A

MANUAL OF TRIGONOMETRY.

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FOR THE USE, MORE ESPECIALLY, OF YOUNG
SAILORS AND OFFICERS IN THE
MERCHANT NAVY.

BY
RICHARD C. BUCK,
LATE OF THE THAMES NAUTICAL TRAINING COLLEGE, H.M.S. "WORCESTER."

SIXTH EDITION, WITH SUPPLEMENT.

LONDON:
CHARLES GRIFFIN & COMPANY, LIMITED;
EXETER STREET, STRAND, W.C. 2.
1923.

Printed in Great Britain
by Bell & Bain, Limited, Glasgow.

AUTHOR'S PREFACE.

THE introduction of Trigonometry into the Examinations for Officers of the Mercantile Marine by the Board of Trade must necessarily disconcert all examinees who have either had no special mathematical training, or who, as is usually the case, have become somewhat rusty in the knowledge acquired during their school life. The announcement of these extensions to the examinations in and after 1898 has, in fact, created a certain amount of consternation among those who have still to obtain certificates of competency.

The object of this Manual is to remove any difficulties which the rising generation of officers may encounter. Having spent nearly a lifetime in tuition, the Author claims to be well acquainted with all those obstacles which beset the Student, who, situated as the Sailor is, must depend so greatly on what he can do without the aid of a teacher. Unusual care (some, perhaps, will say "superfluous care") has been taken to explain every step of the work, and thus to put it in the power of the Student to advance steadily without a single halt, or a sigh that no one is present to explain what to him is a knotty point. With a view to test the accuracy of the above statements, the MS. and proof-sheets of the Manual were placed in the hands of boys of from 14 to 16 years of age, with injunctions to learn therefrom how to solve triangles, &c.; in no case did they fail to master the subject without help.

The feature of the work, then, is its utter simplicity, and thus, it is hoped, its suitability as an elementary work for those who have neither much time, nor much inclination, to add to the many branches of study forced upon them. Any intelligent person may, with a limited amount of perseverance, acquire from it a knowledge of Trigonometry sufficient for all ordinary purposes. The budding Surveyor, the Navigator, and the Astronomer may from it lay down a foundation on which to build the more elaborate superstructure of knowledge so desirable and so necessary to a full grasp of the work entrusted to them.

Another reason which induced the Author to enter into so many minutiæ is the great advantage to be derived from the dissection of formulæ. Nothing can be more serviceable to the Student than the power to grasp and translate, and, if necessary, utilise for calculation, the various formulæ which he meets with in books bearing on professional subjects. The origin of such equations, &c., as he may find useful will probably be far beyond him; but without much difficulty he will be able to put the statements he has before him to some practical purpose.

There are many methods of arriving at the same results in Trigonometry (as in most matters); it was, therefore, necessary to adopt some definite course. That in use on board the Training College, H.M.S. "Worcester," is chosen, not because it is the briefest, or even the simplest, but because after years of testing it has been found very efficient. The adoption of the various methods under different names, as "cosines" for three sides, "sines," "tangents," has been found advantageous, because the rules thus become more firmly fixed in the mind. From the Chapter on Formulæ it will be seen that several other plans might be used.

The explanation of the Ratios, and of their transformation from the earlier to the later form, has been introduced in order to dispel the mists which cloud the mathematically-untrained mind when a value so apparently intangible as the modern ratio is placed before it.

The theoretical part of the subject has, for the most part, been placed after the practical portion, experience suggesting the desirability of this innovation upon the plan followed in the ordinary school text book.

The Author desires to call attention to the approximate character of some of the calculations. It is thought to be sufficient in most cases to work to degrees and minutes only, though the Student is advised to practise working to seconds in order to meet emergencies. Some insight into the "degree of accuracy" of the work is very desirable, and would often save much foolish controversy and waste of time in useless calculations. Since the tables of logarithms are themselves but approximations, they cannot be expected to give exact results in any cases that are not well within their scope of accuracy. All extravagant and disproportionate measurements should be avoided, or, at least, the results obtained should be treated as approximations only. In the exercises occasional cases of the above description are given—*viz.*, extravagant differences in the length of lines, sizes of angles, &c.—to illustrate what is meant by the above. A curious instance of ignorance as to the suitability of measurements for triangulation came before the Author some years ago, and may be briefly mentioned here as a case in point. A trigonometrician (?) stated that he had measured and mapped out the "Solent," taking a **21 yards** base, in the Isle of Wight, and the bearings of objects on the opposite shore of the mainland! Theoretically, this is very good; practically, it is little more than nonsense.

The hints and intermediate calculations given in the Chapter on Problems will, it is hoped, give the student just that degree of support so necessary in the early stages of the application of the solutions of triangles, &c. After working through the problems in this Manual the Student should be equal to taking up successfully any ordinary questions bearing on Trigonometry.

Finally, if the *Manual of Algebra* by the Author is studied at the same time, the Student will find his grasp of Trigonometry materially strengthened. The *Algebra* is made as simple as is consistent with mathematical truth.

The Author desires to thank the Masters and Cadets of the "Worcester" for the assistance they have rendered in checking the calculations contained in the book, and he trusts that the general body of young Officers of the Mercantile Marine will find the manual of real help to them.

GREENHITHE,
June, 1897.

NOTE TO SIXTH EDITION.

THE continued demand for this book is very gratifying, and necessary corrections having been made, the Versine formula added and a short Supplement to give further practice in logarithms for beginners introduced into the Fifth edition, it is now reprinted to meet the continued demand.

June, 1923.

CONTENTS.

TRIGONOMETRY.

TRIGONOMETRY is so manifestly useful to the nautical man who wishes to understand the methods by which he determines his position at sea that no excuse is necessary in advising the careful study of it. Navigation by "Dead Reckoning" is little more than a special application of Plane, and that by "Observation" of the celestial bodies a similar use of Spherical Trigonometry. In the following pages a limited amount of theory is given, in order that the student may see whence his formulæ for solutions of the triangles arise: the practical side will be more fully considered. The subject is taken in as simple a form as is consistent with due regard to mathematical requirements; technicalities are, as far as possible, avoided. Only approximate values are given in most of the results, these being generally sufficient for all but the few cases (pointed out in the work) in which more rigorous treatment is necessary. Calculations to seconds should be practised occasionally, and, as the sum of the angles of a plane triangle amounts to 180°, the student may easily test his powers with any of the plane examples, remembering that the deficiencies of the logarithms themselves will account for a few seconds in the total result. To save space and trouble, many small knotty points are treated in Chapter i., instead of in various parts of the manual; in any difficulty this chapter should be sought.

The rules given for the solution of triangles are not the shortest in every case, but have the advantage of easily impressing themselves on the memory. The formulæ for other methods are given for the benefit of more advanced students.

The tables used are "Riddle's," but any other set should give the same results, or a close approximation to them.

Directions to Students.

The Chapter on Logarithms having been studied sufficiently to enable the student to take out the log. of a number and of a ratio (Sine, &c.), and to find a number or ratio corresponding to a given log., the Chapter on Triangles (practical) should be proceeded with.

The early Chapters on the Definitions, &c., should be read at intervals during the practical work of Calculation. The more advanced Theory may be left until a thorough mastery has been obtained of the solutions of triangles, when this and the "Problems" may go hand in hand. Finally, the student should exercise his ingenuity over the Identities and other formulæ given to test his grasp of the subject. The same example can be used many times in the various rules; thus, in "Cosines," if the answers are used, "Tangents," "Sines," &c., can be composed: the results obtained will approximately agree with the parts given in the exercise.

CHAPTER I.

DEFINITIONS AND SPECIAL POINTS.

Trigonometry, which in its earlier stages treated of little but the measurements of triangles, has now, by the aid of Algebra and Geometry, taken up wider ground, and treats of the relationships existing among angular measurements under every condition.

Plane Triangle.—A figure contained by three straight lines on a plane or flat surface.

Great Circle.—A circle which divides the sphere on which it is drawn into two equal parts; its plane passes through the centre of that sphere.

Small Circle.—A circle which divides the sphere on which it is drawn into two unequal parts: its plane does not pass through the centre of that sphere. It cannot be used in a sph. triangle.

Spherical Triangle.—Formed by the intersection of the arcs of three great circles. (Caution —The arcs must be parts of three great circles: mistakes are often made in this, small circles being used.)

Trigonometrical Angle.—If a straight line revolves in a plane about a fixed point till it takes another or returns to the same position, it is said to trace out a trigonometrical angle, which may be unlimited in size. Thus in Fig. 3 (Chap. iii.), the line CD revolves about C from the initial position CB, taking up the positions CD, CD', &c., and forming angles BCD, BCD' (over 90°), BCD'' (over 180°).

Geometrical Angle.—A plane rectilineal angle is the inclination of two straight lines to one another which meet together, but are not in the same straight line.

Note.—A geometrical angle is limited (less than two right angles); a trigonometrical angle is unlimited (CD might revolve about C continuously).

Degree.—The 360th part of the circumference of a circle; a quadrant thus contains 90 degrees (90°), each degree being divided into 60 minutes, and each minute into 60 seconds. As the angle subtended by a quadrant is a right angle, the latter is also 90°.*

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* A degree $\left(\frac{\text{rt. ang.}}{90}\right)$ is the unit of angular measurement in the sexagesimal system; two other units are used, the grade $\left(\frac{\text{rt. ang.}}{100}\right)$ in the French system, and the radian $\left(\frac{180^\circ}{\pi}\right)$ in circular measure.

π (pi).—The amount (3·1416 nearly) by which the diameter of a circle has to be multiplied to produce the length of the circumference. π* is also used in one method of measuring angles—circular measure—to express two right angles (180°).

s.—The sum of the sides of a triangle is called the perimeter: the semi-perimeter is s. (Note **s** for semi.)

Triangles.—Triangles are generally named by letters of the alphabet, as ABC, &c. In this case the sides are called AB, BC, CA, or, from the opposite angle, c, a, b; the angles are known as BAC or CAB, ABC or CBA, ACB or BCA; or any angle may have some special name—thus, angle ABC may be known as θ (theta), angle BAC as ϕ (phi), &c.

Note that in a triangle the greater side is opposite to the greater angle; equal sides have equal angles opposite to them. This is very useful in the practical work. Triangles showing the given parts should be constructed before the calculation of the required parts is begun; this is especially necessary in right angled and quadrantal triangles.

Angles—Angles greater than 90° require special attention, as the tables of logarithms are only extended to 90°; for any such angles adopt the following plan:—Reduce by 90°, and add co to, or subtract co from the ratio; thus the sine of 95° is the cosine of 5°, the cotangent of 119° 20′ is the tangent of 29° 20′. The complement of an angle is the amount which is required to raise the angle to 90°; the supplement added to the angle produces 180°.

Degree of Accuracy.—Whenever there is seen to be any rapid increase or decrease of some quantity (generally a ratio and the angle it represents), the use of seconds of angular and decimals of linear measurement is strongly advised. The results derived from logarithmic calculations are closely approximate to the truth, provided the quantities used consist of not more than 4 or 5 figures. Some discrepancies in the results will be found when the exercises in one Rule are solved by some other Rule; this would not occur if each calculation was carried to decimals of a second, &c.

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* *i.e.*, π units, as it should be written.

CHAPTER II.

THE CIRCLE: LINES AND ANGLES.

In the circle *BDHF* let the angle *BCD* or *A*, less than 90°, be formed by the revolving line *CD*; from *H*, a point perpendicularly above *C*, and 90° from *B*, draw *HG* parallel to *CB*, meeting *CD* produced in the point *G*; from *D* draw *DE* perpendicular to *CB*; draw *BK* perpendicular to *CB*, meeting *CG* (or *CG* produced if necessary) in *K*. It is found that the lines *DE*, *KB*, *CK*, *CE*, *HG*, *CG*, *EB*, and *EF* increase or decrease when the size of the angle is changed. The student will see at once that, if angle *A* were increased, but still kept less than 90°, the lines *DE*, *KB*, *CK*, and *BE* would increase, whilst the other lines would decrease. Here then we have a relationship between lines and angles, and on this, aided by certain truths gathered from geometry, we base the science of trigonometry. It will be shown presently that the general principles established for angles in the first quadrant (less than 90°) are true for all trigonometrical angles.

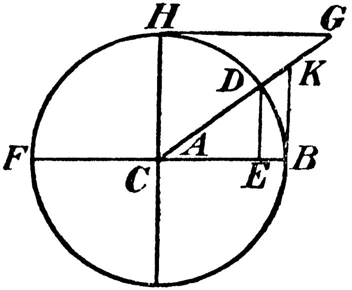

Fig. 1.

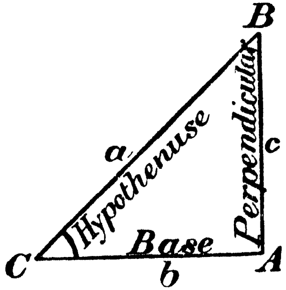

Fig. 2.

To the lines *DE*, &c. (now no longer used, but replaced by equal values), the following names were given :—*DE* the sine of angle *A*, *CE* its cosine, *KB* tangent, *HG* cotangent, *CK* secant, *CG* cosecant, *BE* versed-sine, and *FE* suversed-sine.

It can be geometrically proved that these lines will, in the same sized angle, always have the same proportional value to each other and to the radius of the circle with which they are connected; the radius thus becomes the standard (value one) of their measurement. It can also be shown that the sides of the right-angled triangle *BCA* (Fig. 2) have the same proportions to each other as the sine, tangent, &c., have to the radius in Fig. 1, and that this is equally true of all such triangles. We are thus able to dispense with the circle in forming trigonometrical ratios (the general name for sine, &c.).

In Fig. 1 let the sides of the right-angled triangle *DCE* be measured and compared; it will be found that the length of *DE* divided by the length of *CE* will give the same result as is found by comparing *KB* (tan.) with radius; and so on. The student will strengthen his insight into these valuable elementary points by taking a large circle (with radius a foot), drawing any angle (less than 90°) and the lines of the figure, and testing the above ratios with a scale marked to inches and decimals. Then, if he finds $DE = 4$ inches and $CE = 6$ inches, he will find $KB = 8$ inches; that is *DE* divided by *CE*, or $\frac{4}{6}$ or $\frac{2}{3}$ gives tangent *KB*; for *KB* will be found to be 8 inches, or $\frac{2}{3}$ of the foot radius. And so on with the other ratios.

Let any circle (Fig. 1) be described, and a radius revolve about the centre; if from the end of the radius a straight line fall perpendicularly upon the initial or first position of the radius, the revolving radius shall be called the hypothenuse, the perpendicular line shall be called the perpendicular, and the part of the initial line contained in the triangle formed shall be called the base, with respect to the angle *DCB* described by the revolving radius.

Or, in any right-angled triangle (Fig. 2) the side opposite the right angle is the hypothenuse (H), the side opposite either of the acute angles is the perpendicular (P), and the remaining side is the base (B), with respect to that angle.

From Fig. 1 it may be seen (by geometry and measurement) that the length of the perpendicular, divided by the length of the hypothenuse, gives sine; that is, the ratio of perp. to hyp. is sine, or briefly $\frac{P}{H} = \text{sine}$; similarly $\frac{P}{B} = \text{tangent}$, $\frac{B}{H} = \text{cosine}$, $\frac{B}{P} = \text{cotangent}$, $\frac{H}{P} = \text{cosecant}$, $\frac{H}{B} = \text{secant}$. Versed-sines and suversed-sines have been explained sufficiently for our purposes. These ratios must be committed to memory before the solution of right-angled triangles is attempted.

CHAPTER III.

POSITIVE AND NEGATIVE.

As a trigonometrical angle is formed by the revolution of a line in a plane, and as this may occur in either of two directions—that is, with or against the movement of the hands of a clock—we need some means to distinguish these. It is agreed that when the revolving line moves contrary to the hands, the angle formed shall be termed positive, when with the hands, negative.

It is also necessary to note the directions of other lines: it is allowed that perpendiculars above the initial line shall be positive, those below being negative; that the initial line (a radius) shall be positive, its continuation (forming with it a diameter) shall be negative; that the revolving line shall in every case be positive. Looking at a map we may take lines showing N. latitude as +, those showing S. latitude as −; E. longitude (on the equator) may be taken as +, and W. longitude as −.

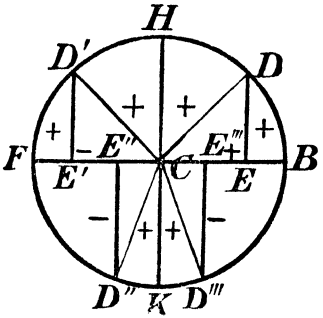

Fig. 3.

Thus in Fig. 3 if we suppose CD to have moved from CB towards H, forming the angle BCD, this angle is positive, the lines CD, DE, and CE are also positive. If CD be supposed to have moved as before till the angle BCD' is formed, this angle is also positive, and the lines are, CD' positive, $D'E'$ positive, CE' negative. If CD moved through H till it took up the position

CD'', thus making the angle *BCD''* (measured by arc *BHD''*), this angle and the line *CD''* would be positive, but *CE''* and *E''D''* would be negative. If *CD* moved from *CB* towards *K*, forming the angle *BCD'''*, this angle would be negative, and the lines would be *CD'''* and *CE'''* positive, and *E'''D'''* negative. *Note.*—The utility of these distinctions will be seen when the values of certain angles have to be considered. The negative angle is, however, so much less important than the positive that we shall confine our attention to the latter.

CHAPTER IV.

RATIOS.

We saw in Chapter ii. how the ratios of trigonometrical angles are formed, and, as an angle and its ratios are dependent on each other, it is easy to see that given the ratio we can find the angle, and *vice versâ.* The ratios for all angles from 0° to 90° have been computed and tabulated in two forms—the natural and logarithmic values (tables of Sines, Cosecants, &c.). These and Traverse tables (in which the parts of right-angled triangles are given) are explained in the works which contain them. In the theory of Plane Trig. (Chapter x.) the methods of calculation of the ratios of certain angles are given.

The following table shows how the ratios of angles from 0° to 360° vary in value, and in character or affection. For angles of more than 360° the values and signs may be found by taking the angle after 360° have been subtracted (once or more times).* Thus the sine of 530° will be the sine of (530° – 360°) or sine 170°; the tangent of 980° will be the tangent of (980° – twice 360°, *i.e.*, 720°), or tan. 260°. It will be shown presently that to meet the requirements of the tables of sines, &c., these angles 170° and 260° may be still further reduced.

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* Till the remainder is less than 360°.

Tabulated Ratios, &c., from 0° to 360°.
Changes in Value and Sign.

Quadrants.	I. 0° to 90°.	II. 90° to 180°.	III. 180° to 270°.	IV. 270° to 360°.
Perp, . .	0 to rad	rad to 0	−0 to −rad	−rad to −0
Base, . .	rad to 0	−0 to −rad	−rad to −0	0 to rad
Hyp, . .	radius	radius	radius	radius
$\frac{\text{Perp}}{\text{Hyp}}$, . . or Sine, . .	$\frac{0}{\text{rad}}$ to $\frac{\text{rad}}{\text{rad}}$ or 0 to 1	$\frac{\text{rad}}{\text{rad}}$ to $\frac{0}{\text{rad}}$ or 1 to 0	$\frac{-0}{\text{rad}}$ to $\frac{-\text{rad}}{\text{rad}}$ or −0 to −1	$\frac{-\text{rad}}{\text{rad}}$ to $\frac{-0}{\text{rad}}$ or −1 to −0
$\frac{\text{Hyp}}{\text{Perp}}$, . . or Cosec, .	$\frac{\text{rad}}{0}$ to $\frac{\text{rad}}{\text{rad}}$ or ∞ to 1	$\frac{\text{rad}}{\text{rad}}$ to $\frac{\text{rad}}{0}$ or 1 to ∞	$\frac{\text{rad}}{-0}$ to $\frac{\text{rad}}{-\text{rad}}$ or −∞ to −1	$\frac{\text{rad}}{-\text{rad}}$ to $\frac{\text{rad}}{-0}$ or −1 to −∞
$\frac{\text{Perp}}{\text{Base}}$, . . or Tan, . .	$\frac{0}{\text{rad}}$ to $\frac{\text{rad}}{0}$ or 0 to ∞	$\frac{\text{rad}}{-0}$ to $\frac{0}{-\text{rad}}$ or −∞ to −0	$\frac{-0}{-\text{rad}}$ to $\frac{-\text{rad}}{-0}$ or 0 to ∞	$\frac{-\text{rad}}{0}$ to $\frac{-0}{\text{rad}}$ or −∞ to −0
$\frac{\text{Base}}{\text{Perp}}$, . . or Cot, . .	$\frac{\text{rad}}{0}$ to $\frac{0}{\text{rad}}$ or ∞ to 0	$\frac{-0}{\text{rad}}$ to $\frac{-\text{rad}}{0}$ or −0 to −∞	$\frac{-\text{rad}}{-0}$ to $\frac{-0}{-\text{rad}}$ or ∞ to 0	$\frac{0}{-\text{rad}}$ to $\frac{\text{rad}}{-0}$ or −0 to −∞
$\frac{\text{Base}}{\text{Hyp}}$, . . or Cos, . .	$\frac{\text{rad}}{\text{rad}}$ to $\frac{0}{\text{rad}}$ or 1 to 0	$\frac{-0}{\text{rad}}$ to $\frac{-\text{rad}}{\text{rad}}$ or −0 to −1	$\frac{-\text{rad}}{\text{rad}}$ to $\frac{-0}{\text{rad}}$ or −1 to −0	$\frac{0}{\text{rad}}$ to $\frac{\text{rad}}{\text{rad}}$ or 0 to 1
$\frac{\text{Hyp}}{\text{Base}}$, . . or Sec, . .	$\frac{\text{rad}}{\text{rad}}$ to $\frac{\text{rad}}{0}$ or 1 to ∞	$\frac{\text{rad}}{-0}$ to $\frac{\text{rad}}{-\text{rad}}$ or −∞ to −1	$\frac{\text{rad}}{-\text{rad}}$ to $\frac{\text{rad}}{-0}$ or −1 to −∞	$\frac{\text{rad}}{0}$ to $\frac{\text{rad}}{\text{rad}}$ or ∞ to 1

Note.—In the first quadrant all ratios are positive.
,, second ,, all but sine and cosec are negative.
,, third ,, all but tan and cot are negative.
,, fourth ,, all but cosine and sec are negative.

CHAPTER V.

PRACTICAL USE OF LOGARITHMS.

THE construction of a logarithm is explained in "Algebra"; how to use tables of logs.* is shown with the tables; here we shall take a few examples to more fully illustrate the use of these aids to calculation.

1. Find the log. of 125946.

The decimal part of the log. of 1259 is 100026; we have now to provide for 46. In column *D*, opposite the log., we find 346; multiply this number by the 46, which gives 15916, and cut off the two right-hand figures (because we multiplied by two figures); we then have 159 to add to log. 100026, making 100185, to which prefix index 5, as the given number consists of six figures, and we find the log of 125946 to be 5·100185.

Note that if the number were 125·946 the index would be 2; if 1·25946, index would be 0; if ·125946, index $\bar{1}$ or negative 1; if ·0000125946, index $\bar{5}$; the decimal part of the log. (the mantissa) would be the same in each case.

2. Find the number corresponding to log. 6·129847.

Index 6 shows we need 7 whole numbers in the answer; the tables give us only 4 of these. From 129847 take the nearest log. below it in value, 129690, and we have 157; to this number add as many noughts as are required to give us our extra three figures—noting that each nought must give us a figure (may be only 0), and divide by the number opposite the log. in the *D* column (here 323). Now 157000 ÷ 323 gives 486 (we could continue the division and obtain decimals in our answer if necessary). The log. 129690 gives 1348 as natural number, and the answer, therefore, is 1348486.

3. Find the log. of $\frac{7}{8}$.

This may be done by converting $\frac{7}{8}$ into a decimal, and taking the log. of that decimal, or—a better method—by subtracting the log. of 8 from the log. of 7; thus log. 7 = 0·845098, log. 8 = 0·903090, subtracting the latter we get $\bar{1}$·942008. (*Note* the

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*In the Chapters following v., the words log., tan., &c., will be written log, tan, without any stop to mark the abbreviations.

negative 1; we have to borrow 1 in the subtraction, and, there being no means of repaying it, we write it negative 1, or 1 to the debit account in the answer. As an example, from log. $\bar{1}\cdot285946$ take log. $\bar{3}\cdot781294$; we get $1\cdot504652$, for we carry one from the mantissa—always positive—which positive 1 with negative 3 gives negative 2; taking negative 2 from negative 1 (in the upper line) by the Algebraic method of subtraction—that is, by changing the sign of the lower line, and collecting or setting off positives against negatives, we get positive 1 as index. Again, add log. $2\cdot951462$ to log. $\bar{3}\cdot128416$; we get $0\cdot079878$ for we have positive 1 over from the mantissa, and this collected with positive 2 and negative 3 evidently gives 0, the negative and positive quantities being equal; as it is addition no change of sign takes place.)

4. Find the logs. of 89^2 (89 squared) and of $\cdot0092^3$ ($\cdot0092$ cubed).

In the first case, the log. of 89 is multiplied by 2, because the second power, or square, is required; in the other case, the log. of $\cdot0092$ is multiplied by 3 because the third power is wanted. Now the log. of 89 is $1\cdot949390$, and this doubled is $3\cdot898780$; the log. of $\cdot0092$ is $\bar{3}\cdot963788$, three times this is $\bar{7}\cdot891364$. (Negative 7 because, although 3 times $\bar{3}$ gives $\bar{9}$, we had a positive 2 to collect with it from the mantissa.)

Note that the numbers indicating powers are multipliers in logs.

5. Find the logs. of $\sqrt[3]{913}$ (cube root of 913) and $\sqrt[5]{\cdot0961}$ (fifth root of $\cdot0961$).

The log. of 913 is $2\cdot960471$; dividing by 3 (cube root) we get $0\cdot986824$ (nearest). The log. of $\cdot0961$ is $\bar{2}\cdot982723$; dividing by 5 we meet with this difficulty, the negative 2 is not divisible by 5, and we cannot say 29 because one number is negative and the other positive. The difficulty is overcome by adding just sufficient negatives to the $\bar{2}$ to enable us to carry on the division, and, in compensation for this, letting the same number of positives precede the mantissa. Thus, adding $\bar{3}$ and $+ 3$ we get $\bar{5} + 3\cdot982723$, and this, divided by 5, gives $\bar{1}\cdot796545$ (nearest).

Note.—Make the negative index exactly divisible by adding negatives, and increase the positive mantissa by a like amount by prefixing the figure added; the 39, not the 3, must be divided by the 5.

6. Find the log. tan. of 21° 19′ 29″.

Taking log. tan. 21° 19′, we proceed to multiply the difference (number in nearest *D* column standing between 21° 19′ and 21° 20′, the next higher minute) by 29; cut off the two right-hand figures, and add what is left to the log. taken out (invariably cut off two figures, but add only for sines, tans., secants, and versines; subtract for "co" sines, &c.).

Thus, log. tan. 21° 19′ = 9·591308; $D = 622$; $622 \times 29 = 18038$, from which cut off 38, and add 180 to log., making log. tan. 21° 19′ 29″ = 9·591488. (This should be $\bar{1}$·591488, but the index-figures in the logs. of all ratios are, by agreement, increased by 10 to get rid of negative signs; note carefully, as the use of this 10 is at times difficult to understand.)

7. Find the arc or angle of which 10 219136 is the log. secant.

Take the nearest log. sec. below this, and subtract it from the one given. It is 10·219032; subtracting we get 104, to this add two noughts (invariably two), and divide by the D number between the minute of log. 10·219032 and the next higher minute.

Thus, $10400 \div 278$ gives 37; these are the seconds in the answer, which is 52° 51′ 37″. (*Note* that the log. below was taken; this is the case for sines, tans., secs., and versines; in other cases, "co" sines, &c., the log. above is used.)

8. Find the natural tangent of 59° 30′ from the log. tan., and the natural cot. from the log. cot.

Log. tan. 59° 30′ = 10·229852, log. cotan. = 9·770148. Subtracting 10 from the index of each, we get 0·229852 and $\bar{1}$·770148; looking these out in the table of logs. of numbers, we find the first gives 1·698, and the second ·589. The index was reduced by 10 because each log. ratio, as before stated, has an index 10 too large.

9. Find the value (by logs. where possible) of the expression

$$\frac{4\sqrt[3]{27} + \frac{5}{8}\text{ tan. }(41^\circ\ 20')^{\frac{2}{3}} + \sqrt{\cdot 9}}{(\cdot 018\sqrt[5]{32} - \text{sec. } 20^\circ \text{ tan. } 45^\circ\ 13')^2}$$

Note that, as addition and subtraction are required in this exercise, and these cannot be performed by logarithms, we must find the value of each term in natural numbers before collecting them. From the form which results the final value may be obtained.

$$\begin{aligned}\text{Thus, log. } 4\sqrt[3]{27} &= \text{log. } 27 \div 3 \text{ (to take cube root)} + \text{log. } 4\\ &= \tfrac{1}{3}\text{ log. } 27 + \text{log. } 4\\ &= \frac{1{\cdot}431364}{3} + 0{\cdot}602060 = {\cdot}477121 + 0{\cdot}602060\\ &= \text{log. } 1{\cdot}079181, \text{ of which the natural number}\\ &\quad\text{is } 12.\end{aligned}$$

$\log \frac{5}{8}(\text{tan. } 41^\circ\ 20')^{\frac{2}{3}} = \frac{2}{3}$ of log. tan. 41° 20′ + log. 5 − log. 8.

$$\begin{array}{l} \bar{1}\cdot944262 + 0\cdot698970 - 0\cdot903090 \\ \qquad\quad 2 \\ \hline 3\,)\bar{1}\cdot888524 \\ \hline \bar{1}\cdot962841 \\ 0\cdot698970 \\ \hline 0\cdot661811 - 0\cdot903090 = \text{log. } \bar{1}\cdot758721 = \text{nat. number } \cdot5738. \end{array}$$

Note that when dividing $\bar{1}\cdot888524$ by 3 we added 2 to the index (sufficient to allow division), and + 2 preceded the mantissa; thus the log became $\bar{3} + 2\cdot888524$ (20 with the mantissa was equal to 2 with the index, because the index takes 10 times as high a position as the first figure of the mantissa); also notice that we deducted 10 from the tabulated log. of log. tan. 41° 20′ to get the actual log.

log. $\sqrt{\cdot9} =$ log. $\cdot9 \div 2 = \bar{1}\cdot954243 \div 2 = \bar{2} + 1\cdot954243 \div 2 = \bar{1}\cdot977121$, and natural number = ·9487.

log. $\cdot018\ \sqrt[5]{32} = \frac{1}{5}$ log. 32 + log. $\cdot018 = \dfrac{5)1\cdot505150}{0\cdot301030} + \bar{2}\cdot255273$

$= \bar{2}\cdot556303$ and natural number = ·036.

(This and $4\sqrt[3]{27}$ would be much easier without logs.)

log. sec. 20° tan. 45° 13′ = 0·027014 + 0·003285 = 0·030299, and natural number = 1·072.

Here again both logs. were reduced by 10 in the index because we wanted the actual, not the tabular, logs., from which to get the natural number.

Our work now stands $\dfrac{12 + \cdot5738 + \cdot9487}{(\cdot036 - 1\cdot072)^2}$ or $\dfrac{13\cdot5225}{(-1\cdot036)^2}$ or log. 13·5225 − twice log. 1·036. Now log. 13·5225 = 1·131057 (130977 + 80 because of the extra 25), and 2 × ·015360 = 0·030720; therefore the result is 1·130337, and the natural number is 12·6.

10. Find the sines of 5″ and 4′ (small angles).

Here the angle is very small, and there will be no great error if we take the sine of 1″ and multiply by 5, instead of finding the sine of 5″;* and sine 1′ × 4 instead of sine 4′. This method is useful occasionally in connection with certain

---

* Or, by logs., the log. sine of 1″ + log. 5.

trigonometrical quantities. The tabulated log. of **sine** 1″ is 4·685575, the actual log. being $\bar{6}\cdot685575$; the log. of ½ **sine** 1″ (tabulated) is 4·384545 (*i.e.*, 4·685575 − ·301030 or log. 2).

EXERCISES.

Find the values of the following expressions (as far as possible by logarithms) :—

(1) $258 \times 27 \div 381$; (2) $\dfrac{\cdot07\,(\cdot79 - \cdot03)}{1\cdot49}$; (3) $(\cdot1)^7$; (4) $\sqrt{49}$; (5) $\dfrac{2+8}{2-8}$;

(6) $\dfrac{\sec 60^\circ}{4}$; (7) $\dfrac{\cos 90^\circ - \frac{1}{2}}{\tan^2 45^\circ + 1}$; (8) $\dfrac{\sqrt{10\cdot01} \times (\cdot099)^{\frac{2}{5}}}{(1\frac{1}{13}) \times 1\cdot1}$; (9) $\sec 45^\circ\ 30'\ 28''$;

(10) $\dfrac{\tan 45^\circ}{\sin^2 45^\circ} - \sqrt[5]{32}$; (11) $\dfrac{89 \times \cdot89}{8\cdot9 \times \cdot089}$; (12) $\dfrac{235 \text{ cosec } 15^\circ\ 20'}{235 + \text{cosec } 15^\circ\ 20'}$;

(13) $\frac{4}{3}\sqrt[3]{45 \text{ cosec } 21^\circ 31'}$; (14) $\dfrac{\frac{1}{10}\sqrt[3]{\cdot027} \times \sqrt[5]{\cdot01024}}{\cdot7 \times (7)^{\frac{1}{2}}}$; (15) $\dfrac{11 \times 1 \times 10 \times 1 \times 01}{10000\,(\cdot1 \times \cdot01)^2}$;

(16) $\dfrac{1258947 \times \cdot8219046 \times \cdot00001}{125\,8947 \times 20547\cdot615}$; (17) $81 \cot 81^\circ + 41 \text{ cosec } 41^\circ + 63 \cos 63^\circ$.

Find x in the following :—(18) tab log $\sin x = 9\cdot512642$; (19) $\dfrac{\cdot2 \tan x}{\cdot16 \cot 35^\circ} = 1\cdot25$; (20) $\dfrac{11 \sin^3 115^\circ}{x^2} = \dfrac{1\cdot56}{\sqrt{\sec 30^\circ}}$.

ANSWERS.

(1) 86; (2) ·035705; (3) ·0000001; (4) ·7; (5) $-1\cdot\dot{6}$; (6) ½; (7) ¼; (8) 1·059; (9) 1·426; (10) 0; (11) 100; (12) 3·722; (13) 6·6254; (14) ·00648; (15) 11; (16) ·000004; (17) 103·925; (18) 19°; (19) 55°; (20) 2·37, &c.

CHAPTER VI.

PRACTICAL SOLUTIONS OF PLANE TRIANGLES.

Example 1.—" Cosines."—Given the three sides of a plane triangle to find any angle; also to find the area of the triangle.

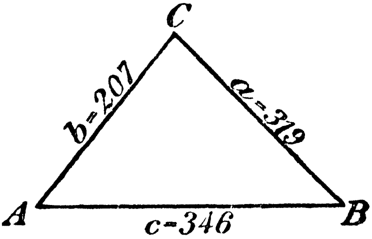

Fig. 4

Let ABC be a triangle having the side BC (or a) = 319 ft., side AC (or b) = 207 ft., and AB (or c) = 346 ft.

Find angles A, B, C; also the area.

Formula to find angle A . . . . $\cos\frac{A}{2}=\sqrt{\frac{s\,(s-a)}{bc}}$,

B . . . $\cos\frac{B}{2}=\sqrt{\frac{s\,(s-b)}{ac}}$,

C . . . . $\cos\frac{C}{2}=\sqrt{\frac{s\,(s-c)}{ab}}$,

To find A $\qquad \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$

$$
\begin{array}{rlrr}
a=319 & & & \\
b=207 & \log s = & 2\cdot639486 & \log b=2\cdot315970 \\
c=346 & \log (s-a) = & 2\cdot068186 & \log c=2\cdot539076 \\
\hline
2)872 & \text{add } 20 & 24\cdot707672 & 4\cdot855046 \\
 & & 4\cdot855046 & \\
\hline
s=436 & & & \\
a=319 & & 2)19\cdot852626 & \\
\hline
s-a=117 & \log\cos\frac{A}{2} = & 9\cdot926313 &
\end{array}
$$

$$\frac{A}{2}=32^\circ\ 26'$$

$$A=64^\circ\ 52'$$

Explanation.—(1) Add a, b, c, and divide by 2 to find s (semi-perimeter). (2) Take a from s, according to formula. (3) Add logs of s and $(s-a)$ together, and increase the index of the sum by 20. (4) Add logs of b and c. (5) Take the sum of logs b and c, from the sum of logs s and $(s-a)$. (6) Divide by 2 for square root. (7) Look out from the tables the angle of which this result is the log **cosine.** (8) This being half the angle, $\left(\frac{A}{2}\right)$, multiply by 2 to find A.

Angle B may be found in a similar manner from its formula that is by using $s-b$ instead of $s-a$, and ac instead of bc; and angle C similarly.

To find B. $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$

$s=436$
$b=207$
$s-b=229$

$\log s\quad\;\;\, = 2{\cdot}639486$
$\log (s-b) = 2{\cdot}359835$
$20+4{\cdot}999321$
$5{\cdot}042867$
$2)19{\cdot}956454$
$\log\cos\frac{B}{2}=\quad 9{\cdot}978227$

$\log a = 2{\cdot}503791$
$\log c = 2{\cdot}539076$
$5{\cdot}042867$

$\frac{B}{2}=17°\ 59'$

$B=35°\ 58'$

To find C. $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

$s=436$
$c=346$
$s-c=\ 90$

$\log s\quad\;\;\, = 2{\cdot}639486$
$\log (s-c) = 1{\cdot}954243$
$24{\cdot}593729$
$4{\cdot}819761$
$2)19{\cdot}773968$
$\log\cos\frac{C}{2}=\quad 9{\cdot}886984$

$\log a = 2{\cdot}503791$
$\log b = 2{\cdot}315970$
$4{\cdot}81976$

$\frac{C}{2}=39°\ 34'$

$C=79°\ 8'$

Note.—Tabular logs are used for the logs of the ratios—that is, the index of each log sine, &c., is increased by 10.

The sum of the three angles should be 180°, but there may be an error of a few minutes, caused by the work being carried only to minutes. The student, after some practice, should work this and other problems to seconds.

Any one angle of the above triangle being found, the other two need not be computed as shown, but might be worked by any of several methods to be given later.

Notice the almost exact resemblance of this work and that used in spherical trigonometry when three sides are given to find an angle.

Various other ways of using three sides to find an angle may be seen by consulting the formulæ in Chapter viii.

AREA.

To find area of this triangle.

Formula. $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

$$\begin{aligned}
\log s &= 2\cdot639486\\
\log (s-a) &= 2\cdot068186\\
\log (s-b) &= 2\cdot359835\\
\log (s-c) &= 1\cdot954243\\
\hline
2)&9\cdot021750\\
\hline
&4\cdot510875\\
&\cdot510813\ (3242)\\
\hline
&134)620(4
\end{aligned}$$

$\therefore$ **Area** $= 32424$ sq. ft.

As the index is 4, we must have five integers (whole numbers) in the area: the extra figure is obtained as explained in the chapter on logarithms. The area is naturally in square or surface measure

Exercises in Plane "Cosines" and Areas.

Given the following Three Sides, find the Three Angles.

	Given.			*Required.*		
1.	$a=201$,	$b=295$,	$c=284$	$A=40^\circ 34'$,	$B=72^\circ 40'$,	$C=66^\circ 46'$
2.	$a=385$,	$b=729$,	$c=591$	$A=31^\circ 46'$,	$B=94^\circ 16'$,	$C=53^\circ 58'$
3.	$a=309$,	$b=285$,	$c=196$	$A=77^\circ 30'$,	$B=64^\circ 14'$,	$C=38^\circ 16'$
4.	$a=291$,	$b=384$,	$c=296$	$A=48^\circ 34'$,	$B=81^\circ 42'$,	$C=49^\circ 44'$
5.	$a=228$,	$b=365$,	$c=450$	$A=30^\circ 16'$,	$B=53^\circ 46'$,	$C=95^\circ 58'$
6.	$a=20{\cdot}1$,	$b=19{\cdot}5$,	$c=24{\cdot}7$	$A=52^\circ 30'$,	$B=50^\circ 20'$,	$C=77^\circ 8'$*
7.	$a=21{\cdot}9$,	$b=21{\cdot}9$,	$c=21{\cdot}9$	$A=60^\circ$,	$B=60^\circ$,	$C=60^\circ$
8.	$a=272$,	$b=250$,	$c=200$	$A=73^\circ 26'$,	$B=61^\circ 46'$,	$C=44^\circ 48'$
9.	$a=124$,	$b=213$,	$c=110$	$A=26^\circ 4'$,	$B=131^\circ$,	$C=22^\circ 56'$
10.	$a=422$,	$b=364$,	$c=386$	$A=68^\circ 24'$,	$B=53^\circ 20'$,	$C=58^\circ 16'$
11.	$a=324$,	$b=527$,	$c=478$	$A=37^\circ 12'$,	$B=79^\circ 38'$,	$C=63^\circ 10'$
12.	$a=982$,	$b=1367$,	$c=836$	$A=45^\circ 36'$,	$B=97^\circ 18'$,	$C=37^\circ 6'$*
13.	$a=1142$,	$b=369$,	$c=934$	$A=115^\circ 38'$,	$B=16^\circ 22'$,	$C=47^\circ 54'$*
14.	$a=13$,	$b=27$,	$c=19$	$A=26^\circ 8'$,	$B=113^\circ 44'$,	$C=40^\circ 6'$
15.	$a=456$,	$b=327$,	$c=368$	$A=81^\circ 46'$,	$B=45^\circ 12'$,	$C=53^\circ$
16.	$a=12$,	$b=8$,	$c=9$	$A=89^\circ 36'$,	$B=41^\circ 48'$,	$C=48^\circ 36'$
17.	$a=139{\cdot}6$,	$b=234{\cdot}5$,	$c=193{\cdot}9$	$A=36^\circ 30'$,	$B=87^\circ 46'$,	$C=55^\circ 42'$
18.	$a=43{\cdot}78$,	$b=36{\cdot}29$,	$c=37{\cdot}13$	$A=73^\circ 12'$,	$B=52^\circ 32'$,	$C=54^\circ 16'$
19.	$a=171{\cdot}4$,	$b=369{\cdot}9$,	$c=412{\cdot}3$	$A=24^\circ 34'$,	$B=63^\circ 44'$,	$C=91^\circ 42'$
20.	$a=589$,	$b=376$,	$c=444$	$A=91^\circ 26'$,	$B=39^\circ 40'$,	$C=48^\circ 54'$
21.	$a=63{\cdot}2$,	$b=91{\cdot}7$,	$c=32{\cdot}3$	$A=22^\circ 52'$,	$B=145^\circ 40'$,	$C=11^\circ 28'$
22.	$a=91$,	$b=102$,	$c=103$	$A=52^\circ 42'$,	$B=63^\circ 6'$,	$C=64^\circ 12'$
23.	$a=105$,	$b=106$,	$c=107$	$A=59^\circ 4'$,	$B=60^\circ$,	$C=60^\circ 56'$
24.	$a=507$,	$b=608$,	$c=409$	$A=55^\circ 46'$,	$B=82^\circ 26'$,	$C=41^\circ 48'$

Given Three Sides to find Area of Triangle.

25.	$a=15$,	$b=25$,	$c=30$	Area = 187·1
26.	$a=100$,	$b=120$,	$c=130$	,, 5700
27.	$a=200$,	$b=300$,	$c=400$	,, 29047
28.	$a=250$,	$b=300$,	$c=350$	,, 36742
29.	$a=565$,	$b=684$,	$c=752$	,, 185535 or 185119*
30.	$a=4878$,	$b=5627$,	$c=6259$	,, 13106696·9

---

* Work approximately, also rigidly to seconds and decimals; note results. Thus, in 13, angle B may be 17° 21′. Using "sines," subtracting the sum of 2 angles from 180°, &c., will slightly alter results.

Example II.—"**Tangents.**"—In a plane triangle **given two sides and the angle between them, required to find the other two angles.**

Given $a = 342$, $b = 291$ (feet or any other measurement, provided both are alike in this respect) and included angle C 51° 24′; find angles A and B.

Formula. $a + b : a - b :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$;

$$\text{or } \tan \tfrac{1}{2}(A - B) = \frac{(a - b) \tan \frac{1}{2}(A + B)}{a + b}.$$

$$\begin{array}{rrr}
a = 342 & 342 & 180^\circ\ \ 0' \\
b = 291 & 291 & C\ \ 51^\circ\ 24' \\
\hline
a + b = 633 & a - b = 51 & 2)128^\circ\ 36' \\
\hline
 & & \tfrac{1}{2}(A + B) = \ 64^\circ\ 18'\,* \\
\end{array}$$

$$\begin{array}{rl}
\log \tan \tfrac{1}{2}(A + B) = & 10{\cdot}317613 \\
\log\ 51 = & 1{\cdot}707570 \\
\hline
 & 12{\cdot}025183 \\
\log 633 = & 2{\cdot}801404 \\
\hline
\log \tan \tfrac{1}{2}(A - B) = & 9{\cdot}223779 \\
\tfrac{1}{2}(A - B) = & 9^\circ\ 30' \\
\end{array}$$

$$\begin{array}{lcr@{\qquad\qquad}lcr}
\tfrac{1}{2}A + \tfrac{1}{2}B & = & 64^\circ\ 18' & \tfrac{1}{2}A + \tfrac{1}{2}B & = & 64^\circ\ 18' \\
\tfrac{1}{2}A - \tfrac{1}{2}B & = & 9^\circ\ 30' & \tfrac{1}{2}A - \tfrac{1}{2}B & = & 9^\circ\ 30' \\
\hline
\text{Add.} & & A = 73^\circ\ 48' & \text{Subtract.} & & B = 54^\circ\ 48' \\
\end{array}$$

Explanation.—(1) Find the sum of given sides. (2) Find difference of the given sides. (3) Take given angle from 180° and halve the remainder, which will give $\frac{1}{2}$ sum of angles required.

---

* Angle C being taken from 180°, the balance must be the other two angles, A and B; whence $\frac{1}{2}(A + B)$.

(4) Add log tan $\frac{1}{2}$ sum of angles to log difference of the sides, and subtract log sum of sides. (5) Find from table the angle of which this is log tan. (6) Find the sum and difference of this angle and that obtained in (3); the sum will be the larger and the difference the smaller of the required angles, which will be named as the sides determine—viz., larger opposite larger, &c.

The remaining side in this triangle will be found by the rule stated in Example III.

AREA.

To find the area of this triangle

Formula. Area $= \frac{1}{2}\,(a\,b\ .\ \sin C)$

$$
\begin{array}{lr}
 & \log a = 2{\cdot}534026 \\
 & \log b = 2{\cdot}463893 \\
 & \log \sin C = 9{\cdot}892940 \\
\hline
\text{(10 rejected from index)} & 4{\cdot}890859 \\
\text{(Subtract)} & \log 2 = 0{\cdot}301030 \\
\hline
 & 4{\cdot}589829 \\
 & \log 3888 = 4{\cdot}589726 \\
\hline
 & 112)1030(9
\end{array}
$$

$\therefore$ Area $=$ 38889 (sq. measure).

Exercises in Plane "Tangents" and "Third side direct."

1. $a=434$, $b=110$, $C=82^\circ\ 16'$: find A, B, c.	$A=83^\circ\ 9'$, $B=14^\circ\ 35'$, $c=432{\cdot}9$
2. $a=314$, $c=267$, $B=34^\circ\ 20'$: find A, C, b.	$A=87^\circ\ 30'$, $C=58^\circ\ 10'$, $b=177{\cdot}3$
3. $b=375$, $c=417$, $A=77^\circ\ 27'$: find B, C, a.	$B=47^\circ\ 29'$, $C=55\ \ 3'$, $a=496{\cdot}6$
4. $a=993$, $b=637$, $C=49^\circ\ 30'$: find A, B, c.	$A=90^\circ\ 36'$, $B=39^\circ\ 54'$, $c=755{\cdot}1$
5. $a=634$, $c=379$, $B=40^\circ\ 39'$: find A, C, b.	$A=103^\circ\ 51'$, $C=35^\circ\ 29'$, $b=125{\cdot}3$
6. $b=937$, $c=386$, $A=27^\circ\ 30'$: find B, C, a.	$B=135^\circ\ 49'$, $C=16^\circ\ 41'$, $a=620{\cdot}8$
7. $a=982$, $b=637$, $C=11^\circ\ 23'$: find A, B, c.	$A=149^\circ\ 12'$, $B=19^\circ\ 24'$, $c=378{\cdot}5$
8. $a=1219$, $c=1210$, $B=32^\circ\ 40'$: find A, C, b.	$A=74^\circ\ 23'$, $C=72^\circ\ 57'$, $b=683{\cdot}1$
9. $a=369$, $b=437$, $C=66^\circ\ 30'$: find A, B, c.	$A=49^\circ\ 25'$, $B=64^\circ\ 5'$, $c=445{\cdot}6$
10. $a=326$, $c=189$, $B=40^\circ\ 30'$: find A, C, b.	$A=105^\circ\ 33'$, $C=33^\circ\ 57'$, $b=219{\cdot}8$
11. $b=98$, $c=364$, $A=87^\circ\ 30'$: find B, C, a.	$C=77^\circ\ 16'$, $B=15^\circ\ 14'$, $a=372{\cdot}6$
12. $a=327$, $c=639$, $B=40^\circ\ 30'$: find A, C, b.	$A=28^\circ\ 33'$, $C=110^\circ\ 57'$, $b=444{\cdot}4$
13. $b=17$, $c=19$, $A=40^\circ\ 30'$: find B, C, a.	$B=61^\circ\ 11'$, $C=78^\circ\ 19'$, $a=12{\cdot}6$
14. $a=132{\cdot}4$, $c=167{\cdot}5$, $B=60^\circ$: find A, C, b.	$A=48^\circ\ 32'$, $C=71^\circ\ 28'$, $b=153$
15. $b=246$, $c=112$, $A=40^\circ\ 20'$: find B, C, a.	$B=115^\circ\ 23'$, $C=24^\circ\ 17'$, $a=176{\cdot}2$
16. $b=427$, $c=324$, $A=40^\circ\ 30'$: find B, C, a	$B=90^\circ\ 9'$, $C=49^\circ\ 21'$, $a=277{\cdot}4$
17. $b=360$, $c=240$, $A=50^\circ\ 29'$: find B, C, a.	$B=87^\circ\ 44'$, $C=41^\circ\ 46'$, $a=277{\cdot}9$
18. $b=32$, $c=18$, $A=110^\circ\ 30'$: find B, C, a.	$B=45^\circ\ 45'$, $C=23^\circ\ 45'$, $a=41{\cdot}86$
19. $a=213{\cdot}4$, $c=193{\cdot}7$, $B=40^\circ\ 47'$: find A, C, b.	$A=77^\circ\ 1'$, $C=62^\circ\ 11'$, $b=143{\cdot}1$
20. $b=112$, $c=96$, $A=78^\circ\ 40'$: find B, C, a.	$B=56^\circ\ 2'$, $C=45^\circ\ 18'$, $a=132{\cdot}4$
21. $a=42$, $c=57$, $B=130^\circ\ 40'$: find A, C, b.	$A=20^\circ\ 41'$, $C=28^\circ\ 39'$, $b=90{\cdot}2$
22. $a=115{\cdot}5$, $c=107{\cdot}8$, $B=49^\circ\ 25'$: find A, C, b.	$A=69^\circ\ 33'$, $C=61^\circ\ 1'$, $b=93{\cdot}6$
23. $b=873{\cdot}1$, $a=793$, $C=9^\circ\ 43'$: find A, B, c.	$A=55^\circ\ 43'$, $B=114^\circ\ 33'$, $c=162$
24. $b=333$, $c=456$, $A=78^\circ\ 9'$: find B, C, a.	$B=40^\circ\ 3'$, $C=61^\circ\ 47'$, $a=506{\cdot}5$
25. $b=13{\cdot}36$, $c=9{\cdot}2$, $A=40^\circ\ 30'$: find B, C, a.	$B=96^\circ\ 18'$, $C=43^\circ\ 11'$, $a=8{\cdot}731$

Given Two Sides and included Angle to find Area.

26. $a=20$, $b=25$, $C=60^\circ$	Area$=216{\cdot}5$.
27. $a=70$, $b=160$, $C=30^\circ$.	,, 2800.
28. $a=100$, $b=25$, $C=40^\circ\ 40'$.	,, 814·5.
29. $a=368$, $b=470$, $C=60^\circ\ 25'$.	,, 75206.
30. $a=5735$, $b=12960$, $C=50^\circ\ 30'$.	,, 28675657.

Example III.—"Sines."—Given two angles and one opposite side, or two sides and one opposite angle; required to find the other opposite part.

First Case.—Given $a=210$, $b=195$, $A=76^\circ\ 20'$. Find B (the other opposite part).

Formula. $a : b :: \sin A : \sin B$; or $\sin B = \dfrac{\sin A \times b}{a}$.

$$\begin{array}{rr}
\log b = & 2{\cdot}290035 \\
\log \sin A = & 9{\cdot}987526 \\
\hline
 & 12{\cdot}277561 \\
\log a = & 2{\cdot}322219 \\
\hline
\log \sin B = & 9{\cdot}955342 \\
\hline
B = & 64^\circ\ 28'
\end{array}$$

Second Case.—Given $A = 51° 45'$, $B = 59° 22'$, $a = 56$; required b (the other opposite part).

Formula. Sin A : sin B : : a : b; or $b = \frac{a \sin B}{\sin A}$.

$$
\begin{aligned}
\log a &= 1{\cdot}748188 \\
\log \sin B &= 9{\cdot}934723 \\
\hline
& 11{\cdot}682911 \\
\log \sin A &= 9{\cdot}895045 \\
\log b &= 1{\cdot}787866 \\
\hline
b &= 61{\cdot}36
\end{aligned}
$$

Explanation—First Case.—(1) Add the log of the side of the same name as the required angle to the log sine of the given angle. (2) Subtract the log of the other given side. (3) Look out the resulting log as sine, and the angle it gives will be the one required.

Second Case.—The arrangement only is somewhat different. In both cases set down the part required in the fourth term of a proportion, as third term write the similar part given, the second term is the one given of the same name as term 4, and the first term is of the same name as the third; remember to use sines of angles.

To finish the solution of the triangle by finding C and c. Since the three angles of a triangle (plane) always equal 180°, whenever two are known their sum can be taken from 180° to find the third angle—thus, Case 1, as $A = 76° 20'$ and $B = 64° 28'$, C will be $180° - (76° 20' + 64° 28')$, or 39° 12'. **This applies only to plane triangles.**

Side c can now be found by a formula similar to that used in the second case—viz.:

$$\sin A : \sin C :: a : c;$$

or

$$\sin B : \sin C :: b : c.$$

Note. : means "is to," and : : means "as."

Adopt "sines" in the case of any triangle where it is possible to use proportion, as in this example—thus, after one angle is found in **Ex. I.**, find the second angle by this proportionate method ("sines"); also, in **Ex. II.**, find the third side by it, after the third angle is known. **Another method of finding the third side** when two angles and the included side are given will be shown presently (**the direct method**).

From the foregoing examples the student sees that any **oblique** (not **right**) angled triangles can be solved by the rule of "**cosines**" (three sides given), "**tangents**" (two sides and included angle given), or "**sines**" (any case which is not necessarily cosines or tangents).* Before proceeding to right-angled triangles we must notice a peculiar case of "**sines**," called the **ambiguous** case. If two sides be given, and the angle opposite the smaller side, there will be doubt as to the results. From Fig. 5 it will be seen that, given b, c and angle C (opposite smaller side), the line AB may take up either of the positions occupied by AB or AB', and the angle opposite b may be ABC (obtuse) or $AB'C$ (acute), for there is not sufficient control given by these parts to determine the position of c. Careful consideration (in problems) will generally enable the student to settle this matter, which we must (for want of space) now leave to him.

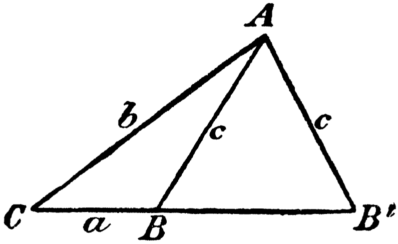

Fig. 5.

Example IV.—"Third Side Direct."—**Given two sides and the included angle to find the third side** direct (without finding the other angles).

Formulæ. $$\text{Tan } \theta = \frac{\text{sum of given sides} \times \tan \frac{\text{given angle}}{2}}{\text{difference of given sides}}$$

$$\text{side} = \frac{\text{sum of given sides} \times \sin \frac{\text{given angle}}{2}}{\sin \theta}$$

---

* One or more *sides* must be among the parts given for the solution of a *plane* triangle.

Thus—Given $b=295$, $c=198$, $A=76^\circ\ 24'$; find a.

$$\text{Tan}\ \theta = \frac{(b+c)\ .\ \tan\frac{A}{2}}{b-c}; \qquad a = \frac{(b+c)\ .\ \sin\frac{A}{2}}{\sin\theta}$$

$b+c$	$b-c$	$\frac{A}{2}$
295	295	
198	198	2)76° 24′
493	97	38° 12′

(1)		(2)	
log 493 =	2·692847	log 493 =	2·692847
log tan 38° 12′ =	9·895932	log sin 38° 12′ =	9·791275
	12·588779		12·484122
log 97 =	1·986772	log sin 75° 58′ =	9·986841
log tan θ =	10·602007	log a =	2·497281
θ =	75° 58′	a =	314·3

Explanation.—(1) Find the sum and difference of the sides, and half the angle. (2) Work both formulæ at the same time as far as the addition. (3) Subtract in (1) and find θ. (4) Subtract log sin θ in (2) and look for the natural number, which will give the required side.

This method often saves much labour, as frequently the side only is needed, not the angles.

Exercises in Plane "Sines" and "Third side direct."

Given Three Parts—Find the other Three.

	Given.			*Required.*		
1.	$a=234$,	$b=167$,	$A=64^\circ\ 30'$.	$B=40\ 6'$,	$C=75^\circ\ 24'$,	$c=250{\cdot}9$.
2.	$c=287$,	$C=40^\circ\ 34'$,	$A=81^\circ\ 2'$.	$B=58^\circ\ 24'$,	$b=375{\cdot}9$,	$a=435{\cdot}9$.
3.	$a=364$.	$b=527$,	$B=49^\circ\ 33'$.	$A=31^\circ\ 43'$,	$C=98^\circ\ 44'$,	$c=684{\cdot}4$.
4.	$a=364$,	$b=186$,	$A=60^\circ\ 30'$.	$B=26^\circ\ 24'$,	$C=93^\circ\ 6'$,	$c=417{\cdot}7$.
5.	$c=489$,	$A=65^\circ\ 31'$,	$B=60^\circ\ 15'$.	$C=54^\circ\ 14'$,	$a=548{\cdot}5$,	$b=523{\cdot}2$.
6.	$b=432{\cdot}4$,	$A=64^\circ\ 30'$,	$B=62^\circ\ 29'$.	$C=53^\circ\ 1'$,	$a=440{\cdot}1$,	$c=389{\cdot}4$.
7.	$c=267$,	$a=165$,	$C=60^\circ\ 45'$.	$A=32^\circ\ 38'$,	$B=86^\circ\ 37'$,	$b=305{\cdot}4$.
8.	$c=426{\cdot}3$,	$A=39^\circ\ 23'$,	$B=76^\circ\ 10'$.	$C=64^\circ\ 27'$,	$a=299{\cdot}8$,	$b=458{\cdot}8$.
9.	$a=72$,	$B=56^\circ\ 20'$,	$A=60^\circ\ 30'$.	$C=63^\circ\ 10'$,	$b=68{\cdot}8$,	$c=73{\cdot}9$.
10.	$b=69$,	$B=61^\circ\ 30'$,	$C=80^\circ\ 25'$.	$A=38^\circ\ 5'$,	$c=77{\cdot}43$,	$a=48{\cdot}42$.

	Given.			Required.	
11.	$c=21\cdot65$,	$a=29\cdot85$,	$B=101^\circ\,32'$.	$\theta=82^\circ\,36'$,	$b=40\cdot23$.
12.	$b=1615$,	$c=2148$,	$A=39^\circ\,16'$.	$\theta=68^\circ\,21'$,	$a=1360$.
13.	$c=6\ 25$,	$a=4185$,	$B=121^\circ\,16'$.	$\theta=83^\circ\,58'$,	$b=9035$.
14.	$b=612$,	$a=31$,	$C=31^\circ\,10'$.	$\theta=17^\circ\,9'$,	$c=585\cdot7$.
15.	$b=12\cdot85$,	$c=12\cdot85$,	$A=126^\circ\,16'$.	$\theta=90^\circ$,	$a=22\cdot93$.*
16.	$b=2168$,	$c=3147$,	$A=112^\circ\,14'$.	$\theta=82^\circ\,57'$,	$a=4446$.
17.	$a=161$,	$b=172$,	$C=29^\circ\,46'$.	$\theta=82^\circ\,55'$,	$c=86\cdot19$.
18.	$a=2001$,	$b=3002$,	$C=90^\circ$.	$\theta=78^\circ\,41$,	$c=3608$.

Example V.—Right-angled Plane Triangles.—These are very important as the calculations in Geo-Navigation depend almost entirely upon them. We saw how the ratios were formed; by these the parts of the triangle are computed.

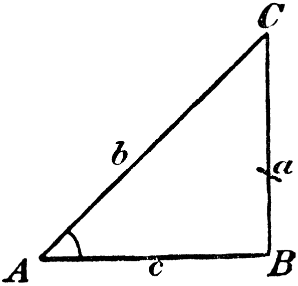

Fig. 6.

Formula for finding a side, $\dfrac{\text{side required}}{\text{side known}}=$ some ratio of given angle.

Formula for finding an angle, $\dfrac{\text{any known side}}{\text{any other known side}}=$ some ratio of required angle.

As before stated, the student can take either of the sides forming the right angle as perpendicular on condition that he uses in his formula the angle opposite to it.

Note.—The right angle is not to be used.

First Case.—Given $a=5$, $B=90^\circ$, $A=51^\circ\ 10'$, to find c (Fig. 6).

---

* To be *thought* out; as the difference of the sides is 0, tan θ must be infinite (tan 90°). Note that *logs cannot be used with zero as multiplier or divisor.*

Formula. $\dfrac{c \text{ (required)}}{a \text{ (known)}} = \text{cotangent } A.$

$$\text{Or, } c = a \,.\, \cot A.$$

$$\begin{aligned} \log a &= 0{\cdot}698970 \\ \log \cot A &= 9{\cdot}905784 \\ \hline \log c &= 0{\cdot}604754 \quad \text{(10 rejected from index).} \\ c &= 4{\cdot}025 \end{aligned}$$

Explanation.—(1) Form a fraction having the side required in the numerator and a given side in the denominator, and state what ratio this equals, remembering that the side opposite the acute angle used is the perpendicular, that opposite the right angle is the hypothenuse, and the other side is the base. (2) Re-write this equation with the unknown quantity only on one side. (3) Add the log of the ratio to the log of the side used —the natural number of the total log (10 rejected) gives the sides required. (4) The other side can be found in a similar way by forming a new equation—and the third angle as before explained (the three angles = 180°).

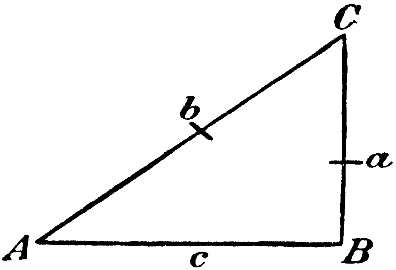

Fig. 7.

Second Case.—Given $B = 90°$, $a = 31$, $b = 45$, to find A (Fig. 7).

Formula. $\operatorname{Sin} A = \dfrac{a \text{ (given)}}{b \text{ (given)}}.$

$$\begin{aligned} \log a &= 11{\cdot}491362 \quad \text{(10 added to index).} \\ \log b &= 1{\cdot}653213 \\ \hline \log \sin A &= 9{\cdot}838149 \\ A &= 43° \; 33' \end{aligned}$$

Explanation.—(1) A fraction (made with the given sides) and the ratio they produce form an equation. (2) Subtract the log of the denominator from the log of the numerator with index increased by 10. (3) The log produced will give the required angle if looked out in the table of log Ratios. (4) The other side can be found as in Case 1, and the angle as before.

Exercises in Right-angled Triangles.

1. $B=90^\circ$, $A=50^\circ\ 20'$, $c=217{\cdot}1$: find a, b. $a=261{\cdot}8$: $b=340$.
2. $C=90^\circ$, $A=40^\circ\ 20'$, $b=100$: find a, c. $a=84{\cdot}91$: $c=131{\cdot}2$.
3. $B=90^\circ$, $C=36^\circ\ 40'$, $c=70$: find a, b. $a=94{\cdot}03$: $b=117{\cdot}2$.
4. $C=90^\circ$, $B=15^\circ\ 32'$, $c=317$: find a, b. $a=305{\cdot}4$: $b=84{\cdot}89$.
5. $B=90^\circ$, $A=21^\circ\ 39'$, $c=290$: find a, b. $a=115{\cdot}1$: $b=312$.
6. $B=90^\circ$, $A=62^\circ\ 32'$, $c=296$: find a, b. $a=569{\cdot}4$: $b=641{\cdot}7$.
7. $C=90^\circ$, $b=69$, $c=137$: find A, a. $A=59^\circ\ 46'$: $a=118{\cdot}4$.
8. $C=90^\circ$, $a=16$, $b=34$: find B, c. $B=64^\circ\ 48'$: $c=37{\cdot}58$.
9. $A=90^\circ$, $a=314$, $b=216$: find B, c. $B=43^\circ\ 28'$: $c=227{\cdot}9$.
10. $C=90^\circ$, $b=3{\cdot}826$, $c=9{\cdot}12$: find A, a. $A=65^\circ\ 12'$: $a=8{\cdot}279$.
11. $C=90^\circ$, $a=46{\cdot}27$, $b=49\ 63$: find A, c. $A=43^\circ$: $c=67{\cdot}85$.
12. $C=90^\circ$, $B=24^\circ\ 51'$, $a=404$: find b, c. $b=187{\cdot}1$: $c=445{\cdot}2$.
13. $C=90^\circ$, $B=28^\circ\ 25'$, $a=313$: find b, c. $b=169\ 3$: $c=355\ 9$.
14. $B=90^\circ$, $c=316$, $a=212{\cdot}4$: find A, b. $A=33^\circ\ 54'$: $b=380\ 8$.
15. $B=90^\circ$, $a=130$, $c=147$: find A, b. $A=41^\circ\ 29'$: $b=196\ 3$.
16. $C=90^\circ$, $c=234$, $b=136$: find A, a. $A=54^\circ\ 28'$: $a=190\ 4$.
17. $B=90^\circ$, $b=267$, $c=167$: find C, a. $C=38^\circ\ 43'$: $a=208{\cdot}3$.
18. $C=90^\circ$, $a=1364$, $b=967$: find A, c. $A=54^\circ\ 40'$: $c=1672$.
19. $A=90^\circ$, $a=346$, $b=234$: find B, c. $B=42^\circ\ 33'$: $c=254{\cdot}9$.
20. $A=90^\circ$, $b=113$, $c=127$: find B, a. $B=41^\circ\ 40'$: $a=170$.
21. $C=90^\circ$, $B=69^\circ\ 30'$, $a=243$: find b, c. $b=649{\cdot}9$: $c=693{\cdot}9$.
22. $A=90^\circ$, $a=314$, $b=216$: find B, c. $B=43^\circ\ 28'$: $c=227{\cdot}9$.
23. $B=90^\circ$, $c=327$, $b=516{\cdot}1$: find A, a. $A=50^\circ\ 41'$: $a=399{\cdot}3$.
24. $B=90^\circ$, $c=11{\cdot}21$, $A=31^\circ 45'$: find b, a. $b=13{\cdot}18$: $a=6{\cdot}937$.

CHAPTER VII.

PRACTICAL SOLUTIONS OF SPHERICAL TRIANGLES.

Example I.—"Cosines."—Given the three sides of a spherical triangle *A B C*, to find any angle.*

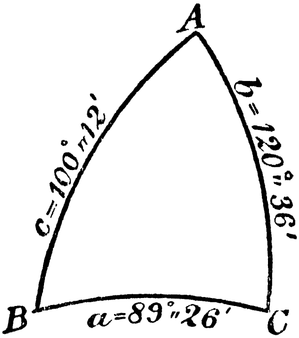

Fig. 8.

Let $a=89°\ 26'$, $b=120°\ 36'$, $c=100°\ 12'$; find A, B, C.

Formula to find A. $\cos\frac{A}{2}=\sqrt{\frac{\sin s\ .\ \sin(s-a)}{\sin b\ .\ \sin c}}$

$$\begin{array}{rlrlr}
a= & 89°\ 26' & \log\sin\ s= & 9{\cdot}624047 & \log\sin b=9{\cdot}934873 \\
b= & 120°\ 36' & \log\sin(s-a)= & 9{\cdot}959654 & \log\sin c=9{\cdot}993081 \\
c= & 100°\ 12' & & \overline{\phantom{39{\cdot}583701}} & \overline{\phantom{19{\cdot}927954}} \\
 & \overline{} & \text{adding } 20 & 39{\cdot}583701 & 19{\cdot}927954 \\
 & 2)310°\ 14' & & 19{\cdot}927954 & \\
 & \overline{} & & \overline{\phantom{19{\cdot}655747}} & \\
s= & 155°\ 7' & & 2)19{\cdot}655747 & \\
s-a= & 65°\ 41' & & \overline{\phantom{19{\cdot}655747}} & \\
 & & \log\cos\frac{A}{2}= & 9{\cdot}827873 & \\
 & & \frac{A}{2}= & 47°\ 43' & \\
 & & A= & 95°\ 26' &
\end{array}$$

---

* A formula for finding a side when the three angles are given will be found in Chapter viii. (spherical triangles only).

To find B. $$\cos\frac{B}{2}=\sqrt{\frac{\sin s\ .\ \sin(s-b)}{\sin a\ .\ \sin c}}$$

$s=155°\ \ 7'$ $\quad$ $\log\sin s=9{\cdot}624047$ $\quad$ $\log\sin a=9{\cdot}999979$

$s-b=\ 34°\ 31'$ $\quad$ $\log\sin(s-b)=9{\cdot}753312$ $\quad$ $\log\sin c=9{\cdot}993081$

$39{\cdot}377359$ $\quad$ $19\ 993060$

$19{\cdot}993060$

$2)19{\cdot}384299$

$$\log\cos\frac{B}{2}=\ 9{\cdot}692149$$

$$\frac{B}{2}=\ 60°\ 31'$$

$$B=121°\ \ 2'$$

To find C. $$\cos\frac{C}{2}=\sqrt{\frac{\sin s\ .\ \sin(s-c)}{\sin a\ .\ \sin b}}$$

$s=155°\ \ 7'$ $\quad$ $\log\sin s=9{\cdot}624047$ $\quad$ $\log\sin a=\ 9{\cdot}999979$

$s-c=\ 54°\ 55'$ $\quad$ $\log\sin(s-c)=9{\cdot}912922$ $\quad$ $\log\sin b=\ 9{\cdot}934873$

$39{\cdot}536969$ $\quad$ $19{\cdot}934852$

$19{\cdot}934852$

$2)19{\cdot}602117$

$$\log\cos\frac{C}{2}=\ 9{\cdot}801058$$

$$\frac{C}{2}=\ 50°\ 46'$$

$$C=101°\ 32'$$

Explanation.—This rule is so similar to the Rule of "Cosines" in plane trigonometry that little remains to be said. As the sides of the triangle are measured in degrees, &c., the logarithms of ordinary numbers are not used; instead of these we have log sines of s, $(s-a)$, &c. In all other respects proceed as in Example I., Chapter vi. The student may find the three angles by "cosines," or he may find the second angle (also the third) by the rule given in Example III., Chapter vii. He cannot find the third angle in a spherical triangle by subtracting the sum of two from 180°, since the angles of a spherical triangle are always greater than 180°, ranging in amount between 180° and 540°.

Note that the three angles of a plane triangle must amount to 180°; those of a spherical triangle must exceed 180°. Unless there are reasons to the contrary it is best, when using both "**cosines**" (plane or spherical) and "**sines**," to **find the largest angle first** by "**cosines**," completing the solution by "**sines**."

Application.—This rule is of service in finding the **time** angle or the **azimuth** angle in astronomy when the sides **co-latitude**, polar distance, and zenith distance are known.

Exercises in "Spherical Cosines."

To give an idea of the variations in results obtained, certain exercises are worked to seconds as well as to minutes only.

	Given.			*Required*		
1.	*a*= 87° 28′,	*b*= 96° 12′,	*c*=100° 8′.	*A*= 88° 32′,	*B*= 95° 50′,	*C*= 99° 56′.
2.	*a*= 78° 14′,	*b*= 68° 26′,	*c*= 72° 12′.	*A*= 84° 4′,	*B*= 70° 54′,	*C*= 75° 20′.
3.	*a*= 36° 36′,	*b*= 28° 28′,	*c*= 24° 24′.	*A*= 89° 20′,	*B*= 53° 4′,	*C*= 43° 52′.
4.	*a*= 31° 29′,	*b*= 38° 46′,	*c*= 37° 23′.	*A*= 52° 10′,	*B*= 71° 14′,	*C*= 66° 38′.
5.	*a*= 86° 9′,	*b*= 73° 23′,	*c*= 49° 36′.	*A*= 99° 20′,	*B*= 71° 24′,	*C*= 48° 52′.
6	*a*= 37° 10′,	*b*= 52° 12′,	*c*= 71° 50°.	*A*= 36° 12′,	*B*= 50° 36′,	*C*=111° 42′.
7.	*a*= 32° 22′,	*b*= 63° 40′,	*c*= 59° 53′.	*A*= 36° 40′,	*B*= 87° 36′,	*C*= 74° 40′.
8.	*a*= 52° 27′,	*b*= 37° 37′,	*c*= 86° 20′.	*A*= 23° 28′,	*B*= 17° 50′,	*C*=149° 56′.
9.	*a*= 49° 10′,	*b*= 58° 25′,	*c*= 56° 42′.	*A*= 59° 4′,	*B*= 74° 56′,	*C*= 71° 20′.
10.	*a*=119° 42′,	*b*=108° 4′,	*c*= 68° 54′.	*A*=115° 38′,	*B*= 99° 22′,	*C*= 75° 32′.
11.	*a*=120° 54′,	*b*=105° 6′,	*c*=108° 42′.	*A*=130° 46′,	*B*=121° 32′,	*C*=123° 16′.
12.	*a*= 64° 21′,	*b*= 80° 39′,	*c*=104° 28′.	*A*= 60° 18′,	*B*= 71° 56′,	*C*=111° 6′.
13.	*a*= 38° 48′,	*b*= 42° 26′,	*c*= 54° 16′.	*A*= 50° 28′,	*B*= 56° 10′,	*C*= 88° 48′.
14.	*a*= 42° 16′,	*b*= 38° 18′,	*c*= 36° 14′.	*A*= 73° 0′,	*B*= 61° 48′,	*C*= 57° 10′
15.	*a*= 72° 20′,	*b*= 64° 18′,	*c*= 56° 16′.	*A*= 85° 12′,	*B*= 70° 28′,	*C*= 60° 26′.
16.	*a*= 84° 24′,	*b*= 72° 18′,	*c*= 60° 18′.	*A*= 93° 40′,	*B*= 72° 48′,	*C*= 60° 36′.
17.	*a*= 89° 26′,	*b*=120° 36′,	*c*=100° 12′,	*A*= 95° 26′,	*B*=121° 2′,	*C*=101° 32′ (101° 31′ 50″).
18.	*a*= 72° 50′,	*b*= 94° 15′,	*c*=120° 40′.	*A*= 72° 32′,	*B*= 84° 40′,	*C*=120° 50′.
19.	*a*=100° 0′,	*b*=115° 0′,	*c*=120° 0′.	*A*=119° 22′,	*B*=126° 40′,	*C*=129° 58′.
20.	*a*= 30° 0′,	*b*=108° 10′,	*c*= 84° 50′.	*A*= 19° 8′,	*B*=141° 32′,	*C*= 40° 44′.
21.	*a*= 38° 20′,	*b*= 38° 20′,	*c*= 38° 20′.	*A*= 63° 54′,	*B*= 63° 54′,	*C*= 63° 54′.
22.	*a*= 1° 10′,	*b*= 1° 10′,	*c*= 2° 4′.	*A*= 27° 40′ (27° 39′ 56″).	*B*= 27° 40′,	*C*=124° 42′.
23.	*a*=179° 20′,	*b*=179° 10′,	*c*= 0° 30′.	*A*=126° 52′,	*B*= 90° 0′,	*C*= 36° 52′.
24.	*a*= 90° 0′,	*b*= 90° 0′,	*c*= 90° 0′.	*A*= 90° 0′,	*B*= 90° 0′,	*C*= 90° 0′.
25.	*a*= 80° 0′,	*b*= 80° 0′,	*c*= 78° 34′.	*A*= 81° 42′,	*B*= 81° 42′,	*C*= 80° 2′.
26.	*a*=100° 47′,	*b*=145° 10′,	*c*= 81° 10′.	*A*= 96° 12′,	*B*=144° 42′,	*C*= 90° 2′.
27.	*a*= 72° 11′,	*b*= 37° 19′,	*c*= 61° 47′.	*A*= 97° 32′ 6″,	*B*= 39° 8′ 34″,	*C*= 66° 34′ 19″.
28.	*a*= 31° 11′ 7″,	*b*= 32° 19′ 18″,	*c*= 33° 15′ 21″.	*A*= 59° 29′ 42″,	*B*= 62° 49′ 42″,	*C*= 65° 50′ 48″.

* Note this very small triangle; the angles approximately worked amount to 180° (it may be considered almost a plane triangle). **Work this to seconds and a slight improvement is seen in the results.**

Example II.—"Tangents."—Case 1.—When **two sides** and included angle are given to find the other **two angles.**

Given $a = 72° 56'$, $b = 86° 12'$, $C = 100° 20'$; find A and B in triangle ABC.

Formulæ.
$$\begin{cases} \text{Tan}\,\dfrac{B+A}{2} = \sec\dfrac{b+a}{2}\,\cos\dfrac{b-a}{2}\,\cot\dfrac{C}{2}. \\ \text{Tan}\,\dfrac{B-A}{2} = \operatorname{cosec}\dfrac{b+a}{2}\,\sin\dfrac{b-a}{2}\,\cot\dfrac{C}{2}. \end{cases}$$

½ sum sides.	½ diff. sides.	½ angle.
86° 12′	86° 12′	2)100° 20′
72° 56′	72° 56′	
2)159° 8′	2)13° 16′	50° 10′
79° 34′	6° 38′	

79° 34′	log sec	10·742102	79° 34′	log cosec	10·007241
6° 38′	log cos	9·997083	6° 38′	log sin	9·062639
50° 10′	log cot	9·921247	50° 10′	log cot	9·921247

(20 rejected) $\log\tan\dfrac{B+A}{2} = 10{\cdot}660432$ $\qquad \log\tan\dfrac{B-A}{2} = 8{\cdot}991127$

$\dfrac{B+A}{2} = 77° 40'$ $\qquad \dfrac{B-A}{2} = 5° 36'$

By addition $B = 83° 16'$ By subtraction $A = 72° 4'$.

Case 2.—When **two angles** and **the included side are given to find** the other **two sides.**

Given $A = 112° 20'$, $B = 100° 8'$, $c = 75°$; find a and b in triangle ABC.

Formulæ.
$$\begin{cases} \text{Tan}\,\dfrac{a+b}{2} = \sec\dfrac{A+B}{2}\,\cos\dfrac{A-B}{2}\,\tan\dfrac{c}{2}. \\ \text{Tan}\,\dfrac{a-b}{2} = \operatorname{cosec}\dfrac{A+B}{2}\,\sin\dfrac{A-B}{2}\,\tan\dfrac{c}{2}. \end{cases}$$

½ sum angles.	½ diff. angles.	½ side.
106° 14′	6° 6′	37° 30′

106° 14′	log sec 10·553541	log cosec 10·017669
6° 6′	„ cos 9·997534	„ sin 9·026386
37° 30′	„ tan 9·884980	„ tan 9·884980

$$\log \tan \frac{a+b}{2} = 10{\cdot}436055 \qquad \log \tan \frac{a-b}{2} = 8{\cdot}929035$$

$$\frac{a+b}{2} = 180^\circ - 69^\circ\ 53' \qquad \frac{a-b}{2} = 4^\circ\ 51'$$

$$= 110^\circ\ 7'$$

By addition $a = 114^\circ\ 58'$ **By subtraction** $b = 105^\circ\ 16'$.

Explanation.—Case 1.—(1) Find half the sum of the given sides. (2) Find half the difference of the given sides. (3) Halve the given angle (mark that this angle is **not taken from 180° in sph. trigy.**). (4) Add log **sec** half sum sides, log **cos** half diff. sides, and log **cot** half angle, reject 20 from index, look out result as log **tan.** (5) Add log **cosec** half sum sides, log **sin** half diff. sides, and log **cot** half angle, reject 20 from index, look out result as log **tan.** (6) Find the sum of the two quantities obtained in (4) and (5), also their difference ; the sum will give the larger angle sought, the difference will give the smaller one (**greater angle opposite greater side, &c.**). In Case 2 the work differs from that in Case 1 thus :—Use angles as sides were used, and sides as angles were used, and instead of the third log in each case being **cot** let it be **tan.** No other changes are required. In both cases the two columns of logs should be taken out simultaneously ; thus when looking for **sec** half sum for one column look for **cosec** ditto for the other column. Should any quantity exceed 90°, reduce it by 90° and take out the balance, adding "**co**" to, or omitting "**co**" from, the ratio to be used. Thus **sec** 101° 20′ becomes **cosec** 11° 20′.

Note.—When the half sum exceeds 90° **the half sum in the result** (the left column answer) **must exceed** 90°—that is, the quantity obtained from the table of logs must be taken from 180°. The **addition and subtraction** to obtain final results must be made **after this** has been attended to.

The remaining side, Case 1, **or angle,** Case 2, may be found by "**sines,**" **Example III., Chapter vii.,** and by other methods explained in this Chapter.

Application.—If co-latitude and the zenith distance and the azimuth of a celestial body are known, the **hour angle, &c.,** may **be determined by Case 1**; if the hour angle and the azimuth of a celestial body are known, and the co-latitude of the observer, **Case 2** will give zen. dist., polar dist., &c.

Exercises in "Spherical Tangents."

	Given.			*Required.*	
1.	$A=100^\circ\,30'$	$B=92^\circ\,36'$	$c=70^\circ$	$a=102^\circ\;3'$	$b=96^\circ\,29'$
2.	$a=45^\circ\,15'$	$b=60^\circ\,35'$	$C=70^\circ$	$A=53^\circ\,30'$	$B=80^\circ\,22'$
3.	$A=120^\circ$	$B=100^\circ$	$c=80^\circ$	$a=121^\circ\,18'$	$b=103^\circ\,40'$
4.	$A=150^\circ$	$B=120^\circ\,20'$	$c=75^\circ\,40'$	$a=149^\circ\;7'$	$b=117^\circ\,37'$
5.	$a=72^\circ\,36'$	$b=60^\circ\,12'$	$C=75^\circ$	$A=81^\circ\,34'$	$B=64^\circ\;6'$
6.	$A=72^\circ\,40'$	$B=50^\circ\,20'$	$c=80^\circ\,40'$	$a=70^\circ\,48'$	$b=49^\circ\,36'$
7.	$A=84^\circ\,12'$	$B=75^\circ\,24'$	$c=75^\circ$	$a=80^\circ\,23'$	$b=73^\circ\,33'$
8.	$A=66^\circ\,30'$	$B=108^\circ\,15'$	$c=20^\circ\,20'$	$a=71^\circ$	$b=78^\circ\,20'$
9.	$B=75^\circ\,40'$	$C=68^\circ\,20'$	$a=75^\circ\,20'$	$b=71^\circ\;7'$	$c=65^\circ\,11'$
10.	$B=60^\circ\,30'$	$C=84^\circ\,12'$	$a=84^\circ\,16'$	$b=60^\circ\;4'$	$c=82^\circ\;8'$
11.	$B=115^\circ\,20'$	$C=100^\circ\;8'$	$a=75^\circ\,30'$	$b=117^\circ\,47'$	$c=105^\circ\,31'$
12.	$A=118^\circ\,40'$	$B=136^\circ\,40'$	$c=80^\circ\,20'$	$a=116^\circ\,46'$	$b=135^\circ\,42'$
13.	$B=134^\circ\,30'$	$C=75^\circ\,50'$	$a=64^\circ\,48'$	$b=133^\circ\,10'$	$c=97^\circ\,28'$
14·	$a=37^\circ\,10'$	$c=71^\circ\,50'$	$B=50^\circ\,36'$	$A=36^\circ\,12'$	$C=111^\circ\,42'$
15.	$b=50^\circ\,20'$	$c=56^\circ\,48'$	$A=130^\circ\,20'$	$B=36^\circ\;2'$	$C=39^\circ\,44'$
16.	$a=179^\circ\,20'$	$b=179^\circ\,10'$	$C=36^\circ\,52'$	$A=126^\circ\,52'$	$B=90^\circ$
17.	$B=89^\circ\,50'$	$C=96^\circ\,36'$	$a=70^\circ\,26'$	$c=96^\circ\,56'$	$b=92^\circ\,10'$
***18.**	$c=1^\circ\,10'$	$a=2^\circ\;4'$	$B=27^\circ\,40'$	$A=124^\circ\,40'$	$C=27^\circ\,40'$
19.	$a=100^\circ$	$b=99^\circ\,58'$	$C=120^\circ$	$A=106^\circ\,41'$	$B=106^\circ\,42'$
***20.**	$A=175^\circ\,10'$	$B=179^\circ\,24'$	$c=2^\circ\,12'$	$a=178^\circ\;2'$	$b=179^\circ\,46'$
***21.**	$a=92^\circ\,10'$	$c=42^\circ\,10'$	$B=90^\circ$	$A=91^\circ\,27'$	$C=42^\circ\,11'$
***22.**	$B=101^\circ\,25'$	$C=84^\circ\,35'$	$a=90^\circ$	$b=101^\circ\,23'$	$c=84^\circ\,41'$
23.	$B=22^\circ\,40'$	$C=31^\circ\,50'$	$a=21^\circ\,40'$	$b=10^\circ\,12'$	$c=14^\circ$
24.	$b=41^\circ\,30'$	$c=56^\circ\,16'$	$A=39^\circ\,30'$	$B=51^\circ\,12'$	$C=102^\circ$
25.	$a=17^\circ\,21'$	$b=25^\circ\,19'$	$C=131^\circ\,20'$	$B=30^\circ\,47'$	$A=20^\circ\,55'$
26.	$a=17^\circ\,21'\,11''$	$b=23^\circ\,19'\,47''$	$C=131^\circ\,19'\;6''$	$B=29^\circ\,36'\,23''$	$A=21^\circ\,50'\,37''$

Example III.—" Sines."—When two angles and a side opposite one of them, or two sides and an angle opposite one of them, are given, to find the other opposite.

Case 1.—Given $a=76^\circ\;12'$, $b=84^\circ\;55'$, $A=75^\circ\;30'$; find B in triangle $A\;B\;C$.

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* Work these to seconds to get results. In No. 20 considerably different results will be found.

Formula. $\sin a : \sin b :: \sin A : \sin B$.

log sin A = 9·985942
„ b = 9·998289

19·984231
* „ a = 9·987279

log sin B = 9·996952
B = 83° 13′. or 96° 47′

Case 2.—Given $A = 103° 15'$, $a = 102° 10'$, $B = 84° 8'$; find b in triangle $A B C$.

Formula. $\sin A : \sin B :: \sin a : \sin b$.

log sin a = 9·990134
„ B = 9·997719

19·987853
„ A = 9·988282

log sin b = 9·999571
b = 87° 27′. or 92° 33′.

(See Note, p. 48.)

Explanation.—This rule differs from "**sines**" in plane trigonometry in only one respect, the use of the log **sine** instead of the log of a number; this arises from the fact that sides of sph. triangles are measured in degrees, &c. (see also proofs of trigonometrical rules). **Note** that the third angle cannot be found by the method used in plane trig.—*i.e.*, by taking the sum of two angles from 180° and thus getting the third angle (and from this, by "**sines**," the third side). As before said, the **three angles do not equal 180°. Methods of completing the solution** of the triangle in which two sides and two opposite angles are known are explained in **Examples IIIa. and IIIb.** A **difficulty** often arises in "**sines**," the quantity taken from the table of logs having to be subtracted from 180°, a process indicated only by the relative proportions of the parts used (**sine is positive for both first and second quadrants**). The **ambiguity** noted in plane trig. also occurs in spherical solutions.

Application.—This rule is most often used in connection with rules "**cosines**" and "**tangents**," no important problem in nautical astronomy depending on it alone for solution.

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* Subtraction may be avoided by adding cosec a to the first 2 logs; cosec is the inversion of sine.

Exercises in "Spherical Sines."

	Given.			*Required.*
1.	$a=119^\circ\ 42'$	$b=108^\circ\ 4'$	$A=115^\circ\ 38'$	$B=99^\circ\ 20'$, or $80^\circ\ 40'$
2.	$a=78^\circ\ 59'$	$b=72^\circ\ 10'$	$A=32^\circ\ 21'$	$B=31^\circ\ 18'$
3.	$a=92^\circ\ 10'$	$b=58^\circ\ 34'$	$B=51^\circ\ 30'$	$A=113^\circ\ 34'$, or $66^\circ\ 26'$
4.	$a=56^\circ\ 16'$	$b=64^\circ\ 10'$	$A=43^\circ\ 12'$	$B=47^\circ\ 48'$, or $132^\circ\ 12'$
5.	$a=49^\circ\ 15'$	$b=52^\circ\ 18'$	$B=74^\circ\ 16'$	$A=67^\circ\ 10'$
6.	$A=84^\circ\ 16'$	$B=50^\circ\ 20'$	$a=38^\circ\ 16'$	$b=28^\circ\ 38'$
7.	$A=64^\circ\ 16'$	$B=78^\circ\ 24'$	$b=50^\circ\ 19'$	$a=45^\circ\ 3'$
8.	$c=86^\circ\ 13'$	$a=79^\circ\ 38'$	$A=80^\circ\ 10'$	$C=88^\circ\ 9'$, or $91^\circ\ 51'$
9.	$b=72^\circ\ 14'$	$c=106^\circ\ 20'$	$B=64^\circ\ 14'$	$C=114^\circ\ 50'$, or $65^\circ\ 10'$
10.	$a=100^\circ\ 45'$	$c=75^\circ\ 30'$	$A=120^\circ\ 40'$	$C=57^\circ\ 57'$
11.	$A=76^\circ\ 42'$	$a=50^\circ\ 14'$	$B=36^\circ\ 8'$	$b=27^\circ\ 46'$
12.	$B=82^\circ\ 14'$	$C=82^\circ\ 20'$	$b=75^\circ\ 12'$	$c=75^\circ\ 15'$, or $104^\circ\ 45'$
13.	$A=85^\circ\ 35'$	$B=72^\circ\ 18'$	$b=65^\circ\ 30'$	$a=72^\circ\ 14'$, or $107^\circ\ 46'$
14.	$a=76^\circ\ 12'$	$b=84^\circ\ 55'$	$A=75^\circ\ 30'$	$B=83^\circ\ 13'$, or $96^\circ\ 47'$
15.	$A=103^\circ\ 15'$	$a=102^\circ\ 10'$	$B=84^\circ\ 8'$	$b=87^\circ\ 27'$, or $92^\circ\ 33'$
16.	$A=84^\circ\ 4'$	$a=78^\circ\ 14'$	$B=70^\circ\ 54'$	$b=68^\circ\ 27'$
17.	$a=120^\circ\ 40'$	$A=172^\circ\ 32'$	$c=120^\circ\ 40'$	$C=172^\circ\ 32'$
18.	$b=115^\circ\ 0'$	$B=126^\circ\ 40'$	$a=120^\circ\ 0'$	$A=129^\circ\ 58'$
19.	$B=89^\circ\ 50'$	$a=70^\circ\ 26'$	$b=92^\circ\ 10'$	$A=70^\circ\ 33'$
20.	$C=31^\circ\ 50'$	$c=14^\circ\ 0'$	$a=21^\circ\ 40'$	$A=53^\circ\ 36'$, or $126^\circ\ 24'$
21.	$B=51^\circ\ 12'$	$A=39^\circ\ 30'$	$b=41^\circ\ 30'$	$a=32^\circ\ 44'$
22.	$a=71^\circ\ 20'$	$A=72^\circ\ 31'$	$B=85^\circ\ 17'$	$b=81^\circ\ 51'$, or $98^\circ\ 9'$
23.	$b=31^\circ\ 15'$	$a=72^\circ\ 10'$	$A=62^\circ\ 40'$	$B=28^\circ\ 56'$
24.	$C=80^\circ\ 31'$	$B=95^\circ\ 50'$	$b=91^\circ\ 20'$	$c=82^\circ\ 24'$

(See Note, p. 48.)

Note 1.—More exercises may be formed by using parts from the spherical "cosines;" approximate results will be obtained.

Note 2.—Mark the ambiguous cases in these exercises. In practical problems (astronomy) little difficulty arises as to ambiguity in results.

Example IIIa.—To Complete the Solution in "Sines."—In spherical "**sines**" only one part can be found direct from the parts given; the remaining side, or remaining angle may be calculated by the following formulæ (derived from sph. "**tangents**"); the sixth part should then be found by "**sines**":—

To find the third angle :—

Now, $\tan\frac{A+B}{2} = \sec\frac{a+b}{2} \, . \, \cos\frac{a-b}{2} \, . \, \cot\frac{C}{2}$ (see "**tangents**").

$$\therefore \qquad \cot\frac{C}{2} = \frac{\tan\frac{A+B}{2}}{\sec\frac{a+b}{2} \, . \, \cos\frac{a-b}{2}}$$

(1) Or, $\qquad \cot\frac{C}{2} = \tan\frac{A+B}{2} \, . \, \cos\frac{a+b}{2} \, . \, \sec\frac{a-b}{2}.$

The two latter steps are explained by the theory of equations, and by $\sec\frac{a+b}{2} \, . \, \cos\frac{a-b}{2}$ as divisors becoming $\cos\frac{a+b}{2} \, . \, \sec\frac{a-b}{2}$ as multipliers.

To find the third side*:—

$$\tan\frac{a+b}{2} = \sec\frac{A+B}{2} \, . \, \cos\frac{A-B}{2} \, . \, \tan\frac{c}{2} \text{ (see "tangents").}$$

$$\therefore \qquad \tan\frac{c}{2} = \frac{\tan\frac{a+b}{2}}{\sec\frac{A+B}{2} \, . \, \cos\frac{A-B}{2}}$$

(2) Or, $\tan\frac{c}{2} = \tan\frac{a+b}{2} \, . \, \cos\frac{A+B}{2} \, . \, \sec\frac{A-B}{2}.$

Explained as above. The third angle may then be found by "sines."

Given $b = 10° \, 4'$, $c = 14° \, 6'$, $B = 22° \, 40'$, $C = 31° \, 50'$; find A and a in triangle ABC.

By (1) $\cot\frac{A}{2} = \tan 27° \, 15' \, . \, \cos 12° \, 5' \, . \, \sec 2° \, 1'$.

$$\begin{aligned} \log \tan 27° \, 15' &= 9{\cdot}711836 \\ \text{,,} \ \cos 12° \ \ 5' &= 9{\cdot}990270 \\ \text{,,} \ \sec \ 2° \ \ 1' &= 10{\cdot}000269 \\ \hline \text{,,} \ \cot\frac{A}{2} \qquad &= 9{\cdot}702375 \end{aligned}$$

$$\therefore \ \frac{A}{2} = 63° \, 15', \ A = 126° \, 30'.$$

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* The third side may also be found by "sines," from the third angle.

Similarly by (2) $\tan\frac{a}{2} = \tan 12^\circ\ 5' . \cos 27^\circ\ 15' . \sec 4^\circ\ 35'$.

From which $a = 21^\circ\ 38'$.

Exercises in finding the "Third Parts" in Sph. "Sines."

By adaptation of the Rule of "Tangents."

	Given.	*Required.*
1.	$a=78^\circ\ 14'$, $b=68^\circ\ 26'$, $A=84^\circ\ 4'$, $B=70^\circ\ 54'$	$c=72^\circ\ 10'$, $C=75^\circ\ 16'$
2.	$a=36^\circ\ 36'$, $b=28^\circ\ 28'$, $A=89^\circ\ 20'$, $B=53^\circ\ 4'$	$c=24^\circ\ 24'$, $C=43^\circ\ 52'$
3.	$b=72^\circ\ 14'$, $c=106^\circ\ 20'$, $B=64^\circ\ 14'$, $C=114^\circ\ 50'$	$A=63^\circ\ 48'$, $a=71^\circ\ 32'$

Note.—More exercises may be obtained by taking parts from "**cosines**" and "**tangents**" (**Examples I. and II.**); the answers will agree with the values in these examples within a minute or two.

Example IIIb.—To complete the Solution in "Sines" (to be studied subsequent to **Example V.**).—In spherical "**sines**" the third angle or the third side may be found (though in a few cases with some little difficulty owing to the perpendicular falling outside the triangle) by letting fall a perpendicular from **the third angular point** on the opposite side. By this means we get **two right-angled spherical** triangles (see Example V.), and are able to compute the two parts into which the third angle or the third side is divided, and from thence to obtain the whole angle or side. If it be supposed that A, B, a, b are known in the sph. triangle ABC, and that a perp. CD fall on AB, the **formulæ** will be :—(1) $\cos A = \cot CA \times \tan AD$, whence $\tan AD = \cos A \times \tan CA$. Similarly (2) $\tan BD = \cos B \times \tan BC$. The **sum** of AD and BD (**difference when perp. falls outside triangle**) gives AB the third side; the third angle may be found by "**sines.**" Or, the third angle being computed by formulæ (3) $\cot ACD = \tan A \times \cos AC$, and (4) $\cot BCD = \tan B \times \cos BC$, the third side may be found by "**sines.**"

*Given $a = 64^\circ\ 21'$, $b = 80^\circ\ 39'$, $A = 60^\circ\ 18'$, $B = 71^\circ\ 56'$, find c.

Formula. (1) $\tan AD = \cos 60^\circ\ 18' . \tan 80^\circ\ 39'$

,, (2) $\tan BD = \cos 71^\circ\ 56' . \tan 64^\circ\ 21'$

$$\begin{aligned} \log \cos 60^\circ\ 18' &= 9{\cdot}695007 \\ ,,\ \tan 80^\circ\ 39' &= 10{\cdot}783432 \\ \hline ,,\ \tan AD &= 10{\cdot}478439 \end{aligned} \qquad \begin{aligned} \log \cos 71^\circ\ 56' &= 9{\cdot}491535 \\ ,,\ \tan 64^\circ\ 21' &= 10{\cdot}318584 \\ \hline ,,\ \tan BD &= 9{\cdot}810119 \end{aligned}$$

$\therefore\ AD = 71^\circ\ 37'$ and $BD = 32^\circ\ 51'$ $\qquad c = AD + BD = 104^\circ\ 28'$

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* **Construct a figure, showing perpendicular.**

Exercises in Finding the "Third Parts" in Sph. "Sines."

By letting fall a perpendicular from the third angle on the third side.

	Given.				*Required.*	
1.	$a=48^\circ\ 30'$	$b=56^\circ\ 28'$	$A=40^\circ\ 36'$	$B=46^\circ\ 26'$	$c=\ 86^\circ\ 48'$,	$C=119^\circ\ 46'$
2.	$a=42^\circ\ 16'$	$b=38^\circ\ 18'$	$A=73^\circ\ \ 0'$	$B=61^\circ\ 48'$	$c=\ 36^\circ\ 16'$,	$C=\ 57^\circ\ 12'$
3.	$a=71^\circ\ 32'$	$b=72^\circ\ 14'$	$A=64^\circ\ \ 2'$	$B=64^\circ\ 14'$	$c=106^\circ\ 16'$,	$C=114^\circ\ 40'$

See notes to preceding exercises as to further examples in finding the "third" parts.

Example IV.—"Third Side Direct."—When two sides and the included angle are given to find the third side (without finding either of the other angles).

Given $a=72^\circ\ 8'$, $b=54^\circ\ 12'$, $C=86^\circ\ 50'$ to find c.

$$\text{Formula.}\quad \begin{array}{l} (1) \\ (2) \end{array} \left\{ \begin{array}{l} \sin\theta = \sqrt{\cos^2\frac{C}{2}\ .\ \sin a\ .\ \sin b.} \\ \sin\frac{c}{2} = \sqrt{\sin\left(\frac{a+b}{2}+\theta\right)\ .\ \sin\left(\frac{a+b}{2}-\theta\right)}^{*} \end{array} \right.$$

$$\begin{array}{r} 72^\circ\ \ 8' \\ 54^\circ\ 12' \\ \hline 2)\underline{126^\circ\ 20'} \end{array} \qquad \begin{array}{r} 2)\underline{86^\circ\ 50'} \end{array}$$

$$\frac{a+b}{2} = 63^\circ\ 10' \qquad \frac{C}{2} = 43^\circ\ 25'$$

$$\begin{array}{lrl} (1)\ \log\cos^2\frac{C}{2} & = & \left\{\begin{array}{l} 9{\cdot}861161 \\ 9{\cdot}861161 \end{array}\right. \\ \quad ,,\ \ \sin\ a & = & 9{\cdot}978533 \\ \quad ,,\ \ ,,\ \ b & = & 9{\cdot}909055 \\ \hline & & 2)\underline{19{\cdot}609910} \\ \quad ,,\ \ \sin\ \theta & = & 9{\cdot}804955 \\ \theta & = & 39^\circ\ 40' \end{array}$$

$$\begin{array}{rl} \frac{a+b}{2}+\theta = & 63^\circ\ 10'+39^\circ\ 40 \\ = & 102^\circ\ 50' \\ \frac{a+b}{2}-\theta = & 63^\circ\ 10'-39^\circ\ 40 \\ = & 23^\circ\ 30' \end{array}$$

$$\begin{array}{rll} \log\sin 102^\circ\ 50' = & 9{\cdot}989014 & (2) \\ ,,\ \ ,,\ \ 23^\circ\ 30' = & 9{\cdot}600700 & \\ \hline & 2)\underline{19{\cdot}589714} & \\ \log\sin\frac{c}{2} = & 9{\cdot}794857 & \\ \frac{c}{2} = & 38^\circ\ 34' & \\ c = & 77^\circ\ \ 8' & \end{array}$$

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* Or $\left(\theta+\frac{a+b}{2}\right)$ and $\left(\theta-\frac{a+b}{2}\right)$ if necessary because θ exceeds $\frac{a+b}{2}$ (see theory).

Explanation.—(1) Halve the given angle. (2) Add twice log cos half given angle (twice, for $\cos^2$) to log sines of given sides. (3) Divide by 2 (square root in logs). (4) Look out as sin, and call the quantity thus found θ (theta). (5) Find half the sum of the given sides. (6) Add θ to and subtract θ from this half sum. (7) Add log sines of the two quantities found in number (6) and divide by 2; look out as sin and double the quantity given by the table; this will be the third side.

Application.—Among many important problems to which this rule may be applied the calculation of distance on a great circle is very important.

Exercises in "Third Side, Direct."

1.	$b=$ 85° 24′,	$c=$ 81° 36′,	$A=$ 90° 40′.	$\theta=$44° 16′,	$a=$ 90° 0′.
2.	$a=$ 89° 36′,	$c=$ 12° 36′,	$B=$ 88° 14′.	$\theta=$19° 35′,	$b=$ 89° 14′.
3.	$a=$ 91° 30′,	$b=$ 92° 24′,	$C=$ 90° 6′.	$\theta=$44° 55′,	$c=$ 90° 2′.
4.	$a=$ 72° 24′,	$c=$ 86° 32′,	$B=$124° 10′.	$\theta=$27° 10′,	$b=$121° 4′.
5.	$a=$ 89° 54′,	$b=$ 89° 20′,	$C=$ 81° 16′.	$\theta=$49° 22′,	$c=$ 81° 16′.
6.	$b=$ 72° 30′,	$c=$109° 26′,	$A=$ 83° 36′.	$\theta=$44° 59′,	$a=$ 90° 0′.
7.	$a=$ 82° 20′,	$c=$101° 30′,	$B=$170° 26′.	$\theta=$ 4° 43′,	$b=$169° 50′.
8.	$a=$ 81° 25′,	$c=$ 94° 12′,	$B=$101° 8′.	$\theta=$39° 6′,	$b=$101° 36′.
9.	$a=$ 81° 39′,	$b=$110° 21′,	$C=$113° 18′.	$\theta=$31° 58′,	$c=$114° 40′.
10.	$a=$ 91° 21′,	$b=$ 89° 45′,	$C=$100° 30′.	$\theta=$39° 45′,	$c=$100° 30′.
11.	$a=$120° 0′,	$c=$ 60° 0′,	$B=$179° 0′.	$\theta=$ 0° 26′,	$b=$179° 8′.
12.	$b=$101° 15′,	$c=$ 79° 47′,	$A=$ 87° 58′.	$\theta=$44° 59′,	$a=$ 90° 2′.
13.	$a=$ 82° 32′,	$b=$ 90° 0′,	$C=$ 92° 12′.	$\theta=$43° 40′,	$c=$ 92° 10′.
14.	$a=$ 60° 0′,	$b=$ 80° 0′,	$C=$100° 0′.	$\theta=$36° 25′,	$c=$ 93° 30′.
*15.	$a=$ 2° 4′,	$c=$ 1° 10′,	$B=$ 27° 40′.	$\theta=$ 1° 31′ (1° 30′ 28″).	$b=$ 1° 8′ (1° 9′ 42″).
16.	$b=$175° 25′,	$c=$175° 43′,	$A=$ 42° 21′.	$\theta=$ 4° 8′ (4° 7′ 52″).	$a=$ 3° 12′ (3° 12′ 42″).
17.	$a=$ 0° 30′,	$b=$ 0° 32′,	$C=$120° 15′.	$\theta=$ 0° 16′ (0° 15′ 27″).	$c=$ 0° 54′ (0° 53′ 44″).
18.	$a=$ 71° 53′,	$c=$ 76° 58′,	$B=$80° 41′ 50″	$\theta=$47° 10′ (47° 10′ 5″).	$b=$ 77° 20′ (77° 18′ 12″)
19.	$a=$ 45° 36′,	$b=$ 71° 20′,	$C=$ 81° 10′.	$\theta=$38° 40′,	$c=$ 70° 52′.
20.	$a=$ 80° 15′,	$c=$ 80° 15′,	$B=$ 80° 16′.	$\theta=$48° 54′,	$b=$ 78° 52′.
21.	$a=$ 60° 0′,	$b=$ 80° 0′,	$C=$ 92° 20′.	$\theta=$39° 46′,	$c=$ 87° 0′.
22.	$a=$ 21° 23′,	$b=$ [illegible]3° 19′,	$C=$130° 24′.	$\theta=$13° 7′,	$c=$ 68° 28′.
23.	$b=$ 31° 30′,	$c=$ 39° 30′,	$A=$145° 14′.	$\theta=$ 9° 55′,	$a=$ 67° 22′.
24.	$b=$ 18° 11′,	$a=$ 76° 19′,	$C=$ 60° 24′.	$\theta=$28° 25′,	$c=$ 68° 0′.

Example V.—Right-angled Triangles.—Given the **right angle and two other parts to find the three remaining parts.**

Note.—*c* and *a touch* the rt. ang., and thus *are not complementary. c, A, b, C, a* are CIRCULAR PARTS (theory).

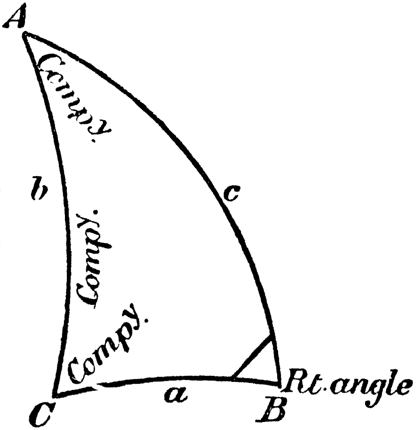

Fig. 9.

Formulæ.—Sin middle part = **product of tans of adjacent parts, or product of cosines of opposite parts.**

(For Examples, see pp. 42 and 43.)

Explanation.—(1) Every triangle contains 3 sides and 3 angles, and if we **omit** to count the **right angle** in the preceding triangle we get 5 parts, which may be looked upon as **arranged in order round a circle** (for convenience only); hence called **circular parts. Two** of these, **sides,** will **form the** right angle, and **three others,** a side and two angles, will be **apart from the right angle**; it is important to notice this because each time one of these three parts is used the **word complement** has to be **affixed to the ratio** written against it. (**The two sides forming the right angle must be written without the word complement.**) (2) The **right angle** must **not** be **used** in the solution.* An **equation** is formed by **combining** the **two given parts** and the **part** the student wishes **to find.**† Of these three **one** will stand in such **a** position that it **has the others right and left of** it (they **must either both touch** it, **or neither** must **touch** it). This **one is** termed **the middle** part; **two which touch it are** called **adjacent parts;**

(*Cont. p.* 44.)

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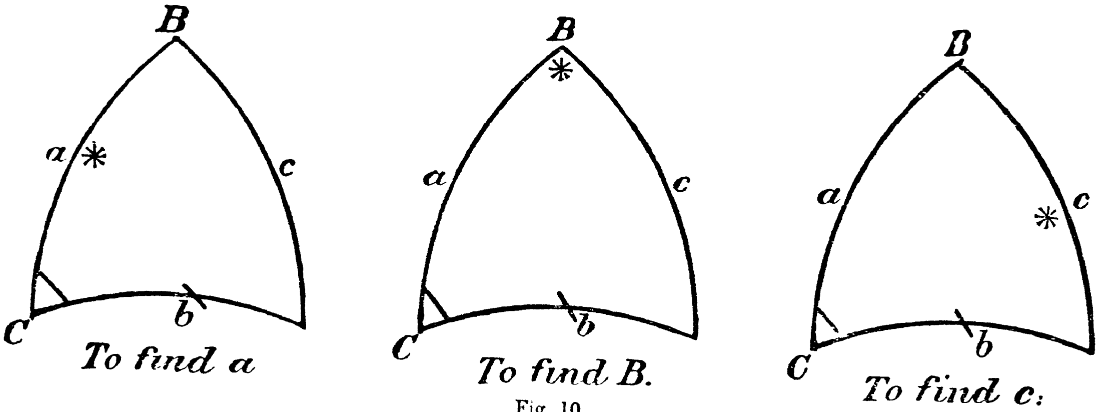
* **Its name is purposely omitted in the Figs. given.**

† **The *given parts only* should be *used,* not any parts found by the student.**



To find a

To find B.

To find c:

Fig. 10.

The required part in each triangle is marked *. The right angle is unnamed.

Example 1.—Given $A = 90°$, $b = 78° 26'$, $C = 38° 44'$, find the parts a, B, c, in triangle ABC (Fig. 10).

$$\text{Sin comp. } C = \tan b \times \underline{\tan \text{comp. } a}.$$

$$\therefore \cot a = \frac{\cos C}{\tan b}. \quad \text{(subtract)}$$

$$\begin{aligned} \log \cos C &= 9{\cdot}892132 \\ \log \tan b &= 10{\cdot}688958 \\ \hline \log \cot a &= 9{\cdot}203174 \end{aligned}$$

$$\therefore a = 80° \ 56'.$$

$$\underline{\text{Sin comp. } B} = \cos \text{comp. } C \times \cos b.$$

$$\therefore \cos B = \sin C \times \cos b. \quad \text{(add)}$$

$$\begin{aligned} \log \sin C &= 9{\cdot}796364 \\ \log \cos b &= 9{\cdot}302132 \\ \hline \log \cos B &= 9{\cdot}098496 \end{aligned}$$

$$\therefore B = 82° \ 48'.$$

$$\text{Sin } b = \tan \text{comp. } C \times \underline{\tan c}.$$

$$\therefore \tan c = \frac{\sin b}{\cot C}. \quad \text{(subtract)}$$

$$\begin{aligned} \log \sin b &= 9{\cdot}991090 \\ \log \cot C &= 10{\cdot}095768 \\ \hline \log \tan c &= 9{\cdot}895322 \end{aligned}$$

$$\therefore c = 38° \ 10'.$$

Example 2.—Given $A = 90°$, $B = 107° 17'$, $C = 112° 12'$; find a, b, c.

Fig. 10 will apply if the parts marked are B and C given, and a, b, c required (*, or any other mark).

$$\underline{\text{Sin comp. } a} = \tan \text{comp. } B \times \tan \text{comp. } C.$$

$$\therefore \cos a = \cot B \times \cot C.$$

(with signs) or $\cos a = \underset{-}{\cot B} \times \underset{-}{\cot C}$.

As $\cot B$ and $\cot C$ are both negative (these angles being greater than 90°) $\cos a$ is positive.

$$a = 82° \ 42'.$$

$$\text{Sin comp. } B = \cos \text{comp. } C \times \underline{\cos b}.$$

$$\therefore \cos b = \frac{\cos B}{\sin C}.$$

$$\text{Or } \cos b = \frac{\overset{-}{\cos B}}{\overset{+}{\sin C}}.$$

As the two signs on the right side give a negative, the result must be taken from 180° to get b.

$$b = 108° \ 43'.$$

$$\text{Sin comp. } C = \cos \text{comp. } B \times \underline{\cos c}.$$

$$\therefore \cos c = \frac{\cos C}{\sin B}.$$

$$\text{Or } \cos c = \frac{\overset{-}{\cos C}}{\overset{+}{\sin B}}.$$

Thus, $\cos c$ is negative, and the quantity found in the tables must be taken from 180°.

$$c = 113° \ 19'.$$

two which are not in touch with it are called opposite parts. The sin of a middle part equals the tan of an adjacent part multiplied into the tan of another adjacent part. If opposite parts are used the tans change to cosines. The word comp. must be written against all parts except the sides enclosing the right angle. (3) An equation having been written down, it is well to underline the part to be found, and to re-arrange the equation, if necessary, getting the unknown on one side, known on the other. Remove the word comp. by adding or subtracting "co," thus comp. sin becomes cos, comp. cosine becomes sin.* (By the theory of equations a multiplier on changing sides becomes a divisor, a divisor becomes a multiplier.) (4) Take out the log ratios and add or subtract as the equation requires; the resulting log will give the part sought when that part is not greater than 90°. (5) To ensure whether or not the quantity taken from the table shall be taken from 180° (that is, whether this quantity or its *supplement* shall be considered the right answer), proceed as follows:—Over each known part in the equation put its sign, and from them form a sign for the unknown part (like signs give plus, unlike minus). Remember that all ratios in the first quadrant are positive; in the second quadrant, sine and cosecant only are positive. When the unknown is negative the supplement of the quantity taken from the table is wanted —*i.e.*, the quantity must be taken from 180°. (6) A difficulty often arises when sin is used since it is positive in both quadrants. This may be got over by noting that the side opposite the right angle and the other two sides control each other; thus if side opp. rt. ang. is over 90° the other two sides must be one over and one under 90° (one is generally known, and thus the other can be determined); if side opp. rt. ang. is under 90°, the other two sides must be either both over or both under 90°. From the two sides containing the right angle the value of the side opp. the rt. ang. can, it is evident, be estimated.

Application.—This rule is serviceable in all cases where one great circle of the heavens cuts another at right angles, as in amplitude problems, in which the meridian and the horizon form the right angle; in ex-meridian altitudes where a perpendicular is drawn on the meridian from the body, &c.

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* The student may like to know that the simple rules by Napier for solving rt. ang. sph. triangles are not true if certain parts are not called complementary (see theory).

Exercises in Right-Angled Spherical Triangles.

	Given.			*Required.*		
1.	$A=90^\circ$,	$b=89^\circ 20'$,	$c=81^\circ 16'$.	$B=89^\circ 20'$,	$a=89^\circ 54'$,	$C=81^\circ 16'$.
2.	$B=90^\circ$,	$a=101^\circ 21'$,	$A=93^\circ 20'$.	$c=16^\circ 52'$,	$C=17^\circ 11'$,	$b=100^\circ 52'$.
3.	$C=90^\circ$,	$a=81^\circ 25'$,	$b=101^\circ 15'$.	$B=101^\circ\ 8'$,	$A=81^\circ 35'$,	$c=94^\circ 40'$.
4.	$A=90^\circ$,	$b=110^\circ 21'$,	$C=113^\circ 18'$.	$a=81^\circ 39'$,	$c=114^\circ 40'$,	$B=108^\circ 38'$.
5.	$B=90^\circ$,	$A=91^\circ 20'$,	$C=100^\circ 30'$.	$b=89^\circ 45'$,	$c=100^\circ 30'$,	$a=91^\circ 21'$.
6.	$C=90^\circ$,	$a=120^\circ\ 0'$,	$B=179^\circ\ 0'$.	$c=60^\circ\ 0'$,	$b=179^\circ\ 8'$,	$A=90^\circ 30'$.
7.	$C=90^\circ$,	$c=81^\circ 10'$,	$a=100^\circ 47'$.	$B=144^\circ 41'$,	$A=96^\circ 13'$,	$b=145^\circ 10'$.
8.	$B=90^\circ$,	$a=71^\circ 36'$,	$A=71^\circ 36'$.	$c=90^\circ\ 0'$,	$b=90^\circ\ 0'$,	$C=90^\circ\ 0'$.
9.	$A=90^\circ$,	$a=91^\circ 20'$,	$b=150^\circ 19'$.	$C=89^\circ 14'$,	$c=88^\circ 28'$,	$B=150^\circ 18'$.
10.	$C=90^\circ$,	$a=72^\circ\ 0'$,	$B=81^\circ 19'$.	$b=80^\circ 53'$,	$c=87^\circ 12'$,	$A=72^\circ 13'$.
11.	$A=90^\circ$,	$B=95^\circ 12'$,	$C=101^\circ 41'$.	$a=88^\circ 55'$,	$b=95^\circ 19'$,	$c=101^\circ 44'$.
12.	$A=90^\circ$,	$B=84^\circ 26'$,	$c=108^\circ 19'$.	$a=91^\circ 50'$,	$b=84^\circ\ 8'$,	$C=108^\circ 13'$.
13.	$B=90^\circ$,	$b=21^\circ\ 6'$,	$c=13^\circ 41'$.	$a=16^\circ 13'$,	$A=50^\circ 53'$,	$C=41^\circ\ 5'$.
14.	$C=90^\circ$,	$A=81^\circ 13'$,	$B=65^\circ 24'$.	$a=80^\circ 2$[illegible]$'$,	$b=65^\circ\ 5'$,	$c=85^\circ 56'$.
15.	$C=90^\circ$,	$a=3^\circ 49'$,	$b=6^\circ 24'$.	$c=7^\circ 27'$,	$B=59^\circ 19'$,	$A=30^\circ 54'$.
16.	$A=90^\circ$,	$b=7^\circ 20'$,	$B=86^\circ 17'$.	$a=7^\circ 21'$,	$c=0^\circ 29'$,	$C=3^\circ 45'$.
17.	$B=90^\circ$,	$c=79^\circ 13'$,	$C=80^\circ 20'$.	$a=63^\circ 26'$,	$b=85^\circ 12'$,	$A=63^\circ 50'$.
18.	$B=90^\circ$,	$A=38^\circ 25'$,	$a=24^\circ 16'$.	$b=41^\circ 24'$,	$c=34^\circ 38'$,	$C=59^\circ 15'$.
19.	$A=90^\circ$,	$a=86^\circ 49'$,	$B=101^\circ 19'$.	$b=101^\circ 45'$,	$c=105^\circ 49'$,	$C=105^\circ 31'$.
20.	$A=90^\circ$,	$a=45^\circ 16'$,	$b=45^\circ 11'$.	$c=3^\circ\ 6'$,	$B=86^\circ 55'$,	$C=4^\circ 22'$.
21.	$B=90^\circ$,	$a=179^\circ 20'$,	$C=36^\circ 52'$.	$b=179^\circ 10'$,	$A=126^\circ 52'$,	$c=0^\circ 30'$.
22.	$B=90^\circ$,	$a=92^\circ 10'$,	$c=42^\circ 10'$.	$C=42^\circ 11'$,	$A=91^\circ 27'$,	$b=91^\circ 36'$.
23.	$C=90^\circ$,	$A=72^\circ 30$,	$B=72^\circ 30'$.	$c=84^\circ 18'$,	$a=71^\circ 37'$,	$b=71^\circ 37'$.
24.	$C=90^\circ$,	$b=50^\circ 50'$,	$A=50^\circ 50'$.	$a=43^\circ 35'$,	$c=62^\circ 47'$,	$B=60^\circ 40'$.

2, 16, 17, and 18 are ambiguous, and there are two sets of answers for these examples. (See Note, p. 48.)

* *Note.*—Think out this simple case **as well as** work it.

Example VI.—Quadrantal Triangles.—Given the quadrantal side and two other parts, to find the three remaining parts.

Formulæ.—Sin middle part = product of tans of adjacent parts, or = product of cosines of opposite parts.

Given $a=90^\circ$, $b=118^\circ 32'$, $c=65^\circ 50'$; **find** the other parts (Fig 12).

B, *c*, *A*, *b*, *C* are circular parts; *B* **and** *C* ***touch*** the quadrant, and are ***not comp.*** parts (theory).

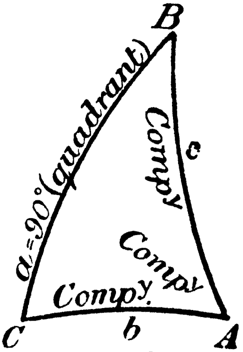
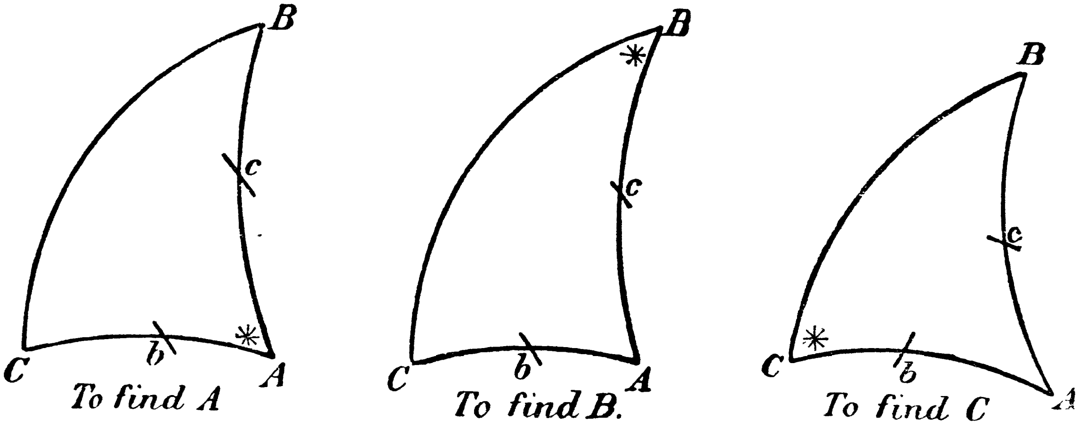


Fig. 11.



To find A

To find B.

To find C

Fig. 12.

$$\underline{\text{Sin comp.}}A = \overset{(-)\ \ -}{\tan}\text{comp.}b \times \overset{+}{\tan}\text{comp.}c.$$

$$\therefore \cos A = \cot b \times \cot c.$$

$$\log \cot b = 9{\cdot}735367$$
$$\log \cot c = 9{\cdot}651974$$

$$\log \cos A = 9{\cdot}387341$$

$$A = 75^\circ\ 53'.$$

$$\overset{-}{\text{Sin}}\text{comp. } b = \overset{+}{\cos}\text{comp.}c \times \underline{\overset{-}{\sin}\text{ comp.}B}.$$

$$\therefore \cos B = \frac{\cos b}{\sin c}.$$

$$\log \cos b = 9{\cdot}679128$$
$$\log \sin c = 9{\cdot}960165$$

$$\log \cos B = 9{\cdot}718963$$

$$B = 180^\circ - 58^\circ\ 26', \text{ or } 121^\circ\ 34'.$$

$$\overset{+}{\text{Sin}}\text{ comp. } c = \overset{+}{\cos}\text{ comp. } b \times \underline{\cos C}.$$

$$\therefore \cos C = \frac{\cos c}{\sin b}.$$

$$\log \cos c = 9{\cdot}612140$$
$$\log \sin b = 9{\cdot}943761$$

$$\log \cos C = 9{\cdot}668379$$

$$C = 62^\circ\ 14'.$$

Explanation.—The student who has become **proficient in the** solving of **right-angled** spherical **triangles** should **find** but **little difficulty** in understanding **quadrantal triangles.** The **same formulæ** serve, the **complementary parts** are those **not in contact with** the **quadrantal side,** just as in rt. ang. sph. triangles the comp. parts are those not in contact with the right angle. One **peculiar difference** between quadrantal and rt. ang. triangles must be noted most carefully. In quadls. :—**When an equation is formed** signs must be put over the ratios of the known parts to determine the sign of the unknown part, and **if there should be two sides or two angles on the same side of the equation an extra negative sign must be put on that side before the sign of the unknown is affixed.** Note also that **sin** comp. must be considered **cosine** of part given, &c., when settling the question of signs. In the specimen triangle (Fig. 11), (1) **the five circular parts are the three angles and the two sides which** (usually) **are not quadrantal**; the name of the quadrantal side is omitted from Fig. 12. (2) In the first equation, A the part to be found stands between b and c the given parts, which are both in contact with it (adjacent). (3) All these parts are **comp. because they do not touch the quadrantal side** a. (4) **Tan** comp. b or **cot** b is negative because b is in the second quadrant; **tan** comp. c or **cot** c is positive, being in the first quadrant, or under 90°. (5) These two signs would produce a negative for **sin** comp. A, but **as there are two sides together in the equation an extra negative must be placed** (seen in the bracket), and this converts **sin** comp. A or **cos** A into a positive quantity. (6) In the second equation **sin** comp. b or **cos** b is negative, and **cos** comp. c or **sin** c is positive, and no extra sign being necessary in this case, we get **sin** comp. B or **cos** B negative, that is, the quantity taken from the table has to be subtracted from 180° for the actual value of B. (7) In the third equation **sin** comp. c or **cos** c is positive, and **cos** comp. b or **sin** b is also positive (**sin** is pos. in both quadrants); therefore, as no extra sign is needed, **cos** C is positive, and the quantity taken from the table is the correct value of C.

Application.—When **a body is on** the **horizon** its zenith distance is 90°, when **on** the **celestial equator** its polar **distance is** 90°; in such cases quadrantal triangles are of service.

Exercises in Quadrantal Triangles.

	Given.			*Required.*		
1.	$a=90^\circ$,	$B=85^\circ 20'$,	$c=81^\circ 36'$.	$A=90^\circ 41'$,	$C=81^\circ 34'$,	$b=85^\circ 23'$.
2.	$b=90^\circ$,	$a=89^\circ 36'$,	$A=88^\circ 10'$.	$C=12^\circ 36'$,	$c=12^\circ 36'$,	$B=91^\circ 47'$.
3.	$c=90^\circ$,	$a=91^\circ 30'$,	$b=92^\circ 24'$.	$C=90^\circ\ 4'$,	$A=91^\circ 30'$,	$B=92^\circ 24'$.
4.	$a=90^\circ$,	$b=72^\circ 30'$,	$C=110^\circ 25'$.	$A=83^\circ 37'$,	$c=109^\circ 26'$,	$B=71^\circ 25'$.
5.	$a=90^\circ$,	$b=101^\circ 15'$,	$C=79^\circ 36'$.	$A=87^\circ 57'$,	$B=101^\circ 26'$,	$c=79^\circ 47'$.
6.	$b=90^\circ$,	$A=81^\circ 26'$,	$C=92^\circ 12'$.	$a=81^\circ 26'$,	$c=92^\circ 11'$,	$B=89^\circ 40'$.
7.	$a=90^\circ$,	$b=110^\circ 21'$,	$C=125^\circ 30'$.	$A=103^\circ 56'$,	$B=114^\circ 39'$,	$c=123^\circ\ 0'$.
8.	$c=90^\circ$,	$A=81^\circ 30'$,	$B=101^\circ 50'$.	$a=81^\circ 41'$,	$b=101^\circ 42'$,	$C=88^\circ 16'$.
9.	$b=90^\circ$,	$B=80^\circ 32'$,	$C=100^\circ 14'$.	$a=22^\circ 31'$,	$c=86^\circ\ 6'$,	$A=22^\circ 13'$.
10.	$c=90^\circ$,	$b=89^\circ 40'$,	$A=106^\circ 24'$.	$a=106^\circ 24'$,	$B=89^\circ 39'$,	$C=89^\circ 54'$.
11.	$a=90^\circ$,	$b=90^\circ\ 0'$,	$c=90^\circ\ 0'$.	$A=90^\circ\ 0'$,	$B=90^\circ\ 0'$,	$C=90^\circ 0'$.*
12.	$b=90^\circ$,	$a=120^\circ\ 0'$,	$c=120^\circ\ 0'$.	$B=109^\circ 28'$,	$A=125^\circ 15'$,	$C=125^\circ 16'$.
13.	$c=90^\circ$,	$b=31^\circ\ 6'$,	$C=101^\circ 20'$.	$a=83^\circ 14'$,	$A=76^\circ 49'$,	$B=30^\circ 26'$.
14.	$b=90^\circ$,	$B=83^\circ\ 4'$,	$C=82^\circ 11'$.	$a=152^\circ 21'$,	$c=86^\circ 23'$,	$A=152^\circ 34'$.
15.	$a=90^\circ$,	$C=84^\circ 35'$,	$B=101^\circ 25'$.	$c=84^\circ 41'$,	$b=101^\circ 22'$,	$A=88^\circ 56'$.

2 is ambiguous, and there are two solutions.

* *Think* out this simple case as well as work it.

Note.—In "Sines" and in Right-angled or Quadrantal Triangles it is sometimes possible to draw two triangles which satisfy the given conditions when the working parts are two sides and a corresponding angle or two angles and a corresponding side.

If a, b, A are given, there is only one triangle if a lies between b and $180-b$, otherwise there are two triangles.

If A, B, a are given, there is only one triangle if A lies between B and $180-B$, otherwise there are two triangles.

The solutions can always be checked by applying two tests :—

(1) The greater side is opposite the greater angle and *vice versâ.*

(2) If the sum of two sides exceeds 180°, then the sum of the opposite angles exceeds 180° and *vice versâ.*

CHAPTER VIII.

FORMULÆ

For the solution of Triangles and other Geometrical **Figures.**—The letters, &c., used are explained below.

I. Plane Triangles.—Three sides given to find an angle:—

(1) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

(2) $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

(3) $bc \,.\, \text{hav } A = \tfrac{1}{2}(a + \overline{b \sim c}) \,.\, \tfrac{1}{2}(a - \overline{b \sim c})$.

To find third side direct—

(4) $a = \dfrac{\sqrt{4bc \text{ hav } A}}{\sin \theta}$; $\tan \theta = \dfrac{\sqrt{4bc \text{ hav } A}}{b \sim c}$.

II. Spherical Triangles.—

(1) $\text{hav } A = \text{cosec } b \,.\, \text{cosec } c \sqrt{\text{hav}\{a + \overline{b \sim c}\} \,.\, \text{hav}\{a - \overline{b \sim c}\}}$.

(2) $\sin \dfrac{A}{2} = \sqrt{\text{cosec } b \,.\, \text{cosec } c \,.\, \sin \tfrac{1}{2}(a + \overline{b \sim c}) \,.\, \sin \tfrac{1}{2}(a - \overline{b \sim c})}$.

(3) $\sin \dfrac{A}{2} = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b \,.\, \sin c}}$.

(4) $\tan \dfrac{A}{2} = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s \,.\, \sin(s-a)}}$.

(5) $\cos \dfrac{a}{2} = \sqrt{\dfrac{\cos(M-B)\cos(M-C)}{\sin B \,.\, \sin C}}$ (Three angles given to find side. $M = \dfrac{A+B+C}{2}$).

(6) $\cot \dfrac{C}{2} = \tan \dfrac{A+B}{2} \,.\, \cos \dfrac{a+b}{2} \,.\, \sec \dfrac{a \sim b}{2}$ (Rule of "Sines" to find third angle).

(7) $\tan \dfrac{c}{2} = \tan \dfrac{a+b}{2} \,.\, \cos \dfrac{A+B}{2} \,.\, \sec \dfrac{A \sim B}{2}$ (Rule of "Sines" to find third side).

III. Miscellaneous:—

(a) Areas, Planes.

(1) Rectangle $= ab$.

(2) Parallelogram $= ab \sin C$.

(3) Regular polygon $= \frac{1}{4} na^2 \cot \frac{180°}{n}$.

(4) Circle $= \pi r^2$ or $\frac{\pi}{4}d^2$.

(5) Ellipse $= \frac{\pi}{4}dd'$.

(6) Trapezoid $= \frac{x+y}{2} \times h$.

(7) Trapezium = sum of the two contained triangles.

(8) Ship's deck or water-plane $= \frac{2}{3}l(1p + 4p_1 + 2p_2 + 4p_3 + \text{\&c.})$.

In this formula the amidship line from stem to stern is divided into an even number of equal parts (the more of these parts the more exact will be the area found); at each division, and at the bow and stern, perpendiculars (p, p_1, p_2, &c.) must be raised (that at the bow will $= 0$) to meet the ship's side. The length of each perpendicular must be multiplied by a special multiplier ("Simpson's First Rule for Areas"), thus, $p \times 1$, $p_1 \times 4$, $p_2 \times 2$, $p_3 \times 4$, $p_4 \times 2$, &c., the final perpendicular being multiplied by 1. The sum of these products should be multiplied by l, the distance between any two perpendiculars; $\frac{1}{3}$ this result gives the area of half the deck, $\frac{2}{3}$ gives the whole deck.

(b) Surfaces not Planes.

(1) Sphere $= \pi d^2$ or cd.

(2) Cone $= \frac{1}{2}\pi dh'$.

(3) Cylinder $= \pi dh$.

(c) Contents.

(1) Cylinder $= \frac{\pi}{4}d^2h$.

(2) Cone $= \frac{1}{3} \cdot \frac{\pi}{4} d^2h$.

(3) Sphere $= \frac{1}{6}\pi d^3 = \frac{4}{3}\pi r^3$.

(4) Prolate spheroid $= \frac{4}{3}\pi ab^2$.

(5) Oblate spheroid $= \frac{4}{3}\pi a^2 b$.

(6) Displacement $= \frac{2}{3} l\,(1p + 4p_1 + 2p_2 + 4p_3 + 2p_4 +$ &c.).

By cutting the immersed body of a ship into fine layers (from flotation plane to keel, getting an even number of equally thick pieces, and thus an odd number of areas; the first being flotation plane, the last the bottom of outer keel) and using the half areas of these water planes, as p, p_1, p_2, p_3, &c., displacement is found (l being thickness of pieces) by the above formula.

Explanation of Letters, &c.

s (half sum of sides); hav (haversine or half versine); M (half sum of angles, spherical trig.); a (side of a figure); b (side adjacent to a in figure)*; n (the number of sides in a figure); d (diameter of a figure; d', the other diameter in an ellipse); c (circumference); h (perpendicular height; h' slant height); C (angle between a and b); x and y (the parallel sides, the head and foot of an upper topsail).

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* In spheroid a = semi-major and b = semi-minor axis.

CHAPTER IX.

IDENTITIES AND EQUATIONS.

(1) $\sec A = \tan A \,.\, \operatorname{cosec} A.$ (2) $\dfrac{\cot A}{\cos A} = \operatorname{cosec} A.$

(3) $\dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}.$ (4) $\cot A \,.\, \tan^2 A = \tan A.$

(5) $\cos A \,.\, \sec A = \sin^2 A + \cos^2 A.$ (6) $\dfrac{\cos A}{\sin A \,.\, \cot A} = 1.$

(7) $\sin^2 A \,.\, \sec A = \dfrac{\tan A}{\operatorname{cosec} A}.$ (8) $\sin^2 A \,.\, \cot^2 A = \cos^2 A.$

(9) $\tan A = \dfrac{\sin 2A}{1 + \cos 2A}.$ (10) $\sin A(\operatorname{cosec} A - \cot A) = 1 - \cos A$

(11) $\operatorname{vers} A \,.\, \sec A = \dfrac{1 - \cos A}{1 - \operatorname{vers} A}.$ (12) $\dfrac{\cos A}{\sin A \,.\, \tan^2 A} = \cot^3 A.$

(13) $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \,.\, \operatorname{cosec}^2 A.$

(14) $\sin (A + B) \,.\, \sin (A - B) = \sin^2 A - \sin^2 B.$

(15) $\tan A + \tan B = \dfrac{\sin (A + B)}{\cos A \,.\, \cos B}.$ (16) $\sec^2 A \,.\, \sin 2A = 2 \tan A$

(17) $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A.$ (18) $1 + \tan A \,.\, \tan \dfrac{A}{2} = \sec A.$

(19) $(\operatorname{cosec} A - \cot A)^2 = \dfrac{1 - \cos A}{1 + \cos A}.$

(20) $\dfrac{\cos^2 A - \cos^2 B}{\cot^2 A - \cot^2 B} = \dfrac{\cos^2 B - \sin^2 A}{\cot^2 A \,.\, \cot^2 B - 1}.$

(21) $\dfrac{1}{\cos A} = \dfrac{\sqrt{1 + \cot^2 A}}{\cot A}.$

(22) $\tan (45^\circ + A) + \tan (45^\circ - A) = 2 \sec 2A$

(23) $\tan A \,.\, \cot \dfrac{A}{2} = 1 + \sec A.$

(24) $\sin A + \sin (60^\circ - A) = \sin (60^\circ + A).$

(25) $\dfrac{\text{cosec } A}{\sec A} + \dfrac{\sec A}{\text{cosec } A} = \sec A \,.\, \text{cosec } A.$

(26) $\cos(2\pi - A) = -\cos(\pi + A).$

(27) $\dfrac{\cot A \,.\, \cot B}{\cot A \,.\, \cot B - 1} = \dfrac{\cos A \,.\, \cos B}{\cos(A + B)}.$

(28) $\sin A - \cos A = \sqrt{2} \,.\, \sin(A - 45^\circ).$

(29) $\dfrac{\sin(A - \mathrm{B})}{\sin A \,.\, \sin B} = \cot B - \cot A.$ (30) $\sec 2A = \dfrac{1 + \tan^2 A}{1 - \tan^2 A}.$

(31) $\dfrac{\tan^2 A}{1 + \tan^2 A} = 1 - \cos^2 A.$ (32) $\sin(45^0 + A) = \cos(45^\circ - A).$

(33) If $\cos A = \frac{1}{2}$, find the value of $\tan A$. *Ans.* $(\pm \sqrt{3})$.

(34) If $\text{cosec } A = 1\frac{2}{5}$, find $\sin A$. ,, $(\frac{5}{7})$.

(35) If $\cos A = \dfrac{\sin^2 \theta}{\cos^2 \theta} + 1$, find $\sec A$. ,, $(\cos^2 \theta)$.

(36) If $\sin^2 A + 5\cos^2 A = 3$, find $\sin A$ and $\tan A$. ,, $(\frac{1}{2}\sqrt{2}$ and $1)$

(37) If $\sin A = \frac{1}{2}$, find $2 \text{ vers}\left(\dfrac{\pi}{2} - A\right)$. ,, (1).

(38) If $\tan A = \dfrac{b}{a}$, prove $\sqrt{\dfrac{a + b}{a - b}} + \sqrt{\dfrac{a - b}{a + b}} = \dfrac{2}{\sqrt{1 - \tan^2 A}}.$

(39) If $\sin 18^\circ = \dfrac{\sqrt{5} - 1}{4}$ find $\tan 36^\circ$. *Ans.* $\sqrt{5 - 2\sqrt{5}}$.

(40) If $A = 15^\circ$, find value for $\tan A + \tan 5A$. ,, (4).

(41) In a plane triangle $A\,B\,C$ show $\cos A + \cos B + \cos C - 1$

$$= 4 \sin\frac{A}{2} \times \sin\frac{B}{2} \times \sin\frac{C}{2}.$$

(42) Prove $\sin\theta\,(1 + 2\cos\theta) = 2\cos\dfrac{\theta}{2} \times \sin\dfrac{3\theta}{2}.$

(43) In a plane triangle $A\,B\,C$ show that

$$\text{area} = \frac{1}{2}(a^2 - b^2)\frac{\sin A \,.\, \sin B}{\sin(A - B)}.$$

(44) In a plane triangle $A\,B\,C$ show that if $a\cos B = b\cos A$, $A\,B\,C$ is isosceles.

(45) Find x in the following equations :—

(1) $\tan x + \cot x = 2$. (2) $\cos 2x + \sin x = 1$.

Ans. (1) 45° or 225°. (2) 0° or 180°; 30° or 150°.

(46) In a right-angled triangle ($C = 90°$) show that

$$\cos \frac{A}{2} = \left(\frac{c+b}{2c}\right)^{\frac{1}{2}}.$$

(47) Prove that in plane triangle ABC, $c = b \cdot \cos A + a \cos B$.

(48) $A + B + C = 90°$,—show that
$\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cdot \cos B \cdot \cos C$.

(49) Prove $\tan^2 \frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)}$ in a plane triangle.

(50) Prove area $= \frac{c^2}{2} \times \frac{\sin A \times \sin B}{\sin C}$ in a plane triangle.

CHAPTER X.

THEORY.

PROOFS OF FORMULÆ USED IN PLANE TRIGONOMETRY.

Class I.—Fundamental Identities.

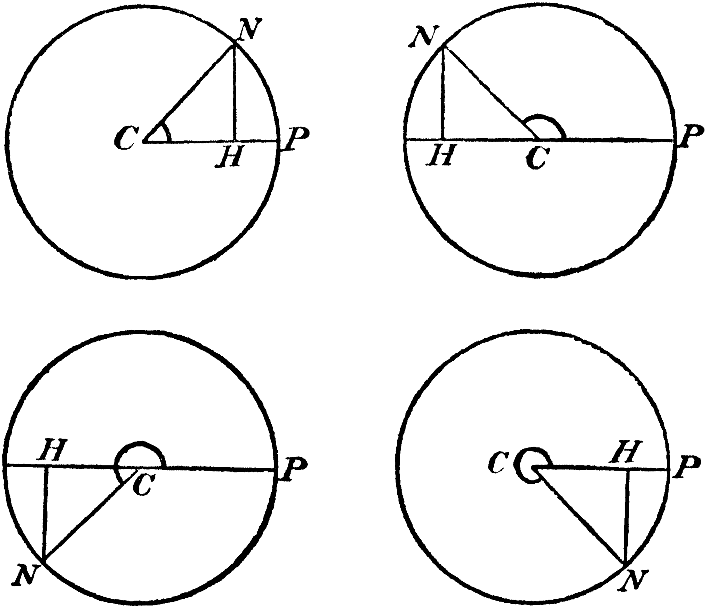

Fig. 13.

Let the straight line CN revolve about the point C from the initial position CP till it takes up the position CN. It has then generated the angle PCN (which is marked in the figure).

From N draw NH perp. to PC or PC produced.

Let the angle *PCN* be called θ, and let the sides of the triangle *CNH* be denoted by c, n, h.

(1)	(2)	(3)
Now $h^2 = n^2 + c^2$.	$h^2 = n^2 + c^2$.	$h^2 = n^2 + c^2$. **Euc. I. 47.**
divide by h^2.	divide by n^2.	divide by c^2.
$\therefore \dfrac{h^2}{h^2} = \dfrac{n^2}{h^2} + \dfrac{c^2}{h^2}.$	$\dfrac{h^2}{n^2} = \dfrac{n^2}{n^2} + \dfrac{c^2}{n^2}.$	$\dfrac{h^2}{c^2} = \dfrac{n^2}{c^2} + \dfrac{c^2}{c^2}.$
i.e.,	*i.e.*,	*i.e.*, (see *Ratios*).
$1 = \cos^2\theta + \sin^2\theta$.	$\sec^2\theta = 1 + \tan^2\theta$.	$\operatorname{cosec}^2\theta = \cot^2\theta + 1$.

These three results are important. They can be re-arranged in several ways; thus :—

$$1 - \sin^2\theta = \cos^2\theta. \quad \sec^2\theta - 1 = \tan^2\theta. \quad \operatorname{cosec}^2\theta - 1 = \cot^2\theta.$$

They should be committed to memory thus :—

(1) Sin squared of any angle plus cos squared of the same angle equals 1.

(2) Sec squared of any angle equals 1 plus the tan squared of the same angle.

(3) Cosec squared of any angle equals 1 plus the cot squared of the same angle.

N.B.—These results apply to all angles :—

$$\therefore \quad \sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1, \quad \sin^2 40^\circ\ 10' + \cos^2 40^\circ\ 10' = 1, \text{ \&c.}$$

The student should test the equations with various angles by using the tables of log sines, &c.; this affords exercise in the use of logs, and fixes the eqns. in the memory.

Class II.—Reciprocals, &c.

From the figure used in Class I.

$$\text{(1)} \quad \sin\theta \times \operatorname{cosec}\theta = \frac{NH}{NC} \times \frac{NC}{NH} = 1.$$

$$\text{(2)} \quad \tan\theta \times \cot\theta = \frac{c}{n} \times \frac{n}{c} \left(\text{or } \frac{NH}{CH} \times \frac{CH}{NH}\right) = 1.$$

$$\text{(3)} \quad \sec\theta \times \cos\theta = \frac{CN}{CH} \times \frac{CH}{CN} = 1.$$

(4) $$\frac{\sin\theta}{\cos\theta}=\frac{\frac{NH}{NC}}{\frac{HC}{NC}}=\frac{NH}{NC}\times\frac{NC}{HC}=\frac{NH}{HC}=\tan\theta.$$

(5) $$\frac{\cos\theta}{\sin\theta}=\frac{\frac{HC}{NC}}{\frac{NH}{NC}}=\frac{HC}{NC}\times\frac{NC}{NH}=\frac{HC}{NH}=\cot\theta,$$

and these values may be re-arranged (and adapted to any angles). Thus,

(1) $\sin\theta\times\operatorname{cosec}\theta=1\quad\therefore\quad\sin\theta=\frac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta=\frac{1}{\sin\theta}$

(2) $\tan A\times\cot A=1\quad\therefore\ \tan A=\frac{1}{\cot A}$ and $\cot A=\frac{1}{\tan A}$

(3) $\sec B\times\cos B=1\quad\therefore\ \sec B=\frac{1}{\cos B}$ and $\cos B=\frac{1}{\sec B}$

Class III.—Complements, Supplements, &c.

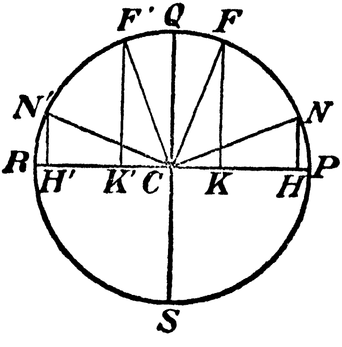

Fig. 14.

(1) Values for ratios of (90° – A).—Let $PQRS$ be a circle, C the centre, PR and QS two diameters at right angles, PCN an angle which we will call A.

At the point C in the straight line CQ make an angle QCF equal to A. From the points F and N on the circumference draw perps. to CP. Then $PCF = 90^\circ - A$.

In the two triangles CFK and CNH, because $CFK = FCQ = NCH$, and $FKC = NHC$, and $CF = CN$.

$$\therefore \qquad CK = NH \text{ and } FK = CH. \qquad \text{(Euc. I. 26.)}$$

$$\sin (90^\circ - A) = \sin PCF = \frac{FK}{FC} = \frac{CH}{CN} = \cos A,$$

$$\tan (90^\circ - A) = \tan PCF = \frac{FK}{KC} = \frac{CH}{HN} = \cot A,$$

and so on for the other ratios.

If two angles make 90° by addition, the sin of one angle equals the cos of the other; *i.e.*, **the sin of an angle is the cos of its complement.**

(2) **Values for ratios of $(90^\circ + A)$.**—In the above figure, make an angle $QCF' = PCN$, draw $F'K'$ perp. to PC produced. Then $PCF' = (90^\circ + A)$.

The triangles $F'K'C$ and NCH may be shown to be equal in all respects, as before.

$$\sin (90^\circ + A) = \sin PCF' = \frac{F'K'}{F'C} = \frac{CH}{CN} = \cos A;\text{*}$$

$$\cos (90^\circ + A) = \cos PCF' = \frac{K'C}{F'C} = \frac{-NH}{CN} = -\sin A;$$

(neg. and pos. lines).

and so on for the other ratios. Note the neg. sign.

(3) **Values for ratios of $(180^\circ - A)$.**—In the above figure make an angle $RCN' = A$.

From N' draw $N'H'$ perp. to PC produced. Then $PCN' = 180^\circ - A$.

The triangles $N'CH'$ and NCH may be shown to be equal in all respects.

$$\sin (180^\circ - A) = \sin PCN' = \frac{N'H'}{N'C} = \frac{NH}{NC} = \sin A.\text{*}$$

$$\cos (180^\circ - A) = \cos PCN' = \frac{H'C}{N'C} = \frac{-HC}{NC} = -\cos A,$$

and so on for the other ratios.

---

* Hence, in practical calculations an angle greater than 90° may be reduced by 90°, or be taken from 180°, the ratio being changed in the former case, but not in the latter. Sin 130° = cos 40°, or sin 50°.

In a similar manner we may obtain :—

(4) Values for ratios of $(180° + A)$.

(5) Values for ratios of $(270° - A)$.

(6) Values for ratios of $(270° + A)$.

(7) Values for ratios of $(360° - A)$

Class IV.—Compound Angles.

Values for ratios of $A + B$.—Let XCY be an angle A, and YCZ be an angle B, then XCZ equals $(A + B)$.

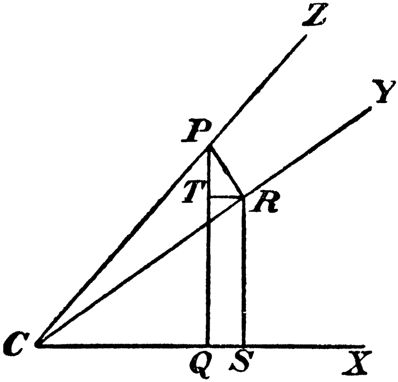

Fig. 15.

In CZ take any point P, from P draw PQ perpendicular to CX, from P draw PR perpendicular to CY, through R draw RS parallel to PQ, through R draw RT parallel to XC.

Now $TPR + TRP = 90°$ (I. 32 and constr.)

and $TRP + TRC = 90°$ (constr.)

$\therefore \quad TPR + TRP = TRP + TRC$;

omit TRP,

$\therefore \quad TPR = TRC$,

and $\quad TRC = RCX = A$ (I. 29)

$\therefore \quad TPR = A$.

$$(1)\ \sin(A+B)=\sin XCZ=\frac{PQ}{PC}=\frac{QT+TP}{PC}=\frac{RS+TP}{PC}$$

$$=\frac{RS}{PC}+\frac{TP}{PC}=\frac{RS\times RC}{RC\times PC}+\frac{TP\times PR}{PR\times PC}*$$

$$=\sin A\times\cos B+\cos A\times\sin B.$$

$$(2)\ \cos(A+B)=\cos XCZ=\frac{CQ}{CP}=\frac{CS-SQ}{CP}=\frac{CS-TR}{CP}$$

$$=\frac{CS}{CP}-\frac{TR}{CP}=\frac{CS\times CR}{CR\times CP}-\frac{TR\times PR}{PR\times CP}$$

$$=\cos A\times\cos B-\sin A\times\sin B.$$

$$(3)\ \tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)} \qquad \text{(ii. 4)}$$

$$=\frac{\sin A\ \cos B+\cos A\ \sin B}{\cos A\ \cos B-\sin A\ \sin B} \qquad \text{(class iv. 1, 2)}$$

divide num. and den. by $\cos A\ \cos B$,

$$\therefore\quad \tan(A+B)=\frac{\dfrac{\sin A\ \cos B}{\cos A\ \cos B}+\dfrac{\cos A\ \sin B}{\cos A\ \cos B}}{\dfrac{\cos A\ \cos B}{\cos A\ \cos B}-\dfrac{\sin A\ \sin B}{\cos A\ \cos B}} \qquad \text{(cancel, \&c.)}$$

$$=\frac{\tan A+\tan B}{1-\tan A\ \tan B}. \qquad \text{(ii. 4)}$$

Hence (to be committed to memory) :—

(1) The sin of the sum of any two angles equals the sin of the first angle into the cos of the second plus the cos of the first into the sin of the second.

(2) The cos of the sum of any two angles equals the cos of the first angle into the cos of the second minus the sin of the first into the sin of the second.

(3) The tan of the sum of any two angles equals the tan of the first angle plus the tan of the second, divided by 1 minus the tan of the first into the tan of the second.

---

* $\frac{RS}{PC}\times\frac{RC}{RC}=\frac{RS\times RC}{RC\times PC}$, &c. The student should carefully note (from Fig. 15) the above fractions give the result $\sin A\times\cos B+\cos A\times\sin B$.

Value for ratios of $(A-B)$.—Let XCY be an angle A, and YCZ be an angle B; then XCZ equals $(A-B)$.

In CZ take any point P, from P draw PQ perp. to CX, from P draw PR perp. to CY, through R draw RS parallel to PQ, through R draw RT parallel to CX, meeting QP produced in T.

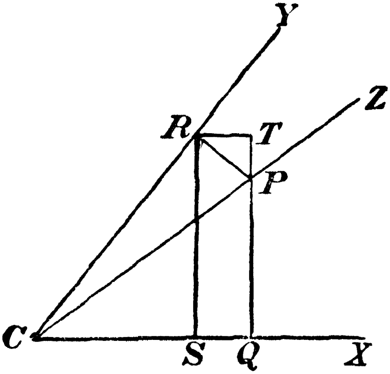

Fig. 16.

Now $TPR + TRP = 90°$ (I. 32 and constr.)

and $TRP + TRY = 90°$ (constr.)

$\therefore \quad TPR + TRP = TRP + TRY$;

omit TRP,

$\therefore \quad TPR = TRY$;

but $TRY = RCX = A$ (I. 29)

$\therefore \quad TPR = A$.

(4) $$\sin(A-B) = \sin XCZ = \frac{PQ}{PC} = \frac{QT - TP}{PC} = \frac{RS - TP}{PC}$$

$$= \frac{RS}{PC} - \frac{TP}{PC} = \frac{RS \times RC}{RC \times PC} - \frac{TP \times PR}{PR \times PC}$$

$$= \sin A \times \cos B - \cos A \times \sin B.$$

(5) $$\cos(A-B) = \cos XCZ = \frac{CQ}{CP} = \frac{CS + QS}{CP} = \frac{CS + RT}{CP}$$

$$= \frac{CS}{CP} + \frac{RT}{CP} = \frac{CS \times CR}{CR \times CP} + \frac{RT \times RP}{RP \times CP}$$

$$= \cos A \times \cos B + \sin A \times \sin B.$$

$$(6)\ \tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)} \qquad \text{(ii. 4)}$$

$$=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B} \qquad \text{(iv. 4, 5)}$$

divide num. and den. by cos A cos B.

$$\therefore\ \tan(A-B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B} \qquad \text{(ii. 4)}$$

Hence (to be committed to memory):—

(4) The sin of the diff. of any two angles equals the sin of the first angle into the cos of the second, minus the cos of the first into the sin of the second.

(5) The cos of the diff. of any two angles equals the cos of the first angle into the cos of the second, plus the sin of the first into the sin of the second.

(6) The tan of the diff. of any two angles equals the tan of the first angle minus the tan of the second divided by 1 plus the tan of the first into the tan of the second.

Values for sin A + sin B, &c.

$$\text{Now } \frac{A+B}{2}+\frac{A-B}{2}=A, \text{ and } \frac{A+B}{2}-\frac{A-B}{2}=B;$$

and if these values be substituted for A and B, we get—

$$\sin A=\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \text{(iv. 1)}$$

$$\sin B=\sin\frac{A+B}{2}\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \text{(iv. 2)}$$

from which by addition—

$$(7)\qquad \sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2};$$

and by subtraction—

$$(8)\qquad \sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Again, by substituting the above values for A and B,

$$\cos A = \cos\frac{A+B}{2}\cos\frac{A-B}{2} - \sin\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \text{(iv. 4)}$$

$$\cos B = \cos\frac{A+B}{2}\cos\frac{A-B}{2} + \sin\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \text{(iv. 5)}$$

from which by addition—

(9) $$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2};$$

and by subtraction—

(10) $$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= 2\sin\frac{B+A}{2}\sin\frac{B-A}{2} \qquad \text{(vi. 8)*}$$

These four results are very important (to be remembered):—

(7) The sum of the sines of two angles = twice sin half sum × cos half diff.

(8) The diff. of the sines of two angles = twice cos half sum × sin half diff.

(9) The sum of the cosines of two angles = twice cos half sum × cos half diff.

(10) The diff. of the cosines of two angles = twice sin half sum × sin half diff. (with order of terms reversed).

From the proofs just obtained the following are easily deduced:—

(11) $$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}} \qquad \text{(iv. 7, 9)}$$

$$= \tan\frac{A+B}{2} \qquad \text{(ii. 4)}$$

(12) $$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}} \qquad \text{(iv. 8, 9)}$$

$$= \tan\frac{A-B}{2} \qquad \text{(ii. 4)}$$

---

* Note how the negative sign of the preceding line is disposed of; examine (by construction) how $-\sin A$ (any angle) $= \sin -A$.

(13) $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}} \qquad \text{(iv. 7, 8)}$$

$$= \tan \frac{A+B}{2} \cot \frac{A-B}{2} \qquad \text{(ii. 4, 5)}$$

$$= \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}} \qquad \text{(ii. 2)}$$

and these results can be adapted to all angles.

Class V.—Multiples and Sub-Multiples of an Angle.

(1) Now $$\sin 2A = \sin (A + A)$$

$$= \sin A \cos A + \cos A \sin A \qquad \text{(iv. 1)}$$

$$= 2 \sin A \cos A;$$

i.e., the sin of any angle equals twice the sine of half the angle × the cos of half the angle (memory).

(2) Now $$\cos 2A = \cos (A + A)$$

$$= \cos A \cos A - \sin A \sin A \qquad \text{(iv. 2)}$$

$$= \cos^2 A - \sin^2 A, \quad . \quad . \quad . \quad \text{(a)}$$

and since $$\cos^2 A = 1 - \sin^2 A \qquad \text{(i. 1)}$$

∴ $$\cos 2A = 1 - \sin^2 A - \sin^2 A$$

$$= 1 - 2 \sin^2 A, \quad . \quad . \quad . \quad . \quad \text{(b)}$$

and since $$\sin^2 A = 1 - \cos^2 A \qquad \text{(i. 1)}$$

∴ $$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$

$$= 2 \cos^2 A - 1, \quad . \quad . \quad . \quad . \quad \text{(c)}$$

These three results are very important (memory):—

2a. The cos of any angle = cos squared of its half minus sin squared of its half.

2b. The cos of any angle = 1 minus twice sin squared of its half.

2c. The cos of any angle = twice cos squared of its half minus 1.

(3) Now
$$\tan 2A = \tan (A + A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} \qquad \text{(iv. 3)}$$
$$= \frac{2 \tan A}{1 - \tan^2 A};$$

i.e., the tan of any angle equals twice the tan of half the angle divided by 1 minus tan squared half the angle (memory).

(4) Now
$$1 + \cos A = 1 + 2 \cos^2 \frac{A}{2} - 1 \qquad \text{(v. 2}c\text{)}$$
$$= 2 \cos^2 \frac{A}{2};$$

i.e., 1 plus cos of any angle equals twice cos squared half the angle (memory).

(5) Now
$$1 - \cos A = 1 - \left(1 - 2 \sin^2 \frac{A}{2}\right) \qquad \text{(v. 2}b\text{)}$$
$$= 1 - 1 + 2 \sin^2 \frac{A}{2}$$
$$= 2 \sin^2 \frac{A}{2};$$

i.e., 1 minus cos of any angle equals twice sin squared half the angle (memory).

Class VI.—Values of the Ratios of Certain Angles.

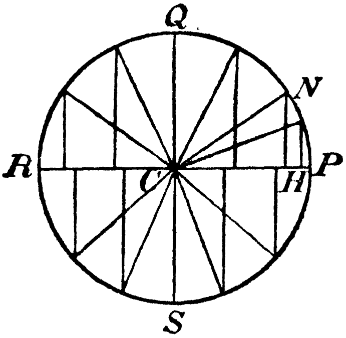

Fig. 17.

Let *PQRS* be a circle, *PR* and *QS* two diameters at right angles.

Let *PCN* be an angle less than 90°. It is evident that :—

(1) **0°.** As the angle decreases the perp. decreases, and the base increases, the hyp. remaining the same, till at 0° the perp. disappears, and the base becomes radius. Hence,

$$\sin 0^\circ = \frac{\text{perp}}{\text{hyp}} = \frac{0}{r} = 0;$$

$$\cos 0^\circ = \frac{\text{base}}{\text{hyp}} = \frac{r}{r} = 1;$$

$$\tan 0^\circ = \frac{\text{perp}}{\text{base}} = \frac{0}{r} = 0.$$

(2) **90°.** As the angle increases from 0° to 90°, so the perp. increases and the base decreases, the hyp. remaining the same, till at 90° the perp. becomes radius and the base disappears. Hence,

$$\sin 90^\circ = \frac{\text{perp}}{\text{hyp}} = \frac{r}{r} = 1;$$

$$\cos 90^\circ = \frac{\text{base}}{\text{hyp}} = \frac{0}{r} = 0;$$

$$\tan 90^\circ = \frac{\text{perp}}{\text{base}} = \frac{r}{0} = \infty \text{ (i.e., infinity).}$$

(3) **180°.** As the angle increases from 90° to 180°, so the perp. decreases and the base increases, the hyp. remaining the same, till at 180° the perp. disappears, and the base becomes radius and is negative. Hence,

$$\sin 180^\circ = \frac{\text{perp}}{\text{hyp}} = \frac{0}{r} = 0;$$

$$\cos 180^\circ = \frac{\text{base}}{\text{hyp}} = \frac{-r}{r} = -1,$$

$$\tan 180^\circ = \frac{\text{perp}}{\text{base}} = \frac{0}{-r} = 0.$$

(4) **270°.** As the angle increases from 180° to 270°, so the perp. increases and the base decreases, the hyp. remaining the same, till at 270° the perp. becomes radius (negative)* and the base disappears. Hence,

$$\sin 270^\circ = \frac{\text{perp}}{\text{hyp}} = \frac{-r}{r} = -1;$$

$$\cos 270^\circ = \frac{\text{base}}{\text{hyp}} = \frac{0}{r} = 0.$$

(5) **360°.** As the angle increases from 270° to 360°, so the perp. decreases and the base increases, the hyp. remaining the same, till at 360° the perp. disappears and the base becomes radius. Hence,

$$\sin 360^\circ = \frac{\text{perp}}{\text{hyp}} = \frac{0}{r} = 0;$$

$$\cos 360^\circ = \frac{\text{base}}{\text{hyp}} = \frac{r}{r} = 1.$$

(See **Ch. iv.** for full results).

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* See Chap. III.

(6) 30° & 60°. Let *PCN* be an equilateral triangle, then each of its angles equals 60° (Euc. I. 32).

Bisect the angle *CNP* by a line cutting the base at *H*, then the angle *CNH* is 30°.

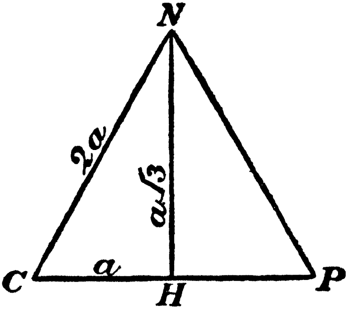

Fig. 18.

Because $CN = PN$, and NH is common, and angle CNH = angle PNH, therefore NH bisects the base at right angles (Euc. I. 4), &c.

Let $CH = a,$

then $CN = 2a;$

and $NH^2 + HC^2 = CN^2,$

i.e., $NH^2 + a^2 = 4a^2,$

$NH^2 = 3a^2,$

∴ $NH = \sqrt{3a^2} = a\sqrt{3}.$

Now $\sin 60° = \sin NCH = \dfrac{NH}{NC} = \dfrac{a\sqrt{3}}{2a} = \dfrac{\sqrt{3}}{2},$

and $\sin 30° = \sin CNH = \dfrac{CH}{CN} = \dfrac{a}{2a} = \dfrac{1}{2}.$

Similarly for any ratio of 60° or 30°.

(7) **45°.** Let HCN be a right-angled isosceles triangle, H the right angle; then $HCN = 45°$ (Euc. I. 32).

Let $CH = a$. then $NH = a$;

and $CN^2 = CH^2 + NH^2$,

i.e., $CN^2 = a^2 + a^2$,

$CN^2 = 2a^2$,

$\therefore$ $CN = a\sqrt{2}$.

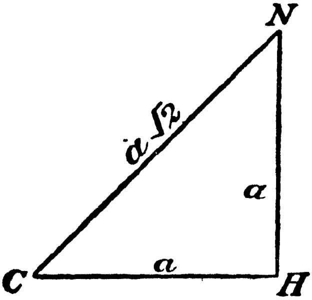

Fig. 19.

Now $$\sin 45° = \sin NCH = \frac{NH}{NC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

and $$\tan 45° = \tan NCH = \frac{NH}{HC} = \frac{a}{a} = 1.$$

Similarly for any ratio of 45°.

(8) Values for ratios of $(90° - A)$, $(90° + A)$, &c., deduced by expansion.*

Thus, $$\begin{aligned} \sin(90° - A) &= \sin 90° \cos A - \cos 90° \sin A && \text{(iv. 4)} \\ &= 1 \times \cos A - 0 \times \sin A && \text{(vi. 2)} \\ &= \cos A. \end{aligned}$$

$$\begin{aligned} \cos(90° + A) &= \cos 90° \cos A - \sin 90° \sin A && \text{(iv. 2)} \\ &= 0 \times \cos A - 1 \times \sin A && \text{(vi. 2)} \\ &= -\sin A. \end{aligned}$$

---

* The student should also prove these identities by *construction*—that is by the *geometrical* method (see Class III.).

$$\begin{aligned}
\sin(180^\circ - A) &= \sin 180^\circ \cos A - \cos 180^\circ \sin A && \text{(iv. 4)}\\
&= 0 \times \cos A - (-1) \times \sin A && \text{(vi. 3)}\\
&= \sin A.
\end{aligned}$$

$$\begin{aligned}
\cos(180^\circ + A) &= \cos 180^\circ \cos A - \sin 180^\circ \sin A && \text{(iv. 2)}\\
&= (-1) \times \cos A - 0 \times \sin A && \text{(vi. 3)}\\
&= -\cos A.
\end{aligned}$$

$$\begin{aligned}
\sin(-A) &= \sin(0^\circ - A)\\
&= \sin 0^\circ \cos A - \cos 0^\circ \sin A && \text{(iv. 4)}\\
&= 0 \cos \times A - 1 \times \sin A && \text{(vi. 1)}\\
&= -\sin A.
\end{aligned}$$

Similarly for values of $(270^\circ - A)$, $(270^\circ + A)$, $(360^\circ - A)$.

The following values are very important:—

$\sin 0^\circ = 0$, $\cos 0^\circ = 1$, $\tan 0^\circ = 0$;

$\sin 90^\circ = 1$, $\cos 90^\circ = 0$, $\cot 90^\circ = 0$;

$\sin(90^\circ - A) = \cos A$, $\cos(90^\circ - A) = \sin A$, $\cot(90^\circ - A) = \tan A$;

$\sin(180^\circ - A) = \sin A$, $\cos(180^\circ - A) = -\cos A$;

$\tan(180^\circ - A) = -\tan A$;

$\sin(-A) = -\sin A$, $\cos(-A) = \cos A$, $\tan(-A) = -\tan A$.

Class VII.—Rules for Solution of Plane Triangles.

(1) Proof of "Rule of Sines."

(a) When the angles are acute.

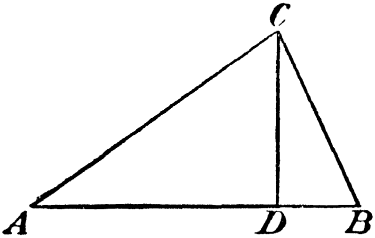

Fig. 20.

Let ABC be a triangle. From C draw CD perp. to AB.

Now $$\frac{\sin A}{\sin B} = \frac{\frac{CD}{CA}}{\frac{CD}{CB}} = \frac{CD}{CA} \times \frac{CB}{CD} = \frac{CB}{CA} = \frac{a}{b}.$$

(b) When one of the angles is obtuse (viz. B),

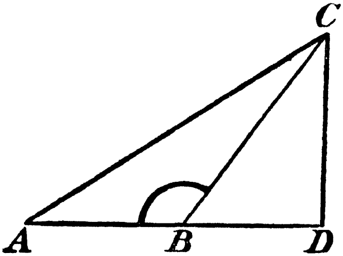

Fig. 21.

Now $$\frac{\sin A}{\sin B} = \frac{\sin A}{\sin(180^\circ - B)} = \frac{\sin A}{\sin CBD} = \frac{\frac{CD}{CA}}{\frac{CD}{CB}} = \frac{CD}{CA} \times \frac{CB}{CD}$$

$$= \frac{CB}{CA} = \frac{a}{b}. \qquad \text{(vi. 8)}$$

This may be stated thus, $a : b :: \sin A : \sin B$, &c.; or, as one side of a triangle is to another, so is the sin of the angle subtended by the former side to the sin of the angle subtended by the latter.

(2) Proof of "Rule of Tangents."

Now $$\frac{a}{b} = \frac{\sin A}{\sin B}, \qquad \text{(vii. 1)}$$

hence, $\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1$ and $\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1$;

i.e., $\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B}$ and $\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}$.

By division $$\frac{\frac{a+b}{b}}{\frac{a-b}{b}} = \frac{\frac{\sin A + \sin B}{\sin B}}{\frac{\sin A - \sin B}{\sin B}};$$

or $$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B},$$

$$\therefore \quad \frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \qquad \text{(iv. 13)}$$

or $$a+b : a-b :: \tan \tfrac{1}{2}(A+B) : \tan \tfrac{1}{2}(A-B).$$

This may be stated thus:—The sum of two sides is to their difference as the tan of half the sum of the subtending angles is to the tan of half the difference of those angles.

(3a) Proofs of "Rule of Cosines."—To find a value for cos of an angle of a triangle in terms of the sides when the angle is acute.

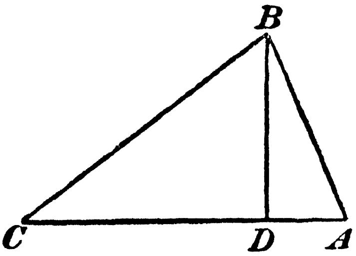

Fig. 22.

Let ABC be a triangle having A acute. From B draw BD perp. to AC.

Then $BC^2 = CA^2 + AB^2 - 2CA \cdot AD$, (Euc. II. 13)

and $AD = AB \cos A$;

hence, $BC^2 = CA^2 + AB^2 - 2CA \cdot AB \cos A$;

i.e., $a^2 = b^2 + c^2 - 2bc \cos A$; $(CA = b,\ AB = c)$

$$2bc \cos A = b^2 + c^2 - a^2;$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

This may be stated thus :—The cos of an angle equals the quotient produced by dividing the sum of the squares of the containing sides minus the square of the subtending side by twice the product of the containing sides.

N.B.—This formula (3a & 3b) is not adapted to logarithmic calculation.

(3b) To find the value for the cos of an angle of a triangle in terms of the sides when the angle is obtuse (Fig. 23).

Let ABC be an obtuse-angled triangle, A the obtuse angle. From B draw BD perp. to CA produced. Then

$$BC^2 = CA^2 + AB^2 + 2CA \cdot AD. \quad \text{(Euc. ii. 12)}$$

Now $AD = BA \cos BAD$,

i.e., $AD = BA\ (-\cos A)$ (vi. 8)

hence, $BC^2 = CA^2 + AB^2 + 2CA \, . \, BA \, (-\cos A)$,

or, removing bracket,

$$BC^2 = CA^2 + AB^2 - 2CA \, . \, BA \, . \cos A;$$

i.e., $a^2 = b^2 + c^2 - 2bc \cos A$,*

$$2bc \cos A = b^2 + c^2 - a^2,$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

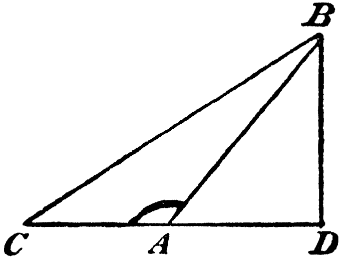

Fig. 23.

(4a) "Rule of Cosines" (logarithmic formula).

Reduce the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ to a logarithmic form.}$$

Now $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, (vii. 3a, b)

hence, $1 + \cos A = 1 + \dfrac{b^2 + c^2 - a^2}{2bc}$,

or $2 \cos^2 \dfrac{A}{2} = \dfrac{2bc + b^2 + c^2 - a^2}{2bc}$, (v. 4)

$$2 \cos^2 \frac{A}{2} = \frac{(b + c)^2 - a^2}{2bc},$$

$$\therefore \quad 2 \cos^2 \frac{A}{2} = \frac{(b + c + a) \, (b + c - a)}{2bc}.$$

---

* This formula (the same as in 3a) should be noticed.

Let $b + c + a = 2s,$*

then $b + c - a = 2s - 2a.$

$$2\cos^2\frac{A}{2} = \frac{2s \cdot 2\,(s-a)}{2bc},$$

$$\cos^2\frac{A}{2} = \frac{s\,(s-a)}{bc},$$

and $$\cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}}.$$

(4b) The sine of half an angle of a triangle in terms of the sides.

Reduce the formula $\cos A = \frac{b^2+c^2-a^2}{2bc}$ to a logarithmic form.

Now $$\cos A = \frac{b^2+c^2-a^2}{2bc} \qquad \textbf{(vii. 3a, b)}$$

hence, $$1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc},$$

or $$2\sin^2\frac{A}{2} = \frac{2bc - b^2 - c^2 + a^2}{2bc}, \qquad \textbf{(v. 5)}$$

$$2\sin^2\frac{A}{2} = \frac{a^2 - (b^2+c^2-2bc)}{2bc},$$

$$2\sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{2bc},$$

∴ $$2\sin^2\frac{A}{2} = \frac{(a-b+c)\,(a+b-c)}{2bc}.$$

Let $a + b + c = 2s,$

then $a - b + c = 2s - 2b,$

and $a + b - c = 2s - 2c.$

$$2\sin^2\frac{A}{2} = \frac{2\,(s-b)\;2\,(s-c)}{2bc},$$

$$\sin^2\frac{A}{2} = \frac{(s-b)\,(s-c)}{bc},$$

and $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}$$

---

* s, semi-perimeter.

(4c) The tangent of half an angle of a triangle in terms of the sides.

By combining the last two results we get

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)\,(s-c)}{bc}}}{\sqrt{\frac{s\,(s-a)}{bc}}}=\sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}\qquad\begin{array}{r}\text{(ii. 4)}\\ \text{(vii. 4a, 4b)}\end{array}$$

(5) **Find the area of a plane triangle.**

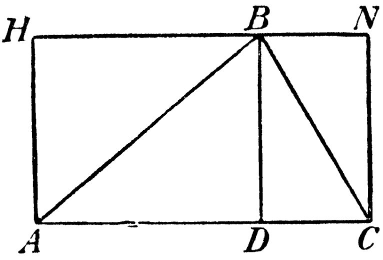

Fig. 24.

Let ABC be a plane triangle. From B draw BD perp. to AC, and complete the rectangle $HBNCA$.

Now area of rectangle $HC = AC \times BD$

$$\therefore\quad \text{area of triangle } ABC = \tfrac{1}{2}\,AC \times BD \quad . \quad . \quad . \quad . \quad (1)$$

(Euc. I. 41)

$$\text{and } BD = BA \sin A$$

$$\therefore\quad \text{area of triangle } ABC = \tfrac{1}{2}\,AC\,.\,BA \sin A$$

$$= \tfrac{1}{2}bc \sin A \quad . \quad . \quad . \quad . \quad (2)$$

$$= \tfrac{1}{2}bc\ .\ 2\sin\frac{A}{2}\cos\frac{A}{2} \qquad \text{(v. 1)}$$

$$= bc\,.\sqrt{\frac{(s-b)\,(s-c)}{bc}}\,.\sqrt{\frac{s(s-a)}{bc}}\,; \qquad \text{(vii. 4b, a)}$$

$$\therefore\qquad \text{area} = \sqrt{s(s-a)\,(s-b)\,(s-c)} \qquad (3)$$

Hence,

(1) area $=$ half one side multiplied by perp. from opposite angle.

(2) area $=$ half product of two sides into the sin of the contained angle.

(3) area $= \sqrt{s(s-a)\,(s-b)\,(s-c)}$.

(6) Obtain a formula for finding the third side of a plane triangle direct from two sides and included angle.

Now $a^2 = b^2 + c^2 - 2bc \cos A$, (vii. 3a, b)

$$a^2 = b^2\left(\cos^2\frac{A}{2} + \sin^2\frac{A}{2}\right) + c^2\left(\cos^2\frac{A}{2} + \sin^2\frac{A}{2}\right)$$

$$- 2bc\left(\cos^2\frac{A}{2} - \sin^2\frac{A}{2}\right) \qquad \begin{matrix}\text{(i. 1)}\\ \text{(v. 2a)}\end{matrix}$$

$$= b^2\cos^2\frac{A}{2} + b^2\sin^2\frac{A}{2} + c^2\cos^2\frac{A}{2} + c^2\sin^2\frac{A}{2}$$

$$- 2bc\cos^2\frac{A}{2} + 2\,b\,c\sin^2\frac{A}{2}$$

$$= \cos^2\frac{A}{2}\,(b^2 + c^2 - 2bc) + \sin^2\frac{A}{2}\,(b^2 + c^2 + 2bc);$$

$$\therefore \quad a^2 = \cos^2\frac{A}{2}\,(b-c)^2 + \sin^2\frac{A}{2}\,(b+c)^2.$$

Divide and multiply by the first term, $\cos^2\frac{A}{2}\,(b-c)^2$.

$$\therefore \quad a^2 = \cos^2\frac{A}{2}(b-c)^2\left\{1 + \frac{\sin^2\frac{A}{2}\,(b+c)^2}{\cos^2\frac{A}{2}\,(b-c)^2}\right\}$$

$$= \cos^2\frac{A}{2}\,(b-c)^2\left\{1 + \frac{\tan^2\frac{A}{2}\,(b+c)^2}{(b-c)^2}\right\} \qquad \text{(ii. 4)}$$

$$\text{Let } \tan^2\theta = -\frac{\tan^2\frac{A}{2}\,(b+c)^2}{(b-c)^2}.$$

Then $$a^2 = \cos^2\frac{A}{2}\,(b-c)^2\,\{1 + \tan^2\theta\}$$

$$a^2 = \cos^2\frac{A}{2}\,(b-c)^2 \sec^2\theta \qquad \text{(i. 2)}$$

$$\therefore \quad a = \cos\frac{A}{2}\,(b-c)\sec\theta,$$

and $$\tan\theta = \frac{\tan\frac{A}{2}\,(b+c)}{(b-c)}$$

Divide and multiply by the second term, $\sin^2\frac{A}{2}\,(b+c)^2$,

and the formulæ become as on p. 25—

$$\tan\theta = \frac{(b+c)\tan\frac{A}{2}}{(b-c)} \text{ and } a = (b+c)\sin\frac{A}{2}\operatorname{cosec}\theta.$$

(7) $\sin^2 A - \sin^2 B$

$$= (\sin A + \sin B)\,(\sin A - \sin B)$$

$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\;2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad \text{(iv. 7, 8)}$$

$$= 2\sin\frac{A+B}{2}\cos\frac{A+B}{2}\;2\sin\frac{A-B}{2}\cos\frac{A-B}{2}$$

$$= \sin(A+B)\,\sin(A-B) \qquad \text{(v. 1)}$$

CHAPTER XI.

THEORY OF SPHERICAL TRIGONOMETRY.

I. **Spherical Trigonometry** treats of the relations which exist between the parts of a solid angle formed by the inclination of three planes, or (supposing that the angular point is at the centre of a sphere) between the sides and angles of the triangle on the surface of the sphere enclosed by the three planes.

A **Sphere** is a solid body bounded by a curved surface every point on which is equidistant from an interior point called the centre of the sphere.

A **Solid Angle** is formed by the meeting at a point of more than two plane angles which are not in the same plane.

A **Great Circle** is formed on a sphere by a plane passing through the centre of that sphere.

A **Spherical Triangle** is the triangular portion of the surface of a sphere enclosed by the arcs of three great circles.

The **angles of a spherical triangle** are formed by the great circles at the points of intersection.

The **Poles of a Circle** are the extremities of that diameter of the sphere which is perpendicular to the plane of the circle.

Of the three plane angles which form a solid angle :—

1. Any one angle is less than two right angles. (Convention.)
2. Any one angle is less than the sum of the other two angles. (Euc. xi. 20).
3. The sum of the three angles is less than four right angles. (Euc. xi. 21).

Now the three arcs forming the sides of the spherical triangle are the measures of the three plane angles forming the solid angle, therefore

1. Any one side of a spherical triangle is less than two right angles.
2. Any one side of a spherical triangle is less than the sum of the other two sides.
3. The sum of the three sides of a spherical triangle is less than four right angles.

II. The boundary of every plane section of a sphere is a circle.

In the sphere *ABDH*—

1. Let the cutting plane *AEB* pass through *C*, the sphere's centre. Then *AEB* shall be a circle.

By definition all straight lines drawn from *C* to the sphere's surface are equal, therefore *AEB* is a circle.

2. Let the cutting plane not pass through the centre, as *HVD*.

Draw *CQ* perpendicular to the plane *HVD*, meeting *HVD* in the point *Q*.

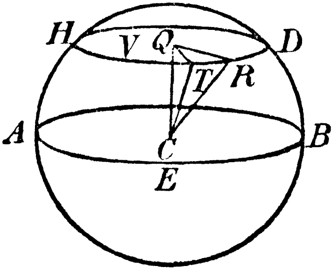

Fig. 25.

Take points *T* and *R* on the curve *HVD*. Join *QT*, *QR*, *CT*, and *CR*.

Because *CQ* is perpendicular to the plane *HVD*, therefore *CQ* is perpendicular to every line drawn in that plane. Hence each of the angles *CQT* and *CQR* is a right angle.

Now $CT^2 = CQ^2 + QT^2$, and $CR^2 = CQ^2 + QR^2$;

but $CT = CR$ (radii of sphere);

therefore $CT^2 = CR^2$;

therefore $CQ^2 + QT^2 = CQ^2 + QR^2$;

therefore $QT^2 = QR^2$, or $QT = QR$.

Similarly it can be shown that all straight lines drawn from *Q* to the curve *HVD* are equal.

Therefore *HVD* is a circle.

III. The Pole of a Great Circle is 90° from every point on the Great Circle.

In Fig. 26 :—

Let *P* be the pole of the great circle *DE*. Take any points *A* and *B* on the great circle ; join these points with *P* by arcs of great circles. Let *C* be the sphere's centre. Join *CA*, *CB*, and *CP*.

Because *P* is the pole of *DE*; therefore *PC* is perpendicular to the plane *ADBE* ; therefore *PCA* is a right angle ; therefore *PA* is 90° (Def).

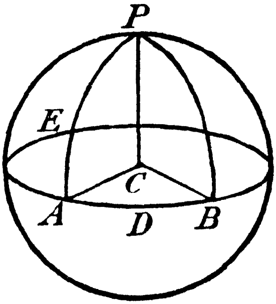

Fig. 26.

For the same reason *PCB* is a right angle ; therefore *PB* is 90°.

Hence the pole of a great circle is 90° from every point on the great circle.

IV. An Angle of a sph. triangle is measured by the inclination of the planes of the great circles forming the angle.

Let *PA* and *PR* the arcs of two great circles meet at the point *P* (Fig. 27).

Then *PCN* is the common section of the two great circles. At *P* draw tangents *PB* and *PV* to the arcs *PA* and *PR*.

Now the angle *VPB* is the inclination of the planes of the great circles *PAN* and *PRN*, since the lines *PB* and *PV* are at right angles to their common section *PCN* (Euc. xi. Def. 6).

But the arcs *PR* and *PA* are also at right angles to *PCN* at the point *P* (Euc. III. 16) ; hence the arcs *PA* and *PR* coincide with the tangents *PB* and *PV* at the point *P*, and therefore the angle *APR* is equal to the rectilineal angle *VPB*. Now the angle *VPB* is the inclination of the planes of the great circles forming the angle *APR*. Therefore, &c.

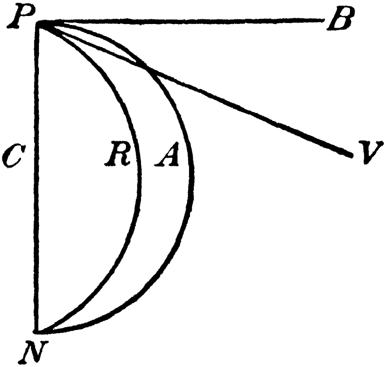

Fig. 27.

V. The angle at the Pole of a Great Circle is measured by the arc of the Great Circle which subtends that angle (Fig. 28).

Let *P* be the pole of a great circle. Draw *PB* and *PA* arcs

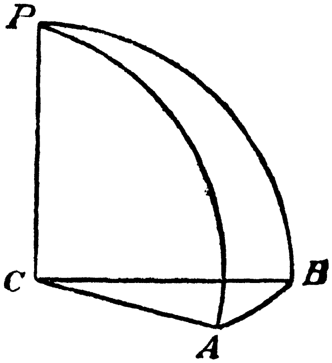

Fig. 28.

of two great circles, cutting the great circle in the points *B* and *A*. Take *C* the centre of the sphere. Join *CA*, *CB*, and *CP*.

Because *AC* and *BC* are perpendicular to *PC*, the common section of the planes *PBC* and *PAC*, therefore the angle *ACB* measures the inclination of the planes; hence the angle *ACB* is equal to the angle *APB*; but since *AB* measures the angle *ACB*, therefore *AB* measures the angle *APB*.

VI. The greater side of a spherical triangle is opposite to the greater angle.

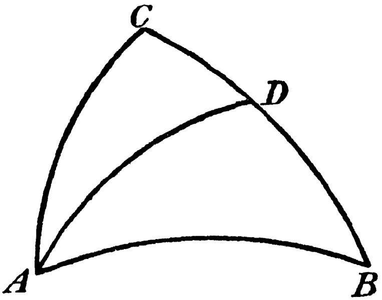

Fig. 29.

Let *ABC* be a spherical triangle, having the angle *CAB* greater than the angle *CBA*. Then *CB* shall be greater than *CA*. Make the angle *BAD* equal to the angle *CBA*. Because the angle *BAD* is equal to the angle *DBA*, therefore *DA* is equal to *DB*. Add *DC* to each of the equal sides; therefore *DA* and *DC* are equal to *DB* and *DC*. But *DA* and *DC* are greater than *AC*, therefore *DB* and *DC* are greater than *AC*. Hence *BC* is greater than *AC*.

VII. Find an expression for the cosine of an angle of a spherical triangle in terms of the sides of the triangle.

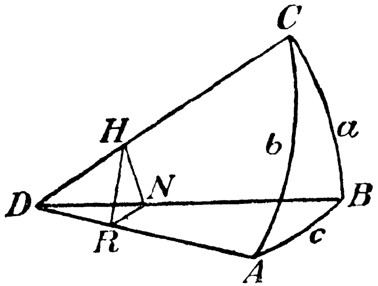

Fig. 30.

Let *ABC* be a spherical triangle formed by the intersection of the arcs of three great circles, whose planes CDB, CDA, and BDA form a solid angle at D the centre of a sphere.

In DC the common section of two planes take a point H. From H, in the plane DCA, draw HR perpendicular to DA. From R, in the plane BDA, draw RN at right angles to DA. Join HN.

$$\begin{array}{llllll} \text{Now } HN^2 = & HD^2 & + & DN^2 & -2HD\,.\,DN\,.\cos HDN & \text{(Pl.*vii. 3a)} \\ \text{and } HN^2 = & HR^2 & + & RN^2 & -2HR\,.\,RN\,.\,\text{co } HRN & \text{subtracting} \end{array}$$

$$0 = HD^2 - HR^2 + DN^2 - RN^2 - 2HD\,.\,DN\,.\cos HDN + 2HR\,.\,RN\,.\cos HRN$$

$$\therefore\ 0 = DR^2 + DR^2 - 2HD\,.\,DN\,.\cos HDN + 2HR\,.\,RN\,.\cos HRN. \qquad \text{(Euc. I. 47)}$$

But HDN measures a, and HRN measures A,

$$\therefore\quad \cancel{2}HD\,.\,DN\,.\cos a = \cancel{2}DR^2 + \cancel{2}HR\,.\,RN\,.\cos A;$$

$$HD\,.\,DN\,.\cos a = DR^2 + HR\,.\,RN\,.\cos A.$$

$$\cos a = \frac{DR\,.\,DR}{HD\,.\,DN} + \frac{HR\,.\,RN}{HD\,.\,DN}\,.\cos A;$$

$$\text{or}\qquad \cos a = \cos b\,.\cos c + \sin b\,.\sin c\,.\cos A;\ \dagger$$

$$\therefore\quad \sin b\,.\sin c\,.\cos A = \cos a - \cos b\,.\cos c$$

$$\text{and } \cos A = \frac{\cos a - \cos b\,.\cos c}{\sin b\,.\sin c}.$$

VIII. Given the sides a, b, c of the spherical triangle ABC, find logarithmic formulæ for computing an angle (A) of the triangle.

$$\text{Now}\qquad \cos A = \frac{\cos a - \cos b\,.\cos c}{\sin b\,.\sin c} \qquad \text{(Sp. vii.)}$$

$$\begin{aligned} \therefore\quad \cos A + 1 &= \frac{\cos a - \cos b\,.\cos c}{\sin b\,.\sin c} + 1 \\ &= \frac{\cos a - \cos b\,.\cos c + \sin b\,.\sin c}{\sin b\,.\sin c} \\ &= \frac{\cos a - (\cos b\,.\cos c - \sin b\,.\sin c)}{\sin b\,.\sin c} \\ &= \frac{\cos a - \cos(b + c)}{\sin b\,.\sin c} \qquad \text{(Pl. iv. 2)} \end{aligned}$$

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* Proofs of Plane Trigonometry.

† N.B.—This may be termed the Fundamental formula in spherical trigonometry.

$$\therefore \cancel{2}\cos^2\frac{A}{2} = \cancel{2}\sin\tfrac{1}{2}(b+c+a)\,.\,\sin\tfrac{1}{2}(b+c-a)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,.\,c.$$

(Pl. v. 4, iv. 10, ii. 1)

Let $\frac{1}{2}(b+c+a)=s$; $\therefore$ $\frac{1}{2}(b+c-a)=s-a$.

$$\therefore \quad \cos^2\frac{A}{2} = \sin s\,.\,\sin(s-a)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c.$$

(1) $\therefore$ $$\cos\frac{A}{2} = \sqrt{\sin s\,.\,\sin(s-a)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c.}$$

Again, $$\cos A = \frac{\cos a - \cos b\,.\,\cos c}{\sin b\,.\,\sin c} \qquad \text{(Sp. vii.)}$$

$$\therefore \quad 1-\cos A = 1 - \left\{\frac{\cos a - \cos b\,.\,\cos c}{\sin b\,.\,\sin c}\right\}$$

$$= \frac{\sin b\,.\,\sin c - \cos a + \cos b\,.\,\cos c}{\sin b\,.\,\sin c}$$

$$= \frac{\cos b\,.\,\cos c + \sin b\,.\,\sin c - \cos a}{\sin b\,.\,\sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b\,.\,\sin c} \qquad \text{(Pl. iv. 5)}$$

$$\therefore \cancel{2}\sin^2\frac{A}{2} = \cancel{2}\sin\tfrac{1}{2}(a+b-c)\,.\,\sin\tfrac{1}{2}(a-b+c)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c.$$

(Pl. v. 5, iv. 10, ii. 1)

Let $\frac{1}{2}(a+b+c)=s$; $\therefore$ $\frac{1}{2}(a+b-c)=s-c$; $\frac{1}{2}(a-b+c)=s-b$.

Then $$\sin^2\frac{A}{2} = \sin(s-c)\,.\,\sin(s-b)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c.$$

(2) and $$\sin\frac{A}{2} = \sqrt{\sin(s-c)\,.\,\sin(s-b)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c.}$$

Now $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} \qquad \text{(Pl. ii. 3)}$$

$$= \frac{\sqrt{\sin(s-c)\,.\,\sin(s-b)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c.}}{\sqrt{\sin s\,.\,\sin(s-a)\,.\,\mathrm{cosec}\,b\,.\,\mathrm{cosec}\,c}}$$

(Sp. viii. 2); (Sp. viii. 1)

$$(3)\ \therefore \qquad \tan\frac{A}{2}=\sqrt{\frac{\sin(s-c)\,.\,\sin(s-b)}{\sin s\,.\,\sin(s-a)}}.$$

IX. To prove in a spherical triangle *ABC*,

$$\sin A : \sin B : : \sin a : : \sin b.$$

Now $\sin A=2\,.\,\sin\frac{A}{2}\,.\,\cos\frac{A}{2}$ (Pl. v. 1)

$$=2\sqrt{\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin b\,.\,\sin c}}\times\sqrt{\frac{\sin s\,.\,\sin(s-a)}{\sin b\,.\,\sin c}}$$

(Sp. viii. 2, 1)

$$=2\frac{\sqrt{\sin(s-b)\,.\,\sin(s-c)\,.\,\sin s\,.\,\sin(s-a)}}{\sin b\,.\,\sin c}\quad .\quad (1)$$

also $\sin B=2\dfrac{\sqrt{\sin(s-a)\,.\,\sin(s-c)\,.\,\sin s\,.\,\sin(s-b)}}{\sin a\,.\,\sin c}$. (2)

Let the Numerator in 1 and $2=N$.

Then $\dfrac{\sin A}{\sin B}=\dfrac{N}{\sin b\,.\,\sin c}\times\dfrac{\sin a\,.\,\sin c}{N}=\dfrac{\sin a}{\sin b}$

$\therefore$ $\sin A : \sin B : : \sin a : \sin b.$

X. In the spherical triangle *ABC*, given *a*, *b*, *C*; prove the formula used in computing *c* (direct).

$$\cos c=\cos a\,.\,\cos b+\sin a\,.\,\sin b\,.\,\cos C \qquad \text{(Sp. viii.)}$$

$$=\cos a\,.\,\cos b+\sin a\,.\,\sin b\,.\,\left(2\cos^2\frac{C}{2}-1\right)$$

(Pl. v. 2c)

$$=\cos a\,.\,\cos b-\sin a\,.\,\sin b+\sin a\,.\,\sin b\,.\,2\cos^2\frac{C}{2}$$

$$=\cos(a+b)+\sin a\,.\,\sin b\,.\,2\cos^2\frac{C}{2} \qquad \text{(Pl. iv. 2)}$$

$$\therefore\quad 1-\cos c=1-\cos(a+b)-\sin a\,.\,\sin b\,.\,2\cos^2\frac{C}{2}$$

and $\cancel{2}\sin^2\frac{c}{2}=2\sin^2\left(\frac{a+b}{2}\right)-\sin a\,.\,\sin b\,.\,\cancel{2}\cos^2\frac{C}{2}$ (Pl. v. 5)

Let $\sin a\,.\,\sin b\,.\,\cos^2\frac{C}{2}=\sin^2\theta.$

Then $$\sin^2\frac{c}{2} = \sin^2\left(\frac{a+b}{2}\right) - \sin^2\theta$$

$$= \sin\left(\frac{a+b}{2} + \theta\right) . \sin\left(\frac{a+b}{2} - \theta\right) \qquad \text{(Pl. vii. 7)}$$

and $$\sin\frac{c}{2} = \sqrt{\sin\left(\frac{a+b}{2} + \theta\right) . \sin\left(\frac{a+b}{2} - \theta\right)}^{\,*}$$

XI. In the spherical triangle ABC, find a formula which connects four successive parts of the triangle; or prove

$$\cot A . \sin B = \cot a . \sin c - \cos c . \cos B.$$

Now $$\cos A = \frac{\cos a - \cos b . \cos c}{\sin b . \sin c}; \qquad \text{(Sp. vii.)}$$

$$\therefore \quad \sin b . \sin c . \cos A = \cos a - \cos b . \cos c;$$

but $$\frac{\sin b}{\sin a} = \frac{\sin B}{\sin A}$$

or $$\sin b = \frac{\sin B . \sin a}{\sin A} \qquad \text{(Sp. ix.)}$$

$$\therefore \quad \frac{\sin B . \sin a . \sin c . \cos A}{\sin A} = \cos a - \cos c\,(\cos a . \cos c + \sin a . \sin c . \cos B) \qquad \text{(Sp. vii.)}$$

$$\sin B . \sin a . \sin c . \cot A = \cos a - \cos a . \cos^2 c - \sin a . \sin c . \cos c . \cos B; \qquad \text{(Pl. ii. 5)}$$

$$\sin B . \sin a . \sin c . \cot A = \cos a\,(1 - \cos^2 c) - \sin a . \sin c . \cos c . \cos B;$$

$$\sin B . \sin a . \sin c . \cot A = \cos a . \sin^2 c - \sin a . \sin c . \cos c . \cos B; \qquad \text{(Pl. i. 1)}$$

$$\therefore \quad \sin B . \cot A = \cot a . \sin c - \cos c . \cos B. \qquad \text{(Pl. ii. 5)}$$

XII. Given a, b, C of the spherical triangle ABC, prove the formulæ for computing the other angles of the triangle, viz.:—

(1) $$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}} . \cot\frac{C}{2}.$$

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[*] Versine formula, see p. 105.

(2) $$\tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cdot\cot\frac{C}{2}.$$

(α) $$\cos A=\frac{\cos a-\cos b\cdot\cos c}{\sin b\cdot\sin c};\qquad\text{(Sp. vii.)}$$

$$\therefore\quad \sin b\cdot\sin c\cdot\cos A=\cos a-\cos b\cdot\cos c$$

$$=\cos a-\cos b\,(\cos a\cdot\cos b+\sin a\cdot\sin b\cdot\cos C)\qquad\text{(Sp. vii.)}$$

$$=\cos a-\cos a\cdot\cos^2 b-\sin a\cdot\sin b\cdot\cos b\cdot\cos C$$

$$=\cos a\,(1-\cos^2 b)-\sin a\cdot\sin b\cdot\cos b\cdot\cos C$$

$$=\cos a\cdot\sin^2 b-\sin a\cdot\sin b\cdot\cos b\cdot\cos C.\qquad\text{(Pl. i. 1)}$$

$$\therefore\quad \sin c\cdot\cos A=\cos a\cdot\sin b-\sin a\cdot\cos b\cdot\cos C,$$

also $$\underline{\sin c\cdot\cos B=\cos b\cdot\sin a-\sin b\cdot\cos a\cdot\cos C.}$$

$$\therefore\quad \sin c\,(\cos A+\cos B)=\sin a\cdot\cos b+\cos a\cdot\sin b-\sin a\cdot\cos b\cdot\cos C-\sin b\cdot\cos a\cdot\cos C$$

$$=\sin a\cdot\cos b+\cos a\cdot\sin b-(\sin a\cdot\cos b+\sin b\cdot\cos a)\cdot\cos C$$

$$=\sin(a+b)-\sin(a+b)\cdot\cos C\qquad\text{(Pl. iv. 1)}$$

$$=\sin(a+b)\,(1-\cos C)$$

$$=\sin(a+b)\cdot 2\sin^2\frac{C}{2}.\qquad\text{(Pl. v. 5)}$$

$$\therefore\quad \cos A+\cos B=\frac{\sin(a+b)\cdot 2\sin^2\frac{C}{2}}{\sin c}.$$

(β) $\quad \dfrac{\sin A}{\sin C} = \dfrac{\sin a}{\sin c}; \qquad \therefore \quad \sin A = \dfrac{\sin a \,.\, \sin C}{\sin c}.$ (Sp. ix.)

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}; \qquad \therefore \quad \sin B = \frac{\sin b \,.\, \sin C}{\sin c}.$$

$\therefore$ $$\sin A + \sin B = \frac{(\sin a + \sin b) \,.\, \sin C}{\sin c},$$

and $$\sin A - \sin B = \frac{(\sin a - \sin b) \,.\, \sin C}{\sin c}.$$

$\left(\frac{\beta}{\alpha}\right)$ Now $\tan \dfrac{A+B}{2}$

$$= \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{(\sin a + \sin b) \,.\, \sin C}{\sin (a+b) \,.\, 2 \sin^2 \frac{C}{2}}$$

$$= \frac{2 \sin \frac{1}{2}(a+b) \,.\, \cos \frac{1}{2}(a-b) \,.\, 2 \sin \frac{1}{2} C \,.\, \cos \frac{1}{2} C}{2 \sin \frac{1}{2}(a+b) \,.\, \cos \frac{1}{2}(a+b) \,.\, 2 \sin \frac{1}{2} C \,.\, \sin \frac{1}{2} C} \quad \begin{matrix}\text{(Pl. iv. 7, v. 1)} \\ \text{(Pl. v. 1)}\end{matrix}$$

$$= \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \,.\, \cot \tfrac{1}{2} C \quad \ldots\ldots\ldots \quad (1); \qquad \text{(Pl. ii. 5)}$$

also $\tan \dfrac{A-B}{2}$

$$= \frac{\sin A - \sin B}{\cos A + \cos B} = \frac{(\sin a - \sin b) \,.\, \sin C}{\sin (a+b) \,.\, 2 \sin^2 \frac{1}{2} C}$$

$$= \frac{2 \cos \frac{1}{2}(a+b) \,.\, \sin \frac{1}{2}(a-b) \,.\, 2 \sin \frac{1}{2} C \,.\, \cos \frac{1}{2} C}{2 \cos \frac{1}{2}(a+b) \,.\, \sin \frac{1}{2}(a+b) \,.\, 2 \sin \frac{1}{2} C \,.\, \sin \frac{1}{2} C} \quad \begin{matrix}\text{(Pl. iv. 8, v. 1)} \\ \text{(Pl. v. 1)}\end{matrix}$$

$$= \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C \quad \ldots\ldots\ldots \quad (2); \qquad \text{(Pl. ii. 5)}$$

Equations (1) and (2) are known as **Napier's 1st and 2nd Analogies.**

XIII. To prove that the angles at the base of an isosceles spherical triangle are equal.

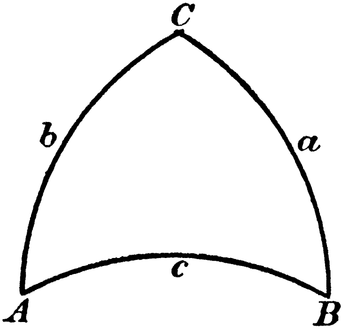

Fig. 31.

Let ABC be an isosceles spherical triangle, having CA equal to CB. Then shall the angle CBA be equal to the angle CAB.

Now $$\cos A = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c},$$

and $$\cos B = \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \sin c}. \qquad \text{(Sp. vii.)}$$

But $$b = a, \quad \therefore \quad \cos B = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}.$$

Hence $\cos A = \cos B$; therefore $A = B$.

XIV. Enunciate and prove the properties of the Polar or Supplemental triangle.

Let ABC be a spherical triangle. Take each of the angular points, A, B, C, as a centre, and with a radius of 90° describe three arcs which intersect at the points A', B', C'. Then $A'B'C'$ is called the Polar or Supplemental triangle, and ABC is called the Primitive triangle.

$A'B'C'$ is called the Polar triangle because each of its angular points is the pole of the side opposite to it in the Primitive triangle.

$A'B'C'$ is called the Supplemental triangle, because each of its sides and angles is the Supplement of the angle and side opposite to it in the Primitive triangle.

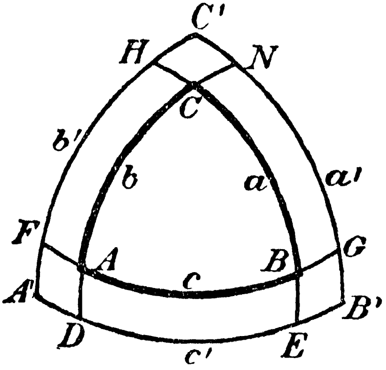

Fig. 32.

Produce AB, BC, and CA both ways, meeting the sides of the triangle $A'B'C'$ in the points F, G, H, E, N, and D.

(1) **To prove** A' **is the Pole of** a.

Because C is the pole of $A'B'$, therefore $CA' = 90°$. (Sp. iii.)

Because B is the pole of $A'C'$, therefore $BA' = 90°$.

Since therefore from A' two arcs, each $90°$, are drawn to CB, A' is the pole of a.

(2) **To prove** $\angle B' = 180° - b$.

$$\text{Now } \angle B' = DN = DC + CN = DC + AN - AC = 90° + 90° - b$$
$$= 180° - b.$$

(3) **To prove** $c' = 180° - C$.

$$\text{Now } c' = A'B' = A'E + EB' = A'E + DB' - DE = 90° + 90° - C$$
$$= 180° - C.$$

XV. Find an expression for the cosine of a side in a spherical triangle in terms of the angles of the triangle.

$$\text{Now} \qquad \cos A' = \frac{\cos a' - \cos b' \cdot \cos c'}{\sin b' \cdot \sin c'}; \qquad \text{(Sp. vii.)}$$

$$\therefore \cos(180^\circ - a) = \frac{\cos(180^\circ - A) \quad \cos(180^\circ - B) . \cos(180^\circ - C)}{\sin(180^\circ - B) . \sin(180^\circ - C)},$$

hence, $$-\cos a = \frac{-\cos A - (-\cos B)(-\cos C)}{\sin B . \sin C}, \qquad \text{(Pl. iii. 3)}$$

and $$-\cos a = \frac{-\cos A - \cos B . \cos C}{\sin B . \sin C};$$

$$\therefore \qquad \cos a = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C}.$$

XVI. **Given the angles, A, B, C, of a spherical triangle; prove a logarithmic formula for computing a.**

$$\cos a = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C}. \qquad \text{(Sp. xv.)}$$

$$\cos a + 1 = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C} + 1,$$

$$= \frac{\cos A + \cos B . \cos C + \sin B . \sin C}{\sin B . \sin C},$$

$$= \frac{\cos A + \cos(B - C)}{\sin B . \sin C}; \qquad \text{(Pl. iv. 5)}$$

$$2 \cos^2 \frac{a}{2} = \cancel{2} \cos \tfrac{1}{2}(A + B - C) . \cos \tfrac{1}{2}(A - B + C) .$$
$$\text{cosec } B . \text{cosec } C. \qquad \text{(Pl. v. 4, iv. 9, ii. 1)}$$

Let $\frac{1}{2}(A + B + C) = M$; then $\frac{1}{2}(A + B - C) = M - C$; and $\frac{1}{2}(A - B + C) = M - B$.

Then $$\cos^2 \frac{a}{2} = \cos(\text{M} - C) . \cos(M - B) . \text{cosec } B . \text{cosec } C,$$

and $$\cos \frac{a}{2} = \sqrt{\cos(M - C) . \cos(M - B) . \text{cosec } B . \text{cosec } C.}$$

XVII. **Given A, B, and c of the spherical triangle, ABC; prove the formulæ for computing a and b, viz. :—**

(3) $$\tan \frac{a + b}{2} = \frac{\cos \frac{A - B}{2}}{\cos \frac{A + B}{2}} . \tan \frac{c}{2};$$

(4) $$\tan\frac{a-b}{2}=\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\,.\tan\frac{c}{2}.$$

Now $$\tan\frac{A'+B'}{2}=\frac{\cos\frac{a'-b'}{2}}{\cos\frac{a'+b'}{2}}\cot\frac{C'}{2};\qquad\text{(Sp. xii.)}$$

$$\therefore\quad \tan\left\{\frac{180^\circ-a+180^\circ-b}{2}\right\}=\frac{\cos\left\{\frac{180^\circ-A-180^\circ+B}{2}\right\}}{\cos\left\{\frac{180^\circ-A+180^\circ-B}{2}\right\}}\,.$$

$$\cot\left\{\frac{180^\circ-c}{2}\right\};\qquad\text{(Sp. xiv.)}$$

$$\tan\left\{180^\circ-\left(\frac{a+b}{2}\right)\right\}=\frac{\cos\left\{\frac{B-A}{2}\right\}}{\cos\left\{180^\circ-\left(\frac{A+B}{2}\right)\right\}}\,.$$

$$\cot\left\{90^\circ-\frac{c}{2}\right\};$$

$$-\tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{-\cos\frac{A+B}{2}}\,.\tan\frac{c}{2};\qquad\text{(Pl. iii. 3, 1, or Pl. vi. 8)}$$

$$\therefore\qquad \tan\frac{a+b}{2}=\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\,.\tan\frac{c}{2}\quad .\ .\quad (3)$$

Now $$\tan\frac{A'-B'}{2} = \frac{\sin\frac{a'-b'}{2}}{\sin\frac{a'+b'}{2}}\cdot\cot\frac{C'}{2}; \qquad \text{(Sp. xiv}$$

$$\therefore \quad \tan\left\{\frac{180^\circ - a - 180^\circ + b}{2}\right\} = \frac{\sin\left\{\frac{180^\circ - A - 180^\circ + B}{2}\right\}}{\sin\left\{\frac{180^\circ - A + 180^\circ - B}{2}\right\}}$$

$$\cot\left\{\frac{180^\circ - c}{2}\right\}; \qquad \text{(Sp. xiv.)}$$

$$\tan\left\{\frac{b-a}{2}\right\} = \frac{\sin\left\{\frac{B-A}{2}\right\}}{\sin\left\{180^\circ - \left(\frac{A-B}{2}\right)\right\}}.$$

$$\cot\left\{90^\circ - \frac{c}{2}\right\}$$

$$-\tan\frac{a-b}{2} = \frac{-\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\cdot\tan\frac{c}{2}; \qquad \text{(Pl.iii.3,1, or Pl.vi.8)}$$

$$\therefore \qquad \tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\cdot\tan\frac{c}{2}. \quad . \quad . \quad (4)$$

Equations (3) and (4) are known as **Napier's 3rd and 4th analogies.**

XVIII. To prove that the three angles of a spherical triangle, ABC, are together greater than two right angles, and less than six right angles.

Let a', b', c' be the sides of the supplemental triangle $A'B'C'$, therefore $A = 180^\circ - a$; $B = 180^\circ - b'$; $C = 180^\circ - c'$ (Sp. xiv.); whence $A + B + C =$ six rt. angles $-(a' + b' + c')$. But $a' + b' + c'$

are less than four rt. angles (Euc. xi. 21); therefore $A+B+C$ are greater than two rt. angles. Again, $A+B+C=$ six rt. angles $-(a'+b'+c')$. But $a'+b'+c'$ must have some value, therefore $A+B+C$ are less than six rt. angles.

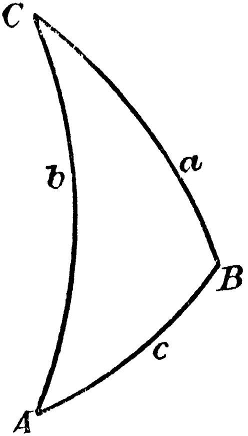

Fig 33.

XIX. To prove the formulæ used in solving a right-angled spherical triangle.

Let ABC be a right-angled spherical triangle, $B=90°$. Then shall

(1) $\cos b=\cos a\,.\cos c.$ (4) $\cos A=\cot b\,.\tan c.$

(2) $\cos b=\cot A\,.\cot C.$ (5) $\sin c=\sin C\,.\sin b.$

(3) $\cos A=\sin C\,.\cos a.$ (6) $\sin c=\cot A\,.\tan a.$

Now $\cos b=\cos a\,.\cos c+\sin a\,.\sin c\,.\cos B.$ (Sp. vii.)

But $\cos B=\cos 90°=0;$ (Pl. vi. 2)

$\therefore$ $\cos b=\cos a\,.\cos c$ . . . . . . (1)

Again, $$\cos b=\frac{\cos B+\cos A\,.\cos C}{\sin A\,.\sin C}.$$ (Sp. xv.)

But $\cos B = \cos 90° = 0;$ (Pl. vi. 2)

$\therefore$ $\cos b = \dfrac{\cos A \,.\, \cos C}{\sin A \,.\, \sin C} = \cot A \,.\, \cot C.$ . . (2)

(Pl. ii. 5)

Now $\cos a = \dfrac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}.$ (Sp. xv.)

But $\cos B = \cos 90° = 0,$

and $\sin B = \sin 90° = 1;$ (Pl. vi. 2)

$\therefore$ $\cos a = \dfrac{\cos A}{\sin C}$, and $\cos A = \sin C \,.\, \cos a.$. (3)

Again, $\cot B \,.\, \sin A = \cot b \,.\, \sin c - \cos c \,.\, \cos A.$ (Sp. xi.)

But $\cot B = \cot 90° = 0;$ (Pl. vi. 2)

$\therefore$ $0 = \cot b \,.\, \sin c - \cos c \,.\, \cos A;$

$\cos c \,.\, \cos A = \cot b \,.\, \sin c;$

$\therefore$ $\cos A = \cot b \,.\, \tan c.$ . . . . . (4)

(Pl. ii. 4)

Now $\dfrac{\sin c}{\sin b} = \dfrac{\sin C}{\sin B}.$

But $\sin B = \sin 90° = 1;$ (Sp. ix., Pl. vi. 2)

$\therefore$ $\dfrac{\sin c}{\sin b} = \dfrac{\sin C}{1}$, and $\sin c = \sin b \,.\, \sin C.$. (5)

Again, $\cot A \,.\, \sin B = \cot a \,.\, \sin c - \cos c \,.\, \cos B.$ (Sp. xi.)

But $\sin B = \sin 90° = 1;$

and $\cos B = \cos 90° = 0;$ (Pl. vi. 2)

$\therefore$ $\cot A = \cot a \,.\, \sin c;$

and $\sin c = \cot A \,.\, \tan a.$ . . . . (6)

Similarly, equations for finding a and C can be determined.

XX. **When one side of a Spherical triangle is 90° the triangle is called a Quadrantal triangle. To prove the formulæ used in solving a Quadrantal triangle :—**

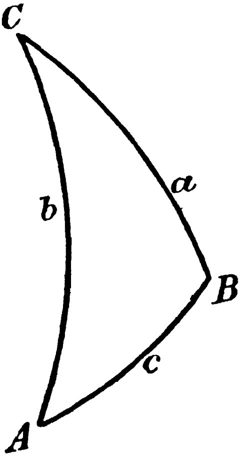

Fig. 34.

Let ABC be a spherical triangle having $a = 90°$. Then shall

(1) $\cos A = -\cos B \cdot \cos C$.

(2) $\cos A = -\cot b \cdot \cot c$.

(3) $\cos c = \cos C \cdot \sin b$.

(4) $\cos c = -\cot A \cdot \tan B$.

(5) $\sin B = \sin b \cdot \sin A$.

(6) $\sin B = \cot c \cdot \tan C$.

Now $$\cos a = \frac{\cos A + \cos B \cdot \cos C}{\sin B \cdot \sin C}.$$

But $\cos a = \cos 90° = 0$; (Sp. xv., Pl. v. 2)

whence $$0 = \frac{\cos A + \cos B \cdot \cos C}{\sin B \cdot \sin C},$$

and $$0 = \cos A + \cos B \cdot \cos C.$$

$\therefore$ $$\cos A = -\cos B \cdot \cos C. \quad . \quad . \quad . \quad . \quad (1)$$

Again, $$\cos A = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}.$$

But $\cos a = \cos 90° = 0$; (Sp. vii., Pl. vi. 2)

whence $\cos A = \dfrac{-\cos b \,.\, \cos c}{\sin b \,.\, \sin c} = -\cot b \,.\, \cot c$. . . (2)

(Pl. ii. 5)

Now $\cos C = \dfrac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}$. (Sp. vii.)

But $\cos a = \cos 90° = 0$;

and $\sin a = \sin 90° = 1$; (Pl. vi. 2)

∴ $\cos C = \dfrac{\cos c}{\sin b}$, and $\cos c = \cos C \,.\, \sin b$. . . (3)

Again, $\cot A \,.\, \sin B = \cot a \,.\, \sin c - \cos c \,.\, \cos B$. (Sp. xi.)

But $\cot a = \cot 90° = 0$; (Pl. vi. 2)

∴ $\cot A \,.\, \sin B = -\cos c \,.\, \cos B$;

whence $\cos c = -\cot A \,.\, \tan B$. . . . . . (4)

(Pl. ii. 4)

Now $\dfrac{\sin B}{\sin A} = \dfrac{\sin b}{\sin a}$.

But $\sin a = \sin 90° = 1$; (Sp. ix., Pl. vi. 2)

∴ $\dfrac{\sin B}{\sin A} = \sin b$, and $\sin B = \sin b \,.\, \sin A$. . . (5)

Again, $\cot C \,.\, \sin B = \cot c \,.\, \sin a - \cos a \,.\, \cos B$. (Sp. xi.)

But $\sin a = \sin 90° = 1$;

and $\cos a = \cos 90° = 0$; (Pl. vi. 2)

∴ $\cot C \,.\, \sin B = \cot c$, and $\sin B = \cot c \,.\, \tan C$. . . (6)

Similarly, equations for finding b and C can be determined.

CHAPTER XII.

One sea mile = one nautical mile = 6080 feet nearly.

The word "knot" should be used to indicate "one sea mile per hour."

PROBLEMS (Explained).

WITH a view to giving the student some idea of the practical application of the triangle solutions which have been already studied, a few examples in surveying, &c., are given, the plans of working them being shown; the actual figuring has been left to the student himself. In the Exercises given later in this Chapter hints as to solution will be found, as this manual is specially designed and prepared for those who cannot always consult a teacher of Trigonometry.

General Notes.

(*a*) The student should examine any problem given for solution to see if **rt.-angled triangles** can be used; should **vertical or perpendicular** lines or **right angles** be mentioned or implied he should avail himself of this class of triangle.

(*b*) **Three adjacent parts** (except in the case of **two angles and the included side in plane trigonometry**) require "**tangents**" for the solution of the figure.

(*c*) **Three sides** being known, to find angles "**cosines**" must be used. In spherical trig. (only) three angles may be used to find sides.

(*d*) The rule of "**sines**" meets such examples as do **not** come under the working of "**cosines**" or "**tangents**," and in practical work this rule plays a most prominent part. Note the possible ambiguity of results (see "**sines**"); this ambiguity seldom occurs.

(*e*) Where the **third side only** is wanted, in cases of "two sides and included angle" being given, the student should by preference find it direct rather than through "**tangents.**"

(*f*) A given problem should first be **planned** out, the student sketching the given parts approximately in both value and position. A clear idea as to the method of solution will in most cases be thus obtained.

A few examples are given below :—

(1*a*) From a "crow's nest," C, 120 ft. above the water, the angle of depression, DCB, of a boat, B, was 25° 20′; find the distance, AB, between the boat and the foot of the mast, AC, on which the "crow's nest" is placed.

Explanation.—The mast may be assumed upright (vertical to surface of sea), as any very small deviation from this position will hardly affect the result. The solution is, therefore, by plane rt.-angled triangle. The angle of depression and the angle of elevation (marked in the figure) are equal by Prop. 29, Book I., Euclid's Elements; therefore angle $CBA = 25° \ 20'$. AB, the required line, acts as the base of the rt.-angled triangle ABC; AC (120 ft.) acts as the perpendicular. The formula is, therefore,

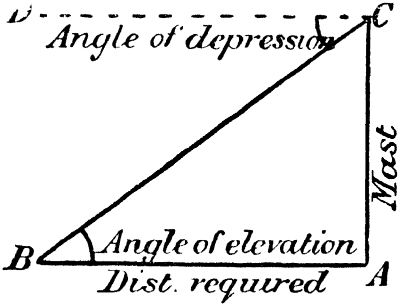

Fig. 35.

$$\frac{AB \text{ (required)}}{AC \text{ (known)}} = \text{cotangent of angle } CBA\,;$$

i.e., $AB = AC \times \cot CBA$, or $AB = 120 \times \cot 25° \ 20'$.
From this we get $AB = 253{\cdot}5$ ft.

The student should actually work the examples given.

(1*b*) In the above figure suppose AC to be the mast of a man-o'-war, and B a target which has to be placed at a distance of 2,000 yds. from it; it is required to find the angle ABC, in order that the officer placing the target may do so when the mast exactly measures (subtends) that angle on his pocket sextant.

Here we have perp. AC and base AB; the formula becomes $\frac{AC}{AB} = \text{tangent } ABC$, from which $ABC = 1° \ 9'$ (better to work for seconds, as the sides vary so much in size), or 1° 8′ 45″.

(1*c*) In **Fig. 35**, suppose C to be a cloud vertically over A, from which B, on a plain, was $5\frac{1}{2}$ miles distant, the angle of elevation of the lower edge of the cloud being 67° 25′; how high is the cloud? Here the formula is $AC = AB \times \tan ABC$.

Note.—As AB is $5\frac{1}{2}$ miles, our answer will be in miles unless we alter this measurement into 11 half miles (ans. in half miles), or into 9680 yds. ($5\frac{1}{2} \times 1760$), and get the answer in yards. Ht. = 23273·8 yds., or 13·2 miles. The student should solve this problem in yards, in feet, and by using $5\frac{1}{2}$ miles direct, as it will give good practice in the use of logarithms.

(2*a*) Fig. 36. From the top of a fort, AB, 150 ft. high, two men-o'-war, C and D, were seen in a direct line from the fort; the angle of depression, EBC, of the nearer was 12° 10′, that of the farther, EBD, being 2° 18′; find how far the ships are distant from the fort and from each other.

$$ACB = EBC = 12^\circ\ 10';\ ADB = EBD = 2^\circ\ 18'.$$

Formulæ. AC (in ABC) $= AB \times \cot ACB$,

and AD (in ADB) $= AB \times \cot ADB$.

AD (dist. of D) - AC (dist. of C) $= CD$ dist. of ships apart.

$$AD = 3734{\cdot}7 \text{ ft.},\quad AC = 695{\cdot}7 \text{ ft.},\quad CD = 3039 \text{ ft.}$$

(2*b*) Fig. 36. Let AB be a fort and a hill, and D a ship, from which the angle of elevation, BDA, of this hill and fort is 30° 10′; let D sail 1 naut. mile directly towards AB to C, where the angle of elevation has increased by 29° 10′. Find the height of the combined fort and hill and the distance of A in its base from C.

Angle $BDA = 30^\circ\ 10'$, angle $ACB = 30^\circ\ 10' + 29^\circ\ 10' = 59^\circ\ 20'$,

angle $CBD = 29^\circ\ 10'$ (that is, the difference between angles ACB and ADB (Euc. I. bk. prop. 32).

By rule "sines" find BC by the formula

sin CBD (op DC) : sin BDC (op. BC) : DC (known) : BC (req.),

from which

$$BC = \frac{DC \times \sin BDC}{\sin CBD} \text{ or } \frac{6080 \text{ ft.} \times \sin 30^\circ\ 10'}{\sin 29^\circ\ 10'}.$$

Knowing BC, the rt.-angled triangle ABC can now be used; from this $AB = BC \times \sin ACB$, and $AC = BC \times \cos ACB$.

$$BC = 6269{\cdot}2 \text{ ft.},\quad AB = 5392{\cdot}4 \text{ ft.},\quad AC = 3197{\cdot}6 \text{ ft.}$$

(2*c*) Fig. 36. From an elevation, *B*, the angle of depression (35°) of a point, *C*, is double that of a point, *D*, which lies in the same direction; *B* is 100 ft. high; how far is *D* from *A* (vertically below *B*)? Angles of depn. = corresponding angles of elevn.; angle *CBD* (see 2*b*) = angle *ACB* − angle *ADB*; but

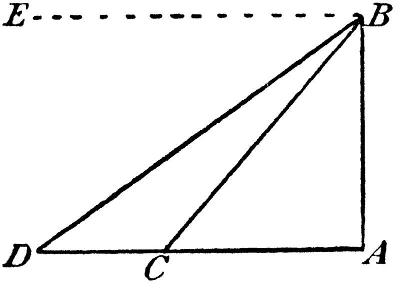

Fig. 36.

$ADB = \frac{1}{2} ACB$; $\therefore CBD = \frac{1}{2} ACB$; *i.e.*, *CDB* (or *ADB*) = *CBD*, and each equals 17° 30′. Side *BC* = side *CD*, because equal angles subtend equal sides, ∴ find *BC* from rt.-ang. triangle *ABC*, and add it to *AC* in the same triangle; the sum = *AD*. *AD* = 317·2 ft.

(3*a*) Fig. 37. A steam vessel, *B*, going 10 knots and making

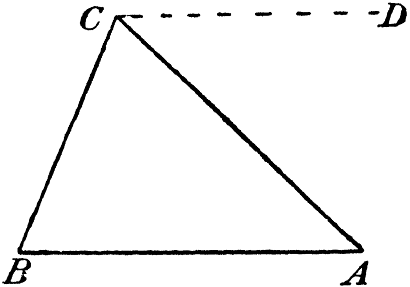

Fig. 37.

direct for the mouth of a harbour, *A*, 15 knots distant, has a steamer, *C*, 3 points on her port bow and 2 knots off, making for the same harbour. At what rate must *C* steam that the two vessels may reach the harbour mouth at the same time?

Since angle *ABC* represents the 3 points (1 point of the compass = 11° 15′) of the question, and *BA* is 15 and *BC* 2 knots, the solution is "tangents" and "sines" to find *AC*, or, much better, *AC* the third side is found direct (plane trigy.). Since dist. 15 is covered by steamer, *B*, in $1\frac{1}{2}$ hours (90 minutes); for *B* goes 10 knots, therefore *CA* must be steamed over in $1\frac{1}{2}$ hours. Find *AC*, and divide by $1\frac{1}{2}$ hours; *i.e.*, multiply by 2 and divide by 3. $CA = 13{\cdot}39$. The rate will be $13{\cdot}39\ K \div 1\frac{1}{2} = 8{\cdot}9\ K$ nearly.

(3*b*) Fig. 37. Let *B* be a pirate craft wishing to fall in with *C* some miles distant from where she first sights her (to catch her alone, beyond help, there being no harbour, &c., near). Suppose she keeps *C* in a certain fixed position, say 3 pts. on her port bow. In this case the problem would be similar to 3*a*, but *B's* rate of steaming would have to be regulated so that the meeting might be "timed" suitably (supposing she steamed faster than *C*).

(3*c*) Fig. 37. As in 3*b*, but let *B's* course and *C's* course be known, due *E.* and *E.S.E.*, respectively, the distance, *BC*, being 2 knots, relative positions as before; to find how many miles each steamer would travel before meeting, the rule of "sines" would apply. Thus :—*BA* lies *W.* to *E.*; *BC* lies 3 pts. to the left of this; that is, *E.* 3 pts. *N.* or *N.E.* by *E.*; *CA* lies *E.S.E.* Any one standing at *C* and putting one arm along *CA* and the other along *CB* (which lies *S.W.* by *W.*; *i.e.*, *N.E.* by *E.* reversed), would find the two arms contained an angle of 11 points. The triangle *ABC* thus contains :— $ABC = 3$ pts., or 33° 45′; $ACB = 11$ pts., or 123° 45′; $BC = 2\ K$. Adding *ABC* and *ACB*, and taking the sum from 180°, we get *CAB*; the student, after studying thus far, can doubtless see that *CA* and *BA* may be found by "sines."

$CA = 2{\cdot}9$ knots. $BA = 4{\cdot}3$ knots.

(4*a*) A dock of the form shown in Fig. 38 has sides measuring, *AB* 550 yds., *BC* 230 yds., *CD* 450 yds., and *AD* 345 yds.; angle $BAD = 69° 18'$. Find its area.

Join *BD*, giving triangles *ABD* and *CBD*.

Area of $ABD = \frac{1}{2}\ BA \times AD \times \sin BAD$ (see Areas).

BD, in triangle *ABD*, is found "direct" to be 536·1.

Area of $CBD = \sqrt{s(s-b)\ (s-c)\ (s-d)}$ (*d*, *b*, and *c* sides).

Add the areas of the two triangles.

Areas :—$ABD = 88750$, $CBD = 51132$, $ABCD = 139882$ sq. yds.

Versine Formula for "Third Side Direct" used in Practice for Computing Altitudes.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\text{now } \operatorname{vers} A = 1 - \cos A$$

$$\text{and } \cos A = 1 - \operatorname{vers} A$$

$$= \cos b \cos c + \sin b \sin c\,(1 - \operatorname{vers} A)$$

$$= \cos b \cos c + \sin b \sin c - \sin b \sin c \operatorname{vers} A$$

$$= \cos (b \sim c) - \sin b \sin c \operatorname{vers} A$$

$$\therefore\ 1 - \cos a = 1 - \cos (b \sim c) + \sin b \sin c \operatorname{vers} A$$

$$\therefore\ \operatorname{vers} a = \operatorname{vers} (b \sim c) + \operatorname{vers} \theta$$

$$\text{where } \operatorname{vers} \theta = \sin b \sin c \operatorname{vers} A.$$

If these formulæ are expressed in the astronomical terms of **zenith dist, latitude, declination,** and **hour angle** (H), they become

$$\text{vers zen dist} = \text{vers (lat and dec)} + \operatorname{vers} \theta$$

and $$\operatorname{vers} \theta = \text{cos lat cos dec vers } H.$$

Lat and **dec** here means **lat** $\sim$ **dec** when **lat** and **dec** have same letter, and means **lat** + **dec** when **lat** and **dec** have different letters.

PROBLEMS AND EXERCISES, WITH HINTS FOR SOLUTIONS (H).

Intermediate calculations are occasionally given to assist the student in testing his work.

1. A headland due north of a ship bore N.N.W. when she had run 7 miles due east; find distance of headland when due north.

H. Plane right-angled triangle; N.N.W. = 2 pts. = 22° 30′. **Ans.** 16·9 miles.

2. The above headland bore E N.E. from a ship which sailed E. by S. 5 miles and then found it bore N. by E. Find its distance when N. by E.; how much further must the ship sail on the same course to get the headland N.?

H. Rt.-ang. triangles; rt. ang. at second position of ship. **Ans.** 3·3 and ·66 miles.

3. The mainmast of a "slaver" subtended 13° from a launch of H M.S. "Tartar" in East African waters. After chasing and gaining (approximately) 1 mile on her, it subtended 39°. How far are the vessels then apart, and about how long will it take to overhaul the slaver, if the launch steams 7 miles per hour and the slaver travels 6 miles per hour?

H. "Sines" and rt.-ang. triangle; 1 mile = 1760 yds. **Ans.** 702 yds.; 24 mins.

4. A steamer 20 miles from port, and going 15 knots (6080 ft. = 1 knot), has a 3 knot current setting broadside across her track. How much will she have to "bear up" to make the shortest run for the port?

H. Rt.-ang. triangle. **Ans.** 11° 19′.

5. If the lead from a vessel going 8 knots "bottomed" in 3 seconds, about how far ahead should it be hove to secure an upright cast?

H. Exercise in logs. only; no triangles. **Ans.** 40·5 ft.

6. How much canvas (approximate) would there be in a jib measuring luff 50 ft., foot 25 ft., after leech 30 ft., taking 5 per cent. of canvas above actual area; canvas, 22 ins. wide?

H. Plane area, three sides given; add 5 per cent.; area = 285. **Ans.** 163 ft.

7. A torpedo catcher 180 ft. long turned in a circle, whose radius equalled three lengths of the vessel. Find the area of this circle, and the difference between this area and that of the square which touches it externally.

H. Area of circle $= \pi r^2$, area of square $4r^2$, $\pi = \frac{22}{7}$ or 3·14159, &c. **Ans.** 916,459 sq. ft., 249,941 sq. ft.

8. A vessel, S, in lat. 20° 20′ N. has a vessel, V, due north of her in lat. 20° 25′ N., and another vessel, Z, due east a naut. mile away (6080 ft.). Find the angle at V subtended by line SZ.

H. Plane rt.-ang. triangle. **Ans.** 11° 19′.

9. A sailing vessel having the wind direct from her port, 30 miles distant, finds she makes 8½ miles per hour when she lies 6½ pts. off the wind, and 9 miles if she falls off ¼ pt. Which will be her best course (two tacks)?

H. Plane rt.-ang. triangles, or "sines." Inspection from "traverse table;" 2 × 51·67 takes 12·15 hrs., 2 × 61·73 takes 13·72 hrs. **Ans.** 6½ pts. from wind.

10. Two ships, A and B, are in the offing, and from the ends of an esplanade 1000 ft. long, lying E and W., they bear—A due S. and E.S.E., and B S. by E. and S.W. by W. Find dist. of A from B, and of each ship from nearest end of esplanade.

H. Rt.-ang. triangle, "sines," "tans," and "sines," or third side direct. **Ans.** A to B, 899·6 ft. A to esp., 414 ft. B to esp., 601 ft.

11. Three objects, A, B, and C, on shore are seen from ship S. At the ship the angles are $ASB = 10° \ 21'$, $BSC = 15° \ 19'$. When the ship had moved 500 yards direct towards C the angles were $AS'B = 12° \ 30'$, and $BS'C = 20° \ 40'$. Find the dist. AB.

H. Plane "sines" and "tans," or third side direct. **Ans.** 412 yds.

12. Required the length of a forestay when the top is 50 ft. from deck, knightheads 70 ft. from mast (rake of mast disregarded), forecastle 7 ft. high, knighthead bolts 1½ ft.

H. Pl. rt.-ang. triangle. **Ans.** 162·7 ft.

N.B.—To allow for collar, add ¼ of perimeter of mast to above result.

13. Sailing along a coast the report of a gun from a fort N.W. of us followed the flash after 4 secs. After sailing E. by N. the interval became 8 secs. Find the distance sailed and bearing of fort from the second station. Sound travels 1090 ft. per sec. (varying with the state of the atmosphere).

H. Plane "sines." **Ans.** 5509 ft. N., 76° 41′ W.

14. Steaming up the Thames, two marks which are used for the "measured mile" bore—inner one W. 13° S., outer one W. 36° S. After going due W. ¼ mile, the marks were in line due S. How far were they apart?

H. Plane rt.-ang. triangles. **Ans.** 218 yds.

15. From a balloon to the eastward 2 miles high the angles of depression of A and B were 76° 20′ and 78° 15′, respectively. B was due S. of A, and AB subtended an angle of 2° 10′ at the balloon. Find the dist. AB and the bearings of balloon from A and B.

H. Rt.-ang. triangles. "Tans" and "sines." Of 4 triangles, one only is on the horizontal plane; cosines. **Ans.** 139 yds. S. 29° 36′ E., S. 25° E.

16. A base line, CD, along the bank of a river ran N. and S. for 1000 yards. A point, G, on the opposite bank bore W. ½ N. from C distant 1½ miles. Find bearing and distance of G from D and width of the stream at G.

H. "Tans" and rt.-ang. triangle. **Ans.** N. 64° 23′, W. 2914 yds., 2628 yds.

17. At A the angle of elevation of C, the top of a shaft bearing S.S.W. and 300 ft. high, was 10° 24′; at B the angle of elevation was 10° 4′, and the shaft bore S.W. Find the direction and distance of B from A.

H. Rt.-ang. triangles and "sines." **Ans.** S. 60° 51′ E., 661 ft.

18. Two headlands, B and C, one on each side of the entrance to a harbour, are 130 and 80 ft. high. From a boat, A, the angles of elevation are 23° 20′ and 15° 18′, respectively. A is 100 ft. from the nearest point of the line uniting B and C across the harbour's mouth. Allowing 30 ft. each side for slope to the water, find the width of the entrance.

H. Four rt.-ang. triangles, 2 vert., 2 horizontal. **Ans.** 274·9 + 284·3 − 60, or 498·2 ft.

19. A column, DE, on an elevation, CD, subtends an angle of 16° 25′, and the statue, EF, surmounting it subtends 1° 10′, to an observer at A in the plane below. At B 200 ft. from A, direct from the objects, the angle of elevation of the top of the statue is 25° 30′, and the angle BAD is 135°. Find the length of the statue.

H. Triangle BFA "sines." triangles AFC, AEC right angled. **Ans.** 6 ft.

20. The equator cuts the ecliptic at an angle of 23° 27′ (nearly). A body on the equator has right ascension 15° 18′. Find its long. and lat.

H. Sph. rt.-ang. triangle. **Ans.** Lat. 6° 2′ N.; long. 14° 5′.

21. Find the extent of the earth's surface, and the cubical miles contained in it.

H. See formula (sphere used) diameter = 7912 miles. **Ans.** 196,742,000 sq. miles; 778,311,000,000 cubic miles.

22. What is the area of an ellipse whose diameters are 30 and 25 ft.?

H. d and d' 30 and 25 (see formula). **Ans.** 589 sq. ft.

23. A spar 40 feet long, tapering to a point from a circular end of 1 ft. radius, is towed (wrongly) point first. Find the area of surface which offers opposition to progress. Also find area of thick end.

H. Surface of cone, &c. (see formulæ). **Ans.** 126 sq. ft.; 3·14 sq. ft.

24. A chord of a circle 16 miles in diameter subtends an angle of 100° at the centre of the circle. Find the length of the corresponding arc.

H. Circumference $= 2\pi r$ and $\frac{\text{arc}}{\text{circumf.}} = \frac{100^\circ}{360^\circ}$. **Ans.** 14 miles.

25. Three objects, A, B, and C, were distant from D 4250, 5000, 2764 yds., and their bearings were N.W., N.E., and N. by W. Find the lengths of AB and AC, and bearing of B from A.

H. Plane "tans" and "sines," or third side direct and "sines." **Ans.** AB 5177, AC 2483, N. 71° 51′ E.

26. The lats. of A and B are 29° 20′ N. and 38° 19′ N., and their d. long. is 14° 18′ W.; find their distance apart on a great circle, and the initial course from A to B.

H. Sph. third side direct, using the co. lats. (90° – lat.) of A and B as sides, and d. long. as included angle. Course by "sines," or sph. "tans" and "sines." **Ans.** 886 miles; N. 49° 6′ W.

27. Compute the dist. of the horizon from a masthead 100 ft. above water.

H. See formula, **Chap. xviii.**, *Algebra*. **Ans.** 12·25 miles.

28. How high should a light be placed that it may be seen from a boat 20 miles off? also from a point 15 ft. above water at 15 miles distance?

H. See **Chap. xviii.**, *Algebra*. **Ans.** 266 ft., 70 ft.

29. A ship sailing N.N.E. 10 knots per hour is in a current setting W. 2 knots per hour. What course will she make good, and at what rate will she travel?

H. Plane "tans" and "sines" (parallelogram of forces). **Ans.** N. by E., nearly 9·4 knots per hour.

30. Required to find the successive distances of four stations, A, B, C, D, from one another, given diagonals AD 5000 ft., BC 4164 ft., and bearings A from C N. ½ W., B from C N.N.E., B from D N.N.W., A from D N.W., and D from C due E.

H. Plane "sines." **Ans.** AC 3553, CD 3188, AB 1967, BD 4164 ft.

31. In the irregular pentagon, ABCDE, given AB 80 ft., AE 75 ft., angle BAE 125°, BC 130 ft., and BCD 101° 3′, CD 101 ft., ED 99 ft.; find area of pentagon.

H. Three triangles; formula for area after finding BE and BD. BE = 137·5, BD 179·2. Areas 2457, 6742, 6443. **Ans.** 15,642 sq. ft.

32. The sun is 92 millions of miles from the earth (approximately), and its semi-diameter subtends an angle of 16′. Find the diameter and volume of the sun.

H. Plane third side direct; formula for volume. **Ans.** 856,400 miles. 329,025,000 000,000,000 cubic miles.

33. The eye for a pair of shrouds is $\frac{1}{4}$ more in measurement than the perimeter of the masthead, which is trimmed square from a mast of diameter 2 ft. Find the number of feet in the eye.

H. Diagonal of square = diameter of mast; rt.-ang. triangles, &c. Ans. 7·07 ft.

34. The sun's dec. is 25° 10′ N., the true alt. 17° 15′, and the lat. of the observer 18° N. Find the hour angle and azimuth of the body (time a.m).

H. Sph. "cosines," using polar dist. (90° − dec.), zen. dist. (90° − alt.) and co.-lat. Ans. N. 68° 28′ E., 5h. 15m. 52s.

35. The difference between the right ascensions of two stars is 1h. 25m.; their declinations are 18° 20′ N. and 25° 40′ N. Find their dist. apart.

H. Sph. "tans;" diff. of R.A. = included angle between arcs showing polar distances. Ans. 20° 55′.

36. The rising amplitude of the sun is E. 30° N. in lat. 43° 20′ N. Find its hour angle and polar dist.

H. Sph. rt.-ang. triangle (right angle at horizon); use co.-amp. Ans. 7h. 26m. 28s., 68° 40′.

37. A body on the horizon bears E.N.E. from an observer in lat. 36° 20′ N. Find its polar dist.

H. Quadrantal triangle (zen. dist. is 90°). Ans. 72° 3′.

38. What is the lat. of the vertex of a great circle between A, lat. 61° 36′ N., long. 51° 35′ W., and B lat. 76° 43′ N., long. 5° 32′ E.?

H. Sph. "tans," find angle A; and rt.-ang. triangle, find PV. Ans. 76° 49′.

39. The angle of elevation of the centre of a balloon 3000 yds. high is 68°, and the diameter subtends 10′. Find the circumference of the balloon.

H. Plane rt.-ang. triangle, find hyp.; and third side direct for diameter. Ans. 30 ft.

40. When the sun's dec. was 20° N. he set 3 hours later than when it was 20° S. Find the lat., and times of setting.

H. Sin amplitude = sin dec. × sec. lat.; lat. = x. Ans. 46° 26′ N.; 7h. 30m. and 4h. 30m.

41. A line joining the tops of two vertical posts 12 ft. apart points direct to the sun's centre, and measures 15 ft. The shadow cast by the sun reaches 6 ft. beyond the smaller post. Find height of posts and alt. of sun.

H. Rt.-ang. triangles. Ans. 13·5 ft. and 4·5 ft., 36° 52′.

42. The sun's alt. on the prime vertical was 21° 30′, and at 6 a.m. it was 20°. Find lat. of observer and dec. of sun.

H. Polar dist. supposed constant, x = co.-lat., y = polar dist.; rt.-ang. sph. triangles and equations. Ans. 75° 1′ and 20° 44′.

43. Two stars, A and B, had, respectively, alts. 75° and 72°, right ascensions 18h. 20m. and 17h. 25m., decs. 30° 20′ N. and 29° 18′ N. Find the lat. of observer.

H. Use polar distances and diff. of R.A. in sph. triangle; then "cosines," then third side direct. Intermediate calculations, to be found by the student, here given as aids:—AB = 11° 58′. PBA 81° 39′, ZBA 56° 18′. Ans. Lat. 45° 10′ N.

44. Two towers, A and B, 300 and 125 ft. high, subtended angles of 2° 10′ and 1° 13′, their bearings being N. ½ E. and E. ½ N., respectively. Find their distance apart and the bearing of B from A.

H. Plane rt.-ang. triangle, "tans," "sines." Ans. 8.06 ft., S. 34° 47′ E.

45. The azimuth of a celestial body is S. 29° W., its alt. 36°, and the lat. of the observer 29° 30′ N. Find polar distance of body.

H. Sph. third side direct; use co.-lat., zen. dist., and azimuth. Ans. 108° 4′.

SUPPLEMENT

ON

INTRODUCTION TO LOGARITHMS.

INTRODUCTION TO LOGARITHMS.

The book of Logarithms, referred to in these pages, is used in the early stages of their work by the Cadets of the "Worcester." Riddle's "Nautical Tables," by Simpkin, Marshall & Co., Paternoster Row, E.C.

Any other book of logarithms can be used.

The word logarithm is written log.

Refer to Riddle's Tables, page 6, where the logarithms of numbers from 1 to 100 are tabulated.

log 4 is 0·602060 and 0·698970 is log of 5
log 9 is 0·954243 and 0·845098 is log of 7
log 13 = 1·113943 and 1·278754 = log 19
log 76 = 1·880814 and 1·929419 = log 85

so that from the table on page 6 we can find the log corresponding to a number, and the number corresponding to a log.

Logarithms can be used for dealing with

(1) Multiplication.
(2) Division.
(3) Such forms as 3^2 and 2^5.
(4) Such forms as $\sqrt{16}$ and $\sqrt[3]{64}$.

Very easy examples to show the use of logs. The answers are obvious.

More difficult work is done in like manner.

1. Multiply 8 by 2.	Multiply 16 by 5.
log 8 = 0·903090	1·204120 = log 16
log 2 = 0·301030	0·698970 = log 5
1·204120	1·903090
Ans. = 16.	**Ans. = 80.**

To multiply numbers, write down the logarithms of the numbers, add the logs, and find in the tables the number corresponding to the final log. Arrange the work as shown. Choose one of the methods.

2. Divide 24 by 3.

log 24 = 1·380211
log 3 = 0·477121

0·903090

Ans. = 8.

Divide 54 by 27.

1·732394 = log 54
1·431364 = log 27

0·301030

Ans. = 2.

To divide numbers, write the logs and subtract as shown in the examples.

3. Find the value of 5^2.

5^2 means 5 × 5.

log 5 = 0·698970
2

1·397940

Ans. = 25.

Find the value of 2^5.

2^5 means 2×2×2×2×2.

0·301030 = log 2
5

1·505150

Ans. = 32.

The first example here could be worked by putting down log 5 **twice** and adding the logs. The second example could be worked by writing log 2 **five** times and adding the logs.

4. Find the value of $\sqrt{16}$.

2) 1·204120 = log 16

0·602060

Ans. = 4.

Find the value of $\sqrt[3]{64}$.

3) 1·806180 = log 64

0·602060

Ans. = 4.

$\sqrt{16}$ means the **square root** of 16, and we have to find the number which, multiplied **twice**, gives 16.

$\sqrt[3]{64}$ means the **cube root** of 64, and we have to find the number which, multiplied three times, gives 64. Here 4 × 4 × 4 is what we have in mind.

To find the square root of a number divide the log of the number by 2 and find the answer. For cube root divide by 3.

Multiply	17 by 4	28 by 3	25 by 4	5 by 2
Divide	24 by 4	54 by 9	10 by 2	100 by 5
Find the value of	2^4	4^2	3^4	4^3
Find the value of	$\sqrt{9}$	$\sqrt{49}$	$\sqrt[3]{27}$*	$\sqrt[3]{8}$

Find the number (represented by x or some other letter) when—

$x = 2 \times 2$ $a = 14 \times 7$ $w = 48 \times 2$

$x = 85 \div 5$ $b = 36 \div 6$* $x = \frac{84}{4}$

$x = 7^2$ $c = 8^2$ $y = 2^3$

$x = \sqrt{81}$* $d = \sqrt{4}$ $z = \sqrt[3]{125}$*

***N.B.**—When dividing log 27 by 3, carry on the division to six decimal places. The last figure in your work will sometimes differ from the last figure in the log book—thus 36 ÷ 6 gives final log 0·778152. For the log when the number exceeds 100, see below.

If some logarithms are known others can easily be found.

log 2 = 0·301030 log 3 = 0·477121 log 10 = 1·000000

From these **three** logs, try to find the logs of 4, 5, 6, 8, 9, 12, etc.

$4 = 2 \times 2$ add log 2 and log 2 and you will get log 4.

$5 = 10 \div 2$ subtract log 2 from log 10 to get log 5.

$6 = 2 \times 3$ add log 2 and log 3 to get log 6.

$8 = 2 \times 2 \times 2$ add log 2, log 2, log 2.

$9 = 3 \times 3$ multiply log 3 by 2.

$12 = 2 \times 2 \times 3$ add logs of 2, 2, 3, or add logs 4 and 3.

The logs of 7 and 11 and numbers that won't factor are not so easily found.

Logarithms are found by Algebra, and they work out to many decimal figures. Thus log 2 = 0·301029994 and more figures, and similarly log 3, log 7, log 11, etc.

In some Tables of Logarithms only four figures after the decimal point are printed, and in other books more decimals are given. Thus we find log 2 = 0·3010, log 2 = 0·30103, log 2 = 0·301030, and log 3 = 0·4771, log 3 = 0·477121, log 3 = 0·4771213.

In Riddle's Tables we find six decimal places. If two students are working the same example, one using a log book with logs to four figures, and the other using six-figure logs, a slight difference in the results would be expected.

Special attention is requested for the following ;—

$\frac{6}{2} = 3$	$\frac{x}{2} = 3$	x is evidently 2×3 or 6.
$8 = 4 \times 2$	$8 = 4 \times x$	x is $8 \div 4$ or 2.
$\frac{6}{8} = \frac{3}{4}$	$\frac{x}{8} = \frac{3}{4}$	x is $\frac{8 \times 3}{4}$ or 6.
$\frac{16}{18} = \frac{8}{9}$	$\frac{16}{y} = \frac{8}{9}$	y is $\frac{16 \times 9}{8}$ or 18.
$\frac{56}{48} = \frac{7}{6}$	$\frac{56}{48} = \frac{a}{6}$	$a = \frac{56 \times 6}{48}$ or 7.
$\frac{76}{78} = \frac{38}{39}$	$\frac{76}{78} = \frac{38}{b}$	$b = \frac{78 \times 38}{76}$ or 39.

If a letter or other symbol takes the place of a number in such cases as the above, the value which the letter represents can be found. Such operations are very often required in practical work, and many blunders are made through lack of attention to these elementary facts, especially by those who have had but little practice in Algebra. From the middle column find by logs the values of x, y, a, b.

Logs of numbers from 100 to 999—that is, logs of numbers with **three** figures.

Log 100 = 2·000000. The **logarithm** consists of two parts, the number to the left of the decimal point and the part to the right of the decimal point. The former is called the **index or** characteristic, and the latter is called the **mantissa.**

All the numbers from 100 to 999 have 2 for the index, and the mantissa is given on pages 7 to 21 in the log book.

Check the following ;—

log 100 = 2·000000
log 101 = 2·004321
log 102 = 2·008600
log 103 = 2·012837
log 111 = 2·045323
log 158 = 2·198657
log 241 = 2·382017

2·113943 is the log of 130
2·117271 = log 131.
2·401401 = log 252
2·403121 = log 253
2·580925 = log 381
2·582063 = log 382
2·875640 = log 751

Logs of numbers from 1000 to 9999—that is, logs of numbers with **four** figures.

The index of all these numbers is 3.

To find the mantissa of 1154, find 115 on page 7, and go across the page to the column marked 4.

The mantissa of 7408 is found in column 8 opposite 740.

$$\begin{array}{rlcrl}
\log 1154 = & 3{\cdot}062206 & \qquad & 3{\cdot}111599 & = \log 1293 \\
\log 7408 = & 3{\cdot}869701 & & 3{\cdot}129045 & = \log 1346
\end{array}$$

N.B.—All whole numbers from 1 to 9 have the index 0
" " 10 to 99 " 1
" " 100 to 999 " 2
" " 1000 to 9999 " 3

The decimal part of the logarithm (the mantissa) is found in the log book.

Work by logs $\dfrac{23 \times 1736}{868}$ and $\dfrac{2233 \times 16}{638}$

$$\begin{array}{rlcrl}
\log 23 = & 1{\cdot}361728 & \qquad & 3{\cdot}348889 & = \log 2233 \\
\log 1736 = & 3{\cdot}239550 & & 1{\cdot}204120 & = \log 16 \\
\cline{2-2}\cline{4-4}
& 4{\cdot}601278 & & 4{\cdot}553009 & \\
\log 868 = & 2{\cdot}938520 & & 2{\cdot}804821 & = \log 638 \\
\cline{2-2}\cline{4-4}
& 1{\cdot}662758 & & 1{\cdot}748188 &
\end{array}$$

Ans. = 46. **Ans. = 56.**

Index 1 in the final log shows that the answer has two figures.

Logs of numbers with decimals and rule for index.

Divide 1054 by 10, the answer is 105·4.
Divide 105·4 by 10, the answer is 10·54.

$$\begin{array}{rlcrl}
\log 1054 = & 3{\cdot}022841 & \qquad & \log 105{\cdot}4 = & 2{\cdot}022841 \\
\log 10 = & 1{\cdot}000000 & & \log 10 = & 1{\cdot}000000 \\
\cline{2-2}\cline{5-5}
\therefore \log 105{\cdot}4 = & 2{\cdot}022841 & & \therefore \log 10{\cdot}54 = & 1{\cdot}022841
\end{array}$$

From this we see that when we divide a number by 10—

(1) The decimal point for the resulting **number** moves one place.

(2) The index of the resulting **log** is less by one, but the mantissa is not changed.

$\log 1054 = 3{\cdot}022841$ $\qquad$ $\log 7008 = 3{\cdot}845594$

$\log 105{\cdot}4 = 2{\cdot}022841$ $\qquad$ $\log 700{\cdot}8 = 2{\cdot}845594$

$\log 10{\cdot}54 = 1{\cdot}022841$ $\qquad$ $\log 70{\cdot}08 = 1{\cdot}845594$

$\log 1{\cdot}054 = 0{\cdot}022841$ $\qquad$ $\log 7{\cdot}008 = 0{\cdot}845594$

and if we continue to divide the number by 10, and reduce the index by 1, the index becomes negative, because one less than nought is minus one, and 1 less than -1 is -2. The minus is generally written **over** the index. One less than $\bar{2}$ is $\bar{3}$.

$\log {\cdot}1054 = \bar{1}{\cdot}022841$ $\qquad$ $\log {\cdot}7008 = \bar{1}{\cdot}845594$

$\log {\cdot}01054 = \bar{2}{\cdot}022841$ $\qquad$ $\log {\cdot}07008 = \bar{2}{\cdot}845594$

$\log {\cdot}001054 = \bar{3}{\cdot}022841$ $\qquad$ $\log {\cdot}007008 = \bar{3}{\cdot}845594$

Hence we can say definitely how to find the index.

If there is a whole number the index is positive (or plus), and is one less than the number of figures in the whole number.

If the number consists entirely of decimals the index is negative (or minus), and is one more than the number of noughts immediately following the decimal point. Thus, log 90·74 is 1·957799, index 1, because there are **two** figures in the whole number, and mantissa the same as for 9074.

Log ·002306 is $\bar{3}{\cdot}362859$, index $\bar{3}$, because there are **two** noughts immediately following the decimal point, and the mantissa is the same as for 2306.

N.B.—The decimal point affects the index but not the mantissa.

$\log 24{\cdot}3 = 1{\cdot}385606$ $\qquad$ $\log 2{\cdot}43 = 0{\cdot}385606$

$\log 6{\cdot}8 = 0{\cdot}832509$ $\qquad$ $\log {\cdot}0068 = \bar{3}{\cdot}832509$

The index may be plus or minus, the mantissa is always plus.

Multiply 72·6 by ·32. $\qquad$ Divide 173·8 by 9939.

$\log 72{\cdot}6 = 1{\cdot}860937$ $\qquad$ $\log 173{\cdot}8 = 2{\cdot}240050$

$\log {\cdot}32 = \bar{1}{\cdot}505150$ $\qquad$ $\log 9939 = 3{\cdot}997343$

$1{\cdot}366087$ $\qquad$ $\bar{2}{\cdot}242707$

Ans. = 23·23 nearly. $\qquad$ **Ans.** = ·01748 nearly.

(The addition and subtraction as in Algebra.)

N.B.—Beginners should omit pp. 121, 122, and half of 123.

Logs of Numbers with more than Four Figures.

Find the logs of 51643 and 516435.

log 51640 = 4·712986	log 516400 = 5·712986
log 51650 = 4·713070	log 516500 = 5·713070
diff. = ·000084	diff. = ·000084
No. changes 10, **mantissa** changes 84.	For 100 the change is 84.
,, 1, ,, $\frac{84}{10}$.	,, 1 ,, $\frac{84}{100}$.
,, 3, ,, $\frac{84 \times 3}{10}$.	,, 35 ,, $\frac{84 \times 35}{100}$.
,, 3, ,, 25.	,, 35 ,, 29.
log 51640 = 4·712986	log 516400 = 5·712986
change for 3 = 25	change for 35 = 29
log 51643 = 4·713011	log 516435 = 5·713015

N.B.—The difference (change) in mantissæ is given in Table XIII. in the right-hand column headed D (for difference).

To find log 51643, write down the mantissa for 51640, multiple 84 (from column D) by 3 (the extra figure), which gives 252, cut off one figure from the end (*i.e.*, divide by 10), and add 25 (the result) to the mantissa as above.

To find log 516435, write down the mantissa for 516400, multiply 84 (from column D) by 35 (the extra figures), which gives 2940, cut off two figures from the end (*i.e.*, divide by 100), and add 29 (the result) to the mantissa as above.

To find log 5164374 same as before but cut off three figures.

For 51·643 the index is 1, the mantissa is 713011 as before.
,, 5164·3 ,, 3, ,, 713011 ,,

To find the number corresponding to a given logarithm when more than four figures are required in the number.

Find the number corresponding to the log 4·551960.

The index 4 suggests five figures in the answer.

Find in the log book the mantissæ next below and next above 551960, and find how much they differ. This difference (122 in this example) is given in the log book under column D. Find also how much the smaller of these differs from the mantissa in question.

·551938 = mantissa for 35640		·551938 = mantissa for 35640.	
·552060 = ,, for 35650		·551960 = ,, given.	
·000122 = diff. for 10		·000022 = diff.	

Hence mantissa changes 122 when change in number = 10.

,, ,, 1 ,, ,, $= \dfrac{10}{122}$.

,, ,, 22 ,, ,, $= \dfrac{10 \times 22}{122}$

,, ,, 22 ,, ,, = 2.*

·551938 = mantissa for 35640
22 = change for 2
·551960 = mantissa for 35642

Hence 4·551960 = log 35642, and this is the answer.
Also 3·551960 = log 3564·2
and 1·551960 = log 35·642

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* This value can be obtained quickly. Here we want one extra figure, because five are necessary and the log book only gives four. Note how this 2 is obtained Multiply the difference 22 by 10, and divide by the number 122 found in column D.

If 2 extra figures are required (*i.e.*, 6 figures in answer), multiply by 100.
If 3 ,, ,, (*i.e.*, 7 ,,), ,, 1,000

Find the number corresponding to the log 5·234126.

The diff. between this mantissa and the one next less 234011 from the log book is 115, the number in column D is 253, and the number for mantissa 234011 is 1714.

We want 2 extra figures, because index 5 means 6 figures.

Multiply 115 by 100 and divide by 253, result 45, and the answer is 171445.

Find the number (to 3 decimal places) if the log is 1·710640.

Here the index 1 shows there are 2 figures in the answer, but we want three decimals in addition—that is, five figures—and as the book supplies four, one extra figure is required.

Find 710640 as a mantissa. The next one less is 710625, opposite 5136, the diff. is 15, and column D is 85.

Multiply 15 by 10 and divide by 85, result 2, so the figures are 51362, and the answer 51·362, because index is 1.

Log is 6·863769, find the number. **Ans.** 7307508.
Log is 5·221193 „ **Ans.** 166415.
Log is 4·700400 „ **Ans.** 50165

In the following, give the answers to 2 decimal places :—

Log is 2·164967, find the number. **Ans.** 146·21.
Log is 3·926269 „ **Ans.** 8438·59.
Log is 4·368297 „ **Ans.** 23350·54.

In this last example the next mantissa is 368287, the diff. is 10, the number 2335, and column D is 186.

Multiply the diff. 10 by 1,000 (for **three** extra figures) and divide by 186. Three figures must be obtained, and they are 054, not 540.

Refer to diagrams on page 55.

Trigonometrical Angle.—If a straight line $C\,P$ revolves in a plane about a point C from one position $C\,P$ to another position $C\,N$, it is said to trace out an angle $P\,C\,N$. The angle is marked in the figure, and may exceed 360°.

What is meant by **Sine of an Angle** ?

If a straight line $C\,P$ revolves in a plane about a point C from one position $C\,P$ to another position $C\,N$, so as to trace out an angle $P\,C\,N$, and if from N (**any** point in the latter line) a perpendicular $N\,H$ is drawn to $C\,P$, produced if necessary, then

$\frac{NH}{NC}$ is called the sine of the angle $P\,C\,N$

$\frac{NC}{NH}$ is the cosecant of the angle $P\,C\,N$

$\frac{CH}{CN}$ is the cosine of the angle $P\,C\,N$

$\frac{CN}{CH}$ is the secant of the angle $P\,C\,N$

$\frac{NH}{HC}$ is the tangent of the angle $P\,C\,N$

$\frac{HC}{NH}$ is the cotangent of the angle $P\,C\,N$

Learn these **thoroughly,** using the diagram in which the angle $P C N$ is less than 90°. (These statements are true for all the diagrams on page 55.)

What is meant by the logarithm of the sine of an angle, and how is it tabulated? Refer to Table XIV. in log book.

Draw a circle with radius 2·2 inches (take diagram, page 55 as a guide), with a protractor make an angle $P C N = 30°$. From N the end of the radius draw $N H$ perp. to $C P$. Measure carefully $N H$ and $N C$.

$N H$ the perp. is 1·1 inches and $N C$ the radius is 2·2 inches, so that $\text{sine}\, P C N = \dfrac{1{\cdot}1 \text{ inches}}{2{\cdot}2 \text{ inches}} = \frac{1}{2} = {\cdot}5$, and $P C N = 30°$

Therefore sine $P C N$, sin 30°, is a mere number, and log sine $P C N$, log sin 30°, is the log of this number.

Subtract log 2 from log 1, or look up log ·5, and log sin 30° is $\bar{1}{\cdot}698970$.

The minus index is difficult for beginners, and requires special type. In Table XIV. a 10 has been added to the index, thus making the index 9, and the tabular log sin 30° 0′ [that is, the log in the Tables (page 52, log book) is 9·698970], so that sine 30°, log sine 30°, tabular log sin 30° are three distinct ideas, and must not be confused. They are often written sin 30°, l sin 30°, L sin 30°.

Make a similar diagram, radius 3 inches, and draw $N H$ perp. as before, but make a different angle, say $P C N = 42°$. Draw carefully. Measure $C N$ and $C H$. $C N = 3$, $C H = 2{\cdot}23$ inches,

so that $\text{secant } 42° = \dfrac{C N}{C H} = \dfrac{3}{2{\cdot}23} = 1{\cdot}34$ nearly,

and secant 42° is a number whose log is 0·127105 nearly; therefore, log sec 42° = 0·127105, and 10 is added in the tables; therefore, L sec 42° = 10·127105 nearly.

N.B.—This does not agree with the value in the log book, page 64, because you cannot draw accurately an angle of 42°, neither can you measure accurately the lengths of the lines.

Hence such logarithms are based on a method in mathematics, and are not based on a diagram as above.

Check the following on pages 52 and 64, log book :—

L sin 30° 0′ = 9·698970	L sec 42° 0′ = 10·128927
L sin 30° 1′ = 9·699189	L sec 42° 1′ = 10·129040
L sin 30° 2′ = 9·699407	L sec 42° 2′ = 10·129154

and observe that when the degrees are at the top of the page then use the names at the top and the minutes down the left-hand column. Pay special attention to this.

$L \sin\ 31^\circ\ 14' = 9{\cdot}714769$
$L\ \mathrm{cosec}\ 31^\circ\ 15' = 10{\cdot}285022$
$L \tan\ 32^\circ\ 20' = 9{\cdot}801396$

$L\ \mathrm{cotan}\ 33^\circ\ 52' = 10{\cdot}173195$
$L \sec\ 34^\circ\ 7' = 10{\cdot}082024$
$L \cos\ 35^\circ\ 29' = 9{\cdot}910776$

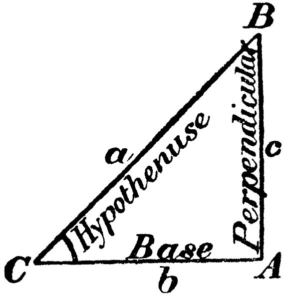
At the top of Table XIV. the degrees from 0° to 44° can be found, and on page 66 at the bottom is 45°, then backwards along the bottom of the pages 46°, 47°, on to 90°.

$L \sin\ 50^\circ\ 0' = 9{\cdot}884254$
$L \sin\ 50^\circ\ 1' = 9{\cdot}884360$
$L\ \mathrm{cosec}\ 51^\circ\ 14' = 10{\cdot}108071$
$L \tan\ 52^\circ\ 23' = 10{\cdot}113190$

$L \cos\ 53^\circ\ 46' = 9{\cdot}771643$
$L \cos\ 53^\circ\ 47' = 9{\cdot}771470$
$L \sec\ 54^\circ\ 52' = 10{\cdot}239969$
$L \cot\ 55^\circ\ 9' = 9{\cdot}842805$

Observe carefully that when the degrees are at the bottom of the page, then use the names at the bottom and the minutes up the right-hand column.

Read the last two pages very carefully, and notice what a vast amount of labour is indicated in Table XIV. The values for the sines of all the angles from 0° to 90° (for every minute) have been calculated, the logarithms have been found, and these logarithms, with a 10 added to every index, have been tabulated in a form easy for reference. Not only sine, but cosecant, tangent, cotangent, secant, and cosine.



Let ABC be a right-angled triangle, $A = 90^\circ$, $\therefore C$ less than 90°.

$$\text{Then sine } C = \frac{BA}{BC}. \quad \text{cosine } C = \frac{CA}{CB}. \quad \text{tangent } C = \frac{BA}{CA}.$$

$$\text{cosec } C = \frac{BC}{BA}. \quad \text{sec } C = \frac{CB}{CA}. \quad \text{cot } C = \frac{CA}{BA}.$$

Turn the book and view the diagram from different positions. If instead of using the full names (BA, BC, CA) for the sides we use a for the side opposite A, b for the side opposite B, and c for the side opposite C, then

$$\sin C = \frac{c}{a}. \qquad \text{cosine } C = \frac{b}{a}. \qquad \tan C = \frac{c}{b}.$$

$$\text{cosec } C = \frac{a}{c}. \qquad \sec C = \frac{a}{b}. \qquad \cot C = \frac{b}{c}.$$

Or, if we substitute the words hypotheneuse (h) for the side opposite the right angle, perpendicular (p) for the side opposite the angle we are using, and base (b) for the other side, then

$$\sin C = \frac{p}{h}. \qquad \text{cosine } C = \frac{b}{h}. \qquad \tan C = \frac{p}{b}.$$

$$\text{cosec } C = \frac{h}{p}. \qquad \sec C = \frac{h}{b}. \qquad \cot C = \frac{b}{p}.$$

Again, if for AB we write "side opposite" the angle, for AC we write "side adjacent" to the angle we are using, and still call CB the hypotheneuse, then

$$\sin C = \frac{\text{opp. side}}{\text{hyp.}}. \qquad \text{cosine } C = \frac{\text{adj. side}}{\text{hyp.}}. \qquad \tan C = \frac{\text{opp. side}}{\text{adj. side}}.$$

$$\text{cosec } C = \frac{\text{hyp}}{\text{opp. side}}. \qquad \sec C = \frac{\text{hyp.}}{\text{adj. side}}. \qquad \cot C = \frac{\text{adj. side}}{\text{opp. side}}.$$

The student should be thoroughly familiar with these terms. Any fraction is called a **Ratio.** ∴ sin, cosec, etc., are ratios.

Multiply 36 by sin 30°.	Multiply 132·4 by sec 42°.
1·556303 = log 36	2·121888 = log 132·4
9·698970 = L sin 30°	10·128927 = L sec 42°
1·255273	2·250815
Ans. = 18.	**Ans.** = 178·2.

We noticed in drawing a diagram that sin 30° = ½, so that 36 × sin 30° is 36 × ½, and the result is 18.

In a second diagram we made sec 42° work out to 1·34, so that 132·4 sec. 42° is 132·4 × 1·34, and this gives 177·4, not an accurate value, because the diagram was imperfect. Notice, we get a correct value by using the logs, if the logs are reliable.

Reproduce the last diagram *A B C*, make *A C* 3·4 inches, and the angle *A C B* 35° 30′ as nearly as you can. Set up *A B* at right angles. Calculate and measure the lengths of *B C* and *A B*.

$$\frac{BC}{AC} = \sec C \qquad \frac{AB}{AC} = \tan C$$

$$\frac{BC}{3{\cdot}4} = \sec 35^\circ\, 30' \qquad \frac{AB}{3{\cdot}4} = \tan 35^\circ\, 30'$$

$$BC = 3{\cdot}4 \times \sec 35^\circ\, 30' \qquad AB = 3{\cdot}4 \times \tan 35^\circ\, 30'$$

$$BC = 4{\cdot}176 \text{ inches} \qquad AB = 2{\cdot}425 \text{ inches}$$

Remember that every index in Table XIV. is 10 more than it should be; therefore, in the four examples above, after adding the logs, this extra 10 has been omitted.

Your figure will probably give *B C* about 4·2 and *A B* about 2·4 inches. Reproduce the last diagram *A B C*. This time let *C A* be the water surface, *B* a masthead, *B A* the height of the masthead above the water, say 160 feet, *C* a boat. It is possible from such a position *C* to measure with a sextant the value of the angle *B C A*. (Get someone to show you a sextant, and explain how to use it to measure an angle like *B C A*.) Suppose the angle *B C A* was 5°. Draw the figure carefully, make $C = 5^\circ$ and *B A* 1·6 inches. Measure and calculate *C B* and *C A*.

$$\frac{CB}{BA} = \operatorname{cosec} C \qquad \frac{CA}{BA} = \operatorname{cotan} C$$

$$CB = 160 \times \operatorname{cosec} 5^\circ \qquad CA = 160 \times \cot 5^\circ.$$

Multiply by adding the logs, and drop the extra 10, Table **XIV**.

$$CB = 1836 \text{ feet} \qquad CA = 1828 \text{ feet}$$

Your figure would give 18·4 inches and 18·3 if it was drawn correctly, but a small error in angle *C*, which should be 5°, will make the other sides inaccurate.

If the angle *C* was 15° and $AB = 160$ feet, then

$$CB = 618 \text{ and } CA = 597.$$

If the angle *C* was 31° and $AB = 160$ feet, then

$$CB = 310 \text{ and } CA = 266.$$

These are the answers to the nearest foot. Measure and calculate them.

Reproduce the diagram $A\,B\,C$. Let $C\,A$ represent the level ground. A boy at the point C is flying a kite B. The length of string $A\,B$ is 93 feet, and the string makes an angle of 54° ($B\,C\,A$) with the ground. Draw $B\,A$ perp. to the ground. Measure and calculate $B\,A$ the height of the kite, and $C\,A$ the length from the boy to the spot vertically under the kite. In your sketch make $A\,B$ 9·3 cm., so that 1 cm. represents 10 feet.

$$\frac{A\,B}{B\,C} = \sin C \qquad \frac{C\,A}{C\,B} = \text{cosine } C$$

$$A\,B = 93 \sin 54^\circ \qquad C\,A = 93 \cos 54^\circ$$

$$A\,B = 75{\cdot}2 \text{ feet} \qquad C\,A = 54{\cdot}7 \text{ feet}$$

To multiply 93 by sin 54° add the logs and drop out 10 from the index. Index 1 shows there must be two figures in the whole number (answer). Because 54° is at the bottom of the page, find sin and cos at the bottom of the page, and use the minutes in the right-hand column. Your sketch should give $A\,B = 7{\cdot}5$ cm, and $C\,A = 5{\cdot}5$ cm. (approx.).

Exercises in the Last Three Examples, using the Same Diagram.

1. $A\,B$ is a lamp post, $A\,C$ is its shadow on the level pavement, and the angle $B\,C\,A$ (as we shall see later) is the altitude of the sun.

If $A\,C = 10$ feet and $B\,C\,A = 37^\circ\ 10'$, find
$B\,C = 12{\cdot}55$ feet and $B\,A = 7{\cdot}581$ feet.

If $A\,C = 23$ feet and $B\,C\,A = 14^\circ\ 20'$, find
$B\,C = 23{\cdot}74$ feet and $B\,A = 5{\cdot}877$ feet.

If $A\,C = 7$ feet and $B\,C\,A = 56^\circ\ 25'$, find
$B\,C = 12{\cdot}66$ feet and $B\,A = 10{\cdot}54$ feet.

2. $A\,B$ is a mast and at C the angle $B\,C\,A$ is measured.

If $A\,B = 160$ feet and $B\,C\,A = 14^\circ\ 20'$, find
$B\,C = 646{\cdot}3$ feet and $A\,C = 626{\cdot}2$ feet.

If $A\,B = 94$ feet and $B\,C\,A = 27^\circ\ 14'$, find
$B\,C = 205{\cdot}4$ feet and $A\,C = 182{\cdot}7$ feet.

If $A\,B = 79$ feet and $B\,C\,A = 47^\circ\ 18'$, find
$B\,C = 107{\cdot}5$ feet and $A\,C = 72{\cdot}9$ feet.

3. BC is a kite string and BCA is the angle made by it with the ground.

If $BC = 93$ feet and $BCA = 42° \; 10'$, find
$AB = 62{\cdot}43$ feet and $AC = 68{\cdot}93$ feet.

If $BC = 127$ feet and $BCA = 52° \; 40'$, find
$AB = 101$ feet and $AC = 77{\cdot}02$ feet

If $BC =$ 22 yards and $BCA = 33° \; 50'$, find
$AB = 12{\cdot}25$ yards and $AC = 18{\cdot}27$ yards.

N.B.—If the measurements taken (10 feet and 37° 10′ in the first question here) are not exact, it is absurd to give the answers as 12·55 feet and 7·581 feet, and it would be better to write 12 feet (nearly) and 7 feet (nearly) Try to measure the length of the shadow of a vertical post. Later we have to consider the best way to deal with approximate values and with reliable values. Rework all these examples and vary the letters in the triangle, not triangle ABC always, but PQR or HNK or XYZ, etc., and instead of side AB or side PQ or side HK or side YZ, write side c or r or n or x respectively.

In the **sine column,** Table XIV., find the following logs, and write down the degrees and minutes :—

$9{\cdot}658284$ (p. 49) $= L \sin 27° \; 5'$
$9{\cdot}749429$ (p. 56) $= L \sin 34° \; 10'$
$9{\cdot}845405$ (p. 66) $= L \sin 44° \; 28'$
$9{\cdot}850745$ (p. 66) $= L \sin 45° \; 10'$
$9{\cdot}875682$ (p. 63) $= L \sin 48° \; 41'$
$9{\cdot}953228$ (p. 48) $= L \sin 63° \; 53'$

$9{\cdot}356984 = L \sin 13° \; 9'$
$9{\cdot}545338 = L \sin 20° \; 33'$
$9{\cdot}842424 = L \sin 14° \; 5'$
$9{\cdot}855711 = L \sin 45° \; 50'$
$9{\cdot}926110 = L \sin 57° \; 31'$
$9{\cdot}995591 = L \sin 81° \; 51'$

On a sheet of paper write only these **logs,** and practise from the logs to get the degrees and minutes quickly. Go over this small group many times till you become expert. In the same way we can deal with the other columns, Table XIV.

N.B.—In all these logs the index is 10 more than its proper value—

$$\frac{10}{2} = 5 \text{ and } \frac{24}{3} = 8,$$

so that **log 2 taken from log 10 gives log 5,**
and **log 3 taken from log 24 gives log 8.**

Similarly, if $\frac{3}{7} = \text{sine } C$ and $\frac{44}{53} = \text{sine } P$,

we can find the logs of sine C and sine P.

$$\begin{array}{ll}
\log 3 = 0{\cdot}477121 & \qquad \log 44 = 1{\cdot}643453 \\
\log 7 = 0{\cdot}845098 & \qquad \log 53 = 1{\cdot}724276 \\
\hline
L \sin C = 9{\cdot}632023 & \qquad L \sin P = 9{\cdot}919177
\end{array}$$

Add 10 mentally to the index in each top line here (see **N.B.** above)

Find these logs in the sine column, and we get (nearest values)

$$C = 25^\circ\, 22' \qquad\qquad P = 56^\circ\, 7'$$

Lay off these fraction values in a diagram. Take 3 cm. and 7 cm. in one diagram, 44 mm. and 53 mm. in another (or 3 and 7 inches, 4·4 and 5·3 inches), and measure the angles with a protractor. The values for C and P should agree with those obtained by calculation.

Draw a triangle $A\ B\ C$—

$$A = 90. \qquad A\,B = 2{\cdot}3 \text{ inches.} \qquad A\,C = 3{\cdot}5 \text{ inches.}$$

and a second triangle—

$$A = 90. \qquad A\,B = 6 \text{ cm.} \qquad A\,C = 4{\cdot}2 \text{ cm.}$$

Measure and calculate angle C in each triangle.

$$\frac{A\,B}{A\,C} = \tan C. \qquad\qquad \frac{A\,B}{A\,C} = \tan C.$$

$$\frac{2{\cdot}3}{3{\cdot}5} = \tan C. \qquad\qquad \frac{6}{4{\cdot}2} = \tan C.$$

$$\begin{array}{ll}
\log 2{\cdot}3 = 0{\cdot}361728 & \qquad \log 6 \;\;\, = 0{\cdot}778151 \\
\log 3{\cdot}5 = 0{\cdot}544068 & \qquad \log 4{\cdot}2 = 0{\cdot}623249 \\
\hline
L \tan C = 9{\cdot}817660 & \qquad L \tan C = 10{\cdot}154902
\end{array}$$

Add 10 to the upper index, and find the logs in tan column.

$$C = 33^\circ\, 19' \qquad\qquad C = 55^\circ\, 1'$$

Reproduce the triangle $A\,B\,C$, make $A = 90^\circ$, $A\,B = 6$ cm $A\,C = 7$ cm. Now try to **calculate** the values of $B\,C\,A$ and $B\,C$.

Divide the two given sides, and we notice the ratio (fraction) is called the tangent of the angle BCA. This step enables us to work out the value of the angle. In the next stage put BC, the side you wish to find, over a given side, and write the name of the fraction (ratio).

$$\frac{AB}{AC} = \tan BCA.$$

$$\frac{6}{7} = \tan BCA.$$

$$\begin{array}{rl} \log 6 = & 0\cdot778151 \\ \log 7 = & 0\cdot845098 \\ \cline{2-2} (10 \text{ added}) & 9\cdot933053 \\ BCA = & 40^\circ\, 36' \end{array}$$

$$\frac{BC}{AB} = \operatorname{cosec} BCA.$$

$$\frac{BC}{6} = \operatorname{cosec} 40^\circ\, 36'$$

$$\begin{array}{rl} \log 6 = & 0\cdot778151 \\ L \operatorname{cosec} BCA = & 10\cdot186570 \\ \cline{2-2} (10 \text{ omitted}) = & 0\cdot964721 \\ BC = & 9\cdot22 \text{ cm.} \end{array}$$

The same results would be obtained if the opening steps were

$$\frac{AC}{AB} = \operatorname{cotan} BCA. \qquad \frac{BC}{AC} = \operatorname{secant} BCA.$$

Draw a triangle ABC. Make $A = 90^\circ$, $AB = 8\cdot7$ cm., $BC = 10\cdot4$ cm. Measure and calculate angle BCA and side CA.

$$\frac{AB}{BC} = \sin BCA.$$

$$\begin{array}{rl} \log 8\cdot7 = & 0\cdot939519 \\ \log 10\cdot4 = & 1\cdot017033 \\ \cline{2-2} & 9\cdot922486 \\ BCA = & 56^\circ\, 47' \end{array}$$

$$\frac{AC}{BC} = \operatorname{cosine} BCA.$$

$$\begin{array}{rl} \log 10\cdot4 = & 1\cdot017033 \\ L \operatorname{cosine} BCA = & 9\cdot738627 \\ \cline{2-2} & 0\cdot755660 \\ AC = & 5\cdot697 \text{ cm.} \end{array}$$

The same results would be obtained if we used

$$\frac{BC}{AB} = \operatorname{cosec} BCA. \qquad \frac{AC}{AB} = \operatorname{cotan} BCA.$$

In ordinary work many obvious steps are omitted.

Draw a triangle ABC, make $A = 90^\circ$, $CB = 54$ mm,

$C A = 39$ mm. Measure and calculate angle $B C A$ and the side $B A$.

(Vary the letters; use C for the angle, a, b, c for the sides.)

$$\frac{C B}{C A} \text{ or } \frac{a}{b} = \sec C. \qquad \frac{B A}{C A} \text{ or } \frac{c}{b} = \tan C.$$

$$\begin{array}{rl} \log a = & 1{\cdot}732394 \\ \log b = & 1{\cdot}591065 \\ \hline L \sec C = & 10{\cdot}141329 \\ C = & 43^\circ\, 46' \end{array} \qquad \begin{array}{rl} \log b = & 1{\cdot}591065 \\ L \tan C = & 9{\cdot}981297 \\ \hline \log c = & 1{\cdot}572362 \\ c = & 37{\cdot}36 \text{ mm.} \end{array}$$

The same results would be obtained if we used

$$\frac{b}{a} = \text{cosine } C. \qquad \frac{c}{a} = \sin C.$$

Exercises in the Last Three Examples, using the Same Diagram.

1. $A B$ is a lamp post, $A C$ its shadow on the level pavement.

If $A B = 14$ feet and $A C = 18$ feet, find
$B C A = 37^\circ\, 52'$ and $B C = 22{\cdot}8$ feet.

If $A B = 300$ cm and $A C = 400$ cm., find
$B C A = 36^\circ\, 52'$ and $B C = 500$ cm.

If $A B = 16{\cdot}2$ feet and $A C = 7{\cdot}4$ feet, find
$B C A = 65^\circ\, 27'$ and $B C = 17{\cdot}81$ feet.

2. A is not accessible from C. Measure any length along $A B$ (at right angles to $A C$) and the distance from C to B. ($A B C$ here may be three points on a level piece of ground.)

If $A B = 42$ feet and $B C = 78$ feet, find
$B C A = 32^\circ\, 35'$ and $A C = 65{\cdot}72$ feet.

If $A B = 100$ feet and $B C = 145$ feet, find
$B C A = 43^\circ\, 36'$ and $A C = 105$ feet.

If $A B = 247$ yards and $B C = 536$ yards, find
$B C A = 27^\circ\, 26'$ and $A C = 475{\cdot}7$ yards.

3. $B\,C$ is a kite string, and A is the point vertically under B.

If $B\,C = 42$ feet and $C\,A = 32$ feet, find
$B\,C\,A = 40°\ 22'$ and $A\,B = 27{\cdot}2$ feet.

If $B\,C = 94$ feet and $C\,A = 39$ feet, find
$B\,C\,A = 65°\ 29'$ and $A\,B = 85{\cdot}52$ feet.

If $B\,C = 82$ feet and $C\,A = 64$ feet, find
$B\,C\,A = 38°\ 42'$ and $A\,B = 51{\cdot}27$ feet.

N.B.—Give approximate answers when measurements are not accurate. Rework these examples by the alternative method, and keep the same values but use different letters for the triangle.

In right-angled triangles, if (in addition to the right angle 90°) two sides are known, or if one side and one angle are known, then the other parts can be found.

The examples above show—

(1) How to get an angle when two sides are known.

(2) How to get a side when one side and one angle are known.

(3) How to find the third angle. If one angle is 90°, the other two angles together must make 90°.

N.B.—Always check your calculated values by taking measurements from a diagram drawn to some convenient scale. Do not use very small diagrams. There is also a method of checking results by **inspection.**