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PREFACE

The purpose of this text is to introduce the student to a wide and extensive body of synthetic geometry. With only the high school courses in plane geometry (including, at best, trigonometry) as prerequisites, the student may become acquainted with a large body of more or less connected geometric doctrine. Beautiful in its simplicity, it concerns the geometry of the triangle and the circle, and requires only the known Euclidean concepts. It is not the purpose of this book to be exhaustive, or even to explore the whole field, but to call the attention of the student to the existence of this geometric doctrine, which, in spite of its elementary character, is by no means well known to mathematicians in general.

A number of problems have been included. Some of these are simple; others require the reproduction of the material in the text, while a few, usually marked by an asterisk or followed by a suggestion, will test the skill of the better students or require a search of the available literature. A key to the solution of the problems has been included, but it is hoped that the student will seriously try the problems first and obtain solutions independent of the key.

A partial bibliography at the end lists a number of the more important references, some of which will no doubt be available in any college library. The author is indebted to all of them, for he has consulted them and found them useful in the preparation of this book.

One of the pleasant tasks of an author is to acknowledge his obligations to those who assisted him. My colleague, Professor Clifford Bell, used this text in manuscript form, made many suggestions for its improvement, and was extremely helpful in reading the entire text in proof. To

Roy H. Luke, I owe a still greater debt of gratitude for preparing the illustrations for the engraver in such a satisfactory manner. And finally I wish to express to Prentice-Hall, Inc., my appreciation for their many courtesies while the book was in process of being published.

PAUL H. DAUS

HISTORICAL INTRODUCTION

The history of demonstrative geometry properly begins with the Greek geometer Thales (640–546 B.C.). His actual contributions to geometrical knowledge were few, but he first recognized the necessity of giving a demonstration based upon a logical sequence of ideas. He was followed by Pythagoras (572–501 B.C.) and his school, whose main contribution was that they studied mathematics from the immaterial or intellectual viewpoint, and thus raised mathematics to the rank of a science. Many geometers followed: Hippocrates (460 B.C.) wrote the first elements of geometry; Zeno (450 B.C.), the founder of the so-called Neo-Pythagorean School, set forth the famous paradoxes that caused mathematicians and philosophers to examine carefully the foundations of mathematics; Democrites, Antiphon, Bryson, Hippias, Archytas, Eudoxus, Menaechmus, and Demostratus gave themselves to the task; and the period terminated in the greatest of them all, Euclid of Alexandria. If not the founder, at least the head of the famous mathematical school of Alexandria (about 300 B.C.), Euclid attempted and succeeded in doing what no other mathematician ever did or can hope to do again: namely, to collect all the essentials of the accumulated mathematical knowledge of his time. He is the greatest textbook writer of all time. The best known of his works, his *Elements*, has gone through innumerable editions. Perhaps he was too successful, for two thousand years later, when new geometrical ideas and modern pedagogy were struggling for recognition, the unrivaled supremacy of Euclid made the assimilation of these ideas extremely difficult. Euclid did not write for high school students, but the English clung tenaciously to the literal translations even when geometry finally forced its way down from the universities to the secondary schools. Lagrange, in the last half of the eighteenth century, separated the theorems from the exercises and tried to organize the material from a modern point of view. But America followed England until lately, when our own texts have become more human.

Closely following Euclid were two famous mathematicians: Apollonius (225 B.C.), who attempted to do for the conic sections what Euclid did for the circle and the triangle, and Archimedes, the most brilliant mathematician of antiquity. Their works are not so well known only because their methods were outshone by those of Descartes' geometry and the calculus of Newton and Leibnitz. From that time on little progress was made in geometrical ideas until the renaissance of pure geometry at the beginning of the nineteenth century. But there are a few names and ideas that should be mentioned as they occur in this course. For three centuries after Archimedes, during which Greece fell and Rome came to power, little progress was made. About 100 A.D. a Roman mathematician, Menelaus, discovered the theorem that bears his name, concerning a triangle and a transversal. Carnot later made this theorem the foundation of an extensive development. The theorem was forgotten, however, and rediscovered 1500 years later, by another Italian mathematician, Giovanni Ceva, who also discovered an important companion theorem. To Pappus (300 A.D.) we owe the development of harmonic division, although the idea was known to Menelaus and probably to Apollonius before him. With Pappus the first epoch of geometrical knowledge closed.

The birth of analytic geometry, the brilliant discovery of Descartes (1635), although he must share the honor with Harriot and Fermat, completely masked the projective geometry of Desargues, which might have started a geometrical revival. When all Europe grasped hastily at these marvelous tools (for analytic geometry paved the way for the calculus), when mathematicians could solve in an instant problems which bothered the ancients for ages—it is no wonder that pure geometry was neglected and that the contributions of a Desargues or a Pascal lay quiescent for 150 years. Two hundred years after Desargues' time, no copies of his work could be found, except a copy fortunately made by one of his pupils, De la Hire. The fault was partly Desargues', for although admired by Descartes, Fermat, and others, he refused to listen to advice and failed to heed Descartes' warnings. He introduced seventy new terms, of which but one, *involution,* lives today, and that owes its life to the fact that it was picked out for the most ridicule by one of his reviewers.

But the youth of analytic geometry and the calculus could not last forever, and about 1800 the renaissance of pure geometry occurred. This developed along two distinct lines: projective geometry, and the geometry of the circle and the triangle. We are at present concerned with the latter. It began with the descriptive geometry of Monge, published in 1794, although developed 30 years earlier. Mascheroni (1800) gave us the geometry of compass constructions, which was systematized later by Adler by means of the geometry of inversion. Carnot, starting with the ideas of Menelaus and Ceva, built up his geometry of transversals. Crelle in 1816 started the modern geometry of the triangle. In 1822 Feuerbach enunciated the famous theorem that goes by his name. In 1840 Steiner gave projective geometry its modern metrical point of view and made extensive additions to the geometry of the circle and the triangle, frequently announcing results without proof, and thus setting problems that excited interest for many years thereafter. In 1873 Lemoine started the subject of geometrography, dealing with a certain phase of construction problems. In 1877 Brocard announced his results about the Brocard points, and since that time the number of new results in that and related connections has been almost transfinite. It is not the purpose of this course to go into these most recent developments, but enough has been mentioned to indicate the tremendous geometric development since the "Heroic Age of Geometry" began.

CONTENTS

1 SIMILAR FIGURES

1. Introduction

In elementary plane geometry we frequently use the simple rigid motions; that is, translation and rotation, with occasional use of reflection. These transformations are characterized by the fact that both distances and angles remain invariant under these motions. These ideas are included in the postulates of Euclid, which imply that figures remain unaltered when moved about, and become the basis of all so-called proofs by the method of superposition. Some use is made in elementary geometry of similar figures, and we assume the student is familiar with the elementary properties of similar triangles. We shall proceed to expand this idea in this chapter.

A number of propositions in elementary geometry deal with problems of concurrency of three lines or collinearity of three points. In Chapter 2 we shall systematize the procedure for the discussion of such problems. In Chapter 3 we shall consider a transformation called inversion. This is one of our most powerful and important tools for the study of the modern geometry of the triangle and circle. Although the technical difficulties of Chapter 3 may appear greater than those of later chapters, it is most desirable to master this transformation at this point as a tool for later work.

2. Division of segments

We shall first recall some elementary results concerning line segments. We adopt the usual conventions of analytic geometry. We assign arbitrarily the positive direction to a line, and if A and B are any two points on the line, we define

$AB = -BA$, where AB designates the directed length of the line segment from A to B. Under such a convention, if A, B, and O are three points in any order on the line,

$$AB = AO + OB = OB - OA.$$

If we take O as the origin and let the distances $OA = a$, and $OB = b$, this relation becomes

$$AB = b - a.$$

If X is a point on the line which divides the segment AB into two segments AX and XB, then their ratio

$$r = \frac{AX}{XB} = \frac{x - a}{b - x}. \tag{1}$$

This ratio is positive if X is between A and B; otherwise it is negative. It has a unique value for each given point X, with the understanding that if A and X coincide, then $r = 0$, and if B and X coincide, then $r = \infty$. Further, if X proceeds to infinity along the line, the limit of r is -1. Conversely, if A and B are any two points on a line, and r is any given ratio, there exists a unique point X such that $AX/XB = r$. For the solution of equation (1) for x yields a unique value

$$x = \frac{a + rb}{1 + r}, \tag{2}$$

except when $r = -1$. If $r = 0$, it is evident that X is identical with A. If we divide both the numerator and denominator of the fraction (2) by r, we see that as r tends to infinity, X tends to B. Finally, if we permit r to approach -1, X becomes the ideal point which we postulate is on the line at infinity.

Two distinct lines in the same plane intersect or are parallel. The fundamental Euclidean parallel postulate for the plane may be stated in the form: *One and only one parallel to a given line passes through a given point not on the line.* In order to avoid the continual and annoying statement of exceptions occasioned by the difference between parallel lines and intersecting lines, it is convenient to interpret the parallel postulate in terms of *ideal points*, and replace the parallel postulate by the following statement: *On a given line there is one and only one ideal point, which lies on every parallel to the given line and is at an infinite distance from every other point of the line.* With this convention, two lines in the plane always intersect, but the ideal point at infinity on any line may be considered only in connection with a limit process.

The set of all ideal points in the plane has the characteristic that any other line contains one and only one point of the set. Because of this property we agree to call this set the *ideal line* at infinity. One must not expect that it has all the properties of an ordinary line, for it also must be considered only in connection with a limit process.

At times it is convenient to begin both segments at X; this merely changes the sign of r. Also, as in the theory of similar triangles, we may be interested in the ratio of one segment to the whole segment, say $AX/AB = k$. The equations which correspond to (1) and (2) are

$$k = \frac{x - a}{b - a}, \qquad x = a + k(b - a), \tag{3}$$

with $k = 0$ corresponding to A, $k = 1$ to B, and $k = \infty$ to the ideal point at infinity. It may be left as an exercise for the student to determine the relations between r and k.

* As a numerical example, it is readily seen by constructing a diagram, that if T is the first trisection point of AB, then $k = AT/AB = 1/3$, and $r = AT/TB = 1/2$. If, however, we extend AB to S, so that B is the first trisection point of AS, then $k = AS/AB = 3$, and $r = AS/SB = -3/2$. If the extension is made in the opposite direction to R, so that A is the first trisection point of BR, then $k = AR/AB = -2$, and $r = AR/RB = -2/3$.

3. Angle between two lines

The distance between two finite points P and Q is uniquely understood so that the distance PQ is a definite, finite, posi-

* Smaller type indicates the less essential portions of this book.

tive or negative quantity, but this uniqueness does not appear when we consider the angle between two lines which intersect in a finite point. The angle from the line p to the line q is the measure of rotation of p about the point of intersection of the lines until p coincides with q. It is apparent that this rotation could be either counterclockwise (positive) or clockwise (negative) as indicated in Fig. 1, so that $\measuredangle\ pq$ is ambiguous; we might also change either of the angles indicated in the figure by multiples of 180° to further complicate this ambiguity. Because of this we cannot say $\measuredangle\ pq = -\measuredangle\ qp$, but we can say that $\measuredangle\ pq + \measuredangle\ qp$ is a multiple of 180°, no matter which of the possible values we use for either $\measuredangle\ pq$ or $\measuredangle qp$. We borrow a term from the Theory of Numbers and then say $\measuredangle\ pq + \measuredangle\ qp$ is congruent to zero modulo 180°, written

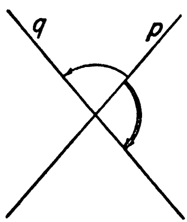

Fig. 1

$$\measuredangle\ pq + \measuredangle\ qp \equiv 0 \pmod{180^\circ}$$

or again

$$\measuredangle\ pq \equiv -\measuredangle\ qp \pmod{180^\circ}.$$

Where there can be no misunderstanding, we may omit the term "modulo 180°" and merely say $\measuredangle\ pq$ is congruent to $-\measuredangle\ qp$.

If a, b, c are any three lines through the same point, then

$$\measuredangle ab \equiv \measuredangle\ ac + \measuredangle\ cb \pmod{180^\circ},$$

but the corresponding statement with equal signs would not necessarily be true.

We shall illustrate this notation by stating a simple but very useful necessary and sufficient condition that four points be concyclic.

If A, B, C, D are four points on a circle, then (see Fig. 2)

$$\measuredangle\ ACB = \measuredangle\ ADB,$$

where the angles are those marked by a single directed arrow. If in the figure on the right, we use the angle which measures the rotation of the half line DA to the half line DB, this angle being negative, our relation becomes

$$\measuredangle\, ACB \equiv \measuredangle\, ADB \pmod{180°}.$$

This relation is obviously correct no matter which of the marked angles at D we use. For example, in Fig. 2b, if

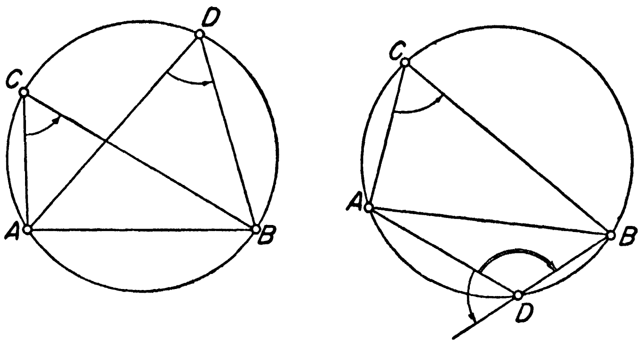

Fig. 2a Fig. 2b

$\measuredangle\, ACB = 75°$, and we take $\measuredangle\, ADB$, as indicated by the double arrow, as $-105°$, we have $75° \equiv -105° \pmod{180°}$, since $75° = -105° + 1 \cdot 180°$.

Since the converse results can be established by the method of *reductio ad absurdum*, the details of which do not concern us just now, we may state the theorem in the following form:

THEOREM. *A necessary and sufficient condition that the points* A, B, C, D *be concyclic is that the angles which* AB *subtends at* C *and* D, *sense being taken into account, be congruent modulo* 180°.

4. Homothetic figures

If O is a fixed point and the point P describes a curve, and if P' is a point on OP such that

$$\frac{OP'}{OP} = k \qquad (k \neq 0 \text{ or } 1),$$

the locus of P' is said to be homothetic (similar and similarly placed) to the locus of P. The point O is called the homothetic center, and k the homothetic ratio or ratio of similitude. The constant k may be positive or negative, and the curves are accordingly directly or inversely homothetic.

PANTOGRAPH. There is a well known elementary linkage, called a pantograph, which enables one to draw mechanically a curve homothetic to another given curve (Fig. 3). It consists essentially of four rods so arranged that the figure $ABCP$ is a parallelogram and the lengths $OA = AP$ and $PC = CP'$. The point O is fixed, and the linkage is pivoted at P, A, B, C. If P describes a curve, then P' describes a curve homothetic to it.

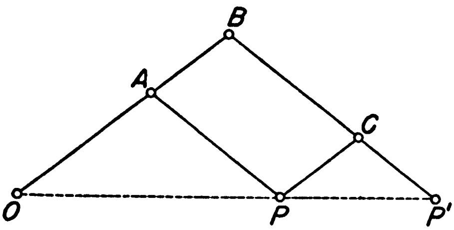

Fig. 3

From the parallelogram we have $AB = PC$ and $AP = BC$, so that $OB = BP'$, and $\measuredangle\ OAP = \measuredangle\ OBP'$. It follows that the isosceles triangles OAP and OBP' are similar, so that $\measuredangle\ POA = \measuredangle\ P'OB$; hence the three points O, P, P' are collinear. Further $OP'/OP = OB/OA = k$. By means of adjustable pivots, the value of k may be varied.

5. Homothetic circles

The student can readily show that the curve homothetic to a line is a line parallel to the given line; and hence the figure homothetic to a triangle is a similar triangle whose sides are parallel to those of the given triangle. (See problems 9, 10, and 11 in exercise 1.)

The converse theorem is more difficult to prove and, as an illustration of a commonly used indirect method, we outline a proof here. The student should draw the diagram for the problem:

If two noncongruent triangles are such that their corresponding sides are parallel, the triangles are homothetic.

We recognize that we must prove that the lines joining corresponding vertices meet in a point O and that $OP/OP' = k$, where P and P' are any pair of corresponding points. We assume that the direct theorem has been proved. Let the triangles be ABC and $A'B'C'$, and let AA' and BB' meet in O. Then

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{AB}{A'B'} = \text{constant}.$$

Construct the triangle $A'B'C''$ homothetic to the triangle ABC with respect to O and the above ratio. Then $B'C''$ is parallel to BC, and hence the lines $B'C'$ and $B'C''$ coincide. Similarly the lines $A'C'$ and $A'C''$ coincide, so C' and C'' coincide and the given triangles are homothetic from O with ratio $AB/A'B'$.

A second indirect method is suggested in the Key to the Solution of the Exercises.

For the circle we have the following elementary theorem:

THEOREM. *The curve homothetic to a circle is a circle and the centers are corresponding points in the transformation.*

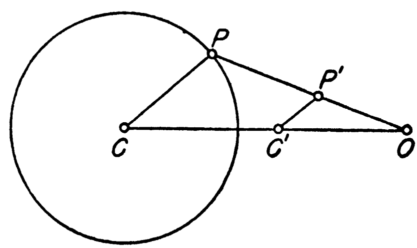

Fig. 4

Let P (Fig. 4) be any point on the circle whose center is C, and P' determined on OP such that $OP'/OP = k$. Draw $P'C'$ parallel to PC. Then

$$\frac{OC'}{OC} = \frac{P'C'}{PC} = \frac{OP'}{OP} = k.$$

But OC and PC are constant, and hence OC' and $P'C'$ are also constant, and P' lies on a circle whose center is C'. The points C and C' are corresponding points in the transformation.

We note that if k is positive, then O is exterior to CC', while if k is negative, O is between C and C'. The student should draw diagrams for the cases when k is positive, when it is negative, and when O is within or outside of the given circle.

Before proceeding further we digress a moment to consider such terms as *converse, necessary and sufficient*, and *locus.* If a theorem has but a single statement in its hypothesis and a single statement in its conclusion, then the result of interchanging the hypothesis and conclusion is called *the converse* theorem. If, however, as often happens, the hypothesis contains more than one condition, expressed or implied, more than one converse theorem may be obtained by replacing different parts of the hypothesis by parts of the conclusion, and the word "converse" thus becomes ambiguous. Clearly, in this case, the precise converse to be proved should be stated.

If statement B implies statement A, then A is a *necessary* condition that B be true. Conversely, if statement A implies statement B, then A is a *sufficient* condition that B be true. If the theorem and its converse are true, we say that either statement is the *necessary and sufficient* condition that the other be true. In the proof of a theorem which involves a number of steps, it may be true that *all the converses* of the intermediate theorems are true and hence the converse of the original theorem may be obtained by retracing the steps. But a mere statement, "retrace the steps," is fraught with dangers and is never satisfactory unless we know that all the converses are true. The algebraic processes are more readily recognized than the geometric ones, and for that reason we need to be especially circumspect, and make sure the steps are really retraceable.

The term "locus of a point" in the sense that we shall use it implies both a necessary and a sufficient condition. To determine the locus requires the proof of a theorem and a converse theorem. We must show that every point on the locus satisfies certain conditions and conversely that every point that satisfies these conditions lies upon the locus.

After this digression we return to the subject of homothetic circles, and prove the following theorem which is in the nature of a converse to the preceding one:

THEOREM. *If* A *and* B *are the centers of two circles of* *unequal radii* a *and* b, *and if* AB *is divided internally at* O_1 *and externally at* O *in the ratio of the radii, then the circles are homothetic with respect to both* O *and* O_1, *and corresponding points lie on the extremities of parallel radii.*

Let

$$\frac{OA}{OB}=\frac{a}{b},$$

and let AP and BP' be parallel radii (Fig. 5a), drawn so

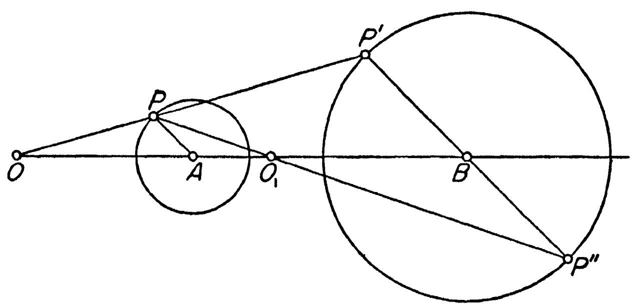

Fig. 5a

that P and P' are on the same side of the line AB. Since $OA/OB = AP/BP'$ and the angles at A and B are equal, it follows that the triangles OAP and OBP' are similar. Hence $\measuredangle AOP$ and $\measuredangle BOP'$ are equal; O, P, and P' are collinear; and $OP/OP' = a/b$—that is, the circles are homothetic with respect to O. Precisely the same argument may be applied to O_1, defined by $AO_1/O_1B = a/b$, with the radii drawn in opposite senses; the details are left to the reader.

It may be remarked that if the radii are equal, the external homothetic center is the ideal point at infinity on the line AB, but the indicated properties follow directly as a limiting case. Hereafter we shall not always point out these limiting cases but imply them and leave their recognition and proof as exercises for the reader. Only when they are important shall we mention them explicitly.

We call attention to the following corollaries: (1) If OP is tangent to one circle, it must also be tangent to the other, and the points of contact lie on the extremities of parallel radii. (2) Since the points O and O_1 are unique, the circles can be homothetic in only two ways. The points are called the homothetic centers of the two circles.

The two given circles may be in three distinct, relative positions—external to each other as in Fig. 5a, intersecting

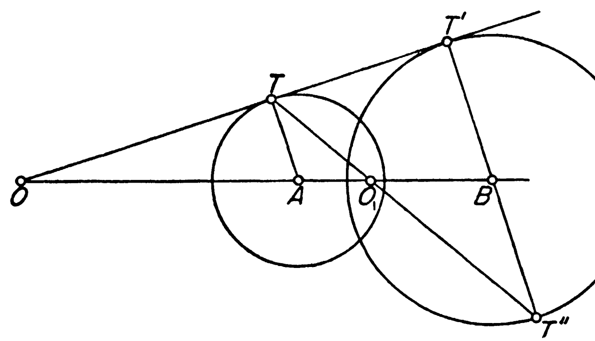

Fig. 5b

as in Fig. 5b, or one inside the other as in Fig. 5c. The first corollary indicates that if the circles have a common tangent, it passes through a homothetic center of the two circles and this center is outside of both circles. If O is inside one circle, it must be inside the other, for, from the relation $OA/OB = a/b$, if $OA < a$, then $OB < b$. Similar statements are true for O_1.

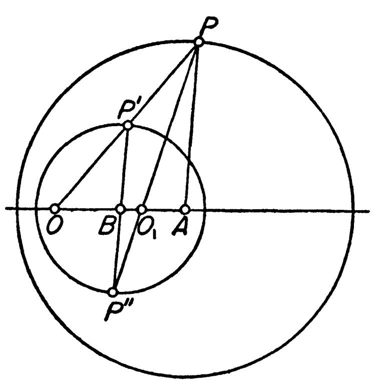

Fig. 5c

These notions suggest a solution of the problem to construct a common tangent to two circles. By means of parallel radii, we may construct the homothetic centers. If one of these is outside the circles, we construct the tangent from this point to one of the circles. We recall how this can be done. Suppose the homothetic center is O. Then the circle that has OA as diameter will intersect the circle that has the center A at the points of contact.

6. Similar figures

Two figures are said to be similar if all corresponding angles are numerically equal. It is a proposition of elementary geometry that this necessitates that corresponding segments are proportional, and conversely. If the angles of both figures are described in the same sense, the figures are *directly similar*; if described in the opposite sense, the figures are *inversely similar*.

By way of review, we recall a second criterion that four points be on a circle.

THEOREM. *If* A, B, C, D *are four points and if* AB *meets* CD *at* O, *then a necessary and sufficient condition that the four points be concyclic is* $\mathrm{OA} \cdot \mathrm{OB} = \mathrm{OC} \cdot \mathrm{OD}$.

Suppose the four points are on a circle. Then (Fig. 6)

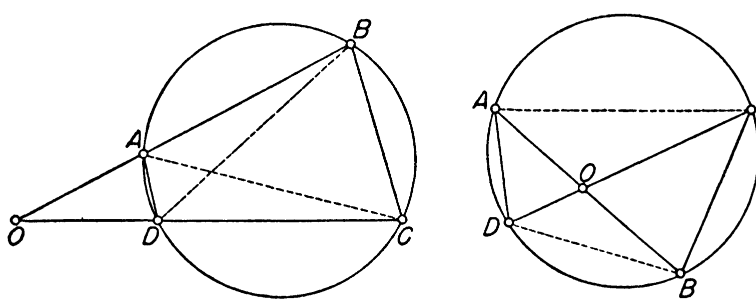

Fig. 6

from the equality of angles, the triangles OAD and OCB are directly similar, so that $OA/OD = OC/OB$ or

$$OA \cdot OB = OC \cdot OD.$$

Conversely, suppose this equation is true and consider the circle determined by A, B, and C. Let OC cut this circle at D'. Then $OA \cdot OB = OC \cdot OD'$, and it follows that $OD = OD'$; hence D and D' coincide, and the four points A, B, C, D are concyclic.

The lines AC and BD are said to be *antiparallel* with respect to AB and CD. The property that the lines AC and BD are antiparallel with respect to AB and CD is equivalent to the above condition that the four points A, B, C, D be concyclic. This condition may be expressed in terms of angles as in section 3. It follows that AD and BC are antiparallel with respect to AB and CD, and likewise AB and CD are antiparallel with respect to AC and BD.

It is obvious that if any or all of the transformations, translations, rotations, reflections, or expansions from a fixed point (homothetic transformations) be performed on a figure, the resulting figure will be similar to the original one. Conversely, we can now prove that if two figures are similar, they can be made to coincide by a series of these transformations, thus indicating that the general similarity transformation includes the four transformations mentioned.

If the two figures are inversely similar, a reflection of one of them on an arbitrary line yields two directly similar figures, and consequently we confine our further attention to this case.

7. Center of similitude

The following theorem is fundamental:

THEOREM. *Any segment* AB *can be carried into any segment* A′B′ *in one and only one way by a transformation which is either* (1) *a translation, or* (2) *a rotation followed by an expansion about the same fixed point.*

CASE 1. AB parallel to $A'B'$. If AB equals $A'B'$, a translation achieves the result; if AB is not equal to $A'B'$, an expansion suffices, as may readily be seen by constructing the simple figures, provided that the segments are not on the same line. If the segments are on the same line, let O be the unique point such that

$$\frac{OA'}{OA} = \frac{A'B'}{AB} = k.$$

Then

$$\frac{OB'}{OB} = \frac{OA' + A'B'}{OA + AB} = \frac{k \cdot OA + k \cdot AB}{OA + AB} = k,$$

and hence an expansion from O suffices. (If $k = 1$, and A is distinct from A', then O is at infinity. If A and A' coincide, then for any value of k, O is at A.)

Case 2. AB not parallel to $A'B'$. Let AB and $A'B'$ meet at P. Let the circles through A, A', P and B, B', P intersect again at O. We see at once from the consideration of circular arcs (Fig. 7a) that the triangles OAB and $OA'B'$ are directly similar, for $\measuredangle\ OAB = \measuredangle\ OA'B'$, and also $\measuredangle\ OBA = \measuredangle\ OB'A'$. Hence if we rotate OAB about O until A falls on OA' and then if we expand OAB with regard to O as center until A coincides with A', B will coincide with B'.

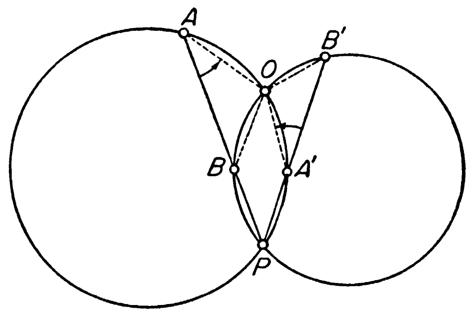

Fig. 7a

Conversely, if there exists a point O about which we may make such a transformation, then the angles AOB and $A'OB'$ are equal and $OA/OA' = OB/OB'$. It follows that $\measuredangle\ BAO = \measuredangle\ B'A'O$, or what is the same thing, $\measuredangle\ PAO \equiv \measuredangle\ PA'O \pmod{180^\circ}$; hence A, A', P, O, and similarly B, B', P, O, are concyclic (section 3), so that our construction determines the point O. The point O is called the *center of similitude*. If a translation is involved, the center of similitude is considered to be at infinity. The fixed ratio is called *the ratio of similitude*. In case there is no rotation (or translation), but merely an expansion, the center of similitude is a homothetic center; that is, the homothetic transformation is a special case of the general similarity transformation in the case where an expansion alone suffices.

If some of the points considered above coincide, the circles take special positions, but the proof holds without essential modification. If one of the given points, say B, coincides with P, the circle BPB' is the circle through B', tangent to AB at B. If the two circles are tangent at P, so that O and

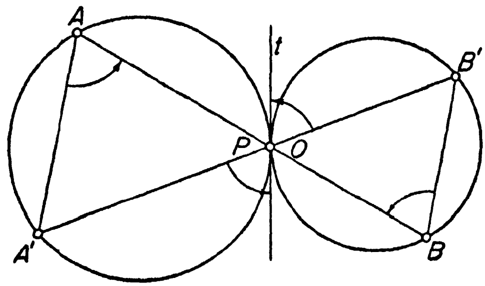

Fig. 7b

P coincide (Fig. 7b), then AA' and BB' are parallel. For a consideration of arcs shows that

$$\measuredangle A'AO \equiv \measuredangle A'PO \equiv \measuredangle B'PO \equiv \measuredangle B'BO,$$

making AA' and BB' parallel lines. The construction for O is still valid; a rotation about O, followed by a homothetic transformation from O, transforms AB into $A'B'$.

Let us now consider two triangles (or polygons) which are directly similar. Then there exists uniquely either (1) a translation or (2) a rotation followed by an expansion about a fixed point, that carries one polygon into the other. For the transformation of the last theorem that carries AB into $A'B'$ must also carry any other vertex C into C' because of the equality of angles. The definitions given above apply equally well to our present configuration, and we may summarize the results by the statement that two directly similar polygons have one, and only one, center of similitude.

An interesting interpretation of the foregoing result is obtained if we consider two maps of the same plane region drawn to different scales. If the maps are laid one upon the other, then there exists one and only one point whose representations in both maps coincide (of course it may fall outside the boundaries of the actual maps). If the maps are drawn to the same scale, either one may be rotated about a fixed point or moved parallel to itself until they coincide.

8. Centers of similitude for two circles; circle of similitude

If we consider a circle as the limiting position of an inscribed polygon, we might consider that any point on the second circle corresponds to a given point on the first circle, and so suspect that there would be an infinite number of centers of similitude. In this connection we have the following theorem:

THEOREM. *The locus of centers of similitude of two non-concentric unequal circles is a circle having as diameter the line joining the two homothetic centers.*

Let P (Fig. 8) be any position of the center of similitude;

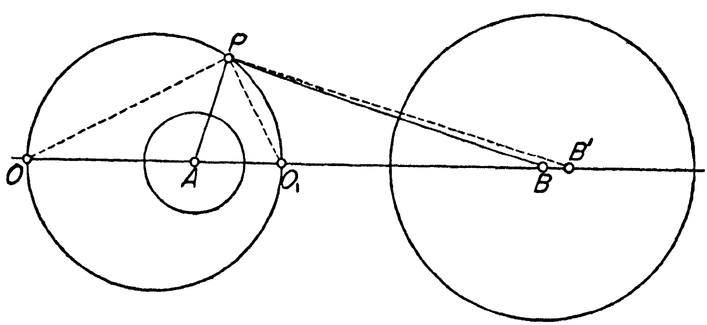

Fig. 8

that is, P is such that a rotation about P followed by a homothetic transformation from P will transform one circle into the other. We must then have

$$\frac{PA}{PB} = \frac{a}{b}.$$

But O and O_1 divide the segment AB externally and internally in this same ratio. Hence, from a proposition of elementary geometry (problem 4, exercise 1), PO and PO_1 are the external and internal bisectors of the angle at P, and so are perpendicular. Consequently P lies on the circle having

OO_1 as diameter. This circle is known as the *circle of similitude.*

Conversely, let P be any point on this circle, so that PO is perpendicular to PO_1, and construct PB' so that PO_1 bisects the angle APB' internally. Then PO is the external bisector of this same angle, and it follows that O and O_1 divide the segment AB' internally and externally in the same ratio. This implies that B and B' coincide, for by combining

$$\frac{AO}{OB} = -\frac{AO_1}{O_1B} \quad \text{and} \quad \frac{AO}{OB'} = -\frac{AO_1}{O_1B'},$$

we obtain

$$\frac{OB}{BO_1} = \frac{OB'}{B'O_1}.$$

Hence (section 2) B and B' coincide so that PO and PO_1 are the angle bisectors of angle APB. Hence $PA/PB = a/b$, and after PA is rotated into the position PB, an expansion from P makes the circles coincide.

If the two circles are equal, since O is at infinity, there is properly no circle of similitude, but the locus of the centers of similitude is the perpendicular bisector of the line segment joining the centers. This line may be considered as the limiting position of the circle of similitude as the radii of the circles approach equality.

Exercise 1. Similar Figures

1. If A and B are two points on a line, trace the change in $r = AX/XB$ as X moves along the line; similarly for $k = AX/AB$. Discuss the relation between r and k.

2. Give a geometric construction for the points that divide a given line segment in the ratios a/b and $-a/b$, where a and b are given positive integers.

3. (a) Euler's Theorem. If A, B, C, D are any four points on a line, prove $DA \cdot BC + DB \cdot CA + DC \cdot AB = 0$.

(b) Restate the theorem beginning with the point A.

4. State and prove the elementary propositions used in section 8 concerning the bisectors of the angles of a triangle.

5. Prove the converse indicated in section 3; namely, that if A, B, C, D are four points such that angle $ACB \equiv$ angle ADB, then the four points are concyclic.

6. If two lines are each antiparallel to a third line with respect to a given pair of lines, prove the first two lines are parallel.

7. If ABC is a triangle inscribed in a circle, prove that any line parallel to the tangent at A is antiparallel to BC with respect to the other sides.

8. Arrange the rods of a pantograph so that the figures will be inversely homothetic. Letter the figure as in the text and follow the proof to see if it still holds.

9. Prove that the curve homothetic to a line is a line.

10. Show that angles remain invariant under a homothetic transformation.

11. (a) Prove that the figure homothetic to a triangle is a similar triangle.

(b) Use this property to prove that the curve homothetic to a circle is a circle.

12. (a) If two triangles are such that their corresponding sides are parallel, the triangles are homothetic.

(b) Apply this result to the triangle formed by joining the midpoints of the sides of a given triangle. What is the homothetic ratio? Where is the homothetic center?

13. Use problem 12(a) to solve this problem: Through a given point draw a line through the inaccessible (that is, not meeting on the paper) intersection of two given lines.

14. In a given triangle inscribe a triangle whose sides are parallel to the sides of a third given triangle. (*Suggestion*: Place one side and use the method of homothetic figures.)

15. (a) If $ABCD$ is a simple convex (nonre-entrant) quadrilateral inscribed in a circle, by considering the triangles formed by joining the vertices to the center, prove that angle A + angle C = angle B + angle D, where angle A = angle DAB, and so forth; that is, prove that the sums of the pairs of opposite angles are equal.

(b) Discuss the same problem if the quadrilateral is re-entrant, showing how the sense of direction is involved.

***16.** If a simple quadrilateral is circumscribed to a circle, using the fact that the tangents from a point to the circle are equal, derive a relation analogous to that of problem 15. Discuss the cases of a nonre-entrant and a re-entrant quadrilateral.

17. Stewart's Theorem. (a) If A, B, C, D are four points on a line, show that

$$DA^2 \cdot BC + DB^2 \cdot CA + DC^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

(b) If A, B, C are collinear and D is any point not on their line, show that the same equation is true. (*Suggestion*: Draw DD' perpendicular to AB. Apply the theorem of Pythagoras and case (a) above.)

***18.** Make a study of the properties of inversely similar figures. (Consult Lachlan, *Modern Pure Geometry*, pages 134 and following.)

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* Problems preceded by an asterisk may require an extensive discussion or a search of the literature. A bibliography is given at the end of this text.

2 CONCURRENCY AND COLLINEARITY

Part 1. Theorems of Menelaus and Ceva

9. Definitions

Three or more points are said to be *collinear* if they lie on a line; three or more lines are said to be *concurrent* if they pass through the same point. The student is already familiar with a number of propositions of elementary geometry involving these notions, proofs of which can be found in any elementary text. Such theorems will also serve as good applications of the theorems of Menelaus and Ceva which follow. The proofs of these theorems given in sections 10 and 11 are due to Jacob Steiner.

10. Menelaus' Theorem

This theorem was discovered by Menelaus of Alexandria (about 100 A.D.), but was forgotten and rediscovered 1600 years later by an Italian mathematician, Giovanni Ceva, who published, in 1678, both this theorem and the one that bears his name.

THEOREM. *Three points* B_1, B_2, B_3 *on the respective sides of a triangle* $A_1A_2A_3$ *are collinear if and only if*

$$\frac{A_1B_3}{B_3A_2} \cdot \frac{A_2B_1}{B_1A_3} \cdot \frac{A_3B_2}{B_2A_1} = -1. \qquad \textbf{(A)}$$

If B_1, B_2, B_3 are collinear (Fig. 9), let a line through A_2 parallel to A_1A_3 cut the *transversal* B_1B_2 in K. The transversal cuts one or three of the sides of the triangle externally and hence one or three of the ratios in (A) are negative; consequently the product of these three ratios is negative. The signs are accounted for in the proportions below.

From the similar triangles B_1A_2K and $B_1A_3B_2$, we have

$$\frac{A_2K}{A_3B_2} = \frac{A_2B_1}{A_3B_1} \quad \text{or} \quad A_2K = -\frac{A_3B_2 \cdot A_2B_1}{B_1A_3}. \tag{1}$$

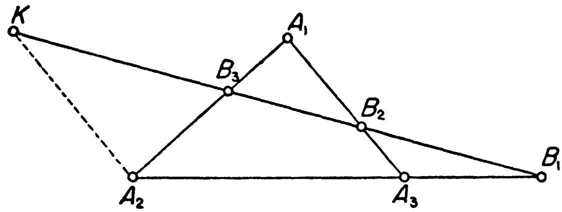


Fig. 9a

From the similar triangles A_2KB_3 and $A_1B_2B_3$, we have

$$\frac{A_2K}{B_2A_1} = \frac{B_3A_2}{A_1B_3} \quad \text{or} \quad A_2K = +\frac{B_2A_1 \cdot B_3A_2}{A_1B_3}. \tag{2}$$

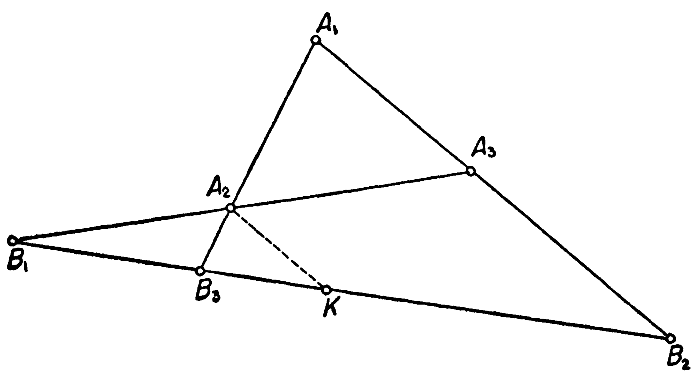


Fig. 9b

Equating the two values of A_2K and simplifying, we obtain the relation (A).

Conversely, if the equation (A) is satisfied for points B_1, B_2, B_3 on the sides of the triangle, let B_1B_2 cut $a_3 = A_1A_2$ in B_3'. Then in addition to equation (A), we have a similar relation with B_3 replaced by B_3'. Hence

$$\frac{A_1B_3'}{B_3'A_2} = \frac{A_1B_3}{B_3A_2},$$

from which by the laws of proportion we find

$$\frac{A_1B_3'}{A_1A_2} = \frac{A_1B_3}{A_1A_2},$$

and B_3 and B_3' coincide, completing the proof.

11. Ceva's Theorem

The three lines which join the three points B_1, B_2, B_3 *on the respective sides of the triangle* $A_1A_2A_3$ *to the opposite vertices are concurrent if and only if*

$$\frac{A_1B_3}{B_3A_2} \cdot \frac{A_2B_1}{B_1A_3} \cdot \frac{A_3B_2}{B_2A_1} = +1. \tag{B}$$

The following proof shows that this and the last theorem are not independent, for this proof is made to depend upon Menelaus' Theorem.

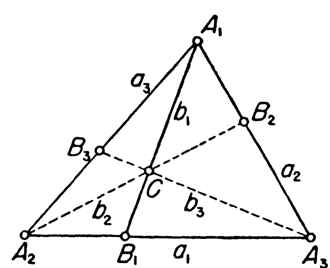


Fig. 10

Suppose that A_iB_i meet in a point C (the Ceva point). Consider A_2B_2 as a transversal of triangle $A_1B_1A_3$. Then (Fig. 10)

$$\frac{A_1C}{CB_1} \cdot \frac{B_1A_2}{A_2A_3} \cdot \frac{A_3B_2}{B_2A_1} = -1. \tag{1}$$

Similarly, using B_3A_3 as a transversal of the triangle $A_1A_2B_1$, we find

$$\frac{A_1B_3}{B_3A_2} \cdot \frac{A_2A_3}{A_3B_1} \cdot \frac{B_1C}{CA_1} = -1. \tag{2}$$

The product of (1) and (2), properly arranged, gives the required result (B). Since the converse may be established by the indirect method as in the last theorem, the details may be omitted.

It frequently happens in the geometry of the triangle that we are interested in the angles rather than the sides. Hence we put (B) in a form using angles. We designate the sides by a_i and the lines through C as b_j, and the symbol a_ib_j

stands for the angle from the line a_i to b_j. If we apply the sine law of trigonometry to the triangles $A_1B_3A_3$ and $A_2B_3A_3$, we find

$$\frac{A_1B_3}{B_3A_2} = \frac{\sin a_2b_3}{\sin b_3a_1} \cdot \frac{\sin \alpha_2}{\sin \alpha_1},$$

where α_i represent the angles of the triangle $A_1A_2A_3$. Similarly

$$\frac{A_2B_1}{B_1A_3} = \frac{\sin a_3b_1}{\sin b_1a_2} \cdot \frac{\sin \alpha_3}{\sin \alpha_2},$$

$$\frac{A_3B_2}{B_2A_1} = \frac{\sin a_1b_2}{\sin b_2a_3} \cdot \frac{\sin \alpha_1}{\sin \alpha_3}.$$

Hence by multiplying these three equations, we find an alternative form of Ceva's criterion. By reversing the direction of all angles it can be put in a form analogous to (B)

$$\frac{\sin a_1b_3}{\sin b_3a_2} \cdot \frac{\sin a_2b_1}{\sin b_1a_3} \cdot \frac{\sin a_3b_2}{\sin b_2a_1} = 1. \tag{C}$$

12. Applications

The concurrency of the bisectors of the angles of a triangle follows immediately from (C). That the altitudes are concurrent follows from the fact that the angles a_1b_3 and b_1a_3 are equal and from similar relations.

That the medians meet in a point follows immediately from (B). The trisection property follows from an application of Menelaus' Theorem to the triangle $A_1B_1A_3$ with A_2B_2 as the transversal and with B_i the midpoints of the sides. Then (Fig. 10)

$$\frac{A_1C}{CB_1} \cdot \frac{B_1A_2}{A_2A_3} \cdot \frac{A_3B_2}{B_2A_1} = \frac{A_1C}{CB_1} \cdot -\frac{1}{2} \cdot 1 = -1.$$

Therefore

$$A_1C = 2CB_1.$$

Other applications will appear in the exercises and later in the text. One interesting application of Menelaus' Theorem appears in the next section.

13. Homothetic centers of three circles

THEOREM. *The six homothetic centers of three circles taken in pairs lie by threes on four straight lines.*

Let us designate the centers of the circles by A, B, C and their radii by a, b, c (Fig. 11). Let X, Y, Z and X', Y', Z'

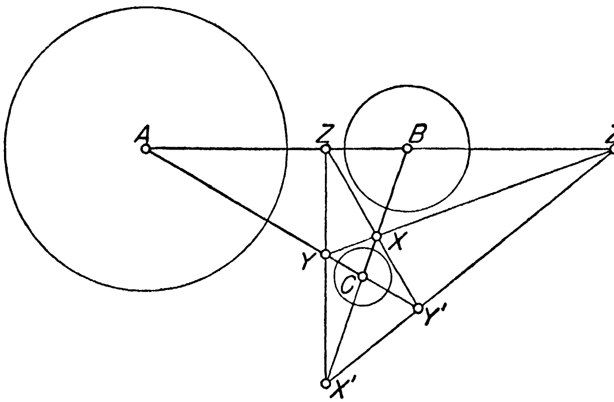


Fig. 11

be the internal and external homothetic centers, X corresponding to the circles B and C, and so forth. Then

$$\frac{AZ}{ZB} = \frac{a}{b}, \qquad \frac{AZ'}{Z'B} = -\frac{a}{b},$$

and so on. If we consider the triangle ABC and any of the four sets of points (X', Y, Z), (X, Y', Z), (X, Y, Z'), and (X', Y', Z'), a direct application of Menelaus' Theorem proves the result. For example, using the first set,

$$\frac{AZ}{ZB} \cdot \frac{BX'}{X'C} \cdot \frac{CY}{YA} = \frac{a}{b} \cdot -\frac{b}{c} \cdot \frac{c}{a} = -1,$$

so that the points are collinear.

If the circle whose center is C is tangent to the other two circles, we may consider the points of contact as the limiting positions of the homothetic centers, say X and Y, and hence we may state the following result:

THEOREM. *If a circle is tangent to two other circles, the line* *joining the points of contact will pass through a homothetic center.*

Another proof is suggested in problem 8, exercise 2—part 1.

Exercise 2.—Part 1. Theorems of Menelaus and Ceva

1. Prove Menelaus' Theorem by drawing perpendiculars from the vertices to the transversal.

2. Prove Ceva's Theorem by drawing a line through A_1 parallel to A_2A_3.

3. State and prove a trigonometric form of Menelaus' Theorem.

4. Use problem 3 to prove that the bisectors of the exterior angles of a triangle meet the opposite sides in three collinear points.

5. If a, b, c, d are any four lines through a point O, show $\sin ab \sin cd + \sin ac \sin db + \sin ad \sin bc = 0$.

6. If a circle cuts the sides of the triangle $A_1A_2A_3$ in P_1, Q_1, P_2, Q_2, and P_3, Q_3, so that A_1P_1, A_2P_2, A_3P_3 are concurrent, prove that A_1Q_1, A_2Q_2, A_3Q_3 are concurrent.

7. The perpendiculars erected at the midpoints of the bisectors of the angles of a triangle meet the sides opposite the corresponding vertices in three collinear points.

8. Use Menelaus' Theorem to prove that if a circle is tangent to two others, the line joining the points of contact will pass through a homothetic center. Consider both internal and external contacts.

9. Generalize Menelaus' Theorem to a polygon and a transversal. Prove a theorem and a correct converse.

10. If M_i are the midpoints of the sides of the triangle $A_1A_2A_3$, and if A_1M_1 meets M_2M_3 at P, and A_3P meets A_1A_2 at Q, prove that Q trisects A_1A_2.

11. If two given circles meet at P and two perpendicular lines through P meet the first circle in B and B', the second circle in C and C', and the line of centers OO' in A and A', prove that $AB/A'B' = AC/A'C'$.

12. (a) If three lines through the vertices of the triangle $A_1A_2A_3$, say A_iB_i, meet at C, prove that

$$A_1C/CB_1 = A_1B_2/B_2A_3 + A_1B_3/B_3A_2.$$

(b) Apply this result to the medians of a triangle.

13. Given three circles, prove that the six lines formed by joining a center of a circle to the homothetic centers of the other pair pass by threes through four points.

14. The tangents to the circumcircle of a triangle $A_1A_2A_3$ drawn at the vertices meet the opposite sides in B_1, B_2, B_3. Show that these points are collinear. (*Suggestions*: Show that $B_1A_2/B_1A_3 = (A_1A_2/A_1A_3)^2$, or evaluate the angles a_1b_3, and so on.)

***15.** Pascal's Theorem. $ABCDEF$ is a simple hexagon inscribed in a circle. The opposite sides AB, DE meet at L; BC, EF at M; CD, FA at N. Prove that L, M, N are collinear. (*Suggestion*: Extend AB, CD, EF to form a triangle XYZ, and then prove that LMN is a transversal of triangle XYZ by considering in turn BC, DE, FA as transversals of this triangle.

***16.** Desargues' Theorem. (a) The triangles ABC and $A'B'C'$ are such that AA', BB', CC' meet in a point O. Prove that the intersections of the corresponding sides are collinear. (*Suggestion*: Consider the transversals $B'C'$, $C'A'$, $A'B'$ in connection with the triangles OBC, OCA, OAB, respectively.)

(b) State and prove the converse.

Part 2. Harmonic Division and Orthogonal Circles

14. Harmonic division

If A, B, C, D are four points on a line such that C and D divide AB internally and externally in the same ratio, so that

$$\frac{AC}{CB} \div \frac{AD}{DB} = -1, \tag{1}$$

then C and D are said to divide the segment AB *harmonically*, and C and D are called *harmonic conjugates* with respect to A and B. Either C or D is said to be the *fourth harmonic* of the other with respect to A and B. If (1) is true, then

$$\frac{CA}{AD} \div \frac{CB}{BD} = -1, \tag{2}$$

so that the relationship between the two pairs AB, CD is mutual, and we say that the points are *four harmonic points* or form a *harmonic range*. We symbolize the relationship by $(AB, CD) = -1$ (read the cross ratio of A, B, C, D is minus one, or A, B, C, D are four harmonic points). The order of the points in the parentheses is important; the comma separates one pair from the other. It is evident from the definition that the pairs AB and CD separate each other, and when we write $(AB, CD) = -1$, we understand that they do not appear on the line in the order A, B, C, D, but appear so that the segments AB and CD overlap.

The name "harmonic" comes from the following property. If $(AB, CD) = -1$, then

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}. \tag{3}$$

The segments AC, AB, AD are thus in harmonic progression, since their reciprocals are in arithmetic progression.

The proof is simple and follows from (1) by expressing all segments with A considered as the origin. Thus

$$\frac{AC}{AB - AC} = -\frac{AD}{AB - AD}. \tag{1'}$$

We clear of fractions, collect terms, and then divide by the product $AB \cdot AC \cdot AD$. The result is equation (3). This is a convenient form for computation to determine D, if A, B, C are given. It shows that D is then uniquely determined, provided, of course, we take D at infinity if C is the midpoint of AB. Additional properties will be developed later in this chapter.

15. Complete quadrilateral

The configuration made up of four lines, no three of which are concurrent, and the six points determined by them is called a *complete quadrilateral*. Any two of the six points which do not both lie on one of the four lines are called *opposite vertices*. The lines joining opposite vertices are called *diagonal lines*, and the points of intersection of the diagonal lines are called *diagonal points*, thus forming the *diagonal* *triangle.* Thus in the theorem of section 13 (Fig. 11), the four lines joining the homothetic centers form a complete quadrilateral; X, X'; Y, Y'; Z, Z' are the pairs of opposite vertices, and ABC is the diagonal triangle.

16. Harmonic property of the complete quadrilateral

Let A_1, A_2; B_1, B_2; C_1, C_2 be the vertices of a complete quadrilateral (Fig. 12). Let B_1B_2 and C_1C_2 cut A_1A_2 in X_1

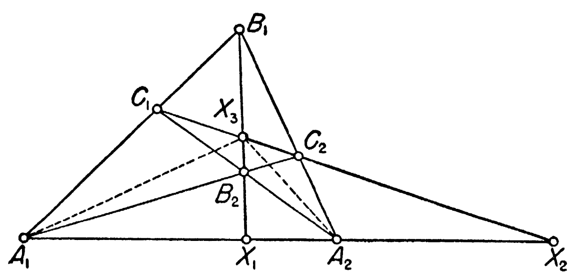


Fig. 12

and X_2. Let us apply Menelaus' and Ceva's Theorems to the triangle $A_1A_2B_1$ with C_1C_2 as a transversal and B_2 as the Ceva point. Then

$$\frac{A_1X_1}{X_1A_2} \cdot \frac{A_2C_2}{C_2B_1} \cdot \frac{B_1C_1}{C_1A_1} = +1, \tag{1}$$

$$\frac{A_1X_2}{X_2A_2} \cdot \frac{A_2C_2}{C_2B_1} \cdot \frac{B_1C_1}{C_1A_1} = -1, \tag{2}$$

$$\frac{A_1X_1}{X_1A_2} \div \frac{A_1X_2}{X_2A_2} = -1, \tag{3}$$

or $(A_1A_2, X_1X_2) = -1$. This result may be stated as the following theorem:

THEOREM. *A complete quadrilateral determines on any diagonal a harmonic range; the vertices on the given diagonal are one pair, and the intersection of this diagonal with the other diagonals form the second pair of points in the harmonic range.*

If in Fig. 12, the third diagonal point is X_3, then

$$(B_1B_2, X_1X_3) = -1,$$

and

$$(C_1C_2, X_2X_3) = -1.$$

17. Complete quadrangle

The configuration made up of four points, no three of which are collinear, and the six lines determined by them taken in pairs is called a *complete quadrangle.* Any two of the six lines which do not have one of the four points in common are called *opposite sides.* The points of intersection of opposite sides are called *diagonal points,* and the lines connecting the diagonal points are *diagonal lines.*

Thus in Fig. 12, $B_1C_1B_2C_2$ represents a complete quadrangle whose diagonal triangle is $A_1A_2X_3$. The six sides of this complete quadrangle are the four sides and two of the diagonals of the complete quadrilateral discussed before. When a complete quadrilateral and its diagonals are drawn, it is possible to pick out a complete quadrangle in more than one way, and it is to be carefully noted that the quadrangle and the quadrilateral have *distinct* diagonal triangles.

The theorem of section 16 may be restated as follows:

THEOREM. *A complete quadrangle determines on any diagonal a harmonic range; the diagonal points on the given diagonal are one pair, and the intersection of this diagonal with the third pair of opposite sides forms the second pair of points in the harmonic range.*

In Fig. 12, quadrangle $B_1C_1B_2C_2$ has diagonal triangle $A_1A_2X_3$ and determines

$$(A_1A_2, X_1X_2) = -1,$$

quadrangle $A_1C_1A_2C_2$ has diagonal triangle $B_1B_2X_2$ and determines

$$(B_1B_2, X_1X_3) = -1,$$

quadrangle $A_1B_1A_2B_2$ has diagonal triangle $C_1C_2X_1$ and determines

$$(C_1C_2, X_2X_3) = -1.$$

If A, B, C are given, to determine the fourth harmonic of C with respect to A and B, we proceed as follows (Fig. 13): We draw any two lines through A, any line through C, cutting the two lines through A in P and Q. We join P and Q to B, thus determining R and S on the lines through A. The line RS cuts AB in the required point D, as determined by the six lines which form the complete quadrangle $PQRS$.

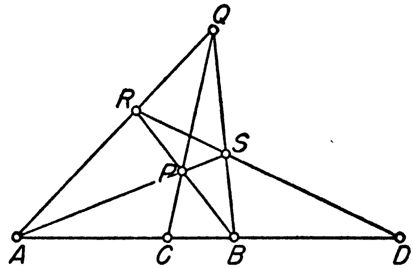


Fig. 13

18. Four harmonic lines

If four harmonic points are joined to a fifth point not on their line, the four lines so formed are called *four harmonic lines* or a *harmonic pencil*. The name is derived from the fact that these lines will be cut by any transversal in four harmonic points.

CASE 1. Let the four harmonic points be (AB, CD) and

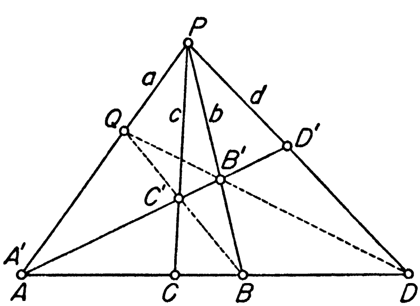


Fig. 14a

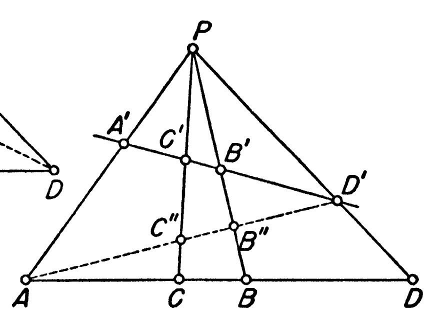


Fig. 14b

join them to P by the lines a, b, c, d. If the transversal passes through one of the points, say A (Fig. 14a), let BC'

cut a in Q. Then as determined by the quadrangle $PQC'B'$, the line QB' passes through D. But the quadrangle $PQBD$ shows that $(A'B', C'D') = -1$.

CASE 2. If A and A' are distinct, let AD' cut the pencil in the points A, C'', B'', D' (Fig. 14b). Then, by case 1, we have $(AB'', C''D') = -1$, and then $(A'B', C'D') = -1$, as was to be proved. Briefly we say that *four harmonic points are invariant under projection.* In terms of concurrent lines, this theorem is equivalent to the following one:

THEOREM. *If two harmonic ranges on different lines are such that the lines joining three pairs of corresponding points are concurrent, then the line joining the fourth pair also passes through this point of concurrence.*

19. Trigonometric formula for a harmonic pencil

In case the vertex of the harmonic pencil is a finite point, it is not difficult to show by writing the trigonometric forms of the theorems of Menelaus and Ceva and applying them to Fig. 14a that

$$\frac{\sin ac}{\sin cb} \div \frac{\sin ad}{\sin db} = -1. \tag{1}$$

From this formula it follows immediately that the bisectors of the angles between two lines and the two lines form a harmonic pencil. This pencil is of a special type, since one pair of conjugate rays are at right angles. In this connection we have the following theorem, which was proved before in section 8.

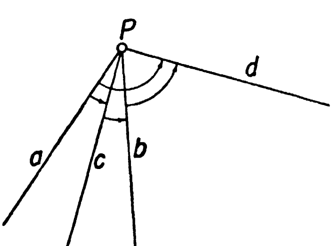


Fig. 15

THEOREM. *If one pair of conjugate rays of a harmonic pencil are at right angles, they bisect the angles between the other pair.*

The proof given in section 8 may be restated in the language of projection and harmonic ranges, but it is perhaps more instructive to proceed as follows. If c is perpendicular to

d (Fig. 15), then

$$\begin{aligned}\measuredangle ca + \measuredangle ad &\equiv 90^\circ \pmod{180^\circ},\\ \measuredangle cb + \measuredangle bd &\equiv 90^\circ \pmod{180^\circ},\\ \sin ad &= \pm\cos ca,\\ \sin bd &= \pm\cos cb.\end{aligned}$$

If we substitute these values in (1), taking due regard of signs when we reverse directions, we find

$$\begin{aligned}\sin ac \cos cb \pm \cos ac \sin cb &= 0,\\ \sin (ac \pm cb) &= 0.\end{aligned}$$

We must use the $-$ sign, since a and b do not coincide, so that

$$\measuredangle ac \equiv \measuredangle cb \pmod{180^\circ},$$

and c is one of the bisectors of $\measuredangle ab$ and d is the other.

20. Orthogonal circles

Two circles are orthogonal if they intersect at right angles. In this case a tangent to either at a point of intersection passes through the center of the other. Conversely, if the tangent to either of the two circles at a point of intersection passes through the center of the other, then the circles are orthogonal. Metrically, this property is equivalent to

$$r_1^2 + r_2^2 = d^2,$$

where d is the distance between their centers, and r_1 and r_2 are the radii of the circles. It follows that d is greater than either radius; further, one and only one circle can be drawn orthogonal to a given circle c and having its center at a given external point O.

21. Orthogonal circles and harmonic division

We return to the equation

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}, \tag{1}$$

previously derived for four harmonic points. Let O be the midpoint of AB, so that $AB = 2\,OB$, $AC = AO + OC = OC + OB$, $AD = OD + OB$. Then

$$\frac{1}{OB} = \frac{1}{OC + OB} + \frac{1}{OD + OB},$$

so that by clearing of fractions and simplifying, we find

$$OC \cdot OD = OB^2. \tag{2}$$

Conversely, if O is the midpoint of AB, and C and D are such that (2) is satisfied, then (AB, CD) are four harmonic points. This result readily follows from the reversal of the above steps, or we may proceed in another way. Let D' be the harmonic conjugate of C with regard to A and B. Then

$$OC \cdot OD' = OB^2, \tag{2'}$$

so that D and D' coincide, and the converse is established.

Draw the circle having AB as diameter (Fig. 16), and

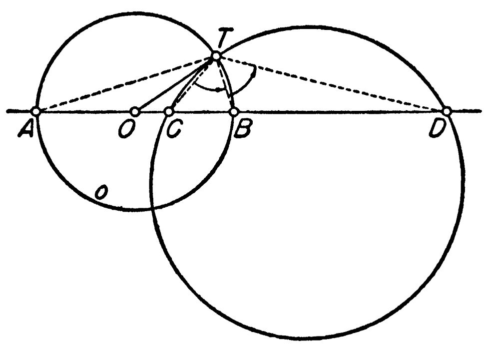


Fig. 16

draw any circle through C and D; then the length of the tangent OT from O to this circle is given by $OC \cdot OD = OT^2$. From (2) it follows that $OT = OB$; and the two circles are orthogonal. We restate our results in the following form:

THEOREM. *If two circles are orthogonal, a diameter of one is cut harmonically by the circles; and, conversely, if a diameter of one is cut harmonically by two circles, the circles are orthogonal.*

We note (Fig. 16) that the four lines joining T to A, B, C,

D are four harmonic lines with TA perpendicular to TB. The bisection property of section 19 thus suggests a simple construction for the fourth harmonic of three given points. Other constructions based on the orthogonality property may readily be devised. For example, to draw a circle through the point D, orthogonal to the circle o, center O, we may proceed as follows: Let DO cut the circle in A and B, and let C be the harmonic conjugate of D with respect to A and B. Any circle through C and D will satisfy the required conditions. The point C may be located by the quadrangle construction of Fig. 13, or as just suggested by Fig. 16. Let T be any point on the circle o, and make $\measuredangle\ CTB = \measuredangle\ BTD$.

22. Harmonic conjugate with respect to two pairs

The relation between harmonic division and orthogonal circles lends itself readily to a solution of the following important problem:

Given two pairs of points A_1, B_1 and A_2, B_2 on a line, find a pair of points C and D which are harmonic conjugates with regard to both given pairs.

Let P be any point not on the given line, and construct

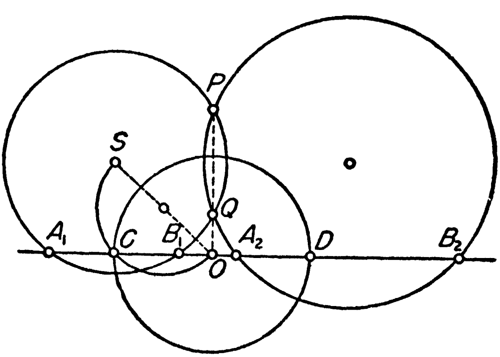


Fig. 17

the circles A_1B_1P and A_2B_2P meeting again at Q (Fig. 17). Let PQ meet the given line at O, so that

$$OA_1 \cdot OB_1 = OP \cdot OQ = OA_2 \cdot OB_2.$$

The circle, center O and radius equal to $(OP \cdot OQ)^{1/2}$, will be orthogonal to both the circles A_1B_1P and A_2B_2P and hence will cut the line in the required points. This circle is perhaps most readily constructed by first locating the intersection of the circle A_1B_1P with the circle drawn on OS as diameter, where S is the center of the circle A_1B_1P. Other constructions are available and should be investigated by the reader.

If the point pairs A_1, B_1 and A_2, B_2 separate each other, then the point Q is such that P and Q are on opposite sides of the line, so that there is no real pair of points to satisfy the problem.

That the pair of points C, D is unique when they exist may be shown. Let us take A_1 as origin; let $A_1B_1 = b_1$, and so on, and $A_1O = x$. Then from the relation

$$OA_1 \cdot OB_1 = OA_2 \cdot OB_2 = r^2,$$

we have

$$(-x)(b_1 - x) = (a_2 - x)(b_2 - x),$$

so that

$$x = \frac{a_2b_2}{a_2 + b_2 - b_1}.$$

Hence O is uniquely determined by the given segments; the radius of the circle is also uniquely determined; and finally C and D are uniquely fixed. A geometric proof of this uniqueness will be given later.

23. Circle of Apollonius

THEOREM. *The locus of a point, the ratio of whose distances from two fixed points is constant* ($\neq 1$), *is a circle.*

Let C and D divide the given line segment AB (Fig. 18) internally and externally in the given ratio k, and consider the circle whose diameter is CD. Let P be any point on this circle. Then the pencil at P is harmonic and since PC is perpendicular to PD, PC and PD bisect the angles between

AP and BP (section 19), and hence from elementary geometry (problem 4, exercise 1),

$$\frac{AP}{PB} = \frac{AC}{CB} = k.$$

Conversely, if P is any point such that $AP/PB = k$, and C and D are determined as above, then PC and PD are the

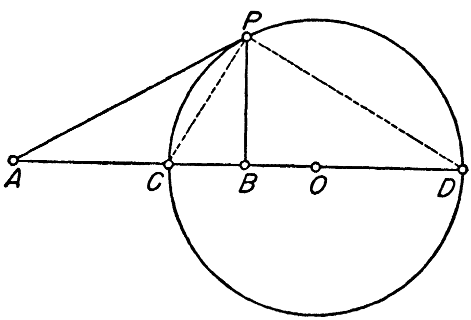


Fig. 18

bisectors of the angles between PA and PB, and since they are perpendicular, P lies on the circle having CD as diameter. This circle is known as the *circle of Apollonius.*

Obviously, if $k = 1$, the locus is the perpendicular bisector of the line segment AB; this may be interpreted as a limiting case of the general theorem and will be included as such in the future.

As a corollary we state a property developed in the proof.

COROLLARY. *If* A, C, B *are three points on a line in the indicated order, then the locus of a point* P *such that the angle* APC *equals angle* CPB *is the circle of Apollonius determined by* A, B *and the ratio* AC/CB.

Another corollary follows readily from the fact that $(AB, CD) = -1$:

COROLLARY. *The circle of Apollonius for any* k *is orthogonal to any circle through* A *and* B, *and in particular to the circle on* AB *as diameter.*

As a numerical illustration, suppose that $AB = 6$ units and we wish to locate the circle of Apollonius for the ratio 2 (see Fig. 18). Then C is determined by the ratio $AC/CB = 2$, so that $AC = 4$. We can find D, for example, from the relation

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD},$$

which becomes

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{AD}.$$

Hence $AD = 12$. The center of the circle is the midpoint of CD, so $AO = (4 + 12)/2 = 8$; the radius is $AO - AC = 4$, and the circle is thus determined. If P is any point on this circle, $\measuredangle APC = \measuredangle CPB$.

The general algebraic case which this example illustrates has been left as an exercise for the student (problem 11, exercise 2—part 2).

Exercise 2.—Part 2. Harmonic Division and Orthogonal Circles

1. If $(AB, CD) = -1$, derive the relation similar to $2/AB = 1/AC + 1/AD$, using C as the origin.

2. Given the parallelogram $ABCD$, and an arbitrary line through A cutting DB at E, BC at F, DC at G, prove $1/AE = 1/AF + 1/AG$.

3. If P, P' divide one diameter of a circle harmonically, and Q, Q' divide another diameter harmonically, then P, Q, P', Q' are concyclic.

4. Construct a circle orthogonal to a given circle and passing through two points.

5. Construct a circle orthogonal to two given circles and through a given point.

6. If two circles with centers C and D intersect at A and B, prove that the angles at which they intersect are equal and that this angle is the supplement of angle CAD. Hence prove that if X is any point on one circle and Y on the other, the circles will be orthogonal if, and only if, $\measuredangle BXA + \measuredangle AYB = 90°$.

7. If A, B, C, D are four points on a line, and if M and N are the midpoints of AB and CD, respectively, prove that a necessary and sufficient condition that $(AB, CD) = -1$, is $MN^2 = MA^2 + NC^2$.

8. A line cuts two circles in the harmonic point pairs A_1, A_2 and B_1, B_2. The radii of the circles are a and b; the distance between their centers is d; and p and q represent the distances from the centers to the line. Prove that

$$2\,pq = a^2 + b^2 - d^2.$$

Hence prove the theorem: If a line cuts two orthogonal circles in a harmonic range, then it is a diameter of one of them.

9. A, B, C, D are four collinear points. Find a point P such that angles APB, BPC, and CPD are equal.

10. (a) Prove that the distances from any point on the circle of similitude of two given circles to the centers of these circles are proportional to the radii and conversely.

(b) Hence prove that the circle of similitude is the locus of points from which two circles subtend equal angles.

11. If A and B are two fixed points and P moves so that $AP/PB = k$ (a constant), show that the radius and center are located by the relations

$$AO = \frac{k^2 AB}{k^2 - 1}, \qquad r = \frac{k\,AB}{k^2 - 1}.$$

Discuss the change in position and magnitude of the locus of P as k increases from 0 to 1 and then increases indefinitely.

***12.** The midpoints of the diagonals of a complete quadrilateral are collinear. (By diagonals we mean here the line segments joining the pairs of opposite vertices.)

3 INVERSION

Part 1. Inverse of a Point, Line, and Circle

24. Inversion

If on the line joining a given fixed point O to a point P, a point P' is taken such that

$$OP \cdot OP' = k,$$

where k is a given constant, then P' is called the inverse of P with respect to O with the modulus k. If k is positive, the square root of k is called the radius of inversion, and the circle with center O, radius $\sqrt{k}$ is called the (real) circle of inversion; we speak of the inversion with respect to this circle. In case k is negative, we may first reflect P on the origin and then invert with respect to the circle whose radius is $\sqrt{-k}$. It is also convenient, in case k is negative, to define the inversion as an inversion with respect to the *imaginary circle* with real center O and imaginary radius $\sqrt{k}$.

From the definition we note that each point has a unique inverse, except the point O, which is an exceptional point. As P approaches O along a given direction, P' recedes to infinity, and so we say that O corresponds to the ideal point at infinity in this direction. Considering all directions, O corresponds to the ideal line at infinity. Points inside the circle of inversion correspond to points outside, and points on the circle of inversion correspond to themselves.

As a limiting case, the inverse of a point with regard to a line is taken as the reflection of the point on the line. This definition is consistent with the general definition, for if the line OP cuts the circle of inversion at C, then

$$OP \cdot OP' = r^2 = (OC + CP)(OC + CP'),$$

$$(r + CP)(r + CP') = r^2.$$

We may rewrite this equation

$$CP + CP' = -\frac{CP \cdot CP'}{r}.$$

Now permit the point C to remain fixed, while the center of the circle of inversion O goes to infinity; r becomes infinite, and the circle becomes a line. Then $CP + CP'$ must tend to zero, and P' in the limit is the reflection of P in the line.

If $OP \cdot OP' = k$, and $OP \cdot OP'' = k'$, we have $OP'/OP'' =$ constant, and the locus of P'' is homothetic to that of P', so that if we are not interested in the size of the figure, as frequently happens, we speak of inversion with respect to a point without reference to the modulus of inversion.

25. Geometric constructions

If the circle of inversion is real and if the point P is outside the circle, and if T and T_1 are the points of contact of the tangents to the circle from P, then TT_1 meets OP at P', while if the point P is inside the circle we reverse the opera-

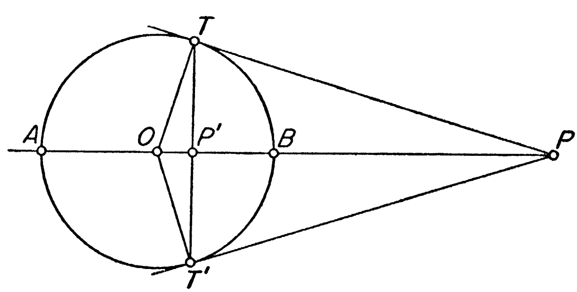


Fig. 19a

tions, recognizing that TT_1 is perpendicular to OP. The proof follows from the fact that the triangles OTP and $OP'T$ are similar, so that (Fig. 19a)

$$OP \cdot OP' = OT^2 = r^2.$$

If the line OP cuts the circle at A and B, then $(AB, PP') = -1$, and any construction for the harmonic conjugate of P with respect to A and B determines P'. (See sections 17 and 21.)

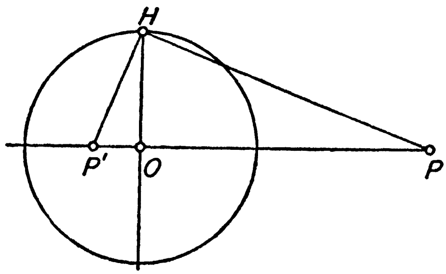


Fig. 19b

If the modulus k is negative, we first find P'', the inverse of P, using the radius $\sqrt{-k}$ and reflect P'' upon O to find P'. Or we may proceed directly as follows (Fig. 19b). Let OH be drawn perpendicular to OP and equal to $\sqrt{-k}$, and construct HP' perpendicular to HP. It is easily seen that

$$OP \cdot OP' = -OH^2 = k.$$

26. Inverse of a line

It is evident that the inverse of a line with respect to a point on it is the line itself.

THEOREM. *The inverse of a line not through the center of inversion is a circle through the center of inversion.*

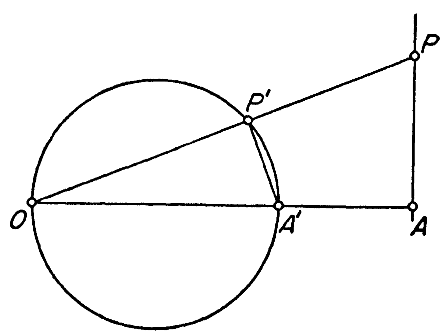


Fig. 20

From O (Fig. 20) draw a perpendicular to the given line, thus determining the point A. Let P be any point on the line and let A' and P' be the inverses of A and P, so that

$$OA \cdot OA' = OP \cdot OP',$$

$$\frac{OA}{OP} = \frac{OP'}{OA'}.$$

Since the angle at O is common to the triangles OAP and $OP'A'$, these triangles are similar; the angle $OP'A'$ is a right angle; and hence P' describes a circle on the fixed segment OA' as diameter.

Conversely, the inverse of a circle through the center of inversion is a line. The details of the proof, which may be obtained by retracing the above steps, are left to the reader.

We offer a second proof of the theorem which puts in evidence the center of the circle (Fig. 21). Let B be the reflec-

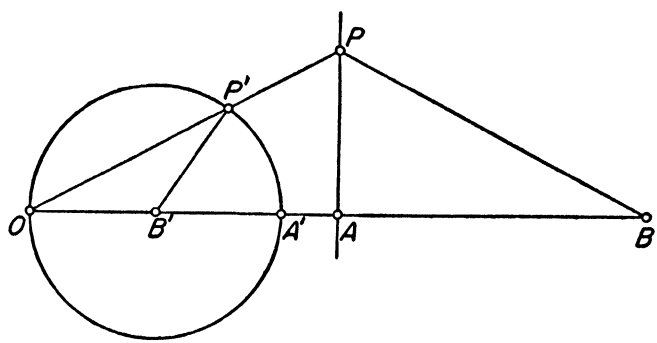


Fig. 21

tion of O upon the given line, P and A, as before, and let inverses be designated by primes. Then, as in the first proof, it is readily seen that the triangles OBP and $OP'B'$ are similar, so that

$$\frac{B'P'}{PB} = \frac{OP'}{OB}.$$

But $PB = OP$, and $OB = 2\,OA$, so that

$$B'P' = \frac{OP \cdot OP'}{2\,OA} = \text{constant},$$

independent of P, and since B' is a fixed point, the locus is a circle; that it passes through O follows from the fact that if P is at A, then P' is at A', and then

$$B'A' = \frac{OA \cdot OA'}{2\,OA} = \frac{OA'}{2}.$$

27. Peaucellier's cell

Peaucellier's cell is a system of linkages* for describing mechanically a curve inverse to a given curve. It was invented in 1873 by a captain in the French army, and because it enables one to draw a straight line without a straight edge, it excited considerable curiosity. It consists essentially of four equal rods forming a rhombus $APBP'$, and two other equal rods, OA and OB (Fig. 22). The link-

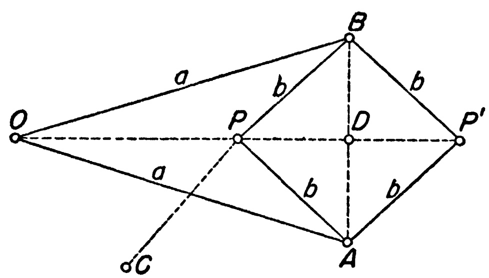


Fig. 22

age is freely hinged at each of the five points, and the point O is fixed. It is apparent from the manner in which it is made that O, P, and P' lie on the perpendicular bisector of AB and so are collinear. Further

$$\begin{aligned} OP \cdot OP' &= (OD - PD)(OD + PD) = OD^2 - PD^2 \\ &= a^2 - DB^2 + DB^2 - b^2 \\ &= a^2 - b^2, \end{aligned}$$

where a and b represent the lengths of the rods, so that $OP \cdot OP'$ is constant and P and P' describe inverse curves.

If now we attach another rod at P and determine C so that $CP = CO$ and fix the point C, then P will be constrained to move along the arc of a circle through O, and P' will describe a straight line.

* For an extended elementary discussion of linkages, see R. C. Yates, "The Story of the Parallelogram," *The Mathematics Teacher*, volume 33 (1940), pages 301–310.

28. Power of a point

If O is any point and if a line through O cuts a circle in the points P and Q, then $OP \cdot OQ$ is a constant p (section 6), called the power of O with regard to the circle. If O is outside the circle, $p = OT^2$, where OT is the length of the tangent from O to the circle; if O is inside the circle, then

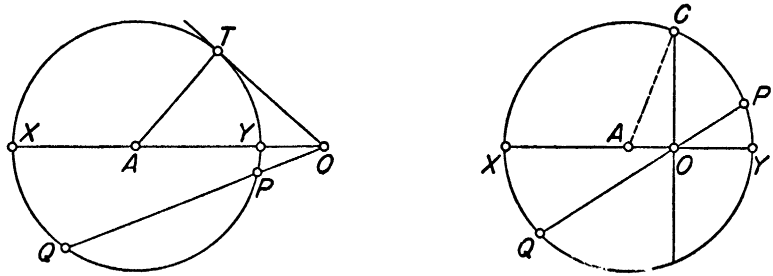


Fig. 23

$p = -OC^2$, where OC is the halfchord (*minimal chord*) perpendicular to the diameter of the circle. In either case (Fig. 23),

$$p = OA^2 - a^2,$$

where A is the center of the circle and a is its radius.

The first part of the result comes from specializing the line OP as the tangent line or the chord perpendicular to AO, as the case may be. The second part comes from consideration of the diameter XY through O. Then, for either case,

$$\begin{aligned} p = OX \cdot OY &= (OA + AX)(OA + AY) \\ &= (OA + a)(OA - a) = OA^2 - a^2. \end{aligned}$$

29. Inverse of a circle; circle of antisimilitude

THEOREM. *The inverse of a circle not through the center of inversion is another circle.*

Let a line through O cut the given circle in P and Q, and represent inverses by primed letters (Fig. 24). Then from the relations,

$$OP \cdot OP' = k, \qquad OQ \cdot OQ' = k, \qquad OP \cdot OQ = p,$$

we find

$$\frac{OP'}{OQ} = \frac{OQ'}{OP} = \frac{k}{p} = r, \text{ a constant.}$$

Hence the locus of P' (or Q') is a circle homothetic to the locus of Q (or P), having O as a homothetic center. The

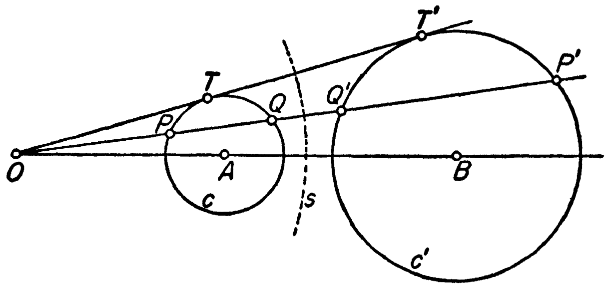


Fig. 24a

circles will be directly or inversely homothetic according as the constant r is positive or negative.

THEOREM. *Any two circles with different centers are mutually inverse with respect to the two homothetic centers, and at least one circle of inversion is real.*

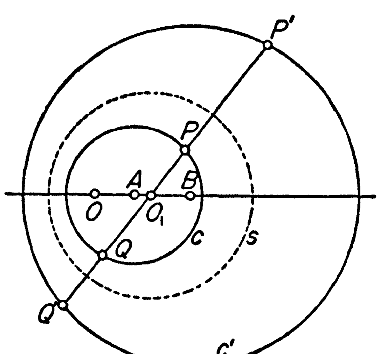


Fig. 24b

The two circles can be homothetic in only two ways (section 5, second corollary) and hence can be inverse only with respect to the two homothetic centers. Let O be either homothetic center (Fig. 24) of the two circles and let a line through O cut the circles in P, Q and Q', P', respectively, so that P and Q' correspond in the homothetic transformation from O. Then

$$\frac{OQ'}{OP} = \frac{OP'}{OQ} = r,$$

and

$$OP \cdot OQ = p.$$

These equations give

$$OP \cdot OP' = OQ \cdot OQ' = pr = \text{a constant, say } k,$$

and the two circles are mutually inverse with respect to either homothetic center.

To prove the second and more important part of the theorem we must examine the sign of $k = pr$. There are three possible cases, and to distinguish them we will use O for the external homothetic center and O_1 for the internal homothetic center.

CASE 1. *Circles mutually external.* Since both O and O_1 are outside both circles, p is positive. For O, r is positive; but for O_1, r is negative. Hence $k = pr$ is positive for the external homothetic center, and the circle of inversion s is real (Fig. 24a).

CASE 2. *Circles intersect.* For O, both p and r are positive; for O_1, both p and r are negative. Hence $k = pr$ is positive for both centers.

CASE 3. *One circle inside the other.* Since both O and O_1 are inside both circles, p is negative. For O, r is positive; but for O_1, r is negative. Hence $k = pr$ is positive for the internal homothetic center, and the circle of inversion s is real (Fig. 24b).

The *real* circles of inversion are known as the circles of *antisimilitude.*

The student should draw the given circles in different positions and locate the circles of antisimilitude, including the special cases indicated below.

If the given circles are equal, then the perpendicular bisector of the segment joining the centers (as in section 24) is the limiting position of a circle of antisimilitude, and the inversion is replaced by a reflection on this line. (Hereafter when using the general term "circle of antisimilitude" we shall include this line as a special case and need not directly discuss this case unless it is of special importance.) If the equal circles intersect, this line is their common chord; in this case the two circles also have a second circle of antisimilitude, whose center is at the intersection of the common chord and the line of centers AB. This circle of antisimilitude also passes through the points common to the circles.

If one of the circles becomes a straight line, the line and circle may still be inverted into each other. The center of inversion must be on the circle and on a diameter of the circle perpendicular to the line. There are again two cases to consider according as the line does or does not intersect the circle. Further details are left for the student in problem 8, exercise 3 —part 1.

30. Circles orthogonal to the circle of inversion

THEOREM. *A circle orthogonal to the circle of inversion inverts into itself, and conversely.*

Let any diameter of the circle of inversion (Fig. 25) cut

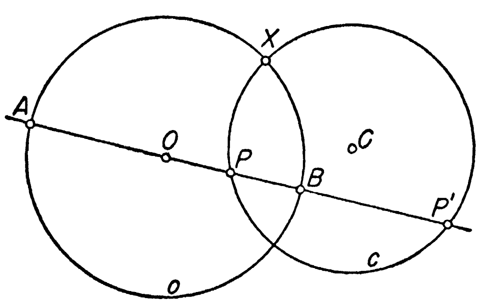


Fig. 25

the circle o in A and B, and the circle c, given orthogonal to o, in the points P and P'. Let a point of intersection be X. Then

$$OP \cdot OP' = OX^2 = r^2,$$

or P and P' are mutually inverse points. The point X inverts into itself. Therefore the circle XPP' inverts into the circle $XP'P$; that is, into itself.

Conversely, any circle other than the circle of inversion that inverts into itself is orthogonal to the circle of inversion. Such a circle must cut the circle of inversion at a point X, for if it is inside (outside) the circle of inversion, its inverse is outside (inside), and could not invert into itself. An arbitrary line through the center of inversion cuts the circle in a pair of inverse points P and P', such that $r^2 = OP \cdot OP' = OX^2$, and the circles are orthogonal. Incidentally, we have proved the following corollary:

Corollary 1. *A circle orthogonal to the circle of inversion cuts any diameter of the circle of inversion in a pair of inverse points; and, conversely, a circle through a pair of mutually inverse points is orthogonal to the circle of inversion.*

Corollary 2. *If two intersecting circles are orthogonal to the circle of inversion, they intersect in a pair of inverse points.*

Suppose one point of intersection is P, and let the line OP cut the circles again in the points P' and P''. Then P' and P'' are both inverses of P and so coincide, proving the desired result.

We note that it is always possible to invert a circle into itself with respect to a given center O. If the center O is

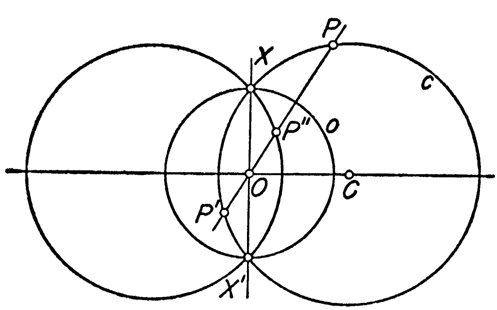


Fig. 26

outside the circle, the circle of inversion is real and is orthogonal to the given circle. If O is inside the given circle, the circle of inversion is imaginary with radius equal to the square root of the modulus of inversion, but the given circle still inverts into itself. If we interpret this inversion (section 24) as a real inversion followed by a reflection (Fig. 26), the given circle cuts the circle of this real inversion in the ends of a diameter—namely, the chord XX' perpendicular to OC—for the given circle and its inverse with respect to this real circle must be of the same size. In this case we have, where $OX = r_2$, $CX = r_1$, and $OC = d$,

$$r_1^2 - r_2^2 = d^2,$$

and we may properly interpret the real circle c and the imaginary circle whose center is at O and whose radius squared is $-r_2^2$, as orthogonal circles.

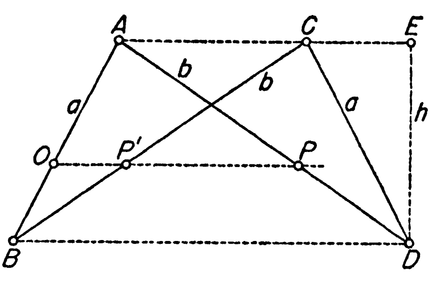
Exercise 3—Part 1. Inverse of a Point, Line, and Circle

1. What is the inverse of a pair of parallel lines? Prove.

2. Prove that the inverse of a circle through the center of inversion is a line not through the center of inversion.

3. Construct a Peaucellier cell out of strips of cardboard. Design one for the case k positive and one for the case k negative.

4. Hart's Inversor. $ABCD$ is a re-entrant quadrilateral formed with four hinged rods AB, BC, CD, and DA with $AB = CD = a$, and $DA = BC = b$. Points O, P, P' are taken on AB, AD, BC, respectively, so that $AO/AB = AP/AD = CP'/CB = k$, and O is fixed. Prove that as the system moves, P and P' describe inverse curves.



(*Suggestions*: (1) Show that O, P, and P' are collinear; (2) show that $c(OP \cdot OP') = AC \cdot BD = b^2 - a^2$, where c is a constant depending upon k.)

5. Find the locus of a point whose power with respect to a given circle is constant.

6. Draw figures for all relative positions of two unequal circles, showing the circles of antisimilitude.

7. Prove the statements in the text concerning the circles of antisimilitude for two equal circles.

8. Discuss the circles of antisimilitude for a line and a circle, proving your statements, and hence show how to invert a given circle into a given line. Consider the cases when the line does or does not intersect the circle.

9. Prove that the inverse of a circle is a circle by determining the ratio $OP'/P'C'$, where C' is the inverse of the center and P' is the inverse of an arbitrary point.

Part 2. Angle and Distance Relations Under Inversion

31. Invariance of the angle between two circles

THEOREM. *The angle between two circles is invariant in magnitude but reversed in direction under inversion.*

Let P be the point of intersection of the given circles. The angle between the given circles is the angle between their respective tangents at a common point P, and hence is equal to the angle between the diameters of the circle of inversion which are parallel to these tangents (Fig. 27).

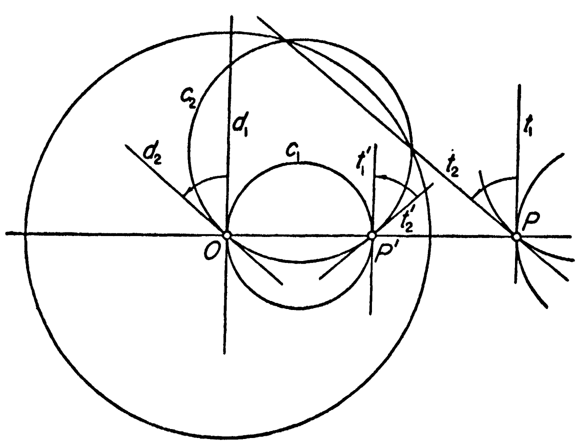


Fig. 27

Each tangent t_i inverts into a circle c_i tangent to these diameters, and hence at O the inverse circles intersect at the angle between the given circles. This angle at O is equal to the angle at P', but oppositely directed, which completes the proof.

This important theorem can be established in various other ways. One method depends upon the following theorem; this method may be applied to any curves whatever.

32. Angle relation under inversion

THEOREM. *If* P′, Q′, R′ *are the inverses of* P, Q, R, *then*

$$\measuredangle P'Q'R' \equiv \measuredangle RQP + \measuredangle POR \text{ (mod } 180°),$$

direction of angles being taken into account.

From the relations $OP \cdot OP' = OQ \cdot OQ'$, it follows that P, P', Q, Q' are concyclic, so that

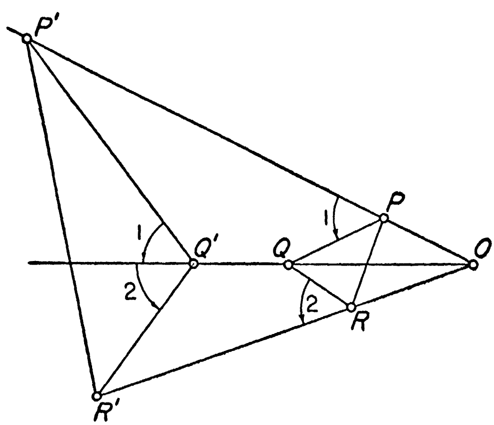


Fig. 28

$$\measuredangle P'Q'Q \equiv \measuredangle P'PQ \text{ (mod } 180°). \tag{1}$$

Similarly

$$\measuredangle QQ'R' \equiv \measuredangle QRR' \text{ (mod } 180°). \tag{2}$$

If now we use the fact that the external angle of a triangle equals the sum of the opposite interior angles, we have

$$\measuredangle P'PQ \equiv \measuredangle OQP + \measuredangle POQ, \tag{3}$$

$$\measuredangle QRR' \equiv \measuredangle RQO + \measuredangle QOR. \tag{4}$$

Hence adding (1) and (2), using (3) and (4), and obvious addition relations, we find

$$\measuredangle P'Q'R' \equiv \measuredangle RQP + \measuredangle POR \pmod{180°}. \tag{5}$$

An alternative form of this equation is

$$\measuredangle P'Q'R' + \measuredangle PQR \equiv \measuredangle POR \pmod{180°}.$$

(*Remark.* In the derivation of equation (5), if we use Fig. 28, the congruence signs could all be replaced by equal signs, but if the figure is drawn with changes in the relative positions of P, Q, R, O, the congruence signs must be used. The proof given is valid for all cases.)

This theorem shows that we may always invert P, Q, R into P', Q', R' so that $\measuredangle P'Q'R'$ is a given constant. For, since $\measuredangle PQR$ and $\measuredangle P'Q'R'$ are given, $\measuredangle POR \equiv$ a constant (mod 180°), and hence lies on a circle determined by this constant angle. Suppose for purpose of illustration, using Fig. 28 as much as possible, we are given $\measuredangle PQR \equiv -60° \equiv 120°$ (mod 180°), and we require that $\measuredangle P'Q'R' \equiv 105°$. Then $\measuredangle POR \equiv 45°$. Hence O lies on a circle having PR as a chord such that $\measuredangle POR \equiv 45°$. There are several elementary ways of constructing this circle. For instance, if we construct $\measuredangle TPR \equiv 45°$ (not shown in the diagram), and make the circle pass through R and P, tangent to TP at P, it will be the required circle. Any point O of this circle may be used as the center of inversion, and after inversion $\measuredangle P'Q'R' \equiv 105°$ (mod 180°).

If now two curves intersect at Q, we take P as an arbitrary point on one curve and R as an arbitrary point on the other. If after we apply equation (5), we permit P and R to approach Q, then angle POR approaches zero and in the limit $\measuredangle P'Q'R' \equiv \measuredangle RQP$, so that the angle between the two curves is invariant in magnitude, but reversed in direction under inversion.

33. Harmonic property under inversion

If (AB, CD) is a harmonic range and if we invert with regard to any point O on their line, then the circles having AB and CD as diameters are orthogonal and will invert into orthogonal circles, so that the harmonic range will invert into a harmonic range (section 21).

This result enables us to prove the following theorem which is a converse of the theorem in section 21; a proof was suggested in problem 8 of exercise 2—part 2.

THEOREM. *If a line is cut harmonically by two orthogonal circles, it must be a diameter of one of them.*

Let the circles c_1 and c_2 be cut by the line u in a harmonic range (AB, CD) (Fig. 29). We invert with regard to the

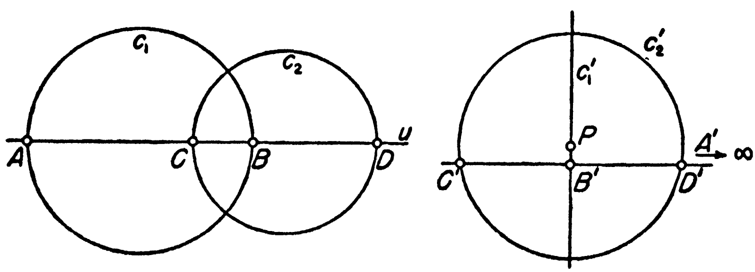


Fig. 29

point A. The circle c_1 inverts into a line c_1' orthogonal to the circle c_2', the inverse of c_2, and hence c_1', must be a diameter of c_2'; that is, it passes through the center, say P, of c_2'. But $(A'B', C'D') = -1$, and since A' is at infinity, B' is the midpoint of $C'D'$, so that either

c_1' is perpendicular to the given line u, or (1)

B' and P are identical, so that the given line u is a diameter of c_2'. (2)

We now invert back again. In case (1) we find that u is orthogonal to c_1, or AB is a diameter of c_1; in case (2), u is a diameter of c_2. We remark that cases (1) and (2) may happen simultaneously, but at any rate at least one of the circles c_1 and c_2 has u as a diameter.

34. Invariance of inverseness under inversion

Suppose that A and B are inverse points with respect to a circle c, and we invert with respect to another point O. Let k_1 and k_2 be two circles through A and B, so that k_1 and k_2 are orthogonal to c. Then after inversion there will be a figure analogous to that already drawn, the circles c, k_1, k_2 inverting into circles c', k_1', k_2'. The circles k_1' and k_2' will be orthogonal to c', and their intersections A' and B', the inverses of A and B with respect to O, will be inverse points with respect to c' (section 30, corollary 2). If A describes a curve and B is its inverse with respect to c, then A' and B' will describe inverse curves with respect to c'. This property may be stated in the form: *The relation of inverseness is invariant under inversion.* We now give an application of this property.

35. Inversion into equal circles

If two circles are inverted with respect to a point on a circle of antisimilitude, they are inverted into equal circles; and, conversely, if two circles are inverted into equal circles, the center of inversion is on a circle of antisimilitude.

For if O is any point on a circle of antisimilitude s of two circles c_1 and c_2, then s inverts into a straight line s', and c_1 and c_2 into circles c_1' and c_2', which are inverse with regard to s' and hence are equal. Conversely, if O is a point such that an inversion with respect to it inverts the circles into equal circles c_1' and c_2', let s' be the perpendicular bisector of the line of centers of c_1' and c_2', so that c_1' and c_2' are inverse with respect to s'. Then by inverting back again, we find c_1 and c_2 are inverse with respect to a circle s which must pass through O, or O is on a circle of antisimilitude.

If we wish to invert three circles c_1, c_2, c_3 into equal circles, it is necessary and sufficient that the center of inversion lie on a circle of antisimilitude of c_1 and c_2, and also on a circle of antisimilitude of c_1 and c_3. Further details of this problem have been left for the student as problem 6, exercise 3 —part 2.

36. Inverse of the center of a circle

Although a circle inverts into a circle, the centers of the circles are not mutually inverse points. To find the inverse of the center of the given circle, we proceed as follows: Let the center of inversion be O (Fig. 30), and the center of the

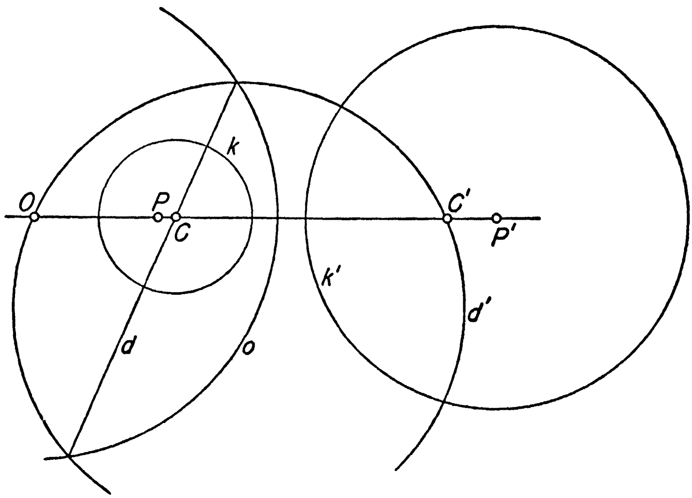


Fig. 30

given circle k be C, and let d be any diameter of k. Invert with respect to O. The line d inverts into a circle d' through O and C', the inverse of C, and orthogonal to the circle k', the inverse of k. Hence C' is the inverse of O with respect to the circle k'. Similarly if the center of k' is P', the inverse of P' with respect to o, say P, is the inverse of O with respect to k.

THEOREM. *The inverse of the center of a circle* k *is the same as the inverse of the center of inversion with respect to the circle* k′, *inverse to the given circle.*

37. Metrical relations

The distances between two points P and Q (Fig. 31) and their inverse points are connected by the equation*

$$P'Q' = QP\frac{r^2}{OP \cdot OQ}. \tag{1}$$

* The formula has been written so that if O, P, Q are collinear, direction is taken into account. If these points are not collinear, no special significance is attached to direction.

This follows readily from the similar triangles OPQ and $OQ'P'$, and the relation $OQ \cdot OQ' = r^2$. Thus

$$\frac{P'Q'}{PQ} = \frac{OQ'}{OP} = \frac{r^2}{OP \cdot OQ}.$$

If O, P, Q are on a line, a different proof must be given, the details of which can be left to the reader.

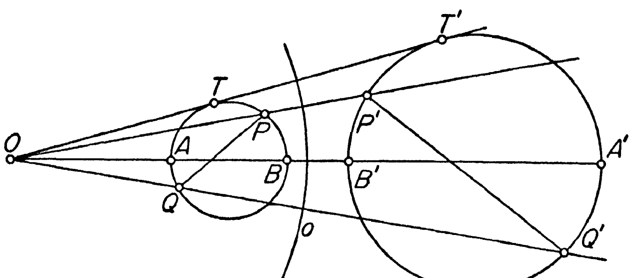


Fig. 31

Likewise the reader can readily show that if inverse points are represented as usual by primed letters, and P, Q, U, V are any four points, then

$$\frac{PU}{UQ} \div \frac{PV}{VQ} = \frac{P'U'}{U'Q'} \div \frac{P'V'}{V'Q'}.$$

If in particular the four points are four harmonic points on a line, and we invert with respect to a point on that line, this equation readily shows that they invert into four harmonic points.

If we apply equation (1) to a diameter AB and disregard direction, we find

$$\begin{aligned} 2R' &= 2R\frac{r^2}{OA \cdot OB} \\ R' &= R\frac{r^2}{p}, \end{aligned} \tag{2}$$

where p is the power of O with regard to the given circle.

Similarly

$$R = R'\frac{r^2}{p'}, \tag{3}$$

so that

$$pp' = r^4. \tag{4}$$

38. Application to inverse of center

We apply section 37 to find the inverse of the center of a given circle. Let C be the center of the given circle k; let C' be its inverse point; and let P' be the center of the circle k' as in Fig. 30. Let the radii be R and R', respectively. Since C and P' are corresponding points in the homothetic transformation from O which transforms k into k', we have

$$\frac{OP'}{OC} = \frac{R'}{R} = \frac{p'}{r^2} \quad \text{and} \quad OC \cdot OC' = r^2.$$

By multiplication of these equations, we find

$$OP' \cdot OC' = p'.$$

We rewrite all segments using P' as the origin, and use the fundamental power relation of section 28,

$$p' = P'O^2 - R'^2,$$

thus finding

$$P'O(P'O - P'C') = P'O^2 - R'^2$$

or

$$P'O \cdot P'C' = R'^2.$$

Hence O and C' are inverse with respect to the circle k' as just proved geometrically above.

39. Ptolemy's Theorem

As another illustration, we prove Ptolemy's original theorem and then consider its extension.

THEOREM. *A necessary and sufficient condition that a simple convex quadrilateral be inscribable in a circle is that* *the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.*

Let the simple convex quadrilateral be $ABCD$ (Fig. 32), and invert with respect to D. Then the circle ABC will invert into a line if, and only if, A, B, C, D are concyclic. If inverses are represented as usual by primes,

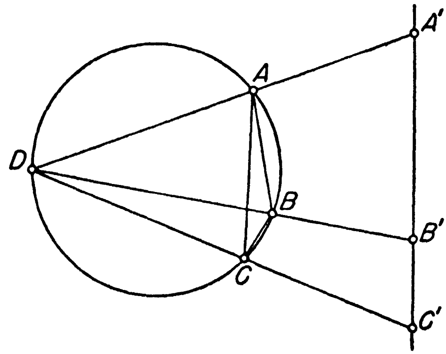


Fig. 32

$$A'B' + B'C' = A'C' \tag{1}$$

if, and only if, A, B, C, D are concyclic. We now make use of equation (1) of section 37, which gives a relation involving the original points,

$$AB\frac{r^2}{DA \cdot DB} + BC\frac{r^2}{DB \cdot DC} = AC\frac{r^2}{DA \cdot DC}. \tag{2}$$

If we clear of fractions, recognizing that sense of direction is not taken into account in Ptolemy's theorem, we find

$$AB \cdot CD + BC \cdot DA = AC \cdot BD \tag{3}$$

as the necessary and sufficient condition that the simple convex quadrilateral be inscribable in a circle.

Extensions. If A, B, C, D are any four points and we apply the procedure above, then

$$A'B' + B'C' + C'A' = 0 \text{ or exceeds } 0, \tag{1'}$$

according as the four points are concyclic or not. Then

$$\pm AB\frac{r^2}{DA \cdot DB} \pm BC\frac{r^2}{DB \cdot DC} \pm CA\frac{r^2}{DC \cdot DA} \tag{2'}$$
$$= 0 \text{ or exceeds } 0,$$

where the signs to be selected depend upon the relative order of the points A, B, C, D. Finally as the extension of Ptolemy's theorem, we have

$$\pm AB \cdot CD \pm AC \cdot DB \pm AD \cdot BC = 0 \text{ or exceeds } 0, \qquad (3)$$

according as $ABCD$ is or is not a cyclic quadrilateral, and where two of the signs are $+$, and the other one $-$, the negative sign corresponding to the diagonals of the simple convex quadrilateral formed from the points.

40. Tripolar coordinates

If a triangle and a point are given, the distances of the point from the vertices and hence their ratios—called the tripolar coordinates of the point—are determined. Let us consider the problem of determining the points corresponding to a given set of ratios.

THEOREM. *There are at most two points in the plane whose distances from three fixed points are proportional to given numbers.*

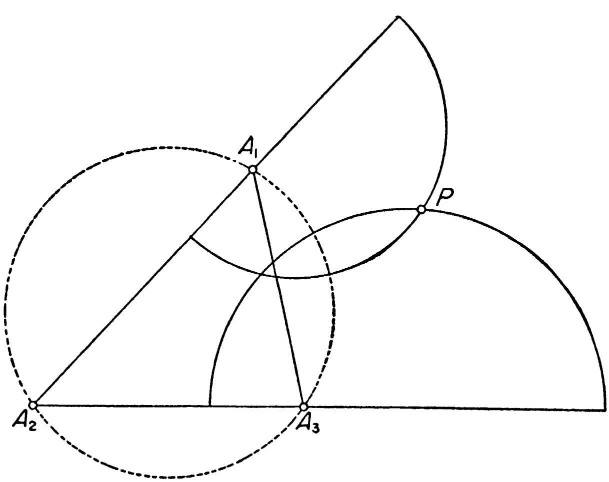


Fig. 33

Let A_1, A_2, A_3 be the given points; we seek a point P such that

$$PA_1 : PA_2 : PA_3 = p_1 : p_2 : p_3,$$

where p_i are given numbers. If such points exist, then P lies on the circle of Apollonius determined by

$$\frac{PA_2}{PA_3} = \frac{p_2}{p_3},$$

a circle which is orthogonal to the circumcircle of the triangle $A_1A_2A_3$ (Fig. 33). Similarly P lies on the circles determined by the ratios

$$\frac{PA_1}{PA_2} = \frac{p_1}{p_2}, \quad \frac{PA_1}{PA_3} = \frac{p_1}{p_3}.$$

If any two circles have points in common, they lie on the third circle; hence at most there are only two such points. These points are inverse with regard to the circumcircle (section 30) and will coincide if, and only if, they are on the circumcircle.

41. Condition for existence of the points with given tripolar coordinates

THEOREM. *If* $\mathrm{A_1}$, $\mathrm{A_2}$, $\mathrm{A_3}$ *are three points, there will exist points* P *such that* $\mathrm{PA_1 : PA_2 : PA_3 = p_1 : p_2 : p_3}$, *if, and only if, the lengths* $\mathrm{p_1 \cdot A_2A_3}$, $\mathrm{p_2 \cdot A_3A_1}$, $\mathrm{p_3 \cdot A_1A_2}$ *have the property that the sum of any two of them is greater than the third; that is, these three products form a possible triangle.*

Let the lengths of $PA_1 = kp_1$, $PA_2 = kp_2$, $PA_3 = kp_3$. If P exists, then it follows as a consequence of Ptolemy's Theorem and its extensions (section 39), that the triangle condition is satisfied for the products $PA_1 \cdot A_2A_3$, $PA_2 \cdot A_3A_1$, $PA_3 \cdot A_1A_2$, and hence for $p_1 \cdot A_2A_3$, $p_2 \cdot A_3A_1$, $p_3 \cdot A_1A_2$, the equality sign holding when the two possible solutions for P coincide.

The proof of the converse is not so direct; let us suppose that the figure is lettered (Fig. 34) so that

$$p_2a_2 + p_3a_3 \geqq p_1a_1,$$

where p_1a_1 is greater than either p_2a_2 or p_3a_3, with P subject to the condition

$$PA_1:PA_2:PA_3 = p_1:p_2:p_3,$$

so that

$$PA_2 \cdot A_1A_3 + PA_3 \cdot A_1A_2 \geqq PA_1 \cdot A_2A_3, \tag{1}$$

it being understood that all segments are considered positive.

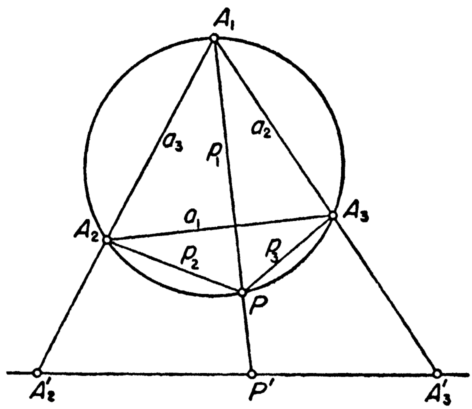


Fig. 34

This can be written after division by $A_1P \cdot A_1A_2 \cdot A_1A_3$,

$$\frac{PA_2}{A_1P \cdot A_1A_2} + \frac{PA_3}{A_1P \cdot A_1A_3} \geqq \frac{A_2A_3}{A_1A_2 \cdot A_1A_3}. \tag{2}$$

If we invert with respect to A_1, we notice that each fraction except for the arbitrary factor r^2, represents the length of the segment joining the corresponding inverse points (section 37, equation (1)), so that

$$A_2'P' + P'A_3' \geqq A_2'A_3'; \tag{3}$$

that is, if the triangle relation holds, (3) must be satisfied. But there always exist real points P' satisfying this relation, with P' being on the line $A_2'A_3'$ if the equality sign holds. Consequently P is real, being on the circumcircle if the equality sign holds. This completes the proof of the theorem.

If one of the coordinates is considered as a variable, we must determine it so that we have a simultaneous solution of

$$p_2a_2 + p_3a_3 \geqq p_1a_1, \quad p_1a_1 + p_3a_3 \geqq p_2a_2, \quad p_1a_1 + p_2a_2 \geqq p_3a_3,$$

if the construction problem is to have real solutions. If one of the sides of the triangle is considered as a variable, we must in addition impose the triangle condition upon the sides, using the inequality sign. For example if $a_1 = 1$, $a_2 = 2$, $a_3 = x$; $p_1 : p_2 : p_3 = 1 : 1 : 2$, then we must have

$$1 + 2 > x, \quad 1 + x > 2, \quad 2 + x > 1;$$

and

$$1 + 2 \geqq 2x, \quad 1 + 2x \geqq 2, \quad 2 + 2x \geqq 1.$$

The first group implies $1 < x < 3$; the second group $1/2 < x < 3/2$. Hence if we use $x = 2$, there will be a triangle but no real points P; if we use $x = 3/2$, there will be one point P; and if we use $x = 5/4$, there will be two points P. The student should draw the configuration corresponding to each of these values of x.

Exercise 3—Part 2. Inversion

1. The tangents to two inverse curves at a pair of corresponding points make equal angles with the radius vector to the points of contact.

2. Draw another figure for section 32, and follow the steps of the proof. What multiple of 180° is involved in your figure?

3. If P', Q', R' are the inverses of P, Q, R, show that the triangles $P'Q'R'$ and PQR are never inversely similar, and are directly similar if, and only if, each is inscribed in a circle concentric with the circle of inversion. (*Hint*: $\measuredangle PQR \equiv \measuredangle POR/2$.)

4. There exists in general an inversion by which the inverses of three given points P, Q, R are the vertices of a triangle similar to a second given triangle ABC. Construct the center of inversion. (*Suggestion*: Determine the point O by section 32.)

5. If P, Q, R, S are any four points, show that

$$\measuredangle PQR + \measuredangle RSP \equiv \measuredangle R'Q'P' + \measuredangle P'S'R'.$$

Hence prove that the inverse of a circle is a circle.

6. Show that three circles can be inverted into equal circles by means of not more than eight centers of inversion, but that there may be no such points.

7. Prove that equation (1) of section 37 is valid if O, P, Q are collinear.

8. Ptolemy's First Theorem. If $ABCD$ is a simple convex quadrilateral inscribed in a circle, prove

$$AC \cdot BD = AB \cdot CD + BC \cdot DA$$

by inverting the circle into a line and applying Euler's Theorem (problem 3, exercise 1).

9. Ptolemy's Second Theorem. If $ABCD$ is a simple convex quadrilateral inscribed in a circle, prove

$$\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{BA \cdot BC + DA \cdot DC}$$

by inverting with respect to D and applying Stewart's Theorem (problem 17, exercise 1).

10. Prove by inversion that if the circumcircles of the triangles ABC and ABD cut orthogonally, then the circumcircles of the triangles CDA and CDB also cut orthogonally.

11. Find the points at which three given circles subtend equal angles. Discuss the condition for the existence of such points (problem 10, exercise 2—part 2).

12. If two circles intersect, prove that their two circles of antisimilitude are orthogonal.

13. Given an equilateral triangle $A_1A_2A_3$, determine x so that the problem of constructing a point P such that $PA_1 : PA_2 : PA_3 = 1:3:x$ will have (a) two solutions, (b) one solution, (c) no solution. Include sketches drawn to scale.

14. Construct a triangle $A_1A_2A_3$ having $A_2A_3 = A_1A_2 = 1$, and such that the problem of constructing a point P with $PA_1 : PA_2 : PA_3 = 1:3:4$ will have (a) two, (b) one, (c) no solution. Include sketches drawn to scale.

***15.** Make a study of two circles intersecting at a given angle. (See Coolidge, *Treatise on the Geometry of the Circle and Sphere*, for problems 15 and 16.)

***16.** Make a study of circles which bisect a given circle, that is, pass through the ends of a diameter of the given circle.

4 OTHER TRANSFORMATIONS

Part 1. Poles and Polars

42. Poles and polars

The subject of poles and polars properly belongs to the domain of projective geometry and can hardly be adequately treated by elementary methods. But in view of its close relation to inversion, harmonic division, and orthogonal circles, we give it some consideration.

If D is the inverse of C with respect to the circle o, center O, then the line through D perpendicular to OC is defined as the *polar* of C with respect to the circle. The point D is determined by the relations

$$OC \cdot OD = OA^2 \text{ or } (AB, CD) = -1.$$

The point C is called the *pole* of this line through D. Every point except the center of the circle has a definite polar, and any line not passing through the center has a definite pole. The polar of a point on the circle is the tangent line, and the pole of a tangent line is the point of contact.

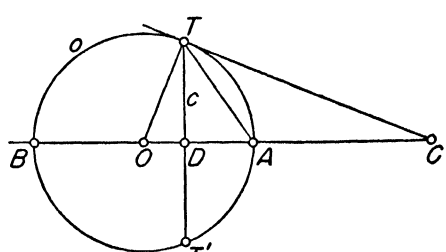


Fig. 35

If C is outside the circle, the polar of C is the chord of contact of the tangents from C to the circle (compare section 25). For let the points of contact be T, T' (Fig. 35), and let TT' cut OC in D. Then OC is perpendicular to TT', and it follows easily from the similar triangles ODT and OTC that

$$OC \cdot OD = OT^2 = OA^2,$$

so that TT' is the polar of C. A corresponding statement can be made if C is inside the circle, reversing the construction to find the inverse point, but now the perpendicular must be constructed.

For completeness the polar of the center of the circle is defined to be the line at infinity, and the pole of a diameter is the point at infinity in the direction perpendicular to the diameter.

43. Incidence relation

As an illustration of the use of orthogonal circles, we prove several properties of poles and polars.

THEOREM. *If* A *is on the polar of* B, *then* B *is on the polar of* A.

Let the inverse of B be B', and let the polar of B be b; let A be any point on b (Fig. 36). Draw the circle ABB'. It

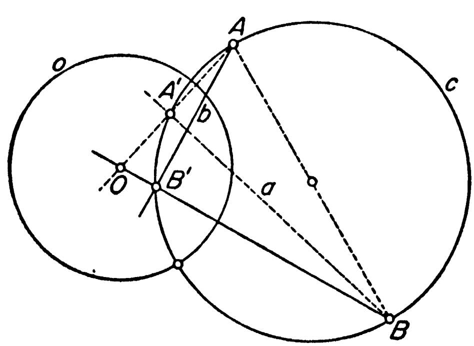


Fig. 36

has AB as a diameter and is orthogonal to the circle o, and hence cuts OA in the inverse of A, say A'. But the angle $BA'A$ is a right angle since it is inscribed in a semicircle. Therefore $A'B$ is the polar of A, or B is on the polar of A.

This theorem is its own converse, for the names of the points are immaterial; it is known as the *incidence relation* of poles and polars.

COROLLARY. *If four points lie on a line, their polars pass through a point; if the four points form a harmonic range, their polars form a harmonic pencil.*

For if the points lie on the line u, their polars pass through U, the pole of u. If we join four harmonic points to O, we obtain a harmonic pencil. The polars form a pencil at U such that the angle between any two polars equals the angle between the corresponding lines of the pencil at O, and hence the pencil at U is also harmonic.

44. Harmonic property of poles and polars

THEOREM. *If* P *is the pole of* p *with respect to a circle, then any line through* P *which cuts the circle is cut harmonically by* P, p *and the circle.*

Let the inverse of P be P' and let the polar be p (Fig. 37). Let the arbitrary line cut the circle in A, B, and cut p in Q. Since $\measuredangle\ QP'P = 90°$, these three points lie on a circle having PQ as a diameter. This circle is orthogonal to the given circle, since P and P' are inverse points. Hence $(AB, PQ) = -1$. Conversely, if Q is determined by $(AB, PQ) = -1$, Q is on p.

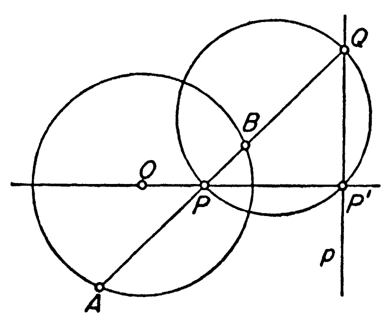


Fig. 37

We thus see that the polar of P appears as the locus of the harmonic conjugate of P with respect to the points in which a variable secant through P cuts the circle.

45. Inscribed quadrangle

Let any two secants through P (Fig. 38) cut the circle in the vertices A, B and C, D of a complete quadrangle. Let

X and Y be the harmonic conjugates of P with respect to A, B and C, D, respectively. Let AD meet BC in R, and AC meet BD in Q. Then the complete quadrangles $AQBR$ and $CQDR$ show that QR passes through X and Y, and hence is the polar of P. Thus the polar of P appears as the diagonal opposite P of any inscribed quadrangle which has P as a diagonal point. The use of this result yields the simplest way of constructing the polar of a point, since it requires a straight edge only.

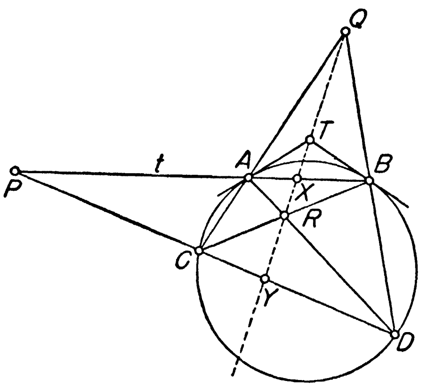


Fig. 38

46. Intersection of tangents

Let the tangents at A and B (Fig. 38) meet at T. Then the line AB is the polar of T, and since P is on t, the polar of P passes through T. Hence we have a third property to determine the polar of P. The polar of P is the locus of intersections of tangents drawn at points where a variable secant through P cuts the circle.

47. Conjugate points and lines

Two points such that each lies on the polar of the other are said to be *conjugate points* (*polar conjugates*), and two lines each of which passes through the pole of the other are said to be *conjugate lines*. The harmonic property can be restated in the following terms:

THEOREM. *If the line joining two conjugate points cuts the circle, the four points so determined form a harmonic range.*

Let the conjugate points be A, B, and their polars a, b. Let AB $(= c)$ cut the circle in X and Y. The harmonic property is evident. Draw the tangents at X and Y. From the incidence relation, they meet at C, the pole of c. By projection the pencil (xy, ab) is harmonic, and since all steps are reversible, we can state another theorem.

THEOREM. *If tangents to the circle can be drawn from the intersection of two conjugate lines, the four lines so determined form a harmonic pencil.*

Two triangles are polar conjugates with regard to a circle when the sides of one are the polars of the vertices of the other. A triangle is *self-polar* if each vertex is the pole of the opposite side. In Fig. 39, ABC is a self-polar triangle, and from the quadrangle property, we can readily prove that the diagonal triangle of an inscribed quadrangle is self-polar; hence, in Fig. 38, triangle PQR is self-polar.

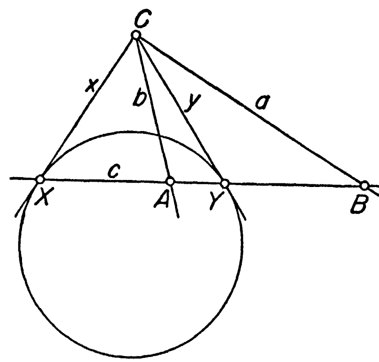


Fig. 39

Exercise 4—Part 1. Poles and Polars

1. If four points of a circle project to a fifth point of the circle in four harmonic lines, they project to every point of the circle in four harmonic lines. Prove. What happens if the vertex is taken as one of the given points?

2. Four points as in problem 1 are called *four harmonic points on a circle.* Prove that two polar conjugate chords cut the circle in four harmonic points on the circle.

3. TP and TQ are two tangents to a circle. PR is a diameter of the circle, and QN is perpendicular to PR. Prove that the pencil $Q(TN, PR)$ is harmonic and that QN is bisected by TR.

4. If two fixed tangents, TA and TB, to a circle c meet a variable tangent in P and Q, prove that the angle PCQ is constant. Hence prove that if four tangents to a circle cut a fifth tangent harmonically, they cut every tangent harmonically. What happens if the base of the range is taken as one of the tangents?

5. Four tangents as in problem 4 are called *four harmonic tangents to a circle.* Prove that the tangents drawn from two polar conjugate points outside a circle are four harmonic tangents to a circle.

6. PQ is a diameter of a circle c; c_1 is a circle orthogonal to c. Prove that the polar of P with respect to c_1 passes through Q. Conversely, if PQ is a diameter of c and the polar of P with respect to c_1 passes through Q, then c is orthogonal to c_1. Note that c_1 need not intersect the line PQ.

7. If three circles are orthogonal to another circle, prove that the locus of a point whose polars with respect to the three given circles are concurrent is this common orthogonal circle.

8. Find the locus of points whose polars with regard to two given circles are perpendicular.

9. A line meets two given nonintersecting circles in two pairs of points. Find the two points on this line such that each is the intersection of the polars of the other with regard to the two circles.

10. Salmon's Theorem. The distances from the center of a circle to any two points are proportional to the distances from each point to the polar of the other.

11. Given a triangle, construct a circle with respect to which the triangle is self-polar.

12. Invert the construction for the polar of O with respect to a circle, using O as the center of inversion and a radius equal to the square root of the power of O with regard to the circle. What theorem results?

Part 2. Trilinear Polar. Isogonal Conjugates

48. Trilinear polar

Another transformation involving harmonic division which correlates a point with a line follows: Let $A_1A_2A_3$ be any triangle (Fig. 40), and let Q be any point. Let A_iQ meet the opposite sides in P_i, and let the harmonic conjugate of P_i with regard to the vertices on that side be Q_i. Then the points Q_1, Q_2, Q_3 are collinear.

From the quadrangle $A_1P_2QP_3$, the line P_2P_3 passes through Q_1. Since A_1 is common to both harmonic ranges (A_1A_2, P_3Q_3) and (A_1A_3, P_2Q_2), and since P_2P_3 and A_2A_3 meet at Q_1, it follows from section 18 that Q_2Q_3 also passes through Q_1.

Conversely, if any line q cuts the sides of a triangle in Q_i, and if P_i is the harmonic conjugate of Q_i with regard to the vertices on that side, then the lines A_iP_i are concurrent at Q. For as before (section 18), P_3, P_2, and Q_1 are collinear, and if A_2P_2 and A_3P_3 meet at Q, then A_1Q must pass through P_1.

Thus we see in general that to each point of the plane corresponds a line called its *trilinear polar*, and to each line corresponds a point called its *trilinear pole*. If Q is on a side, but not at a

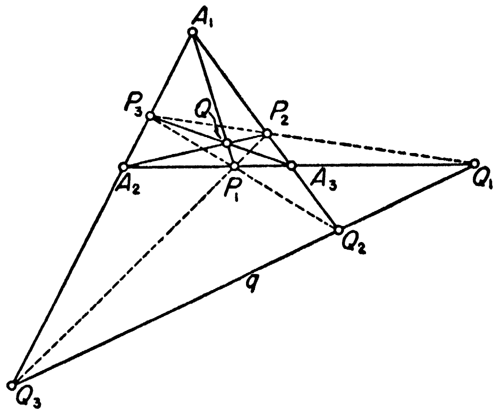


Fig. 40

vertex, its trilinear polar is that side; but if Q is at a vertex, say A_1, then the line A_1Q and the point P_1 are indeterminate, and the trilinear polar is not defined. Similarly, if q passes through a vertex, its trilinear pole is that vertex, unless q is a side, in which case its trilinear pole is not defined.

It is easily shown by using Menelaus' Theorem or convenient complete quadrangles that not only are Q_1, Q_2, Q_3 and Q_1, P_2, P_3 collinear, but so are Q_2, P_1, P_3 and Q_3, P_1, P_2, which makes possible a simple construction of the trilinear polar of a given point.

Likewise not only are A_1P_1, A_2P_2, A_3P_3 concurrent, but so are each of the sets A_1P_1, A_2Q_2, A_3Q_3; A_1Q_1, A_2P_2, A_3Q_3; and A_1Q_1, A_2Q_2, A_3P_3.

49. Isogonal conjugates

If two lines through the vertex of an angle are symmetrical with regard to a bisector of that angle, they are said to be *isogonal*, and one line is said to be the isogonal conjugate of the other. Each line through the vertex of an angle has a definite isogonal conjugate; each angle bisector is self-isogonal. The fundamental theorem of isogonal conjugates is the following:

THEOREM. *If three lines, one through each vertex of a triangle, are concurrent, their isogonal conjugates are also concurrent.*

The corresponding points are said to be *isogonal conjugate points*. The proof is an immediate application of the trigonometric form of Ceva's Theorem, the details of which we leave to the student.

Thus in general we find for each point P a unique point P', and the process is reversible. There are certain exceptions. To any point on the side of the triangle corresponds the opposite vertex, so that the point that corresponds to a vertex is indeterminate; but as a moving point approaches a vertex along a given direction, its isogonal conjugate approaches a limiting position on the opposite side—namely, the point on the opposite side in the direction isogonal to the given direction. Each of the four points of intersection of the angle bisectors are self-conjugate and these are the only self-conjugate points.

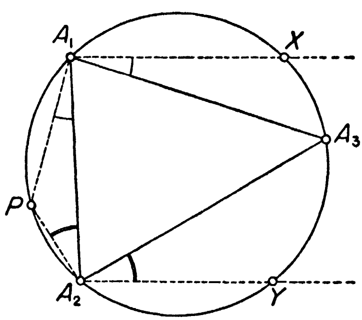


Fig. 41

THEOREM. *The isogonal conjugate of a point on the circumcircle of a triangle is at infinity, and conversely.*

For if P is on the circumcircle of the triangle $A_1A_2A_3$ (Fig. 41), and the isogonal conjugates of PA_1 and PA_2 meet the circumcircle in X and Y, then, from a consideration of arcs,

$$\measuredangle A_2PA_1 \equiv \measuredangle \alpha_1 + \measuredangle \alpha_2 \text{ (mod } 180°\text{)}.$$

But

$$\measuredangle PA_1A_2 \equiv \measuredangle A_3A_1X,$$

and

$$\measuredangle PA_2A_1 \equiv \measuredangle A_3A_2Y,$$

and since the sum of the angles of the triangle A_1A_2P is 180°, we find

$$\measuredangle A_2A_1X + \measuredangle YA_2A_1 \equiv 0° \pmod{180°},$$

and the lines are parallel, so that P' is at infinity. Conversely, if P' is at infinity, we may retrace our steps and show that P is on the circumcircle.

50. Angle property

Another proof of this last theorem depends upon the fundamental angle formula relating the angles at P and P'.

THEOREM. *If* P *and* P′ *are isogonal conjugates with regard to the triangle* $A_1A_2A_3$, *then*

$$\measuredangle A_2PA_3 + \measuredangle A_2P'A_3 \equiv \measuredangle A_2A_1A_3 \pmod{180°}.$$

The multiple of 180° involved depends upon the positions of P and P' with regard to both the circumcircle and the line A_2A_3. In Fig. 43 we have illustrated two cases; the same proof, however, applies to all cases. The student should draw a diagram with both points inside the triangle, and should also consider the corresponding equations when using the sides A_1A_2 and A_1A_3.

The proof is given in terms of arcs. We recognize that if

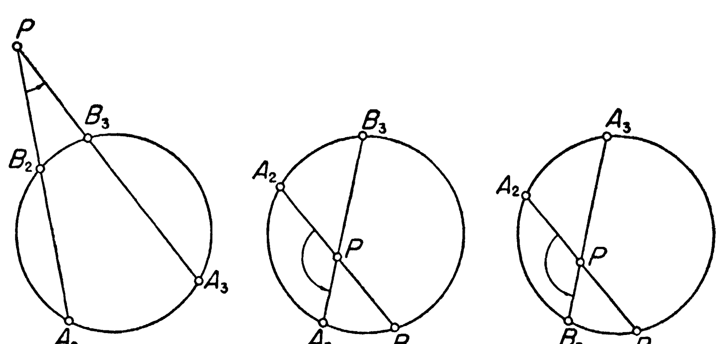


Fig. 42

A and B are two points on the circle, the term *arc AB* is ambiguous, but all such arcs are congruent modulo 360°, or the half-arcs are congruent modulo 180°, if direction is taken into account. The student may readily verify by simple diagrams (Fig. 42) that if $l_2 = A_2B_2$ and $l_3 = A_3B_3$ are any two secants of a circle meeting at P, then

$$\measuredangle l_2l_3 = \measuredangle P \equiv \frac{1}{2} \text{ arcs } (A_2A_3 + B_2B_3) \pmod{180°},$$

independent of the position of P and of the order of the points, so that not only has uniqueness been restored, but

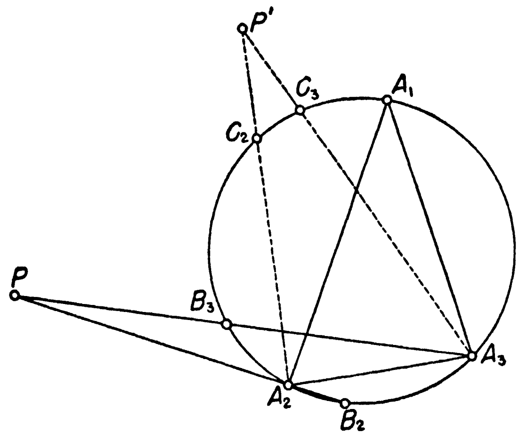


Fig. 43a

also several elementary theorems have been combined into one.

In the center diagram, the result is immediate; for the left diagram we must reverse one direction and the corresponding sign; for the right diagram, we recognize that arcs $(A_2B_3 + B_2A_3) \equiv$ arcs $(A_2A_3 + A_3B_3 + B_2B_3 + B_3A_3) \equiv$ arcs $(A_2A_3 + B_3A_3)$.

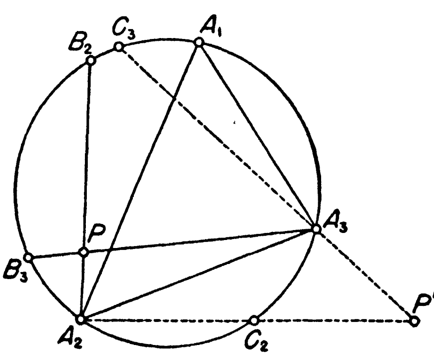


Fig. 43b

Let us now return to the proof of the theorem.

Let PA_2 and PA_3 cut the circle in B_2 and B_3, respectively; and let their isogonal conjugates cut the circle in C_2 and C_3, respectively (Fig. 43). Let us designate the angles under consideration by $\measuredangle P$, $\measuredangle P'$, and $\measuredangle \alpha_1$. Then, modulo 180°,

$$\measuredangle P \equiv \frac{1}{2} \text{ arcs } (A_2A_3 + B_2B_3), \tag{1}$$

$$\measuredangle P' \equiv \frac{1}{2} \text{ arcs } (A_2A_3 + C_2C_3). \tag{2}$$

The isogonal relations imply

$$\text{arc } B_2A_3 \equiv \text{arc } A_1C_2, \qquad \text{arc } A_2B_3 \equiv \text{arc } C_3A_1. \tag{3}$$

From the identity

$$\text{arc } B_2B_3 \equiv \text{arcs } (B_2A_3 + A_3A_2 + A_2B_3) \tag{4}$$

and equations (3), we have

$$\begin{aligned} \frac{1}{2} \text{ arc } B_2B_3 &\equiv \frac{1}{2} \text{ arcs } (A_1C_2 + A_3A_2 + C_3A_1) \\ &\equiv \frac{1}{2} \text{ arcs } (A_3A_2 + C_3C_2) \\ &\equiv -\measuredangle P' \end{aligned}$$

from equation (2). Hence equation (1) gives

$$\measuredangle P + \measuredangle P' \equiv \measuredangle \alpha_1$$

or

$$\measuredangle A_2PA_3 + \measuredangle A_2P'A_3 \equiv \measuredangle A_2A_1A_3 \pmod{180°}. \tag{5}$$

In view of formula (5), if P is on the circumcircle so that $\measuredangle P \equiv \alpha_1$, we see that $\measuredangle P' \equiv 0 \pmod{180°}$, and P' is at infinity, and conversely.

COROLLARY. *It follows that if a point traces a circle through two vertices of a triangle, so that* $\measuredangle A_2PA_3$ *is constant, then its isogonal conjugate traces another circle through the same two vertices.*

51. Distance relation

THEOREM. *The distances from the sides of a triangle to isogonal conjugate points are inversely proportional.*

Let the isogonal conjugate points be P and Q, and the feet of the perpendiculars from P and Q be B_1, B_2, B_3 and C_1, C_2, C_3, and the lengths of the perpendiculars be p_1, p_2, p_3; q_1, q_2, q_3, respectively (Fig. 44). Then from the similar triangles PA_2B_1 and QA_2C_3, and the triangles PA_2B_3 and QA_2C_1, we find

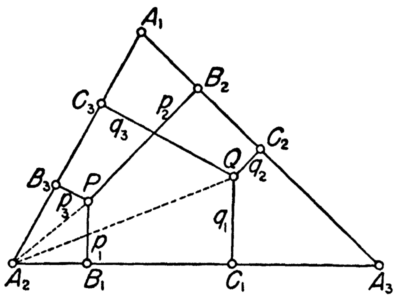


Fig. 44

$$\frac{p_1}{q_3} = \frac{A_2P}{A_2Q} = \frac{p_3}{q_1}.$$

Similarly for other pairs, so that

$$p_1q_1 = p_2q_2 = p_3q_3.*$$

Exercise 4—Part 2. Trilinear Polar. Isogonal Conjugates

1. Prove that the external bisectors of the angles of a triangle meet the opposite sides in three collinear points. How is this line related to the incenter? Where are the lines corresponding to the excenters?

2. Prove the last statement of section 48 that A_1P_1, A_2Q_2, A_3Q_3, and so on, are concurrent. Where are the trilinear polars of these points of concurrence?

3. (a) If three lines, one through each vertex of a triangle, are concurrent, their isogonal conjugates are concurrent.

(b) If three lines one through each vertex of a triangle meet the opposite sides in three collinear points, then their isogonal conjugates likewise meet the opposite sides in three points that are collinear.

---

* This relation forms the basis for an analytical study of isogonal conjugates, for it is easily recognized that these distances may be taken as trilinear coordinates of the points. The transformation is one that transforms a line into a conic circumscribing the triangle of reference. The interested student may consult D. M. Y. Sommerville, *Analytical Conics*, London: G. Bell and Sons, 1924, for further details.

4. Prove the angle relation of isogonal conjugates using the side A_1A_2. For selected angles at P and P' in your diagram, what multiple of $180°$ is involved?

5. Make freehand sketches of the isogonal conjugates (**a**) of a line that does not pass through a vertex, (**b**) of a circle which passes through two vertices.

6. If P_1 and Q_1 are points on the side A_2A_3 of the triangle $A_1A_2A_3$ such that $A_2P_1 = Q_1A_3$, then A_1P_1 and A_1Q_1 are called *isotomic* conjugate lines. State and prove the theorems that correspond to problems 3 (a) and 3 (b).

*Investigate further the transformation thus obtained.

5 COAXAL CIRCLES

Part 1. Radical Axis. Coaxal Circles

52. Radical axis

The locus of a point having equal powers (section 28) with regard to two nonconcentric circles is a straight line perpendicular to the line of centers. This line is called the *radical axis* of the two circles.

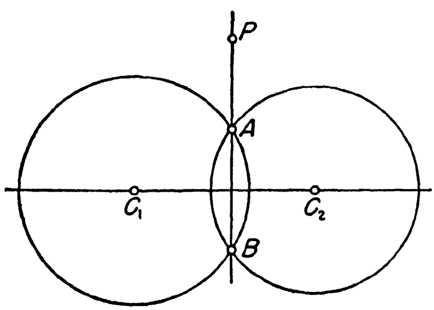


Fig. 45

CASE 1. The circles intersect (Fig. 45). We prove that the locus is their common chord. Let the circles have centers C_1 and C_2. Let A and B be their common points and let P be any point on AB. It is evident that the power of P with regard to either circle is $PA \cdot PB$. Conversely, if P is any point such that $p_1 = p_2$, let PA meet the circles in B_1 and B_2, respectively. Then

$$PA \cdot PB_1 = p_1 = p_2 = PA \cdot PB_2, \qquad PB_1 = PB_2,$$

and B_1 and B_2 coincide, so that P is on the common chord of the circles, which is a line perpendicular to the line of centers.

We digress a moment to reconsider the problem of section 22, to find a pair of points which separate two given pairs harmonically (Fig. 17). The line PQ is the radical axis of the two arbitrary circles through A_1, B_1 and A_2, B_2, respectively; hence the point O has equal powers with regard to these circles. But the point O has the same power with regard to any circle through A_1, B_1 or A_2, B_2. Hence no matter what other two circles we use, their radical axis will pass through O, and the power of O will be positive if, and only if, O is external to both segments (that is, if the segments do not overlap), in which case there is a unique pair of points which satisfy the required condition.

CASE 2. The circles do not intersect. Let P be a point such that the powers of P with regard to both circles are equal. Let the circles cut the line of centers in A_1, B_1, A_2, B_2 (Fig. 46). The circle with center P and radius equal to

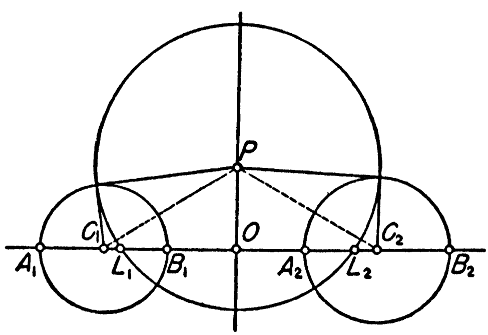


Fig. 46

the tangent to either circle will be orthogonal to both circles. Let it cut C_1C_2 in L_1 and L_2. Then

$$(L_1L_2, A_1B_1) = -1, \quad (L_1L_2, A_2B_2) = -1,$$

or L_1 and L_2 are the points which are harmonic conjugates with respect to the two given pairs, the existence and uniqueness of which we have just again considered. Hence P lies on the perpendicular bisector of L_1L_2 for every position of P such that $p_1 = p_2$. Conversely, if L_1, L_2 is the pair of points which divides A_1, B_1 and A_2, B_2 harmonically, and P is any point on the perpendicular bisector of L_1L_2, then the circle with center P and radius PL, will be orthogonal to both circles, so that it has equal powers with regard to both circles, which concludes the proof.

53. Metrical property

These two cases are forerunners of the distinctions which must later be made in this chapter, and so the above proof was presented first. A single proof which takes care of both cases simultaneously is easily given. Let P have equal powers with respect to both circles and let PO be perpendicular to C_1C_2. Let the distances $C_1O = d_1$, $OC_2 = d_2$, and $C_1C_2 = d$, and let r_1 and r_2 be the radii of the circles. Then from the right triangles in Fig. 46, and the power property, we find

$$p_1 = PC_1^2 - r_1^2 = PC_2^2 - r_2^2 = p_2, \tag{1}$$

$$PO^2 + d_1^2 - r_1^2 = PO^2 + d_2^2 - r_2^2, \tag{2}$$

$$d_1^2 - d_2^2 = r_1^2 - r_2^2, \tag{3}$$

where

$$d_1 + d_2 = d. \tag{4}$$

Without proceeding further, we note that this result is independent of P and depends only on the location of O as determined by (3) and (4). Conversely, since the steps in the proof are reversible, if O is determined by (3) and (4), any point on the line through O perpendicular to C_1C_2 will have equal powers with respect to both circles. If the circles intersect, the common points have zero power with regard to both circles, and hence the common chord is the radical axis. The further details of the solution of (3) and (4) for d_1 and d_2 we leave to the reader.

54. Radical center

The radical axes of three circles taken in pairs are concurrent in a point called the *radical center*. The proof is left to the reader.

This property enables us readily to construct the radical axis of two circles that do not intersect by reducing the problem to the intersecting type. For if c_1 and c_2 do not intersect, draw a circle intersecting both of them. The two common chords will intersect in a point A on the required radical axis. A second point B is found similarly, so that AB is the required radical axis.

If the radical center of three circles lies outside of one circle, it must lie outside of all three, and it is the center of a circle cutting all three orthogonally, called the *radical circle* of the three circles. If we invert with respect to any point on the radical circle, the circles invert into circles whose centers are collinear.

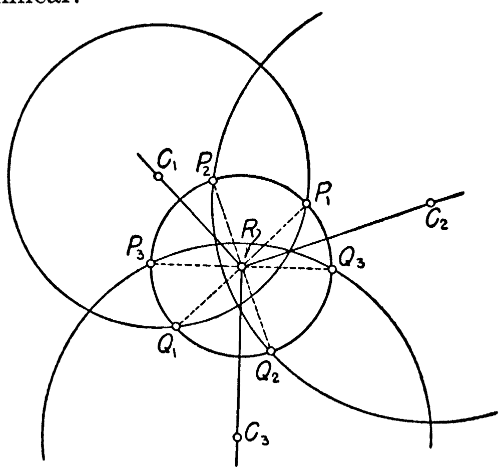


Fig. 47a

If the radical center R lies within one, and hence within all three, it is the center of a circle cut by all three in pairs of diametrically opposite points P_i, Q_i (Fig. 47a); the radius of the circle is $\sqrt{-p}$ (p in this case being negative); and the diameters will be those perpendicular to the lines joining R to the centers of the circles. The circle with center R and radius $\sqrt{p}$ may be thought of as an imaginary circle orthogonal to all three circle, for an inversion with respect to the real circle, followed by re-

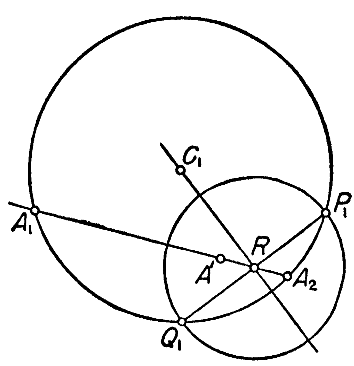


Fig. 47b

flection on R, leaves each of the circles invariant. For if any chord through R cuts one of the given circles in A_1 and A_2, and A' is the inverse of A_1, we have (Fig. 47b)

$$RA_1 \cdot RA_2 = +p,$$
$$RA_1 \cdot RA' = -p,$$

from which it follows that the reflection of A' on R is A_2, so that the given circle remains fixed.

55. Coaxal circles

If a system of circles is such that every pair has the same radical axis, the circles are said to be *coaxal.* The system is determined by any two circles, for they determine the radical axis. If the two circles intersect, then every circle through the points of intersection is obviously a circle of the system; and, conversely, if a circle belongs to the system, it must intersect every circle of the system in points on the radical axis, and hence at the two points of intersection. For completeness we shall always consider the radical axis as a member of the system. The circles of such a system all have their centers on a line and pass through two real points (the second property implies the first). It is known as a system of the intersecting type.

Let us now consider the circles of the nonintersecting type. First we note that if all the circles have the same radical axis, the line joining the centers of all pairs must be perpendicular to the radical axis, and hence the centers all lie on a line. Second, suppose the radical axis is determined by c_1 and c_2 and that they cut the line of centers in A_1, B_1, and A_2, B_2 (Fig. 48). Further let L_1, L_2 be the points (called *limiting points*) which are harmonic conjugates of both pairs, and let O be the center of the circle having L_1L_2 as diameter. (This circle, for reasons that appear below, is known as the *construction circle.*) Now circles c_1 and c_2 are orthogonal to the construction circle o, and their radical axis is the perpendicular bisector of L_1L_2. If c_3 is any other circle whose center is on L_1L_2 and which is orthogonal to o, then c_1 and c_3 will have the perpendicular bisector of L_1L_2 as their radical axis, since L_1, L_2 will be the unique pair of points which are harmonic conjugates of the pairs of points where the circles intersect

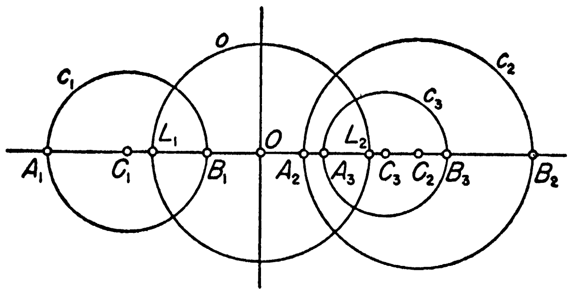


Fig. 48

the line of centers. Conversely, if c_3 is any circle with its center on the line L_1L_2, such that the radical axis of c_1 and c_3 is the perpendicular bisector of L_1L_2, then c_3 is orthogonal to the circle o. For since O has equal powers with regard to c_1 and c_3 and since o is orthogonal to c_1,

$$p_1 = OL_1^2 = p_3,$$

and circle o is orthogonal to c_3.

It should be noted that the circle o could be any circle through L_1 and L_2, for the proof holds without further modification.

Consequently, a system of the nonintersecting type appears as all circles that have the properties: (1) their centers are on a line and (2) the circles are orthogonal to a given circle which cuts this line.

To construct a system of this type, we may begin by drawing the construction circle. We construct the tangent at any point of it, and the point where this tangent cuts the line of centers will be the center of a circle of the system. If the system is to have two given points L_1, L_2 as limiting points, the circle on L_1L_2 as diameter is the construction circle. If the system is to be determined by two given circles which do not intersect, we may find L_1 and L_2 by the method of Fig. 17 and then proceed, or we may first construct the radical axis and find the point O, where it meets the line of centers. Draw the tangent from O to one of the given circles (section 5), and then draw the construction circle.

It is evident that the limiting points L_1 and L_2 may be considered point circles of the system and that they are inverse with regard to any circle of the system. Conversely, if L_1 and L_2 are inverse with regard to every circle of a family, the circles of the family must have their centers on L_1L_2, and each circle must be orthogonal to the circle on L_1L_2 as diameter; that is, the circles form a coaxal system of the nonintersecting type.

The center of every circle of the system is external to the segment L_1L_2, and any circle of the system cuts the axis of the system in two points on the same side of O, since $OA_i \cdot OB_i = OL_1^2$. Consequently, no circle cuts the radical axis or another circle of the system, for then it would cut the radical axis.

56. Conjugate systems

With every system of coaxal circles there is associated another system of the opposite type; the radical axis of one system is the line of centers of the other, and every circle of either system cuts every circle of the other system orthogonally.

If L_1 and L_2 are any two points, then all circles c_i, which pass through L_1 and L_2, form a system of the intersecting type, and all circles k_i, which have L_1 and L_2 as inverse points, form a system of the nonintersecting type; the center of every c lies on the perpendicular bisector of L_1L_2, and every k is orthogonal to every c. Two such systems are said to be conjugate.

Since the point pair is uniquely determined by any two intersecting circles, we have as an alternative criterion for a coaxal system of the nonintersecting type: *all circles that are orthogonal to two intersecting circles form a system of the nonintersecting type.*

57. Special cases

Besides the two fundamental types considered above, we may consider other types. As a limiting case of either type, we have the set of all circles tangent to a line at the same point, the line being the radical axis of the system. Its conjugate system is a system of the same type.

If two circles are concentric, they have, properly speaking, no radical axis; but for completeness, we define their radical axis as the line at infinity, which is consistent with the formulas for locating the distance of the radical axis from the common center (section 53). All concentric circles form a coaxal system, and, for reasons that will appear as we proceed, we take the pencil of lines whose vertex is the common center as the conjugate system orthogonal to it.

58. Circles of a coaxal system through a point

THEOREM. *Through any point of the plane, not a limiting point, passes one, and only one, circle of each of two conjugate coaxal systems.*

The proof and construction are indicated simultaneously. Suppose that L_1 and L_2 are the common points of the system of the intersecting type (Fig. 49a). Then P, L_1, L_2 determine a unique circle (the line L_1L_2 included), except when P is one of these points. The circle drawn with center on L_1L_2, orthogonal to c at P, belongs to the conjugate system; and, conversely, the circle of the conjugate system through P is orthogonal to c. If P is at a limiting point, it is the circle of the nonintersecting system, but the circle of the intersecting system is indeterminate.

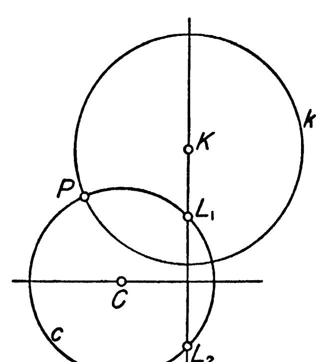


Fig. 49a

59. Circles of a coaxal system orthogonal or tangent to a circle

We may determine a circle of a coaxal system subject to various conditions. We state the following theorems:

THEOREM. *If a circle belongs to neither of two conjugate coaxal systems, there is not more than one circle of either system orthogonal to it.*

The proof and construction of the circle belonging to one system depend upon the determination of the radical center of the given circle with the circles of the conjugate system.

THEOREM. *If a circle (or line) belongs to neither of two conjugate coaxal systems, there are not more than two circles of either system tangent to it.*

The construction involves the circle of the conjugate system which is orthogonal to the given circle or line.

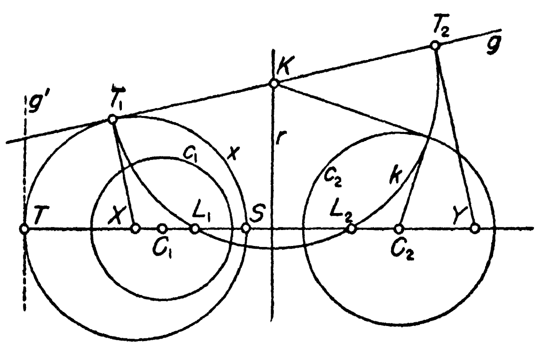


Fig. 49b

The details and elaborations of most of these constructions we leave as exercises for the student, but the following specific case illustrates the analysis, synthesis, and discussion of such problems.

Given c_1 and c_2, two nonintersecting circles, and a line g, not their radical axis, construct the circles coaxal with c_1 and c_2 tangent to g.

Analysis. Suppose that the figure has been completed, showing the line of centers and the radical axis r, of c_1 and c_2, and suppose that x is the required circle with T_1 the point of contact of x with g (Fig. 49b). If g and r meet in K, then the circle, center K, radius KT_1, belongs to the system conjugate to that determined by c_1 and c_2, and hence the point T_1 can be found. T_1X, perpendicular to g, will determine X, the center of the required circle which has XT_1 as radius.

Construction. Construct the radical axis of c_1 and c_2 by one of the several known methods. Let g and r intersect at K. Construct the circle k, center K, orthogonal to c_2, cutting g in T_1 and T_2. The perpendiculars to g at T_1 and T_2 determine the centers X and Y, and radii XT_1 and YT_2 of the required circles.

Discussion. In general, since g and r will meet in one and only one finite point K, there will be two solutions. If, however, the given line g' and r are parallel, the circle k is replaced by the line C_1C_2. There is only one finite point of contact T on the line C_1C_2, and the point X cannot be determined as above. If S is the harmonic conjugate of T with respect to the limiting points L_1, L_2, the circle on ST as diameter is the required circle.

Exercise 5—Part 1. Radical Axis. Coaxal Circles

1. Prove that the radical axis of two circles bisects a common tangent of the circles.

(a) Hence suggest a construction of the radical axis.
(b) When does the construction fail?
(c) Modify the construction for one circle and a point.
(d) Give another construction when the point is within the circle.

2. (a) If two circles are mutually inverse, prove that the tangents at inverse points meet on the radical axes of the two circles.

(b) Hence suggest a construction for the radical axes of two circles when one is within the other.

3. (a) Complete the solution for d_1 and d_2 of the equations of section 53.

(b) Show that the statement, "the radical axis of two concentric circles is the line at infinity," is consistent with these equations.

4. (a) The polars of a fixed point P with respect to a system of coaxal circles with real limiting points all pass through a fixed point; namely, that point on the circle through P and the limiting points, which is at the other extremity of the diameter through P.

(b) State and prove a corresponding theorem for a system of the intersecting type.

5. Find the locus of the inverse of a given point with regard to the circles of a coaxal system.

6. The polar of P with regard to a circle center O is the radical axis of the given circle and the circle on OP as diameter. (*Suggestion*: Let the radical axis cut OP in N; express all distances from O, using the equal power property of N.)

7. (a) If A and P are any two points on the circle c, and if A', P', c' are their inverses, prove that AP and $A'P'$ meet on the radical axis of c and c'.

(b) Hence prove that if A, B, C, D are four harmonic points on a circle (problem 1, exercise 4—part 1), they invert into four harmonic points on a circle.

In problems 8, 9, 10, 11 consider both types of coaxal systems and determine the number of solutions.

8. Construct a circle of a coaxal system orthogonal to a given line.

9. Construct a circle of a coaxal system orthogonal to a given circle.

10. Construct a circle of a coaxal system tangent to a given line.

11. Construct a circle of a coaxal system tangent to a given circle.

Part 2. Coaxal Systems

60. Inverse of a coaxal system

THEOREM. *The inverse of a coaxal system of circles is a coaxal system of circles.*

Let L_1, L_2 be the limiting points, O the center of inversion, and c and k the circles (section 58) of each system through O (Fig. 50, left). Represent the systems by c_i and k_i. The system c_i inverts into all circles through the inverses of L_1

and L_2; the system k_i inverts into a system of circles orthogonal to c_i' and hence is a coaxal system. The special circles c and k invert into perpendicular lines c' and k', which are

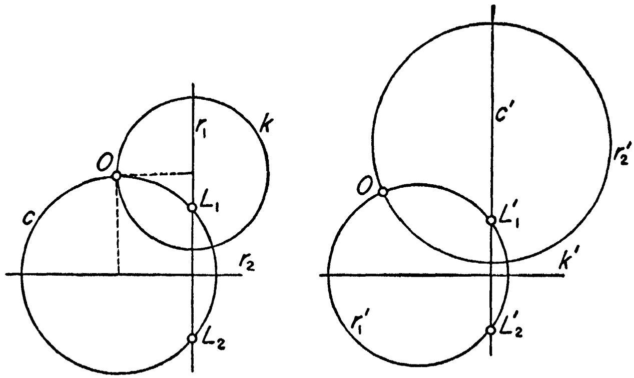


Fig. 50

the radical axes of the new systems. The radical axes of the old systems invert into the circles of the new system which pass through O.

If O takes special positions, the inverse system becomes specialized, but the only case of importance is when O is a limiting point, say L_1. Then the system c_i inverts into a system of lines through L_2'. The system k_i inverts into a system of concentric circles whose common center is L_2', for the orthogonal relations persist under inversion. Or, otherwise, since L_1 and L_2 are mutually inverse with respect to any circle of the system k_i, the same relation holds after inversion (section 34). But L_1' is at infinity, and hence L_2' is the center of every k_i'.

61. Circle of Apollonius for a variable ratio

We present in the next few sections several examples of coaxal circles that we have already met. The circles of Apollonius (section 23) for a variable k all have their centers on a line and are orthogonal to any circle through A and B, and so form a coaxal system. Conversely, every circle of the system that has A and B as limiting points may be interpreted as a circle of Apollonius.

62. Two circles and a circle of antisimilitude

THEOREM. *A circle of antisimilitude of two given circles is coaxal with them.*

If the circles intersect, both circles of antisimilitude pass through the common points. If the circles c_1 and c_2 do not intersect, we consider (Fig. 51) the system k_i conjugate to

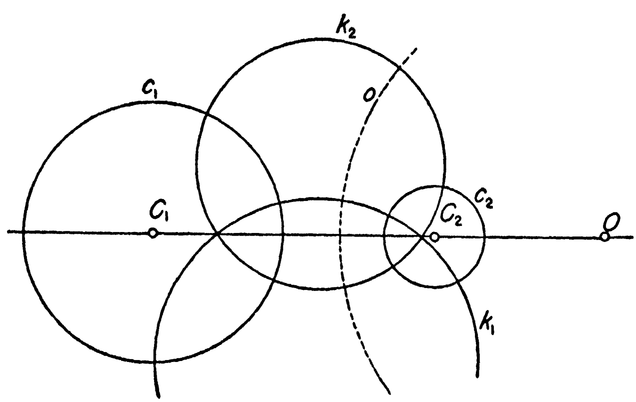


Fig. 51

the coaxal system determined by c_1 and o, where o is the circle of antisimilitude of c_1 and c_2. We invert with respect to the circle of antisimilitude. The system k_i inverts into itself, since every k_i is orthogonal to o; the circle o is invariant, and c_1 inverts into c_2. Since orthogonality is invariant, c_2, as well as c_1 and o, is orthogonal to k_i; and since the centers O, C_1, and C_2 are collinear, the circles are coaxal.

63. Casey's power theorem

Before proceeding to the next illustration, we first prove the following theorems:

THEOREM. *The difference of the powers of a point with* *regard to two nonconcentric circles is proportional to the distance from the point to their radical axis.*

Let the distance from a point P to the radical axis of the circles with centers C_1, C_2 and radii r_1 and r_2 be x. Let PN

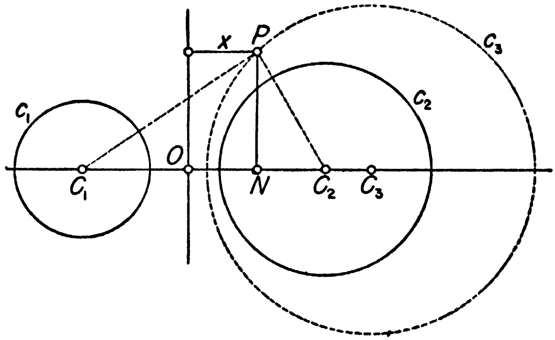


Fig. 52

be perpendicular to C_1C_2, so that $ON = x$ (Fig. 52). Then

$$\begin{aligned} p_1 - p_2 &= (PC_1)^2 - r_1^2 - (PC_2)^2 + r_2^2 \\ &= C_1N^2 + PN^2 - NC_2^2 - PN^2 - r_1^2 + r_2^2 \\ &= (C_1O + ON)^2 - (NO + OC_2)^2 - r_1^2 + r_2^2 \\ &= (C_1O^2 - r_1^2) - (OC_2^2 - r_2^2) + 2\,ON(C_1O + OC_2). \end{aligned}$$

Since O is on the radical axis, the first two parentheses are equal, so that

$$p_1 - p_2 = 2(C_1C_2)x = cx. \tag{1}$$

This result is due to Casey, and is called the Casey Power Theorem.

THEOREM. *The locus of a point whose powers with regard to two fixed circles are in a constant ratio is a circle coaxal with them.*

Suppose, first, that in Fig. 52, P moves on a circle c_3, coaxal with c_1 and c_2; we apply equation (1) to the circles c_1, c_3 and c_2, c_3. Then, since $p_3 = 0$, we have

$$p_1 = 2(C_1C_3)x, \qquad p_2 = 2(C_2C_3)x,$$

so that p_1/p_2 is independent of the variable x, or

$$\frac{p_1}{p_2} = k. \tag{2}$$

Conversely, if P is a point satisfying (2), construct the circle through P belonging to the coaxal system determined by c_1 and c_2 (section 58). Let its center be C_3. Then

$$p_1 = 2(C_1C_3)x, \qquad p_2 = 2(C_2C_3)x,$$

so that

$$\frac{p_1}{p_2} = k = \frac{C_1C_3}{C_2C_3}.$$

The point C_3 is thus completely fixed by k, independent of the position of P, and since there is only one circle of the system with its center at a fixed point, P lies on a circle coaxal with c_1 and c_2, thus completing the theorem.

If $k = p_1/p_2 = 1$, we get the radical axis. (Although the above proof does not apply, the statement is obvious.) If $k = -1$, we get the circle of the coaxal system whose center is the midpoint of C_1C_2; this circle has been called the *radical circle of the two given circles.** If $k = \pm r_1/r_2$ (see problem 10a below), we get the circle (or circles) of antisimilitude, and in the next section we show that if $k = r_1^2/r_2^2$, we get the circle of similitude.

64. Coaxal systems involving the circle of similitude

THEOREM. *The circle of similitude of two circles is coaxal with them.*

For if P is on the circle of similitude of circles with centers C_1 and C_2, then (section 8),

$$\frac{PC_1}{PC_2} = \frac{r_1}{r_2},$$

so that

* Some confusion may be occasioned by this term although it already appears in the literature. This circle must not be confused with the radical circle of three given circles (section 54). The author recommends the more suggestive term *midaxal circle* instead of radical circle here.

$$\frac{p_1}{p_2} = \frac{PC_1^2 - r_1^2}{PC_2^2 - r_2^2} = \left(\frac{r_1}{r_2}\right)^2 = k,$$

and the last theorem applies directly.

The above proof depends essentially upon the Casey Power Theorem, which was introduced chiefly for this purpose. We now give a second proof which does not depend upon that theorem.

Let the given circles (Fig. 53) have centers A and B, and

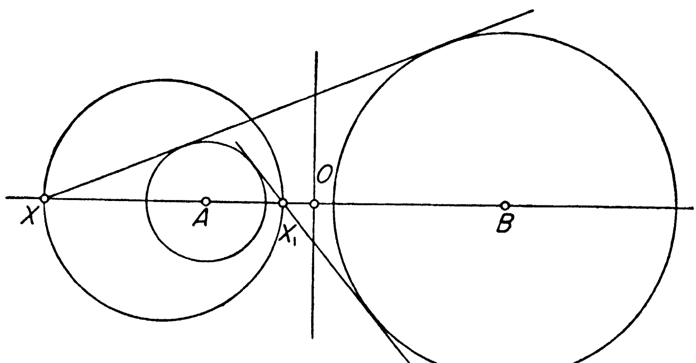


Fig. 53

radii r and s, respectively. Let X and X_1 be the external and internal homothetic centers, and let O be the intersection of the radical axis and AB. Then, since O has equal powers with respect to the circles,

$$p = a^2 - r^2 = b^2 - s^2, \tag{1}$$

where $OA = a$, $OB = b$. Let $OX = x$, $OX_1 = x_1$. Then as in section 2, we may solve for x and x_1, from the relations $AX/XB = -r/s$, $AX_1/X_1B = +r/s$, and find

$$x = \frac{as - br}{s - r}, \qquad x_1 = \frac{as + br}{s + r}. \tag{2}$$

Consequently the power of O with respect to the circle of similitude—namely, $p' = xx_1$—becomes

$$p' = \frac{a^2s^2 - b^2r^2}{s^2 - r^2}. \tag{3}$$

If we solve (1) for a^2 and b^2 and substitute in (3), we find

$$p' = \frac{(r^2 + p)s^2 - (s^2 + p)r^2}{s^2 - r^2} = p.$$

Hence the three circles have the same radical axis or are coaxal.

We next apply this result to the circles of similitude of three circles taken in pairs.

THEOREM. *The circles of similitude of three given circles taken in pairs are coaxal.*

Let us call the centers A, B, C, the radii a, b, c, and the homothetic centers X, X'; Y, Y'; Z, Z' (Fig. 54). Each of

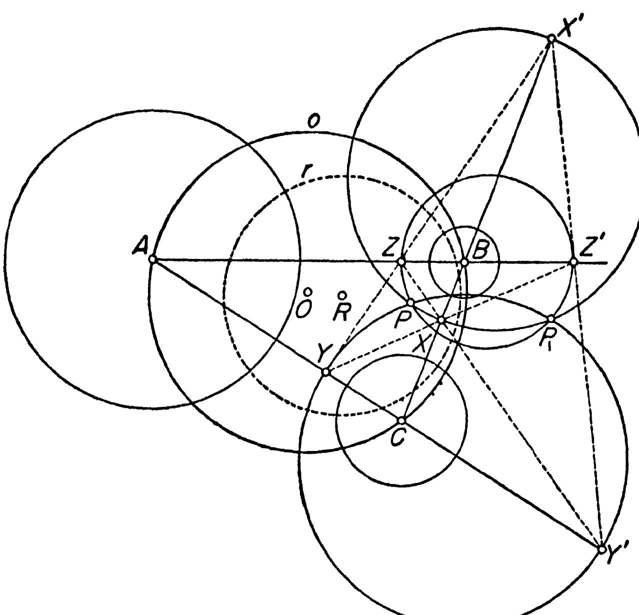


Fig. 54

the circles of similitude is orthogonal to the circumcircle o of the triangle ABC (section 61), since it is the circle of Apollonius determined by the ratio of the corresponding radii. If the radical circle of the given circles is real, it is orthogonal to all three of the given circles and hence to any circle coaxal with any pair of the three given circles. Thus the circles of similitude are also orthogonal to the radical circle r, as well as to o. Hence the three circles of similitude are coaxal. If the circles o and r do not intersect as in the diagram, the circles of similitude belong to a system of the intersecting type; if the radical circle and the circumcircle do intersect, the circles of similitude form a system of the other type.

If the radical circle is not real, so that the radical center lies inside all three circles whose centers are A, B, C, then any two of the circles of similitude intersect, as may readily be seen by constructing the diagram, and the points of intersection, because of the relations $PA : PB : PC = a : b : c$, lie on the three circles of similitude, as already shown in section 40.

65. Involution on a line

If O is a fixed point on a line, and if A and B are variable points subject to the condition

$$OA \cdot OB = k,$$

the points are said to be paired in involution; O is called the center of the involution and corresponds to the infinitely distant point on the line. We discuss three cases according as k is positive, negative, or zero.

Case k *positive.* For each point A there corresponds a unique point B; the corresponding points lie on the same side of O; and no two pairs of points mutually separate each other. There exist two points which correspond to themselves, and these points are harmonic conjugates with respect to any two pairs of the system (Fig. 55a). The involution may be constructed by means of a coaxal system of circles of either type determined by any circles through two pairs of corresponding points. The radical axis passes through the center of the involution.

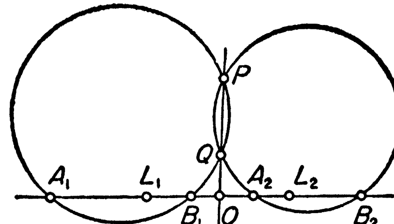


Fig. 55a

Case k *negative.* For each point A there corresponds a unique point B, and the corresponding points lie on opposite sides of O (Fig. 55b). Two pairs of points mutually separate each other, for if OA_2 is numerically less than OA_1, OB_2 is numerically greater than OB_1. There are no real self-corresponding points. The involution may be constructed by means of a coaxal system of circles of the intersecting type, determined by any circles through two pairs of corresponding points. The radical axis passes through the center of the involution.

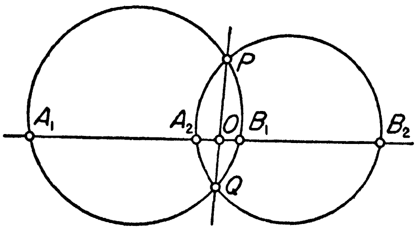


Fig. 55b

Case k *zero.* In this case the correspondence is not unique; the point corresponding to every point is O; and the involution may be represented by a coaxal system of the tangent type with the given line through the point of tangency (Fig. 55c).

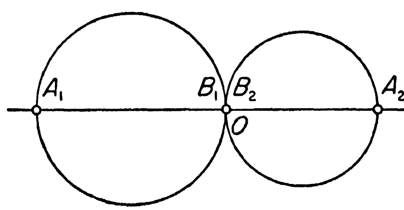


Fig. 55c

All three types of involution may be associated with one coaxal system of the intersecting type. The fundamental theorem covering this situation is as follows:

THEOREM. *A straight line cuts a coaxal system of circles in an involution. The type of involution depends upon whether the intersection of the line and the radical axis is outside, inside, or on two circles of the system. The self-corresponding points, if they exist, are the points of contact of the two circles of the system tangent to the line* (section 59).

66. Involution of lines through a point

If the involution on a line is projected to any point not on the line, we obtain an *involution of lines.* In Fig. 56 we have an involution of points on the line u, and if we project it to the point P, we obtain an involution of lines at P, with corresponding lines being a_1, b_1 and a_2, b_2. If the circles PA_1B_1 and PA_2B_2 meet again at Q, then an arbitrary circle through P and Q cuts the line u in a new pair of points in the involution on u, and hence determines a new pair of lines in the involution of lines at P. There is no line of the pencil that may be singled out from all the

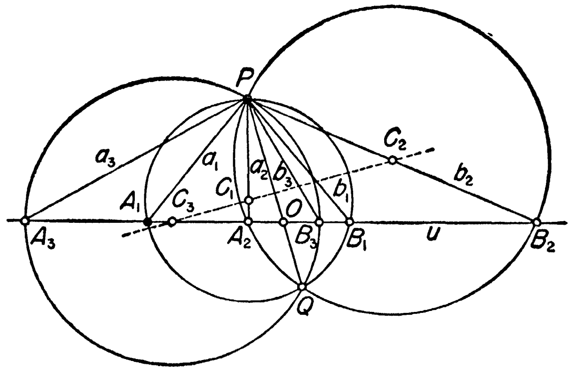


Fig. 56

others, but there is usually one pair that may be singled out.

THEOREM. *In an involution of lines on a finite point there is always one pair of corresponding lines which are at right angles. If more than one pair are at right angles, then all pairs of corresponding lines are at right angles.*

For if the line C_1C_2 cuts u in C_3 (Fig. 56), then the circle of the coaxal system with center at C_3 determines the points A_3 and B_3 and hence a_3 and b_3, which are at right angles. All circles of the system have their centers on C_1C_2, and in general C_1C_2 and u have only one point in common. If, however, C_1C_2 and u coincide, then every circle through P and Q cuts u in a pair of points that subtend a right angle at P. This is just the situation that arises if the point P is the intersection of circles drawn on A_1B_1 and A_2B_2 as diameters, for then the points C_1 and C_2 lie on u; a_1, b_1 and a_2, b_2 are at right angles, and then every pair a_i, b_i will also be at right angles.

67. Poincaré's universe

In *Science and Hypothesis*,* Henri Poincaré conceives of a universe contained within a great sphere, in which the dimensions of objects diminish as they move to increasing distances from the center in such a way that the geodesics (shortest path between

* See Henri Poincaré, *Science and Hypothesis*, translated by George B. Halsted. New York: Science Press, 1921, pages 75 ff.

two points) are arcs of circles orthogonal to the boundary. To the inhabitants, such a universe would appear infinite in extent, and the geodesics would appear as straight lines. In many respects the geometry of this universe is the same as that of our own, but in some fundamental respects it is very different. Let us consider a map of a plane section through the center of this universe. Let the center of the boundary circle be C. A geodesic will be that part of a circle orthogonal to the circle C which lies within the circle; points on the boundary are an infinite distance away. The sum of the angles of a triangle is *less* than 180° as may be visualized by taking one of the vertices at the center, as the triangle ABC in Fig. 57.

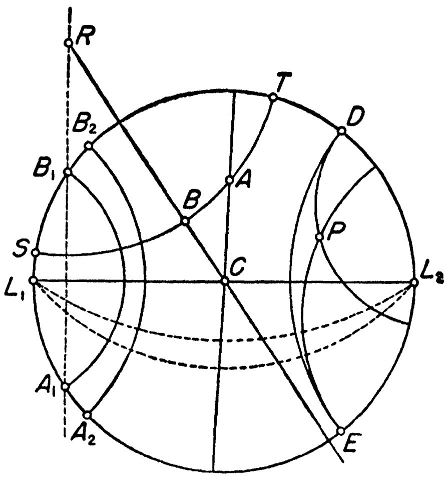


Fig. 57

Suppose that P is any point and we draw two geodesics through P intersecting the boundary in D and E and then draw the geodesic DE. Any geodesic through P within the angle DPE will cut DE; others will not cut DE; the positions PD and PE correspond to our concept of parallels.

Consider the geodesic L_1L_2 taken as a diameter for convenience, and consider the coaxal system of circles which have L_1 and L_2 as limiting points. They are geodesics of the universe, and are perpendicular to the geodesic L_1L_2. As is evident in the diagram, they *diverge* at either side of L_1L_2. The circles of the system through L_1 and L_2 cut the geodesics A_iB_i at right angles, and hence correspond to *equidistant* curves. It is apparent that the properties of parallelism and equidistance are not the same.

Geodesics which do not intersect and are not coaxal have one and only one common perpendicular; that is, the circle which is orthogonal to three circles. Here the radical center is outside the circle C and the radical circle R is real. For example, the geodesics A_1B_1 and CB have the unique common perpendicular ST.

Enough has been said to indicate the possibilities. A study of non-Euclidean geometry is necessary to appreciate their full significance. The recent advances in physics have indicated that perhaps our own universe can best be appreciated in terms analogous to those of this fancied domain.

Exercise 5—Part 2. Coaxal Systems

1. What is the inverse of a coaxal system with respect to (a) a circle of the system, (b) a circle of the conjugate system.

2. Discuss the inverse of a coaxal system of the tangent type for all positions of the center of inversion.

3. The segment between the points of contact of a common tangent to two circles of a coaxal system of the nonintersecting type subtends a right angle at a limiting point.

4. If three circles are coaxal, the common tangent of two of them is divided harmonically by the third. Prove. Also state and prove a converse.

5. Use this last exercise to prove that two circles and their circle of similitude are coaxal. What limitation does this method have?

6. (a) If three noncoaxal, nonintersecting circles are given, then the three pairs of limiting points which they determine two by two are concyclic.

(b) Use this property to show that the three circles of antisimilitude are coaxal.

7. Construct a diagram corresponding to Fig. 54, where the circumcircle and the radical circle intersect. Show the three circles of similitude of the three given circles.

8. Prove that two nonintersecting circles and their circle of antisimilitude are coaxal by inverting with respect to a point on this latter circle.

9. Two circles intersect at A. A variable line through A meets the circles at P and Q, respectively, and R is determined so that PR/RQ = a given constant. Prove that the locus of R is a circle.

10. (a) Show that the ratio of the powers with regard to two given circles of any point on a circle of antisimilitude is $\pm r_1/r_2$.

(b) Use this property to prove the coaxal properties of the circles of antisimilitude of three given circles taken in pairs. Discuss its limitations.

11. The pairs of points A_1, A_2 and B_1, B_2 separate each other. In the involution determined by them, determine the pair of points which are harmonic conjugates of A_1, A_2.

6 NOTABLE POINTS AND CIRCLES CONNECTED WITH A TRIANGLE

Part 1. Circumcenter. Centroid. Incenter. Orthocenter

68. Notation

In this chapter we shall discuss quite briefly the properties of some of the best-known points connected with a triangle and the lines and circles associated with them. We shall adopt the following notation:

A_1, A_2, A_3, so lettered that they appear in counterclockwise order, are the vertices.

a_1, a_2, a_3 are the opposite sides; a_i will be used to represent both the side and the length A_jA_k.

$s = (a_1 + a_2 + a_3)/2$, one-half the perimeter; Δ = area of the triangle.

α_1, α_2, α_3 are the angles measured in a positive sense.

O_1, O_2, O_3 represent the midpoints of the sides; O is the circumcenter, R is the radius of the circumcircle.

G, the centroid or median point, represents the intersection of the medians; K, its isogonal conjugate, is the symmedian point.

H_1, H_2, H_3 are the feet of the altitudes; h_1, h_2, h_3 are the lengths of the altitudes; and H is the orthocenter. The triangle $H_1H_2H_3$ is called the *pedal triangle* of H.

If P is any point, and P_i is the foot of the perpendicular upon a_i, then $P_1P_2P_3$ is the pedal triangle of P.

N is the nine-point center.

I, I_1, I_2, I_3 stand for the incenter and excenters.

r, r_1, r_2, r_3 are the radii of the incircles and excircles.

X_1, X_2, X_3 represent the points of contact of the incircle with the sides; and X_i', X_i'', X_i''', the points of contact of the excircles on the side a_i.

69. Circumcenter

We recall some of the elementary properties of the circumcenter. Additional results may be found in any of the standard works.

The perpendicular bisectors of the sides intersect in a point O, called the *circumcenter*; the circle with center O and radius OA_i is the *circumcircle*.

$$\measuredangle A_2OO_1 \equiv \measuredangle O_1OA_3 \equiv \alpha_1 \pmod{180°}, \tag{1}$$

with similar relations for α_2 and α_3. When $\measuredangle A_2OO_1$ is taken as the smallest positive angle from the line OA_2 to the

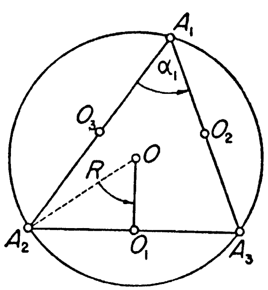


Fig. 58a

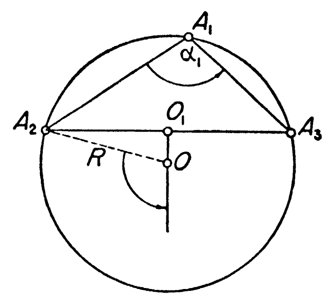


Fig. 58b

line OO_1 (Fig. 58), equality signs may be used instead of congruence signs.

$$\frac{a_1}{\sin \alpha_1} = \frac{a_2}{\sin \alpha_2} = \frac{a_3}{\sin \alpha_3} = 2R. \tag{2}$$

$$\Delta = \frac{a_2a_3 \sin \alpha_1}{2} = 2R^2 \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 = \frac{a_1a_2a_3}{4R}. \tag{3}$$

70. Centroid

The lines joining A_iO_i are called the medians. The only important elementary property for our purpose is well known.

(a) The medians of a triangle meet in a point, called the centroid G, and each median is trisected by this point. A proof using Menelaus' Theorem was given in section 12. Elementary proofs are numerous.

(b) Each median divides the given triangle into two triangles, with equal areas, and the three medians divide the given triangle into six triangles, each with the same area. The three triangles GA_1A_2, GA_2A_3, GA_3A_1 (Fig. 59) have equal directed areas, and, conversely, if G is such that these directed areas are equal, it is the centroid. Hence the perpendiculars $GM_i = g_i$ to the three sides are inversely proportional to the sides, for

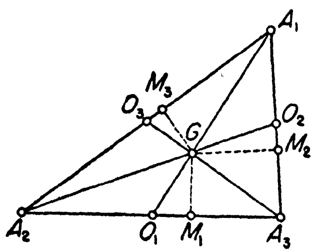


Fig. 59

$$a_1g_1 = a_2g_2 = a_3g_3 = \frac{2\Delta}{3},$$

or

$$g_1 : g_2 : g_3 = \frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3}.$$

This relation involving the trilinear coordinates of G forms the basis for an analytical study of the point G.

71. Symmedians

The isogonal conjugate line of a median is called a *symmedian.* It follows immediately that the three symmedians meet in a point called the *symmedian point,* the isogonal conjugate of the centroid. We shall develop a few of the more important properties as an illustration of the properties of isogonal conjugates.

(a) The distances from the symmedian point to the sides are proportional to the sides.

From the last relation of section 70 and the distance property for isogonal conjugates (section 51), we have

$$k_1 : k_2 : k_3 = a_1 : a_2 : a_3, \tag{1}$$

where k_i is the distance from the symmedian point K to the side a_i. Conversely, a point within the triangle satisfying (1) is the symmedian point.

It may be remarked that by methods foreign to our point of view it may be shown that the point within the triangle such that the sum of the squares of its distances from the sides is a minimum is the symmedian point. The methods of differential calculus for a minimum applied to the conditions

$$S = x^2 + y^2 + z^2, \qquad a_1x + a_2y + a_3z = 2\Delta,$$

lead to $x:y:z = a_1:a_2:a_3$. Or the same result may be derived by considering the algebraic identity

$$(a_1^2 + a_2^2 + a_3^2)(x^2 + y^2 + z^2) \equiv (a_1x + a_2y + a_3z)^2 \\ + (a_1y - a_2x)^2 + (a_2z - a_3y)^2 + (a_3x - a_1z)^2,$$

and by showing that the last three quantities are all zero for S to be a minimum.

(b) If P is any point on the symmedian A_1L_1, then its distances from the sides are proportional to the lengths of the corresponding sides.

For

$$\frac{d_3}{d_2} = \frac{k_3}{k_2} = \frac{a_3}{a_2}.$$

If P is at L_1 (Fig. 60), we have in addition from the right

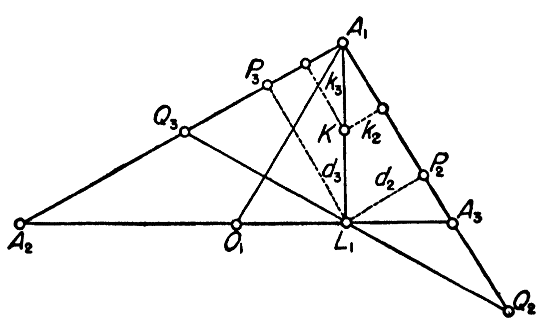


Fig. 60

triangles $L_1A_2P_3$ and $L_1P_2A_3$,

$$A_2L_1 = \frac{d_3}{\sin \alpha_2}, \qquad L_1A_3 = \frac{d_2}{\sin \alpha_3},$$

so that

$$\frac{A_2L_1}{L_1A_3} = \frac{d_3}{d_2} \cdot \frac{\sin \alpha_3}{\sin \alpha_2} = \frac{a_3^2}{a_2^2}. \tag{3}$$

The symmedian through a vertex divides the opposite side in segments which are proportional to the squares of the adjacent sides.

A median of a triangle bisects all segments parallel to one side. The corresponding statement for the symmedian is:

(c) A symmedian of a triangle bisects all segments antiparallel to one side.

It is apparent that if we prove this result for one line antiparallel (section 6) to a side, it will be true for all such lines. We select the antiparallel through L_1 as the special case. Let this line cut the sides in Q_2 and Q_3, respectively. Then from the similar triangles $L_1A_2Q_3$ and $L_1Q_2A_3$ and the sine law, we have

$$\frac{A_2L_1}{Q_3L_1} = \frac{L_1Q_2}{L_1A_3} = \frac{\sin \alpha_3}{\sin \alpha_2} = \frac{a_3}{a_2},$$

so that

$$A_2L_1 = \frac{a_3}{a_2}\, Q_3L_1, \qquad L_1A_3 = \frac{a_2}{a_3}\, L_1Q_2.$$

These relations when combined with (3) yield

$$Q_3L_1 = L_1Q_2.$$

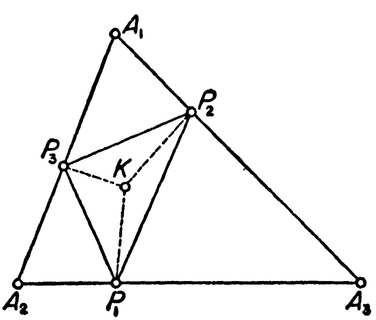


Fig. 61

Conversely, the midpoint of any line antiparallel to a side lies on a symmedian, as may be seen by retracing the steps or using the method of *reductio ad absurdum*.

(d) The symmedian point is the centroid of its pedal triangle.

Let the pedal triangle of

K be $P_1P_2P_3$ and consider the areas of the triangles KP_2P_3, KP_3P_1, KP_1P_2,

$$\Delta_1 = \frac{k_2k_3}{2}\sin\alpha_1,$$

$$\Delta_2 = \frac{k_3k_1}{2}\sin\alpha_2,$$

$$\Delta_3 = \frac{k_1k_2}{2}\sin\alpha_3.$$

These are all equal to $Ca_1a_2a_3$, where C is a constant, in light of (1) above and (2) of section 69. Hence K is the centroid of triangle $P_1P_2P_3$.

72. Exmedians and exsymmedians

The line through a vertex parallel to the opposite side is called an *exmedian*; its isogonal conjugate an *exsymmedian*. The concurrency properties of medians and exmedians or of symmedians and exsymmedians and their related harmonic properties, with obvious definitions of the exmedian points (G_1, G_2, G_3) and the exsymmedian points (K_1, K_2, K_3), are left for the reader to determine. The most important property is the following:

THEOREM. *The exsymmedian at a vertex is tangent to the circumcircle.*

As shown in Fig. 62

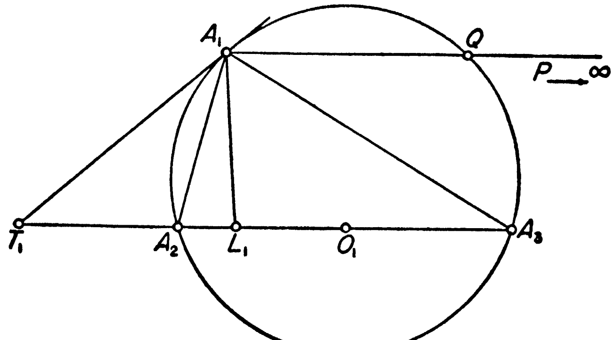


Fig. 62

$$\measuredangle T_1A_1A_2 = \measuredangle A_3A_1Q,$$

and these angles are measured by half the equal arcs A_1A_2 and A_3Q, making A_1T_1 the tangent at A_1. Or we may point out that the line at infinity is transformed into the circumcircle and that A_1P_∞ must be transformed into the tangent at A_1, since P_∞ is on both the side and the line at infinity (section 51).

73. Incenter and excenters

By elementary means or as an application of the trigo-

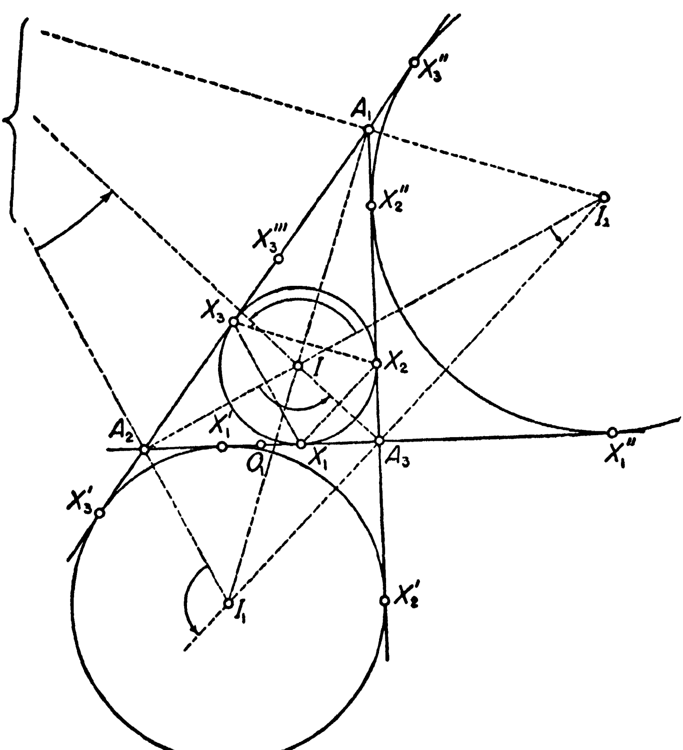


Fig. 63

nometric form of Ceva's Theorem, it is easy to prove the following known results.

(a) The interior bisectors of the angles of a triangle are concurrent at I; the bisector of an interior angle and those of the exterior angles at the other two vertices are likewise concurrent. The point I is the orthocenter of the triangle $I_1I_2I_3$.

(b) The following angle relations are established without difficulty from the figure (Fig. 63):

$$\measuredangle A_2IA_3 = 90° + \frac{\alpha_1}{2} = \measuredangle I_2II_3. \tag{1}$$

$$\measuredangle A_2I_2A_3 = \measuredangle A_2I_3A_3 = \frac{\alpha_1}{2}. \tag{2}$$

$$\measuredangle A_2I_1A_3 = 180° - \frac{(\alpha_2 + \alpha_3)}{2} = 180° - I_2I_1I_3 = 90° + \frac{\alpha_1}{2}, \tag{3}$$

so that the angles within the triangle $I_1I_2I_3$ are $90° - \alpha_i/2$, or $(\alpha_k + \alpha_j)/2$, respectively. Since X_1X_3 is perpendicular to A_2I, and so on, and from (1),

$$\measuredangle X_2X_1X_3 = 90° - \frac{\alpha_1}{2}. \tag{4}$$

Indeed the triangles $X_1X_2X_3$ and $I_1I_2I_3$ are not only similar but also homothetic (The homothetic center is not shown in the diagram).

(c) The area of the triangle is readily found to be

$$\Delta = rs = r_1(s - a_1) = r_2(s - a_2) = r_3(s - a_3),$$

from which we easily find

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

The number of similar formulas is very large indeed.*

74. Distance between contact points

Each of the four circles determines on any side a point of contact. The lengths of the segments so determined may be

* The reader who is interested should consult the following papers by J. S. Mackay, in the *Proceedings of the Edinburgh Mathematical Society*: volume 12, pages 86–105, and volume 13, pages 103–104; and volume 13, pages 37–102.

obtained algebraically, by virtue of the theorem that tangents to any circle from an external point are equal. If we neglect the algebraic signs, which depend upon the relative sizes of the angles, we have the following relations:

$$\begin{aligned}
s &= A_1X_2' = A_1X_3' = A_2X_1'' = A_2X_3'' = A_3X_1''' = A_3X_2''',\\
s - a_1 &= A_1X_2 = A_1X_3 = A_2X_1''' = A_2X_3''' = A_3X_1'' = A_3X_2'',\\
s - a_2 &= A_1X_2''' = A_1X_3''' = A_2X_1 = A_2X_3 = A_3X_1' = A_3X_2',\\
s - a_3 &= A_1X_2'' = A_1X_3'' = A_2X_1' = A_2X_3' = A_3X_1 = A_3X_2.
\end{aligned} \tag{1}$$

By combining these relations, taking direction into account, but again neglecting the algebraic signs, we find additional relations. For example

$$\begin{aligned}
X_1X_1' &= a_3 - a_2,\\
X_1X_1'' &= a_2,\\
X_1X_1''' &= a_3,\\
X_1''X_1''' &= a_2 + a_3,
\end{aligned} \tag{2}$$

and so forth.

If we introduce the point O_1 as origin, then

$$\begin{aligned}
O_1X_1 &= \frac{a_3 - a_2}{2} = O_1X_1',\\
O_1X_1'' &= \frac{a_3 + a_2}{2} = O_1X_1'''.
\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}
O_2X_2 &= O_2X_2'' = \frac{a_3 - a_1}{2},\\
O_2X_2' &= O_2X_2''' = \frac{a_3 + a_1}{2}.
\end{aligned} \tag{4}$$

$$\begin{aligned}
O_3X_3 &= O_3X_3''' = \frac{a_2 - a_1}{2},\\
O_3X_3' &= O_3X_3'' = \frac{a_2 + a_1}{2}.
\end{aligned} \tag{5}$$

75. Geometric corollaries

The following geometric corollaries are derived as an immediate application of Ceva's Theorem.

COROLLARY 1. *The lines* A_1X_1, A_2X_2, A_3X_3 *are concurrent (Gergonne point).*

COROLLARY 2. *The lines* A_1X_1', A_2X_2'', A_3X_3''' *are concurrent (Nagel point).*

That the Gergonne and Nagel points are isotomic conjugates follows from the equations (3), (4), and (5) above. (For the definition of isotomic conjugates see problem 6, exercise 4—part 2.)

COROLLARY 3. *More generally, the lines connecting the vertices with the points of contact of the inscribed and escribed circles are concurrent three by three at eight points, which form four pairs of isotomic conjugates.*

76. Circumcircle and excircles

Let the internal and external bisectors of α_1 cut the circumcircle

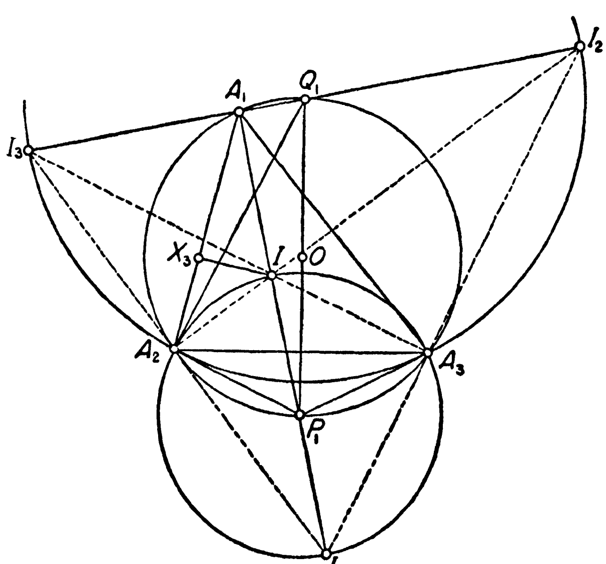


Fig. 64

in P_1 and Q_1. Then P_1 and Q_1 bisect the major and minor arcs A_2A_3, and P_1Q_1 is a diameter of the circumcircle (Fig. 64).

THEOREM. *The circles on* II_1 *and* I_2I_3 *as diameters pass through* A_2 *and* A_3, *have their centers at* P_1 *and* Q_1, *respectively, and are orthogonal.*

The first part of the theorem follows from the fact that the bisectors are perpendicular; the second part, from the facts that the midpoints of the arcs A_2A_3, P_1 and Q_1, lie on II_1 and I_2I_3, respectively, and on the perpendicular bisector of A_2A_3. The third part follows from the relation

$$P_1Q_1^2 = P_1A_2^2 + Q_1A_2^2$$
$$= \rho_1^2 + \rho_2^2,$$

where ρ_1 and ρ_2 are the radii of the circles and P_1Q_1 is the distance between their centers. Many algebraic relations follow from this theorem and the corresponding diagram.

$$II_1 = 2\rho_1 = \frac{a_1}{\cos(\alpha_1/2)} = 4R\sin(\alpha_1/2). \tag{1}$$

$$I_2I_3 = 2\rho_2 = \frac{a_1}{\sin(\alpha_1/2)} = 4R\cos(\alpha_1/2). \tag{2}$$

Now $r = A_2I\sin(\alpha_2/2)$, where $A_2I = II_1\sin IA_3A_2 = II_1\sin(\alpha_3/2)$, so that

$$r = 4R\sin(\alpha_1/2)\sin(\alpha_2/2)\sin(\alpha_3/2), \tag{3}$$

which relates the circumradius and inradius. Another relation between them is given in the next section.

77. In- and circumscribed triangle

From Fig. 64, we see that triangles $Q_1P_1A_2$ and A_1IX_3 are similar,

$$A_2P_1 = IP_1, \quad Q_1P_1 = 2R, \quad IX_3 = r,$$

so that from

$$\frac{Q_1P_1}{IA_1} = \frac{A_2P_1}{IX_3},$$

we find

$$-2Rr = IA_1 \cdot IP_1,$$

where the right member is the power of I with regard to the circumcircle. Hence

$$2Rr = R^2 - d^2, \text{ or } \frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}, \tag{1}$$

where d is the distance OI.

Similarly we can prove

$$2Rr_i = d_i^2 - R^2, \text{ or } \frac{1}{d_i - R} - \frac{1}{d_i + R} = \frac{1}{r_i}, \tag{2}$$

where $d_i = OI_i$, and now $d_i > R$.

An interesting geometric proof of (1) which has the advantage that it enables us to demonstrate readily the converse will now be given (Fig. 65). Let us invert the triangle and

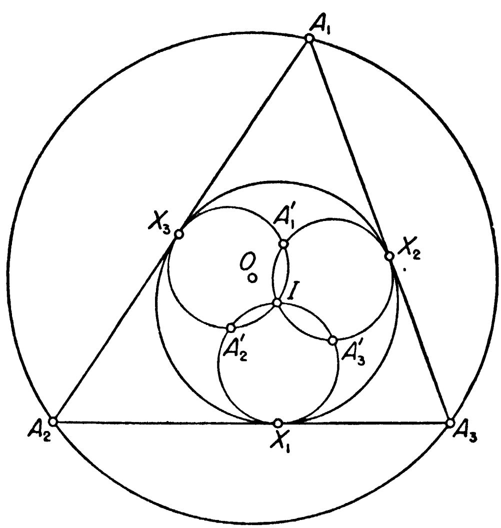


Fig. 65

the circumcircle with regard to the incircle. The sides invert into three equal circles having IX_i as diameters. The circumcircle inverts into the circle through their three points of intersection other than I, namely A_i'. It is easily shown that the following lemma is true:

LEMMA. *If three equal circles pass through a point, the circle through their other three intersections is equal to them.*

In Fig. 66, let the three equal circles have I in common and meet again at A_1', A_2', A_3'. Define C as the intersection of the lines $A_2'C$ parallel to C_1A_3', and $A_3'C$ parallel to C_1A_2', making $A_2'C_1A_3'C$ a rhombus of side, say c. Then since IC_2

is parallel and equal to both C_1A_3' and C_3A_1', $A_1'C_3A_2'C$ is a rhombus of side c, so that C is the center of a circle through A_1', A_2', A_3' with radius c, proving the lemma.

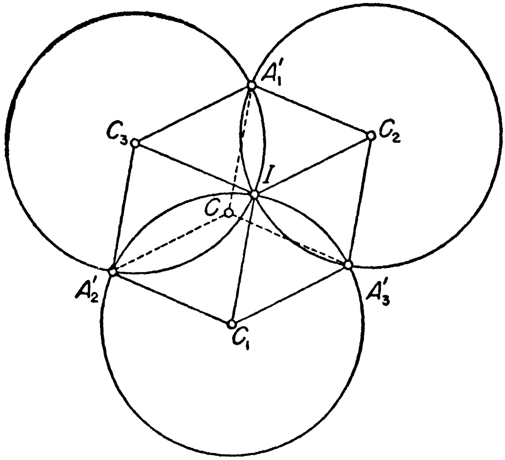


Fig. 66

We now apply the formula for the radius of a circle after inversion with respect to a circle of radius r, and find

$$\frac{r}{2} = R\frac{r^2}{p} = R\,\frac{r^2}{R^2 - d^2},$$

or

$$2Rr = R^2 - d^2.$$

Conversely, if two circles with center O and radius R, and center I and radius r, are so placed that

$$R^2 - OI^2 = 2Rr,$$

then a triangle can be drawn which is inscribed to the first circle and circumscribed to the second; and this may be done in an infinite number of ways.

We effect an inversion with respect to the second circle, and suppose the first circle c is inverted into a circle c' with radius R' (Fig. 67). Then

$$R' = \frac{r^2}{R^2 - OI^2}\,R,$$

and hence, comparing this with the hypothesis,

$$R' = \frac{r}{2}.$$

Let A_1' be any point on the circle c', and let the two circles of diameter r, passing through I and A_1', cut the circle c'

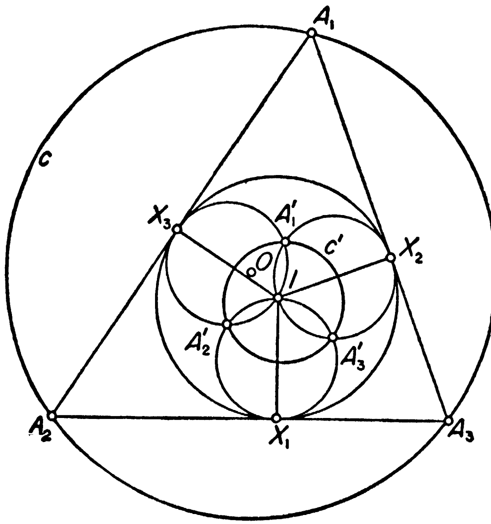


Fig. 67

again at A_2' and A_3'. Then, by the lemma used above, the circle $A_2'A_3'I$ will also have the diameter r, and hence the three circles through I will be tangent to the circle whose center is I and whose radius is r at points which we call X_1, X_2, X_3. Let us invert back again. The circle c' inverts into the circle c with center O and radius R, and the other three circles of radius $r/2$ invert into straight lines with A_1, A_2, A_3 on c, and hence the theorem is established. Since A_1' was arbitrary, we can construct the triangle in an infinite number of ways.

78. Orthocenter

The altitudes of a triangle meet in a point called the orthocenter. Simple proofs can be obtained by drawing lines through the vertices parallel to the opposite sides, or using the trigonometric form of Ceva's Theorem. As an immediate consequence of this fact and simple angle relations, it follows that the following six sets of points are concyclic: $(H,\ H_i,\ H_j,\ A_k)$ and $(H_i,\ H_j,\ A_i,\ A_j)$. For example (A_2, H, H_1, H_3) and (A_1, A_3, H_1, H_3) are shown concyclic in Fig. 68a.

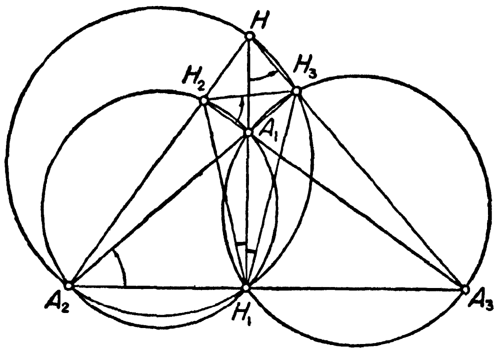


Fig. 68a

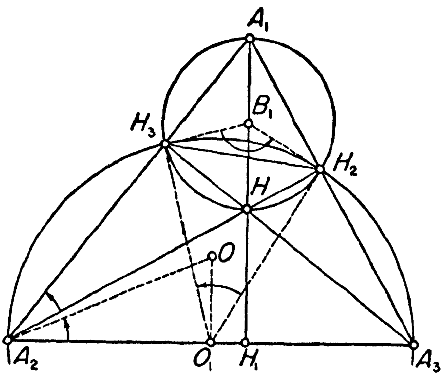


Fig. 68b

79. Angle properties

Many angle relations can be read from the figure. Among them are:

$$\measuredangle H_2A_2A_1 \equiv 90^\circ - \alpha_1 \equiv \measuredangle H_1A_2O \text{ (see Fig. 68b).}$$

(a) Consequently H and O are isogonal conjugates.

$$\measuredangle A_1H_2H_3 \equiv \measuredangle A_1HH_3 \equiv \alpha_2 \text{ (see Fig. 68a).}$$

(b) Consequently H_2H_3 and A_2A_3 are antiparallel, and the triangles $A_1A_2A_3$ and $A_1H_2H_3$ are inversely similar.

$$\measuredangle H_2H_1A_1 \equiv \measuredangle A_1H_1H_3 \text{ (see Fig. 68a).}$$

(c) Consequently the altitudes and sides of the given triangle bisect the interior and exterior angles of the pedal triangle of H, and the pencil at H_1 is harmonic.

From the triangles A_1HH_3 and $A_1A_3H_3$ we find

$$A_1H = \frac{A_1H_3}{\sin \alpha_2} = \frac{a_2 \cos \alpha_1}{\sin \alpha_2} = 2R \cos \alpha_1 = 2\,OO_1. \tag{1}$$

Similarly from triangles A_2HH_1 and $A_2A_1H_1$ we find

$$HH_1 = \frac{A_2H_1}{\tan \alpha_3} = \frac{a_3 \cos \alpha_2 \cos \alpha_3}{\sin \alpha_3} = 2R \cos \alpha_2 \cos \alpha_3. \tag{2}$$

So that, disregarding directions,

$$HA_1 \cdot HH_1 = 4R^2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3. \tag{3}$$

Consequently

$$HA_1 \cdot HH_1 = HA_2 \cdot HH_2 = HA_3 \cdot HH_3. \tag{4}$$

(d) That is, the products of the segments of the respective altitudes are equal; H has equal powers with regard to the circles on the sides as diameter and is the radical center of these three circles. This statement alone, of course, is easily proved independent of this equation.

(e) The circles on A_2A_3 and A_1H as diameters are orthogonal to each other at H_2 and H_3.

Both circles pass through H_2 and H_3, and the angles at the centers of these circles, say B_1 and O_1 (Fig. 68b), are congruent to $2\alpha_1$ and $180° - 2\alpha_1$, respectively. For as a central angle in one circle we have

$$\measuredangle H_3B_1H_2 \equiv 2\alpha_1;$$

and from the isosceles triangles $H_3O_1A_2$ and $A_3O_1H_2$, we find

$$\measuredangle H_3O_1A_2 \equiv 180° - 2\alpha_2, \quad \text{and} \quad \measuredangle A_3O_1H_2 \equiv 180° - 2\alpha_3,$$

leaving

$$\measuredangle H_2O_1H_3 \equiv 180° - 2\alpha_1.$$

Consequently in the quadrilateral $O_1H_2B_1H_3$, the angles at H_2 and H_3 are each equal to 90°.

80. Altitude and circumcircle

THEOREM. *If the altitude* A_1H_1 *meets the circumcircle at* H_1', *then* $HH_1 = H_1H_1'$.

For at once triangles H_1HA_2 and $H_1H_1'A_2$ are proved congruent (Fig. 69).

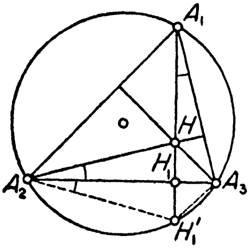


Fig. 69

It is apparent that not only is H the orthocenter of triangle $A_1A_2A_3$, but any one of the four points is the orthocenter of the triangle formed from the other three. Four such points form an *orthocentric* system. The last theorem leads to the following one:

THEOREM. *The four circumcircles of an orthocentric system are equal.*

For the triangle A_2HA_3 is congruent to the triangle $A_2H_1'A_3$, whose circumcircle is the same as that of the triangle $A_1A_2A_3$. The circumcircle of A_2HA_3 is thus the reflection on A_2A_3 of the circumcircle of $A_2A_1A_3$. Similarly for the other triangles.

81. Gauss-Bodenmiller theorem

Equation (4) in section 79 leads to even more general results than stated there, for H has the same power with respect to (that is, is the radical center for) all circles which pass through the ends of an altitude. In other words, if B_1, B_2, B_3 are any points whatever on the respective sides of a triangle, the circles on A_iB_i as diameters have H as their radical center. The most important case (Fig. 70) occurs when B_1, B_2, B_3 are collinear, so that the configuration is a complete quadrilateral, and the circles in question are those on the diagonals A_1B_1, A_2B_2, A_3B_3 as diameters (Only one circle is indicated in the diagram). Now if H is the orthocenter of triangle $A_1A_2A_3$, then it is a radical center of these three circles. Now consider the triangle $A_1B_2B_3$ and the transversal $B_1A_2A_3$; let its orthocenter be H'. Then H' is also a radical center of the same three circles, and similarly for points H'' and H'''. Now these orthocenters are not the same, and hence the three circles are coaxal, having HH' as their radical axis. Likewise the midpoints of these diagon-

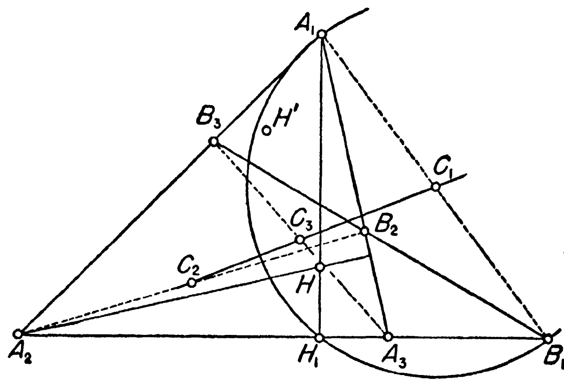


Fig. 70

als, the centers of the circles, are collinear. We state these results formally as follows:

THEOREM OF GAUSS-BODENMILLER. *The circles on the diagonals of a complete quadrilateral as diameters are coaxal.*

THEOREM. *The orthocenters of the four triangles of a complete quadrilateral are collinear on the radical axis of the aforementioned circles.*

Exercise 6—Part 1. Notable Points Connected With a Triangle

1. If A, B, C are three collinear points and D is a point not on their line, then the circumcenters of the triangle BCD, CAD, ABD are concyclic with D.

2. Given three line segments as the medians, construct the triangle.

3. Prove that the sum of the medians is greater than 3/4 of the perimeter and less than the perimeter.

4. (a) State and prove the concurrency properties of medians and exmedians.

(b) State and prove the concurrency properties of symmedians and exsymmedians.

5. (a) Prove that a median, exmedian, and pair of sides form a harmonic pencil.

(b) State and prove the corresponding result for a symmedian and exsymmedian.

6. (a) The line joining the midpoint of a side to the midpoint of the altitude upon that side passes through the symmedian point.

(b) Hence locate the symmedian point without first finding the symmedians.

7. Verify some of the relations between the points of contact given in section 74.

8. The four feet of the perpendiculars from one vertex to the bisectors of the angles at the other vertices of a triangle are four collinear points.

9. If we invert with regard to the incircle of a triangle $A_1A_2A_3$, prove that the inverse of A_1 is the midpoint of X_2X_3, and hence prove that the circle through A_1', A_2', A_3' has the radius $r/2$.

10. Derive equations (2) of section 77 for $i = 1$, either by the method of section 76 or by inversion as in section 77.

11. Prove the concurrency of the altitudes by each of the two methods suggested in section 78.

12. Prove that $h_1 = A_1H_1 = a_2a_3/(2R) = a_2 \sin \alpha_3 = a_3 \sin \alpha_2$. Hence prove

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r},$$

$$\frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{h_3} = \frac{1}{r_3}.$$

*13. Of all triangles inscribed in a given acute triangle, the pedal triangle of H has the minimum perimeter.

Part 2. Nine-Point Circle. Polar Circle. Feuerbach's Theorem

82. Nine-point circle

THEOREM. *The nine points, the feet of the altitudes, the midpoints of the sides, and the midpoints of the segments con-*

necting the vertices to the orthocenter lie on a circle which is homothetic to the circumcircle with the orthocenter and centroid as homothetic centers.

Let the points be O_i, H_i and C_i. The line O_3C_1 is parallel to A_2H, and the line O_3O_1 is parallel to A_1A_3 (Fig. 71a). Hence $\measuredangle O_1O_3C_1=90°$. Similarly $\measuredangle C_1O_2O_1 = 90°$. The circle on O_1C_1 as diameter passes through O_2, O_3 and also H_1. Otherwise stated and generalized, H_i, C_i lie on the circle through O_1, O_2, O_3, so that the nine points in question are concyclic.

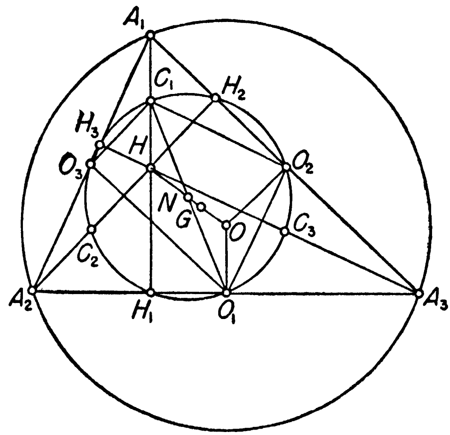


Fig. 71a

Because of the importance of the homothetic properties, we give two independent proofs.

FIRST PROOF. Since $HC_i = HA_i/2$, the circle $C_1C_2C_3$ (nine-point circle) is homothetic to the circle $A_1A_2A_3$ (circumcircle) from H with the ratio $1/2$. Since the triangle $O_1O_2O_3$ is homothetic to the triangle $A_1A_2A_3$ from G with ratio $-1/2$, the circumcircles of these triangles are homothetic, with G as the homothetic center.

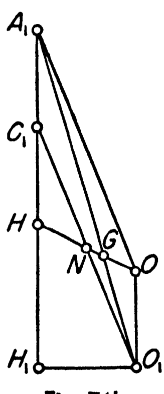


Fig. 71b

SECOND PROOF. This proof puts into evidence the center of the nine-point circle. Let it be N. Since N lies on the perpendicular bisectors of H_iO_i, it must lie at the midpoint of HO, or $HN/HO = 1/2$. Further $NC_1 = OA_1/2 = R/2$, so that the nine-point circle is homothetic to the circumcircle with homothetic center H and ratio $1/2$. Let A_1O_1 cut HO in G. Then from the similar triangles C_1HN and O_1ON (Fig. 71b), we find $O_1O = HC_1$. From the similar triangles A_1HG and O_1OG, we find $A_1G = 2GO_1$, which shows that G is the centroid. Also

$$GO = \frac{HO}{3} = \frac{2NO}{3} = \frac{2(GO - GN)}{3},$$

so that $GN/GO = -1/2$, showing that the circles are inversely homothetic with respect to the centroid.

Euler Line. Incidentally we have proved the following property of the so-called Euler line:

COROLLARY. *The points* O, G, N, H *are collinear and form a harmonic range.*

Let us consider as an illustrative example what happens if we fix the circumcircle c, and then fix A_2 and A_3, but let A_1 vary. Then O and O_1 remain fixed, while the points G, H, N vary (Fig. 71b). Since $O_1G/O_1A_1 = 1/3$, the locus of G must be a circle g homothetic to c, with homothetic center O_1 and ratio 1/3. The center of g is at a trisection point of O_1O, say G_0, and its radius is $R/3$. Next, since O is fixed and it is easily seen that $OH/OG = 3$, the locus of H is a circle homothetic to g, with homothetic center O and ratio 3. Hence the radius of this circle is R, and its center lies at H_0, such that $OH_0 = 3OG_0 = 2OO_1$.

We leave as an exercise for the student the determination of the locus of N in a similar manner and by other methods (problem 1, exercise 6—part 2).

Nine-Point Circle of an Orthocentric System. We remark that the four triangles of an orthocentric system have the same nine-point circle. The midpoints of the sides of HA_2A_3, for example, are O_1, C_2, C_3.

83. Polar circle

The polar circle of a triangle is the circle whose center is H and whose radius is given by (section 79),

$$\rho^2 = HA_1 \cdot HH_1 = -4R^2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3,$$

the negative sign being introduced to take care of directions. If all angles of the triangle are acute, H lies within the triangle, the indicated power of H is negative, and the polar circle is imaginary. If the triangle has an obtuse angle, the circle is real (Fig. 72). In an orthocentric system, three of the four triangles are obtuse, and the corresponding polar circles are real; the fourth triangle is acute and the corresponding polar circle has a real center and a radius which is the square root of a negative number. We have

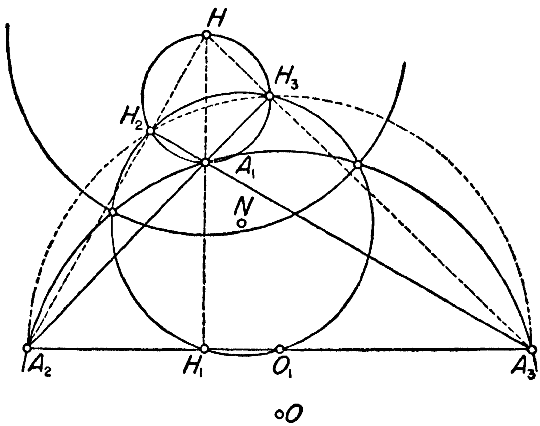


Fig. 72

already pointed out (section 24) that an inversion with regard to this imaginary circle can be given a real geometric interpretation. The next theorems which we state have a real existence when the triangle has an obtuse angle and need this additional interpretation if all angles are acute.

84. Theorems involving the polar circle

(a) Each vertex and the foot of the corresponding altitude (A_i, H_i) are inverse with regard to the polar circle (Fig. 72).

THEOREM. *The circumcircle and the nine-point circle are mutually inverse with regard to the polar circle, and the three circles are coaxal.*

(b) Each side is the polar of the opposite vertex.

THEOREM. *The triangle* $A_1A_2A_3$ *is self-polar with regard to the polar circle.*

(c) The circle on A_2A_3 as diameter is invariant under an inversion with respect to the polar circle, and hence is orthogonal to it. The inverse of the side A_2A_3 is the circle on HA_1 as diameter. Consequently, we have another proof of section 79e.

THEOREM. *The circles on* A_iA_j *and* HA_k *as diameters are orthogonal.*

85. Polar circles of a complete quadrilateral

More generally, any circle passing through A_i and H_i is invariant under inversion with regard to the polar circle and hence orthogonal to it. That is, if B_i is any point on a_i, the polar circle is orthogonal to the circle having A_iB_i as diameter. In the important special case when B_1, B_2, B_3 are collinear (see section 81), we are led to the following theorem:

THEOREM. *The polar circles of the four triangles of a complete quadrilateral belong to that coaxal system of circles which is conjugate to that of the circle on the diagonals as diameters.*

(We note that at least two and possibly four of the triangles may be obtuse, the corresponding polar circles being real.)

86. Polar circles of an orthocentric system

As previously remarked, an orthocentric system determines three real and one imaginary polar circles. In this connection we have the following result:

THEOREM. *Any two polar circles of an orthocentric system are orthogonal, and their common chord (radical axis) is the altitude from the third vertex.*

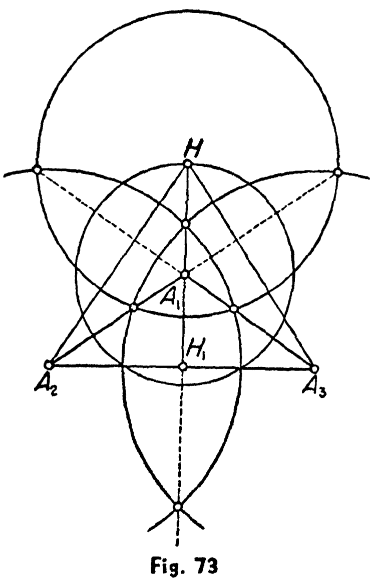


Fig. 73

Suppose that α_1 is obtuse, and let ρ_2 and ρ_3 be the radii of the polar circles with centers A_2, A_3, respectively. Then

$$\rho_2^2 = A_2H_1 \cdot A_2A_3,$$

$$\rho_3^2 = H_1A_3 \cdot A_2A_3,$$

so that by addition

$$\rho_2^2 + \rho_3^2 = A_2A_3^2,$$

the condition for orthogonality. Thus (see Fig. 73), the polar circle with center at H is orthogonal to the polar circles with centers at A_2 and A_3; and since HH_1 is perpendicular to A_2A_3, it is the radical axis of these two circles. Further, A_1 is the radical center of these three orthogonal circles and the center of the imaginary polar circle which may be considered orthogonal to all three; or the circle with center A_1, radius $= (-A_1H \cdot A_1H_1)^{1/2}$, will be cut by each of the three real polar circles in the ends of a diameter (section 54 or section 30).

87. Polar axis

The corresponding sides of the triangles $H_1H_2H_3$ and $A_1A_2A_3$ meet in three collinear points, say P_1, P_2, P_3, this line being the trilinear polar of H (section 48). It is called the *polar axis.* We have the following group of theorems:

(a) *The polar axis is the radical axis of the circumcircle and the nine-point circle.*

Let H_1H_2 meet A_1A_2 in P_3 (Fig. 74); since H_1, H_2, A_1, A_2 are

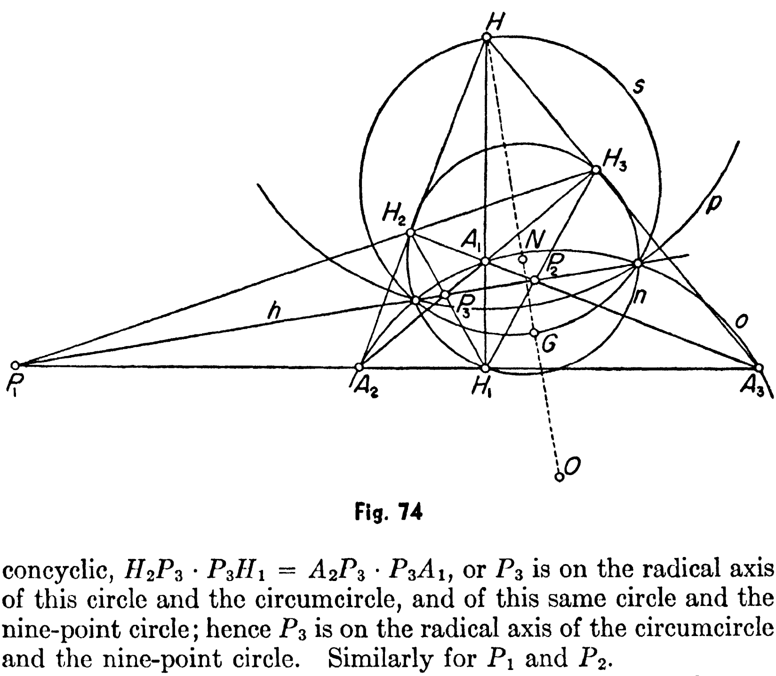


Fig. 74

concyclic, $H_2P_3 \cdot P_3H_1 = A_2P_3 \cdot P_3A_1$, or P_3 is on the radical axis of this circle and the circumcircle, and of this same circle and the nine-point circle; hence P_3 is on the radical axis of the circumcircle and the nine-point circle. Similarly for P_1 and P_2.

(b) *The polar axis is perpendicular to the Euler line* ON.

(c) *The coaxal system determined by the circumcircle and the nine-point circle is of the intersecting or nonintersecting type according as the triangle is obtuse or acute.*

(d) *The circle on* HG *as diameter is a member of this system,* since it is the circle of similitude of the two given circles (section 64).

(e) *The polar circle, if it exists, belongs to this system,* being their circle of antisimilitude (section 84a and section 62).

88. Feuerbach's theorem

We conclude this chapter with two proofs of Feuerbach's justly famous theorem.

FEUERBACH'S THEOREM. *The nine-point circle is tangent to the inscribed and to each of the escribed circles.*

It was first stated and proved by Feuerbach in 1822. A few years later it was discovered independently and stated by Steiner, and in the last hundred years, it has been rediscovered many times and numerous new proofs have been developed. None of them is essentially simple; those that we present seem quite direct and, like most proofs, involve directly or indirectly the subject of inversion.

The following proof is due to Fontene (1907) and has the advantage over many of the others of showing where the points of contact are.

Let I_2, I_3 be the centers of two of the escribed circles, and t_4 the fourth common tangent to these circles (Fig. 75). We

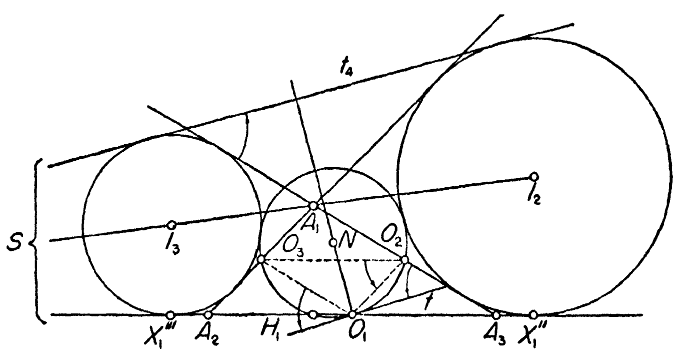


Fig. 75

will prove the desired result by showing that t_4 is the inverse of the nine-point circle. The vertex A_1 is one homothetic center; let S be the other one. Then

$$(SA_1,\ I_3I_2) = -1.$$

Projecting by perpendiculars upon A_2A_3, we get

$$(SH_1, X_1'''X_1'') = -1.$$

The tangent t at O_1 to the nine-point circle makes with O_1O_3, and hence with A_1A_3, an angle equal to α_2, and so is parallel to t_4. For, from equal arcs,

$$\measuredangle O_3O_1, t \equiv \measuredangle O_3O_2O_1 \equiv \alpha_2,$$

and since I_2I_3 is the axis of symmetry for the entire configuration,

$$\measuredangle A_1A_3, t_4 \equiv \alpha_2.$$

Hence t_4 is perpendicular to the line O_1N. Now O_1 is the midpoint of $X_1'''X_1''$ (section 74), for

$$X_1'''O_1 = O_1X_1'' = \frac{a_2 + a_3}{2}.$$

Let us invert with respect to O_1 as center and radius $(a_2 + a_3)/2$. The inverse of the nine-point circle is a line perpendicular to O_1N. Since the circle contains H_1, its inverse must pass through S, the harmonic conjugate of H_1 with regard to X_1'' and X_1'''. Therefore t_4 is the inverse of the nine-point circle. Now the circle of inversion is orthogonal to the excircles with centers at I_2 and I_3, and so they invert into themselves; and since t_4 is tangent to them, inverting back again, we have the nine-point circle tangent to the two excircles. The points of contact are on the lines joining O_1 to the points of contact of t_4 with the excircles.

Similarly, it can be shown that the nine-point circle is tangent to I and I_1, completing the proof. The student is urged to draw the figure and follow through the argument again. It will be almost identical with that given if proper care is taken in labeling the figure.

89. Casey's criterion

The second proof depends upon a special case of the following theorem. The theorem is important in its own right since it applies to any four circles tangent to a fifth. We will not give the theorem in its most general form, but state the criterion in a form suitable for our purpose, limiting ourselves to mutually external circles.

THEOREM. *The necessary and sufficient condition that four mutually external circles be tangent to a fifth is that a relation of the type*

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0 \tag{1}$$

be satisfied, where t_{ij} *represents the length of the common tangential segment to two circles.* (i) *All the* t_{ij}*'s are to be direct tangential segments; or* (ii) *two of them with no common subscript number (as* t_{13} *and* t_{42}*) denote direct tangents and the other four transverse tangents; or* (iii) *those which lack one subscript denote direct and those which include it denote transverse tangential segments.*

The proof may be made to depend upon the following lemma.

LEMMA. *If two circles are mutually external, then* $t_{12}^2/(r_1r_2)$ *is invariant under inversion, where* t_{12} *is the length of a common tangential segment, and* r_1 *and* r_2 *are the radii of the circles.*

Let the given circles have centers C_1 and C_2 and construct a line through C_2 parallel to the common tangent (Fig. 76), and a

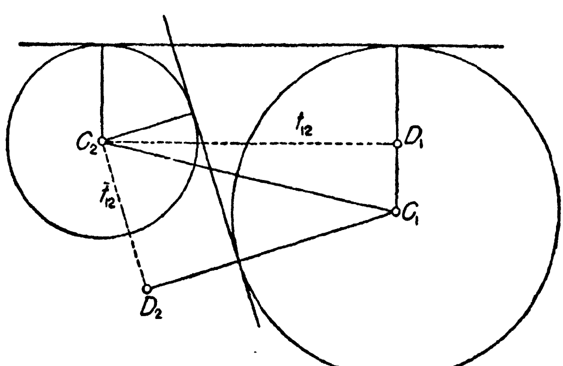


Fig. 76

line through C_1 perpendicular to this common tangent. Then it is readily seen that

$$t_{12}^2 = d^2 - (r_1 - r_2)^2,$$
$$\bar{t}_{12}^2 = d^2 - (r_1 + r_2)^2,$$

where $d = C_1C_2$. If now we invert with respect to a point O and radius r, we know that the circles c_1' and c_2' are homothetic to c_1 and c_2, respectively, with centers of c_i' and c_i corresponding points in a homothetic transformation with ratio r_i/r_i'. Hence if d' is the distance between the new centers,

$$\frac{d}{d'} = \frac{r_1}{r_1'} = \frac{r_2}{r_2'} = \frac{r_1 \pm r_2}{r_1' \pm r_2'} = \text{constant},$$

in view of equations (2) and (3) of section 37. Hence, since the constant will factor out,

$$\frac{t_{12}^2}{r_1r_2} = \frac{d^2 - (r_1 - r_2)^2}{r_1r_2} = \frac{d'^2 - (r_1' - r_2')^2}{r_1'r_2'} = \frac{t_{12}'^2}{r_1'r_2'},$$

with a similar result for the transverse tangent $\bar{t}_{12}$.* Thus the lemma is proved.

We return to the direct theorem in Casey's criterion. Suppose that c_5 is tangent to four mutually external circles c_1, c_2, c_3, c_4. We may invert c_5 into a line and c_i into mutually external circles c_i'. We make use of Euler's identity that if P_1, P_2, P_3, P_4 are points on a line, then

$$P_1P_2 \cdot P_3P_4 + P_1P_3 \cdot P_4P_2 + P_1P_4 \cdot P_3P_2 = 0.$$

We thus find (Fig. 77a represents one possible arrangement)

$$t_{12}'t_{34}' \pm t_{13}'t_{42}' \pm t_{14}'t_{23}' = 0, \tag{2}$$

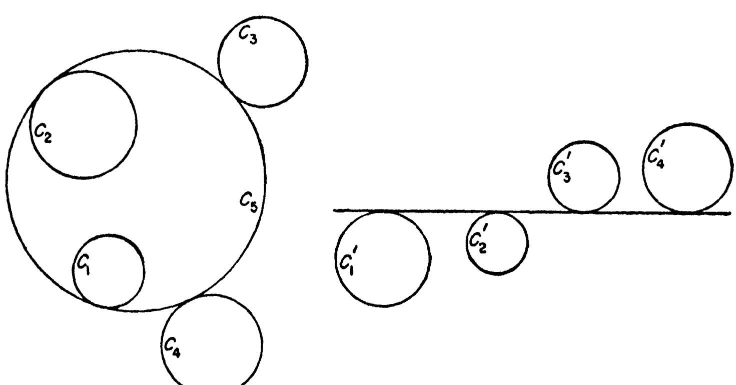


Fig. 77a

* If the circles are not external, we may define $d^2 - (r_1 - r_2)^2$ and $d^2 - (r_1 + r_2)^2$ as the direct and transverse power of one circle with respect to the other; and in the theory that follows, we may replace the lengths of the tangents by the square roots of these powers.

where the t''s are direct or transverse according as the circles are on the same or opposite sides of the line. If the four circles are on the same side of the line, or two are on each side of the line, or three on one side and the other one on the other side, we get the three cases indicated in the original statement of the theorem. The algebraic signs depend upon the order of the contact points on the line. For example, for the particular case of Fig. 77a (right), we get

$$t'_{12}t'_{34} - \bar{t}'_{13}\bar{t}'_{42} + \bar{t}'_{14}\bar{t}'_{23} = 0. \tag{3}$$

If we divide by the square root of the products of the new diameters, and invert back again, multiplying the old diameters out again, we get Casey's criterion, equation (1),

$$t_{12}t_{43} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.$$

Now conversely, for it is actually the converse that we want to use in Feuerbach's Theorem, suppose that four mutually external circles satisfy an equation of this type. The proof involves several steps.

First, let c_1 be the smallest circle. We begin by diminishing the radius of c_1 by r_1, and diminish or increase the other radii by r_1 according as t_{1i} is a direct or transverse tangent (Fig. 77b).

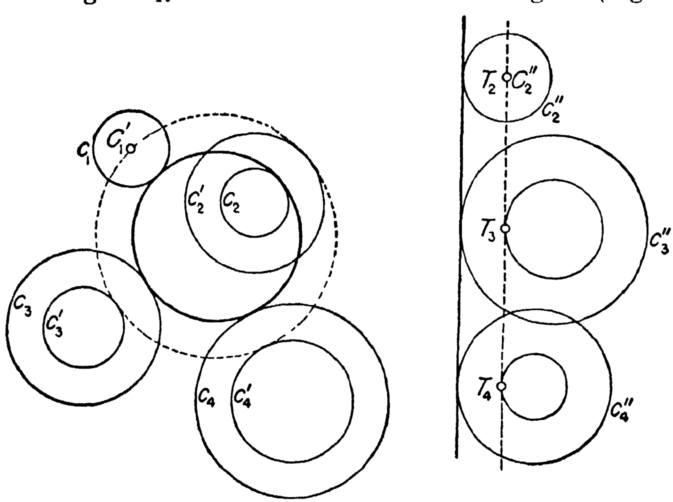


Fig. 77b

Call these new circles, which are still mutually external, c_i', where c_1' is a point. Then, since the length of the tangent is not changed by this transformation,

$$t'_{12}t'_{34} \pm t'_{13}t'_{42} \pm t'_{14}t'_{23} = 0. \tag{4}$$

Second, we perform an inversion with regard to C_1' and an arbitrary radius R, obtaining circles c_2'', c_3'', c_4''. Then for tangents which involve the subscript 1, we have relations of the type,

$$t_{12}' = R\left(\frac{r_2'}{r_2''}\right)^{1/2}, \tag{5}$$

obtained from the relations between the radii after inversion (section 37); and for those that do not involve the subscript 1, we have relations of the type

$$t_{23}' = t_{23}''\left(\frac{r_2'r_3'}{r_2''r_3''}\right)^{1/2} \tag{6}$$

as given by the last lemma. Consequently, substitution in (4) gives

$$t_{34}'' \pm t_{42}'' \pm t_{23}'' = 0. \tag{7}$$

Third, suppose that c_2'' is the smallest of these circles. Diminish (or increase) each circle by r_2''. The last relation still holds with seconds replaced by thirds, but c_2''' is now a point, and it remains to show that C_2'' (or c_2''') is on a common tangent to c_3''' and c_4'''. Let T_3T_4 be the common tangent to c_3''' and c_4''', so that $T_3T_4 = t_{34}'''$, and on this line mark off a point T_2, so that $T_4T_2 = \pm t_{42}'''$, according to the sign in (7). Then $T_{23} = \pm t_{23}'''$. Now the locus of a point from which the tangent to either circle has the given value is a circle concentric with the corresponding circle, and these two new circles intersect once and only once on each common tangent. Therefore T_2 or its symmetrical point on the other common tangent of the same type is the point C_2''; but in either case, the circles c_2''', c_3''', c_4''', and hence c_2'', c_3'', c_4'' are tangent to a line. By reversing the inversion, c_1', c_2', c_3', c_4' and finally, c_1, c_2, c_3, c_4 are tangent to a circle, which was to be proved.

90. Second proof of Feuerbach's theorem

Feuerbach's Theorem follows readily from Casey's criterion. We apply it to the three point circles O_1, O_2, O_3 and, in turn, to the incircle or excircles with centers I, I_1, I_2, I_3, using the relations of section 74. For example, if the triangle is so labeled that $a_1 \geqq a_2 \geqq a_3$, then

$$O_1X_1 = \frac{a_2 - a_3}{2}, \qquad O_2O_3 = \frac{a_1}{2}, \text{ and so on,}$$

so that

$$t_{12}t_{34} - t_{13}t_{42} + t_{14}t_{23}$$

becomes

$$O_1O_2 \cdot O_3X_3 - O_1O_3 \cdot O_2X_2 + O_1X_1 \cdot O_2O_3$$

$$= \frac{a_3}{2} \cdot \frac{a_1 - a_2}{2} - \frac{a_2}{2} \cdot \frac{a_1 - a_3}{2} + \frac{a_2 - a_3}{2} \cdot \frac{a_1}{2} = 0,$$

thus proving that the nine-point circle and the incircle are tangent. By a slight modification we can establish the balance of the theorem.

Exercise 6—Part 2. Nine-Point Circle, Polar Circle, Feuerbach's Theorem

1. Given the base A_2A_3 and the angle α_1 of the triangle $A_1A_2A_3$, find the locus of the nine-point center. Determine the locus in magnitude and position.

2. Show that the nine-point circle of $A_1A_2A_3$ is also the nine-point circle of A_1A_2H, and so on, and thus locate sixteen circles which are tangent to it.

3. Discuss and draw a diagram for the nine-point circle when the triangle is (a) a right triangle; (b) an equilateral triangle; (c) an obtuse angle triangle.

4. Show that the circumcircle is the nine-point circle of $I_1I_2I_3$ and hence determine the corresponding bisection properties.

5. Show that the nine-point circles of $O_1O_2O_3$ and $A_1O_2O_3$ are tangent at the midpoint of O_3O_2.

6. The polar axis of a right triangle is defined as the tangent to the circumcircle at the vertex of the right angle. Show that this definition is consistent with the general case and discuss the properties of the polar circle, the polar axis, and the coaxal system of circles related to them (see sections 83, 84, 87).

7. Discuss the configuration involving the polar axis when the triangle is acute (section 87).

8. If the excircles are c_1, c_2, c_3 and the incircle is c_4, and the lengths of the sides satisfy the inequality $a_3 > a_2 > a_1$, prove that a circle exists tangent to the four circles by showing $\bar{t}_{41}t_{23} - \bar{t}_{42}t_{13} + \bar{t}_{43}t_{12} = 0$. What can be said about the nature of the contacts?

9. Prove that the nine-point circle is tangent to I and I_1 by the method of section 88 (Fontene's Proof).

10. Complete the details of the proof of Feuerbach's Theorem in section 90.

11. What does Casey's criterion become if the four circles become point circles?

12. Casey's criterion for three points and a circle as used in section 90 can be stated as follows: If O_iX_i are the lengths of the tangents to a circle k from any three points O_i, then the circle $o = O_1O_2O_3$ is tangent to k, if and only if,

$$O_1X_1 \cdot O_2O_3 \pm O_2X_2 \cdot O_1O_3 \pm O_3X_3 \cdot O_1O_2 = 0.$$

Complete the details of the following proof, which does not depend upon the general criterion. If o and k are tangent at P, apply the theorem concerning the ratio of powers for coaxal circles (section 63) to k and the point circle P, and then apply Ptolemy's First Theorem (section 39); for the converse, let P be such that $O_1P : O_2P : O_3P = O_1X_1 : O_2X_2 : O_3X_3$, and reverse the operations to prove that P, o, and k are coaxal.

7 MIQUEL POINT AND SIMSON LINE

91. Miquel point

THEOREM. *If a point is marked on each side of the triangle, then the three circles, each through a vertex and the points on the adjacent side, are concurrent.*

Let the points be B_i. Let two of the circles $A_1B_2B_3$ and $A_2B_3B_1$ meet at M, and let $MB_i = b_i$. Then in terms of angles (Fig. 78),

$$\measuredangle b_2a_2 \equiv \measuredangle b_3a_3, \tag{1}$$

$$\measuredangle b_3a_3 \equiv \measuredangle b_1a_1 \pmod{180^\circ}, \tag{2}$$

the first relation coming from the circle $A_1B_2B_3$ and the second from the circle $A_2B_3B_1$. Consequently

$$\measuredangle b_2a_2 = \measuredangle MB_2A_3 \equiv \measuredangle MB_1A_3 = \measuredangle b_1a_1 \pmod{180^\circ}, \tag{3}$$

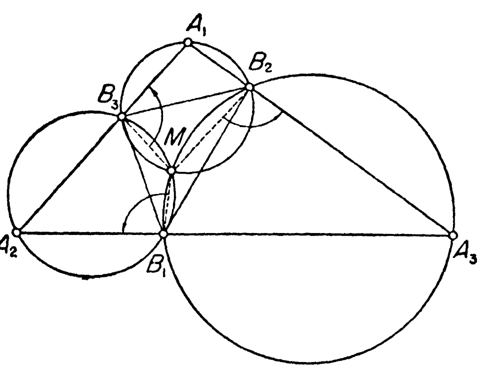


Fig. 78

and the points B_1, B_2, A_3, M are concyclic, or the three circles in question are concurrent at M.

This theorem was explicitly stated and proved by Miquel in 1838, although its truth was probably recognized much earlier. For the sake of definiteness, the theorem will be called the Miquel Theorem and his name will be associated with the point M, the triangle $B_1B_2B_3$, the three circles, and other related properties.

92. Angle property

Incidentally in the proof of the theorem, we proved the following property:

THEOREM. *The lines from the Miquel point to the vertices of a Miquel triangle make equal angles with the respective sides of the given triangle, sense taken into account.*

Conversely, it is easy to show that if three lines through a point make equal angles with the sides, three Miquel circles are determined.

93. Miquel equation

The fundamental angle property may be put in another form, called the *Miqucl equation*:

$$\measuredangle A_2MA_3 \equiv \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 \pmod{180°}. \qquad (4)$$

For

$$\begin{aligned}\measuredangle A_2MA_3 &\equiv \measuredangle A_2MB_1 + \measuredangle B_1MA_3\\ &\equiv \measuredangle A_2B_3B_1 + \measuredangle B_1B_2A_3.\end{aligned}$$

But since these last two angles are exterior angles of the simple quadrilateral $A_1B_3B_1B_2$, their sum is equal to that of the opposite interior angles, so that

$$\begin{aligned}\measuredangle A_2MA_3 &\equiv \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3,\\ &\equiv \alpha_1 + \beta_1, \pmod{180°}.\end{aligned}$$

94. Miquel triangles of a point

Suppose now that M is any point in the plane and the lines MB_i make equal angles, directions taken into account, with the sides A_jA_k (Fig. 79), thus obtaining a Miquel triangle of M. Let the lines rotate rigidly about M, thus obtaining a set of Miqucl triangles of the point M.

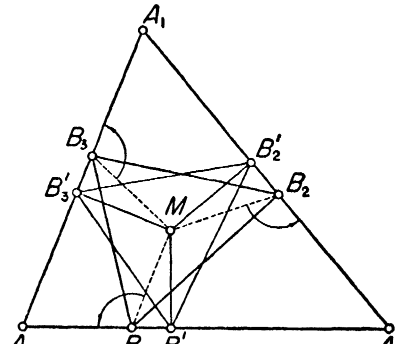


Fig. 79

THEOREM. *All the Miquel triangles of a given point* M *are directly similar, with* M *as the self-corresponding point.*

Since the angles at M (A_2MA_3, and so on) and angles α_i are fixed, the angles β_i remain constant, and so the triangles are directly similar. Since a rigid motion about M would make the triangles homothetic, M is the unique self-corresponding point (center of similitude).

Conversely, from the same angle relation, it follows that if two or more directly similar triangles are drawn with corresponding vertices on the sides of a given triangle, they have the same Miquel point, which is their center of similitude.

As an illustrative example, suppose we are given a triangle $A_1A_2A_3$ and a point B_1 on A_2A_3, and we wish to determine the point M and its Miquel triangle $B_1B_2B_3$ so that the angle at B_1 shall equal a given constant. We first consider the pedal triangle and the relation $\measuredangle A_2MA_3 \equiv \alpha_1 + \beta_1$. The point M must lie on a circle determined by this sum and the chord A_2A_3, and we may select any point on this circle as M. We next construct the pedal triangle $P_1P_2P_3$ of M. If we determine B_2 and B_3 so that $\measuredangle B_1MP_1 = \measuredangle B_2MP_2 = \measuredangle B_3MP_3$, the triangle $B_1B_2B_3$ is the required triangle.

95. Miquel configuration under inversion

It is interesting to note that if we invert the Miquel configuration with respect to any one of the seven points A_i, B_i, M, we obtain a Miquel configuration, with B_i', A_i', M as the Miquel point of the new configuration. For example (Fig. 78), if we invert with respect to A_1, the lines A_1A_2, A_1A_3 and the circle $A_1B_2B_3M$ invert into the sides of the triangle $A_1B_2'B_3'$; the line $A_2A_3B_1$ and the circles $A_2B_3MB_1$ and $A_3B_2MB_1$ invert into three circles having B_1' in common, such that two of them pass through each of the points A_2', A_3', M' on the sides of the triangle $A_1B_2'B_3'$. Similar statements may be made for B_1. If we invert with respect to M, the circles through M invert into sides of the triangle $B_1'B_2'B_3'$, and the sides of the triangle invert into circles through M, forming a new Miquel configuration. This result can be stated in the form of a converse of the original Miquel Theorem.

THEOREM. *If three circles are concurrent at a point, it is possible to start at any point of one of the circles and draw a triangle whose vertices lie on the circles and whose sides pass through the corresponding intersections; all triangles thus drawn are similar.*

The first part follows from inversion with respect to M, and the second from the fact that corresponding angles are measured by the same arcs.

96. The Miquel configuration and isogonal conjugates

We prove the following statements:

THEOREM. (a) *If a circle cuts the sides of the triangle* $A_1A_2A_3$ *in* P_1, Q_1, P_2, Q_2, P_3, Q_3 *in any order, then*

$$\measuredangle P_2P_1P_3 + \measuredangle Q_2Q_1Q_3 + \measuredangle A_2A_1A_3 \equiv 0 \pmod{180^\circ}.$$

(b) *The Miquel points* P *and* Q *of the triangles* $P_1P_2P_3$ *and* $Q_1Q_2Q_3$ *are isogonal conjugates.* (c) *The angle from* a_1 *to* PP_i *equals the angle from* QQ_i *to* a_i*; for example,*

$$\measuredangle PP_1A_2 + \measuredangle QQ_1A_2 \equiv 0 \pmod{180^\circ}.$$

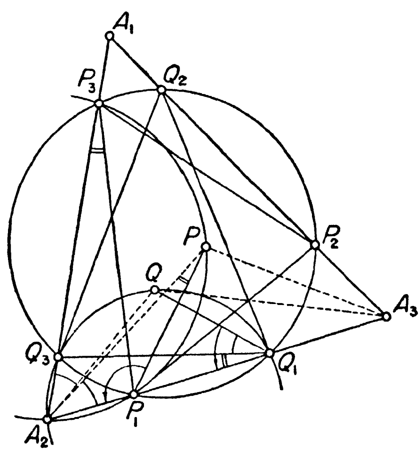


Fig. 80

PROOF OF (a). We express the three angles in terms of arcs (Fig. 80) and then add. This gives, where we use single letters to designate the angles,

$$\measuredangle P_1 + \measuredangle Q_1 + \measuredangle \alpha_1$$
$$\equiv \frac{1}{2} \text{arcs } (P_2P_3 + Q_2Q_3 + (Q_3P_2 + P_3Q_2))$$
$$\equiv \frac{1}{2} \text{arcs } (P_2P_3 + Q_2Q_3 + Q_3Q_2 + Q_2P_2 + P_3P_2 + P_2Q_2)$$
$$\equiv 0, \pmod{180^\circ},$$

which proves (a).

PROOF OF (b). From the angle property of the Miquel configuration, we have

$$\measuredangle A_2PA_3 \equiv \measuredangle\alpha_1 + \measuredangle P_1,$$

$$\measuredangle A_2QA_3 \equiv \measuredangle\alpha_1 + \measuredangle Q_1.$$

These combined with (a) above give

$$\measuredangle A_2PA_3 + \measuredangle A_2QA_3 \equiv \alpha_1 \pmod{180°},$$

and since similar relations hold for the other sides, it follows that P and Q are isogonal conjugates (section 50).

PROOF OF (c). We break the angle QQ_1A_2 into two parts. Since A_2, Q_3, Q, Q_1 are concyclic,

$$\measuredangle QQ_1Q_3 = \measuredangle QA_2Q_3, \tag{1}$$

and

$$\measuredangle QA_2Q_3 \equiv \measuredangle P_1A_2P, \tag{2}$$

because of the isogonal conjugate relation. By considering the circles $P_1Q_1P_3Q_3$ and $PP_1A_2P_3$, we have

$$\measuredangle Q_3Q_1A_2 \equiv \measuredangle A_2P_3P_1 \equiv \measuredangle A_2PP_1. \tag{3}$$

Since the sum of the angles of the triangle A_2PP_1 is 180°, we have

$$\measuredangle PP_1A_2 + \measuredangle QQ_1A_2 \equiv 0 \pmod{180°}.$$

Conversely, if the Miquel triangles of two isogonal conjugate points P *and* Q *are so constructed that* PP_1 *and* QQ_1 *make angles with the sides that satisfy relations of the type*

$$\measuredangle PP_1A_2 + \measuredangle QQ_1A_2 \equiv 0 \pmod{180°},$$

then they have the same circumcircle.

PROOF. Let the circumcircle of $P_1P_2P_3$ cut the sides again in Q_i' and let the Miquel point of $Q_1'Q_2'Q_3'$ be Q'. Then P and Q' are isogonal conjugates, so that Q and Q' coincide. Further, from the angle relations (c), Q_i and Q_i' coincide, and the theorem follows.

97. Pedal triangle

Obviously, the pedal triangle of M (that triangle such that the lines joining M to the vertices B_i are perpendicular to the sides of the triangle $A_1A_2A_3$) is one of the Miquel triangles, and indeed it is the most important one. The angles of the pedal triangle of a given point are determined by the equation of section 93,

$$\beta_1 \equiv \measuredangle A_2MA_3 - \alpha_1.$$

The sides of the pedal triangle may be determined as in Fig.

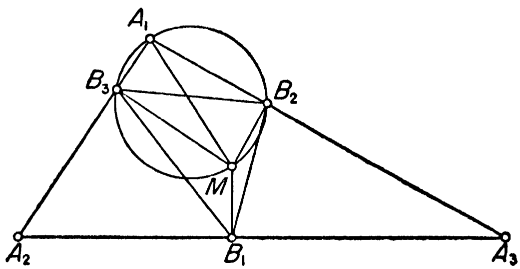


Fig. 81

81, using the circle on A_1M as diameter,

$$b_1 = B_2B_3 = A_1M \sin \alpha_1 = A_1M \cdot \frac{a_1}{2R}, \tag{1}$$

and so on, where R, as usual, is the circumradius. This follows from the fact that A_1M is a diameter of the circle $A_1B_2MB_3$, and B_2B_3 subtends at its center an angle equal to $2\alpha_1$. Consequently, the sides of the pedal triangle are proportional to certain multiples of the sides of the given triangle.

If the triangle is any other Miquel triangle, an additional proportionality factor enters, but always

$$b_1 : b_2 : b_3 = d_1a_1 : d_2a_2 : d_3a_3, \tag{2}$$

where d_i is the distance from M to A_i. It follows from (2) that a Miquel triangle will be similar to the given triangle, in the sense $P_1P_2P_3$ is directly similar to $A_1A_2A_3$, when and only when $d_1 = d_2 = d_3$, or when M is the circumcenter O, in which case the pedal circle is the nine-point circle.

98. Miquel triangle a line

Simson Line. A case of special interest arises when the Miquel triangle reduces to a line. We have at once from the fundamental equation $\alpha_1 + \beta_1 \equiv \measuredangle A_2MA_3$, the following theorem:

THEOREM. *The Miquel point of a set of collinear points on the sides of a triangle is on the circumcircle; and, conversely, a Miquel triangle of a point on the circumcircle is a straight line.*

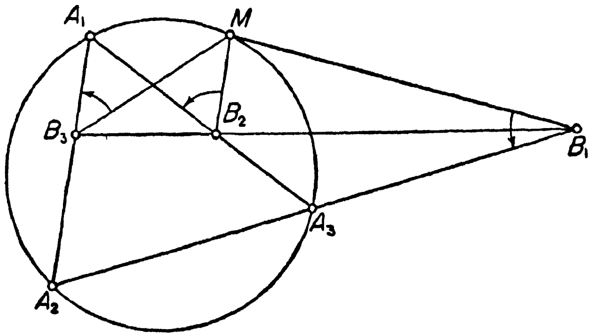


Fig. 82

For if

$$\measuredangle B_2B_1B_3 \equiv 0,$$

then (Fig. 82)

$$\measuredangle A_2MA_3 \equiv \measuredangle A_2A_1A_3 \text{ (mod } 180°),$$

and conversely.

In particular, if we use the pedal triangle and permit it to become a line, our results can be stated as follows:

THEOREM. *The feet of the perpendiculars to the sides of a triangle from a given point are collinear, if, and only if, the point is on the circumcircle of the triangle.*

This line (we use the letters P and P_i instead of M and B_i in Fig. 83) is called the *pedal line* or *Simson line* of the point with regard to the triangle.

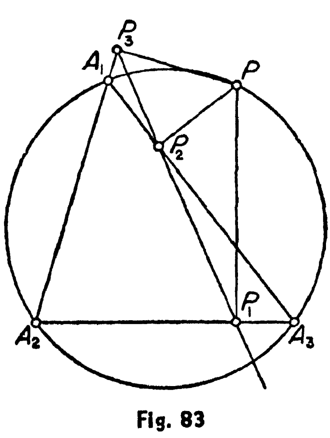


Fig. 83

This theorem was erroneously attributed to Robert Simson, and through subsequent repetition, the error was perpetuated, so that his name is still in use. In reality the theorem was probably never known to Robert Simson, but was first discovered by William Wallace in 1797. Its history is given in detail in a paper by J. S. Mackay.*

99. Simson line and the nine-point circle

THEOREM. *The Simson line of any point of the circumcircle bisects the line joining the point to the orthocenter, and the point of bisection lies on the nine-point circle.*

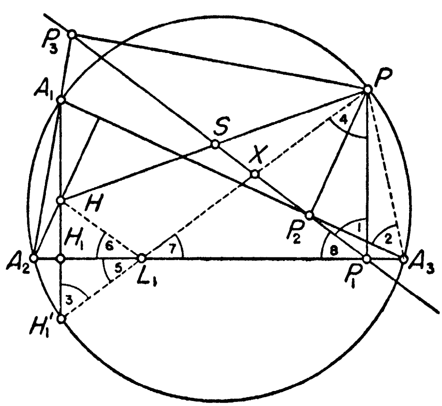


Fig. 84

There seems to be no very simple proof of this important theorem.

Let the altitude A_1H_1 cut the circumcircle at H_1'. Let PH_1' cut A_2A_3 at L_1, and the Simson line at X. We first show that the Simson line is parallel to HL_1 and bisects PL_1 at X The angles indicated in the diagram (Fig. 84) as 1, 2, 3, 4 are all equal:

$\measuredangle 1 = \measuredangle 2$, from the cyclic quadrangle $PP_2P_1A_3$.

$\measuredangle 2 = \measuredangle 3$, as determined from the circumcircle.

$\measuredangle 3 = \measuredangle 4$, as determined by the parallel lines A_1H_1' and PP_1.

Hence the triangle PXP_1 is isosceles, so that $XP = XP_1$. Further, the angles indicated by 5, 6, 7, 8 are also all equal, $\measuredangle$ 5 and $\measuredangle$ 8 being the complements of $\measuredangle$ 3 and $\measuredangle$ 1, respectively. Therefore, HL_1 is parallel to the Simson line and $L_1X = XP_1 = XP$. Thus, since the Simson line bisects one side of the triangle HL_1P and is parallel to HL_1, it also bisects HP at S.

* See J. S. Mackay, in *Proceedings of the Edinburgh Mathematical Society*, volume 9 (1890), pages 83–91.

Since the nine-point circle is homothetic to the circumcircle with respect to H, with the ratio $1/2$, S lies on the nine-point circle.

100. Direction of the Simson line

The direction of the Simson line is given by the next theorem, and several corollaries may be found from the last theorem.

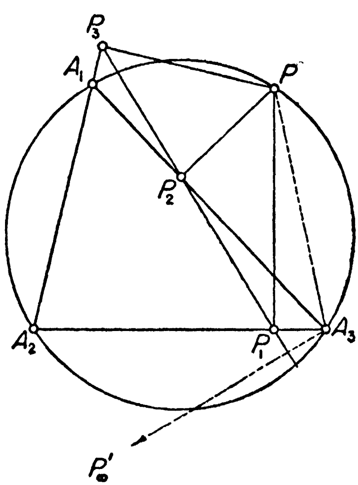


Fig. 85

THEOREM. *The Simson line of a point on the circumcircle is perpendicular to the isogonal conjugate of the line joining the point to a vertex.*

We already know (section 49) that the isogonal conjugates of PA_i are all parallel, and so we need consider only one vertex, say A_3 (Fig. 85). Since

$$\measuredangle A_2A_3P'_\infty \equiv \measuredangle PA_3A_1 \equiv \measuredangle PP_1P_2,$$

and since PP_1 is perpendicular to A_2A_3, it follows that P_1P_2 is perpendicular to $A_3P'_\infty$, as was to be proved.

Thus to find the Simson line parallel to a given direction, we construct $A_3P'_\infty$ perpendicular to the given direction, and find where the isogonal conjugate of $A_3P'_\infty$ cuts the circumcircle at P. The Simson line of P is the required one.

These corollaries follow immediately:

COROLLARY. *The angle between the Simson lines of two points on the circumcircle is measured by one-half the arc between the points.*

COROLLARY. *The Simson lines of diametrically opposite points are mutually perpendicular and intersect on the nine-point circle.*

The last part of this statement is proved as follows. Let the given diameter be PQ and let the midpoints of HP and HQ be S and T, respectively (Fig. 86). Then from the homothetic property of the nine-point circle and the circumcircle, ST is a diameter of the nine-point circle. But the Simson lines pass through S and T, respectively, and are perpendicular; hence they meet on the nine-point circle.

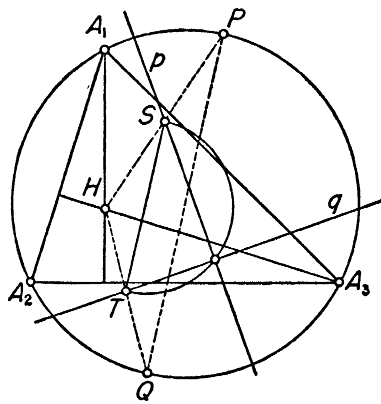


Fig. 86

101. Miquel point of a complete quadrilateral

THEOREM. *The circumcircles of the four triangles formed by four lines are concurrent.*

If we fix our attention on any one of the four triangles, say $A_1B_1C_1$, with the fourth side as a transversal (Fig. 87), then three of the circles (c_1, c_2, c_3) are Miquel circles of the triangle $A_1B_1C_1$,

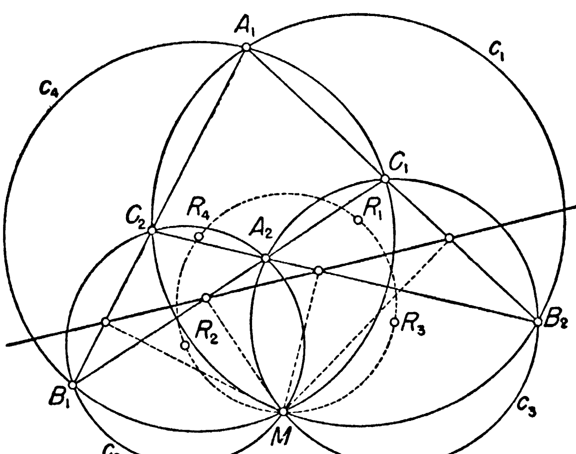


Fig. 87

and hence meet at M. But by section 98, M lies on the circumcircle of $A_1B_1C_1$ or c_4, so that the four circles are concurrent. We call this point the Miquel point of the complete quadrilateral.

102. Simson line of a complete quadrilateral

THEOREM. *The Miquel point of a complete quadrilateral is the only point such that the feet of the perpendiculars from the point to the sides are collinear.*

Any such point must lie on the circumcircle of each triangle and hence is M; the feet of the perpendiculars lie on the Simson line of each triangle, and by considering two triangles, we find all four such points collinear. The line is known as the Simson line of the complete quadrilateral (Fig. 87).

103. Centers of the Miquel circles of a complete quadrilateral

Let us invert the configuration of Fig. 87 with respect to M. We get a second configuration entirely analogous to the given one, the four lines inverting into four circles through M, and the four circles into lines, say l_i into c_i' and c_i into l_i'. The feet of the perpendiculars M_i' from M to the four new lines are on the Simson line of the new quadrilateral. The four reflections of M in the

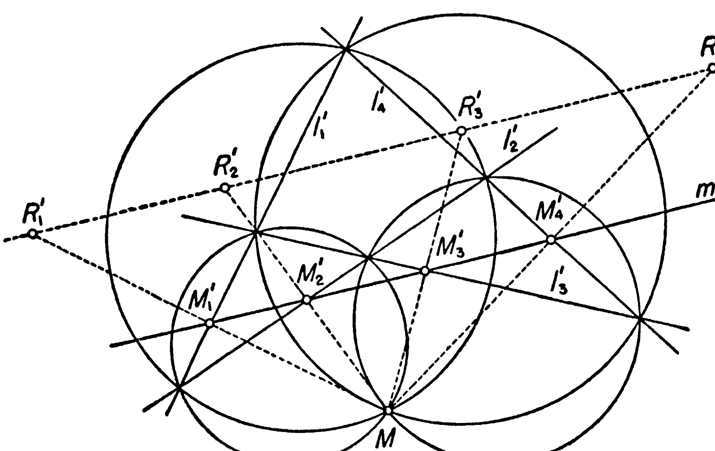


Fig. 88

lines l_i', say R_i', are also collinear (Fig. 88). But R_i' is the inverse of the center of circle c_i (section 36). Inverting back again, we have the following theorem (Fig. 87):

THEOREM. *The centers of the circumcircles of the four triangles formed by four lines lie on a circle through the Miquel point of the four lines.*

104. Simson line and line of orthocenters

We have proved that the orthocenters of the four triangles are collinear (section 81), and that the line joining an orthocenter to M is bisected by the Simson line (section 97). Consequently, we have the following theorem:

THEOREM. *The Simson line and the line of orthocenters of a complete quadrilateral are parallel and the Simson line lies midway between the line of orthocenters and the Miquel point.*

In Fig. 88, the orthocenters H, H', H'', H''' lie on the line containing the R_i'.

Exercise 7. Miquel Point and Simson Line

1. Draw the Miquel configuration with M outside the triangle and follow the steps of section 91 and section 93. In particular state what multiple of 180° is involved in your diagram if all angles are taken as positive.

2. Find a point P whose pedal triangle is equilateral.

3. By means of similar triangles (section 51), prove that two isogonal conjugate points have the same pedal circle.

4. The feet of the perpendiculars from H_i to the opposite sides are six concyclic points.

5. If P is a point on the circumcircle of triangle $A_1A_2A_3$, and if the perpendicular from P to A_2A_3 meets the circumcircle in P_1', then A_1P_1' is parallel to the Simson line of P.

6. If in problem 5, A_1P is parallel to A_2A_3, then the Simson line of P is parallel to A_1O.

7. If the altitude from A_1 cuts the circumcircle at H_1', then the Simson line of H_1' is parallel to the tangent to the circumcircle at A_1.

8. If the altitude from A_i cuts the circumcircle at B_i, then the Simson line of A_1 with respect to the triangle $B_1B_2B_3$ is parallel to A_2A_3. (*Suggestion:*

$$\measuredangle B_3P_2P_1 = \cdots = \measuredangle \alpha_1 = \cdots = \measuredangle B_3B_1,\ A_2A_3.)$$

9. Show that the Simson lines of three points on the circumcircle form a triangle similar to the triangle formed by the three given points.

10. (a) Find the three points whose Simson lines are perpendicular to a set of three concurrent lines, one through each vertex.

(b) Discuss the special configuration when the point of concurrency is either O or H.

8 RULER AND COMPASS CONSTRUCTIONS

Part 1. Tangent Circles

105. Tangent circles

In this chapter we propose to consider certain famous problems of construction. Because these problems are frequently associated with tangent circles, we shall first consider a few definitions and elementary properties in that connection.

Two circles are said to be *externally tangent* or *internally tangent* according as they lie on the opposite or the same side of their common tangent at the point of contact. If the circles are externally tangent, the distance between their centers is the sum of their radii; if internally tangent, this distance is the difference of their radii, and conversely.

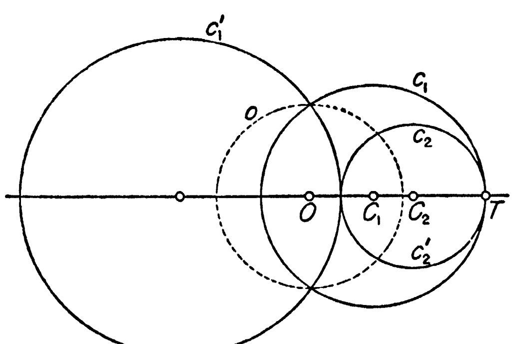


Fig. 89

If two tangent circles are subject to an inversion, the inverse circles will still be tangent, but the type of contact depends upon the position of the center of inversion. Since the radii are related by

$$R_i' = R_i \cdot \frac{r^2}{p_i},$$

the type of contact depends upon p_i. If O is outside both or inside both, the type of contact is unchanged; but if O is inside one and outside the other, then p_1 and p_2 have opposite signs, and the type of contact will be changed. Thus in Fig. 89, c_1 and c_2 have internal contact; O is taken within c_1 and outside of c_2; and the circle of inversion o is taken orthogonal to c_2 for convenience. The circles c_1' and c_2' are externally tangent.

106. Circles tangent to two circles

(a) If a circle x is tangent to two circles c_1 and c_2 (Fig. 90), then (1) the line joining the points of contact passes

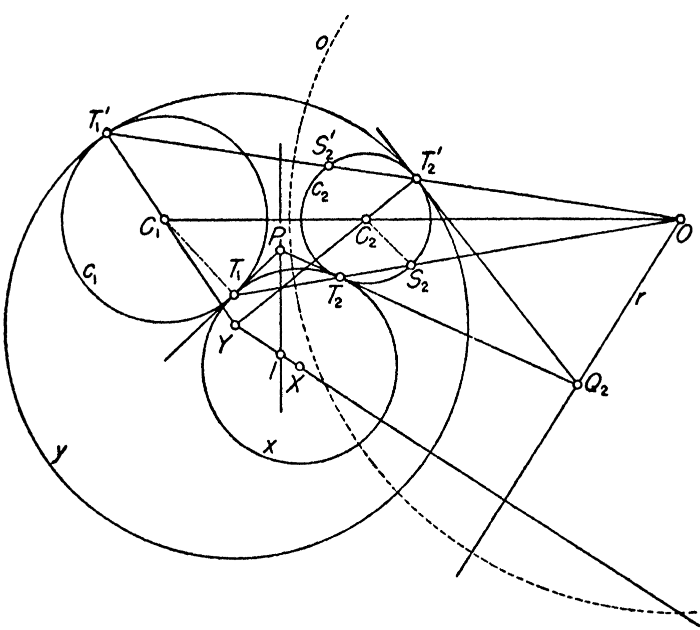


Fig. 90

through the external or internal homothetic center according as x has like or unlike contacts with c_1 and c_2 (problem 8, exercise 2–part 1). (2) If the contact points are T_1 and T_2, and if T_1T_2 cuts c_2 again at S_2, then T_1 and S_2 are corresponding points in the homothetic transformation from O; the radii C_1T_1 and C_2S_2 are parallel; and the tangents at T_1 and S_2 are parallel. (3) The points T_1 and T_2 are mutually inverse with respect to the center O and circle of inversion o, and if this circle is real, it is orthogonal to x. (4) If the tangents at T_1 and T_2 meet at P, then P is the radical center of c_1, c_2, and x; and, in particular, P lies on the radical axis of c_1 and c_2. (5) Further, P is the pole with regard to x of the line T_1T_2, so that the pole of the radical axis of c_1 and c_2 with respect to x lies on T_1T_2 (not shown in Fig. 90).

(**b**) If the two circles x and y are tangent to c_1 and c_2 in such a way that they both have the same type of contact with c_1 as with c_2 (for example, in Fig. 90, x is tangent externally to c_1 and c_2, while y is tangent internally to both of them), or both x and y each have opposite types of contact with c_1 and c_2, then the lines joining the corresponding points of contact pass through the same homothetic center. (1) If the points of contact are T_1, T_2, T_1', T_2', then T_1T_2 and $T_1'T_2'$ meet at a homothetic center of c_1 and c_2, say O, and T_1T_1' and T_2T_2' meet at a homothetic center of x and y, say I. (2) The circles c_1 and c_2 are mutually inverse with respect to O; the circles x and y are mutually inverse with respect to I; and if the corresponding circles of inversion are real, x and y are orthogonal to o, and c_1 and c_2 are orthogonal to i. (3) The four points of contact are concyclic (since $OT_1 \cdot OT_2 = OT_1' \cdot OT_2'$), and the homothetic center O lies on the radical axis r of x and y, while I lies on the radical axis of c_1 and c_2. (4) If the tangents at T_1 and T_2 meet at P, and those at T_1' and T_2' meet at P', and if the tangents at T_1 and T_1' meet at Q_1, and those at T_2 and T_2' meet at Q_2, then $PP'I$ is the radical axis of c_1 and c_2, and Q_1Q_2O is the radical axis of x and y. (5) Hence, as in (a), the pole of PI with respect to x lies on T_1T_2, and the pole of OQ_1 with respect to

c_1 lies on $T_1T'_1$, with analogous statements for the pole of PI with respect to y, and for the pole of OQ_1 with respect to c_2.

For example, in Fig. 90, the external homothetic center of c_1 and c_2 lies on OQ_2, the radical axis of x and y. The internal homothetic center of x and y, say I, lies on the radical axis of c_1 and c_2, so that PI is the radical axis of c_1 and c_2. Also the pole of PI with respect to x lies on T_1T_2 and the pole of OQ_2 with respect to c_2 lies on $T_2T'_2$.

107. Systems of circles tangent to two circles

To construct a circle tangent to c_1 and c_2, we may begin with any point T_1 on c_1 and construct two circles tangent to

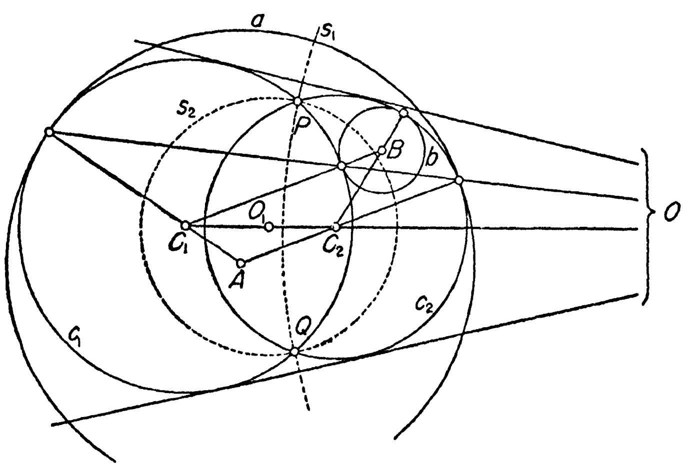


Fig. 91

c_1 and c_2, the points of contact on c_2 being found by joining T_1 to the homothetic centers. We thus obtain two one-parameter families of circles, and we now consider these systems. We separate the cases in which c_1 and c_2 intersect from those in which they do not intersect. In each case we simplify the figures by inversion.

If the two circles intersect, we may invert them into a pair of straight lines. The circles tangent to these two lines (Fig. 92) form two distinct sets with their centers on the bisectors of the angles. Through each point not on one of the lines pass two circles of the same set (reflect the point on the bisector and apply section 59); if the point is on a bisec-

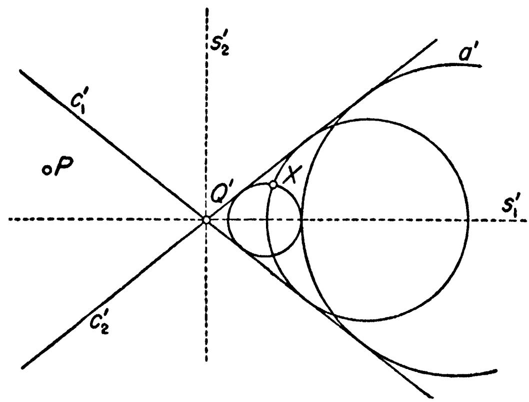


Fig. 92

tor, the circles are tangent at that point. The center of inversion lies within one of the angles and through it pass two circles of a set. Hence when we invert back again, we get the following result (Fig. 91):

THEOREM. *The circles tangent to two intersecting circles consist of two systems, one having like contacts with both the given circles (circles* a_i*) including the common tangent lines; the other having unlike contacts with the given circles (circles* b_i*). Either system has a common orthogonal circle* (s_1 *for* a_i, s_2 *for* b_i), *which is coaxal with the given circles and indeed is a circle of antisimilitude; the circles of a system are tangent to one another along this common orthogonal circle.*

If the two circles do not intersect, an inversion with respect to a limiting point inverts them into concentric circles. In the inverse figure (Fig. 93b) we can recognize two sets of circles. One set lies between c_1' and c_2', having unlike contacts with them (b_i'); the other set encircles c_2' (in this figure) and has like contact with both c_1' and c_2' (a_i'). The circles of the first set (b_i') have a common orthogonal circle s', which is concentric with c_1' and c_2', having the property that c_1' and c_2'

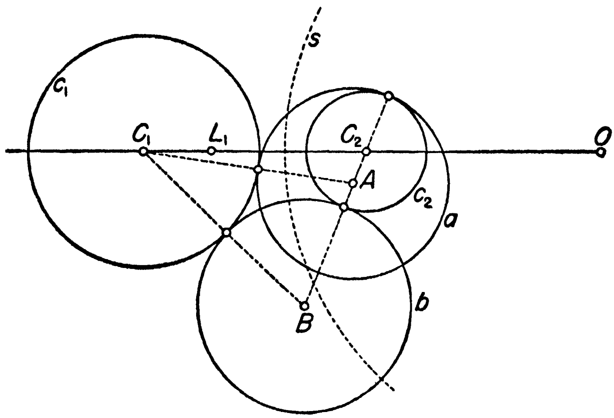


Fig. 93a

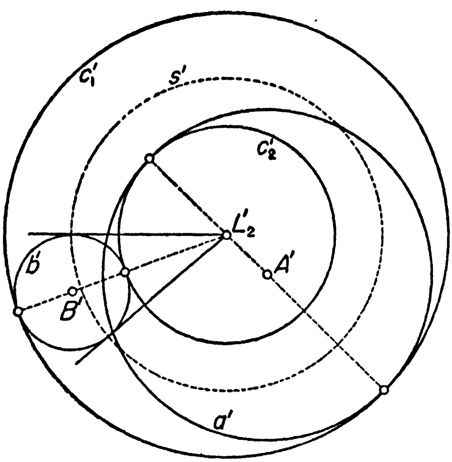


Fig. 93b

are mutually inverse with respect to s'. Through each point of s' pass two tangent circles of the set b_i'. The circles of the second set (a_i') have no (real) common orthogonal circle, but all of them cut a fixed circle in the ends of a diameter (section 54).

If we invert back again, recognizing that the types of contact will be changed in our diagram, we have the following results (Fig. 93a):

THEOREM. *The circles tangent to two nonintersecting circles consist of two systems. There are in general two or no circles of each system through a point. If the given circles* c_1 *and* c_2

are mutually external, the system having like contact with them (b_1) *has a common orthogonal circle* s, *which is coaxal with* c_1 *and* c_2 *and is their circle of antisimilitude, and the circles of the system are tangent along this circle. The other system* (a_1) *has unlike contacts with* c_1 *and* c_2, *and has no real common orthogonal circle. If* c_1 *and* c_2 *are not mutually external, the roles of the systems are interchanged.*

108. Steiner chains

A Steiner chain of circles is a finite set of circles each tangent to two fixed nonintersecting circles, such that each circle of the set is tangent to two others of the set.

We again invert the fixed circles into concentric circles (Fig. 93b). If we start with a circle of the set between c_1' and c_2', we may construct a circle of the same set tangent to it and continue this process. It is possible, but not inevit-

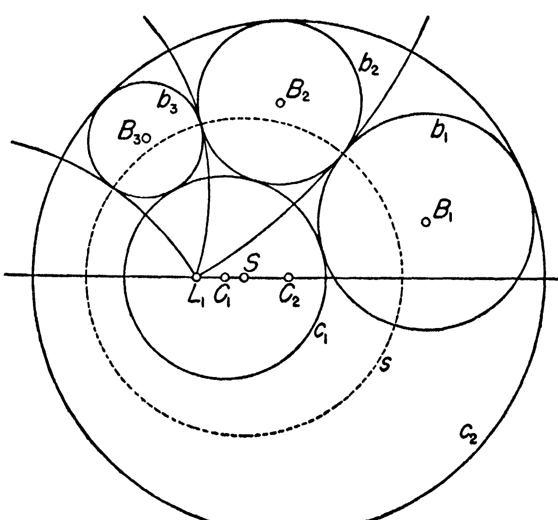


Fig. 94

able, that eventually the nth circle will be tangent to the first. This will happen after m circuits about the common center L_2', if, and only if, the angles subtended by b' at L_2'—

that is, the arc of s' between successive points of contact—is commensurable with 360°, being $360° \cdot (m/n)$. The property of forming a Steiner chain is invariant under a rotation about L_2', and also under inversion. Hence if we invert back again (Fig. 94), we get the following properties:

THEOREM. *If two circles admit a Steiner chain, they admit an infinite number, and any one of the circles of that tangent system which has a common orthogonal circle is a member of a chain. If* L_1 *and* L_2 *are the limiting points of the coaxal system determined by the given circles, then circles of the conjugate system can be drawn tangent to any two adjacent circles of the chain at their point of contact, and the angle at* L_1 *between these circles is* $360° \cdot (m/n)$.

109. Problem of Apollonius

We come next to the most famous problem of elementary geometry, the problem of Apollonius: to construct a circle tangent to three given circles. The number of circles tangent to three given (noncoaxal) circles is finite. Since the contact of the required circle with each of the given ones may be either external or internal, we anticipate that we may have as many as eight solutions. A consideration of the various cases will show that there will be eight solutions in some cases and that in other cases there may be no real solution. We include in the general problem the cases when some of the given circles are point circles or straight lines.

The number of ways of solving the problem is very great. According to Simon,* some 70 articles dealing with this problem appeared in the nineteenth century. First we consider methods similar to those used by Apollonius, but expressed in modern terminology, and then alternative solutions involving inversion.

PROBLEM 1. Construct a circle through two points and tangent to a circle [symbolized by (P_2C_1)].

* See Maximilian Simon, *Ueber die Entwickelung der Elementargeometrie im XIX Jahrhundredt.* Berlin: B. G. Teubner, 1906.

This problem has already been considered (section 59), but we repeat certain details. It is clear that to have a real solution, the points must not be separated by the circle. Let the given points be P_1 and P_2, and the given circle have its center at O (Fig. 95a). Let an arbitrary circle c through

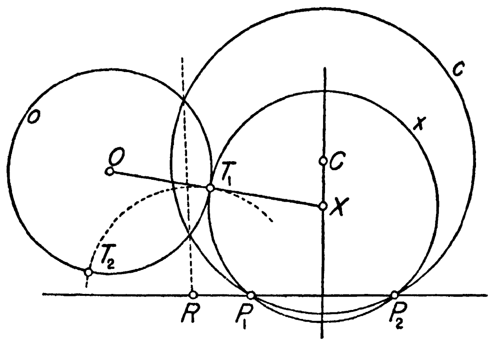


Fig. 95a

P_1 and P_2 be drawn; the radical axis of o and c cuts P_1P_2 in the radical center R of o, c, and the required circles. The points of contact, T_1 and T_2, can be found by drawing a circle, center R, orthogonal to the circle c. The center of a required circle is the intersection of OT_1 and the perpendicular bisector of P_1P_2.

Or we may proceed in another way. Invert with respect

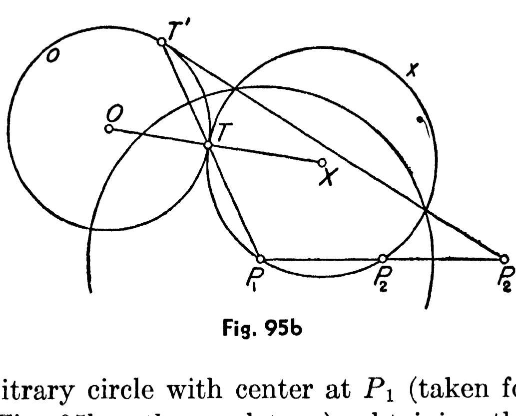


Fig. 95b

to an arbitrary circle with center at P_1 (taken for convenience in Fig. 95b orthogonal to o), obtaining the circle o'

(now identical with o) and the point P_2'. Draw the tangents from P_2' to o' and invert back again. Actually, instead of inverting back, we note that P_1, T', and T are collinear and that the required circle x passes through P_1, P_2, and T, its center X being the intersection of OT and the perpendicular bisector of P_1P_2.

Both methods indicate that if the circle does not separate the points, there will be two solutions of the problem.

If the circle becomes a line (P_2L_1), no changes need be made in the solution. In this connection see section 65.

PROBLEM 2. Construct a circle through a point tangent to two circles (P_1C_2).

We make use of section 106. Let O be a homothetic center of the given circles c_1 and c_2, selected according to the type of contacts desired. To be explicit, suppose P is outside two mutually external circles and we desire external

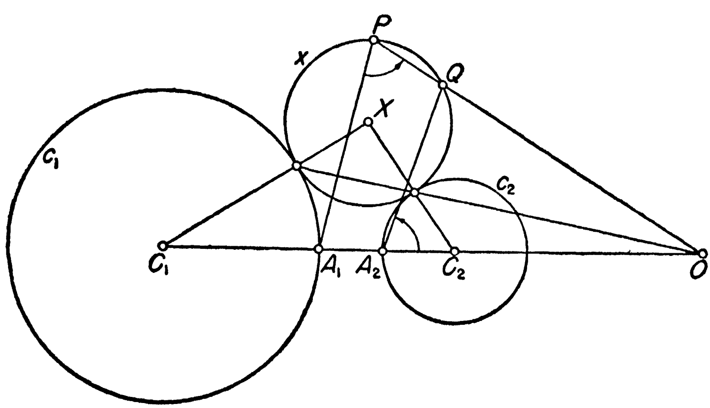


Fig. 96

contact with both (Fig. 96), so that O is the external homothetic center.

Let the corresponding circle of inversion have the radius r, and let Q be the inverse of P with regard to this circle; that is,

$$OP \cdot OQ = r^2 = OA_1 \cdot OA_2,$$

where A_1 and A_2 are a pair of inverse points, taken for convenience on the line of centers. The circle through P, Q, tangent to c_1 is the required circle (Problem 1). To find the point Q, we note that the triangles OPA_1 and OA_2Q are similar, or that A_1P and A_2Q are antiparallel. Hence we may either construct the angle at A_2 equal to angle A_1PO, or the circle A_1A_2P, to determine Q.

A closely allied solution, which at least sounds simpler, may be given. Invert with respect to P and any convenient circle. (It could be taken as the circle with center P, and orthogonal to c_1, so that c_1 remains fixed.) Draw the common tangents and invert back again. This puts into marked evidence the four possible solutions of the problem. The actual inversion back again may be avoided by using the same artifice used in the last problem.

If one or both the circles become straight lines, the solution is still valid. For $(P_1C_1L_1)$ we recall that the homothetic center is the end of a diameter of the given circle drawn perpendicular to the given line; and for (P_1L_2), that the circle of antisimilitude is a bisector of the angle between the lines. For $(P_1C_1L_1)$ there may be four real solutions, but for (P_1L_2) there are only two, since in the inverse figure, the two circles intersect and have but two (real) common tangents.

PROBLEM 3. Construct a circle tangent to three circles (C_3).

We assume for the sake of definiteness that the three circles are mutually external, so that there are effectively eight real solutions. Let C_1 be the center and r_1 the radius of the smallest circle, and reduce this circle to a point; reduce or increase the radii of the other circles by r_1 according as the required circle has the same type of contact with c_2 and c_3 as with c_1. In Fig. 97, we show the circles x and y with like contacts with all three circles, and hence our first operation is to decrease all three radii. We next construct the circles tangent to c_2' and c_3' and through C_1, and then appropriately increase or decrease the radii of these circles (decrease for x, increase for y).

If one or two of the circles become straight lines, (C_2L_1)

or (C_1L_2), there is no essential difference in the construction if we merely interpret the increase or decrease of the radius as a translation into a position parallel to the given line.

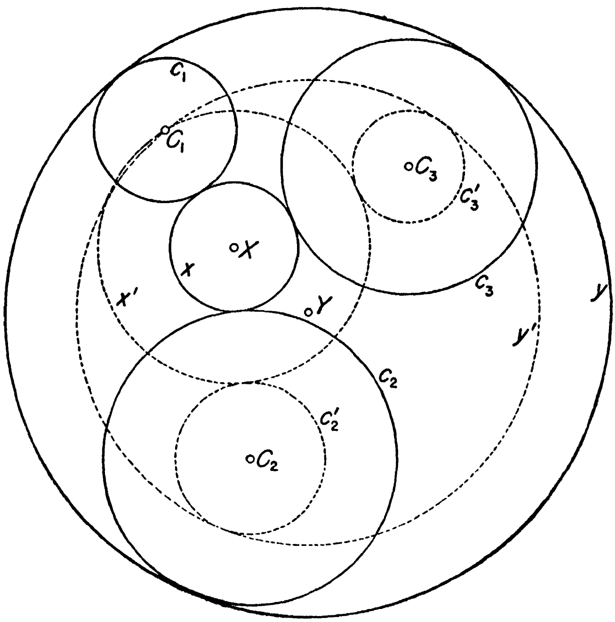


Fig. 97

The details and figure as well as a consideration of the number of solutions, we leave as an exercise for the reader.

110. Geometrography

In order to answer the question, "Which of the various solutions is the simplest?" it is necessary to set up arbitrarily certain criteria. The best known and least undesirable tests for the simplicity of a geometric construction are those devised by Emile Lemoine. Three distinct operations are recognized for the compass, and two for the ungraduated ruler:

(1) To place one point of the compass in a given position.
(2) To place one point of the compass on a given line.

(3) To draw a circle.
(4) To place the ruler on a point.
(5) To draw a line.

The sum of the number of times all of these operations are performed is called the *simplicity* of the construction; the sum of the number of times (1), (2), and (4) are performed is called its *exactitude.* Lemoine recognized that these names are ill chosen and at best that they yield but the roughest of tests. A better term instead of "simplicity" might have been "measure of complication," and it is apparent that the true exactness of a construction would depend upon the angles involved in the diagram. However, these terms at least afford a method of comparison. Hence we proceed to examine some of the constructions of the last section with reference to these criteria.

The problem (P_2C_1) (Fig. 95a) first involves the construction of a circle through two given points; this operation requires the drawing of three circles (or arcs) with the same compass opening ($S6$, $E3$). To find the radical center requires two lines each through two given points ($S6$, $E4$). To find the points of contact of the tangents from R to the circle o adds ($S11$, $E7$) as follows: We draw the line RO ($S3$, $E2$), and bisect it, using the circle o as part of this construction ($S5$, $E3$), and draw the circle on RO as diameter ($S3$, $E2$), which gives us the points of contact. The line OT_1 meets the perpendicular bisector of P_1P_2 in X. Since the points to determine this bisector would be found when finding c, this now involves ($S6$, $E4$), and the drawing of the required circle adds ($S3$, $E2$). Hence the construction of one of the required circles gives ($S32$, $E20$), and the construction of the other circle adds ($S6$, $E4$) for a total of ($S38$, $E24$).

If we use the second method (Fig. 95b), the circle with center P_1 orthogonal to the given circle gives ($S16$, $E10$); a similar construction for the inverse of P_2 (section 25) adds ($S19$, $E12$) (this may be reduced greatly by other methods), and

($S13$, $E8$) is added for finding the points of contact of the tangents from P_2' to o. The balance of the construction, as in the other method, adds ($S15$, $E10$) for a total score of ($S63$, $E40$). This method, then, according to these criteria, is much more complex than the other one.

Enough has been said to indicate the possibilities. By ingenious devices the Lemoine numbers can frequently be reduced; the following numbers given by Lemoine indicate the relative complication of the constructions. The first construction for (P_1C_2) has numbers ($S145$, $E94$); the second has numbers ($S139$, $E95$); and the solution of the general problem of Apollonius has ($S508$, $E336$) for all solutions.

Exercise 8—Part 1. Tangent Circles.

1. Consider in detail the systems of circles tangent to two given tangent circles.

2. If c_1 and c_2 are concentric circles with center O, and if a_i and b_i represent the tangent systems to c_1 and c_2, prove that the angle subtended by a at O is equal to the angle of intersection of two circles of the b-system whose centers are collinear with O. [*Suggestion*: radii $= (r \pm r')/2$.]

3. (a) Discuss in more detail than given in the text the solution of the problem (P_1C_2) using the method of inversion.

(b) What are the Lemoine numbers for your solution?

4. Construct the solutions of the problem (P_1C_2) when the required circle is tangent externally to one circle and tangent internally to the other using the first method of section 109 for P_1C_2.

5. Consider problems 3 and 4 when one of the given circles is a line.

6. Solve the problem (P_1L_2) by the following methods and compare the corresponding Lemoine numbers.

(a) Reflect the point on the bisector of the angle between the lines.

(b) Construct a circle tangent to the lines and perform an expansion.

7. Construct the circle tangent externally to c_1 and c_2 and internally to c_3, where c_1 is the smallest circle.

8. Construct a circle tangent to two lines and orthogonal to a third circle.

9. Construct a circle tangent to two circles and orthogonal to a third circle.

10. Construct a circle tangent to two lines and a circle c using the method of homothetic figures. Discuss the maximum number of solutions and their possible existence. (*Suggestion*: Suppose x is the required circle and x' an arbitrary circle homothetic to it. Consider the homothetic centers of x, x', and c.)

***11.** If circles x and y are each tangent to c_1 and c_2, prove that the tangents at the four points of contact are tangent to a circle. (First prove the converse of problem 16, exercise 1.)

***12.** Given three circles tangent externally two by two. Construct the circles tangent to all three of them by first inverting with respect to the point of contact of c_2 and c_3, so that c_1 inverts into itself.

***13.** In problem 12 show that (a) the radius of inversion is given by

$$r^2 = \frac{4r_1 r_2 r_3}{r_2 + r_3},$$

and (b) if x and y are the radii of the required circles,

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3}.$$

Part 2. Gergonne's Solution. Compass Constructions

111. Gergonne's construction

As a second solution of the problem of Apollonius, we give the neatest and most famous solution of all, that due to Gergonne (1817). Without loss of generality we again consider that the circles are mutually external, so that there are eight solutions. We find the required circles in pairs, one circle having the opposite type of contact from the other with each of the three given circles. Let one such pair be x and y.

(1) Since x and y are tangent to both c_i and c_j with appropriate contacts, a homothetic center of c_i and c_j will be on the radical axis of x and y (section 106b). It will be an external or internal homothetic center according as x has the same or

different types of contact with c_i and c_j (the contacts of y are opposite to those of x). Hence the four lines which contain the six homothetic centers (section 13) will be the radical axes of the four pairs of required circles. In Fig. 98, x has

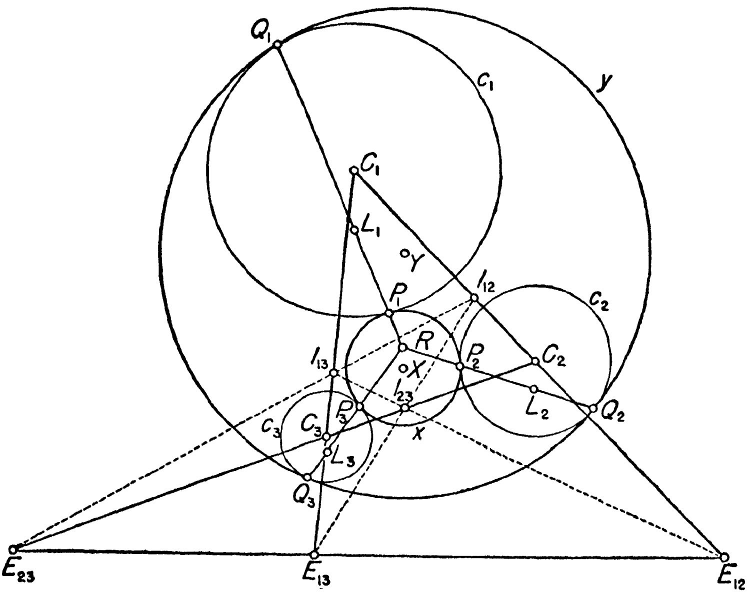


Fig. 98

external contact and y internal contact with each of the three given circles, and hence the line of the three external homothetic centers is the radical axis of x and y.

(2) Since c_1 and c_2 each have unlike contacts with x and y, the internal homothetic center of x and y will be on the radical axis of c_1 and c_2. Similarly, it is on the radical axis of c_1 and c_3, so that the radical center R of the circles c_1, c_2, c_3 will be the internal homothetic center of x and y. Hence R will lie on the line joining the points of contact P_i, Q_i of c_i with x and y.

(3) This line likewise contains the pole L_i with respect to c_i of the radical axis of x and y (section 106a) and is thus the line RL_i. Hence we have the following construction:

Gergonne's Construction. *Find the poles with respect to the given circles of a line containing a triad of homothetic centers of the circles taken in pairs. The lines connecting these poles with the radical center of the three circles will meet these circles in the points of contact with one pair of the circles sought.**

If all constructions are carried out without simplifications, the Lemoine numbers are (S479, E318). These numbers may be greatly reduced; Lemoine cites one construction involving S154.

112. Compass constructions

It is natural that the discussions that we have just had should bring up the question of what constructions can be made with ruler and compass. It is singular that geometry fails to answer this question, and we must depend upon analysis for the following answer. If a problem has been stated in algebraic language, then the construction can be made with ruler and compass if, and only if, the algebraic problem can be reduced to the solution of linear and quadratic equations. But geometry does enable us to discuss an interesting corollary to this proposition. Every construction that can be made with ruler and compass can be made with compass alone (of course, drawing a line is excepted). The idea was originally due to Mascheroni,† a machine designer, who recognized that in mechanical drawing greater accuracy can be attained by using the compass rather than the straight edge. Mascheroni's original procedure was later shortened and systematized by Adler through the aid of inversion, although some of Mascheroni's constructions

* The most systematic attempt ever made to reduce to a uniform method the solution of all problems involving the construction of circles was made by Wilhelm Fiedler in his *Cyklography*. Leipzig: B. G. Teubner, 1882. The method is beyond the scope of this book, but the interested reader will find a brief account of the method and its application to the problem of Apollonius in the treatise by Coolidge mentioned in the Bibliography. It is worthy of note that this general method leads one to Gergonne's construction.

† See Lorenzo Mascheroni, *La Geometria del Compasso*. Pavio: Pietro Galeazzi, 1797.

are simpler than the inversion theory requires. We shall first discuss the problem using inversion and then briefly discuss Mascheroni's methods.

There are two fundamental problems to solve:

(1) Determine the points of intersection of a circle and of a line given by two of its points.

(2) Determine the points of intersection of two lines given by two pairs of points.

It is evident that if we invert with regard to an arbitrary point, these problems reduce to finding intersections of two circles, and we need to show that by the compass alone we can draw the inverse figure, for then we can find the intersections and invert back again. To do this we must solve two auxiliary problems.

(3) Construct A', the inverse of A with regard to a given circle o of radius R.

Case 1. $OA > R/2$. Draw the circle with center A and radius AO cutting o at C and D (Fig. 99). Let the circles with centers C and D and radius R meet again at A'. The points O, A, A' are equidistant from C and D and hence on a line. The triangles OAC and OCA' are similar isosceles triangles, so that $OA/OC = OC/OA'$, or $OA \cdot OA' = OC^2 = R^2$.

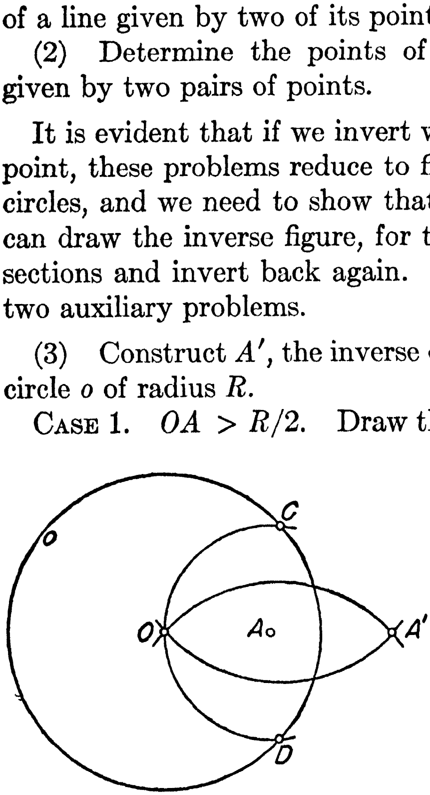


Fig. 99

It is interesting to compare the Lemoine numbers for this construction ($S9$, $E6$) with those for the construction of section 25 ($S19$, $E12$), and note the decided gain as well as the true gain in accuracy.

Case 2. $OA < R/2$. If $OA < R/2$, the above construction fails, but we modify it as follows (Fig. 100). Construct A_1, such that $OA_1 = 2\,OA$, as follows. With center A and radius OA draw the circle; with this same radius strike three arcs in succession beginning with O, and so determine the point diametrically opposite O. If necessary, we may con-

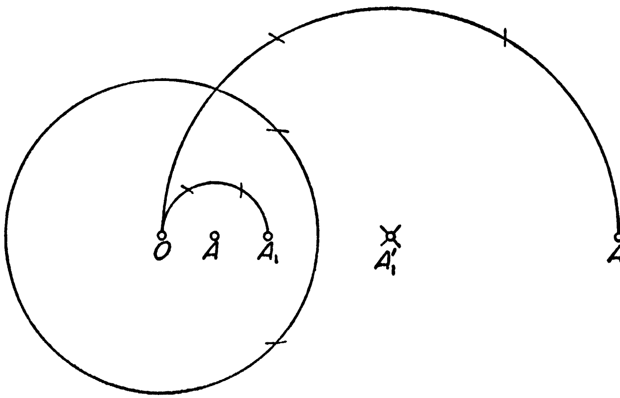


Fig. 100

struct the point A_n, so that $OA_n = 2^n \cdot OA$. We construct the inverse point A_n', and double OA_n' n times to find A'. Since

$$\begin{aligned} OA_n' \cdot OA_n &= R^2, \\ OA_n &= 2^n \cdot OA, \\ OA' &= 2^n \cdot OA_n', \end{aligned}$$

it readily follows that A and A' are inverse points.

(4) Find the center of a circle given by three points on it.

ANALYSIS. Let the given points be A, B, C; call the center of the required circle O, and let AD be a diameter (Fig. 101). If we invert with respect to the circle with center A and radius AB, B inverts into itself; let the inverses of the other points be represented by primed letters. The circle o will invert into the line BC' perpendicular to AD at the point D', and the point O' will be the reflection of A on the line BC' (section 36).

CONSTRUCTION. Draw the circle with center A and radius AB; construct C', the inverse of C with regard to this circle. Find the reflection O' of A upon the line BC' by means of

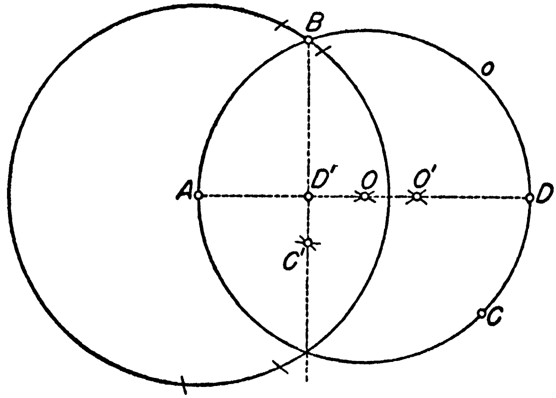


Fig. 101

circles with centers at B and C'. Find O, the inverse of O', the required center. The construction involves the drawing of nine circles with ($S27$, $E18$); the ordinary ruler and compass construction involves ($S12$, $E7$).

113. Examples of compass constructions

It now follows that every construction that can be made by ruler and compass can be made by compass alone. As an illustration consider the following problems, the first of which is used in the solution of the second.

(1) Construct a circle, center C, orthogonal to a circle o.

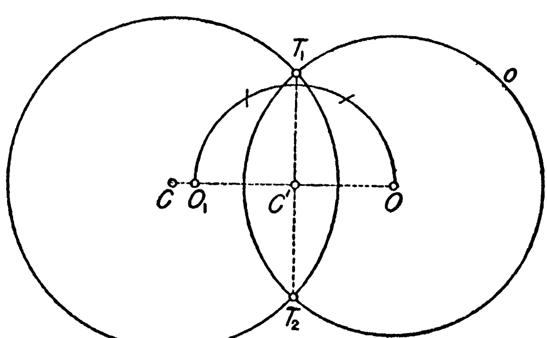


Fig. 102

Analysis. Assume that the construction has been made (Fig. 102) and that the points of intersection are T_1, T_2. If

OC and T_1T_2 meet at C', then C' is the inverse of C with respect to o, and T_1T_2 is the perpendicular bisector of OO_1, where O_1 is the reflection of O on the point C'; and, further, O_1T_1 equals the radius of the given circle.

CONSTRUCTION. Construct C', the inverse of C with respect to o (3 circles). Reflect O on C', to obtain O_1 (4 circles). With center O_1 and radius of the given circle determine T_1. The circle with center C and radius CT_1 is the required circle (2 circles). A total of nine circles is required.

(2) Construct a circle x through the point P, orthogonal to the circle a and tangent to the circle b.

ANALYSIS. Suppose the construction has been made as in Fig. 103. Then since a and x are orthogonal, the circle x

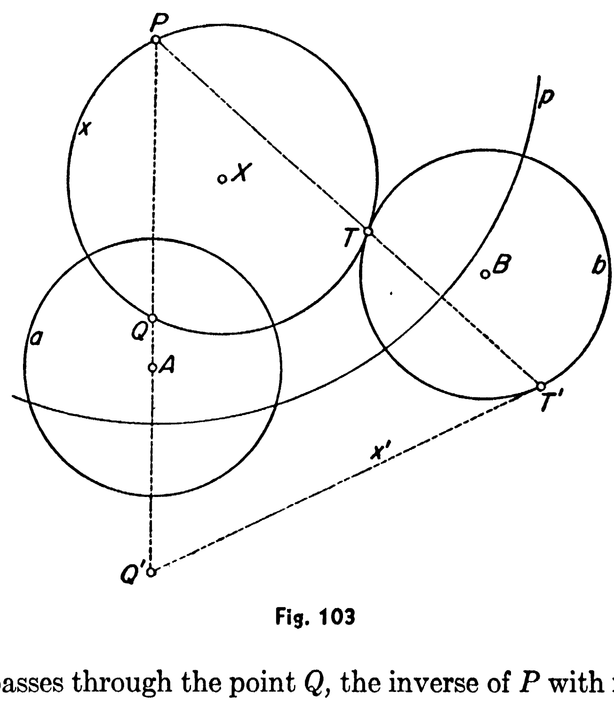


Fig. 103

also passes through the point Q, the inverse of P with respect to the circle a. We use a method similar to the construction given in Fig. 95b. If we invert with respect to the circle p, center P, orthogonal to b, x will invert into a straight line

$Q'T'$ tangent to b. But the point T' lies on a circle, center Q', orthogonal to b. Hence the following construction:

CONSTRUCTION. Construct Q the inverse of P with respect to the circle a (3 circles); construct p, center P, orthogonal to b (9 circles); construct Q', the inverse of Q with respect to p (3 circles); determine the point T', where the circle q', center Q', orthogonal to b, cuts b (8 circles, since the circle q' itself is not needed). Find the inverse T of T' with respect to p (3 circles). The circle through P, Q, T is a required circle (10 circles). The construction involves a total of 36 circles.

There are two points T', and hence two solutions. To obtain both solutions requires at most 13 more circles, but this number may be reduced by using circles already drawn. Of course if P and Q are separated by b, there would be no solution.

114. Mascheroni's method

The constructions indicated by the above theory can often be simplified. The general solution of problem (1), section 112 (to find the intersection of the line joining A and B with the circle c, center C) involves the finding of A', B', the center of the circle $A'B'C$, and the drawing of this circle (16 circles in all). Mascheroni's original construction uses the idea of reflection (a special inversion) and is much simpler (Fig. 104). We let D be any point on c and construct D_1 and C_1, the reflections of D and C upon the line AB, using A and B as centers. The circle with center C_1 and radius C_1D_1 cuts c in the required points; only 5 circles are used in all.

Both of these constructions fail if AB passes through C. In that case different constructions can be found, and the reader who is interested may find elaborations of the whole theme in the books by Hudson, de Lanascol, and Fourrey mentioned in the Bibliography at the end of this text. (See also the solutions of problems 9 and 12 at the end of this chapter.)

Mascheroni's solution of problem (2), section 112 (to find the intersection of two lines) is more complex and depends

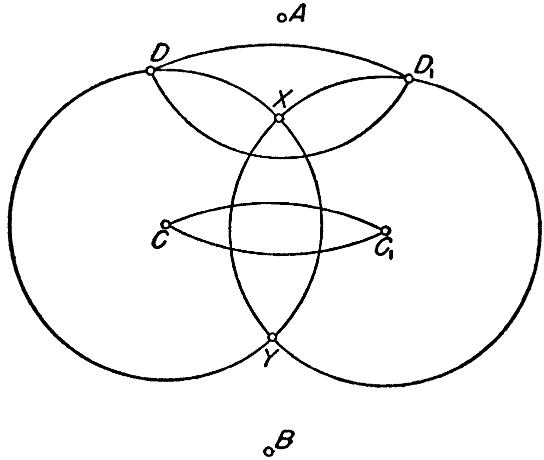


Fig. 104

upon the theory of similar figures. We must first solve the following auxiliary problem:

Construct the fourth proportional to a, b, c; that is, construct d, so that

$$\frac{d}{c} = \frac{b}{a}.$$

Draw two concentric circles with radii a and b and select P and P' arbitrarily on them (Fig. 105). Construct $PQ = c$, and $QQ' = PP'$, selecting Q' so that the triangles OPP' and OQQ' are directly similar. It follows that the isosceles triangles OPQ and $OP'Q'$ are directly similar, so that $d/c = b/a$.

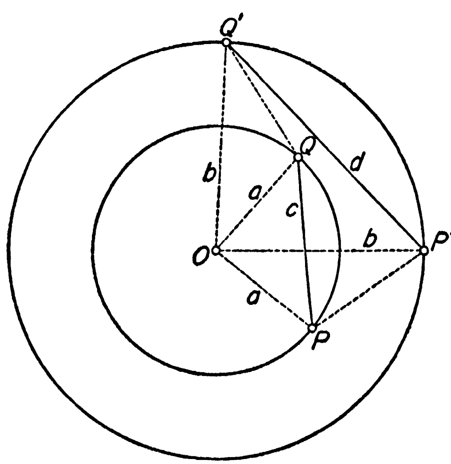


Fig. 105

The construction fails if $c > 2a$, but a doubling process similar to the one used before eliminates this difficulty.

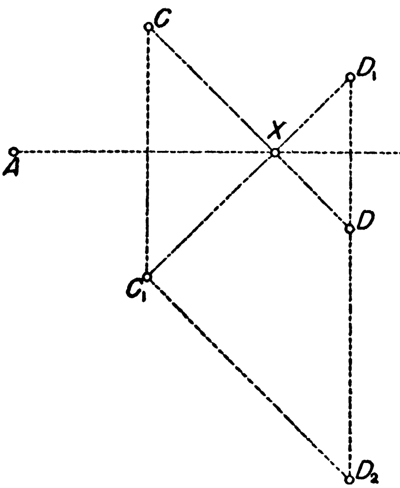


Fig. 106

Let us now seek the intersection X of the lines determined by A, B and C, D. Suppose that C_1 and D_1 are the reflections of C and D upon AB, and that D_2 is the fourth vertex of the parallelogram C_1CDD_2 (Fig. 106), C_1, D_1, D_2 being readily found by compass alone. Then CD and C_1D_1 meet at X, and D, D_1, D_2 are collinear. It follows that $DX = D_1X$ satisfies the proportion

$$\frac{DX}{D_2C_1} = \frac{DD_1}{D_2D_1},$$

so that X can be constructed.

115. Constructions involving imaginaries

It may happen in making ruler and compass constructions that the points of intersection of a line and a circle or of two circles may be imaginary, yet the final result is real. We recognize that the problem of drawing a circle through their imaginary intersections is merely one of drawing a circle of their coaxal system. This of course is accomplished by means of the conjugate coaxal system.

We also recall that a coaxal system of circles determines on its axis an involution. If O is the center of the involution, then $OA_i \cdot OB_i = k$. If k is positive, there is a pair of self-corresponding points which are harmonic conjugates with respect to every pair; if k is negative, the involution defines a pair of conjugate imaginary points on the axis u (Fig. 107). The involution on the axis is uniquely determined by any circle of the conjugate system, and the imaginary points on u are the common points of u and every circle k_i of the conjugate system.

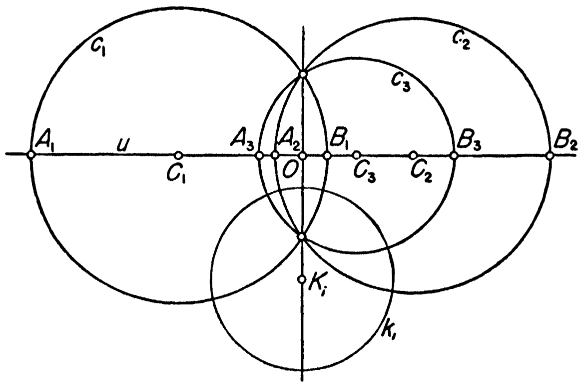


Fig. 107

In the involution on u there is one, and only one, pair of points which are harmonic conjugates with regard to A_1 and B_1. They are the points A_3, B_3, in which a circle of the c system, drawn orthogonal to c_1, cuts u. These ideas and the principles of projection enable us to solve problems of the type indicated.

As an example consider the linear construction of the polar of P given in section 44. We draw two lines cutting the circle in A, B and C, D. The points of intersection Q and R of AC, BD and AD, BC determine the required polar. If the given lines u_1 and u_2 do not cut the circle c in real points, nevertheless the construction may still be construed as valid. We proceed as follows (Fig. 108):

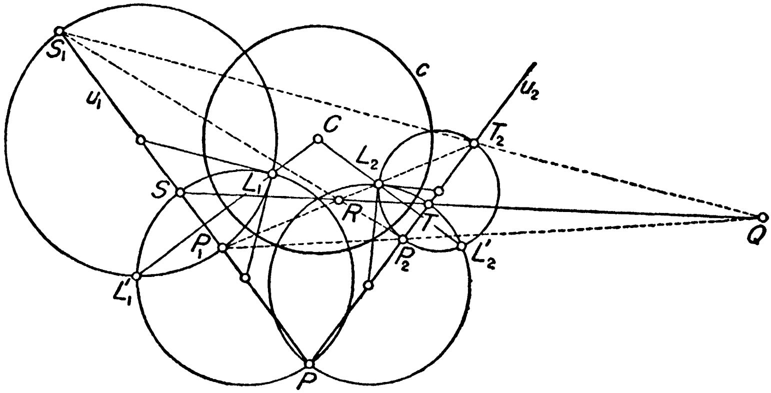


Fig. 108

Let the limiting points of the coaxal system determined by c and u_1 be L_1, L_1'. Draw the circle which passes through L_1, L_1', P, and the circle through L_1, L_1' orthogonal to this circle, thus obtaining the harmonic range (PS, P_1S_1). Proceed similarly for c and u_2, obtaining the harmonic range (PT, P_2T_2). Then the lines ST, S_1T_2, P_1P_2 meet in Q, and ST, S_1P_2, P_1T_2 meet in R due to the invariance of four harmonic points under projection. But Q and R are the desired points, for the involution at Q defines a pair of imaginary lines at Q projecting to PSP_1S_1 as well as to PTP_2T_2, or Q (and likewise R) lies at the intersection of the lines joining the imaginary points of intersection of u_1 and u_2 with c.

116. Prospectus

We have by no means exhausted the geometry of the circle and the triangle. We have said nothing about the extensive subject of the Brocard configurations; there are a host of points and circles and related theorems connected with the triangle which bear the names of famous mathematicians; there are famous construction problems like that of Malfatti, the solution of which is due to Steiner, although the first proof was given by Hart. The problem of making constructions by means of a ruler and one fixed circle has not been discussed.

The extension of our results to space represents a large and none-too-well-known body of material. Much, but not all, of the theory of concurrency and collinearity carries over easily to three dimensions. The geometry of lines in three dimensions is intrinsically more difficult than in the plane, and hence the geometry of the tetrahedron not only contains results which are direct analogues of those for the triangle, but it also contains results which have no planar counterpart. The theory of inversion with respect to a sphere, of pole and polar with respect to a sphere, of coaxal spheres, and problems of construction are all matters which have both direct analogues and distinct properties from the corresponding theory in plane geometry.

The study of geometry on surfaces other than the plane, especially the sphere, has been extensively developed and indeed the study of spherical geometry bridges a large gap between the study of plane Euclidean and non-Euclidean geometries. But these are things the student must now investigate for himself; the literature is extensive. It would be difficult to discover new results in the field which we have covered, but the domain of elementary geometry has by no means been exhausted. It is one branch of mathematics which at intervals throughout the ages seems to rise again and again with a new fascination and a renewed vigor.

Exercise 8—Part 2. Gergonne's Solution. Compass Constructions.

1. Draw a circle through two points cutting a given circle at a given angle.

2. Draw a circle through a point cutting each of two given circles at given angles.

*3. Discuss the system of circles which cut two given circles at given angles.

4. Determine a point A on the radical axis of two given circles such that the tangents from A to the two circles are perpendicular.

5. Use Gergonne's construction to find the circle which is tangent externally to two circles and tangent internally to a third.

6. Poncelet's Construction for the Problem of Apollonius. Let P_1 be any point on c_1 and P_1E_{12} cut c_2 in P_2, where P_1 and P_2 correspond in the inversion from E_{12}. And let P_1E_{13} cut c_3 in P_3, with a similar restriction on P_3. Let the circle $p = P_1P_2P_3$ cut c_1 again at Q_1, and P_1Q_1 meet $E_{12}E_{13}$ in L_1, and let a tangent from L_1 to c_1 have the contact point T_1. Then T_1 is a contact point of a required circle x. Similarly for T_2 and T_3. Justify this construction.

7. Discuss Gergonne's solution for P_1C_2.

8. Discuss Gergonne's solution for $P_1C_1L_1$.

9. If C is the center of a given circle and A any point, by means of compass alone, using the method of inversion, find the intersections of AC with the circle.

10. (a) Give an analysis and make the construction for the following problem. Enumerate the number of circles needed at

each step: Construct by compass alone a circle through a point orthogonal to two given circles.

(b) Will your construction work if the point is on one of the given circles? If not, how could you adjust it?

11. Solve as in problem 10 this problem: Construct by compass alone a circle through a point P, orthogonal to the circle c, and having its center on a line given by two points A, B.

12. Discuss Mascheroni's method of bisecting an arc PQ: Let C be the center of the circle and complete the parallelograms $PQCQ_1$ and $CPQP_1$; construct E so that $Q_1E = P_1E = QQ_1$. Then the circle, center Q_1, radius CE, passes through the required point.

***13.** Discuss the following method of solving the problem of Apollonius: Let a_{12} be a circle with respect to which c_1 and c_2 are mutually inverse, and let a_{13} be defined similarly. The circle orthogonal to a_{12} and a_{13} tangent to c_1 is the required circle. (Note that a_{ij} need not be real.)

***14.** If a line u_1 cuts the circle c_1 in A_1, B_1, and if u_2 cuts the circle c_2 in A_2, B_2, the lines A_1A_2, B_1B_2 and A_1B_2, B_1A_2 meet in Q and R. If the points A_1, B_1, A_2, B_2 are imaginary, the points Q and R are still real; construct them.

KEY TO THE SOLUTION OF THE EXERCISES

Suggestions to the student

There is no one general method or procedure for solving problems in geometry, but certain suggestions may be made. (1) A figure should *always* be drawn. Sometimes it may be a rough sketch, but very often accurately drawn diagrams will offer possible clues to the solution. This is especially true in construction problems. (2) Unless the method of attack is straightforward and directly apparent as in exercises of an algebraic nature, try to make a clear analysis of the problem, studying how we might prove (or construct) the required result if we could prove (or construct) statement A; how statement A would follow if we could prove statement B, and so on. In the main, the problems have been arranged to correspond directly with the text, and hence the tools available should be kept clearly in mind. (3) When a correct analysis has been made, make a direct synthesis of the problem. Put all steps in logical order; see that all statements of the hypothesis have been used; be careful not to confuse a theorem with its converse. Examination of the details may lead to consideration of special cases.

The key

It is not the purpose of this key to give a complete analysis or solution of any given problem, but rather to indicate the salient points of the argument. In all cases it is expected that the student will actually supply the missing details. Nor is it to be inferred that the solution indicated in this key is always the only one or the best. The student is strongly urged to find alternative solutions. Where the solution is too obvious or too mechanical, comments have been omitted.

The real test of the student's ability in geometry is the solution of problems with a minimum amount of help from this key.

Exercise 1

1. As X moves from ∞ in the direction of A to B, r changes from -1 to 0, through positive values to ∞, from $-\infty$ to -1. $k = r/(r+1)$.

2. On an arbitrary line through A lay off a equal segments to C, and from C lay off b of these segments, in the direction of AC or CA, according to sign, to point D. Then the line through C parallel to DB determines the required point. Prove by similar triangles.

3. $AB = b - a$, and so forth. Substitute and expand.

4. If the bisector is PX, draw the line through A parallel to PB cutting PX in B'. Triangle PAB' is isosceles, and triangles $AB'X$ and BPX are similar. The converse is proved by using the steps in reverse order or by the method of *reductio ad absurdum.*

5. Let the circle ABC cut AD and BD in X and Y. From a consideration of arcs, show that the arc $XY = 0$.

6. Concyclic property shows equality of angles.

8. O must be between A and B. The balance of the figure is constructed precisely as before.

9. Let A be a fixed point and P a variable point on the given line, and let A' and P' be the transformed points. Prove that $A'P'$ is parallel to AP independent of the position of P.

10. Use the parallel property of problem 9.

11. (a) Use problem 10. (b) Consider an inscribed triangle ABC with A and B fixed and C variable.

12. (a) Let AA' and BB' meet at O; AA' and CC' at O'. Show by means of similar triangles (3 sets) that O and O' coincide. (b) $-1/2$; intersection of the medians.

13. Make triangle $A'B'P'$ homothetic (problem 12) to ABP with P given and A, B, and A' arbitrary.

14. If ABC is the first triangle, XYZ the second, construct $X'Z'$ parallel to XZ and complete triangle $X'Y'Z'$. Expand from B.

15. The sums of the pairs of opposite angles are equal for both cases if angles are measured in the proper manner. For example, angle C = angle DCB.

16. The sums or differences of the pairs of opposite sides are equal according as the quadrilateral is nonre-entrant or re-entrant. We indicate the details for the second case. Suppose the quadrilateral is $ABCD$, with positive sense assigned to each line as we proceed around the quadrilateral. Let the points of contact of successive sides be X, Y, Z, U. Then $AB - CD = (AX + XB) - \cdots = (UA + YB) - \cdots = (DU + UA) - \cdots$, and so on.

17. (a) Compare problem 3; take D as the origin to simplify calculations. (b) Apply the theorem of Pythagoras and then case (a), using the foot of the perpendicular from D to the line.

18. See reference given. For a statement of theorems without proof consult Johnson, *Modern Geometry*, pages 26–27.

Exercise 2—Part 1

1. Disregarding signs, $A_2B_1/B_1A_3 = d_2/d_3$, and so on, where the d's represent the lengths of the perpendiculars.

2. Four sets of similar triangles are involved.

3. Let $A_iB_i = b_i$, and the sides opposite A_i be a_i; then the left member of equation (C), section 11, equals -1.

4. The angles involved are congruent modulo 180° in pairs.

5. Cut by a transversal and apply Euler's Theorem (problem 3, exercise 1). From the sine law show $AB = ab \sin ab/h$, where $a = OA$, $b = OB$, and h is distance from O to the transversal, and so on.

6. Ceva's theorem with the conditions that (P_1, Q_1, P_3, Q_3), (P_1, Q_1, P_2, Q_2), (P_2, Q_2, P_3, Q_3) are concyclic.

7. Use problem 3, expressing the angles involved in terms of the angles of the triangle.

8. If the circles have centers A_1, A_2, A_3 and the points of contact are B_3 and B_2, with B_3B_2 meeting A_2A_3 in B_1, then from Menelaus' Theorem and

$$A_1B_3/B_3A_2 = \pm a_1/a_2, \quad A_1B_2/B_2A_3 = \pm a_1/a_3$$

(the signs depending on the type of contact), we can show that B_1 is a homothetic center.

9. If $A_1A_2 \ldots A_n$ is a simple polygon, and if a transversal cuts the side A_kA_{k+1} in B_k, then

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \ldots \cdot \frac{A_nB_n}{B_nA_1} = (-1)^n. \qquad (1)$$

Prove by mathematical induction. Assume another vertex A_{n+1}, and apply (1) and Menelaus' Theorem. The condition (1) is *not* sufficient. A correct converse is: if (1) is true and if $n - 1$ of the points B_k are collinear, then the n points are collinear.

10. Use A_3Q as a transversal of the triangle $A_1A_2M_1$.

11. Use OO' as a transversal of the triangles PBB' and PCC'.

12. Apply Menelaus' Theorem to triangle $A_1B_1A_3$ and transversal A_2B_2, to triangle $A_1B_1A_2$ and transversal A_3B_3, and add after making adjustments in signs. Write in the form

$$\frac{A_1C}{CB_1} \cdot \frac{B_1A_3}{A_2A_3} = \frac{A_1B_3}{B_3A_2}, \quad \frac{A_1C}{CB_1} \cdot \frac{A_2B_1}{A_2A_3} = \frac{A_1B_2}{B_2A_3}.$$

13. Using the notation of section 13, and letting $AX = x$, $AX' = x'$, and so forth, consider the sets (x, y, z), (x, y', z'), (x', y, z'), (x', y', z) and apply Ceva's Theorem.

14. The triangles $A_1B_1A_3$ and $A_2B_1A_1$ are similar. From the ratios of the *three* pairs of corresponding sides, the suggested equation follows. Two other equations follow similarly. Or from a consideration of arcs, $a_1b_3 \equiv \alpha_1$, $b_3a_2 \equiv \alpha_2$, and so on.

15. Besides the given suggestion, use the fact that the product of the segments of a secant to a circle is constant.

16. Follow suggestion. Indirect method for converse, or the converse may be proved in the following dual fashion. Let the sides of the triangles be a, a', b, b', c, c', so that a and a', b and b', and c and c' meet in three points on the line o. Let $AA' = p$, $BB' = q$, $CC' = r$. Take $A' = b'c'$ as the Ceva point of the triangle obc, C' for abo, B' for aoc. Then

$$\frac{\sin cp}{\sin pb} \cdot \frac{\sin bb'}{\sin b'c} \cdot \frac{\sin oc'}{\sin c'c} = 1,$$

$$\frac{\sin br}{\sin ra} \cdot \ldots = 1,$$

$$\frac{\sin aq}{\sin qc} \cdot \ldots = 1.$$

The product gives the condition that p, q, r be concurrent.

Exercise 2—Part 2

2. Project (DB, OP_∞) from C upon the line, thus giving $(GF, AH) = -1$, with $AH = 2AE$.

3. Draw the circle $PP'Q$; the circles are orthogonal; therefore the circle $PP'Q$ passes through Q'.

4. Let the points be P and Q, and let P' be such that P and P' divide the diameter through P harmonically; then the circle $P'PQ$ is the required circle. One solution.

5. Let P_i, $(i = 1, 2)$, be such that P and P_i divide the diameter PO_i harmonically; the circle PP_1P_2 is the required circle.

6. The supplementary property follows from the fact that the radius is perpendicular to the tangent; the equality of angles, from the congruent triangles CAD, CBD. If $\measuredangle A = \measuredangle B = 90°$, then $\measuredangle C + \measuredangle D = 2(\measuredangle X + \measuredangle Y)$, and conversely.

7. Geometric Proof: Consider the circles on AB and CD as diameters. If $(AB, CD) = -1$, the triangle MXN is a right triangle, and $MX^2 + NX^2 = MN^2$ and conversely, with $MX = MA$, and so on.

Analytic Proof:
$MA^2 = MC \cdot MD = (MN - CN)(MN + CN) = MN^2 - NC^2$. Transpose. For converse use *reductio ad absurdum* or retrace steps.

8. From Problem 7, with M and N the midpoints of A_1A_2 and B_1B_2, $MN^2 = MA_1^2 + NB_1^2$. From right triangles we have

$$MN^2 = d^2 - (p - q)^2, \quad MA_1^2 = a^2 - p^2, \quad NB_1^2 = b^2 - q^2,$$

giving

$$2pq = a^2 + b^2 - d^2.$$

If the two circles are orthogonal, $a^2 + b^2 = d^2$, so that $pq = 0$, and at least one of the circles has the line as a diameter.

9. Intersection of the circles of Apollonius determined by the ratios $k_1 = AB/BC$, $k_2 = BC/CD$.

10. (a) Implied in the definition; see section 8. (b) From similar triangles; direct theorem and converse required for problem 10b.

11. $\underline{A \quad C \; B \; O \quad D}$ From $AC/CB = k/1$, we find

$$AC/AB = k/(k + 1).$$

Similarly,

$$AD/AB = -k/(-k + 1).$$

$$AO = (AC + AD)/2, \quad CO = r = CA + AO.$$

12. See Johnson, page 152; Durell, page 118; or Graustein, page 81. Other proofs are given later in this text.

Exercise 3—Part 1

1. Tangent circles; both through O with centers on the same line.

2. Reverse the steps of section 26.

4. (1) By congruent triangles, BD parallel to AC; then from the given ratios, OP is parallel to BD, and OP' to AC.

(2) $OP/BD = k, \quad OP'/AC = 1 - k,$

giving

$$c \cdot OP \cdot OP' = AC \cdot BD,$$

where c is a constant depending upon k. Let the perpendicular from D to AC give E, and let $DE = h$. Then

$$h^2 = b^2 - AE^2 = a^2 - CE^2,$$

so that

$$b^2 - a^2 = AE^2 - CE^2 = (AE - CE)(AE + CE) = AC \cdot BD.$$

5. $p = PC^2 - r^2$; PC constant; concentric circle.

8. O is the end of the diameter perpendicular to the line. If the line cuts the circle, there are two solutions with circles of inversion through the common points. If the line does not cut the circle, there is one solution with the radius of inversion found by construction of a mean proportional.

9. From similar triangles show that $OP'/P'C'$ is constant.

Exercise 3—Part 2

1. Angles OPQ and $OP'Q'$ are equal with P and Q tending to coincidence.

3. (a) Not inversely similar since angle POR is not zero.

(b) If PQR is inscribed in a circle concentric with the circle of inversion, so is $P'Q'R'$, and suggested equation and corresponding equation in primes are established. Conversely, if corresponding angles are equivalent in direction and magnitude, then from section 32, the suggested equation and concyclic property follow.

4. O lies on a circle subtending a fixed angle on the segment PR; also on another circle subtending a fixed angle on the segment PQ. For details of construction of such circles, consult a text on plane geometry.

5. (a) Equation (5) of section 32 using Q and S. (b) Apply condition for concylic points (section 3).

6. Let a_{12}, b_{12} be the circles of antisimilitude of c_1 and c_2; a_{13}, b_{13} of c_1 and c_3. Any of the eight points they determine will suffice. Any such point is also on a circle of antisimilitude of c_2 and c_3, so that there are not more than eight. Some of the points may not be real, and some may be coincident.

7. $P'Q' = OQ' - OP' = r^2/OQ - r^2/OP$, and so on.

8. After inversion with respect to a point on the arc DA,

$$A'B' \cdot C'D' + A'D' \cdot B'C' = A'C' \cdot B'D',$$

where all segments are written positively. Apply section 37(a).

9. After inversion,

$$DA'^2 \cdot B'C' + DC'^2 \cdot A'B' = DB'^2 \cdot A'C' + B'C' \cdot A'C' \cdot A'B',$$

where all segments are written positively. Replace DA' by r^2/DA, and $B'C'$ by $(r^2 \cdot BC)/(DB \cdot DC)$, and so on, where now all distances are undirected.

10. Invert with respect to A. In the inverse figure the triangle $B'C'D'$ has a right angle at B', so the circle $B'C'D'$ is orthogonal to the line $C'D'$. Invert back again.

11. Intersection of two of the circles of similitude of the circles taken in pairs; equivalent to $PC_1 : PC_2 : PC_3 = r_1 : r_2 : r_3$. The segments $r_1 \cdot c_3c_2$, $r_2 \cdot c_3c_1$, $r_3 \cdot c_1c_2$ must form a triangle.

12. First proof: Invert with respect to a point of intersection. Second proof: Invert with respect to one of the circles of antisimilitude.

13. $1 + 3 \geqq x$, $1 + x \geqq 3$, $3 + x \geqq 1$ have simultaneous solutions for positive x, if $2 \leqq x \leqq 4$. For equal signs there is one solution; for inequality signs, two solutions.

14. If $x = A_1A_3$ (a positive number), then 1, x, 1 must form a possible triangle and so must 1, $3x$, 4. The first implies that x is less than 2; the second gives for real solutions, $1 + 3x \geqq 4$ or $5 \geqq 3x$, according as to how the sides are paired. Hence for $x = 1$ or $5/3$ there is one solution, and in the diagram the circles

of Apollonius meet on the circumcircle of the triangle. For x between 1 and 5/3 there are two solutions and outside that range none.

Exercise 4—Part 1

1. (a) Pencils are equiangular. (b) One line of the pencil is the tangent to retain the angle property.

2. Let the conjugate lines be a and b, meeting at C; let a cut the circle at U and V, and b cut the circle at X and Y. Then (section 47), $(BC, UV) = -1$, and these points project to X in four harmonic lines with the line BX as the tangent. Hence (XY, UV) are four harmonic points on the circle.

3. (1) PR and QN are conjugate lines. (2) Project upon PR, to T, and then upon QN, recognizing that QN is parallel to TP.

4. (1) $\measuredangle PCQ = 1/2\measuredangle ACB = (180° - T)/2$.
(2) Let the four tangents determine $(P_1P_2, P_3P_4) = -1$. Then

$$(P_1P_2, P_3P_4) = (p_1p_2, p_3p_4) = (q_1q_2, q_3q_4) = (Q_1Q_2, Q_3Q_4) = -1,$$

where $p_1 = CP_1$, and so on. (3) One point of the range is the point of tangency.

5. Polarize the solution of problem 2, or draw the tangent at one of the points of contact and apply problem 4.

6. Let PC_1 cut c_1 in X and Y and c in R. Then $(PR, XY) = -1$, and $\measuredangle R = 90°$, so that $RQ = p$; conversely, if $p = RQ$, then $\measuredangle R = 90°$, PR is a diameter of c_1 with $(PR, XY) = -1$, so that c and c_1 are orthogonal.

7. Apply problem 6 (direct and converse statements).

8. The locus is the circle on C_1C_2 as diameter; prove a theorem and its converse.

9. They are the points which are harmonic conjugates with respect to two pairs of points.

10. Let the center of the circle be O, the points A and B, with inverses A' and B', and let the distances be AC and BD; let the feet of the perpendiculars from A to OB and B to OA be A_1 and B_1, respectively. Then

$$OA \cdot OA' = OB \cdot OB', \quad OA \cdot OB_1 = OB \cdot OA_1.$$

Subtract and use $BD = B_1O - A'O$, and so forth.

11. Let the given triangle be ABC (abc), and the center of the required circle o be O. Then AO and CO are perpendicular to a and c. If AB cuts the required circle in X, Y, then $(AB, XY) = -1$, and o is orthogonal to the circle on AB as diameter, and hence may be constructed. For a real solution, O must be outside the triangle, so the triangle must be obtuse, in which case there is a unique solution.

12. Let the diameter OC cut the circle in A, B, and the polar of O in D. Then $(OD, AB) = -1$. After inversion $(O'D', BA) = -1$ with O' at infinity, so that $D' = C$. Hence we can state the theorem concerning the inverse of the center of the circle.

Exercise 4—Part 2

1. Trilinear polar of the incenter. The trilinear polars of the excenters are the lines joining the intersections of the internal bisectors of the angles with sides.

2. Since Q_1, P_2, P_3 are collinear, A_1P_1, A_2Q_2, A_3Q_3 are concurrent at its trilinear pole (Refer to Fig. 40). Similarly A_1Q_1, A_2P_2, A_3Q_3 meet at the trilinear pole of $P_1Q_2P_3$, and A_1Q_1, A_2Q_2, A_3P_3 meet at the trilinear pole of $P_1P_2Q_3$.

3. Use the trigonometric form of Ceva's Theorem for (**a**), of Menelaus' Theorem for (**b**).

5. For a line we get a conic through the three vertices; the tangents at the vertices are also easily obtained; the points at infinity on the conic are the transforms of the points where the line meets the circumcircle. For a circle through two vertices we get another circle through the same vertices.

6. For statement replace *isogonal* in problem 3 by *isotomic*. In the proof use Ceva's and Menelaus' Theorems.

The correspondence is unique except for points on the triangle of reference. The median and ex-median points are invariant. The line at infinity corresponds to a certain ellipse which circumscribes the triangle of reference and is inscribed in the ex-medial triangle. For further study consult Sommerville, *Analytical Conics*.

Exercise 5—Part 1

1. (**a**) Bisect two common tangents. For one construction of the common tangents see section 5. (**b**) Failure occurs when c_2 is within c_1. (**c**) Common tangent is tangent from the point. (**d**) Point is one limiting point; harmonic property determines the other.

2. (a) If points are P and P' and tangents meet at T, triangle PTP' is isosceles due to invariance of angle under inversion. (b) By means of parallel radii construct a pair of inverse points (using either homothetic center) and the tangents at them.

3. (a) $d_1 = (r_1^2 - r_2^2 + d^2)/(2d)$; (b) as d tends to 0, d_1 tends to ∞.

4. (a) See problem 6, exercise 4—part 1; (b) State in terms of the conjugate system. Proof similar to that in (a).

5. If P_i' is the inverse of P, it lies on the circle k of the conjugate system which passes through P. Conversely, C_iP cuts k in inverse points.

6. Same proof for P inside or outside circle. In the equation, $NA \cdot NB = NP \cdot NO$, where AB is a diameter of c, express all distances using O as origin.

7. (a) The points A, P, A', P' are concyclic, and hence AP meets $A'P'$ at a point R having equal powers with regard to c and c'. (b) Apply (a) to PA, PB, PC, and PD. (Proof due to M. E. Fouyé.)

8. The given line cuts the line of centers in required center.

9. For the circle of the intersecting system, construct R the radical center of L_1, L_2 and the given circle g. RL_1 is required radius. For circle of nonintersecting system, construct S, the radical center of the intersecting system and g. The circle whose center is S and which is orthogonal to g is the required circle.

10. If the line g cuts the radical axis in P, the circle with center P of the conjugate system cuts g in the required points of contact, and perpendiculars at these points determine the centers. Two solutions are possible.

11. Construct the circle k of the conjugate system orthogonal to the given circle g (problem 9). If k and g intersect at P and Q, these are the points of contact; PG and QG determine the centers.

Exercise 5—Part 2

1. System invariant. In (1) circles interchanged in pairs; in (2) each circle invariant.

2. System inverts into one of the same type, except when O is point of contact, when we obtain a set of parallel lines.

3. Circle on common tangent as diameter belongs to the conjugate system and hence passes through the limiting points.

4. Use problem 3. Converse: If the centers of three circles are collinear and if a common tangent of two of them is divided harmonically, the circles are coaxal. Proof essentially retraces steps.

5. $(OO', AB) = -1 = (OP, A'B')$ by orthogonal projection, where P is on the circle of similitude and A' and B' are the points of contact. Circles may have no common tangent.

6. (a) Radical circle passes through the six points. (b) Radical circle is orthogonal to c_i and c_j, and hence to circle of antisimilitude a_{ij}; centers of a_{ij} are collinear.

8. Circles invert into equal circles which are the reflections of each other on the inverse of the circle of antisimilitude or c_1', c_2' have s' as radical axis. Invert back again.

9. Apply Casey's Power Theorem directly.

10. (a) Circles are coaxal, so that $p_1/p_2 = k = C_1O/C_2O = \pm r_1/r_2$, $+$ if O is external, $-$ if O is internal homothetic center (see section 63).

(b) If two circles of antisimilitude intersect, then $p_1/p_2 = \pm r_1/r_2$, $p_1/p_3 = \pm r_1/r_3$. This implies $p_2/p_3 = \pm r_2/r_3$, showing that their common points lie on a third circle of antisimilitude. Method fails if circles of antisimilitude do not intersect unless we appeal to the *principle of continuity.*

11. Construct the circle of the coaxal system which determines the involution that is orthogonal to the circle having A_1A_2 as diameter.

Exercise 6—Part 1

1. First proof: Let the centers of the circles be O_i; O_1O_2 and O_2O_3 are perpendicular bisectors of DC and DA, respectively. Hence $\measuredangle\, O_3DO_1 = \measuredangle\, O_3O_2O_1$. Second proof: Ptolemy's theorem can be used.

2. Analysis: Extend A_1G to G' making $AG = GG'$. Then diagonals of A_2GA_3G' bisect each other making it a parallelogram, so that triangle A_2GG' has sides equal to 2/3 of the given medians and is uniquely determined. From it construct $A_1A_2A_3$.

3. (1) From triangle A_2GA_3, $2m_2/3 + 2m_3/3 > a_1$, with similar relations for a_2 and a_3; add. (2) Complete the parallelogram of which $A_1A_2A_3$ is half; $2m < a_2 + a_3$, and so on.

4. (a) Parallelograms $A_1A_2A_3G_i$ show that two exmedians and a median are concurrent at each G_i (exmedian point), and

(b) hence their isogonal conjugates are also concurrent (exsymmedian points).

5. (b) Consider the isogonal conjugate of the pencil at a vertex in part (a).

6. From 4(b), show (A_1L_1, KK_1) is a harmonic range; project to O_1 and cut by transversal A_1H_1, proved parallel to O_1K_1.

7. (1) $A_1X_2' = A_1X_3'$, (2) $A_1X_2' = a_3 + A_2X_1'$, (3) $A_1X_3' = a_2 + A_3X_1'$. One half of (2) plus (3) gives $A_1X_2' = s$; this combined with (2) gives $A_2X_1' = s - a_3$.

8. First solution: The bisectors at A_2 and the perpendiculars to them form a rectangle whose diagonal (1) passes through O_3 (2) parallel to A_2A_3. Second solution: Parallelism alone can be used, proving angle relations from concyclic quadrangles, and so forth. Third solution: Menelaus' Theorem also can be applied.

9. X_2X_3 is the polar of A_1, and $A_1A_1'I$ is perpendicular bisector of X_2X_3. Triangle $A_1'A_2'A_3'$ is homothetic to $X_1X_2X_3$ with ratio 1/2.

10. In first proof replace I by I_1, X_3 by X_3', r by r_1. In proof by inversion, invert with respect to excircle.

11. (1) The altitudes become perpendicular bisectors of the new triangle. (2) See section 12.

12. Besides relations given we use $\Delta = rs = a_1a_2a_3/(4R)$, and so forth.

13. If P_1 and P_2 are on the same side of a line and P_1' is the reflection of P_1, and if $P_1'P_2$ meets the line at C, then $P_1C + CP_2$ is the shortest path from P_1 to P_2 which meets the line, and the line is the external bisector of the angle from P_1C to P_2C. Hence if $P_1P_2P_3$ is inscribed in $A_1A_2A_3$, and if a minimum exists, the sides a_i bisect the external angles of $P_1P_2P_3$ if it is to have a minimum perimeter. If the triangle is acute, the existence of a minimum is intuitively evident.

We have proved that the pedal triangle of H has this bisection property. To complete the proof we must show, conversely, that a triangle having this bisection property is the pedal triangle of H. To do this we first prove that the equal angles must be the angles of the triangle $A_1A_2A_3$. This involves the angle sum of four triangles, $A_1P_2P_3$, and so on. Next we construct perpendiculars at P_1 and P_3 to form a new triangle $A_1'A_2A_3'$, with a new pedal triangle $P_1P_2'P_3$, and then prove by angle relations that they coincide with the old ones.

Exercise 6—Part 2

1. A circle with center O_1 and radius $=$ 1/2 circumradius. First solution: $r = R/2 = a_1/(4 \sin \alpha_1)$; all nine-point circles go through O_1. Second solution: G describes a circle homothetic to circumcircle with respect to O_1, ratio 1/3; N describes a circle homothetic to g with ratio 3/2, placing center at O_1 and radius $= R/2$.

2. The sixteen circles are the incircles and excircles of the four triangles $A_1A_2A_3$, A_1A_2H, A_2A_3H, A_3A_1H.

4. $A_1A_2A_3$ is the pedal triangle of I for the triangle $I_1I_2I_3$. Hence the circumcircle bisects the six segments I_iI_j, II_i. (Compare section 76.)

5. The triangles $O_1O_2O_3$ and $A_1O_3O_2$ are homothetic with respect to M_1, the midpoint of O_2O_3. The tangent is perpendicular to M_1N_i.

6. If angle $A_1 = 90°, A_1 = H = H_2 = H_3$, while the tangent at A_1 meets A_2A_3 in the harmonic conjugate of H_1, which makes the tangent the polar axis. The circles o and n are homothetic with regard to A_1; the polar circle is the point A_1; and properties (a), (b), (d), (e) of section 87 now follow.

7. The circles o and n do not intersect; properties (a), (b), (d) of section 87 hold; the polar circle is imaginary.

8. $\bar{t}_{41} = a_3 - a_2$, $\bar{t}_{42} = a_3 - a_1$, $\bar{t}_{43} = a_2 - a_1$; $t_{23} = a_2 + a_3$, and so on (section 74). External contact with c_1, c_2, c_3; internal contact with c_4.

9. $(SA_1, II_1) = -1 = (SH_1, X_1X_1')$. $\measuredangle O_1O_3, t = \measuredangle A_1A_3, t = \measuredangle O_3O_2O_1 = \alpha_2 = \measuredangle A_1A_3, t_4$. Hence t_4 is perpendicular to O_1N. Invert with regard to circle with center O_1, radius $(a_3 - a_2)/2$.

10. If the sides are labeled in order of magnitude a_1, a_2, a_3, then the criterion can be written

$$\pm a_1 \cdot O_1X_1^k \pm a_2 \cdot O_2X_2^k \pm a_3 \cdot O_3X_3^k = 0,$$

where the superscript k is to be omitted for I, and replaced by $'$, $''$, $'''$, for I_1, I_2, I_3, respectively. For I the middle sign is $-$; for I_1 the middle sign is $-$; for I_2 the third sign is $-$; for I_3 the first sign is $-$. For other relations see (3) of section 74.

11. Ptolemy's First Theorem.

12. Since the powers of O_i with regard to k and P are $O_iX_i^2$ and O_iP^2, we have

$$O_1P : O_2P : O_3P = O_1X_1 : O_2X_2 : O_3X_3.$$

From Ptolemy's Theorem

$$O_1P \cdot O_2O_3 \pm O_2P \cdot O_1O_3 \pm O_3P \cdot O_1O_2 = 0,$$

and hence the direct theorem. Reversing the steps proves P, o, and k coaxal, but this can happen only if o and k are tangent.

Exercise 7

2. First solution: Determine P from two equations of the type

$$\measuredangle\, A_2PA_3 \equiv \measuredangle\, A_2A_1A_3 + 60°.$$

Second solution: Construct any equilateral triangle inscribed in the given triangle by the method of homothetic figures and find its Miquel point.

3. From similar triangles (section 51), we have

$$A_3P_1 \cdot A_3Q_1 = A_3P_2 \cdot A_3Q_2, \qquad A_2P_1 \cdot A_2Q_1 = A_2P_3 \cdot A_2Q_3,$$

so that P_1, Q_1, P_2, Q_2 and P_1, Q_1, P_3, Q_3 are concyclic. If these circles do not coincide, P_1Q_1 is their radical axis, which must contain the point A_1, since $A_1P_2 \cdot A_1Q_2 = A_1P_3 \cdot A_1Q_3$. This is impossible, so that the six points are concyclic. Or we may prove the circles $P_1Q_1P_2Q_2$ and $P_1Q_1P_3Q_3$ are identical by showing each has its center at R, the midpoint of PQ, and radius RP_1.

4. Let the feet of the perpendiculars in cyclic order be P_1, Q_1, P_2, Q_2, P_3, Q_3. Since H_3, P_3, Q_2, H_2 are concyclic,

$$A_1P_3 \cdot A_1H_3 = A_1Q_2 \cdot A_1H_2.$$

From similar triangles

$$A_1Q_3/A_1H_3 = A_1H_1/A_1H = A_1P_2/A_1H_2.$$
$$A_1P_3 \cdot A_1Q_3 = A_1P_2 \cdot A_1Q_2,$$

and so on. The rest of the proof follows as in problem 3.

5. $\measuredangle\, P_2P_1P = \measuredangle\, P_2A_3P = \measuredangle\, A_1P_1'P$.

6. If A_1P is parallel to A_2A_3, then A_1, O, P_1' are collinear.

7. If $P_1 = H_1'$, then A_1 and P_1' coincide, and in the interpretation of the equation in problem 5, A_1P_1' must be considered the tangent.

8. $\measuredangle\, B_3P_2P_1 = \measuredangle\, B_3A_1P_1 = \measuredangle\, B_3A_1A_2 + \measuredangle\, A_2A_1P_1$.

$$\measuredangle\, B_3A_1A_2 = 1/2 \text{ arc } B_3A_2,$$

$$\measuredangle\, A_2A_1P_1 = \measuredangle\, A_3B_3B_2 \text{ (by perpendiculars)}$$

$$= \measuredangle\, A_3A_2B_2 = \measuredangle\, B_1A_1A_3 = 1/2 \text{ arc } B_1A_3.$$

Therefore

$$\measuredangle\, B_3P_2P_1 = 1/2 \text{ arcs } (B_3A_2 + B_1A_3) = \measuredangle\, B_3B_1,\, A_2A_3,$$

so that the lines are parallel.

9. Let the three points be B_i and their Simson lines be b_i. Then the angle at $B_i = 1/2$ arc $B_jB_k =$ angle b_jb_k.

10. Let the given lines meet at P; let the isogonal conjugate of P be Q; and let A_iQ cut the circle again in B_i. These are the required points, for the Simson line of B_i is perpendicular to A_iP. If $P = H$, then $Q = O$, A_iB_i is a diameter, and the Simson lines are the sides a_i. If $P = O$, then $Q = H$, and $B_i = H_i'$. See problem 7.

Exercise 8—Part 1

1. The circles tangent to two tangent circles consist of two sets, (1) those belonging to their coaxal system, (2) those orthogonal to their nonzero circle of antisimilitude.

2. Let the center of a be A, OT be tangent to a, and the intersection of b_1 and b_2 be P. Then $OA = (r_1 + r_2)/2 = B_1P$; $OB_1 = (r_1 - r_2)/2$; so that the triangles POB_1 and OAT are congruent.

3. Assume that P, c_1, c_2 and the centers C_1 and C_2 are given. Invert with respect to the circle p, center P, and orthogonal to c_1. To construct p requires ($S16$, $E10$). Invert c_2 into c_2'; obtain c_2' from the inverses of the points A_2, B_2, where PC_2 cuts c_2; the drawing of PC_2, the inverses A_2', B_2', the bisection of $A_2'B_2'$ and the drawing of c_2' total ($S51$, $E32$). Draw a common tangent to c_1 and c_2'. We do not require the tangent but merely the points of contact. To decrease one circle and find a point of contact of the tangent from a point to a circle and extend to the circles c_1 and c_2' to obtain T_1' and T_2' requires a total of ($S31$, $E21$). PT_1' cuts c_1 again at T_1; PT_2' cuts c_2 in T_2; C_1T_1 and C_2T_2 meet at X, the center of the required circle; the radius is PX. These details add ($S15$, $E10$). Hence these particular constructions total ($S113$, $E73$) to obtain one circle through P tangent to c_1 and c_2. These numbers may be modified considerably by variation of the details.

4. We construct Q so that $O_1P \cdot O_1Q = O_1A_1 \cdot O_1A_2$, where O_1 is the internal homothetic center and A_1 and A_2 are mutually inverse points with respect to O_1. The circle through P and Q tangent externally (internally) to c_1 will be internally (externally) tangent to c_2.

5. No essential modifications.

6. (a) Bisector ($S9$, $E5$), reflection ($S6$, $E3$), (P_2L_1) using the bisector already drawn ($S40$, $E23$), total ($S55$, $E31$). (b) Bisector as in (a), arbitrary tangent circle ($S12$, $E7$), homothetic expansions ($S31$, $E20$), total ($S52$, $E32$). (These might be reduced by special devices.)

7. Reduce c_1 to a point; decrease c_2 by r_1 to c_2', and increase c_3 by r_1 to c_3'. Construct the circle x' through C_1 tangent externally to c_2' and internally to c_3' as in problem 4. Decrease x' by r_1 to x.

8. A required circle is orthogonal to the given circle and to the bisector of l_1l_2, and hence belongs to the conjugate coaxal system tangent to a given line (exercise 5, part 1, problems 8, 9). For further details see the author's note in the *American Mathematical Monthly*, volume 34 (1927), page 359.

9. Let the required circle be orthogonal to c_1. First solution: Invert with respect to a point on c_1; reflect on c_1'; and reduce two circles to a point. Second solution: Invert with respect to C_1, and solve the problem (C_3). Third solution: Invert with respect to a circle of antisimilitude of c_2 and c_3, and apply section 59.

10. The homothetic center of x and c is the contact point T; that of x and x' is the intersection of the lines at P. Let O be a homothetic center of x' and c. Then P, O, T are collinear. Hence construct O; let PO cut c in T, and CT will meet the angle bisector in the center of x.

The circle x' can be taken in either of two independent positions; there are two homothetic centers; and each such center determines two points T, making eight possible solutions. Solutions can appear only in sections of the plane through which c passes.

11. If the indicated converse is not true, draw DC' tangent to the circle constructed tangent to three sides; apply the direct theorem, and show this leads to a contradiction in that one side of the triangle DCC' would equal the difference of the other two. For the given problem (see section 106b for notation),

$$\begin{aligned} PQ_2 + P'Q_1 &= PT_2 + T_2Q_2 + P'T_1' + T_1'Q_1 \\ &= T_1P + (Q_2T_2' + T_2'P') + Q_1T_1 = Q_2P' + Q_1P. \end{aligned}$$

12. The circles c_2 and c_3 invert into parallel lines tangent to c_1. The inverses of the required circles, say x' and y', are easily constructed.

13. (a) Let OC_2C_3 cut c_2 and c_2' at A_2 and A_2', and c_3 and c_3'

at A_3 and A_3'. Then, disregarding signs, (1) $2r_3 \cdot OA_3' = r^2 = 2r_2 \cdot OA_2'$. (2) $OA_2' + OA_3' = 2r_1$, from which r^2 is easily found.

(b) Apply formula for radii after inversion:

$$x = \frac{r^2 r_1}{p_1}, \quad y = \frac{r^2 r_1}{p_2},$$

giving

$$\frac{1}{x} + \frac{1}{y} = \frac{p_1 + p_2}{r^2 r_1}, \tag{1}$$

where

$$p_1 = OX'^2 - r_1^2, \quad p_2 = OY'^2 - r_1^2, \quad r^2 = OC_1^2 - r_1^2.$$

We apply the cosine law to triangles OC_1X' and OC_1Y' to find

$$OX'^2 + OY'^2 = 2OC_1^2 + 8r_1^2 = 2r^2 + 10r_1^2,$$

so that

$$p_1 + p_2 = 2r^2 + 8r_1^2. \tag{2}$$

The elimination of r^2 from (1), (2), and the result of (a) gives the required formula.

Exercise 8—Part 2

1. Invert with respect to P_1. All lines that cut c' at a given angle are tangent to a circle b' concentric with c'; two of them pass through P_2'. Invert back again.

2. As in problem 1, we find two circles b_1' and b_2', whose common tangents invert back again into the required circles.

3. Invert two intersecting circles into two lines and two nonintersecting circles into concentric circles. For details and properties see Johnson, page 128, or Papelier, volume 6, page 73.

4. Analysis: Let the points of contact be T_1 and T_2. Since the tangents meet on the radical axis, the line T_1T_2 must pass through a homothetic center O, being inverse points with respect to O. Since the tangents are perpendicular, OT_2 cuts c_2 at 45°, and is tangent to a circle c_2' as in problem 1.
Construction: Construct circle c_2', such that its tangents cut c_2 at 45°; through a homothetic center draw a tangent to c_2', cutting c_1 and c_2 in points properly selected for T_1 and T_2. The tangents at T_1 and T_2 meet at A. Eight possible solutions.

6. Suppose x is required circle with contact points T_1, T_2, T_3. Then T_1T_2 passes through E_{12}, and $E_{12}P_1 \cdot E_{12}P_2 = E_{12}T_1 \cdot E_{12}T_2$,

so E_{12} is on the radical axis of p and x. Similarly for E_{13}, so $E_{12}E_{13}$ is the radical axis of p and x. If t_1 is tangent to c_1 at T_1, P_1Q_1 and t_1 meet on the radical axis of p and x, since L_1 is a radical center of p, x, and c_1. Each line of homothetic centers determines a pair of circles.

7. There are now only two lines of homothetic centers, and we need only the poles with respect to the circles. Four solutions.

8. There are now only two lines of homothetic centers, and only one pole to determine the point of contact T with the circle. To locate the center of a required circle we use the perpendicular bisector of PT and C_1T. Four solutions.

9. First solution: Invert with respect to a circle through A and C meeting c at B and D (3 circles). Find the inverse of an arbitrary point E on c (3 circles). Find both centers (18 circles) and draw the circles OAC and BDE' (2 circles) meeting at X' and Y'. Invert back again (6 circles). (32 circles in all.) Second solution: A simpler solution is to take the first circle through A and A', the inverse of A with respect to c. (21 circles in all.)

10. (a) The orthogonality implies that the required circle x intersect PC_i in the inverse of P with respect to c_i. Hence construct the inverse P_1 of P with respect to c_1, and P_2 the inverse of P with respect to c_2. The circle PP_1P_2 is x. (16 circles.)

(b) This construction fails if P is on c_1. However, since x is orthogonal to c_1, it contains P_2', the inverse of P_2 with respect to c_1. Hence the circle PP_2P_2' is x. (16 circles.) Note that this construction is valid for both cases.

11. Proceed as in problem 10, except that now P_2 is the reflection of P on AB. The second method always works (15 circles). A poor suggestion would be to locate X as the intersection of the perpendicular bisector of PP_1 and AB, for the reader may count up the number of circles required.

12. Justification: Let radius be r; let $PQ = d$; and X the required midpoint. Then $Q_1X^2 = d^2 + r^2$. From the construction C, X, E are collinear and

$$CE^2 = Q_1Q^2 - d^2 = \left(\frac{3d}{2}\right)^2 + \left[r^2 - \left(\frac{d}{2}\right)^2\right] - d^2 = d^2 + r^2.$$

The construction requires but 6 circles. Problem 9 can be reduced to problem 12 by drawing a circle whose center is A.

13. If x is a solution, and if a_{12} and a_{13} are circles of antisimilitude, then x is orthogonal to a_{12} and a_{13} [section 106 a (3)], and belongs to their conjugate coaxal system. Hence x can be constructed as in section 59. If a_{ij} is not real, the circle $\bar{a}_{ij}$ is real, and orthogonality is replaced by cutting $\bar{a}_{ij}$ in the ends of a diameter (section 30). For this and related constructions involving bisecting a circle, see a paper by the author in the *American Mathematical Monthly*, volume 47 (1940), pages 519–529.

14. For details of construction, see a paper by the author in the *American Mathematical Monthly*, volume 33 (1926), pages 265–269.

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INDEX

INDEX

(THE NUMBERS REFER TO PAGES.)

P

Q

R

S