1, 2, \dots$) is said to be uniformly convergent to a limit $a \in R$, if there exist a positive element $l \in R$ and a sequence of numbers $\varepsilon_{\nu} \downarrow_{\nu=1}^{\infty} 0$ and μ_{ν} ($\nu = 1, 2, \dots$) such that

$$|a_{\mu} - a| \leq \varepsilon_{\nu} l \quad \text{for every } \mu \geq \mu_{\nu}, \nu = 1, 2, \dots.$$

With this definition it is evident that if a sequence of elements $a_{\nu} \in R$ ($\nu = 1, 2, \dots$) is uniformly convergent to a limit $a \in R$, then we have $\lim_{\nu \rightarrow \infty} a_{\nu} = a$.

Theorem 6.1. If a lattice ordered linear space R is archimedean, then $\lim_{\nu \rightarrow \infty} \alpha_{\nu} = \alpha$, $\lim_{\nu \rightarrow \infty} a_{\nu} = a$ implies $\lim_{\nu \rightarrow \infty} \alpha_{\nu} a_{\nu} = \alpha a$.

Proof. From assumption we conclude that there exists a positive number γ such that $|\alpha_{\nu}| \leq \gamma$ for every $\nu = 1, 2, \dots$. As

$$|\alpha_{\nu} a_{\nu} - \alpha a| \leq |\alpha_{\nu}| |a_{\nu} - a| + |\alpha_{\nu} - \alpha| |a|$$

$$\leq \gamma |a_{\nu} - a| + |\alpha_{\nu} - \alpha| |a|,$$

$$\lim_{\nu \rightarrow \infty} |a_{\nu} - a| = 0, \quad \lim_{\nu \rightarrow \infty} |\alpha_{\nu} - \alpha| |a| = 0,$$

we obtain by Theorem 5.12

$$\lim_{\nu \rightarrow \infty} |\alpha_{\nu} a_{\nu} - \alpha a| = 0,$$

and hence $\lim_{\nu \rightarrow \infty} \alpha_{\nu} a_{\nu} = \alpha a$ by Theorem 5.13.

A semi-ordered linear space R is said to be continuous, if every sequence of positive elements $a_{\nu} \in R$ ($\nu = 1, 2, \dots$) has

the greatest lower bound $\bigwedge_{\nu=1}^{\infty} a_{\nu}$.

Theorem 6.2. If a semi-ordered linear space R is continuous, then to every upper bounded sequence of elements $a_{\nu} \in R$ ($\nu = 1, 2, \dots$) there exists the least upper bound $\bigvee_{\nu=1}^{\infty} a_{\nu}$, and to every lower bounded sequence of elements $a_{\nu} \in R$ ($\nu = 1, 2, \dots$) there exists the greatest lower bound $\bigwedge_{\nu=1}^{\infty} a_{\nu}$.

Proof. If $a_{\nu} \leq a$ for every $\nu = 1, 2, \dots$, then we have obviously

$$a - a_{\nu} \geq 0 \quad \text{for every } \nu = 1, 2, \dots$$

Since R is continuous by assumption, there exists $\bigwedge_{\nu=1}^{\infty} (a - a_{\nu})$ by definition, and hence there exists $\bigvee_{\nu=1}^{\infty} a_{\nu}$ by Theorems 2.2 and 2.4. We also can prove likewise the other assertion.

Theorem 6.3. If a semi-ordered linear space R is continuous, then R is lattice ordered and archimedean.

Proof. Since $\{a, 0\}$ is upper bounded for every $a \in R$, if R is continuous, then there exists a^+ by the previous theorem, that is, R is lattice ordered by definition. Furthermore, to any positive element $a \in R$ there exists by the previous theorem a positive element $b \in R$ such that $\frac{1}{\nu} a \downarrow_{\nu=1}^{\infty} b$. Then we have $\frac{1}{2\nu} a \downarrow_{\nu=1}^{\infty} \frac{1}{2} b$ by Theorem 5.4, and hence $b = \frac{1}{2} b$ by Theorem 5.1, that is, $b = 0$. Therefore R is archimedean by definition.

Theorem 6.4. If a semi-ordered linear space R is continuous, then in order that a sequence of elements $a_{\nu} \in R$ ($\nu = 1, 2, \dots$) be convergent, it is necessary and sufficient that there exist $R \ni \ell_{\nu} \downarrow_{\nu=1}^{\infty} 0$ and μ_{ν} ($\nu = 1, 2, \dots$) such that

$$|a_{\mu} - a_{\rho}| \leq \ell_{\nu} \quad \text{for every } \mu, \rho \geq \mu_{\nu}, \nu = 1, 2, \dots$$

Proof. If $\lim_{\nu \rightarrow \infty} a_{\nu} = a$, then there exist by definition $R \ni \ell_{\nu} \downarrow_{\nu=1}^{\infty} 0$ and μ_{ν} ($\nu = 1, 2, \dots$) such that

$$|a_{\mu} - a| \leq \ell_{\nu} \quad \text{for every } \mu \geq \mu_{\nu}, \nu = 1, 2, \dots$$

For every natural numbers $\mu, \rho \geq \mu_{\nu}$ we have then

$$|a_{\mu} - a_{\rho}| \leq |a_{\mu} - a| + |a_{\rho} - a| \leq 2\ell_{\nu}$$

and further $2l_\nu \downarrow_{\nu=1}^\infty 0$ by Theorem 5.3.

Conversely, if there exist $R \ni l_\nu \downarrow_{\nu=1}^\infty 0$ and μ_ν ($\nu = 1, 2, \dots$) such that

$$|a_\mu - a_\rho| \leq l_\nu \quad \text{for every } \mu, \rho \geq \mu_\nu, \nu = 1, 2, \dots,$$

then we have obviously for every natural numbers $\mu, \rho \geq \mu_\nu$

$$a_\rho - l_\nu \leq a_\mu \leq a_\rho + l_\nu.$$

Thus the sequence a_ν ($\nu = 1, 2, \dots$) is upper bounded. Since

R is continuous by assumption, there exists $\bigcup_{\mu \in \mu_\nu} a_\mu$ by Theorem 6.2, and we have for every $\mu, \rho \geq \mu_\nu$, $\nu = 1, 2, \dots$

$$a_\rho - l_\nu \leq \bigcup_{\mu \in \mu_\nu} a_\mu \leq a_\rho + l_\nu.$$

Therefore, putting $a = \bigcap_{\nu=1}^\infty \left\{ \bigcup_{\mu \in \mu_\nu} a_\mu \right\}$, we obtain similarly

$$a_\rho - l_\nu \leq a \leq a_\rho + l_\nu,$$

that is, $|a_\rho - a| \leq l_\nu$ for every $\rho \geq \mu_\nu$, $\nu = 1, 2, \dots$, and

hence $\lim_{\nu \rightarrow \infty} a_\nu = a$ by definition.

Theorem 6.5. In a continuous semi-ordered linear space R

if $\sum_{\nu=1}^\infty a_\nu$ is convergent and $a_\nu \geq l_\nu \geq 0$ for every $\nu = 1, 2, \dots$, then $\sum_{\nu=1}^\infty l_\nu$ is convergent too and we have

$$\sum_{\nu=1}^\infty a_\nu \geq \sum_{\nu=1}^\infty l_\nu \geq 0.$$

Proof. If $a = \sum_{\nu=1}^\infty a_\nu$, then we have by assumption

$$0 \leq l_1 + l_2 + \dots + l_\nu \leq a \quad \text{for every } \nu = 1, 2, \dots$$

Consequently we obtain by Theorems 6.2 and 5.6 that the series

$$\sum_{\nu=1}^\infty l_\nu \text{ is convergent and } 0 \leq \sum_{\nu=1}^\infty l_\nu \leq a.$$

If $\sum_{\nu=1}^\infty |a_\nu|$ is convergent, then we shall say that a series $\sum_{\nu=1}^\infty a_\nu$ is absolutely convergent.

Theorem 6.6. If a semi-ordered linear space R is continuous, then every absolutely convergent series $\sum_{\nu=1}^\infty a_\nu$ is convergent and $\left| \sum_{\nu=1}^\infty a_\nu \right| \leq \sum_{\nu=1}^\infty |a_\nu|$.

Proof. Since $0 \leq a_\nu^+ \leq |a_\nu|$, $0 \leq a_\nu^- \leq |a_\nu|$, if $\sum_{\nu=1}^\infty |a_\nu|$ is absolutely convergent, then both $\sum_{\nu=1}^\infty a_\nu^+$ and $\sum_{\nu=1}^\infty a_\nu^-$ are convergent by the previous theorem. Consequently $\sum_{\nu=1}^\infty a_\nu$ is convergent by Theorem 5.15, and we have

$$|\sum_{v=1}^{\infty} a_v| = |\sum_{v=1}^{\infty} a_v^+ - \sum_{v=1}^{\infty} a_v^-| \leq \sum_{v=1}^{\infty} a_v^+ + \sum_{v=1}^{\infty} a_v^- = \sum_{v=1}^{\infty} |a_v|.$$

A semi-ordered linear space R is said to be universally continuous, if to every system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) there exists the greatest lower bound $\bigcap_{\lambda \in \Lambda} a_\lambda$. It is evident that if R is universally continuous, then R is continuous.

By similar methods used to prove Theorem 6.2, we can prove:

Theorem 6.7. If a semi-ordered linear space R is universally continuous, then to every upper bounded system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) there exists the least upper bound $\bigcup_{\lambda \in \Lambda} a_\lambda$, and to every lower bounded system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) there exists the greatest lower bound $\bigcap_{\lambda \in \Lambda} a_\lambda$.

§7 Projectors

Let R be a continuous semi-ordered linear space in the sequel. By virtue of Theorem 6.2, to every positive elements p and $a \in R$ there exists uniquely an element $c \in R$ such that

$$a \wedge np \uparrow_{n=1}^{\infty} c.$$

This element c will be denoted by $[p]a$ Corresponding to an arbitrary element $p \in R$, putting

$$[p]x = [1p]x^+ - [1p]x^- \quad (x \in R),$$

we obtain an operator $[p]$ on R , which will be called the projector of an element $p \in R$.

Theorem 7.1. Projectors $[p]$ are linear:

$$[p](\alpha x + \beta y) = \alpha [p]x + \beta [p]y$$

for every elements $x, y \in R$ and for every real numbers α, β .

Proof. For every positive elements a and $b \in R$ we have by definition

$$a \wedge n[1p] \uparrow_{n=1}^{\infty} [p]a, \quad b \wedge n[1p] \uparrow_{n=1}^{\infty} [p]b$$

and hence by Theorem 5.3

$$(a \wedge \vee |p|) + (b \wedge \vee |p|) \uparrow_{\nu=1}^{\infty} [p]a + [p]b.$$

On the other hand we have by Theorem 2.6

$$(a \wedge \vee |p|) + (b \wedge \vee |p|) = (a+b) \wedge (a \vee |p|) \wedge (b \vee |p|) \wedge 2 \vee |p|,$$

and consequently we obtain

$$(a+b) \wedge \vee |p| \leq (a \wedge \vee |p|) + (b \wedge \vee |p|) \leq (a+b) \wedge 2 \vee |p|.$$

Since we have by Theorems 5.5 and 5.7

$$\lim_{\nu \rightarrow \infty} \{(a+b) \wedge \vee |p|\} = \lim_{\nu \rightarrow \infty} \{(a+b) \wedge 2 \vee |p|\} = [p](a+b),$$

we obtain therefore by Theorem 5.12

$$\lim_{\nu \rightarrow \infty} \{(a \wedge \vee |p|) + (b \wedge \vee |p|)\} = [p](a+b),$$

and hence by Theorem 5.5

$$[p](a+b) = [p]a + [p]b$$

for every positive elements a and $b \in R$.

For arbitrary elements x and $y \in R$, putting

$$a = (x^+ + y^+) - (x + y)^+,$$

we obtain a positive element $a \in R$ by Theorem 3.10.

Since

$$(x^+ + y^+) - (x^- + y^-) = (x + y)^+ - (x + y)^-,$$

we have then further

$$a = (x^- + y^-) - (x + y)^-.$$

As proved just now, we obtain hence

$$[p]x^+ + [p]y^+ = [p]a + [p](x + y)^+,$$

$$[p]x^- + [p]y^- = [p]a + [p](x + y)^-.$$

Therefore we have by definition

$$[p]x + [p]y = [p](x + y)$$

for arbitrary elements x and $y \in R$.

For every positive element $a \in R$ and for every positive number α , we have by definition

$$(\alpha a) \wedge \vee |p| \uparrow_{\nu=1}^{\infty} [p]\alpha a,$$

and hence we obtain by Theorems 2.3 and 5.3

$$a \wedge \frac{\nu}{\alpha} |p| \uparrow_{\nu=1}^{\infty} \frac{1}{\alpha} [p]\alpha a.$$

Since to any $\mu = 1, 2, \dots$ there exists a natural number such that $\frac{\nu}{\alpha} \geq \mu$, we see easily by definition

$$a \wedge \mu | p | \uparrow_{\mu=1}^{\infty} [p]a \leq \frac{1}{\alpha} [p]a.$$

On the other hand, since to any $\nu = 1, 2, \dots$ there exists a natural number μ such that $\mu \geq \frac{\nu}{\alpha}$, we obtain similarly

$$[p]a \geq \frac{1}{\alpha} [p]a.$$

Consequently we have for every positive element $a \in R$ and positive number α

$$\alpha [p]a = [p]\alpha a.$$

For every positive number α , since $(\alpha x)^+ = \alpha x^+$, $(\alpha x)^- = \alpha x^-$ by Theorem 3.9, we obtain hence for every element $x \in R$

$$[p](\alpha x) = [p]\alpha x^+ - [p]\alpha x^- = \alpha([p]x^+ - [p]x^-) = \alpha [p]x.$$

For an arbitrary element $x \in R$ we have by definition

$$[p](-x) = [p]x^- - [p]x^+ = -[p]x.$$

Therefore we have for every real number α

$$[p]\alpha x = \alpha [p]x \quad (x \in R)$$

We have obviously by definition that every projector $[p]$ is positive:

$[p]a \geq 0$ for every positive element $a \in R$,
and hence $[p]a \geq [p]b$ for $a \geq b$.

Theorem 7.2. Projectors $[p]$ are idempotent: $[p][p] = [p]$.

Proof. For any positive element $a \in R$ we have by definition

and by Theorem 3.2

$$\begin{aligned} [p]([p]a) &= \bigvee_{\nu=1}^{\infty} ([p]a \wedge \nu | p |) = \bigvee_{\nu=1}^{\infty} \{ (\bigvee_{\mu=1}^{\infty} (a \wedge \mu | p |)) \wedge \nu | p | \} \\ &= \bigvee_{\nu=1}^{\infty} \{ \bigvee_{\mu=1}^{\infty} (a \wedge \mu | p | \wedge \nu | p |) \} = \bigvee_{\nu=1}^{\infty} (a \wedge \nu | p |) = [p]a. \end{aligned}$$

Therefore we obtain by definition and Theorem 7.1

$$[p]([p]x) = [p]([p]x^+ - [p]x^-) = [p]x^+ - [p]x^- = [p]x$$

for arbitrary element $x \in R$.

Theorem 7.3. For every positive elements a , and $b \in R$

we have

$$[p](a \wedge b) = [p]a \wedge b = a \wedge [p]b.$$

Proof. For positive elements $a, b \in R$ we have by definition

and Theorem 3.2

$$\begin{aligned}
 [p]a \wedge b &= \left\{ \bigcup_{i=1}^{\infty} (a \wedge \vee |p|) \right\} \wedge b \\
 &= \bigcup_{i=1}^{\infty} (a \wedge b \wedge \vee |p|) = [p](a \wedge b).
 \end{aligned}$$

Theorem 7.4. For every element $x \in R$ we have

$$\begin{aligned}
 ([p]x)^+ &= [p]x^+, \quad ([p]x)^- = [p]x^-, \\
 |[p]x| &= [p]|x|.
 \end{aligned}$$

Proof. Since $x^+ \wedge x^- = 0$, we have by the previous theorem

$$[p]x^+ \wedge [p]x^- = 0.$$

As $[p]x = [p]x^+ - [p]x^-$ by definition, we obtain hence by

Theorem 3.8

$$([p]x)^+ = [p]x^+, \quad ([p]x)^- = [p]x^-,$$

and consequently $|[p]x| = [p]|x|$ by definition and Theorem 7.1.

Theorem 7.5. $|[p]x| \leq |x|$ for every element $x \in R$.

Proof. We have by definition and by the previous theorem

$$|[p]x| = [p]|x| = \bigcup_{i=1}^{\infty} (|x| \wedge \vee |p|) \leq |x|.$$

Theorem 7.6. If $x = \bigcup_{\lambda \in \Lambda} x_\lambda$, then we have

$$[p]x = \bigcup_{\lambda \in \Lambda} [p]x_\lambda.$$

If $x = \bigwedge_{\lambda \in \Lambda} x_\lambda$, then we have

$$[p]x = \bigwedge_{\lambda \in \Lambda} [p]x_\lambda.$$

Proof. If $x = \bigcup_{\lambda \in \Lambda} x_\lambda$, then we have $\bigcap_{\lambda \in \Lambda} (x - x_\lambda) = 0$ by

Theorems 2.2 and 2.4. Since

$$0 \leq [p]x - [p]x_\lambda = [p](x - x_\lambda) \leq x - x_\lambda$$

by Theorems 7.1, 7.2 and 7.5, we have hence

$$\bigcap_{\lambda \in \Lambda} ([p]x - [p]x_\lambda) = 0,$$

and consequently $[p]x = \bigcup_{\lambda \in \Lambda} [p]x_\lambda$ by Theorems 2.2 and 2.4.

We also can prove likewise the other assertion.

Theorem 7.7. Projectors $[p]$ are continuous: $\lim_{\nu \rightarrow \infty} x_\nu = x$ implies $\lim_{\nu \rightarrow \infty} [p]x_\nu = [p]x$.

Proof. If $|x_\mu - x| \leq \ell_\nu$ for $\mu \geq \mu_\nu$, $\nu = 1, 2, \dots$ and $R \ni \ell_\nu \downarrow_{\nu \rightarrow \infty} 0$, then we have by Theorems 7.1, 7.4 and 7.5

$$|[p]x_\mu - [p]x| = [p]|x_\mu - x| \leq |x_\mu - x| \leq \ell_\nu$$

for every $\mu \geq \mu_\nu$, $\nu = 1, 2, \dots$. Therefore we obtain the assertion.

Theorem 7.8. $p \perp x - [p]x$ for every elements $p, x \in R$.

Proof. For a positive element $a \in R$ we have by definition

and by Theorems 2.4 and 3.3

$$\begin{aligned} |p| \wedge (a - [p]a) &= |p| \wedge \{a - \bigvee_{i=1}^{\infty} (a \wedge \nu |p|)\} \\ &= \{(|p| \wedge \bigvee_{i=1}^{\infty} (a \wedge \nu |p|)) \wedge a\} - \bigvee_{i=1}^{\infty} (a \wedge \nu |p|) \\ &= \bigvee_{i=1}^{\infty} \{ (a + |p|) \wedge (\nu + 1) |p| \wedge a \} - \bigvee_{i=1}^{\infty} (a \wedge \nu |p|) \\ &= \bigvee_{i=1}^{\infty} (a \wedge (\nu + 1) |p|) - \bigvee_{i=1}^{\infty} (a \wedge \nu |p|) = 0, \end{aligned}$$

and hence $p \perp a - [p]a$ for $a \geq 0$. For an arbitrary element $x \in R$ we have therefore

$$p \perp x^+ - [p]x^+ \text{ and } p \perp x^- - [p]x^-,$$

and consequently $p \perp x - [p]x$ by Theorems 4.2 and 4.3.

Theorem 7.9. $[p]x \perp y$ implies $x \perp [p]y$.

Proof. We have by Theorems 7.3 and 7.4

$$|[p]x| \wedge |y| = [p]|x| \wedge |y| = |x| \wedge [p]|y| = |x| \wedge |[p]y|,$$

and hence $[p]x \perp y$ is equivalent to $x \perp [p]y$.

Theorem 7.10. $[p]x \perp y - [p]y$ for every elements $x, y \in R$.

Proof. We have by Theorems 7.1 and 7.3

$$[p](y - [p]y) = [p]y - [p]([p]y) = 0,$$

and obviously $x \perp 0$ for every element $x \in R$. Therefore

we obtain $[p]x \perp y - [p]y$ by the previous theorem.

Theorem 7.11. $[p]p = p$ for every element $p \in R$.

Proof. As $0 \leq p^+ \leq |p|$, we have by definition

$$[p]p^+ = \bigvee_{i=1}^{\infty} (p^+ \wedge \nu |p|) = p^+.$$

We also obtain likewise $[p]p^- = p^-$, and hence by definition

$$[p]p = [p]p^+ - [p]p^- = p.$$

Theorem 7.12. We have $[p]x = 0$ if and only if $p \perp x$.

Proof. If $p \perp x$, then we have $\nu p \perp x$ for every

$\nu = 1, 2, \dots$ by Theorem 4.2, and hence we obtain by definition

and by Theorem 7.4

$$|[p]x| = [p]|x| = \bigvee_{i=1}^{\infty} (|x| \wedge \nu |p|) = 0,$$

that is, $[p]x = 0$. Conversely, if $[p]x = 0$, then we have

by Theorem 7.8

$$p \perp x - [p]x = x.$$

By definition we have obviously that $[0] = 0$. Conversely, if $[p] = 0$, then we have $p = 0$ by Theorem 7.11. Thus we have:

Theorem 7.13. We have $[p] = 0$ if and only if $p = 0$.

If p is a complete element of R , then we have by Theorem 7.8

$$x - [p]x = 0 \quad \text{for every element } x \in R,$$

that is, $[p] = 1$. Conversely, if $[p] = 1$, then for every element $x \perp p$ we have by Theorem 7.12

$$x = [p]x = 0,$$

and hence p is a complete element of R . Therefore we have:

Theorem 7.14. We have $[p] = 1$ if and only if p is a complete element of R .

Theorem 7.15. $[p] = [\alpha p]$ for every real number $\alpha \neq 0$.

Proof. If $\alpha \neq 0$, then for a positive element $a \in R$ we have by definition

$$[\alpha p]a = \bigcup_{n=1}^{\infty} (a \cap \nu | \alpha | | p |) = \bigcup_{n=1}^{\infty} (a \cap \nu | p |) = [p]a,$$

and hence $[\alpha p] = [p]$ for $\alpha \neq 0$ by definition.

Theorem 7.16. For every element $a \in R$ we have

$$[a^+]a = a^+, \quad (1 - [a^+])a = -a^-,$$

$$[a^-]a = -a^-, \quad (1 - [a^-])a = a^+.$$

Proof. Since $a^+ \wedge a^- = 0$, we have by Theorems 7.11 and 7.12

$$[a^+]a = [a^+]a^+ - [a^+]a^- = a^+,$$

$$(1 - [a^+])a = (a^+ - [a^+]a^+) - (a^- - [a^+]a^-) = -a^-.$$

We also can prove likewise the other assertion.

Remark. We have proved independently Theorems 7.8 and 7.11.

However, we obtain Theorem 7.11, putting $x = p$ in Theorem 7.8.

and we conclude immediately Theorem 7.8 from Theorems 7.10 and 7.11.

§8 Calculus of projectors

Now we shall consider relations between different projectors.

We will prove first the formula

$$(1) \quad [p][q] = [[p]q] = [p] \wedge [q].$$

For every positive element $a \in R$ we have by definition and by Theorems 7.6 and 7.3

$$[p][q]a = [p] \bigcup_{i=1}^{\infty} (a \wedge [q]i) = \bigcup_{i=1}^{\infty} (a \wedge [p]q i) = [[p]q]a,$$

and hence $[p][q] = [[p]q]$ by definition. For any positive element $a \in R$ we have by Theorem 7.3

$$[p][q]a = [p]a \wedge [q]a.$$

Since $a \wedge [p]i \uparrow_{i=1}^{\infty} [p]a$, $a \wedge [q]i \uparrow_{i=1}^{\infty} [q]a$ by definition, we obtain by Theorem 5.9

$$a \wedge ([p] \wedge [q]) = (a \wedge [p]) \wedge (a \wedge [q]) \uparrow_{i=1}^{\infty} [p]a \wedge [q]a.$$

Therefore we have $[p] \wedge [q]a = [p][q]a$ for every positive element $a \in R$, and consequently $[p][q] = [p] \wedge [q]$ by definition.

By virtue of the formula (1) we have that projectors are mutually commutative: $[p][q] = [q][p]$. Furthermore we have obviously:

Theorem 8.1. $[p][q] = 0$ if and only if $p \perp q$.

For two projectors $[p]$ and $[q]$ we shall write

$$[p] \geq [q] \quad \text{or} \quad [q] \leq [p],$$

if $[p]x \geq [q]x$ for every positive element $x \in R$. With this definition it is obvious that

$$1') \quad [p] \geq [p] \text{ for every element } p \in R,$$

$$2') \quad [p] \geq [q], [q] \geq [p] \text{ implies } [p] = [q],$$

$$3') \quad [p] \geq [q], [q] \geq [r] \text{ implies } [p] \geq [r].$$

Theorem 8.2. $[p] \geq [q]$ is equivalent to each one of the following two conditions:

$$1) \quad [p][q] = [q],$$

$$2) \quad [p]q = q.$$

Proof. If $[p] \geq [q]$, then we have by Theorems 7.2 and 7.3 for every positive element $a \in \mathcal{R}$

$$[q]a \geq [p]([q]a) \geq [q]([q]a) = [q]a,$$

and hence $[q] \geq [p][q] \geq [q]$ by definition. Consequently we obtain $[q] = [p][q]$ by 2'), if $[p] \geq [q]$. If $[p][q] = [q]$, then we obtain by Theorem 7.11

$$[p]q = [p][q]q = [q]q = q.$$

Furthermore, if $[p]q = q$, then we have for any positive $a \in \mathcal{R}$

$$[p]a \geq [p][q]a = [p]q a = [q]a$$

by Theorems 7.2, 7.5 and by the formula (1), and hence $[p] \geq [q]$ by definition.

Theorem 8.3. $|p| \geq |q|$ implies $[p] \geq [q]$.

Proof. If $|p| \geq |q|$, then we have by the formula (1)

$$[p][q] = [|p| \wedge |q|] = [|q|] = [q],$$

and hence $[p] \geq [q]$ by the previous theorem.

For a system of projectors $[p_\lambda]$ ($\lambda \in \Lambda$), if there exists a projector $[p]$ such that

$$1) \quad [p] \geq [p_\lambda] \text{ for every } \lambda \in \Lambda,$$

$$2) \quad [q] \geq [p_\lambda] \text{ for every } \lambda \in \Lambda \text{ implies } [q] \geq [p],$$

then $[p]$ is called the least upper bound of a system $[p_\lambda]$ ($\lambda \in \Lambda$) and we shall write

$$[p] = \bigvee_{\lambda \in \Lambda} [p_\lambda].$$

The uniqueness of such $[p]$ is obvious by definition. Similarly for a system of projectors $[p_\lambda]$ ($\lambda \in \Lambda$), if there exists a projector $[p]$ such that

$$1) \quad [p] \leq [p_\lambda] \text{ for every } \lambda \in \Lambda,$$

$$2) \quad [q] \leq [p_\lambda] \text{ for every } \lambda \in \Lambda \text{ implies } [q] \leq [p],$$

then $[p]$ is called the greatest lower bound of a system $[p_\lambda]$ ($\lambda \in \Lambda$) and we shall write

$$[p] = \bigwedge_{\lambda \in \Lambda} [p_\lambda].$$

With this definition we have

$$(2) \quad [p] \wedge [q] = [p][q].$$

Indeed, we have by the formula (1) and by Theorem 8.3

$$[p] \geq [p][q] \quad \text{and} \quad [q] \geq [p][q].$$

If $[r] \leq [p]$ and $[r] \leq [q]$, then we have by Theorem 7.3

$$[r]a \leq [p]a \wedge [q]a = [p][q]a$$

for every positive element $a \in R$, and hence $[r] \leq [p][q]$.

$$(3) \quad [p] \vee [q] = [1p \vee 1q] = [1p + 1q].$$

Because, we have obviously by Theorem 8.3

$$[1p \vee 1q] \geq [p] \quad \text{and} \quad [1p \vee 1q] \geq [q].$$

For any positive element $a \in R$ we have by definition

$$a \wedge \vee 1p \uparrow_{n=1}^{\infty} [p]a, \quad a \wedge \vee 1q \uparrow_{n=1}^{\infty} [q]a,$$

and hence by Theorem 5.9

$$(a \wedge \vee 1p) \vee (a \wedge \vee 1q) \uparrow_{n=1}^{\infty} [p]a \vee [q]a.$$

Since $(a \wedge \vee 1p) \vee (a \wedge \vee 1q) = a \wedge \vee (1p \vee 1q)$ by Theorems 3.3 and 2.3, we obtain thus by definition

$$[1p \vee 1q]a = [p]a \vee [q]a.$$

Therefore, if $[r] \geq [p]$ and $[r] \geq [q]$, then we have

$$[r]a \geq [p]a \vee [q]a = [1p \vee 1q]a$$

for every positive element $a \in R$, and hence $[r] \geq [1p \vee 1q]$.

Since $1p \vee 1q \leq 1p + 1q \leq 2(1p \vee 1q)$ by Theorem 3.1, we have by Theorems 8.3 and 7.15

$$[1p \vee 1q] \leq [1p + 1q] \leq [2(1p \vee 1q)] = [1p \vee 1q],$$

and hence $[1p \vee 1q] = [1p + 1q]$.

As proved just now, we have furthermore:

$$(4) \quad ([p] \wedge [q])a = [p]a \wedge [q]a \quad \text{for every element } a \geq 0,$$

$$(5) \quad ([p] \vee [q])a = [p]a \vee [q]a \quad \text{for every element } a \geq 0.$$

For every projectors $[p]$, $[q]$ and for every real numbers

$$\alpha, \beta \quad \text{we obtain a linear operator } \alpha[p] + \beta[q] \quad \text{on } R \quad \text{as}$$

$$(\alpha[p] + \beta[q])x = \alpha[p]x + \beta[q]x \quad (x \in R).$$

We have then

$$(6) \quad ([p] \vee [q]) + ([p] \wedge [q]) = [p] + [q].$$

In fact, for every positive element $a \in R$ we have by the formulas (4), (5) and Theorem 3.1

$$([p] \vee [q])a + ([p] \wedge [q])a \\ = ([p]a \vee [q]a) + ([p]a \wedge [q]a) = [p]a + [q]a,$$

and hence we conclude (6) by definition of projectors.

Theorem 8.4. $[p+q] = [p] + [q]$ if and only if $p \perp q$.

Proof. If $p \perp q$, then we have by Theorem 8.1

$$[p] \wedge [q] = [p][q] = 0,$$

and hence by the formulas (6), (3) and by Theorem 4.5

$$[p] + [q] = [p] \vee [q] = [p+q] = [p+q] = [p+q].$$

Conversely, if $[p] + [q]$ is a projector, then we have by Theorem 7.3

$$[p] + [q] = ([p] + [q])([p] + [q]) = [p] + [q] + 2[p][q],$$

and hence $[p][q] = 0$, that is, $p \perp q$ by Theorem 8.1.

Since $[q]p \perp p - [q]p$ by Theorem 7.10, we obtain by Theorem 8.4 and by the formula (1)

$$[p] = [q]p + [p - [q]p] = [p][q] + [p - [q]p].$$

Therefore we have

$$(7) \quad [p] - [p][q] = [p - [q]p].$$

Consequently we obtain by Theorem 8.2

$$(8) \quad [p] - [q] = [p - [q]p] \quad \text{for} \quad [p] \geq [q].$$

Theorem 8.5. If $|p| = \bigcup_{\lambda \in \Lambda} |p_\lambda|$, then we have

$$[p]a = \bigcup_{\lambda \in \Lambda} [p_\lambda]a \quad \text{for} \quad a \geq 0.$$

Proof. For every positive element $a \in R$ we have by

Theorems 3.3 and 2.5

$$\bigcup_{\lambda \in \Lambda} (a \wedge |p|) = \bigcup_{\lambda \in \Lambda} \left\{ \bigcup_{\lambda \in \Lambda} (a \wedge |p_\lambda|) \right\} = \bigcup_{\lambda \in \Lambda} \left\{ \bigcup_{\lambda \in \Lambda} (a \wedge |p_\lambda|) \right\}.$$

Therefore we obtain $[p]a = \bigcup_{\lambda \in \Lambda} [p_\lambda]a$ for $a \geq 0$ by definition.

Theorem 8.6. For every sequence of elements $p_\nu \in R$ ($\nu = 1, 2, \dots$) there exists the greatest lower bound $\bigwedge_{\nu=1}^{\infty} [p_\nu]$ and

$$\left(\bigwedge_{\nu=1}^{\infty} [p_\nu] \right) a = \bigwedge_{\nu=1}^{\infty} [p_\nu]a \quad \text{for} \quad a \geq 0.$$

Proof. Putting $q = \bigwedge_{\nu=1}^{\infty} (|p_\nu| - [p_\nu]|p_\nu|)$, we have by

the previous theorem

$$[q]a = \bigcup_{\nu=1}^{\infty} [p_{\nu}]a \quad \text{for } a \geq 0.$$

As $[p_{\nu}]a = [p_{\nu}]p_{\nu}a$ by the formula (7),

we have then by Theorems 2.2, 2.4 and 3.2

$$\begin{aligned} [p_{\nu}]a - [q]a &= \bigcap_{\nu=1}^{\infty} [p_{\nu}]p_{\nu}a \\ &= \bigcap_{\nu=1}^{\infty} ([p_{\nu}]a \wedge [p_{\nu}]a) = \bigcap_{\nu=1}^{\infty} [p_{\nu}]a. \end{aligned}$$

Since $0 \leq q \leq p_1$, we have $[q] \leq [p_1]$ by Theorem 8.3, and

hence we obtain by the formula (8)

$$[p_1 - [q]p_1]a = \bigcap_{\nu=1}^{\infty} [p_{\nu}]a \quad \text{for } a \geq 0.$$

This relation yields obviously $[p_1 - [q]p_1] \leq [p_{\nu}]$ for every

$\nu = 1, 2, \dots$. If $[r] \leq [p_{\nu}]$ for all $\nu = 1, 2, \dots$, then we have for any positive element $a \in R$

$$[r]a \leq [p_{\nu}]a \quad \text{for every } \nu = 1, 2, \dots,$$

and hence $[r]a \leq [p_1 - [q]p_1]a$ for $a \geq 0$, that is, we obtain

$[r] \leq [p_1 - [q]p_1]$. Therefore we have by definition

$$[p_1 - [q]p_1] = \bigcap_{\nu=1}^{\infty} [p_{\nu}].$$

Theorem 8.7. For a sequence of projectors $[p_{\nu}]$ ($\nu = 1, 2, \dots$), if there exists a projector $[p_0]$ such that $[p_{\nu}] \geq [p_0]$ for every $\nu = 1, 2, \dots$, then there exists the least upper bound

$\bigcup_{\nu=1}^{\infty} [p_{\nu}]$ and we have

$$(\bigcup_{\nu=1}^{\infty} [p_{\nu}])a = \bigcup_{\nu=1}^{\infty} [p_{\nu}]a \quad \text{for } a \geq 0.$$

Proof. Putting $p = \bigcup_{\nu=1}^{\infty} [p_{\nu}]p_0$, we have by Theorem 8.5

$$[p]a = \bigcup_{\nu=1}^{\infty} [p_{\nu}]p_0a \quad \text{for } a \geq 0.$$

As $[p_{\nu}]p_0 = [p_{\nu}][p_0] = [p_{\nu}]$ by the formula (1) and Theorem 8.2, we obtain thus

$$[p]a = \bigcup_{\nu=1}^{\infty} [p_{\nu}]a \quad \text{for } a \geq 0.$$

From this relation we can conclude easily that $[p] = \bigcup_{\nu=1}^{\infty} [p_{\nu}]$, similarly as in Proof of the previous theorem.

Theorem 8.8. If $[p] = \bigcup_{\nu=1}^{\infty} [p_{\nu}]$, then we have for every projector $[q]$

$$[p][q] = \bigcup_{\nu=1}^{\infty} [p_{\nu}][q].$$

Proof. By virtue of the previous theorem we have

$$[p]a = \bigvee_{\nu=1}^{\infty} [p_{\nu}]a \quad \text{for } a \geq 0,$$

and hence by Theorems 7.6 and 8.7 for every positive element $a \in R$

$$[q][p]a = \bigvee_{\nu=1}^{\infty} ([q][p_{\nu}]a) = (\bigvee_{\nu=1}^{\infty} [q][p_{\nu}])a.$$

Therefore we have $[q][p] = \bigvee_{\nu=1}^{\infty} [q][p_{\nu}]$ by definition of projectors.

Theorem 8.9. If $[p] = \bigvee_{\nu=1}^{\infty} [p_{\nu}]$, then we have for every projector $[q]$

$$[p] \vee [q] = \bigvee_{\nu=1}^{\infty} ([p_{\nu}] \vee [q]).$$

Proof. By virtue of Theorem 8.6 we have

$$[p]a = \bigvee_{\nu=1}^{\infty} [p_{\nu}]a \quad \text{for } a \geq 0,$$

and hence by Theorem 3.3 for every positive element $a \in R$

$$[p]a \vee [q]a = \bigvee_{\nu=1}^{\infty} ([p_{\nu}]a \vee [q]a).$$

Therefore we obtain by the formula (5) and by Theorem 8.6

$$([p] \vee [q])a = \bigvee_{\nu=1}^{\infty} \{([p_{\nu}] \vee [q])a\} = \bigvee_{\nu=1}^{\infty} ([p_{\nu}] \vee [q])a$$

for $a \geq 0$, and consequently $[p] \vee [q] = \bigvee_{\nu=1}^{\infty} ([p_{\nu}] \vee [q])$.

For a sequence of projectors $[p_{\nu}]$ ($\nu = 1, 2, \dots$) we define $[p_{\nu}] \uparrow_{\nu=1}^{\infty}$ to mean $[p_1] \leq [p_2] \leq \dots$, and $[p_{\nu}] \downarrow_{\nu=1}^{\infty}$ to mean $[p_1] \geq [p_2] \geq \dots$. If $[p_{\nu}] \uparrow_{\nu=1}^{\infty}$ and $[p] = \bigvee_{\nu=1}^{\infty} [p_{\nu}]$, then we shall write $[p_{\nu}] \uparrow_{\nu=1}^{\infty} [p]$; and if $[p_{\nu}] \downarrow_{\nu=1}^{\infty}$ and $[p] = \bigwedge_{\nu=1}^{\infty} [p_{\nu}]$, then we shall write $[p_{\nu}] \downarrow_{\nu=1}^{\infty} [p]$. Then we see by Theorem 8.6 that

$$(9) \quad [p_{\nu}]a \uparrow_{\nu=1}^{\infty} [p]a \quad \text{for all } a \geq 0,$$

if and only if $[p_{\nu}] \uparrow_{\nu=1}^{\infty} [p]$; and by Theorem 8.7 that

$$(10) \quad [p_{\nu}]a \downarrow_{\nu=1}^{\infty} [p]a \quad \text{for all } a \geq 0,$$

if and only if $[p_{\nu}] \downarrow_{\nu=1}^{\infty} [p]$.

Theorem 8.10. If $[p_{\nu}] \uparrow_{\nu=1}^{\infty} [p]$ and $[q_{\nu}] \uparrow_{\nu=1}^{\infty} [q]$, then we have

$$[p_{\nu}] \wedge [q_{\nu}] \uparrow_{\nu=1}^{\infty} [p] \wedge [q] \quad \text{and} \quad [p_{\nu}] \vee [q_{\nu}] \uparrow_{\nu=1}^{\infty} [p] \vee [q].$$

If $[p_{\nu}] \downarrow_{\nu=1}^{\infty} [p]$ and $[q_{\nu}] \downarrow_{\nu=1}^{\infty} [q]$, then we have

$$[p_{\nu}] \wedge [q_{\nu}] \downarrow_{\nu=1}^{\infty} [p] \wedge [q] \quad \text{and} \quad [p_{\nu}] \vee [q_{\nu}] \downarrow_{\nu=1}^{\infty} [p] \vee [q].$$

Proof. If $[p_{\nu}] \uparrow_{\nu=1}^{\infty} [p]$ and $[q_{\nu}] \uparrow_{\nu=1}^{\infty} [q]$, then we have for any positive element $a \in R$ both

$$[p_\nu]a \uparrow_{\nu=1}^\infty, [p]a \quad \text{and} \quad [q_\nu]a \uparrow_{\nu=1}^\infty, [q]a,$$

and hence by Theorem 5.9 and by the formula (4)

$([p_\nu] \cap [q_\nu])a = [p_\nu]a \cap [q_\nu]a \uparrow_{\nu=1}^\infty, [p]a \cap [q]a = ([p] \cap [q])a$
for all $a \geq 0$. Consequently we obtain by the formula (9)

$$[p_\nu] \cap [q_\nu] \uparrow_{\nu=1}^\infty, [p] \cap [q].$$

We also obtain likewise $[p_\nu] \cup [q_\nu] \uparrow_{\nu=1}^\infty, [p] \cup [q]$. By similar method we also can prove the other assertion.

Theorem 8.11. If $[p_\nu] \uparrow_{\nu=1}^\infty, [p]$, or if $[p_\nu] \downarrow_{\nu=1}^\infty, [p]$, then we have

$$\lim_{\nu \rightarrow \infty} [p_\nu]a = [p]a \quad \text{for all } a \in R.$$

Proof. If $[p_\nu] \uparrow_{\nu=1}^\infty, [p]$, then we have by Theorem 8.7

$$[p_\nu]a^+ \uparrow_{\nu=1}^\infty, [p]a^+, \quad [p_\nu]a^- \uparrow_{\nu=1}^\infty, [p]a^-,$$

and hence we obtain by Theorem 5.5 and 5.8

$$\lim_{\nu \rightarrow \infty} [p_\nu]a = [p]a^+ - [p]a^- = [p]a.$$

We also can dispose likewise of the case $[p_\nu] \downarrow_{\nu=1}^\infty, [p]$.

Theorem 8.13. If $\bigcap_{\lambda \in \Lambda} [p_\lambda]p = 0$ for a positive element $p \in R$, then we have

$$\bigcap_{\lambda \in \Lambda} [p_\lambda][p]a = 0 \quad \text{for all } a \geq 0.$$

Proof. If $\bigcap_{\lambda \in \Lambda} [p_\lambda]p = 0$ for a positive element $p \in R$ then we have by Theorems 2.2 and 2.4

$$\bigcup_{\lambda \in \Lambda} (p - [p_\lambda]p) = p,$$

and hence we have by Theorem 8.5

$$[p]a = \bigcup_{\lambda \in \Lambda} [p - [p_\lambda]p]a \quad \text{for all } a \geq 0.$$

On the other hand we have by the formula (7)

$$[p - [p_\lambda]p]a = [p]a - [p_\lambda][p]a.$$

Therefore we obtain $\bigcap_{\lambda \in \Lambda} [p_\lambda][p]a = 0$ for all $a \geq 0$ by Theorems 2.2 and 2.4.

As an immediate consequence of Theorem 8.12 we have:

Theorem 8.13. $[p_\nu]p \downarrow_{\nu=1}^\infty, 0$ implies $[p_\nu][p] \downarrow_{\nu=1}^\infty, 0$.

Theorem 8.14. For every elements a and $b \in R$ we have

$$[(\nu|a + [a]b)^+] \uparrow_{\nu=1}^\infty, [a].$$

Proof. It is obvious by Theorem 8.3 that

$$[(\nu|a| + [a]f)^+] \uparrow_{\nu=1}^{\infty}.$$

We have by Theorem 7.16

$$(1 - [(\nu|a| + [a]f)^+]) (\nu|a| + [a]f) \leq 0,$$

and hence by Theorem 7.5

$$(1 - [(\nu|a| + [a]f)^+])|a| \leq -\frac{1}{\nu}([a] - [a][(\nu|a| + [a]f)^+])f \leq \frac{1}{\nu}|f|.$$

Passing to the limit, we obtain from this relation

$$[(\nu|a| + [a]f)^+]|a| \uparrow_{\nu=1}^{\infty} |a|.$$

Since we have by the formula (1) and by Theorem 7.4

$$[[\nu|a| + [a]f]^+]|a| = [a][\nu|a| + [a]f]^+ = [\nu|a| + [a]f]^+,$$

we obtain therefore our assertion by Theorem 8.5.

Theorem 8.15. For every elements a and $f \in \mathcal{R}$ we have

$$[(\nu|a| + [a]f)^-] \downarrow_{\nu=1}^{\infty} 0$$

Proof. It is obvious by Theorem 8.3 that

$$[(\nu|a| + [a]f)^-] \downarrow_{\nu=1}^{\infty}.$$

We have by Theorem 7.16

$$[(\nu|a| + [a]f)^-](\nu|a| + [a]f) \leq 0,$$

and hence by Theorem 7.5

$$[(\nu|a| + [a]f)^-]|a| \leq -\frac{1}{\nu}[(\nu|a| + [a]f)^-][a]f \leq \frac{1}{\nu}|f|.$$

Passing to the limit, we obtain from this relation

$$[(\nu|a| + [a]f)^-]|a| \downarrow_{\nu=1}^{\infty} 0,$$

and hence $[(\nu|a| + [a]f)^-] \downarrow_{\nu=1}^{\infty} 0$ by Theorem 8.13, because we

have by the formula (1) and by Theorems 7.4 and 7.11

$$[(\nu|a| + [a]f)^-][a] = [(\nu|a| + [a]f)^-].$$

CHAPTER II
FIRST SPECTRAL THEORY

§9 Stieltjes integral

Let R be a continuous semi-ordered linear space in the sequel. A system of elements $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) is said to be of bounded variation in a closed interval $[\alpha, \beta]$, if there exists a positive element $\ell \in R$ such that we have

$$\sum_{\nu=1}^k |a_{\lambda_\nu} - a_{\lambda_{\nu-1}}| \leq \ell$$

for every partition $\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_k = \beta$ of $[\alpha, \beta]$.

With this definition we see easily that if both systems a_λ and $b_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) are of bounded variation in a closed interval $[\alpha, \beta]$, then $\delta a_\lambda + \delta b_\lambda$ ($\alpha \leq \lambda \leq \beta$) is of bounded variation in $[\alpha, \beta]$ for every real numbers δ and δ . We see further by Theorem 7.5 that if a system of elements $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) is of bounded variation in a closed interval $[\alpha, \beta]$, then $[p]a_\lambda$ ($\alpha \leq \lambda \leq \beta$) is of bounded variation in $[\alpha, \beta]$ for every projector $[p]$.

A system of elements $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) is said to be increasing if $a_\lambda \leq a_\rho$ for $\lambda < \rho$, or to be decreasing if $a_\lambda \geq a_\rho$ for $\lambda < \rho$. With this definition it is evident that if a system of elements $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) is increasing or decreasing in $[\alpha, \beta]$, then it is of bounded variation in $[\alpha, \beta]$.

If a system $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) is of bounded variation in $[\alpha, \beta]$, then corresponding to every real number λ such that $\alpha \leq \lambda \leq \beta$, there exists uniquely an element $c_\lambda \in R$ such that we have

$$\lim_{\nu \rightarrow \infty} a_{\lambda_\nu} = c_\lambda$$

for every sequence of real numbers $\lambda > \lambda_\nu \uparrow_{\nu=1}^{\infty} \lambda$. Because, for any sequence of real numbers $\lambda > \lambda_\nu \uparrow_{\nu=1}^{\infty} \lambda$ the sequence of elements

$$\sum_{\nu=1}^k |a_{\lambda_{\nu+1}} - a_{\lambda_\nu}| \quad (k = 1, 2, \dots)$$

is upper bounded, since a_λ ($\alpha \leq \lambda \leq \beta$) is of bounded variation in $[\alpha, \beta]$ by assumption. Consequently the series

$$\sum_{\nu=1}^{\infty} (a_{\lambda_{\nu+1}} - a_{\lambda_\nu})$$

is absolutely convergent, and hence naturally convergent by Theorem 6.6. Therefore we have

$$\lim_{\nu \rightarrow \infty} a_{\lambda_\nu} = a_{\lambda_1} + \sum_{\nu=1}^{\infty} (a_{\lambda_{\nu+1}} - a_{\lambda_\nu}),$$

that is, the sequence a_{λ_ν} ($\nu = 1, 2, \dots$) is convergent for every sequence $\lambda > \lambda_\nu \uparrow_{\nu=1}^{\infty} \lambda$. For every two sequences $\lambda > \lambda_\nu \uparrow_{\nu=1}^{\infty} \lambda$ and $\lambda > \mu_\nu \uparrow_{\nu=1}^{\infty} \lambda$ there exists obviously an sequence $\lambda > \kappa_\nu \uparrow_{\nu=1}^{\infty} \lambda$ such that each one of λ_ν and μ_ν ($\nu = 1, 2, \dots$) is a partial sequence of κ_ν ($\nu = 1, 2, \dots$). Then we have by Theorem 5.7

$$\lim_{\nu \rightarrow \infty} a_{\lambda_\nu} = \lim_{\nu \rightarrow \infty} a_{\kappa_\nu} = \lim_{\nu \rightarrow \infty} a_{\mu_\nu}.$$

Such a uniquely determined element c_λ will be denoted by $a_{\lambda-0}$.

We also can prove likewise that to every real number λ such that $\alpha \leq \lambda < \beta$, there exists uniquely an element $d_\lambda \in R$ such that we have

$$\lim_{\nu \rightarrow \infty} a_{\lambda_\nu} = d_\lambda$$

for every sequence of real numbers $\lambda < \lambda_\nu \downarrow_{\nu=1}^{\infty} \lambda$, and such an uniquely determined element d_λ will be denoted by $a_{\lambda+0}$.

A system of elements $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is said to be of bounded variation, if it is of bounded variation in every closed interval. If for every sequence of real numbers $\lambda_\nu \uparrow_{\nu=1}^{\infty} +\infty$ the corresponding sequence of elements a_{λ_ν} ($\nu = 1, 2, \dots$) is convergent, then the limit $\lim_{\nu \rightarrow \infty} a_{\lambda_\nu}$ is the same for every sequence $\lambda_\nu \uparrow_{\nu=1}^{\infty} +\infty$, as proved just above. Such $\lim_{\nu \rightarrow \infty} a_{\lambda_\nu}$ will be denoted by $a_{+\infty}$. It is similar for the notation $a_{-\infty}$, if it exists.

Let $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) be of bounded variation in a closed interval $[\alpha, \beta]$ and

$$\sum_{i=1}^k |a_{\lambda_i} - a_{\lambda_{i-1}}| \leq l$$

for every partition $\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_k = \beta$ of $[\alpha, \beta]$

Let a function $\varphi(\lambda)$ ($\alpha \leq \lambda \leq \beta$) be finite and continuous in $[\alpha, \beta]$. Since a closed interval $[\alpha, \beta]$ is compact as a point set, we see easily that every finite continuous function $\varphi(\lambda)$ on $[\alpha, \beta]$ is uniformly continuous, that is, to any positive number ε there exists a positive number δ such that

$$|\lambda - \rho| \leq 2\delta \quad \text{implies} \quad |\varphi(\lambda) - \varphi(\rho)| \leq \varepsilon.$$

For two partition

$$\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_\kappa = \beta, \quad \lambda_\nu - \lambda_{\nu-1} \leq \delta,$$

$$\alpha = \rho_0 < \rho_1 < \dots < \rho_\sigma = \beta, \quad \rho_\mu - \rho_{\mu-1} \leq \delta,$$

and for $\lambda_{\nu-1} \leq \xi_\nu \leq \lambda_\nu$, $\rho_{\mu-1} \leq \eta_\mu \leq \rho_\mu$ ($\nu = 1, 2, \dots, \kappa$; $\mu = 1, 2, \dots, \sigma$)

we obtain easily

$$\left| \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu)(\lambda_\nu - \lambda_{\nu-1}) - \sum_{\mu=1}^{\sigma} \varphi(\eta_\mu)(\rho_\mu - \rho_{\mu-1}) \right| \leq \varepsilon \ell$$

by the usual method in the theory of integral. Therefore we

can recognize by Theorem 6.4 that there exists an element $C \in \mathcal{R}$ such that

$$\lim_{\mu \rightarrow \infty} \sum_{\nu=1}^{\kappa_\mu} \varphi(\xi_{\mu,\nu})(\lambda_{\mu,\nu} - \lambda_{\mu,\nu-1}) = C,$$

if $\alpha = \lambda_{\mu,0} < \lambda_{\mu,1} < \dots < \lambda_{\mu,\kappa_\mu} = \beta$, $\lambda_{\mu,\nu} - \lambda_{\mu,\nu-1} \leq \varepsilon_\mu$,

$\lim_{\mu \rightarrow \infty} \varepsilon_\mu = 0$ and $\lambda_{\mu,\nu-1} \leq \xi_{\mu,\nu} \leq \lambda_{\mu,\nu}$ ($\nu = 1, 2, \dots, \kappa_\mu$; $\mu = 1, 2, \dots$).

such an element C is called the integral of $\varphi(\lambda)$ by a system

$a_\lambda \in \mathcal{R}$ ($\alpha \leq \lambda \leq \beta$) in $[\alpha, \beta]$ and denoted by

$$\int_\alpha^\beta \varphi(\lambda) da_\lambda \quad \text{or} \quad \int_\alpha^\beta \varphi(\lambda) d_\lambda a_\lambda.$$

With this definition we have obviously

$$\int_\alpha^\beta \varphi(\lambda) da_\lambda = \int_\alpha^\gamma \varphi(\lambda) da_\lambda + \int_\gamma^\beta \varphi(\lambda) da_\lambda$$

for $\alpha \leq \gamma \leq \beta$, adopting the convention

$$\int_\gamma^\gamma \varphi(\lambda) da_\lambda = 0.$$

Let a system $a_\lambda \in \mathcal{R}$ ($-\infty < \lambda < +\infty$) be of bounded variation.

For a finite continuous function $\varphi(\lambda)$ ($-\infty < \lambda < +\infty$), if to every sequence of real numbers $\beta_\nu \uparrow_{\nu=1}^\infty +\infty$ there exists the limit

$$\lim_{\nu \rightarrow \infty} \int_\alpha^{\beta_\nu} \varphi(\lambda) da_\lambda,$$

then we see easily that this limit is uniquely determined, that is, there exists the limit

$$\lim_{\beta \rightarrow +\infty} \int_{\alpha}^{\beta} \varphi(\lambda) d a_{\lambda}.$$

In this case we define the integral

$$\int_{\alpha}^{+\infty} \varphi(\lambda) d a_{\lambda}$$

by such limit, and we shall say that this integral is convergent.

It is similar for the integral

$$\int_{-\infty}^{\alpha} \varphi(\lambda) d a_{\lambda}$$

if it is convergent. If both integrals

$$\int_{-\infty}^0 \varphi(\lambda) d a_{\lambda} \quad \text{and} \quad \int_0^{+\infty} \varphi(\lambda) d a_{\lambda}$$

are convergent, then we define the integral of $\varphi(\lambda)$ by a system

$a_{\lambda} \in \mathcal{R} \quad (-\infty < \lambda < +\infty)$ as

$$\int_{-\infty}^{+\infty} \varphi(\lambda) d a_{\lambda} = \int_{-\infty}^0 \varphi(\lambda) d a_{\lambda} + \int_0^{+\infty} \varphi(\lambda) d a_{\lambda},$$

and we shall say that this integral is convergent, or that $\varphi(\lambda)$

is integrable by a system $a_{\lambda} \quad (-\infty < \lambda < +\infty)$.

With this definition we have obviously that if a function $\varphi(\lambda)$ is integrable by a system $a_{\lambda} \in \mathcal{R} \quad (-\infty < \lambda < +\infty)$, then we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \varphi(\lambda) d a_{\lambda} &= \lim_{\substack{\alpha \rightarrow -\infty \\ \beta \rightarrow +\infty}} \int_{\alpha}^{\beta} \varphi(\lambda) d a_{\lambda} \\ &= \int_{-\infty}^{\gamma} \varphi(\lambda) d a_{\lambda} + \int_{\gamma}^{+\infty} \varphi(\lambda) d a_{\lambda} \end{aligned}$$

for every real number γ .

If a system of elements $a_{\lambda} \in \mathcal{R} \quad (\alpha \leq \lambda \leq \beta)$ is of bounded variation in $[\alpha, \beta]$, then, putting

$$a_{\lambda} = \begin{cases} a_{\beta} & \text{for } \lambda > \beta, \\ a_{\alpha} & \text{for } \lambda < \alpha, \end{cases}$$

we obtain a system of bounded variation $a_{\lambda} \quad (-\infty < \lambda < +\infty)$.

For such $a_{\lambda} \quad (-\infty < \lambda < +\infty)$, it is evident that every finite continuous function $\varphi(\lambda) \quad (-\infty < \lambda < +\infty)$ is integrable by $a_{\lambda} \quad (-\infty < \lambda < +\infty)$ and we have

$$\int_{-\infty}^{+\infty} \varphi(\lambda) d a_{\lambda} = \int_{\alpha}^{\beta} \varphi(\lambda) d a_{\lambda}.$$

§10 Properties of integral

For a system of bounded variation $a_\lambda \in \mathcal{R}$ ($-\infty < \lambda < +\infty$) we have obviously by definition

$$(1) \quad \int_a^b da_\lambda = a_\beta - a_\alpha.$$

Furthermore we see easily that if both functions $\varphi(\lambda)$ and $\psi(\lambda)$ are integrable by a_λ ($-\infty < \lambda < +\infty$), then $\alpha\varphi(\lambda) + \beta\psi(\lambda)$ is integrable by a_λ ($-\infty < \lambda < +\infty$) for every real numbers α and β , and we have

$$(2) \quad \int_{-\infty}^{+\infty} (\alpha\varphi(\lambda) + \beta\psi(\lambda)) da_\lambda = \alpha \int_{-\infty}^{+\infty} \varphi(\lambda) da_\lambda + \beta \int_{-\infty}^{+\infty} \psi(\lambda) da_\lambda,$$

and that if a function $\varphi(\lambda)$ is integrable by a system a_λ as well as by a system b_λ ($-\infty < \lambda < +\infty$), then $\varphi(\lambda)$ is integrable by the system $\alpha a_\lambda + \beta b_\lambda$ ($-\infty < \lambda < +\infty$) for every real numbers α and β , and we have

$$(3) \quad \int_{-\infty}^{+\infty} \varphi(\lambda) d(\alpha a_\lambda + \beta b_\lambda) = \alpha \int_{-\infty}^{+\infty} \varphi(\lambda) da_\lambda + \beta \int_{-\infty}^{+\infty} \varphi(\lambda) db_\lambda.$$

By virtue of Theorem 7.7 we obtain obviously that if a function $\varphi(\lambda)$ is integrable by a system $a_\lambda \in \mathcal{R}$ ($-\infty < \lambda < +\infty$), then $\varphi(\lambda)$ is integrable by the system $[p]a_\lambda$ ($-\infty < \lambda < +\infty$) for every projector $[p]$, and we have

$$(4) \quad \int_{-\infty}^{+\infty} \varphi(\lambda) d[p]a_\lambda = [p] \int_{-\infty}^{+\infty} \varphi(\lambda) da_\lambda.$$

As $a_{\lambda-0} - a_{\lambda+0} = (a_\lambda - a_p) + (a_p - a_{p-0}) - (a_\lambda - a_{\lambda-0})$, we see easily by definition that

$$(5) \quad \int_a^b \varphi(\lambda) da_{\lambda-0} = \int_a^b \varphi(\lambda) da_\lambda + \varphi(\alpha)(a_\alpha - a_{\alpha-0}) - \varphi(\beta)(a_\beta - a_{\beta-0}),$$

and furthermore we obtain likewise

$$(6) \quad \int_a^b \varphi(\lambda) da_{\lambda+0} = \int_a^b \varphi(\lambda) da_\lambda - \varphi(\alpha)(a_{\alpha+0} - a_\alpha) + \varphi(\beta)(a_{\beta+0} - a_\beta).$$

If a system $a_\lambda \in \mathcal{R}$ ($-\infty < \lambda < +\infty$) is of bounded variation, then the system $a_{-\lambda}$ ($-\infty < \lambda < +\infty$) is obviously of bounded variation too, and we have by definition

$$(7) \quad \int_a^b \varphi(\lambda) d_\lambda a_{-\lambda} = - \int_{-b}^{-a} \varphi(-\lambda) d_\lambda a_\lambda.$$

If a function $\omega(\lambda)$ is finite, continuous, and increasing in a closed interval $[\alpha, \beta]$, then for any system of bounded variation a_λ ($-\infty < \lambda < +\infty$), the system $a_{\omega(\lambda)}$ ($\alpha \leq \lambda \leq \beta$) is

of bounded variation in $[\alpha, \beta]$ too, and we have obviously by definition

$$(8) \quad \int_{\alpha}^{\beta} \varphi(\omega(\lambda)) d_{\lambda} a_{\omega(\lambda)} = \int_{\omega(\alpha)}^{\omega(\beta)} \varphi(\lambda) d\lambda.$$

Next we will consider integrals by an increasing system $a_{\lambda} \in \mathcal{R}$ ($-\infty < \lambda < +\infty$), that is, $a_{\lambda} \leq a_{\rho}$ for $\lambda < \rho$.

We see immediately by definition:

Theorem 10.1. If $\varphi(\lambda) \geq \psi(\lambda)$ for $\alpha \leq \lambda \leq \beta$, then we have for any increasing system $a_{\lambda} \in \mathcal{R}$ ($-\infty < \lambda < +\infty$)

$$\int_{\alpha}^{\beta} \varphi(\lambda) d a_{\lambda} \geq \int_{\alpha}^{\beta} \psi(\lambda) d a_{\lambda}.$$

Theorem 10.2. For a finite continuous function $\varphi(\lambda)$ ($-\infty < \lambda < +\infty$), if there exists a function $\psi(\lambda)$ such that

$$|\varphi(\lambda)| \leq \psi(\lambda) \quad (-\infty < \lambda < +\infty)$$

and $\psi(\lambda)$ is integrable by an increasing system $a_{\lambda} \in \mathcal{R}$ ($-\infty < \lambda < +\infty$), then $\varphi(\lambda)$ is integrable by this system a_{λ} ($-\infty < \lambda < +\infty$) too and we have

$$\left| \int_{-\infty}^{+\infty} \varphi(\lambda) d a_{\lambda} \right| \leq \int_{-\infty}^{+\infty} \psi(\lambda) d a_{\lambda}.$$

Proof. For any sequence of real numbers $0 \leq \alpha_{\nu} \uparrow_{\nu=\infty} +\infty$, we have by assumption

$$\int_0^{+\infty} \psi(\lambda) d a_{\lambda} = \sum_{\nu=1}^{\infty} \int_{\alpha_{\nu}}^{\alpha_{\nu+1}} \psi(\lambda) d a_{\lambda} + \int_0^{\alpha_1} \psi(\lambda) d a_{\lambda}.$$

Since we have by the previous theorem

$$\pm \int_{\alpha_{\nu}}^{\alpha_{\nu+1}} \varphi(\lambda) d a_{\lambda} \leq \int_{\alpha_{\nu}}^{\alpha_{\nu+1}} \psi(\lambda) d a_{\lambda},$$

we obtain by the formula §4(1)

$$\left| \int_{\alpha_{\nu}}^{\alpha_{\nu+1}} \varphi(\lambda) d a_{\lambda} \right| \leq \int_{\alpha_{\nu}}^{\alpha_{\nu+1}} \psi(\lambda) d a_{\lambda} \quad \text{for every } \nu = 1, 2, \dots$$

Thus we have by Theorems 6.5 and 6.6 that

$$\int_0^{+\infty} \varphi(\lambda) d a_{\lambda} = \sum_{\nu=1}^{\infty} \int_{\alpha_{\nu}}^{\alpha_{\nu+1}} \varphi(\lambda) d a_{\lambda} + \int_0^{\alpha_1} \varphi(\lambda) d a_{\lambda}$$

is convergent and

$$\left| \int_0^{+\infty} \varphi(\lambda) d a_{\lambda} \right| \leq \int_0^{+\infty} \psi(\lambda) d a_{\lambda}$$

We also can prove likewise that

$$\left| \int_{-\infty}^0 \varphi(\lambda) d a_{\lambda} \right| \leq \int_{-\infty}^0 \psi(\lambda) d a_{\lambda},$$

and the left side is convergent. Therefore $\varphi(\lambda)$ is integrable by a_{λ} ($-\infty < \lambda < +\infty$) and we have the inequality described in the theorem.

If a system $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is increasing, then both system $a_{\lambda+0}$ and $a_{\lambda-0}$ ($-\infty < \lambda < +\infty$) are obviously increasing too.

Theorem 10.3. For an increasing system $a_\lambda \in R$ ($-\infty < \lambda < +\infty$), if $|\varphi(\lambda)|$ is integrable by one of systems a_λ , $a_{\lambda+0}$ and $a_{\lambda-0}$ ($-\infty < \lambda < +\infty$), then a finite continuous function $\varphi(\lambda)$ is integrable by each of a_λ , $a_{\lambda+0}$, $a_{\lambda-0}$ ($-\infty < \lambda < +\infty$), and

$$\int_{-\infty}^{+\infty} \varphi(\lambda) da_\lambda = \int_{-\infty}^{+\infty} \varphi(\lambda) da_{\lambda-0} = \int_{-\infty}^{+\infty} \varphi(\lambda) da_{\lambda+0}.$$

Proof. By the formulas (5), (6) and by the previous theorem we need only to prove that if $|\varphi(\lambda)|$ is integrable by one of a_λ , $a_{\lambda+0}$, and $a_{\lambda-0}$ ($-\infty < \lambda < +\infty$), then for any sequence $\alpha_\nu \uparrow_{\nu \rightarrow \infty} +\infty$ we have

$$\lim_{\nu \rightarrow \infty} |\varphi(\alpha_\nu)| (a_{\alpha_\nu} - a_{\alpha_\nu-0}) = \lim_{\nu \rightarrow \infty} |\varphi(\alpha_\nu)| (a_{\alpha_\nu+0} - a_{\alpha_\nu}) = 0,$$

and for any sequence $\alpha_\nu \downarrow_{\nu \rightarrow \infty} -\infty$ we have

$$\lim_{\nu \rightarrow \infty} |\varphi(\alpha_\nu)| (a_{\alpha_\nu} - a_{\alpha_\nu-0}) = \lim_{\nu \rightarrow \infty} |\varphi(\alpha_\nu)| (a_{\alpha_\nu+0} - a_{\alpha_\nu}) = 0.$$

However, these relations hold obviously, since we have by definition

$$\begin{aligned} & |\varphi(\alpha_\nu)| (a_{\alpha_\nu} - a_{\alpha_\nu-0}) \vee |\varphi(\alpha_\nu)| (a_{\alpha_\nu+0} - a_{\alpha_\nu}) \\ & \leq \int_{\alpha_\nu-1}^{\alpha_\nu+1} |\varphi(\lambda)| da_\lambda \cap \int_{\alpha_\nu-1}^{\alpha_\nu+1} |\varphi(\lambda)| da_{\lambda \pm 0} \end{aligned}$$

§11 Resolutions of elements

For two elements a and $f \in R$, if $a \perp f - a$, then a is said to be a part of f , and we shall write $a < f$ or $f > a$. With this definition we have obviously that $a > a$ and $a > 0$ for every element $a \in R$.

$$(1) \quad a < f, \quad f < a \quad \text{implies} \quad a = f.$$

Because, $a \perp f - a$, $f \perp a - f$ implies $a - f \perp a - f$ by Theorems 4.2 and 4.3, and hence further $a - f = 0$ by definition.

We have obviously by Theorem 4.5

$$(2) \quad a < f \quad \text{implies} \quad |a| \leq |f|.$$

Therefore $a < f \perp c - f$ implies $a \perp c - f$ by Theorem 4.1,

and hence further by Theorem 4.3

$$a \perp (c-b) + (b-a) = c-a,$$

that is, we have

$$(3) \quad a < b, \quad b < c \text{ implies } a < c.$$

We have obviously by Theorem 7.10 for every projector $[p]$

$$(4) \quad [p]a < a.$$

$$(5) \quad a < b \text{ implies } a = [a]b.$$

Because, $a \perp b-a$ implies $[a](b-a) = 0$ by Theorem 7.12,

and hence further $[a]b = [a]a = a$ by Theorem 7.11.

We have obviously by Theorem 4.3

$$(6) \quad a < b, \quad a \perp c \text{ implies } a < b+c.$$

$$(7) \quad a < c, \quad b < c, \quad a \perp b \text{ implies } a+b < c.$$

Because, $a < c, b < c, a \perp b$ implies by Theorem 8.4 and by the formula (5)

$$[a+b]c = [a]c + [b]c = a+b,$$

and hence $a+b < c$ by the formula (4).

We obtain immediately by definition

$$(8) \quad a < b \text{ implies } b-a < b.$$

We have obviously by Theorem 4.2 for every real number α

$$(9) \quad a < b \text{ implies } \alpha a < \alpha b.$$

$$(10) \quad a < b < c \text{ implies } c-a > c-b \text{ and } c-a > b-a.$$

Indeed, $a < b < c$ implies $c > b > b-a$ by the formula (8),

and hence by the formula (2)

$$c-b \perp b-a = (c-a) - (c-b), \quad (c-a) - (b-a) = c-b \perp b-a.$$

We obtain immediately by Theorem 7.6 for every projector $[p]$

$$(11) \quad a < b \text{ implies } [p]a < [p]b,$$

and we have by the formula (2) and by Theorem 8.3

$$(12) \quad u < b \text{ implies } [a] \leq [b].$$

We have obviously by Theorem 5.9

Theorem 11.1. If $\lim_{\nu \rightarrow \infty} a_\nu = a$ and $a_\nu < b$ for every $\nu = 1, 2, \dots$, then we have $a < b$. If $\lim_{\nu \rightarrow \infty} a_\nu = a$

and $a_\nu > f$ for every $\nu = 1, 2, \dots$, then we have $a > f$.

A system of elements $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is called a resolution of an element $a \in R$, if

- 1) $a_\lambda < a_\rho$ for $\lambda < \rho$,
- 2) a_λ is left side continuous: $a_{\lambda-0} = a_\lambda$,
- 3) $a_{+\infty} = a$, $a_{-\infty} = 0$.

It is obvious by Theorem 11.1 that if $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is a resolution of an element $a \in R$, then we have $a_\lambda < a$ for every real number λ .

We obtain immediately by the formula (11):

Theorem 11.2. If a system $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is a resolution of an element $a \in R$, then $[p]a_\lambda$ ($-\infty < \lambda < +\infty$) is a resolution of $[p]a$ for every projector $[p]$.

We have obviously by the formula (9):

Theorem 11.3. If a system $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is a resolution of an element $a \in R$, then αa_λ ($-\infty < \lambda < +\infty$) is a resolution of αa for every real number α .

We obtain further by the formulas (6) and (7):

Theorem 11.4. For a resolution a_λ ($-\infty < \lambda < +\infty$) of an element $a \in R$ and for a resolution b_λ ($-\infty < \lambda < +\infty$) of an element $b \in R$, if $a \perp b$, then $a_\lambda + b_\lambda$ ($-\infty < \lambda < +\infty$) is a resolution of $a + b$.

We can prove easily by the formula (10) and by Theorem 11.1:

Theorem 11.5. If a system $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) is a resolution of an element $a \in R$, then $a - a_{-\lambda+0}$ ($-\infty < \lambda < +\infty$) is a resolution of $a \in R$ too.

Since every part of a positive element is positive by the formula (5), we have by the formula (8):

Theorem 11.6. Every resolution of a positive element is increasing.

§12 Resolutions of projectors

A system of projectors $[p_\lambda] \quad (-\infty < \lambda < +\infty)$ is said to be monotone increasing, if

$$[p_\lambda] \leq [p_\rho] \quad \text{for } \lambda < \rho.$$

If $[p_\lambda] \quad (-\infty < \lambda < +\infty)$ is monotone increasing, then to any real number λ there exists the least upper bound $\bigcup_{\rho < \lambda} [p_\rho]$. Because, to any sequence of real numbers $\lambda > \alpha_\nu \uparrow_{\nu=1}^\infty \lambda$ there exists by Theorem 8.7 a projector $[p]$ such that $[p_{\alpha_\nu}] \uparrow_{\nu=1}^\infty [p]$. Then, since to any real number $\rho < \lambda$ there exists $\alpha_\nu > \rho$, we have hence

$$[p] \geq [p_\rho] \quad \text{for every real number } \rho < \lambda.$$

On the other hand, if $[q] \geq [p_\rho]$ for every real number $\rho < \lambda$, then we have naturally $[q] \geq [p_{\alpha_\nu}]$ for every $\nu = 1, 2, \dots$, and hence $[q] \geq [p]$. Therefore we obtain $[p] = \bigcup_{\rho < \lambda} [p_\rho]$.

We also can prove likewise that to any real number λ there exists the greatest lower bound $\bigcap_{\rho > \lambda} [p_\rho]$, and we have

$$[p_{\alpha_\nu}] \downarrow_{\nu=1}^\infty \bigcap_{\rho > \lambda} [p_\rho]$$

for every sequence of real numbers $\lambda < \alpha_\nu \downarrow_{\nu=1}^\infty \lambda$. Therefore we have:

Theorem 12.1. If a system of projectors $[p_\lambda] \quad (-\infty < \lambda < +\infty)$ is monotone increasing, then we have

$$[p_{\alpha_\nu}] \uparrow_{\nu=1}^\infty \bigcup_{\rho < \lambda} [p_\rho]$$

for every sequence of real numbers $\lambda > \alpha_\nu \uparrow_{\nu=1}^\infty \lambda$; and

$$[p_{\alpha_\nu}] \downarrow_{\nu=1}^\infty \bigcap_{\rho > \lambda} [p_\rho]$$

for every sequence of real numbers $\lambda < \alpha_\nu \downarrow_{\nu=1}^\infty \lambda$.

A monotone increasing system of projectors $[p_\lambda] \quad (-\infty < \lambda < +\infty)$ is called a resolution of a projector $[p]$, if

$$1) \quad [p_\lambda] = \bigcup_{\rho < \lambda} [p_\rho] \quad \text{for every real number } \lambda,$$

$$2) \quad [p] = \bigcup_{-\infty < \lambda < +\infty} [p_\lambda], \quad 0 = \bigcap_{-\infty < \lambda < +\infty} [p_\lambda].$$

Theorem 12.2. If a system $a_\lambda \in R \quad (-\infty < \lambda < +\infty)$ is a resolution of an element $a \in R$, then $[a_\lambda] \quad (-\infty < \lambda < +\infty)$

is a resolution of the projector $[a]$ and we have for every real number λ

$$[a_{\lambda+0}] = \bigcap_{p > \lambda} [a_p].$$

Proof. For a resolution a_λ ($-\infty < \lambda < +\infty$) of an element $a \in R$, $[a_\lambda]$ ($-\infty < \lambda < +\infty$) is obviously monotone increasing by the formula §11(2) and by Theorem 8.3. As $a_{\lambda-0} = a_\lambda$ by definition, we have by Theorem 5.10

$$|a_{\alpha_\nu}| \uparrow_{\nu=1}^{\infty} |a_\lambda| \quad \text{for } \lambda > \alpha_\nu \uparrow_{\nu=1}^{\infty} \lambda,$$

and hence by Theorem 8.5

$$[a_{\alpha_\nu}] \uparrow_{\nu=1}^{\infty} [a_\lambda] \quad \text{for } \lambda > \alpha_\nu \uparrow_{\nu=1}^{\infty} \lambda.$$

Consequently we have $[a_\lambda] = \bigcup_{p < \lambda} [a_p]$ by the previous theorem.

We also can prove likewise

$$[a] = \bigcup_{-\infty < \lambda < +\infty} [a_\lambda].$$

For any sequence of real numbers $\lambda < \alpha_\nu \downarrow_{\nu=1}^{\infty} \lambda$, since

$$\lim_{\nu \rightarrow \infty} a_{\alpha_\nu} = a_{\lambda+0},$$

we have $|a| - |a_{\alpha_\nu}| \uparrow_{\nu=1}^{\infty} |a| - |a_{\lambda+0}|$, and hence by Theorem 8.5

$$[|a| - |a_{\alpha_\nu}|] \uparrow_{\nu=1}^{\infty} [|a| - |a_{\lambda+0}|].$$

On the other hand we have by the formula §8(8)

$$[|a| - |a_{\alpha_\nu}|] = [|a| - [a_{\alpha_\nu}]] = [a] - [a_{\alpha_\nu}],$$

$$[|a| - |a_{\lambda+0}|] = [|a| - [a_{\lambda+0}]] = [a] - [a_{\lambda+0}].$$

Consequently we obtain $[a_{\alpha_\nu}] \downarrow_{\nu=1}^{\infty} [a_{\lambda+0}]$. From this relation

we conclude easily that $[a_{\lambda+0}] = \bigcap_{p > \lambda} [a_p]$. Furthermore we

can prove likewise $0 = \bigcap_{-\infty < \lambda < +\infty} [a_\lambda]$.

We have obviously by Theorems 8.6, 8.7 and the formula §11(12):

Theorem 12.3. If a system $[p_\lambda]$ ($-\infty < \lambda < +\infty$) is a resolution of a projector $[p]$, then $[p_\lambda]a$ ($-\infty < \lambda < +\infty$) is a resolution of $[p]a$ for every element $a \in R$.

Theorem 12.4. Corresponding to a resolution a_λ ($-\infty < \lambda < +\infty$) of an element $a \in R$, there exists uniquely a resolution $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of the projector $[a]$ such that

$$[p_\lambda]a = a_\lambda \quad \text{for every real number } \lambda.$$

Proof. For a resolution a_λ ($-\infty < \lambda < +\infty$) of an element $a \in R$, putting $p_\lambda = a_\lambda$, we see easily by Theorem 12.2 that $[p_\lambda]$ ($-\infty < \lambda < +\infty$) is a resolution of the projector $[a]$ and satisfies the condition described in the theorem. Conversely, if for a resolution $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of the projector $[a]$ we have $[p_\lambda]a = a_\lambda$ for every real number λ , then we obtain by the formula §8(1) and by Theorem 8.2

$$[a_\lambda] = [[p_\lambda]a] = [p_\lambda][a] = [p_\lambda],$$

since $[p_\lambda] \leq [a]$. Therefore such resolution $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of the projector $[a]$ is uniquely determined.

Recalling Theorem 8.10 we obtain immediately by definition:

Theorem 12.5. For a resolution $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of a projector $[p]$, we obtain a resolution $[q][p_\lambda]$ ($-\infty < \lambda < +\infty$) of the projector $[q][p]$ for every projector $[q]$.

Theorem 12.6. Let a_λ ($-\infty < \lambda < +\infty$) be a resolution of an element $a \in R$. If a function $\varphi(\lambda)$ ($-\infty < \lambda < +\infty$) is integrable by a_λ ($-\infty < \lambda < +\infty$), then we have for $-\infty \leq \alpha \leq \beta \leq +\infty$

$$\int_\alpha^\beta \varphi(\lambda) da_\lambda = [a_\beta - a_\alpha] \int_{-\infty}^{+\infty} \varphi(\lambda) da_\lambda.$$

Proof. For $-\infty \leq \alpha \leq \beta \leq +\infty$, as $a_\alpha < a_\beta$ by assumption, we have by Theorem 8.4

$$[a_\beta - a_\alpha] = [a_\beta] - [a_\alpha].$$

Since $\varphi(\lambda)$ is integrable by a_λ ($-\infty < \lambda < +\infty$) by assumption, we have by the formula §10(4)

$$([a_\beta] - [a_\alpha]) \int_{-\infty}^{+\infty} \varphi(\lambda) da_\lambda = \int_{-\infty}^{+\infty} \varphi(\lambda) d_\lambda ([a_\beta] - [a_\alpha]) a_\lambda.$$

On the other hand we have by the formula §11(5)

$$[a_\lambda]a = a_\lambda \quad \text{for every real number } \lambda,$$

and further by Theorem 8.2 and by the formula §11(12)

$$([a_\beta] - [a_\alpha])[a_\lambda] = \begin{cases} [a_\beta] - [a_\alpha] & \text{for } \lambda \geq \beta, \\ [a_\lambda] - [a_\alpha] & \text{for } \alpha \leq \lambda \leq \beta, \\ 0 & \text{for } \lambda \leq \alpha. \end{cases}$$

Therefore we obtain by the formula §10(3)

$$\begin{aligned} & \int_{-\infty}^{+\infty} \varphi(\lambda) d\lambda ([a_\beta] - [a_\alpha]) a_\lambda \\ &= \int_{\alpha}^{\beta} \varphi(\lambda) d\lambda ([a_\lambda] a - [a_\alpha] a) = \int_{\alpha}^{\beta} \varphi(\lambda) d a_\lambda. \end{aligned}$$

§13 Integral representation

As a principal theorem in the first spectral theory we have

Theorem 13.1. To every elements a and $f \in R$ there exists uniquely a resolution a_λ ($-\infty < \lambda < +\infty$) of a such that

$$[a] f = \int_{-\infty}^{+\infty} \lambda d a_\lambda.$$

Proof. We shall consider first the case where $a \geq 0$.

In this case, if we put

$$a_\lambda = [p_\lambda] a, \quad p_\lambda = (\lambda a - [a] f)^+ \quad (-\infty < \lambda < +\infty),$$

then we can prove that a_λ ($-\infty < \lambda < +\infty$) satisfies our requirement.

For $\lambda < \beta$, since $0 \leq p_\lambda \leq p_\beta$, we have $[p_\lambda] \leq [p_\beta]$ by

Theorem 8.3, and hence we obtain by Theorem 8.2

$$a_\lambda = [p_\lambda] a = [p_\lambda] [p_\beta] a = [p_\lambda] a_\beta.$$

Consequently we have by the formula §11(4)

$$a_\lambda \prec a_\beta \quad \text{for } \lambda < \beta.$$

For every sequence of real numbers $\lambda > \alpha_\nu \uparrow_{\nu=1}^{\infty} \lambda$, since $p_{\alpha_\nu} \uparrow_{\nu=1}^{\infty} p_\lambda$ by Theorems 6.1 and 5.10, we obtain by Theorem 8.5

$$[p_{\alpha_\nu}] \uparrow_{\nu=1}^{\infty} [p_\lambda],$$

and hence $a_{\alpha_\nu} \uparrow_{\nu=1}^{\infty} a_\lambda$. Consequently we have $a_{\lambda-0} = a_\lambda$.

By virtue of Theorem 8.14 we have obviously $[p_\nu] \uparrow_{\nu=1}^{\infty} [a]$, and hence

$$a_\nu = [p_\nu] a \uparrow_{\nu=1}^{\infty} a.$$

From this relation we conclude $a_{+\infty} = a$, because a_λ ($-\infty < \lambda < +\infty$) is increasing. Since

$$p_{-\nu} = (-\nu a - [a] f)^+ = (\nu a + [a] f)^-,$$

we have $[p_{-\nu}] \downarrow_{\nu=1}^{\infty} 0$ by Theorem 8.15, and hence $a_{-\nu} \downarrow_{\nu=1}^{\infty} 0$.

From this relation we conclude $a_{-\infty} = 0$. Therefore a_λ ($-\infty < \lambda < +\infty$) is a resolution of a .

Since we have by Theorem 8.7

$$[p_\lambda](\lambda a - [a]t) \geq 0,$$

$$(1 - [p_\lambda])(\lambda a - [a]t) \leq 0,$$

we obtain by the formulas §8(1) and §8(8) respectively

$$(*) \quad \lambda a_\lambda \geq [p_\lambda][a]t = [a_\lambda]t,$$

$$(**) \quad \lambda(a - a_\lambda) \leq ([a] - [p_\lambda][a])t = [a - a_\lambda]t.$$

For $\lambda < \beta$, since $a_\beta - a_\lambda \prec a_\beta$ by the formula §11(8), we conclude from (*) by the formulas §11(5), §11(12) and by Theorem 8.4

$$\begin{aligned} \beta(a_\beta - a_\lambda) &= \beta[a_\beta - a_\lambda]a_\beta \geq [a_\beta - a_\lambda][a_\beta]t \\ &= [a_\beta - a_\lambda]t = ([a_\beta] - [a_\lambda])t, \end{aligned}$$

and, since $a_\beta - a_\lambda \prec a - a_\lambda$ by the formula §11(10), we conclude from (**) by the same formulas and Theorem

$$\begin{aligned} \lambda(a_\beta - a_\lambda) &= \lambda[a_\beta - a_\lambda](a - a_\lambda) \leq [a_\beta - a_\lambda][a - a_\lambda]t \\ &= [a_\beta - a_\lambda]t = ([a_\beta] - [a_\lambda])t. \end{aligned}$$

Consequently we have for $\lambda < \beta$

$$\lambda(a_\beta - a_\lambda) \leq [a_\beta]t - [a_\lambda]t \leq \beta(a_\beta - a_\lambda).$$

Thus, for every partition $\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_n = \beta$ of $[\alpha, \beta]$, we have

$$\sum_{\nu=1}^n \lambda_{\nu-1}(a_{\lambda_\nu} - a_{\lambda_{\nu-1}}) \leq [a_\beta]t - [a_\alpha]t \leq \sum_{\nu=1}^n \lambda_\nu(a_{\lambda_\nu} - a_{\lambda_{\nu-1}}),$$

and hence we obtain by definition of integral

$$\int_\alpha^\beta \lambda da_\lambda = [a_\beta]t - [a_\alpha]t.$$

From this relation we conclude by Theorem 12.2

$$\int_{-\infty}^{+\infty} \lambda da_\lambda = \lim_{\beta \rightarrow +\infty} ([a_\beta]t - [a_\alpha]t) = [a]t.$$

Next we shall prove the uniqueness of such resolution a_λ

($-\infty < \lambda < +\infty$) of a . If

$$[a]t = \int_{-\infty}^{+\infty} \lambda dC_\lambda$$

for another resolution C_λ ($-\infty < \lambda < +\infty$) of a , then we have by the formulas §10(1) and §10(2)

$$\begin{aligned}
\mu a - [a]f &= \int_{-\infty}^{+\infty} (\mu - \lambda) d\lambda c_\lambda \\
&= \int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda + \int_{\mu}^{+\infty} (\mu - \lambda) d\lambda c_\lambda \\
&= \int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda - \int_{\mu}^{+\infty} (\lambda - \mu) d\lambda c_\lambda.
\end{aligned}$$

As c_λ ($-\infty < \lambda < +\infty$) is increasing by Theorem 11.6, we have by Theorem 10.1

$$\int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda \geq 0, \quad \int_{\mu}^{+\infty} (\lambda - \mu) d\lambda c_\lambda \geq 0.$$

Since we have by Theorem 12.6

$$\begin{aligned}
\int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda &= [c_\mu] \int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda, \\
\int_{\mu}^{+\infty} (\lambda - \mu) d\lambda c_\lambda &= [a - c_\mu] \int_{\mu}^{+\infty} (\lambda - \mu) d\lambda c_\lambda,
\end{aligned}$$

and further $[c_\mu][a - c_\mu] = 0$ by Theorem 8.1, we obtain by Theorem 7.3

$$\int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda \cap \int_{\mu}^{+\infty} (\lambda - \mu) d\lambda c_\lambda = 0.$$

Therefore we have by Theorem 3.8

$$p_\mu = (\mu a - [a]f)^+ = \int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda.$$

From this relation we conclude by Theorem 12.6

$$[c_\mu] p_\mu = \int_{-\infty}^{\mu} (\mu - \lambda) d\lambda c_\lambda = p_\mu.$$

Thus we have by Theorem 8.2

$$[c_\mu] \geq [p_\mu] \quad \text{for every real number } \mu.$$

On the other hand, for any positive number ε we have by Theorem 10.1 and by the formula §10(1)

$$p_\mu \geq \int_{-\infty}^{\mu-\varepsilon} (\mu - \lambda) d\lambda c_\lambda \geq \int_{-\infty}^{\mu-\varepsilon} \varepsilon d\lambda c_\lambda = \varepsilon c_{\mu-\varepsilon} \geq 0,$$

and hence by Theorems 8.3 and 7.15

$$[p_\mu] \geq [\varepsilon c_{\mu-\varepsilon}] = [c_{\mu-\varepsilon}].$$

Since $[c_\mu] = \bigcup_{\varepsilon > 0} [c_{\mu-\varepsilon}]$ by Theorem 12.2, we obtain hence

$$[p_\mu] = [c_\mu] \quad \text{for every real number } \mu.$$

Consequently we have by the formula §11(5) for every real number

$$a_\mu = [p_\mu]a = [c_\mu]a = c_\mu.$$

Now we can prove the general case. For two arbitrary elements a and $b \in \mathcal{R}$, we obtain uniquely a resolution c_λ ($-\infty < \lambda < +\infty$) of the positive part a^+ of a and a

resolution e_λ $(-\infty < \lambda < +\infty)$ of the negative part a^- of a such that

$$[a^+]f = \int_{-\infty}^{+\infty} \lambda d c_\lambda, \quad [a^-](-f) = \int_{-\infty}^{+\infty} \lambda d e_\lambda$$

as proved just now. Putting

$$a_\lambda = c_\lambda - e_\lambda \quad (-\infty < \lambda < +\infty),$$

we obtain then a resolution a_λ $(-\infty < \lambda < +\infty)$ of a by Theorems 11.3 and 11.4, and we have by the formula §10(3)

$$[a]f = [a^+]f + [a^-]f = \int_{-\infty}^{+\infty} \lambda d a_\lambda.$$

Such a resolution a_λ $(-\infty < \lambda < +\infty)$ of a is determined uniquely. Because, if

$$[a]f = \int_{-\infty}^{+\infty} \lambda d f_\lambda$$

for another resolution f_λ $(-\infty < \lambda < +\infty)$ of a , then we have by the formulas §10(4) and §10(3)

$$[a^+]f = \int_{-\infty}^{+\infty} \lambda d [a^+]f_\lambda, \quad [a^-](-f) = \int_{-\infty}^{+\infty} \lambda d [a^-](-f_\lambda).$$

Since $[a^+]f_\lambda$ $(-\infty < \lambda < +\infty)$ is a resolution of a^+ by Theorem 11.2, $[a^-](-f_\lambda)$ $(-\infty < \lambda < +\infty)$ is a resolution of a^- by Theorems 11.2 and 11.3, we obtain

$$c_\lambda = [a^+]f_\lambda, \quad e_\lambda = [a^-](-f_\lambda) \quad (-\infty < \lambda < +\infty)$$

by the uniqueness of such resolutions c_λ and e_λ $(-\infty < \lambda < +\infty)$, and consequently we have for every real number λ

$$a_\lambda = c_\lambda - e_\lambda = [a^+]f_\lambda + [a^-]f_\lambda = [a]f_\lambda = f_\lambda.$$

Theorem 12.4 says that to a resolution a_λ $(-\infty < \lambda < +\infty)$ of an element $a \in R$ there exists uniquely a resolution $[p_\lambda]$ $(-\infty < \lambda < +\infty)$ of the projector $[a]$ such that

$$[p_\lambda]a = a_\lambda \quad (-\infty < \lambda < +\infty).$$

Therefore we obtain by Theorem 13.1 that corresponding to every pair of elements a and $f \in R$ there exists uniquely a resolution $[p_\lambda]$ $(-\infty < \lambda < +\infty)$ of the projector $[a]$ such that

$$[a]f = \int_{-\infty}^{+\infty} \lambda d_\lambda [p_\lambda]a.$$

Such a resolution $[p_\lambda]$ $(-\infty < \lambda < +\infty)$ of $[a]$ is called the

spectral system of an element $b \in R$ by an element $a \in R$.

In Proof of Theorem 31.1 we find easily that

$$c_\lambda = [(\lambda a^+ - [a^+]b)^+]a^+, \quad e_\lambda = [(\lambda a^- + [a^-]b)^+]a^-,$$

$$a_\lambda = c_\lambda - e_\lambda \quad (-\infty < \lambda < +\infty).$$

Since we have by Theorems 7.4, 7.16 and by the formula §8(1)

$$[(\lambda a^+ - [a^+]b)^+]a^+ = [(\lambda a - b)^+][a^+]a,$$

$$[(\lambda a^- + [a^-]b)^+]a^- = [(-\lambda[a^-]a + [a^-]b)^+][a^-](-a)$$

$$= -[(\lambda a - b)^-][a^-]a,$$

we obtain hence for every real number λ

$$a_\lambda = ([(\lambda a - b)^+][a^+] + [(\lambda a - b)^-][a^-])a.$$

Therefore we have:

Theorem 13.2. The spectral system of an element $b \in R$ by an element $a \in R$ is given by

$$[(\lambda a - b)^+][a^+] + [(\lambda a - b)^-][a^-] \quad (-\infty < \lambda < +\infty).$$

Theorem 13.3. For the spectral system $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of an element $b \in R$ by an element $a \in R$, if $a \geq 0$, then we have

$$[a]b^+ = \int_0^{+\infty} \lambda d[p_\lambda]a, \quad [a]b^- = - \int_0^{+\infty} \lambda d[p_\lambda]a,$$

$$[a]|b| = \int_{-\infty}^{+\infty} |\lambda| d[p_\lambda]a.$$

Proof. As proved just in Proof of Theorem 13.1, we have

by assumption

$$[a]b = \int_0^{+\infty} \lambda d[p_\lambda]a - \int_{-\infty}^0 (-\lambda) d[p_\lambda]a,$$

$$\int_0^{+\infty} \lambda d[p_\lambda]a \cap \int_{-\infty}^0 (-\lambda) d[p_\lambda]a = 0.$$

Therefore we obtain by Theorem 3.8

$$[a]b^+ = ([a]b)^+ = \int_0^{+\infty} \lambda d[p_\lambda]a,$$

and further by the formula §10(7)

$$[a]b^- = ([a]b)^- = \int_{-\infty}^0 (-\lambda) d[p_\lambda]a$$

$$= - \int_0^{+\infty} \lambda d[p_{-\lambda}]a = \int_0^{+\infty} \lambda d([a] - [p_{-\lambda}])a.$$

From these relations we conclude immediately

$$[a]|b| = \int_{-\infty}^{+\infty} |\lambda| d[p_\lambda]a.$$

§14 Relations between spectral systems

In the following we shall consider relations between spectral systems of different elements only by a positive element $a \in R$. However, Theorems 14.1 and 14.4 hold for an arbitrary element $a \in R$ (cf. §23).

Theorem 14.1. For the spectral system $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of an element $b \in R$ by a positive element $a \in R$, if we put

$$[q_\lambda] = [a] - \bigcap_{\rho > -\lambda} [p_\rho] \quad (-\infty < \lambda < +\infty),$$

then $[q_\lambda]$ ($-\infty < \lambda < +\infty$) is the spectral system of $-b$ by a .

Proof. As $[a]b = \int_{-\infty}^{+\infty} \lambda d[p_\lambda]a$ by assumption, we have by the formulas §10(1), §10(3), and §10(7)

$$[a](-b) = \int_{-\infty}^{+\infty} \lambda d([a] - [p_\lambda])a = \int_{-\infty}^{+\infty} \lambda d([a] - [p_{-\lambda}])a.$$

Putting $c_\lambda = ([a] - [p_{-\lambda}])a$ ($-\infty < \lambda < +\infty$), we obtain hence by Theorem 10.3

$$[a](-b) = \int_{-\infty}^{+\infty} \lambda d c_{\lambda-0},$$

and $c_{\lambda-0}$ ($-\infty < \lambda < +\infty$) is a resolution of a by Theorem 11.5.

Furthermore we see easily by Theorems 2.2 and 2.4 that we have for every real number λ

$$[c_{\lambda-0}] = \bigcup_{\rho < \lambda} ([a] - [p_{-\rho}]) = [a] - \bigcap_{\rho > -\lambda} [p_\rho].$$

We have obviously by Theorem 13.3:

Theorem 14.2. For the spectral system $[p_\lambda]$ ($-\infty < \lambda < +\infty$) of an element $b \in R$ by a positive element $a \in R$, putting

$$[q_\lambda] = \begin{cases} [p_\lambda] & \text{for } \lambda > 0, \\ 0 & \text{for } \lambda \leq 0, \end{cases}$$

we obtain the spectral system $[q_\lambda]$ ($-\infty < \lambda < +\infty$) of b^+ by a .

Theorem 14.3. For a positive number α , if $[p_\lambda]$ ($-\infty < \lambda < +\infty$) is the spectral system of an element $b \in R$ by an element $a \in R$, then $[p_\alpha]$ ($-\infty < \lambda < +\infty$) is the spectral system of αb by a .

Proof. Since $[a]b = \int_{-\infty}^{+\infty} \lambda d[p_\lambda]a$ by assumption, we have by the formula §10(8)

$$[a](\alpha b) = \int_{-\infty}^{+\infty} \alpha \lambda d[p_\lambda]a = \int_{-\infty}^{+\infty} \lambda d[p_{\frac{\alpha}{2}}]a,$$

and we see further easily by definition that the system $[p_{\frac{\alpha}{2}}]$ ($-\infty < \lambda < +\infty$) is a resolution of the projector $[a]$.

Therefore $[p_{\frac{\alpha}{2}}]$ ($-\infty < \lambda < +\infty$) is the spectral system of αb by a .

Theorem 14.4. If two systems $[p_\lambda]$ and $[q_\lambda]$ ($-\infty < \lambda < +\infty$) are respectively the spectral systems of elements b and $c \in R$ by a positive element $a \in R$, then putting

$[r_\lambda] = \bigcup_{\sigma + \rho < \lambda} [p_\sigma][q_\rho]$ for rational numbers σ, ρ , we obtain the spectral system $[r_\lambda]$ ($-\infty < \lambda < +\infty$) of $b+c$ by a

Proof. By virtue of Theorem 13.2 we need only to prove that, putting for every real number λ

$$p_\lambda = (\lambda a - [a]b)^+,$$

$$q_\lambda = (\lambda a - [a]c)^+,$$

$$r_\lambda = (\lambda a - [a](b+c))^+,$$

we have $[r_\lambda] = \bigcup_{\sigma + \rho < \lambda} [p_\sigma][q_\rho]$ for rational numbers σ, ρ .

For such systems p_λ, q_λ , and r_λ ($-\infty < \lambda < +\infty$), we obtain by Theorem 7.16 for every real numbers σ, ρ , and λ

$$\sigma [p_\sigma]a \geq [p_\sigma]b,$$

$$\rho [q_\rho]a \geq [q_\rho]c,$$

$$\lambda (1 - [r_\lambda])a \leq (1 - [r_\lambda])[a](b+c),$$

and hence we have

$$\begin{aligned} \lambda [p_\sigma][q_\rho](1 - [r_\lambda])a &\leq [p_\sigma][q_\rho](1 - [r_\lambda])(b+c) \\ &= (1 - [r_\lambda])([p_\sigma][q_\rho]b + [p_\sigma][q_\rho]c) \\ &\leq (1 - [r_\lambda])(\sigma [p_\sigma][q_\rho]a + \rho [p_\sigma][q_\rho]a) \\ &= (\sigma + \rho)(1 - [r_\lambda])[p_\sigma][q_\rho]a, \end{aligned}$$

that is, $\{\lambda - (\sigma + \rho)\} [p_\sigma][q_\rho](1 - [r_\lambda])a \leq 0$. Thus, if $\lambda > \sigma + \rho$, then we have

$$[p_\sigma][q_\rho](1 - [r_\lambda])a = 0,$$

since $a \geq 0$ by assumption. This relation yields by the formula §8(1) and by Theorem 8.2

$$[p_\sigma][q_\rho](1 - [r_\lambda]) = 0,$$

since $[p_\sigma][q_\rho](1 - [r_\lambda]) \leq [a]$ obviously. Therefore we have by Theorem 8.2

$$[p_\sigma][q_\rho] \leq [r_\lambda] \quad \text{for } \lambda > \sigma + \rho.$$

On the other hand we have by Theorem 7.16

$$\sigma(1 - [p_\sigma])a \leq (1 - [p_\sigma])[a]b,$$

$$\rho(1 - [q_\rho])a \leq (1 - [q_\rho])[a]c,$$

$$\lambda[r_\lambda]a \geq [r_\lambda](b + c),$$

and hence we obtain likewise

$$\{(\sigma + \rho) - \lambda\}(1 - [p_\sigma])(1 - [q_\rho])[r_\lambda]a \leq 0.$$

Thus, if $\lambda < \sigma + \rho$, then we have $(1 - [p_\sigma])(1 - [q_\rho])[r_\lambda]a = 0$, and we conclude likewise

$$(1 - [p_\sigma])(1 - [q_\rho])[r_\lambda] = 0.$$

Therefore we have by Theorem 8.2

$$[r_\lambda](1 - [p_\sigma]) \leq [q_\rho] \quad \text{for } \lambda < \sigma + \rho.$$

For any rational number $\varepsilon > 0$, we have by Theorem 5.15

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} \sum_{\nu=-n}^n ([p_{\nu\varepsilon}] - [p_{(\nu-1)\varepsilon}])a \\ &= \bigcup_{\nu=-\infty}^{+\infty} (1 - [p_{(\nu-1)\varepsilon}])[p_{\nu\varepsilon}]a, \end{aligned}$$

since $[p_\lambda] (-\infty < \lambda < +\infty)$ is a resolution of the projector $[a]$.

For a rational number δ such that

$$\lambda - \varepsilon < \delta < \lambda,$$

since $\lambda - 2\varepsilon < (\nu-1)\varepsilon + (\delta - \nu\varepsilon)$, we have hence by Theorem 7.6

$$\begin{aligned} [r_{\lambda-2\varepsilon}]a &= \bigcup_{\nu=-\infty}^{+\infty} [r_{\lambda-2\varepsilon}](1 - [p_{(\nu-1)\varepsilon}])[p_{\nu\varepsilon}]a \\ &\leq \bigcup_{\nu=-\infty}^{+\infty} [q_{\delta-\nu\varepsilon}][p_{\nu\varepsilon}]a, \end{aligned}$$

and hence we have by the formula §8(1) and by Theorem 8.5

$$[r_{\lambda-2\varepsilon}] = [r_{\lambda-2\varepsilon}][a] \leq \bigcup_{\nu=-\infty}^{+\infty} [q_{\delta-\nu\varepsilon}][p_{\nu\varepsilon}].$$

This relation yields obviously

$$[r_{\lambda-2\varepsilon}] \leq \bigcup_{\sigma+\rho<\lambda} [p_\sigma][q_\rho] \quad \text{for rational numbers } \sigma, \rho.$$

Here a rational number $\varepsilon > 0$ may be arbitrary, and we have

$$[r_\lambda] = \bigcup_{\nu=1}^{\infty} [r_{\lambda-\frac{1}{\nu}}],$$

since $[r_\lambda] (-\infty < \lambda < +\infty)$ is a resolution of the projector $[a]$.

Therefore we conclude

$[r_\lambda] = \bigcup_{\sigma+p \leq \lambda} [p_\sigma][q_p]$ for rational numbers σ, p
for every real number λ

As shown just above, we can discuss the relations between spectral systems, but it is difficult and complicated. We will however recognize after that it will become easy and simple, if we apply the second spectral theory.

CHAPTER III

SECOND SPECTRAL THEORY§15 Maximal ideals

Let R be a continuous semi-ordered linear space in the sequel. A set of projectors $\mathcal{F}$ is called an ideal, if

- 1) $\mathcal{F} \ni 0$,
- 2) $\mathcal{F} \ni [p]$, $[p] \leq [q]$ implies $\mathcal{F} \ni [q]$,
- 3) $\mathcal{F} \ni [p]$, $\mathcal{F} \ni [q]$ implies $\mathcal{F} \ni [p][q]$.

An ideal $\mathcal{F}$ is said to be maximal, if there exists no other ideal including $\mathcal{F}$.

From the postulates 1) and 2) we conclude obviously that every ideal $\mathcal{F}$ satisfies the condition:

- (F) $[p_1][p_2] \dots [p_n] \neq 0$ for every finite number of projectors $[p_1], [p_2], \dots, [p_n] \in \mathcal{F}$.

Conversely we have:

Theorem 15.1. If a set of projectors $\mathcal{F}$ satisfies the condition (F), then there exists a maximal ideal including $\mathcal{F}$.

Proof. By virtue of Maximal theorem we see easily that there exists a maximal set of projectors $\mathcal{F}_0$ which include $\mathcal{F}$ and satisfies the condition (F). Such $\mathcal{F}_0$ is a maximal ideal. In fact, we have obviously $\mathcal{F}_0 \ni 0$. If $\mathcal{F}_0 \ni [p]$ and $[p] \leq [q]$, then we have $\mathcal{F}_0 \ni [q]$. Because, if $\mathcal{F}_0 \not\ni [q]$, then the set of projectors $\{\mathcal{F}_0, [q]\}$ satisfies obviously the condition (F), contradicting that $\mathcal{F}_0$ is a maximal set of projectors satisfying the condition (F). Finally, if $\mathcal{F}_0 \ni [p]$, $\mathcal{F}_0 \ni [q]$, then we have $\mathcal{F}_0 \ni [p][q]$. Because, if $\mathcal{F}_0 \not\ni [p][q]$, then the set of projectors $\{\mathcal{F}_0, [p][q]\}$ satisfies obviously the condition (F), contradicting that $\mathcal{F}_0$ is a maximal set of projectors satisfying the condition (F). Therefore $\mathcal{F}_0$ is an ideal by definition. Furthermore $\mathcal{F}_0$ is maximal, because every ideal satisfies the

condition (F).

Theorem 15.2. In order that an ideal $\mathfrak{f}$ be maximal, it is necessary and sufficient that to any projector $[q] \notin \mathfrak{f}$ there exists a projector $[p] \in \mathfrak{f}$ for which we have $[p][q] = 0$.

Proof. Let $\mathfrak{f}$ be an ideal. If there exists a projector $[q] \notin \mathfrak{f}$ such that

$$[p][q] \neq 0 \quad \text{for every projector } [p] \in \mathfrak{f},$$

then the set of projectors $\{\mathfrak{f}, [q]\}$ satisfies obviously the condition (F), and hence there exists by the previous theorem a maximal ideal including $\{\mathfrak{f}, [q]\}$. Therefore $\mathfrak{f}$ is not maximal.

Conversely, if $\mathfrak{f}$ is not maximal, then there exists by definition an ideal $\mathfrak{f}_0$ which includes $\mathfrak{f}$ and does not coincide with $\mathfrak{f}$, that is, there exists a projector $[q]$ such that $[q] \in \mathfrak{f}_0$ but $[q] \notin \mathfrak{f}$. For such a projector $[q]$ we have obviously by definition of ideal

$$[p][q] \neq 0 \quad \text{for every projector } [p] \in \mathfrak{f}.$$

Theorem 15.3. Let $\mathfrak{f}$ be a maximal ideal. If

$$\mathfrak{f} \ni [p_1] \vee [p_2] \vee \dots \vee [p_\kappa],$$

then there exists ν for which we have $\mathfrak{f} \ni [p_\nu]$.

Proof. If $\mathfrak{f} \ni [p_\nu]$ for every $\nu = 1, 2, \dots, \kappa$, then there exist by the previous theorem projectors $[q_\nu] \in \mathfrak{f}$ ($\nu = 1, 2, \dots, \kappa$) such that

$$[p_\nu][q_\nu] = 0 \quad \text{for every } \nu = 1, 2, \dots, \kappa.$$

For such projectors $[q_\nu]$ ($\nu = 1, 2, \dots, \kappa$), putting

$$[q] = [q_1][q_2] \dots [q_\kappa],$$

we have $[q] \in \mathfrak{f}$ by the postulate 3), and naturally

$$[p_\nu][q] = 0 \quad \text{for every } \nu = 1, 2, \dots, \kappa.$$

Consequently we have by Theorem 8.8

$$([p_1] \vee [p_2] \vee \dots \vee [p_\kappa])[q] = 0;$$

and hence $[p_1] \vee [p_2] \vee \dots \vee [p_\kappa] \notin \mathfrak{f}$ by the postulates 1) and 3).

Corresponding to every projector $[p]$ we define $\mathcal{U}_{[p]}$ to mean the set of all maximal ideals $\mathfrak{f} \ni [p]$, and naturally

$$\mathcal{U}_{[0]} = 0.$$

With this definition we have obviously:

- (1) $\mathfrak{f} \in \mathcal{U}_{[p]}$ if and only if $\mathfrak{f} \ni [p]$.
 (2) $\mathcal{U}_{[p]} \mathcal{U}_{[q]} = \mathcal{U}_{[p][q]}$.

Because, if $\mathfrak{f} \in \mathcal{U}_{[p]} \mathcal{U}_{[q]}$, then we have $\mathfrak{f} \ni [p], \ni [q]$ by the formula (1), and hence $\mathfrak{f} \ni [p][q]$ by the postulate 3), that is, $\mathfrak{f} \in \mathcal{U}_{[p][q]}$ by the formula (1). Conversely, if $\mathfrak{f} \in \mathcal{U}_{[p][q]}$, then we have $\mathfrak{f} \ni [p][q]$, and hence $\mathfrak{f} \ni [p], \ni [q]$ by the postulate 2), that is, $\mathfrak{f} \in \mathcal{U}_{[p]} \mathcal{U}_{[q]}$ by the formula (1).

By virtue of Theorem 8.3 and by the postulate 2), we have

$$\mathfrak{f} \ni [p] \vee [q]$$

if and only if $\mathfrak{f} \ni [p]$ or $\mathfrak{f} \ni [q]$. Therefore we have by the formula (1)

$$(3) \quad \mathcal{U}_{[p]} \dot{+} \mathcal{U}_{[q]} = \mathcal{U}_{[p] \vee [q]}.$$

$\mathcal{U}_{[p]} \supset \mathcal{U}_{[q]}$ is obviously equivalent to $\mathcal{U}_{[p]} \mathcal{U}_{[q]} = \mathcal{U}_{[q]}$; and $[p] \geq [q]$ is equivalent to $[p][q] = [q]$ by Theorem 8.2. Furthermore $\mathcal{U}_{[p][q]} = \mathcal{U}_{[q]}$ implies $[p][q] = [q]$. Because, if $[q] - [q][p] \neq 0$, then, since $[q] - [q][p]$ is a projector by the formula §8(7), there exists by Theorem 15.1 a maximal ideal $\mathfrak{f} \ni [q] - [q][p]$. For such $\mathfrak{f}$, since

$$[q] - [q][p] \leq [q], \quad [q][p]([q] - [q][p]) = 0,$$

we have $\mathfrak{f} \ni [q]$ by the postulate 2), but $\mathfrak{f} \not\ni [q][p]$ by the postulates 1) and 3), that is, $\mathfrak{f} \in \mathcal{U}_{[q]}$ but $\mathfrak{f} \notin \mathcal{U}_{[p][q]}$ by the formula (1).

Therefore we obtain by the formula (2):

$$(4) \quad \mathcal{U}_{[p]} \supset \mathcal{U}_{[q]} \text{ if and only if } [p] \geq [q].$$

Recalling Theorem 8.1 we have obviously by the formulas (2) and (4):

$$(5) \quad \mathcal{U}_{[p]} \mathcal{U}_{[q]} = 0 \text{ if and only if } [p][q] = 0, \text{ that is, } p \perp q.$$

Since $[p][q]([p] - [p][q]) = 0$, we have by the

formula §8(6)

$$[p][q] \cup ([p] - [p][q]) = [p],$$

and hence we obtain by the formulas (3) and (5)

$$\mathcal{U}_{[p]} = \mathcal{U}_{[p][q]} \dot{+} \mathcal{U}_{[p] - [p][q]},$$

$$\mathcal{U}_{[p][q]} \mathcal{U}_{[p] - [p][q]} = 0.$$

Therefore we have

$$(6) \quad \mathcal{U}_{[p]} - \mathcal{U}_{[p]} \mathcal{U}_{[q]} = \mathcal{U}_{[p] - [p][q]}.$$

§16 Proper spaces

Considering every maximal ideal as a point, we obtain a space E , which consists of all maximal ideals. We can introduce a topology into this space E such that the totality of $\mathcal{U}_{[p]}$ for all projectors $[p]$ constitutes a neighbourhood system, because the set of all $\mathcal{U}_{[p]}$ satisfies the neighbourhood conditions by the formulas §15(1) and §15(2). This topological space E is called the proper space of a continuous semi-ordered linear space R , and $\mathcal{U}_{[p]}$ is called a neighbourhood in the proper space E .

Theorem 16.1. The proper space E is a Hausdorff space.

Proof. If two maximal ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are different, then there exists obviously a projector $[p_1]$ such that $\mathfrak{P}_1 \ni [p_1]$ but $\mathfrak{P}_2 \not\ni [p_1]$. For such a projector $[p_1]$ we obtain by Theorem 15.2 a projector $[p_2] \in \mathfrak{P}_2$ such that $[p_1][p_2] = 0$. Then we have by the formulas §15(1) and §15(5)

$$\mathfrak{P}_1 \in \mathcal{U}_{[p_1]}, \quad \mathfrak{P}_2 \in \mathcal{U}_{[p_2]}, \quad \mathcal{U}_{[p_1]} \mathcal{U}_{[p_2]} = 0.$$

Therefore E is a Hausdorff space by definition.

Theorem 16.2. Every neighbourhood $\mathcal{U}_{[p]}$ is compact, and hence $\mathcal{U}_{[p]}$ is open and closed at the same time for every $[p]$.

Proof. If $\mathcal{U}_{[p]} \subset \sum_{\lambda \in \Lambda} \mathcal{U}_{[p_\lambda]}$, then we have

$$\prod_{\lambda \in \Lambda} \{ \mathcal{U}_{[p]} - \mathcal{U}_{[p]} \mathcal{U}_{[p_\lambda]} \} = 0,$$

and hence by the formula §15(6)

$$(*) \quad \prod_{\lambda \in \Lambda} \mathcal{U}_{[p] - [p][p_\lambda]} = 0.$$

If $\prod_{\nu=1}^k \mathcal{U}_{[p] - [p][p_{\lambda_\nu}]} \neq 0$ for every finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_k \in \Lambda$, then we see easily by the formula §15(2) that the set of projectors

$$[p] - [p][p_\lambda] \quad (\lambda \in \Lambda)$$

satisfies the condition (F) in §15, and hence there exists by Theorem 15.1 a maximal ideal $\mathfrak{P}$ such that

$$\mathfrak{P} \ni [p] - [p][p_\lambda] \quad \text{for every } \lambda \in \Lambda,$$

that is, $\mathfrak{P} \in \prod_{\lambda \in \Lambda} \mathcal{U}_{[p] - [p][p_\lambda]}$ by the formula §15(1), contradicting the relation (*). Therefore there exists a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_k \in \Lambda$ such that

$$\prod_{\nu=1}^k \mathcal{U}_{[p] - [p][p_{\lambda_\nu}]} = 0,$$

that is, $\mathcal{U}_{[p]} \subset \sum_{\nu=1}^k \mathcal{U}_{[p_{\lambda_\nu}]}$. Thus $\mathcal{U}_{[p]}$ is compact by definition. Since the proper space E is a Hausdorff space by the previous theorem, $\mathcal{U}_{[p]}$ is hence closed by Theorem 1, and $\mathcal{U}_{[p]}$ is open obviously by definition of topology of E .

By virtue of this theorem we find that the proper space E is a locally compact Hausdorff space.

Theorem 16.3. In the proper space E of R , if a compact set A is included in an open set B° , then there exists a projector $[p]$ such that

$$A \subset \mathcal{U}_{[p]} \subset B^\circ.$$

Proof. Since the set of all $\mathcal{U}_{[p]}$ constitutes a neighbourhood system of E , to every open set B° there exists a system $\mathcal{U}_{[p_\lambda]}$ ($\lambda \in \Lambda$) such that

$$B^\circ = \sum_{\lambda \in \Lambda} \mathcal{U}_{[p_\lambda]}.$$

Since $A \subset B^\circ$ and A is compact by assumption, there exists then a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_k \in \Lambda$ for which we have

$$A \subset \sum_{\nu=1}^k \mathcal{U}_{[p_{\lambda_\nu}]} \subset B^\circ.$$

For such $\lambda_1, \lambda_2, \dots, \lambda_k \in \Lambda$, putting $[p] = [p_{\lambda_1}] \vee [p_{\lambda_2}] \vee \dots \vee [p_{\lambda_k}]$, we have by the formula §15(3)

$$\mathcal{U}_{[p]} = \sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]},$$

and hence $A \subset \mathcal{U}_{[p]} \subset B^{\circ}$.

From this theorem we conclude immediately:

Theorem 16.4. To an open compact set A in the proper space E of R there exists uniquely a projector $[p]$ for which we have $A = \mathcal{U}_{[p]}$.

Theorem 16.5. For a sequence of projectors $[p_{\nu}]$ ($\nu = 1, 2, \dots$) we have

$$\mathcal{U}_{[p]} = \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]} \right)^{-}$$

if and only if $[p] = \bigvee_{\nu=1}^{\infty} [p_{\nu}]$.

Proof. If $[p] = \bigvee_{\nu=1}^{\infty} [p_{\nu}]$, then we have obviously

$$[p] \geq [p_{\nu}] \quad \text{for every } \nu = 1, 2, \dots,$$

and hence by the formula §15(4)

$$\mathcal{U}_{[p]} \supset \sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]}.$$

Since $\mathcal{U}_{[p]}$ is closed by Theorem 16.2, we obtain hence

$$\mathcal{U}_{[p]} \supset \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]} \right)^{-}.$$

Furthermore, for any neighbourhood $\mathcal{U}_{[q]} \subset \mathcal{U}_{[p]} - \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]} \right)^{-}$, we have by the formulas §15(4) and §15(5)

$$[q] \leq [p], \quad [q][p_{\nu}] = 0 \quad \text{for every } \nu = 1, 2, \dots,$$

and consequently by Theorems 8.2 and 8.8

$$[q] = [q][p] = \bigvee_{\nu=1}^{\infty} [q][p_{\nu}] = 0.$$

Therefore we must have

$$\mathcal{U}_{[p]} = \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]} \right)^{-}.$$

Conversely, from this relation we conclude by the formula §15(4)

$$[p] \geq [p_{\nu}] \quad \text{for every } \nu = 1, 2, \dots,$$

and hence there exists by Theorem 8.7 a projector $[q]$ for which

$$[q] = \bigvee_{\nu=1}^{\infty} [p_{\nu}].$$

For such $[q]$ we have

$$\mathcal{U}_{[q]} = \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]} \right)^{-} = \mathcal{U}_{[p]}$$

as proved just now. Consequently we have $[q] = [p]$ by the formula §15(4).

Theorem 16.6. For a sequence of projectors $[p_\nu]$ ($\nu = 1, 2, \dots$) we have

$$\mathcal{U}_{[p]} = \left(\prod_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]} \right)^*$$

if and only if $[p] = \bigcap_{\nu=1}^{\infty} [p_\nu]$.

Proof. $\mathcal{U}_{[p]} = \left(\prod_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]} \right)^*$ is equivalent to

$$\mathcal{U}_{[p,]} - \mathcal{U}_{[p]} = \left\{ \sum_{\nu=1}^{\infty} (\mathcal{U}_{[p,]} - \mathcal{U}_{[p,]} \mathcal{U}_{[p_\nu]}) \right\}^-,$$

and hence by the formula §15(6) equivalent to

$$\mathcal{U}_{[p,] - [p]} = \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p,] - [p,] [p_\nu]} \right)^-.$$

$[p] = \bigcap_{\nu=1}^{\infty} [p_\nu]$ is equivalent to

$$[p,] - [p] = \bigcup_{\nu=1}^{\infty} ([p,] - [p,] [p_\nu]),$$

because for every positive element $a \in \mathcal{R}$ we have by Theorems 2.2, 2.4, 8.6, and 8.7

$$\bigcup_{\nu=1}^{\infty} ([p,] - [p,] [p_\nu])a = [p,]a - \bigcap_{\nu=1}^{\infty} [p,] [p_\nu]a = ([p,] - \bigcap_{\nu=1}^{\infty} [p_\nu])a.$$

Therefore this theorem is reduced to the previous theorem.

Theorem 16.7. For every $\mathcal{G}$ -open set A in the proper space E of $\mathcal{R}$, its closure A^- is open too.

Proof. We suppose that

$$A = B_1 \dot{+} B_2 \dot{+} \dots$$

for a sequence of closed sets B_ν ($\nu = 1, 2, \dots$) and A is open. For any neighbourhood $\mathcal{U}_{[p]}$, $B_\nu \mathcal{U}_{[p]}$ is compact by Theorem 16.2, and hence there exists by Theorem 15.6 a projector $[p_\nu]$ such that

$$B_\nu \mathcal{U}_{[p]} \subset \mathcal{U}_{[p_\nu]} \subset A \mathcal{U}_{[p]} \quad (\nu = 1, 2, \dots).$$

Then we have obviously

$$A \mathcal{U}_{[p]} = \sum_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]}.$$

Since we have by the formula §15(4)

$$[p] \geq [p_\nu] \quad \text{for every } \nu = 1, 2, \dots,$$

there exists by Theorem 8.7 a projector $[q]$ for which

$$[q] = \bigcup_{\nu=1}^{\infty} [p_\nu].$$

For such $[q]$ we have by Theorem 16.5

$$\mathcal{U}_{[p]} = \left(\sum_{p \in R} \mathcal{U}_{[p]} \right)^- = (A \mathcal{U}_{[p]})^- = A^- \mathcal{U}_{[p]},$$

since $\mathcal{U}_{[p]}$ is open and closed at the same time. Therefore

$A^- \mathcal{U}_{[p]}$ is open for every neighbourhood $\mathcal{U}_{[p]}$. On the

other hand we have naturally

$$A^- = \sum_{p \in R} A^- \mathcal{U}_{[p]},$$

and consequently A^- is open too.

§17 Proper values

Let E be the proper space of R . For every element $a \in R$ we have by Theorems 8.1 and 8.4

$$[a] = [a^+] + [a^-], \quad [a^+][a^-] = 0,$$

and hence by the formulas §15(3) and §15(5)

$$\mathcal{U}_{[a]} = \mathcal{U}_{[a^+]} \dot{+} \mathcal{U}_{[a^-]}, \quad \mathcal{U}_{[a^+]} \mathcal{U}_{[a^-]} = 0.$$

Therefore for any point $p \in E$ we have one and only one of the three cases:

$$1) p \in \mathcal{U}_{[a^+]}, \quad 2) p \notin \mathcal{U}_{[a]}, \quad 3) p \in \mathcal{U}_{[a^-]}.$$

Corresponding to each case we define the proper value (a, p) of an element $a \in R$ at p as

$$(a, p) = \begin{cases} +\infty & \text{for } p \in \mathcal{U}_{[a^+]}, \\ 0 & \text{for } p \notin \mathcal{U}_{[a]}, \\ -\infty & \text{for } p \in \mathcal{U}_{[a^-]}. \end{cases}$$

Since $\mathcal{U}_{[0]} = 0$, we have naturally

$$(1) \quad (0, p) = 0,$$

and we see at once by definition

$$\begin{aligned} (2) \quad (a, p) &\geq 0 && \text{if and only if } p \notin \mathcal{U}_{[a^-]}, \\ (3) \quad (a, p) &\leq 0 && \text{if and only if } p \in \mathcal{U}_{[a^+]}, \\ (4) \quad (a, p) &\neq 0 && \text{if and only if } p \in \mathcal{U}_{[a]}. \end{aligned}$$

Since we have by Theorem 7.4 and by the formula §8(1)

$$[[p]a]^\pm = [p][a^\pm], \quad [[p]a] = [p][a],$$

we obtain by the formula §15(2)

$$\mathcal{U}_{[(p)a]^+} = \mathcal{U}_{[p]} \mathcal{U}_{[a]^+}, \quad \mathcal{U}_{[(p)a]} = \mathcal{U}_{[p]} \mathcal{U}_{[a]}.$$

Therefore we have by definition

$$(5) \quad (a, p) = ([p]a, p) \quad \text{for } p \in \mathcal{U}_{[p]}.$$

If $a \geq b$, then we have by Theorem 3.7

$$a^+ \geq b^+, \quad a^- \leq b^-,$$

and hence by Theorem 8.3

$$[a^+] \geq [b^+], \quad [a^-] \leq [b^-],$$

and consequently further by the formula §15(4)

$$\mathcal{U}_{[a^+]} \supset \mathcal{U}_{[b^+]}, \quad \mathcal{U}_{[a^-]} \subset \mathcal{U}_{[b^-]}.$$

From these relations we conclude by definition that

$$(a, p) = 0 \quad \text{implies } p \in \mathcal{U}_{[b^+]}, \quad \text{namely } (b, p) \leq 0;$$

$$(a, p) = -\infty \quad \text{implies } (b, p) = -\infty.$$

Therefore we have

$$(6) \quad a \geq b \quad \text{implies } (a, p) \geq (b, p).$$

For every positive number α we have by Theorems 3.9 and 7.5

$$[(\alpha a)^+] = [\alpha a^+] = [\alpha^+], \quad [\alpha a] = [\alpha],$$

$$[(-\alpha a)^+] = [(\alpha a)^-] = [\alpha^-], \quad [-\alpha a] = [\alpha].$$

Consequently we have by definition

$$(7) \quad (\alpha a, p) = \alpha (a, p)$$

for every real number α , adopting the convention $0(\pm\infty) = 0$.

$$(8) \quad (a+b, p) = (a, p) + (b, p)$$

if the right side has any sense. Because, if

$$(a, p) = +\infty, \quad (b, p) \geq 0,$$

then we have $p \in \mathcal{U}_{[a^+]}$ by definition and $p \in \mathcal{U}_{[b^-]}$ by the formula (2), and hence further by the formula §15(6)

$$p \in \mathcal{U}_{[a^+]} - \mathcal{U}_{[a^+]} \mathcal{U}_{[b^-]} = \mathcal{U}_{[a^+] - [a^+][b^-]}.$$

Then we have by the formula (5)

$$(a+b, p) = ([a^+] - [a^+][b^-])(a+b), p).$$

As $([a^+] - [a^+][b^-])(a+b) = ([a^+] - [a^+][b^-])(a^+ + b^+)$ by Theorem 7.16, we have hence by the formulas (5) and (6)

$$(a+b, p) = (a^+ + b^+, p) \geq (a^+, p) = +\infty,$$

since $f \in \mathcal{U}_{[a+]}$ by assumption. Therefore our assertion (8) is obtained in the case: $(a, f) = +\infty$, $(b, f) \geq 0$.

In the case: $(a, f) = -\infty$, $(b, f) \leq 0$, we have by the formula (7)

$$(-a, f) = +\infty, \quad (-b, f) \geq 0,$$

and hence by the previous case

$$(a+b, f) = -(-a-b, f) = -(-a, f) - (-b, f) = (a, f) + (b, f)$$

In the final case: $(a, f) = (b, f) = 0$, we have by definition

$$f \in \mathcal{U}_{[a]}, \quad f \in \mathcal{U}_{[b]},$$

and hence by the formulas §15(3) and §8(3)

$$f \in \mathcal{U}_{[a]} \dot{+} \mathcal{U}_{[b]} = \mathcal{U}_{[a] \cup [b]} = \mathcal{U}_{[|a|+|b|]}.$$

On the other hand, as $|a| + |b| \geq |a+b|$, we have by Theorem 8.3 and by the formula §15(4)

$$\mathcal{U}_{[|a|+|b|]} \supset \mathcal{U}_{[a+b]}.$$

Consequently we have $f \in \mathcal{U}_{[a+b]}$, namely $(a+b, f) = 0$ by definition.

By virtue of the formula (6) we have obviously

$$(a \cup b, f) \geq \text{Max} \{ (a, f), (b, f) \}.$$

If $(a, f) \leq 0$, $(b, f) \leq 0$, then we have by the formula (3)

$$f \in \mathcal{U}_{[a+]}, \quad f \in \mathcal{U}_{[b+]},$$

and hence by the formulas §15(3) and §(3)

$$f \in \mathcal{U}_{[a+]} \dot{+} \mathcal{U}_{[b+]} = \mathcal{U}_{[a+] \cup [b+]} = \mathcal{U}_{[a \cup b+]}.$$

Since $a^+ \cup b^+ = (a \cup b)^+$ by Theorem 3.2, we obtain then

$$f \in \mathcal{U}_{[(a \cup b)^+]},$$

that is, $(a \cup b, f) \leq 0$ by the formula (3).

Furthermore, if $(a, f) = (b, f) = 0$, then we have by the formulas §15(3) and §8(1)

$$f \in \mathcal{U}_{[a-]} \mathcal{U}_{[b-]} = \mathcal{U}_{[a-] \cap [b-]} = \mathcal{U}_{[a \cap b-]}.$$

Since we have by Theorems 3.3 and 2.2

$$a^- \cap b^- = ((-a) \cup 0) \cap ((-b) \cup 0) = ((-a) \cap (-b)) \cup 0 = (a \cup b)^-,$$

we obtain then $f \in \mathcal{U}_{[(a \cup b)^-]}$, that is, $(a \cup b, f) = -\infty$ by definition. Therefore we have

$$(9) \quad (a \vee b, p) = \text{Max} \{ (a, p), (b, p) \}.$$

Thus we obtain further by the formula (7)

$$\begin{aligned} (a \wedge b, p) &= -((-a) \vee (-b), p) \\ &= -\text{Max} \{ (-a, p), (-b, p) \} = \text{Min} \{ (a, p), (b, p) \}, \end{aligned}$$

that is, we have

$$(10) \quad (a \wedge b, p) = \text{Min} \{ (a, p), (b, p) \}.$$

Theorem 17.1. For a point $p \in U_{[a]}$, if

$$(\lambda, a + b, p) = (a, p)$$

for some real number λ , then we also have

$$(\lambda a + b, p) = (a, p) \quad \text{for every } \lambda > \lambda_0.$$

Proof. If $(\lambda, a + b, p) \neq (a, p)$ for some real number $\lambda_0 > \lambda$, then, since by assumption

$$(\lambda, a + b, p) = (a, p) \neq 0$$

we conclude easily by definition

$$(\lambda, a + b, p) - (\lambda, a + b, p) = (a, p),$$

and hence by the formulas (7) and (8)

$$(\lambda, -\lambda, p)(a, p) = (a, p) \neq 0,$$

contradicting the assumption $\lambda, -\lambda, < 0$.

Theorem 17.2. For a point $p \in U_{[a]}$, if

$$(\lambda, a + b, p) = -(a, p)$$

for some real number λ , then we also have

$$(\lambda a + b, p) = -(a, p) \quad \text{for every } \lambda < \lambda_0.$$

Proof. For a point $p \in U_{[a]}$, if

$$(\lambda, a + b, p) = -(a, p),$$

then we have by the formula (7)

$$(-\lambda, a - b, p) = (a, p),$$

and hence by the previous theorem

$$(-\lambda a - b, p) = (a, p) \quad \text{for } -\lambda > -\lambda_0,$$

that is, we have

$$(\lambda a + b, p) = -(a, p) \quad \text{for every } \lambda < \lambda_0.$$

Theorem 17.3. If $f \in U_{[a]}$, then we have

$$(\lambda a + b, f) \neq 0$$

for all real numbers λ up to at most one value of λ

Proof. If $(\lambda_1 a + b, f) = (\lambda_2 a + b, f) = 0$, then we have by the formulas (7) and (8)

$$(\lambda_1 - \lambda_2)(a, f) = 0,$$

and hence $\lambda_1 = \lambda_2$, since $(a, f) \neq 0$ by assumption.

§18 Relative spectra

By virtue of Theorems 17.1, 17.2, and 17.3 we see easily that if $f \in U_{[a]}$, then to every element $b \in R$ there exists uniquely ξ , which may be $\pm \infty$ as well as a real number, such that

$$(\lambda a - b, f) = \begin{cases} (a, f) & \text{for } \lambda > \xi, \\ -(a, f) & \text{for } \lambda < \xi. \end{cases}$$

Such uniquely determined ξ is called the relative spectrum of an element $b \in R$ by an element $a \in R$ at a point f and denoted by

$$\left(\frac{b}{a}, f \right)$$

The relative spectrum $\left(\frac{b}{a}, f \right)$ is defined therefore only for $f \in U_{[a]}$.

With this definition we have obviously:

Theorem 18.1. If $(\lambda a - b, f) = (a, f) \neq 0$, then we have

$$\lambda \geq \left(\frac{b}{a}, f \right).$$

If $(\lambda a - b, f) = -(a, f) \neq 0$, then we have $\lambda \leq \left(\frac{b}{a}, f \right)$.

Theorem 18.2. For every real number α we have

$$\left(\frac{\alpha a}{a}, f \right) = \alpha \quad (f \in U_{[a]})$$

Proof. Since we have by the formula §17(7)

$$(\lambda a - \alpha a, f) = (\lambda - \alpha)(a, f)$$

we obtain obviously

$$(\lambda a - \alpha a, f) = \begin{cases} (a, f) & \text{for } \lambda > \alpha, \\ -(a, f) & \text{for } \lambda < \alpha, \end{cases}$$

and hence $(\frac{\alpha a}{a}, f) = \alpha$ by definition.

Theorem 18.3. $(t, f) = 0$ implies

$$(\frac{t}{a}, f) = 0 \quad \text{for } f \in \mathcal{U}_{[a]}.$$

Proof. If $(t, f) = 0$, then we have by the formulas

§17(7) and §17(8)

$$(\lambda a - t, f) = \lambda(a, f).$$

Therefore we have obviously

$$(\lambda a - t, f) = \begin{cases} (a, f) & \text{for } \lambda > 0, \\ -(a, f) & \text{for } \lambda < 0, \end{cases}$$

and hence $(\frac{t}{a}, f) = 0$ for $f \in \mathcal{U}_{[a]}$ by definition.

Theorem 18.4. For every point $f \in \mathcal{U}_{[p]} \mathcal{U}_{[a]}$ we have

$$(\frac{t}{a}, f) = (\frac{[p]t}{a}, f) = (\frac{t}{[p]a}, f) = (\frac{[p]t}{[p]a}, f).$$

Proof. For $f \in \mathcal{U}_{[p]} \mathcal{U}_{[a]}$ we have by the formula §17(5)

$$(\lambda a - t, f) = (\lambda[p]a - [p]t, f),$$

$$(a, f) = ([p]a, f),$$

and hence we obtain by definition

$$(\frac{t}{a}, f) = (\frac{[p]t}{[p]a}, f).$$

By virtue of this relation we obtain further

$$(\frac{[p]t}{a}, f) = (\frac{[p][p]t}{[p]a}, f) = (\frac{[p]t}{[p]a}, f),$$

$$(\frac{t}{[p]a}, f) = (\frac{[p]t}{[p][p]a}, f) = (\frac{[p]t}{[p]a}, f),$$

since $[p][p] = [p]$ by Theorem 7.2.

Theorem 18.5. For every real number α we have

$$(\frac{\alpha t}{a}, f) = \alpha(\frac{t}{a}, f) \quad (f \in \mathcal{U}_{[a]})$$

adopting the convention $0(\pm\infty) = 0$.

Proof. We shall consider first the case: $\alpha > 0$. In this case, if

$$\lambda > \alpha(\frac{t}{a}, f),$$

then we have naturally $\frac{\lambda}{\alpha} > (\frac{t}{a}, f)$, and hence by definition

$$(\frac{\lambda}{\alpha} a - t, f) = (a, f).$$

Therefore we obtain by the formula §17(7) that

$$(\lambda a - \alpha b, p) = (a, p) \quad \text{if } \lambda > \alpha \left(\frac{b}{a}, p\right).$$

We also can prove likewise that

$$(\lambda a - \alpha b, p) = -(a, p) \quad \text{if } \lambda < \alpha \left(\frac{b}{a}, p\right).$$

Consequently we have by definition

$$\left(\frac{\alpha b}{a}, p\right) = \alpha \left(\frac{b}{a}, p\right).$$

in the case: $\alpha > 0$. We also can dispose likewise of the case: $\alpha < 0$. The final case: $\alpha = 0$ is evident by Theorem 18.3.

Theorem 18.6. For a point $p \in U_{[a]}$, we have

$$\left(\frac{b+c}{a}, p\right) = \left(\frac{b}{a}, p\right) + \left(\frac{c}{a}, p\right)$$

if the right side has any sense.

Proof. If $\lambda > \left(\frac{b}{a}, p\right) + \left(\frac{c}{a}, p\right)$, then there exists obviously real numbers σ and ρ such that

$$\sigma > \left(\frac{b}{a}, p\right), \quad \rho > \left(\frac{c}{a}, p\right), \quad \lambda = \sigma + \rho.$$

For such σ and ρ we have by definition

$$(\sigma a - b, p) = (\rho a - c, p) = (a, p),$$

and hence by the formula §17(8)

$$(\lambda a - (b+c), p) = ((\sigma a - b) + (\rho a - c), p) = (a, p).$$

We also can prove likewise that if $\lambda < \left(\frac{b}{a}, p\right) + \left(\frac{c}{a}, p\right)$ then we have

$$(\lambda a - (b+c), p) = -(a, p).$$

Therefore we obtain $\left(\frac{b+c}{a}, p\right) = \left(\frac{b}{a}, p\right) + \left(\frac{c}{a}, p\right)$, by definition, if the right side has a sense.

Theorem 18.7. For every real number $\alpha \neq 0$ we have

$$\left(\frac{\alpha b}{a}, p\right) = \left(\frac{b}{a}, p\right) \quad (p \in U_{[a]}).$$

Proof. Since we have by the formula §17(7)

$$(\lambda \alpha a - \alpha b, p) = \alpha (\lambda a - b, p),$$

$$(\alpha a, p) = \alpha (a, p),$$

we obtain $\left(\frac{\alpha b}{a}, p\right) = \left(\frac{b}{a}, p\right)$ by definition, if $\alpha \neq 0$.

Theorem 18.8. For every point $p \in U_{[a+]}$, $b \geq c$ implies $(\frac{b}{a}, p) \geq (\frac{c}{a}, p)$.

Proof. For a point $p \in U_{[a+]}$, if $\lambda > (\frac{b}{a}, p)$, then we have by definition

$$(\lambda a - b, p) = (a, p) = +\infty.$$

For $c \leq b$, since $\lambda a - c \geq \lambda a - b$, we obtain hence by the formula §17(6)

$$(\lambda a - c, p) = +\infty = (a, p),$$

and consequently $\lambda \geq (\frac{c}{a}, p)$ by Theorem 18.1. Therefore we have

$$(\frac{b}{a}, p) \geq (\frac{c}{a}, p) \quad \text{for } b \geq c.$$

Theorem 18.9. For every point $p \in U_{[a+]}$ we have

$$(\frac{b \vee c}{a}, p) = \text{Max} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \},$$

$$(\frac{b \wedge c}{a}, p) = \text{Min} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \}.$$

Proof. For every point $p \in U_{[a+]}$ we have obviously by the previous theorem

$$(\frac{b \vee c}{a}, p) \geq \text{Max} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \}.$$

If $\lambda > \text{Max} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \}$, then we have by definition

$$(\lambda a - b, p) = (\lambda a - c, p) = (a, p),$$

and hence by the formula §17(10)

$$((\lambda a - b) \wedge (\lambda a - c), p) = (a, p).$$

Since $(\lambda a - b) \wedge (\lambda a - c) = \lambda a - (b \vee c)$ by Theorems 2.2 and 2.4, we obtain then

$$(\lambda a - (b \vee c), p) = (a, p),$$

and consequently $\lambda \geq (\frac{b \vee c}{a}, p)$ by Theorem 18.1. Therefore we have for every point $p \in U_{[a+]}$

$$(\frac{b \vee c}{a}, p) = \text{Max} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \}.$$

From this relation we can conclude by Theorem 18.5

$$\begin{aligned} (\frac{b \wedge c}{a}, p) &= -(\frac{(b \vee c)}{a}, p) \\ &= -\text{Max} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \} \\ &= \text{Min} \{ (\frac{b}{a}, p), (\frac{c}{a}, p) \}. \end{aligned}$$

Theorem 18.10. For every point $\mathcal{P} \in \mathcal{U}_{[a]}$ we have

$$|(\frac{b}{a}, \mathcal{P})| = |(\frac{|b|}{a}, \mathcal{P})| = |(\frac{b}{|a|}, \mathcal{P})| = (\frac{|b|}{|a|}, \mathcal{P}).$$

Proof. By virtue of Theorem 18.6 we have

$$(\frac{b}{a}, \mathcal{P}) = (\frac{b^+}{a}, \mathcal{P}) - (\frac{b^-}{a}, \mathcal{P}),$$

$$(\frac{|b|}{a}, \mathcal{P}) = (\frac{b^+}{a}, \mathcal{P}) + (\frac{b^-}{a}, \mathcal{P}),$$

and further by Theorem 18.3

$$(\frac{b^+}{a}, \mathcal{P}) = 0 \quad \text{or} \quad (\frac{b^-}{a}, \mathcal{P}) = 0 \quad (\mathcal{P} \in \mathcal{U}_{[a]}),$$

since $\mathcal{U}_{[b^+]} \cap \mathcal{U}_{[b^-]} = \emptyset$ by the formula §15(5). Therefore we have

$$|(\frac{b}{a}, \mathcal{P})| = |(\frac{|b|}{a}, \mathcal{P})| \quad (\mathcal{P} \in \mathcal{U}_{[a]}).$$

For $\mathcal{P} \in \mathcal{U}_{[a^+]}$ we have by Theorem 18.4

$$(\frac{b}{a}, \mathcal{P}) = (\frac{b}{[a^+]a}, \mathcal{P}) = (\frac{b}{[a^+]|a|}, \mathcal{P}) = (\frac{b}{|a|}, \mathcal{P}),$$

since $[a^+]a = a^+ = [a^+]|a|$ by Theorem 7.12; and for $\mathcal{P} \in \mathcal{U}_{[a^-]}$

we have by Theorems 18.4 and 18.7

$$(\frac{b}{a}, \mathcal{P}) = (\frac{b}{[a^-]a}, \mathcal{P}) = (\frac{b}{-[a^-]|a|}, \mathcal{P}) = -(\frac{b}{|a|}, \mathcal{P}),$$

since $[a^-]a = -a^- = -[a^-]|a|$ by Theorem 7.12. Therefore we have

$$|(\frac{b}{a}, \mathcal{P})| = |(\frac{b}{|a|}, \mathcal{P})| \quad (\mathcal{P} \in \mathcal{U}_{[a]})$$

Since $(\frac{|b|}{|a|}, \mathcal{P}) \geq 0$ by Theorems 18.8 and 18.3, we obtain hence

$$|(\frac{b}{a}, \mathcal{P})| = |(\frac{|b|}{|a|}, \mathcal{P})| = (\frac{|b|}{|a|}, \mathcal{P}).$$

Theorem 18.11. $|b| \leq \alpha |a|$ implies

$$|(\frac{b}{a}, \mathcal{P})| \leq \alpha \quad (\mathcal{P} \in \mathcal{U}_{[a]}).$$

Proof. If $|b| \leq \alpha |a|$, then we have by Theorems 18.10,

18.8, and 18.2

$$|(\frac{b}{a}, \mathcal{P})| = (\frac{|b|}{|a|}, \mathcal{P}) \leq (\frac{\alpha |a|}{|a|}, \mathcal{P}) = \alpha.$$

Theorem 18.12. For a point $\mathcal{P} \in \mathcal{U}_{[a]} \cap \mathcal{U}_{[b]}$ we have

$$(\frac{c}{a}, \mathcal{P}) = (\frac{c}{b}, \mathcal{P})(\frac{b}{a}, \mathcal{P})$$

if the right side has any sense.

Proof. By virtue of Theorems 18.5 and 18.7 we can assume

without loss of generality that

$$(\frac{c}{b}, \mathcal{P}) \geq 0, \quad (\frac{b}{a}, \mathcal{P}) \geq 0.$$

We shall consider first the case: $(a, \mathcal{P}) = (b, \mathcal{P})$. In this

case, if for a real number λ

$$\lambda > (\frac{c}{f}, f)(\frac{b}{a}, f),$$

then there exist real numbers ρ and σ such that

$$\rho > (\frac{c}{f}, f), \quad \sigma > (\frac{b}{a}, f), \quad \lambda = \rho\sigma,$$

and we have by definition

$$(\rho b - c, f) = (b, f), \quad (\sigma a - b, f) = (a, f).$$

These relations yield by the formulas §17(7) and §17(8)

$$\begin{aligned} (\lambda a - c, f) &= (\rho(\sigma a - b) + (\rho b - c), f) \\ &= \rho(a, f) + (b, f) = (a, f). \end{aligned}$$

If $0 < \lambda < (\frac{c}{f}, f)(\frac{b}{a}, f)$, then there exist real numbers

ρ and σ such that

$$0 < \rho < (\frac{c}{f}, f), \quad 0 < \sigma < (\frac{b}{a}, f), \quad \lambda = \rho\sigma,$$

and we have by definition

$$(\rho b - c, f) = -(b, f), \quad (\sigma a - b, f) = -(a, f).$$

From these relations we conclude likewise

$$(\lambda a - c, f) = -(a, f).$$

If $(\frac{c}{f}, f)(\frac{b}{a}, f) = 0$, then we have naturally, by assumption

$$0 \leq (\frac{b}{a}, f) < +\infty,$$

and hence to any negative number λ there exist obviously real

numbers ρ and σ such that

$$\rho < 0, \quad \sigma > (\frac{b}{a}, f), \quad \lambda = \rho\sigma.$$

Then we have by definition

$$(\rho b - c, f) = -(b, f), \quad (\sigma a - b, f) = (a, f);$$

These relations yield by the formulas §17(7) and §17(8)

$$(\lambda a - c, f) = \rho(a, f) - (b, f) = -(a, f),$$

since $\rho < 0$ and $(a, f) = (b, f)$ by assumption. Therefore

we have by definition

$$(\frac{c}{a}, f) = (\frac{c}{f}, f)(\frac{b}{a}, f).$$

In the other case: $(a, f) = -(b, f)$, we have by Theorem 18.1

$$(\frac{b}{a}, f) = 0,$$

because $(\sigma a - b, f) = (a, f)$ by the formula §17(7). Therefore

we have by assumption

$$0 \leq (\frac{c}{\varepsilon}, p) < +\infty, \quad (\frac{c}{\varepsilon}, p)(\frac{b}{a}, p) = 0.$$

For any positive number λ , putting

$$p < 0, \quad \sigma < 0, \quad \lambda = p\sigma,$$

we have by definition

$$(p\ell - c) = -(\ell, p), \quad (\sigma a - b, p) = -(a, p).$$

These relations yield by the formulas §17(7) and §17(8)

$$(\lambda a - c, p) = -p(a, p) - (\ell, p) = (a, p),$$

because $-p(a, p) = -(\ell, p) = (a, p)$ by assumption.

To any negative number λ there exist obviously real numbers

p and σ such that

$$p > (\frac{c}{\ell}, p), \quad \sigma < 0, \quad \lambda = p\sigma,$$

and we have by definition

$$(p\ell - c, p) = (\ell, p), \quad (\sigma a - b, p) = -(a, p).$$

From these relations we conclude by the formulas §17(7) and §17(8)

$$(\lambda a - c, p) = -p(a, p) + (\ell, p) = -(a, p),$$

because $-p(a, p) = (\ell, p) = -(a, p)$ by assumption. Therefore

we have by definition

$$(\frac{c}{a}, p) = 0.$$

§19 Relative spectra as functions of p

Every relative spectrum $(\frac{b}{a}, p)$ may be considered as a function on $\mathcal{U}_{[a]}$ in the proper space E of R

Theorem 19.1. For a positive element $a \in R$, if

$[p] \leq [a]$ and

$$(\frac{b}{a}, p) \geq 0 \quad \text{for every point } p \in \mathcal{U}_{[p]},$$

then we have $[p]b \geq 0$.

Proof. If $(\frac{b}{a}, p) \geq 0$ for every $p \in \mathcal{U}_{[p]}$, then for any positive number ε we have by definition

$(-\varepsilon a - b, f) = -(a, f) = -\infty$ for every $f \in \mathcal{U}_{[p]}$,
 since $a \geq 0$ by assumption, that is, $f \in \mathcal{U}_{[p]}$ implies

$$f \in \mathcal{U}_{[(-\varepsilon a - b)^-]}$$

by definition. Therefore we have obviously

$$\mathcal{U}_{[p]} \subset \mathcal{U}_{[(-\varepsilon a - b)^-]},$$

and hence by the formula §15(4)

$$[p] \leq [(-\varepsilon a - b)^-].$$

Consequently we obtain by Theorems 8.2, 7.9, and 7.12

$$p = [(-\varepsilon a - b)^-] p \perp (-\varepsilon a - b)^+.$$

This relation yields by Theorem 7.12

$$[p](-\varepsilon a - b) = -[p](-\varepsilon a - b)^- \leq 0,$$

that is, we have for any positive number ε

$$-[p]b \leq \varepsilon [p]a.$$

Therefore we obtain by Theorem 6.3

$$-[p]b \leq 0, \text{ namely } [p]b \geq 0.$$

Theorem 19.2. Every relative spectrum $(\frac{b}{a}, f)$ is a continuous function on $\mathcal{U}_{[a]}$.

Proof. For an arbitrary point $f_0 \in \mathcal{U}_{[a]}$, if

$$(\frac{b}{a}, f_0) < \lambda,$$

then we have by definition

$$(\lambda a - b, f_0) = (a, f_0).$$

In the case: $(a, f_0) = +\infty$, we have then by definition

$$f_0 \in \mathcal{U}_{[a^+]} \mathcal{U}_{[(\lambda a - b)^+]},$$

$$(\lambda a - b, f) = (a, f) \text{ for every } f \in \mathcal{U}_{[a^+]} \mathcal{U}_{[(\lambda a - b)^+]},$$

and hence by Theorem 18.1

$$(\frac{b}{a}, f) \leq \lambda \text{ for every } f \in \mathcal{U}_{[a^+]} \mathcal{U}_{[(\lambda a - b)^+]}.$$

In the other case: $(a, f_0) = -\infty$, we also obtain likewise

$$f_0 \in \mathcal{U}_{[a^-]} \mathcal{U}_{[(\lambda a - b)^-]},$$

$$(\frac{b}{a}, f) \leq \lambda \text{ for every } f \in \mathcal{U}_{[a^-]} \mathcal{U}_{[(\lambda a - b)^-]}.$$

Therefore $(\frac{b}{a}, f)$ is upper semi-continuous at f_0 . Since the point $f_0 \in \mathcal{U}_{[a]}$ may be arbitrary, $(\frac{b}{a}, f)$ is hence

upper semi-continuous in $\mathcal{U}_{[a]}$.

Since $(\frac{-\ell}{a}, \mathfrak{f})$ is upper semi-continuous in $\mathcal{U}_{[a]}$, as proved just now, we see easily by Theorem 18.5 that $(\frac{\ell}{a}, \mathfrak{f})$ is lower semi-continuous in $\mathcal{U}_{[a]}$. Therefore $(\frac{\ell}{a}, \mathfrak{f})$ is continuous in $\mathcal{U}_{[a]}$.

Theorem 19.3. Every relative spectrum $(\frac{\ell}{a}, \mathfrak{f})$ is almost finite in $\mathcal{U}_{[a]}$, that is, $(\frac{\ell}{a}, \mathfrak{f})$ is finite in an open set being dense in $\mathcal{U}_{[a]}$.

Proof. Since $(\frac{\ell}{a}, \mathfrak{f})$ is continuous in $\mathcal{U}_{[a]}$ by the previous theorem, the point set

$$A = \{ \mathfrak{f} : |(\frac{\ell}{a}, \mathfrak{f})| < +\infty \}$$

is an open set included in $\mathcal{U}_{[a]}$.

For any neighbourhood $\mathcal{U}_{[\mathfrak{p}]} \subset \mathcal{U}_{[a]} - A^-$ we have naturally

$$|(\frac{\ell}{a}, \mathfrak{f})| = +\infty \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[\mathfrak{p}]},$$

and hence by Theorems 18.6, 18.5, 18.2, and 18.10

$$(\frac{|\ell| - \nu|a|}{|a|}, \mathfrak{f}) = (\frac{|\ell|}{|a|}, \mathfrak{f}) - \nu = |(\frac{\ell}{a}, \mathfrak{f})| - \nu = +\infty$$

for every $\mathfrak{f} \in \mathcal{U}_{[\mathfrak{p}]}$ and $\nu = 1, 2, \dots$. From this relation we conclude by Theorem 19.1

$$[p](|\ell| - \nu|a|) \geq 0,$$

that is, $\frac{1}{\nu}[p]|\ell| \geq [p]|a|$ for every $\nu = 1, 2, \dots$, and hence

$[p]|a| = 0$ by Theorem 6.3. As $[p] \leq [a]$ by the formula §15(4), we have consequently by Theorem 8.2 and the formula §8(1)

$$[p] = [p][a] = [p]|a| = 0,$$

that is, $\mathcal{U}_{[\mathfrak{p}]} = 0$. Since $\mathcal{U}_{[a]} - A^-$ is obviously open, we obtain therefore $A^- = \mathcal{U}_{[a]}$, that is, A is dense in $\mathcal{U}_{[a]}$.

Theorem 19.4. For a positive element $a \in R$, if

$$[p] \leq [a],$$

$$(\frac{\ell}{a}, \mathfrak{f}) \geq (\frac{c}{a}, \mathfrak{f}) \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[\mathfrak{p}]},$$

then we have $[p]\ell \geq [p]c$.

Proof. By virtue of the previous theorem there exists an open set A being dense in $\mathcal{U}_{[a]}$ such that

$$|(\frac{f}{a}, p)| < +\infty \quad \text{for every } p \in A.$$

Then for every point $p \in A \cap \mathcal{U}_{[p]}$ we have by Theorems 18.5 and 18.6

$$(\frac{f-c}{a}, p) = (\frac{f}{a}, p) - (\frac{c}{a}, p) \geq 0.$$

Since $(\frac{f-c}{a}, p)$ is continuous in $\mathcal{U}_{[a]}$ by Theorem 19.2, and the point set $A \cap \mathcal{U}_{[p]}$ is dense in $\mathcal{U}_{[p]}$, we have further

$$(\frac{f-c}{a}, p) \geq 0 \quad \text{for every } p \in \mathcal{U}_{[p]},$$

and hence by Theorem 19.1

$$[p](f-c) \geq 0, \text{ namely } [p]f \geq [p]c.$$

Theorem 19.5. For a projector $[p] \leq [a]$ if

$$(\frac{f}{a}, p) = (\frac{c}{a}, p) \quad \text{for every } p \in \mathcal{U}_{[p]},$$

then we have $[p]f = [p]c$.

Proof. By assumption we have naturally

$$(\frac{f}{a}, p) = (\frac{c}{a}, p) \quad \text{for every } p \in \mathcal{U}_{[p]} \cap \mathcal{U}_{[a^+]},$$

$$(\frac{f}{a}, p) = (\frac{c}{a}, p) \quad \text{for every } p \in \mathcal{U}_{[p]} \cap \mathcal{U}_{[a^-]}.$$

For every $p \in \mathcal{U}_{[p]} \cap \mathcal{U}_{[a^+]}$, since $[a^+]a = a^+$, we have then by Theorem 18.4

$$(\frac{f}{a^+}, p) = (\frac{c}{a^+}, p),$$

and hence $[p][a^+]f = [p][a^+]c$ by the previous theorem.

Since $[a^-]a = -a^-$, we also have by Theorems 18.4, 18.5, 18.7

$$(\frac{f}{a^-}, p) = (\frac{c}{a^-}, p) \quad \text{for every } p \in \mathcal{U}_{[p]} \cap \mathcal{U}_{[a^-]},$$

and hence $[p][a^-]f = [p][a^-]c$ by the previous theorem.

Therefore we obtain $[p]f = [p]c$, because we have by Theorems 8.2 and 8.4

$$[p]([a^+] + [a^-]) = [p][a] = [p].$$

Since $|(\frac{f}{a}, p)| = |(\frac{f}{|a|}, p)|$ by Theorem 18.10, we obtain immediately by Theorems 19.4, 18.11, and 18.4:

Theorem 19.6. $|(\frac{f}{a}, p)| \leq \alpha$ for every $p \in \mathcal{U}_{[a]} \cap \mathcal{U}_{[p]}$,

if and only if $|[a][p]f| \leq \alpha [p]|a|$.

Theorem 19.7. For a positive element $a \in R$ we have

$$\mathcal{U}_{[(\lambda a - f)^+][a]} = \{ p : (\frac{f}{a}, p) < \lambda \}^-,$$

$$\mathcal{U}_{[(\lambda a - b)^+][a]} = \{f : (\frac{b}{a}, f) > \lambda\}^-.$$

Proof. If $(\frac{b}{a}, f) < \lambda$, then, as $a \geq 0$ by assumption, we have by definition

$$(\lambda a - b, f) = (a, f) = +\infty,$$

and hence $f \in \mathcal{U}_{[(\lambda a - b)^+][a]} \cap \mathcal{U}_{[a]}$. Thus we conclude

$$\{f : (\frac{b}{a}, f) < \lambda\} \subset \mathcal{U}_{[(\lambda a - b)^+][a]},$$

and consequently

$$\{f : (\frac{b}{a}, f) < \lambda\}^- \subset \mathcal{U}_{[(\lambda a - b)^+][a]}.$$

For every point $f \in \mathcal{U}_{[(\lambda a - b)^+][a]}$, we have by definition

$$(\lambda a - b, f) = +\infty = (a, f)$$

and hence $(\frac{b}{a}, f) \leq \lambda$ by Theorem 18.1. Thus we conclude for every real number λ

$$\mathcal{U}_{[(\lambda a - b)^+][a]} \subset \{f : (\frac{b}{a}, f) \leq \lambda\}.$$

Therefore we obtain

$$\begin{aligned} \{f : (\frac{b}{a}, f) < \lambda\} &= \bigcup_{\nu=1}^{\infty} \{f : (\frac{b}{a}, f) \leq \lambda - \frac{1}{\nu}\} \\ &> \bigcup_{\nu=1}^{\infty} \mathcal{U}_{[(\lambda - \frac{1}{\nu})a - b]^+[a]}. \end{aligned}$$

On the other hand, since $((\lambda - \frac{1}{\nu})a - b)^+ \uparrow_{\nu=1}^{\infty} (\lambda a - b)^+$ by Theorem 5.10, we have by Theorems 8.5 and 8.8

$$[(\lambda - \frac{1}{\nu})a - b]^+[a] \uparrow_{\nu=1}^{\infty} [(\lambda a - b)^+][a],$$

and consequently by Theorem 15.5

$$\begin{aligned} \mathcal{U}_{[(\lambda a - b)^+][a]} &= \left(\bigcup_{\nu=1}^{\infty} \mathcal{U}_{[(\lambda - \frac{1}{\nu})a - b]^+[a]} \right)^- \\ &\subset \{f : (\frac{b}{a}, f) < \lambda\}^-. \end{aligned}$$

Therefore we obtain

$$\mathcal{U}_{[(\lambda a - b)^+][a]} = \{f : (\frac{b}{a}, f) < \lambda\}^-.$$

From this relation we conclude further by Theorem 18.5

$$\begin{aligned} \mathcal{U}_{[(\lambda a - b)^-][a]} &= \mathcal{U}_{[(-\lambda a + b)^+][a]} \\ &= \{f : (\frac{-b}{a}, f) < -\lambda\}^- \\ &= \{f : (\frac{b}{a}, f) > \lambda\}^-. \end{aligned}$$

§20 Integral

Let $\varphi(\mathcal{F})$ be a bounded continuous function on a neighbourhood $\mathcal{U}_{[p]}$. To any positive number ε there exists a system of neighbourhoods $\mathcal{U}_{[p_\lambda]}$ ($\lambda \in \Lambda$) such that

$$\operatorname{osc}_{\mathcal{F} \in \mathcal{U}_{[p_\lambda]}} \varphi(\mathcal{F}) \leq \varepsilon \quad \text{for every } \lambda \in \Lambda,$$

$$\mathcal{U}_{[p]} = \sum_{\lambda \in \Lambda} \mathcal{U}_{[p_\lambda]}.$$

Since $\mathcal{U}_{[p]}$ is compact by Theorem 16.2, there exists hence a finite number of projectors $[q_\nu]$ ($\nu = 1, 2, \dots, \kappa$) such that

$$\operatorname{osc}_{\mathcal{F} \in \mathcal{U}_{[q_\nu]}} \varphi(\mathcal{F}) \leq \varepsilon \quad \text{for every } \nu = 1, 2, \dots, \kappa,$$

$$\mathcal{U}_{[p]} = \mathcal{U}_{[q_1]} \dot{+} \mathcal{U}_{[q_2]} \dot{+} \dots \dot{+} \mathcal{U}_{[q_\kappa]}.$$

For such $[q_\nu]$ ($\nu = 1, 2, \dots, \kappa$), putting

$$[p_1] = [q_1], \quad [p_\nu] = [q_\nu](1 - ([p_1] \vee \dots \vee [p_{\nu-1}])) \quad (\nu = 2, 3, \dots, \kappa),$$

we obtain obviously a partition of the projector $[p]$:

$$[p] = [p_1] + [p_2] + \dots + [p_\kappa],$$

such that $\operatorname{osc}_{\mathcal{F} \in \mathcal{U}_{[p_\nu]}} \varphi(\mathcal{F}) \leq \varepsilon$ for every $\nu = 1, 2, \dots, \kappa$. Here

$$[p] = [p_1] + [p_2] + \dots + [p_\kappa]$$

implies $[p_\nu][p_\mu] = 0$ for $\nu \neq \mu$, as stated in Theorem 8.4.

For any two partitions of $[p]$:

$$[p] = \sum_{\nu=1}^{\kappa} [p_\nu], \quad \operatorname{osc}_{\mathcal{F} \in \mathcal{U}_{[p_\nu]}} \varphi(\mathcal{F}) \leq \varepsilon \quad (\nu = 1, 2, \dots, \kappa),$$

$$[p] = \sum_{\mu=1}^{\sigma} [q_\mu], \quad \operatorname{osc}_{\mathcal{F} \in \mathcal{U}_{[q_\mu]}} \varphi(\mathcal{F}) \leq \varepsilon \quad (\mu = 1, 2, \dots, \sigma),$$

and for $\mathcal{F}_\nu \in \mathcal{U}_{[p_\nu]}, \mathcal{F}_\mu \in \mathcal{U}_{[q_\mu]}$ ($\nu = 1, 2, \dots, \kappa; \mu = 1, 2, \dots, \sigma$),

we have

$$\begin{aligned} \sum_{\nu=1}^{\kappa} \varphi(\mathcal{F}_\nu)[p_\nu]a - \sum_{\mu=1}^{\sigma} \varphi(\mathcal{F}_\mu)[q_\mu]a \\ = \sum_{\nu, \mu} \{ \varphi(\mathcal{F}_\nu) - \varphi(\mathcal{F}_\mu) \} [p_\nu][q_\mu]a, \end{aligned}$$

since $[p_\nu] = \sum_{\mu=1}^{\sigma} [p_\nu][q_\mu], [q_\mu] = \sum_{\nu=1}^{\kappa} [p_\nu][q_\mu]$. If

$$[p_\nu][q_\mu] \neq 0,$$

then we have $\mathcal{U}_{[p_\nu]} \mathcal{U}_{[q_\mu]} \neq \emptyset$ by the formula §15(5), and hence

$$|\varphi(\mathcal{F}_\nu) - \varphi(\mathcal{F}_\mu)| \leq 2\varepsilon$$

for such ν and μ . Therefore we obtain

$$\begin{aligned} \left| \sum_{\nu=1}^{\kappa} \varphi(\mathcal{F}_\nu)[p_\nu]a - \sum_{\mu=1}^{\sigma} \varphi(\mathcal{F}_\mu)[q_\mu]a \right| \\ \leq 2\varepsilon \sum_{\nu, \mu} [p_\nu][q_\mu]|a| = 2\varepsilon [p]|a|. \end{aligned}$$

By virtue of Theorem 6.4 we conclude hence that to every element $a \in R$ there exists uniquely an element $b \in R$ such that

$$\lim_{\mu \rightarrow \infty} \sum_{\nu=1}^{x_\mu} \varphi(\xi_{\mu,\nu}) [p_{\mu,\nu}] a = b$$

for every sequence of partitions of $[p]$:

$$[p] = \sum_{\nu=1}^{x_\mu} [p_{\mu,\nu}] \quad (\mu = 1, 2, \dots)$$

such that

$$\begin{aligned} \sup_{\xi \in \mathcal{U}_{[p_{\mu,\nu}]}} \varphi(\xi) &\leq \varepsilon_\mu \quad (\nu = 1, 2, \dots, x_\mu; \mu = 1, 2, \dots), \\ \lim_{\mu \rightarrow \infty} \varepsilon_\mu &= 0, \end{aligned}$$

and for arbitrary $\xi_{\mu,\nu} \in \mathcal{U}_{[p_{\mu,\nu}]}$ ($\nu = 1, 2, \dots, x_\mu; \mu = 1, 2, \dots$).

Such an element $b \in R$ is called the integral of a function $\varphi(\xi)$ by an element $a \in R$ in $\mathcal{U}_{[p]}$ and denoted by

$$\int_{[p]} \varphi(\xi) d\xi a,$$

that is,

$$\int_{[p]} \varphi(\xi) d\xi a = \lim_{\varepsilon \rightarrow 0} \sum_{\nu=1}^x \varphi(\xi_\nu) [p_\nu] a$$

for

$$[p] = [p_1] + [p_2] + \dots + [p_x],$$

$$\sup_{\xi \in \mathcal{U}_{[p_\nu]}} \varphi(\xi) \leq \varepsilon, \quad \xi_\nu \in \mathcal{U}_{[p_\nu]} \quad (\nu = 1, 2, \dots, x),$$

and we have for such a partition of $[p]$

$$\left| \sum_{\nu=1}^x \varphi(\xi_\nu) [p_\nu] a - \int_{[p]} \varphi(\xi) d\xi a \right| \leq 2\varepsilon [p] |a|.$$

For integral of bounded continuous functions we have following properties. It is obvious by definition:

$$(1) \quad \int_{[p]} d\xi a = [p] a.$$

Recalling Theorem 7.7, since

$$\begin{aligned} [q] \sum_{\nu=1}^x \varphi(\xi_\nu) [p_\nu] a &= \sum_{\nu=1}^x \varphi(\xi_\nu) [p_\nu] [q] a, \\ \sum_{\nu=1}^x [p_\nu] [q] &= [p] [q], \end{aligned}$$

we have obviously by definition

$$(2) \quad [q] \int_{[p]} \varphi(\xi) d\xi a = \int_{[p]} \varphi(\xi) d\xi [q] a = \int_{[p][q]} \varphi(\xi) d\xi a.$$

We have for $[p] = [q] + [r]$

$$(3) \quad \int_{[p]} \varphi(\xi) d\xi a = \int_{[q]} \varphi(\xi) d\xi a + \int_{[r]} \varphi(\xi) d\xi a.$$

Because we have by the formula (2)

$$\int_{[q]} \varphi(f) dfa = [q] \int_{[p]} \varphi(f) dfa, \quad \int_{[r]} \varphi(f) dfa = [r] \int_{[p]} \varphi(f) dfa.$$

We obtain immediately by definition

$$(4) \quad \int_{[p]} (\alpha \varphi(f) + \beta \psi(f)) dfa = \alpha \int_{[p]} \varphi(f) dfa + \beta \int_{[p]} \psi(f) dfa,$$

$$(5) \quad \int_{[p]} \varphi(f) dfa (\alpha a + \beta b) = \alpha \int_{[p]} \varphi(f) dfa + \beta \int_{[p]} \varphi(f) dfa b.$$

Since we have by Theorem 4.5

$$\left| \sum_{\nu=1}^N \varphi(f_\nu) [p_\nu] a \right| = \sum_{\nu=1}^N |\varphi(f_\nu)| [p_\nu] |a|,$$

we conclude by definition

$$(6) \quad \left| \int_{[p]} \varphi(f) dfa \right| = \int_{[p]} |\varphi(f)| dfa |a|.$$

We have obviously by definition:

Theorem 20.1. For a positive element $a \in R$, if

$$\varphi(f) \geq \psi(f) \quad \text{for every } f \in \mathcal{U}_{[p]},$$

then we have

$$\int_{[p]} \varphi(f) dfa \geq \int_{[p]} \psi(f) dfa.$$

By virtue of this theorem and the formulas (1), (6) we obtain immediately:

Theorem 20.2. If $|\varphi(f)| \leq \alpha$ for every $f \in \mathcal{U}_{[p]}$,

then we have

$$\left| \int_{[p]} \varphi(f) dfa \right| \leq \alpha [p] |a|.$$

We see easily by definition:

Theorem 20.3. If $[p] = \sum_{\mu=1}^m [p_{\mu, \nu}]$ ($\mu=1, 2, \dots$),

$$\sup_{f \in \mathcal{U}_{[p_{\mu, \nu}]}} \varphi(f) \leq \frac{1}{\mu}, \quad \varphi(f_{\mu, \nu}) = \inf_{f \in \mathcal{U}_{[p_{\mu, \nu}]}} \varphi(f), \quad f_{\mu, \nu} \in \mathcal{U}_{[p_{\mu, \nu}]},$$

$$[p_{\mu, \nu}][p_{\mu, \sigma}] = [p_{\mu, \sigma}] \quad \text{or} \quad 0 \quad \text{for } \mu > \nu,$$

then we have for every positive element $a \in R$

$$\sum_{\mu=1}^m \varphi(f_{\mu, \nu}) [p_{\mu, \nu}] a \uparrow_{\mu=1}^{\infty} \int_{[p]} \varphi(f) dfa.$$

Furthermore we have obviously by definition:

Theorem 20.4. If $\varphi(f) \geq 0$ for every $f \in \mathcal{U}_{[p]}$, then

$a \geq b$ implies

$$\int_{[p]} \varphi(f) dfa \geq \int_{[p]} \varphi(f) dfa b.$$

§21 Integration of unbounded continuous functions

We shall now consider integration of unbounded continuous functions. Let $\varphi(\mathfrak{p})$ be a continuous function on a neighbourhood $\mathcal{U}_{[\mathfrak{p}]}$, but not necessarily bounded. Putting for $\nu=1, 2, \dots$

$$\varphi_\nu(\mathfrak{p}) \begin{cases} \nu & \text{for } \varphi(\mathfrak{p}) > \nu, \\ \varphi(\mathfrak{p}) & \text{for } -\nu \leq \varphi(\mathfrak{p}) \leq \nu, \\ -\nu & \text{for } \varphi(\mathfrak{p}) < -\nu, \end{cases}$$

we obtain a sequence of bounded continuous functions $\varphi_\nu(\mathfrak{p})$ ($\nu=1, 2, \dots$) on $\mathcal{U}_{[\mathfrak{p}]}$. If there exists the limit

$$b = \lim_{\nu \rightarrow \infty} \int_{[\mathfrak{p}]} \varphi_\nu(\mathfrak{p}) d\mathfrak{p} a,$$

then we shall say that $\varphi(\mathfrak{p})$ is integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[\mathfrak{p}]}$, or that the integral

$$b = \int_{[\mathfrak{p}]} \varphi(\mathfrak{p}) d\mathfrak{p} a$$

is convergent.

With this definition we obtain immediately by the formula §20(2):

Theorem 21.1. If $\varphi(\mathfrak{p})$ is integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[\mathfrak{p}]}$, then we have for every projector $[q]$

$$[q] \int_{[\mathfrak{p}]} \varphi(\mathfrak{p}) d\mathfrak{p} a = \int_{[\mathfrak{p}]} \varphi(\mathfrak{p}) d\mathfrak{p} [q] a = \int_{[\mathfrak{p}][q]} \varphi(\mathfrak{p}) d\mathfrak{p} a,$$

and all appearing integrals are convergent.

We see easily by the formula §20(6):

Theorem 21.2. If $\varphi(\mathfrak{p})$ is integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[\mathfrak{p}]}$, then $|\varphi(\mathfrak{p})|$ is integrable by $|a|$ in $\mathcal{U}_{[\mathfrak{p}]}$ and

$$\left| \int_{[\mathfrak{p}]} \varphi(\mathfrak{p}) d\mathfrak{p} a \right| = \int_{[\mathfrak{p}]} |\varphi(\mathfrak{p})| d\mathfrak{p} |a|.$$

Conversely we have:

Theorem 21.3. If $|\varphi(\mathfrak{p})|$ is integrable by $|a|$ in $\mathcal{U}_{[\mathfrak{p}]}$, then $\varphi(\mathfrak{p})$ is integrable by a in $\mathcal{U}_{[\mathfrak{p}]}$.

Proof. For a continuous function $\varphi(\mathfrak{p})$ on $\mathcal{U}_{[\mathfrak{p}]}$, since

$$\{\mathfrak{p} : \varphi(\mathfrak{p}) > 0\} = \bigcup_{t=1}^{\infty} \{\mathfrak{p} : \varphi(\mathfrak{p}) \geq \frac{1}{t}\},$$

the point set $\{\mathfrak{p} : \varphi(\mathfrak{p}) > 0\}$ is σ -open, and hence its

closure is open by Theorem 16.7. Therefore there exists by

Theorem 16.4 a projector $[q]$ such that

$$\mathcal{U}_{[q]} = \mathcal{U}_{[p]} \{f: \varphi(f) > 0\}^-.$$

For such $[q]$ we have obviously

$$\varphi(f) \begin{cases} \geq 0 & \text{for } f \in \mathcal{U}_{[q]}, \\ \leq 0 & \text{for } f \in \mathcal{U}_{[p]} - \mathcal{U}_{[q]}. \end{cases}$$

Thus, for the sequence of bounded continuous functions $\varphi_\nu(f)$ ($\nu = 1, 2, \dots$) defined just above, we have by the formulas (3), (2), (4), and (5) in §20

$$\begin{aligned} \int_{[p]} \varphi_\nu(f) d\mathfrak{f} a &= \int_{[q]} |\varphi_\nu(f)| d\mathfrak{f} a - \int_{[p] - [q]} |\varphi_\nu(f)| d\mathfrak{f} a \\ &= [q] \int_{[p]} |\varphi_\nu(f)| d\mathfrak{f} a - ([p] - [q]) \int_{[p]} |\varphi_\nu(f)| d\mathfrak{f} a \\ &= (2[q] - [p]) \int_{[p]} |\varphi_\nu(f)| d\mathfrak{f} ([a^+] |a| - [a^-] |a|) \\ &= (2[q] - [p]) ([a^+] - [a^-]) \int_{[p]} |\varphi_\nu(f)| d\mathfrak{f} |a|. \end{aligned}$$

Since projectors are continuous by Theorem 7.7, we obtain hence by assumption

$$\lim_{\nu \rightarrow \infty} \int_{[p]} \varphi_\nu(f) d\mathfrak{f} a = (2[q] - [p]) ([a^+] - [a^-]) \int_{[p]} |\varphi(f)| d\mathfrak{f} |a|.$$

Theorem 21.4. If $\varphi(f)$ is integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[p]}$, then we have for $[p_\nu] \uparrow_{\nu=1}^\infty [p]$

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f) d\mathfrak{f} a = \int_{[p]} \varphi(f) d\mathfrak{f} a.$$

Proof. By virtue of Theorem 21.1 we have for $[p_\nu] \uparrow_{\nu=1}^\infty [p]$

$$\int_{[p_\nu]} \varphi(f) d\mathfrak{f} a = [p_\nu] \int_{[p]} \varphi(f) d\mathfrak{f} a,$$

and hence we obtain by Theorem 8.7

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f) d\mathfrak{f} a = [p] \int_{[p]} \varphi(f) d\mathfrak{f} a = \int_{[p]} \varphi(f) d\mathfrak{f} a.$$

Theorem 21.5. For a continuous function $\varphi(f)$ on $\mathcal{U}_{[p]}$, if there exists a sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [p]$ such that $\varphi(f)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and the sequence of elements

$$\left| \int_{[p_\nu]} \varphi(f) d\mathfrak{f} a \right| \quad (\nu = 1, 2, \dots)$$

is upper bounded, then $\varphi(f)$ is integrable by a in $\mathcal{U}_{[p]}$.

Proof. For the sequence of bounded continuous functions $\varphi_\nu(f)$ ($\nu = 1, 2, \dots$) defined at first, we have by the formulas §20(2), §20(6) and by Theorem 20.1

$$\begin{aligned} [p_\mu] \int_{[p]} |q_\nu(x)| d\beta |a| &= \int_{[p_\mu]} |q_\nu(x)| d\beta |a| \\ &\leq \int_{[p_\mu]} |q(x)| d\beta |a| = \left| \int_{[p_\mu]} q(x) d\beta a \right|, \end{aligned}$$

since $q(x)$ is bounded in $\mathcal{U}_{[p_\mu]}$ by assumption. Therefore there exists by assumption a positive element $\ell \in R$ such that

$$\left| \int_{[p_\mu]} q(x) d\beta a \right| \leq \ell \quad \text{for every } \mu = 1, 2, \dots$$

For such ℓ we obtain hence by Theorem 8.7

$$\int_{[p]} |q_\nu(x)| d\beta |a| \leq \ell \quad \text{for every } \nu = 1, 2, \dots,$$

since $[p_\mu] \uparrow_{\mu=1}^\infty [p]$ by assumption. Therefore $|q(x)|$ is integrable by $|a|$ in $\mathcal{U}_{[p]}$ by definition, and consequently $q(x)$ is integrable by a in $\mathcal{U}_{[p]}$ by Theorem 21.3.

Theorem 21.6. If $q(x)$ is integrable by an element $a \in R$ in $\mathcal{U}_{[p]}$, then $q(x)$ is almost finite in $\mathcal{U}_{[p][a]}$.

Proof. If $q(x)$ is integrable by an element $a \in R$ in $\mathcal{U}_{[p]}$, then $|q(x)|$ is integrable by $|a|$ in $\mathcal{U}_{[p]}$ by Theorem 21.2, and for the sequence of bounded continuous functions $q_\nu(x)$ ($\nu = 1, 2, \dots$) defined already, we have by definition

$$\int_{[p]} |q_\nu(x)| d\beta |a| \uparrow_{\nu=1}^\infty \int_{[p]} |q(x)| d\beta |a|.$$

For a projector $[q] \leq [p][a]$, if

$$|q(x)| = +\infty \quad \text{for every } x \in \mathcal{U}_{[q]},$$

then we have obviously

$$|q_\nu(x)| = \nu \quad \text{for every } x \in \mathcal{U}_{[q]},$$

and hence by the formulas (3), (4), (1) in §20 and Theorem 20.1

$$\begin{aligned} \int_{[p]} |q(x)| d\beta |a| &\geq \int_{[p]} |q_\nu(x)| d\beta |a| \\ &\geq \int_{[q]} |q_\nu(x)| d\beta |a| \geq \int_{[q]} \nu d\beta |a| = \nu [q]|a|. \end{aligned}$$

From this relation we conclude by Theorem 6.3 that $[q]|a| = 0$, and consequently by Theorem 8.2

$$[q] = [q][a] = [[q]|a|] = 0,$$

since $[q] \leq [a]$ by assumption. Therefore $q(x)$ is almost finite in $\mathcal{U}_{[p][a]}$ by definition.

Theorem 21.7. If a continuous function $q(x)$ is almost finite in $\mathcal{U}_{[p]}$, then there exists a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$$

such that $\varphi(f)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu=1, 2, \dots$

Proof. By virtue of Theorem 15.3 there exists a sequence of projectors $[p_\nu]$ ($\nu=1, 2, \dots$) such that

$$\{f: |\varphi(f)| \leq \nu\} \mathcal{U}_{[p_\nu]} \subset \mathcal{U}_{[p_\nu]} \subset \{f: |\varphi(f)| < \nu+1\} \mathcal{U}_{[p]}.$$

For such $[p_\nu]$ ($\nu=1, 2, \dots$) we have obviously $[p_\nu] \uparrow_{\nu=1}^{\infty}$, and

$$\left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]}\right)^- = \mathcal{U}_{[p]},$$

because $\varphi(f)$ is almost finite in $\mathcal{U}_{[p]}$ by assumption.

Therefore we obtain by Theorem 15.5

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$$

Furthermore it is evident by the construction of $[p_\nu]$ that

$$|\varphi(f)| < \nu+1 \quad \text{for every } f \in \mathcal{U}_{[p_\nu]}.$$

Theorem 21.8. For a projector $[p]$, if a continuous function $\varphi(f)$ on $\mathcal{U}_{[p]}$ is integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[p][a]}$, then $\varphi(f)$ is integrable by a further in $\mathcal{U}_{[p]}$.

Proof. For the sequence of bounded continuous functions $\varphi_\nu(f)$ ($\nu=1, 2, \dots$) defined already, we have by the formula §20(2)

$$\int_{[p][a]} \varphi_\nu(f) dfa = \int_{[p]} \varphi_\nu(f) dfa.$$

Consequently $\varphi(f)$ is integrable by a in $\mathcal{U}_{[p]}$ by definition, if $\varphi(f)$ is integrable by a in $\mathcal{U}_{[p][a]}$.

By virtue of Theorems 21.4-8 we obtain obviously:

Theorem 21.9. In order that a continuous function $\varphi(f)$ on $\mathcal{U}_{[p]}$ be integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[p]}$, it is necessary and sufficient that there exists a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p][a]$$

such that $\varphi(f)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu=1, 2, \dots$, and there exists the limit

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f) dfa,$$

and then this limit coincides with the integral

$$\int_{[p]} \varphi(f) dfa.$$

§22 Integral representation

Now we can consider relation between relative spectra and integrals. We will prove first the following

Theorem 22.1. If a continuous function $\varphi(\mathfrak{f})$ on $\mathcal{U}_{\mathfrak{C}\mathfrak{P}}$ is integrable by an element $a \in R$ in $\mathcal{U}_{\mathfrak{C}\mathfrak{P}}$ and

$$\mathfrak{b} = \int_{\mathfrak{C}\mathfrak{P}} \varphi(\mathfrak{f}) d\mathfrak{f} a,$$

then we have

$$\left(\frac{\mathfrak{b}}{a}, \mathfrak{f}\right) = \varphi(\mathfrak{f}) \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{\mathfrak{C}\mathfrak{P}}[a].$$

Proof. We shall consider first the case where $\varphi(\mathfrak{f})$ is bounded in $\mathcal{U}_{\mathfrak{C}\mathfrak{P}}$ and

$$\mathfrak{b} = \int_{\mathfrak{C}\mathfrak{P}} \varphi(\mathfrak{f}) d\mathfrak{f} a.$$

For an arbitrary point $\mathfrak{f}_0 \in \mathcal{U}_{\mathfrak{C}\mathfrak{P}}[a]$, since $\varphi(\mathfrak{f})$ is continuous, to any positive number ε there exists a projector $[\mathfrak{q}]$ such that $\mathfrak{f}_0 \in \mathcal{U}_{[\mathfrak{q}]}$ and

$$|\varphi(\mathfrak{f}) - \varphi(\mathfrak{f}_0)| \leq \varepsilon \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[\mathfrak{q}]}. \quad (1)$$

For such $[\mathfrak{q}]$ we have by the formulas (1), (2), (4), (6) in §20 and by Theorem 20.1

$$\begin{aligned} |[\mathfrak{q}]\mathfrak{b} - \varphi(\mathfrak{f}_0)[\mathfrak{q}]a| &= \left| \int_{\mathfrak{C}\mathfrak{P}} [\mathfrak{q}] \{ \varphi(\mathfrak{f}) - \varphi(\mathfrak{f}_0) \} d\mathfrak{f} a \right| \\ &= \int_{\mathfrak{C}\mathfrak{P}} [\mathfrak{q}] |\varphi(\mathfrak{f}) - \varphi(\mathfrak{f}_0)| d\mathfrak{f} |a| \leq \varepsilon \int_{\mathfrak{C}\mathfrak{P}} [\mathfrak{q}] d\mathfrak{f} |a| \\ &= \varepsilon [\mathfrak{p}][\mathfrak{q}]|a| \leq \varepsilon [\mathfrak{q}]|a|. \end{aligned} \quad (2)$$

Thus we obtain by Theorem 18.11

$$\left| \left(\frac{[\mathfrak{q}]\mathfrak{b} - \varphi(\mathfrak{f}_0)[\mathfrak{q}]a}{[\mathfrak{q}]a}, \mathfrak{f}_0 \right) \right| \leq \varepsilon.$$

Since we have by Theorems 18.2, 18.4, and 18.6

$$\left(\frac{[\mathfrak{q}]\mathfrak{b} - \varphi(\mathfrak{f}_0)[\mathfrak{q}]a}{[\mathfrak{q}]a}, \mathfrak{f}_0 \right) = \left(\frac{\mathfrak{b}}{a}, \mathfrak{f}_0 \right) - \varphi(\mathfrak{f}_0),$$

we obtain therefore

$$\left| \left(\frac{\mathfrak{b}}{a}, \mathfrak{f}_0 \right) - \varphi(\mathfrak{f}_0) \right| \leq \varepsilon.$$

As the positive number ε may be arbitrary, we conclude hence

$$\left(\frac{\mathfrak{b}}{a}, \mathfrak{f}_0 \right) = \varphi(\mathfrak{f}_0) \quad \text{for every } \mathfrak{f}_0 \in \mathcal{U}_{\mathfrak{C}\mathfrak{P}}[a].$$

Next we shall consider the general case. If $\varphi(\mathfrak{f})$ is integrable by $a \in R$ in $\mathcal{U}_{\mathfrak{C}\mathfrak{P}}$ and

$$b = \int_{[p]} \varphi(f) d f a,$$

then there exists by Theorem 21.9 a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p][a]$$

such that $\varphi(f)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$

Then we have by Theorem 21.1

$$\int_{[p_\nu]} \varphi(f) d f a = [p_\nu] b \quad (\nu = 1, 2, \dots)$$

and hence by Theorem 18.4

$$\left(\frac{b}{a}, f\right) = \left(\frac{[p_\nu] b}{a}, f\right) = \varphi(f) \quad \text{for every } f \in \mathcal{U}_{[p_\nu]},$$

as proved just now. Consequently we have

$$\left(\frac{b}{a}, f\right) = \varphi(f) \quad \text{for every } f \in \sum_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]}.$$

On the other hand we have by Theorem 15.5

$$\mathcal{U}_{[p][a]} = \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]}\right)^-,$$

and both $\left(\frac{b}{a}, f\right)$ and $\varphi(f)$ are continuous in $\mathcal{U}_{[p][a]}$.

Therefore we obtain

$$\left(\frac{b}{a}, f\right) = \varphi(f) \quad \text{for every } f \in \mathcal{U}_{[p][a]}.$$

Theorem 22.2. For every pair of elements a and $b \in R$ the relative spectrum $\left(\frac{b}{a}, f\right)$ is integrable by a in $\mathcal{U}_{[a]}$ and we have

$$[a] b = \int_{[a]} \left(\frac{b}{a}, f\right) d f a.$$

Proof. Since $\left(\frac{b}{a}, f\right)$ is almost finite in $\mathcal{U}_{[a]}$ by Theorem 19.3, there exists by Theorem 21.7 a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [a]$$

such that $\left(\frac{b}{a}, f\right)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$

For such $[p_\nu]$ ($\nu = 1, 2, \dots$), if we put

$$b_\nu = \int_{[p_\nu]} \left(\frac{b}{a}, f\right) d f a,$$

then we have by the previous theorem

$$\left(\frac{b_\nu}{a}, f\right) = \left(\frac{b}{a}, f\right) \quad \text{for every } f \in \mathcal{U}_{[p_\nu]},$$

and consequently $[p_\nu] b = [p_\nu] b_\nu$ by Theorem 19.5. Since we have obviously by the formula §20(2)

$$[p_\nu] b_\nu = b_\nu,$$

we obtain hence by Theorem 8.11

$$\lim_{\nu \rightarrow \infty} b_\nu = [a]b.$$

Therefore $(\frac{b}{a}, f)$ is integrable by a in $\mathcal{U}_{[a]}$ by Theorem 21.9 and we have

$$[a]b = \int_{[a]} (\frac{b}{a}, f) dfa.$$

§23 Relation between first and second spectral theories

We shall consider relation between the spectral system $[a_\lambda]$ ($-\infty < \lambda < +\infty$) of an element $b \in \mathcal{R}$ by an element $a \in \mathcal{R}$ and the relative spectrum $(\frac{b}{a}, f)$ in the sequel. We will prove first:

Theorem 23.1. The spectral system $[a_\lambda]$ ($-\infty < \lambda < +\infty$) of an element $b \in \mathcal{R}$ by an element $a \in \mathcal{R}$ is given by

$$\mathcal{U}_{[a_\lambda]} = \{f : (\frac{b}{a}, f) < \lambda\}^-.$$

Proof. By virtue of Theorem 19.6 we have

$$\mathcal{U}_{[(\lambda a^+ - b)^+][a^+]} = \{f : (\frac{b}{a^+}, f) < \lambda\}^-,$$

$$\mathcal{U}_{[(-\lambda a^- - b)^-][a^-]} = \{f : (\frac{b}{a^-}, f) > -\lambda\}^-.$$

Since we have by Theorem 18.4 for every point $f \in \mathcal{U}_{[a^+]}$

$$(\frac{b}{a^+}, f) = (\frac{b}{[a^+]a}, f) = (\frac{b}{a}, f),$$

we obtain obviously

$$\{f : (\frac{b}{a^+}, f) < \lambda\}^- = \mathcal{U}_{[a^+]} \{f : (\frac{b}{a}, f) < \lambda\}^-.$$

Since we have by Theorems 18.4, 18.5, 18.7 for every $f \in \mathcal{U}_{[a^-]}$

$$(\frac{b}{a^-}, f) = (\frac{b}{[-a^-]a}, f) = -(\frac{b}{a}, f),$$

we obtain further

$$\{f : (\frac{b}{a^-}, f) > -\lambda\}^- = \mathcal{U}_{[a^-]} \{f : (\frac{b}{a}, f) < \lambda\}^-.$$

Therefore we conclude by the formulas §20(3) and §20(4)

$$\begin{aligned} & \mathcal{U}_{[(\lambda a^+ - b)^+][a^+] + [(-\lambda a^- - b)^-][a^-]} \\ &= \{f : (\frac{b}{a}, f) < \lambda\}^-. \end{aligned}$$

Furthermore, recalling Theorem 13.2, we have by Theorems 7.4, 7.16 and by the formula §8(1)

$$\begin{aligned}
& [(\lambda a^+ - \ell)^+] [a^+] + [(-\lambda a^- - \ell)^-] [a^-] \\
& = [(\lambda [a^+] a - [a^+] \ell)^+] + [(\lambda [a^-] a - [a^-] \ell)^-] \\
& = [(\lambda a - \ell)^+] [a^+] + [(\lambda a - \ell)^-] [a^-] = [a_\lambda].
\end{aligned}$$

For the spectral system $[a_\lambda]$ ($-\infty < \lambda < +\infty$) of an element $\ell \in R$ by an element $a \in R$, if a_λ ($-\infty < \lambda < +\infty$) is a resolution of a , that is, if

$$a_\lambda = [a_\lambda] a \quad (-\infty < \lambda < +\infty),$$

then we have by Theorem 12.2

$$[a_{\lambda+0}] = \bigcap_{\varepsilon > 0} [a_{\lambda+\varepsilon}] \quad (-\infty < \lambda < +\infty).$$

Theorem 23.2. For the spectral system $[a_\lambda]$ ($-\infty < \lambda < +\infty$)

of an element $\ell \in R$ by an element $a \in R$ such that

$$[a_{\lambda+0}] = \bigcap_{\varepsilon > 0} [a_{\lambda+\varepsilon}] \quad (-\infty < \lambda < +\infty),$$

we have

$$\begin{aligned}
(\lambda a - \ell, f) &= (a, f) \neq 0 && \text{if and only if } f \in \mathcal{U}_{[a_\lambda]}; \\
(\lambda a - \ell, f) &= -(a, f) \neq 0 && \text{if and only if } f \in \mathcal{U}_{[a]} - \mathcal{U}_{[a_{\lambda+0}]}; \\
(\lambda a - \ell, f) &= 0, (a, f) \neq 0 && \text{if and only if } f \in \mathcal{U}_{[a_{\lambda+0}]} - \mathcal{U}_{[a]}.
\end{aligned}$$

Proof. $(\lambda a - \ell, f) = (a, f) \neq 0$ is by definition equivalent to

$$f \in \mathcal{U}_{[(\lambda a - \ell)^+]} \mathcal{U}_{[a^+]} \dot{+} \mathcal{U}_{[(\lambda a - \ell)^-]} \mathcal{U}_{[a^-]}.$$

On the other hand we have by the formulas §15(2), §15(3) and by Theorem 13.2

$$\mathcal{U}_{[(\lambda a - \ell)^+]} \mathcal{U}_{[a^+]} \dot{+} \mathcal{U}_{[(\lambda a - \ell)^-]} \mathcal{U}_{[a^-]} = \mathcal{U}_{[a_\lambda]}.$$

Therefore we have $(\lambda a - \ell, f) = (a, f) \neq 0$ if and only if $f \in \mathcal{U}_{[a_\lambda]}$.

If $(\lambda a - \ell, f) = 0$, $(a, f) \neq 0$, then we have by definition

$$f \in \mathcal{U}_{[(\lambda a - \ell)^+]} \mathcal{U}_{[a^+]} \dot{+} \mathcal{U}_{[(\lambda a - \ell)^-]} \mathcal{U}_{[a^-]}.$$

Then there exists obviously a projector $[p]$ for which

$$\mathcal{U}_{[p]} \mathcal{U}_{[(\lambda a - \ell)^+]} \mathcal{U}_{[a^+]} = 0, \quad f \in \mathcal{U}_{[p]} \subset \mathcal{U}_{[a]}.$$

For such $[p]$ we have by the formulas §15(4) and §15(5)

$$[p][\lambda a - \ell] = 0, \quad [p] \leq [a],$$

and hence by the formula §8(1)

$$[p](\lambda a - \ell) = 0, \quad \text{namely} \quad [p]\ell = \alpha[p]a.$$

On the other hand we have by the formula §10(4)

$$[p]b = \int_{-\infty}^{+\infty} \lambda d_{\lambda} [p][a_{\lambda}]a,$$

and $[p][a_{\lambda}]$ ($-\infty < \lambda < +\infty$) is the spectral system of $[p]b$ by $[p]a$. By virtue of the uniqueness of spectral system we obtain hence

$$[p][a_{\lambda}] = \begin{cases} [p] & \text{for } \lambda > \alpha, \\ 0 & \text{for } \lambda \leq \alpha, \end{cases}$$

and consequently

$$[p]([a_{\alpha+0}] - [a_{\alpha}]) = [p].$$

From this relation we conclude by the formulas §15(2) and §15(4)

$$p \in \mathcal{U}_{[p]} \subset \mathcal{U}_{[a_{\alpha+0}]} - \mathcal{U}_{[a_{\alpha}]}.$$

Conversely, if $p \in \mathcal{U}_{[a_{\alpha+0}]} - \mathcal{U}_{[a_{\alpha}]}$, then we have obviously $p \in \mathcal{U}_{[a]}$, but $p \notin \mathcal{U}_{[\alpha a - b]}$, because we have by the formula §10(4)

$$\begin{aligned} ([a_{\alpha+0}] - [a_{\alpha}])b &= \int_{-\infty}^{+\infty} \lambda d_{\lambda} ([a_{\alpha+0}] - [a_{\alpha}])[a_{\lambda}]a \\ &= \alpha ([a_{\alpha+0}] - [a_{\alpha}])a, \end{aligned}$$

that is,

$$([a_{\alpha+0}] - [a_{\alpha}])(\alpha a - b) = 0,$$

which yields by the formula §15(5)

$$(\mathcal{U}_{[a_{\alpha+0}]} - \mathcal{U}_{[a_{\alpha}]})\mathcal{U}_{[\alpha a - b]} = 0.$$

From the fact obtained just now, we conclude finally that

$$(\lambda a - b, p) = -(a, p) \neq 0$$

is equivalent to

$$p \in \mathcal{U}_{[a]}, \quad p \notin \mathcal{U}_{[a_{\lambda}]} \div (\mathcal{U}_{[a_{\lambda+0}]} - \mathcal{U}_{[a_{\lambda}]}) ,$$

and hence we have $(\lambda a - b, p) = -(a, p) \neq 0$ if and only if

$$p \in \mathcal{U}_{[a]} - \mathcal{U}_{[a_{\lambda+0}]} ,$$

because we have obviously

$$\mathcal{U}_{[a]} - \mathcal{U}_{[a_{\lambda+0}]} = \mathcal{U}_{[a_{\lambda}]} \div (\mathcal{U}_{[a_{\lambda+0}]} - \mathcal{U}_{[a_{\lambda}]}) .$$

Theorem 23.3. For the spectral system $[a_{\lambda}]$ ($-\infty < \lambda < +\infty$) of an element $b \in \mathcal{R}$ by an element $a \in \mathcal{R}$ we have

$$\left(\frac{t}{a}, f\right) = \begin{cases} \lambda & \text{if and only if } f \in \prod_{\varepsilon > 0} (\mathcal{U}_{[a_{\lambda+\varepsilon}]} - \mathcal{U}_{[a_{\lambda-\varepsilon}]}); \\ +\infty & \text{if and only if } f \in \prod_{-\infty < \lambda < +\infty} (\mathcal{U}_{[a_{\lambda}]} - \mathcal{U}_{[a_{\lambda+1}]}); \\ -\infty & \text{if and only if } f \in \prod_{-\infty < \lambda < +\infty} \mathcal{U}_{[a_{\lambda}]} . \end{cases}$$

Proof. If $\left(\frac{t}{a}, f\right) = \lambda$, then for any positive number

ε we have by definition

$$((\lambda + \varepsilon)a - t, f) = (a, f) \neq 0,$$

$$((\lambda - \varepsilon)a - t, f) = -(a, f) \neq 0,$$

and hence by the previous theorem

$$f \in \mathcal{U}_{[a_{\lambda+\varepsilon}]} (\mathcal{U}_{[a_{\lambda}]} - \mathcal{U}_{[a_{\lambda-\varepsilon}]}) = \mathcal{U}_{[a_{\lambda+\varepsilon}]} - \mathcal{U}_{[a_{\lambda-\varepsilon}]}$$

for every positive number ε .

Conversely, for any $f \in \prod_{\varepsilon > 0} (\mathcal{U}_{[a_{\lambda+\varepsilon}]} - \mathcal{U}_{[a_{\lambda-\varepsilon}]})$ we have by the previous theorem

$$((\lambda + \varepsilon)a - t, f) = (a, f) \neq 0,$$

$$((\lambda - \varepsilon)a - t, f) = -(a, f) \neq 0$$

for every positive number ε , and hence $\left(\frac{t}{a}, f\right) = \lambda$ by definition,

$\left(\frac{t}{a}, f\right) = +\infty$ is by definition equivalent to

$$(\lambda a - t, f) = -(a, f) \neq 0 \quad \text{for every real number } \lambda.$$

Therefore we have $\left(\frac{t}{a}, f\right) = +\infty$ by the previous theorem if and only if

$$f \in \mathcal{U}_{[a]} - \mathcal{U}_{[a_{\lambda}]} \quad \text{for every real number } \lambda.$$

We also can prove likewise that we have $\left(\frac{t}{a}, f\right) = -\infty$ if and only if

$$f \in \mathcal{U}_{[a_{\lambda}]} \quad \text{for every real number } \lambda.$$

If we make use of Theorem 23.1, then we also can prove

Theorems 14.1 and 14.4 for an arbitrary element $a \in \mathcal{R}$:

For the spectral system $[a_{\lambda}]$ ($-\infty < \lambda < +\infty$) of an element

$t \in \mathcal{R}$ by an element $a \in \mathcal{R}$ such that

$$[a_{\lambda+\varepsilon}] = \bigcap_{\varepsilon > 0} [a_{\lambda+\varepsilon}] \quad (-\infty < \lambda < +\infty)$$

we have by Theorems 23.1 and 16.6

$$\begin{aligned}
U_{[a]} - U_{[a - \lambda + 0]} &= U_{[a]} - \left(\prod_{i=1}^{\infty} \{ f : (\frac{f}{a}, f) < -\lambda + \frac{1}{i} \}^{\circ} \right)^{\circ} \\
&= U_{[a]} \left(\sum_{i=1}^{\infty} \{ f : (\frac{f}{a}, f) < -\lambda + \frac{1}{i} \}^{\circ} \right)^{-} \\
&= \left(\sum_{i=1}^{\infty} \{ f : (\frac{f}{a}, f) \geq -\lambda + \frac{1}{i} \}^{\circ} \right)^{-} \\
&= \left(\sum_{i=1}^{\infty} \{ f : (\frac{-f}{a}, f) \leq \lambda - \frac{1}{i} \}^{\circ} \right)^{-} \\
&= \{ f : (\frac{-f}{a}, f) < \lambda \}^{-},
\end{aligned}$$

and hence $[a] - [a - \lambda + 0]$ ($-\infty < \lambda < +\infty$) is the spectral system of $-f$ by a by Theorem 23.1.

For the spectral systems $[p_{\lambda}]$ and $[q_{\lambda}]$ ($-\infty < \lambda < +\infty$) respectively of elements f and $c \in R$ by an element $a \in R$, putting

$$[r_{\lambda}] = \bigcup_{\sigma + \rho < \lambda} [p_{\sigma}] [q_{\rho}] \quad \text{for rational numbers } \sigma, \rho,$$

we have by Theorems 23.1, 16.4, and 18.6

$$\begin{aligned}
U_{[r_{\lambda}]} &= \left(\sum_{\sigma + \rho < \lambda} \{ f : (\frac{f}{a}, f) < \sigma \}^{-} \{ f : (\frac{c}{a}, f) < \rho \}^{-} \right)^{-} \\
&= \left(\sum_{\sigma + \rho < \lambda} \{ f : (\frac{f}{a}, f) < \sigma \} \{ f : (\frac{c}{a}, f) < \rho \} \right)^{-} \\
&= \{ f : (\frac{f+c}{a}, f) < \lambda \}^{-},
\end{aligned}$$

and hence $[r_{\lambda}]$ ($-\infty < \lambda < +\infty$) is the spectral system of $f + c$ by a by Theorem 23.1.

CHAPTER IV
SEMI-ORDERED RINGS

§24 Fundamental definition

A linear space R is said to be a ring, if to every elements a and $b \in R$ there is defined a product $ab \in R$ subject to the postulates:

- 1) $a(bc) = (ab)c$,
- 2) $a(b+c) = ab+ac$, $(b+c)a = ba+ca$,
- 3) $(\alpha a)b = a(\alpha b) = \alpha ab$ for every real number α .

Putting $b=c=0$, we conclude immediately from 2)

$$(1) \quad a0 = 0a = 0.$$

for every $a \in R$, if R is a ring.

A ring R is said to be a semi-ordered ring, if R is a semi-ordered linear space subject to the postulate:

- 4) $a \geq 0$, $b \geq 0$ implies $ab \geq 0$.

If R is a semi-ordered ring, then we obtain at once by the postulates 2) and 4)

$$(2) \quad a \geq b \text{ implies } ac \geq bc, \quad ca \geq cb \text{ for } c \geq 0.$$

Therefore we have for a system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) and for an element $b \geq 0$

$$(3) \quad \left(\bigcap_{\lambda \in \Lambda} a_\lambda\right)b \leq \bigcap_{\lambda \in \Lambda} a_\lambda b, \quad b\left(\bigcap_{\lambda \in \Lambda} a_\lambda\right) \leq \bigcap_{\lambda \in \Lambda} ba_\lambda,$$

if both sides of each inequality have any sense.

If a semi-ordered ring R is lattice ordered, continuous, or universally continuous as a semi-ordered linear space, then R^* is said to be lattice ordered, continuous, or universally continuous respectively.

If a semi-ordered ring R is lattice ordered, then we have for every elements a and $b \in R$

$$(4) \quad |ab| \leq |a||b|.$$

Because, as $a = a^+ - a^-$, $b = b^+ - b^-$, we have by the postulate 2)

$$a \ell = (a^+ \ell^+ + a^- \ell^-) - (a^+ \ell^- + a^- \ell^+),$$

and hence further by the postulate 4) and by the formula §4(3)

$$\begin{aligned} |a \ell| &\leq a^+ \ell^+ + a^- \ell^- + a^+ \ell^- + a^- \ell^+ \\ &= (a^+ + a^-)(\ell^+ + \ell^-) = |a| |\ell|. \end{aligned}$$

§25 Semi-normal rings

A semi-ordered ring R is said to be semi-normal, if R is continuous and we have

$$[a \ell] \leq [a][\ell]$$

for every positive elements a and $\ell \in R$.

Let R be a semi-normal ring in the sequel. Then we have

$$(1) \quad a \wedge \ell = 0 \quad \text{implies} \quad a \ell = 0.$$

Because, if $a \wedge \ell = 0$, then we have by the formula §8(1)

$$[a \ell] \leq [a][\ell] = [a \wedge \ell] = 0,$$

and hence $a \ell = 0$ by Theorem 7.11.

For every element $a \in R$, as $a^+ \wedge a^- = 0$ by Theorem 3.6, we have by the formula (1)

$$a^+ a^- = a^- a^+ = 0.$$

Since $a^+ \geq 0$, $a^- \geq 0$, we obtain consequently

$$\begin{aligned} a^2 &= (a^+ - a^-)^2 = a^+ a^+ - a^+ a^- - a^- a^+ + a^- a^- \\ &= a^+ a^+ + a^- a^- \geq 0. \end{aligned}$$

Therefore we have

$$(2) \quad a^2 \geq 0 \quad \text{for every } a \in R.$$

$$(3) \quad a \wedge \ell = 0 \text{ implies } ac \wedge \ell = ca \wedge \ell = 0 \text{ for } c \geq 0.$$

Because, if $a \wedge \ell = 0$, then we have $[a][\ell] = 0$ by Theorem 8.1 and hence by the formula §8(1) for every $c \geq 0$

$$[ac \wedge \ell] = [ac][\ell] \leq [a][c][\ell] = 0.$$

From this relation we conclude $ac \wedge \ell = 0$ by Theorem 7.11.

We also conclude likewise $ca \wedge \ell = 0$.

$$(4) \quad ac \wedge \ell c = (a \wedge \ell)c, \quad ca \wedge \ell c = c(a \wedge \ell) \text{ for } c \geq 0.$$

Because, as $(a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0$ by Theorem 2.4, we have by the formula (3) for every $c \geq 0$

$$(a - (a \wedge b))c \wedge (b - (a \wedge b))c = 0,$$

and hence $ac \wedge bc = (a \wedge b)c$ by Theorem 2.4. We also can prove likewise the other equality.

$$(5) \quad ac \vee bc = (a \vee b)c, \quad ca \vee cb = c(a \vee b) \quad \text{for } c \geq 0.$$

Because we have by the formula (4) and Theorem 2.2 for every $c \geq 0$

$$\begin{aligned} -(ac \vee bc) &= (-ac) \wedge (-bc) \\ &= ((-a) \wedge (-b))c = -(a \vee b)c. \end{aligned}$$

We also can prove likewise the other relation.

$$(6) \quad (af)^+ = a^+f^+ + a^-f^-, \quad (af)^- = a^+f^- + a^-f^+.$$

Since $af = (a^+f^+ + a^-f^-) - (a^+f^- + a^-f^+)$, we need only to prove by Theorem 3.6 that

$$(a^+f^+ + a^-f^-) \wedge (a^+f^- + a^-f^+) = 0.$$

Indeed, as $a^+ \wedge a^- = f^+ \wedge f^- = 0$ by Theorem 3.6, we obtain by the formula (3)

$$a^+f^+ \wedge a^-f^+ = 0, \quad a^+f^+ \wedge a^+f^- = 0,$$

and hence $a^+f^+ \wedge (a^+f^- + a^-f^+) = 0$ by Theorem 4.3. We also obtain likewise

$$a^-f^- \wedge (a^+f^- + a^-f^+) = 0,$$

and consequently by Theorem 4.3

$$(a^+f^+ + a^-f^-) \wedge (a^+f^- + a^-f^+) = 0.$$

From the formula (6) we conclude immediately

$$(7) \quad |af| = |a||f|.$$

Thus, for every elements a and $f \in \mathcal{R}$ we have by definition

$$[af] = [|af|] = [|a||f|] \leq [a][f],$$

that is, we have

$$(8) \quad [af] \leq [a][f] \quad \text{for every } a, f \in \mathcal{R}.$$

By the formulas (3) and (7) we have obviously

$$(9) \quad a \perp f \text{ implies } ac \perp f, \quad ca \perp f \quad \text{for every } c \in \mathcal{R}$$

For every projector $[p]$ we have

$$(10) \quad ([p]a)t = a([p]t) = [p](at).$$

Because, as $a = [p]a + (a - [p]a)$, we have

$$at = ([p]a)t + (a - [p]a)t,$$

and hence further

$$[p](at) = [p]\{([p]a)t\} + [p]\{(a - [p]a)t\}.$$

On the other hand we have by the formulas (8) and §8(1)

$$([p]a)t \leq [p]a[pt] = [p]a[t] \leq [p],$$

$$[p]((a - [p]a)t) \leq [p][a - [p]a][t] = 0.$$

Consequently we have by Theorem 8.12 and by the formula §8(1)

$$[p]\{([p]a)t\} = ([p]a)t, \quad [p]\{(a - [p]a)t\} = 0.$$

Therefore we obtain $[p](at) = ([p]a)t$. We also can conclude

likewise $[p](at) = a([p]t)$ from the relation

$$[p](at) = [p]\{a([p]t)\} + [p]\{a(t - [p]t)\}.$$

Theorem 25.1. If $a = \bigcap_{\lambda \in A} a_\lambda$, then we have

$$at = \bigcap_{\lambda \in A} a_\lambda t, \quad ta = \bigcap_{\lambda \in A} ta_\lambda \quad \text{for every } t \geq 0.$$

If $a = \bigcup_{\lambda \in A} a_\lambda$, then we have

$$at = \bigcup_{\lambda \in A} a_\lambda t, \quad ta = \bigcup_{\lambda \in A} ta_\lambda \quad \text{for every } t \geq 0.$$

Proof. If $\bigcap_{\lambda \in A} a_\lambda = 0$, then we have obviously for any $t \geq 0$

$$a_\lambda t \geq 0 \quad \text{for every } \lambda \in A.$$

For any positive number ε and for an element $\lambda_0 \in A$, putting

$$p_\lambda = (a_\lambda - \varepsilon a_{\lambda_0})^+ \quad (\lambda \in A),$$

we obtain by Theorem 7.16

$$[p_\lambda](a_\lambda - \varepsilon a_{\lambda_0}) = (a_\lambda - \varepsilon a_{\lambda_0})^+ \geq 0,$$

and hence

$$[p_\lambda]a_{\lambda_0} \leq \frac{1}{\varepsilon} [p_\lambda]a_\lambda \leq \frac{1}{\varepsilon} a_\lambda.$$

Since $\bigcap_{\lambda \in A} a_\lambda = 0$ by assumption, we obtain thus by Theorem 2.3

$$\bigcap_{\lambda \in A} [p_\lambda]a_{\lambda_0} = 0,$$

and consequently by Theorem 8.12

$$\bigcap_{\lambda \in A} [p_\lambda](a_{\lambda_0})(a_{\lambda_0}t) = 0 \quad \text{for every element } t \geq 0.$$

On the other hand we also have by Theorem 7.16

$$(1 - [p_\lambda])(a_\lambda - \varepsilon a_{\lambda_0}) \leq 0,$$

that is,

$$(1 - [p_\lambda])a_\lambda \leq \varepsilon(1 - [p_\lambda])a_{\lambda_0} \leq \varepsilon a_{\lambda_0}.$$

For a positive element $t \in \mathcal{R}$, if

$$x \leq a_\lambda t \quad \text{for every } \lambda \in \Lambda,$$

then we have naturally by the formula (4)

$$x \leq (a_\lambda \wedge a_{\lambda_0})t \quad \text{for every } \lambda \in \Lambda.$$

Therefore we have for every $\lambda \in \Lambda$

$$\begin{aligned} x &\leq \{[p_\lambda](a_\lambda \wedge a_{\lambda_0}) + (1 - [p_\lambda])(a_\lambda \wedge a_{\lambda_0})\}t \\ &\leq \{[p_\lambda]a_{\lambda_0} + \varepsilon a_{\lambda_0}\}t = ([p_\lambda]a_{\lambda_0})t + \varepsilon a_{\lambda_0}t. \end{aligned}$$

Since we have by the formula (10)

$$([p_\lambda]a_{\lambda_0})t = [p_\lambda][a_{\lambda_0}](a_{\lambda_0}t),$$

we obtain hence

$$\bigcap_{\lambda \in \Lambda} ([p_\lambda]a_{\lambda_0})t = 0,$$

and consequently $x \leq \varepsilon a_{\lambda_0}t$ for any positive number ε

Therefore we have $x \leq 0$ by Theorem 6.3, that is, $\bigcap_{\lambda \in \Lambda} a_\lambda = 0$ implies

$$\bigcap_{\lambda \in \Lambda} a_\lambda t = 0 \quad \text{for every element } t \geq 0.$$

If $a = \bigcap_{\lambda \in \Lambda} a_\lambda$, then we have by Theorem 2.4

$$\bigcap_{\lambda \in \Lambda} (a_\lambda - a) = 0,$$

and then we obtain

$$\bigcap_{\lambda \in \Lambda} (a_\lambda - a)t = 0 \quad \text{for every element } t \geq 0,$$

as proved just now. From this relation we conclude by Theorem 2.4

$$at = \bigcap_{\lambda \in \Lambda} a_\lambda t \quad \text{for every element } t \geq 0.$$

If $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, then we have by Theorems 2.2 and 2.4

$$\bigcap_{\lambda \in \Lambda} (a - a_\lambda) = 0,$$

and then we conclude likewise

$$at = \bigcup_{\lambda \in \Lambda} a_\lambda t \quad \text{for every element } t \geq 0.$$

We also can prove likewise that $a = \bigcap_{\lambda \in \Lambda} a_\lambda$ implies

$$ta = \bigcap_{\lambda \in \Lambda} ta_\lambda \quad \text{for every element } t \geq 0,$$

and $a = \bigcup_{\lambda \in \Lambda} a_\lambda$ implies

$$ta = \bigcup_{\lambda \in \Lambda} ta_\lambda \quad \text{for every element } t \geq 0.$$

Theorem 25.2. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, $\lim_{\nu \rightarrow \infty} b_\nu = b$, then we have

$$\lim_{\nu \rightarrow \infty} a_\nu b_\nu = ab.$$

Proof. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, $\lim_{\nu \rightarrow \infty} b_\nu = b$, then there exist by definition sequence of elements $l_\nu \downarrow_{\nu=\infty} 0$, $k_\nu \downarrow_{\nu=\infty} 0$ and a sequence of natural numbers μ_ν ($\nu = 1, 2, \dots$) such that

$$|a_\mu - a| \leq l_\nu, \quad |b_\mu - b| \leq k_\nu \quad \text{for } \mu \geq \mu_\nu, \nu = 1, 2, \dots$$

Then we have by the formula (7) for $\mu \geq \mu_\nu$

$$\begin{aligned} |a_\mu b_\mu - ab| &\leq |a_\mu b_\mu - ab_\mu| + |ab_\mu - ab| \\ &= |a_\mu - a| |b_\mu| + |a| |b_\mu - b| \\ &\leq l_\nu (k_\nu + |b|) + |a| k_\nu, \end{aligned}$$

since $|b_\mu| \leq k_\nu + |b|$ for every $\mu \geq \mu_\nu$. As we have by the previous theorem

$$l_\nu (k_\nu + |b|) \downarrow_{\nu=\infty} 0, \quad |a| k_\nu \downarrow_{\nu=\infty} 0,$$

we obtain by Theorem 5.8

$$l_\nu (k_\nu + |b|) + |a| k_\nu \downarrow_{\nu=\infty} 0,$$

and hence $\lim_{\nu \rightarrow \infty} a_\nu b_\nu = ab$ by definition.

For every element $a \in R$ we define $a^\nu \in R$ ($\nu = 1, 2, \dots$) as

$$a^1 = a, \quad a^\nu = a^{\nu-1} a \quad \text{for } \nu = 2, 3, \dots$$

With this definition we have obviously by Theorem 25.2:

Theorem 25.3. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, then we have

$$\lim_{\nu \rightarrow \infty} a_\nu^\mu = a^\mu \quad \text{for every } \mu = 1, 2, \dots$$

§26 Null factors, normal factors

Let R be a semi-normal ring in the sequel. An element $n \in R$ is said to be a null factor, if $n^2 = 0$.

Theorem 26.1. Every null factor $n \in K$ satisfies

$$xn = nx = 0 \quad \text{for every } x \in K.$$

Proof. For an arbitrary element $x \in K$, if we put

$$q_\nu = ([n]|x| - \nu|n|)^+ \quad \text{for } \nu = 1, 2, \dots,$$

then we have $[n] \geq [q_\nu] \downarrow_{\nu=1}^\infty$ by Theorem 8.3, and hence there exists by Theorem 8.6 a projector $[q]$ such that

$$[q_\nu] \downarrow_{\nu=1}^\infty [q].$$

Since we have by Theorem 7.16

$$[q_\nu]([n]|x| - \nu|n|) = ([n]|x| - \nu|n|)^+ \geq 0,$$

we obtain by Theorem 7.5

$$[q_\nu]|n| \leq \frac{1}{\nu} [q_\nu][n]|x| \leq \frac{1}{\nu} |x|.$$

On passing to the limit, we obtain $[q]|n| = 0$, and hence we conclude by the formula §8(1) and by Theorem 8.2

$$[q] = [q][n] = [[q]|n|] = 0,$$

that is, we have $[q_\nu] \downarrow_{\nu=1}^\infty 0$.

Since we have by Theorem 7.16

$$(1 - [q_\nu])([n]|x| - \nu|n|) \leq 0,$$

we obtain by Theorem 8.2

$$(*) \quad ([n] - [q_\nu])|x| \leq (1 - [q_\nu])(\nu|n|) \leq \nu|n|.$$

As $n^2 = 0$ by assumption, we have therefore by the formulas §25(7) and §25(10)

$$\begin{aligned} |nx| &= |n|([n]|x|) = |n|([q_\nu]|x|) + |n|([n] - [q_\nu])|x| \\ &\leq [q_\nu]|n||x| + \nu|n|^2 = [q_\nu]|n||x|. \end{aligned}$$

Hence the passage to the limit yields $nx = 0$. We also can prove likewise $xn = 0$.

In this proof we conclude easily from (*) that

$$([n] - [q_\nu])|x|^2 \leq \nu^2|n|^2 = \nu^2|n^2| = 0,$$

and consequently $([n] - [q_\nu])|x|$ is a null factor too. Therefore we obtain by Theorem 26.1, proved just now, and by the formulas §25(7) and §25(10)

$$([q_\nu]|x| + ([n] - [q_\nu])|x|)^2 = ([q_\nu]|x|)^2 = [q_\nu]|x|^2.$$

Since we have by the formulas (10) and (7) in §25

$$[n]|x|^2 = ([n]|x|)^2 = ([q_\nu]|x| + ([n] - [q_\nu])|x|)^2,$$

we obtain finally by the formula §8(1) and by Theorem 7.5

$$[n][x^2] = [([n]|x^2)] = [\varphi_n][x^2] \leq [\varphi_n],$$

and then the passage to the limit yields $[n][x^2] = 0$ Therefore we conclude by the formula §8(1)

Theorem 26.2. Every null factor $n \in R$ satisfies

$$[n]x^2 = [x^2]n = 0 \quad \text{for every } x \in R.$$

Theorem 26.3. $([a] - [a^2])a$ is a null factor for every element $a \in K$.

Proof. Since $[a^2] \leq [a][a] = [a]$ for every element $a \in R$ by definition, we have by Theorems 8.2 and 7.11

$$[a]a^2 = a^2 = [a^2]a^2.$$

Therefore we obtain by the formula §25(10)

$$([a] - [a^2])a^2 = ([a] - [a^2])a^2 = 0,$$

that is, $([a] - [a^2])a$ is a null factor by definition.

An element $a \in R$ is said to be a normal factor, if $[a^2] = [a]$.

Theorem 26.4. $[a^2]a$ is a normal factor for every element $a \in R$.

Proof. Since $[a^2] \leq [a][a] = [a]$, we have by the formulas §25(10), §8(1) and by Theorem 7.11

$$[[a^2]a]^2 = [[a^2]a^2] = [a^2] = [a^2][a] = [[a^2]a].$$

Therefore $[a^2]a$ is a normal factor by definition.

Theorem 26.5. An element $a \in R$ is a normal factor, if there exists a pair of elements $b, c \in R$ such that $[a] \leq [bc]$.

Proof. For arbitrary elements $b, c \in R$ we have by Theorems 26.1 and 26.3

$$bc = ([b^2]b)c + ([b] - [b^2])bc = ([b^2]b)c,$$

and hence we obtain by the formula §8(1) and by Theorem 7.5

$$[bc] = ([b^2]b)c \leq [[b^2]b][c] = [b^2][b][c] \leq [b^2].$$

Therefore if $[a] \leq [bc]$, then we have

$$\begin{aligned} [a] - [a^2] &= [a]([a] - [a^2]) \\ &\leq [b^2]([a] - [a^2]) = [b^2]([a] - [a^2])a. \end{aligned}$$

As $[a^2]([a] - [a^2])a = 0$ by Theorems 26.2 and 26.3, we obtain hence $[a] - [a^2] = 0$, that is, a is a normal factor by definition.

Theorem 26.6. An element $a \in R$ is a normal factor, if and only if a is orthogonal to every null factor.

Proof. If $a \in R$ is a normal factor, then we have by Theorem 26.2 for every null factor n

$$[a]n = [a^2]n = 0,$$

and hence a is orthogonal to n by Theorem 7.12. Conversely, if $a \in R$ is orthogonal to every null factor, then, as $([a] - [a^2])a$ is a null factor by Theorem 26.3, we have by Theorems 8.2 and 7.12

$$[a] - [a^2] = [[a]([a] - [a^2])a] = 0,$$

and hence a is a normal factor by definition.

Theorem 26.7. An element $a \in R$ is a null factor, if and only if a is orthogonal to every normal factor.

Proof. If $n \in R$ is a null factor, then we have by Theorem 26.2 for every normal factor a

$$[a]n = [a^2]n = 0,$$

and hence n is orthogonal to every normal factor by Theorem 7.12. Conversely, if $n \in R$ is orthogonal to every normal factor, then, since $[n^2]n$ is a normal factor by Theorem 26.4, we have by Theorems 8.2 and 7.12

$$[n^2] = [n][n^2] = [[n][n^2]n] = 0$$

and hence $n^2 = 0$, that is, n is a null factor by definition.

Theorem 26.8. If both a and $b \in R$ are normal factors, then we have

$$[ab] = [a][b]$$

Proof. If both a and $b \in R$ are normal factors, then we see easily by Theorems 26.6, 4.1, and 4.4 that $|a| \cap |b|$ is a normal factor too. Since we have by the formula §25(7)

$$(|a| \cap |b|)^2 \leq |a||b| = |ab|,$$

we obtain hence by Theorem 8.3

$$[|a| \wedge |b|] = [(|a| \wedge |b|)^2] \subseteq [ab].$$

On the other hand we have by the formula §8(1)

$$[|a| \wedge |b|] = [a][b] \supseteq [ab].$$

Consequently we obtain $[a][b] = [ab]$.

A subset M of R is said to be a closed subring of R , if

1) $a, b \in M$ implies $ab \in M$ and $\alpha a + \beta b \in M$

for every real numbers α, β ;

2) $a = \bigcup_{\lambda \in \Lambda} a_\lambda$, $a_\lambda \in M$ ($\lambda \in \Lambda$) implies $a \in M$;

3) $a = \bigcap_{\lambda \in \Lambda} a_\lambda$, $a_\lambda \in M$ ($\lambda \in \Lambda$) implies $a \in M$.

With this definition we see easily that if M is a closed subring of R , then M is itself a semi-ordered ring and for every element $p \in M$ the projector $[p]$ in R coincides with the projector $[p]$ in M . Therefore every closed subring of a semi-normal ring is itself a semi-normal ring.

For two closed subring M and N of R , if every element of M is orthogonal to every element of N and to every element $a \in R$ there exist $m \in M$ and $n \in N$ such that

$$a = m + n,$$

then R is said to be a direct sum of M and N . If R is a direct sum of M and N , then to any element $a \in R$ there exist uniquely $m \in M$ and $n \in N$ such that $a = m + n$. Because, if

$$m_1 + n_1 = m + n, \quad m_1, m \in M, \quad n_1, n \in N,$$

then we have $M \ni m_1 - m = n - n_1 \in N$, and hence

$$|m_1 - m| = |n - n_1| = |m_1 - m| \wedge |n - n_1| = 0,$$

that is, $m_1 = m$ and $n_1 = n$.

Theorem 26.9. The set of all normal factors M and the set of all null factors N are closed subrings of R , and R is a direct sum of M and N .

Proof. By virtue of Theorems 4.2, 4.3, 4.4, and the formula

§25(9) we conclude immediately from Theorem 26.6 that M is a closed subring of R . We see likewise from Theorem 26.7 that N is a closed subring of R . Furthermore for every element $a \in R$ we have

$$a = [a^2]a + ([a] - [a^2])a.$$

Here $[a^2]a$ is a normal factor by Theorem 26.4, and $([a] - [a^2])a$ is a null factor by Theorem 26.3. Therefore R is a direct sum of M and N by definition.

§27 Normal rings

A semi-ordered ring R is said to be normal, if R is continuous and we have for every positive elements $a, b \in R$

$$[ab] = [a][b].$$

With this definition, every normal ring is obviously semi-normal, and hence we have by the formula §25(7)

$$[ab] = [a][b]$$

for every elements $a, b \in R$, if R is a normal ring.

From Theorems 26.8 and 26.9 we conclude immediately:

Theorem 27.1. For a semi-normal ring R , the set of all normal factors constitutes a normal ring.

Theorem 27.2. A normal ring is the same as a semi-normal ring consisting only of normal factors.

Proof. If R is a normal ring, then R is semi-normal and we have by the formula §25(7) for every element $a \in R$

$$[a^2] = [|a^2|] = [|a|^2] = [a][a] = [a],$$

that is, every element $a \in R$ is a normal factor by definition.

Conversely, if a semi-normal ring R consists only of normal factors, then R is normal by the previous theorem.

Let R be a normal ring in the sequel. Then we have for proper values:

Theorem 27.3. For every elements $a, b \in R$ we have

$$(ab, p) = (a, p)(b, p)$$

adopting the convention $0(\pm\infty) = (\pm\infty)0 = 0$.

Proof. In the case: $(a, p) = 0$, we have $p \in \mathcal{U}_{[a]}$ by definition, and since

$$\mathcal{U}_{[ab]} = \mathcal{U}_{[a][b]} \subset \mathcal{U}_{[a]},$$

we obtain hence $p \in \mathcal{U}_{[ab]}$, that is, $(ab, p) = 0$. Thus we see that the assertion of the theorem is valid in this case.

We also can dispose likewise of the case: $(b, p) = 0$.

We proceed next to the case: $(a, p) = (b, p) = +\infty$. In this case, since we have by definition

$$p \in \mathcal{U}_{[a+]}, \quad p \in \mathcal{U}_{[b+]},$$

and by the formula §25(6)

$$(ab)^+ = a^+b^+ + a^-b^- \geq a^+b^+,$$

we conclude by Theorem 8.3 and by formulas (2), (4) in §15

$$p \in \mathcal{U}_{[a+]} \mathcal{U}_{[b+]} = \mathcal{U}_{[a+][b+]} = \mathcal{U}_{[a^+b^+]} \subset \mathcal{U}_{[(ab)^+]},$$

that is, $(ab, p) = +\infty$. Consequently the assertion of the theorem is valid again in this case.

In general, if $(a, p) = \varepsilon(+\infty)$, $(b, p) = \delta(+\infty)$, where $\varepsilon, \delta = \pm 1$, then we have by the formula §17(7)

$$(\varepsilon a, p) = (\delta b, p) = +\infty.$$

Thus, applying the preceding case to this case, we obtain by repeated use of the formula §17(7)

$$\varepsilon \delta (a, p)(b, p) = (\varepsilon a, p)(\delta b, p) = (\varepsilon \delta ab, p) = \varepsilon \delta (ab, p).$$

From this relation we also can conclude the assertion of the theorem.

Theorem 27.4. For every point $p \in \mathcal{U}_{[a][c]}$ we have

$$\left(\frac{bc}{ac}, p\right) = \left(\frac{c}{a}, p\right) = \left(\frac{b}{c}, p\right).$$

Proof. For every point $p \in \mathcal{U}_{[a][c]}$, putting $\xi = \left(\frac{b}{c}, p\right)$, we have by definition

$$(\lambda a - b, p) = \begin{cases} (a, p) & \text{for } \lambda > \xi, \\ -(a, p) & \text{for } \lambda < \xi; \end{cases}$$

Making the product with $(c, \mathfrak{f})$ we obtain from this relation

$$(\lambda ac - \mathfrak{f}c, \mathfrak{f}) = \begin{cases} (ac, \mathfrak{f}) & \text{for } \lambda > \xi, \\ -(ac, \mathfrak{f}) & \text{for } \lambda < \xi. \end{cases}$$

in accordance with the previous theorem. Hence we have

$$\xi = \left(\frac{\mathfrak{f}c}{ac}, \mathfrak{f} \right),$$

and the first assertion of the theorem is verified. We also can dispose likewise of the other assertion.

Theorem 27.5. For every point $\mathfrak{f} \in \bigcup_{[a_1] \cup [a_2]}$ we have

$$\left(\frac{\mathfrak{f}_1 \mathfrak{f}_2}{a_1 a_2}, \mathfrak{f} \right) = \left(\frac{\mathfrak{f}_1}{a_1}, \mathfrak{f} \right) \left(\frac{\mathfrak{f}_2}{a_2}, \mathfrak{f} \right),$$

if the right side has a meaning.

Proof. We assume that the right side of our asserted equality has a sense. Then naturally

$$(a_1, \mathfrak{f}) \neq 0, \quad (a_2, \mathfrak{f}) \neq 0,$$

and we have by Theorem 27.3 for $(\mathfrak{f}_1, \mathfrak{f}) \neq 0$

$$(a_1 a_2, \mathfrak{f}) \neq 0, \quad (\mathfrak{f}_1 a_2, \mathfrak{f}) \neq 0.$$

Therefore, applying the previous theorem to our case, we obtain

$$\left(\frac{\mathfrak{f}_1 a_2}{a_1 a_2}, \mathfrak{f} \right) = \left(\frac{\mathfrak{f}_1}{a_1}, \mathfrak{f} \right), \quad \left(\frac{\mathfrak{f}_1 \mathfrak{f}_2}{\mathfrak{f}_1 a_2}, \mathfrak{f} \right) = \left(\frac{\mathfrak{f}_2}{a_2}, \mathfrak{f} \right)$$

for $(\mathfrak{f}_1, \mathfrak{f}) \neq 0$. Consequently, recalling Theorem 18.12, we have the relation

$$\left(\frac{\mathfrak{f}_1 \mathfrak{f}_2}{a_1 a_2}, \mathfrak{f} \right) = \left(\frac{\mathfrak{f}_1 \mathfrak{f}_2}{\mathfrak{f}_1 a_2}, \mathfrak{f} \right) \left(\frac{\mathfrak{f}_1 a_2}{a_1 a_2}, \mathfrak{f} \right) = \left(\frac{\mathfrak{f}_1}{a_1}, \mathfrak{f} \right) \left(\frac{\mathfrak{f}_2}{a_2}, \mathfrak{f} \right)$$

in the case: $(\mathfrak{f}_1, \mathfrak{f}) \neq 0$. Concerning the case: $(\mathfrak{f}_1, \mathfrak{f}) = 0$,

since $(\mathfrak{f}_1 \mathfrak{f}_2, \mathfrak{f}) = 0$ by Theorem 27.3, we have by Theorem 18.3

$$\left(\frac{\mathfrak{f}_1 \mathfrak{f}_2}{a_1 a_2}, \mathfrak{f} \right) = \left(\frac{\mathfrak{f}_1}{0}, \mathfrak{f} \right) = 0.$$

Hence our assertion is verified.

§28 Absolute spectra

Let R be a normal ring in the following, and let E be the proper space of R . The absolute spectrum of an element $\hat{a} \in R$ at a point $\mathfrak{f} \in E$ is defined as

$$\left(\frac{a}{1}, f\right) = \begin{cases} \left(\frac{a^2}{1}, f\right) & \text{for } f \in \mathcal{U}_{[a]}, \\ 0 & \text{for } f \notin \mathcal{U}_{[a]}. \end{cases}$$

With this definition the absolute spectrum $\left(\frac{a}{1}, f\right)$ is defined all over the proper space E , while the relative spectrum $\left(\frac{a^2}{a}, f\right)$ is defined only on the neighbourhood $\mathcal{U}_{[a]}$ associated with the projector $[a]$.

Recalling Theorems 19.2 and 19.3 we obtain:

Theorem 28.1. The absolute spectrum $\left(\frac{a}{1}, f\right)$ is continuous all over E and the totality of points f for which

$$\left(\frac{a}{1}, f\right) = +\infty \text{ or } -\infty$$

is nowhere dense in $\mathcal{U}_{[a]}$.

We shall enumerate properties of absolute spectra in the sequel.

Theorem 28.2. For every point $f \in \mathcal{U}_{[a]}$ we have

$$\left(\frac{a}{1}, f\right) = \left(\frac{af}{f}, f\right) = \left(\frac{fa}{f}, f\right).$$

Proof. In the case: $(a, f) = 0$, since

$$(af, f) = (fa, f) = 0$$

by Theorem 27.3, we have obviously by Theorem 18.3

$$\left(\frac{a}{1}, f\right) = \left(\frac{af}{f}, f\right) = \left(\frac{fa}{f}, f\right) = 0 \quad \text{for } f \in \mathcal{U}_{[a]}.$$

In the case: $(a, f) \neq 0$, we see easily by Theorem 27.4 that we have for every point $f \in \mathcal{U}_{[a]}$

$$\left(\frac{af}{f}, f\right) = \left(\frac{a^2 f}{af}, f\right) = \left(\frac{a^2}{a}, f\right) = \left(\frac{a}{1}, f\right),$$

and similarly

$$\left(\frac{fa}{f}, f\right) = \left(\frac{fa^2}{fa}, f\right) = \left(\frac{a^2}{a}, f\right) = \left(\frac{a}{1}, f\right).$$

Theorem 28.3. For every projector $[p]$ we have

$$\left(\frac{[p]a}{1}, f\right) = \left(\frac{a}{1}, f\right) \text{ for } f \in \mathcal{U}_{[p]}.$$

Proof. For every point $f \in \mathcal{U}_{[p]}$ we have by the previous theorem

$$\left(\frac{[p]a}{1}, f\right) = \left(\frac{([p]a)p}{p}, f\right) = \left(\frac{ap}{p}, f\right) = \left(\frac{a}{1}, f\right),$$

since $([p]a)p = a([p]p) = ap$ by the formula §25(10) and by Theorem 7.11.

Theorem 28.4. We have

$$\left(\frac{\alpha a}{1}, f\right) = \alpha \left(\frac{a}{1}, f\right) \quad (-\infty < \alpha < +\infty),$$

$$\left(\frac{a+b}{1}, f\right) = \left(\frac{a}{1}, f\right) + \left(\frac{b}{1}, f\right),$$

$$\left(\frac{a}{1}, f\right) = \left(\frac{a}{1}, f\right) \left(\frac{1}{1}, f\right),$$

if the right side has any sense.

Proof. To any point $f \in E$ there exists obviously an element $c \in R$ for which $(c, f) \neq 0$. For such $c \in R$ we have by the previous theorem and by Theorem 18.5

$$\left(\frac{\alpha a}{1}, f\right) = \left(\frac{\alpha ac}{c}, f\right) = \alpha \left(\frac{ac}{c}, f\right) = \alpha \left(\frac{a}{1}, f\right),$$

and further by Theorem 18.6

$$\left(\frac{a+b}{1}, f\right) = \left(\frac{ac+bc}{c}, f\right) = \left(\frac{a}{1}, f\right) + \left(\frac{b}{1}, f\right).$$

Recalling Theorem 27.5, we obtain by repeated use of Theorem 28.2

$$\left(\frac{a}{1}, f\right) = \left(\frac{cac}{c^2}, f\right) = \left(\frac{c}{c}, f\right) \left(\frac{a}{1}, f\right) = \left(\frac{a}{1}, f\right) \left(\frac{1}{1}, f\right).$$

Theorem 28.5. If $\left(\frac{a}{1}, f\right) \geq \left(\frac{b}{1}, f\right)$ for every point $f \in E$, then we have $a \geq b$.

Proof. If $\left(\frac{a}{1}, f\right) \geq \left(\frac{b}{1}, f\right)$ for every $f \in E$, then we have by Theorem 28.2 for any positive element $c \in R$

$$\left(\frac{ac}{c}, f\right) \geq \left(\frac{bc}{c}, f\right) \quad \text{for } f \in U_{cc}.$$

From this relation we conclude by Theorem 19.4

$$[c]ac \geq [c]bc, \text{ namely } [c](a-b)c \geq 0.$$

Putting $c = (a-b)^-$, this inequality yields by Theorem 7.16

$$-\{(a-b)^-\}^2 \geq 0.$$

Since $\{(a-b)^-\}^2 \geq 0$ by the formula §25(2), we obtain hence

$$\{(a-b)^-\}^2 = 0,$$

and consequently

$$[(a-b)^-] = [\{(a-b)^-\}^2] = 0.$$

From this relation we conclude $(a-b)^- = 0$, and hence $a-b \geq 0$, that is, $a \geq b$.

Theorem 28.6. $a \geq b$ implies

$$\left(\frac{a}{1}, f\right) \geq \left(\frac{b}{1}, f\right) \quad \text{for every point } f \in E$$

Proof. If $a \geq b$, then we have obviously $ac \geq bc$ for

every positive element $c \in R$, and then by Theorem 18.8

$$\left(\frac{ac}{c}, f\right) \geq \left(\frac{fc}{c}, f\right) \quad \text{for } f \in \mathcal{U}_{[c]}.$$

By virtue of Theorem 28.2, this inequality yields

$$\left(\frac{a}{1}, f\right) \geq \left(\frac{f}{1}, f\right) \quad \text{for } f \in \mathcal{U}_{[c]}.$$

Since to any point $f \in E$ there exists obviously a positive element $c \in R$ such that $f \in \mathcal{U}_{[c]}$, our assertion is verified.

Theorem 28.7. If a continuous function $\varphi(f)$ on E satisfies

$$\left(\frac{a}{1}, f\right) \geq \varphi(f) \geq \left(\frac{f}{1}, f\right)$$

over E for some elements $a, b \in R$, then there exists an element $c \in R$ for which

$$\varphi(f) = \left(\frac{c}{1}, f\right)$$

over E , and such $c \in R$ is uniquely determined by $\varphi(f)$

Proof. We shall consider first the case: $b = 0$, that is, we suppose that a continuous function $\varphi(f)$ on E satisfies

$$\left(\frac{a}{1}, f\right) \geq \varphi(f) \geq 0 \quad \text{for every } f \in E$$

for some positive element $a \in R$. We now put

$$A_\nu = \{f : \nu > \left(\frac{a}{1}, f\right) > \frac{1}{\nu}\} \quad \text{for } \nu = 1, 2, \dots$$

Since the absolute spectrum $\left(\frac{a}{1}, f\right)$ is a continuous function on E by Theorem 28.1, the point set A_ν is open and obviously by definition included in the neighbourhood $\mathcal{U}_{[a]}$ associated with the projector $[a]$. As $\mathcal{U}_{[a]}$ is compact by Theorem 16.2, the closure A_ν^- of A_ν also is compact and clearly included in $A_{\nu+1}$. Hence there exists by Theorem 16.3 a projector $[p_\nu]$ such that

$$A_\nu^- \subset \mathcal{U}_{[p_\nu]} \subset A_{\nu+1} \quad (\nu = 1, 2, \dots).$$

For such $[p_\nu]$ we have obviously

$$\mathcal{U}_{[p_1]} \subset \mathcal{U}_{[p_2]} \subset \dots, \quad \mathcal{U}_{[p_\nu]} \subset \mathcal{U}_{[a]} \quad (\nu = 1, 2, \dots),$$

and for any neighbourhood $\mathcal{U}_{[p]} \subset \mathcal{U}_{[a]} - \left(\bigcup_{\nu=1}^{\infty} \mathcal{U}_{[p_\nu]}\right)^-$ we have

$$\left(\frac{a}{1}, f\right) = +\infty \text{ or } 0 \quad \text{for every } f \in \mathcal{U}_{[p]}.$$

Since $\left(\frac{a}{1}, f\right)$ is continuous over $\mathcal{U}_{[p]}$ and the totality of

the points $\mathfrak{f}$ for which $(\frac{a}{1}, \mathfrak{f}) = +\infty$, is nowhere dense in E by Theorem 28.1, we must have

$$(\frac{a}{1}, \mathfrak{f}) = 0 \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[p]}.$$

Since $\mathcal{U}_{[p]a} \subset \mathcal{U}_{[p]}$ by Theorem 8.2 and by the formula §15(4), we obtain hence by Theorem 28.3

$$(\frac{[p]a}{1}, \mathfrak{f}) = 0 \quad \text{for every } \mathfrak{f} \in E,$$

and consequently $[p]a = 0$ by Theorem 28.5. This relation yields by the formulas §8(1), §15(4) and by Theorem 8.2

$$[p] = [p][a] = [[p]a] = 0.$$

Therefore we have

$$\mathcal{U}_{[a]} = \left(\sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]} \right)^-.$$

Recalling Theorem 16.5 and the formula §15(4), we obtain hence

$$[p_{\nu}] \uparrow_{\nu=1}^{\infty} [a].$$

By definition of A_{ν} we have obviously

$$\nu+1 > (\frac{a}{1}, \mathfrak{f}) > \frac{1}{\nu+1} \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[p_{\nu}]},$$

and hence by assumption

$$1 \geq \frac{\varphi(\mathfrak{f})}{(\frac{a}{1}, \mathfrak{f})} \geq 0 \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[p_{\nu}]}.$$

We obtain thus an element $c_{\nu} \in R$ as

$$c_{\nu} = \int_{[p_{\nu}]} \frac{\varphi(\mathfrak{f})}{(\frac{a}{1}, \mathfrak{f})} d\mathfrak{f} a \quad (\nu = 1, 2, \dots).$$

For such $c_{\nu} \in R$ we have obviously by Theorem 20.1 and by the formulas (1), (3) in §20

$$0 \leq c_1 \leq c_2 \leq \dots, \quad c_{\nu} \leq a \quad (\nu = 1, 2, \dots).$$

Therefore there exists by Theorem 6.2 an element $c \in R$ for which

$$c_{\nu} \uparrow_{\nu=1}^{\infty} c \quad \text{For such } c \in R \text{ we have by Theorem 7.7 and by}$$

the formula §20(2)

$$[p_{\mu}]c = \lim_{\nu \rightarrow \infty} [p_{\mu}]c_{\nu} = \int_{[p_{\mu}]} \frac{\varphi(\mathfrak{f})}{(\frac{a}{1}, \mathfrak{f})} d\mathfrak{f} a \quad (\mu = 1, 2, \dots),$$

since $[p_{\mu}][p_{\nu}] = [p_{\mu}]$ for $\nu \geq \mu$ by Theorem 8.2. Thus we have

by Theorem 22.1

$$(\frac{c}{a}, \mathfrak{f}) = \frac{\varphi(\mathfrak{f})}{(\frac{a}{1}, \mathfrak{f})} \quad \text{for every } \mathfrak{f} \in \mathcal{U}_{[p_{\mu}]}.$$

This relation yields by Theorems 28.2 and 27.5 for every $\mathfrak{f} \in \sum_{\nu=1}^{\infty} \mathcal{U}_{[p_{\nu}]}$

$$\begin{aligned}\varphi(\beta) &= (\frac{c}{a}, \beta)(\frac{a}{1}, \beta) = (\frac{c}{a}, \beta)(\frac{a^2}{a}, \beta) \\ &= (\frac{c a^2}{a^2}, \beta) = (\frac{c}{1}, \beta).\end{aligned}$$

Since both $\varphi(\beta)$ and $(\frac{c}{1}, \beta)$ are continuous over E by Theorem 28.1, since we have obviously by definition

$$(\frac{a}{1}, \beta) = \varphi(\beta) = (\frac{c}{1}, \beta) = 0 \quad \text{for } \beta \in U_{[a]},$$

and since $U_{[a]} = (\sum_{p=1}^{\infty} U_{[p]})^-$ as proved just above, we have thus

$$\varphi(\beta) = (\frac{c}{1}, \beta) \quad \text{for every } \beta \in E.$$

Now we can prove the general case:

$$(\frac{a}{1}, \beta) \geq \varphi(\beta) \geq (\frac{b}{1}, \beta) \quad \text{for } \beta \in E.$$

From this relation we conclude by Theorem 28.6

$$(\frac{a^+}{1}, \beta) \geq \text{Max} \{ (\frac{a}{1}, \beta), 0 \} \geq \text{Max} \{ \varphi(\beta), 0 \} \geq 0,$$

$$(\frac{b^-}{1}, \beta) \geq \text{Max} \{ -(\frac{b}{1}, \beta), 0 \} \geq \text{Max} \{ -\varphi(\beta), 0 \} \geq 0.$$

Both functions $\text{Max} \{ \varphi(\beta), 0 \}$ and $\text{Max} \{ -\varphi(\beta), 0 \}$ are obviously continuous over E . By the case just proved, we obtain two elements c_1 and $c_2 \in R$ such that for every $\beta \in E$

$$(\frac{c_1}{1}, \beta) = \text{Max} \{ \varphi(\beta), 0 \},$$

$$(\frac{c_2}{1}, \beta) = \text{Max} \{ -\varphi(\beta), 0 \}.$$

Putting $c = c_1 - c_2$, we have by Theorem 28.4

$$(\frac{c}{1}, \beta) = (\frac{c_1}{1}, \beta) - (\frac{c_2}{1}, \beta) = \varphi(\beta).$$

The uniqueness of such $c \in R$ is obvious by Theorem 28.5.

Theorem 28.8. For every elements $a, b \in R$ we have

$$(\frac{a \vee b}{1}, \beta) = \text{Max} \{ (\frac{a}{1}, \beta), (\frac{b}{1}, \beta) \},$$

$$(\frac{a \wedge b}{1}, \beta) = \text{Min} \{ (\frac{a}{1}, \beta), (\frac{b}{1}, \beta) \}.$$

Proof. By virtue of Theorem 28.6 we have obviously

$$(\frac{a \vee b}{1}, \beta) \geq \text{Max} \{ (\frac{a}{1}, \beta), (\frac{b}{1}, \beta) \} \geq (\frac{a \wedge b}{1}, \beta).$$

Since $\text{Max} \{ (\frac{a}{1}, \beta), (\frac{b}{1}, \beta) \}$ is a continuous function on E by Theorem 28.1, there exists by Theorem 28.7 an element $c \in R$ such that

$$(\frac{c}{1}, \beta) = \text{Max} \{ (\frac{a}{1}, \beta), (\frac{b}{1}, \beta) \}.$$

For such $c \in R$ we see easily by Theorem 28.5 that $c = a \vee b$.

We also can dispose likewise of the other assertion.

As an immediate consequence of this theorem we obtain:

Theorem 28.9. For every element $a \in R$ we have

$$|(\frac{a}{1}, \mathfrak{f})| = |(\frac{1a}{1}, \mathfrak{f})|.$$

Theorem 28.10. Every semi-normal ring R is commutative, that is, we have $a\mathfrak{f} = \mathfrak{f}a$ for every elements $a, \mathfrak{f} \in R$.

Proof. If R is a normal ring, then we have by Theorems 28.4 and 28.1

$$(\frac{a\mathfrak{f}}{1}, \mathfrak{f}) = (\frac{a}{1}, \mathfrak{f})(\frac{\mathfrak{f}}{1}, \mathfrak{f}) = (\frac{\mathfrak{f}}{1}, \mathfrak{f})(\frac{a}{1}, \mathfrak{f}) = (\frac{\mathfrak{f}a}{1}, \mathfrak{f})$$

over E up to a point set being nowhere dense in E . Since both $(\frac{a\mathfrak{f}}{1}, \mathfrak{f})$ and $(\frac{\mathfrak{f}a}{1}, \mathfrak{f})$ are continuous over E , we have hence

$$(\frac{a\mathfrak{f}}{1}, \mathfrak{f}) = (\frac{\mathfrak{f}a}{1}, \mathfrak{f})$$

throughout E , and consequently $a\mathfrak{f} = \mathfrak{f}a$ by Theorem 28.5.

If R is a semi-normal ring, then to every elements $a, \mathfrak{f} \in R$ there exist by Theorem 26.9 normal factors m_1, m_2 and null factors n_1, n_2 such that

$$a = m_1 + n_1, \quad \mathfrak{f} = m_2 + n_2.$$

Then we have by Theorem 26.1

$$a\mathfrak{f} = m_1 m_2, \quad \mathfrak{f}a = m_2 m_1.$$

On the other hand, since the set of all normal factors constitutes a normal ring by Theorem 27.1, we have

$$m_1 m_2 = m_2 m_1$$

as proved just now, and hence $a\mathfrak{f} = \mathfrak{f}a$.

§29 Normality

It will be discussed in the sequel under what conditions a continuous semi-ordered ring may be semi-normal or normal. Throughout this section we shall be concerned with a continuous semi-ordered ring R alone.

An element $a \in R$ is said to be a continuous factor, if

we have

$$\lim_{\nu \rightarrow \infty} a l_\nu = \lim_{\nu \rightarrow \infty} l_\nu a = 0$$

for every sequence $l_\nu \downarrow_{\nu=1}^{\infty} 0$.

With this definition, Theorem 25.2 says that a semi-normal ring consists only of continuous factors.

If $|a|$ is a continuous factor, then $\lim_{\nu \rightarrow \infty} l_\nu = l$ implies

$$\lim_{\nu \rightarrow \infty} a l_\nu = a l \quad \text{and} \quad \lim_{\nu \rightarrow \infty} l_\nu a = l a.$$

Because, if $\lim_{\nu \rightarrow \infty} l_\nu = l$, then there exist by definition a sequence of elements $l_\nu \downarrow_{\nu=1}^{\infty} 0$ and a sequence of natural numbers μ_ν ($\nu = 1, 2, \dots$) such that

$$|l_{\mu_\nu} - l| \leq l_\nu \quad \text{for } \mu \geq \mu_\nu, \nu = 1, 2, \dots,$$

and then we have by the formula §24(4) for $\mu \geq \mu_\nu$

$$|a l_\mu - a l| \leq |a| |l_\mu - l| \leq |a| l_\nu,$$

$$|l_\mu a - l a| \leq |l_\mu - l| |a| \leq l_\nu |a|,$$

while $|a| l_\nu \downarrow_{\nu=1}^{\infty} 0$, $l_\nu |a| \downarrow_{\nu=1}^{\infty} 0$ by assumption.

An element $a \in R$ is said to be a bounded factor, if there exists a positive number α such that we have both .

$$|a| p \leq \alpha p \quad \text{and} \quad p |a| \leq \alpha p$$

for every positive element $p \in R$, and such α is called a bound of a bounded factor a .

With this definition we see easily that every bounded factor is a continuous factor, and that the totality of bounded factors is a subring of R , i.e. from bounded factors $a, b \in R$ we obtain again a bounded factor $\alpha a + \beta b$ for every real numbers α, β , and a bounded factor $a b$ by the formula §24(4), and furthermore, if $a \in R$ is a bounded factor, then every element $b \in R$ satisfying $|b| \leq |a|$ is a bounded factor too.

Theorem 29.1. In a normal ring R , in order that an element $a \in R$ be a bounded factor, it is necessary and sufficient that the absolute spectrum $(\frac{a}{1}, \beta)$ be bounded over the proper space E of R .

Proof. Let $a \in R$ be a bounded factor and let α be a bound of a . To any point $f \in E$ there exists obviously a positive element $c \in R$ such that $f \in U_{\{c\}}$. For such an element $c \in R$, since $|a|c \leq \alpha c$ by assumption, we obtain by Theorems 28.9, 28.2, and 18.11

$$|(\frac{a}{1}, f)| = (\frac{|a|}{1}, f) = (\frac{|a|c}{c}, f) \leq \alpha.$$

Therefore $(\frac{a}{1}, f)$ is bounded over E .

Conversely, if there exists a positive number α such that

$$|(\frac{a}{1}, f)| \leq \alpha \quad \text{for every } f \in E,$$

then we have by Theorems 28.4 and 28.9 for every element $c \geq 0$

$$(\frac{c|a|}{1}, f) = (\frac{|a|c}{1}, f) = (\frac{|a|}{1}, f)(\frac{c}{1}, f) \leq (\frac{\alpha c}{1}, f),$$

and consequently by Theorem 28.5

$$c|a| = |a|c \leq \alpha c,$$

that is, a is a bounded factor by definition.

Theorem 29.2. A semi-normal ring is the same as a continuous semi-ordered ring consisting only of continuous factors which are limits of some convergent sequences of bounded factors.

Proof. In a semi-normal ring, every element is a continuous factor as remarked already. We can prove furthermore that every element is a limit of some convergent sequence of bounded factors. In proving this fact, since every null factor is obviously a bounded factor by Theorem 26.1, we need only to consider the case of a normal ring R by virtue of Theorems 26.7 and 27.1. To every positive $a \in R$ there exists by Theorem 28.7 a sequence of positive elements $c_\nu \in R$ ($\nu = 1, 2, \dots$) for which

$$(\frac{c_\nu}{1}, f) = \text{Min} \{ (\frac{a}{1}, f), \nu \} \quad (\nu = 1, 2, \dots).$$

For such c_ν ($\nu = 1, 2, \dots$) we have obviously by Theorem 28.5

$$a \geq c_\nu \uparrow_{\nu=1}^{\infty}.$$

Thus there exists by Theorem 6.2 an element $c \in R$ such that

$$c_\nu \uparrow_{\nu=1}^{\infty} c \leq a.$$

By virtue of Theorem 28.6 we conclude then from $c_\nu \leq c \leq a$ that

we have for every point $f \in E$.

$$\left(\frac{a}{1}, f\right) = \lim_{\nu \rightarrow \infty} \left(\frac{c_\nu}{1}, f\right) = \left(\frac{c}{1}, f\right),$$

and hence $c = a$ by Theorem 28.5. Since all c_ν ($\nu = 1, 2, \dots$) are bounded factors by the previous theorem, it is proved that every positive element is a limit of some convergent sequence of bounded factors. On the other hand, every element $a \in R$ may be written in the form $a = a^+ - a^-$, and hence our assertion remains valid for arbitrary elements.

Next we shall prove that a continuous semi-ordered ring subject to the condition described in the theorem is a semi-normal ring. For this purpose we must first remark that if a positive $a \in R$ is a bounded factor, then we have for every positive $f \in R$

$$[af] \subseteq [f] \quad \text{and} \quad [fa] \subseteq [f],$$

because, if $af \leq \alpha f$ and $fa \leq \alpha f$, then we obtain by Theorems 8.3 and 7.15

$$[af] \subseteq [\alpha f] = [f] \quad \text{and} \quad [fa] \subseteq [\alpha f] = [f].$$

Secondly, if a positive $a \in R$ is a limit of a convergent sequence of bounded factors $c_\nu \in R$ ($\nu = 1, 2, \dots$), then, putting

$$a_\nu = (c_1^+ \cup c_2^+ \cup \dots \cup c_\nu^+) \cap a \quad (\nu = 1, 2, \dots),$$

we obtain again bounded factors a_ν ($\nu = 1, 2, \dots$) and

$$0 \leq a_\nu \uparrow_{\nu=1}^{\infty} a,$$

because we have obviously

$$0 \leq a_\nu \leq |c_1| + |c_2| + \dots + |c_\nu| \quad (\nu = 1, 2, \dots),$$

and further by Theorem 5.9

$$a \geq \bigcup_{\nu=1}^{\infty} a_\nu \quad \lim_{\nu \rightarrow \infty} (c_\nu^+ \cap a) = a^+ \cap a = a.$$

For every positive $f \in R$, since f is a continuous factor by assumption, we have hence

$$0 \leq a_\nu f \uparrow_{\nu=1}^{\infty} af \quad \text{and} \quad 0 \leq fa_\nu \uparrow_{\nu=1}^{\infty} fa.$$

Consequently we obtain by Theorem 8.5

$$[a_\nu f] \uparrow_{\nu=1}^{\infty} [af] \quad \text{and} \quad [fa_\nu] \uparrow_{\nu=1}^{\infty} [fa].$$

As all elements a_ν ($\nu = 1, 2, \dots$) are bounded factors, we have

$$[a, b] \leq [b] \text{ and } [b, a] \leq [b],$$

as remarked just above. Therefore we have for every positive element $b \in R$

$$[a, b] \leq [b] \text{ and } [b, a] \leq [b],$$

if $0 \leq a \in R$ is a limit of some convergent sequence of bounded factors.

Since every element of R may be a limit of some convergent sequence of bounded factors by assumption, we have for every positive elements $a, b \in R$

$$[a, b] \leq [a] \text{ and } [a, b] \leq [b],$$

as proved just now, and consequently by Theorem 8.2

$$[a, b] = [a][a, b] \leq [a][b],$$

as we wish to prove.

Since every bounded factor is a continuous factor, we conclude immediately from Theorem 29.2:

Theorem 29.3. A continuous semi-ordered ring is semi-normal, if it consists only of bounded factors.

We shall state consecutively algebraic conditions under which a continuous semi-ordered ring be normal. In a normal ring we have by Theorem 8.1 that the relation $a \wedge b = 0$ holds equally as well as $a \wedge b = 0$ for positive elements a and b .

Conversely we have:

Theorem 29.4. A continuous semi-ordered ring R is normal if the relation $a \wedge b = 0$ is equivalent to $a \wedge b = 0$ for positive elements $a, b \in R$.

Proof. Let a and b be arbitrary positive elements of R . Since we have by Theorem 7.10

$$(a \wedge b - [a]a \wedge b) \wedge a = 0,$$

we obtain by assumption

$$(a \wedge b - [a]a \wedge b)a = 0,$$

and hence naturally

$$(a \wedge b - [a]a \wedge b)a \wedge b = 0.$$

This relation yields by assumption

$$(ab - [a]ab) \wedge ab = 0,$$

and consequently, since $ab \geq ab - [a]ab \geq 0$, we obtain

$$ab - [a]ab = 0, \text{ namely } ab = [a]ab.$$

Therefore we have $[ab] \leq [a]$ by Theorem 8.2. We also can prove likewise $[ab] \leq [b]$, and consequently by Theorem 8.2

$$[ab] \leq [a][b]$$

for every positive element $a, b \in R$, that is, our ring R is semi-normal by definition. Furthermore our semi-normal ring

R does not contain any null factor except 0 , because the relation $a^2 = 0$ implies $|a|^2 = 0$ by the formula §25(7), and consequently by assumption

$$|a| = |a| \wedge |a| = 0.$$

Thus our ring R is normal by Theorems 26.9 and 27.2.

Theorem 29.5. A normal ring is the same as a continuous semi-ordered ring subject to the conditions:

- 1) $a \wedge b = 0$ implies $ab = 0$,
- 2) $a^2 = 0$ implies $a = 0$.

Proof. A normal ring satisfies obviously the condition 1) and 2) by the formula §25(1) and by Theorem 27.2. Therefore we need only to prove the converse. For positive elements a and $b \in R$, since

$$(a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0,$$

we obtain by the condition 1)

$$(a - (a \wedge b))(b - (a \wedge b)) = 0,$$

that is,

$$ab - (a \wedge b)b - a(a \wedge b) + (a \wedge b)^2 = 0.$$

If $a \wedge b = 0$, then, since

$$0 \leq (a \wedge b)b \leq ab, \quad 0 \leq a(a \wedge b) \leq ab,$$

we obtain hence $(a \wedge b)^2 = 0$ This relation yields by the condition 2)

$$a \wedge b = 0.$$

Therefore a continuous semi-ordered ring subject to the conditions 1) and 2) satisfies the condition in the previous theorem, that is, our assertion is reduced to the previous theorem.

Theorem 29.6. A continuous semi-ordered ring is normal, if

1) $a \wedge b = 0$ implies $ac \wedge bc = ca \wedge cb = 0$ for $c \geq 0$,

2) to every positive element there exists a positive element b such that

$$[a] \leq [ab][ba].$$

Proof. Let a and b be arbitrary positive elements.

For any positive element x satisfying $x \wedge a = 0$, we have by the condition 1)

$$a(b+c) \wedge x(b+c) = 0 \quad \text{for } c \geq 0,$$

and consequently

$$ab \wedge xc = 0 \quad \text{for } c \geq 0,$$

since $0 \leq b \leq b+c$ and $0 \leq c \leq b+c$. Therefore we obtain by Theorem 8.1

$$[ab][xc] = 0 \quad \text{for } c \geq 0,$$

if $x \wedge a = 0$. In this relation we can select by the condition

2) $C \in R$ such that $[x] \leq [xc]$. For such C we obtain obviously that

$$[ab][x] = 0 \quad \text{if } x \wedge a = 0.$$

Since $(ab - [a]ab) \wedge a = 0$ by Theorem 7.10, putting

$$x = ab - [a]ab,$$

we obtain hence by the formula §8(1) and by Theorem 7.11

$$ab - [a]ab = [ab](ab - [a]ab) = 0,$$

that is, $ab = [a]ab$, and consequently $[ab] \leq [a]$ by Theorem 8.2.

We also can prove likewise $[ab] \leq [b]$. Therefore we have by Theorem 8.2

$$[ab] \leq [a][b]$$

for every positive elements a and b . In consequence our ring is semi-normal by definition. Furthermore our ring is normal. For the condition 2) shows by Theorem 26.5 that every element is normal.

Finally we consider the case where a continuous semi-ordered ring has a unit factor, that is, an element e satisfying

$$ae = ea = a \quad \text{for every element } a.$$

If a semi-normal ring has a unit factor e , then it consists only of normal factors by Theorem 26.5, since

$$[ae] = [a] \quad \text{for every element } a.$$

Therefore we have the following

Theorem 29.7. A semi-normal ring possessing a unit factor is normal.

Theorem 29.8. If a normal ring has a unit factor e then we have

$$\left(\frac{e}{1}, p\right) = 1 \quad \text{for every point } p \in E.$$

Proof. If a normal ring has a unit factor e , then we have obviously for every element a

$$[a] = [ae] \leq [e],$$

and hence e is a complete element, because $e \perp a$ implies by Theorems 8.2 and 8.1

$$[a] = [a][e] = 0.$$

Therefore we have by definition $U_{[e]} = E$ and further by Theorems 28.2 and 18.2 for every point $p \in E$

$$\left(\frac{e}{1}, p\right) = \left(\frac{e^2}{e}, p\right) = \left(\frac{e}{e}, p\right) = 1.$$

Theorem 29.9. A continuous semi-ordered ring possessing a positive unit factor e is normal, if and only if

$$a \wedge e = 0 \quad \text{implies} \quad a = 0,$$

that is, if and only if e is a complete element.

Proof. If a normal ring has a unit factor e , then e is a complete element, as proved already in Proof of Theorem 29.8.

Conversely, if a continuous semi-ordered ring has the properties described in the theorem, then $[p]e$ is a continuous factor for every projector $[p]$, because $\ell_v \downarrow_{v=1}^{\infty}, 0$ implies

$$0 \leq ([p]e)\ell_v \leq e\ell_v = \ell_v \downarrow_{v=1}^{\infty}, 0,$$

$$0 \leq \ell_v([p]e) \leq \ell_v e = \ell_v \downarrow_{v=1}^{\infty}, 0,$$

since $0 \leq [p]e \leq e$ by Theorem 7.5.

For every projectors $[p]$ and $[q]$ we have obviously

$$[p]e = ([p]e)e = ([p]e)([q]e) + ([p]e)(e - [q]e),$$

and further by Theorem 7.8

$$0 \leq |q| \wedge ([p]e)(e - [q]e) \leq |q| \wedge (e - [q]e) = 0.$$

Consequently we obtain by Theorem 7.12

$$[p][q]e = [q]([p]e)([q]e).$$

On the other hand, since $([p]e)([q]e) \leq e([q]e) = [q]e$, we obtain by Theorems 8.3 and 8.2

$$([p]e)([q]e) \leq [q]e \leq [q].$$

Consequently we have for every projectors $[p]$ and $[q]$

$$[p][q]e = ([p]e)([q]e).$$

Now, recalling definition of integral, we have by Theorems 22.2 and 21.1

$$\begin{aligned} [p]a &= \int_{[e]} \left(\frac{a}{e}, f\right) d_f [p]e = \int_{[e]} \left(\frac{a}{e}, f\right) ([p]e)(d_f e) \\ &= ([p]e) \int_{[e]} \left(\frac{a}{e}, f\right) d_f e = ([p]e)a, \end{aligned}$$

that is, $[p]a = ([p]e)a$ for every projector $[p]$. Hence we obtain by the formula §8(1) and by Theorem 7.11

$$\begin{aligned} [a]f &= [([a]a)f] = [([a]e)a]f \\ &= [a]([a]f) = [a][a]f \leq [a]. \end{aligned}$$

Since we also can prove likewise $[a]f \leq [f]$, we obtain further by Theorem 8.2

$$[a]f \leq [a][f].$$

Thus our ring is semi-normal by definition, and furthermore normal by Theorem 29.7.

By virtue of Theorem 29.6 we have obviously:

Theorem 29.10. A continuous semi-ordered ring possessing
a positive unit factor is normal, if and only if $a \wedge b = 0$
implies

$$a c \wedge b c = c a \wedge c b = 0 \quad \text{for } c \geq 0.$$

CHAPTER V

EXTENSIONS§30 Cut extensions of semi-ordered linear spaces

Let R and $\hat{R}$ be semi-ordered linear spaces. If there exists a mapping of R into $\hat{R}$ which assigns to every element $a \in R$ an element $a^{\hat{R}} \in \hat{R}$ such that

- 1) $(\alpha a + \beta b)^{\hat{R}} = \alpha a^{\hat{R}} + \beta b^{\hat{R}}$ for every $a, b \in R$,
- 2) $a^{\hat{R}} \geq 0$ if and only if $a \geq 0$,

then $\hat{R}$ is said to be an extension of R by an extending correspondence

$$R \ni a \longrightarrow a^{\hat{R}} \in \hat{R}.$$

If $\hat{R}$ is an extension of R by an extending correspondence $R \ni a \longrightarrow a^{\hat{R}} \in \hat{R}$, then we see easily that if

$$a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}} \quad \text{or} \quad a^{\hat{R}} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$$

in $\hat{R}$ for a system of elements $a_{\lambda} \in R$ ($\lambda \in \Lambda$), then we have

$$a = \bigcup_{\lambda \in \Lambda} a_{\lambda} \quad \text{or} \quad a = \bigcap_{\lambda \in \Lambda} a_{\lambda}$$

respectively, because we have by the postulates 1) and 2) that

$$a^{\hat{R}} \geq b^{\hat{R}} \text{ in } \hat{R} \text{ if and only if } a \geq b \text{ in } R.$$

If an extension $\hat{R}$ of a semi-ordered linear space R by an extending correspondence $R \ni a \longrightarrow a^{\hat{R}} \in \hat{R}$ satisfies further

- 3) $\hat{R}$ is universally continuous,
- 4) $\bigcap_{\lambda \in \Lambda} a_{\lambda} = 0$, $a_{\lambda} \in R$ ($\lambda \in \Lambda$), implies $\bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}} = 0$ in $\hat{R}$,
- 5) to every element $\hat{a} \in \hat{R}$ there exists a system of element $a_{\lambda} \in R$ ($\lambda \in \Lambda$), such that

$$\hat{a} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}},$$

then $\hat{R}$ is said to be a cut extension of R .

If $\hat{R}$ is a cut extension of a semi-ordered linear space R by an extending correspondence $R \ni a \longrightarrow a^{\hat{R}} \in \hat{R}$, then $\hat{R}$ is archimedean by the postulate 3) and by Theorem 7.3, and hence we have for every positive element $a \in R$

$$\frac{1}{\nu} a^{\hat{R}} \downarrow_{\nu=1}^{\infty} 0,$$

which yields $\frac{1}{\nu} a \downarrow_{\nu=1}^{\infty} 0$ by the postulates 1) and 2), as remarked already. Therefore R is archimedean too. If $a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$ in R , then we have by Theorems 2.2 and 2.4

$$\bigcap_{\lambda \in \Lambda} (a - a_{\lambda}) = 0,$$

and then $\bigcap_{\lambda \in \Lambda} (a^{\hat{R}} - a_{\lambda}^{\hat{R}}) = 0$ by the postulates 1) and 4), that is,

$$a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$$

by Theorems 2.2 and 2.4. Consequently $a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$ in R implies $a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$. We also can prove likewise that $a = \bigcap_{\lambda \in \Lambda} a_{\lambda}$ in R implies $a^{\hat{R}} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$. To every element $\hat{a} \in \hat{R}$ there exists by the postulate 5) a system of elements $a_{\lambda} \in R$ ($\lambda \in \Lambda$) such that

$$-\hat{a} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}},$$

and hence we have by the postulate 1) and by Theorem 2.2

$$\hat{a} = \bigcup_{\lambda \in \Lambda} (-a_{\lambda})^{\hat{R}}.$$

Therefore we obtain the following

Theorem 30.1. If a semi-ordered linear space R has a cut extension $\hat{R}$, then R is archimedean, and for the extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ we have that

$a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$ or $a = \bigcap_{\lambda \in \Lambda} a_{\lambda}$ in R implies $a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$ or $a^{\hat{R}} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$ respectively, and that to every element $\hat{a} \in \hat{R}$ there exist two system of elements a_{λ} and $b_{\lambda} \in R$ ($\lambda \in \Lambda$) such that

$$\hat{a} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}} = \bigcup_{\lambda \in \Lambda} b_{\lambda}^{\hat{R}}.$$

For semi-ordered linear space R and $\hat{R}$, if there exists a mapping of R onto $\hat{R}$ which assigns to every element $a \in R$ an element $a^{\hat{R}} \in \hat{R}$ such that

$$(\alpha a + \beta b)^{\hat{R}} = \alpha a^{\hat{R}} + \beta b^{\hat{R}},$$

$a^{\hat{R}} \geq 0$ if and only if $a \geq 0$, and to every element, $\hat{a} \in \hat{R}$ there exists an element $a \in R$ for which $a^{\hat{R}} = \hat{a}$, then R is said to be isomorphic to $\hat{R}$ by a correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$.

Here the correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ is one-to-one, since we have $a^{\hat{R}} = 0$ if and only if $a = 0$. Therefore, if R is isomorphic to $\hat{R}$, then $\hat{R}$ is isomorphic to R by the inverse correspondence.

Theorem 30.2. If a semi-ordered linear space R has a cut extension, then its cut extension is uniquely determined up to an isomorphism.

Proof. Let $\hat{R}$ and $\tilde{R}$ be cut extensions of R by extending correspondences

$$R \ni a \rightarrow a^{\hat{R}} \in \hat{R} \quad \text{and} \quad R \ni a \rightarrow a^{\tilde{R}} \in \tilde{R}$$

respectively. By virtue of the previous theorem, to every element $\hat{a} \in \hat{R}$ there exist two systems of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) and $b_\gamma \in R$ ($\gamma \in \Gamma$) such that

$$\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\hat{R}}$$

In general, if $\bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\hat{R}}$, then we have by Theorems 2.2 and 2.4

$$\bigcap_{\lambda, \gamma} (a_\lambda - b_\gamma)^{\hat{R}} = 0,$$

and hence $\bigcap_{\lambda, \gamma} (a_\lambda - b_\gamma) = 0$. From this relation we conclude by the postulate 3)

$$\bigcap_{\lambda, \gamma} (a_\lambda - b_\gamma)^{\tilde{R}} = 0,$$

that is, $\bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\tilde{R}}$ by Theorems 2.2 and 2.4. Therefore to every element $\hat{a} \in \hat{R}$ there exists uniquely an element $\hat{a}^{\tilde{R}} \in \tilde{R}$ such that

$$\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\hat{R}}, \quad a_\lambda \in R (\lambda \in \Lambda), \quad b_\gamma \in R (\gamma \in \Gamma),$$

implies $\hat{a}^{\tilde{R}} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\tilde{R}}$. For this correspondence

$$\hat{R} \ni \hat{a} \rightarrow \hat{a}^{\tilde{R}} \in \tilde{R},$$

we also can prove likewise that $\bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\tilde{R}}$ implies $\bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\hat{R}}$, and hence further that to every element $\tilde{a} \in \tilde{R}$ there exists an element $\hat{a} \in \hat{R}$ for which $\hat{a}^{\tilde{R}} = \tilde{a}$.

If $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}}$, $a_\lambda \in R$ ($\lambda \in \Lambda$), then we have by Theorem 2.2

$$-\hat{a} = \bigcap_{\lambda \in \Lambda} (-a_\lambda)^{\hat{R}},$$

and hence by the fact as proved just above

$$(-\hat{a})^{\tilde{R}} = \bigcup_{\lambda \in \Lambda} (-a_\lambda)^{\tilde{R}} = - \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} = -\hat{a}^{\tilde{R}},$$

and furthermore we have for every positive number α

$$(\alpha \hat{a})^{\tilde{R}} = \bigcap_{\lambda \in \Lambda} (\alpha a_\lambda)^{\tilde{R}} = \alpha \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} = \alpha \hat{a}^{\tilde{R}},$$

since $\alpha \hat{a} = \bigcap_{\lambda \in \Lambda} (\alpha a_\lambda)^{\hat{R}}$ by Theorem 2.3.

If $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}}$, $\hat{b} = \bigcap_{\mu \in \Gamma} b_\mu^{\hat{R}}$, then we have by Theorem 2.6

$$\hat{a} + \hat{b} = \bigcap_{\lambda, \mu} (a_\lambda + b_\mu)^{\hat{R}},$$

and hence we have further

$$(\hat{a} + \hat{b})^{\tilde{R}} = \bigcap_{\lambda, \mu} (a_\lambda + b_\mu)^{\tilde{R}} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} + \bigcap_{\mu \in \Gamma} b_\mu^{\tilde{R}} = \hat{a}^{\tilde{R}} + \hat{b}^{\tilde{R}}.$$

Therefore we obtain for every elements $\hat{a}, \hat{b} \in \hat{R}$

$$(\alpha \hat{a} + \beta \hat{b})^{\tilde{R}} = \alpha \hat{a}^{\tilde{R}} + \beta \hat{b}^{\tilde{R}}.$$

If $\hat{a} \geq 0$ and $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}}$, $a_\lambda \in R$ ($\lambda \in \Lambda$), then we have obviously $a_\lambda \geq 0$ for every $\lambda \in \Lambda$, and hence

$$\hat{a}^{\tilde{R}} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} \geq 0.$$

Conversely, if $\hat{a}^{\tilde{R}} \geq 0$, $\hat{a} \in \hat{R}$, then, for $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}}$ we have by definition

$$\hat{a}^{\tilde{R}} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} \geq 0,$$

and hence $a_\lambda \geq 0$ for every $\lambda \in \Lambda$, which yields $\hat{a} \geq 0$ by definition.

Therefore $\hat{R}$ is isomorphic to $\tilde{R}$ by the correspondence $\hat{R} \ni \hat{a} \rightarrow \hat{a}^{\tilde{R}} \in \tilde{R}$ and we have

$$(a^{\hat{R}})^{\tilde{R}} = a^{\tilde{R}} \quad \text{for every } a \in R.$$

We shall prove next that if a semi-ordered linear space R is archimedean, then R has a cut extension. For this purpose we consider lower bounded sets of elements from R . For two lower bounded sets of elements A and B , we define $A \leq B$ to mean that every lower bound of A is a lower bound of B . With this definition we have obviously:

$$(1) \quad A \leq A \quad \text{for every lower bounded set } A,$$

$$(2) \quad A \leq B, \quad B \leq C \quad \text{implies } A \leq C.$$

$A > B$ implies $A \leq B$ by definition, but we have not necessarily $A > B$, even if $A \leq B$

For a lower bounded set of elements A , the set of elements

$$\sum_{A \leq x} x$$

is called a cut generated by A and denoted by

$$(A) \quad \text{or} \quad (x : x \in A).$$

Concerning cuts, we see easily that

$$(3) \quad (A) > (B) \text{ if and only if } A \leq B,$$

and hence $(A) > (B)$ is equivalent to $(A) \leq (B)$. Consequently we have

$$(4) \quad (A) = (B) \text{ if and only if } A \leq B \text{ and } B \leq A \text{ at the same time.}$$

For every element $a \in R$ we obtain a cut (a) generated by the set $\{a\}$ composed of a single element a , that is,

$$(a) = \{x : x \geq a\}.$$

For a lower bounded set A we have by definition $(a) \leq A$ if and only if a is a lower bound of A . Therefore, to every two cuts (A) and (B) there exists an element $a \in R$ for which

$$(a) \leq (A) \text{ and } (a) \leq (B).$$

$$(5) \quad A \leq B \text{ implies for every lower bounded set } C$$

$$(x+z : x \in A, z \in C) \leq (y+z : y \in B, z \in C).$$

Because, if $a \leq x+z$ for every $x \in A$ and $z \in C$, then we have for any $z \in C$

$$a - z \leq x \quad \text{for every } x \in A,$$

and hence $a - z \leq y$ for every $y \in B$, since $A \leq B$ by assumption. Consequently, if $a \leq x+z$ for every $x \in A$ and $z \in C$ then we have $a \leq y+z$ for every $y \in B$ and $z \in C$.

By virtue of the formula (5) we see easily that $A \leq B$ and $C \leq D$ imply

$$(x+u : x \in A, u \in C) \leq (y+v : y \in B, v \in D),$$

and hence we have by the formula (4) that if $(A) = (B)$ and $(C) = (D)$, then we have

$$(x+u : x \in A, u \in C) = (y+v : y \in B, v \in D).$$

Therefore we can define the addition for two cuts (A) and (B) as

$$(A) + (B) = (x+y : x \in A, y \in B).$$

With this definition we have obviously:

$$(6) \quad (A) + (B) = (B) + (A),$$

$$(7) \quad \{(A) + (B)\} + (C) = (A) + \{(B) + (C)\}.$$

We have obviously by definition for every cut (A)

$$(A) + (0) = (A).$$

We shall now prove that to every cut (A) there exists uniquely a cut (B) such that

$$(A) + (B) = (0).$$

For any cut (A) , if we put

$$B = \{-x : (x) \leq A\},$$

then we have obviously $x + y \geq 0$ for every $x \in A$ and $y \in B$, that is,

$$(x+y : x \in A, y \in B) \geq (0).$$

On the other hand, if $a \leq x + y$ for every $x \in A$ and $y \in B$, then for any $y \in B$ we have

$$a - y \leq x \quad \text{for every } x \in A,$$

that is, $(a-y) \in A$ for every $y \in B$. Consequently $y \in B$ implies $y - a \in B$, and hence $y \in B$ implies

$$y - \nu a \in B \quad \text{for every } \nu = 1, 2, \dots$$

Therefore, for any $x \in A$ and $y \in B$ we have

$$a \leq \frac{1}{\nu} (x+y) \quad \text{for every } \nu = 1, 2, \dots$$

Since R is archimedean by assumption and $x+y \geq 0$ for every $x \in A$ and $y \in B$, we obtain hence $a \leq 0$, and consequently by the formula (4)

$$(x+y : x \in A, y \in B) = (0),$$

that is, $(A) + (B) = (0)$ by definition. Such (B) is uniquely determined by (A) . Because, if

$$(A) + (C) = (0)$$

for another cut (C) , then we have by the formulas (6) and (7)

$$\begin{aligned}(C) &= (0) + (C) = \{(A) + (B)\} + (C) \\ &= \{(A) + (C)\} + (B) = (0) + (B) = (B).\end{aligned}$$

Therefore such a cut (B) will be denoted by $-(A)$. With this notation we have by the formulas (6) and (7) for every two cuts (A) and (B)

$$(B) + \{(A) + \{-(B)\}\} = (A) + \{(B) + \{-(B)\}\} = (A) + (0) = (A).$$

$$(8) \quad A \leq B \text{ implies for every positive number } \alpha \\ (\alpha x : x \in A) \leq (\alpha y : y \in B).$$

Because, if $a \leq \alpha x$ for every $x \in A$, then we have $\frac{1}{\alpha} a \leq x$ for every $x \in A$, and hence $\frac{1}{\alpha} a \leq y$ for every $y \in B$, since $A \leq B$ by assumption. Consequently, if $a \leq \alpha x$ for every $x \in A$, then we have $a \leq \alpha y$ for every $y \in B$.

We see easily by the formulas (4)' and (8) that $(A) = (B)$ implies for every positive number α

$$(\alpha x : x \in A) = (\alpha y : y \in B).$$

Therefore we define $\alpha(A)$ for a real number α to mean

$$\alpha(A) = \begin{cases} (\alpha x : x \in A) & \text{for } \alpha \geq 0, \\ (-\alpha)\{-(A)\} & \text{for } \alpha < 0. \end{cases}$$

With this definition we have obviously the following relations for every positive numbers α and β

$$(9) \quad \alpha(A) + \alpha(B) = \alpha \{(A) + (B)\},$$

$$(10) \quad \alpha \{\beta(A)\} = \alpha \beta(A),$$

$$(11) \quad 1(A) = (A).$$

We have further

$$(12) \quad \alpha(A) + \beta(A) = (\alpha + \beta)(A).$$

Because we have by the formula (3) for every cut (A)

$$\alpha(A) + \beta(A) = (\alpha x + \beta y : x, y \in A) \leq (\alpha + \beta)(A).$$

Consequently we have also $\alpha \{-(A)\} + \beta \{-(A)\} \leq (\alpha + \beta)\{-(A)\}$.

This relation yields by the formulas (5) and (9)

$$(\alpha + \beta)(A) \leq \alpha(A) + \beta(A).$$

We see easily further that the formulas (9), (10), and (12) remain valid for arbitrary real numbers α , β . Therefore the set of all cuts $\hat{R}$ constitutes a linear space. Furthermore $\hat{R}$ is a semi-ordered linear space by the formulas (5) and (8).

$\hat{R}$ is an extension of R by the correspondence

$$R \ni a \longrightarrow (a) \in \hat{R}.$$

In fact, we have by definition

$$(\alpha a + \beta b) = \alpha(a) + \beta(b),$$

and $a \geq 0$ is equivalent to $(a) \geq (0)$ by the formula (3).

$\hat{R}$ is universally continuous. Because, if

$$A_\lambda \geq (a) \quad \text{for every } \lambda \in A,$$

then we have by definition that

$$(A) \geq \left(\sum_{\lambda \in A} A_\lambda \right) \quad \text{for every } \lambda \in A,$$

and that $(A_\lambda) \geq (X)$ for every $\lambda \in A$ implies $\left(\sum_{\lambda \in A} A_\lambda \right) \geq (X)$.

Therefore $\hat{R}$ is universally continuous and

$$(13) \quad \left(\sum_{\lambda \in A} A_\lambda \right) = \bigcap_{\lambda \in A} (A_\lambda),$$

if there exists an element $a \in R$ such that $A_\lambda \geq (a)$ for every $\lambda \in A$. As an immediate consequence from (13) we have for every cut (A)

$$(14) \quad (A) = \bigcap_{x \in A} (x).$$

If $\bigcap_{x \in A} x = 0$ in R , then we have $(A) = (0)$ by the formula (4), and hence by the formula (14)

$$\bigcap_{x \in A} (x) = (0) \quad \text{in } \hat{R}.$$

Therefore $\hat{R}$ is a cut extension of R by the correspondence

$$R \ni a \longrightarrow (a) \in \hat{R}.$$

Thus we have proved:

Theorem 30.3. If a semi-ordered linear space R is archimedean, then R has a cut extension.

§31 Cut extension of semi-ordered rings

Let R and $\hat{R}$ be semi-ordered rings. If $\hat{R}$ is a cut extension of R as a semi-ordered linear space by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, and further if

$$0 \leq \hat{a} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}, \quad 0 \leq \hat{b} = \bigcap_{\gamma \in \Gamma} b_{\gamma}^{\hat{R}} \quad \text{in } \hat{R}$$

implies $\hat{a} \hat{b} = \bigcap_{\lambda, \gamma} (a_{\lambda} b_{\gamma})^{\hat{R}}$, then $\hat{R}$ is said to be a cut extension of a semi-ordered ring R by an extending correspondence

$$R \ni a \rightarrow a^{\hat{R}} \in \hat{R}.$$

Let a semi-ordered ring $\hat{R}$ be a cut extension of a semi-ordered ring R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$. Then we have obviously by definition for every positive elements $a, b \in \hat{R}$

$$a^{\hat{R}} b^{\hat{R}} = (ab)^{\hat{R}},$$

and hence for arbitrary elements $a, b \in R$

$$\begin{aligned} a^{\hat{R}} b^{\hat{R}} &= (a^{+\hat{R}} - a^{-\hat{R}})(b^{+\hat{R}} - b^{-\hat{R}}) \\ &= (a^{+}b^{+})^{\hat{R}} + (a^{-}b^{-})^{\hat{R}} - (a^{+}b^{-})^{\hat{R}} - (a^{-}b^{+})^{\hat{R}} \\ &= (a^{+}b^{+} + a^{-}b^{-} - a^{+}b^{-} - a^{-}b^{+})^{\hat{R}} = (ab)^{\hat{R}}. \end{aligned}$$

If $\bigcap_{\lambda \in \Lambda} \hat{a}_{\lambda} = 0$, $\hat{a}_{\lambda} \in \hat{R}$ ($\lambda \in \Lambda$), then, putting

$$\hat{a}_{\lambda} = \bigcap_{\gamma \in \Gamma} a_{\lambda, \gamma}^{\hat{R}}, \quad a_{\lambda, \gamma} \in R \quad (\lambda \in \Lambda, \gamma \in \Gamma),$$

we have $\bigcap_{\lambda, \gamma} a_{\lambda, \gamma}^{\hat{R}} = 0$ by Theorem 2.5, and hence for every positive element $\hat{b} \in \hat{R}$, putting

$$\hat{b} = \bigcap_{\delta \in \Delta} b_{\delta}^{\hat{R}}, \quad b_{\delta} \in R \quad (\delta \in \Delta),$$

we obtain by definition

$$\bigcap_{\lambda, \gamma, \delta} (a_{\lambda, \gamma} b_{\delta})^{\hat{R}} = 0.$$

On the other hand we have by definition

$$\bigcap_{\gamma, \delta} (a_{\lambda, \gamma} b_{\delta})^{\hat{R}} = \hat{a}_{\lambda} \hat{b},$$

and consequently we obtain by Theorem 2.5

$$\bigcap_{\lambda, \gamma, \delta} (a_{\lambda, \gamma} b_{\delta})^{\hat{R}} = \bigcap_{\lambda} \{ \bigcap_{\gamma, \delta} (a_{\lambda, \gamma} b_{\delta})^{\hat{R}} \} = \bigcap_{\lambda \in \Lambda} \hat{a}_{\lambda} \hat{b}.$$

Therefore $\bigcap_{\lambda \in \Lambda} \hat{a}_{\lambda} = 0$ implies $\bigcap_{\lambda \in \Lambda} \hat{a}_{\lambda} \hat{b} = 0$ for every positive element $\hat{b} \in \hat{R}$. We also can prove likewise that $\bigcap_{\lambda \in \Lambda} \hat{a}_{\lambda} = 0$ implies $\bigcap_{\lambda \in \Lambda} \hat{b} \hat{a}_{\lambda} = 0$ for every positive element $\hat{b} \in \hat{R}$. If

$$\bigcap_{\lambda \in \Lambda} a_\lambda = 0, \quad a_\lambda \in R \quad (\lambda \in \Lambda),$$

then we have $\bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = 0$ by the postulate 4) in §30, and hence

$$\bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} \ell^{\hat{R}} = \bigcap_{\lambda \in \Lambda} \ell^{\hat{R}} a_\lambda \hat{R} = 0$$

for every positive element $\ell \in R$, as proved just now. Consequently we have that

$$\bigcap_{\lambda \in \Lambda} a_\lambda = 0, \quad a_\lambda \in R \quad (\lambda \in \Lambda),$$

implies $\bigcap_{\lambda \in \Lambda} a_\lambda \ell = \bigcap_{\lambda \in \Lambda} \ell a_\lambda = 0$ for every positive element $\ell \in R$.

Thus we obtain:

Theorem 31.1. If a semi-ordered ring R has a cut extension $\hat{R}$, then R satisfies the condition that

$$\bigcap_{\lambda \in \Lambda} a_\lambda = 0 \text{ implies } \bigcap_{\lambda \in \Lambda} a_\lambda \ell = \bigcap_{\lambda \in \Lambda} \ell a_\lambda = 0 \text{ for } \ell \geq 0,$$

$\hat{R}$ also satisfies the same condition, and for the extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ we have

$$a^{\hat{R}} \ell^{\hat{R}} = (a \ell)^{\hat{R}} \quad (a, \ell \in R).$$

A semi-ordered ring R is said to be isomorphic to a semi-ordered ring $\hat{R}$ by a correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, if R is isomorphic to $\hat{R}$ as a semi-ordered linear space by this correspondence and

$$(a \ell)^{\hat{R}} = a^{\hat{R}} \ell^{\hat{R}} \quad \text{for every } a, \ell \in R.$$

Theorem 31.2. If a semi-ordered ring R has a cut extension, then its cut extension is uniquely determined up to an isomorphism.

Proof. If both semi-ordered rings $\hat{R}$ and $\tilde{R}$ are cut extensions of R respectively by correspondences

$$R \ni a \rightarrow a^{\hat{R}} \in \hat{R} \quad \text{and} \quad R \ni a \rightarrow a^{\tilde{R}} \in \tilde{R},$$

then there exists by Theorem 30.2 a correspondence $\hat{R} \ni \hat{a} \rightarrow \hat{a}^{\tilde{R}} \in \tilde{R}$ such that $\hat{R}$ is isomorphic to $\tilde{R}$ as a semi-ordered linear space by this correspondence,

$$(a^{\hat{R}})^{\tilde{R}} = a^{\tilde{R}} \quad \text{for every } a \in R,$$

and $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}}, a_\lambda \in R (\lambda \in \Lambda)$, implies $\hat{a}^{\tilde{R}} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\tilde{R}}.$

To every positive element $\hat{a} \in \hat{R}$ there exists by the postulate 5) in §30 a system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) for which $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R}$. If

$$0 \leq \hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R}, \quad 0 \leq \hat{b} = \bigcap_{\gamma \in \Gamma} b_\gamma \hat{R}$$

then we have by definition $\hat{a} \hat{b} = \bigcap_{\lambda, \gamma} (a_\lambda b_\gamma) \hat{R}$, and hence by the fact remarked just above

$$(\hat{a} \hat{b}) \hat{R} = \bigcap_{\lambda, \gamma} (a_\lambda b_\gamma) \hat{R} = \hat{a} \hat{R} \hat{b} \hat{R},$$

since we obtain $\hat{a} \hat{R} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R}$, $\hat{b} \hat{R} = \bigcap_{\gamma \in \Gamma} b_\gamma \hat{R}$ by the same reason. Therefore $\hat{R}$ is isomorphic to $\hat{R}$ as a semi-ordered ring by the same correspondence.

Theorem 31.3. If a semi-ordered ring R is archimedean and satisfies the condition:

$$\bigcap_{\lambda \in \Lambda} a_\lambda = 0 \text{ implies } \bigcap_{\lambda \in \Lambda} a_\lambda b = \bigcap_{\lambda \in \Lambda} b a_\lambda = 0 \text{ for } b \geq 0,$$

then R has a cut extension.

Proof. Since R is archimedean by assumption, by virtue of Theorem 30.3 R has a cut extension $\hat{R}$ as a semi-ordered linear space by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$. Then to every element $\hat{a} \in \hat{R}$ there exists by Theorem 30.1 two systems of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) and $b_\gamma \in R$ ($\gamma \in \Gamma$) such that

$$\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = \bigcup_{\gamma \in \Gamma} b_\gamma \hat{R}.$$

If $\bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = \bigcup_{\gamma \in \Gamma} b_\gamma \hat{R}$, $a_\lambda, b_\gamma \in R$ ($\lambda \in \Lambda$, $\gamma \in \Gamma$), then we have $\bigcap_{\lambda, \gamma} (a_\lambda - b_\gamma) \hat{R} = 0$ by Theorems 2.2 and 2.4, and hence $\bigcap_{\lambda, \gamma} (a_\lambda - b_\gamma) = 0$ by the postulate 2) in §30. This relation yields by assumption for every positive element $c \in R$

$$\bigcap_{\lambda, \gamma} (a_\lambda - b_\gamma) c = 0,$$

and consequently by Theorem 30.1

$$\bigcap_{\lambda, \gamma} ((a_\lambda c)^{\hat{R}} - (b_\gamma c)^{\hat{R}}) = 0,$$

that is, $\bigcap_{\lambda \in \Lambda} (a_\lambda c)^{\hat{R}} = \bigcup_{\gamma \in \Gamma} (b_\gamma c)^{\hat{R}}$ by Theorems 2.2 and 2.4, since $\hat{R}$ is universally continuous by the postulate 3) in §30. Therefore

$$\bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = \bigcup_{\gamma \in \Gamma} b_\gamma \hat{R}, \quad a_\lambda, b_\gamma \in R \quad (\lambda \in \Lambda, \gamma \in \Gamma)$$

implies $\bigcap_{\lambda \in \Lambda} (a_\lambda c)^{\hat{R}} = \bigcup_{\gamma \in \Gamma} (b_\gamma c)^{\hat{R}}$ for every positive element $c \in R$.

If $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = \bigcap_{\lambda \in \Lambda} c_\lambda \hat{R}$, $0 \leq t_\delta \in R$ ($\delta \in \Delta$), then, since we can put by Theorem 30.1

$$\hat{a} = \bigcup_{\lambda \in \Lambda} d_\lambda \hat{R},$$

and since we have for every $\delta \in \Delta$

$$\bigcap_{\lambda \in \Lambda} (a_\lambda t_\delta) \hat{R} = \bigcup_{\lambda \in \Lambda} (d_\lambda t_\delta) \hat{R} = \bigcap_{\lambda \in \Lambda} (c_\lambda t_\delta) \hat{R},$$

as proved just now, we obtain by Theorem 2.5

$$\bigcap_{\lambda, \delta} (a_\lambda t_\delta) \hat{R} = \bigcap_{\delta \in \Delta} \left\{ \bigcap_{\lambda \in \Lambda} (a_\lambda t_\delta) \hat{R} \right\} = \bigcap_{\delta \in \Delta} \left\{ \bigcap_{\lambda \in \Lambda} (c_\lambda t_\delta) \hat{R} \right\} = \bigcap_{\lambda, \delta} (c_\lambda t_\delta) \hat{R}.$$

Consequently if $0 \leq \hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = \bigcap_{\lambda \in \Lambda} c_\lambda \hat{R}$, $0 \leq \hat{t} = \bigcap_{\delta \in \Gamma} t_\delta \hat{R} = \bigcap_{\delta \in \Gamma} d_\delta \hat{R}$, then we can conclude

$$\bigcap_{\lambda, \delta} (a_\lambda t_\delta) \hat{R} = \bigcap_{\lambda, \delta} (c_\lambda t_\delta) \hat{R} = \bigcap_{\lambda, \delta} (c_\lambda d_\delta) \hat{R}.$$

Therefore we can define the product $\hat{a} \hat{t}$ for every positive elements $\hat{a}$, $\hat{t} \in \hat{R}$ as

$$\hat{a} \hat{t} = \bigcap_{\lambda, \delta} (a_\lambda t_\delta) \hat{R}$$

for $\hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R}$, $\hat{t} = \bigcap_{\delta \in \Gamma} t_\delta \hat{R}$. With this definition we see easily that for positive elements of $\hat{R}$ we have the following relations:

$$(\hat{a} \hat{t}) \hat{c} = \hat{a} (\hat{t} \hat{c}),$$

and $\alpha (\hat{a} \hat{t}) = (\alpha \hat{a}) \hat{t} = \hat{a} (\alpha \hat{t})$ for every real number $\alpha \geq 0$.

If $0 \leq \hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R} = \bigcup_{\lambda \in \Lambda} d_\lambda \hat{R}$, then we have for every positive element $\hat{t} \in \hat{R}$

$$\hat{a} \hat{t} \hat{R} = \bigcap_{\lambda \in \Lambda} (a_\lambda \hat{t}) \hat{R} = \bigcup_{\lambda \in \Lambda} (d_\lambda \hat{t}) \hat{R},$$

as proved just above, and consequently we have for every positive elements $\hat{t}$, $\hat{c} \in \hat{R}$ by Theorem 2.6

$$\begin{aligned} \hat{a} (\hat{t} \hat{R} + \hat{c} \hat{R}) &= \bigcap_{\lambda \in \Lambda} (a_\lambda (\hat{t} + \hat{c})) \hat{R} \geq \bigcap_{\lambda, \delta \in \Lambda} (a_\lambda \hat{t} + a_\lambda \hat{c}) \hat{R} \\ &= \bigcap_{\lambda \in \Lambda} (a_\lambda \hat{t}) \hat{R} + \bigcap_{\lambda \in \Lambda} (a_\lambda \hat{c}) \hat{R} = \hat{a} \hat{t} \hat{R} + \hat{a} \hat{c} \hat{R} \\ &= \bigcup_{\lambda \in \Lambda} (d_\lambda \hat{t}) \hat{R} + \bigcup_{\lambda \in \Lambda} (d_\lambda \hat{c}) \hat{R} \geq \bigcup_{\lambda \in \Lambda} (d_\lambda (\hat{t} + \hat{c})) \hat{R} = \hat{a} (\hat{t} \hat{R} + \hat{c} \hat{R}). \end{aligned}$$

Therefore we have for every positive elements $\hat{t}$, $\hat{c} \in \hat{R}$

$$\hat{a} (\hat{t} \hat{R} + \hat{c} \hat{R}) = \hat{a} \hat{t} \hat{R} + \hat{a} \hat{c} \hat{R} \quad \text{for } 0 \leq \hat{a} \in \hat{R}.$$

In general, if $0 \leq \hat{a} = \bigcap_{\lambda \in \Lambda} a_\lambda \hat{R}$, $0 \leq \hat{t} = \bigcap_{\delta \in \Gamma} t_\delta \hat{R}$, $0 \leq \hat{c} = \bigcap_{\delta \in \Delta} c_\delta \hat{R}$, then we have hence by Theorems 2.5 and 2.6

$$\begin{aligned}
\hat{a}(\hat{\ell} + \hat{c}) &= \bigcap_{\lambda, \delta} (a_{\lambda}(\ell_{\delta} + c_{\delta}))^{\hat{R}} \\
&= \bigcap_{\lambda, \delta} \left\{ \bigcap_{\lambda \in A} (a_{\lambda}(\ell_{\delta} + c_{\delta}))^{\hat{R}} \right\} = \bigcap_{\lambda, \delta} (\hat{a}(\ell_{\delta}^{\hat{R}} + c_{\delta}^{\hat{R}})) \\
&= \bigcap_{\lambda, \delta} (\hat{a} \ell_{\delta}^{\hat{R}} + \hat{a} c_{\delta}^{\hat{R}}) = \bigcap_{\lambda \in R} \hat{a} \ell_{\delta}^{\hat{R}} + \bigcap_{\delta \in A} \hat{a} c_{\delta}^{\hat{R}} \\
&= \bigcap_{\lambda, \delta} (a_{\lambda} \ell_{\delta})^{\hat{R}} + \bigcap_{\lambda, \delta} (a_{\lambda} c_{\delta})^{\hat{R}} = \hat{a} \hat{\ell} + \hat{a} \hat{c}.
\end{aligned}$$

Consequently we have for positive elements of $\hat{R}$

$$\hat{a}(\hat{\ell} + \hat{c}) = \hat{a} \hat{\ell} + \hat{a} \hat{c}.$$

We also can prove likewise for positive elements of $\hat{R}$

$$(\hat{\ell} + \hat{c})\hat{a} = \hat{\ell}\hat{a} + \hat{c}\hat{a}.$$

If $\bigcap_{\lambda \in A} \hat{a}_{\lambda} = 0$, then we have for every positive $\hat{\ell} \in \hat{R}$

$$\bigcap_{\lambda \in A} \hat{a}_{\lambda} \hat{\ell} = \bigcap_{\lambda \in A} \hat{\ell} \hat{a}_{\lambda} = 0.$$

Because, for $\hat{a}_{\lambda} = \bigcap_{\delta \in A} a_{\lambda, \delta}^{\hat{R}}$, $a_{\lambda, \delta} \in R$ ($\lambda \in A$, $\delta \in A$), if $\bigcap_{\lambda \in A} \hat{a}_{\lambda} = 0$, then we have by Theorem 2.5

$$\bigcap_{\lambda, \delta} a_{\lambda, \delta}^{\hat{R}} = 0,$$

and hence $\bigcap_{\lambda, \delta} a_{\lambda, \delta} = 0$ by the postulate 2) in §30. From this relation we conclude by assumption for every positive $\ell \in R$

$$\bigcap_{\lambda, \delta} a_{\lambda, \delta} \ell = \bigcap_{\lambda, \delta} \ell a_{\lambda, \delta} = 0,$$

and hence by the postulate 4) in §30

$$\bigcap_{\lambda, \delta} (a_{\lambda, \delta} \ell)^{\hat{R}} = \bigcap_{\lambda, \delta} (\ell a_{\lambda, \delta})^{\hat{R}} = 0.$$

Therefore, for every positive element $\hat{\ell} \in \hat{R}$, putting

$$\hat{\ell} = \bigcap_{\delta \in R} \ell_{\delta}^{\hat{R}},$$

we obtain by Theorem 2.5

$$\bigcap_{\lambda \in A} \hat{a}_{\lambda} \hat{\ell} = \bigcap_{\lambda \in A} \left\{ \bigcap_{\delta \in R} (a_{\lambda, \delta} \ell_{\delta})^{\hat{R}} \right\} = \bigcap_{\delta \in R} \left\{ \bigcap_{\lambda \in A} (a_{\lambda, \delta} \ell_{\delta})^{\hat{R}} \right\} = 0,$$

and similarly $\bigcap_{\lambda \in A} \hat{\ell} \hat{a}_{\lambda} = 0$.

Now we can define the product $\hat{a} \hat{\ell}$ of arbitrary elements $\hat{a}$ and $\hat{\ell} \in \hat{R}$ as

$$\hat{a} \hat{\ell} = (\hat{a}^+ \hat{\ell}^+ + \hat{a}^- \hat{\ell}^-) - (\hat{a}^+ \hat{\ell}^- + \hat{a}^- \hat{\ell}^+).$$

With this definition we shall prove that $\hat{R}$ constitutes a ring.

Since $\hat{a}^+ \wedge \hat{a}^- = 0$ by Theorem 3.6, we have

$$\hat{a}^+ \hat{\ell}^+ \wedge \hat{a}^- \hat{\ell}^+ = \hat{a}^+ \hat{\ell}^- \wedge \hat{a}^- \hat{\ell}^- = 0,$$

as proved just above. From $\hat{\ell}^+ \wedge \hat{\ell}^- = 0$ we conclude likewise

$$\hat{a}^+ \hat{\ell}^+ \wedge \hat{a}^+ \hat{\ell}^- = \hat{a}^- \hat{\ell}^+ \wedge \hat{a}^- \hat{\ell}^- = 0.$$

Therefore we obtain by Theorem 4.3

$$(\hat{a}^+ \hat{\ell}^+ + \hat{a}^- \hat{\ell}^-) \wedge (\hat{a}^+ \hat{\ell}^- + \hat{a}^- \hat{\ell}^+) = 0,$$

and consequently by Theorem 3.8

$$(\hat{a} \hat{\ell})^+ = \hat{a}^+ \hat{\ell}^+ + \hat{a}^- \hat{\ell}^-, \quad (\hat{a} \hat{\ell})^- = \hat{a}^+ \hat{\ell}^- + \hat{a}^- \hat{\ell}^+.$$

These relations enable us to conclude

$$\begin{aligned} ((\hat{a} \hat{\ell}) \hat{c})^+ &= (\hat{a}^+ \hat{\ell}^+ + \hat{a}^- \hat{\ell}^-) \hat{c}^+ + (\hat{a}^+ \hat{\ell}^- + \hat{a}^- \hat{\ell}^+) \hat{c}^- \\ &= \hat{a}^+ (\hat{\ell}^+ \hat{c}^+ + \hat{\ell}^- \hat{c}^-) + \hat{a}^- (\hat{\ell}^+ \hat{c}^- + \hat{\ell}^- \hat{c}^+) = (\hat{a} (\hat{\ell} \hat{c}))^+, \end{aligned}$$

and similarly $((\hat{a} \hat{\ell}) \hat{c})^- = (\hat{a} (\hat{\ell} \hat{c}))^-$. Therefore we obtain for arbitrary elements of $\hat{R}$

$$(\hat{a} \hat{\ell}) \hat{c} = \hat{a} (\hat{\ell} \hat{c}).$$

We have obviously by definition for arbitrary elements of $\hat{R}$

$$\hat{a} \hat{\ell} = \hat{a}^+ \hat{\ell} - \hat{a}^- \hat{\ell} = \hat{a} \hat{\ell}^+ - \hat{a} \hat{\ell}^-.$$

Since $\hat{a}^+ + \hat{\ell}^+ + (\hat{a} + \hat{\ell})^- = \hat{a}^- + \hat{\ell}^- + (\hat{a} + \hat{\ell})^+$, we obtain

$$\hat{a}^+ \hat{c}^+ + \hat{\ell}^+ \hat{c}^+ + (\hat{a} + \hat{\ell})^- \hat{c}^+ = \hat{a}^- \hat{c}^+ + \hat{\ell}^- \hat{c}^+ + (\hat{a} + \hat{\ell})^+ \hat{c}^+,$$

as proved already for positive elements of $\hat{R}$, and consequently

$$\hat{a} \hat{c}^+ + \hat{\ell} \hat{c}^+ = (\hat{a} + \hat{\ell}) \hat{c}^+,$$

as remarked just now. We also can prove likewise

$$\hat{a} \hat{c}^- + \hat{\ell} \hat{c}^- = (\hat{a} + \hat{\ell}) \hat{c}^-.$$

Hence we obtain for arbitrary elements of $\hat{R}$

$$\hat{a} \hat{c} + \hat{\ell} \hat{c} = (\hat{a} + \hat{\ell}) \hat{c}.$$

We also can prove likewise

$$\hat{c} \hat{a} + \hat{c} \hat{\ell} = \hat{c} (\hat{a} + \hat{\ell}).$$

We can prove further by the similar methods

$$(\alpha \hat{a}) \hat{\ell} = \hat{a} (\alpha \hat{\ell}) = \alpha \hat{a} \hat{\ell}$$

for arbitrary elements $\hat{a}, \hat{\ell} \in \hat{R}$ and for every real number α .

Therefore $\hat{R}$ constitutes a ring by definition. Furthermore it

is evident by the definition of the product of positive elements

that $\hat{R}$ is a cut extension of R as a semi-ordered ring by the same correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$

Theorem 31.4. If a semi-ordered ring R is archimedean and satisfies the condition that to every positive element $a \in R$ there exists a positive number α such that

$$ax \leq \alpha x \text{ and } xa \leq \alpha x \quad \text{for } 0 \leq x \in R$$

then R has a cut extension, and its cut extension is a semi-normal ring consisting only of bounded factors.

Proof. If $\bigcap_{\lambda \in \Lambda} a_\lambda = 0$, $a_\lambda \in R$ ($\lambda \in \Lambda$), then, since to every positive element $b \in R$ there exists by assumption a positive number β such that

$$a_\lambda b \leq \beta a_\lambda \text{ and } b a_\lambda \leq \beta a_\lambda \quad \text{for every } \lambda \in \Lambda,$$

we conclude easily by Theorem 2.3 for every positive $b \in R$

$$\bigcap_{\lambda \in \Lambda} a_\lambda b = \bigcap_{\lambda \in \Lambda} b a_\lambda = 0.$$

Therefore R has a cut extension $\hat{R}$ by the previous theorem.

To every element $\hat{a} \in \hat{R}$ there exists by Theorem 30.1 a positive element $a \in R$ such that $|\hat{a}| \leq a \hat{R}$ for the extending correspondence $R \ni a \rightarrow a \hat{R} \in \hat{R}$. Corresponding to such $a \in R$ there exists by assumption a positive number α such that

$$ax \leq \alpha x \text{ and } xa \leq \alpha x \quad \text{for } 0 \leq x \in R.$$

Then we have by the postulate 2) in §30

$$|\hat{a}| x \hat{R} \leq \alpha x \hat{R} \quad \text{and} \quad x \hat{R} |\hat{a}| \leq \alpha x \hat{R} \quad \text{for } 0 \leq x \in R.$$

For every positive element $\hat{b} \in \hat{R}$, putting by Theorem 30.1

$$\hat{b} = \bigcap_{\lambda \in \Lambda} b_\lambda \hat{R}, \quad b_\lambda \in R \quad (\lambda \in \Lambda),$$

we obtain hence by Theorem 2.3

$$|\hat{a}| \hat{b} \leq \alpha \bigcap_{\lambda \in \Lambda} b_\lambda \hat{R} = \alpha \hat{b},$$

$$\hat{b} |\hat{a}| \leq \alpha \bigcap_{\lambda \in \Lambda} b_\lambda \hat{R} = \alpha \hat{b}.$$

Therefore every element $\hat{a} \in \hat{R}$ is a bounded factor of $\hat{R}$ by definition. Accordingly $\hat{R}$ is a semi-normal ring by Theorem 29.3.

§32 Complete semi-ordered linear spaces

A semi-ordered linear space R is said to be complete, if R is continuous and satisfies the condition that the series $\sum_{\nu=1}^{\infty} a_{\nu}$ is convergent for every orthogonal sequence $a_{\nu} \in R$ ($\nu = 1, 2, \dots$).

Theorem 32.1. If a semi-ordered linear space R is complete, then to every sequence of elements $p_{\nu} \in R$ ($\nu = 1, 2, \dots$) there exists $\bigcup_{\nu=1}^{\infty} [p_{\nu}]$ and we have

$$\left(\bigcup_{\nu=1}^{\infty} [p_{\nu}] \right) a = \bigcup_{\nu=1}^{\infty} [p_{\nu}] a \text{ for } 0 \leq a \in R.$$

Proof. By virtue of Theorem 8.7 we need only to prove that to every sequence of elements $p_{\nu} \in R$ ($\nu = 1, 2, \dots$) there exists an element $p_0 \in R$ for which $[p_{\nu}] \leq [p_0]$ ($\nu = 1, 2, \dots$).

For every sequence of elements $p_{\nu} \in R$ ($\nu = 1, 2, \dots$), putting

$$q_1 = |p_1|, \quad q_{\nu} = (1 - [|p_1| + \dots + |p_{\nu-1}|])(|p_1| + \dots + |p_{\nu}|)$$

for $\nu = 2, 3, \dots$, we have by the formula §8(8) and Theorem 8.3

$$[q_{\nu}] = [|p_1| + \dots + |p_{\nu}|] - [|p_1| + \dots + |p_{\nu-1}|]$$

for $\nu = 2, 3, \dots$. Then, since $[q_{\nu}][q_{\mu}] = 0$ for $\nu \neq \mu$ by Theorem 8.2, q_{ν} ($\nu = 1, 2, \dots$) is an orthogonal sequence, and hence we can put

$$p_0 = \sum_{\nu=1}^{\infty} q_{\nu},$$

since R is complete by assumption. For such $p_0 \in R$ we have by Theorems 8.3 and 8.4

$$[p_0] \geq [q_1] + \dots + [q_{\nu}] = [|p_1| + \dots + |p_{\nu}|] \geq [p_{\nu}]$$

for every $\nu = 1, 2, \dots$, as we wish to prove.

Theorem 32.2. If a semi-ordered linear space R is complete, then for every element $a \in R$, every almost finite continuous function $\varphi(\mathfrak{P})$ on $\mathcal{U}_{[a]}$ is integrable by a in $\mathcal{U}_{[a]}$.

Proof. If a continuous function $\varphi(\mathfrak{P})$ is almost finite in $\mathcal{U}_{[a]}$, then there exists by Theorem 21.7 a sequence of projectors $[p_{\nu}] \uparrow_{\nu=1}^{\infty} [a]$ such that $\varphi(\mathfrak{P})$ is bounded in $\mathcal{U}_{[p_{\nu}]}$ for

every $\nu = 1, 2, \dots$. For such $[p_\nu]$ ($\nu = 1, 2, \dots$) we have by the formula §20(3), putting $p_0 = 0$ for convention,

$$\int_{[p_\nu]} \varphi(f) d\mathfrak{f} a = \sum_{\mu=1}^{\nu} \int_{[p_\mu] - [p_{\mu-1}]} \varphi(f) d\mathfrak{f} a \quad (\nu = 1, 2, \dots)$$

and $\int_{[p_\mu] - [p_{\mu-1}]} \varphi(f) d\mathfrak{f} a$ ($\mu = 1, 2, \dots$) is an orthogonal sequence, because we have by the formula §20(2)

$$\int_{[p_\mu] - [p_{\mu-1}]} \varphi(f) d\mathfrak{f} a = ([p_\mu] - [p_{\mu-1}]) \int_{[p_\mu] - [p_{\mu-1}]} \varphi(f) d\mathfrak{f} a,$$

and $([p_\nu] - [p_{\nu-1}])([p_\mu] - [p_{\mu-1}]) = 0$ for $\nu \neq \mu$ by Theorem 8.2. Since

R is complete by assumption, there exists then the limit

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f) d\mathfrak{f} a = \sum_{\nu=1}^{\infty} \int_{[p_\nu] - [p_{\nu-1}]} \varphi(f) d\mathfrak{f} a,$$

and consequently $\varphi(f)$ is integrable by a in $\mathcal{U}_{[a]}$ by Theorem 21.9.

We also can prove by the similar methods:

Theorem 32.3. If a semi-ordered linear space R is complete, then for every resolution $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) of an arbitrary element $a \in R$, every finite continuous function $\varphi(\lambda)$ ($-\infty < \lambda < +\infty$) is integrable by a_λ ($-\infty < \lambda < +\infty$).

A semi-ordered linear space R is said to be universally complete, if R is universally continuous and satisfies the condition that every orthogonal system of positive elements is upper bounded.

With this definition, if a semi-ordered linear space R is universally complete, then to every orthogonal system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) there exists $\bigcup_{\lambda \in \Lambda} a_\lambda$, and hence for every orthogonal sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) the series $\sum_{\nu=1}^{\infty} |a_\nu|$ is convergent by Theorem 5.15, and consequently the series $\sum_{\nu=1}^{\infty} a_\nu$ is convergent by Theorem 6.6. Therefore, if a semi-ordered linear space R is universally complete, then R is complete..

Theorem 32.4. If a semi-ordered linear space R is universally complete, then R has a complete element.

Proof. By virtue of Theorem 4.6 there exists a complete orthogonal system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$). Since

R is universally complete by assumption, there exists by definition an element $c \in R$ for which $c \geq a_\lambda$ for every $\lambda \in \Lambda$. Such $c \in R$ is a complete element, because $c \perp x$ implies by Theorem 4.1 $x \perp a_\lambda$ for every $\lambda \in \Lambda$, and hence $x = 0$, since $a_\lambda \in R$ ($\lambda \in \Lambda$) is a complete system.

Theorem 32.5. If a semi-ordered linear space R is complete, universally continuous, and has a complete element, then R is universally complete.

Proof. Let $c \in R$ be a complete element of R . For an arbitrary orthogonal system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$), putting

$$a_{\lambda, \nu} = [(\nu | c | - a_\lambda)^+ - ((\nu-1) | c | - a_\lambda)^+] a_\lambda \quad (\nu = 1, 2, \dots),$$

we obtain again an orthogonal system of positive elements

$$a_{\lambda, \nu} \in R \quad (\lambda \in \Lambda, \nu = 1, 2, \dots)$$

by Theorems 7.3 and 8.2, since we have by Theorem 8.3

$$[(\nu | c | - a_\lambda)^+] \geq [((\nu-1) | c | - a_\lambda)^+] \quad (\nu = 1, 2, \dots).$$

For such $a_{\lambda, \nu}$ ($\lambda \in \Lambda, \nu = 1, 2, \dots$), since we have by Theorem 7.16

$$[(\nu | c | - a_\lambda)^+] (\nu | c | - a_\lambda) \geq 0,$$

we obtain for every $\lambda \in \Lambda$

$$a_{\lambda, \nu} \leq [(\nu | c | - a_\lambda)^+] a_\lambda \leq \nu | c |,$$

and hence we can put by Theorem 6.7

$$b_\nu = \bigcup_{\lambda \in \Lambda} a_{\lambda, \nu} \quad (\nu = 1, 2, \dots),$$

since R is universally continuous by assumption. Then

b_ν ($\nu = 1, 2, \dots$) is an orthogonal sequence by Theorem 4.4,

and consequently we can put further

$$a = \sum_{\nu=1}^{\infty} b_\nu,$$

since R is complete by assumption. For such a we have

obviously $a \geq a_{\lambda, \nu}$ for every $\nu = 1, 2, \dots$, and hence by Theorem 5.15 for every $\nu = 1, 2, \dots$

$$[(\nu | c | - a_\lambda)^+] a_\lambda = \bigcup_{\mu=1}^{\nu} a_{\lambda, \mu} \leq a.$$

On the other hand we have by Theorem 8.14

$$[(\nu|c| - a_\lambda)^+] \uparrow_{\nu=1}^{\infty} [c].$$

Since $[c] = 1$ by Theorem 7.14, we obtain hence by Theorem 8.7

$$[(\nu|c| - a_\lambda)^+] a_\lambda \uparrow_{\nu=1}^{\infty} a_\lambda,$$

and consequently $a \geq a_\lambda$ for every $\lambda \in \Lambda$. Therefore R is universally complete by definition.

§33 Completions

Let R be a continuous semi-ordered linear space. An extension $\hat{R}$ of R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ is said to be a completion of R by this correspondence, if

- 1) $\hat{R}$ is complete,
- 2) $a^{\hat{R}} \perp b^{\hat{R}}$ if and only if $a \perp b$,
- 3) $a = \sum_{\nu=1}^{\infty} a_\nu$ for an orthogonal sequence $a_\nu \in R$ ($\nu=1, 2, \dots$) implies

$$a^{\hat{R}} = \sum_{\nu=1}^{\infty} a_\nu^{\hat{R}},$$

- 4) to every element $\hat{a} \in \hat{R}$ there exists an orthogonal sequence $a_\nu \in R$ ($\nu=1, 2, \dots$) for which

$$\hat{a} = \sum_{\nu=1}^{\infty} a_\nu^{\hat{R}}.$$

Theorem 33.1. Let $\hat{R}$ be a completion of a continuous semi-ordered linear space R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$. If

$$a^{\hat{R}} \geq \hat{f} \geq 0, \quad a \in R, \quad \hat{f} \in \hat{R},$$

then there exists an element $b \in R$ for which $\hat{f} = b^{\hat{R}}$. If

$$a = \bigcup_{\lambda \in \Lambda} a_\lambda \quad \text{or} \quad a = \bigcap_{\lambda \in \Lambda} a_\lambda \quad \text{in } R,$$

then we have respectively

$$a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}} \quad \text{or} \quad a^{\hat{R}} = \bigcap_{\lambda \in \Lambda} a_\lambda^{\hat{R}} \quad \text{in } \hat{R}.$$

Proof. To every $\hat{f} \in \hat{R}$ there exists by the postulate 4)

an orthogonal sequence $b_\nu \in R$ ($\nu=1, 2, \dots$) such that

$$\hat{f} = \sum_{\nu=1}^{\infty} b_\nu^{\hat{R}}.$$

If $a^{\hat{R}} \geq \hat{b} \geq 0$, $a \in R$, then we have by Theorems 7.2, 7.5, 7.12

$$a^{\hat{R}} \geq [b_{\nu}^{\hat{R}}] \hat{b} = b_{\nu}^{\hat{R}} \geq 0,$$

and hence for every $\mu = 1, 2, \dots$

$$a^{\hat{R}} \geq \sum_{\nu=1}^{\mu} b_{\nu}^{\hat{R}} \geq 0.$$

These relations yield by the postulates 1) and 2) in §30

$$a \geq b_{\nu} \geq 0, \quad a \geq \sum_{\nu=1}^{\mu} b_{\nu} \quad (\mu = 1, 2, \dots),$$

and hence $\sum_{\nu=1}^{\infty} b_{\nu}$ is convergent, since R is continuous by assumption. Putting

$$b = \sum_{\nu=1}^{\infty} b_{\nu},$$

we have then by the postulates 2) and 3)

$$b^{\hat{R}} = \sum_{\nu=1}^{\infty} b_{\nu}^{\hat{R}} = \hat{b}.$$

For $\bigcap_{\lambda \in A} a_{\lambda} = 0$, $a_{\lambda} \in R$ ($\lambda \in A$), we have obviously $a_{\lambda}^{\hat{R}} \geq 0$ for every $\lambda \in A$. If

$$0 \leq \hat{b} \leq a_{\lambda}^{\hat{R}} \quad \text{for every } \lambda \in A,$$

then there exists an element $b \in R$ for which $\hat{b} = b^{\hat{R}}$, as proved just now. For such $b \in R$ we have obviously

$$0 \leq b \leq a_{\lambda} \quad \text{for every } \lambda \in A,$$

and hence $b = 0$. Therefore $\bigcap_{\lambda \in A} a_{\lambda} = 0$, $a_{\lambda} \in R$ ($\lambda \in A$), implies

$$\bigcap_{\lambda \in A} a_{\lambda}^{\hat{R}} = 0.$$

If $a = \bigcup_{\lambda \in A} a_{\lambda}$ in R , then we have by Theorems 2.2 and 2.4

$$\bigcap_{\lambda \in A} (a - a_{\lambda}) = 0,$$

and hence $\bigcap_{\lambda \in A} (a^{\hat{R}} - a_{\lambda}^{\hat{R}}) = 0$, as proved just now. Consequently

$a = \bigcup_{\lambda \in A} a_{\lambda}$, $a_{\lambda} \in R$ ($\lambda \in A$), implies

$$a^{\hat{R}} = \bigcup_{\lambda \in A} a_{\lambda}^{\hat{R}}.$$

We also can prove likewise that $a = \bigcap_{\lambda \in A} a_{\lambda}$ in R implies

$$a^{\hat{R}} = \bigcap_{\lambda \in A} a_{\lambda}^{\hat{R}}.$$

As an immediate consequence from this theorem we obtain:

Theorem 33.2. If $\hat{R}$ is a completion of a continuous semi-

ordered linear space R by an extending correspondence

$R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, then we have for every element $a \in R$

$$a^{+\hat{R}} = a^{\hat{R}+}, \quad a^{-\hat{R}} = a^{\hat{R}-}, \quad |a|^{\hat{R}} = |a^{\hat{R}}|.$$

Theorem 33.3. If $\hat{R}$ is a completion of a continuous semi-ordered linear space R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, then for any complete orthogonal system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) in R we obtain again a complete orthogonal system $a_\lambda^{\hat{R}} \in \hat{R}$ ($\lambda \in \Lambda$) in $\hat{R}$.

Proof. It is evident by definition that for a complete orthogonal system of elements $a_\lambda \in R$ ($\lambda \in \Lambda$) in R we obtain again an orthogonal system $a_\lambda^{\hat{R}} \in \hat{R}$ ($\lambda \in \Lambda$) in $\hat{R}$. If

$$\hat{e} \perp a_\lambda^{\hat{R}} \quad \text{for every } \lambda \in \Lambda,$$

then there exists by definition an orthogonal sequence of elements $b_\nu \in R$ ($\nu = 1, 2, \dots$) for which

$$\hat{e} = \sum_{\nu=1}^{\infty} b_\nu^{\hat{R}},$$

and we have by Theorems 5.15 and 4.1

$$b_\nu^{\hat{R}} \perp a_\lambda^{\hat{R}} \quad \text{for every } \lambda \in \Lambda, \nu = 1, 2, \dots$$

These relations yield by the postulate 2)

$$b_\nu \perp a_\lambda \quad \text{for every } \lambda \in \Lambda, \nu = 1, 2, \dots$$

Since $a_\lambda \in R$ ($\lambda \in \Lambda$) is a complete system in R by assumption, we obtain hence $b_\nu = 0$ for every $\nu = 1, 2, \dots$, and consequently $\hat{e} = 0$. Therefore $a_\lambda^{\hat{R}} \in \hat{R}$ ($\lambda \in \Lambda$) is also a complete system in $\hat{R}$.

Theorem 33.4. Every continuous semi-ordered linear space has a completion uniquely up to an isomorphism.

Proof. Let R be a continuous semi-ordered linear space. Between two orthogonal sequence of finite or countable elements

a_ν and $b_\nu \in R$ ($\nu = 1, 2, \dots$) we define equivalence

$$\{a_\nu\}_\nu \sim \{b_\nu\}_\nu$$

to mean that to every projector $[p]$ there exists a sequence of projectors $[p_\mu] \uparrow_{\mu=1}^\infty [p]$ such that

$$\sum_{\mu=1}^\infty [p_\mu] a_\nu = \sum_{\mu=1}^\infty [p_\mu] b_\nu \quad \text{for every } \nu = 1, 2, \dots$$

With this definition the equivalence conditions are satisfied, i.e.

$$(1) \quad \{a_\nu\}_\nu \sim \{a_\nu\}_\nu.$$

Because, to every projector $[p]$; putting

$$[p_\mu] = \bigcup_{\nu=1}^{\infty} [a_\nu][p] + ([p] - \bigcup_{\nu=1}^{\infty} [a_\nu][p]) \quad (\mu=1, 2, \dots),$$

we have by Theorems 8.4 and 7.12

$$[p_\mu] \uparrow_{\mu=1}^{\infty} [p],$$

$$\sum_{\nu=1}^{\infty} [p_\mu] a_\nu = [p] \sum_{\nu=1}^{\infty} a_\nu \quad (\mu=1, 2, \dots).$$

$$(2) \quad \{a_\nu\}_\nu \sim \{b_\nu\}_\nu \text{ implies obviously } \{b_\nu\}_\nu \sim \{a_\nu\}_\nu.$$

$$(3) \quad \{a_\nu\}_\nu \sim \{b_\nu\}_\nu, \{b_\nu\}_\nu \sim \{c_\nu\}_\nu \text{ implies } \{a_\nu\}_\nu \sim \{c_\nu\}_\nu.$$

Because, to every projector $[p]$ there exist by definition

$$[p_\mu] \uparrow_{\mu=1}^{\infty} [p] \text{ and } [q_\mu] \uparrow_{\mu=1}^{\infty} [p]$$

such that we have for every $\mu=1, 2, \dots$

$$\sum_{\nu=1}^{\infty} [p_\mu] a_\nu = \sum_{\nu=1}^{\infty} [p_\mu] b_\nu \text{ and } \sum_{\nu=1}^{\infty} [q_\mu] b_\nu = \sum_{\nu=1}^{\infty} [q_\mu] c_\nu,$$

and then we obtain by Theorem 8.10 $[p_\mu][q_\mu] \uparrow_{\mu=1}^{\infty} [p]$ and

$$\sum_{\nu=1}^{\infty} [p_\mu][q_\mu] a_\nu = \sum_{\nu=1}^{\infty} [p_\mu][q_\mu] b_\nu = \sum_{\nu=1}^{\infty} [p_\mu][q_\mu] c_\nu.$$

Therefore we can consider classification by this equivalence.

For orthogonal sequences $\{a_\nu\}_\nu$, $\{b_\nu\}_\nu$, $\{c_\nu\}_\nu$ and real numbers α , β , if to every projector $[p]$ there exists a sequence of projectors $[p_\mu] \uparrow_{\mu=1}^{\infty} [p]$ such that we have

$$\alpha \sum_{\nu=1}^{\infty} [p_\mu] a_\nu + \beta \sum_{\nu=1}^{\infty} [p_\mu] b_\nu = \sum_{\nu=1}^{\infty} [p_\mu] c_\nu$$

for every $\mu=1, 2, \dots$, then we shall write

$$\alpha \{a_\nu\}_\nu + \beta \{b_\nu\}_\nu \sim \{c_\nu\}_\nu.$$

With this definition we have that

$$\alpha \{a_\nu\}_\nu + \beta \{b_\nu\}_\nu \sim \{c_\nu\}_\nu, \quad \alpha \{a'_\nu\}_\nu + \beta \{b'_\nu\}_\nu \sim \{c'_\nu\}_\nu,$$

$$\{a_\nu\}_\nu \sim \{a'_\nu\}_\nu, \quad \{b_\nu\}_\nu \sim \{b'_\nu\}_\nu$$

implies $\{c_\nu\}_\nu \sim \{c'_\nu\}_\nu$. Because to every projector $[p]$ there exist by definition sequences of projectors

$$[p_\mu], [q_\mu], [p'_\mu], [q'_\mu] \uparrow_{\mu=1}^{\infty} [p]$$

such that

$$\alpha \sum_{\nu=1}^{\infty} [p_\mu] a_\nu + \beta \sum_{\nu=1}^{\infty} [p_\mu] b_\nu = \sum_{\nu=1}^{\infty} [p_\mu] c_\nu,$$

$$\alpha \sum_{\nu=1}^{\infty} [q_\mu] a'_\nu + \beta \sum_{\nu=1}^{\infty} [q_\mu] b'_\nu = \sum_{\nu=1}^{\infty} [q_\mu] c'_\nu,$$

$$\sum_{\nu=1}^{\infty} [p'_\mu] a_\nu = \sum_{\nu=1}^{\infty} [p'_\mu] a'_\nu, \quad \sum_{\nu=1}^{\infty} [q'_\mu] b_\nu = \sum_{\nu=1}^{\infty} [q'_\mu] b'_\nu.$$

and hence by Theorem 8.10 $[p_\mu][q_\mu][p'_\mu][q'_\mu] \uparrow_{\mu=1}^\infty [p]$ and

$$\sum_{\mu=1}^\infty [p_\mu][q_\mu][p'_\mu][q'_\mu]c_\mu = \sum_{\mu=1}^\infty [p_\mu][q_\mu][p'_\mu][q'_\mu]c'_\mu.$$

Furthermore we can prove that to every orthogonal sequences $\{a_\nu\}$, $\{b_\nu\}$ and real numbers α , β there exists an orthogonal sequence $\{c_\nu\}$ such that

$$\alpha\{a_\nu\} + \beta\{b_\nu\} \sim \{c_\nu\}.$$

For two orthogonal sequences $\{a_\nu\}$, $\{b_\nu\}$, let $[p_\nu]$ ($\nu = 1, 2, \dots$) be a sequence of projectors composed of all projectors

$$[a_\nu][b_\mu], \quad [a_\nu] - \bigcup_{\mu=1}^\infty [a_\nu][b_\mu], \quad [b_\mu] - \bigcup_{\nu=1}^\infty [a_\nu][b_\mu]$$

for all $\nu, \mu = 1, 2, \dots$. Then we see easily that

$$[p_\nu][p_\mu] = 0 \quad \text{for } \nu \neq \mu,$$

$$[a_\nu] = \bigcup_{\mu=1}^\infty [a_\nu][p_\mu], \quad [b_\nu] = \bigcup_{\mu=1}^\infty [b_\nu][p_\mu] \quad (\nu = 1, 2, \dots),$$

and further $[p_\mu]a_\nu = 0$, $[p_\mu]b_\nu = 0$ for every $\nu = 1, 2, \dots$

except for at most one of ν respectively. Consequently we have by Theorem 8.11 and by the formula §8(6)

$$a_\nu = \sum_{\mu=1}^\infty [p_\mu]a_\nu, \quad b_\nu = \sum_{\mu=1}^\infty [p_\mu]b_\nu \quad (\nu = 1, 2, \dots).$$

For such $[p_\nu]$ ($\nu = 1, 2, \dots$), putting

$$c_\mu = \alpha \sum_{\nu=1}^\infty [p_\mu]a_\nu + \beta \sum_{\nu=1}^\infty [p_\mu]b_\nu \quad (\mu = 1, 2, \dots),$$

we obtain an orthogonal sequence $\{c_\nu\}$ by Theorems 4.2, 4.3, and 7.9. For every projector $[p]$, putting

$$[q_f] = \bigcup_{\mu=1}^\infty [p_\mu][p] + ([p] - \bigcup_{\mu=1}^\infty [p_\mu][p]) \quad (f = 1, 2, \dots),$$

we have by the formula §8(3) and Theorem 8.4 $[q_f] \uparrow_{f=1}^\infty [p]$ and

$$\begin{aligned} \sum_{\mu=1}^\infty [q_f]c_\mu &= \alpha \sum_{\mu=1}^\infty \sum_{\nu=1}^\infty [p][p_\mu]a_\nu + \beta \sum_{\mu=1}^\infty \sum_{\nu=1}^\infty [p][p_\mu]b_\nu \\ &= \alpha \sum_{\nu=1}^\infty [q_f]a_\nu + \beta \sum_{\nu=1}^\infty [q_f]b_\nu, \end{aligned}$$

since $[p_\nu]([p] - \bigcup_{\mu=1}^\infty [p_\mu][p]) = 0$ for every $\nu = 1, 2, \dots$. Consequently we have

$$\alpha\{a_\nu\} + \beta\{b_\nu\} \sim \{c_\nu\}.$$

Therefore we see easily that the set of all equivalent classes of orthogonal sequences $\hat{R}$ constitute a linear space such that

$$\hat{R} \ni \hat{a} \ni \{a_\nu\}, \quad \hat{R} \ni \hat{b} \ni \{b_\nu\}$$

implies for every real numbers α, β

$$\hat{R} \ni \alpha \hat{a} + \beta \hat{b} \ni \{a_\nu\}_\nu \sim \{\alpha a_\nu\}_\nu + \{\beta b_\nu\}_\nu.$$

For two orthogonal sequences $\{a_\nu\}_\nu, \{b_\nu\}_\nu$ we shall write

$$\{a_\nu\}_\nu \geq \{b_\nu\}_\nu,$$

if to every projector $[p]$ there exists a sequence of projectors

$[p_\mu] \uparrow_{\mu=1}^\infty [p]$ such that

$$\sum_{\mu=1}^\infty [p_\mu] a_\nu \geq \sum_{\mu=1}^\infty [p_\mu] b_\nu \quad (\mu = 1, 2, \dots).$$

With this definition we can prove easily by the similar methods used just above that

$$(4) \quad \{a_\nu\}_\nu \sim \{b_\nu\}_\nu \text{ implies } \{a_\nu\}_\nu \geq \{b_\nu\}_\nu,$$

$$(5) \quad \{a_\nu\}_\nu \geq \{b_\nu\}_\nu, \{b_\nu\}_\nu \geq \{a_\nu\}_\nu \text{ implies } \{a_\nu\}_\nu \sim \{b_\nu\}_\nu,$$

$$(6) \quad \{a_\nu\}_\nu \geq \{b_\nu\}_\nu \text{ implies } \{a_\nu\}_\nu + \{c_\nu\}_\nu \geq \{b_\nu\}_\nu + \{c_\nu\}_\nu,$$

$$(7) \quad \{a_\nu\}_\nu \geq \{b_\nu\}_\nu \text{ implies } \alpha \{a_\nu\}_\nu \geq \alpha \{b_\nu\}_\nu \text{ for positive number } \alpha.$$

For every orthogonal sequence of positive elements $\{a_\nu\}_\nu$ we have $\{a_\nu\}_\nu \geq \{0\}$. Because, for every projector $[p]$, putting

$$[p_\mu] = \bigvee_{\nu=1}^\mu [a_\nu][p] + ([p] - \bigvee_{\nu=1}^\mu [a_\nu][p]) \quad (\mu = 1, 2, \dots),$$

we have $[p_\mu] \uparrow_{\mu=1}^\infty [p]$ and

$$\sum_{\mu=1}^\infty [p_\mu] a_\nu = [p](a_1 + \dots + a_\mu) \geq 0 \quad (\mu = 1, 2, \dots).$$

Conversely, if $\{a_\nu\}_\nu \geq \{0\}$, then we have $a_\nu \geq 0$ for every $\nu = 1, 2, \dots$. Because, to every a_p there exists by definition a sequence of projectors $[p_\mu] \uparrow_{\mu=1}^\infty [a_p]$ such that

$$\sum_{\mu=1}^\infty [p_\mu] a_\nu \geq 0.$$

Then, since $[p_\mu] a_p = \sum_{\nu=1}^\infty [p_\mu] a_\nu$ by Theorems 8.1 and 8.2, passing to the limit, we obtain $a_p \geq 0$ by Theorems 7.11 and 8.11.

For every orthogonal sequence $\{a_\nu\}_\nu$ we obtain by Theorem 4.1 an orthogonal sequence $\{a_\nu^+\}_\nu$ and we see easily by definition that

$$\{a_\nu^+\}_\nu \geq \{a_\nu\}_\nu \quad \text{and} \quad \{a_\nu^+\}_\nu \geq \{0\}.$$

If $\{b_\nu\}_\nu \geq \{a_\nu\}_\nu$ and $\{b_\nu\}_\nu \geq \{0\}$ for an orthogonal sequence $\{b_\nu\}_\nu$, then to every projector $[p]$ there exists by definition a sequence of projectors $[p_\mu] \uparrow_{\mu=1}^\infty [p]$ such that

$$\sum_{\mu=1}^\infty [p_\mu] b_\nu \geq \sum_{\mu=1}^\infty [p_\mu] a_\nu \quad (\mu = 1, 2, \dots).$$

Then, since $\sum_{\mu=1}^\infty [p_\mu] b_\nu \geq 0$, we obtain by Theorems 5.15 and 7.4

$$\sum_{\nu=1}^{\infty} [p_{\mu}] b_{\nu} \geq (\sum_{\nu=1}^{\infty} [p_{\mu}] a_{\nu})^{+} = \sum_{\nu=1}^{\infty} [p_{\mu}] a_{\nu}^{+},$$

and consequently $\{b_{\nu}\}_{\nu} \geq \{a_{\nu}^{+}\}_{\nu}$ by definition. Therefore we obtain that $\hat{R}$ constitutes a lattice ordered linear space by the semi-order defined just now such that $\hat{R} \ni \hat{a} \ni \{a_{\nu}\}_{\nu}$ implies

$$\hat{a}^{+} \ni \{a_{\nu}^{+}\}_{\nu}, \quad \hat{a}^{-} \ni \{a_{\nu}^{-}\}_{\nu}$$

$$|\hat{a}| \ni \{|a_{\nu}|\}_{\nu} \sim \{a_{\nu}^{+}, a_{\nu}^{-}\}_{\nu}$$

Furthermore we can prove that $\hat{R}$ is continuous. For this purpose we must remark that for two orthogonal sequences $\{a_{\nu}\}_{\nu}$ $\{b_{\nu}\}_{\nu}$, if $\{a_{\nu}\}_{\nu} \geq \{b_{\nu}\}_{\nu} \geq \{0\}$, then we have

$$a_{\nu} \geq \sum_{\mu=1}^{\infty} [a_{\nu}] b_{\mu} \geq 0 \quad (\nu = 1, 2, \dots),$$

$$b_{\mu} = \sum_{\nu=1}^{\infty} [a_{\nu}] b_{\mu} \quad (\mu = 1, 2, \dots),$$

$$\{\sum_{\mu=1}^{\infty} [a_{\nu}] b_{\mu}\}_{\nu} \sim \{b_{\nu}\}_{\nu}.$$

In fact, if $\{a_{\nu}\}_{\nu} \geq \{b_{\nu}\}_{\nu} \geq \{0\}$ and $\sum_{\nu=1}^{\infty} [p] a_{\nu}$ is convergent for a projector $[p]$, then there exists by definition a sequence of projectors $[p_{\mu}] \uparrow_{\mu=1}^{\infty} [p]$ such that

$$\sum_{\nu=1}^{\infty} [p_{\mu}] a_{\nu} \geq \sum_{\nu=1}^{\infty} [p_{\mu}] b_{\nu} \geq 0,$$

and hence we have by Theorems 5.15 and 2.5

$$\begin{aligned} \sum_{\nu=1}^{\infty} [p] a_{\nu} &= \bigcup_{\mu=1}^{\infty} [p] a_{\nu} = \bigcup_{\mu=1}^{\infty} \{\bigcup_{\nu=1}^{\infty} [p_{\mu}] a_{\nu}\} \\ &\geq \bigcup_{\mu=1}^{\infty} \{\bigcup_{\nu=1}^{\infty} [p_{\mu}] b_{\nu}\} = \bigcup_{\mu=1}^{\infty} [p] b_{\nu} = \sum_{\nu=1}^{\infty} [p] b_{\nu}, \end{aligned}$$

that is, if $\{a_{\nu}\}_{\nu} \geq \{b_{\nu}\}_{\nu} \geq \{0\}$ and $\sum_{\nu=1}^{\infty} [p] a_{\nu}$ is convergent, then we have

$$\sum_{\nu=1}^{\infty} [p] a_{\nu} \geq \sum_{\nu=1}^{\infty} [p] b_{\nu} \geq 0.$$

Putting $p = a_{\nu}$ in this relation, we obtain by Theorems 7.11, 8.1

$$a_{\nu} \geq \sum_{\mu=1}^{\infty} [a_{\nu}] b_{\mu} \geq 0.$$

Since $\sum_{\nu=1}^{\infty} [a_{\nu}] b_{\mu} = \bigcup_{\nu=1}^{\infty} [a_{\nu}] b_{\mu} \leq b_{\mu}$ by Theorem 5.15, putting

$$p = b_{\mu} - \sum_{\nu=1}^{\infty} [a_{\nu}] b_{\mu},$$

we have by Theorem 8.1

$$[a_{\mu}] p = [a_{\mu}] b_{\mu} - \sum_{\nu=1}^{\infty} [a_{\mu}] [a_{\nu}] b_{\mu} = 0,$$

and hence by Theorem 7.12

$$[p] a_{\mu} = 0 \quad \text{for every } \mu = 1, 2, \dots$$

Consequently we obtain $[p] b_{\mu} = 0$, and hence $[b_{\mu}] p = 0$ by

Theorem 7.12. Therefore we have by Theorem 7.11

$$b_\mu - \sum_{\nu=1}^{\infty} [a_\nu] b_\mu = [b_\mu] p = 0,$$

that is, $b_\mu = \sum_{\nu=1}^{\infty} [a_\nu] b_\mu$ for every $\mu = 1, 2, \dots$.

For every projector $[p]$, putting

$$[p_\mu] = \sum_{\nu=1}^{\infty} [p][b_\nu] + ([p] - \sum_{\nu=1}^{\infty} [p][b_\nu]) \quad (\mu=1, 2, \dots),$$

we have $[p_\mu] \uparrow_{\mu=1}^{\infty} [p]$ and by the fact proved just now

$$\begin{aligned} \sum_{\nu=1}^{\infty} \{ [p_\mu] \sum_{\rho=1}^{\infty} [a_\rho] b_\rho \} &= \sum_{\nu=1}^{\infty} \sum_{\rho=1}^{\infty} [a_\rho] [p] b_\rho \\ &= \sum_{\rho=1}^{\infty} [p] b_\rho = \sum_{\nu=1}^{\infty} [p_\mu] b_\nu. \end{aligned}$$

Consequently $\{ \sum_{\nu=1}^{\infty} [a_\nu] b_\rho \}_\nu \sim \{ b_\nu \}_\nu$ by definition.

Conversely, for an orthogonal sequence $\{a_\nu\}_\nu$, if

$$a_\nu \geq b_\nu \geq 0 \quad \text{for every } \nu = 1, 2, \dots,$$

then we have $\{a_\nu\}_\nu \geq \{b_\nu\}_\nu \geq \{0\}$. Because, for every projector $[p]$, putting

$$[p_\mu] = \sum_{\nu=1}^{\infty} [p][a_\nu] + ([p] - \sum_{\nu=1}^{\infty} [p][a_\nu]) \quad (\mu=1, 2, \dots),$$

we have $[p_\mu] \uparrow_{\mu=1}^{\infty} [p]$ and for every $\mu = 1, 2, \dots$

$$\sum_{\nu=1}^{\infty} [p_\mu] a_\nu = \sum_{\nu=1}^{\infty} [p] a_\nu \geq \sum_{\nu=1}^{\infty} [p_\mu] b_\nu \geq 0.$$

If $\{a_\nu\}_\nu \geq \{a_{\mu,\nu}\}_\nu \geq \{0\}$ for every $\mu = 1, 2, \dots$, then there exists a sequence of orthogonal sequences $\{b_{\mu,\nu}\}_\nu$ ($\mu = 1, 2, \dots$) such that

$$\{b_{\mu,\nu}\}_\nu \sim \{a_{\mu,\nu}\}_\nu \quad \text{for every } \mu = 1, 2, \dots,$$

$$a_\nu \geq b_{\mu,\nu} \geq 0 \quad \text{for every } \nu, \mu = 1, 2, \dots,$$

as remarked just above. Putting $c_\nu = \sum_{\mu=1}^{\infty} b_{\mu,\nu}$ ($\nu = 1, 2, \dots$),

we obtain then an orthogonal sequence $\{c_\nu\}_\nu$ such that

$$\{b_{\mu,\nu}\}_\nu \geq \{c_\nu\}_\nu \geq \{0\} \quad \text{for every } \mu = 1, 2, \dots$$

as proved just now. On the other hand, if

$$\{b_{\mu,\nu}\}_\nu \geq \{d_\nu\}_\nu \geq \{0\} \quad \text{for every } \mu = 1, 2, \dots,$$

then there exists an orthogonal sequence $\{e_\nu\}_\nu$ such that

$$\{e_\nu\}_\nu \sim \{d_\nu\}_\nu, \quad a_\nu \geq e_\nu \geq 0 \quad \text{for every } \nu = 1, 2, \dots,$$

as remarked already. For such $\{e_\nu\}_\nu$ we have by the fact remarked just above

$$b_{\mu,\nu} \geq \sum_{\rho=1}^{\infty} [b_{\mu,\rho}] e_\rho, \quad e_\nu = \sum_{\rho=1}^{\infty} [b_{\mu,\rho}] e_\rho.$$

Since $e_{\mu, \nu} \in e_{\nu} = 0$ for $\nu \neq \mu$, we obtain hence by Theorem 7.12 for every $\mu, \nu = 1, 2, \dots$

$$e_{\mu, \nu} \geq [e_{\mu, \nu}] e_{\nu} = e_{\nu},$$

and consequently $e_{\nu} \geq e_{\nu} \geq 0$ for every $\nu = 1, 2, \dots$. These relations yield $\{c_{\nu}\}_{\nu} \geq \{d_{\nu}\}_{\nu}$ by the relations (4) and (5). Therefore $\hat{R}$ is continuous by definition.

Corresponding to every element $a \in R$ we shall denote by $a^{\hat{R}}$ the equivalent class containing $\{a\}$. Then we see easily by definition that we have for every elements $a, b \in R$ and for every real numbers α, β

$$(\alpha a + \beta b)^{\hat{R}} = \alpha a^{\hat{R}} + \beta b^{\hat{R}},$$

and $a^{\hat{R}} \geq 0$ if and only if $a \geq 0$, as proved just above. Therefore $\hat{R}$ is by definition an extension of R by the correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$. Furthermore we have proved already that

$$a^{\hat{R}+} = (a^{\hat{R}})^+, \quad a^{\hat{R}-} = (a^{\hat{R}})^-, \quad |a^{\hat{R}}| = |a|^{\hat{R}}$$

for every element $a \in R$. Since $a \wedge b = a - (a - b)^+$ by Theorem 3.12, we obtain hence

$$\begin{aligned} (a \wedge b)^{\hat{R}} &= a^{\hat{R}} - (a - b)^{\hat{R}+} \\ &= a^{\hat{R}} - (a^{\hat{R}} - b^{\hat{R}})^+ = a^{\hat{R}} \wedge b^{\hat{R}}. \end{aligned}$$

Consequently we have $a^{\hat{R}} \perp b^{\hat{R}}$ if and only if $a \perp b$.

For an orthogonal sequence of positive elements $\{a_{\nu}\}_{\nu}$, if $\hat{a} \ni \{a_{\nu}\}_{\nu}$, then we have $\hat{a} \geq a_{\nu}^{\hat{R}}$ for every $\nu = 1, 2, \dots$, as proved already. If

$$\{b_{\nu}\}_{\nu} \geq \{a_{\mu}\}_{\mu} \geq \{0\} \quad \text{for every } \mu = 1, 2, \dots,$$

then for every projector $[p]$, putting

$$[p_p] = \bigcup_{\nu=1}^p [p][b_{\nu}] + ([p] - \bigcup_{\nu=1}^p [p][b_{\nu}]) \quad (p = 1, 2, \dots),$$

we have $[p_p] \uparrow_{p=1}^{\infty} [p]$ and for every $p = 1, 2, \dots$

$$\sum_{\nu=1}^{\infty} [p_p] b_{\nu} = \sum_{\nu=1}^p [p] b_{\nu},$$

as proved already, and hence we obtain

$$\sum_{\nu=1}^{\infty} [p_p] b_{\nu} \geq [p_p] a_{\mu} \quad \text{for every } \mu = 1, 2, \dots,$$

as proved just above. Consequently we have

em 5.15

$$\sum_{\nu=1}^{\infty} [p_\nu] e_\nu \geq \bigcup_{\mu=1}^{\infty} [p_\mu] a_\mu = \sum_{\nu=1}^{\infty} [p_\nu] a_\nu,$$

and hence $\{e_\nu\} \geq \{a_\nu\}$ by definition. Therefore for an orthogonal sequence of positive elements $\{a_\nu\}$, if $\hat{a} \ni \{a_\nu\}$, then we have

$$\hat{a} = \bigcup_{\nu=1}^{\infty} a_\nu \hat{R} = \sum_{\nu=1}^{\infty} a_\nu \hat{R}.$$

For an arbitrary orthogonal sequence $\{a_\nu\}$, if $\hat{a} \ni \{a_\nu\}$, then, since $\hat{a}^+ \ni \{a_\nu^+\}$, $\hat{a}^- \ni \{a_\nu^-\}$, we have thus

$$\hat{a}^+ = \sum_{\nu=1}^{\infty} a_\nu^+ \hat{R}, \quad \hat{a}^- = \sum_{\nu=1}^{\infty} a_\nu^- \hat{R},$$

and consequently $\hat{a} = \sum_{\nu=1}^{\infty} (a_\nu^+ \hat{R} - a_\nu^- \hat{R}) = \sum_{\nu=1}^{\infty} a_\nu \hat{R}$.

If $a = \sum_{\nu=1}^{\infty} a_\nu$ for an orthogonal sequence $\{a_\nu\}$, then we have obviously by definition

$$\{a\} \sim \{a_\nu\},$$

that is, $a \hat{R} \ni \{a_\nu\}$, and consequently $a \hat{R} = \sum_{\nu=1}^{\infty} a_\nu \hat{R}$, as proved just now.

For an orthogonal sequence $\hat{a}_\mu \in \hat{R}$ ($\mu=1, 2, \dots$), if

$$\hat{a}_\mu \ni \{a_{\mu, \nu}\} \quad (\mu=1, 2, \dots),$$

then we have for every $\mu=1, 2, \dots$

$$\hat{a}_\mu = \sum_{\nu=1}^{\infty} a_{\mu, \nu} \hat{R},$$

as proved just above, and hence by Theorem 5.15

$$|\hat{a}_\mu| = \sum_{\nu=1}^{\infty} |a_{\mu, \nu}| \hat{R} = \bigcup_{\nu=1}^{\infty} |a_{\mu, \nu}| \hat{R}.$$

Since $\hat{a}_\mu \perp \hat{a}_\rho$ for $\mu \neq \rho$ by assumption, we obtain hence an orthogonal sequence $\{|a_{\mu, \nu}|\}_{\mu, \nu}$, and we have

$$\{|a_{\rho, \nu}|\}_{\nu} \leq \{|a_{\mu, \nu}|\}_{\mu, \nu} \quad (\rho=1, 2, \dots),$$

as remarked already. Accordingly $\sum_{\mu=1}^{\infty} |\hat{a}_\mu|$ is convergent by Theorem 4.5, and consequently $\sum_{\mu=1}^{\infty} \hat{a}_\mu$ is convergent by Theorem 6.6. Therefore $\hat{R}$ is complete, and hence $\hat{R}$ is a completion of R by definition.

Finally we shall prove the uniqueness of completion up to an isomorphism. Let $\hat{R}$ and $\tilde{R}$ be completions of a continuous semi-ordered linear space R respectively by extending correspondence

$$R \ni a \rightarrow a^{\hat{R}} \in \hat{R} \quad \text{and} \quad R \ni a \rightarrow a^{\tilde{R}} \in \tilde{R}.$$

To every element $\hat{a} \in \hat{R}$ there exists by the postulate 4) an orthogonal sequence $a_\nu \in R$ ($\nu = 1, 2, \dots$) such that

$$\hat{a} = \sum_{\nu=1}^{\infty} a_\nu \hat{R},$$

and then there exists by the postulate 1) an element $\tilde{a} \in \tilde{R}$ such that

$$\tilde{a} = \sum_{\nu=1}^{\infty} a_\nu \tilde{R}.$$

If $\hat{a} = \sum_{\nu=1}^{\infty} a_\nu \hat{R} = \sum_{\nu=1}^{\infty} b_\nu \hat{R}$ for two orthogonal sequences $a_\nu \in R$ and $b_\nu \in R$ ($\nu = 1, 2, \dots$), then we have by Theorems 5.15 and 33.2

$$\hat{a}^+ = \bigcup_{\nu=1}^{\infty} a_\nu^+ \hat{R} = \bigcup_{\nu=1}^{\infty} b_\nu^+ \hat{R}, \quad \hat{a}^- = \bigcup_{\nu=1}^{\infty} a_\nu^- \hat{R} = \bigcup_{\nu=1}^{\infty} b_\nu^- \hat{R},$$

and hence we obtain by Theorems 3.3 and 33.1

$$\begin{aligned} a_\mu^+ \hat{R} &= a_\mu^+ \hat{R} \cap \left(\bigcup_{\nu=1}^{\infty} a_\nu^+ \hat{R} \right) \\ &= a_\mu^+ \hat{R} \cap \left(\bigcup_{\nu=1}^{\infty} b_\nu^+ \hat{R} \right) = \bigcup_{\nu=1}^{\infty} (a_\mu^+ \cap b_\nu^+) \hat{R}. \end{aligned}$$

Since $\hat{R}$ is an extension of R by the correspondence $R \ni a \rightarrow a \hat{R} \in \hat{R}$, we obtain therefore for every $\mu = 1, 2, \dots$

$$a_\mu^+ = \bigcup_{\nu=1}^{\infty} (a_\mu^+ \cap b_\nu^+),$$

and hence by Theorem 33.1

$$a_\mu^+ \tilde{R} = \bigcup_{\nu=1}^{\infty} (a_\mu^+ \tilde{R} \cap b_\nu^+ \tilde{R}) \quad (\mu = 1, 2, \dots).$$

we also can prove likewise

$$b_\mu^+ \tilde{R} = \bigcup_{\nu=1}^{\infty} (b_\mu^+ \tilde{R} \cap a_\nu^+ \tilde{R}) \quad (\mu = 1, 2, \dots).$$

Consequently we obtain by Theorems 2.5 and 3.15

$$\begin{aligned} \sum_{\mu=1}^{\infty} a_\mu^+ \tilde{R} &= \bigcup_{\mu=1}^{\infty} a_\mu^+ \tilde{R} = \bigcup_{\mu, \nu} (a_\mu^+ \tilde{R} \cap b_\nu^+ \tilde{R}) \\ &= \bigcup_{\nu=1}^{\infty} b_\nu^+ \tilde{R} = \sum_{\nu=1}^{\infty} b_\nu^+ \tilde{R}. \end{aligned}$$

We also can prove likewise

$$\sum_{\nu=1}^{\infty} a_\nu^- \tilde{R} = \sum_{\nu=1}^{\infty} b_\nu^- \tilde{R}.$$

In consequence, we have proved that $\sum_{\nu=1}^{\infty} a_\nu \hat{R} = \sum_{\nu=1}^{\infty} b_\nu \hat{R}$ for two orthogonal sequences $a_\nu \in R$ and $b_\nu \in R$ ($\nu = 1, 2, \dots$) implies

$$\sum_{\nu=1}^{\infty} a_\nu \tilde{R} = \sum_{\nu=1}^{\infty} b_\nu \tilde{R}.$$

Therefore we obtain a correspondence $\hat{R} \ni \hat{a} \rightarrow \hat{a} \tilde{R} \in \tilde{R}$ such that

$$\hat{a} = \sum_{\nu=1}^{\infty} a_\nu \hat{R}$$

for an orthogonal sequence $a_\nu \in R$ ($\nu = 1, 2, \dots$) implies

$$\hat{a} \tilde{R} = \sum_{\nu=1}^{\infty} a_\nu \tilde{R}.$$

Furthermore $\hat{R}$ is isomorphic to $\tilde{R}$ by this correspondence.

Indeed we have obviously that

$$(\alpha \hat{a})^{\tilde{R}} = \alpha^{\tilde{R}} \quad \text{for every element } a \in R,$$

and that $\hat{a}^{\tilde{R}} \geq 0$ if and only if $\hat{a} \geq 0$. Hence we need only to prove that

$$(\alpha \hat{a} + \beta \hat{b})^{\tilde{R}} = \alpha \hat{a}^{\tilde{R}} + \beta \hat{b}^{\tilde{R}}.$$

As proved already first, to every two orthogonal sequences

$\{a_\nu\}_\nu, \{b_\nu\}_\nu$ there exists a sequence of projectors $[p_\mu]$ ($\mu = 1, 2, \dots$) such that

$$[p_\nu][p_\mu] = 0 \quad \text{for } \nu \neq \mu,$$

$$a_\nu = \sum_{\mu=1}^{\infty} [p_\mu] a_\nu, \quad b_\nu = \sum_{\mu=1}^{\infty} [p_\mu] b_\nu \quad (\nu = 1, 2, \dots),$$

and both series $\sum_{\mu=1}^{\infty} [p_\mu] a_\nu$ and $\sum_{\mu=1}^{\infty} [p_\mu] b_\nu$ are convergent for every $\mu = 1, 2, \dots$. If

$$\hat{a} = \sum_{\nu=1}^{\infty} a_\nu \hat{R},$$

then we have by Theorems 5.15 and 2.5

$$\begin{aligned} \hat{a}^+ &= \sum_{\nu=1}^{\infty} a_\nu^+ \hat{R} = \bigcup_{\nu=1}^{\infty} \left\{ \bigcup_{\mu=1}^{\infty} ([p_\mu] a_\nu^+) \hat{R} \right\} \\ &= \bigcup_{\mu=1}^{\infty} \left\{ \bigcup_{\nu=1}^{\infty} ([p_\mu] a_\nu^+) \hat{R} \right\} = \sum_{\mu=1}^{\infty} \left\{ \sum_{\nu=1}^{\infty} ([p_\mu] a_\nu^+) \hat{R} \right\}, \end{aligned}$$

and similarly

$$\hat{a}^- = \sum_{\mu=1}^{\infty} \left\{ \sum_{\nu=1}^{\infty} ([p_\mu] a_\nu^-) \hat{R} \right\}.$$

Consequently we obtain

$$\hat{a} = \sum_{\mu=1}^{\infty} \left\{ \sum_{\nu=1}^{\infty} ([p_\mu] a_\nu) \hat{R} \right\} = \sum_{\mu=1}^{\infty} \left(\sum_{\nu=1}^{\infty} [p_\mu] a_\nu \right) \hat{R},$$

since $\sum_{\nu=1}^{\infty} [p_\mu] a_\nu$ is convergent. Furthermore, if

$$\hat{b} = \sum_{\nu=1}^{\infty} b_\nu \hat{R},$$

then we obtain likewise

$$\hat{b} = \sum_{\mu=1}^{\infty} \left(\sum_{\nu=1}^{\infty} [p_\mu] b_\nu \right) \hat{R}.$$

Therefore we have for every real numbers α, β

$$\begin{aligned} \alpha \hat{a} + \beta \hat{b} &= \sum_{\mu=1}^{\infty} \left\{ \alpha \left(\sum_{\nu=1}^{\infty} [p_\mu] a_\nu \right) \hat{R} + \beta \left(\sum_{\nu=1}^{\infty} [p_\mu] b_\nu \right) \hat{R} \right\} \\ &= \sum_{\mu=1}^{\infty} \left(\alpha \sum_{\nu=1}^{\infty} [p_\mu] a_\nu + \beta \sum_{\nu=1}^{\infty} [p_\mu] b_\nu \right) \hat{R}, \end{aligned}$$

and hence

$$(\alpha \hat{a} + \beta \hat{b})^{\tilde{R}} = \sum_{\mu=1}^{\infty} \left(\alpha \sum_{\nu=1}^{\infty} [p_\mu] a_\nu + \beta \sum_{\nu=1}^{\infty} [p_\mu] b_\nu \right)^{\tilde{R}},$$

since $[p_\nu][p_\mu] = 0$ for $\nu \neq \mu$. On the other hand we have

$$\begin{aligned} & \sum_{n=1}^{\infty} (\alpha \sum_{i=1}^n [p_i] a_i + \beta \sum_{i=1}^n [p_i] b_i)^{\tilde{K}} \\ &= \alpha \sum_{n=1}^{\infty} (\sum_{i=1}^n [p_i] a_i)^{\tilde{K}} + \beta \sum_{n=1}^{\infty} (\sum_{i=1}^n [p_i] b_i)^{\tilde{K}} = \alpha \hat{a}^{\tilde{K}} + \beta \hat{b}^{\tilde{K}}. \end{aligned}$$

Consequently $(\alpha \hat{a} + \beta \hat{b})^{\tilde{K}} = \alpha \hat{a}^{\tilde{K}} + \beta \hat{b}^{\tilde{K}}$, as we wish to prove.

§34 Universal completion

Let R be a universally continuous semi-ordered linear space. An extension $\hat{R}$ of R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ is said to be a universal completion of R , if

- 1) $\hat{R}$ is universally complete,
- 2) $a^{\hat{R}} \perp b^{\hat{R}}$ if and only if $a \perp b$,
- 3) $a = \bigcup_{\lambda \in \Lambda} a_{\lambda}$ for an orthogonal system of positive elements $a_{\lambda} \in R$ ($\lambda \in \Lambda$) implies

$$a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}},$$

- 4) to every positive element $\hat{a} \in \hat{R}$ there exists an orthogonal system of positive $a_{\lambda} \in R$ ($\lambda \in \Lambda$) such that

$$\hat{a} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}$$

Corresponding to Theorems 33.1 and 33.2 we obtain likewise:

Theorem 34.1. Let $\hat{R}$ be a universal completion of a universally continuous semi-ordered linear space R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$ If

$$a^{\hat{R}} \geq \hat{b} \geq 0, \quad a \in R, \quad \hat{b} \in \hat{R}$$

then there exists a positive element $b \in R$ such that $\hat{b} = b^{\hat{R}}$.

If

$$a = \bigcup_{\lambda \in \Lambda} a_{\lambda} \quad \text{or} \quad a = \bigcap_{\lambda \in \Lambda} a_{\lambda},$$

then we have respectively

$$a^{\hat{R}} = \bigcup_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}} \quad \text{or} \quad a^{\hat{R}} = \bigcap_{\lambda \in \Lambda} a_{\lambda}^{\hat{R}}.$$

And for every element $a \in R$ we have

$$a^{+\hat{R}} = a^{\hat{R}+}, \quad a^{-\hat{R}} = a^{\hat{R}-}, \quad |a|^{\hat{R}} = |a^{\hat{R}}|.$$

Corresponding to Theorem 33.3 we can prove likewise:

Theorem 34.2. If $\hat{R}$ is a universal completion of a univer-

sally continuous semi-ordered linear space R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$, then for every complete orthogonal system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) in R we obtain again a complete orthogonal system $a_\lambda^{\hat{R}} \in \hat{R}$ ($\lambda \in \Lambda$) in $\hat{R}$.

Theorem 34.3. If a semi-ordered linear space R is universally continuous and has a complete orthogonal sequence $a_\nu \in R$ ($\nu = 1, 2, \dots$), then the completion of R is a universal completion of R .

Proof. Let $\hat{R}$ be a completion of R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$. Since $a_\nu \in R$ ($\nu = 1, 2, \dots$) is a complete orthogonal sequence by assumption, the corresponding $a_\nu^{\hat{R}} \in \hat{R}$ ($\nu = 1, 2, \dots$) is a complete orthogonal sequence in $\hat{R}$ by Theorem 33.3, and hence, putting

$$\hat{c} = \sum_{\nu=1}^{\infty} a_\nu^{\hat{R}},$$

we obtain a complete element $\hat{c} \in \hat{R}$ in $\hat{R}$.

$\hat{R}$ is universally continuous. Because, for any system of positive elements $\hat{b}_\lambda \in \hat{R}$ ($\lambda \in \Lambda$) we have obviously

$$0 \leq \hat{b}_\lambda \wedge \mu |a_\nu|^{\hat{R}} \leq |\mu a_\nu|^{\hat{R}} \quad (\lambda \in \Lambda; \mu, \nu = 1, 2, \dots),$$

and hence there exists by Theorem 33.1 $b_{\lambda, \mu, \nu} \in R$ such that

$$b_{\lambda, \mu, \nu}^{\hat{R}} = \hat{b}_\lambda \wedge \mu |a_\nu|^{\hat{R}} \quad (\lambda \in \Lambda; \mu, \nu = 1, 2, \dots).$$

Since R is universally continuous by assumption, there exists then

$$b_{\mu, \nu} = \bigcap_{\lambda \in \Lambda} b_{\lambda, \mu, \nu} \quad (\mu, \nu = 1, 2, \dots),$$

and we have by Theorem 33.1

$$b_{\mu, \nu}^{\hat{R}} = \bigcap_{\lambda \in \Lambda} (\hat{b}_\lambda \wedge \mu |a_\nu|^{\hat{R}}) \quad (\mu, \nu = 1, 2, \dots).$$

For such $b_{\mu, \nu} \in R$ we have obviously $b_{\mu, \nu}^{\hat{R}} \leq \hat{b}_\lambda$ for every μ , $\nu = 1, 2, \dots$, and hence we obtain an element $\hat{b} \in \hat{R}$ as

$$\hat{b} = \bigcup_{\mu, \nu} b_{\mu, \nu}^{\hat{R}} \leq \hat{b}_\lambda \quad \text{for every } \lambda \in \Lambda.$$

If $0 \leq \hat{x} \leq \hat{b}_\lambda$ for every $\lambda \in \Lambda$, then we have obviously

$$\hat{x} \wedge \mu |a_\nu|^{\hat{R}} \leq \hat{b}_\lambda \wedge \mu |a_\nu|^{\hat{R}} \quad \text{for every } \lambda \in \Lambda,$$

and consequently

$$\hat{x} \wedge \mu |a_\nu|^{\hat{R}} \leq b_{\mu, \nu}^{\hat{R}} \quad \text{for every } \mu, \nu = 1, 2, \dots$$

From this relation we conclude by definition

$$\bigcup_{\mu, \nu} (\hat{x} \wedge \mu | a_\nu | \hat{R}) \leq \hat{\ell}.$$

On the other hand we have by Theorems 2.5 and 8.5

$$\begin{aligned} \bigcup_{\mu, \nu} (\hat{x} \wedge \mu | a_\nu | \hat{R}) &= \bigcup_{\nu=1}^{\infty} \{ \bigcup_{\mu=1}^{\infty} (\hat{x} \wedge \mu | a_\nu | \hat{R}) \} \\ &= \bigcup_{\nu=1}^{\infty} [| a_\nu | \hat{R}] \hat{x} = \hat{x}, \end{aligned}$$

since $|\hat{c}| = \bigcup_{\nu=1}^{\infty} |a_\nu| \hat{R}$ by Theorem 5.15 and $[\hat{c}] = I$ by Theorem 7.14.

Consequently $0 \leq \hat{x} \leq \hat{c}_\lambda$ for every $\lambda \in A$ implies $\hat{x} \leq \hat{\ell}$, and

hence $\hat{\ell} = \bigcap_{\lambda \in A} \hat{c}_\lambda$. Therefore $\hat{R}$ is universally continuous by

definition. Accordingly $\hat{R}$ is universally complete by Theorem

32.5. Furthermore we see easily by Theorem 33.1 that $\hat{R}$ is a universal completion of R by the correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$.

Theorem 34.4. Every universally continuous semi-ordered.

linear space has a universal completion uniquely within an isomorphism.

Proof

Let R be a universally continuous semi-ordered linear space. By virtue of Theorem 4.6 there exists a complete orthogonal system of positive elements $c_\lambda \in R$ ($\lambda \in A$). Corresponding to every $c \in R$ we shall denote by $[c]R$ the set of all elements $[c]x$ for $x \in R$. Then we see easily by Theorems 7.1, 7.2, and 7.6 that $[c_\lambda]R$ is itself a universally continuous semi-ordered linear space. By virtue of Theorem 33.4 there exists then a completion $\hat{R}_\lambda$ of $[c_\lambda]R$ by an extending correspondence $[c_\lambda]R \ni a \rightarrow a^{\hat{R}_\lambda} \in \hat{R}_\lambda$. Since c_λ is a complete element in $[c_\lambda]R$ by Theorem 7.12, we see easily by Theorem 34.3 that $\hat{R}_\lambda$ is a universal completion of $[c_\lambda]R$ by the same correspondence.

We shall now consider systems of elements $\hat{a}_\lambda \in \hat{R}_\lambda$ ($\lambda \in A$), denoting it by $\{\hat{a}_\lambda\}_\lambda$, and set the following definitions:

$$\{\hat{a}_\lambda\}_\lambda = \{\hat{\ell}_\lambda\}_\lambda \quad \text{if and only if } \hat{a}_\lambda = \hat{\ell}_\lambda \text{ for every } \lambda \in A;$$

$$\alpha \{\hat{a}_\lambda\}_\lambda + \beta \{\hat{\ell}_\lambda\}_\lambda = \{\alpha \hat{a}_\lambda + \beta \hat{\ell}_\lambda\}_\lambda \quad \text{for every real numbers } \alpha, \beta;$$

$$\{\hat{a}_\lambda\}_\lambda \geq \{\hat{\ell}_\lambda\}_\lambda \quad \text{if and only if } \hat{a}_\lambda \geq \hat{\ell}_\lambda \text{ for every } \lambda \in A.$$

Then we see easily that the totality of systems $\{a_\lambda\}_\lambda$ constitutes

a universally continuous semi-ordered linear space $\hat{R}$, and that

$$\{\hat{a}_\lambda\}_\lambda = \bigcup_{\gamma \in \Gamma} \{\hat{a}_{\gamma, \lambda}\}_\lambda \quad \text{or} \quad \{\hat{a}_\lambda\}_\lambda = \bigcap_{\gamma \in \Gamma} \{\hat{a}_{\gamma, \lambda}\}_\lambda$$

if and only if respectively

$$\hat{a}_\lambda = \bigcup_{\gamma \in \Gamma} \hat{a}_{\gamma, \lambda} \quad \text{or} \quad \hat{a}_\lambda = \bigcap_{\gamma \in \Gamma} \hat{a}_{\gamma, \lambda} \quad \text{for every } \lambda \in \Lambda.$$

Hence we have for every system $\{\hat{a}_\lambda\}_\lambda$

$$\{\hat{a}_\lambda\}_\lambda^+ = \{\hat{a}_\lambda^+\}_\lambda, \quad \{\hat{a}_\lambda\}_\lambda^- = \{\hat{a}_\lambda^-\}_\lambda, \quad \|\{\hat{a}_\lambda\}_\lambda\| = \|\{\hat{a}_\lambda^+\}_\lambda\|$$

and we have $\{\hat{a}_\lambda\}_\lambda \perp \{\hat{b}_\lambda\}_\lambda$ if and only if

$$\hat{a}_\lambda \perp \hat{b}_\lambda \quad \text{for every } \lambda \in \Lambda.$$

$\hat{R}$ is universally complete. In fact, for every orthogonal

system of positive elements $\{\hat{a}_{\gamma, \lambda}\}_\lambda \in \hat{R}$ ($\gamma \in \Gamma$), $\hat{a}_{\gamma, \lambda}$ ($\gamma \in \Gamma$)

is an orthogonal system of positive elements in $\hat{R}_\lambda$ for every

$\lambda \in \Lambda$, and hence there exists $\bigcup_{\gamma \in \Gamma} \hat{a}_{\gamma, \lambda}$ ($\lambda \in \Lambda$), for which

we have obviously by definition

$$\bigcup_{\gamma \in \Gamma} \{\hat{a}_{\gamma, \lambda}\}_\lambda \geq \{\hat{a}_{\gamma, \lambda}\}_\lambda \quad \text{for every } \gamma \in \Gamma.$$

that is, $\hat{R}$ is universally complete by definition.

Corresponding to every element $a \in R$ we obtain uniquely

$$\{[c_\lambda]a\}_{\hat{R}_\lambda} \in \hat{R}$$

which will be denoted by $a^{\hat{R}}$. Then we see easily that

$$(\alpha a + \beta b)^{\hat{R}} = \alpha a^{\hat{R}} + \beta b^{\hat{R}} \quad \text{for every elements } a, b \in R,$$

and that $a^{\hat{R}} \geq 0$ if and only if $a \geq 0$, since $c_\lambda \in R$ ($\lambda \in \Lambda$)

is a complete orthogonal system in R . Accordingly $\hat{R}$ is an extension of R by the correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$.

Since $\hat{R}_\lambda$ is a universal completion of $[c_\lambda]R$ for every

$\lambda \in \Lambda$, to every positive element $\{\hat{a}_\lambda\}_\lambda \in \hat{R}$ there exists an

orthogonal system of positive elements $a_{\gamma, \lambda} \in [c_\lambda]R$ ($\gamma \in \Gamma$, $\lambda \in \Lambda$)

for which

$$\hat{a}_\lambda = \bigcup_{\gamma \in \Gamma} a_{\gamma, \lambda}^{\hat{R}_\lambda} \quad \text{for every } \lambda \in \Lambda,$$

and then we have

$$\{\hat{a}_\lambda\}_\lambda = \bigcup_{\gamma \in \Gamma} \{a_{\gamma, \lambda}^{\hat{R}_\lambda}\}_\lambda = \bigcup_{\gamma, \lambda} a_{\gamma, \lambda}^{\hat{R}},$$

as remarked just above.

If $a = \bigcup_{\gamma \in \Gamma} a_\gamma$ for a system of elements $a_\gamma \in R$ ($\gamma \in \Gamma$),

then we have by Theorem 7.6

$$[c_\lambda]a = \bigcup_{\gamma \in \Gamma} [c_\lambda]a_\gamma \quad \text{for every } \lambda \in \Lambda,$$

and hence by Theorem 34.1

$$([c_\lambda]a)^{\hat{R}_\lambda} = \bigcup_{\gamma \in \Gamma} ([c_\lambda]a_\gamma)^{\hat{R}_\lambda} \quad \text{for every } \lambda \in \Lambda.$$

Consequently we obtain

$$a^{\hat{R}} = \{([c_\lambda]a)^{\hat{R}_\lambda}\}_\lambda = \bigcup_{\gamma \in \Gamma} \{([c_\lambda]a_\gamma)^{\hat{R}_\lambda}\}_\lambda = \bigcup_{\gamma \in \Gamma} a_\gamma^{\hat{R}},$$

as remarked already. Therefore $\hat{R}$ is a universal completion of R by the correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$.

Finally we shall prove the uniqueness of universal completion up to an isomorphism. Let $\hat{R}$ and $\tilde{R}$ be universal completions of a universally continuous semi-ordered linear space R respectively by extending correspondences

$$R \ni a \rightarrow a^{\hat{R}} \in \hat{R} \quad \text{and} \quad R \ni a \rightarrow a^{\tilde{R}} \in \tilde{R}.$$

To every positive element $\hat{a} \in \hat{R}$ there exists by definition an orthogonal system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) for which

$$\hat{a} = \bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}},$$

and then there exists by definition a positive element $\tilde{a} \in \tilde{R}$ for which

$$\tilde{a} = \bigcup_{\lambda \in \Lambda} a_\lambda^{\tilde{R}}.$$

If $\bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\hat{R}}$ for two orthogonal systems $a_\lambda \in R$ ($\lambda \in \Lambda$) and $b_\gamma \in R$ ($\gamma \in \Gamma$), then we have by Theorems 3.3 and 34.1

$$a_p^{\hat{R}} = a_p^{\hat{R}} \cap \left(\bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}} \right) = a_p^{\hat{R}} \cap \left(\bigcup_{\gamma \in \Gamma} b_\gamma^{\hat{R}} \right) = \bigcup_{\gamma \in \Gamma} (a_p \cap b_\gamma)^{\hat{R}}$$

for every $p \in \Lambda$, and hence

$$a_p = \bigcup_{\gamma \in \Gamma} (a_p \cap b_\gamma) \quad \text{for every } p \in \Lambda.$$

We also obtain likewise

$$b_\gamma = \bigcup_{\lambda \in \Lambda} (a_\lambda \cap b_\gamma) \quad \text{for every } \gamma \in \Gamma.$$

Consequently we have by Theorems 2.5 and 3.15

$$\bigcup_{\lambda \in \Lambda} a_\lambda^{\tilde{R}} = \bigcup_{\lambda, \gamma} (a_\lambda \cap b_\gamma)^{\tilde{R}} = \bigcup_{\gamma \in \Gamma} b_\gamma^{\tilde{R}}.$$

Thus we obtain a correspondence $\hat{R} \ni \hat{a} \rightarrow \hat{a}^{\tilde{R}} \in \tilde{R}$ such that

$$\hat{a} = \bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}}$$

for an orthogonal system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$)

implies

$$\hat{a} \tilde{R} = \bigcup_{\lambda \in \Lambda} a_{\lambda} \tilde{R},$$

and for an arbitrary element $\hat{a} \in \hat{R}$

$$\hat{a} \tilde{R} = \hat{a} + \tilde{R} - \hat{a} - \tilde{R}.$$

$\hat{R}$ is isomorphic to $\tilde{R}$ by this correspondence. In fact, we have obviously that

$$(a \hat{R}) \tilde{R} = a \tilde{R} \quad \text{for every } a \in R,$$

and that $\hat{a} \tilde{R} \geq 0$ if and only if $\hat{a} \geq 0$. Thus we need only to prove that

$$(\alpha \hat{a} + \beta \hat{b}) \tilde{R} = \alpha \hat{a} \tilde{R} + \beta \hat{b} \tilde{R}$$

for every elements $\hat{a}, \hat{b} \in \hat{R}$ and for every real numbers α, β .

If $\hat{a} = \bigcup_{\lambda \in \Lambda} a_{\lambda} \hat{R}$ for an orthogonal system of positive elements $a_{\lambda} \in R$ ($\lambda \in \Lambda$), then we have by Theorem 2.3 for every positive number α

$$\alpha \hat{a} = \bigcup_{\lambda \in \Lambda} \alpha a_{\lambda} \hat{R} = \bigcup_{\lambda \in \Lambda} (\alpha a_{\lambda}) \hat{R},$$

and hence by definition

$$(\alpha \hat{a}) \tilde{R} = \bigcup_{\lambda \in \Lambda} (\alpha a_{\lambda}) \tilde{R} = \alpha \bigcup_{\lambda \in \Lambda} a_{\lambda} \tilde{R} = \alpha \hat{a} \tilde{R}.$$

Consequently we have by Theorem 3.9 for an arbitrary element $\hat{a} \in \hat{R}$ and for every positive number α

$$(\alpha \hat{a}) \tilde{R} = (\alpha \hat{a} +) \tilde{R} - (\alpha \hat{a} -) \tilde{R} = \alpha (\hat{a} + \tilde{R} - \hat{a} - \tilde{R}) = \alpha \hat{a} \tilde{R}.$$

Since $(-\hat{a}) \tilde{R} = \hat{a} - \tilde{R} - \hat{a} + \tilde{R} = -\hat{a} \tilde{R}$ by definition, we have hence

$$(\alpha \hat{a}) \tilde{R} = \alpha \hat{a} \tilde{R}$$

for an arbitrary element $\hat{a} \in \hat{R}$ and for an arbitrary real number α .

For two orthogonal systems of positive elements $a_{\lambda} \in R$ ($\lambda \in \Lambda$) and $b_{\gamma} \in R$ ($\gamma \in \Gamma$), denoting by $[p_{\delta}]$ ($\delta \in \Delta$) the totality of projectors

$$[a_{\lambda}][b_{\gamma}], \quad [a_{\lambda}] - [\bigcup_{\gamma \in \Gamma} [b_{\gamma}] a_{\lambda}], \quad [b_{\gamma}] - [\bigcup_{\lambda \in \Lambda} [a_{\lambda}] b_{\gamma}]$$

for every $\lambda \in \Lambda$, $\gamma \in \Gamma$, we see easily that

$$[p_{\delta}][p_{\rho}] = 0 \quad \text{for } \delta \neq \rho,$$

$$a_{\lambda} = \bigcup_{\delta \in \Delta} [p_{\delta}] a_{\lambda}, \quad b_{\gamma} = \bigcup_{\delta \in \Delta} [p_{\delta}] b_{\gamma} \quad \text{for every } \lambda \in \Lambda, \gamma \in \Gamma,$$

and both $\bigcup_{\lambda \in \Lambda} [p_{\delta}] a_{\lambda}$ and $\bigcup_{\gamma \in \Gamma} [p_{\delta}] b_{\gamma}$ exist in R for every $\delta \in \Delta$.

Furthermore if $\hat{a} = \bigcup_{\lambda \in \Lambda} a_\lambda \hat{R}$, $\hat{b} = \bigcup_{\gamma \in \Gamma} b_\gamma \hat{R}$, then we have by Theorems 34.1 and 2.5

$$\begin{aligned}\hat{a} &= \bigcup_{\lambda, \delta} ([p_\delta] a_\lambda) \hat{R} = \bigcup_{\delta \in \Delta} \left(\bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda \right) \hat{R}, \\ \hat{b} &= \bigcup_{\gamma, \delta} ([p_\delta] b_\gamma) \hat{R} = \bigcup_{\delta \in \Delta} \left(\bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma \right) \hat{R}.\end{aligned}$$

These relations yields by Theorem 2.6

$$\hat{a} + \hat{b} = \bigcup_{\delta \in \Delta} \left(\bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma \right) \hat{R},$$

because we have by Theorem 3.1 for $\delta \neq \delta'$

$$\begin{aligned}& \bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_{\delta'}] b_\gamma \\ & \leq \bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma + \bigcup_{\lambda \in \Lambda} [p_{\delta'}] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_{\delta'}] b_\gamma \\ & = \left(\bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma \right) \cup \left(\bigcup_{\lambda \in \Lambda} [p_{\delta'}] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_{\delta'}] b_\gamma \right),\end{aligned}$$

since we have for $\delta \neq \delta'$

$$\left(\bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma \right) \cap \left(\bigcup_{\lambda \in \Lambda} [p_{\delta'}] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_{\delta'}] b_\gamma \right) = 0.$$

On the other hand we have by definition

$$\begin{aligned}\hat{a} \tilde{R} &= \bigcup_{\delta \in \Delta} \left(\bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda \right) \tilde{R}, \\ \hat{b} \tilde{R} &= \bigcup_{\delta \in \Delta} \left(\bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma \right) \tilde{R},\end{aligned}$$

and we can prove likewise

$$\hat{a} \tilde{R} + \hat{b} \tilde{R} = \bigcup_{\delta \in \Delta} \left(\bigcup_{\lambda \in \Lambda} [p_\delta] a_\lambda + \bigcup_{\gamma \in \Gamma} [p_\delta] b_\gamma \right) \tilde{R}.$$

Therefore we obtain by definition

$$\hat{a} \tilde{R} + \hat{b} \tilde{R} = (\hat{a} + \hat{b}) \tilde{R}$$

for every positive elements $\hat{a}, \hat{b} \in \hat{R}$.

For arbitrary elements $\hat{a}, \hat{b} \in \hat{R}$ we have obviously

$$\hat{a}^+ + \hat{b}^+ + (\hat{a} + \hat{b})^- = \hat{a}^- + \hat{b}^- + (\hat{a} + \hat{b})^+,$$

and hence we obtain

$$\hat{a} + \tilde{R} + \hat{b} + \tilde{R} + (\hat{a} + \hat{b})^{-\tilde{R}} = \hat{a}^{-\tilde{R}} + \hat{b}^{-\tilde{R}} + (\hat{a} + \hat{b})^{+\tilde{R}},$$

as proved just now. Consequently we have

$$\begin{aligned}\hat{a} \tilde{R} + \hat{b} \tilde{R} &= \hat{a}^{+\tilde{R}} - \hat{a}^{-\tilde{R}} + \hat{b}^{+\tilde{R}} - \hat{b}^{-\tilde{R}} \\ &= (\hat{a} + \hat{b})^{+\tilde{R}} - (\hat{a} + \hat{b})^{-\tilde{R}} = (\hat{a} + \hat{b}) \tilde{R},\end{aligned}$$

as we wish to prove.

CHAPTER VI

C-SPACES§35 C-functions

Let E be a locally compact Hausdorff space. A bounded continuous function φ on E is said to be a C-function on E if the point set

$$\{x : |\varphi(x)| \geq \varepsilon\}$$

is compact for every positive number ε .

Theorem 35.1. For a compact set A and for an open set B , if $A \subset B$, then there exists a C-function φ on E such that

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin B, \end{cases}$$

and $0 \leq \varphi(x) \leq 1$ for every point $x \in E$.

Proof. Since E is a locally compact Hausdorff space by assumption, for a compact set A and an open set B , if $A \subset B$, then there exists a finite number of open sets U_ν ($\nu = 1, 2, \dots, \kappa$) such that the closure U_ν^- is compact for every $\nu = 1, 2, \dots, \kappa$, and

$$A \subset \sum_{\nu=1}^{\kappa} U_\nu \subset B.$$

Then, putting $U = \sum_{\nu=1}^{\kappa} U_\nu$, we obtain an open set U such that the closure U^- is compact and $A \subset U \subset B$.

Since E is regular by Theorem 2 in Introduction, we see easily by Extension theorem that there exists a continuous function φ on E such that

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin U, \end{cases}$$

and $0 \leq \varphi(x) \leq 1$ for every point $x \in E$. Such a continuous function φ is obviously by definition a C-function on E satisfying our requirement.

Theorem 35.2. If a sequence of C-functions φ_ν ($\nu = 1, 2, \dots$) is uniformly convergent to a function φ on E , that is, if to any positive number ε there exists ν_0 such that we have

$$|\varphi_\nu(x) - \varphi(x)| \leq \varepsilon \quad \text{for every } \nu \geq \nu_0 \text{ and } x \in E,$$

then φ is a C-function on E too.

Proof. We see easily by definition that φ is a bounded continuous function on E . To every positive number ε there exists by assumption ν such that

$$|\varphi_\nu(x) - \varphi(x)| \leq \frac{1}{2} \varepsilon \quad \text{for every point } x \in E.$$

Then we have obviously

$$|\varphi_\nu(x)| \geq |\varphi(x)| - \frac{1}{2} \varepsilon \quad \text{for every point } x \in E.$$

This relation yields obviously

$$\{x : |\varphi_\nu(x)| \geq \frac{1}{2} \varepsilon\} \supset \{x : |\varphi(x)| \geq \varepsilon\}.$$

Since φ_ν is a C-function on E by assumption, the point set

$$\{x : |\varphi(x)| \geq \varepsilon\}$$

is hence compact for every positive number ε . Therefore φ is a C-function on E too.

The totality of C-functions on a locally compact Hausdorff space E is called the C-space on E . If E is a compact Hausdorff space, then the C-space on E consists of all bounded continuous functions on E .

Let C be the C-space on a locally compact Hausdorff space E . For every C-functions $\varphi, \psi \in C$ and for every real numbers α, β , putting

$$(\alpha\varphi + \beta\psi)(x) = \alpha\varphi(x) + \beta\psi(x) \quad \text{for every point } x \in E,$$

we obtain $\alpha\varphi + \beta\psi \in C$, because $\alpha\varphi + \beta\psi$ is obviously a bounded continuous function on E and we have for any positive number ε

$$\{x : |(\alpha\varphi + \beta\psi)(x)| \geq \varepsilon\}$$

$$\subset \{x : |\varphi(x)| \geq \frac{\varepsilon}{|\alpha| + |\beta|}\} \cup \{x : |\psi(x)| \geq \frac{\varepsilon}{|\alpha| + |\beta|}\}.$$

Thus C constitutes a linear space. Furthermore, for every

C-functions $\varphi, \psi \in \mathcal{C}$, putting

$$\varphi\psi(x) = \varphi(x)\psi(x) \quad \text{for every point } x \in E,$$

we obtain $\varphi\psi \in \mathcal{C}$, since we have for every positive number ε

$$\{x: |\varphi\psi(x)| \geq \varepsilon\} \subset \{x: |\psi(x)| \geq \frac{\varepsilon}{\sup_{x \in E} |\varphi(x)|}\}.$$

Therefore $\mathcal{C}$ constitutes a ring.

We shall define $\varphi \geq \psi$ for C-functions $\varphi, \psi \in \mathcal{C}$ to mean

$$\varphi(x) \geq \psi(x) \quad \text{for every point } x \in E.$$

With this definition we have obviously that $\mathcal{C}$ is a semi-ordered ring. Furthermore we see easily that $\mathcal{C}$ is lattice ordered and we have

$$\varphi \vee \psi(x) = \max\{\varphi(x), \psi(x)\},$$

$$\varphi \wedge \psi(x) = \min\{\varphi(x), \psi(x)\}$$

for every point $x \in E$, and consequently

$$|\varphi|(x) = |\varphi(x)| \quad \text{for every point } x \in E.$$

Corresponding to every C-function $\varphi \in \mathcal{C}$ we define the norm

$\|\varphi\|$ as

$$\|\varphi\| = \sup_{x \in E} |\varphi(x)|.$$

With this definition we have obviously:

Theorem 35.3. Concerning the norm of the C-space $\mathcal{C}$ on

E , we have

- 1) $0 \leq \|\varphi\| < +\infty$ for every $\varphi \in \mathcal{C}$,
- 2) $\|\varphi\| = 0$ implies $\varphi = 0$, i.e. $\varphi(x) = 0$ for every $x \in E$
- 3) $0 \leq \varphi \leq \psi$ implies $\|\varphi\| \leq \|\psi\|$,
- 4) $\|\varphi\| = \| |\varphi| \|$,
- 5) $\|\alpha\varphi\| = |\alpha| \|\varphi\|$ for every real number α ,
- 6) $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$,
- 7) $\|\varphi \vee \psi\| = \max\{\|\varphi\|, \|\psi\|\}$ for $\varphi, \psi \geq 0$,
- 8) $\|\varphi\psi\| \leq \|\varphi\| \|\psi\|$.

Theorem 35.4. For a system of positive C-functions $\varphi_\lambda \in \mathcal{C}$

($\lambda \in \Lambda$), if there exists a C-function $\psi \in \mathcal{C}$ such that $\psi \geq \varphi_\lambda$ for every $\lambda \in \Lambda$, then there exists a positive C-function $\varphi \in \mathcal{C}$

such that $\varphi \geq \varphi_\lambda$ for every $\lambda \in A$ and

$$\|\varphi\| = \sup_{\lambda \in A} \|\varphi_\lambda\|.$$

Proof. We have obviously by definition

$$\alpha = \sup_{\lambda \in A, x \in E} |\varphi_\lambda(x)| = \sup_{\lambda \in A} \|\varphi_\lambda\| \leq \|\varphi\|.$$

Putting $\varphi(x) = \min\{\varphi(x), \alpha\}$ for every $x \in E$, we see easily by definition that $\varphi \in C$,

$$\varphi \geq \varphi_\lambda \quad \text{for every } \lambda \in A,$$

and $\|\varphi\| = \alpha$.

We see easily by definition of the norm that

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \varphi\| = 0$$

is equivalent to that the sequence $\varphi_\nu \in C$ ($\nu = 1, 2, \dots$) is uniformly convergent to $\varphi \in C$ in E . Therefore we conclude immediately from Theorem 35.2:

Theorem 35.5. The C-space C on E is norm complete, i.e. for a sequence of C-functions $\varphi_\nu \in C$ ($\nu = 1, 2, \dots$), if to any positive number ε there exists ν_0 such that

$$\|\varphi_\nu - \varphi_\mu\| \leq \varepsilon \quad \text{for every } \nu, \mu \geq \nu_0,$$

then there exists a C-function $\varphi \in C$ for which

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \varphi\| = 0$$

§36 Ortho-normal manifolds

A subset A of a linear space R is said to be a linear manifold of R , if $x, y \in A$ implies for every real numbers α, β

$$\alpha x + \beta y \in A.$$

Let R be a lattice ordered linear space. A subset A of R is said to be a lattice manifold of R , if $x, y \in A$ implies both

$$x \vee y \in A \quad \text{and} \quad x \wedge y \in A$$

A subset A of R is said to be a linear lattice manifold of R , if A is a linear manifold as well as a lattice one of R

For a subset $A \neq \emptyset$ of R , the totality of the elements which are orthogonal to every element of A is said to be the orthogonal complement of A and denoted by $A^\perp$. With this definition we have obviously the following relations:

- (1) $R^\perp = \{0\}$, $\{0\}^\perp = R$,
- (2) $A \supset B$ implies $A^\perp \subset B^\perp$,
- (3) $A \subset A^{\perp\perp}$,
- (4) $AA^\perp \subset \{0\}$.

Since $A \subset A^{\perp\perp}$ by the formula (3), we obtain $A^\perp \supset A^{\perp\perp\perp}$ by the formula (2). On the other hand we have by the formula (3)

$$A^\perp \subset (A^\perp)^{\perp\perp} = A^{\perp\perp\perp}.$$

Consequently we obtain:

$$(5) \quad A^\perp = A^{\perp\perp\perp}.$$

We conclude immediately from definition:

$$(6) \quad (A \dot{+} B)^\perp = A^\perp B^\perp.$$

A subset A of R is said to be an ortho-normal manifold of R , if A is an orthogonal complement of some subset of R that is, if there exists a subset B of R for which $A = B^\perp$. We see easily by the formula (5) that a subset A of R is an ortho-normal manifold of R if and only if $A = A^{\perp\perp}$. Furthermore it is obvious by Theorems 4.2, 4.3, and 4.4 that every ortho-normal manifold of R is a linear lattice manifold of R .

Let C be the C-space on a locally compact Hausdorff space

E. Now we shall consider ortho-normal manifold of C .

Theorem 36.1. To every ortho-normal manifold A of C there exists uniquely an open set P in E such that A consists only of all C-functions on P , that is, only of all C-functions $\varphi \in C$ which vanish in the complement P' , and such an open set P is regularly open.

Proof. For every C-functions $\varphi \in A$ and $\psi \in A^\perp$ we have by definition

$$\min\{|\varphi(x)|, |\psi(x)|\} = 0 \quad \text{for every point } x \in E.$$

Hence, if we put

$$N_\psi = \{x : |\psi(x)| > 0\} \quad \text{for every } \psi \in A^\perp,$$

then every C-function $\varphi \in A$ vanished in the point set N_ψ for every $\psi \in A^\perp$ and consequently in the union $\sum_{\psi \in A^\perp} N_\psi$, which is an open set, since every C-function is continuous over E .

Conversely, if a C-function $\varphi \in C$ vanished in $\sum_{\psi \in A^\perp} N_\psi$, then we have obviously by definition

$$\varphi \in A^{\perp\perp} = A,$$

since A is an ortho-normal manifold of C by assumption.

Therefore, putting

$$P = \left(\sum_{\psi \in A^\perp} N_\psi \right)^{-}$$

we obtain a regularly open set P such that A consists only of all C-functions on P . Furthermore we see easily by Theorem 35.1 that to every point $y \in P$ there exists a C-function $\varphi \in C$ such that

$$\varphi(y) = 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{for every point } x \in P',$$

and that to every point $y \in P^-$ there exists a C-function $\varphi \in C$ such that

$$\varphi(y) = 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{for every point } x \in P^-.$$

Accordingly there exists no other open set than P such that A be composed of all C-functions on it.

By virtue of Theorem 36.1, corresponding to every ortho-normal manifold A of C there exists uniquely an open set P such that A is composed only of all C-functions on P . Such an open set P is called the characteristic set of an ortho-normal manifold A of C and denoted by E_A .

Theorem 36.2. To every regularly open set P in E there exists uniquely an ortho-normal manifold A of the C-space C such that the characteristic set E_A of A coincides with P

Proof. Putting $Q = P^{-'}$, we have

$$Q^{-} = P^{-' -} = P^{-\circ'} = P',$$

since P is regularly open by assumption. Let B be the totality of C -functions which vanish in P , and let A be the totality of C -functions which vanish in Q . Then it is obvious by definition that every C -function of A is orthogonal to every C -function of B , since every C -function of A vanishes further in Q^{-} and $Q^{-} = P'$. To every point $y \in Q$ there exists by Theorem 35.1 a C -function φ on E such that

$$\varphi(y) = 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{for every point } x \in Q',$$

and such a C -function φ is contained in B since $Q' = P^{-}$. Therefore, if a C -function φ is orthogonal to every C -function of B , then φ must vanish in Q , that is, $\varphi \in A$. Accordingly we have $A = B^{\perp}$ by definition, and hence A is an ortho-normal manifold of the C -space C on E . Furthermore it is evident by definition that the characteristic set E_A of A coincides with P . The uniqueness of such A is obvious by definition.

In this Proof we have proved further:

Theorem 36.3. For every ortho-normal manifold A of the C -space C we have

$$E_{A^{\perp}} = E_A^{-'}.$$

We have obviously by definition:

Theorem 36.4. For two ortho-normal manifolds A and B of the C -space C we have $A \supset B$ if and only if $E_A \supset E_B$.

Theorem 36.5. For every two ortho-normal manifolds A and B of the C -space C , the intersection AB is again an ortho-normal manifold of C and we have

$$E_{AB} = E_A E_B.$$

Proof. It is obvious by the formula (6) that the intersection AB is again an ortho-normal manifold of the C -space

$\mathcal{C}$ Furthermore we have by the previous the rem

$$E_{AB} \subset E_A E_B.$$

To every point $y \in E_A E_B$ there exists by Theorem 35.1 a $\mathcal{C}$ -function φ such that

$$\varphi(y) = 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{for every point } x \notin E_A E_B.$$

Such φ is contained obviously in A as well as in B , and consequently in the intersection AB . Accordingly $y \in E_A E_B$ implies $y \in E_{AB}$ by definition. Therefore we obtain $E_{AB} = E_A E_B$.

From this theorem we conclude immediately:

Theorem 36.6. For two ortho-normal manifolds A and B of the $\mathcal{C}$ -space $\mathcal{C}$ we have

$$AB = \{0\}$$

if and only if $E_A E_B = 0$.

For two ortho-normal manifolds A and B of the $\mathcal{C}$ -space

$\mathcal{C}$ we shall write

$$A > B \quad \text{or} \quad B < A$$

if A contains a $\mathcal{C}$ -function φ_0 such that to every $\mathcal{C}$ -function $\psi \in B$ there exists a positive number α for which we have

$$|\psi(x)| \leq \alpha \varphi_0(x) \quad \text{for every point } x \in E.$$

If $A > B$, then we have $A \supset B$, because we see easily by Theorems 4.1 and 4.2 that every $\mathcal{C}$ -function of $A^\perp$ is orthogonal to every $\mathcal{C}$ -function of B , if $A > B$.

Theorem 36.7. For two ortho-normal manifolds A and B of the $\mathcal{C}$ -space $\mathcal{C}$ we have $A < B$ if and only if the closure E_A^- of the characteristic set E_A is compact and included in the characteristic set E_B of B

Proof. If $A < B$, then we have $E_A \subset E_B$ by Theorem 36.4, and there exists by definition a $\mathcal{C}$ -function $\psi_0 \in B$ such that to every $\mathcal{C}$ -function $\varphi \in A$ there exists a positive number α for which

$$|\varphi(x)| \leq \alpha \psi_0(x) \quad \text{for every point } x \in E.$$

Corresponding to such ψ_0 there exists a positive number ε such that

$$\psi_0(x) \geq \varepsilon \quad \text{for every point } x \in E_A.$$

Because, if there exists a sequence of points $x_\nu \in E_A$ ($\nu = 1, 2, \dots$) such that

$$0 < \psi_0(x_\nu) \leq \frac{1}{\nu^4} \quad \text{for every } \nu = 1, 2, \dots,$$

then there exists by Theorem 35.1 a sequence of C-functions φ_ν ($\nu = 1, 2, \dots$) such that

$$\varphi_\nu(x) = \begin{cases} \sqrt{\psi_0(x_\nu)} & \text{for } x = x_\nu, \\ 0 & \text{for } x \in E_A, \end{cases}$$

and $0 \leq \varphi_\nu(x) \leq \sqrt{\psi_0(x_\nu)}$ for every point $x \in E$. For such φ_ν ($\nu = 1, 2, \dots$) we have obviously

$$0 \leq \varphi_\nu(x) \leq \frac{1}{\nu^2} \quad \text{for every point } x \in E,$$

and hence the series $\sum_{\nu=1}^{\infty} \varphi_\nu(x)$ is uniformly convergent in E .

Accordingly, putting

$$\varphi(x) = \sum_{\nu=1}^{\infty} \varphi_\nu(x),$$

we obtain by Theorem 35.2 a C-function φ , and we have obviously for every $\nu = 1, 2, \dots$

$$\varphi(x_\nu) \geq \varphi_\nu(x_\nu) = \sqrt{\psi_0(x_\nu)} \geq \nu^2 \psi_0(x_\nu).$$

On the other hand we have $\varphi(x) = 0$ for every point $x \in E_A$, and hence $\varphi \in A$, contradicting our assumption that there exists a positive number α for which we have

$$|\varphi(x)| \leq \alpha \psi_0(x) \quad \text{for every point } x \in E.$$

Therefore there exists a positive number ε such that

$$\psi_0(x) \geq \varepsilon \quad \text{for every point } x \in E_A.$$

Then we have further

$$\psi_0(x) \geq \varepsilon \quad \text{for every point } x \in E_A^-,$$

since ψ_0 is a continuous function on E . Consequently we have

$$E_B \supset \{x : |\psi_0(x)| \geq \varepsilon\} \supset E_A^-.$$

Since the point set $\{x : |\psi_0(x)| \geq \varepsilon\}$ is compact, E_A^- is

hence compact too.

Conversely, if E_A^- is compact and included in E_B , then there exists a C-function ψ_0 by Theorem 35.1 such that

$$\psi_0(x) = \begin{cases} 1 & \text{for } x \in E_A^-, \\ 0 & \text{for } x \in E_B, \end{cases}$$

and $0 \leq \psi_0(x) \leq 1$ for every point $x \in E$. For such ψ_0 we have obviously $\psi_0 \in B$ and

$$|\varphi(x)| \leq \|\varphi\| \psi_0(x) \quad \text{for every } \varphi \in A,$$

since $\varphi \in A$ implies $\varphi(x) = 0$ for every point $x \in E_A$.

As an immediate consequence from Theorem 36.7 we have:

Theorem 36.8. For an ortho-normal manifold A of the C-space C we have $A < C$ if and only if the closure E_A^- of the characteristic set E_A is compact.

§37 Isomorphisms between two C-spaces

If a locally compact Hausdorff space E is homeomorphic to a locally compact Hausdorff space $\hat{E}$ by a correspondence $E \ni x \rightarrow x^{\hat{E}} \in \hat{E}$, and ω is a continuous function on E subject to the condition

$$0 < \inf_{x \in E} \omega(x) \leq \sup_{x \in E} \omega(x) < +\infty,$$

then we see easily that the C-space $\hat{C}$ on $\hat{E}$ is isomorphic to the C-space C on E as a semi-ordered linear space by the correspondence

$$\hat{C} \ni \hat{\varphi} \rightarrow \hat{\varphi}^C \in C : \hat{\varphi}^C(x) = \omega(x) \hat{\varphi}(x^{\hat{E}}) \quad \text{for } x \in E.$$

Conversely we have:

Theorem 37.1. If the C-space C on a locally compact Hausdorff space E is isomorphic to the C-space $\hat{C}$ on a locally compact Hausdorff space $\hat{E}$ as a semi-ordered linear space by a correspondence $C \ni \varphi \rightarrow \varphi^{\hat{C}} \in \hat{C}$, then E is homeomorphic to $\hat{E}$ by a correspondence $E \ni x \rightarrow x^{\hat{E}} \in \hat{E}$ such that

$$\varphi^{\hat{E}}(x^{\hat{E}}) = \omega(x) \varphi(x) \quad (x \in E, \varphi \in C)$$

for a continuous function $\omega(x)$ on E subject to the condition

$$0 < \inf_{x \in E} \omega(x) \leq \sup_{x \in E} \omega(x) < +\infty.$$

Proof. Corresponding to every subset A of C , putting

$$A^{\hat{E}} = \{ \varphi^{\hat{E}} : \varphi \in A \}$$

we obtain a subset $A^{\hat{E}}$ of $\hat{C}$. Since C is isomorphic to $\hat{C}$ as a semi-ordered linear space by the indicated correspondence, we see easily that if A is an ortho-normal manifold of C , then $A^{\hat{E}}$ is again an ortho-normal manifold of $\hat{C}$, and conversely that to every ortho-normal manifold $\hat{A}$ of $\hat{C}$ there exists uniquely an ortho-normal manifold A of C for which $\hat{A} = A^{\hat{E}}$. Furthermore we have $A^{\hat{E}} > B^{\hat{E}}$ if and only if $A > B$. Therefore we have by Theorem 36.2 that to every regularly open set P in E there exists uniquely a regularly open set $\hat{P}$ in $\hat{E}$ such

$$E_A = P \quad \text{implies} \quad E_{A^{\hat{E}}} = \hat{P},$$

i.e., if P is the characteristic set of an ortho-normal manifold A of C , then $\hat{P}$ is the characteristic set of $A^{\hat{E}}$. Such $\hat{P}$ will be denoted by $P^{\hat{E}}$ in the sequel. Then we see easily that to every regularly open set $\hat{P}$ in $\hat{E}$ there exists uniquely a regularly open set P in E for which $\hat{P} = P^{\hat{E}}$.

With this correspondence $P \rightarrow P^{\hat{E}}$ between regularly open sets P in E and $P^{\hat{E}}$ in $\hat{E}$ we have by Theorem 36.5

$$(PQ)^{\hat{E}} = P^{\hat{E}} Q^{\hat{E}},$$

and further by Theorem 36.6 that

$$P^{\hat{E}} Q^{\hat{E}} = 0 \quad \text{if and only if} \quad PQ = 0.$$

Furthermore we have by Theorem 36.8 that the closure $P^{\hat{E}-}$ of $P^{\hat{E}}$ is compact if and only if the closure P^- is compact, and by Theorem 36.7 that for regularly open sets P and Q with compact closures we have $P^{\hat{E}} > Q^{\hat{E}}$ if and only if $P > Q$.

For an arbitrary point $x \in E$, let P_λ ($\lambda \in A$) be the

system of all regularly open sets containing x whose closures are compact. Then, to every $\lambda \in A$ there exists obviously an element $p \in A$ for which $p_p < p_\lambda$. For every finite number of elements $\lambda_\nu \in A$ ($\nu = 1, 2, \dots, \kappa$) we have obviously

$$p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_\kappa} \neq 0,$$

and hence $p_{\lambda_1}^{\hat{E}} p_{\lambda_2}^{\hat{E}} \dots p_{\lambda_\kappa}^{\hat{E}} \neq 0$. Since the closure $p_\lambda^{\hat{E}}$ is compact for every $\lambda \in A$, as remarked just above, we have hence

$$\prod_{\lambda \in A} p_\lambda^{\hat{E}} = \prod_{\lambda \in A} p_\lambda^{\hat{E}-} \neq 0,$$

because to any $\lambda \in A$ there exists an element $f \in A$ for which $p_f^{\hat{E}} < p_\lambda^{\hat{E}}$, as remarked just now. Furthermore $\prod_{\lambda \in A} p_\lambda^{\hat{E}}$ consists nly of a single point. Because, for any point

$$\hat{x} \in \prod_{\lambda \in A} p_\lambda^{\hat{E}},$$

if we denote by $\hat{Q}_\gamma$ ($\gamma \in \Gamma$) the system of all regularly open sets containing $\hat{x}$ whose closures are compact, then there exists a system of regularly open sets Q_γ ($\gamma \in \Gamma$) in E such that $\hat{Q} = Q_\gamma^{\hat{E}}$ for every $\gamma \in \Gamma$, and we also can prove likewise that

$$\prod_{\gamma \in \Gamma} Q_\gamma = \prod_{\gamma \in \Gamma} Q_\gamma^- \neq 0.$$

The system Q_γ ($\gamma \in \Gamma$) includes obviously all p_λ ($\lambda \in A$), and consequently Q_γ ($\gamma \in \Gamma$) must coincide with p_λ ($\lambda \in A$). Therefore $\prod_{\lambda \in A} p_\lambda^{\hat{E}}$ is composed only of a single point $\hat{x}$ which will be denoted by $x^{\hat{E}}$ corresponding to the point $x \in E$.

Then we see easily that the correspondence

$$E \ni x \rightarrow x^{\hat{E}} \in \hat{E}$$

is one-to-one, and that for a regularly open set P with a compact closure P^- we have $P \ni x$ if and only if $p^{\hat{E}} \ni x^{\hat{E}}$. Thus E is homeomorphic to $\hat{E}$ by this correspondence $E \ni x \rightarrow x^{\hat{E}} \in \hat{E}$.

Let y be an arbitrary point of E . If $\psi(y) > 0$ for a C-function $\psi \in \mathcal{C}$, then there exist obviously a positive number ε and a regularly open set P with a compact closure P^- such that $\psi(y) > \varepsilon$, $P \ni y$, and

$$\psi(x) \geq \varepsilon \quad \text{for every point } x \in P.$$

For the ortho-normal manifold A of C such that $E_A = P$, we have then that corresponding to every $\varphi \in A$ there exists a positive number α such that

$$\alpha \psi^+ \geq |\varphi|,$$

and hence $\alpha \psi^{\hat{E}+} \geq |\varphi^{\hat{E}}|$. Consequently, to every C-function $\hat{\varphi} \in A^{\hat{E}}$ there exists a positive number α such that

$$\alpha \psi^{\hat{E}+} \geq |\hat{\varphi}|.$$

Since $E_A^{\hat{E}} = P^{\hat{E}}$ as remarked already, we have thus

$$\psi^{\hat{E}+}(y^{\hat{E}}) > 0,$$

and hence naturally $\psi^{\hat{E}}(y^{\hat{E}}) > 0$. Conversely if $\psi^{\hat{E}}(y^{\hat{E}}) > 0$, then we can prove likewise that $\psi(y) > 0$.

If $\psi(y) < 0$ for a C-function $\psi \in C$, then we have naturally $-\psi(y) > 0$, and hence

$$-\psi^{\hat{E}}(y^{\hat{E}}) > 0, \quad \text{namely } \psi^{\hat{E}}(y^{\hat{E}}) < 0,$$

as proved just now. Conversely, we see likewise that

$$\psi^{\hat{E}}(y^{\hat{E}}) < 0 \quad \text{implies} \quad \psi(y) < 0.$$

Consequently we have $\psi^{\hat{E}}(y^{\hat{E}}) = 0$ if and only if $\psi(y) = 0$.

If $\varphi(y) - \alpha \psi(y) = 0$ for two C-functions $\varphi, \psi \in C$ and for a real number α , then we also have

$$\varphi^{\hat{E}}(y^{\hat{E}}) - \alpha \psi^{\hat{E}}(y^{\hat{E}}) = 0,$$

as proved just now. Therefore, corresponding to every point

$x \in E$ there exists a positive number $\omega(x)$ such that

$$\varphi^{\hat{E}}(x^{\hat{E}}) = \omega(x) \varphi(x) \quad \text{for every } \varphi \in C.$$

To every regularly open set P with the compact closure P^- , there exists by Theorem 35.1 a C-function $\varphi_0 \in C$ such that

$$\varphi_0(x) = 1 \quad \text{for every point } x \in P^-.$$

For such φ_0 we have obviously

$$\omega(x) = \varphi_0^{\hat{E}}(x^{\hat{E}}) \quad \text{for every point } x \in P,$$

and $\varphi_0^{\hat{E}}(x^{\hat{E}})$ is continuous in E as a function of x , since E is homeomorphic to $\hat{E}$ by the correspondence $E \ni x \rightarrow x^{\hat{E}} \in \hat{E}$.

Consequently $\omega(x)$ is a continuous function on E . Furthermore $\omega(x)$ is bounded. Because, if

$$\omega(x_\nu) \geq \nu^3 \quad (\nu = 1, 2, \dots)$$

for some sequence of points $x_\nu \in E$ ($\nu = 1, 2, \dots$), then there exists by Theorem 35.1 a sequence of C-functions $\varphi_\nu \in \mathcal{C}$ ($\nu = 1, 2, \dots$) such that

$$\varphi_\nu(x_\nu) = 1 \text{ and } 0 \leq \varphi_\nu(x) \leq 1 \text{ for every point } x \in E.$$

For such φ_ν ($\nu = 1, 2, \dots$), the series $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \varphi_\nu(x)$ is uniformly convergent in E , and hence, putting

$$\varphi(x) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \varphi_\nu(x) \quad (x \in E),$$

we obtain by Theorem 35.2 a C-function φ , and we have for every $\nu = 1, 2, \dots$

$$\varphi(\hat{x}_\nu) = \omega(x_\nu) \varphi(x_\nu) \geq \nu^3 \frac{1}{\nu^2} \varphi_\nu(x_\nu) = \nu,$$

contradicting that φ is a C-function on $\hat{E}$.

Considering $\varphi(x) = \frac{1}{\omega(x)} \varphi(\hat{x})$, we also can prove likewise that $\frac{1}{\omega(x)}$ is bounded in E . Therefore we have

$$0 < \inf_{x \in E} \omega(x) \leq \sup_{x \in E} \omega(x) < +\infty$$

Theorem 37.2. If the C-space $\mathcal{C}$ on a locally compact Hausdorff space E is isomorphic to the C-space $\hat{\mathcal{C}}$ on a locally compact Hausdorff space $\hat{E}$ as a semi-ordered linear space by a correspondence $\mathcal{C} \ni \varphi \rightarrow \varphi \hat{\in} \hat{\mathcal{C}}$ such that

$$\|\varphi\| = \|\varphi \hat{\in}\| \quad \text{for every } \varphi \in \mathcal{C},$$

then E is homeomorphic to $\hat{E}$ by a correspondence $E \ni x \rightarrow x \hat{\in}$ $\in \hat{E}$ such that

$$\varphi(x) = \varphi \hat{\in}(x \hat{\in}) \quad \text{for every } \varphi \in \mathcal{C} \text{ and } x \in E.$$

Proof. By virtue of the previous theorem there exists a correspondence $E \ni x \rightarrow x \hat{\in} \in \hat{E}$ by which E is homeomorphic to $\hat{E}$ and we have for some continuous positive function ω on E

$$\varphi \hat{\in}(x \hat{\in}) = \omega(x) \varphi(x) \text{ for every } \varphi \in \mathcal{C} \text{ and } x \in E.$$

We need hence only to prove that $\omega(x) = 1$ for every point $x \in E$. To every point $y \in E$ there exists by Theorem 35.1 a C-function $\varphi \in C$ such that

$$\varphi(y) = 1 \quad \text{and} \quad 0 \leq \varphi(x) \leq 1 \quad \text{for every point } x \in E.$$

For such $\varphi \in C$ we have obviously by assumption

$$\omega(y) = \omega(y) \varphi(y) = \varphi \hat{C}(y \hat{E}) \leq \|\varphi \hat{C}\| = \|\varphi\| \leq 1.$$

Therefore we obtain that $\omega(x) \leq 1$ for every point $x \in E$.

Considering

$$\varphi(x) = \frac{1}{\omega(x)} \varphi \hat{C}(x \hat{E}),$$

we also can prove likewise that $\frac{1}{\omega(x)} \leq 1$ for every point $x \in E$. Consequently $\omega(x) = 1$ for every point $x \in E$, as we wish to prove.

Theorem 37.3. If the C-space C on a locally compact Hausdorff space E is isomorphic to the C-space $\hat{C}$ on a locally compact Hausdorff space $\hat{E}$ as a semi-ordered ring by a correspondence $C \ni \varphi \rightarrow \varphi \hat{C} \in \hat{C}$, then E is homeomorphic to $\hat{E}$ by a correspondence $E \ni x \rightarrow x \hat{E} \in \hat{E}$ such that

$$\varphi(x) = \varphi \hat{C}(x \hat{E}) \quad \text{for every } \varphi \in C \text{ and } x \in E.$$

Proof. We have naturally by assumption that C is isomorphic to $\hat{C}$ as a semi-ordered linear space by the indicated correspondence. Thus there exists by Theorem 37.1 a correspondence $E \ni x \rightarrow x \hat{E} \in \hat{E}$ by which E is homeomorphic to $\hat{E}$ and we have for some positive continuous function ω on E

$$\varphi \hat{C}(x \hat{E}) = \omega(x) \varphi(x) \quad \text{for every } \varphi \in C.$$

To every point $y \in E$ there exists by Theorem 35.1 a C-function $\varphi \in C$ such that $\varphi(y) = 1$. For such $\varphi \in C$ we have

$$\begin{aligned} \varphi \hat{C}(y \hat{E}) \varphi \hat{C}(y \hat{E}) &= \omega(y) \omega(y), \\ (\varphi \varphi) \hat{C}(y \hat{E}) &= \omega(y) \varphi(y) \varphi(y) = \omega(y). \end{aligned}$$

Since C is isomorphic to $\hat{C}$ as a semi-ordered ring by the indicated correspondence, we have further

$$(\varphi \varphi) \hat{C}(y \hat{E}) = \varphi \hat{C}(y \hat{E}) \varphi \hat{C}(y \hat{E}),$$

and consequently we obtain

$$\omega(y) = \omega(y)\omega(y) \quad \text{for every point } y \in E.$$

From this relation we conclude easily that $\omega(x) = 1$ for every point $x \in E$.

§38 Approximation theorems

Let C be the C -space on a locally compact Hausdorff space E . For a subset A of C , a function ψ on E is said to be uniformly approximated by A if to any positive number ε there exists a C -function $\varphi \in A$ such that

$$|\psi(x) - \varphi(x)| \leq \varepsilon \quad \text{for every point } x \in E,$$

that is, $\|\psi - \varphi\| \leq \varepsilon$. Every C -function $\varphi \in A$ is naturally uniformly approximated by A . If every C -function of the C -space C is uniformly approximated by A , then A is said to be norm dense in C .

Theorem 38.1. If a linear lattice manifold A of the C -space C on a locally compact Hausdorff space E satisfies the condition that to every different points x and $y \in E$ there exists a C -function $\varphi \in A$ such that

$$\varphi(x) = 1, \quad \varphi(y) = 0,$$

then A is norm dense in C .

Proof. Let x_0 be an arbitrary point of E and let P be an arbitrary closed set such that $P \ni x_0$. It is obvious by assumption that there exists a C -function $\varphi_0 \in A$ such that $\varphi_0(x_0) = 1$. Corresponding to every positive number ε , if we put

$$P_\varepsilon = \{x : |\varphi_0(x)| \geq \varepsilon\},$$

then P_ε is compact, and hence the intersection PP_ε is compact too. Corresponding to every point $y \in PP_\varepsilon$ there exists by assumption a C -function $\varphi_y \in A$ such that

$$\varphi_y(x_0) = 1, \quad \varphi_y(y) = 0.$$

Since φ_y is continuous over E , there exists then an open set $U_y \ni y$ such that

$$\varphi_y(x) < \varepsilon \quad \text{for every point } x \in U_y.$$

For such U_y we have obviously $PP_0 \subset \sum_{y \in PP_0} U_y$. As PP_0 is compact, there exists hence a finite number of points $y_1, y_2, \dots, y_n$ for which

$$PP_0 \subset U_{y_1} \cup U_{y_2} \cup \dots \cup U_{y_n}.$$

For such $y_1, y_2, \dots, y_n$, putting

$$\varphi_1 = \varphi_0 \wedge \varphi_{y_1} \wedge \varphi_{y_2} \wedge \dots \wedge \varphi_{y_n},$$

we obtain $\varphi_1 \in A$, $\varphi_1(x_0) = 1$, and

$$\varphi_1(x) < \varepsilon \quad \text{for every point } x \in P.$$

Since φ_1 is continuous over E , there exists an open set

$U \ni x_0$ such that

$$\varphi_1(x) < 1 + \varepsilon \quad \text{for every point } x \in U.$$

Then, as proved just now, there exists a C-function $\varphi_0 \in A$ such that

$$\varphi_0(x_0) = 1 \quad \text{and} \quad \varphi_0(x) < \varepsilon \quad \text{for every point } x \in U.$$

For such $\varphi_0 \in A$, if we put

$$\varphi = (\varphi_1 \wedge \varphi_0) \vee 0,$$

then we have $\varphi \in A$, $\varphi(x_0) = 1$,

$$\varphi(x) < \varepsilon \quad \text{for every point } x \in P,$$

and $0 \leq \varphi(x) \leq 1 + \varepsilon$ for every point $x \in E$.

Next, let P be a compact set and let Q be a closed set such that $PQ = \emptyset$. To every point $y \in P$ and every positive number ε there exists a C-function $\varphi_y \in A$ such that

$$\varphi_y(y) = 1 + \frac{1}{2}\varepsilon,$$

$$\varphi_y(x) < \varepsilon \quad \text{for every point } x \in Q,$$

and $0 \leq \varphi_y(x) \leq 1 + \varepsilon$ for every point $x \in E$, as proved just now. Since φ_y is continuous over E , there exists an open set $U_y \ni y$ such that

$$\varphi_{y_j}(x) > 1 \quad \text{for every point } x \in U_{y_j}.$$

As P is compact by assumption, there exists then a finite number of points $y_1, y_2, \dots, y_n$ such that

$$P \subset \sum_{j=1}^n U_{y_j}.$$

For such $y_1, y_2, \dots, y_n$, if we put

$$\varphi = \varphi_{y_1} \cup \varphi_{y_2} \cup \dots \cup \varphi_{y_n},$$

then we have

$$\varphi(x) > 1 \quad \text{for } x \in P,$$

$$\varphi(x) < \varepsilon \quad \text{for } x \in Q,$$

and $0 \leq \varphi(x) \leq 1 + \varepsilon$ for every point $x \in E$.

Let $\psi \in C$ be a positive C-function. To any positive number ε there exists obviously a finite number of positive numbers

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n,$$

such that $\alpha_\nu - \alpha_{\nu-1} < \varepsilon$ ($\nu = 1, 2, \dots, n$) and

$$0 \leq \psi(x) \leq \alpha_n \quad \text{for every point } x \in E$$

Since the point set

$$\{x : \psi(x) \geq \alpha_\nu\} \quad (\nu = 1, 2, \dots, n)$$

is compact, and the point set

$$\{x : \psi(x) \leq \alpha_{\nu-1}\} \quad (\nu = 1, 2, \dots, n)$$

is closed, we see easily, as proved just above, that there exists a sequence of C-functions $\varphi_\nu \in A$ ($\nu = 1, 2, \dots, n$) such that

$$\varphi_\nu(x) > 1 \quad \text{for } \psi(x) \geq \alpha_\nu,$$

$$\varphi_\nu(x) < \frac{\varepsilon}{\alpha_n} \quad \text{for } \psi(x) \leq \alpha_{\nu-1},$$

and $0 \leq \varphi_\nu(x) \leq 1 + \frac{\varepsilon}{\alpha_n}$ for every point $x \in E$. For such

$\varphi_\nu \in A$ ($\nu = 1, 2, \dots, n$), if we put

$$\varphi(x) = \sum_{\nu=1}^n (\alpha_\nu - \alpha_{\nu-1}) \varphi_\nu(x) \quad (x \in E),$$

then we have $\varphi \in A$, and that $\alpha_{\mu-1} \leq \psi(x) \leq \alpha_\mu$ implies

$$\begin{aligned} \sum_{\nu=1}^{\mu-1} (\alpha_\nu - \alpha_{\nu-1}) &\leq \varphi(x) \\ &\leq \sum_{\nu=1}^{\mu-1} (\alpha_\nu - \alpha_{\nu-1}) \left(1 + \frac{\varepsilon}{\alpha_n}\right) + \sum_{\nu=\mu}^n (\alpha_\nu - \alpha_{\nu-1}) \frac{\varepsilon}{\alpha_n}, \end{aligned}$$

that is, $\alpha_{\mu-1} \leq \varphi(x) \leq \alpha_\mu + \varepsilon$ if $\alpha_{\mu-1} \leq \psi(x) \leq \alpha_\mu$.

Consequently we have for every point $x \in E$

$$|\varphi(x) - \varphi(x)| \leq 2\varepsilon,$$

since $\alpha_\mu - \alpha_{\mu-1} < \varepsilon$ for every $\mu = 1, 2, \dots, \kappa$ by assumption. Therefore every positive C-function $\varphi \in C$ is uniformly approximated by A . For an arbitrary C-function $\varphi \in C$, since we have

$$\varphi = \varphi^+ - \varphi^-, \quad \varphi^+, \varphi^- \in C,$$

and both φ^+ and φ^- are uniformly approximated by A , as proved just now, we see easily that φ is uniformly approximated by A too.

Theorem 38.2. If a subring A of the C-space C on a locally compact Hausdorff space E satisfies the condition that to every pair of different points x and $y \in E$ there exists a C-function $\varphi \in A$ such that

$$\varphi(x) \neq \varphi(y), \quad \varphi(x) \neq 0,$$

then A is norm dense in C .

Proof. Let B be the totality of C-functions which are uniformly approximated by A . Then we see easily that B is a subring of C , and further that if a C-function $\varphi \in C$ is uniformly approximated by B , then φ is uniformly approximated by A , and consequently $\varphi \in B$. We shall prove that B is a lattice manifold of C . For this purpose we will prove that putting

$$\alpha_0 = 1, \quad \alpha_\nu = \frac{2\nu-1}{2\nu} \alpha_{\nu-1} \quad \text{for } \nu = 2, 3, \dots,$$

we have for $-1 \leq \lambda \leq 1$

$$|\lambda| = \sum_{\nu=0}^{\infty} \alpha_\nu (1-\lambda^2)^\nu \lambda^2,$$

and this series is uniformly convergent for $-1 \leq \lambda \leq 1$.

In fact, since

$$\alpha_\nu (1-\lambda^2)^\nu \lambda^2 \geq 0 \quad \text{for } -1 \leq \lambda \leq 1,$$

if we put without consideration of convergence

$$\gamma = \sum_{\nu=0}^{\infty} \alpha_{\nu} (1-\lambda^2)^{\nu} \lambda^2,$$

then we have for $-1 \leq \lambda \leq 1$

$$\gamma^2 = \sum_{\kappa=0}^{\infty} \left(\sum_{\nu+\mu=\kappa, \nu \geq 0, \mu \geq 0} \alpha_{\nu} \alpha_{\mu} \right) (1-\lambda^2)^{\kappa} \lambda^4.$$

On the other hand, since $2\nu\alpha_{\nu} = (2\nu-1)\alpha_{\nu-1}$, by assumption, we have

$$\begin{aligned} (2\kappa) \sum_{\nu+\mu=\kappa, \nu \geq 0, \mu \geq 0} \alpha_{\nu} \alpha_{\mu} &= \sum_{\nu+\mu=\kappa, \nu \geq 0, \mu \geq 0} (2\nu+2\mu) \alpha_{\nu} \alpha_{\mu} \\ &= \sum_{\nu+\mu=\kappa, \nu \geq 1, \mu \geq 0} (2\nu-1) \alpha_{\nu-1} \alpha_{\mu} + \sum_{\nu+\mu=\kappa, \nu \geq 0, \mu \geq 1} (2\mu-1) \alpha_{\nu} \alpha_{\mu-1} \\ &= \sum_{\nu+\mu=\kappa-1, \nu \geq 0, \mu \geq 0} (2\nu+1) \alpha_{\nu} \alpha_{\mu} + \sum_{\nu+\mu=\kappa-1, \nu \geq 0, \mu \geq 0} (2\mu+1) \alpha_{\nu} \alpha_{\mu} \\ &= (2\kappa) \sum_{\nu+\mu=\kappa-1, \nu \geq 0, \mu \geq 0} \alpha_{\nu} \alpha_{\mu}, \end{aligned}$$

and consequently $\sum_{\nu+\mu=\kappa, \nu \geq 0, \mu \geq 0} \alpha_{\nu} \alpha_{\mu} = \alpha_0^2 = 1$. Thus we obtain

$$\gamma^2 = \sum_{\kappa=0}^{\infty} (1-\lambda^2)^{\kappa} \lambda^4 = \lambda^2,$$

and hence $\gamma = |\lambda|$, since $\gamma \geq 0$. Furthermore, since all functions of λ

$$|\lambda|, \quad \alpha_{\nu} (1-\lambda^2)^{\nu} \lambda^2 \quad (\nu = 0, 1, 2, \dots)$$

are positive continuous functions for $-1 \leq \lambda \leq 1$, we see easily that the indicated series is uniformly convergent for $-1 \leq \lambda \leq 1$.

To every C-function $\varphi \in \mathcal{B}$ there exists obviously a positive number α such that

$$|\alpha \varphi(x)| \leq 1 \quad \text{for every point } x \in E.$$

As proved just now, we have then

$$|\alpha \varphi(x)| = \sum_{\nu=0}^{\infty} \alpha_{\nu} (1-\alpha^2 \varphi(x)^2)^{\nu} \varphi(x)^2,$$

and this series is uniformly convergent in E . Therefore

$|\alpha \varphi|$ is uniformly approximated by $\mathcal{B}$, and hence $|\alpha \varphi| \in \mathcal{B}$.

Consequently we have $|\varphi| \in \mathcal{B}$ for every C-function $\varphi \in \mathcal{B}$.

Since we have obviously

$$\varphi^+ = \frac{1}{2} (|\varphi| + \varphi),$$

we obtain hence that $\mathcal{B}$ is a linear lattice manifold of $\mathcal{C}$.

To every two different points y_1 and $y_2 \in E$, there exists by assumption a C-function $\varphi \in \mathcal{B}$ such that

$$\varphi(y_1) \neq \varphi(y_2), \quad \varphi(y_1) \neq 0.$$

For such $\varphi \in B$, if we put

$$\psi(x) = \frac{1}{\varphi(y_1)(\varphi(y_1) - \varphi(y_2))} (\varphi(x)^2 - \varphi(y_2)\varphi(x)),$$

then we have obviously $\psi \in B$, $\psi(y_1) = 1$, and $\psi(y_2) = 0$.

Therefore we obtain by the previous theorem that every C-function on E is uniformly approximated by B and consequently by A .

Theorem 38.3. Let C be the C-space on a locally compact Hausdorff space E and let $\hat{C}$ be the C-space on a locally compact Hausdorff space $\hat{E}$. If a linear lattice manifold A being norm dense in C is isomorphic to a linear lattice manifold $\hat{A}$ being norm dense in $\hat{C}$ by a correspondence $A \ni \varphi \rightarrow \varphi \hat{A} \in \hat{A}$ such that

$$\|\varphi\| = \|\varphi \hat{A}\| \quad \text{for every } \varphi \in A,$$

then C is isomorphic to $\hat{C}$ by a correspondence $C \ni \varphi \rightarrow \varphi \hat{C} \in \hat{C}$ such that

$$\|\varphi\| = \|\varphi \hat{C}\| \quad \text{for every } \varphi \in C$$

and $\varphi \hat{C} = \varphi \hat{A}$ for every C-function $\varphi \in A$.

Proof. To every C-function $\varphi \in C$ there exists by assumption a sequence of C-functions $\varphi_\nu \in A$ ($\nu = 1, 2, \dots$) such that

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \varphi\| = 0.$$

For such $\varphi_\nu \in A$ ($\nu = 1, 2, \dots$) we have obviously

$$\lim_{\nu, \mu \rightarrow \infty} \|\varphi_\nu - \varphi_\mu\| = 0,$$

and hence by assumption

$$\lim_{\nu, \mu \rightarrow \infty} \|\varphi_\nu \hat{A} - \varphi_\mu \hat{A}\| = 0.$$

Hence there exists by Theorem 35.5 a C-function $\hat{\varphi} \in \hat{C}$ for which

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu \hat{A} - \hat{\varphi}\| = 0.$$

Therefore we see easily that there exists a correspondence $C \ni \varphi \rightarrow \varphi \hat{C} \in \hat{C}$ such that

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \varphi\| = 0, \quad \varphi_\nu \in A \quad (\nu = 1, 2, \dots)$$

implies $\lim_{\nu \rightarrow \infty} \|\varphi_\nu \hat{A} - \varphi \hat{C}\| = 0$. For such a correspondence we see easily that

$$(\alpha\varphi + \beta\psi)\hat{e} = \alpha\varphi\hat{e} + \beta\psi\hat{e}$$

$$\|\varphi\hat{e}\| = \|\varphi\| \quad \text{for every } \varphi \in \mathcal{C},$$

$$\varphi\hat{e} = \varphi\hat{A} \quad \text{for every } \varphi \in A,$$

and to every C-function $\hat{\varphi} \in \hat{\mathcal{C}}$ there exists a C-function $\varphi \in \mathcal{C}$ for which $\hat{\varphi} = \varphi\hat{e}$. Therefore we need only to prove that we have $\varphi\hat{e} \geq 0$ if and only if $\varphi \geq 0$. To every C-function $\varphi \in \mathcal{C}$ there exists a sequence of C-functions $\varphi_\nu \in A$ ($\nu = 1, 2, \dots$) such that

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \varphi\| = 0.$$

Then, since $|\varphi_\nu^+ - \varphi^+| \leq |\varphi_\nu - \varphi|$ by the formula §4(5), we have

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu^+ - \varphi^+\| = 0, \quad \varphi_\nu^+ \in A \quad (\nu = 1, 2, \dots).$$

Therefore, to every positive C-function $\varphi \in \mathcal{C}$, there exists a sequence of positive C-functions $\varphi_\nu \in A$ ($\nu = 1, 2, \dots$) for which

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \varphi\| = 0,$$

and hence $\lim_{\nu \rightarrow \infty} \|\varphi_\nu\hat{A} - \varphi\hat{e}\| = 0$. Then, as $\varphi_\nu\hat{A} \geq 0$ ($\nu = 1, 2, \dots$) by assumption, we obtain $\varphi\hat{e} \geq 0$. Conversely, if $\varphi\hat{e} \geq 0$, then we also can prove likewise $\varphi \geq 0$.

Theorem 38.4. Let $\mathcal{C}$ be the C-space on a locally compact Hausdorff space E and let $\hat{\mathcal{C}}$ be the C-space on a locally compact Hausdorff space $\hat{E}$. If a subring A being norm dense in $\mathcal{C}$ is isomorphic to a subring $\hat{A}$ being norm dense in $\hat{\mathcal{C}}$ as a semi-ordered ring by a correspondence $A \ni \varphi \rightarrow \varphi\hat{A} \in \hat{A}$, then $\mathcal{C}$ is isomorphic to $\hat{\mathcal{C}}$ by a correspondence $\mathcal{C} \ni \varphi \rightarrow \varphi\hat{e} \in \hat{\mathcal{C}}$ such that

$$\varphi\hat{e} = \varphi\hat{A} \quad \text{for every } \varphi \in A.$$

Proof. For an arbitrary C-function $\psi \in A$ we have obviously

$$\psi\varphi \leq \|\psi\|\varphi, \quad -\psi\varphi \leq \|\psi\|\varphi$$

for every positive C-function $\varphi \in A$. Therefore we have

$$\psi\hat{A}\varphi\hat{A} \leq \|\psi\|\varphi\hat{A}, \quad -\psi\hat{A}\varphi\hat{A} \leq \|\psi\|\varphi\hat{A}$$

for every positive C-function $\varphi \in A$. To every point $\hat{y} \in \hat{E}$

there exists by Theorem 35.1 a C-function $\hat{\varphi} \in \hat{\mathcal{C}}$ such that $\hat{\varphi}(\hat{y}) \neq 0$. Since $\hat{A}$ is norm dense in $\hat{\mathcal{C}}$ by assumption, there exists hence a C-function $\varphi \in A$ for which $\varphi \hat{A}(\hat{y}) \neq 0$.

For such $\varphi \in A$ we have obviously

$$\varphi^2 \geq 0 \quad \text{and} \quad (\varphi^2) \hat{A}(\hat{y}) \neq 0.$$

Consequently we obtain for every point $\hat{y} \in \hat{E}$

$$\varphi \hat{A}(\hat{y}) \leq \|\varphi\|, \quad -\varphi \hat{A}(\hat{y}) \leq \|\varphi\|,$$

that is, $\|\varphi \hat{A}\| \leq \|\varphi\|$. We also can prove likewise $\|\varphi\| \leq \|\varphi \hat{A}\|$.

Thus we have

$$\|\varphi \hat{A}\| = \|\varphi\| \quad \text{for every } \varphi \in A.$$

Therefore we obtain by the similar methods as used in Proof of the previous theorem that there exists a correspondence $\mathcal{C} \ni \varphi \rightarrow \varphi \hat{\mathcal{C}} \in \hat{\mathcal{C}}$ such that

$$(\alpha \varphi + \beta \psi) \hat{\mathcal{C}} = \alpha \varphi \hat{\mathcal{C}} + \beta \psi \hat{\mathcal{C}},$$

$$(\varphi \psi) \hat{\mathcal{C}} = \varphi \hat{\mathcal{C}} \psi \hat{\mathcal{C}},$$

$$\|\varphi \hat{\mathcal{C}}\| = \|\varphi\| \quad \text{for every } \varphi \in \mathcal{C},$$

$$\varphi \hat{\mathcal{C}} = \varphi \hat{A} \quad \text{for every } \varphi \in A,$$

and to every C-function $\hat{\varphi} \in \hat{\mathcal{C}}$ there exists a C-function $\varphi \in \mathcal{C}$ for which $\hat{\varphi} = \varphi \hat{\mathcal{C}}$.

For a positive C-function $\varphi \in \mathcal{C}$ we have obviously $\sqrt{\varphi} \in \mathcal{C}$ and hence there exists by assumption a sequence of C-functions $\varphi_\nu \in A$ ($\nu = 1, 2, \dots$) such that

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu - \sqrt{\varphi}\| = 0.$$

From this relation we conclude easily

$$\lim_{\nu \rightarrow \infty} \|\varphi_\nu^2 - \varphi\| = 0, \quad \varphi_\nu^2 \in A \quad (\nu = 1, 2, \dots),$$

and hence $\lim_{\nu \rightarrow \infty} \|(\varphi_\nu \hat{A})^2 - \varphi \hat{\mathcal{C}}\| = 0$. Since $(\varphi_\nu \hat{A})^2 \geq 0$ for every $\nu = 1, 2, \dots$, we obtain consequently $\varphi \hat{\mathcal{C}} \geq 0$.

We also can prove likewise that $\varphi \hat{\mathcal{C}} \geq 0$ implies $\varphi \geq 0$.

Therefore $\mathcal{C}$ is isomorphic to $\hat{\mathcal{C}}$ as a semi-ordered ring by this correspondence $\mathcal{C} \ni \varphi \rightarrow \varphi \hat{\mathcal{C}} \in \hat{\mathcal{C}}$.

§39 Characterizations of C-spaces

Let R be a lattice ordered linear space. A functional $\|a\|$ ($a \in R$) on R is said to be a norm on R , if it satisfies the norm conditions:

- 1) $0 \leq \|a\| < +\infty$ for every element $a \in R$,
- 2) $\|a\| = 0$ implies $a = 0$,
- 3) $\|\alpha a\| = |\alpha| \|a\|$ for every real number α ,
- 4) $\|a + b\| \leq \|a\| + \|b\|$,
- 5) $|a| \leq |b|$ implies $\|a\| \leq \|b\|$.

A semi-ordered linear space R is said to be a normed semi-ordered linear space, if it is lattice ordered and associated with a norm. We see immediately by Theorem 35.3 that the norm of the C-space on a locally compact Hausdorff space satisfies the norm conditions. Every normed semi-ordered linear space is archimedean, because, if

$$\frac{1}{\nu} a \geq b \geq 0 \quad \text{for every } \nu = 1, 2, \dots,$$

then we have by the conditions 3) and 5) for every $\nu = 1, 2, \dots$

$$\frac{1}{\nu} \|a\| = \|\frac{1}{\nu} a\| \geq \|b\|.$$

Consequently we obtain $\|b\| = 0$, which yields $b = 0$ by the condition 2).

Theorem 39.1. If a normed semi-ordered linear space R satisfies the condition.

- (C₁) for every upper bounded system of positive elements $\alpha_\lambda \in R$ ($\lambda \in \Lambda$), denoting by B the totality of its upper bounds we have

$$\sup_{\lambda \in \Lambda} \|\alpha_\lambda\| = \inf_{x \in B} \|x\|,$$

then R is isomorphic to a linear lattice manifold A being dense in the C-space on a locally compact Hausdorff space E by a correspondence $R \ni a \rightarrow \varphi_a \in A$ such that

$$\|a\| = \|\varphi_a\| \quad \text{for every element } a \in R,$$

and such a space E is uniquely determined up to a homeomorphism.

Proof. By virtue of Theorem 30.3, R has a cut extension $\tilde{R}$ by an extending correspondence $R \ni a \rightarrow a\tilde{R} \in \tilde{R}$, and furthermore by virtue of Theorem 34.4, $\tilde{R}$ has a universal completion $\hat{R}$ by a correspondence $\tilde{R} \ni \tilde{a} \rightarrow \tilde{a}\hat{R} \in \hat{R}$. Then $\hat{R}$ is obviously an extension of R by the extending correspondence $R \ni a \rightarrow a\hat{R} \in \hat{R}$: $a\hat{R} = (a\tilde{R})\hat{R}$ for every element $a \in R$, and we see by Theorems 30.1 and 34.1 that

$$a = \bigcup_{\lambda \in \Lambda} a_\lambda \text{ implies } a\hat{R} = \bigcup_{\lambda \in \Lambda} a_\lambda \hat{R}.$$

Furthermore we have by definition of cut extension and universal completion that to every positive $\hat{a} \in \hat{R}$ there exists a system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) such that $\hat{a} = \bigcup_{\lambda \in \Lambda} a_\lambda \hat{R}$.

We shall prove first that putting

$$D = \{x : \|x\| \leq 1, 0 \leq x \in R\},$$

there exists $\bigcup_{x \in D} x\hat{R}$ in $\hat{R}$, that is, the set of elements

$$\{x\hat{R} : x \in D\}$$

is upper bounded in $\hat{R}$. By virtue of Theorem 32.4, $\hat{R}$ has a complete positive element $\hat{c}$. If we put

$$\hat{p}_\nu = \bigcup_{x \in D} [(x\hat{R} - \nu\hat{c})^+] \hat{c} \quad \text{for } \nu = 0, 1, 2, \dots,$$

then we have by Theorem 8.3 $0 \leq \hat{p}_\nu \downarrow_{\nu=1}^\infty$, and hence $[\hat{p}_\nu] \downarrow_{\nu=1}^\infty$.

Furthermore we have

$$[\hat{p}_\nu] \downarrow_{\nu=1}^\infty 0.$$

Because, there exists by Theorem 8.6 a projector $[\hat{p}]$ for which

$$[\hat{p}_\nu] \downarrow_{\nu=1}^\infty [\hat{p}].$$

If $[\hat{p}]\hat{c} \geq a\hat{R} \geq 0$ for a positive element $a \in R$, then, since we have by Theorem 7.16

$$[(x\hat{R} - \nu\hat{c})^+](x\hat{R} - \nu\hat{c}) \geq 0,$$

we obtain by Theorem 7.5

$$x\hat{R} \geq [(x\hat{R} - \nu\hat{c})^+](\nu\hat{c}) \geq [(x\hat{R} - \nu\hat{c})^+](\nu a)\hat{R},$$

and consequently

$$\bigcup_{x \in D} \{x\hat{R} \wedge (\nu a)\hat{R}\} \geq \bigcup_{x \in D} [(x\hat{R} - \nu\hat{c})^+](\nu a)\hat{R}.$$

On the other hand we have by Theorem 8.5

$$\begin{aligned} \bigcup_{x \in D} [(x^{\hat{R}} - \nu \hat{C})^+] (\nu a)^{\hat{R}} &= [\hat{p}_\nu] (\nu a)^{\hat{R}} \\ &\geq [\hat{p}] (\nu a)^{\hat{R}} = (\nu a)^{\hat{R}} \\ &\leq \bigcup_{x \in D} \{x \wedge (\nu a)\}^{\hat{R}} = \bigcup_{x \in D} \{x^{\hat{R}} \wedge (\nu a)^{\hat{R}}\}, \end{aligned}$$

since $[\hat{p}] \leq [\hat{p}] \hat{C} \leq [a^{\hat{R}}]$ by Theorem 8.3. Thus we obtain

$$(\nu a)^{\hat{R}} = \bigcup_{x \in D} \{x \wedge (\nu a)\}^{\hat{R}},$$

and consequently

$$\nu a = \bigcup_{x \in D} \{x \wedge (\nu a)\} \quad \text{for every } \nu = 1, 2, \dots$$

Since $\|x \wedge (\nu a)\| \leq \|x\| \leq 1$ for $x \in D$, we conclude hence by the condition (C₁) that

$$\|\nu a\| \leq 1 \quad \text{for every } \nu = 1, 2, \dots$$

This relation yields obviously $a = 0$. Therefore we have

$[\hat{p}] \hat{C} = 0$, because there exists a system of positive elements

$a_\lambda \in \mathcal{R}$ ($\lambda \in \Lambda$) such that

$$[\hat{p}] \hat{C} = \bigcup_{\lambda \in \Lambda} a_\lambda^{\hat{R}},$$

as remarked just above. Since $\hat{C}$ is a complete element of $\hat{\mathcal{R}}$ we obtain consequently $[\hat{p}] = 0$, that is,

$$[\hat{p}_\nu] \downarrow_{\nu=1}^\infty 0.$$

As $\hat{\mathcal{R}}$ is complete, putting

$$\hat{C}_0 = \sum_{\nu=1}^\infty \nu ([\hat{p}_{\nu-1}] - [\hat{p}_\nu]) \hat{C},$$

we obtain a positive element $\hat{C}_0 \in \hat{\mathcal{R}}$. Since $[\hat{p}_\nu] \downarrow_{\nu=1}^\infty 0$ and

$$[\hat{p}_0] \geq [x^{\hat{R}}] [\hat{C}] = [x^{\hat{R}}] \quad \text{for every element } x \in D,$$

we have for every element $x \in D$

$$x^{\hat{R}} = \sum_{\nu=1}^\infty ([\hat{p}_{\nu-1}] - [\hat{p}_\nu]) x^{\hat{R}}.$$

On the other hand we have by Theorems 7.5 and 8.2

$$([\hat{p}_{\nu-1}] - [\hat{p}_\nu]) (x^{\hat{R}} - \nu \hat{C})$$

$$= [\hat{p}_{\nu-1}] (1 - [\hat{p}_\nu]) (1 - [(x^{\hat{R}} - \nu \hat{C})^+]) (x^{\hat{R}} - \nu \hat{C}) \leq 0,$$

since $[\hat{p}_\nu] \geq [(x^{\hat{R}} - \nu \hat{C})^+]$ by Theorem 8.3. Consequently we

obtain for every element $x \in D$ and for every $\nu = 1, 2, \dots$

$$([\hat{p}_{\nu-1}] - [\hat{p}_\nu]) x^{\hat{R}} \leq \nu ([\hat{p}_{\nu-1}] - [\hat{p}_\nu]) \hat{C}.$$

Therefore we have

$$x^{\hat{R}} \leq \hat{C}_0 \quad \text{for every element } x \in D.$$

Now we can put thus

$$\hat{e} = \bigcup_{x \in D} x^{\hat{R}}.$$

Recalling the second spectral theory, if we put

$$\varphi_a(\mathfrak{f}) = \left(\frac{a^{\hat{R}}}{\varepsilon}, \mathfrak{f} \right) \quad \text{for every element } a \in R,$$

then $\varphi_a(\mathfrak{f})$ is a continuous function on $\mathcal{U}_{[\hat{e}]}$. Furthermore

$\varphi_a(\mathfrak{f})$ is bounded in $\mathcal{U}_{[\hat{e}]}$ for every element $a \in R$. Because for any positive number $\alpha \geq \|a\|$ we have obviously

$$\| \frac{1}{\alpha} a \| \leq 1 \quad \text{and hence} \quad \frac{1}{\alpha} |a| \in D.$$

Accordingly we obtain $|a^{\hat{R}}| \leq \alpha \hat{e}$, and consequently by Theorem 18.11

$$|\varphi_a(\mathfrak{f})| = \left| \left(\frac{a^{\hat{R}}}{\varepsilon}, \mathfrak{f} \right) \right| \leq \alpha \quad \text{for every point } \mathfrak{f} \in \mathcal{U}_{[\hat{e}]}. \quad \text{Therefore we conclude for every element } a \in R$$

$$\sup_{\mathfrak{f} \in \mathcal{U}_{[\hat{e}]}} |\varphi_a(\mathfrak{f})| \leq \|a\|.$$

Conversely, if $\alpha > \sup_{\mathfrak{f} \in \mathcal{U}_{[\hat{e}]}} |\varphi_a(\mathfrak{f})|$, then we have by Theorem 19.6

$$\frac{1}{\alpha} |a^{\hat{R}}| \leq \hat{e},$$

and hence we obtain by Theorem 3.3

$$\frac{1}{\alpha} |a^{\hat{R}}| = \bigcup_{x \in D} \{ x^{\hat{R}} \cap \frac{1}{\alpha} |a^{\hat{R}}| \} = \bigcup_{x \in D} \{ x \cap \frac{1}{\alpha} |a| \}^{\hat{R}}$$

This relation yields obviously

$$\frac{1}{\alpha} |a| = \bigcup_{x \in D} \{ x \cap \frac{1}{\alpha} |a| \},$$

and consequently we obtain by the condition (C₁)

$$\frac{1}{\alpha} \|a\| = \left\| \frac{1}{\alpha} |a| \right\| \leq 1,$$

since $\|x \cap \frac{1}{\alpha} |a|\| \leq 1$ for every element $x \in D$. Therefore we have

$$\|a\| = \sup_{\mathfrak{f} \in \mathcal{U}_{[\hat{e}]}} |\varphi_a(\mathfrak{f})| \quad (a \in R).$$

We see easily by Theorems 18.5, 18.6, and 18.9 that

$$\varphi_{\alpha a + \beta b}(\mathfrak{f}) = \alpha \varphi_a(\mathfrak{f}) + \beta \varphi_b(\mathfrak{f}),$$

$$\varphi_{a \wedge b}(\mathfrak{f}) = \max \{ \varphi_a(\mathfrak{f}), \varphi_b(\mathfrak{f}) \},$$

$$\varphi_{a \vee b}(\mathfrak{f}) = \min \{ \varphi_a(\mathfrak{f}), \varphi_b(\mathfrak{f}) \}.$$

For a point $\mathfrak{f}_0 \in \mathcal{U}_{[\hat{e}]}$, if there exists an element $a \in R$ for which $\varphi_a(\mathfrak{f}_0) \neq 0$, then to every positive number $\varepsilon < 1$ there exists a positive element $c \in R$ such that

$$\mathcal{G}_c(\mathcal{F}_0) \geq 1 - \varepsilon \text{ and } \|C\| \leq 1.$$

In fact, it is evident by assumption that there exists a positive element $a \in R$ for which $\mathcal{G}_a(\mathcal{F}_0) = 1$. For such $a \in R$, since $\|x \wedge a\| \leq 1$ for every element $x \in D$, there exists by the condition (G₁) a positive element $C_0 \in R$ such that

$$\|C_0\| < \frac{1}{1 - \varepsilon},$$

$$x \wedge a \leq C_0 \quad \text{for every element } x \in D.$$

For such $C_0 \in R$ we have by Theorem 3.3

$$\widehat{C_0} \cdot a^{\widehat{R}} = \bigcup_{x \in D} (x \wedge a)^{\widehat{R}} \leq \widehat{C_0} \cdot \widehat{R},$$

and hence by Theorem 18.9

$$\mathcal{G}_c(\mathcal{F}_0) \geq \min\left\{1, \left(\frac{a^{\widehat{R}}}{\widehat{C_0} \cdot \widehat{R}}, \mathcal{F}_0\right)\right\} = 1.$$

Therefore, putting $C = (1 - \varepsilon) C_0$, we have

$$\mathcal{G}_C(\mathcal{F}_0) \geq 1 - \varepsilon \quad \text{and} \quad \|C\| < 1.$$

$\mathcal{U}_{[\widehat{E}]}$ may be considered as a compact Hausdorff space. For two points $\mathcal{F}_1$ and $\mathcal{F}_2 \in \mathcal{U}_{[\widehat{E}]}$ we define $\mathcal{F}_1 \sim \mathcal{F}_2$ to mean that

$$\mathcal{G}_a(\mathcal{F}_1) = \mathcal{G}_a(\mathcal{F}_2) \quad \text{for every element } a \in R.$$

Then we see easily that this equivalence relation satisfies the equivalence conditions, and we obtain by this equivalence a partition space *) E from $\mathcal{U}_{[\widehat{E}]}$, i.e., for any $X \in E$ we have

$$\mathcal{G}_a(\mathcal{F}_1) = \mathcal{G}_a(\mathcal{F}_2)$$

for every two points $\mathcal{F}_1, \mathcal{F}_2 \in X$ and for every element $a \in R$ and to every two different $X_1, X_2 \in E$ there exists an element $a \in R$ such that

$$\mathcal{G}_a(\mathcal{F}_1) \neq \mathcal{G}_a(\mathcal{F}_2) \quad \text{for } \mathcal{F}_1 \in X_1, \mathcal{F}_2 \in X_2.$$

We see further by definition of partition space that putting

$$\mathcal{G}_a(X) = \mathcal{G}_a(\mathcal{F}) \quad \text{for } \mathcal{F} \in X \in E,$$

we obtain a continuous function $\mathcal{G}_a(X)$ on E for every element $a \in R$.

Since $\mathcal{U}_{[\widehat{E}]}$ is compact, the partition space E is compact too. Furthermore E is a Hausdorff space. Because

*) See the page 213.

to every different points X_1 and $X_2 \in E$ there exists an element $a \in R$ for which

$$\varphi_a(X_1) \neq \varphi_a(X_2).$$

In the case: $\varphi_a(X_1) > \varphi_a(X_2)$, considering a real number α such that

$$\varphi_a(X_1) > \alpha > \varphi_a(X_2),$$

we obtain two disjoint open sets

$$\{X: \varphi_a(X) > \alpha\} \ni X_1 \quad \text{and} \quad \{X: \varphi_a(X) < \alpha\} \ni X_2,$$

and it is similar for the other case: $\varphi_a(X_1) < \varphi_a(X_2)$.

If the partition space E contains a point X_0 such that

$$\varphi_a(X_0) = 0 \quad \text{for every element } a \in R,$$

then, taking off such a point X_0 from E we obtain a locally compact Hausdorff space E_0 such that to every point $X \in E_0$ there exists an element $a \in R$ for which $\varphi_a(X) \neq 0$. If

E does not contain such point, then we suppose that E_0 coincides with E . Then we see easily that $\varphi_a(X)$ is a C-function on E_0 , and that to every two different points X_1 and $X_2 \in E_0$ there exists an element $a \in R$ for which

$$\varphi_a(X_1) \neq \varphi_a(X_2) \quad \text{and} \quad \varphi_a(X_1) \neq 0$$

It is evident by definition of E_0 that the totality of C-functions φ_a on E_0 for every element $a \in R$ constitutes a linear lattice manifold A of the C-space on E_0 , and that R is isomorphic to A by the correspondence $R \ni a \rightarrow \varphi_a \in A$ such that

$$\|a\| = \|\varphi_a\| \quad \text{for every element } a \in R.$$

To every two different points X_1 and $X_2 \in E_0$ there exists an element $a \in R$ for which

$$\varphi_a(X_1) \neq \varphi_a(X_2).$$

For such $a \in R$, if we put

$$\alpha = \varphi_a(X_1), \quad \beta = \varphi_a(X_2),$$

then to a positive number

$$\varepsilon < \frac{|\alpha - \beta|}{|\alpha| + |\beta|}$$

there exist positive elements a_1 and $a_2 \in R$ such that

$$\varphi_{a_1}(X_1) \geq 1 - \varepsilon, \quad \|a_1\| \leq 1,$$

$$\varphi_{a_2}(X_2) \geq 1 - \varepsilon, \quad \|a_2\| \leq 1,$$

as proved already. For such $a_1, a_2 \in R$, putting $\ell = a_1 \vee a_2$, we obtain an element $\ell \in R$ for which we have $\|\ell\| \leq 1$ by the condition (C_1) and hence further

$$1 \geq \varphi_\ell(X_1) \geq \varphi_{a_1}(X_1) \geq 1 - \varepsilon,$$

$$1 \geq \varphi_\ell(X_2) \geq \varphi_{a_2}(X_2) \geq 1 - \varepsilon.$$

Consequently we have

$$\begin{aligned} & |\alpha \varphi_\ell(X_2) - \beta \varphi_\ell(X_1)| \\ & \geq |\alpha - \beta| - |\alpha|(1 - \varphi_\ell(X_2)) - |\beta|(1 - \varphi_\ell(X_1)) \\ & \geq |\alpha - \beta| - \varepsilon(|\alpha| + |\beta|) > 0. \end{aligned}$$

Thus, putting $c = \frac{1}{\alpha \varphi_\ell(X_2) - \beta \varphi_\ell(X_1)} (\varphi_\ell(X_2)a - \beta \ell)$, we obtain an element $c \in R$ for which we have

$$\varphi_c(X_1) = 1, \quad \varphi_c(X_2) = 0.$$

Therefore A is norm dense in the C -space on E , by Theorem 38.1. By virtue of Theorems 38.3 and 37.2, it is evident that such a space E_0 is uniquely determined up to a homeomorphism.

Theorem 39.2. If a normed semi-ordered linear space R satisfies the condition:

(C_2) every upper bounded system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$) has an upper bound $a \in R$ such that

$$\|a\| = \sup_{\lambda \in \Lambda} \|a_\lambda\|,$$

then R is isomorphic to a linear lattice manifold A being norm dense in the C -space on a locally compact Hausdorff space E by a correspondence $R \ni a \rightarrow \varphi_a \in A$ such that

$$\|a\| = \|\varphi_a\| \quad \text{for every element } a \in R,$$

and to every compact set P in E there exists a positive element $a \in R$ for which

$$\|a\| = 1 \quad \text{and} \quad \varphi_a(x) = 1 \quad \text{for every point } x \in P.$$

Such a space E is uniquely determined up to a homeomorphism.

Proof. The condition (C_2) is obviously stronger than the condition (C_1) . Therefore we have by the previous theorem that R is isomorphic to a linear lattice manifold A being norm dense in the C -space C on a locally compact Hausdorff space E by a correspondence $R \ni a \rightarrow \varphi_a \in A$ such that

$$\|a\| = \|\varphi_a\| \quad \text{for every element } a \in R,$$

and such a space E is uniquely determined up to a homeomorphism. Let P be a compact set in E . Since A is norm dense in C , we see easily by Theorem 35.1 that there exists a positive element $c \in R$ such that

$$\varphi_c(x) \geq 1 \quad \text{for every point } x \in P,$$

and furthermore that to every positive number ε there exists a positive element $a_\varepsilon \in R$ such that

$$\varphi_{a_\varepsilon}(x) \geq 1 - \varepsilon \quad \text{for every point } x \in P,$$

and $0 \leq \varphi_{a_\varepsilon}(x) \leq 1$ for every point $x \in E$. For such c and $a_\varepsilon \in R$, the system of positive elements

$$c \wedge a_\varepsilon \in R \quad (\varepsilon > 0)$$

has obviously an upper bound c , and hence there exists by the condition (C_2) a positive element $a \in R$ such that

$$\|a\| \leq 1 \quad \text{and} \quad a \geq c \wedge a_\varepsilon \quad \text{for every positive number } \varepsilon,$$

since $\|c \wedge a_\varepsilon\| \leq \|a_\varepsilon\| \leq 1$ for every positive number ε

For such $a \in R$ we have obviously for every point $x \in P$

$$\varphi_a(x) \geq \min \{ \varphi_c(x), \varphi_{a_\varepsilon}(x) \} \geq 1 - \varepsilon.$$

Making ε tend to 0, we obtain hence

$$\varphi_a(x) \geq 1 \quad \text{for every point } x \in P.$$

On the other hand, as $\|a\| \leq 1$, we have $\varphi_a(x) \leq 1$ for every point $x \in E$. Consequently we obtain

$$\|a\| = 1 \quad \text{and} \quad \varphi_a(x) = 1 \quad \text{for every point } x \in P.$$

Theorem 39.3. If a normed semi-ordered linear space R satisfies the condition:

(C₃) there exists $e = \bigcup_{\|x\| \leq 1} x$ and $\|e\| = 1$,
then R is isomorphic to a linear lattice manifold A being
norm dense in the C-space on a compact Hausdorff space E by
a correspondence $R \ni a \rightarrow \varphi_a \in A$ such that

$$\|a\| = \|\varphi_a\| \quad \text{for every element } a \in R,$$

and $\varphi_e(x) = 1$ for every point $x \in E$. Such a space E
is uniquely determined up to a homeomorphism.

Proof. The condition (C₃) is stronger than the condition (C₂).
 Indeed, if R satisfies the condition (C₃), then for any upper bounded system of positive elements $a_\lambda \in R$ ($\lambda \in \Lambda$), putting

$$\alpha = \sup_{\lambda \in \Lambda} \|a_\lambda\|,$$

we have $\|\alpha e\| = \alpha$ and $a_\lambda \leq \alpha e$ for every $\lambda \in \Lambda$, that is, R satisfies the condition (C₂). Therefore we have by the previous theorem that R is isomorphic to a linear lattice manifold A being norm dense in the C-space on a locally compact Hausdorff space E by a correspondence $R \ni a \rightarrow \varphi_a \in A$ such that

$$\|a\| = \|\varphi_a\| \quad \text{for every element } a \in R,$$

and such a space E is uniquely determined up to a homeomorphism, Furthermore to every point $y \in E$ there exists a positive element $a \in R$ such that

$$\varphi_a(y) = 1 \quad \text{and} \quad \|a\| = 1.$$

For such $a \in R$ we have naturally $e \geq a$, and hence

$$\varphi_e(y) \geq \varphi_a(y) = 1.$$

Consequently we have $\varphi_e(y) \geq 1$ for every point $y \in E$. On the other hand, as $\|e\| = 1$, we have

$$\varphi_e(x) \leq 1 \quad \text{for every point } x \in E,$$

and hence $\varphi_e(x) = 1$ for every point $x \in E$. Therefore E must be compact, since φ_e is a C-function on E .

§40 Characterizations as semi-ordered rings

Let $\mathcal{C}$ be the C-space on a locally compact Hausdorff space

E For any C-function $\psi \in \mathcal{C}$ we have obviously

$$\psi \varphi \leq \|\psi\| \varphi \quad \text{and} \quad -\psi \varphi \leq \|\psi\| \varphi$$

for every positive C-function $\varphi \in \mathcal{C}$; and for a C-function $\psi \in \mathcal{C}$

if $\psi \varphi \geq 0$ for every positive C-function $\varphi \in \mathcal{C}$, then we have

$\psi \geq 0$. Conversely we have:

Theorem 40.1. If a semi-ordered ring R is archimedean and satisfies the conditions:

(B) to any positive element $a \in R$ there exists a positive number α such that we have for every positive element $x \in R$

$$x a \leq \alpha x \quad \text{and} \quad a x \leq \alpha x,$$

(P) $a \geq 0$, if we have for every positive element $x \in R$

$$a x \geq 0 \quad \text{and} \quad x a \geq 0,$$

then R is isomorphic to a subring A being norm dense in the C-space on a locally compact Hausdorff space E . Such a space E is uniquely determined up to a homeomorphism.

Proof. By virtue of Theorem 31.4, R has a cut extension composed only of bounded factors. Let $\hat{R}$ be a cut extension

of R by an extending correspondence $R \ni a \rightarrow a^{\hat{R}} \in \hat{R}$.

Since $\hat{R}$ consists only of bounded factors, we see by Theorem 29.3.

that $\hat{R}$ is a semi-normal ring. Let $\hat{N}$ be the totality of normal factors in $\hat{R}$.

Then, by virtue of Theorem 27.1, $\hat{N}$ is a normal ring and we have by Theorem 26.4

$$[(a^{\hat{R}})^2] a^{\hat{R}} \in \hat{N} \quad \text{for every element } a \in R.$$

Corresponding to every element $a \in R$ we consider the absolute spectrum

$$\varphi_a(\mathcal{F}) = \left(\frac{[(a^{\hat{R}})^2] a^{\hat{R}}}{1}, \mathcal{F} \right)$$

on the proper space $\hat{E}$ of $\hat{N}$. Then $\varphi_a(\mathcal{F})$ is a C-function

on $\hat{E}$. Because, we see easily by Theorem 29.1 that $\varphi_a(\mathcal{F})$

is a bounded continuous function on $\hat{E}$ and that for any positive number α the point set

$$\{ \mathcal{F} : |\mathcal{G}_a(\mathcal{F})| \geq \alpha \}$$

is compact, since $\mathcal{G}_a(\mathcal{F}) = 0$ for every point $\mathcal{F} \in \bigcup_{[(a\hat{R})^2]}.$

Now we shall prove that $\mathcal{R}$ is isomorphic to the totality of C-functions $\mathcal{G}_a$ associated with every element $a \in \mathcal{R}$ by the correspondence $\mathcal{R} \ni a \rightarrow \mathcal{G}_a$. For every element $a \in \mathcal{R}$ and for every real number α we have by Theorem 28.4

$$\mathcal{G}_{\alpha a}(\mathcal{F}) = \left(\frac{[(\alpha a\hat{R})^2](\alpha a\hat{R})}{1}, \mathcal{F} \right) = \alpha \left(\frac{[(a\hat{R})^2]a\hat{R}}{1}, \mathcal{F} \right) = \alpha \mathcal{G}_a(\mathcal{F}).$$

For two arbitrary elements a and $b \in \mathcal{R}$ we have by Theorems 26.2 and 26.3

$$\begin{aligned} & [(|a\hat{R}| + |b\hat{R}|)^2] a\hat{R} \\ &= [(|a\hat{R}| + |b\hat{R}|)^2] [(|a\hat{R}|^2) a\hat{R} + (|a\hat{R}| - |a\hat{R}|^2) a\hat{R}] = [(|a\hat{R}|^2)] a\hat{R}, \end{aligned}$$

and similarly

$$\begin{aligned} & [(|a\hat{R}| + |b\hat{R}|)^2] b\hat{R} = [(|b\hat{R}|^2)] b\hat{R}, \\ & [(|a\hat{R}| + |b\hat{R}|)^2] (a\hat{R} + b\hat{R}) = [(|a\hat{R} + b\hat{R}|^2)] (a\hat{R} + b\hat{R}). \end{aligned}$$

Therefore we obtain by Theorem 28.4

$$\begin{aligned} \mathcal{G}_a(\mathcal{F}) + \mathcal{G}_b(\mathcal{F}) &= \left(\frac{[(|a\hat{R}|^2)] a\hat{R} + [(|b\hat{R}|^2)] b\hat{R}}{1}, \mathcal{F} \right) \\ &= \left(\frac{[(|a\hat{R}| + |b\hat{R}|)^2] (a\hat{R} + b\hat{R})}{1}, \mathcal{F} \right) \\ &= \left(\frac{[(|a\hat{R} + b\hat{R}|^2)] (a\hat{R} + b\hat{R})}{1}, \mathcal{F} \right) = \mathcal{G}_{a+b}(\mathcal{F}). \end{aligned}$$

Recalling Theorem 7.12 we see by Theorems 26.2 and 26.6 that both $(a\hat{R})^2$ and $(b\hat{R})^2$ are normal factors in $\hat{\mathcal{R}}$. Therefore we have by Theorems 26.8, 28.10 and by the formula §25(10)

$$[(|a\hat{R}|^2)] a\hat{R} [(|b\hat{R}|^2)] b\hat{R} = [(|a\hat{R}|^2)] [(|b\hat{R}|^2)] a\hat{R} b\hat{R} = [(|a\hat{R} b\hat{R}|^2)] a\hat{R} b\hat{R},$$

and consequently we obtain by Theorem 28.4

$$\begin{aligned} \mathcal{G}_a(\mathcal{F}) \mathcal{G}_b(\mathcal{F}) &= \left(\frac{[(|a\hat{R}|^2)] a\hat{R} [(|b\hat{R}|^2)] b\hat{R}}{1}, \mathcal{F} \right) \\ &= \left(\frac{[(|a b\hat{R}|^2)] (a b\hat{R})}{1}, \mathcal{F} \right) = \mathcal{G}_{ab}(\mathcal{F}). \end{aligned}$$

Finally it is left to prove that we have

$$\varphi_a(\mathfrak{f}) \geq 0 \quad \text{for every point } \mathfrak{f} \in \hat{E},$$

if and only if $a \geq 0$. For every positive element $a \in R$, as $[(a\hat{R})^2]a\hat{R} \geq 0$, we have by Theorem 28.6 $\varphi_a(\mathfrak{f}) \geq 0$ for every point $\mathfrak{f} \in \hat{E}$. Conversely, if $\varphi_a(\mathfrak{f}) \geq 0$ for every point $\mathfrak{f} \in \hat{E}$, then we have by Theorem 28.5

$$[(a\hat{R})^2]a\hat{R} \geq 0.$$

This inequality yields $x a = a x \geq 0$ for every positive element $x \in R$, since we have by Theorems 26.1, 26.3, and 28.10

$$\begin{aligned} (x a)\hat{R} &= (a x)\hat{R} = a\hat{R} x\hat{R} \\ &= [(a\hat{R})^2]a\hat{R} + ([a\hat{R}] - [(a\hat{R})^2])a\hat{R} x\hat{R} \\ &= ([a\hat{R})^2]a\hat{R} x\hat{R} \geq 0. \end{aligned}$$

Hence we obtain $a \geq 0$ by assumption (F), as we wish to prove.

Identifying all points, at which φ_a takes the same value for every element $a \in R$, and removing all points, at which φ_a vanishes for every element $a \in R$, we obtain a partition space E from $\hat{E}$. We can prove similarly as in Proof of Theorem 39.1 that this partition space is again a locally compact Hausdorff space and that φ_a remains again as a C-function on E for every element $a \in R$. Furthermore it is evident by the construction of E that to every pair of different points x_1 and $x_2 \in E$ there exists an element $a \in R$ for which

$$\varphi_a(x_1) \neq \varphi_a(x_2), \quad \varphi_a(x_1) \neq 0.$$

Thus we see by Theorem 38.2 that the totality of C-functions φ_a for every element $a \in R$ is norm dense in the C-space on E , and by Theorems 38.4 and 37.3 that such a space E is uniquely determined up to a homeomorphism.

In this proof, the assumption (F) is used only to prove that $\varphi_a(\mathfrak{f}) \geq 0$ for every point $\mathfrak{f} \in \hat{E}$ implies $a \geq 0$. Instead of the assumption (F), if we suppose that to every positive element $a \in R$ there exists two positive elements b and $c \in R$ for which $bc \geq a$, then $\varphi_a(\mathfrak{f}) \geq 0$ for every point $\mathfrak{f} \in \hat{E}$ implies

$a \geq 0$ by Theorem 28.5, because $a^{\hat{R}}$ is by Theorem 26.5 a normal factor of $\hat{R}$ and consequently $a^{\hat{R}}$ is contained in $\hat{N}$.

Hence we also obtain the following

Theorem 40.2. If a semi-ordered ring is archimedean and satisfies the conditions:

(B) to any positive element $a \in R$ there exists a positive number α such that we have for every positive element $x \in R$

$$xa \leq \alpha x \quad \text{and} \quad ax \leq \alpha x,$$

(N) to every positive element $a \in R$ there exist two positive elements b and $c \in R$ such that $bc \geq a$,

then R is isomorphic to a subring being norm dense in the C-space on a locally compact Hausdorff space E , and such a space E is uniquely determined up to a homeomorphism.

Theorem 40.3. If a semi-ordered ring R possessing a unit factor e is archimedean and satisfies the condition that to every positive element $a \in R$ there exists a positive number α for which $a \leq \alpha e$, then R is isomorphic to a subring being norm dense in the C-space on a compact Hausdorff space E , and such a space E is uniquely determined up to a homeomorphism.

Proof. We have obviously by assumption that $e \geq 0$, and both the conditions (B) and (N) in the previous theorem are satisfied, since $e^2 = e$. Therefore R is isomorphic to a subring A being norm dense in the C-space on a locally compact Hausdorff space E by a correspondence $R \ni a \rightarrow \varphi_a \in A$, and such a space E is uniquely determined up to a homeomorphism. Since A is norm dense in the C-space, to every point $X \in E$ there exists an element $a \in R$ for which $\varphi_a(X) \neq 0$. As we have

$$\varphi_a(X) \varphi_e(X) = \varphi_{ae}(X) = \varphi_a(X)$$

for every element $a \in R$ and for every point $X \in E$, we obtain therefore

$$\varphi_e(X) = 1 \quad \text{for every point } X \in E.$$

Since $\mathcal{G}_c$ is a C-function on E , the space E must be compact.

*) For a partition of an abstract space E :

$$E = \sum_{\lambda \in A} X_\lambda, \quad X_\lambda X_\rho = 0 \quad \text{for } \lambda \neq \rho,$$

we obtain an abstract space $\bar{E}$ identifying X_λ ($\lambda \in A$), that is, considering every X_λ as a point of $\bar{E}$. This space $\bar{E}$ is called a partition space of an abstract space E . If E is a topological space, then the topology of a partition space $\bar{E}$ of E is defined such that a point set $\bar{A}$ in $\bar{E}$ is open if and only if the union $\sum_{X \in \bar{A}} X$ is an open set in E . With this definition we see easily that if a topological space E is compact, then every partition space of E is compact too.

CHAPTER VII

DILATATORS§41 σ -universal spaces, universal spaces

A topological space E is said to be σ -universal, if the closure of every σ -open set in E is open.

Theorem 41.1. To every sequence of positive continuous functions $\varphi_\nu(x)$ ($\nu = 1, 2, \dots$) on a σ -universal space E there exists a positive continuous function $\varphi_0(x)$ on E such that we have for every point $x \in E$

$\varphi_0(x) \leq \varphi_\nu(x)$ for every $\nu = 1, 2, \dots$, and $\varphi_0(x) \geq \varphi(x)$ for every point $x \in E$, if a continuous function $\varphi(x)$ on E satisfies for every point $x \in E$

$$\varphi(x) \leq \varphi_\nu(x) \quad \text{for every } \nu = 1, 2, \dots.$$

Proof. As $\varphi_\nu(x)$ is by assumption a continuous function on E for every $\nu = 1, 2, \dots$, the point set

$$\{x : \varphi_\nu(x) < \alpha\}$$

is σ -open for every positive number α , since

$$\{x : \varphi_\nu(x) < \alpha\} = \bigcup_{\mu=1}^{\infty} \{x : \varphi_\nu(x) \leq \alpha - \frac{1}{\mu}\}.$$

Thus, putting

$$E_\alpha = \bigcup_{\nu=1}^{\infty} \{x : \varphi_\nu(x) < \alpha\},$$

we obtain a σ -open set E_α , and hence its closure E_α^- is open, since the space E is σ -universal by assumption:

Putting

$$\varphi_0(x) = \begin{cases} \inf_{E_\alpha \ni x} \alpha & \text{for } x \in \bigcup_{\alpha > 0} E_\alpha^-, \\ 0 & \text{for } x \notin \bigcup_{\alpha > 0} E_\alpha^-, \end{cases}$$

we obtain a function $\varphi_0(x)$ on E . For such $\varphi_0(x)$ we have obviously

$$\begin{aligned} \{x : \varphi_0(x) < \alpha\} &= \bigcup_{\mu=1}^{\infty} E_{\alpha - \frac{1}{\mu}}^-, \\ \{x : \varphi_0(x) \leq \alpha\} &= \bigcap_{\mu=1}^{\infty} E_{\alpha + \frac{1}{\mu}}^-. \end{aligned}$$

Since E_α^- is open and closed for every positive number α we conclude therefore that $\varphi_0(x)$ is a continuous function over E . Furthermore we have for every $\mu = 1, 2, \dots$

$$\{x: \varphi_0(x) < \alpha\} \supset E_{\alpha - \frac{1}{\mu}}^- \supset \{x: \varphi_\nu(x) < \alpha - \frac{1}{\mu}\},$$

and hence for every $\nu = 1, 2, \dots$

$$\{x: \varphi_0(x) < \alpha\} \supset \bigcup_{\mu=1}^{\infty} \{x: \varphi_\nu(x) < \alpha - \frac{1}{\mu}\} = \{x: \varphi_\nu(x) < \alpha\}.$$

Consequently we have for every point $x \in E$

$$\varphi_0(x) \leq \varphi_\nu(x) \quad \text{for every } \nu = 1, 2, \dots$$

For a continuous function $\varphi(x)$ on E , if we have for every point $x \in E$

$$\varphi(x) \leq \varphi_\nu(x) \quad \text{for every } \nu = 1, 2, \dots,$$

then we have for every $\nu, \mu = 1, 2, \dots$

$$\{x: \varphi(x) < \alpha - \frac{1}{\mu}\} \supset \{x: \varphi_\nu(x) < \alpha - \frac{1}{\mu}\},$$

and hence for every $\mu = 1, 2, \dots$

$$\{x: \varphi(x) \leq \alpha - \frac{1}{\mu}\} \supset E_{\alpha - \frac{1}{\mu}}^-.$$

Since the point set $\{x: \varphi(x) \leq \alpha - \frac{1}{\mu}\}$ is closed, we obtain hence

$$\{x: \varphi(x) \leq \alpha - \frac{1}{\mu}\} \supset E_{\alpha - \frac{1}{\mu}}^-,$$

and consequently for every $\mu = 1, 2, \dots$

$$\{x: \varphi(x) < \alpha\} \supset E_{\alpha - \frac{1}{\mu}}^-.$$

This relation yields at once

$$\{x: \varphi(x) < \alpha\} \supset \bigcup_{\mu=1}^{\infty} E_{\alpha - \frac{1}{\mu}}^- = \{x: \varphi_0(x) < \alpha\}.$$

Therefore we have $\varphi(x) \leq \varphi_0(x)$ for every point $x \in E$, as we wish to prove.

Theorem 41.2. Let A be a σ -open set in a σ -universal space E . Every continuous function on A has a continuous extension over E , that is, to any continuous function $\varphi(x)$ on A there exists a continuous function $\psi(x)$ on E such that

$$\varphi(x) = \psi(x) \quad \text{for every point } x \in A.$$

Proof. We shall prove first that the point set

$$\{x: \varphi(x) < \alpha\}$$

is σ -open for every real number α . Since $\varphi(x)$ is a

continuous function on A , there exists a sequence of closed sets B_ν ($\nu = 1, 2, \dots$) for which

$$\{x : \varphi(x) \leq \alpha - \frac{1}{\nu}\} = AB_\nu \quad (\nu = 1, 2, \dots),$$

As A is σ -open by assumption, if we put

$$A = \sum_{\mu=1}^{\infty} A_\mu$$

for closed sets A_μ ($\mu = 1, 2, \dots$), then we have

$$\{x : \varphi(x) < \alpha\} = \sum_{\nu, \mu} A_\mu B_\nu,$$

and hence $\{x : \varphi(x) < \alpha\}$ is σ -open.

Putting

$$\psi(x) = \begin{cases} \inf_{\sigma \ni x} \{ \sup_{y \in \sigma \cap A} \varphi(y) \} & \text{for } x \in A^-, \\ 0 & \text{for } x \notin A^-, \end{cases}$$

we obtain a function $\psi(x)$ on E , for which we have obviously

$$\psi(x) = \varphi(x) \quad \text{for every point } x \in A,$$

since $\varphi(x)$ is a continuous function on A . As E is

σ -universal by assumption, A^- is open and closed. Thus we need only to prove that $\psi(x)$ is continuous in A^- .

If $\alpha > \psi(x_0)$ for a point $x_0 \in A^-$, then there exists an open set $\mathcal{U}^* \ni x_0$, such that

$$\sup_{y \in \mathcal{U}^* \cap A} \varphi(y) < \alpha,$$

and then naturally $\psi(x) < \alpha$ for every point $x \in \mathcal{U}^* \cap A^-$. Therefore $\psi(x)$ is upper semi-continuous in A^- .

If $\alpha < \psi(x_0)$ for a point $x_0 \in A^-$, then putting

$$B = \{x : \varphi(x) < \alpha\}$$

we have $x_0 \in B^-$. Because B is σ -open as proved just above, and hence we have that the closure B^- is open and closed, and

$$\sup_{y \in B^- \cap A} \varphi(y) \leq \alpha.$$

Therefore we have $x_0 \in A^- B^-$. On the other hand $A^- B^-$ is open and

$$\varphi(y) \geq \alpha \quad \text{for every point } y \in (A^- B^-) \cap A.$$

Consequently we obtain $\psi(x) \geq \alpha$ for every point $x \in A^- B^-$.

Thus $\psi(x)$ is lower semi-continuous in A^- , and hence $\psi(x)$ is continuous in A^- .

Recalling Theorem 16.7 we have obviously:

Theorem 41.3. The proper space of every continuous semi-ordered linear space is σ -universal.

A topological space E is said to be universal, if the closure of every open set in E is open again.

Corresponding to Theorem 41.1 we can prove by similar methods:

Theorem 41.4. To any system of positive continuous functions $\varphi_\lambda(x)$ ($\lambda \in A$) on a universal space E there exists a positive continuous function $\varphi_0(x)$ on E such that we have for every point $x \in E$

$$\varphi_0(x) \leq \varphi_\lambda(x) \quad \text{for every } \lambda \in A,$$

and $\varphi_0(x) \geq \varphi(x)$ for every point $x \in E$, if we have for every point $x \in E$

$$\varphi(x) \leq \varphi_\lambda(x) \quad \text{for every } \lambda \in A$$

for a continuous function $\varphi(x)$ on E .

We also can prove likewise as Theorem 41.2:

Theorem 41.5. Let A be an open set in a universal space E . Every continuous function on A has a continuous extension over E .

Theorem 41.6. The proper space of every universally continuous semi-ordered linear space is universal.

Proof. Let R be a universally continuous semi-ordered linear space and let E be its proper space. We suppose that A is an arbitrary open set in E . For every element $p \in R$, putting

$$q = \bigcup_{\sigma(x) < A \sigma(p)} [x] |p|,$$

we obtain a positive element $q \in R$. For such $q \in R$ we have obviously that

$$\sigma(x) < A \sigma(p) \text{ implies } [x] |p| \leq q.$$

This relation yields by Theorems 8.3, 8.2 and the formula §15(4)

$$\mathcal{U}_{[q]} \supset \sum_{\mathcal{U}_{[x]} \subset A \mathcal{U}_{[p]}} \mathcal{U}_{[x]} = A \mathcal{U}_{[p]}.$$

Since $\mathcal{U}_{[q]}$ is closed, we obtain hence

$$\mathcal{U}_{[q]} \supset (A \mathcal{U}_{[p]})^-.$$

On the other hand, if $A \mathcal{U}_{[p]} \mathcal{U}_{[q]} = 0$, then we have by the formula §15(5) that

$$\mathcal{U}_{[x]} \subset A \mathcal{U}_{[p]} \quad \text{implies} \quad [x][y] | p = 0,$$

and consequently by Theorem 7.6

$$[y]q = \bigcup_{\mathcal{U}_{[x]} \subset A \mathcal{U}_{[p]}} [y][x] | p = 0,$$

that is, $\mathcal{U}_{[y]} \mathcal{U}_{[q]} = 0$ by Theorems 8.1, 7.12, and by the formula §15(5). Therefore

$$A \mathcal{U}_{[p]} \mathcal{U}_{[q]} = 0 \quad \text{implies} \quad \mathcal{U}_{[y]} \mathcal{U}_{[q]} = 0,$$

and hence $(A \mathcal{U}_{[p]})^{-'} \subset \mathcal{U}_{[q]}^{-'}$. Thus we obtain

$$A^{-} \mathcal{U}_{[p]} = (A \mathcal{U}_{[p]})^{-} = \mathcal{U}_{[q]},$$

and hence $A^{-} \mathcal{U}_{[p]}$ is open for every element $p \in R$. Since

$$A^{-} = \sum_{p \in R} A^{-} \mathcal{U}_{[p]},$$

we conclude therefore that A^{-} is open.

§42 Calculus of almost finite continuous functions

Let E be a σ -universal space. If a continuous function $\varphi(x)$ on E is almost finite in E , that is, if $\varphi(x)$ is finite in an open set which is dense in E , then the point set

$$\{x : |\varphi(x)| < +\infty\}$$

is σ -open and dense in E , since

$$\{x : |\varphi(x)| < +\infty\} = \bigcup_{\nu=1}^{\infty} \{x : |\varphi(x)| \leq \nu\}.$$

Let both $\varphi(x)$ and $\psi(x)$ be almost finite continuous functions on E . For the σ -open dense sets

$$A = \{x : |\varphi(x)| < +\infty\}, \quad B = \{x : |\psi(x)| < +\infty\},$$

if we put

$$\omega(x) = \varphi(x) + \psi(x) \quad \text{for } x \in AB,$$

then $\omega(x)$ is obviously a finite continuous function on AB . Since AB is also a σ -open dense set, we obtain by Theorem 41.2 a continuous extension $\tilde{\omega}(x)$ of $\omega(x)$ over E . Since AB is dense in E , such a continuous extension $\tilde{\omega}(x)$ is almost finite and uniquely determined by $\varphi(x)$ and $\psi(x)$. Therefore to every two almost finite continuous functions $\varphi(x)$ and $\psi(x)$ on E there exists uniquely an almost finite continuous function $\omega(x)$ on E such that we have

$$\omega(x) = \varphi(x) + \psi(x)$$

for every point $x \in E$ if the right side has any sense. We define the sum $\varphi + \psi$ of two almost finite continuous functions φ and ψ on E by such an almost finite continuous function ω . We also can define likewise the product $\varphi\psi$ of two almost finite continuous functions φ and ψ on E . Then we see easily that the totality of almost finite continuous functions on E constitutes a ring.

For two almost finite continuous functions φ and ψ on E we define $\varphi \geq \psi$ to mean that

$$\varphi(x) \geq \psi(x) \quad \text{for every point } x \in E.$$

With this definition we conclude by Theorem 41.1 that the totality of almost finite continuous functions on E is a continuous semi-ordered ring, and for every two almost finite continuous functions φ and ψ we have for every point $x \in E$

$$\varphi \vee \psi(x) = \text{Max} \{ \varphi(x), \psi(x) \},$$

$$\varphi \wedge \psi(x) = \text{Min} \{ \varphi(x), \psi(x) \}.$$

This continuous semi-ordered ring contains the constant 1 on E as a unit factor, which is obviously a complete element. Hence this ring is normal by Theorem 29.9. Furthermore this ring is complete as a semi-ordered linear space. Because for an orthogonal sequence of positive almost finite continuous functions φ_ν ($\nu = 1, 2, \dots$) on E , putting

$$A_\nu = \{x : \varphi_\nu(x) > 0\} \quad (\nu = 1, 2, \dots),$$

we obtain a sequence of mutually disjoint $\mathcal{G}$ -open sets A_ν ($\nu = 1, 2, \dots$). Then, since

$$\sum_{\nu=1}^{\infty} A_\nu$$

is also a $\mathcal{G}$ -open set, we see easily by Theorem 41.2 that there exists a positive almost finite continuous function φ on E such that

$$\varphi(x) = \varphi_\nu(x) \quad \text{for every point } x \in A_\nu$$

for every $\nu = 1, 2, \dots$. For such φ we have obviously

$$\varphi \geq \varphi_\nu \quad \text{for every } \nu = 1, 2, \dots,$$

and hence there exists $\bigcup_{\nu=1}^{\infty} \varphi_\nu$ as we wish to prove. Therefore we have proved:

Theorem 42.1. The totality of almost finite continuous functions on a $\mathcal{G}$ -universal space constitutes a complete normal ring possessing a unit factor.

If a space E is universal, then we see by Theorem 41.4 that the totality of almost finite continuous functions on E is universally continuous as a semi-ordered linear space. Furthermore we can prove likewise as in proving Theorem 42.1 that it is universally complete. Therefore we have:

Theorem 42.2. The totality of almost finite continuous functions on a universal space constitutes a universally complete normal ring possessing a unit factor.

Let R be a continuous semi-ordered linear space. Since the proper space E of R is $\mathcal{G}$ -universal by Theorem 41.3, we can consider the sum $\varphi + \psi$ and the product $\varphi\psi$ for every almost finite continuous functions φ and ψ on E .

Theorem 42.3. If both almost finite continuous functions φ and ψ are integrable by an element $a \in R$ in $\mathcal{U}_{(p)}$, then the sum $\varphi + \psi$ also is integrable by a in $\mathcal{U}_{(p)}$ and

$$\int_{(p)} (\varphi + \psi)(f) df a = \int_{(p)} \varphi(f) df a + \int_{(p)} \psi(f) df a.$$

Proof. Putting $\ell = \int_{[p]} \varphi(x) d\beta a$, $c = \int_{[p]} \psi(x) d\beta a$, we have by Theorem 22.1

$$\left(\frac{\ell}{a}, \beta\right) = \varphi(\beta), \quad \left(\frac{c}{a}, \beta\right) = \psi(\beta) \quad \text{for } \beta \in \mathcal{U}_{[p]}[a],$$

and hence by Theorem 18.6

$$\left(\frac{\ell+c}{a}, \beta\right) = \varphi(\beta) + \psi(\beta) = (\varphi + \psi)(\beta)$$

in an open set being dense in $\mathcal{U}_{[p]}[a]$. Since both $\left(\frac{\ell+c}{a}, \beta\right)$ and $\varphi + \psi$ are continuous in $\mathcal{U}_{[p]}[a]$, we obtain hence

$$\left(\frac{\ell+c}{a}, \beta\right) = (\varphi + \psi)(\beta) \quad \text{for every point } \beta \in \mathcal{U}_{[p]}[a],$$

and consequently by Theorems 22.2 and 21.1

$$\ell + c = [p][a](\ell + c) = \int_{[p]} (\varphi + \psi)(\beta) d\beta a.$$

Theorem 42.4. If $\ell = \int_{[p]} \psi(\beta) d\beta a$ for an almost finite continuous function ψ on $\mathcal{U}_{[p]}$, then in order that an almost finite continuous function φ on $\mathcal{U}_{[p]}$ be integrable by ℓ in $\mathcal{U}_{[p]}$, it is necessary and sufficient that the product $\varphi\psi$ be integrable by a in $\mathcal{U}_{[p]}$, and in this case we have

$$\int_{[p]} \varphi \psi(\beta) d\beta a = \int_{[p]} \varphi(\beta) d\beta \ell.$$

Proof. If φ is integrable by ℓ in $\mathcal{U}_{[p]}$, then, putting

$$c = \int_{[p]} \varphi(\beta) d\beta \ell,$$

we have by Theorem 22.1

$$\left(\frac{c}{\ell}, \beta\right) = \varphi(\beta) \quad \text{for every point } \beta \in \mathcal{U}_{[p]}[\ell].$$

Since $\ell = \int_{[p]} \psi(\beta) d\beta a$ by assumption, we also have

$$\left(\frac{\ell}{a}, \beta\right) = \psi(\beta) \quad \text{for every point } \beta \in \mathcal{U}_{[p]}[a].$$

As $[\ell] \subseteq [a]$ by Theorem 21.1, we obtain then

$$\left(\frac{c}{\ell}, \beta\right) \left(\frac{\ell}{a}, \beta\right) = \varphi \psi(\beta)$$

in an open set being dense in $\mathcal{U}_{[p]}[\ell]$, and hence by Theorem 18.12

$$\left(\frac{c}{a}, \beta\right) = \varphi \psi(\beta) \quad \text{for every point } \beta \in \mathcal{U}_{[p]}[\ell].$$

Since $[c] \subseteq [\ell] \subseteq [a]$ by Theorem 21.1, for every point

$$\beta \in \mathcal{U}_{[p]}[a] \quad \text{but } \beta \notin \mathcal{U}_{[p]}[\ell],$$

we have by Theorem 18.3

$$\left(\frac{c}{a}, \beta\right) = 0, \quad \psi(\beta) = \left(\frac{\ell}{a}, \beta\right) = 0,$$

and consequently we obtain

$$\left(\frac{c}{a}, f\right) = \varphi \psi(f) \quad \text{for every point } f \in \mathcal{U}_{[p][a]}.$$

Therefore we have by Theorems 22.2 and 21.8

$$C = [p][a]C = \int_{[p]} \varphi \psi(f) df a.$$

Conversely, if the product $\varphi \psi$ is integrable by a in $\mathcal{U}_{[p]}$, then, putting

$$C = \int_{[p]} \varphi \psi(f) df a,$$

we have by Theorem 22.1

$$\left(\frac{c}{a}, f\right) = \varphi \psi(f) \quad \text{for every point } f \in \mathcal{U}_{[p][a]},$$

and hence by Theorem 18.12

$$\left(\frac{c}{\ell}, f\right)\left(\frac{\ell}{a}, f\right) = \varphi(f)\psi(f)$$

in an open set being dense in $\mathcal{U}_{[p][\ell]}$, since $[\ell] \subseteq [a]$ as remarked just above. On the other hand we see easily that

$$0 < |(\frac{\ell}{a}, f)| < +\infty$$

in an open set being dense in $\mathcal{U}_{[p][\ell]}$. Therefore, since

$$\left(\frac{\ell}{a}, f\right) = \psi(f) \quad \text{for every point } f \in \mathcal{U}_{[p][\ell]},$$

as remarked already, we obtain

$$\left(\frac{c}{\ell}, f\right) = \varphi(f)$$

in an open set being dense in $\mathcal{U}_{[p][\ell]}$, and hence by Theorems 22.2 and 21.8

$$[p][\ell]C = \int_{[p]} \varphi(f) df \ell,$$

that is, $\varphi(f)$ is integrable by ℓ in $\mathcal{U}_{[p]}$.

§43 Dilatators

Let R be a continuous semi-ordered linear space. For a linear manifold D of R , an operator $Tx \in R$ ($x \in D$) is said to be a linear operator with domain D , if we have

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for every elements $x, y \in D$ and for every real numbers α, β .

A linear operator T with domain D is called a dilatator in R , if

1) D is dense in R : to every element $a \in R$ there exists a sequence of elements $a_\nu \in D$ ($\nu = 1, 2, \dots$) such that

$$\lim_{\nu \rightarrow \infty} a_\nu = a,$$

2) for every projector $[p]$, $a \in D$ implies $[p]a \in D$ and

$$T[p]a = [p]Ta,$$

3) T is closed: $\lim_{\nu \rightarrow \infty} a_\nu = a$, $a_\nu \in D$ ($\nu = 1, 2, \dots$)

and $\lim_{\nu \rightarrow \infty} Ta_\nu = b$ imply $a \in D$ and $Ta = b$.

Let T be a dilatator with domain D in the sequel.

Theorem 43.1. For an element $a \in D$, if an almost finite continuous function $\varphi(p)$ on a neighbourhood $\mathcal{U}_{[p]}$ is integrable by a and by Ta in $\mathcal{U}_{[p]}$, then we have

$$\int_{[p]} \varphi(p) d\varphi a \in D,$$

$$T \int_{[p]} \varphi(p) d\varphi a = \int_{[p]} \varphi(p) d\varphi Ta.$$

Proof. If $\varphi(p)$ is a bounded continuous function on $\mathcal{U}_{[p]}$,

then we have by the definition of integral

$$\lim_{\xi \rightarrow 0} \sum_{\nu=1}^{\infty} \varphi(p_\nu) [p_\nu] a = \int_{[p]} \varphi(p) d\varphi a,$$

$$\lim_{\xi \rightarrow 0} \sum_{\nu=1}^{\infty} \varphi(p_\nu) [p_\nu] Ta = \int_{[p]} \varphi(p) d\varphi Ta,$$

for partitions of $[p]$:

$$[p] = [p_1] + [p_2] + \dots + [p_n],$$

$$\sup_{p \in \mathcal{U}_{[p]}} \varphi(p) \leq \xi, \quad p_\nu \in \mathcal{U}_{[p_\nu]} \quad (\nu = 1, 2, \dots, n).$$

On the other hand we have by the postulate 2)

$$T \sum_{\nu=1}^{\infty} \varphi(p_\nu) [p_\nu] a = \sum_{\nu=1}^{\infty} \varphi(p_\nu) [p_\nu] Ta,$$

and hence we obtain by the postulate 3)

$$\int_{[p]} \varphi(p) d\varphi a \in D,$$

$$T \int_{[p]} \varphi(p) d\varphi a = \int_{[p]} \varphi(p) d\varphi Ta.$$

In general, if $\varphi(p)$ is an almost finite continuous function on $\mathcal{U}_{[p]}$, then there exists by Theorem 21.7 a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$$

such that $\varphi(p)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$.

For such $[p_\nu]$ ($\nu = 1, 2, \dots$), putting

$$b_\nu = \int_{[p_\nu]} \varphi(p) d\varphi a. \quad (\nu = 1, 2, \dots),$$

we obtain $t_\nu \in D$ and

$$T t_\nu = \int_{[p_\nu]} \varphi(\xi) d\xi T a \quad (\nu = 1, 2, \dots),$$

as proved just now. Since $\varphi(\xi)$ is integrable by a and by $T a$ in $\mathcal{U}_{[p]}$ by assumption, we have then by Theorem 21.4

$$\begin{aligned} \lim_{\nu \rightarrow \infty} t_\nu &= \int_{[p]} \varphi(\xi) d\xi a, \\ \lim_{\nu \rightarrow \infty} T t_\nu &= \int_{[p]} \varphi(\xi) d\xi T a, \end{aligned}$$

and hence we obtain by the postulate 3)

$$\begin{aligned} \int_{[p]} \varphi(\xi) d\xi a &\in D, \\ T \int_{[p]} \varphi(\xi) d\xi a &= \int_{[p]} \varphi(\xi) d\xi T a. \end{aligned}$$

Theorem 43.2. For domain D of a dilatator, $a \in D$,

$|a| \geq |b|$ implies $b \in D$, and hence $a \in D$ implies $a^+, a^-, |a| \in D$.

Proof. If $|b| \leq |a|$, then we have by Theorems 22.2, 18.11

$$\begin{aligned} b &= \int_{[a]} \left(\frac{b}{a}, \xi \right) d\xi a, \\ \left| \left(\frac{b}{a}, \xi \right) \right| &\leq 1 \quad \text{for every point } \xi \in \mathcal{U}_{[a]}. \end{aligned}$$

Therefore, if $a \in D$, $|b| \leq |a|$, then we have $b \in D$ by the previous theorem.

Theorem 43.3. For every elements $a, b \in D$ we have

$$[a] T b = \int_{[a]} \left(\frac{b}{a}, \xi \right) d\xi T a$$

Proof. Since $\left(\frac{b}{a}, \xi \right)$ is an almost finite continuous function on $\mathcal{U}_{[a]}$ by Theorems 19.1 and 19.2, there exists by Theorem 21.7 a sequence of projectors

$$[p_\nu] \uparrow \overset{\sim}{1}, [a]$$

such that $\left(\frac{b}{a}, \xi \right)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$. For such $[p_\nu]$ ($\nu = 1, 2, \dots$) we have by Theorem 43.1

$$T [p_\nu] b = \int_{[p_\nu]} \left(\frac{b}{a}, \xi \right) d\xi T a \quad (\nu = 1, 2, \dots),$$

since $[p_\nu] b = \int_{[p_\nu]} \left(\frac{b}{a}, \xi \right) d\xi a$ by Theorem 22.2. On the other hand we have by the postulate 2)

$$T [p_\nu] b = [p_\nu] T b \quad (\nu = 1, 2, \dots),$$

and hence by Theorem 21.9

$$[a] T b = \int_{[a]} \left(\frac{b}{a}, \xi \right) d\xi T a.$$

Theorem 43.4. To every positive element $p \in R$ there exists a positive element $a \in D$ such that

$$[a] = [p] \text{ and } |Ta| \leq p$$

Proof. Since D is dense in R by the postulate 1), to every positive element $p \in R$, there exists a sequence of elements $a_\nu \in D$ ($\nu = 1, 2, \dots$) for which

$$\lim_{\nu \rightarrow \infty} a_\nu = p.$$

For such $a_\nu \in D$ ($\nu = 1, 2, \dots$) we see easily by Theorem 43.2 that putting

$$b_1 = |a_1| \wedge p, \quad b_\nu = (1 - \bigvee_{\mu=1}^{\nu-1} [a_\mu])(|a_\nu| \wedge p) \quad (\nu = 2, 3, \dots),$$

we obtain an orthogonal sequence of positive elements $b_\nu \in D$ ($\nu = 1, 2, \dots$) such that

$$\sum_{\nu=1}^{\infty} b_\nu \leq p \quad \text{and} \quad [\sum_{\nu=1}^{\infty} b_\nu] = [p],$$

since $\lim_{\nu \rightarrow \infty} |a_\nu| = p$ and

$$[b_1 + \dots + b_\nu] = [p] \bigvee_{\mu=1}^{\nu-1} [a_\mu] \quad (\nu = 1, 2, \dots).$$

Putting $p_\nu = (|Tb_\nu| - p)^+$ ($\nu = 1, 2, \dots$), we have then by Theorem 7.16

$$[p_\nu](|Tb_\nu| - p) \geq 0, \quad (1 - [p_\nu])(|Tb_\nu| - p) \leq 0,$$

that is,

$$[p_\nu]p \leq [p_\nu]|Tb_\nu|, \quad (1 - [p_\nu])p \geq (1 - [p_\nu])|Tb_\nu|.$$

Furthermore we have obviously $p_\nu \leq |Tb_\nu|$, and hence by Theorem 8.3

$$[p_\nu] \leq [Tb_\nu] \quad (\nu = 1, 2, \dots).$$

Therefore we have by Theorem 18.11 for every $\nu = 1, 2, \dots$

$$0 \leq \left(\frac{p}{|Tb_\nu|}, f \right) \leq 1 \quad \text{for every point } f \in \mathcal{U}_{[p_\nu]}.$$

Hence we conclude by Theorem 43.1 that putting

$$c_\nu = \int_{[p_\nu]} \left(\frac{p}{|Tb_\nu|}, f \right) d_f b_\nu \quad (\nu = 1, 2, \dots),$$

we obtain a sequence of positive elements $c_\nu \in D$ ($\nu = 1, 2, \dots$) and

$$Tc_\nu = \int_{[p_\nu]} \left(\frac{p}{|Tb_\nu|}, f \right) d_f Tb_\nu \quad (\nu = 1, 2, \dots).$$

For such $c_\nu \in D$ ($\nu = 1, 2, \dots$), since by Theorems 20.1, 22.2,

and by the formula §20(6)

$$c_\nu \leq [p_\nu] b_\nu,$$

$$|Tc_\nu| = \int_{[p_\nu]} \left(\frac{p}{|Tb_\nu|}, \mathfrak{F} \right) d\mathfrak{g} \quad |Tb_\nu| = [p_\nu] p,$$

we have

$$\begin{aligned} \sum_{\nu=1}^{\infty} ((1-[p_\nu])b_\nu + c_\nu) &\leq \sum_{\nu=1}^{\infty} b_\nu \leq p, \\ \sum_{\nu=1}^{\infty} (|T(1-[p_\nu])b_\nu| + |Tc_\nu|) &\leq \sum_{\nu=1}^{\infty} [b_\nu]((1-[p_\nu])p + [p_\nu]p) = \sum_{\nu=1}^{\infty} [b_\nu]p \leq p, \end{aligned}$$

and hence putting

$$a = \sum_{\nu=1}^{\infty} ((1-[p_\nu])b_\nu + c_\nu),$$

we obtain by the postulate 3) that $a \in D$ and

$$|Ta| = \left| \sum_{\nu=1}^{\infty} (T(1-[p_\nu])b_\nu + Tc_\nu) \right| \leq p.$$

On the other hand, since by the postulate 2)

$$\begin{aligned} [p_\nu] &\leq [Tb_\nu] = [[b_\nu]Tb_\nu] \leq [b_\nu], \\ p &\geq b_\nu \quad (\nu = 1, 2, \dots), \end{aligned}$$

we see easily that

$$\left(\frac{p}{|Tb_\nu|}, \mathfrak{F} \right) > 0$$

in an open set being dense in $\mathcal{U}_{[p_\nu]}$, and hence

$$[c_\nu] = [p_\nu][b_\nu] = [p_\nu] \quad (\nu = 1, 2, \dots).$$

From this relation we conclude by Theorems 5.15, 8.5, and 8.4

$$[a] = \bigcup_{\nu=1}^{\infty} ((1-[p_\nu])[b_\nu] + [c_\nu]) = \bigcup_{\nu=1}^{\infty} [b_\nu] = [p].$$

Theorem 43.5. To every element $p \in R$ there exists a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$$

such that $[p_\nu]p \in D$ for every $\nu = 1, 2, \dots$.

Proof. By virtue of the previous theorem, to every element $p \in R$ there exists an element $a \in D$ for which $[a] = [p]$. For such $a \in D$, if we put

$$p_\nu = (\nu|a| - |p|)^+ \quad (\nu = 1, 2, \dots),$$

then we have by Theorem 7.16

$$[p_\nu](\nu|a| - |p|) \geq 0 \quad (\nu = 1, 2, \dots),$$

and hence $[p_\nu]p \leq \nu|a|$ for every $\nu = 1, 2, \dots$. Consequently we obtain by Theorem 43.2

$$[p_\nu]p \in D \quad \text{for every } \nu = 1, 2, \dots$$

and by Theorem 8.14

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [a] = [p].$$

Theorem 43.6. For every almost finite continuous function $\varphi(\xi)$ on the proper space E of R , denoting by D the totality of such elements $a \in R$ that $\varphi(\xi)$ is integrable by a in $\mathcal{U}_{[a]}$, we obtain a dilatator T with domain D as

$$Ta = \int_{[a]} \varphi(\xi) d\xi a \quad (a \in D).$$

Proof. Since $\varphi(\xi)$ is almost finite by assumption, to every element $p \in R$ there exists by Theorem 21.7 a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$$

such that $\varphi(\xi)$ is bounded in $\mathcal{U}_{[p_\nu]}$. For such $[p_\nu]$ ($\nu = 1, 2, \dots$), $\varphi(\xi)$ is obviously integrable by $[p_\nu]p$ in $\mathcal{U}_{[p_\nu]}$, and hence

$$[p_\nu]p \in D \quad \text{for every } \nu = 1, 2, \dots$$

As $\lim_{\nu \rightarrow \infty} [p_\nu]p = p$ by Theorem 8.11, D is dense in R by definition.

By virtue of Theorem 21.1 we have obviously for every projector $[p]$ that $a \in D$ implies $[p]a \in D$ and

$$T[p]a = [p]Ta.$$

For every element a and $t \in D$ we obtain a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [|a| + |t|]$$

such that $\varphi(\xi)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$, as remarked just above. Then we have by Theorem 21.1 and by the formula §20(5) for every real numbers α, β

$$\begin{aligned} [p_\nu](\alpha Ta + \beta Tt) &= \alpha \int_{[p_\nu]} \varphi(\xi) d\xi a + \beta \int_{[p_\nu]} \varphi(\xi) d\xi t \\ &= \int_{[p_\nu]} \varphi(\xi) d\xi (\alpha a + \beta t), \end{aligned}$$

and hence

$$\begin{aligned}\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(\xi) d\xi (\alpha a + \beta b) &= [\alpha a + \beta b] (\alpha T a + \beta T b) \\ &= \alpha T [\alpha a + \beta b] a + \beta T [\alpha a + \beta b] b = \alpha T a + \beta T b.\end{aligned}$$

Therefore we see by Theorem 21.9 that $\varphi(\xi)$ is integrable by

$\alpha a + \beta b$ in $\mathcal{U}_{[\alpha a + \beta b]}$, and hence naturally in $\mathcal{U}_{[\alpha a + \beta b]}$, since

$$[\alpha a + \beta b] \geq [\alpha a + \beta b].$$

Consequently we obtain that $a, b \in D$ implies for every real numbers α, β

$$\alpha a + \beta b \in D,$$

$$T(\alpha a + \beta b) = \alpha T a + \beta T b.$$

If $\lim_{\mu \rightarrow \infty} a_\mu = a$, $a_\mu \in D$ ($\mu = 1, 2, \dots$), and $\lim_{\mu \rightarrow \infty} T a_\mu = b$, then there exists a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty}, [\alpha a + \beta b]$$

such that $\varphi(\xi)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and we have by Theorem 7.7

$$\begin{aligned}\lim_{\mu \rightarrow \infty} [p_\nu] a_\mu &= [p_\nu] a, \\ \lim_{\mu \rightarrow \infty} T [p_\nu] a_\mu &= [p_\nu] b.\end{aligned}$$

If we suppose that

$$|\varphi(\xi)| \leq \alpha_\nu \quad \text{for every point } \xi \in \mathcal{U}_{[p_\nu]},$$

then we have by the formula §20(6) and by Theorem 20.1 for every element $x \in D$

$$|T[p_\nu]x| = \int_{[p_\nu]} |\varphi(\xi)| d\xi |x| \leq \alpha_\nu [p_\nu] |x|,$$

and hence for every $\nu, \mu = 1, 2, \dots$

$$|T[p_\nu]a_\mu - T[p_\nu]a| \leq \alpha_\nu [p_\nu] |a_\mu - a|.$$

Consequently we obtain for every $\nu = 1, 2, \dots$

$$[p_\nu] b = \lim_{\mu \rightarrow \infty} T[p_\nu] a_\mu = T[p_\nu] a = \int_{[p_\nu][a]} \varphi(\xi) d\xi a.$$

Therefore we see by Theorem 21.1 that $\varphi(\xi)$ is integrable by a in $\mathcal{U}_{[a]}$ and

$$b = [\alpha a + \beta b] b = \int_{[a]} \varphi(\xi) d\xi a,$$

that is, $a \in D$ and $T a = b$. Thus we have proved that T

is a dilatator with domain D

Corresponding to every almost finite continuous function $\varphi(p)$ on the proper space E of R we obtain by Theorem 43.6 uniquely a dilatator T . This dilatator T will be denoted by $\int \varphi(p) d\mathfrak{p}$.

Conversely we have:

Theorem 43.7. To every dilatator T with domain D there exists uniquely an almost finite continuous function $\varphi(p)$ on the proper space E of R such that

$$T = \int \varphi(p) d\mathfrak{p}$$

and we have for every element $a \in D$

$$\left(\frac{Ta}{a}, p\right) = \varphi(p) \quad \text{for every point } p \in \mathcal{U}_{[a]}.$$

Proof. For every elements $a, b \in D$ we have by Theorems 43.3 and 22.1

$$\left(\frac{Tb}{Ta}, p\right) = \left(\frac{b}{a}, p\right) \quad \text{for every point } p \in \mathcal{U}_{[Ta]},$$

since $[Ta] = [[a]Ta] \subseteq [a]$ by the postulate 2). As we have by Theorem 18.12

$$\left(\frac{Tb}{b}, p\right) = \left(\frac{Tb}{Ta}, p\right) \left(\frac{Ta}{b}, p\right) = \left(\frac{Ta}{a}, p\right)$$

in an open set being dense in $\mathcal{U}_{[a][b][Ta]}$, we obtain hence

$$\left(\frac{Tb}{b}, p\right) = \left(\frac{Ta}{a}, p\right) \quad \text{for every point } p \in \mathcal{U}_{[a][b][Ta]}.$$

We also obtain likewise

$$\left(\frac{Tb}{b}, p\right) = \left(\frac{Ta}{a}, p\right) \quad \text{for every point } p \in \mathcal{U}_{[a][b][Tb]}.$$

If $p \in \mathcal{U}_{[a][b]}$ but $p \notin \mathcal{U}_{[a][b][Ta]} \cap \mathcal{U}_{[a][b][Tb]}$, then we have by Theorem 18.3

$$\left(\frac{Tb}{b}, p\right) = \left(\frac{Ta}{a}, p\right) = 0$$

Therefore we obtain for every elements $a, b \in D$

$$\left(\frac{Tb}{b}, p\right) = \left(\frac{Ta}{a}, p\right) \quad \text{for every point } p \in \mathcal{U}_{[a][b]}.$$

Since to every element $p \in R$ there exists by Theorem 43.4 an element $a \in D$ for which $[a] = [p]$, there exists hence a function

$\varphi(p)$ on E such that we have for every element $a \in D$

$$\varphi(p) = \left(\frac{Ta}{a}, p\right) \quad \text{for every point } p \in \mathcal{U}_{[a]}$$

This function $\varphi(\mathfrak{f})$ is obviously continuous and almost finite in E by Theorems 19.2 and 19.3, and we have by Theorem 22.2 for every element $a \in D$

$$Ta = \int_{[a]} \varphi(\mathfrak{f}) d\mathfrak{f} a.$$

Furthermore if the integral

$$\int_{[p]} \varphi(\mathfrak{f}) d\mathfrak{f} p$$

is convergent for an element $p \in R$, then there exists by Theorem 43.5 a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$$

such that $[p_\nu]p \in D$ for every $\nu = 1, 2, \dots$, and we have

$$T[p_\nu]p = \int_{[p_\nu]} \varphi(\mathfrak{f}) d\mathfrak{f} [p_\nu]p = \int_{[p_\nu]} \varphi(\mathfrak{f}) d\mathfrak{f} p$$

for every $\nu = 1, 2, \dots$, as proved just now. Accordingly

we obtain by Theorem 21.4

$$\begin{aligned} \lim_{\nu \rightarrow \infty} [p_\nu]p &= p, \\ \lim_{\nu \rightarrow \infty} T[p_\nu]p &= \int_{[p]} \varphi(\mathfrak{f}) d\mathfrak{f} p, \end{aligned}$$

and hence by the postulate 3)

$$p \in D, \quad Tp = \int_{[p]} \varphi(\mathfrak{f}) d\mathfrak{f} p.$$

Therefore we have by definition

$$T = \int \varphi(\mathfrak{f}) d\mathfrak{f}.$$

Conversely, if $T = \int \varphi(\mathfrak{f}) d\mathfrak{f}$ for an almost finite continuous function $\varphi(\mathfrak{f})$ on E , then we have by definition for every element $a \in D$

$$Ta = \int_{[a]} \varphi(\mathfrak{f}) d\mathfrak{f} a,$$

and hence by Theorem 22.1

$$\left(\frac{Ta}{a}, \mathfrak{f}\right) = \varphi(\mathfrak{f}) \quad \text{for every point } \mathfrak{f} \in \mathcal{U}_{[a]}.$$

Since to every element $p \in R$ there exists by Theorem 43.4 an element $a \in D$ for which $[a] = [p]$, we can recognize hence the uniqueness of such a function $\varphi(\mathfrak{f})$.

Corresponding to every dilatator T in R we obtain by Theorem 43.7 uniquely an almost finite continuous function $\varphi(\mathfrak{f})$ on the proper space E of R for which

$$T = \int \varphi(\mathfrak{f}) d\mathfrak{f}.$$

Such a function $\varphi(\mathfrak{f})$ is called the spectral function of a dilatator T in R .

§44 Calculus of dilatators

Let R be a continuous semi-ordered linear space. To every two dilatators T_1 with domain D_1 and T_2 with domain D_2 there exists uniquely a dilatator T_3 such that

$$T_3 a = T_1 a + T_2 a \quad \text{for every element } a \in D_1, D_2.$$

Because, denoting by $\varphi_1(\mathfrak{f})$ and $\varphi_2(\mathfrak{f})$ the spectral functions respectively of T_1 and T_2 , we obtain by Theorem 43.6 a dilatator T_3 as

$$T_3 = \int (\varphi_1 + \varphi_2)(\mathfrak{f}) d\mathfrak{f}.$$

For such a dilatator T_3 we see easily by Theorem 42.3 that

$$T_3 a = T_1 a + T_2 a \quad \text{for every element } a \in D_1, D_2.$$

Conversely, we suppose that this relation holds for a dilatator T_3 . To every positive element $p \in R$ there exists by Theorem 43.4 positive elements $a_1 \in D_1$ and $a_2 \in D_2$ such that

$$[a_1] = [a_2] = [p].$$

For such a_1 and a_2 , putting $a = a_1 \wedge a_2$, we obtain by Theorem 43.2 and by the formula §8(1)

$$a \in D_1, D_2, \quad [a] = [a_1] \wedge [a_2] = [p].$$

Hence for the spectral function $\varphi_3(\mathfrak{f})$ of T_3 we have by definition

$$\varphi_3(\mathfrak{f}) = \left(\frac{T_3 a}{a}, \mathfrak{f} \right)$$

$$= \left(\frac{T_1 a}{a}, \mathfrak{f} \right) + \left(\frac{T_2 a}{a}, \mathfrak{f} \right) = \varphi_1(\mathfrak{f}) + \varphi_2(\mathfrak{f})$$

in an open set being dense in $\mathcal{U}_{[p]}$, and consequently

$$\varphi_3(\mathfrak{f}) = (\varphi_1 + \varphi_2)(\mathfrak{f}) \quad \text{for every point } \mathfrak{f} \in \mathcal{U}_{[p]}.$$

Since p may be an arbitrary positive element of R , we obtain thus

$$\varphi_3 = \varphi_1 + \varphi_2.$$

Therefore such a dilatator T_3 is uniquely determined. Such T_3 is called the sum of T_1 and T_2 , and denoted by

$$T_1 + T_2$$

As proved just now, the spectral function of the sum $T_1 + T_2$ coincides with the sum $\varphi_1 + \varphi_2$ for the spectral functions $\varphi_1(\mathfrak{p})$ and $\varphi_2(\mathfrak{p})$ respectively of T_1 and T_2 .

For every two dilatators T_1 and T_2 we obtain by Theorem 43.6 a dilatator T_3 as

$$T_3 = \int \varphi_1 \varphi_2(\mathfrak{p}) d\mathfrak{p}$$

for the spectral function $\varphi_1(\mathfrak{p})$ and $\varphi_2(\mathfrak{p})$ respectively of T_1 and T_2 . For such a dilatator T_3 we have by Theorem 42.4

$$T_3 a = T_2(T_1 a),$$

if the right side has any sense. Conversely, if this relation holds for some dilatator T_3 , then, for the domain D_1 and D_2 respectively of T_1 and T_2 , to every positive element $p \in R$ there exist by Theorem 43.2 positive elements $a \in D_1$ and $b \in D_2$ such that

$$[a] = [b] = [p], \quad |T_1 a| \leq b.$$

For such $a \in D_1$, $T_2(T_1 a)$ has a sense by Theorem 43.2, and we have by Theorem 42.4

$$T_3 a = T_2(T_1 a) = \int_{[a]} \varphi_2(\mathfrak{p}) d\mathfrak{p} T_1 a = \int_{[a]} \varphi_1 \varphi_2(\mathfrak{p}) d\mathfrak{p} a.$$

Consequently, for the spectral function $\varphi_3(\mathfrak{p})$ of T_3 we have by Theorem 22.1

$$\varphi_3(\mathfrak{p}) = \left(\frac{T_3 a}{a}, \mathfrak{p} \right) = \varphi_1 \varphi_2(\mathfrak{p})$$

for every point $\mathfrak{p} \in \mathcal{U}_{[a]} = \mathcal{U}_{[p]}$. Since p may be an arbitrary positive element of R , we obtain hence by definition

$$T_3 = \int \varphi_1 \varphi_2(\mathfrak{p}) d\mathfrak{p}.$$

Therefore to every two dilatators T_1 and T_2 there exists uniquely a dilatator T_3 such that

$$T_3 a = T_2(T_1 a)$$

if the right side has any sense. Such a dilatator T_3 is called the product of T_2 and T_1 , and denoted by

$$T_2 T_1$$

Then, as proved just now, the spectral function of the product $T_2 T_1$ coincides with the product $\varphi_2 \varphi_1(\beta)$ for the spectral functions $\varphi_1(\beta)$ and $\varphi_2(\beta)$ respectively of T_1 and T_2 .

Every real number may be considered as a dilatator with domain R . Now we see easily that the totality of dilatators in R constitutes a ring, which is isomorphic to the ring of all almost finite continuous functions on the proper space E of R by the correspondence $T \rightarrow \varphi_T$ for the spectral function $\varphi_T(\beta)$ of a dilatator T

For a dilatator T with domain D we define $T \geq 0$ to mean that

$$Ta \geq 0 \quad \text{for every positive element } a \in D.$$

If the spectral function $\varphi_T(\beta)$ of T is positive, that is, if

$$\varphi_T(\beta) \geq 0 \quad \text{for every point } \beta \in E,$$

then we have obviously $T \geq 0$ by Theorem 20.1. Conversely

$T \geq 0$ implies

$$\varphi_T(\beta) \geq 0 \quad \text{for every point } \beta \in E.$$

Because, if $\varphi_T(\beta_0) < 0$ for some point $\beta_0 \in E$, then there exist a positive number ε and a neighbourhood $U_{[\beta_0]} \ni \beta_0$ such that

$$\varphi_T(\beta) \leq -\varepsilon \quad \text{for every point } \beta \in U_{[\beta_0]},$$

since $\varphi_T(\beta)$ is continuous over E . By virtue of Theorem

43.4 there exists further a positive element $a \in D$ such that

$[a] = [p]$, and Ta has a sense. For such $a \in D$ we have

by Theorem 20.1

$$Ta = \int_{[a]} \varphi_T(\beta) d\beta a \leq -\varepsilon a,$$

contradicting that $T \geq 0$ and $[a] = [p] \neq 0$.

Therefore we have $T \geq 0$ if and only if the spectral func-

tion of T is positive. By virtue of Theorem 42.1, the totality of almost finite continuous functions on the proper space E of R constitutes a complete normal ring possessing a unit factor. Thus we have:

Theorem 44.1. Totality of dilators in R constitutes a complete normal ring possessing a unit factor and is isomorphic to the semi-ordered ring consisting of all almost finite continuous functions on the proper space E of R by the correspondence $T \rightarrow \varphi_T$ for the spectral function $\varphi_T(\mathfrak{P})$ of a dilator T , that is, we have

$$\varphi_{\alpha T_1 + \beta T_2} = \alpha \varphi_{T_1} + \beta \varphi_{T_2}, \quad \varphi_{T_1 T_2} = \varphi_{T_1} \varphi_{T_2},$$

$$\varphi_{T_1 \cup T_2} = \varphi_{T_1} \cup \varphi_{T_2}, \quad \varphi_{T_1 \cap T_2} = \varphi_{T_1} \cap \varphi_{T_2},$$

$$\varphi_{T^+} = \varphi_T^+, \quad \varphi_{T^-} = \varphi_T^-, \quad \varphi_{|T|} = |\varphi_T|.$$

Theorem 44.2. For two dilators T_1 and T_2 , if $|T_1| \leq |T_2|$, then the domain D_2 of T_2 is included in the domain D_1 of T_1 .

Proof. Denoting by $\varphi_T(\mathfrak{P})$ the spectral function of a dilator T in R , we have by Theorem 43.7 that $\varphi_{T_2}(\mathfrak{P})$ is integrable by every element $a \in D_2$ in $\mathcal{U}_{[a]}$. Since $|T_1| \leq |T_2|$ by assumption, we have by the previous theorem

$$|\varphi_{T_1}| = \varphi_{|T_1|} \leq \varphi_{|T_2|} = |\varphi_{T_2}|,$$

and hence we see by Theorems 20.1, 21.2, and 21.3 that $\varphi_{T_1}(\mathfrak{P})$ is integrable by every element $a \in D_2$ in $\mathcal{U}_{[a]}$. Therefore we obtain $D_2 \subset D_1$ by Theorem 43.7.

By virtue of Theorems 19.2 and 19.3, for every pair of elements a and $p \in R$, putting

$$\varphi(\mathfrak{P}) = \begin{cases} (\frac{a}{p}, \mathfrak{P}) & \text{for } \mathfrak{P} \in \mathcal{U}_{[p]}, \\ 0 & \text{for } \mathfrak{P} \notin \mathcal{U}_{[p]}, \end{cases}$$

we obtain an almost finite continuous function $\varphi(\mathfrak{P})$ on E , and hence there exists by Theorem 43.6 uniquely a dilator, whose spectral function coincides with $\varphi(\mathfrak{P})$. This dilator will be denoted by $[a|p]$, that is,

$$[a|p] = \int \varphi(\mathfrak{f}) d\mathfrak{f},$$

if we put

$$\varphi(\mathfrak{f}) = \begin{cases} (\frac{a}{p}, \mathfrak{f}) & \text{for } \mathfrak{f} \in \mathcal{U}_{[p]}, \\ 0 & \text{for } \mathfrak{f} \notin \mathcal{U}_{[p]}. \end{cases}$$

With this definition we have obviously

$$(1) \quad [0|p] = [a|0] = 0,$$

and by Theorem 22.2

$$(2) \quad [a|p]p = a.$$

Recalling Theorem 18.4, we obtain at once

$$(3) \quad [a|p] = [[p]a|p] = [a|[a]p] = [[p]a|[a]p].$$

By virtue of Theorem 19.5, we have obviously:

Theorem 44.3. we have

$$[a|p] = [l|p]$$

if and only if $[p]a = [p]l$.

Every projector $[p]$ is obviously a dilatator by definition

and

$$(4) \quad [p] = [p|p],$$

since we have

$$[p]a = \int_{[p]} d\mathfrak{f} a \quad \text{for every element } a \in R,$$

and by Theorem 18.2

$$(\frac{p}{p}, \mathfrak{f}) = 1 \quad \text{for every point } \mathfrak{f} \in \mathcal{U}_{[p]},$$

that is, the spectral function of $[p]$ is the characteristic function of the point set $\mathcal{U}_{[p]}$ in E .

Furthermore we have for every projector $[q]$

$$(5) \quad [a|p][q] = [[q]a|p].$$

Because, we have by Theorems 18.4 and 18.2

$$(\frac{[q]a}{p}, \mathfrak{f}) = (\frac{a}{p}, \mathfrak{f})(\frac{q}{q}, \mathfrak{f}) \quad \text{for every point } \mathfrak{f} \in \mathcal{U}_{[p][q]},$$

and hence the spectral function of $[[q]a|p]$ is the product of the spectral function of $[a|p]$ and the characteristic function of the point set $\mathcal{U}_{[q]}$ in E .

Recalling Theorems 18.5, 18.6, and 18.9, we see easily:

Theorem 44.4. We have

$$\alpha[a|p] + \beta[b|p] = [\alpha a + \beta b|p],$$

and if $p \geq 0$, then we have

$$[a|p] \vee [b|p] = [a \vee b|p],$$

$$[a|p] \wedge [b|p] = [a \wedge b|p],$$

$$[a|p]^+ = [a^+|p], \quad [a|p]^- = [a^-|p], \quad |[a|p]| = [|a||p].$$

Since we have by Theorem 18.10

$$|(\frac{a}{p}, \mathfrak{F})| = (\frac{|a|}{|p|}, \mathfrak{F}) \quad \text{for every point } \mathfrak{F} \in U_{cp},$$

we obtain by Theorem 44.1

$$(6) \quad |[a|p]| = [|a||p|].$$

A dilatator T is said to be principal, if there exists a projector $[p]$ such that

$$T[p] = T.$$

For every pair of elements a and $p \in R$, the dilatator $[a|p]$ is principal, since we have by the formulas (3) and (5)

$$[a|p][p] = [[p]a|p] = [a|p].$$

If a dilatator T is principal, then there exists by Theorem 43.4 a positive element $p \in R$ such that $T[p] = T$ and T_p has a sense.

Theorem 44.5. For a principal dilatator T , if $T[p] = T$ and T_p has a sense, then we have

$$T = [Tp|p].$$

Proof. Since $T[p] = T$ by assumption, for the spectral function $\varphi_T(\mathfrak{F})$ of T we have by definition

$$\varphi_T(\mathfrak{F}) = \begin{cases} (\frac{Tp}{p}, \mathfrak{F}) & \text{for } \mathfrak{F} \in U_{cp}, \\ 0 & \text{for } \mathfrak{F} \notin U_{cp}. \end{cases}$$

Consequently we have $T = [Tp|p]$ by definition.

Theorem 44.6. For a dilatator T , if Ta has a sense, then we have for every element $p \in R$

$$T[a|p] = [Ta|p].$$

Proof. We have by the formulas (3) and (5)

$$T[a|p][p] = T[[p]a|p] = T[a|p].$$

Since Ta has a sense by assumption, we have

$$T([a|p]p) = T[p]a = [p]Ta.$$

Therefore we obtain by the previous theorem

$$T[a|p] = [[p]Ta|p] \doteq [Ta|p].$$

As an immediate consequence from Theorem 44.6 we have

$$(7) \quad [a|q][q|p] = [[q]a|p] = [a|p][q].$$

Theorem 44.7. $[a|p] \geq T \geq 0$ implies $T = [Tp|p]$.

Proof. Since we have by the formulas (3) and (5)

$$[a|p](1-[p]) = 0 \quad \text{and} \quad 1-[p] \geq 0.$$

if $[a|p] \geq T \geq 0$, then we have

$$T(1-[p]) = 0,$$

that is, $T[p] = T$. Furthermore Tp has a sense by Theorem 44.2, and hence we obtain by Theorem 44.5

$$T = [Tp|p].$$

Theorem 44.8. $\lim_{\nu \rightarrow \infty} a_\nu = a$ implies $\lim_{\nu \rightarrow \infty} [a_\nu|p] = [a|p]$
for every element $p \in R$

Proof. If $\lim_{\nu \rightarrow \infty} a_\nu = a$, then there exist by definition a sequence of elements $\ell_\nu \downarrow_{\nu=1}^\infty 0$ and μ_ν ($\nu = 1, 2, \dots$) such that

$$|a_\mu - a| \leq \ell_\nu \quad \text{for } \mu \geq \mu_\nu, \nu = 1, 2, \dots,$$

and we have by Theorem 44.4 and the formula (6) for $\mu \geq \mu_\nu$

$$|[a_\mu|p] - [a|p]| = |[a_\mu - a||p]| \leq [\ell_\nu |p|].$$

Therefore we need only to prove $[\ell_\nu |p|] \downarrow_{\nu=1}^\infty 0$. If

$$0 \leq T \leq [\ell_\nu |p|] \quad \text{for every } \nu = 1, 2, \dots,$$

then we have for every $\nu = 1, 2, \dots$

$$T|p| \leq [\ell_\nu |p|]|p| = [p]\ell_\nu \leq \ell_\nu,$$

and consequently we obtain $T|p| = 0$. On the other hand, we have by the previous theorem

$$T = [T|p||p|],$$

and hence $T = 0$. Thus we have $[\ell_\nu |p|] \downarrow_{\nu=1}^\infty 0$.

§45 Dilatator rings

Let R be a continuous semi-ordered linear space. By virtue of Theorem 44.1 the totality of dilatators in R constitutes a complete normal ring possessing a unit factor, which will be called the dilatator ring of R . As stated in Theorem 44.1, the dilatator ring of R is isomorphic to the complete normal ring consisting of all almost finite continuous functions on the proper space E of R by the correspondence $T \rightarrow \varphi_T$ for the spectral function $\varphi_T(p)$ of a dilatator T .

We shall consider first projectors in the dilatator ring.

Theorem 45.1. For every dilatator T we have

$$[[a|p]]T = T[a|p].$$

Proof. Since the dilatator ring is normal, we have by the formula §25(10)

$$\begin{aligned} [[a|p]]T &= ([[a|p]]1)T \\ &= ([[a|p]][p])T + ([[a|p]](1-[p]))T. \end{aligned}$$

On the other hand we have by the formulas (4), (5), (6) in §44 and by Theorem 44.8

$$\begin{aligned} [[a|p]][p] &= [[|a| \vee |p|]][|p| \vee |p|] \\ &= \lim_{\rightarrow} \{[|p| \vee |p|] \wedge \vee [|a| \vee |p|]\} \\ &= \lim_{\rightarrow} [|p| \wedge \vee |a| \vee |p|] \\ &= [[a|p| \vee |p|]] = [a][p], \end{aligned}$$

and by the formulas §44(3), §44(5), §8(1), and by Theorem 26.8

$$\begin{aligned} [[a|p]](1-[p]) &= [[a|p]][1-[p]] \\ &= [[a|p](1-[p])] = [[a|p][p](1-[p])] = 0. \end{aligned}$$

Consequently we obtain $[[a|p]]T = [a][p]T$.

As a special case of Theorem 45.1 we have by the formula §44(4):

Theorem 45.2. For every dilatator T we have

$$[[p]]T = T[p].$$

Theorem 45.3. We have $[[p]] \geq [[q]]$ if and only if $[p] \geq [q]$.

Proof. $[p] \geq [q]$ implies $[[p]] \geq [[q]]$ by Theorem 8.3.

Conversely, if $[[p]] \geq [[q]]$, then we have by Theorem 45.2

$$[p] = [[p]]1 \geq [[q]]1 = [q].$$

Since the dilatator ring is normal, we have:

Theorem 45.4. For every projectors $[p]$ and $[q]$ we have

$$[[p]][[q]] = [[p][q]].$$

Theorem 45.5. To every dilatator T and to every projector $[p]$ there exists a projector $[q]$ such that

$$[[p]][T] = [[q]].$$

Proof. By virtue of Theorem 43.4 there exists an element

$a \in R$ such that $[a] = [p]$ and Ta has a sense. For such $a \in R$, since

$$(T[p])[a] = T[p], \quad (T[p])a = Ta,$$

we have by Theorem 44.4

$$T[p] = [Ta | a].$$

Therefore we obtain by Theorems 45.1, 45.2, and the formula §8(1)

$$\begin{aligned} [[p]][T] &= [T[p]] = [[Ta | a]] \\ &= [[Ta][a]] = [[a]Ta] = [[Ta]]. \end{aligned}$$

Theorem 45.6. The proper space E of R is homeomorphic to the dense open set

$$\sum_{p \in R} \mathcal{U}_{[[p]]}$$

in the proper space $\hat{E}$ of the dilatator ring of R by a correspondence $E \ni \mathfrak{f} \rightarrow \mathfrak{f}^D \in \hat{E}$ such that

$$\{\mathfrak{f}^D : \mathfrak{f} \in \mathcal{U}_{[[p]]}\} = \mathcal{U}_{[[p]]} \quad \text{for every point } p \in R.$$

For such a correspondence we have

$$\left(\frac{T}{1}, \mathfrak{f}^D\right) = \mathcal{V}_T(\mathfrak{f}) \quad \text{for every point } \mathfrak{f} \in E$$

for the spectral function $\mathcal{V}_T(\mathfrak{f})$ of a dilatator T

Proof. For every point $\mathfrak{f} \in E$ we have by Theorems 45.3 and 45.4

$$\prod_{\mathcal{U}_{[[p]]} \ni \mathfrak{f}} \mathcal{U}_{[[p]]} \neq \emptyset,$$

since $\mathcal{U}_{[[p]]}$ is compact. Furthermore we see easily by Theorem 45.5 that the intersection

$$\prod_{\mathcal{U}_{[p]} \ni \mathcal{F}} \mathcal{U}_{[p]}$$

is composed only of a single point, which will be denoted by $\mathcal{F}^0$ corresponding to a point $\mathcal{F} \in E$. Conversely, for every point

$$\mathcal{Q} \in \sum_{p \in R} \mathcal{U}_{[p]}$$

we see likewise that the intersection

$$\prod_{\mathcal{U}_{[p]} \ni \mathcal{Q}} \mathcal{U}_{[p]}$$

is composed only of a single point $\mathcal{F}$, and then we have obviously

$\mathcal{F}^0 = \mathcal{Q}$. For such a correspondence $E \ni \mathcal{F} \rightarrow \mathcal{F}^0 \in \hat{E}$, it is evident that

$$\{\mathcal{F}^0 : \mathcal{F} \in \mathcal{U}_{[p]}\} = \mathcal{U}_{[p]} \text{ for every element } p \in R,$$

and hence E is homeomorphic to the open set

$$\sum_{p \in R} \mathcal{U}_{[p]}$$

in $\hat{E}$ by this correspondence.

To every projector $[T] \neq 0$ in the dilatator ring there exists an element $p \in R$ for which $Tp \neq 0$. For such $p \in R$ we have by Theorems 44.4, 45.1, and 45.2

$$\begin{aligned} [T][[p]] &= [T[p]] = [[Tp|p]] \\ &= [[Tp][p]] = [[p]Tp] = [[Tp]] \neq 0. \end{aligned}$$

Thus $\sum_{p \in R} \mathcal{U}_{[p]}$ is dense in $\hat{E}$.

For an arbitrary point $\mathcal{F}_0 \in E$, if

$$\left(\frac{T}{1}, \mathcal{F}_0^0\right) > \alpha,$$

then there exists by Theorems 19.2 and 45.5 a projector $[p]$ such that $\mathcal{U}_{[p]} \ni \mathcal{F}_0$ and

$$\left(\frac{T}{1}, \mathcal{F}^0\right) > \alpha \quad \text{for every point } \mathcal{F} \in \mathcal{U}_{[p]}.$$

For such $[p]$ we have by Theorem 19.4

$$[[p]]T \geq \alpha [[p]]1,$$

and hence by Theorem 45.2

$$T[p] \geq \alpha [p].$$

By virtue of Theorem 43.4 there exists a positive element $a \in R$ such that $[a] = [p]$ and Ta has a sense, and for such $a \in R$ we obtain thus

$$\tau a = (\tau[p])a \geq \alpha[p]a = \alpha a.$$

This relation yields by Theorems 18.4 and 18.8

$$\mathcal{G}_\tau(\mathcal{J}_0) = \left(\frac{\tau a}{a}, \mathcal{J}_0\right) \geq \alpha.$$

Therefore we conclude $\left(\frac{\tau}{\tau}, \mathcal{J}_0^0\right) \leq \mathcal{G}_\tau(\mathcal{J}_0)$ for every point $\mathcal{J}_0 \in E$.

We also can prove likewise that

$$\left(\frac{\tau}{\tau}, \mathcal{J}^0\right) \geq \mathcal{G}_\tau(\mathcal{J}) \quad \text{for every point } \mathcal{J} \in E.$$

Consequently we have $\left(\frac{\tau}{\tau}, \mathcal{J}^0\right) = \mathcal{G}_\tau(\mathcal{J})$ for every point $\mathcal{J} \in E$.

Theorem 45.7. If R has a complete positive element p , then the dilatator ring of R is a completion of R by an extending correspondence

$$R \ni a \rightarrow [a|p].$$

Proof. We see easily by Theorems 44.3 and 44.4 that the dilatator ring of R is an extension of R by the indicated correspondence. The dilatator ring is complete by Theorem 44.1. Since we conclude by Theorem 44.4

$$[a|p] \cap [t|p] = [a \cap t | p],$$

we obtain by Theorem 44.3 that we have

$$[a|p] \perp [t|p]$$

if and only if $a \perp t$, that is, the postulate 2) in §33 is satisfied. Furthermore the postulate 3) in §33 is obviously satisfied by Theorem 44.8.

To every dilatator τ there exists by Theorem 43.5 a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^\infty [p] = 1,$$

such that $\tau([p_\nu]p)$ has a sense for every $\nu=1, 2, \dots$.

For such $[p_\nu]$ ($\nu=1, 2, \dots$), putting $[p_0]=0$, we have by Theorem 44.5 for every $\nu=1, 2, \dots$

$$\tau([p_\nu] - [p_{\nu-1}]) = [\tau([p_\nu]p - [p_{\nu-1}]p) | p].$$

On the other hand we conclude by Theorem 45.2

$$\sum_{\nu=1}^\infty \tau([p_\nu] - [p_{\nu-1}]) = \tau[p_\nu] = [[p_\nu] \tau$$

for every $v \neq 1, 2, \dots$, and further by Theorem 8.5

$$[[p_v]] \uparrow_{v=1}^{\infty} [1] = 1.$$

Consequently we obtain

$$T = \sum_{v=1}^{\infty} [T([p_v]p - [p_{v-1}]p) | p],$$

that is, the postulate 4) in §33 is satisfied too. Therefore the dilatator ring of R is a completion of R by the indicated correspondence as defined in §33.

Theorem 45.8. If R is universally continuous, then the dilatator ring of R is a universal completion of R by an extending correspondence

$$R \ni a \rightarrow a^D = \bigcup_{\lambda \in \Lambda} [a^+ | p_\lambda] - \bigcup_{\lambda \in \Lambda} [a^- | p_\lambda]$$

for a complete orthogonal system of positive elements $p_\lambda \in R$ ($\lambda \in \Lambda$) in R .

Proof. As R is universally continuous by assumption, we can conclude by Theorems 41.6, 42.2, and 44.1 that the dilatator ring of R is universally complete. Since $p_\lambda \in R$ ($\lambda \in \Lambda$) is a complete orthogonal system in R , we have obviously

$$a = \bigcup_{\lambda \in \Lambda} [p_\lambda]a \quad \text{for } 0 \leq a \in R,$$

and hence we conclude easily that

$$1 = \bigcup_{\lambda \in \Lambda} [p_\lambda]$$

in the dilatator ring. Accordingly we obtain by Theorem 8.5

$$T = \bigcup_{\lambda \in \Lambda} [[p_\lambda]]T \quad \text{for every dilatator } T \geq 0.$$

For every positive elements a and $b \in R$ we have

$$[a | p_\lambda] \wedge [b | p_\mu] = 0 \quad \text{for } \lambda \neq \mu,$$

since we conclude by Theorem 45.1 and by the formulas (3) and (5) in §44 for $\lambda \neq \mu$

$$[[a | p_\lambda]] [[b | p_\mu]] = [a][p_\lambda][[p_\mu]b | p_\mu] = [a][0 | p_\mu] = 0.$$

Therefore we obtain by Theorems 44.4, 3.1, and 2.6 for every positive elements a and $b \in R$

$$\begin{aligned} \bigcup_{\lambda \in \Lambda} [a+b | p_\lambda] &= \bigcup_{\lambda \in \Lambda} ([a | p_\lambda] + [b | p_\lambda]) \\ &= \bigcup_{\lambda, \mu, \nu \in \Lambda} \{([a | p_\lambda] + [b | p_\lambda]) \vee [a | p_\mu] \vee [b | p_\nu]\} \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{\lambda, p, \sigma \in A} \{([a | p_\lambda] + [b | p_\lambda]) \cup ([a | p_p] + [b | p_\sigma])\} \\
&= \bigcup_{p, \sigma \in A} \{[a | p_p] + [b | p_\sigma]\} \\
&= \bigcup_{\lambda \in A} [a | p_\lambda] + \bigcup_{\lambda \in A} [b | p_\lambda],
\end{aligned}$$

that is, we have for every positive elements a and $b \in R$

$$(a + b)^D = a^D + b^D.$$

For arbitrary elements a and $b \in R$, since

$$(a + b)^+ + a^- + b^- = (a + b)^- + a^+ + b^+,$$

we obtain, as proved just now,

$$(a + b)^+D + a^{-D} + b^{-D} = (a + b)^-D + a^{+D} + b^{+D},$$

and hence by definition

$$\begin{aligned}
(a + b)^D &= (a + b)^+D - (a + b)^-D \\
&= (a^{+D} + b^{+D}) - (a^{-D} + b^{-D}) = a^D + b^D.
\end{aligned}$$

For every positive number α we have by Theorems 44.4 and 2.3

$$\begin{aligned}
\bigcup_{\lambda \in A} [\alpha a^+ | p_\lambda] &= \alpha \bigcup_{\lambda \in A} [a^+ | p_\lambda], \\
\bigcup_{\lambda \in A} [\alpha a^- | p_\lambda] &= \alpha \bigcup_{\lambda \in A} [a^- | p_\lambda],
\end{aligned}$$

and hence $(\alpha a)^D = \alpha a^D$. Furthermore it is evident by definition

$$(-a)^D = -a^D.$$

Therefore we have in general

$$(\alpha a + \beta b)^D = \alpha a^D + \beta b^D.$$

Since we see by Theorems 3.4 and 44.4 that we have

$$\left(\bigcup_{\lambda \in A} [a^+ | p_\lambda]\right) \cap \left(\bigcup_{\lambda \in A} [a^- | p_\lambda]\right) = \bigcup_{\lambda \in A} [a^+ \wedge a^- | p_\lambda] = 0,$$

we obtain by Theorem 3.8

$$\begin{aligned}
a^{D+} &= \bigcup_{\lambda \in A} [a^+ | p_\lambda] = a^{+D}, \\
a^{D-} &= \bigcup_{\lambda \in A} [a^- | p_\lambda] = a^{-D},
\end{aligned}$$

and hence $|a^D| = |a|^D$ for every element $a \in R$.

It is evident that $a \geq 0$ implies $a^D \geq 0$. Conversely,

if $a^D \geq 0$, then we have

$$a^{-D} = a^{D-} = 0,$$

and hence $[a^- | p_\lambda] = 0$ for every $\lambda \in A$. This relation yields

by Theorem 44.3

$$[p_\lambda] a^- = 0 \quad \text{for every } \lambda \in A,$$

and hence $a^- = \bigcup_{\lambda \in \Lambda} [p_\lambda] a^- = 0$, that is, $a \geq 0$. Therefore the dilatator ring of R is an extension of R by the indicated correspondence $R \ni a \rightarrow a^D$.

We have further by Theorem 3.12

$$\begin{aligned} |a^D| \wedge |b^D| &= |a|^D \wedge |b|^D = |a|^D - (|a|^D - |b|^D)^+ \\ &= |a|^D - (|a| - |b|)^+{}^D = (|a| \wedge |b|)^D, \end{aligned}$$

and hence $a^D \perp b^D$ if and only if $a \perp b$, that is, the postulate 2) in §34 is satisfied.

If $a = \bigcup_{\gamma \in \Gamma} a_\gamma$ for a system of positive elements $a_\gamma \in R$ ($\gamma \in \Gamma$), then we have by Theorem 44.4

$$[a_\gamma | p_\lambda] \leq [a | p_\lambda] \quad \text{for every } \gamma \in \Gamma,$$

and hence, putting $T = \bigcup_{\gamma \in \Gamma} [a_\gamma | p_\lambda]$, we obtain

$$0 \leq [a_\gamma | p_\lambda] \leq T \leq [a | p_\lambda] \quad \text{for every } \gamma \in \Gamma.$$

Then we have by Theorem 44.7

$$T = [T p_\lambda | p_\lambda],$$

and by the formula §44(2)

$$[p_\lambda] a_\gamma \leq T p_\lambda \leq [p_\lambda] a \quad \text{for every } \gamma \in \Gamma.$$

Since $[p_\lambda] a = \bigcup_{\gamma \in \Gamma} [p_\lambda] a_\gamma$ by Theorem 7.6, we obtain hence

$$T p_\lambda = [p_\lambda] a,$$

and consequently by the formula §44(3)

$$T = [T p_\lambda | p_\lambda] = [[p_\lambda] a | p_\lambda] = [a | p_\lambda].$$

Therefore if $a = \bigcup_{\gamma \in \Gamma} a_\gamma$ for a system of positive elements $a_\gamma \in R$ ($\gamma \in \Gamma$), then we have

$$[a | p_\lambda] = \bigcup_{\gamma \in \Gamma} [a_\gamma | p_\lambda] \quad \text{for every } \lambda \in \Lambda$$

in the dilatator ring, and consequently we obtain by Theorem 2.5

$$a^D = \bigcup_{\lambda \in \Lambda} [a | p_\lambda] = \bigcup_{\lambda \in \Lambda} \bigcup_{\gamma \in \Gamma} [a_\gamma | p_\lambda] = \bigcup_{\gamma \in \Gamma} a_\gamma^D.$$

Thus the postulate 3) in §34 is satisfied.

Let T be an arbitrary positive dilatator. To every $\lambda \in \Lambda$ there exists by Theorem 43.5 a sequence of projectors

$$[p_{\lambda, \nu}] \uparrow_{\nu=1}^{\infty} [p_\lambda]$$

such that $T([p_{\lambda, \nu}] p_\lambda)$ has a sense for every $\nu = 1, 2, \dots$

For such $[p_{\lambda, \nu}]$ ($\nu = 1, 2, \dots$), putting $[p_{\lambda, 0}] = 0$, we have

$$[[p_{\lambda}]]T = \bigcup_{\nu=1}^{\infty} [T([p_{\lambda, \nu}]p_{\lambda} - [p_{\lambda, \nu-1}]p_{\lambda}) | p_{\lambda}]$$

as proved already in Proof of the previous theorem. Since

$$T = \bigcup_{\lambda \in \Lambda} [[p_{\lambda}]]T$$

as established just above, we obtain hence

$$T = \bigcup_{\lambda \in \Lambda, \nu=1, 2, \dots} (T([p_{\lambda, \nu}]p_{\lambda} - [p_{\lambda, \nu-1}]p_{\lambda}))^D,$$

while the system of positive elements

$$T([p_{\lambda, \nu}]p_{\lambda} - [p_{\lambda, \nu-1}]p_{\lambda}) \in R \quad (\lambda \in \Lambda, \nu = 1, 2, \dots)$$

is an orthogonal system, because

$$([p_{\lambda, \nu}] - [p_{\lambda, \nu-1}])p_{\lambda} \in R \quad (\lambda \in \Lambda, \nu = 1, 2, \dots)$$

is an orthogonal system of positive elements, and further we have by Theorems 7.9 and 8.1

$$Ta = [a]Ta \perp [b]Tb = Tb,$$

if $a \perp b$ and both Ta , Tb have any sense. Therefore the postulate 4) in §34 is satisfied too, and consequently the dilatator ring of R is a universal completion of R by the indicated correspondence $R \ni a \rightarrow a^D$.

CHAPTER VIIICONTINUOUS LINEAR FUNCTIONALS§46 Bounded linear functionals

Let R be a lattice ordered linear space. A real valued functional $L(a)$ ($a \in R$) is said to be a linear functional on R , if we have

$$L(\alpha a + \beta b) = \alpha L(a) + \beta L(b)$$

for every elements $a, b \in R$ and for every real numbers α, β .

A linear functional L on R is said to be bounded, if we have for every positive element $a \in R$

$$\sup_{0 \leq x \leq a} |L(x)| < +\infty.$$

The totality of bounded linear functionals on R is called the associated space of R and denoted by $\tilde{R}$.

For two linear functionals L and F on R we define $\alpha L + \beta F$ as a linear functional on R such that

$$(\alpha L + \beta F)(a) = \alpha L(a) + \beta F(a) \text{ for every element } a \in R.$$

With this definition we see easily that the associated space $\tilde{R}$ of R constitutes a linear space.

We define further $L \geq F$ to mean that

$$L(a) \geq F(a) \text{ for every positive element } a \in R.$$

Then we can prove that the associated space $\tilde{R}$ of R constitutes a universally continuous semi-ordered linear space. For this purpose we shall prove first:

Theorem 46.1. If a functional L is defined only for positive elements of R such that

$$L(\alpha a + \beta b) = \alpha L(a) + \beta L(b)$$

for every positive elements $a, b \in R$ and for every positive numbers α, β , then putting for arbitrary elements $x \in R$

$$\tilde{L}(x) = L(x^+) - L(x^-)$$

we obtain a linear functional $\tilde{L}$ on R .

Proof. For a positive number α , since by Theorem 3.9

$$(\alpha a)^+ = \alpha a^+, \quad (\alpha a)^- = \alpha a^-$$

we have by assumption for an arbitrary element $a \in R$

$$\tilde{L}(\alpha a) = L(\alpha a^+) - L(\alpha a^-) = \alpha \{L(a^+) - L(a^-)\} = \alpha \tilde{L}(a).$$

Furthermore we have by assumption

$$\tilde{L}(-a) = L((-a)^+) - L((-a)^-) = L(a^-) - L(a^+) = -\tilde{L}(a).$$

Consequently we obtain for every element $a \in R$ and for every real number α

$$\tilde{L}(\alpha a) = \alpha \tilde{L}(a).$$

For two arbitrary elements a and $b \in R$, since by Theorem 3.6

$$a^+ + b^+ + (a+b)^- = a^- + b^- + (a+b)^+,$$

we have by assumption

$$L(a^+) + L(b^+) + L((a+b)^-) = L(a^-) + L(b^-) + L((a+b)^+).$$

From this relation we conclude by assumption

$$\begin{aligned} \tilde{L}(a+b) &= L((a+b)^+) - L((a+b)^-) \\ &= L(a^+) + L(b^+) - L(a^-) - L(b^-) = \tilde{L}(a) + \tilde{L}(b). \end{aligned}$$

Therefore $\tilde{L}$ is a linear functional on R .

Let $\tilde{R}$ be the associated space of R in the sequel. For an arbitrary element $\tilde{a} \in \tilde{R}$, if we put

$$P(a) = \sup_{0 \leq x \leq a} \tilde{a}(x) \quad \text{for } 0 \leq a \in R,$$

then we have obviously for every positive number α

$$P(\alpha a) = \alpha P(a).$$

For every two positive elements $a, b \in R$ we have

$$P(a+b) = \sup_{0 \leq x \leq a+b} \tilde{a}(x) \geq \sup_{0 \leq x \leq a, 0 \leq y \leq b} \tilde{a}(x+y),$$

since $0 \leq x \leq a, 0 \leq y \leq b$ implies $0 \leq x+y \leq a+b$. On the other hand, if $0 \leq z \leq a+b$, then putting

$$x = a \wedge z, \quad y = z - x,$$

we obtain $0 \leq x \leq a, z = x + y$, and

$$\begin{aligned} 0 \leq y &= z - x = z - (a \wedge z) \\ &= (z - a) \vee 0 \leq b \vee 0 = b. \end{aligned}$$

Therefore we conclude for every positive elements $a, b \in R$

$$\begin{aligned} P(a+b) &= \sup_{0 \leq x \leq a, 0 \leq y \leq b} \tilde{\alpha}(x+y) \\ &= \sup_{0 \leq x \leq a} \tilde{\alpha}(x) + \sup_{0 \leq y \leq b} \tilde{\alpha}(y) = P(a) + P(b). \end{aligned}$$

For an arbitrary element $a \in R$, putting

$$P(a) = P(a^+) - P(a^-),$$

we obtain by Theorem 46.1 a linear functional P on R . This linear functional P on R satisfies obviously

$$P \geq \tilde{\alpha} \quad \text{and} \quad P \geq 0.$$

For a linear functional L on R , if $L \geq \tilde{\alpha}$ and $L \geq 0$, then we have obviously

$$L(a) \geq L(x) \geq \tilde{\alpha}(x) \quad \text{for } a \geq x \geq 0,$$

and hence for every positive element $a \in R$

$$L(a) \geq \sup_{0 \leq x \leq a} \tilde{\alpha}(x) = P(a),$$

that is, $L \geq P$. Consequently we have by definition

$$P = \tilde{\alpha} \vee 0 = \tilde{\alpha}^+ \in \tilde{R}.$$

Therefore the associated space $\tilde{R}$ of R is lattice ordered, and we have for every element $\tilde{\alpha} \in \tilde{R}$

$$(1) \quad \tilde{\alpha}^+(a) = \sup_{0 \leq x \leq a} \tilde{\alpha}(x) \quad \text{for every positive } a \in R.$$

From this relation we conclude by definition

$$\tilde{\alpha}^-(a) = (-\tilde{\alpha})^+(a) = \sup_{0 \leq x \leq a} \tilde{\alpha}(-x),$$

and hence

$$|\tilde{\alpha}|(a) = \sup_{0 \leq x \leq a, 0 \leq y \leq a} \tilde{\alpha}(x-y).$$

On the other hand, $0 \leq x \leq a, 0 \leq y \leq a$ implies

$$|x-y| = (x-y) \vee (y-x) \leq a,$$

and $|z| \leq a$ implies

$$0 \leq z^+ \leq a, \quad 0 \leq z^- \leq a, \quad z = z^+ - z^-.$$

Therefore we have

$$(2) \quad |\tilde{\alpha}|(a) = \sup_{|x| \leq a} \tilde{\alpha}(x) \quad \text{for every positive } a \in R.$$

Since $\tilde{\alpha} \vee \tilde{\beta} = (\tilde{\alpha} - \tilde{\beta})^+ + \tilde{\beta}$ by Theorem 3.12, we obtain by

the formula (1)

$$\tilde{\alpha} \vee \tilde{\beta}(a) = \sup_{0 \leq x \leq a} \{ \tilde{\alpha}(x) - \tilde{\beta}(x) \} + \tilde{\beta}(a)$$

$$= \sup_{0 \leq x \leq a} \{ \tilde{a}(x) + \tilde{f}(a-x) \}$$

that is,

$$(3) \quad \tilde{a} \vee \tilde{f}(a) = \sup_{a=x+y, x, y \geq 0} \{ \tilde{a}(x) + \tilde{f}(y) \} \quad \text{for } 0 \leq a \in R.$$

Since $\tilde{a} \wedge \tilde{f} = - \{ (-\tilde{a}) \vee (-\tilde{f}) \}$, we conclude from this relation

$$(4) \quad \tilde{a} \wedge \tilde{f}(a) = \inf_{a=x+y, x, y \geq 0} \{ \tilde{a}(x) + \tilde{f}(y) \} \quad \text{for } 0 \leq a \in R.$$

A system of elements $\tilde{a}_\lambda \in \tilde{R}$ ($\lambda \in \Lambda$) is said to be increasing and denoted by $\tilde{a}_\lambda \uparrow_{\lambda \in \Lambda}$, if to every two elements λ_1 and $\lambda_2 \in \Lambda$ there exists an element $\lambda_0 \in \Lambda$ for which

$$\tilde{a}_{\lambda_1} \vee \tilde{a}_{\lambda_2} \leq \tilde{a}_{\lambda_0}.$$

If $\tilde{a}_\lambda \uparrow_{\lambda \in \Lambda}$ and $\tilde{a} = \bigcup_{\lambda \in \Lambda} \tilde{a}_\lambda$, then we shall write

$$\tilde{a}_\lambda \uparrow_{\lambda \in \Lambda} \tilde{a}.$$

Similarly, a system of elements $\tilde{a}_\lambda \in \tilde{R}$ ($\lambda \in \Lambda$) is said to be decreasing and denoted by $\tilde{a}_\lambda \downarrow_{\lambda \in \Lambda}$, if to every two elements

λ_1 and $\lambda_2 \in \Lambda$ there exists an element $\lambda_0 \in \Lambda$ for which

$$\tilde{a}_{\lambda_1} \wedge \tilde{a}_{\lambda_2} \geq \tilde{a}_{\lambda_0}.$$

If $\tilde{a}_\lambda \downarrow_{\lambda \in \Lambda}$ and $\tilde{a} = \bigcap_{\lambda \in \Lambda} \tilde{a}_\lambda$, then we shall write

$$\tilde{a}_\lambda \downarrow_{\lambda \in \Lambda} \tilde{a}.$$

Theorem 46.2. If $\tilde{R} \ni \tilde{a}_\lambda \uparrow_{\lambda \in \Lambda}$ and

$$\sup_{\lambda \in \Lambda} \tilde{a}_\lambda(a) < +\infty \quad \text{for every positive } a \in R,$$

then there exists an element $\tilde{a} \in \tilde{R}$ for which $\tilde{a}_\lambda \uparrow_{\lambda \in \Lambda} \tilde{a}$ and

$$\sup_{\lambda \in \Lambda} \tilde{a}_\lambda(a) = \tilde{a}(a) \quad \text{for every positive } a \in R.$$

Proof. If we put

$$L(a) = \sup_{\lambda \in \Lambda} \tilde{a}_\lambda(a) \quad \text{for every positive } a \in R,$$

then we have obviously for every positive element $a \in R$ and for every positive number α

$$L(\alpha a) = \sup_{\lambda \in \Lambda} \tilde{a}_\lambda(\alpha a) = \alpha \sup_{\lambda \in \Lambda} \tilde{a}_\lambda(a) = \alpha L(a),$$

and for every two positive elements a and $b \in R$

$$\begin{aligned} L(a+b) &= \sup_{\lambda \in \Lambda} \{ \tilde{a}_\lambda(a) + \tilde{a}_\lambda(b) \} \\ &= \sup_{\lambda, \rho \in \Lambda} \{ \tilde{a}_\lambda(a) + \tilde{a}_\rho(b) \} \\ &= \sup_{\lambda \in \Lambda} \tilde{a}_\lambda(a) + \sup_{\rho \in \Lambda} \tilde{a}_\rho(b) = L(a) + L(b), \end{aligned}$$

since to every two elements φ and $\lambda \in \Lambda$ there exists an element $\sigma \in \Lambda$ for which

$$\tilde{\alpha}_\lambda(a) + \tilde{\alpha}_\varphi(b) \leq \tilde{\alpha}_\sigma(a) + \tilde{\alpha}_\sigma(b).$$

Therefore, putting for an arbitrary element $a \in R$

$$L(a) = L(a^+) - L(a^-),$$

we obtain by Theorem 46.1 a linear functional L on R . For such L we have naturally

$$L(a) = \sup_{\lambda \in \Lambda} \tilde{\alpha}_\lambda(a) \quad \text{for every positive } a \in R.$$

Consequently we have $(L - \tilde{\alpha}_\lambda)(a) \geq 0$ for every positive element $a \in R$ and hence $L - \tilde{\alpha}_\lambda$ is a bounded linear functional on R by definition. Since $\tilde{\alpha}_\lambda$ is naturally bounded, L is a bounded linear functional on R , that is, $L \in \tilde{R}$.

Furthermore, if $P \geq \alpha_\lambda$ for every $\lambda \in \Lambda$ for a linear functional P on R , then we have for every positive element $a \in R$

$$P(a) \geq \sup_{\lambda \in \Lambda} \tilde{\alpha}_\lambda(a) = L(a)$$

Therefore we conclude $L = \bigcup_{\lambda \in \Lambda} \tilde{\alpha}_\lambda$ by definition.

For every system of positive elements $\tilde{\alpha}_\lambda \in \tilde{R}$ ($\lambda \in \Lambda$), the system of all elements

$$(-\tilde{\alpha}_{\lambda_1}) \vee (-\alpha_{\lambda_2}) \vee \dots \vee (-\tilde{\alpha}_{\lambda_\nu})$$

for every finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_\nu \in \Lambda$ ($\nu = 1, 2, \dots$)

is obviously increasing and composed only of negative elements.

Consequently there exists by Theorem 46.2 an element $\tilde{\alpha} \in \tilde{R}$ for which

$$\tilde{\alpha} = \bigcup_{\lambda \in \Lambda} (-\tilde{\alpha}_\lambda), \text{ namely } -\tilde{\alpha} = \bigcap_{\lambda \in \Lambda} \tilde{\alpha}_\lambda.$$

Therefore we have:

Theorem 46.3. The associated space $\tilde{R}$ of R is a universally continuous semi-ordered linear space.

As an immediate consequence from Theorem 46.2 we obtain:

$$(5) \quad \tilde{\alpha}_\lambda \uparrow_{\lambda \in \Lambda} \tilde{\alpha} \text{ implies } \tilde{\alpha}(a) = \sup_{\lambda \in \Lambda} \tilde{\alpha}_\lambda(a) \text{ for } 0 \leq a \in R,$$

$$(6) \quad \tilde{\alpha}_\lambda \downarrow_{\lambda \in \Lambda} \tilde{\alpha} \text{ implies } \tilde{\alpha}(a) = \inf_{\lambda \in \Lambda} \tilde{\alpha}_\lambda(a) \text{ for } 0 \leq a \in R.$$

Theorem 46.4. When R is a continuous semi-ordered linear

space, for a system of elements $\tilde{\alpha}_\lambda \in \tilde{R}$ ($\lambda \in \Lambda$), if

$$\sup_{\lambda \in \Lambda} |\tilde{\alpha}_\lambda(a)| < +\infty \quad \text{for every positive } a \in R,$$

then we have

$$\sup_{\lambda \in \Lambda} |\tilde{\alpha}_\lambda|(a) < +\infty \quad \text{for every positive } a \in R,$$

that is,

$$\sup_{\lambda \in \Lambda} \left\{ \sup_{|x| \leq a} |\tilde{\alpha}_\lambda(x)| \right\} < +\infty \quad \text{for every positive } a \in R.$$

Proof. If for some positive element $a \in R$ we have

$$\sup_{\lambda \in \Lambda} \alpha_\lambda = +\infty, \quad \alpha_\lambda = \left\{ \sup_{|x| \leq a} |\tilde{\alpha}_\lambda(x)| \right\}^{\frac{1}{2}},$$

then we can determine $\lambda_\nu \in \Lambda$ and $x_\nu \in R$ ($\nu = 0, 1, 2, \dots$) consecutively such that

$$\begin{aligned} \alpha_{\lambda_\nu} &\geq \delta \alpha_{\lambda_{\nu-1}} \geq 1, \\ \alpha_{\lambda_\nu} &\geq \sup_{\lambda \in \Lambda} |\tilde{\alpha}_\lambda(\sum_{\mu=1}^{\nu-1} \frac{1}{\alpha_{\lambda_{\mu-1}}} x_\mu)|, \end{aligned}$$

and

$$|\tilde{\alpha}_{\lambda_\nu}(x_\nu)| \geq \frac{1}{2} \alpha_{\lambda_\nu}^2, \quad |x_\nu| \leq a.$$

Then, since $|\frac{1}{\alpha_{\lambda_{\nu-1}}} x_\nu| \leq \frac{1}{2} a$, there exists by Theorem 6.6 an element $x \in R$ such that

$$x = \sum_{\nu=1}^{\infty} \frac{1}{\alpha_{\lambda_{\nu-1}}} x_\nu,$$

and we have for every $\nu = 1, 2, \dots$

$$\begin{aligned} x &= \sum_{\mu=1}^{\nu-1} \frac{1}{\alpha_{\lambda_{\mu-1}}} x_\mu + \frac{1}{\alpha_{\lambda_{\nu-1}}} x_\nu + \sum_{\mu=\nu}^{\infty} \frac{1}{\alpha_{\lambda_\mu}} x_{\mu+1}, \\ \left| \sum_{\mu=\nu}^{\infty} \frac{1}{\alpha_{\lambda_\mu}} x_{\mu+1} \right| &\leq \frac{1}{\alpha_{\lambda_\nu}} \sum_{\mu=\nu}^{\infty} \frac{1}{2^{\mu-\nu}} a \leq \frac{2}{\alpha_{\lambda_\nu}} a. \end{aligned}$$

Therefore we obtain for every $\nu = 1, 2, \dots$

$$\begin{aligned} |\tilde{\alpha}_{\lambda_\nu}(x)| &\geq |\tilde{\alpha}_{\lambda_\nu}(\frac{1}{\alpha_{\lambda_{\nu-1}}} x_\nu)| \\ &- |\tilde{\alpha}_{\lambda_\nu}(\sum_{\mu=1}^{\nu-1} \frac{1}{\alpha_{\lambda_{\mu-1}}} x_\mu)| - |\tilde{\alpha}_{\lambda_\nu}(\sum_{\mu=\nu}^{\infty} \frac{1}{\alpha_{\lambda_\mu}} x_{\mu+1})| \\ &\geq \frac{1}{\alpha_{\lambda_{\nu-1}}} (\frac{1}{2} \alpha_{\lambda_\nu}^2) - \alpha_{\lambda_\nu} - \frac{2}{\alpha_{\lambda_\nu}} \alpha_{\lambda_\nu}^2 \\ &\geq \frac{\delta}{\alpha_{\lambda_\nu}} (\frac{1}{2} \alpha_{\lambda_\nu}^2) - 3 \alpha_{\lambda_\nu} = \alpha_{\lambda_\nu} \geq \delta^\nu, \end{aligned}$$

contradicting the assumption: $\sup_{\lambda \in \Lambda} |\tilde{\alpha}_\lambda(x)| < +\infty$.

As an immediate consequence from Theorem 46.4 we have:

Theorem 46.5. when R is continuous, for a sequence of
elements $\tilde{\alpha}_\nu \in \tilde{R}$ ($\nu = 1, 2, \dots$), if $\tilde{\alpha}_\nu(x)$ ($\nu = 1, 2, \dots$) is
convergent for every element $x \in R$, then there exists an element
 $\tilde{\alpha} \in \tilde{R}$ for which

$$\tilde{a}(x) = \lim_{\nu \rightarrow \infty} \tilde{a}_\nu(x)$$

for every element $x \in R$.

Theorem 46.6. When R is continuous, for an orthogonal
sequence of elements $\tilde{a}_\nu \in \tilde{R}$ ($\nu = 1, 2, \dots$), if the series

$$\sum_{\nu=1}^{\infty} \tilde{a}_\nu(x)$$

is convergent for every positive element $x \in R$, then the series

$$\sum_{\nu=1}^{\infty} \tilde{a}_\nu$$

is convergent in $\tilde{R}$.

Proof. Since by assumption

$$\sup_{\sigma=1,2,\dots} \left| \sum_{\nu=1}^{\sigma} \tilde{a}_\nu(x) \right| < +\infty \quad \text{for every positive } x \in R,$$

we obtain by Theorem 46.6

$$\sup_{\sigma=1,2,\dots} \left| \sum_{\nu=1}^{\sigma} \tilde{a}_\nu \right|(x) < +\infty \quad \text{for every positive } x \in R.$$

On the other hand, we have by Theorem 4.5

$$\left| \sum_{\nu=1}^{\sigma} \tilde{a}_\nu \right| = \sum_{\nu=1}^{\sigma} |\tilde{a}_\nu|.$$

Consequently $\sum_{\nu=1}^{\infty} |\tilde{a}_\nu|(x)$ is convergent for every positive element $x \in R$. Then we recognize by Theorem 46.2 that the series

$$\sum_{\nu=1}^{\infty} |\tilde{a}_\nu|$$

is convergent in $\tilde{R}$, and hence $\sum_{\nu=1}^{\infty} \tilde{a}_\nu$ is convergent by Theorem 6.6.

Let R be a continuous semi-ordered linear space in the sequel. Let $[p]$ be an arbitrary projector in R . For every element $\tilde{a} \in \tilde{R}$ we define a linear functional $\tilde{a}[p]$ on R as

$$\tilde{a}[p](x) = \tilde{a}([p]x) \quad (x \in R).$$

With this definition we have obviously that $\tilde{a}[p] \in \tilde{R}$ for every element $\tilde{a} \in \tilde{R}$. By virtue of the formula (1) we conclude for every positive element $a \in R$

$$(\tilde{a}[p])^+(a) = \sup_{0 \leq x \leq a} \tilde{a}([p]x) = \sup_{0 \leq x \leq [p]a} \tilde{a}(x) = \tilde{a}^+[p](a),$$

and hence we obtain

$$(7) \quad (\tilde{a}[p])^+ = \tilde{a}^+[p], \quad (\tilde{a}[p])^- = \tilde{a}^-[p],$$

because

$$(\tilde{a}[p])^- = (-\tilde{a}[p])^+ = (-\tilde{a})^+[p] = \tilde{a}^-[p].$$

Consequently we have by definition

$$(8) \quad |\tilde{\alpha}[p]| = |\tilde{\alpha}|[p].$$

Recalling Theorem 3.12 we obtain by the formula (7)

$$(9) \quad (\tilde{\alpha} \vee \tilde{\ell})[p] = \tilde{\alpha}[p] \vee \tilde{\ell}[p], \quad (\tilde{\alpha} \wedge \tilde{\ell})[p] = \tilde{\alpha}[p] \wedge \tilde{\ell}[p].$$

For positive elements $\tilde{\alpha}, \tilde{\ell} \in \tilde{\mathcal{R}}$, we see easily by the formula (4) that

$$\tilde{\alpha}[p] \wedge (\tilde{\ell} - \tilde{\ell}[p]) = 0,$$

and hence we conclude by Theorems 4.5 and 3.3

$$\tilde{\alpha}[p] \wedge \tilde{\ell} = \tilde{\alpha}[p] \wedge \{\tilde{\ell}[p] \vee (\tilde{\ell} - \tilde{\ell}[p])\} = (\tilde{\alpha}[p] \wedge \tilde{\ell}[p]) \vee 0.$$

Therefore we obtain by the formula (9)

$$(10) \quad \tilde{\alpha}[p] \wedge \tilde{\ell} = \tilde{\alpha} \wedge \tilde{\ell}[p] = (\tilde{\alpha} \wedge \tilde{\ell})[p] \quad \text{for } 0 \leq \tilde{\alpha}, \tilde{\ell} \in \tilde{\mathcal{R}}.$$

From the formulas (5) and (6) we conclude immediately:

$$(11) \quad \begin{aligned} \tilde{\alpha}_\lambda \uparrow_{\lambda \in \Lambda} \tilde{\alpha} & \text{ implies } \tilde{\alpha}_\lambda[p] \uparrow_{\lambda \in \Lambda} \tilde{\alpha}[p]; \\ \tilde{\alpha}_\lambda \downarrow_{\lambda \in \Lambda} \tilde{\alpha} & \text{ implies } \tilde{\alpha}_\lambda[p] \downarrow_{\lambda \in \Lambda} \tilde{\alpha}[p]. \end{aligned}$$

For every positive element $\tilde{\ell} \in \tilde{\mathcal{R}}$, since by definition

$$\begin{aligned} \tilde{\ell} \wedge \vee |\tilde{\alpha}| \uparrow_{\lambda \in \Lambda}^\infty [\tilde{\alpha}] \tilde{\ell}, \\ \tilde{\ell}[p] \wedge \vee |\tilde{\alpha}| \uparrow_{\lambda \in \Lambda}^\infty [\tilde{\alpha}][\tilde{\ell}[p]], \\ \tilde{\ell} \wedge \vee |\tilde{\alpha}[p]| \uparrow_{\lambda \in \Lambda}^\infty [\tilde{\alpha}[p]] \tilde{\ell}, \end{aligned}$$

we conclude by the formulas (10) and (11) that we have

$$(12) \quad ([\tilde{\alpha}]\tilde{\ell})[p] = [\tilde{\alpha}](\tilde{\ell}[p]) = [\tilde{\alpha}[p]]\tilde{\ell}$$

for every element $\tilde{\alpha} \in \tilde{\mathcal{R}}$, and hence this relation remains valid for an arbitrary element $\tilde{\ell} \in \tilde{\mathcal{R}}$ by the formula (7).

§47 Continuous linear functionals

Let $\mathcal{R}$ be a continuous semi-ordered linear space in the sequel. A linear functional L on $\mathcal{R}$ is said to be continuous, if

$$R \ni a_\nu \downarrow_{\nu \in \Lambda}^\infty 0 \text{ implies } \lim_{\nu \rightarrow \infty} L(a_\nu) = 0.$$

With this definition we see at once that if a linear functional L on $\mathcal{R}$ is continuous, then $R \ni a_\nu \uparrow_{\nu \in \Lambda}^\infty a$ or $R \ni a_\nu \downarrow_{\nu \in \Lambda}^\infty a$ implies

$$\lim_{\nu \rightarrow \infty} L(a_\nu) = L(a).$$

Theorem 47.1. If a linear functional L on R is continuous, then L is bounded, that is, $L \in \widetilde{R}$.

Proof. If there is a positive element $a \in R$ for which

$$\sup_{0 \leq x \leq a} |L(x)| = +\infty,$$

then we can determine $x_\nu \in R$ ($\nu = 1, 2, \dots$) consecutively such that

$$|L(x_\nu)| \geq 2^{\nu+1} \left| L\left(\sum_{\mu=1}^{\nu-1} \frac{1}{2^\mu} x_\mu\right) \right|,$$

$$|L(x_\nu)| \geq \nu 2^{\nu+1}, \quad 0 \leq x_\nu \leq a.$$

Then there exists by Theorem 6.2 a positive element $x \in R$ for which

$$x = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} x_\nu \leq \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} a = a,$$

and we have for every $\nu = 1, 2, \dots$

$$\begin{aligned} \left| L\left(\sum_{\mu=1}^{\nu} \frac{1}{2^\mu} x_\mu\right) \right| &\geq \left| L\left(\frac{1}{2^\nu} x_\nu\right) \right| - \left| L\left(\sum_{\mu=1}^{\nu-1} \frac{1}{2^\mu} x_\mu\right) \right| \\ &\geq \frac{1}{2^{\nu+1}} |L(x_\nu)| \geq \nu, \end{aligned}$$

contradicting the assumption: $\lim_{\nu \rightarrow \infty} L\left(\sum_{\mu=1}^{\nu} \frac{1}{2^\mu} x_\mu\right) = L(x)$.

Theorem 47.2. For a positive element $\tilde{a} \in \widetilde{R}$, if

$[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ implies

$$\lim_{\nu \rightarrow \infty} \tilde{a}([p_\nu]a) = 0 \quad \text{for every positive } a \in R,$$

then $\tilde{a}$ is a continuous linear functional on R .

Proof. For a sequence of elements $R \ni a_\nu \downarrow_{\nu=1}^{\infty} 0$ and for a positive number ε , putting

$$p_\nu = (\varepsilon a_1 - a_\nu)^+ \quad (\nu = 1, 2, \dots),$$

we obtain $[p_\nu] \uparrow_{\nu=1}^{\infty} [a_1]$ by Theorems 8.5 and 7.15, as $p_\nu \uparrow_{\nu=1}^{\infty} \varepsilon a_1$.

On the other hand we have by Theorem 7.16 for every $\nu = 1, 2, \dots$

$$[p_\nu](\varepsilon a_1 - a_\nu) \geq 0,$$

that is, $[p_\nu]a_\nu \leq \varepsilon [p_\nu]a_1 \leq \varepsilon a_1$, and hence we conclude for every $\nu = 1, 2, \dots$

$$\begin{aligned} \tilde{a}(a_\nu) &= \tilde{a}([p_\nu]a_\nu) + \tilde{a}([a_1] - [p_\nu])a_\nu \\ &\leq \varepsilon \tilde{a}(a_1) + \tilde{a}([a_1] - [p_\nu])a_1. \end{aligned}$$

Since $[a_1] - [p_\nu] \downarrow_{\nu=1}^{\infty} 0$, we obtain by assumption

$$\lim_{\nu \rightarrow \infty} \tilde{\alpha}([a_\nu] - [p_\nu]a_1) = 0.$$

Consequently we conclude

$$\overline{\lim}_{\nu \rightarrow \infty} \tilde{\alpha}(a_\nu) \leq \varepsilon \tilde{\alpha}(a_1).$$

As a positive number ε may be arbitrary, we obtain hence

$$\lim_{\nu \rightarrow \infty} \tilde{\alpha}(a_\nu) = 0.$$

Theorem 47.3. If a positive element $\tilde{a} \in \tilde{\mathcal{R}}$ is a continuous linear functional on $\mathcal{R}$, then $\tilde{a}(p) = 0$ for a positive element $p \in \mathcal{R}$ implies $\tilde{a}[p] = 0$ in $\tilde{\mathcal{R}}$

Proof. If $\tilde{a}(p) = 0$ for a positive element $p \in \mathcal{R}$, then we have for every positive element $a \in \mathcal{R}$

$$\tilde{a}(a \wedge \nu p) = 0 \quad \text{for every } \nu = 1, 2, \dots,$$

since $0 \leq \tilde{a}(a \wedge \nu p) \leq \nu \tilde{a}(p) = 0$. As $\tilde{a}$ is continuous by assumption, we obtain hence

$$\tilde{a}([p]a) = 0 \quad \text{for every positive } a \in \mathcal{R},$$

since $a \wedge \nu p \uparrow_{\nu=1}^{\infty} [p]a$.

Theorem 47.4. If an element $\tilde{a} \in \tilde{\mathcal{R}}$ is continuous as a linear functional on $\mathcal{R}$, then to every positive element $a \in \mathcal{R}$ there exist projectors $[p]$ and $[q]$ such that

$$\tilde{a}^+[a] = \tilde{a}[p], \quad \tilde{a}^-[a] = -\tilde{a}[q],$$

$$[a] = [p] + [q], \quad [p][q] = 0.$$

Proof. If we put

$$\alpha = \sup_{p \in \mathcal{R}} \tilde{a}([p]a),$$

then we have $0 \leq \alpha < +\infty$ by the formula §46(1). If we have

$$\tilde{a}([p]a) > \alpha - \varepsilon, \quad \tilde{a}([q]a) > \alpha - \delta$$

for two projectors $[p]$, $[q]$ and for two positive numbers ε , δ , then, since by assumption

$$\tilde{a}([p][q]a) \leq \alpha,$$

we obtain by the formulas §8(3) and §8(6)

$$\begin{aligned} \tilde{a}([(p] \vee [q])a) &= \tilde{a}([(p] + [q] - [p][q])a]) \\ &> (\alpha - \varepsilon) + (\alpha - \delta) - \alpha = \alpha - (\varepsilon + \delta). \end{aligned}$$

There exists by assumption a sequence of projectors $[p_\nu] \leq [a]$

($\nu = 1, 2, \dots$) such that

$$\tilde{\alpha}([p_\nu]a) > \alpha - \frac{1}{2^\nu} \quad \text{for every } \nu = 1, 2, \dots$$

For such $[p_\nu]$ ($\nu = 1, 2, \dots$), putting

$$q_\nu = \bigcup_{\mu=\nu}^{\infty} [p_\mu]a \quad (\nu = 1, 2, \dots),$$

we have $[a] \supseteq [q_\nu] \downarrow_{\nu=1}^{\infty}$ by Theorem 8.3. Since $\tilde{\alpha}$ is continuous by assumption, and we have

$$([p_\nu] \cup [p_{\nu+1}] \cup \dots \cup [p_{\nu+\mu}])a \uparrow_{\mu=1}^{\infty} [q_\nu]a,$$

by Theorem 8.5, we obtain naturally

$$\tilde{\alpha}([q_\nu]a) = \lim_{\mu \rightarrow \infty} \tilde{\alpha}([p_\nu] \cup [p_{\nu+1}] \cup \dots \cup [p_{\nu+\mu}]a).$$

On the other hand, we have for every $\mu = 1, 2, \dots$

$$\tilde{\alpha}([p_\nu] \cup [p_{\nu+1}] \cup \dots \cup [p_{\nu+\mu}]a) > \alpha - \left(\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}} + \dots + \frac{1}{2^{\nu+\mu}} \right),$$

as established just above. Consequently we obtain

$$\tilde{\alpha}([q_\nu]a) \geq \alpha - \left(\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}} + \dots \right) = \alpha - \frac{1}{2^{\nu-1}}.$$

By virtue of Theorem 8.6, there exists a projector $[p_0]$ for which

$$[q_\nu] \downarrow_{\nu=1}^{\infty}, [p_0],$$

and then, as $[q_\nu]a \downarrow_{\nu=1}^{\infty}, [p_0]a$, we obtain hence

$$\tilde{\alpha}([p_0]a) \geq \alpha,$$

since $\tilde{\alpha}$ is continuous by assumption. On the other hand we have naturally by assumption

$$\tilde{\alpha}([p_0]a) \leq \alpha.$$

Consequently we obtain

$$\tilde{\alpha}([p_0]a) = \alpha.$$

Then we have for every projector $[p]$

$$\tilde{\alpha}([p_0])([p]a) = \alpha - \tilde{\alpha}([p_0](1-[p])a) \geq 0.$$

For every positive element $b \in R$, since we have by Theorems 22.2 and 18.8

$$[a]b = \int_{\alpha,1} \left(\frac{t}{a}, t \right) dt a, \\ \left(\frac{t}{a}, t \right) \geq 0 \quad \text{for every point } t \in U_{\alpha,1},$$

we conclude hence by Theorems 20.3 and 21.9

$$\tilde{\alpha}([p_0])(b) = \tilde{\alpha}([p_0])([a]b) \geq 0.$$

Therefore we obtain $\tilde{\alpha}[p_0] \geq 0$ in $\tilde{\mathcal{R}}$.

Putting $[q_0] = [a] - [p_0]$, we have by assumption for every projector $[p]$

$$-\tilde{\alpha}[q_0]([p]a) = \alpha - \tilde{\alpha}([p_0] + [q_0])[p]a \geq 0,$$

and hence we can conclude likewise that $-\tilde{\alpha}[q_0] \geq 0$. Furthermore we have by the formula §46(10)

$$\tilde{\alpha}[p_0] \wedge (-\tilde{\alpha}[q_0]) = \{\tilde{\alpha}[p_0] \wedge (-\tilde{\alpha}[q_0])\}[p_0][q_0] = 0.$$

Since $\tilde{\alpha}[a] = \tilde{\alpha}[p_0] - (-\tilde{\alpha}[q_0])$, we obtain therefore by Theorem 3.8 and by the formula §46(7)

$$\begin{aligned}\tilde{\alpha}[p_0] &= (\tilde{\alpha}[a])^+ = \tilde{\alpha}^+[a], \\ -\tilde{\alpha}[q_0] &= (\tilde{\alpha}[a])^- = \tilde{\alpha}^-[a].\end{aligned}$$

Theorem 47.5. If a linear functional $\tilde{\alpha} \in \tilde{\mathcal{R}}$ is continuous, then all $\tilde{\alpha}^+$, $\tilde{\alpha}^-$, and $|\tilde{\alpha}|$ are continuous. If a linear functional $\tilde{\alpha} \in \tilde{\mathcal{R}}$ is continuous and $|\tilde{\alpha}| \geq |\tilde{\ell}|$, $\tilde{\ell} \in \tilde{\mathcal{R}}$, then $\tilde{\ell}$ is continuous too.

Proof. If $\tilde{\alpha} \in \tilde{\mathcal{R}}$ is continuous, then to every sequence of elements $R \ni a_\nu \downarrow_{\nu=1}^\infty 0$ there exists by the previous theorem a projector $[p] \leq [a_1]$ such that

$$\tilde{\alpha}^+[a_1] = \tilde{\alpha}[p],$$

and then, since $[p]a_\nu \downarrow_{\nu=1}^\infty 0$ by Theorem 7.7, we obtain

$$\lim_{\nu \rightarrow \infty} \tilde{\alpha}^+(a_\nu) = \lim_{\nu \rightarrow \infty} \tilde{\alpha}([p]a_\nu) = 0.$$

Therefore $\tilde{\alpha}^+$ is continuous, if $\tilde{\alpha}$ is continuous. Furthermore, if $\tilde{\alpha}$ is continuous, then $\tilde{\alpha}^-$ and $|\tilde{\alpha}|$ also are continuous, since

$$\tilde{\alpha}^- = (-\tilde{\alpha})^+, \quad |\tilde{\alpha}| = \tilde{\alpha}^+ + \tilde{\alpha}^-.$$

If $\tilde{\alpha} \in \tilde{\mathcal{R}}$ is continuous, and $|\tilde{\alpha}| \geq |\tilde{\ell}|$, $\tilde{\ell} \in \tilde{\mathcal{R}}$, then

$$R \ni a_\nu \downarrow_{\nu=1}^\infty 0$$

implies $\lim_{\nu \rightarrow \infty} \tilde{\ell}(a_\nu) = 0$, because

$$\begin{aligned}|\tilde{\ell}(a_\nu)| &= |\tilde{\ell}^+(a_\nu) - \tilde{\ell}^-(a_\nu)| \\ &\leq |\tilde{\ell}^+(a_\nu)| \leq |\tilde{\alpha}|(a_\nu).\end{aligned}$$

Theorem 47.6. For a system of continuous linear functionals
 $\tilde{\alpha}_\lambda \in \tilde{K}$ ($\lambda \in A$), if $\tilde{\alpha} \uparrow_{\lambda \in A} \tilde{\alpha}_\lambda$ or $\tilde{\alpha}_\lambda \downarrow_{\lambda \in A} \tilde{\alpha}$, then $\tilde{\alpha}$ is
continuous too.

Proof. If $\tilde{\alpha}_\lambda \uparrow_{\lambda \in A} \tilde{\alpha}$, then for every sequence of elements
 $R \ni a_\nu \downarrow_{\nu=\infty} 0$,
 to any positive number ε there exists an element $\lambda_0 \in A$ by the
 formula §46(5) such that

$$\tilde{\alpha}(a_1) \leq \tilde{\alpha}_{\lambda_0}(a_1) + \varepsilon$$

As $\tilde{\alpha} - \tilde{\alpha}_{\lambda_0} \geq 0$ by assumption, we have then

$$\begin{aligned} |\tilde{\alpha}(a_\nu)| &\leq |\tilde{\alpha}_{\lambda_0}(a_\nu)| + (\tilde{\alpha} - \tilde{\alpha}_{\lambda_0})(a_\nu) \\ &\leq |\tilde{\alpha}_{\lambda_0}(a_\nu)| + (\tilde{\alpha} - \tilde{\alpha}_{\lambda_0})(a_1) \leq |\tilde{\alpha}_{\lambda_0}(a_\nu)| + \varepsilon, \end{aligned}$$

and hence $\lim_{\nu \rightarrow \infty} |\tilde{\alpha}(a_\nu)| \leq \varepsilon$, since $\tilde{\alpha}_{\lambda_0}$ is continuous by assumption.
 Here a positive number ε may be arbitrary. Consequently we obtain

$$\lim_{\nu \rightarrow \infty} |\tilde{\alpha}(a_\nu)| = 0.$$

Therefore $\tilde{\alpha}$ is continuous by definition. We also can dispose
 likewise of the case: $\tilde{\alpha}_\lambda \downarrow_{\lambda \in A} \tilde{\alpha}$.

Theorem 47.7. If a linear functional $\tilde{\alpha} \in \tilde{K}$ is continuous,
then $\lim_{\nu \rightarrow \infty} x_\nu = x_0$ implies

$$\lim_{\nu \rightarrow \infty} \tilde{\alpha}(x_\nu) = \tilde{\alpha}(x_0).$$

Proof. If $\lim_{\nu \rightarrow \infty} x_\nu = x_0$, then there exist by definition
 a sequence of elements $\ell_\nu \downarrow_{\nu=\infty} 0$ and μ_ν ($\nu = 1, 2, \dots$) such
 that

$$|x_\mu - x| \leq \ell_\nu \quad \text{for } \mu \geq \mu_\nu, \quad \nu = 1, 2, \dots,$$

and then we have for $\mu \geq \mu_\nu$, $\nu = 1, 2, \dots$

$$\begin{aligned} |\tilde{\alpha}(x_\mu) - \tilde{\alpha}(x)| &= |\tilde{\alpha}(x_\mu - x)| \\ &\leq |\tilde{\alpha}^+(x_\mu - x)| + |\tilde{\alpha}^-(x_\mu - x)| \leq \tilde{\alpha}^+(|x_\mu - x|) + \tilde{\alpha}^-(|x_\mu - x|) \leq |\tilde{\alpha}(\ell_\nu)|. \end{aligned}$$

Since $|\tilde{\alpha}|$ is also continuous by Theorem 47.5, we conclude hence
 further

$$\lim_{\nu \rightarrow \infty} \tilde{\alpha}(x_\mu) = \tilde{\alpha}(x).$$

Theorem 47.8. For a sequence of continuous functionals

$\tilde{\alpha}_\nu \in \tilde{\mathcal{R}}$ ($\nu = 1, 2, \dots$), if the sequence of real numbers $\tilde{\alpha}_\nu(x)$ ($\nu = 1, 2, \dots$) is convergent for every element $x \in \mathcal{R}$, then,
for every sequence of elements

$$\mathcal{R} \ni a_\mu \downarrow_{\mu=1}^\infty 0,$$

to every positive number ε there exists μ such that

$$\sup_{0 \leq x \leq a_\mu} |\tilde{\alpha}_\nu(x)| \leq \varepsilon \quad \text{for every } \nu = 1, 2, \dots,$$

and we obtain a continuous linear functional $\tilde{\alpha} \in \tilde{\mathcal{R}}$ as

$$\tilde{\alpha}(x) = \lim_{\nu \rightarrow \infty} \tilde{\alpha}_\nu(x) \quad (x \in \mathcal{R}).$$

Proof. We suppose that there exist

$$\varepsilon > 0, \quad \lambda_\nu \uparrow_{\nu=1}^\infty +\infty, \quad \rho_\nu \uparrow_{\nu=1}^\infty +\infty$$

such that we have for every $\mu = 1, 2, \dots$

$$\lim_{\nu \rightarrow \infty} \left\{ \sup_{0 \leq x \leq a_\mu} |\tilde{\alpha}_{\lambda_\nu}(x) - \tilde{\alpha}_{\rho_\nu}(x)| \right\} > \varepsilon.$$

Then, putting

$$\tilde{\ell}_\nu = \tilde{\alpha}_{\lambda_\nu} - \tilde{\alpha}_{\rho_\nu} \quad (\nu = 1, 2, \dots),$$

we obtain by assumption

$$\lim_{\nu \rightarrow \infty} \tilde{\ell}_\nu(x) = 0 \quad \text{for every element } x \in \mathcal{R}.$$

We can determine natural numbers μ_f , ν_f , and elements $x_f \in \mathcal{R}$ ($f = 1, 2, \dots$) consecutively as follows: being determined until $f-1$, we determine first μ_f such that $\mu_f > \mu_{f-1}$,

$$\sup_{0 \leq x \leq a_{\mu_f}} |\tilde{\ell}_{\nu_\lambda}(x)| < \frac{1}{4} \varepsilon \quad \text{for all } \lambda = 1, 2, \dots, f-1,$$

since all $|\tilde{\ell}_\nu|$ are continuous by Theorem 47.5; next ν_f such that $\nu_f > \nu_{f-1}$,

$$|\tilde{\ell}_{\nu_f}(\sum_{\lambda=1}^{f-1} ((x_\lambda \vee a_{\mu_{\lambda+1}}) - a_{\mu_{\lambda+1}}))| < \frac{1}{4} \varepsilon,$$

and further by assumption such that

$$\sup_{0 \leq x \leq a_{\mu_f}} |\tilde{\ell}_{\nu_f}(x)| > \varepsilon;$$

and finally $x_f \in \mathcal{R}$ such that

$$|\tilde{\ell}_{\nu_f}(x_f)| > \varepsilon, \quad 0 \leq x_f \leq a_{\mu_f}.$$

Then, since $a_{\mu_{f+1}} \leq x_f \vee a_{\mu_{f+1}} \leq a_{\mu_f}$ and $a_{\mu_f} \downarrow_{f=1}^\infty 0$, we obtain by Theorem 6.2 an element $x \in \mathcal{R}$ as

$$x = \sum_{f=1}^\infty ((x_f \vee a_{\mu_{f+1}}) - a_{\mu_{f+1}}).$$

For every $x = 1, 2, \dots$, since by Theorem 3.1

$$(x \vee a_{\mu_{x+1}}) - a_{\mu_{x+1}} = x - (x \wedge a_{\mu_{x+1}}),$$

we have then

$$x = \sum_{p=1}^{x-1} ((x_p \vee a_{\mu_{p+1}}) - a_{\mu_{p+1}}) + x - (x \wedge a_{\mu_{x+1}}) + \sum_{p=x+1}^{\infty} ((x_p \vee a_{\mu_{p+1}}) - a_{\mu_{p+1}}),$$

and furthermore

$$0 \leq \sum_{p=x+1}^{\infty} ((x_p \vee a_{\mu_{p+1}}) - a_{\mu_{p+1}}) \leq x_{x+1} \vee a_{\mu_{x+2}} \leq a_{\mu_{x+1}},$$

$$0 \leq x \wedge a_{\mu_{x+1}} \leq a_{\mu_{x+1}}.$$

Consequently we obtain for every $x = 1, 2, \dots$

$$\begin{aligned} |\tilde{f}_{\nu x}(x)| &\geq -|\tilde{f}_{\nu x}(\sum_{p=1}^{x-1} ((x_p \vee a_{\mu_{p+1}}) - a_{\mu_{p+1}}))| \\ &\quad + |\tilde{f}_{\nu x}(x_x)| - |\tilde{f}_{\nu x}(x \wedge a_{\mu_{x+1}})| - |\tilde{f}_{\nu x}(\sum_{p=x+1}^{\infty} ((x_p \vee a_{\mu_{p+1}}) - a_{\mu_{p+1}}))| \\ &\geq -\frac{\varepsilon}{4} + \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}, \end{aligned}$$

contradicting $\lim_{x \rightarrow \infty} \tilde{f}_{\nu x}(x) = 0$.

Therefore to every positive number ε there exist natural numbers μ_0 and ν_0 such that

$$\sup_{0 \leq x \leq a_{\mu_0}} |\tilde{f}_p(x) - \tilde{a}_\nu(x)| < \frac{1}{2} \varepsilon \quad \text{for } p, \nu \geq \nu_0.$$

On the other hand, since every $|\tilde{a}_\nu|$ is continuous by Theorem 47.5, there exists a natural number μ_1 such that

$$\sup_{0 \leq x \leq a_{\mu_1}} |\tilde{a}_\nu(x)| < \frac{1}{2} \varepsilon \quad \text{for } 1 \leq \nu \leq \nu_0.$$

Putting $\mu = \max\{\mu_0, \mu_1\}$, we have then

$$\sup_{0 \leq x \leq a_\mu} |\tilde{a}_\nu(x)| < \varepsilon \quad \text{for every } \nu = 1, 2, \dots.$$

From this relation we conclude easily that if we put

$$L(x) = \lim_{\nu \rightarrow \infty} \tilde{a}_\nu(x) \quad (x \in R),$$

then L is a continuous linear functional on R .

§48 Characteristic sets

Let R be a continuous semi-ordered linear space and let $\tilde{R}$ be the associated space of R . Corresponding to every element $\tilde{a} \in \tilde{R}$, the closed set

$$(\sum_{\alpha(p)=0} \mathcal{U}_{[p]})'$$

in the proper space E of R is called the characteristic set of $\tilde{\alpha}$ and denoted by $C_{\tilde{\alpha}}$.

With this definition we obtain the following formulas:

$$(1) \quad C_{\tilde{\alpha}} = C_{|\tilde{\alpha}|}.$$

Because we have by the formula §46(8) for every projector $[p]$

$$|\tilde{\alpha}[p]| = |\tilde{\alpha}|[p].$$

$$(2) \quad C_{\tilde{\alpha}} = 0 \quad \text{if and only if} \quad \tilde{\alpha} = 0.$$

Indeed, $\tilde{\alpha} = 0$ implies obviously $C_{\tilde{\alpha}} = 0$. If $C_{\tilde{\alpha}} = 0$, then we have for every element $a \in R$

$$U_{[a]} \subset \sum_{\tilde{\alpha}[p]=0} U_{[p]}.$$

Since $U_{[a]}$ is compact, there exists then a finite number of elements $p_\nu \in R$ ($\nu = 1, 2, \dots, \kappa$) such that

$$U_{[a]} \subset \sum_{\nu=1}^{\kappa} U_{[p_\nu]}, \quad \tilde{\alpha}[p_\nu] = 0 \quad \text{for } \nu = 1, 2, \dots, \kappa.$$

From this relation we conclude by the formulas §15(3), §15(4), and by Theorem 8.2

$$([p_1] \vee [p_2] \vee \dots \vee [p_\kappa])[a] = [a],$$

and further by the formulas §8(2) and §8(6)

$$\tilde{\alpha}([p_1] \vee [p_2] \vee \dots \vee [p_\kappa]) = 0.$$

Consequently we obtain $\tilde{\alpha}[a] = 0$.

$$(3) \quad C_{\tilde{\alpha}} U_{[p]} = 0 \quad \text{if and only if} \quad \tilde{\alpha}[p] = 0.$$

Because, if $C_{\tilde{\alpha}} U_{[p]} = 0$, then we have by definition

$$U_{[p]} \subset \sum_{\tilde{\alpha}[p]=0} U_{[p]},$$

and hence $\tilde{\alpha}[p] = 0$ as proved just now. Conversely, $\tilde{\alpha}[p] = 0$ implies obviously by definition

$$C_{\tilde{\alpha}} U_{[p]} = 0.$$

$$(4) \quad C_{\tilde{\alpha}} \subset U_{[p]} \quad \text{if and only if} \quad \tilde{\alpha}[p] = \tilde{\alpha}.$$

Because $C_{\tilde{\alpha}} \subset U_{[p]}$ is equivalent to

$$C_{\tilde{\alpha}} U_{(1-[p])[a]} = 0 \quad \text{for every element } a \in R,$$

that is, by the formula (3)

$$\tilde{\alpha}(1-[p])[a] = 0 \quad \text{for every element } a \in R.$$

$$(5) \quad C_{\tilde{\alpha}} U_{[p]} = C_{\tilde{\alpha}[p]}.$$

Because $C_{\tilde{\alpha}[p]} U_{[p]} = 0$ is by the formula (3) equivalent to

$$\tilde{\alpha}[p][p] = 0,$$

and hence by repeating use of the formula (3) equivalent to

$$(C_{\tilde{\alpha}} U_{[p]}) U_{[p]} = C_{\tilde{\alpha}} U_{[p][p]} = 0.$$

A linear functional $\tilde{\alpha} \in \tilde{R}$ said to be complete in $[p]R$, if

$$\tilde{\alpha}[a] = 0 \text{ implies } [a][p] = 0.$$

Then we have obviously by the formula (3)

$$(6) \quad C_{\tilde{\alpha}} > U_{[p]} \text{ if and only if } \tilde{\alpha} \text{ is complete in } [p]R.$$

$$(7) \quad C_{\tilde{\alpha}} C_{\tilde{\beta}} = 0 \text{ implies } \tilde{\alpha} \perp \tilde{\beta}.$$

In fact, if $C_{\tilde{\alpha}} C_{\tilde{\beta}} = 0$, then, since both $C_{\tilde{\alpha}}$ and $C_{\tilde{\beta}}$ are closed by definition, to every projector $[a]$ there exist by

Theorem 16.3 two projectors $[p]$ and $[q]$ such that

$$[a] = [p] + [q], \quad [p][q] = 0,$$

$$C_{\tilde{\alpha}} U_{[q]} = 0, \quad C_{\tilde{\beta}} U_{[p]} = 0.$$

Then we have by the formula (3)

$$\tilde{\alpha}[q] = \tilde{\beta}[p] = 0.$$

This relation yields

$$\tilde{\alpha}[a] = \tilde{\alpha}[p], \quad \tilde{\beta}[a] = \tilde{\beta}[q],$$

and hence we obtain by the formulas (8), (9), and (10) in §46

$$\begin{aligned} (|\tilde{\alpha}| \wedge |\tilde{\beta}|)[a] &= |\tilde{\alpha}[a]| \wedge |\tilde{\beta}[a]| \\ &= |\tilde{\alpha}[p]| \wedge |\tilde{\beta}[q]| = (|\tilde{\alpha}| \wedge |\tilde{\beta}|)[p][q] = 0. \end{aligned}$$

Theorem 48.1. If both $\tilde{\alpha}$ and $\tilde{\beta} \in \tilde{R}$ are continuous,

then $\tilde{\alpha} \perp \tilde{\beta}$ is equivalent to

$$C_{\tilde{\alpha}} C_{\tilde{\beta}} = 0.$$

Proof. To every element $a \in R$ there exists by Theorem

47.4 a projector $[p] \leq [a]$ such that

$$(|\tilde{\alpha}| - |\tilde{\beta}|)^+[a] = (|\tilde{\alpha}| - |\tilde{\beta}|)[p].$$

If $|\tilde{\alpha}| \wedge |\tilde{\beta}| = 0$, then we have by Theorem 3.12

$$|\tilde{\alpha}| = (|\tilde{\alpha}| - |\tilde{\beta}|)^+,$$

and hence for such $[p]$

$$|\tilde{\alpha}|[a] = (|\tilde{\alpha}| - |\tilde{\beta}|)[p],$$

that is, $|\tilde{\alpha}|([a] - [p]) = -|\tilde{\ell}|[p]$. Since

$$|\tilde{\alpha}|([a] - [p]) \geq 0, \quad |\tilde{\ell}|[p] \geq 0,$$

we conclude therefore

$$|\tilde{\alpha}|([a] - [p]) = |\tilde{\ell}|[p] = 0.$$

Consequently we obtain by the formula (3)

$$C_{\tilde{\alpha}} C_{\tilde{\ell}} U_{[a]} = C_{\tilde{\alpha}} C_{\tilde{\ell}} (U_{[p]} + U_{[a] - [p]}) = 0.$$

Therefore $|\tilde{\alpha}| \wedge |\tilde{\ell}| = 0$ implies $C_{\tilde{\alpha}} C_{\tilde{\ell}} = 0$, if both $\tilde{\alpha}$ and $\tilde{\ell} \in \tilde{R}$ are continuous.

We have obviously by definition:

$$(8) \quad C_{\tilde{\alpha}} > C_{\tilde{\ell}} \text{ if and only if } \tilde{\alpha}[p] = 0 \text{ implies } \tilde{\ell}[p] = 0.$$

Since the associated space $\tilde{R}$ of R is by Theorem 46.3 a universally continuous semi-ordered linear space, we can consider projectors in $\tilde{R}$.

$$(9) \quad [\tilde{\alpha}] \geq [\tilde{\ell}] \text{ implies } C_{\tilde{\alpha}} > C_{\tilde{\ell}}.$$

In fact, if $[\tilde{\alpha}] \geq [\tilde{\ell}]$, then we have $[\tilde{\alpha}][\tilde{\ell}] = |\tilde{\ell}|$ by Theorem 8.2, and hence by definition

$$|\tilde{\ell}| \wedge \vee |\tilde{\alpha}| \uparrow_{\tilde{R}} = |\tilde{\ell}|.$$

Consequently we can conclude by the formulas (10) and (11) in §46 that $\tilde{\alpha}[p] = 0$ implies $\tilde{\ell}[p] = 0$, and hence $C_{\tilde{\alpha}} > C_{\tilde{\ell}}$ by the formula (8).

Theorem 48.2. If both $\tilde{\alpha}$ and $\tilde{\ell} \in \tilde{R}$ are continuous,
then $[\tilde{\alpha}] \geq [\tilde{\ell}]$ is equivalent to

$$C_{\tilde{\alpha}} > C_{\tilde{\ell}}.$$

Proof. Since $(1 - [\tilde{\alpha}])\tilde{\ell} \perp \tilde{\alpha}$ by Theorems 7.10 and 7.11, we have by the previous theorem

$$C_{(1 - [\tilde{\alpha}])\tilde{\ell}} C_{\tilde{\alpha}} = 0.$$

If $C_{\tilde{\alpha}} > C_{\tilde{\ell}}$, then we have hence

$$C_{(1 - [\tilde{\alpha}])\tilde{\ell}} = 0,$$

since $C_{(1 - [\tilde{\alpha}])\tilde{\ell}} < C_{\tilde{\ell}}$ by the formula (8). Consequently we obtain by the formula (2)

$$[\tilde{\alpha}]\tilde{\ell} = \tilde{\ell},$$

that is, $[\tilde{a}] \geq [\tilde{x}]$ by Theorem 8.2.

By virtue of the formulas (8) and §46(9) we have obviously

$$(10) \quad C_{\tilde{a}} C_{\tilde{x}} \supset C_{|\tilde{a}| \wedge |\tilde{x}|}.$$

Theorem 48.3. If both $\tilde{a}$ and $\tilde{x} \in \tilde{R}$ are continuous,
then we have

$$C_{\tilde{a}} C_{\tilde{x}} = C_{|\tilde{a}| \wedge |\tilde{x}|}.$$

Proof. If $C_{|\tilde{a}| \wedge |\tilde{x}|} \cup_{[p]} = 0$, then we have

$$(|\tilde{a}| \wedge |\tilde{x}|)[p] = 0$$

by the formula (2), and hence by the formula §46(10)

$$|\tilde{a}| \wedge |\tilde{x}[p]| = 0.$$

Therefore we obtain by Theorem 48.1 and by the formula (5) that

$$C_{|\tilde{a}| \wedge |\tilde{x}|} \cup_{[p]} = 0 \text{ implies } C_{\tilde{a}} C_{\tilde{x}} \cup_{[p]} = 0,$$

and hence $C_{\tilde{a}} C_{\tilde{x}} \subset C_{|\tilde{a}| \wedge |\tilde{x}|}$, if both $\tilde{a}$ and $\tilde{x} \in \tilde{R}$ are continuous. Consequently we obtain our assertion by the formula (1

$$(11) \quad C_{|\tilde{a}| \wedge |\tilde{x}|} = C_{[\tilde{a}]\tilde{x}}.$$

Because we have by the formula §8(1)

$$[|\tilde{a}| \wedge |\tilde{x}|] = [\tilde{a}][\tilde{x}] = [[\tilde{a}]\tilde{x}],$$

and hence we obtain the assertion (11) by the formula (9).

By virtue of Theorem 48.3 and the formula (11), we conclude:

Theorem 48.4. If both $\tilde{a}$ and $\tilde{x} \in \tilde{R}$ are continuous,
then we have

$$C_{\tilde{a}} C_{\tilde{x}} = C_{[\tilde{a}]\tilde{x}} = C_{[\tilde{x}]\tilde{a}}.$$

Since $|\tilde{a}| \vee |\tilde{x}| \leq |\tilde{a}| + |\tilde{x}| \leq 2(|\tilde{a}| \vee |\tilde{x}|)$, we obtain by the formulas (8) and §46(9)

$$C_{|\tilde{a}| \vee |\tilde{x}|} = C_{|\tilde{a}| + |\tilde{x}|}.$$

By virtue of the formula (9) we have obviously

$$C_{\tilde{a}} + C_{\tilde{x}} \subset C_{|\tilde{a}| + |\tilde{x}|}.$$

On the other hand, if $(C_{\tilde{a}} + C_{\tilde{x}}) \cup_{[p]} = 0$, then we have by the formula (3)

$$\tilde{a}[p] = \tilde{x}[p] = 0,$$

and hence by repeated use of the formula (3)

$$C_{[\tilde{\alpha}] + [\tilde{\beta}]} U_{[p]} = 0.$$

Therefore we have:

$$(12) \quad C_{\tilde{\alpha}} + C_{\tilde{\beta}} = C_{[\tilde{\alpha}] + [\tilde{\beta}]} = C_{[\tilde{\alpha}] \cup [\tilde{\beta}]}.$$

Theorem 48.5. If both $\tilde{\alpha}$ and $\tilde{\beta} \in \tilde{R}$ are continuous,

then we have

$$C_{\tilde{\beta}} - C_{\tilde{\alpha}} C_{\tilde{\beta}} = C_{(1 - [\tilde{\alpha}])} \tilde{\beta}.$$

Proof. Since we have by Theorem 7.10

$$[\tilde{\beta}] = [\tilde{\alpha}] [\tilde{\beta}] + (1 - [\tilde{\alpha}]) [\tilde{\beta}],$$

$$[\tilde{\alpha}] [\tilde{\beta}] \perp (1 - [\tilde{\alpha}]) [\tilde{\beta}],$$

we obtain by the formula (12) and by Theorems 48.1, 48.4

$$C_{\tilde{\beta}} = C_{\tilde{\alpha}} C_{\tilde{\beta}} + C_{(1 - [\tilde{\alpha}])} \tilde{\beta},$$

$$C_{\tilde{\alpha}} C_{\tilde{\beta}} C_{(1 - [\tilde{\alpha}])} \tilde{\beta} = 0.$$

Theorem 48.6. If both $\tilde{\alpha}$ and $\tilde{\beta} \in \tilde{R}$ are continuous,

then to every element $a \in R$ there exists a projector $[p] \leq [a]$ such that

$$C_{\tilde{\alpha}} C_{\tilde{\beta}} U_{[a]} = C_{\tilde{\alpha}} U_{[p]} = C_{\tilde{\beta}} U_{[p]}.$$

Proof. By virtue of the previous theorem, both

$$C_{\tilde{\alpha}} - C_{\tilde{\alpha}} C_{\tilde{\beta}} \quad \text{and} \quad C_{\tilde{\beta}} - C_{\tilde{\alpha}} C_{\tilde{\beta}}$$

are closed sets and have no common point with $C_{\tilde{\alpha}} C_{\tilde{\beta}}$. To every element $a \in R$, since the point set

$$C_{\tilde{\alpha}} C_{\tilde{\beta}} U_{[a]}$$

is compact, there exists by Theorem 16.3 a projector $[p]$ such that

$$C_{\tilde{\alpha}} C_{\tilde{\beta}} U_{[a]} \subset U_{[p]} \subset (C_{\tilde{\alpha}} - C_{\tilde{\alpha}} C_{\tilde{\beta}})' (C_{\tilde{\beta}} - C_{\tilde{\alpha}} C_{\tilde{\beta}})' U_{[a]}.$$

For such $[p]$ we have obviously $[p] \leq [a]$ and further

$$U_{[p]} C_{\tilde{\alpha}} = U_{[p]} C_{\tilde{\beta}} = U_{[p]} C_{\tilde{\alpha}} C_{\tilde{\beta}} = U_{[a]} C_{\tilde{\alpha}} C_{\tilde{\beta}},$$

since we can conclude from the inequality obtained just now

$$U_{[p]} (C_{\tilde{\alpha}} - C_{\tilde{\alpha}} C_{\tilde{\beta}}) = U_{[p]} (C_{\tilde{\beta}} - C_{\tilde{\alpha}} C_{\tilde{\beta}}) = 0.$$

§49 Integration

Let R be a continuous semi-ordered linear space, and let $\tilde{R}$ be the associated space of R . If a continuous function $\varphi(\xi)$ is bounded in a neighbourhood $\mathcal{U}_{[p]}$ of the proper space E of R , then to every positive number ε there exists a partition of the projector $[p] : [p] = [p_1] + [p_2] + \dots + [p_\kappa]$, such that

$$\sup_{\xi \in \mathcal{U}_{[p_\nu]}} \varphi(\xi) < \varepsilon \quad \text{for every } \nu = 1, 2, \dots, \kappa,$$

and we have for any point $\xi_\nu \in \mathcal{U}_{[p_\nu]}$ ($\nu = 1, 2, \dots, \kappa$)

$$\left| \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) [p_\nu] a - \int_{[p]} \varphi(\xi) d\xi a \right| \leq \varepsilon [p] |a|$$

as established in §20. Then we have for every element $\tilde{a} \in \tilde{R}$

$$\left| \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) \tilde{a}([p_\nu] a) - \tilde{a} \left(\int_{[p]} \varphi(\xi) d\xi a \right) \right| \leq \varepsilon |\tilde{a}|([p] |a|),$$

since we have for every element $x \in R$

$$|\tilde{a}(x)| = |\tilde{a}^+(x^+) - \tilde{a}^-(x^+) - \tilde{a}^+(x^-) + \tilde{a}^-(x^-)| \leq |\tilde{a}|(|x|).$$

Accordingly we obtain

$$\lim_{\varepsilon \rightarrow 0} \sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) \tilde{a}([p_\nu] a) = \tilde{a} \left(\int_{[p]} \varphi(\xi) d\xi a \right)$$

for every partition of $[p]$ such that $[p] = [p_1] + [p_2] + \dots + [p_\kappa]$,

osc $\varphi(\xi) < \varepsilon$ ($\nu = 1, 2, \dots, \kappa$), and for every point $\xi_\nu \in \mathcal{U}_{[p_\nu]}$.

This limit is called the integral of a bounded continuous function

$\varphi(\xi)$ by $\tilde{a}(d\xi a)$ in $\mathcal{U}_{[p]}$ and denoted by $\int_{[p]} \varphi(\xi) \tilde{a}(d\xi a)$.

With this definition we have obviously:

Theorem 49.1. For a bounded continuous function $\varphi(\xi)$ on $\mathcal{U}_{[p]}$ we have $\int_{[p]} \varphi(\xi) \tilde{a}(d\xi a) = \tilde{a} \left(\int_{[p]} \varphi(\xi) d\xi a \right)$.

Theorem 49.2. For the characteristic set C_α of $\tilde{a}$, if a bounded continuous function $\varphi(\xi)$ coincides with a bounded continuous function $\psi(\xi)$ in $C_\alpha \mathcal{U}_{[p]}$, that is, if $\varphi(\xi) = \psi(\xi)$ for every point $\xi \in C_\alpha \mathcal{U}_{[p]}$, then we have

$$\int_{[p]} \varphi(\xi) \tilde{a}(d\xi a) = \int_{[p]} \psi(\xi) \tilde{a}(d\xi a).$$

Proof. If we determine $\xi_\nu \in \mathcal{U}_{[p_\nu]}$ ($\nu = 1, 2, \dots, \kappa$) such that $\xi_\nu \in C_\alpha \mathcal{U}_{[p_\nu]}$ for $C_\alpha \mathcal{U}_{[p_\nu]} \neq 0$, then we have

$$\sum_{\nu=1}^{\kappa} \varphi(\xi_\nu) \tilde{a}([p_\nu] a) = \sum_{\nu=1}^{\kappa} \psi(\xi_\nu) \tilde{a}([p_\nu] a),$$

because, if $\varphi(\xi_\nu) \neq \psi(\xi_\nu)$, then we have $C_\alpha \mathcal{U}_{[p_\nu]} = 0$ and

hence $\tilde{\alpha}[p_\nu] = 0$ by the formula §48(3). Therefore we obtain our assertion by definition.

By virtue of Extension theorem in Introduction, every bounded continuous function $\varphi(\mathfrak{f})$ on $C_\infty \mathcal{U}_{[p]}$ has a continuous extension over $\mathcal{U}_{[p]}$, that is, there exists a bounded continuous function $\psi(\mathfrak{f})$ on $\mathcal{U}_{[p]}$ for which $\psi(\mathfrak{f}) = \varphi(\mathfrak{f})$ for every $\mathfrak{f} \in C_\infty \mathcal{U}_{[p]}$. Therefore we can define by Theorem 49.2 the integral of a bounded continuous function $\varphi(\mathfrak{f})$ on $C_\infty \mathcal{U}_{[p]}$ by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$ as

$$\int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) = \int_{[p]} \psi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a)$$

for a bounded continuous extension $\psi(\mathfrak{f})$ of $\varphi(\mathfrak{f})$ over $\mathcal{U}_{[p]}$.

If a continuous function $\varphi(\mathfrak{f})$ is 'almost finite' in $\mathcal{U}_{[p]}$, then there exists by Theorem 21.7 a sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [p]$ such that $\varphi(\mathfrak{f})$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$.

If there exists a real number α such that we have

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) = \alpha$$

for every such sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [p]$, then $\varphi(\mathfrak{f})$ is said to be integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$, and we shall write

$$\alpha = \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a).$$

With this definition, every bounded continuous function $\varphi(\mathfrak{f})$ on $\mathcal{U}_{[p]}$ is not necessarily integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$, even if it has by definition always the integral by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$. However it is evident by definition that if a bounded continuous function $\varphi(\mathfrak{f})$ on $\mathcal{U}_{[p]}$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$, then $\varphi(\mathfrak{f})$ has the same integral.

Theorem 49.3. If $[p_\nu] \downarrow_{\nu=1}^\infty 0$ implies $\lim_{\nu \rightarrow \infty} |\tilde{\alpha}([p_\nu]|a)| = 0$, then every bounded continuous function $\varphi(\mathfrak{f})$ on $\mathcal{U}_{[p]}$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$.

Proof. Since $\varphi(\mathfrak{f})$ is bounded in $\mathcal{U}_{[p]}$ by assumption, we suppose that $|\varphi(\mathfrak{f})| \leq \alpha$ for every point $\mathfrak{f} \in \mathcal{U}_{[p]}$. For every sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [p]$, we have obviously $[p] - [p_\nu] \downarrow_{\nu=1}^\infty 0$, and further by Theorem 49.1

$$\begin{aligned} & \left| \int_{[p]} \varphi(\xi) \tilde{a}(d\xi a) - \int_{[p_n]} \varphi(\xi) \tilde{a}(d\xi a) \right| \\ &= \left| \tilde{a} \left(\int_{[p] - [p_n]} \varphi(\xi) d\xi a \right) \right| \leq \alpha |\tilde{a}|([p] - [p_n])|a|, \end{aligned}$$

since we conclude by Theorem 20.2

$$\left| \int_{[p] - [p_n]} \varphi(\xi) d\xi a \right| \leq \alpha ([p] - [p_n])|a|.$$

Therefore we have for every sequence of projectors $[p_n] \uparrow_{n=1}^{\infty} [p]$

$$\lim_{n \rightarrow \infty} \int_{[p_n]} \varphi(\xi) \tilde{a}(d\xi a) = \int_{[p]} \varphi(\xi) \tilde{a}(d\xi a).$$

Theorem 49.4. If an almost finite continuous function $\varphi(\xi)$ is integrable by $\tilde{a}(d\xi a)$ in $\mathcal{U}_{[p]}$ as well as in $\mathcal{U}_{[q]}$ and $[p][q] = 0$, then $\varphi(\xi)$ is integrable by $\tilde{a}(d\xi a)$ in $\mathcal{U}_{[p] + [q]}$ and

$$\int_{[p] + [q]} \varphi(\xi) \tilde{a}(d\xi a) = \int_{[p]} \varphi(\xi) \tilde{a}(d\xi a) + \int_{[q]} \varphi(\xi) \tilde{a}(d\xi a).$$

Proof. If $\varphi(\xi)$ is bounded in $\mathcal{U}_{[p_n]}$ and

$$[p_n] \uparrow_{n=1}^{\infty} [p], [p] + [q],$$

then $\varphi(\xi)$ is bounded obviously in $\mathcal{U}_{[p_n][p]}$ as well as in

$\mathcal{U}_{[p_n][q]}$, and we have by Theorem 8.10

$$[p_n][p] \uparrow_{n=1}^{\infty} [p], [p_n][q] \uparrow_{n=1}^{\infty} [q].$$

Consequently we obtain by definition

$$\lim_{n \rightarrow \infty} \int_{[p_n]} \varphi(\xi) \tilde{a}(d\xi a) = \int_{[p]} \varphi(\xi) \tilde{a}(d\xi a) + \int_{[q]} \varphi(\xi) \tilde{a}(d\xi a),$$

since we have by Theorem 49.1 and the formula §20(3)

$$\int_{[p_n]} \varphi(\xi) \tilde{a}(d\xi a) = \int_{[p_n][p]} \varphi(\xi) \tilde{a}(d\xi a) + \int_{[p_n][q]} \varphi(\xi) \tilde{a}(d\xi a),$$

and $\varphi(\xi)$ is by assumption integrable by $\tilde{a}(d\xi a)$ in $\mathcal{U}_{[p]}$ as well as in $\mathcal{U}_{[q]}$.

By virtue of Theorem 49.1 and the formula §20(5) we can prove likewise:

Theorem 49.5. If an almost finite continuous function $\varphi(\xi)$ is integrable by $\tilde{a}(d\xi a)$ as well as by $\tilde{a}(d\xi b)$ in $\mathcal{U}_{[p]}$, then $\varphi(\xi)$ is integrable by $\tilde{a}(d\xi(\alpha a + \beta b))$ in $\mathcal{U}_{[p]}$ for every real numbers α, β , and we have

$$\int_{[p]} \varphi(\xi) \tilde{a}(d\xi(\alpha a + \beta b)) = \alpha \int_{[p]} \varphi(\xi) \tilde{a}(d\xi a) + \beta \int_{[p]} \varphi(\xi) \tilde{a}(d\xi b).$$

Theorem 49.6. If an almost finite continuous function $\varphi(\xi)$ is integrable by $\tilde{a}(d\xi a)$ as well as by $\tilde{b}(d\xi a)$ in $\mathcal{U}_{[p]}$,

then $\varphi(f)$ is integrable by $(\alpha\tilde{a} + \beta\tilde{x})(dga)$ in $\mathcal{U}_{[p]}$ for every real numbers α, β , and we have

$$\int_{[p]} \varphi(f)(\alpha\tilde{a} + \beta\tilde{x})(dga) = \alpha \int_{[p]} \varphi(f)\tilde{a}(dga) + \beta \int_{[p]} \varphi(f)\tilde{x}(dga).$$

Theorem 49.7. If an almost finite continuous function

$\varphi(f)$ is integrable by $\tilde{a}(dga)$ in $\mathcal{U}_{[p]}$, then for every projector $[q]$, $\varphi(f)$ is integrable

by $\tilde{a}(dga)$ in $\mathcal{U}_{[p][q]}$,

by $\tilde{a}(dga)$ in $\mathcal{U}_{[p]}$,

by $\tilde{a}[q](dga)$ in $\mathcal{U}_{[p]}$,

and we have further

$$\int_{[p][q]} \varphi(f)\tilde{a}(dga) = \int_{[p]} \varphi(f)\tilde{a}(dga) = \int_{[p]} \varphi(f)\tilde{a}[q](dga).$$

Proof. If $\varphi(f)$ is bounded in $\mathcal{U}_{[p]}$ as well as in $\mathcal{U}_{[q]}$

for every $\nu = 1, 2, \dots$ and

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p][q], \quad [q_\nu] \uparrow_{\nu=1}^{\infty} [p](1 - [q]),$$

then $\varphi(f)$ is obviously bounded in $\mathcal{U}_{[p_\nu] + [q_\nu]}$ for every $\nu = 1, 2, \dots$, and we have by Theorem 8.10

$$[p_\nu] + [q_\nu] \uparrow_{\nu=1}^{\infty} [p].$$

We have further by Theorem 45.1 and by the formula §20(3)

$$\int_{[p_\nu] + [q_\nu]} \varphi(f)\tilde{a}(dga) = \int_{[p_\nu]} \varphi(f)\tilde{a}(dga) + \int_{[q_\nu]} \varphi(f)\tilde{a}(dga),$$

and by assumption for every such $[p_\nu]$ and $[q_\nu]$ ($\nu = 1, 2, \dots$)

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu] + [q_\nu]} \varphi(f)\tilde{a}(dga) = \int_{[p]} \varphi(f)\tilde{a}(dga).$$

Therefore there exists the limit **)

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f)\tilde{a}(dga),$$

and this limit is the same for every such $[p_\nu]$ ($\nu = 1, 2, \dots$),

that is, $\varphi(f)$ is integrable by $\tilde{a}(dga)$ in $\mathcal{U}_{[p][q]}$.

If $\varphi(f)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p],$$

then $\varphi(f)$ is naturally bounded in $\mathcal{U}_{[p_\nu][q]}$ and we have by Theorem 8.10

$$[p_\nu][q] \uparrow_{\nu=1}^{\infty} [p][q].$$

**) See the page 312.

Since we have by Theorem 49.1 and by the formula §20(2)

$\int_{[p_v] \cap [q]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) = \int_{[p_v]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}[q]a) = \int_{[p_v]} \varphi(\mathfrak{f}) \tilde{\alpha}([q](d\mathfrak{f}a))$,
and $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}[q]$ as proved just now, we can conclude that $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}[q]a)$ as well as by $\tilde{\alpha}([q](d\mathfrak{f}a))$ in $\mathcal{U}_{[p]}$ and we have

$$\int_{[p] \cap [q]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) = \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}[q]a) = \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}([q](d\mathfrak{f}a)).$$

Theorem 49.8. If both almost finite continuous functions
 $\varphi(\mathfrak{f})$ and $\psi(\mathfrak{f})$ are integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$ for $\tilde{\alpha} \geq 0$,
 $a \geq 0$, and

$$\varphi(\mathfrak{f}) \geq \psi(\mathfrak{f}) \quad \text{for every point } \mathfrak{f} \in \mathcal{U}_{[p]},$$

then we have

$$\int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) \geq \int_{[p]} \psi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a).$$

Proof. We see easily by Theorem 8.10 that there exists a sequence of projectors $[p_v] \uparrow_{v=1}^{\infty} [p]$ such that both $\varphi(\mathfrak{f})$ and $\psi(\mathfrak{f})$ are bounded in $\mathcal{U}_{[p_v]}$ for every $v=1, 2, \dots$. For such $[p_v]$ ($v=1, 2, \dots$) we conclude by Theorems 49.1 and 20.1 from the assumption

$$\int_{[p_v]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) \geq \int_{[p_v]} \psi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a),$$

and hence we obtain our assertion.

Theorem 49.9. For $0 \leq \tilde{\alpha} \in \tilde{\mathcal{R}}$, in order that an almost
finite continuous function $\varphi(\mathfrak{f})$ be integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in
 $\mathcal{U}_{[p]}$, it is necessary and sufficient that $|\varphi(\mathfrak{f})|$ be inter-
grable by $\tilde{\alpha}(d\mathfrak{f}|a|)$ in $\mathcal{U}_{[p]}$, and then we have

$$\left| \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) \right| \leq \int_{[p]} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|).$$

Proof. Since the point set

$$\{\mathfrak{f} : \varphi(\mathfrak{f}) > 0\} = \bigcup_{\frac{1}{v}} \{\mathfrak{f} : \varphi(\mathfrak{f}) \geq \frac{1}{v}\}$$

is $\mathcal{G}$ -open, the closure $\{\mathfrak{f} : \varphi(\mathfrak{f}) > 0\}^-$ is open by Theorem 16.7, and hence there exists by Theorem 16.4 a projector $[q]$ for which

$$\mathcal{U}_{[q]} = \{\mathfrak{f} : \varphi(\mathfrak{f}) > 0\}^- \cap \mathcal{U}_{[p]}.$$

For such $[q]$ we have obviously

$$\varphi(\mathfrak{f}) \begin{cases} \geq 0 & \text{for } \mathfrak{f} \in \mathcal{U}_{\{q\}}, \\ \leq 0 & \text{for } \mathfrak{f} \in \mathcal{U}_{[p]} - \mathcal{U}_{\{q\}}. \end{cases}$$

If $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$, then we have by Theorem 49.8 that $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}[a^+]a)$ as well as by $\tilde{\alpha}(d\mathfrak{f}[a^-]a)$ in $\mathcal{U}_{\{q\}}$ and in $\mathcal{U}_{[p]} - \mathcal{U}_{\{q\}}$. Hence we conclude by Theorems 49.4 and 49.5 that $|\varphi(\mathfrak{f})|$ is integrable by $\tilde{\alpha}(d\mathfrak{f}|a|)$ in $\mathcal{U}_{[p]}$ and we have

$$\begin{aligned} \left| \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) \right| &= \left| \int_{\{q\}} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) + \int_{[p] - \{q\}} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}(-a)) \right| \\ &\leq \int_{\{q\}} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|) + \int_{[p] - \{q\}} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|) \\ &= \int_{[p]} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|). \end{aligned}$$

Conversely, if $|\varphi(\mathfrak{f})|$ is integrable by $\tilde{\alpha}(d\mathfrak{f}|a|)$ in $\mathcal{U}_{[p]}$, then we see likewise that $|\varphi(\mathfrak{f})|$ is integrable by $\tilde{\alpha}(d\mathfrak{f}[a^+]|a|)$ as well as by $\tilde{\alpha}(d\mathfrak{f}[a^-]|a|)$ in $\mathcal{U}_{\{q\}}$ and in $\mathcal{U}_{[p]} - \mathcal{U}_{\{q\}}$, and hence $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$.

Theorem 49.10. For $0 \leq \tilde{\alpha} \in \tilde{\mathcal{R}}$, if an almost finite continuous function $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$, then $[p_\nu] \downarrow_{\nu=1}^\infty 0$ implies

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu][p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) = 0.$$

Proof. If $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f}a)$ in $\mathcal{U}_{[p]}$, then $|\varphi(\mathfrak{f})|$ is integrable by $\tilde{\alpha}(d\mathfrak{f}|a|)$ in $\mathcal{U}_{[p]}$ by the previous theorem and we have

$$\left| \int_{[p_\nu][p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f}a) \right| \leq \int_{[p_\nu][p]} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|).$$

If we consider a sequence of projectors $[q_\nu] \uparrow_{\nu=1}^\infty [p]$ such that $|\varphi(\mathfrak{f})|$ is bounded in $\mathcal{U}_{[q_\nu]}$ for every $\nu = 1, 2, \dots$, then, since $[p_\nu] \downarrow_{\nu=1}^\infty 0$ by assumption, we have by Theorem 8.10

$$[q_\nu]([p] - [p_\nu][p]) \uparrow_{\nu=1}^\infty [p],$$

and hence by definition

$$\lim_{\nu \rightarrow \infty} \int_{[q_\nu]([p] - [p_\nu][p])} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|) = \int_{[p]} |\varphi(\mathfrak{f})| \tilde{\alpha}(d\mathfrak{f}|a|).$$

On the other hand we have by Theorems 49.5 and 49.9

$$\begin{aligned}
& \int_{[p]} |\varphi(\xi)| \tilde{\alpha}(d\xi|a) - \int_{[q_v]([p] - [p_v][p])} |\varphi(\xi)| \tilde{\alpha}(d\xi|a) \\
&= \int_{([p] - [q_v][p]) + [p][p_v][q_v]} |\varphi(\xi)| \tilde{\alpha}(d\xi|a) \\
&\geq \int_{[p_v][p]} |\varphi(\xi)| \tilde{\alpha}(d\xi|a) \geq 0,
\end{aligned}$$

since

$$([p] - [q_v][p]) + [p][p_v][q_v]$$

$$\geq ([p_v][p] - [p_v][q_v][p]) + [p_v][q_v][p] = [p_v][p].$$

Therefore we obtain $\lim_{v \rightarrow \infty} \int_{[p_v][p]} |\varphi(\xi)| \tilde{\alpha}(d\xi|a) = 0$, and hence

$$\lim_{v \rightarrow \infty} \int_{[p_v][p]} \varphi(\xi) \tilde{\alpha}(d\xi|a) = 0.$$

Theorem 49.11. For $0 \leq \tilde{\alpha} \in \tilde{\mathcal{R}}$, if an almost finite continuous function $\varphi(\xi)$ is integrable by $\tilde{\alpha}(d\xi|a)$ in $\mathcal{U}_{[p]}$ and if an almost finite continuous function $\psi(\xi)$ on $\mathcal{U}_{[p]}$ coincides with $\varphi(\xi)$ in the characteristic set $C_{\tilde{\alpha}}$ of $\tilde{\alpha}$, that is, if

$$\psi(\xi) = \varphi(\xi) \quad \text{for every point } \xi \in C_{\tilde{\alpha}} \mathcal{U}_{[p]},$$

then $\psi(\xi)$ is integrable by $\tilde{\alpha}(d\xi|a)$ in $\mathcal{U}_{[p]}$ and

$$\int_{[p]} \psi(\xi) \tilde{\alpha}(d\xi|a) = \int_{[p]} \varphi(\xi) \tilde{\alpha}(d\xi|a).$$

Proof. If $\varphi(\xi)$ is bounded in $\mathcal{U}_{[p]}$ and

$$|\varphi(\xi)| \leq \alpha \quad \text{for every point } \xi \in \mathcal{U}_{[p]},$$

then we have by assumption

$$\{\xi : |\varphi(\xi)| \leq \alpha\} \supset C_{\tilde{\alpha}} \mathcal{U}_{[p]},$$

and hence there exists by Theorem 16.3 a projector $[q]$ for which

$$\{\xi : |\varphi(\xi)| < \alpha + 1\} \mathcal{U}_{[p]} \supset \mathcal{U}_{[q]} \supset C_{\tilde{\alpha}} \mathcal{U}_{[p]}.$$

For such $[q]$, since $\varphi(\xi)$ is bounded in $\mathcal{U}_{[q]}$, we have by Theorem 49.2

$$\int_{[q]} \psi(\xi) \tilde{\alpha}(d\xi|a) = \int_{[q]} \varphi(\xi) \tilde{\alpha}(d\xi|a),$$

and further by definition

$$\int_{[p] - [q]} \psi(\xi) \tilde{\alpha}(d\xi|a) = \int_{[p] - [q]} \varphi(\xi) \tilde{\alpha}(d\xi|a) = 0,$$

since $\tilde{\alpha}([p] - [q]) = 0$ by the formula §49(3). Therefore, if

$\psi(\xi)$ is bounded in $\mathcal{U}_{[p]}$, then we have

$$\int_{[p]} \psi(\xi) \tilde{\alpha}(d\xi|a) = \int_{[p]} \varphi(\xi) \tilde{\alpha}(d\xi|a).$$

In general, if $\psi(p)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p],$$

then we have for every $\nu = 1, 2, \dots$

$$\int_{[p_\nu]} \psi(p) \tilde{\alpha}(d\mathfrak{p} a) = \int_{[p_\nu]} \varphi(p) \tilde{\alpha}(d\mathfrak{p} a),$$

as proved just now. Since we have by the previous theorem

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(p) \tilde{\alpha}(d\mathfrak{p} a) = \int_{[p]} \varphi(p) \tilde{\alpha}(d\mathfrak{p} a),$$

we obtain hence by definition that $\psi(p)$ is integrable by

$\tilde{\alpha}(d\mathfrak{p} a)$ in $\mathcal{U}_{[p]}$ and

$$\int_{[p]} \psi(p) \tilde{\alpha}(d\mathfrak{p} a) = \int_{[p]} \varphi(p) \tilde{\alpha}(d\mathfrak{p} a).$$

If an almost finite continuous function $\varphi(p)$ is integrable by $\tilde{\alpha}(d\mathfrak{p} a)$ in $\mathcal{U}_{[a]}$ for every element $a \in R$, then $\varphi(p)$ is said to be integrable by $\tilde{\alpha}$. If $\varphi(p)$ is integrable by an element $\tilde{\alpha} \in \tilde{R}$, then, putting

$$L(a) = \int_{[a]} \varphi(p) \tilde{\alpha}(d\mathfrak{p} a) \quad \text{for every element } a \in R,$$

we obtain by Theorem 49.5 a linear functional L on R , which will be denoted by

$$\int \varphi(p) \tilde{\alpha} d\mathfrak{p}$$

Theorem 49.12. If an almost finite continuous function

$\varphi(p)$ is integrable by a positive element $\tilde{\alpha} \in \tilde{R}$, then

$$\int \varphi(p) \tilde{\alpha} d\mathfrak{p}$$

is a continuous linear functional on R .

Proof. If $\varphi(p)$ is integrable by a positive element $\tilde{\alpha} \in \tilde{R}$, then we see by Theorem 49.9 that $|\varphi(p)|$ is integrable by $\tilde{\alpha}$ and

$$|\int \varphi(p) \tilde{\alpha} d\mathfrak{p}(a)| \leq \int |\varphi(p)| \tilde{\alpha} d\mathfrak{p}(|a|)$$

for every element $a \in R$. Accordingly we obtain by the formula §46(2)

$$|\int \varphi(p) \tilde{\alpha} d\mathfrak{p}| \leq \int |\varphi(p)| \tilde{\alpha} d\mathfrak{p}.$$

Furthermore we see easily by Theorems 49.10 and 49.7 that for every sequence of projectors $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ we have

$$\lim_{\nu \rightarrow \infty} \int |\varphi(p)| \tilde{\alpha} d\mathfrak{p}([p_\nu] a) = \lim_{\nu \rightarrow \infty} \int_{[p_\nu][a]} |\varphi(p)| \tilde{\alpha}(d\mathfrak{p} a) = 0,$$

and hence $\int (\varphi(f)) \tilde{a} d\mu$ is continuous by Theorem 47.2. Consequently we see by Theorem 47.5 that $\int \varphi(f) \tilde{a} d\mu$ is continuous.

§50 Integration by continuous linear functionals

Now we shall consider integration of almost finite continuous functions by a continuous linear functional $\tilde{a} \in \tilde{\mathcal{R}}$. For a continuous continuous linear functional $\tilde{a} \in \tilde{\mathcal{R}}$, since $|\tilde{a}|$ is also continuous by Theorem 47.5, we see easily by Theorem 49.5 that every bounded continuous function is integrable by $\tilde{a}(d\mu a)$.

Furthermore we have:

Theorem 50.1. If an almost finite continuous function $\varphi(f)$ is integrable by an element $a \in \mathcal{R}$ in $\mathcal{U}_{[p]}$, then $\varphi(f)$ is integrable by $\tilde{a}(d\mu a)$ in $\mathcal{U}_{[p]}$ for every continuous linear functional $\tilde{a} \in \tilde{\mathcal{R}}$, and

$$\int_{[p]} \varphi(f) \tilde{a}(d\mu a) = \tilde{a} \left(\int_{[p]} \varphi(f) d\mu a \right).$$

Proof. If $\varphi(f)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [p],$$

then, since $\varphi(f)$ is integrable by $a \in \mathcal{R}$ in $\mathcal{U}_{[p]}$ by assumption, we have by Theorem 21.4

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f) d\mu a = \int_{[p]} \varphi(f) d\mu a,$$

and hence by Theorems 49.1 and 47.7

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi(f) \tilde{a}(d\mu a) = \lim_{\nu \rightarrow \infty} \tilde{a} \left(\int_{[p_\nu]} \varphi(f) d\mu a \right) = \tilde{a} \left(\int_{[p]} \varphi(f) d\mu a \right).$$

Therefore $\varphi(f)$ is integrable by $\tilde{a}(d\mu a)$ in $\mathcal{U}_{[p]}$ by definition.

Recalling Theorem 22.2, we conclude from Theorem 50.1:

Theorem 50.2. For every continuous $\tilde{a} \in \tilde{\mathcal{R}}$, $(\frac{f}{a}, f)$ is integrable by $\tilde{a}(d\mu a)$ in $\mathcal{U}_{[a]}$ and

$$\int_{[a]} \left(\frac{f}{a}, f \right) \tilde{a}(d\mu a) = \tilde{a}([a]f).$$

Theorem 50.3. If a linear functional $\tilde{\alpha} \in \tilde{\mathcal{R}}$ is continuous, then in order that an almost finite continuous function $\varphi(\mathfrak{f})$ be integrable by $\tilde{\alpha}(d\mathfrak{f} a)$ in $\mathcal{U}_{[p]}$, it is necessary and sufficient that $|\varphi(\mathfrak{f})|$ be integrable by $|\tilde{\alpha}|(d\mathfrak{f} |a|)$ in $\mathcal{U}_{[p]}$, and then we have

$$\left| \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f} a) \right| \leq \int_{[p]} |\varphi(\mathfrak{f})| |\tilde{\alpha}|(d\mathfrak{f} |a|).$$

Proof. Since $\tilde{\alpha} \in \tilde{\mathcal{R}}$ is continuous, to every element $a \in \mathcal{R}$ there exist by Theorem 47.3 two projectors $[p_0]$ and $[q_0]$ such that

$$[a] = [p_0] + [q_0], \quad [p_0][q_0] = 0,$$

$$\tilde{\alpha}^+[a] = \tilde{\alpha}[p_0], \quad \tilde{\alpha}^-[a] = -\tilde{\alpha}[q_0].$$

If $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f} a)$ in $\mathcal{U}_{[p]}$, then we see by Theorem 49.7 that $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}[p_0](d\mathfrak{f} a)$ as well as by $\tilde{\alpha}[q_0](d\mathfrak{f} a)$ in $\mathcal{U}_{[p]}$, and

$$\int_{[p_0][p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f} a) = \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}[p_0](d\mathfrak{f} a) = \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}^+(d\mathfrak{f} a),$$

$$\int_{[q_0][p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f} a) = \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}[q_0](d\mathfrak{f} a) = - \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}^-(d\mathfrak{f} a).$$

Hence $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}^+(d\mathfrak{f} a)$ as well as by $\tilde{\alpha}^-(d\mathfrak{f} a)$ in $\mathcal{U}_{[p]}$, and consequently we obtain by Theorem 49.10 that $|\varphi(\mathfrak{f})|$ is integrable by $\tilde{\alpha}^+(d\mathfrak{f} |a|)$ as well as by $\tilde{\alpha}^-(d\mathfrak{f} |a|)$ in $\mathcal{U}_{[p]}$ and

$$\left| \int_{[p_0][p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f} a) \right| \leq \int_{[p]} |\varphi(\mathfrak{f})| \tilde{\alpha}^+(d\mathfrak{f} |a|),$$

$$\left| \int_{[q_0][p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f} a) \right| \leq \int_{[p]} |\varphi(\mathfrak{f})| \tilde{\alpha}^-(d\mathfrak{f} |a|).$$

Therefore $|\varphi(\mathfrak{f})|$ is integrable by $|\tilde{\alpha}|(d\mathfrak{f} |a|)$ in $\mathcal{U}_{[p]}$ by virtue of Theorem 49.6 and

$$\left| \int_{[p]} \varphi(\mathfrak{f}) \tilde{\alpha}(d\mathfrak{f} a) \right| \leq \int_{[p]} |\varphi(\mathfrak{f})| |\tilde{\alpha}|(d\mathfrak{f} |a|).$$

Conversely if $|\varphi(\mathfrak{f})|$ is integrable by $|\tilde{\alpha}|(d\mathfrak{f} |a|)$ in $\mathcal{U}_{[p]}$, then we see by Theorem 49.10 that $\varphi(\mathfrak{f})$ is integrable by $|\tilde{\alpha}|(d\mathfrak{f} a)$ in $\mathcal{U}_{[p]}$, and hence by Theorem 49.7 that $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}[p_0](d\mathfrak{f} a)$ as well as by $\tilde{\alpha}[q_0](d\mathfrak{f} a)$ in $\mathcal{U}_{[p]}$. Since

$$|\tilde{\alpha}|[p_0] = \tilde{\alpha}^+[p_0] = \tilde{\alpha}[p_0],$$

$$|\tilde{\alpha}|[q_0] = \tilde{\alpha}^-[q_0] = -\tilde{\alpha}[q_0],$$

we can conclude by Theorem 49.6 that $\varphi(\mathfrak{f})$ is integrable by $\tilde{\alpha}(d\mathfrak{f} a)$

in $\mathcal{U}_{[p]}$.

Theorem 50.4. For an almost finite continuous function $\varphi(x)$, if there exists a sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [p]$ such that $\varphi(x)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu=1, 2, \dots$ and

$$\sup_{\nu=1, 2, \dots} \int_{[p_\nu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)| < +\infty$$

for a continuous $\tilde{\alpha} \in \tilde{\mathcal{R}}$, then $\varphi(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$.

Proof. If $\varphi(x)$ is bounded in $\mathcal{U}_{[p_\mu]}$ for every $\mu=1, 2, \dots$ for another sequence of projectors $[q_\mu] \uparrow_{\mu=1}^\infty [p]$, then, since $|\tilde{\alpha}|$ is continuous by Theorem 47.5, we have by Theorem 50.1

$$\lim_{\mu \rightarrow \infty} \int_{[p_\mu][q_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)| = \int_{[q_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)|.$$

As $\int_{[p_\mu][q_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)| \leq \int_{[p_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)|$, we obtain hence

$$\lim_{\mu \rightarrow \infty} \int_{[q_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)| \leq \lim_{\nu \rightarrow \infty} \int_{[p_\nu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)|.$$

We also can prove likewise

$$\lim_{\mu \rightarrow \infty} \int_{[p_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)| \leq \lim_{\mu \rightarrow \infty} \int_{[q_\mu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)|.$$

Therefore $|\varphi(x)|$ is integrable by $|\tilde{\alpha}|(dx|a)|$ in $\mathcal{U}_{[p]}$ by definition, and consequently by the previous theorem $\varphi(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$.

Theorem 50.5. For two almost finite continuous functions $\varphi(x)$ and $\psi(x)$, if

$$|\varphi(x)| \geq |\psi(x)| \quad \text{for every point } x \in \mathcal{U}_{[p]},$$

and $\varphi(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$, then $\psi(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$ for a continuous $\tilde{\alpha} \in \tilde{\mathcal{R}}$, if $|\tilde{\alpha}| \geq |\tilde{\beta}|$ in $\tilde{\mathcal{R}}$ and $|a| \geq |b|$ in $\mathcal{R}$.

Proof. If $\varphi(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$, then $|\varphi(x)|$ is integrable by $|\tilde{\alpha}|(dx|a)|$ in $\mathcal{U}_{[p]}$ by virtue of Theorem 50.3, and there exists a sequence of projectors

$$[p_\nu] \uparrow_{\nu=1}^\infty [p]$$

such that $\varphi(x)$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu=1, 2, \dots$ and

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} |\varphi(x)| |\tilde{\alpha}|(dx|a)| < +\infty.$$

If $|\tilde{\alpha}| \geq |\tilde{\beta}|$, $|a| \geq |b|$, then we have by Theorems 49.1 and 20.4

for every $\nu = 1, 2, \dots$

$$\begin{aligned} \int_{[p_\nu]} |\varphi(x)| |\tilde{\alpha}(dx|a)| &= |\tilde{\alpha}| \left(\int_{[p_\nu]} |\varphi(x)| dx|a| \right) \\ &\geq |\tilde{\alpha}| \left(\int_{[p_\nu]} |\varphi(x)| dx|b| \right) \geq |\tilde{\alpha}| \left(\int_{[p_\nu]} |\psi(x)| dx|b| \right) \\ &= \int_{[p_\nu]} |\psi(x)| |\tilde{\alpha}(dx|b)|. \end{aligned}$$

Consequently we see by Theorem 50.4 that $\psi(x)$ is integrable by $\tilde{\alpha}(dx|b)$ in $\mathcal{U}_{[p]}$, since $\tilde{\alpha}$ is continuous by Theorem 47.5.

Theorem 50.6. If both almost finite continuous functions $\varphi(x)$ and $\psi(x)$ are integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$ for a continuous $\tilde{\alpha} \in \tilde{\mathcal{R}}$, then $(\varphi + \psi)(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$ and

$$\int_{[p]} (\varphi + \psi)(x) \tilde{\alpha}(dx|a) = \int_{[p]} \varphi(x) \tilde{\alpha}(dx|a) + \int_{[p]} \psi(x) \tilde{\alpha}(dx|a).$$

Proof. Since both $|\varphi(x)|$ and $|\psi(x)|$ are integrable by $|\tilde{\alpha}|(dx|a)$ in $\mathcal{U}_{[p]}$ by virtue of Theorem 50.3, there exists by Theorem 8.10 a sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [p]$ such that both $\varphi(x)$ and $\psi(x)$ are bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \int_{[p_\nu]} |\varphi(x)| |\tilde{\alpha}(dx|a)| &< +\infty, \\ \lim_{\nu \rightarrow \infty} \int_{[p_\nu]} |\psi(x)| |\tilde{\alpha}(dx|a)| &< +\infty. \end{aligned}$$

Since $|(\varphi + \psi)(x)| \leq |\varphi(x)| + |\psi(x)|$ for every point $x \in \mathcal{U}_{[p_\nu]}$ and by Theorems 49.1, 20.1

$\int_{[p_\nu]} |(\varphi + \psi)(x)| |\tilde{\alpha}(dx|a)| \leq \int_{[p_\nu]} |\varphi(x)| |\tilde{\alpha}(dx|a)| + \int_{[p_\nu]} |\psi(x)| |\tilde{\alpha}(dx|a)|$, we see by Theorem 50.4 that $(\varphi + \psi)(x)$ is integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[p]}$. Furthermore, since we have by Theorem 49.1 and by the formula §20(4)

$\int_{[p_\nu]} (\varphi + \psi)(x) \tilde{\alpha}(dx|a) = \int_{[p_\nu]} \varphi(x) \tilde{\alpha}(dx|a) + \int_{[p_\nu]} \psi(x) \tilde{\alpha}(dx|a)$, we obtain by definition

$$\int_{[p]} (\varphi + \psi)(x) \tilde{\alpha}(dx|a) = \int_{[p]} \varphi(x) \tilde{\alpha}(dx|a) + \int_{[p]} \psi(x) \tilde{\alpha}(dx|a).$$

Theorem 50.7. For a continuous $\tilde{\alpha} \in \tilde{\mathcal{R}}$, in order that an almost finite continuous function $\varphi(x)$ be integrable by $\tilde{\alpha}(dx|b)$ in $\mathcal{U}_{[a]}$, it is necessary and sufficient that $\varphi(x) \left(\frac{b}{a}, x \right)$ be integrable by $\tilde{\alpha}(dx|a)$ in $\mathcal{U}_{[a]}$, and then we have

$$\int_{[a]} \varphi(f) \tilde{\alpha}(df b) = \int_{[a]} \varphi(f) \left(\frac{f}{a}, f\right) \tilde{\alpha}(df a).$$

Proof. Since both $\varphi(f)$ and $(\frac{f}{a}, f)$ are almost finite, there exists by Theorems 21.7 and 8.10 a sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [a]$ such that both $\varphi(f)$ and $(\frac{f}{a}, f)$ are bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$. For such $[p_\nu]$ ($\nu = 1, 2, \dots$), since we have by Theorems 22.2 and 18.10

$$\begin{aligned} [p_\nu] b &= \int_{[p_\nu]} \left(\frac{f}{a}, f\right) df a, \\ [p_\nu] |b| &= \int_{[p_\nu]} \left|\left(\frac{f}{a}, f\right)\right| df |a|, \end{aligned}$$

we obtain by Theorem 42.4

$$\begin{aligned} \int_{[p_\nu]} \varphi(f) \left(\frac{f}{a}, f\right) df a &= \int_{[p_\nu]} \varphi(f) df b, \\ \int_{[p_\nu]} \left|\varphi(f) \left(\frac{f}{a}, f\right)\right| df |a| &= \int_{[p_\nu]} |\varphi(f)| df |b|, \end{aligned}$$

and hence by Theorem 49.1

$$\begin{aligned} \int_{[p_\nu]} \varphi(f) \left(\frac{f}{a}, f\right) \tilde{\alpha}(df a) &= \int_{[p_\nu]} \varphi(f) \tilde{\alpha}(df b), \\ \int_{[p_\nu]} \left|\varphi(f) \left(\frac{f}{a}, f\right)\right| \tilde{\alpha}(df |a|) &= \int_{[p_\nu]} |\varphi(f)| \tilde{\alpha}(df |b|). \end{aligned}$$

Therefore we see easily by Theorems 50.4 and 50.3 that $\varphi(f)$ is integrable by $\tilde{\alpha}(df b)$ in $\mathcal{U}_{[a]}$ if and only if $\varphi(f) \left(\frac{f}{a}, f\right)$ is integrable by $\tilde{\alpha}(df a)$ in $\mathcal{U}_{[a]}$, and then we have by definition

$$\int_{[a]} \varphi(f) \left(\frac{f}{a}, f\right) \tilde{\alpha}(df a) = \int_{[a]} \varphi(f) \tilde{\alpha}(df b).$$

By virtue of Theorems 50.3 and 49.10 we have obviously:

Theorem 50.8. If an almost finite continuous function

$\varphi(f)$ is integrable by $\tilde{\alpha}(df a)$ in $\mathcal{U}_{[p]}$ for a continuous functional $\tilde{\alpha} \in \tilde{\mathcal{R}}$, then $[p_\nu] \downarrow_{\nu=1}^\infty 0$ implies

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu] \cap [p]} \varphi(f) \tilde{\alpha}(df a) = 0.$$

Therefore we can prove by similar methods as used in Theorem 49.11:

Theorem 50.9. If an almost finite continuous function

$\varphi(f)$ is integrable by $\tilde{\alpha}(df a)$ in $\mathcal{U}_{[p]}$ for a continuous $\tilde{\alpha} \in \tilde{\mathcal{R}}$, and if an almost finite continuous function $\psi(f)$ coincides with $\varphi(f)$ in the characteristic set $C_{\tilde{\alpha}}$ of $\tilde{\alpha}$, that is, if

$$\psi(f) = \varphi(f) \quad \text{for every point } f \in C_{\tilde{\alpha}} \cap \mathcal{U}_{[p]},$$

then $\psi(f)$ is integrable by $\tilde{\alpha}(df a)$ in $\mathcal{U}_{[p]}$ and

$$\int_{[p]} \psi(p) \tilde{\alpha}(d\beta a) = \int_{[p]} \varphi(p) \tilde{\alpha}(d\beta a).$$

By virtue of this theorem, if a continuous function $\varphi(p)$ on $C_{\tilde{\alpha}} U_{[p]}$ for the characteristic set $C_{\tilde{\alpha}}$ of $\tilde{\alpha}$ has a continuous extension $\psi(p)$ over $U_{[p]}$ such that $\psi(p)$ is integrable by $\tilde{\alpha}(d\beta a)$ in $U_{[p]}$, then we shall say that $\varphi(p)$ is integrable by $\tilde{\alpha}(d\beta a)$ in $U_{[p]}$ and we define its integral as

$$\int_{[p]} \varphi(p) \tilde{\alpha}(d\beta a) = \int_{[p]} \psi(p) \tilde{\alpha}(d\beta a).$$

Recalling Theorems 50.3 and 50.8, we can prove likewise as Theorem 49.12:

Theorem 50.10. If a continuous function $\varphi(p)$ is integrable by a continuous $\tilde{\alpha} \in \tilde{R}$, then

$$\int \varphi(p) \tilde{\alpha} d\beta$$

is a continuous linear functional on R .

§51 Relative spectra

Let R be a continuous semi-ordered linear space and let E be the proper space of R . For a point $p \in E$ and for a projector $[p_0]$ such that $U_{[p_0]} \ni p$, if a function $\varphi([p])$ of projector $[p]$ is defined for every projector $[p]$ such that $U_{[p_0]} \supset U_{[p]} \ni p$, then we define the upper limit of $\varphi([p])$ at a point $p \in E$ as

$$\overline{\lim}_{[p] \rightarrow p} \varphi([p]) = \inf_{U_{[p_0]} \ni p} \left\{ \sup_{U_{[p_0]} \ni p} \varphi([p_0][p](x)) \right\},$$

and the lower limit of $\varphi([p])$ at a point $p \in E$ as

$$\underline{\lim}_{[p] \rightarrow p} \varphi([p]) = \sup_{U_{[p_0]} \ni p} \left\{ \inf_{U_{[p_0]} \ni p} \varphi([p_0][p](x)) \right\}.$$

With this definition we have obviously

$$\overline{\lim}_{[p] \rightarrow p} \varphi([p]) \geq \underline{\lim}_{[p] \rightarrow p} \varphi([p]).$$

If $\overline{\lim}_{[p] \rightarrow p} \varphi([p]) = \underline{\lim}_{[p] \rightarrow p} \varphi([p])$, then this coincided value is denoted by

$$\lim_{[p] \rightarrow p} \varphi([p])$$

We shall consider only continuous linear functionals on R in the sequel. The totality of continuous linear functionals on R will be denoted by $\tilde{R}^c$. Then we see easily by Theorems 47.1 and 47.5 that $\tilde{R}^c$ is a linear lattice manifold of the associated space $\tilde{R}$ of R . Furthermore we have by Theorem 47.6 that $\tilde{R}^c$ is itself a universally continuous semi-ordered linear space, that $\tilde{a} \in \tilde{R}^c$ implies $\tilde{a}[p] \in \tilde{R}^c$ for every projector $[p]$ in R , and that $[\tilde{a}]\tilde{f}$ is the same in $\tilde{R}^c$ as well as in $\tilde{R}$ for every elements $\tilde{a}, \tilde{f} \in \tilde{R}^c$.

Let E^c be the proper space of $\tilde{R}^c$. We also denote by $\mathcal{U}_{[\tilde{a}]}$ the neighbourhood in E^c associated with a projector $[\tilde{a}]$ in $\tilde{R}^c$.

Theorem 51.1. Corresponding to every point $\tilde{f} \in \sum_{\tilde{a} \in \tilde{R}^c} C_{\tilde{a}}$ for the characteristic sets $C_{\tilde{a}}$ of $\tilde{a}$, there exists uniquely a point $\tilde{f}^c \in E^c$ such that we have $\tilde{f} \in C_{\tilde{a}}$ if and only if $\tilde{f}^c \in \mathcal{U}_{[\tilde{a}]}$, and every characteristic set $C_{\tilde{a}}$ is homeomorphic to the open set

$$\sum_{p \in R} \mathcal{U}_{[\tilde{a}[p]]}$$

being dense in $\mathcal{U}_{[\tilde{a}]}$ by the correspondence $C_{\tilde{a}} \ni \tilde{f} \rightarrow \tilde{f}^c \in \mathcal{U}_{[\tilde{a}]}$.

Proof. Since $C_{\tilde{a}} C_{\tilde{f}} = C_{[\tilde{a}]\tilde{f}}$ by Theorem 48.4, if

$$C_{\tilde{a}} \ni \tilde{f} \quad \text{and} \quad C_{\tilde{f}} \ni \tilde{g},$$

then we have by the formula §8(1)

$$C_{[\tilde{a}]\tilde{f}} \ni \tilde{g} \quad \text{and} \quad [[\tilde{a}]\tilde{f}] = [\tilde{a}][\tilde{f}].$$

As every neighbourhood $\mathcal{U}_{[\tilde{a}]}$ is compact, we obtain hence

$$\prod_{C_{\tilde{a}} \ni \tilde{f}} \mathcal{U}_{[\tilde{a}]} \neq \emptyset.$$

Furthermore this intersection is composed only of a single point.

Because, if $\mathcal{U}_{[\tilde{f}]} \ni q$ for a point

$$q \in \prod_{C_{\tilde{a}} \ni \tilde{f}} \mathcal{U}_{[\tilde{a}]},$$

then we have obviously

$$\mathcal{U}_{[\tilde{f}]} \mathcal{U}_{[\tilde{a}]} \neq \emptyset \quad \text{for} \quad C_{\tilde{a}} \ni \tilde{f},$$

and hence we conclude by Theorem 48.1 that

$$C_{\mathcal{X}} \ni \mathcal{F} \text{ implies } C_{\mathcal{X}} C_{\mathcal{X}} \neq 0.$$

Consequently $U_{\{p\}} \ni \mathcal{F}$ implies $C_{\mathcal{X}} U_{\{p\}} \neq 0$, since for $C_{\mathcal{X}} \ni \mathcal{F}$ we have by the formula §48(5)

$$C_{\mathcal{X}_{\{p\}}} = C_{\mathcal{X}} U_{\{p\}} \ni \mathcal{F},$$

and hence $C_{\mathcal{X}} C_{\mathcal{X}_{\{p\}}} \neq 0$. Therefore we obtain that

$$U_{\{\mathcal{F}\}} \ni \mathcal{Q} \text{ implies } C_{\mathcal{X}} \ni \mathcal{F}.$$

Hence we conclude that to every point $\mathcal{F} \in \sum_{\mathcal{X} \in \mathcal{X}^c} C_{\mathcal{X}}$ there exists uniquely a point $\mathcal{F}^c \in E^c$ such that

$$\{\mathcal{F}^c\} = \prod_{C_{\mathcal{X}} \ni \mathcal{F}} U_{\{\mathcal{X}\}}.$$

and we have $C_{\mathcal{X}} \ni \mathcal{F}$ if and only if $U_{\{\mathcal{X}\}} \ni \mathcal{F}^c$.

For a point $\mathcal{Q} \in E^c$, if

$$U_{\{\mathcal{X}_{\{p\}}\}} \ni \mathcal{Q} \text{ and } U_{\{\mathcal{X}_{\{q\}}\}} \ni \mathcal{Q}$$

then, since we have by the formulas §8(1) and §46(12)

$$[\mathcal{X}_{\{p\}}][\mathcal{X}_{\{q\}}] = [[\mathcal{X}_{\{p\}}]\mathcal{X}_{\{q\}}] = [\mathcal{X}_{\{p\}}[\mathcal{X}_{\{q\}}]],$$

we obtain $U_{\{[\mathcal{X}_{\{p\}}][\mathcal{X}_{\{q\}}]\}} \ni \mathcal{Q}$. As $C_{\mathcal{X}_0} U_{\{p\}}$ is compact, we have therefore

$$\prod_{U_{\{\mathcal{X}_0, \{p\}\}} \ni \mathcal{Q}} C_{\mathcal{X}_0} U_{\{p\}} \neq 0 \text{ for every point } \mathcal{Q} \in U_{\{\mathcal{X}_0, \{p\}\}}.$$

Furthermore this intersection is composed only of a single point.

Because, if

$$U_{\{q\}} \ni \mathcal{F} \text{ for a point } \mathcal{F} \in \prod_{U_{\{\mathcal{X}_0, \{p\}\}} \ni \mathcal{Q}} C_{\mathcal{X}_0} U_{\{p\}},$$

then we have $C_{\mathcal{X}_0} U_{\{p\}} U_{\{q\}} \neq 0$ for $U_{\{\mathcal{X}_0, \{p\}\}} \ni \mathcal{Q}$, and hence we conclude that

$$U_{\{[\mathcal{X}_0, \{p\}]\}} \ni \mathcal{Q} \text{ implies } U_{\{[\mathcal{X}_0, \{p\}][\mathcal{X}_0, \{q\}]\}} \neq 0,$$

since we have by the formula §48(5) and by Theorem 48.4

$$C_{\mathcal{X}_0} U_{\{p\}} U_{\{q\}} = C_{\mathcal{X}_0, \{p\}} C_{\mathcal{X}_0, \{q\}} = C_{[\mathcal{X}_0, \{p\}][\mathcal{X}_0, \{q\}]}$$

On the other hand, for any projector $[\mathcal{X}]$ such that $U_{\{\mathcal{X}\}} \ni \mathcal{Q}$ we have naturally

$$U_{\{[\mathcal{X}_0, \{p\}]\}} U_{\{\mathcal{X}\}} \ni \mathcal{Q},$$

and there exists a projector $[p]$ such that

$$C_{\mathcal{X}_0} C_{\mathcal{X}} U_{\{p\}} = C_{\mathcal{X}_0} U_{\{p\}}$$

by virtue of Theorem 48.6. For such $[p]$ we obtain by the formula §48(5) and by Theorem 48.4

$$C_{[\mathcal{E}]\bar{\alpha}_{[p]}} = C_{\bar{\alpha}_{[p]}},$$

and hence by Theorem 48.2

$$U_{[\bar{\alpha}_{[p]}]} = U_{[[\mathcal{E}]\bar{\alpha}_{[p]}]} = U_{[\bar{\alpha}_{[p]}]} U_{[\mathcal{E}]} \ni q.$$

Consequently we obtain

$$U_{[\mathcal{E}]} U_{[\bar{\alpha}_{[q]}]} \supset U_{[\bar{\alpha}_{[p]}]} U_{[\bar{\alpha}_{[q]}]} \neq 0 \quad \text{for every } U_{[\mathcal{E}]} \ni q.$$

Therefore we conclude now that

$$U_{[q]} \ni \mathcal{F} \text{ implies } U_{[\bar{\alpha}_{[q]}]} \ni q,$$

and hence

$$\{\mathcal{F}\} = \bigcap_{U_{[\bar{\alpha}_{[p]}]} \ni q} C_{\bar{\alpha}_{[p]}}.$$

For such a point $\mathcal{F}$ we have obviously by definition that

$$U_{[\bar{\alpha}_{[p]}]} \ni q \text{ implies } U_{[\bar{\alpha}_{[p]}]} \ni \mathcal{F}^c.$$

Furthermore to every $U_{[\mathcal{E}]} \ni q$ there exists a projector $[p]$ such that

$$U_{[\mathcal{E}]} \supset U_{[\bar{\alpha}_{[p]}]} \ni q$$

as proved just above. Therefore we can conclude that $q = \mathcal{F}^c$.

Thus to every point $q \in \sum_{p \in R} U_{[\bar{\alpha}_{[p]}]}$, there exists uniquely a point $\mathcal{F} \in C_{\bar{\alpha}}$ such that $q = \mathcal{F}^c$, and we have

$$C_{\bar{\alpha}} U_{[p]} \ni \mathcal{F} \text{ if and only if } U_{[\bar{\alpha}_{[p]}]} \ni \mathcal{F}^c.$$

Furthermore to every $U_{[\mathcal{E}]} \ni q$ there exists a projector $[p]$ such that

$$U_{[\mathcal{E}]} \supset U_{[\bar{\alpha}_{[p]}]} \ni q,$$

as proved just above. Therefore $C_{\bar{\alpha}}$ is homeomorphic to the open set

$$\sum_{p \in R} U_{[\bar{\alpha}_{[p]}]}$$

by the correspondence $C_{\bar{\alpha}} \ni \mathcal{F} \rightarrow \mathcal{F}^c \in E^c$ obtained just now.

Here the open set $\sum_{p \in R} U_{[\bar{\alpha}_{[p]}]}$ is dense in $U_{[\bar{\alpha}]}$, because if

$$U_{[\bar{\alpha}_{[p]}]} U_{[\mathcal{E}]} = 0 \quad \text{for every element } p \in R,$$

then we have by the formulas §46(12) and §15(5) for every $p \in R$

$$([\tilde{x}] \tilde{a})(p) = [\tilde{x}](\tilde{a}(p)) = 0,$$

and hence $[\tilde{x}] \tilde{a} = 0$, that is, $U_{[\tilde{a}]} U_{[\tilde{x}]} = 0$ by the formula §15(5).

Since $\tilde{R}^c$ is a universally continuous semi-ordered linear space, we can consider relative spectra in $\tilde{R}^c$. For every elements $\tilde{a}$ and $\tilde{x} \in \tilde{R}^c$ we define the relative spectrum of $\tilde{x}$ by $\tilde{a}$ at a point $p \in C_{\tilde{a}}$ as the relative spectrum of $\tilde{x}$ by $\tilde{a}$ at the point p^c corresponding to p by Theorem 51.1, and denoted by $(\frac{\tilde{x}}{\tilde{a}}, p)$, that is,

$$(\frac{\tilde{x}}{\tilde{a}}, p) = (\frac{\tilde{x}}{\tilde{a}}, p^c) \quad (p \in C_{\tilde{a}})$$

for the correspondence $C_{\tilde{a}} \ni p \rightarrow p^c \in U_{[\tilde{a}]}$ in Theorem 51.1.

With this definition we conclude immediately from Theorems 19.2 and 19.3:

Theorem 51.2. $(\frac{\tilde{x}}{\tilde{a}}, p)$ is an almost finite continuous function on $C_{\tilde{a}}$ and for every real number $\alpha \neq 0$ we have

$$(\frac{\alpha \tilde{x}}{\tilde{a}}, p) = \alpha (\frac{\tilde{x}}{\tilde{a}}, p) = \alpha^2 (\frac{\tilde{x}}{\alpha \tilde{a}}, p).$$

Theorem 51.3. If $(\frac{\tilde{x}}{\tilde{a}}, p) = (\frac{\tilde{c}}{\tilde{a}}, p)$ for every point $p \in C_{\tilde{a}}$, then we have $[\tilde{a}] \tilde{x} = [\tilde{a}] \tilde{c}$.

Proof. From the assumption we conclude by Theorem 51.1 and by definition that we have

$$(\frac{\tilde{x}}{\tilde{a}}, q) = (\frac{\tilde{c}}{\tilde{a}}, q)$$

in an open set being dense in $U_{[\tilde{a}]}$, and consequently this relation holds for every point $q \in U_{[\tilde{a}]}$. Hence we obtain by Theorem 19.5

$$[\tilde{a}] \tilde{x} = [\tilde{a}] \tilde{c}.$$

We obtain immediately by Theorems 18.9, 18.5, and 18.6:

Theorem 51.4. For a positive element $\tilde{a} \in \tilde{R}^c$ we have

$$(\frac{\tilde{x} \vee \tilde{c}}{\tilde{a}}, p) = \text{Max} \{ (\frac{\tilde{x}}{\tilde{a}}, p), (\frac{\tilde{c}}{\tilde{a}}, p) \},$$

$$(\frac{\tilde{x} \wedge \tilde{c}}{\tilde{a}}, p) = \text{Min} \{ (\frac{\tilde{x}}{\tilde{a}}, p), (\frac{\tilde{c}}{\tilde{a}}, p) \},$$

$$(\frac{\alpha \tilde{x} + \beta \tilde{c}}{\tilde{a}}, p) = \alpha (\frac{\tilde{x}}{\tilde{a}}, p) + \beta (\frac{\tilde{c}}{\tilde{a}}, p),$$

if the right side has any sense.

Theorem 51.5. For every point $\mathfrak{P} \in C_{\tilde{\alpha}} \mathcal{U}_{[a]}$ we have

$$\lim_{[p] \rightarrow \mathfrak{P}} \frac{\tilde{\mathcal{F}}([p]a)}{\tilde{\alpha}([p]a)} = \left(\frac{\tilde{\mathcal{F}}}{\tilde{\alpha}}, \mathfrak{P} \right).$$

Proof. We suppose first $\mathfrak{P} \in C_{\tilde{\alpha}^+} \mathcal{U}_{[a^+]}$. Since by

Theorem 48.1 $C_{\tilde{\alpha}^+} C_{\tilde{\alpha}^-} = 0$, there exists a projector $[p_0]$ such that

$$\mathfrak{P} \in \mathcal{U}_{[p_0]} \subset \mathcal{U}_{[a^+]} \quad \text{and} \quad \mathcal{U}_{[p_0]} C_{\tilde{\alpha}^-} = 0.$$

For such $[p_0]$ we have

$$\tilde{\alpha}([p_0][p]a) > 0 \quad \text{for every } \mathcal{U}_{[p]} \ni \mathfrak{P}.$$

Because we have $\tilde{\alpha}^-[p_0] = 0$ by the formula §48(3) and $[p_0]a^- = 0$ by the formula §15(5), and hence

$$\tilde{\alpha}([p_0][p]a) = \tilde{\alpha}^+([p_0][p]a^+) \geq 0.$$

If $\tilde{\alpha}^+([p_0][p]a^+) = 0$, then we have by Theorem 47.3

$$\tilde{\alpha}^+[p_0][p](a^+) \ni 0,$$

and consequently by the formula §48(3)

$$C_{\tilde{\alpha}^+} \mathcal{U}_{[p_0]} \mathcal{U}_{[p]} \mathcal{U}_{[a^+]} = 0,$$

contradicting the assumption $\mathcal{U}_{[p]} \ni \mathfrak{P}$.

As $\mathfrak{P} \in C_{\tilde{\alpha}^+} \mathcal{U}_{[a^+]} = C_{\tilde{\alpha}^+[a^+]}$ by assumption, we have $\mathfrak{P}^c \in \mathcal{U}_{[\tilde{\alpha}^+]}$ for the correspondence $\mathfrak{P} \rightarrow \mathfrak{P}^c \in \mathcal{U}_{[\tilde{\alpha}]}$ in Theorem 51.1. If

$$\left(\frac{\tilde{\mathcal{F}}}{\tilde{\alpha}}, \mathfrak{P} \right) = \left(\frac{\tilde{\mathcal{F}}}{\tilde{\alpha}}, \mathfrak{P}^c \right) < \lambda,$$

then we have by definition of relative spectra

$$\mathcal{U}_{[(\lambda\tilde{\alpha} - \tilde{\mathcal{F}})^+]} \ni \mathfrak{P}^c,$$

and hence $C_{(\lambda\tilde{\alpha} - \tilde{\mathcal{F}})^+} \ni \mathfrak{P}$. Therefore there exists a projector $[q_0]$ such that

$$\mathfrak{P} \in \mathcal{U}_{[q_0]} \subset \mathcal{U}_{[p_0]} \quad \text{and} \quad \mathcal{U}_{[q_0]} C_{(\lambda\tilde{\alpha} - \tilde{\mathcal{F}})^-} = 0,$$

as proved just above. For such $[q_0]$ we have by the formula §48(3)

$$(\lambda\tilde{\alpha} - \tilde{\mathcal{F}})[q_0] = (\lambda\tilde{\alpha} - \tilde{\mathcal{F}})^+[q_0] \geq 0.$$

As $\mathcal{U}_{[q_0]} \subset \mathcal{U}_{[p_0]} \subset \mathcal{U}_{[a^+]}$, we obtain by Theorems 8.2 and 7.16

$$[q_0]a = [q_0][a^+]a = [q_0]a^+,$$

and consequently for every projector $[p]$

$$(\lambda\tilde{\alpha} - \tilde{\mathcal{F}})([q_0][p]a) \geq 0,$$

that is,

$$\lambda\tilde{\alpha}([q_0][p]a) \geq \tilde{\mathcal{F}}([q_0][p]a).$$

Since $\tilde{a}([q_0][p]a) > 0$ for every $U_{[p]} \ni p$ as proved at first, we have hence

$$\lambda \geq \frac{\tilde{f}([q_0][p]a)}{\tilde{a}([q_0][p]a)} \quad \text{for every } U_{[p]} \ni p.$$

Consequently we obtain

$$\lambda \geq \overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]a)}{\tilde{a}([p]a)}$$

for every real number $\lambda > (\frac{\tilde{f}}{\tilde{a}}, p)$. Therefore we have

$$(\frac{\tilde{f}}{\tilde{a}}, p) \geq \overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]a)}{\tilde{a}([p]a)}.$$

Since we have by Theorem 51.2

$$(\frac{\tilde{f}}{\tilde{a}}, p) = -(\frac{-\tilde{f}}{\tilde{a}}, p),$$

and by definition

$$\overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]a)}{\tilde{a}([p]a)} = -\overline{\lim}_{[p] \rightarrow p} \frac{-\tilde{f}([p]a)}{\tilde{a}([p]a)},$$

and further as proved just now

$$(\frac{-\tilde{f}}{\tilde{a}}, p) \geq \overline{\lim}_{[p] \rightarrow p} \frac{-\tilde{f}([p]a)}{\tilde{a}([p]a)},$$

we obtain also

$$(\frac{\tilde{f}}{\tilde{a}}, p) \leq \overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]a)}{\tilde{a}([p]a)}.$$

Accordingly we have for every point $p \in C_{\tilde{a}} + U_{[a]}$

$$\overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]a)}{\tilde{a}([p]a)} = (\frac{\tilde{f}}{\tilde{a}}, p).$$

In general, if $p \in C_{\tilde{a}} U_{[a]}$, then putting $\varepsilon = \pm 1$, $\delta = \pm 1$,

we have

$$p \in C_{(\varepsilon \tilde{a})} + U_{[(\delta a)]},$$

and hence as established just now

$$\overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]\delta a)}{\varepsilon \tilde{a}([p]\delta a)} = (\frac{\tilde{f}}{\varepsilon \tilde{a}}, p).$$

Therefore we have by Theorem 51.2

$$\overline{\lim}_{[p] \rightarrow p} \frac{\tilde{f}([p]a)}{\tilde{a}([p]a)} = (\frac{\tilde{f}}{\tilde{a}}, p) \text{ for every point } p \in C_{\tilde{a}} U_{[a]}.$$

We obtain immediately by this theorem:

Theorem 51.6. We have for every point $p \in C_{\tilde{a}} U_{[p]}$

$$(\frac{\tilde{f}([p]a)}{\tilde{a}}, p) = (\frac{\tilde{f}}{\tilde{a}([p]a)}, p) = (\frac{\tilde{f}}{\tilde{a}}, p).$$

If $p \in C_{\tilde{a}}$ but $\notin U_{[p]}$, then we have

$$(\frac{\tilde{f}([p]a)}{\tilde{a}}, p) = 0.$$

Theorem 51.7. If a continuous function $\varphi(p)$ on $C_{\tilde{a}}$ is

integrable by an element $\tilde{a} \in \tilde{R}^c$ and

$$\tilde{f} = \int \varphi(\mathfrak{p}) \tilde{\alpha} d\mathfrak{p},$$

then we have $[\tilde{f}] \leq [\tilde{\alpha}]$ and

$$\left(\frac{\tilde{f}}{\tilde{\alpha}}, \mathfrak{p}\right) = \varphi(\mathfrak{p}) \quad \text{for every point } \mathfrak{p} \in C_{\tilde{\alpha}}.$$

Proof. To every point $\mathfrak{p}_0 \in C_{\tilde{\alpha}+}$ there exists obviously a positive element $\alpha \in R$ such that

$$\mathfrak{p}_0 \in U_{[\alpha]} \quad \text{and} \quad U_{[\alpha]} C_{\tilde{\alpha}-} = 0.$$

As $\varphi(\mathfrak{p})$ is integrable by $\tilde{\alpha}$, there exists an almost finite continuous function $\psi(\mathfrak{p})$ on $U_{[\alpha]}$ such that $\psi(\mathfrak{p})$ is integrable by $\tilde{\alpha}(d\mathfrak{p} \alpha)$ in $U_{[\alpha]}$ and

$$\psi(\mathfrak{p}) = \varphi(\mathfrak{p}) \quad \text{for every point } \mathfrak{p} \in C_{\tilde{\alpha}} U_{[\alpha]}.$$

If $\varphi(\mathfrak{p}_0) = \psi(\mathfrak{p}_0) < \lambda$, then there exists a projector $[p_0] \leq [\alpha]$ such that $U_{[p_0]} \ni \mathfrak{p}_0$ and

$$\psi(\mathfrak{p}) < \lambda \quad \text{for every point } \mathfrak{p} \in U_{[p_0]}.$$

For such $[p_0]$, since $\tilde{\alpha}[\alpha] = \tilde{\alpha}^+[\alpha]$ by the formula §48(3),

$[p] \leq [p_0]$ implies by Theorems 49.8 and 49.1

$$\begin{aligned} \tilde{f}([p]\alpha) &= \int_{[p]} \psi(\mathfrak{p}) \tilde{\alpha}(d\mathfrak{p} \alpha) = \int_{[p]} \psi(\mathfrak{p}) \tilde{\alpha}^+(d\mathfrak{p} \alpha) \\ &\leq \lambda \int_{[p]} \tilde{\alpha}^+(d\mathfrak{p} \alpha) = \lambda \tilde{\alpha}^+([p]\alpha) = \lambda \tilde{\alpha}([p]\alpha). \end{aligned}$$

Since $\tilde{\alpha}([p]\alpha) > 0$ for every $U_{[p]} \ni \mathfrak{p}_0$ as proved in Proof of Theorem 51.5, we obtain hence

$$\frac{\tilde{f}([p]\alpha)}{\tilde{\alpha}([p]\alpha)} \leq \lambda \quad \text{for } \mathfrak{p}_0 \in U_{[p]} \subset U_{[p_0]},$$

and consequently for every real number $\lambda > \varphi(\mathfrak{p}_0)$

$$\lim_{[p] \rightarrow \mathfrak{p}_0} \frac{\tilde{f}([p]\alpha)}{\tilde{\alpha}([p]\alpha)} \leq \lambda.$$

Therefore we conclude by Theorem 51.5

$$\varphi(\mathfrak{p}) \geq \left(\frac{\tilde{f}}{\tilde{\alpha}}, \mathfrak{p}\right) \quad \text{for every point } \mathfrak{p} \in C_{\tilde{\alpha}+}.$$

On the other hand we also have

$$-\tilde{f} = \int (-\varphi)(\mathfrak{p}) \tilde{\alpha} d\mathfrak{p},$$

and hence we obtain

$$-\varphi(\mathfrak{p}) \geq \left(\frac{-\tilde{f}}{\tilde{\alpha}}, \mathfrak{p}\right) \quad \text{for every point } \mathfrak{p} \in C_{\tilde{\alpha}+},$$

as proved just now. Consequently we have

$$\varphi(\mathfrak{p}) = \left(\frac{\tilde{f}}{\tilde{\alpha}}, \mathfrak{p}\right) \quad \text{for every point } \mathfrak{p} \in C_{\tilde{\alpha}+}.$$

For every point $\mathfrak{p} \in C_{\tilde{\alpha}-}$ we have $\mathfrak{p} \in C_{(-\tilde{\alpha})+}$ and

$$-\tilde{\mathcal{E}} = \int \varphi(\mathcal{F})(-\alpha) d\mathcal{F}.$$

Accordingly we also obtain for every point $\mathcal{F} \in C_{\mathcal{K}}^-$

$$\varphi(\mathcal{F}) = \left(\frac{-\tilde{\mathcal{E}}}{-\alpha}, \mathcal{F} \right) = \left(\frac{\tilde{\mathcal{E}}}{\alpha}, \mathcal{F} \right),$$

and it is evident by the formula §48(12) that $C_{\mathcal{K}} = C_{\mathcal{K}^+} + C_{\mathcal{K}^-}$.

Since $\tilde{\alpha}[\mathcal{P}] = 0$ implies $\tilde{\mathcal{E}}[\mathcal{P}] = 0$ by Theorem 49.7, we have by the formula §48(8)

$$C_{\mathcal{K}} > C_{\tilde{\mathcal{E}}},$$

and hence $[\tilde{\alpha}] \geq [\tilde{\mathcal{E}}]$ by Theorem 48.2.

Theorem 51.8. For every elements $\tilde{\alpha}, \tilde{\mathcal{E}} \in \tilde{\mathcal{R}}^c$ the relative spectrum $(\frac{\tilde{\mathcal{E}}}{\tilde{\alpha}}, \mathcal{F})$ is integrable by $\tilde{\alpha}$ and

$$[\tilde{\alpha}]\tilde{\mathcal{E}} = \int \left(\frac{\tilde{\mathcal{E}}}{\tilde{\alpha}}, \mathcal{F} \right) \tilde{\alpha} d\mathcal{F}.$$

Proof. To every element $a \in \mathcal{R}$ there exists by Theorem 51.2 a sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [a]$ such that $(\frac{\tilde{\mathcal{E}}}{\tilde{\alpha}}, \mathcal{F})$ is bounded in $C_{\mathcal{K}} \mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$ and

$$\sum_{\nu=1}^\infty C_{\mathcal{K}} \mathcal{U}_{[p_\nu]}$$

is dense in $C_{\mathcal{K}} \mathcal{U}_{[a]}$. For such $[p_\nu]$ ($\nu = 1, 2, \dots$), we see easily that there exists a sequence of bounded continuous functions $\varphi_\nu(\mathcal{F})$ ($\nu = 1, 2, \dots$) such that

$$\varphi_\nu(\mathcal{F}) = \left(\frac{\tilde{\mathcal{E}}}{\tilde{\alpha}}, \mathcal{F} \right) \quad \text{for every point } \mathcal{F} \in C_{\mathcal{K}} \mathcal{U}_{[p_\nu]},$$

$$\varphi_{\nu+1}(\mathcal{F}) = \varphi_\nu(\mathcal{F}) \quad \text{for every point } \mathcal{F} \in \mathcal{U}_{[p_{\nu+1}]}.$$

By virtue of Theorem 41.2 we see then that there exists a continuous function $\varphi_0(\mathcal{F})$ on $\mathcal{U}_{[a]}$ such that we have for every $\nu = 1, 2, \dots$

$$\varphi_0(\mathcal{F}) = \varphi_\nu(\mathcal{F}) \quad \text{for every point } \mathcal{F} \in \mathcal{U}_{[p_{\nu+1}]}.$$

Since every $\varphi_\nu(\mathcal{F})$ is bounded in $\mathcal{U}_{[p_\nu]}$, $\varphi_0(\mathcal{F})$ is obviously almost finite in $\mathcal{U}_{[a]}$ and

$$\varphi_0(\mathcal{F}) = \left(\frac{\tilde{\mathcal{E}}}{\tilde{\alpha}}, \mathcal{F} \right) \quad \text{for every point } \mathcal{F} \in C_{\mathcal{K}} \mathcal{U}_{[a]}.$$

For every sequence of projectors $[p_\nu] \uparrow_{\nu=1}^\infty [a]$ such that $\varphi_0(\mathcal{F})$ is bounded in $\mathcal{U}_{[p_\nu]}$ for every $\nu = 1, 2, \dots$, putting

$$\tilde{\mathcal{E}}_\nu = \int \varphi_0(\mathcal{F}) \tilde{\alpha}[p_\nu] d\mathcal{F} \quad (\nu = 1, 2, \dots),$$

we have by the previous theorem $[\tilde{\ell}_\nu] \subseteq [\tilde{\alpha}[p_\nu]]$ and

$$\left(\frac{\tilde{\ell}_\nu}{\tilde{\alpha}[p_\nu]}, \mathcal{F}\right) = \varphi_0(\mathcal{F}) \quad \text{for every point } \mathcal{F} \in C_{\tilde{\alpha}} \cup [p_\nu].$$

Accordingly we obtain by Theorem 51.6

$$\left(\frac{\tilde{\ell}_\nu}{\tilde{\alpha}}, \mathcal{F}\right) = \left(\frac{\tilde{\ell}[p_\nu]}{\tilde{\alpha}}, \mathcal{F}\right) \quad \text{for every point } \mathcal{F} \in C_{\tilde{\alpha}},$$

and hence by Theorem 51.3 for every $\nu = 1, 2, \dots$

$$[\tilde{\alpha}] \tilde{\ell}[p_\nu] = [\tilde{\alpha}] \tilde{\ell}_\nu = \tilde{\ell}_\nu.$$

This relation yields by Theorem 49.7

$$[\tilde{\alpha}] \tilde{\ell}[p_\nu](a) = \int_{[a]} \varphi_0(\mathcal{F}) \tilde{\alpha}[p_\nu](d\mathcal{F}a) = \int_{[p_\nu]} \varphi_0(\mathcal{F}) \tilde{\alpha}(d\mathcal{F}a),$$

and hence

$$\lim_{\nu \rightarrow \infty} \int_{[p_\nu]} \varphi_0(\mathcal{F}) \tilde{\alpha}(d\mathcal{F}a) = [\tilde{\alpha}] \tilde{\ell}(a).$$

Therefore $\varphi_0(\mathcal{F})$ is integrable by $\tilde{\alpha}(d\mathcal{F}a)$ in $\mathcal{U}_{[a]}$ and

$$\int_{[a]} \varphi_0(\mathcal{F}) \tilde{\alpha}(d\mathcal{F}a) = [\tilde{\alpha}] \tilde{\ell}(a).$$

Consequently we see by definition that $\left(\frac{\tilde{\ell}}{\tilde{\alpha}}, \mathcal{F}\right)$ is integrable by $\tilde{\alpha}$ and

$$[\tilde{\alpha}] \tilde{\ell} = \int \left(\frac{\tilde{\ell}}{\tilde{\alpha}}, \mathcal{F}\right) \tilde{\alpha} d\mathcal{F}.$$

§52 Permissible dilatators

Let R be a continuous semi-ordered linear space. For a linear functional L on R , a dilatator T with domain D in R is said to be permissible to L , if

$$\lim_{\nu \rightarrow \infty} a_\nu = 0, \quad a_\nu \in D \quad (\nu = 1, 2, \dots)$$

implies $\lim_{\nu \rightarrow \infty} L(Ta_\nu) = 0$.

Theorem 52.1. If a dilatator T with domain D is permissible to a linear functional L on R , then there exists uniquely a continuous linear functional $\tilde{L}$ on R such that

$$\tilde{L}(a) = L(Ta) \quad \text{for every element } a \in D.$$

Such a continuous linear functional $\tilde{L}$ will be denoted by $L T$ in the sequel.

Proof. Since D is by definition dense in R , to every element $a \in R$ there exists a sequence of elements $a_\nu \in D$ ($\nu = 1,$

2, ...) such that

$$\lim_{\nu \rightarrow \infty} a_\nu = a.$$

Conversely if $a = \lim_{\nu \rightarrow \infty} a_\nu$, $a_\nu \in D$ ($\nu = 1, 2, \dots$), then for every two subsequences a_{μ_ν} and a_{ρ_ν} ($\nu = 1, 2, \dots$) we have obviously

$$\lim_{\nu \rightarrow \infty} (a_{\mu_\nu} - a_{\rho_\nu}) = 0, \quad a_{\mu_\nu} - a_{\rho_\nu} \in D \quad (\nu = 1, 2, \dots),$$

and hence

$$\lim_{\nu \rightarrow \infty} \{L(Ta_{\mu_\nu}) - L(Ta_{\rho_\nu})\} = 0,$$

since T is by assumption permissible to L . Consequently the sequence $L(Ta_\nu)$ ($\nu = 1, 2, \dots$) is convergent, and we see easily that the limit

$$\lim_{\nu \rightarrow \infty} L(Ta_\nu)$$

is uniquely determined corresponding to the limit $\lim_{\nu \rightarrow \infty} a_\nu$, that is,

$$\lim_{\nu \rightarrow \infty} a_\nu = \lim_{\nu \rightarrow \infty} b_\nu; \quad a_\nu, b_\nu \in D \quad (\nu = 1, 2, \dots)$$

implies $\lim_{\nu \rightarrow \infty} L(Ta_\nu) = \lim_{\nu \rightarrow \infty} L(Tb_\nu)$. Therefore, putting

$$\tilde{L}(a) = \lim_{\nu \rightarrow \infty} L(Ta_\nu) \quad \text{for } a = \lim_{\nu \rightarrow \infty} a_\nu, \quad a_\nu \in D,$$

we obtain a functional $\tilde{L}$ on R . It is evident that such $\tilde{L}$ is a linear functional on R . We can prove furthermore that $\tilde{L}$ is continuous. In fact, to every sequence of elements

$a_\nu \downarrow_{\nu=1}^\infty 0$ there exist $a_{\nu, \mu} \in D$ ($\nu, \mu = 1, 2, \dots$) such that

$$\lim_{\mu \rightarrow \infty} a_{\nu, \mu} = a_\nu \quad \text{for every } \nu = 1, 2, \dots$$

For such $a_{\nu, \mu} \in D$ ($\nu, \mu = 1, 2, \dots$) we have by Theorem 5.9

$$\lim_{\mu \rightarrow \infty} (a_{\nu, \mu}^+ \wedge a_\nu) = a_\nu^+ \wedge a_\nu = a_\nu,$$

and by Theorem 43.12

$$a_{\nu, \mu}^+ \wedge a_\nu \in D \quad (\nu = 1, 2, \dots).$$

Accordingly we have

$$\tilde{L}(a_\nu) = \lim_{\mu \rightarrow \infty} L(T(a_{\nu, \mu}^+ \wedge a_\nu)) \quad (\nu = 1, 2, \dots),$$

and hence there exists a sequence μ_ν ($\nu = 1, 2, \dots$) such that

$$|\tilde{L}(a_\nu) - L(T(a_{\nu, \mu_\nu}^+ \wedge a_\nu))| < \frac{1}{\nu} \quad (\nu = 1, 2, \dots).$$

As $0 \leq a_{\nu, \mu_\nu}^+ \wedge a_\nu \leq a_\nu$, we have obviously

$$\lim_{\nu \rightarrow \infty} a_{\nu}^+ \wedge a_{\nu} = 0,$$

and hence we obtain $\lim_{\nu \rightarrow \infty} \tilde{L}(a_{\nu}) = 0$. Therefore $\tilde{L}$ is continuous. The uniqueness of such $\tilde{L}$ is evident by Theorem 47.7.

Let $\tilde{R}^c$ be the totality of continuous linear functionals on R as denoted in the previous section.

Theorem 52.2. In order that a dilatator T in R be permissible to an element $\tilde{a} \in \tilde{R}^c$, it is necessary and sufficient that the spectral function $\varphi_T(\mathfrak{P})$ of T be integrable by $\tilde{a}$, and then we have

$$\tilde{a} T = \int \varphi_T(\mathfrak{P}) \tilde{a} d\mathfrak{P}.$$

Proof. By virtue of Theorem 43.7 the spectral function $\varphi_T(\mathfrak{P})$ of T is almost finite and

$$T = \int \varphi_T(\mathfrak{P}) d\mathfrak{P}.$$

For every sequence of projectors $[p_{\nu}] \uparrow_{\nu=1}^{\infty} [a]$ such that $\varphi_T(\mathfrak{P})$ is bounded in $\mathcal{U}_{[p_{\nu}]}$ for every $\nu = 1, 2, \dots$, we have by Theorem 49.1

$$\int_{[p_{\nu}]} \varphi_T(\mathfrak{P}) \tilde{a} d\mathfrak{P} a = \tilde{a} \left(\int_{[p_{\nu}]} \varphi_T(\mathfrak{P}) d\mathfrak{P} a \right) = \tilde{a} (T[p_{\nu}]a).$$

If T is permissible to $\tilde{a}$, then we have by the previous theorem and Theorem 47.7

$$\lim_{\nu \rightarrow \infty} \tilde{a} (T[p_{\nu}]a) = \lim_{\nu \rightarrow \infty} \tilde{a} T([p_{\nu}]a) = \tilde{a} T(a),$$

since $\lim_{\nu \rightarrow \infty} [p_{\nu}]a = a$ by Theorem 8.11. Then $\varphi_T(\mathfrak{P})$ is by definition integrable by $\tilde{a}$ and

$$\tilde{a} T = \int \varphi_T(\mathfrak{P}) \tilde{a} d\mathfrak{P}.$$

Conversely, if $\varphi_T(\mathfrak{P})$ is integrable by $\tilde{a}$, then, putting

$$\tilde{L} = \int \varphi_T(\mathfrak{P}) \tilde{a} d\mathfrak{P},$$

we have $\tilde{L} \in \tilde{R}^c$ by Theorem 50.10, and if $\varphi_T(\mathfrak{P})$ is integrable by an element $a \in R$ in $\mathcal{U}_{[a]}$, that is, if Ta has a sense, then we have by Theorem 50.1

$$\tilde{L}(a) = \int_{[a]} \varphi_T(\mathfrak{P}) \tilde{a} d\mathfrak{P} a = \tilde{a} (Ta).$$

Since $\lim_{\nu \rightarrow \infty} a_{\nu} = 0$ implies $\lim_{\nu \rightarrow \infty} \tilde{L}(a_{\nu}) = 0$ by Theorem 47.7, we conclude hence by definition that T is permissible to $\tilde{a}$ if $\varphi_T(\mathfrak{P})$ is integrable by $\tilde{a}$.

Theorem 52.3. To every elements $\tilde{a}$, $\tilde{b} \in \tilde{R}^c$ and every projector $[p]$ in R there exists a principal dilatator T such that T is permissible to $\tilde{a}$ and

$$[\tilde{a}] \tilde{b} [p] = \tilde{a} T.$$

Proof. As proved in proof of Theorem 51.8, there exists an almost finite continuous function $\varphi(p)$ on the proper space E of R such that

$$\varphi(p) = \begin{cases} (\frac{\tilde{b}}{\tilde{a}}, p) & \text{for } p \in C_{\tilde{a}} \cup [p], \\ 0 & \text{for } p \in U_{[p]}, \end{cases}$$

and we have

$$[\tilde{a}] \tilde{b} [p] = \int \varphi(p) \tilde{a} d p.$$

Putting $T = \int \varphi(p) d p$, we obtain by Theorem 43.6 a dilatator T whose spectral function coincides with $\varphi(p)$. By virtue of the previous theorem, such a dilatator T is permissible to $\tilde{a}$ and we have

$$\tilde{a} T = \int \varphi(p) \tilde{a} d p = [\tilde{a}] \tilde{b} [p].$$

Furthermore, since $\varphi(p) = 0$ for $p \in U_{[p]}$, we have obviously $T[p] = T$, and hence T is principal by definition.

Theorem 52.4. In order that a dilatator T in R be permissible to an element $\tilde{a} \in \tilde{R}^c$, it is necessary and sufficient that there exists an element $\tilde{b} \in \tilde{R}^c$ such that we have for the spectral function $\varphi_T(p)$ of T

$$(\frac{\tilde{b}}{\tilde{a}}, p) = \varphi_T(p) \quad \text{for every point } p \in C_{\tilde{a}},$$

and then $[\tilde{a}] \tilde{b} = \tilde{a} T$.

Proof. If there exists an element $\tilde{b} \in \tilde{R}^c$ such that

$$(\frac{\tilde{b}}{\tilde{a}}, p) = \varphi_T(p) \quad \text{for every point } p \in C_{\tilde{a}},$$

then we see by Theorems 51.8 and 50.9 that $\varphi_T(p)$ is permissible by $\tilde{a}$ and

$$[\tilde{a}] \tilde{b} = \int \varphi_T(p) \tilde{a} d p.$$

Accordingly we have by Theorem 52.3 that T is permissible to $\tilde{a}$ and

$$[\tilde{a}] \tilde{T} = \tilde{a} T.$$

Conversely, if T is permissible to $\tilde{a}$, then we have by Theorem 52.2

$$\tilde{a} T = \int \varphi_T(\mathfrak{P}) \tilde{a} d\mathfrak{P}.$$

Consequently we obtain by Theorem 51.7

$$\left(\frac{\tilde{a} T}{\tilde{a}}, \mathfrak{P} \right) = \varphi_T(\mathfrak{P}) \quad \text{for every point } \mathfrak{P} \in C_{\tilde{a}}.$$

Theorem 52.5. To every dilatator T in R , putting

$$\tilde{T} \tilde{a} = \tilde{a} T$$

we obtain a dilatator $\tilde{T}$ in $\tilde{R}^c$, whose domain is composed of all elements $\tilde{a} \in \tilde{R}^c$ to which T is permissible.

Proof. The spectral function $\varphi_T(\mathfrak{P})$ of T is an almost finite continuous function on the proper space E of R by virtue of Theorem 43.7. Furthermore $\varphi_T(\mathfrak{P})$ is almost finite in $C_{\tilde{a}}$ for every element $\tilde{a} \in \tilde{R}^c$. Because, if $\varphi_T(\mathfrak{P})$ is not almost finite in $C_{\tilde{a}}$ for some element $\tilde{a} \in \tilde{R}^c$, then, since $\varphi_T(\mathfrak{P})$ is continuous, there exists a projector $[P]$ such that $C_{\tilde{a}} \mathcal{U}_{[P]} \neq 0$ and

$$|\varphi_T(\mathfrak{P})| = +\infty \quad \text{for every point } \mathfrak{P} \in C_{\tilde{a}} \mathcal{U}_{[P]}.$$

Then there exists by Theorem 16.3 a sequence of projectors $[P_\nu]$ ($\nu = 1, 2, \dots$) such that

$$\{\mathfrak{P} : |\varphi_T(\mathfrak{P})| \leq \nu\} \mathcal{U}_{[P]} \subset \mathcal{U}_{[P_\nu]} \subset \{\mathfrak{P} : |\varphi_T(\mathfrak{P})| < \nu+1\} \mathcal{U}_{[P]}.$$

For such $[P_\nu]$ ($\nu = 1, 2, \dots$), since $\varphi_T(\mathfrak{P})$ is almost finite in $\mathcal{U}_{[P]}$, we obtain by Theorem 16.4

$$[P_\nu] \uparrow_{\nu=1}^{\infty} [P],$$

and hence we have obviously for every $\nu = 1, 2, \dots$

$$C_{\tilde{a}} \mathcal{U}_{[P_\nu]} = 0.$$

Consequently we obtain by the formula §48(3)

$$\tilde{a} [P_\nu] = 0 \quad \text{for every } \nu = 1, 2, \dots,$$

and by Theorems 47.7 and 8.11

$$\tilde{a} [P](a) = \lim_{\nu \rightarrow \infty} \tilde{a} ([P_\nu] a) \quad \text{for every element } a \in R.$$

Therefore we conclude $\tilde{a} [P] = 0$, and hence by the formula §48(3)

$$C_{\alpha} \cup_{[p]} = 0,$$

contradicting assumption.

For the proper space E^c of $\tilde{R}^c$ we have by Theorem 51.1 a correspondence

$$\sum_{\alpha \in \tilde{R}^c} C_{\alpha} \ni p \rightarrow p^c \in E^c,$$

by which C_{α} is homeomorphic to the open set $\sum_{p \in R} U_{[\alpha(p)]}$ being dense in $U_{[\alpha]}$. Since $\varphi_T(p)$ is continuous and almost finite in C_{α} as proved just now, putting

$$\psi(p^c) = \varphi_T(p) \quad \text{for } p \in \sum_{\alpha \in \tilde{R}^c} C_{\alpha},$$

we obtain an almost finite continuous function $\psi(p^c)$ on the set

$$\sum_{p \in R, \alpha \in \tilde{R}^c} U_{[\alpha(p)]}$$

which is open and dense in E^c .

Since $\tilde{R}^c$ is universally continuous as remarked in §51, the proper space E^c of $\tilde{R}^c$ is universal by Theorem 41.6. Therefore we see by Theorem 41.5 that $\psi(p^c)$ has a continuous extension over E^c . Thus there exists an almost finite continuous function $\psi(\tilde{p})$ on E^c such that we have for every element $\tilde{\alpha} \in \tilde{R}^c$

$$\psi(p^c) = \varphi_T(p) \quad \text{for every point } p \in C_{\alpha}.$$

For such $\psi(\tilde{p})$, putting

$$\tilde{\tau} = \int \psi(\tilde{p}) d\tilde{p},$$

we obtain by Theorem 43.6 a dilatator $\tilde{\tau}$ in $\tilde{R}^c$.

If $\tilde{\tau}\tilde{\alpha}$ has a sense for an element $\tilde{\alpha} \in \tilde{R}^c$, then we have by Theorem 22.1

$$\left(\frac{\tilde{\tau}\tilde{\alpha}}{\tilde{\alpha}}, p\right) = \left(\frac{\tilde{\tau}\tilde{\alpha}}{\tilde{\alpha}}, p^c\right) = \psi(p^c) = \varphi_T(p)$$

for every point $p \in C_{\alpha}$. Then we have by the previous theorem that τ is permissible to $\tilde{\alpha}$ and

$$\tilde{\alpha}\tau = [\tilde{\alpha}]\tilde{\tau}\tilde{\alpha} = \tilde{\tau}(\tilde{\alpha})\tilde{\alpha} = \tilde{\tau}\tilde{\alpha}.$$

Conversely, if τ is permissible to $\tilde{\alpha}$, then there exists by the previous theorem an element $\tilde{\ell} \in \tilde{R}^c$ such that

$$\left(\frac{\tilde{\ell}}{\tilde{\alpha}}, p\right) = \varphi_T(p) \quad \text{for every point } p \in C_{\alpha}.$$

For such $\tilde{f} \in \tilde{R}^c$ we have for every point $\mathcal{P} \in C_{\tilde{\alpha}}$

$$\psi(\mathcal{P}^c) = \varphi_{\tau}(\mathcal{P}) = \left(\frac{\tilde{f}}{\tilde{\alpha}}, \mathcal{P}\right) = \left(\frac{\tilde{f}}{\tilde{\alpha}}, \mathcal{P}^c\right),$$

and hence

$$\psi(\tilde{\mathcal{P}}) = \left(\frac{\tilde{f}}{\tilde{\alpha}}, \tilde{\mathcal{P}}\right) \quad \text{for every point } \tilde{\mathcal{P}} \in \mathcal{U}_{[\tilde{\alpha}]}.$$

Therefore we see by Theorem 22.2 that $\psi(\tilde{\mathcal{P}})$ is integrable by $\tilde{\alpha}$ in $\mathcal{U}_{[\tilde{\alpha}]}$, and consequently by Theorem 43.6 that $\tilde{\tau}\tilde{\alpha}$ has a sense.

§53 Universally continuous linear functionals

Let R be a universally continuous semi-ordered linear space in the sequel. A linear functional L on R is said to be universally continuous, if $R \ni a_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ implies

$$\inf_{\lambda \in \Lambda} |L(a_{\lambda})| = 0.$$

Here the notation $a_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ means that to every two elements λ_1 and $\lambda_2 \in \Lambda$ there exists an element $\lambda_0 \in \Lambda$ for which

$$a_{\lambda_0} \leq a_{\lambda_1} \wedge a_{\lambda_2},$$

and $\bigcap_{\lambda \in \Lambda} a_{\lambda} = 0$, as used already in §46.

If a linear functional L on R is universally continuous, then for every system of elements $R \ni a_{\lambda} \downarrow_{\lambda \in \Lambda} 0$, to any positive number ε there exists an element $\lambda_0 \in \Lambda$ such that $a_{\lambda} \leq a_{\lambda_0}$ implies

$$|L(a_{\lambda})| < \varepsilon.$$

Because, if to every element $\lambda \in \Lambda$ there exists an element $\lambda_0 \in \Lambda$ such that

$$|L(a_{\lambda_0})| \geq \varepsilon, \quad a_{\lambda_0} \leq a_{\lambda},$$

then, putting $\Lambda_0 = \{\lambda : |L(a_{\lambda})| \geq \varepsilon\}$, we have obviously $a_{\lambda} \downarrow_{\lambda \in \Lambda_0} 0$ but

$$|L(a_{\lambda})| \geq \varepsilon \quad \text{for every } \lambda \in \Lambda_0,$$

contradicting the assumption that L is universally continuous. Therefore, if a linear functional L on R is universally con-

tinuous, then $R \ni a_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies

$$\inf_{\lambda \in \Lambda} \left\{ \sup_{a_p \leq a_\lambda} |L(a_p)| \right\} = 0.$$

From this fact we conclude immediately:

Theorem 53.1. Every universally continuous linear functional on R is continuous, and hence bounded.

The totality of universally continuous linear functionals on R is called the conjugate space of R and denoted by $\bar{R}$. By virtue of Theorem 53.1, the conjugate space $\bar{R}$ of R is included in the associated space $\tilde{R}$ of R . Furthermore $\bar{R}$ is a linear lattice manifold of $\tilde{R}$. Indeed, for every two elements $\bar{a}$ and $\bar{b} \in \bar{R}$, $R \ni a_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies for every real numbers α, β

$$\inf_{\lambda \in \Lambda} |\alpha \bar{a}(a_\lambda) + \beta \bar{b}(a_\lambda)| \leq |\alpha| \inf_{\lambda \in \Lambda} \left\{ \sup_{a_p \leq a_\lambda} |\bar{a}(a_p)| \right\} + |\beta| \inf_{\lambda \in \Lambda} \left\{ \sup_{a_p \leq a_\lambda} |\bar{b}(a_p)| \right\} = 0,$$

that is, we have $\alpha \bar{a} + \beta \bar{b} \in \bar{R}$ for every real numbers α, β . As every linear functional $\bar{a} \in \bar{R}$ is continuous by Theorem 53.1, to every element $a \in R$ there exists by Theorem 47.4 a projector $[p] \leq [a]$ for which

$$\bar{a}^+[a] = \bar{a}[p].$$

Since $R \ni a_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $[p]a_\lambda \downarrow_{\lambda \in \Lambda} 0$ by Theorem 7.6, putting $a = a_{\lambda_0}$ for some element $\lambda_0 \in \Lambda$, we have

$$\inf_{\lambda \in \Lambda} |\bar{a}^+(a_\lambda)| \leq \inf_{a_\lambda \leq a_{\lambda_0}} |\bar{a}^+(a_\lambda)| = \inf_{a_\lambda \leq a_{\lambda_0}} |\bar{a}([p]a_\lambda)| = 0.$$

Hence $\bar{a} \in \bar{R}$ implies $\bar{a}^+ \in \bar{R}$, and consequently further $\bar{a}^- \in \bar{R}$ and $|\bar{a}| \in \bar{R}$.

Theorem 53.2. If $\bar{R} \ni \bar{a}_\lambda \uparrow_{\lambda \in \Lambda}$ and

$$\sup_{\lambda \in \Lambda} \bar{a}_\lambda(a) < +\infty \quad \text{for every positive } a \in R,$$

then there exists an element $\bar{a} \in \bar{R}$ for which $\bar{a}_\lambda \uparrow_{\lambda \in \Lambda} \bar{a}$ and

$$\bar{a}(a) = \sup_{\lambda \in \Lambda} \bar{a}_\lambda(a) \quad \text{for every positive } a \in R.$$

Proof. Recalling Theorem 46.2, we conclude from assumption that there exists a bounded linear functional $\tilde{a}$ on R such that

$$\bar{a}(a) = \sup_{\lambda \in A} \bar{a}_\lambda(a) \quad \text{for every positive } a \in R.$$

If $R \ni a_\gamma \downarrow_{\gamma \in \Gamma} 0$, then for a fixed $\gamma_0 \in \Gamma$, to any positive number ε there exists an element $\lambda_0 \in A$ for which

$$\bar{a}(a_{\gamma_0}) \leq \bar{a}_{\lambda_0}(a_{\gamma_0}) + \varepsilon.$$

As $\bar{a} - \bar{a}_{\lambda_0} \geq 0$, we have for $a_\gamma \leq a_{\gamma_0}$

$$\begin{aligned} |\bar{a}(a_\gamma)| &\leq |\bar{a}_{\lambda_0}(a_\gamma)| + (\bar{a} - \bar{a}_{\lambda_0})(a_\gamma) \\ &\leq |\bar{a}_{\lambda_0}(a_\gamma)| + (\bar{a} - \bar{a}_{\lambda_0})(a_{\gamma_0}) \leq |\bar{a}_{\lambda_0}(a_{\gamma_0})| + \varepsilon, \end{aligned}$$

and hence

$$\inf_{\gamma \in \Gamma} |\bar{a}(a_\gamma)| \leq \inf_{a_\gamma \leq a_{\gamma_0}} |\bar{a}_{\lambda_0}(a_\gamma)| + \varepsilon = \varepsilon,$$

since $\bar{a}_{\lambda_0}$ is universally continuous by assumption. As a positive number ε may be arbitrary, we obtain hence

$$\inf_{\gamma \in \Gamma} |\bar{a}(a_\gamma)| = 0.$$

Therefore $\bar{a}$ is universally continuous, that is, $\bar{a} \in \bar{R}$, and we have obviously by definition

$$\bar{a}_\lambda \uparrow_{\lambda \in A} \bar{a}.$$

From this theorem we conclude immediately:

Theorem 53.3. The conjugate space $\bar{R}$ of R constitutes a universally continuous semi-ordered linear space, and

$\bar{R} \ni \bar{a}_\lambda \uparrow_{\lambda \in A} \bar{a}$ implies

$$\bar{a}(a) = \sup_{\lambda \in A} \bar{a}_\lambda(a) \quad \text{for every positive } a \in R;$$

$\bar{R} \ni \bar{a}_\lambda \downarrow_{\lambda \in A} \bar{a}$ implies

$$\bar{a}(a) = \inf_{\lambda \in A} \bar{a}_\lambda(a) \quad \text{for every positive } a \in R.$$

Theorem 53.4. For every positive element $\bar{a} \in \bar{R}$,

$R \ni a_\lambda \uparrow_{\lambda \in A} a$ implies

$$\bar{a}(a) = \sup_{\lambda \in A} \bar{a}(a_\lambda),$$

and $R \ni a_\lambda \downarrow_{\lambda \in A} a$ implies

$$\bar{a}(a) = \inf_{\lambda \in A} \bar{a}(a_\lambda).$$

Proof. If $R \ni a_\lambda \uparrow_{\lambda \in A} a$, then we have by Theorems

2.2 and 2.4

$$a - a_\lambda \downarrow_{\lambda \in A} 0,$$

and hence by definition for every positive element $\bar{a} \in \bar{R}$

$$\inf_{\lambda \in A} \bar{a}(a - a_\lambda) = 0,$$

that is, $\bar{a}(a) = \sup_{\lambda \in A} \bar{a}(a_\lambda)$. We also can prove likewise the other assertion.

Theorem 53.5. For every element $\bar{a} \in \bar{R}$, the characteristic set $C_{\bar{a}}$ of $\bar{a}$ is open and closed.

Proof. We have by definition

$$C'_{|\bar{a}|} = \sum_{|\bar{a}|[p]=0} \sigma_{[p]},$$

and hence for every element $a \in R$

$$C'_{|\bar{a}|} \sigma_{[a]} = \sum_{|\bar{a}|[p]=0} \sigma_{[a][p]}.$$

Putting $p_0 = \bigcup_{|\bar{a}|[p]=0} [p]a$, we obtain by Theorem 8.5 for every positive element $x \in R$

$$[p_0]x = \bigcup_{|\bar{a}|[p]=0} [p][a]x,$$

and hence by the previous theorem

$$|\bar{a}|([p_0]x) = \sup_{|\bar{a}|[p]=0} |\bar{a}|([p][a]x) = 0.$$

Therefore we have $|\bar{a}|[p_0] = 0$. If $|\bar{a}|[p] = 0$ for a projector $[p]$, then we have by Theorem 8.3

$$[a] \geq [p_0] \geq [p][a],$$

and hence by the formula §15(4)

$$\sigma_{[a]} \supset \sigma_{[p_0]} \supset \sigma_{[p][a]}.$$

Accordingly we obtain

$$\sum_{|\bar{a}|[p]=0} \sigma_{[a][p]} = \sigma_{[p_0]}.$$

From this relation we conclude that $C'_{|\bar{a}|} \sigma_{[a]}$ is open and closed, and hence the point set

$$C_{|\bar{a}|} \sigma_{[a]} = \sigma_{[a]} - C'_{|\bar{a}|} \sigma_{[a]}$$

is also open and closed. Since we have

$$C_{|\bar{a}|} = \sum_{a \in R} C_{|\bar{a}|} \sigma_{[a]},$$

we see now that $C_{|\bar{a}|}$ is open. On the other hand, $C_{|\bar{a}|}$ is closed by definition, and $C_{\bar{a}} = C_{|\bar{a}|}$ by the formula §48(1).

Therefore $C_{\bar{a}}$ is open and closed.

Theorem 53.6. If $C_{\bar{a}} = \sigma_{[a]}$, then we have

$$[\bar{a}]\bar{x} = \bar{x}[a] \quad \text{for every element } \bar{x} \in \bar{R}.$$

Proof. If $C_{\bar{\alpha}} = U_{[a]}$, then we have by Theorem 48.4 and by the formula §48(5) for every element $\bar{x} \in \bar{R}$

$$C_{[a]\bar{x}} = C_{\bar{\alpha}} C_{\bar{x}} = C_{\bar{x}} U_{[a]} = C_{\bar{x}[a]},$$

and hence $[[\bar{a}]\bar{x}] = [\bar{x}[a]]$ by Theorem 48.2. Accordingly we obtain by the formulas §8(1), §46(12) for every element

$$\begin{aligned} [\bar{a}]\bar{x} &= [\bar{a}][\bar{x}]\bar{x} = [[\bar{a}]\bar{x}]\bar{x} \\ &= [\bar{x}[a]]\bar{x} = [\bar{x}]\bar{x}[a] = \bar{x}[a]. \end{aligned}$$

As defined already in §48, a linear functional L on R is said to be complete in R , if

$$L[p] = 0 \quad \text{implies} \quad [p] = 0.$$

A universally continuous semi-ordered linear space R is said to be regular, if there exists a linear functional $\bar{\alpha} \in \bar{R}$ such that $\bar{\alpha}$ is complete in R . In this case, such $\bar{\alpha} \in \bar{R}$ is a complete element of $\bar{R}$, because $C_{\bar{\alpha}}$ coincides with the whole space E by the formula §48(6), and hence $[\bar{\alpha}] \neq 1$ by Theorems 48.2 and 8.2.

A projector $[p]$ in a universally continuous semi-ordered linear space R is said to be regular, if there exists an element $\bar{\alpha} \in \bar{R}$ such that $\bar{\alpha}$ is complete in $[p]R$, that is, if $[p]R$ is regular as a universally continuous semi-ordered linear space. R is said to be locally regular, if every projector in R is regular.

A universally continuous semi-ordered linear space R is said to be semi-regular, if there exists a complete system of elements $p_{\lambda} \in R$ ($\lambda \in \Lambda$) such that $[p_{\lambda}]$ is regular for every $\lambda \in \Lambda$, that is the same, if to any projector $[p] \neq 0$ in R there exists an element $\bar{\alpha} \in \bar{R}$ for which $\bar{\alpha}[p] \neq 0$. Because if $\bar{\alpha}[p] \neq 0$, then, since $C_{\bar{\alpha}}$ is open by Theorem 53.5, there exists an element $a \in R$ for which

$$0 \neq U_{[a]} \subset C_{\bar{\alpha}} U_{[p]} = C_{\bar{\alpha}[p]},$$

and hence $[a]$ is regular by the formula §48(6). We also can

define that R is semi-regular, if the union

$$\sum_{\bar{\alpha} \in \bar{R}} C_{\bar{\alpha}}$$

is dense in the proper space E of R .

§54 Canonical spaces

Let R be a universally continuous semi-ordered linear space and let $\bar{R}$ be its conjugate space. Recalling Theorem 51.1, we can prove likewise that corresponding to every point

$$f \in \sum_{\bar{\alpha} \in \bar{R}} C_{\bar{\alpha}}$$

there exists uniquely a point $f^c \in \bar{E}$ for the proper space $\bar{E}$ of $\bar{R}$ such that we have $f^c \in U_{[\bar{\alpha}]}$ if and only if $f \in C_{\bar{\alpha}}$; and every characteristic set $C_{\bar{\alpha}}$ is homeomorphic to the open set

$$\sum_{p \in R} U_{[\bar{\alpha}(p)]}$$

being dense in $U_{[\bar{\alpha}]}$ by the correspondence $C_{\bar{\alpha}} \ni f \rightarrow f^c \in U_{[\bar{\alpha}]}$.

Since every characteristic set $C_{\bar{\alpha}}$ is open by Theorem 53.6, the system of point sets $C_{\bar{\alpha}(p)}$ ($\bar{\alpha} \in \bar{R}$, $p \in R$) constitutes a neighbourhood system of

$$\sum_{\bar{\alpha} \in \bar{R}} C_{\bar{\alpha}}.$$

The system of point sets $U_{[\bar{\alpha}(p)]}$ ($\bar{\alpha} \in \bar{R}$, $p \in R$) is obviously a neighbourhood system of

$$\sum_{\bar{\alpha} \in \bar{R}, p \in R} U_{[\bar{\alpha}(p)]}.$$

Furthermore $C_{\bar{\alpha}(p)}$ is homeomorphic to $U_{[\bar{\alpha}(p)]}$ by the correspondence

$$C_{\bar{\alpha}(p)} \ni f \rightarrow f^c \in U_{[\bar{\alpha}(p)]}.$$

Therefore we have:

Theorem 54.1. Corresponding to every point $f \in \sum_{\bar{\alpha} \in \bar{R}} C_{\bar{\alpha}}$ there exists uniquely a point f^c in the proper space $\bar{E}$ of $\bar{R}$ such that we have $f^c \in U_{[\bar{\alpha}]}$ if and only if $f \in C_{\bar{\alpha}}$, and the open set

$$\sum_{\bar{\alpha} \in \bar{R}} C_{\bar{\alpha}}$$

is homeomorphic to the open set

being dense in $\bar{E}$ by such correspondence $\mathcal{F} \rightarrow \mathcal{F}^c$.

Corresponding to Theorem 51.5 we can prove likewise:

Theorem 54.2. For every point $\mathcal{F} \in C_{\bar{a}} \cup \{a\}$ we have

$$\lim_{[\mathcal{P}] \rightarrow \mathcal{F}} \frac{\bar{a}([\mathcal{P}]a)}{\bar{a}([\mathcal{P}]a)} = \left(\frac{\bar{a}}{a}, \mathcal{F}^c \right)$$

for the correspondence $\mathcal{F} \rightarrow \mathcal{F}^c$ in Theorem 54.1.

Theorem 54.3. For every point $\mathcal{F} \in C_{\bar{a}} \cup \{a\}$ we have

$$\lim_{[\mathcal{P}] \rightarrow \mathcal{F}} \frac{\bar{a}([\mathcal{P}]b)}{\bar{a}([\mathcal{P}]a)} = \left(\frac{b}{a}, \mathcal{F} \right).$$

Proof. In the case: $\mathcal{F}_0 \in C_{\bar{a}} + \cup \{a\}$, there exists a projector $[p_0]$ such that $\mathcal{F}_0 \in \mathcal{U}_{[p_0]} \subset C_{\bar{a}} + \cup \{a\}$ and

$$\bar{a}([p]a) > 0 \quad \text{for } \mathcal{F}_0 \in \mathcal{U}_{[p]} \subset \mathcal{U}_{[p_0]},$$

as proved already in Theorem 51.5. If $(\frac{b}{a}, \mathcal{F}_0) < \lambda$, then,

since $(\frac{b}{a}, \mathcal{F})$ is continuous by Theorem 19.2, there exists a projector $[q]$ such that $\mathcal{F}_0 \in \mathcal{U}_{[q]} \subset \mathcal{U}_{[p_0]}$ and

$$\left(\frac{b}{a}, \mathcal{F} \right) < \lambda \quad \text{for every point } \mathcal{F} \in \mathcal{U}_{[q]}.$$

For such $[q]$ we obtain $[q]b \leq \lambda[q]a$ by Theorem 19.4, since

$$[q]a = [q][p_0]a = [q][p_0]a^+ = [q]a^+.$$

As $\mathcal{U}_{[q]} \subset C_{\bar{a}} +$, we have $\bar{a}^{-}[q] = 0$ by Theorem 48.1 and by the formula §48(3), and hence $\bar{a}[q] = \bar{a}^+[q]$. Thus we obtain

$$\bar{a}([q][p]b) \leq \lambda \bar{a}([q][p]a) \quad \text{for every element } p \in R,$$

and hence

$$\frac{\bar{a}([q][p]b)}{\bar{a}([q][p]a)} \leq \lambda \quad \text{for every } \mathcal{U}_{[p]} \ni \mathcal{F}_0.$$

This relation yields by definition

$$\lim_{[\mathcal{P}] \rightarrow \mathcal{F}_0} \frac{\bar{a}([\mathcal{P}]b)}{\bar{a}([\mathcal{P}]a)} \leq \lambda.$$

Therefore we conclude

$$\lim_{[\mathcal{P}] \rightarrow \mathcal{F}_0} \frac{\bar{a}([\mathcal{P}]b)}{\bar{a}([\mathcal{P}]a)} \leq \left(\frac{b}{a}, \mathcal{F}_0 \right).$$

Since this relation holds for every element $b \in R$, we also have

$$\lim_{[\mathcal{P}] \rightarrow \mathcal{F}_0} \frac{\bar{a}([\mathcal{P}](-b))}{\bar{a}([\mathcal{P}]a)} \leq \left(\frac{-b}{a}, \mathcal{F}_0 \right),$$

that is,

$$\lim_{[\mathcal{P}] \rightarrow \mathcal{F}_0} \frac{\bar{a}([\mathcal{P}]b)}{\bar{a}([\mathcal{P}]a)} \geq \left(\frac{b}{a}, \mathcal{F}_0 \right).$$

Consequently we obtain for every point $\mathfrak{f}_0 \in C_{\bar{a}} \cup U_{[a]}$

$$\lim_{[p] \rightarrow \mathfrak{f}_0} \frac{\bar{a}([p]b)}{\bar{a}([p]a)} = \left(\frac{b}{a}, \mathfrak{f}_0 \right).$$

In general, to every point $\mathfrak{f} \in C_{\bar{a}} \cup U_{[a]}$, we can determine $\varepsilon = \pm 1$ and $\delta = \pm 1$ such that

$$\mathfrak{f} \in C_{(\varepsilon \bar{a})} \cup U_{[(\delta a)]},$$

and then we have as proved just now

$$\lim_{[p] \rightarrow \mathfrak{f}} \frac{\varepsilon \bar{a}([p]b)}{\varepsilon \bar{a}([p]\delta a)} = \left(\frac{b}{\delta a}, \mathfrak{f} \right).$$

From this relation we obtain obviously our assertion.

In the following, we suppose that $\mathcal{R}$ is semi-regular. Then

$$\sum_{\bar{a} \in \bar{\mathcal{R}}} C_{\bar{a}}$$

is an open set being dense in the proper space E of $\mathcal{R}$ by definition, and furthermore homeomorphic to the open set

$$\sum_{\bar{a} \in \bar{\mathcal{R}}, p \in \mathcal{R}} U_{[\bar{a}([p])]}$$

being dense in the proper space $\bar{E}$ of $\bar{\mathcal{R}}$ by the correspondence

$$\sum_{\bar{a} \in \bar{\mathcal{R}}} C_{\bar{a}} \ni \mathfrak{f} \rightarrow \mathfrak{f}^c \in \sum_{\bar{a} \in \bar{\mathcal{R}}, p \in \mathcal{R}} U_{[\bar{a}([p])]},$$

as stated in Theorem 54.1.

Corresponding to every almost finite continuous function

$\varphi(\mathfrak{f})$ on E , since $\varphi(\mathfrak{f})$ is naturally almost finite and continuous in the open set $\sum_{\bar{a} \in \bar{\mathcal{R}}} C_{\bar{a}}$, there exists by Theorem 41.5 an almost finite continuous function $\varphi^c(\bar{\mathfrak{f}})$ on $\bar{E}$ uniquely such that

$$\varphi^c(\mathfrak{f}^c) = \varphi(\mathfrak{f}) \quad \text{for every point } \mathfrak{f} \in \sum_{\bar{a} \in \bar{\mathcal{R}}} C_{\bar{a}}.$$

Similarly, to every almost finite continuous function $\psi(\bar{\mathfrak{f}})$ on $\bar{E}$ there exists uniquely an almost finite continuous function $\psi(\mathfrak{f})$ on E such that $\varphi^c(\bar{\mathfrak{f}}) = \psi(\bar{\mathfrak{f}})$ in $\bar{E}$.

Therefore there exists by Theorems 43.6 and 43.7 a one-to-one correspondence $T \leftrightarrow \bar{T}$ between dilatators T in $\mathcal{R}$ and $\bar{T}$ in $\bar{\mathcal{R}}$ such that

$$\left(\frac{T a}{a}, \mathfrak{f} \right) = \left(\frac{\bar{T} \bar{a}}{\bar{a}}, \bar{\mathfrak{f}}^c \right) \quad \text{for every point } \mathfrak{f} \in C_{\bar{a}} \cup U_{[a]},$$

if both $T a$ and $\bar{T} \bar{a}$ have any sense. By virtue of Theorem 44.1, the dilatator ring of $\mathcal{R}$ is isomorphic to the dilatator

ring of $\bar{R}$ by this correspondence $T \leftrightarrow \bar{T}$. Hence we can identify the corresponding dilatators $T \leftrightarrow \bar{T}$ and denote it by $\hat{T}$, that is, $\hat{T}$ is a dilatator in R as well as a dilatator in $\bar{R}$ and we have

(1) $(\frac{\hat{T}a}{a}, p) = (\frac{\hat{T}\bar{a}}{\bar{a}}, p^c)$ for every point $p \in C_{\bar{a}} \cup \{a\}$, if both $\hat{T}a$ and $\hat{T}\bar{a}$ have meaning.

Since we have by Theorem 54.2 for every point $p \in C_{\bar{a}} \cup \{a\}$

$$\lim_{p \rightarrow p} \frac{\hat{T}\bar{a}(p)a}{\bar{a}(p)a} = (\frac{\hat{T}\bar{a}}{\bar{a}}, p^c),$$

we conclude from the formula (1) by Theorems 51.5 and 51.8

$$(\hat{T}\bar{a})[a] = [\bar{a}](\hat{T}\bar{a})[a] = \int (\frac{\hat{T}a}{a}, p) \bar{a}[a] dp,$$

and hence by Theorem 50.1

$$\hat{T}\bar{a}(a) = \int_{[a]} (\frac{\hat{T}a}{a}, p) \bar{a}(dp a) = \bar{a}(\hat{T}a).$$

Therefore we have

$$(2) \quad \hat{T}\bar{a}(a) = \bar{a}(\hat{T}a),$$

if both $\hat{T}\bar{a}$ and $\hat{T}a$ have any sense.

If both $\hat{T}a$ and $\hat{T}\bar{a}$ have meaning, then, since we have by Theorem 45.2 as dilatators

$$|\hat{T}[p]| \leq |\hat{T}|$$

for every projector $[p]$ in R , $(\hat{T}[p])\bar{a}$ also has a sense by Theorem 44.2, and we obtain by the formula (2)

$$\hat{T}\bar{a}([p]a) = \bar{a}((\hat{T}[p])a) = (\hat{T}[p])\bar{a}(a).$$

As the domain of $\hat{T}$ is dense in R by definition, we have hence by Theorem 47.7

$$(3) \quad (\hat{T}\bar{a})[p] = (\hat{T}[p])\bar{a}$$

if $\hat{T}\bar{a}$ has any sense. As a special case we have

$$(4) \quad \bar{a}[p] = [p]\bar{a}$$

for every elements $\bar{a} \in \bar{R}$ and $p \in R$.

By virtue of Theorems 41.6, 42.2, and 44.1, the totality of dilatators $\hat{T}$ constitutes a universally complete semi-ordered linear space, whose proper space will be called the canonical space of R and $\bar{R}$. We see by Theorems 32.4, 41.6 that the

canonical space of R is a universal compact Hausdorff space.

By virtue of Theorem 45.6, the proper space E of R is homeomorphic to the open set

$$\sum_{p \in R} U_{[p]}$$

being dense in the canonical space $\hat{E}$ by a correspondence

$E \ni p \rightarrow p^D \in \hat{E}$ such that we have

$$(5) \quad \left(\frac{\hat{\tau}}{1}, p^D \right) = \left(\frac{\hat{\tau}a}{a}, p \right) \quad \text{for every point } p \in U_{[a]},$$

if $\hat{\tau}a$ has any sense. This dense open set $\sum_{p \in R} U_{[p]}$ in the canonical space $\hat{E}$ will be denoted by the same notation E as the proper space E of R .

It is similar for the proper space $\bar{E}$ of $\bar{R}$, i.e., $\bar{E}$ is homeomorphic to an open set $\bar{E}$ being dense in the canonical space $\hat{\bar{E}}$ by a correspondence $\bar{E} \ni \bar{p} \rightarrow \bar{p}^D \in \hat{\bar{E}}$ such that

$$(6) \quad \left(\frac{\hat{\tau}}{1}, \bar{p}^D \right) = \left(\frac{\hat{\tau}\bar{a}}{\bar{a}}, \bar{p} \right) \quad \text{for every point } \bar{p} \in U_{[\bar{a}]}$$

if $\hat{\tau}\bar{a}$ has a sense.

For every point $p \in \sum_{\alpha \in R} C_{\alpha}$ we have by the formulas (1), (5), and (6) for every dilatator $\hat{\tau}$

$$\left(\frac{\hat{\tau}}{1}, (p^c)^D \right) = \left(\frac{\hat{\tau}}{1}, p^D \right),$$

and hence we obtain by Theorem 32.2

$$(7) \quad (p^c)^D = p^D \quad \text{for every point } p \in \sum_{\alpha \in R} C_{\alpha}.$$

Conversely, if $\bar{p}^D = p^D$ for some points $\bar{p} \in \bar{E}$ and $p \in E$, then we have by the formulas (1), (5), (6) for every dilatator $\hat{\tau}$

$$\left(\frac{\hat{\tau}\bar{a}}{\bar{a}}, \bar{p} \right) = \left(\frac{\hat{\tau}a}{a}, p \right)$$

if $U_{[\bar{a}]} \ni \bar{p}$, $U_{[a]} \ni p$ and both $\hat{\tau}\bar{a}$, $\hat{\tau}a$ have meaning.

If we suppose that $\bar{p} \in U_{[\bar{a}]}$ but $p \notin C_{\bar{a}}$, then there exists a projector $[p]$ such that

$$p \in U_{[p]} \quad \text{and} \quad U_{[p]} C_{\bar{a}} = 0.$$

For such $[p]$ we have $\bar{a}[p] = 0$ by the formula §48(3), and hence, putting $\hat{\tau} = [p]$, we obtain by the formula (3)

$$0 = \left(\frac{\bar{a}[p]}{\bar{a}}, \bar{p} \right) = \left(\frac{[p]\bar{a}}{\bar{a}}, \bar{p} \right) = \left(\frac{[p]a}{a}, p \right)$$

contradicting that we have $\left(\frac{[p]a}{a}, p \right) = 1$ by Theorems 18.2 and 18.4.

Therefore $\bar{p} \in \mathcal{U}[\bar{a}]$ implies $p \in C_{\bar{a}}$. It is evident that there exists an element $\bar{a} \in \bar{R}$ for which $\bar{p} \in \mathcal{U}[\bar{a}]$. For such $\bar{a} \in \bar{R}$ we have hence $p \in C_{\bar{a}}$, and further

$$\bar{p}^D = p^D = (p^c)^D$$

as proved just now. From this relation we conclude $\bar{p} = p^c$, since the correspondence $\bar{p} \rightarrow \bar{p}^D$ is one-to-one. Therefore we have

$$(8) \quad E\bar{E} = \sum_{\bar{a} \in \bar{R}} C_{\bar{a}} = \sum_{\bar{a} \in \bar{R}, p \in R} \mathcal{U}[\bar{a}[p]]$$

in the canonical space $\hat{E}$

We have obviously by the formula (7)

$$(9) \quad C_{\bar{a}} = \sum_{p \in R} \mathcal{U}[\bar{a}[p]]$$

in the canonical space $\hat{E}$, and further by the formula (8)

$$(10) \quad E\mathcal{U}[\bar{a}] = C_{\bar{a}},$$

since we have $p^c \in \mathcal{U}[\bar{a}]$ if and only if $p \in C_{\bar{a}}$.

Since we conclude by the formulas (10) and §48(5)

$$\mathcal{U}[\bar{a}]\mathcal{U}[a] = \mathcal{U}[\bar{a}]E\mathcal{U}[a] = C_{\bar{a}}\mathcal{U}[a] = C_{\bar{a}[a]} = E\mathcal{U}[\bar{a}[a]],$$

we obtain by the formula (9)

$$(11) \quad \mathcal{U}[\bar{a}]\mathcal{U}[a] = \mathcal{U}[\bar{a}[a]],$$

and hence naturally

$$(12) \quad E\mathcal{U}[a] = \sum_{\bar{a} \in \bar{R}} \mathcal{U}[\bar{a}[a]].$$

Theorem 54.4. We have

$$[\bar{a}]\bar{x} = \bar{x}[a] \quad \text{for every element } \bar{x} \in \bar{R},$$

if and only if $\mathcal{U}[\bar{a}] = \mathcal{U}[a]$ in the canonical space $\hat{E}$.

Proof. If $\mathcal{U}[\bar{a}] = \mathcal{U}[a]$ in the canonical space $\hat{E}$, then we have by the formula (10)

$$C_{\bar{a}} = E\mathcal{U}[\bar{a}] = \mathcal{U}[a],$$

and hence $[\bar{a}]\bar{x} = \bar{x}[a]$ by Theorem 53.6. Conversely, if

$$[\bar{a}]\bar{x} = \bar{x}[a] \quad \text{for every element } \bar{x} \in \bar{R},$$

then we have $\bar{a} = [\bar{a}]\bar{a} = \bar{a}[a]$, and hence $C_{\bar{a}} \subset \mathcal{U}[a]$ by the formula §48(4). On the other hand, if $\bar{a}[p] = 0$ for a projector $[p]$, then we have by the formula §46(12) for every $\bar{x} \in \bar{R}$

$$\bar{x}[a][p] = [\bar{x}] \bar{x}[p] = [\bar{x}[p]] \bar{x} = 0,$$

and hence $[a][p] = 0$; since R is semi-regular by assumption. Consequently we have $C_{\bar{x}} \supset U_{[a]}$ by the formula §48(6). Therefore we conclude $C_{\bar{x}} = U_{[a]}$ and further by the formula (9)

$$C_{\bar{x}} \subset U_{[\bar{x}]} = U_{[\bar{x}[a]]} \subset C_{\bar{x}}$$

in the canonical space $\hat{E}$. Thus we obtain $U_{[\bar{x}]} = U_{[a]}$ in the canonical space $\hat{E}$.

For every point $\hat{p} \in U_{[a]}$ in the canonical space $\hat{E}$, we define the relative spectrum $(\frac{p}{a}, \hat{p})$ as the same, considering $\hat{p}$ as a point in E , that is,

$$(\frac{p}{a}, \hat{p}) = (\frac{p}{a}, p) \quad \text{for } \hat{p} = p^D, \quad p \in U_{[a]}.$$

It is similar for every point $\hat{p} \in U_{[\bar{a}]}$ in $\hat{E}$, that is,

$$(\frac{\bar{p}}{\bar{a}}, \hat{p}) = (\frac{\bar{p}}{\bar{a}}, \bar{p}) \quad \text{for } \hat{p} = \bar{p}^D, \quad \bar{p} \in U_{[\bar{a}]}.$$

Theorem 54.5. For every point $\hat{p} \in U_{[\bar{a}]} U_{[a]}$ in the canonical space $\hat{E}$ we have

$$\lim_{[p] \rightarrow \hat{p}} \frac{\bar{p}([p]a)}{\bar{a}([p]a)} = \lim_{[\bar{p}] \rightarrow \hat{p}} \frac{[\bar{p}] \bar{p}(a)}{[\bar{p}] \bar{a}(a)} = (\frac{\bar{p}}{\bar{a}}, \hat{p}).$$

Proof. By virtue of Theorem 54.2, we have obviously

$$\lim_{[p] \rightarrow \hat{p}} \frac{\bar{p}([p]a)}{\bar{a}([p]a)} = (\frac{\bar{p}}{\bar{a}}, \hat{p}).$$

To every projector $[p]$ in R such that $U_{[p]} \subset U_{[\bar{a}]} U_{[a]}$ in $\hat{E}$ there exists by Theorem 16.4 a projector $[\bar{p}]$ in $\bar{R}$ such that $U_{[\bar{p}]} = U_{[p]}$ in $\hat{E}$. For such $[\bar{p}]$ we have by the previous theorem

$$[\bar{p}] \bar{x} = \bar{x}[p] \quad \text{for every element } \bar{x} \in \bar{R}.$$

Conversely, we see likewise that to every projector $[\bar{p}]$ in $\bar{R}$ such that $U_{[\bar{p}]} \subset U_{[\bar{a}]} U_{[a]}$ in $\hat{E}$ there exists a projector $[p]$ in R for which

$$[\bar{p}] \bar{x} = \bar{x}[p] \quad \text{for every element } \bar{x} \in \bar{R}.$$

Therefore we have by definition

$$\lim_{[p] \rightarrow \hat{p}} \frac{[\bar{p}] \bar{p}(a)}{[\bar{p}] \bar{a}(a)} = \lim_{[p] \rightarrow \hat{p}} \frac{\bar{p}([p]a)}{\bar{a}([p]a)}.$$

Recalling Theorem 54.3, we also can prove likewise:

Theorem 54.6. For every point $\hat{p} \in U_{[\bar{a}]} U_{[a]}$ in the

canonical space $\hat{E}$ we have

$$\lim_{\{p\} \rightarrow \hat{f}} \frac{\bar{\alpha}(\{p\}t)}{\bar{\alpha}(\{p\}a)} = \lim_{\{p\} \rightarrow \hat{f}} \frac{[\bar{f}] \bar{\alpha}(t)}{[\bar{f}] \bar{\alpha}(a)} = \left(\frac{t}{a}, \hat{f} \right).$$

§55 Reflexivity

Let R be a universally continuous semi-ordered linear space and let $\bar{R}$ be its conjugate space. Since $\bar{R}$ also constitutes by Theorem 53.3 a universally continuous semi-ordered linear space, we can consider further the conjugate space $\bar{\bar{R}}$ of $\bar{R}$.

Corresponding to every element $a \in R$, putting

$$(1) \quad a^{\bar{R}}(\bar{x}) = \bar{x}(a) \quad \text{for every element } \bar{x} \in \bar{R},$$

we obtain a functional $a^{\bar{R}}$ on $\bar{R}$. This functional $a^{\bar{R}}$ is obviously linear. We can prove further that $a^{\bar{R}}$ is universally continuous. In fact, if $\bar{R} \ni \bar{a}_\lambda \downarrow_{\lambda \in \Lambda} 0$, then, since

$$|a^{\bar{R}}(\bar{a}_\lambda)| = |\bar{a}_\lambda(a)| \leq \bar{a}_\lambda(1),$$

we obtain by Theorem 53.3

$$\inf_{\lambda \in \Lambda} |a^{\bar{R}}(\bar{a}_\lambda)| = 0.$$

Therefore $a^{\bar{R}}$ is universally continuous, that is, we have

$$a^{\bar{R}} \in \bar{\bar{R}} \quad \text{for every element } a \in R.$$

This fact shows that the conjugate space $\bar{R}$ of R is semi-regular.

We have obviously by the relation (1)

$$(2) \quad (\alpha a + \beta t)^{\bar{R}} = \alpha a^{\bar{R}} + \beta t^{\bar{R}},$$

$$(3) \quad a \geq 0 \text{ implies } a^{\bar{R}} \geq 0.$$

If R is semi-regular, then we have:

$$(4) \quad a^{\bar{R}} \geq 0 \text{ implies } a \geq 0.$$

Because, if $a^- \neq 0$, then there exists an element $\bar{a} \in \bar{R}$ for which $\bar{a}[a^-] \neq 0$, since R is semi-regular by assumption.

For such $\bar{a} \in \bar{R}$ we have by Theorem 47.3

$$|\bar{a}|a^- > 0,$$

and hence by the relation (1)

$$a^{\bar{R}}(|\bar{a}|[a^-]) = |\bar{a}|([a^-]a) = -|\bar{a}|a^- < 0,$$

contradicting the assumption $a^{\bar{R}} \geq 0$.

The canonical space $\hat{E}$ of R and $\bar{R}$ may be considered by definition as the canonical space of $\bar{R}$ and $\bar{\bar{R}}$, if R is semi-regular. In this sense we have:

Theorem 55.1. If R is semi-regular, then for every element $a \in R$ we have

$$\mathcal{U}_{[a]} = \mathcal{U}_{[a^{\bar{R}}]}$$

in the canonical space $\hat{E}$ and

$$\left(\frac{f}{a}, \hat{f}\right) = \left(\frac{f^{\bar{R}}}{a^{\bar{R}}}, \hat{f}\right) \quad \text{for every point } \hat{f} \in \mathcal{U}_{[a]}.$$

Proof. If $a^{\bar{R}}[\bar{f}] = 0$ for some element $\bar{f} \in \bar{R}$, then we have by the formula (1) for every element $\bar{x} \in \bar{R}$

$$[\bar{f}] \bar{x}(a) = a^{\bar{R}}([\bar{f}] \bar{x}) = 0.$$

Putting $\bar{x} = |\bar{f}|[a^+] - |\bar{f}|[a^-]$, we have then

$$[\bar{f}] \{ |\bar{f}|([a^+]a) - |\bar{f}|([a^-]a) \} = 0,$$

that is, $|\bar{f}|([a]) = 0$, and hence $\bar{f}[a] = 0$ by Theorem 47.3.

Conversely, if $\bar{f}[a] = 0$ for some element $\bar{f} \in \bar{R}$, then we have by the formulas (1) and §46(12) for every element $\bar{x} \in \bar{R}$

$$a^{\bar{R}}[\bar{f}](\bar{x}) = [\bar{f}] \bar{x}(a) = [\bar{f}[a]] \bar{x} = 0,$$

and hence $a^{\bar{R}}[\bar{f}] = 0$. Therefore we have $a^{\bar{R}}[\bar{f}] = 0$ if and only if $\bar{f}[a] = 0$.

By virtue of the formula §48(3), $a^{\bar{R}}[\bar{f}] = 0$ is equivalent to

$$C_{a^{\bar{R}}} \mathcal{U}_{[\bar{f}]} = 0$$

in the proper space $\bar{E}$ of $\bar{R}$, and by virtue of the formula §54(11), $\bar{f}[a] = 0$ is equivalent to

$$\mathcal{U}_{[a]} \mathcal{U}_{[\bar{f}]} = 0$$

in the canonical space $\hat{E}$. Therefore we have

$$\mathcal{U}_{[a]} = (\mathcal{U}_{[a]})^- = (C_{a^{\bar{R}}})^-$$

in the canonical space $\hat{E}$. Since we have by the formula §54(10)

$$C_{a^{\bar{R}}} = \bar{E} \mathcal{U}_{[a^{\bar{R}}]}$$

in $\hat{E}$ and $\bar{E}$ is dense in $\hat{E}$, we obtain hence in $\hat{E}$

$$\mathcal{U}_{[a]} = (C_{a^{\bar{R}}})^- = (\bar{E} \mathcal{U}_{[a^{\bar{R}}]})^- = \mathcal{U}_{[a^{\bar{R}}]}.$$

For every point $\hat{f} \in \mathcal{U}_{[a]} \mathcal{U}_{[\bar{a}]}$ we have by Theorem 54.6

$$\left(\frac{t}{a}, \hat{f}\right) = \lim_{[\bar{f}] \rightarrow \hat{f}} \frac{([\bar{f}] \bar{a}(t))}{([\bar{f}] \bar{a}(a))},$$

and for every point $\hat{f} \in \mathcal{U}_{[a^{\bar{R}}]} \mathcal{U}_{[\bar{a}]}$ we have by Theorem 54.5

$$\left(\frac{t^{\bar{R}}}{a^{\bar{R}}}, \hat{f}\right) = \lim_{[\bar{f}] \rightarrow \hat{f}} \frac{t^{\bar{R}}([\bar{f}] \bar{a})}{a^{\bar{R}}([\bar{f}] \bar{a})}.$$

On the other hand, we have by the relation (1)

$$a^{\bar{R}}([\bar{f}] \bar{a}) = [\bar{f}] \bar{a}(a), \quad t^{\bar{R}}([\bar{f}] \bar{a}) = [\bar{f}] \bar{a}(t),$$

and $\mathcal{U}_{[a^{\bar{R}}]} = \mathcal{U}_{[a]}$ in $\hat{E}$, as proved just now. Consequently we obtain

$$\left(\frac{t}{a}, \hat{f}\right) = \left(\frac{t^{\bar{R}}}{a^{\bar{R}}}, \hat{f}\right)$$

for every point $\hat{f} \in \sum_{a \in R} \mathcal{U}_{[a]} \mathcal{U}_{[\bar{a}]}$. As

$$\left(\sum_{a \in R} \mathcal{U}_{[a]} \mathcal{U}_{[\bar{a}]}\right)^- = (\bar{E} \mathcal{U}_{[\bar{a}]})^- = \mathcal{U}_{[\bar{a}]}$$

in $\hat{E}$, we obtain hence

$$\left(\frac{t}{a}, \hat{f}\right) = \left(\frac{t^{\bar{R}}}{a^{\bar{R}}}, \hat{f}\right) \quad \text{for every point } \hat{f} \in \mathcal{U}_{[\bar{a}]},$$

since relative spectra are continuous by Theorem 19.2.

Making use of the notation

$$R^{\bar{R}} = \{a^{\bar{R}} : a \in R\}$$

we have:

Theorem 55.2. If R is semi-regular, then $R^{\bar{R}}$ is a linear lattice manifold of $\bar{R}$ and complete in $\bar{R}$, and further if

$$|\bar{a}| \leq |a^{\bar{R}}|, \quad \bar{a} \in \bar{R}, \quad a \in R,$$

then there exists an element $t \in K$ for which $\bar{a} = t^{\bar{R}}$.

Proof. If $|\bar{a}| \leq |a^{\bar{R}}|$, $\bar{a} \in \bar{R}$, $a \in R$, then we have by Theorem 18.11

$$\left|\left(\frac{\bar{a}}{a^{\bar{R}}}, \hat{f}\right)\right| \leq 1 \quad \text{for every point } \hat{f} \in \mathcal{U}_{[\bar{a}]},$$

and hence we obtain an element $t \in R$ as

$$t = \int_{[\bar{a}]} \left(\frac{\bar{a}}{a^{\bar{R}}}, \hat{f}\right) d\hat{f} a.$$

For such $t \in R$ we have by the previous theorem and Theorem 22.1

for every point $\hat{f} \in \mathcal{U}_{[\bar{a}]}$

$$\left(\frac{t^{\bar{R}}}{a^{\bar{R}}}, \hat{f}\right) = \left(\frac{t}{a}, \hat{f}\right) = \left(\frac{\bar{a}}{a^{\bar{R}}}, \hat{f}\right),$$

and hence we obtain by Theorem 19.5

$$[a^{\bar{R}}] t^{\bar{R}} = [a^{\bar{R}}] \bar{a}.$$

On the other hand, as $[l] \subseteq [a]$, we have $\mathcal{U}_{[l]} \subset \mathcal{U}_{[a]}$, and hence by the previous theorem

$$\mathcal{U}_{[l^{\bar{R}}]} \subset \mathcal{U}_{[a^{\bar{R}}]}.$$

Consequently we have $[a^{\bar{R}}] l^{\bar{R}} = l^{\bar{R}}$ by Theorem 8.2. Furthermore $|\bar{a}| \leq |a^{\bar{R}}|$ implies $[\bar{a}] \subseteq [a^{\bar{R}}]$ by Theorem 8.3, and hence $[a^{\bar{R}}] \bar{a} = \bar{a}$ by Theorem 8.2. Finally we obtain thus $l^{\bar{R}} = \bar{a}$.

From the fact proved just now, we conclude immediately that $\mathcal{R}^{\bar{R}}$ is a linear lattice manifold of $\bar{\mathcal{R}}$. Furthermore we have for every element $a \in \mathcal{R}$

$$(5) \quad a^{\bar{R}+} = a + \bar{R}, \quad a^{\bar{R}-} = a - \bar{R}.$$

Indeed, as proved just now, there exists an element $l \in \mathcal{R}$ for which $a^{\bar{R}+} = l^{\bar{R}}$. Then we have by the formula (2)

$$l^{\bar{R}} \geq 0, \quad (l - a)^{\bar{R}} \geq 0,$$

and hence we obtain by the formula (4)

$$l \geq 0, \quad l - a \geq 0,$$

that is, $l \geq a^+$. This relation yields by the formula (3)

$$l^{\bar{R}} \geq a^{\bar{R}+} \geq a^{\bar{R}+},$$

and consequently $a^{\bar{R}+} = a^{\bar{R}+}$. Then we have furthermore

$$a^{\bar{R}-} = (-a)^{\bar{R}+} = (-a)^{\bar{R}+} = a^{\bar{R}-}.$$

From the formula (5) we conclude easily:

$$(6) \quad |a^{\bar{R}}| = |a|^{\bar{R}},$$

$$(7) \quad a^{\bar{R}} \cup l^{\bar{R}} = (a \cup l)^{\bar{R}}, \quad a^{\bar{R}} \cap l^{\bar{R}} = (a \cap l)^{\bar{R}}.$$

If an element $\bar{a} \in \bar{\mathcal{R}}$ is orthogonal to every element $a^{\bar{R}}$ ($a \in \mathcal{R}$), then we have by the formula §15(5)

$$\mathcal{U}_{[\bar{a}]} \mathcal{U}_{[a^{\bar{R}}]} = 0 \quad \text{for every element } a \in \mathcal{R}$$

in the canonical space $\hat{E}$. If $\mathcal{U}_{[\bar{p}]} \subset \mathcal{U}_{[\bar{a}]}$ in $\hat{E}$ for a projector $[\bar{p}]$ in $\bar{\mathcal{R}}$, then we have naturally

$$\mathcal{U}_{[\bar{p}]} \mathcal{U}_{[a^{\bar{R}}]} = 0 \quad (a \in \mathcal{R})$$

in $\hat{E}$, and hence by the formula §54(11) for every element $a \in \mathcal{R}$

$$\bar{p}(a) = a^{\bar{R}}(\bar{p}) = a^{\bar{R}}\bar{p} = 0,$$

that is, $\bar{p} = 0$. Therefore we conclude $\mathcal{U}_{[\bar{a}]} = 0$, and hence

$\bar{a} = 0$. Consequently $R^{\bar{R}}$ is complete in $\bar{R}$.

A universally continuous semi-ordered linear space R is said to be weakly monotone complete, if to every increasing system of positive elements $R \ni a_\lambda \uparrow_{\lambda \in A}$ such that

$$\sup_{\lambda \in A} \bar{a}(a_\lambda) < +\infty \quad \text{for every positive } \bar{a} \in \bar{R},$$

there exists an element $a \in R$ for which $a_\lambda \uparrow_{\lambda \in A} a$.

With this definition we have:

Theorem 55.3. The conjugate space $\bar{R}$ of every universally continuous semi-ordered linear space R is weakly monotone complete.

Proof. If $\bar{R} \ni \bar{a}_\lambda \uparrow_{\lambda \in A}$ and

$$\sup_{\lambda \in A} \bar{a}(\bar{a}_\lambda) < +\infty \quad \text{for every positive } \bar{a} \in \bar{R}$$

for the conjugate space $\bar{R}$ of R , then we have naturally for every positive element $a \in R$

$$\sup_{\lambda \in A} \bar{a}_\lambda(a) = \sup_{\lambda \in A} a^{\bar{R}}(\bar{a}_\lambda) < +\infty,$$

since $\bar{R} \ni a^{\bar{R}} \geq 0$ by the formula (3). Consequently there exists by Theorem 53.2 an element $\bar{a} \in \bar{R}$ for which $\bar{a}_\lambda \uparrow_{\lambda \in A} \bar{a}$. Therefore $\bar{R}$ is by definition weakly monotone complete.

A universally continuous semi-ordered linear space R is said to be reflexive, if R is isomorphic to the conjugate space $\bar{\bar{R}}$ of its conjugate space $\bar{R}$ by the correspondence indicated in the formula (1): $R \ni a \rightarrow a^{\bar{R}} \in \bar{\bar{R}}$.

Theorem 55.4. In order that a universally continuous semi-ordered linear space R be reflexive, it is necessary and sufficient that R be weakly monotone complete.

Proof. Since the necessity is evident by the previous theorem, we shall prove the sufficiency. We suppose that R is weakly monotone complete. Then R is semi-regular. Because, if

$$\bar{a}[p] = 0 \quad \text{for every element } \bar{a} \in \bar{R},$$

then we have $\nu|p| \uparrow_{\nu=1}^\infty$ and

$$\bar{a}(\nu|p|) = 0 \quad \text{for every element } \bar{a} \in \bar{R}.$$

Consequently there exists by assumption an element $b \in R$ for which

$\nu |p| \tau_{\nu} = 1$, ℓ . For such $\ell \in R$ we have obviously

$$|p| \leq \frac{1}{\ell} \ell \quad \text{for every } \nu = 1, 2, \dots,$$

and hence $p = 0$ by Theorem 6.3. Therefore R is semi-regular by definition.

For every positive element $\bar{a} \in \bar{R}$, putting

$$A = \{a : a^{\bar{R}} \leq \bar{a}, 0 \leq a \in R\},$$

we obtain an increasing system of positive elements A in R by virtue of Theorem 55.2. For such A , since we have for every positive element $\bar{a} \in \bar{R}$

$$\sup_{a \in A} \bar{a}(a) = \sup_{a \in A} a^{\bar{R}}(\bar{a}) \leq \bar{a}(\bar{a}),$$

there exists by assumption an element $a_0 \in R$ for which

$$a_0 = \bigcup_{a \in A} a.$$

For such $a_0 \in R$ we have by Theorem 53.4 for every positive element $\bar{a} \in \bar{R}$

$$\bar{a}(a_0) = \sup_{a \in A} \bar{a}(a),$$

and hence $a_0^{\bar{R}}(\bar{a}) \leq \bar{a}(\bar{a})$ for every positive element $\bar{a} \in \bar{R}$,

that is, $a_0^{\bar{R}} \leq \bar{a}$. By virtue of Theorem 55.2, to every positive element $x \in R$ there exists a positive element $\ell_0 \in R$ such that

$$(\bar{a} - a_0^{\bar{R}}) \wedge x^{\bar{R}} = \ell_0^{\bar{R}}.$$

For such $\ell_0 \in R$ we have obviously

$$\bar{a} \geq a_0^{\bar{R}} + \ell_0^{\bar{R}} = (a_0 + \ell_0)^{\bar{R}},$$

and hence $a_0 + \ell_0 \in A$ by definition of A . Consequently we have naturally

$$a_0 \geq a_0 + \ell_0,$$

that is, $0 \geq \ell_0$, and hence we conclude $\ell_0 = 0$. Therefore

$\bar{a} - a_0^{\bar{R}}$ is orthogonal to every element $x^{\bar{R}}$ of $R^{\bar{R}}$.

On the other hand, $R^{\bar{R}}$ is by Theorem 55.2 complete in $\bar{R}$.

Hence we obtain $\bar{a} = a_0^{\bar{R}}$.

Thus we have proved that to every positive element $\bar{a} \in \bar{R}$ there exists a positive element $a \in R$ for which $a^{\bar{R}} = \bar{a}$.

Accordingly we see easily by the formulas (2), (3), and (4) that $\mathcal{R}$ is isomorphic to $\bar{\mathcal{R}}$ by the correspondence $\mathcal{R} \ni \alpha \rightarrow \alpha^{\bar{\mathcal{R}}} \in \bar{\mathcal{R}}$.

From Theorems 55.3 and 55.4 we conclude immediately:

Theorem 55.5. The conjugate space $\bar{\mathcal{R}}$ of every universally continuous semi-ordered linear space $\mathcal{R}$ is reflexive.

**) For two sequences of real numbers α_ν and β_ν ($\nu = 1, 2, \dots$), if for every subsequences α_{ρ_ν} and β_{σ_ν} ($\nu = 1, 2, \dots$) the sequence $\alpha_{\rho_\nu} + \beta_{\sigma_\nu}$ ($\nu = 1, 2, \dots$) is convergent, then both sequences α_ν and β_ν ($\nu = 1, 2, \dots$) are convergent.

NOTE

The numbers given in the bibliographical references relate to the list of cited works which will be found in Bibliography.

§2. The notion of semi-ordered linear spaces is essentially due to Riesz (1), but it was considered first by Kantorovich (1) in an abstract form.

§3. Theorem 3.3 is a generalization of the distributive law:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c), \quad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c),$$

which was proved in different manners by Dedekind (1), by Freudenthal (1), by Birkhoff (1), and by Nakano (1). The proof of Nakano (1) coincides essentially with that of Dedekind (1). The proof cited in the text is due to Nakano (8).

§4. The formula (5) is due to Birkhoff (1). We can prove the formula (6) without use of the formula (5). In fact, we have obviously

$$a \vee c \leq (b + |a - b|) \vee (c + |a - b|) = (b \vee c) + |a - b|,$$

and hence

$$(a \vee c) - (b \vee c) \leq |a - b|.$$

Similarly we obtain

$$(b \vee c) - (a \vee c) \leq |a - b|.$$

Consequently we have by the formula (1)

$$|(a \vee c) - (b \vee c)| \leq |a - b|.$$

Replacing a , b , c respectively by $-a$, $-b$, $-c$, we obtain from this relation

$$|(a \wedge c) - (b \wedge c)| \leq |a - b|.$$

§5. Convergence of a sequence of elements was defined first by Kantorovitch (2) and by Steen (1) in such form that we have $\lim_{\nu \rightarrow \infty} a_\nu = a$, if $a = \bigcap_{\mu=1}^{\infty} \{ \bigcup_{\nu=\mu}^{\infty} a_\nu \} = \bigcap_{\mu=1}^{\infty} \{ \bigcap_{\nu=\mu}^{\infty} a_\nu \}$. This definition is equivalent to the definition in the text, which was adopted first by Nakano (1).

There are furthermore three essentially different notions of convergence. Star convergence by Kantorovitch (2): A sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be star convergent to an element $a \in R$ and denoted by

$$s\text{-}\lim_{\nu \rightarrow \infty} a_\nu = a$$

if every subsequence from a_ν ($\nu = 1, 2, \dots$) contains a subsequence which is convergent to a . Individual convergence by Nakano (15): a sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be individually convergent to an element $a \in R$ and denoted by

$$\text{ind-}\lim_{\nu \rightarrow \infty} a_\nu = a,$$

if we have for every elements x and $y \in R$

$$\lim_{\nu \rightarrow \infty} (a_\nu \wedge x) \vee y = (a \wedge x) \vee y.$$

Star individual convergence by Nakano (16): a sequence of elements $a_\nu \in R$ ($\nu = 1, 2, \dots$) is said to be star individually convergent to an element $a \in R$ and denoted by

$$s\text{-ind-}\lim_{\nu \rightarrow \infty} a_\nu = a,$$

if $(a_\nu \wedge x) \vee y$ ($\nu = 1, 2, \dots$) is star convergent to $(a \wedge x) \vee y$ for every elements x and $y \in R$.

For example, the so-called L_p -space ($p \geq 1$), namely the totality of measurable functions $\varphi(t)$ such that

$$\int_0^1 |\varphi(t)|^p dt < +\infty$$

constitutes a semi-ordered linear space, if we define $\varphi \geq \psi$ to mean that we have $\varphi(t) \geq \psi(t)$ for $0 \leq t \leq 1$ up to a point set of measure zero. In the L_p -space, $s\text{-}\lim_{\nu \rightarrow \infty} \varphi_\nu = \varphi$ is equivalent to

$$\lim_{\nu \rightarrow \infty} \int_0^1 |\varphi_\nu(t) - \varphi(t)|^p dt = 0;$$

$\text{ind-lim}_{\nu \rightarrow \infty} q_\nu = q$ is equivalent to

$$\lim_{\nu \rightarrow \infty} q_\nu(t) = q(t)$$

for $0 \leq t \leq 1$ up to a point set of measure zero; and

$$\text{s-ind-lim}_{\nu \rightarrow \infty} q_\nu = q$$

is equivalent to that the measures of point sets

$$\{t : |q_\nu(t) - q(t)| \geq \varepsilon\} \quad (\nu = 1, 2, \dots)$$

are convergent to zero for every positive number ε .

§6. The notion of uniform convergence was introduced by Kantorovitch (2). Theorem 6.4 is due to Kantorovitch (2). Theorem 6.4, established by Nakano (1), is a generalization of Cauchy's convergence theorem in classical analysis.

§7. Notion of projection in semi-ordered linear spaces was introduced in many different manners, by Riesz (1), Freudenthal (1), Maeda (1), Nakano (1), Bochner and Phillips (1). Projectors were defined by Nakano (1) and discussed precisely. Projection operators were most generally defined and discussed by Nakano (16).

§8. If R is universally continuous, then we can prove likewise as in classical analysis that to every system of bounded variation $a_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) there exist two increasing systems b_λ and $c_\lambda \in R$ ($\alpha \leq \lambda \leq \beta$) such that

$$a_\lambda = b_\lambda - c_\lambda \quad (\alpha \leq \lambda \leq \beta).$$

§13. The existence of a resolution $a_\lambda \in R$ ($-\infty < \lambda < +\infty$) in Theorem 13.1 was proved first by Freudenthal (1) in the case $a \geq 0$ without use of projectors. The uniqueness of such a resolution was proved by Nakano (1). Proof cited in the text is due to Nakano (1). A merit of this Proof is to enable us to get a explicit form of spectral system as Theorem 13.2.

§15. Representation of a Boolean algebra was discussed first by Stone (1) and simplified by Wallman (1). This method was applied to projectors to define the proper space.

§18. Relative spectra were defined by Nakano (2) and discussed precisely. Chapter III is due to Nakano (2).

§28. Absolute spectra were defined by Nakano (14) and discussed precisely. By means of absolute spectra we can deduce the first spectral theory in semi-ordered rings, established first by Steen (1), as discussed in §23. Chapter IV is due to Nakano (14)

§30. It is wellknown that the notion of cut extension was introduced by Dedekind to define the real numbers from the rational numbers. This idea was generalized first by MacNeille (1) to semi-ordered sets, proving that every Boolean algebra may be extended by cuts to a Boolean algebra again. This method was applied by Clifford (1) to semi-ordered abelian groups, and further by Nakano (2) to archimedean semi-ordered linear spaces.

§31. Cut extension of semi-ordered rings was suggested already by Nakano (2) and discussed precisely by Nakano (14).

§33. The notion of completeness was introduced first by Nakano (1), as terminology "vollkommen", and completions were established as dilatator extension, c.f. §45.

§36. Relations between ortho-normal manifolds in a C-space and their characteristic sets were obtained by Nakano (5).

§38. Theorem 36.1 was proved by Nakano (5). While the

case for the C -space on a compact Hausdorff space was stated independently by M. and S. Krein (1) without proof. Theorem 38.2 was proved first by Stone (2) and after independently by Gelfand and Silov (1) for the C -space on a compact Hausdorff space. An elementary proof cited in the text is due to Nakano (14).

§39. Theorem 39.1 was proved in different manners by Nakano (4), (5), and (6). Theorem 39.3 was directly proved first by M. and S. Krein (1), and independently stated by Kakutani (1) without proof.

§40. Theorem 40.3 was stated first by Stone (3) without proof, under stronger assumption, and then proved by Nakano (2), and by Vernikoff, S. Krein, and Tovbin (1). Generalization of it, namely Theorems 40.1 and 40.2 are due to Nakano (14).

§41. Theorems 41.1 and 41.4 are due to Nakano (3). Theorems 41.2 and 41.5 are due to Nakano (8).

§44. Dilatators were defined by Nakano (1) and precisely discussed by Nakano (1) and (8).

§45. If $\mathcal{R}$ has a complete element, then the dilatator ring may be considered as a completion of $\mathcal{R}$, as stated in Theorem 45.7. Therefore we can deduce the first spectral theory in $\mathcal{R}$ from that in the dilatator rings. Conversely it is evident that the first spectral theory in semi-ordered rings is obtained as that in semi-ordered linear spaces. Furthermore, we see easily that we can introduce a product into $\mathcal{R}$ by dilatator rings, which was attempted first by Vulich (1) by a direct method.

Note

§48. Characteristic sets of linear functionals were defined first by Nakano (8) for continuous linear functionals and then generalized by Nakano (16).

§51. We also can define relative spectra $(\frac{\mathcal{F}}{\mathcal{E}}, \mathcal{F})$ conversely by Theorem 51.5, c.f. Nakano (8).

§52. Theorem 52.3, obtained by Nakano (8), is a generalization of Radon-Nikodym's Theorem in abstract integral, c.f. Saks (1).

§53. The notion of conjugate spaces was introduced by Nakano (8) and discussed precisely.

§55. Theorem 55.4 was proved first by Nakano (8), and then proved without use of canonical spaces by Nakano (16). Furthermore conjugate spaces were discussed precisely by Nakano (16).

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