UNIVERSAL
LIBRARY

OU_154539

UNIVERSAL
LIBRARY

Osmania University Library

Call No. 512.5 M16C Accession No. 13185

Author macmahon, Major Percy, A

Title Combinatory Analysis

This book should be returned on or before the date last marked below.

COMBINATORY ANALYSIS

BY

MAJOR PERCY A. MACMAHON, F.R.S., D.Sc., LL.D.

(LATE ROYAL ARTILLERY)
OF ST JOHN'S COLLEGE, CAMBRIDGE

VOLUME I

Cambridge:
at the University Press
1915

Cambridge:
PRINTED BY JOHN CLAY, M.A.
AT THE UNIVERSITY PRESS

INTRODUCTION

THE object of this work is, in the main, to present to mathematicians an account of theorems in combinatory analysis which are of a perfectly general character, and to shew the connexion between them by as far as possible bringing them together as parts of a general doctrine. It may appeal also to others whose reading has not been very extensive. They may not improbably find here some new points of view and suggestions which may prompt them to original investigation in a fascinating subject.

Little attempt has been hitherto made either to make a general attack upon the territory to be won or to coordinate and arrange the ground that has been already gained. The combinatory analysis as considered in this work occupies the ground between algebra, properly so called, and the higher arithmetic. The methods employed are distinctly algebraical and not arithmetical. The essential connecting link between algebra and arithmetic is found in the circumstance that a particular case of algebraical multiplication involves arithmetical addition. Thus the multiplication of a^x and a^y, where a, x and y are numerical magnitudes, involves the addition of the magnitudes x and y. When these are integers we have the addition which is effective in combinatory analysis. This link was forged by Euler for use in the theory of the partitions of numbers. It is used here for the most general theory of combinations of which the partition of numbers is a particular case. The theory of the partition of numbers belongs partly to algebra and partly to the higher arithmetic. The former aspect is treated here. It is remarkable that in the international organization of the subject-matter of mathematics "Partitions" is considered to be a part of the Theory of Numbers, which is an alternative name for the Higher Arithmetic, whereas it is essentially a subdivision of Combinatory Analysis which is not considered to be within the purview of the Theory of Numbers. The fact is that up to the point of determining the real and enumerating Generating Functions the theory is essentially algebraical, and it is only when the actual evaluation of the coefficients in the generating functions is taken up that the methods and ideas of the Higher Arithmetic may become involved. Much has been accomplished in respect of various combinations of entities between which there are no similarities. Such researches are only included in this treatise

when they arise as particular cases of a general theory which is concerned with entities between which there may be any number of similarities.

Laplace was the first mathematician to employ enumerating generating functions. He required them for his researches in the Theory of Probabilities. He may be said to have invented the method of generating functions. The Theory of Probabilities is to a large extent concerned with and dependent upon the enumeration of combinations subject to conditions, and almost every theorem in combinatory analysis has its application to that theory. In the notation of Laplace, which is not adhered to here, if $F(x)$ be a number which depends upon the integer x a generating function

$$\sum_{x=0}^{x=\infty} F(x)\, t^x$$

is constructed, and Laplace is frequently able to carry out the summation and to thus present the value of $F(x)$ for all integral values of x as the coefficient of t^x in the expansion of a definite and compact function of t. He further considers a number $F(x_1 x_2 \ldots x_n)$ which depends upon the ordered succession of integers $x_1, x_2, \ldots x_n$ and constructs the generating function

$$\sum_{x_1}\sum_{x_2}\ldots\sum_{x_n} F(x_1 x_2 \ldots x_n)\, t_1^{x_1} t_2^{x_2} \ldots t_n^{x_n},$$

each x_s being summed from zero to infinity so that when the series is summable the number $F(x_1 x_2 \ldots x_n)$ is given as the coefficient of

$$t_1^{x_1} t_2^{x_2} \ldots t_n^{x_n}$$

in the ascending expansion of a function of $t_1, t_2, \ldots t_n$ which does not involve the magnitudes $x_1, x_2, \ldots x_n$ explicitly.

The principal generating function in this work is derived from that of Laplace by substituting for the product $t_1^{x_1} t_2^{x_2} \ldots t_n^{x_n}$ the symmetric function

$$\Sigma t_1^{x_1} t_2^{x_2} \ldots t_n^{x_n},$$

which is written for brevity $(x_1 x_2 \ldots x_n)$.

The great advantage of this method is that the algebra is freed from all consideration of the magnitudes $t_1, t_2, \ldots t_n$ and the attention is concentrated upon the succession of numbers $x_1, x_2, \ldots x_n$ which appear, to the exclusion of other magnitudes, in the magnitude $F(x_1 x_2 \ldots x_n)$ which is under examination. It follows as a matter of course that the summed or compact form of generating function is also a symmetric function and can be treated by the powerful methods which appertain to symmetric function theory.

The next point to notice is that the importation of a symmetric function generating function arises in quite a natural manner. It is in no way forced into the theory. It is shewn in Section I that the general theory of combination is essentially involved in the algebra of monomial symmetric functions. Every multiplication of monomial symmetric functions, the result being exhibited as a linear function of such functions, involves a theorem in

combinations. In many cases the summed form of generating function can be written down from reasoning in which intuition is largely drawn upon. The method of symmetric function differential operators admits of the handling of the generating function in a most satisfactory manner. The simple circumstance that it is possible to design a differential operator which has the effect of transforming the function $(x_1 x_2 \ldots x_n)$ into another precisely equivalent to it, but with the magnitude x_s missing, suffices to establish the important and dominating rôle that the differential calculus can assume in Combinatory Analysis.

An important part of the subject is studied as a Theory of Distributions and much that in idea is valuable and suggestive has been derived from a little book, Whitworth's *Choice and Chance.*

In Section II the simple theory of symmetric functions is extended so that the point of departure is no longer an integer but the partition of an integer, and we have to do not with the partitions of a number but with the separations of a partition. This extension is of much import to the theory of distributions and also enables many theorems of algebraical reciprocity to be derived in an intuitive manner. It also leads in Chapter I of the Section to a law of algebraic expressibility to which special attention may be directed. In proceeding from the Partition to the Separation quite new notions come to the front. Thus the idea of Groups of Separations has no analogue in Partitions. A formula is established in Chapter II analogous to that of Girard (sometimes erroneously associated with the name of Waring) which permits the expression of a sum of powers as a linear function of the separations of any partition of the number which is the exponent in the sum. A pair of symmetric function tables which enjoy row and column symmetry is established for every partition of a number. Hammond's operators d, D are given an extended field of operation and, in the enlarged theory, are shewn to be important instruments for multiplying and evaluating the symmetric functions that present themselves. In Chapter IV binomial coefficients are treated as symmetric functions denoted by partitions with zero parts, and the formation of symmetrical Tables involving them is explained.

Section III is devoted to certain points in the Theory of Permutations that are of value in theories of combination or distribution. The enumeration of Combinations and Permutations is treated from the high point of view supplied by the Theory of Symmetric Functions. In Chapter II a theorem is established which I have ventured to term "a Master Theorem," from the masterly and rapid fashion in which it deals with various questions otherwise troublesome to solve. Many illustrations of its power are given. In particular in Chapter III it is shewn to supply instantly the solution of the generalized "Problème des Rencontres." In finding expressions for the sum of powers of binomial coefficients it is singularly effective.

In Chapter V the reader is introduced to "Lattice Permutations." They are in themselves interesting but are of importance principally because they are shewn in Volume II to supply the key to the solutions of certain questions which involve the arrangement, under given conditions, of numbers in spaces of two and three dimensions; numbers not arranged at points in a straight line but at the crossing points of lines which compose lattices in two and three dimensions. This connexion with lattices determined the nomenclature.

The notion of the Indices of Permutations presented itself in the same connexion and in Chapter VI the question is discussed without particular reference to the Indices of Lattice Permutations. This is carried out in Volume II when the enlarged theory of partitions is under view.

Section IV is entirely devoted to the compositions of numbers. The word "composition" is used in a different sense from that assigned to it in the higher arithmetic. Following Glaisher the word connotes a partition in which account is taken of the order of occurrence of the parts. Thus the partition a, b, c of the number $a+b+c$ would have six compositions $abc, acb, bac, bca, cab, cba$, involving the parts a, b, c. Attention may be drawn to the developments which arise from the consideration of what I have termed "Simon Newcomb's problem" in Chapter IV. The new symmetric functions $h_{pqr\ldots}$, $a_{pqr\ldots}$ present themselves and are briefly discussed. They appear in analysis for the first time, and in addition to themselves possessing elegant properties they give practically the complete solution of the problem under examination. In Section V the subject of the perfect partitions of numbers is dealt with as a necessary preliminary to the discussion of arrangements upon a chess board. The latter, as involving conditions to be satisfied by the numbers appearing in a row or in a column, are of the Magic Square nature. We here see a new and interesting rôle of the Differential Calculus. At first sight the operations of that calculus appear to be remote from the enumeration of arrangements in two dimensions. In fact the whole subject of Magic Squares and connected arrangements of numbers appears at first sight to occupy a position which is completely isolated from other departments of pure mathematics. The object of Chapters II and III is to establish connecting links where none previously existed. This is accomplished by selecting a certain differential operation and a certain algebraical function; shewing that the operation upon the function can be dissected into a number of distinct operations, each of which may be given a graphical representation. When the operations and the functions are suitably chosen these graphical representations have the form of the arrangements which we desire to enumerate. As the simplest example we may choose the operation $\frac{d}{dx}$ and the function x^n. We observe that if x^n be written out in the form $xxx\ldots x$ the operation is equivalent to writing unity for one of the x's in all possible ways and adding the results together.

We obtain nx^{n-1} because one x can be selected in n different ways. So operating n times successively with $\frac{d}{dx}$ we dissect the operation into $n!$ distinct operations and these give rise to $n!$ distinct diagrams. For consider a square of n^2 compartments; we may place a unit at the intersection of the rth row and cth column to indicate that when $\frac{d}{dx}$ was operating for the rth time one process was to substitute unity for the cth x counting from the left. So we obtain $n!$ diagrams which possess the property that n units appear one in each row and one in each column. We thus by the operation upon the function enumerate the diagrams with this property.

The "Problème des Rencontres" already referred to above can be discussed in the same manner. The reader will be familiar with the old question of the letters and envelopes. A given number of letters are written to different persons and the envelopes correctly addressed but the letters are placed at random in the envelopes. The question is to find the probability that not one letter is put into the right envelope. The enumeration connected with this probability question is the first step that must be taken in the solution of the famous problem of the Latin Square. The question is to place n different letters $a, b, c, \ldots$ in each row of a square of n^2 compartments in such wise that, one letter being in each compartment, each column involves the whole of the letters. The number of arrangements is required. The question is famous because, from the time of Euler to that of Cayley inclusive, its solution was regarded as being beyond the powers of mathematical analysis. It is solved without difficulty by the method of differential operators of which we are speaking. In fact it is one of the simplest examples of the method which is shewn to be capable of solving questions of a much more recondite character.

In Section VI the Theory of Distributions is directly applied to the enumeration of the Partitions of Multipartite Numbers.

The second Volume is devoted to various generalizations of the Theory of Partitions and to discussions which arise therefrom.

In the present Volume there appears a certain amount of original matter which has not before been published. It involves the author's preliminary researches in combinatory theory which have been carried out during the last thirty years. For the most part it is original work, which however owes much to valuable papers by Cayley, Sylvester, and Hammond.

To the original papers reference is made in regard to:

Section I. *Quarterly Journal of Mathematics*, No. 85, 1886, "The Law of Symmetry and other theorems in Symmetric Functions."

Section II. *Proceedings London Mathematical Society*, Vol. XIX. 1887, "Symmetric Functions and the Theory of Distributions."

American Journal of Mathematics, Vol. XI. et seq. 1888–1890, "Memoirs on a New Theory of Symmetric Functions."

Section III. *Phil. Trans. R. S.* 1894, A, "A certain class of Generating Functions in the Theory of Numbers."

Trans. Camb. Phil. Soc. Vol. XXI. No. xviii. "The Problem of Derangement in the Theory of Permutations."

Quarterly Journal of Mathematics, No. 131, 1902, "The sums of the Powers of the Binomial Coefficients."

Messenger of Mathematics, New Series, No. 281, 1894, "Self-Conjugate Permutations."

American Journal of Mathematics, Vol. XXXV. No. 3, 1913, "The Indices of Permutations and the derivation therefrom of functions of a single variable associated with the Permutations of any assemblage of objects."

Section IV. *Phil. Trans. R. S.* 1893, A, "Memoir on the Compositions of Numbers."

Phil. Trans. R. S. 1907, A, "Second Memoir on the Compositions of Numbers."

Section V. *Messenger of Mathematics*, Vol. XX. 1891, "The theory of perfect partitions and the compositions of multipartite numbers."

Trans. Camb. Phil. Soc. Vol. XVI. Part iv. 1898, "A new method in Combinatory Analysis with applications to Latin Squares and associated questions."

Phil. Trans. R. S. Vol. 194, A, 1900, "Combinatorial Analysis—the foundations of a new theory."

Section VI. *Trans. Camb. Phil. Soc.* Vol. XXII. No. 1, "On Compound Denumeration."

P. A. M.

January 1915.

TABLE OF CONTENTS

SECTION I

SYMMETRIC FUNCTIONS

CHAPTER I

ELEMENTARY THEORY

ART.		PAGE
1, 2.	Definitions connected with Partitions	1
3, 4.	Elementary symmetric functions and homogeneous product sums .	2
5.	Symmetric functions whose partitions involve a given number of parts. Formula of Girard and its generalization	4
6.	Expression of the functions a_w and h_w by means of the functions s_w	6
7.	Expression of many-part functions by means of the functions s_w .	7

CHAPTER II

CONNEXION WITH THE THEORY OF DISTRIBUTIONS

8.	Definitions and formulation of the chief Problems	8
9.	Definition of the Distribution Functions	9
10.	Theory of the Distribution into Parcels when the number of Parcels is equal to the number of Objects	10
11, 12.	Proof of a "First Law of Symmetry" in Symmetrical Algebra. .	11
13, 14.	A restricted Distribution leading to a "Second Law of Symmetry".	13
15.	The "Inclusive Law of Symmetry"	14
16—18.	Theory of the Distribution into Parcels when the latter are all different and fewer in number than the Objects	14
19.	The Distribution into Groups when the Parcels are all different and fewer in number than the Objects	17

CHAPTER III

THE DISTRIBUTION INTO PARCELS AND GROUPS IN GENERAL

20, 21.	Parcels of type (m) where $m<n$, n being the number of Objects .	19
22.	Distribution into Groups of type (m)	22
23.	Distribution into Parcels of type $(m_1 m_2 \ldots m_r)$	24
24.	Distribution into Groups of type $(m_1 m_2 \ldots m_r)$	25

CHAPTER IV

THE OPERATORS OF THE THEORY OF DISTRIBUTIONS

ART.		PAGE
25.	Some interpretations of formulas as Theorems of Distributions.	26
26.	Hammond's operators d_λ, D_λ	27
27.	The Algebra of the Operators. Correspondence with the Algebra of Quantity	28
28.	Sylvester's Theorem and its Generalization	30
29, 30.	Operation of D_λ upon a Product of Symmetric Functions. Expression of D_λ as a sum of partition Operators	32
31.	Operators adapted for use when the operand is expressed in terms of homogeneous product sums	35

CHAPTER V

APPLICATIONS OF THE OPERATORS d AND D

32.	Connexion between the symmetric functions $(\kappa\lambda^l\mu^m\ldots)$, $(\lambda^l\mu^m\ldots)$	38
33.	Calculation of symmetric functions	39
34, 35.	Hammond's proof of the "Second Law of Symmetry" and of its converse	40
36.	Generalization of the foregoing	41
37.	Applications of the operators when the operand is expressed in terms of homogeneous product sums. Proof by the operators of the "First Law of Symmetry"	41
38.	General "Law of Symmetry"	42
39.	Examples of the mode of operation of D and Δ upon a product of symmetric functions.	43

SECTION II

GENERALIZATION OF THE THEORY OF SECTION I

CHAPTER I

THE THEORY OF SEPARATIONS

40, 41.	The Separation of a Partition	45
42.	Characteristics of a Separation. The multiplicity	46
43.	The enumeration of the Separations of a Partition. Connexion with the Partitions of Multipartite Numbers.	46
44.	Groups of Separations	47
45.	Theory of a restricted distribution which is a generalization of previous results. The type of a distribution	49
46.	Generalization of previous laws of symmetry	51
47.	Law of Expressibility in Symmetrical Algebra	53

CHAPTER II

GENERALIZATION OF GIRARD'S FORMULA

ART.		PAGE
48.	The expression of a sum of powers by means of the separations of a given monomial symmetric function	55
49.	Properties of the coefficients in the expressions of s_n with particular reference to Groups of Separations	58

CHAPTER III

THE DIFFERENTIAL OPERATORS OF THE THEORY OF SEPARATIONS

50.	Adaptation of the operator d_λ to the Separation Theory . . .	62
51.	Relations between the partition-d and the partition-D operators .	66
52.	The operator which selects all separations of a given partition in all possible ways from a product of symmetric functions, one separate only being taken from each factor	68

CHAPTER IV

A CALCULUS OF BINOMIAL COEFFICIENTS

53.	The partition notation of binomial coefficients. The operator D_0. Its effect upon a coefficient and upon a product of coefficients, through the partitions of zero. Connexion with the Calculus of Finite Differences. The splitting up of D_0 into a sum of operators .	74
54.	Connexion with a Theory of Distributions. Law of Symmetry and formation of Symmetrical Tables of Binomial Coefficients . .	76
55.	Remarks upon the association of D_0 with the partitions of zero .	77

CHAPTER V

THE THEORY OF THREE IDENTITIES

56—59.	Further correspondences between the algebras of operation and quantity leading to new laws of symmetry	79
60.	Further Theory of Distributions	83

SECTION III

PERMUTATIONS

CHAPTER I

THE ENUMERATION OF PERMUTATIONS

ART.		PAGE
61.	Combinations	87
62.	Permutations	90

CHAPTER II

THE THEORY OF PERMUTATIONS

63—66.	A Master Theorem	93
67.	General Application of the Master Theorem	98

CHAPTER III

THE THEORY OF DISPLACEMENTS

68.	"Problème des Rencontres"	99
69—72.	Algebraic Theory	100
73—77.	Investigation by the Master Theorem	104
78—81.	Investigation of the general case	109
82, 83.	Application of the Elementary Notions of the Theory of Substitutions	112

CHAPTER IV

OTHER APPLICATIONS OF THE MASTER THEOREM

84, 85.	Certain permutations enumerated by the sum of the nth powers of the binomial coefficients	115
86.	Case where the guiding determinant is of a chess-board pattern	117
87, 88.	Self-conjugate permutations	118
89—91.	Applications to the sum of the nth powers and other functions of binomial coefficients	119
92.	Concluding remarks upon the Master Theorem	122

CHAPTER V

LATTICE PERMUTATIONS

ART.		PAGE
93.	Definition	124
94.	Connexion with two-dimensional arrays of different numbers	125
95.	Correspondence between the permutations of conjugate assemblages	126
96.	Correspondence between permutations and one-dimensional arrays of different numbers	127
97.	Enumeration of the lattice permutations of the assemblage $a^p\beta^q$. The redundant generating function	127
98, 99.	The exact generating function. Prime and composite lattice permutations	128
100, 101.	Property deduced from the difference equation. Expression of $(pp;)$ as a sum of squares	130
102.	Enumeration in connexion with the assemblage $a^p\beta^q\gamma^r$. The redundant generating function	131
103.	Enumeration in the general case. The allied question in probabilities. A remarkable property of numbers	133

CHAPTER VI

THE INDICES OF PERMUTATIONS

104.	Definitions of the Greater, Equal and Lesser Indices and of the associated Classes	135
105.	The generating function which enumerates the permutations which possess a given greater index. The form of Σx^p	136
106, 107.	Important property of Σx^p. Proof that $\Sigma x^p = \Sigma x^r$	136
108.	The expression of Σx^{p+q}	137
109.	The particular case of the assemblage $a^i\beta^j$. Expressions for Σx^q and Σx^{p+r}	138
110, 111.	The Parity of the Greater Index. Analogous property in respect of any prime modulus	139
112.	Average Value of the Greater Index	141
113.	Average Value of the Equal Index	142
114.	Average Values of the Square and of Higher Powers of the Greater Index	143
115, 116.	Average Values from another point of view	144
117.	Average Value of the Class of a Permutation *quâ* the greater index	146
118.	Average Value of the νth power of an Index-Component	146
119.	Average Values connected with the Equal Index	147
120.	Properties of Index-Components	148

SECTION IV

THEORY OF THE COMPOSITIONS OF NUMBERS

CHAPTER I

UNIPARTITE NUMBERS

ART.		PAGE
121—124.	Elementary algebraic theory of the Compositions of Unipartite Numbers	150
125—128.	Graphical method of investigation. Conjugate, inverse and self-inverse compositions. Inverse-conjugate compositions. Correspondence between self-inverse and inverse-conjugate compositions	151
129.	A second graphical method of representation. Zig-zag graphs	153

CHAPTER II

MULTIPARTITE NUMBERS

130.	The Generating Function which enumerates the composition	155
131.	Application of the Master Theorem of Section III to produce an important redundant generating function	156
132.	Connexion with the Theory of Permutations	157
133, 134.	Compositions in which no figure greater than unity appears and their connexion with permutations	158
135—140.	Compositions with a prescribed number of parts. Study of the generating function with the aid of Hammond's operators. Deduction of formulas which enable the enumeration to be carried out. Numerical verifications	160
141.	The graphical representation of the compositions of bipartite numbers	165
142, 143.	The Line of Route of a composition through a graph	167
144—146.	The number of different lines of route. Essential nodes upon a line of route and determination of the number of lines of route which possess a given number of essential nodes	169
147.	Enumeration by means of lines of route. Principal compositions	171
148.	Inverse bipartite compositions	172

CHAPTER III

THE GRAPHICAL REPRESENTATION OF THE COMPOSITIONS OF TRIPARTITE AND MULTIPARTITE NUMBERS

149, 150.	The formation of the graphs. Essential nodes of different characters along a line of route	175
151, 152.	Special investigation of the tripartite case. The redundant and real generating functions	178
153.	Generalization of the idea of "composition." Correspondence with the "Theory of Trees"	180
154, 155.	Combinations of order k. Generating function which enumerates such combinations. Application to the determination of the number of permutations, of a given assemblage, which possess a given number of major contacts	182

CHAPTER IV

SIMON NEWCOMB'S PROBLEM

ART.		PAGE
156.	Statement of the problem. Connexion with the Theory of Compositions. Definition of $N(abc...)$ and its evaluation	187
157, 158.	Simple theorems connected with the numbers $N(abc...)$	189
159, 160.	The Multiplication Theorem	192
161—166.	Applications of the Multiplication Theorem	193

CHAPTER V

GENERALIZATION OF THE FOREGOING THEORY

167.	The auxiliary generating function	197
168.	Investigation of a true generating function. The new function $h_{abcd...}$. Determinant expression of the function. Simple numerical results	199
169—172.	Properties of the functions $h_{p_1 p_2 ...}$, $a_{p_1 p_2 ...}$	203
173.	Expression of h_n in the form a_{1^n}. Important relation between the functions $h_{p_1 p_2 ...}$, $a_{q_1 q_2 ...}$ when $p_1 p_2 ...$, $q_1 q_2 ...$ are zig-zag conjugate compositions	204
174.	Expression of s_n by means of the new functions. Employment of Differential Operators to produce relations between the numbers under examination	206
175—177.	Generalization of the Conjugate Law of Chapter IV	208
178.	Theory of the compositions containing a given number of parts	209
179.	The generating function of the number $N_{m,(pqr...)}$	212
180.	Proof that $N_{m,(p^k)} = N_{(k-1)p-m+2,(p^k)}$	212
181, 182.	Examination of the generating function by means of differential operators. Special relations satisfied by the operators for the particular operands	213
	Numerical applications and tabular results	214

SECTION V

DISTRIBUTIONS UPON A CHESS BOARD, TO WHICH IS PREFIXED A CHAPTER ON PERFECT PARTITIONS

CHAPTER I

THEORY OF THE PERFECT PARTITIONS OF NUMBERS

183.	Definitions of perfect and subperfect partitions	217
184.	Perfect partitions of numbers of the form $p^a - 1$ as depending upon the compositions of a	218
185.	Perfect partitions of numbers of the form $p_1^{a_1} p_2^{a_2} p_3^{a_3} ... - 1$ as depending upon the compositions of the multipartite number $a_1 a_2 a_3 ...$	220
186.	Illustrative examples	221
187.	The general form of a perfect partition. A generating function	222
188, 189.	The perfect partitions, of all numbers, which contain a given number of parts. The theory of subperfect partitions	222

CHAPTER II

ARRANGEMENTS UPON A CHESS BOARD

ART.		PAGE
190.	General ideas. Dissection of the operation of ∂_x upon x^n into n distinct selective operations. Dissection of the operation of ∂_x^n upon x^n into $n!$ distinct operations and the graphical representation of the dissection	224
191—196.	The two methods of research. Examples drawn from elementary theorems of the differential calculus, and the corresponding diagrams. The operator, the operand, and the *modus operandi*. Examples. Diagrams derived from the permutations of any assemblage of objects	225
197, 198.	The problem of placing units in the compartments of a lattice of m rows and l columns, not more than one unit in each compartment, so that there may be $\mu_1, \mu_2, \ldots \mu_m$ units in the successive rows and $\lambda_1, \lambda_2, \ldots \lambda_l$ units in the successive columns. A method of reciprocation	232
199.	The same problem with the removal of the restriction as to the magnitude of the numbers which appear in the compartments	234
200.	The same problem when the magnitude of the number is restricted so as not to exceed k	235
201.	Theorems of symmetry which are manifest from the investigation. Connexion between every case of symmetric function multiplication and a theory of distribution on a lattice	236
202, 203.	Connexion with the Ferrers-graph of the partition of a number. A general question of distribution leading to a refined law of algebraic symmetry	236
204—206.	The operators of the Calculus of Finite Differences. Distributions based upon the partitions of zero	241

CHAPTER III

THE THEORY OF THE LATIN SQUARE

207.	Historical account of the problem	246
208.	Construction of operator and operand effective for the purpose. Application of perfect partitions. Enumeration of Latin Squares and of Reduced Latin Squares	248
209, 210.	Determination of the number of squares of the first five orders. Literal illustrations	250
211.	Definition of Latin Rectangles. Their enumeration	251
212, 213.	Agreement of the foregoing with the Theory of the Latin Square	252
214, 215.	The "Problème des ménages"	253
216.	Discordant permutations. Enumeration of permutations which are discordant with any given set of permutations	254
217.	Direct generalization of the "Problème des ménages".	255
218.	Application of the Master Theorem of Section III to the question of discordant permutations	256
219—223.	Generalized Latin Rectangles and Squares	257
224.	Removal of the restriction as to one letter only appearing in each compartment	260

SECTION VI

THE ENUMERATION OF THE PARTITIONS OF MULTIPARTITE NUMBERS

CHAPTER I

BIPARTITE NUMBERS

ART. PAGE

225, 226. The enumeration of the partitions which appertain to a given group. Example of the bipartite number (33). Symbolic factorization of the enumerating expression 264

227, 228. Enumeration of the partitions into a definite number of parts. Example of the bipartite number (44). Separation of the operator from the operand in the enumerating expression 269

229, 230. Enumeration of partitions which contain dissimilar parts. The same when the number of parts is given. Example of the bipartite number (33) 272

CHAPTER II

TRIPARTITE AND OTHER MULTIPARTITE NUMBERS

231, 232. The enumeration by the theory of distributions of the partitions which appertain to a given group. The evaluation of the enumerating expression. Example of the tripartite number (322) . . . 277

233. The partitions of tripartite numbers which contain a definite number of parts. Example of the tripartite number (322) 281

234, 235. Enumeration of partitions which involve dissimilar parts. Evaluation of the enumerating expression. Example of the tripartite number (322). The same when the number of parts is given . . . 282

236, 237. The case of multipartite numbers in general. Algebraic properties of the functions which present themselves 285

TABLES

Products of Homogeneous Product Sums in terms of Monomial Symmetric Functions 288

Products of Elementary Symmetric Functions in terms of Monomial Symmetric Functions 289

Monomial Symmetric Functions in terms of Elementary Symmetric Functions . 290

The Simplest Distribution Functions 291

Symmetrical Tables of Binomial Coefficients 292

Enumeration of Compositions of Multipartite Numbers. 293

Values of a_{pqr}. and h_{pqr}.. 295

Symmetric Function Tables associated with Partitions of Simple Numbers . 296

SECTION I

SYMMETRIC FUNCTIONS

CHAPTER I

ELEMENTARY THEORY

1. The theory will be developed with particular reference to the combinatory analysis and the theory of the partitions of numbers.

A partition of an integer number n is understood to be any collection of positive integers whose sum is equal to n. The order in which the integers are written out is immaterial, but we may decide, for convenience, to take the integers in descending order of magnitude. This will invariably be done and the integers will be termed "parts" of the partition. The parts will in general also be enclosed in brackets ( ). The sum of the parts, viz. the number n, will be called "the partible number" or the "weight of the partition." A partition which has m parts is called an m-part partition.

Ex. gr. (432), (6111) are partitions of 9 and are 3-part and 4-part respectively.

It is usual to indicate repetitions of the same part by exponents, so that (6111) is written (61^3).

For the present we shall only be concerned with partitions of positive integers into positive parts; zero not being included as a possible part. At a later stage we shall have to consider partitions of positive integers, zero, and negative integers into positive, zero and negative parts.

2. A partition of a number n may be regarded as specifying a distribution of n similar objects into a number of parcels, each parcel comprising one or more of the objects. Thus (432) specifies a distribution of 9 similar objects into three parcels which contain 4, 3 and 2 of the objects respectively, and since however we permute the parcels the partition is not affected we may say that the distribution is of similar objects into similar parcels. It is sometimes convenient not to regard the succession of descending integers as a partition of a single (i.e. unipartite) number, but as being itself a multipartite number which specifies a number of objects which

are not all similar. Thus we may regard (432) as denoting 9 objects, 4 of one kind, 3 of a second, 2 of a third.

3. The symmetric functions, that now come under consideration, are algebraic, rational and integral. Any such function of n quantities

$$\alpha_1, \alpha_2, \dots \alpha_n$$

which remains unchanged however these quantities may be interchanged or permuted is termed a symmetric function of the n quantities.

We write

$$(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots + (-)^n a_n$$

identically, so that

$$\begin{aligned} \alpha_1 + \alpha_2 + \dots + \alpha_n &= \Sigma\, \alpha_1 = a_1, \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \dots + \alpha_{n-1}\alpha_n &= \Sigma\, \alpha_1\alpha_2 = a_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \alpha_1\alpha_2 \dots \alpha_n &= \Sigma\, \alpha_1\alpha_2 \dots \alpha_n = a_n; \end{aligned}$$

and it appears that the quantities $a_1, a_2, \dots a_n$ are symmetric functions of the quantities α. It will be observed that these symmetric functions of the quantities α, on the sinister of the identities, are each of them specified or given by a single term, since to obtain the remaining terms we have merely to interchange the quantities in a sufficient number of ways. For this reason the Σ notation is expressive. In general we may have under view the symmetric function

$$\Sigma\, \alpha_1^{p_1} \alpha_2^{p_2} \dots \alpha_s^{p_s},$$

the whole function consisting of a number of terms similar to the one attached to the sign of summation; in each term we may take $p_1, p_2, \dots p_s$ to be in descending order of magnitude, but the quantities α, therein appearing, will be any s selected therefrom in any permutation.

It thus appears that the succession of numbers $p_1 p_2 \dots p_s$ in descending order of magnitude is a sufficient specification of the symmetric function and, if

$$p_1 + p_2 + \dots + p_s = w,$$

we may regard the function as denoted by the partition $(p_1 p_2 \dots p_s)$ of the number w. We shall accordingly speak of a symmetric function

$$(p_1 p_2 \dots p_s)$$

of the quantities α.

The simple symmetric functions a introduced above are called elementary and, since they are denoted by the partitions

$$(1), (1^2), (1^3), \dots (1^n),$$

they are sometimes called unitary symmetric functions.

As repetitions of parts may occur a more general form of symmetric function is

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots p_s^{\pi_s}),$$

s of course being limited by the number n.

It is frequently convenient to regard n as being indefinitely large, and then we may write

$$(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)\dots = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \dots.$$

Further put

$$\frac{1}{(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)\dots} = \frac{1}{1 - a_1 x + a_2 x^2 - a_3 x^3 + \dots}$$
$$= 1 + h_1 x + h_2 x^2 + h_3 x^3 + \dots,$$

and we find by expanding the left-hand side in ascending powers of x and subsequently equating coefficients of like powers of x

$$h_1 = \Sigma\alpha_1 = (1),$$
$$h_2 = \Sigma\alpha_1^2 + \Sigma\alpha_1\alpha_2 = (2) + (1^2),$$
$$h_3 = \Sigma\alpha_1^3 + \Sigma\alpha_1^2\alpha_2 + \Sigma\alpha_1\alpha_2\alpha_3 = (3) + (21) + (1^3),$$
$$h_4 = \Sigma\alpha_1^4 + \Sigma\alpha_1^3\alpha_2 + \Sigma\alpha_1^2\alpha_2^2 + \Sigma\alpha_1^2\alpha_2\alpha_3 + \Sigma\alpha_1\alpha_2\alpha_3\alpha_4$$
$$= (4) + (31) + (2^2) + (21^2) + (1^4),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

wherein it will be noted that h_s is the sum of a number of symmetric functions, each of which is denoted by a partition of the number s and that in fact h_s is the sum of the whole of such symmetric functions; h_s is called the "homogeneous product sum" of weight s of the quantities α.

4. The symmetric functions h are related to the symmetric functions a in a simple and important manner: for, writing $-x$ for x in the relation

$$\frac{1}{1 - a_1 x + a_2 x^2 - a_3 x^3 + \dots} = 1 + h_1 x + h_2 x^2 + h_3 x^3 + \dots,$$

it becomes equivalent to

$$\frac{1}{1 - h_1 x + h_2 x^2 - h_3 x^3 + \dots} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \dots,$$

a relation derivable from the former by leaving x unchanged and simply interchanging the symbols a and h.

It follows at once that any relation connecting the quantities a with the quantities h remains a valid relation after the interchange of the symbols a and h.

We can forthwith express h_s in terms of the quantities a; in the relation

$$\frac{1}{1 - a_1 x + a_2 x^2 - a_3 x^3 + \dots} = 1 + h_1 x + h_2 x^2 + h_3 x^3 + \dots$$

it is merely necessary to expand the left-hand side by the multinomial theorem to find

$$h_s = \Sigma\,(-)^{s+\Sigma\lambda}\frac{(\Sigma\lambda)!}{\lambda_1!\,\lambda_2!\ldots\lambda_s!}a_1^{\lambda_1}a_2^{\lambda_2}\ldots a_s^{\lambda_s};$$

and thence by interchange of symbols

$$a_s = \Sigma\,(-)^{s+\Sigma\lambda}\frac{(\Sigma\lambda)!}{\lambda_1!\,\lambda_2!\ldots\lambda_s!}h_1^{\lambda_1}h_2^{\lambda_2}\ldots h_s^{\lambda_s}.$$

These simple symmetric functions are so important in the general theory of Distributions, and in combinatory analysis generally, that a further study of their beautiful properties will be necessary at a later stage of the work.

The simple formulæ are

$$\begin{aligned}
h_1 &= a_1,\\
h_2 &= a_1^2 - a_2,\\
h_3 &= a_1^3 - 2a_1a_2 + a_3,\\
h_4 &= a_1^4 - 3a_1^2a_2 + a_2^2 + 2a_1a_3 - a_4,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and these are readily written down *currente calamo* because the numerical value of the coefficient of any a product is equal to the number of permutations of which that product is susceptible.

5. The symmetric functions, next in order of simplicity, which now present themselves for consideration, are the successive sums of powers of the quantities $\alpha_1, \alpha_2, \ldots \alpha_n$, viz. in the usual notation

$$\begin{aligned}
s_1 &= \Sigma\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n = (1),\\
s_2 &= \Sigma\alpha^2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = (2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
s_m &= \Sigma\alpha^m = \alpha_1^m + \alpha_2^m + \ldots + \alpha_n^m = (m).
\end{aligned}$$

These are one-part symmetric functions from the partition point of view. We may then proceed to consider the sums of the two-part, three-part, etc. symmetric functions of given weight, and it is convenient to denote by

$$S_{pq}, \quad \text{where} \quad p \geqslant q,$$

the sums of those symmetric functions which have a weight p and q parts. Thus we have

$$\begin{aligned}
S_{m1} &= s_m = (m),\\
S_{22} &= (1^2),\\
S_{32} &= (21),\\
S_{42} &= (31) + (2^2),\\
&\vdots\\
S_{63} &= (41^2) + (321) + (2^3),\\
&\text{etc.}
\end{aligned}$$

The function S_{pq} is examined as follows:

Write $$f(x) = (1-\alpha_1 x)(1-\alpha_2 x)\dots(1-\alpha_n x),$$

and observe the identity

$$\left(1-\frac{\alpha_1\xi}{1-\alpha_1 x}\right)\left(1-\frac{\alpha_2\xi}{1-\alpha_2 x}\right)\dots\left(1-\frac{\alpha_n\xi}{1-\alpha_n x}\right)=\frac{f(x+\xi)}{f(x)}.$$

The left-hand side may be written

$$1-\xi\Sigma\frac{\alpha_1}{1-\alpha_1 x}+\xi^2\Sigma\frac{\alpha_1\alpha_2}{(1-\alpha_1 x)(1-\alpha_2 x)}-\xi^3\Sigma\frac{\alpha_1\alpha_2\alpha_3}{(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)}+\dots,$$

wherein the coefficient of $(-)^m\xi^m$ is equal on expansion to

$$S_{m,m}+S_{m+1,m}\,x+S_{m+2,m}\,x^2+\dots+S_{m+p,m}\,x^p+\dots,$$

n being supposed indefinitely great.

Moreover the right-hand side is by Taylor's Theorem

$$1+\xi\frac{1}{f(x)}\frac{df}{dx}+\xi^2\frac{1}{2!f(x)}\frac{d^2f}{dx^2}+\dots+\xi^m\frac{1}{m!f(x)}\frac{d^mf}{dx^m}+\dots,$$

whence, equating coefficients of like powers of ξ,

$$S_{11}+S_{21}\,x+S_{31}\,x^2+\dots=-\frac{1}{f(x)}\frac{df}{dx},$$

$$S_{22}+S_{32}\,x+S_{42}\,x^2+\dots=+\frac{1}{2!f(x)}\frac{d^2f}{dx^2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$S_{mm}+S_{m+1,m}\,x+S_{m+2,m}\,x^2+\dots=(-)^m\frac{1}{m!f(x)}\frac{d^mf}{dx^m}.$$

Now $$f(x)=1-a_1x+a_2x^2-a_3x^3+\dots,$$

$$\frac{d^mf}{dx^m}=(-)^m\left\{m!\,a_m-\frac{(m+1)!}{1!}a_{m+1}\,x+\frac{(m+2)!}{2!}a_{m+2}\,x^2+\dots\right\}.$$

Hence

$$S_{mm}+S_{m+1,m}x+S_{m+2,m}x^2+\dots=\frac{a_m-\binom{m+1}{1}a_{m+1}\,x+\binom{m+2}{2}a_{m+2}\,x^2-\dots}{1-a_1\,x+a_2\,x^2-a_3\,x^3+\dots},$$

and we have only to expand the right-hand side by the multinomial theorem to reach the result

$$S_{w,m}=\Sigma\,(-)^{w+m+k-1}\frac{(k-1)!\left\{p_m+\binom{m+1}{1}p_{m+1}+\binom{m+2}{2}p_{m+2}+\dots\right\}}{p_1!\,p_2!\,p_3!\dots}a_1^{p_1}a_2^{p_2}a_3^{p_3}\dots,$$

where $$p_1+p_2+p_3+\dots=k,$$

$$p_1+2p_2+3p_3+\dots=w.$$

From this we deduce, when $m=1$,

$$s_w=S_{w1}=\Sigma\,(-)^{w+k}\frac{(k-1)!\,w}{p_1!\,p_2!\,p_3!\dots}a_1^{p_1}a_2^{p_2}a_3^{p_3}\dots,$$

which is the Girard* formula for the expression of the one-part symmetric functions in terms of the elementary functions.

Observe that the formula, from which the expression for $S_{w,m}$ was derived, yields, by clearing of fractions and subsequently equating coefficients of like powers of x, the relation

$$S_{wm} - a_1 S_{w-1m} + a_2 S_{w-2m} - \ldots + (-)^{w-m} a_{w-m} S_{mm} = (-)^{w-m} \binom{w}{m} a_w;$$

and, herein putting $m = 1$, we obtain the Newtonian relation

$$s_w - a_1 s_{w-1} + a_2 s_{w-2} - \ldots + (-)^w w a_w = 0.$$

The formula given above for $S_{w,m}$ is one generalization, of Girard's well-known result, out of many that will appear as this work proceeds. From the formula for s_w we find

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1^2 - 2a_2, \\ s_3 &= a_1^3 - 3a_1 a_2 + 3a_3, \\ s_4 &= a_1^4 - 4a_1^2 a_2 + 2a_2^2 + 4a_1 a_3 - 4a_4, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and it must be noted that, regarding an a product

$$a_1^{p_1} a_2^{p_2} a_3^{p_3} \ldots$$

as being denoted by the partition $(1^{p_1} 2^{p_2} 3^{p_3} \ldots)$ of w, the expression of s_w, by means of the elementary functions a, involves every partition of w.

6. We may now express a_w by means of the one-part functions s.

For

$$\log\{(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)\ldots\} = \Sigma(-\alpha x - \tfrac{1}{2}\alpha^2 x^2 - \tfrac{1}{3}\alpha^3 x^3 - \ldots),$$

or

$$\log(1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots) = -s_1 x - \tfrac{1}{2} s_2 x^2 - \tfrac{1}{3} s_3 x^3 - \ldots,$$

or

$$1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots = \exp(-s_1 x - \tfrac{1}{2} s_2 x^2 - \tfrac{1}{3} s_3 x^3 - \ldots);$$

and now expansion of the right-hand side, the use of the multinomial theorem and comparison of coefficients, yields

$$a_w = \Sigma(-)^{w+\Sigma p} \frac{s_1^{p_1} s_2^{p_2} s_3^{p_3} \ldots}{1^{p_1}.\, 2^{p_2}.\, 3^{p_3} \ldots p_1! p_2! p_3! \ldots},$$

wherein

$$p_1 + 2p_2 + 3p_3 + \ldots = w.$$

We can also express the one-part functions s in terms of the homogeneous product sums h, for

$$\frac{1}{(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)\ldots} = 1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots;$$

so that, taking logarithms,

$$s_1 x + \tfrac{1}{2} s_2 x^2 + \tfrac{1}{3} s_3 x^3 + \ldots = \log(1 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots);$$

* **Girard**, *Invention Nouvelle en l'Algèbre*, Amsterdam, 1629. The formula is often erroneously ascribed to Waring who gave it without proof in 1782.

and expanding the right-hand side, applying the multinomial theorem and comparing coefficients, we find

$$s_w = \Sigma\,(-)^{k-1}\frac{(k-1)!\,w}{p_1!\,p_2!\,p_3!\,\dots}\,h_1^{p_1}h_2^{p_2}h_3^{p_3}\dots, \quad \text{where} \quad p_1+2p_2+\dots=w,$$
$$\text{also} \qquad p_1+p_2+\dots=k.$$

Moreover

$$1+h_1x+h_2x^2+h_3x^3+\dots=\exp\left(s_1x+\tfrac{1}{2}s_2x^2+\tfrac{1}{3}s_3x^3+\dots\right)$$

leads similarly to

$$h_w=\Sigma\,\frac{s_1^{p_1}\,s_2^{p_2}\,s_3^{p_3}\dots}{1^{p_1}.2^{p_2}.3^{p_3}\dots\,p_1!\,p_2!\,p_3!\,\dots},$$

which should be compared with the expression of a_w in terms of the one-part function.

7. Symmetric functions having two parts, three parts, etc. in their partitions are readily expressible in terms of the functions s_w; for it is easy to see that

$$s_\lambda s_\mu=(\lambda)(\mu)=(\lambda\mu)+(\lambda+\mu),$$
$$s_\lambda s_\mu s_\nu=(\lambda)(\mu)(\nu)=(\lambda\mu\nu)+(\lambda+\mu,\nu)+(\lambda+\nu,\mu)+(\lambda,\mu+\nu)+(\lambda+\mu+\nu),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

by ordinary algebraic multiplication, λ, μ, ν being all different, and thence

$$(\lambda\mu)=s_\lambda s_\mu-s_{\lambda+\mu},$$
$$(\lambda\mu\nu)=s_\lambda s_\mu s_\nu-s_{\lambda+\mu}s_\nu-s_{\lambda+\nu}s_\mu-s_{\mu+\nu}s_\lambda+2s_{\lambda+\mu+\nu};$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

but a modification is necessary when there are any equalities between the numbers λ, μ, ν, ... Ex. gr.

$$2!\,(\lambda^2)=s_\lambda^2-s_{2\lambda},$$
$$3!\,(\lambda^3)=s_\lambda^3-3s_{2\lambda}s_\lambda+2s_{3\lambda},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

In actual practice there are easier ways of calculating the many-part functions and the general formula is of little importance.

From the above it is clear that, of a given weight w, there are as many monomial symmetric functions of weight w as there are partitions of the number w; and that there are (1) as many a products of weight w, (2) as many h products of weight w as there are partitions of the number w. A table may be constructed exhibiting every monomial function of a given weight in terms of the a or of the h products of the same weight. Conversely every a product and every h product of weight w may be exhibited as a linear function of the monomial symmetric functions of weight w.

CHAPTER II

CONNEXION WITH THE THEORY OF DISTRIBUTIONS

8. The reader is assumed to be acquainted with the elementary theory of the permutations and combinations of any number of letters or things not necessarily all of the same sort.

Distribution is the separation of a series of elements into a series of classes; in the general problem the things to be distributed may be of any species, viz.: there may be n things of which p are of one kind, q of a second, r of a third, etc., where $p + q + r + \ldots = n$; it is then convenient to speak of the things as "objects of type $(pqr\ldots)$" so that the objects are defined as to species by a particular partition of the number n; again, the classes into which the objects are to be distributed may be of any species, and this consideration leads us to speak of "classes of type $(p_1 q_1 r_1 \ldots)$" where $p_1 + q_1 + r_1 + \ldots = n_1$ the number of classes; the partition $(p_1 q_1 r_1 \ldots)$ of the number n_1 here defines the classes in regard to species, indicating p_1 classes of one kind, q_1 of a second, r_1 of a third, etc.

If no attention be paid to the arrangement of the objects in a class, whatever be the species of the objects the distribution is said to be one into "parcels"; each parcel is a class of unarranged objects.

If however permutations are permissible amongst objects in the same class, the distribution is said to be one into "groups"; each group is a class of arranged objects.

Two chief problems may be enunciated as follows:

I. To determine the number of distributions of objects of type $(pqr\ldots)$ into parcels of type $(p_1 q_1 r_1 \ldots)$.

II. To determine the number of distributions of objects of type $(pqr\ldots)$ into groups of type $(p_1 q_1 r_1 \ldots)$.

Further we may discuss each of these problems when the distributions are subjected to certain restrictions.

In general the number of objects distributed n will not be less than the number of classes n_1. The partition of a number n into m parts is clearly

the theory of the distributions of objects of type (n) into parcels of type (m), no parcel being empty; in fact the objects are all similar and the classes are all similar. The theory of *compositions* in which the parts are permutable so that for instance (21) and (12) are different compositions of 3, is clearly the distributions of objects of type (n) into parcels of type (1^m), no parcel being empty. The permutation of objects specified by $(pqr\ldots)$ is the distribution of such objects into parcels specified by $(1^{p+q+r+\ldots})$. In general, unless otherwise specified, empty parcels are out of the question.

From the definitions it will appear that when the parcels and objects are equinumerous the group merges into the parcel, since then no permutation or arrangement is possible within the parcel. This also happens when the objects are all similar whatever be the number of the classes.

The permutations of objects of type $(pqr\ldots)$ which were shewn above to be the distribution of objects of type $(pqr\ldots)$ into parcels of type $(1^{p+q+r+\ldots})$ may be also described as the distribution of objects of type $(pqr\ldots)$ into the group of type (1). In general however the parcel and group theories are quite distinct, the identity of theory being only at certain points.

The Distribution Functions.

9. Let $P\{(pqr\ldots), (p_1q_1r_1\ldots)\}$ denote the number of ways of distributing objects of type $(pqr\ldots)$ into parcels of type $(p_1q_1r_1\ldots)$, where

$$p+q+r+\ldots=n, \qquad p_1+q_1+r_1+\ldots=n_1 \text{ and } n \geqslant n_1.$$

Now take the number $P\{(pqr\ldots), (p_1q_1r_1\ldots)\}$ as the coefficient of the algebraic symmetric function $(pqr\ldots)$ and we define as follows:

Definition. The Distribution Function of n objects into parcels of type $(p_1q_1r_1\ldots)$ is the algebraic expression

$$\sum_n P\{(pqr\ldots), (p_1q_1r_1\ldots)\}.(pqr\ldots),$$

the summation being for every partition $(pqr\ldots)$ of the number n.

Similarly let $G\{(pqr\ldots), (p_1q_1r_1\ldots)\}$ denote the number of ways of distributing objects of type $(pqr\ldots)$ into groups of type $(p_1q_1r_1\ldots)$ and we define as follows:

Definition. The Distribution Function of n objects into groups of type $(p_1q_1r_1\ldots)$ is the algebraic expression

$$\sum_n G\{(pqr\ldots), (p_1q_1r_1\ldots)\}.(pqr\ldots),$$

the summation being for every partition $(pqr\ldots)$ of the number n.

Thus the Distribution Functions are linear functions of the monomial symmetric functions of weight n, and depend upon the assigned partition $(p_1q_1r_1\ldots)$ of the number n_1.

Theory of the Distribution into Parcels when the number of parcels is equal to the number of objects. Ex. gr. $n_1 = n$.

10. We consider the homogeneous product sums h_1, h_2, h_3 ... and in particular the product

$$h_{p_1} h_{q_1} h_{r_1} \ldots.$$

Take a simple case. If we had a product $h_4 h_3$ appertaining to the four quantities α, β, γ, δ we may write it

$$\left\{\begin{array}{c} \alpha^4+\beta^4+\gamma^4+\delta^4 \\ +\alpha^3\beta+\alpha\beta^3+\alpha^3\gamma+\alpha\gamma^3+\alpha^3\delta+\alpha\delta^3+\beta^3\gamma+\beta\gamma^3+\beta^3\delta+\beta\delta^3+\gamma^3\delta+\gamma\delta^3 \\ +\alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2 \\ +\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta+\beta^2\alpha\gamma+\beta^2\alpha\delta+\beta^2\gamma\delta+\gamma^2\alpha\beta+\gamma^2\alpha\delta+\gamma^2\beta\delta+\delta^2\alpha\beta+\delta^2\alpha\gamma+\delta^2\beta\gamma \\ +\alpha\beta\gamma\delta \end{array}\right\}$$

multiplied by

$$\left\{\begin{array}{c} \alpha^3+\beta^3+\gamma^3+\delta^3 \\ +\alpha^2\beta+\alpha\beta^2+\alpha^2\gamma+\alpha\gamma^2+\alpha^2\delta+\alpha\delta^2+\beta^2\gamma+\beta\gamma^2+\beta^2\delta+\beta\delta^2+\gamma^2\delta+\gamma\delta^2 \\ +\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta \end{array}\right\}.$$

Seeking herein the coefficients of $\alpha^3\beta^2\gamma\delta$, a term which arises when the multiplication is carried out, we find that the term is formed in eleven ways, viz.

$\alpha^3\beta \,.\, \beta\gamma\delta$	$\alpha^3\gamma \,.\, \beta^2\delta$	$\alpha^3\delta \,.\, \beta^2\gamma$	$\alpha^2\beta^2 \,.\, \alpha\gamma\delta$
$\alpha^2\beta\gamma \,.\, \alpha\beta\delta$	$\alpha^2\beta\delta \,.\, \alpha\beta\gamma$	$\alpha^2\gamma\delta \,.\, \alpha\beta^2$	$\beta^2\alpha\gamma \,.\, \alpha^2\delta$
$\beta^2\alpha\delta \,.\, \alpha^2\gamma$	$\beta^2\gamma\delta \,.\, \alpha^3$	$\alpha\beta\gamma\delta \,.\, \alpha^2\beta$	

We have clearly here a distribution of the seven quantities

$$\alpha,\ \alpha,\ \alpha,\ \beta,\ \beta,\ \gamma,\ \delta,$$

into seven parcels, four of which are of one kind and the remaining three of a second kind; ex. gr. in the first product $\alpha^3\beta \,.\, \beta\gamma\delta$ we have the quantities α, α, α, β placed in four similar parcels since any permutation of the four quantities α, α, α, β is permissible; and similarly we have the quantities β, γ, δ in three similar parcels differing in kind from the former four parcels since permutations are not permitted *between* the quantities α, α, α, β and the quantities β, γ, δ.

Hence in obtaining the portion $11\alpha^3\beta^2\gamma\delta$ of the product $h_4 h_3$ we have in reality distributed objects of type (321^2) into parcels of type (43) in 11 different ways. We have in fact found that

$$P\{(321^2), (43)\} = 11.$$

Moreover we see that every other term of the symmetric function (321^2) appears after multiplication with the same coefficient 11. In other words, if the product $h_4 h_3$ be developed in a series of monomial symmetric functions, the function (321^2) will appear with the coefficient 11.

Thus one term of the distribution function

$$\Sigma P\{(pqr\ldots), (43)\} . (pqr\ldots)$$

where $p+q+r+\ldots=7$, is

$$11\,(321^2).$$

From this reasoning it appears that

$$\Sigma P\{(pqr\ldots), (43)\} . (pqr\ldots) = h_4 h_3$$

where $p+q+r+\ldots=7$.

By parity of reasoning we see that in general

$$\sum_n P\{(pqr\ldots), (p_1 q_1 r_1\ldots)\} . (pqr\ldots) = h_{p_1} h_{q_1} h_{r_1}\ldots.$$

We have merely to multiply out the product $h_{p_1} h_{q_1} h_{r_1}\ldots$ so as to obtain a linear function of monomial symmetric functions in order to arrive at a complete account of the distribution of n objects, of whatever type, into n parcels of type $(p_1 q_1 r_1\ldots)$.

It will be seen later that actual multiplication is not necessary for the attainment of this end.

This result is so simple as to be quite satisfactory.

11. Having thus shewn the power of symmetric functions in solving a notable problem in "Distributions," we at once proceed to obtain an elegant algebraic theorem as an immediate consequence of intuition in the Theory of Distributions.

We will denote parcels by capital letters and objects by small letters, so that one distribution of objects of type (321^2) into parcels of type (43) is shewn by the scheme

$$\begin{array}{ccccccc} A & A & A & A & B & B & B \\ a & a & a & b & b & c & d \end{array}.$$

If we now change the capital letters into small ones and the small letters into capital ones we obtain the scheme

$$\begin{array}{ccccccc} A & A & A & B & B & C & D \\ a & a & a & a & b & b & b \end{array}',$$

which denotes one distribution of objects of type (43) into parcels of type (321^2); and since we may always operate in this manner we ascertain that there is a one-to-one correspondence between the distributions of objects of type (43) into parcels of type (321^2).

Hence $$P\{(321^2), (43)\} = P\{(43), (321^2)\}.$$

The reasoning is evidently quite general; so that also

$$P\{(pqr\ldots), (p_1q_1r_1\ldots)\} = P\{(p_1q_1r_1\ldots), (pqr\ldots)\}.$$

It is clear that when, as in the present instance, the number of objects is equal to the number of parcels so that always one object is in each parcel, we are at liberty to regard an object as *attached* to a parcel or a parcel as *attached* to an object. The fact is that in this particular case of distribution the notion of a "parcel" is not essential and indeed has the effect of obscuring the true point of view. We should regard the parcels as being a second set of objects so that we have two sets of objects before us. We are concerned with the number of ways of making n pairs of objects by taking, for each pair, one object from each set. The interchange of object and parcel is thus intuitively valid. The n pairs of objects may be looked upon as a single set of n two-fold objects and as such we may be concerned with the similarities that present themselves in the two-fold objects. Such similarities may, as in the case of one-fold objects, be specified by a partition of n, say $(p_2q_2r_2\ldots)$. We may say that the particular distribution that has been reached is of type $(p_2q_2r_2\ldots)$. Each distribution will in this manner be associated with a partition of n. Altogether we have before us three partitions of the same number, viz.: those specifying each set of objects and that specifying the distribution.

Since $$P\{(pqr\ldots), (p_1q_1r_1\ldots)\}$$

is equal to the coefficient of the symmetric function $(pqr\ldots)$ in the development of the algebraic function

$$h_{p_1}h_{q_1}h_{r_1}\ldots$$

we reach at once a theorem of symmetry in the algebra of symmetric functions which is now stated.

First Law of Symmetry.

12. "In a Table which expresses products $h_ph_qh_r\ldots$ in terms of monomial symmetric functions $(p_1q_1r_1\ldots)$ for all partitions $(pqr\ldots)$ of the number n, the coefficient of the symmetric function $(p_1q_1r_1\ldots)$ in the expression of $h_ph_qh_r\ldots$ is equal to the coefficient of the symmetric function $(pqr\ldots)$ in the expression of $h_{p_1}h_{q_1}h_{r_1}\ldots$, where $(pqr\ldots)$, $(p_1q_1r_1\ldots)$ are any two partitions of the number n."

This interesting law of symmetry apparently cannot be proved without some difficulty in any other way and is a good example of the power of the notions of the Theory of Distributions in establishing elegant theorems in algebra. From the Table above mentioned we can by solving a number of linear equations express the monomial symmetric functions in terms of the products $h_{p_1}h_{q_1}h_{r_1}\ldots$, and the Elementary Theory of Determinants shews that the symmetrical character of this second Table follows necessarily from the

symmetry of the first Table. Thus in such a table the coefficient of the product $h_{p_1}h_{q_1}h_{r_1}\ldots$ in the expression of the symmetric function $(pqr\ldots)$ is equal to the coefficient of the product $h_p h_q h_r \ldots$ in the expression of the symmetric function $(p_1q_1r_1\ldots)$, where $(pqr\ldots)$, $(p_1q_1r_1\ldots)$ are *any* two partitions of the number n.

It is however not necessary to rely upon the Theory of Determinants for the proof of the symmetry of the second Table; later on in this work another proof will be given with the assistance of differential operators.

A restricted Distribution.

13. We pass on to consider the Distribution subject to the restriction "that parcels of the same kind must not contain more than one object of a particular kind"; we mean that if one parcel A is attached to an object a no other parcel A is to contain an object a.

In the unrestricted distribution we considered the terms of $h_{p_1}, h_{q_1}, h_{r_1}$, etc., all written out at length and observed that we might regard any one term of h_{p_1} as representing a distribution of p_1 objects into similar parcels. In the case before us these p_1 objects must be all different—no two must be alike—and so we reject from h_{p_1} all terms except those in $\Sigma\alpha_1\alpha_2\ldots\alpha_{p_1}$. In like manner the functions h_{q_1}, h_{r_1}, ... reduce to $\Sigma\alpha_1\alpha_2\ldots\alpha_{q_1}$, $\Sigma\alpha_1\alpha_2\ldots\alpha_{r_1}$, ... respectively. In other words $h_{p_1}, h_{q_1}, h_{r_1}, \ldots$ must be replaced by $a_{p_1}, a_{q_1}, a_{r_1}, \ldots$ respectively, and the resulting distribution function is

$$a_{p_1}a_{q_1}a_{r_1}\ldots.$$

Theorem. The Distribution function for the distribution of objects of type $(pqr\ldots)$ into parcels of type $(p_1q_1r_1\ldots)$, subject to the restriction that no two similar objects shall appear in similar parcels, is

$$a_{p_1}a_{q_1}a_{r_1}\ldots.$$

If no two similar objects can appear in similar parcels it follows that no two similar parcels can be attached to or associated with similar objects. Hence in the scheme we may replace capitals by small letters and small letters by capitals.

We accordingly find that there are as many ways of distributing objects of type $(pqr\ldots)$ into parcels of type $(p_1q_1r_1\ldots)$ as of distributing objects of type $(p_1q_1r_1\ldots)$ into parcels of type $(pqr\ldots)$—both distributions being subject to the restriction that two similar objects are not to appear in similar parcels. Thus when the symmetric function product $a_p a_q a_r\ldots$ is expressed in terms of monomial symmetric functions the coefficient of $(p_1q_1r_1\ldots)$ is equal to the coefficient of $(pqr\ldots)$ when $a_{p_1}a_{q_1}a_{r_1}\ldots$ is so expressed.

We thus reach the

Second Law of Symmetry.

14. "In a Table which expresses products of elementary symmetric functions $a_p a_q a_r \ldots$ in terms of monomial symmetric functions $(p_1 q_1 r_1 \ldots)$ for all partitions $(pqr\ldots)$ of the number n, the coefficient of the symmetric function $(p_1 q_1 r_1 \ldots)$ in the expression of $a_p a_q a_r \ldots$ is equal to the coefficient of the symmetric function $(pqr\ldots)$ in the expression of $a_{p_1} a_{q_1} a_{r_1} \ldots$, where $(pqr\ldots)$, $(p_1 q_1 r_1 \ldots)$ are *any* two partitions of the number n."

The inverse Table which expresses the monomial symmetric functions in terms of the products $a_p a_q a_r \ldots$ is, from what has been said above, also symmetrical.

The reader will readily gather from the above that we may make a more general restriction still govern the distribution.

If we delete from the functions $h_{p_1}, h_{q_1}, h_{r_1}, \ldots$ all terms which contain any of the letters $\alpha_1, \alpha_2, \alpha_3, \ldots$ more than t times we shall obtain functions which may be written

$$t_{p_1}, t_{q_1}, t_{r_1} \ldots,$$

corresponding to the restriction that not more than t similar objects can appear in similar parcels. Parity of reasoning then establishes that

$$t_{p_1} t_{q_1} t_{r_1} \ldots$$

is the Distribution Function for such a distribution and that the coefficient of $t_{p_1} t_{q_1} t_{r_1} \ldots$ in the expression of the symmetric function $(pqr\ldots)$ by means of $t_1, t_2, t_3, \ldots$ is equal to the coefficient of the $t_p t_q t_r \ldots$ in the like expression of the symmetric function $(p_1 q_1 r_1 \ldots)$. We are thus led to the

Inclusive Law of Symmetry.

15. "In a Table which expresses products $t_p t_q t_r \ldots$ in terms of monomial symmetric functions $(p_1 q_1 r_1 \ldots)$ for all partitions $(pqr\ldots)$ of the number n, the coefficient of the symmetric function $(p_1 q_1 r_1 \ldots)$ in the expression of $t_p t_q t_r \ldots$ is equal to the coefficient of the symmetric function $(pqr\ldots)$ in the expression of $t_{p_1} t_{q_1} t_{r_1} \ldots$, where $(pqr\ldots)$, $(p_1 q_1 r_1 \ldots)$ are any two partitions of the number n."

The inverse Tables naturally enjoy the same symmetry.

The Inclusive Law of Symmetry involves the First Law when $t = \infty$ and the Second Law when $t = 1$.

Theory of the Distribution into Parcels specified by (1^m) *when* $m < n$.

16. In this case no two of the parcels are alike.

Suppose that the number of objects is six and that there are three parcels of different colours—say red, black and white.

Red	Black	White
h_4	h_1	h_1
h_3	h_2	h_1
h_2	h_2	h_2

It is clear that the parcels, in some order, will contain either 4, 1, 1 or 3, 2, 1 or 2, 2, 2 objects respectively, since an empty parcel is out of the question. When a parcel contains p objects the objects will be denoted by one of the terms of which the function h_p is composed.

Hence when the parcels contain, in some order, 4, 1, 1 objects respectively we are concerned with the product $h_4h_1^2$ and since moreover $h_4h_1h_1$ has three permutations we are concerned with the term $3h_4h_1^2$; similarly when the parcels contain, in some order, 3, 2, 1 objects we have the term $6h_3h_2h_1$ and when the parcels contain 2, 2, 2 objects, the term h_2^3.

Hence for this simple case the distribution function must be

$$3h_4h_1^2 + 6h_3h_2h_1 + h_2^3.$$

The reasoning is of general application and establishes the fact that the Distribution Function of n objects into Parcels of type (1^m) is

$$\Sigma \frac{(\Sigma\pi)!}{\pi_1!\,\pi_2!\,\pi_3!\,\dots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \dots,$$

where $\Sigma p\pi = n$, $\Sigma\pi = m$.

This function is the coefficient of x^n in

$$(h_1x + h_2x^2 + h_3x^3 + \dots)^m.$$

Theorem. "The number of ways of distributing objects of type $(pqr\dots)$ into exactly m different parcels—that is into parcels no two of which are alike—is equal to the coefficient of the symmetric function $(pqr\dots)$ in the development of the function

$$\Sigma \frac{m!}{\pi_1!\,\pi_2!\,\pi_3!\,\dots} h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \dots,$$

where $p + q + r + \dots = \Sigma p\pi = n$; $\Sigma\pi = m$."

17. The number $P\{(pqr\dots), (1^m)\}$, whose generating function has been found, can be evaluated.

We require the coefficient of $(pqr\dots)x^n$ in the expansion of u^m, where

$$u = h_1x + h_2x^2 + h_3x^3 + \dots$$

and
$$1 + u = (1 - \alpha_1x)^{-1}(1 - \alpha_2x)^{-1}(1 - \alpha_3x)^{-1} \dots;$$

now
$$u^m = (1 + u - 1)^m$$
$$= (1+u)^m - \binom{m}{1}(1+u)^{m-1} + \binom{m}{2}(1+u)^{m-2} - \dots;$$

and the coefficient of $(pqr\ldots)\,x^n$ in

$$(1+u)^s \quad \text{or} \quad (1-\alpha_1 x)^{-s}(1-\alpha_2 x)^{-s}(1-\alpha_3 x)^{-s}\ldots$$

is

$$\binom{p+s-1}{s-1}\binom{q+s-1}{s-1}\binom{r+s-1}{s-1}\ldots;$$

so that

$$P\{(pqr\ldots),(1^m)\}$$

$$=\binom{p+m-1}{m-1}\binom{q+m-1}{m-1}\binom{r+m-1}{m-1}\ldots$$

$$-\binom{m}{1}\binom{p+m-2}{m-2}\binom{q+m-2}{m-2}\binom{r+m-2}{m-2}\ldots$$

$$+\binom{m}{2}\binom{p+m-3}{m-3}\binom{q+m-3}{m-3}\binom{r+m-3}{m-3}\ldots$$

$$-\ldots \text{ to } m \text{ terms.}$$

We now make the restriction that not more than t similar objects are to occur in any one parcel. With a previous notation we require the coefficient of $(pqr\ldots)\,x^n$ in the development of

$$(t_1x+t_2x^2+t_3x^3+\ldots)^m.$$

Putting $u=t_1x+t_2x^2+t_3x^3+\ldots$ we see that

$$1+u=\frac{1-\alpha^{t+1}x^{t+1}}{1-\alpha x}\cdot\frac{1-\beta^{t+1}x^{t+1}}{1-\beta x}\cdot\frac{1-\gamma^{t+1}x^{t+1}}{1-\gamma x}\ldots;$$

and

$$(1+u)^m=\prod_{\alpha}\left\{1-\binom{m}{1}\alpha^{t+1}x^{t+1}+\binom{m}{2}\alpha^{2t+2}x^{2t+2}-\ldots\right\}$$

$$\times\left\{1+\binom{m}{1}\alpha x+\binom{m+1}{2}\alpha^2x^2+\ldots\right\}.$$

In this product the coefficient of $(pqr\ldots)\,x^n$, where $p+q+r+\ldots=n$, is

$$\left\{\binom{p+m-1}{m-1}-\binom{m}{1}\binom{p+m-t-2}{m-1}+\binom{m}{2}\binom{p+m-2t-3}{m-1}-\ldots\right\}$$

$$\times\left\{\binom{q+m-1}{m-1}-\binom{m}{1}\binom{q+m-t-2}{m-1}+\binom{m}{2}\binom{q+m-2t-3}{m-1}-\ldots\right\}$$

$$\times\left\{\binom{r+m-1}{m-1}-\binom{m}{1}\binom{r+m-t-2}{m-1}+\binom{m}{2}\binom{r+m-2t-3}{m-1}-\ldots\right\}$$

$$\times\ldots\ldots\ldots;$$

and, since $u^m=(1+u)^m-\binom{m}{1}(1+u)^{m-1}+\binom{m}{2}(1+u)^{m-2}-\ldots,$

we find that we may enunciate as follows:

Theorem. "The number of distributions of objects of type $(pqr\ldots)$ into parcels of type (1^m), subject to the restriction that not more than t similar objects may appear in any one parcel, is

$$F(m)-\binom{m}{1}F(m-1)+\binom{m}{2}F(m-2)-\ldots \text{ to } m \text{ terms,}$$

where

$$F(m) = \left\{\binom{p+m-1}{m-1} - \binom{m}{1}\binom{p+m-t-2}{m-1} + \binom{m}{2}\binom{p+m-2t-3}{m-1} - \dots\right\}$$
$$\times \left\{\binom{q+m-1}{m-1} - \binom{m}{1}\binom{q+m-t-2}{m-1} + \binom{m}{2}\binom{q+m-2t-3}{m-1} - \dots\right\}$$
$$\times \left\{\binom{r+m-1}{m-1} - \binom{m}{1}\binom{r+m-t-2}{m-1} + \binom{m}{2}\binom{r+m-2t-3}{m-1} - \dots\right\}$$
$$\times \dots\dots\dots"$$

18. In the particular case, $t = 1$, the Distribution Function is the coefficient of x^n in

$$(a_1x + a_2x^2 + a_3x^3 + \dots)^m;$$

and the number sought is

$$F(m) - \binom{m}{1}F(m-1) + \binom{m}{2}F(m-2) - \dots,$$

where

$$F(m) = \left\{\binom{p+m-1}{m-1} - \binom{m}{1}\binom{p+m-3}{m-1} + \binom{m}{2}\binom{p+m-5}{m-1} - \dots\right\}$$
$$\times \left\{\binom{q+m-1}{m-1} - \binom{m}{1}\binom{q+m-3}{m-1} + \binom{m}{2}\binom{q+m-5}{m-1} - \dots\right\}$$
$$\times \left\{\binom{r+m-1}{m-1} - \binom{m}{1}\binom{r+m-3}{m-1} + \binom{m}{2}\binom{r+m-5}{m-1} - \dots\right\}$$
$$\times \dots\dots\dots.$$

The Distribution into Groups of type (1^m) *where* $m < n$.

19. Consider the expansion of

$$a_1^n = h_1^n = (\alpha_1 + \alpha_2 + \alpha_3 + \dots)^n.$$

It consists of products of the quantities $\alpha_1, \alpha_2, \alpha_3 \dots$ of degree n, taken in all possible ways, repetitions and permutations being alike allowable. On this understanding the expansion consists of a number of terms each with coefficient unity. Suppose any such term to be

$$\alpha_2\alpha_1\alpha_1\alpha_3\alpha_2\alpha_3\alpha_1\dots$$

and place dividing lines in any $m-1$ of the $n-1$ spaces between the letters, thus dividing the permutation into m compartments. This may be carried out in

$$\binom{n-1}{m-1} \text{ ways.}$$

To each of these ways corresponds a distribution of the objects included in the permutation under consideration into m Groups, no two of which are similar.

Hence the Distribution Function is

$$\binom{n-1}{m-1} a_1^n;$$

and
$$\Sigma G\{(pqr\ldots), (1^m)\} \cdot (pqr\ldots) = \binom{n-1}{m-1} a_1^n;$$

and also
$$G\{(pqr\ldots), (1^m)\} = \binom{n-1}{m-1} \frac{n!}{p!\,q!\,r!\ldots},$$

where
$$p+q+r+\ldots = n.$$

It will be observed that the distribution function is the coefficient of x^n in the development of $\left(\frac{a_1 x}{1-a_1 x}\right)^m$, or of

$$(A_1 x + A_2 x^2 + A_3 x^3 + \ldots)^m,$$

wherein
$$A_s = a_1^s;$$

so that $A_1 = (1)$; $A_2 = (2) + 2(1^2)$; $A_3 = (3) + 3(21) + 6(1^3)$; etc.

The coefficient of x^n in $(A_1 x + A_2 x^2 + A_3 x^3 + \ldots)^m$ is

$$\Sigma \frac{m!}{k_1!\,k_2!\,k_3!\ldots} A_1^{k_1} A_2^{k_2} A_3^{k_3} \ldots,$$

where
$$k_1 + 2k_2 + 3k_3 + \ldots = n,$$
$$k_1 + k_2 + k_3 + \ldots = m;$$

and this shews that we may associate the m functions which make up the product $A_1^{k_1} A_2^{k_2} A_3^{k_3}\ldots$ in all their permutations with the m compartments. It follows that, if the distribution is subject to the condition that not more than t similar objects are to appear in the same compartment or group, we must delete from the expressions of $A_1, A_2, A_3, \ldots$, in terms of monomial symmetric functions, all of those functions whose partitions involve numbers greater than t. Denote these deleted functions by

$$T_1, T_2, T_3, \ldots;$$

and we find that the Distribution Function, for the supposed restriction, is the coefficient of x^n in the expansion of

$$(T_1 x + T_2 x^2 + T_3 x^3 + \ldots)^m.$$

In particular when $t=1$, we are concerned with the coefficient of x^n in the expansion of

$$(a_1 x + 2!\,a_2 x^2 + 3!\,a_3 x^3 + \ldots)^m.$$

If the restriction be that "a group must comprise similar objects" we are concerned with the expansion of

$$(s_1 x + s_2 x^2 + s_3 x^3 + \ldots)^m.$$

CHAPTER III

THE DISTRIBUTION INTO PARCELS AND GROUPS IN GENERAL

The Distribution into Parcels of type (m) where $m < n$.

20. We are here concerned with the partitions of multipartite numbers into exactly m parts. This is one of the main divisions of the subject and much will be said about it later on in the work. At present it will be considered merely from the symmetric function point of view.

Consider

$$h_1 + h_2 + h_3 + h_4 + \ldots$$

to be written out at length in terms of

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \ldots.$$

If these products represent the objects distributed in a parcel it is clear that the m parcels *may*, each of them, contain any term of

$$h_1 + h_2 + h_3 + h_4 + \ldots;$$

write then

$$\frac{1}{(1-\alpha_1 ax)(1-\alpha_2 ax)(1-\alpha_3 ax)\ldots}$$
$$\times \frac{1}{(1-\alpha_1^2 ax^2)(1-\alpha_2^2 ax^2)(1-\alpha_3^2 ax^2)\ldots(1-\alpha_1\alpha_2 ax^2)(1-\alpha_1\alpha_3 ax^2)(1-\alpha_2\alpha_3 ax^2)\ldots}$$
$$\times \frac{1}{(1-\alpha_1^3 ax^3)(1-\alpha_2^3 ax^3)\ldots(1-\alpha_1^2\alpha_2 ax^3)\ldots(1-\alpha_1\alpha_2\alpha_3 ax^3)\ldots}$$
$$\times \ldots\ldots\ldots,$$

where in the sth algebraic fraction we have a denominator factor corresponding to every separate term of h_s; ex. gr. if such a term be σ the denominator factor is $(1-\sigma ax^s)$; we see that the distribution function is the coefficient of $a^m x^n$ in the development of the expression, n of course being the number of the objects under distribution. The expression is conveniently written

$$\begin{aligned}
&(1 + H_1^1 ax + H_2^1 a^2x^2 + H_3^1 a^3x^3 + \ldots)\\
&\times (1 + H_1^2 ax^2 + H_2^2 a^2x^4 + H_3^2 a^3x^6 + \ldots)\\
&\times (1 + H_1^3 ax^3 + H_2^3 a^2x^6 + H_3^3 a^3x^9 + \ldots)\\
&\times \ldots\ldots\ldots\\
&\times (1 + H_1^s ax^s + H_2^s a^2x^{2s} + H_3^s a^3x^{3s} + \ldots)\\
&\times \ldots\ldots\ldots,
\end{aligned}$$

where of course $H_w^1 = h_w$ and the remaining functions H are for discussion.

21. H_p^s is the homogeneous product sum, p together, of the whole of the terms of h_s.

Form the equation whose roots are the several terms of h_s, viz.

$$x^k - j_1 x^{k-1} + j_2 x^{k-2} - \ldots = 0,$$

where $$k = \binom{n+s-1}{s}, \quad \text{and} \quad j_1 = H_s^1 = H_1^s = h_s;$$

and let σ_k denote the sum of the kth powers of its roots.

Let also the partitions in ( ) and [ ], respectively, denote symmetric functions of the roots of

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0,$$
$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0.$$

We can establish the two results

$$(k) = (-)^{k+1}[k],$$
$$[k^s] = (-)^{s(k+1)}\sigma_k.$$

The first of these is well known; in regard to the second take the identity

$$\frac{1}{\left(1-\frac{\alpha_1}{x}\right)\left(1-\frac{\alpha_2}{x}\right)\left(1-\frac{\alpha_3}{x}\right)\ldots} = \left(1+\frac{u_1}{x}\right)\left(1+\frac{u_2}{x}\right)\left(1+\frac{u_3}{x}\right)\ldots,$$

where $u_1, u_2, u_3, \ldots$ are the roots of the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0,$$

and n is supposed indefinitely great.

Let $\rho_1, \rho_2, \ldots \rho_k$ be the kth roots of unity, so that

$$\prod_{s=1}^{s=k} \frac{1}{\left(1-\frac{\rho_s\alpha_1}{x}\right)\left(1-\frac{\rho_s\alpha_2}{x}\right)\left(1-\frac{\rho_s\alpha_3}{x}\right)\ldots} = \prod_{s=1}^{s=k}\left(1+\frac{\rho_s u_1}{x}\right)\left(1+\frac{\rho_s u_2}{x}\right)\left(1+\frac{\rho_s u_3}{x}\right)\ldots;$$

or

$$\frac{1}{\left(1-\frac{\alpha_1^k}{x^k}\right)\left(1-\frac{\alpha_2^k}{x^k}\right)\left(1-\frac{\alpha_3^k}{x^k}\right)\ldots}$$
$$= \left\{1+(-)^{k+1}\frac{u_1^k}{x^k}\right\}\left\{1+(-)^{k+1}\frac{u_2^k}{x^k}\right\}\left\{1+(-)^{k+1}\frac{u_3^k}{x^k}\right\}\ldots;$$

or

$$\frac{1}{1-\frac{(k)}{x^k}+\frac{(k^2)}{x^{2k}}-\frac{(k^3)}{x^{3k}}+\ldots}$$
$$= 1+(-)^{k+1}\frac{[k]}{x^k}+(-)^{2(k+1)}\frac{[k^2]}{x^{2k}}+\ldots+(-)^{s(k+1)}\frac{[k^s]}{x^{sk}}+\ldots.$$

The coefficient of $\frac{1}{x^{sk}}$ in the sinister of this identity is the homogeneous

product sum of order s of the quantities $\alpha_1^k, \alpha_2^k, \alpha_3^k, \ldots$; it is thus equal to σ_k, the sum of the kth powers of the roots of

$$x^k - j_1 x^{k-1} + j_2 x^{k-2} - \ldots = 0.$$

Hence $$\sigma_k = (-)^{s(k+1)} [k^s],$$

leading to the relations

$$H_1^s = \sigma_1 = [1^s],$$

$$H_2^s = \frac{1}{2!}(\sigma_1^2 + \sigma_2) = \frac{1}{2!}\{[1^s]^2 + (-)^s [2^s]\},$$

$$H_3^s = \frac{1}{3!}\{[1^s]^3 + (-)^s 3\,[1^s][2^s] + 2\,[3^s]\},$$

and so forth. The law is identical with that which obtains in the expression of the elementary symmetric functions in terms of the sums of powers, with the exception that the signs are all positive when s is even.

The distribution function when multiplied out is

$$\Sigma\Sigma H_{t_1}^{\tau_1} H_{t_2}^{\tau_2} H_{t_3}^{\tau_3} \ldots a^m x^n;$$

where $$\Sigma\tau t = n, \quad \Sigma t = m;$$

and $$H_t^s = \Sigma (-)^{s(3p_2+5p_4+\ldots)} \frac{[1^s]^{p_1}[2^s]^{p_2}[3^s]^{p_3}\ldots}{1^{p_1}.\,2^{p_2}.\,3^{p_3}\ldots p_1!\,p_2!\,p_3!\ldots} *,$$

where $$p_1 + 2p_2 + 3p_3 + \ldots = t.$$

As the symmetric functions in brackets [ ] refer to the roots of

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0,$$

the Distribution Function may be calculated in terms of h functions.

Ex. gr. The Distribution Function of 4 objects into parcels of type (2) is

$$H_2^2 + H_1^2 H_1^1 = \frac{1}{2!}(h_2^2 + h_2^2 - 2h_1h_3 + 2h_4) + h_1h_3 = h_2 + h_4$$

$$= 2\,(4) + 3\,(31) + 4\,(2^2) + 5\,(21^2) + 7\,(1^4);$$

where the numbers 2, 3, 4, 5, 7 are in correspondence with distributions

$aaa\|a$	$aaa\|b$	$aab\|b$	$aab\|c$	$abc\|d$ $ab\|cd$
$aa\|aa$	$aab\|a$	$abb\|a$	$aac\|b$	$abd\|c$ $ac\|bd$
	$aa\|ab$	$aa\|bb$	$abc\|a$	$acd\|b$ $ad\|bc$
		$ab\|ab$	$aa\|bc$	$bcd\|a$
			$ab\|ac$	
2	3	4	5	7

or with the partitions of the multipartite numbers

$$(4),\quad (31),\quad (22),\quad (211),\quad (1111),$$

into exactly two parts.

* Compare Art. 6.

The Distribution into Groups of type (m).

22. We have to take account in the distribution into Parcels of type (m) of permutations amongst the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ which occur in the denominator factors of the first written form of distribution function. Thus the present distribution function is

$$\frac{1}{(1-\alpha_1 ax)(1-\alpha_2 ax)(1-\alpha_3 ax)\ldots}$$

$$\times \frac{1}{(1-\alpha_1^2 ax^2)(1-\alpha_2^2 ax^2)(1-\alpha_3^2 ax^2)\ldots(1-\alpha_1\alpha_2 ax^2)^2(1-\alpha_1\alpha_3 ax^2)^2(1-\alpha_2\alpha_3 ax^2)^2\ldots}$$

$$\times \frac{1}{(1-\alpha_1^3 ax^3)(1-\alpha_2^3 ax^3)\ldots(1-\alpha_1^2\alpha_2 ax^3)^3\ldots(1-\alpha_1\alpha_2\alpha_3 ax^3)^6\ldots}$$

$$\times \ldots\ldots\ldots .$$

This may be written

$$\begin{aligned}
&(1 + K_1^1 ax + K_2^1 a^2x^2 + K_3^1 a^3x^3 + \ldots)\\
&\times (1 + K_1^2 ax^2 + K_2^2 a^2x^4 + K_3^2 a^3x^6 + \ldots)\\
&\times (1 + K_1^3 ax^3 + K_2^3 a^2x^6 + K_3^3 a^3x^9 + \ldots)\\
&\times \ldots\ldots\ldots\\
&\times (1 + K_1^s ax^s + K_2^s a^2x^{2s} + K_3^s a^3x^{3s} + \ldots)\\
&\times \ldots\ldots\ldots .
\end{aligned}$$

The denominator factors correspond to the terms of

$$h_1 + h_1^2 + h_1^3 + \ldots + h_1^s + \ldots$$

and we have the functions K for discussion.

Let $x^w - k_1 x^{w-1} + k_2 x^{w-2} - \ldots = 0$ be the equation having for its roots the several quantities of which K_t^s is the homogeneous product sum of order t.

Then $$k_1 = h_1^s = K_1^s.$$

Further let σ_t denote the sum of the tth powers of the roots of this equation.

If partitions in ( ) refer to symmetric functions of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0,$$

we have $$\sigma_t = (\alpha_1^t + \alpha_2^t + \alpha_3^t + \ldots)^s = (t)^s;$$

so that $$K_2^s = \frac{1}{2!}\{(1)^{2s} + (2)^s\},$$

$$K_3^s = \frac{1}{3!}\{(1)^{3s} + 3\,(1)^s(2)^s + 2\,(3)^s\},$$

and so forth.

Hence finally, transforming to symmetric functions of the roots of the equation

$$x^n - h_1 x^{n-1} + h_2 x^{n-2} - \ldots = 0$$

and denoting its symmetric functions by partitions in brackets [ ],

$$K_1^s = [1]^s,$$

$$K_2^s = \frac{1}{2!}\{[1]^{2s} + (-)^s [2]^s\},$$

$$K_3^s = \frac{1}{3!}\{[1]^{3s} + (-)^s 3\,[1]^s [2]^s + 2\,[3]^s\},$$

.....

$$K_t^s = \Sigma\,(-)^{s(3p_2+5p_4+\ldots)} \frac{[1]^{sp_1}[2]^{sp_2}[3]^{sp_3}\ldots}{1^{p_1}.2^{p_2}.3^{p_3}\ldots p_1!\,p_2!\,p_3!\ldots},$$

where $$p_1 + 2p_2 + 3p_3 + \ldots = t,$$

the Distribution Function is then

$$\Sigma\Sigma K_{t_1}^{\tau_1} K_{t_2}^{\tau_2} K_{t_3}^{\tau_3} \ldots$$

where $$\Sigma\tau t = n,\ \Sigma t = m.$$

It will be noticed that in this case we can as readily express the functions K in terms of a as in terms of h functions.

Ex. gr. The distribution function of 4 objects into groups of type (2) is

$$K_2^2 + K_1^3 K_1^1 = \tfrac{1}{2}\{(1)^4 + (2)^2\} + (1)^3(1)$$
$$= 2a_1^4 - 2a_1^2 a_2 + 2a_2^2 = 2h_1^4 - 2h_1^2 h_2 + 2h_2^2$$
$$= 2\,(4) + 6\,(31) + 10\,(22) + 18\,(211) + 36\,(1111);$$

where the coefficients are in correspondence with the distributions

$aaa\|a$ 1;	$aaa\|b$ 1;	$aab\|b$ 3;	$aab\|c$ 3;	$\overline{abc\|d}$ 6;	$\overline{ab\|cd}$ 4;
$aa\|aa$ 1;	$aab\|a$ 3;	$abb\|a$ 3;	$aac\|b$ 3;	$abd\|c$ 6;	$ac\|bd$ 4;
	$aa\|ab$ 2;	$aa\|bb$ 1;	$abc\|a$ 6;	$acd\|b$ 6;	$ad\|bc$ 4;
		$ab\|ab$ 3;	$aa\|bc$ 2;	$bcd\|a$ 6;	
			$ab\|ac$ 4;		
2;	6;	10;	18;	36;	

the meaning, in the above, of $ab|ab$ 3 being that, by permutations of objects inside the parcels, three arrangements are obtainable, viz.:

$$ab|ab, \quad ab|ba, \quad ba|ba,$$

$ba|ab$ being *omitted* because the parcels are similar and therefore interchangeable; thus $ba|ab$ and $ab|ba$ yield *one* not *two* distributions.

Distribution into Parcels of type $(m_1 m_2 \dots m_r)$.

23. First suppose $r = 2$, so that we are concerned with m_1 similar parcels of one kind and m_2 similar parcels of another kind. Of the n objects we may have m_1 objects only distributed among the m_1 similar parcels, one object in each parcel, and the remaining $n - m_1$ objects distributed among the m_2 similar parcels; or we may have $m_1 + k$ objects distributed among the m_1 similar parcels and $n - m_1 - k$ objects among the m_2 similar parcels, where of course $k \geqslant n - m_1 - m_2$. It thence follows that the Distribution Function is the coefficient of $\mu_1^{m_1}\mu_2^{m_2}x^n$ in the product

$$\prod_{s=1}^{s=\infty}(1 + H_1^s\mu_1 x^s + H_2^s\mu_1^2 x^{2s} + \dots)(1 + H_1^s\mu_2 x^s + H_2^s\mu_2^2 x^{2s} + \dots),$$

and it easily also follows that the Distribution Function of n objects into parcels of type $(m_1 m_2 \dots m_r)$ is the coefficient of $\mu_1^{m_1}\mu_2^{m_2}\dots\mu_r^{m_r}x^n$ in the product

$$\prod_{s=1}^{s=\infty}(1 + H_1^s\mu_1 x^s + H_2^s\mu_1^2 x^{2s} + \dots)(1 + H_1^s\mu_2 x^s + H_2^s\mu_2^2 x^{2s} + \dots)\dots \dots(1 + H_1^s\mu_r x^s + H_2^s\mu_r^2 x^{2s} + \dots).$$

As an example consider the distribution of 6 objects into parcels of ue (22). The coefficient of $\mu_1^2\mu_2^2 x^6$ in the generating function is

$$2H_2^2 + 2H_1^1H_2^1H_1^3 + H_1^{12}H_1^{22}$$
$$= 2h_2^3 + 2h_2h_4 + h_1^2h_2^2$$
$$= (51) + 14(51) + 26(42) + 40(41^2) + 30(3^2) + 64(321) + 104(31^3) + 87(2^3) + 140(2^21^2) + 232(21^4) + 390(1^6).$$

aking capital letters to denote the parcels, we find that the arrangements of 6 similar objects are shewn by

A	*A*	*B*	*B*
aaa	*a*	*a*	*a*
a	*a*	*a*	*aaa*
aa	*aa*	*a*	*a*
a	*a*	*aa*	*aa*
a	*aa*	*a*	*aa*

viz.: they are 5 in number as given by the formula. Similarly if the objects be of type (51) we have, as required, 14 arrangements

A A B B	*A A B B*	*A A B B*	*A A B B*	*A A B B*
aab a a a	*a a a aab*	*ab aa a a*	*a a aa ab*	*b aa a aa*
aaa b a a	*a a b aaa*	*aa aa a b*	*a b aa aa*	*a ab a aa*
aaa a b a	*a b a aaa*			*a aa b aa*
				a aa a ab

and similarly all of the coefficients of the function are verified.

Distribution into Groups of type $(m_1 m_2 \dots m_r)$.

24. The same reasoning that has just been applied to Parcels will convince the reader that the Distribution Function is the coefficient of

$$\mu_1^{m_1} \mu_2^{m_2} \dots \mu_r^{m_r} x^n$$

in the development of the Generating Function

$$\prod_{s=1}^{s=\infty} (1 + K_1^s \mu_1 x^s + K_2^s \mu_1^2 x^{2s} + \dots)(1 + K_1^s \mu_2 x^s + K_2^s \mu_2^2 x^{2s} + \dots) \dots \\ \dots (1 + K_1^s \mu_r x^s + K_2^s \mu_r^2 x^{2s} + \dots)$$

and we then find that the Distribution Function for 6 objects into Groups (22) is

$$2K_2^1 K_2^2 + 2K_1^1 K_2^1 K_1^3 + K_1^2 K_1^2$$

which is

$$h_1^6 + 4h_1^4 h_2 - 4h_1^2 h_2^2 + 4h_2^3$$

or $5(6) + 22(51) + 51(42) + 94(41^2) + 64(3^2) + 176(321) + 336(31^3)$
$+ 258(2^3) + 488(2^2 1^2) + 936(21^4) + 1800(1^6)$,

and as before the reader will have no difficulty in verifying these coefficients by means of the actual distributions.

CHAPTER IV

THE OPERATORS OF THE THEORY OF DISTRIBUTIONS

25. It will be gathered from the preceding chapters that the Theory of Symmetric Functions is essentially and deeply involved in the Theory of Distributions. The two theories might almost be considered as identical since relations in the former theory are interpretable as theorems in the latter.

As an illustration I give interpretations of two formulas that have been set forth in the foregoing chapters.

(1) Take the formula

$$(-)^n a_n = \Sigma \frac{(-)^{\Sigma\lambda}(\Sigma\lambda)!}{\lambda_1!\lambda_2!\lambda_3!\ldots} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_3} \ldots,$$

where $\Sigma s\lambda_s = n$, and remember that, when $\Sigma\lambda = m$, the right-hand side, taken with positive sign, is the distribution function corresponding to the distribution of n objects into parcels of type (1^m).

Theorem I. "Considering n objects of any kinds whatever the number of distinct ways of distributing them into an even number of parcels, no two of which are alike, is equal to the number of ways of distributing them into an uneven number of such parcels, except when the objects are such that no two are alike; in this case the former number is in excess or in defect of the latter number by unity according as the number of objects is even or uneven."

(2) Take the formula

$$(-)^n h_n = \Sigma \frac{(-)^{\Sigma\lambda}(\Sigma\lambda)!}{\lambda_1!\lambda_2!\lambda_3!\ldots} a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3} \ldots$$

and remember that, when $\Sigma\lambda = m$, the right-hand side, taken with positive sign, is the distribution function corresponding to the distribution of n objects into parcels of type (1^m), the distribution being subject to the restriction that no parcel may contain two similar objects.

Theorem II. "Considering n objects of any kinds whatever and distributions subject to the restriction that no parcel may contain two similar objects, the number of distributions into an even number of parcels, no two

of which are alike, is in excess or in defect by unity of the number of distributions into an uneven number of such parcels according as n, the number of objects, is even or uneven."

26. To further develop the Theory of Symmetric Functions it is necessary now to introduce the differential operators of Hammond*.

Hammond's Operators.

We consider the relation

$$(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) = x^n - a_1x^{n-1} + a_2x^{n-2} - \dots + (-)^n a_n,$$

which connects the quantities α with the elementary symmetric functions a.

If we introduce a new quantity μ by multiplying each side by $x-\mu$ we find

$$(x-\mu)(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$
$$= x^{n+1} - (a_1+\mu)x^n + (a_2+\mu a_1)x^{n-1} - (a_3+\mu a_2)x^{n-2} + \dots;$$

and if we suppose, for the old relation, the symmetric function equality

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots) = \phi(a_1, a_2, a_3, \dots) = \phi$$

then, for the new relation,

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots) + \mu^{p_1}(p_1^{\pi_1-1}p_2^{\pi_2}p_3^{\pi_3}\dots) + \mu^{p_2}(p_1^{\pi_1}p_2^{\pi_2-1}p_3^{\pi_3}\dots) + \mu^{p_3}(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3-1}\dots) + \dots$$
$$= \phi(a_1+\mu, a_2+\mu a_1, a_3+\mu a_2, \dots)$$
$$= \phi + \mu d_1\phi + \frac{\mu^2}{2!}(d_1^2)\phi + \frac{\mu^3}{3!}(d_1^3)\phi + \dots,$$

where
$$d_1 = \frac{d}{da_1} + a_1\frac{d}{da_2} + a_2\frac{d}{da_3} + \dots$$

and (d_1^s) indicates that the multiplication $d_1 \times d_1 \times d_1 \times \dots$ is a symbolic multiplication as usual in Taylor's expansion.

(d_1^s) is thus an operator of order s and is in contrast to $(d_1)^s$ which will be held to denote s successive operations of the linear operator d_1.

Writing further $\frac{1}{s!}(d_1^s) = D_s$ we find, by a comparison of the coefficients of like powers of μ,

$$D_{p_1}(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots) = (p_1^{\pi_1-1}p_2^{\pi_2}p_3^{\pi_3}\dots); \quad D_{p_2}(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots) = (p_1^{\pi_1}p_2^{\pi_2-1}p_3^{\pi_3}\dots);$$
$$D_s(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots) = 0;$$

if s is not included among the parts $p_1, p_2, p_3, \dots$;

and
$$D_s(s) = 1.$$

* "On the Calculation of Symmetric Functions" by J. Hammond, M.A., *Proc. Lond. Math. Soc.*, Vol. XIII. p. 79. 1882.

"On the use of certain Differential Operators in the Theory of Equations" by J. Hammond, *Proc. Lond. Math. Soc.*, Vol. XIV. p. 119. 1883.

The effect of D_s upon any symmetric function denoted by a partition is to obliterate from such partition a part s. It is usual to term the D's obliterating operators. It will be noticed that the operators D obey the commutative law, viz.: $D_pD_q=D_qD_p$, and in their combinations with constants and with each other they also obey the other ordinary laws of algebraic quantity.

We will write in general

$$d_\lambda = \frac{d}{da_\lambda} + a_1\frac{d}{da_{\lambda+1}} + a_2\frac{d}{da_{\lambda+2}} + \ldots,$$

and then the operators D and d are those with which we shall be principally concerned.

27. Let us first study the effect of two successive operations of d_1. A first operation of d_1 upon an operand f yields a sum of terms similar to

$$a_s\frac{df}{da_{s+1}};$$

a second operation of d_1 upon this term yields a sum of terms similar to

$$a_r\frac{d}{da_{r+1}}\left(a_s\frac{df}{da_{s+1}}\right),$$

which is

$$a_ra_s\frac{d^2f}{da_{r+1}da_{s+1}} + a_r\frac{da_s}{da_{r+1}}\frac{df}{da_{s+1}};$$

the second term vanishing unless $r+1=s$. Observe that the first term is arrived at by symbolic multiplication of $a_r\dfrac{d}{da_{r+1}}$ and $a_s\dfrac{d}{da_{s+1}}$ and that the second term, when it is not zero, is arrived at by operating with $a_r\dfrac{d}{da_{r+1}}$ upon $a_s\dfrac{df}{da_{s+1}}$, regarding the latter as a function of the symbol of quantity a_s *only.*

It follows that we may write

$$(d_1)^2 = (d_1^2) + d_1 \dagger d_1,$$

the symbol $\dagger$ indicating that d_1 which precedes it operates upon d_1 which succeeds it on the supposition that the operand d_1 is an explicit function of the symbols of quantity only and not of the differential inverses. Since $d_1 \dagger d_1 = d_2$ we have

$$(d_1)^2 = (d_1^2) + d_2;$$

and generally

$$(d_\lambda)(d_\mu) = (d_\lambda d_\mu) + d_{\lambda+\mu}.$$

A like theorem obtains when the operand is an operator of any order.

Thus

$$(d_\lambda)(D_\kappa) = \frac{1}{\kappa!}(d_\lambda)(d_1^\kappa) = \frac{1}{\kappa!}(d_\lambda d_1^\kappa) + \frac{1}{(\kappa-1)!}(d_{\lambda+1}d_1^{\kappa-1}),$$

or

$$(d_\lambda)(D_\kappa) = (d_\lambda D_\kappa) + (d_{\lambda+1}D_{\kappa-1}).$$

We shall see that the algebra of these operators is parallel with the algebra of symmetric functions, D_κ corresponding to a_κ or (1^κ), and d_λ to s_λ or (λ).

Thus the relation above $(d_1)^2 = (d_1^2) + d_2$ may be written

$$d_2 = D_1^2 - 2D_2,$$

wherein we disuse the bracket in the case of the D operators so that D_1^2 denotes the operation D_1 twice performed. This operator relation is to be compared with the symmetric function relation

$$(2) = a_1^2 - 2a_2 = (1)^2 - 2(1^2).$$

Also the operator relation

$$(d_\lambda)(D_\kappa) = (d_\lambda D_\kappa) + (d_{\lambda+1} D_{\kappa-1})$$

is to be compared with the relation

$$(\lambda)(1^\kappa) = (\lambda, 1^\kappa) + (\lambda + 1, 1^{\kappa-1})$$

in symmetric functions.

Now if we take the expression

$$d_\lambda - (d_{\lambda-1})(D_1) + (d_{\lambda-2})(D_2) - \ldots + (-)^{\lambda-1}(d_1)(D_{\lambda-1}),$$

and make use of the formula

$$(d_\lambda)(D_\kappa) = (d_\lambda D_\kappa) + (d_{\lambda+1} D_{\kappa-1}),$$

we at once reduce it to $(-)^{\lambda-1}\lambda D_\lambda$.

Hence $$d_\lambda - (d_{\lambda-1})(D_1) + (d_{\lambda-2})(D_2) - \ldots + (-)^\lambda \lambda D_\lambda = 0;$$

and we compare this with Newton's Rule for the sums of powers of quantities, viz.:

$$s_\lambda - s_{\lambda-1}a_1 + s_{\lambda-2}a_2 - \ldots + (-)^\lambda \lambda a_\lambda = 0.$$

We find that
$$d_1 = D_1,$$
$$d_2 = D_1^2 - 2D_2,$$
$$d_3 = D_1^3 - 3D_1D_2 + 3D_3,$$
$$d_4 = D_1^4 - 4D_1^2D_2 + 2D_2^2 + 4D_1D_3 - 4D_4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and in general $d_\lambda = [\lambda]$, $D_\lambda = [1^\lambda]$ where the brackets [ ] denote symmetric functions of the *fictitious* quantities which are roots of the equation

$$1 - y^{-1}D_1 + y^{-2}D_2 - y^{-3}D_3 + \ldots = 0.$$

The differential operators corresponding to the other symmetric functions in square brackets may be found. Thus, from a formula above,

$$(d_\lambda d_\mu) = (d_\lambda)(d_\mu) - d_{\lambda+\mu};$$

and since $$\Sigma a_1^\lambda a_2^\mu = s_\lambda s_\mu - s_{\lambda+\mu},$$

we see that $$(d_\lambda d_\mu) = [\lambda\mu].$$

Moreover, the easily obtainable results

$$(d_\lambda^2) = (d_\lambda)^2 - d_{2\lambda},$$
$$(d_\lambda^3) = (d_\lambda)^3 - 3(d_\lambda)(d_{2\lambda}) + 2d_{3\lambda},$$
$$(d_\lambda d_\mu^2) = (d_\lambda)(d_\mu)^2 - (d_\lambda)(d_{2\mu}) - 2(d_\mu)(d_{\lambda+\mu}) + 2d_{\lambda+2\mu},$$
$$(d_\lambda d_\mu d_\nu) = (d_\lambda)(d_\mu)(d_\nu) - (d_\lambda)(d_{\mu+\nu}) - (d_\mu)(d_{\lambda+\nu}) - (d_\nu)(d_{\lambda+\mu}) + 2d_{\lambda+\mu+\nu},$$

when compared with the expressions of $\Sigma\alpha_1^\lambda\alpha_2^\lambda$, $\Sigma\alpha_1^\lambda\alpha_2^\lambda\alpha_3^\lambda$, $\Sigma\alpha_1^\lambda\alpha_2^\mu\alpha_3^\mu$, $\Sigma\alpha_1^\lambda\alpha_2^\mu\alpha_3^\nu$, by means of the sums of the powers of the quantities α, shew that the similarity of the algebras of quantity and operation leads without doubt to the result

$$\frac{1}{l!\,m!\,n!\ldots}(d_\lambda^l d_\mu^m d_\nu^n \ldots) = [\lambda^l \mu^m \nu^n \ldots].$$

The operators d_λ form an algebraic group in the sense that their algebra is wholly comprised within the limits of the group. This arises from the fact

$$d_\lambda \dagger d_\mu = d_{\lambda+\mu},$$

where the right-hand side depends merely upon the sum $\lambda + \mu$ and not upon the actual values of λ and μ.

28. Consider any linear function of the operators of the system, viz.:

$$u_1 = m_1 d_1 + m_2 d_2 + m_3 d_3 + \ldots;$$

and put $u_s = u_1 \dagger u_{s-1}$; so that there arises a system of linear operators $u_1, u_2, u_3, \ldots$.

We find
$$\begin{aligned} u_2 = u_1 \dagger u_1 = m_1^2 d_2 &+ m_1 m_2 d_3 + m_1 m_3 d_4 + m_1 m_4 d_5 + \ldots \\ &+ m_2 m_1 d_3 + \quad m_2^2 d_4 + m_2 m_3 d_5 + \ldots \\ &+ m_3 m_1 d_4 + m_3 m_2 d_5 + \ldots \\ &+ m_4 m_1 d_5 + \ldots \end{aligned}$$

so that if we write $u_1 = m_1 d + m_2 d^2 + m_3 d^3 + m_4 d^4 + \ldots,$

where symbolically d_s is written d^s, we have

$$u_2 = (m_1 d + m_2 d^2 + m_3 d^3 + m_4 d^4 + \ldots)^2;$$

and in general
$$u_s = (m_1 d + m_2 d^2 + m_3 d^3 + m_4 d^4 + \ldots)^s;$$
$$u_s \dagger u_t = u_{s+t};$$

from which it may be gathered that the operators $u_1, u_2, u_3, \ldots$ also form a group in the above stated sense.

Moreover
$$(u_1)^2 = (u_1^2) + u_2,$$
$$(u_1)^3 = (u_1^3) + 3(u_2 u_1) + u_3,$$
$$(u_1)^4 = (u_1^4) + 6(u_2 u_1^2) + 3(u_2^2) + 4(u_3 u_1) + u_4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(u_1)^s = \Sigma \frac{s!}{(\lambda!)^l(\mu!)^m \ldots l!\,m!\ldots}(u_\lambda^l u_\mu^m \ldots);$$

wherein
$$l\lambda + m\mu + \ldots = s.$$

Hence by addition

$$1+u_1+\frac{(u_1)^2}{2!}+\frac{(u_1)^3}{3!}+\ldots=\sum_{s=0}^{s=\infty}\Sigma\frac{(u_\lambda^l u_\mu^m\ldots)}{(\lambda!)^l(\mu!)^m\ldots l!\,m!\ldots}.$$

Observe that on the right-hand side the operators are of order s.

Now consider $$\overline{\exp}\left(u_1+\frac{u_2}{2!}+\frac{u_3}{3!}+\ldots\right),$$

the bar over exp denoting that in the exponential expansion the multiplications of operators imply algebraic combinations and not successive operations.

A simple application of the multinomial theorem yields the formula

$$\overline{\exp}\left(u_1+\frac{u_2}{2!}+\frac{u_3}{3!}+\ldots\right)=\sum_{s=0}^{s=\infty}\Sigma\frac{(u_\lambda^l u_\mu^m\ldots)}{(\lambda!)^l(\mu!)^m\ldots l!\,m!\ldots};$$

and thence the important result

$$\exp u_1=\overline{\exp}\left(u_1+\frac{u_2}{2!}+\frac{u_3}{3!}+\ldots\right),$$

where on the left-hand side, on expansion, the multiplications, or powers of u_1, denote successive operations.

Since $$u_1=m_1d_1+m_2d_2+m_3d_3+\ldots$$

if $$u_1+\frac{u_2}{2!}+\frac{u_3}{3!}+\ldots=M_1d_1+M_2d_2+M_3d_3+\ldots,$$

we must have

$$\exp(m_1d_1+m_2d_2+m_3d_3+\ldots)=\overline{\exp}(M_1d_1+M_2d_2+M_3d_3+\ldots),$$

wherein the coefficients M are connected with the coefficients m in such wise that, y being an arbitrary quantity,

$$\exp(m_1y+m_2y^2+m_3y^3+\ldots)=1+M_1y+M_2y^2+M_3y^3+\ldots.$$

To see how this is observe that

$$1+u_1+\frac{u_2}{2!}+\frac{u_3}{3!}+\ldots=1+M_1d_1+M_2d_2+M_3d_3+\ldots$$

may be symbolically written

$$\exp(m_1d+m_2d^2+m_3d^3+\ldots)=1+M_1d+M_2d^2+M_3d^3+\ldots;$$

and that we may replace the symbol d by an arbitrary quantity y.

We derive the relations $$M_s=\Sigma\frac{m_\lambda^l m_\mu^m\ldots}{l!\,m!\ldots}$$

where $$l\lambda+m\mu+\ldots=s;$$

$$m_s=\Sigma(-)^{s+l+m+\ldots}\frac{(l+m+\ldots-1)!\,s}{l!\,m!\ldots}M_\lambda^l M_\mu^m\ldots;$$

so that the theorem assumes two forms.

(i) If the coefficients m be arbitrary

$$\exp(m_1 d_1 + m_2 d_2 + m_3 d_3 + \ldots)$$
$$= \overline{\exp}\left(m_1 d_1 + \frac{m_1^2 + 2m_2}{2!} d_2 + \frac{m_1^3 + 6m_2 m_1 + 6m_3}{3!} d_3 + \ldots\right);$$

(ii) If the coefficients M be arbitrary

$$\overline{\exp}(M_1 d_1 + M_2 d_2 + M_3 d_3 + \ldots)$$
$$= \exp\{M_1 d_1 - \tfrac{1}{2}(M_1^2 - 2M_2) d_2 + \tfrac{1}{3}(M_1^3 - 3M_2 M_1 + 3M_3) d_3 - \ldots\}.$$

As particular cases; we have

$$\exp d_1 = \overline{\exp}\left(d_1 + \frac{d_2}{2!} + \frac{d_3}{3!} + \ldots\right),$$

which is Sylvester's Theorem.

Also
$$\exp d_\lambda = \overline{\exp}\left(d_\lambda + \frac{1}{2!} d_{2\lambda} + \frac{1}{3!} d_{3\lambda} + \ldots\right);$$
$$\overline{\exp} d_\lambda = \exp(d_\lambda - \tfrac{1}{2} d_{2\lambda} + \tfrac{1}{3} d_{3\lambda} - \ldots).$$

We say that an operator is of weight λ when the effect of the operation is to reduce the weight of the operand by the number λ. The two operators so far considered, viz. d_λ and D_λ, are called linear-weight and obliterating-weight operators respectively. In the sequel we shall arrive at partition operators and at this point we may proceed to the theory of partition-obliterating operators from a consideration of the mode of operation of the weight-obliterating operator upon a product of symmetric functions each of which is symbolized by a partition. We shall thus advance, as will be seen in the sequel, a very important step in the theory of distributions and also obtain a powerful means for the handling of symmetric functions.

Operation of D_λ upon a Product of Symmetric Functions.

29. Let $\phi_s, \quad s = 1, 2, 3, \ldots m,$

denote a symmetric function, symbolized by a single partition, expressed in terms of the elementary functions $a_1, a_2, a_3, \ldots$; and put

$$\phi = \phi_1 \phi_2 \phi_3 \ldots \phi_m.$$

If to the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_m$, from which the functions are derived, we add a new quantity μ, we have seen above, Art. 26, that

$$\phi_s \text{ becomes } (1 + D_1\mu + D_2\mu^2 + \ldots)\phi_s;$$

hence
$$(1 + D_1\mu + D_2\mu^2 + \ldots)\phi$$
$$= (1 + D_1\mu + D_2\mu^2 + \ldots)\phi_1$$
$$\times (1 + D_1\mu + D_2\mu^2 + \ldots)\phi_2$$
$$\times (1 + D_1\mu + D_2\mu^2 + \ldots)\phi_3$$
$$\times \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\times (1 + D_1\mu + D_2\mu^2 + \ldots)\phi_m.$$

We may now multiply out the right-hand side and then equate coefficients of like powers of μ. We shall thus obtain on the left-hand side $D_\lambda \phi$ suppose and on the right-hand side a sum of products such as

$$D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 . D_{\lambda_3}\phi_3 \dots D_{\lambda_m}\phi_m ;$$

wherein clearly $(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_m)$ is some partition of λ into m *or fewer* parts. Restricting attention for the moment to this particular partition of λ, it is obvious that we have a sum

$$\Sigma D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 . D_{\lambda_3}\phi_3 . \dots D_{\lambda_m}\phi_m ;$$

the sign of summation extending to the whole of the permutations of the factors

$$\phi_1, \phi_2, \phi_3, \dots \phi_m .$$

We have a similar sum of products for every other partition of λ into m or fewer parts; so that we may write

$$D_\lambda \phi = \Sigma\Sigma D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 . D_{\lambda_3}\phi_3 \dots D_{\lambda_s}\phi_s . \phi_{s+1} \dots \phi_m ; \qquad s \leqslant m ;$$

and the double summation is in respect of

(i) every partition of λ into m or fewer parts;

(ii) every permutation of every selection of s out of the m factors $\phi_1, \phi_2, \phi_3, \dots \phi_m$.

It will be noted that many of the products will as a rule vanish because

$$D_p \theta = 0,$$

unless the partition denoted by θ contains a part p.

Hence the operation D_λ is performed upon $\phi_1 \phi_2 \phi_3 \dots \phi_m$ by abstracting every partition of λ in all possible ways from the product, one part at most being taken from each partition.

We may conveniently write

$$\Sigma D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 . D_{\lambda_3}\phi_3 \dots D_{\lambda_s}\phi_s . \phi_{s+1} \dots \phi_m$$
$$= D_{(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_s)}\, \phi_1 \phi_2 \phi_3 \dots \phi_m ;$$

and name $D_{(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_s)}$ a partition-obliterating operator.

Then $$D_\lambda \phi_1 \phi_2 \phi_3 \dots \phi_m = (\Sigma D_{(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_s)})\, \phi_1 \phi_2 \phi_3 \dots \phi_m ,$$

and $$D_\lambda = \Sigma D_{(\lambda_1 \lambda_2 \lambda_3 \dots \lambda_s)} ,$$

the summation being in respect of *every* partition of λ; for those partition operators which have more than m parts will not add anything to the result. We have thus expressed the weight-obliterating operator as a sum of partition-obliterating operators; and this is one step towards the generalization of the Theory of Symmetric Functions in which numbers are replaced by partitions of numbers.

Example: $D_4 = D_{(4)} + D_{(31)} + D_{(2^2)} + D_{(21^2)} + D_{(1^4)}$

so that if the operand is symbolized by a single partition

$$D_4 \equiv D_{(4)}.$$

It has been mentioned that in operating with a partition-obliterating operator upon a product, it is necessary to pick out the partition from the product in all possible ways, one part at most being taken from each factor. It is this circumstance that makes the partition-obliterating operator a valuable instrument in the Theory of Distributions.

30. The operation of D_λ upon a product of symmetric functions has been connected with the partitions of λ and with every way of picking out such partitions, one part from each factor, from the product. This has necessitated a summation in regard to all permutations of the factors of the product. We can get rid of such permutations by associating λ with its compositions (see Art. 8) instead of with its partitions. Thus suppose $\lambda = 3$ and the operand $\phi_1\phi_2\phi_3\phi_4$; we will preserve the order of these factors and associate the number 3 with its compositions into *exactly* four parts (there being four factors) where zero is admitted as a part. It is convenient to represent zero in this connexion by the symbol ω and then the operation of D_3 will be apparent in the scheme

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_1	ϕ_2	ϕ_3	ϕ_4
3	ω	ω	ω	1	2	ω	ω
ω	3	ω	ω	1	ω	2	ω
ω	ω	3	ω	1	ω	ω	2
ω	ω	ω	3	ω	1	2	ω
2	1	ω	ω	ω	1	ω	2
2	ω	1	ω	ω	ω	1	2
2	ω	ω	1	1	1	1	ω
ω	2	ω	1	1	1	ω	1
ω	2	1	ω	1	ω	1	1
ω	ω	2	1	ω	1	1	1

where for example the seventh row of figures indicates that a part 2 is to be deleted from ϕ_1 and a part 1 from ϕ_4; ϕ_2 and ϕ_3 being unaffected by the symbol ω. It will be noted that the operation of D_3 is split up into 20 parts, 20 being the number of compositions of 3 into *exactly* four parts, zero being admissible as a part. The symbol ω has no effect; it is merely introduced in order to shew the law.

Generally the number of compositions of λ into exactly m parts, zero parts admissible and $m \leqq \lambda$, is equal to the coefficient of x^λ in the expansion of

$$(1 + x + x^2 + x^3 + \ldots)^m$$

which is $\binom{m+\lambda-1}{\lambda}$.

This idea of associating the operation of λ with the compositions of λ and so splitting up the operation upon a product $\phi_1\phi_2 \dots \phi_m$ into

$$\binom{m+\lambda-1}{\lambda}$$

portions without any permutations of the factors of the operand enables us at once to generalize the theorem.

For the theorem may be written

$$D_\lambda(\phi_1\phi_2 \dots \phi_m) = \Sigma D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 \dots D_{\lambda_m}\phi_m,$$

where the order $\phi_1, \phi_2, \dots \phi_m$ remains unchanged and the summation is in regard to every *composition* $(\lambda_1\lambda_2 \dots \lambda_m)$ of λ into exactly m parts, zero being admissible as a part. We now operate upon the representative term on the dexter, viz.:

$$D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 \dots D_{\lambda_m}\phi_m$$

with D_μ through the medium of the compositions of μ into *exactly* m parts, zero being admissible as a part. Thus

$$D_\mu\{D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 . \dots . D_{\lambda_m}\phi_m\} = \Sigma D_{\lambda_1}D_{\mu_1}\phi_1 . D_{\lambda_2}D_{\mu_2}\phi_2 . \dots . D_{\lambda_m}D_{\mu_m}\phi_m;$$

leading to

$$D_\lambda D_\mu(\phi_1\phi_2 \dots \phi_m) = \Sigma\Sigma D_{\lambda_1}D_{\mu_1}\phi_1 . D_{\lambda_2}D_{\mu_2}\phi_2 . \dots . D_{\lambda_m}D_{\mu_m}\phi_m.$$

On the right-hand side there will obviously be

$$\binom{m+\lambda-1}{\lambda}\binom{m+\mu-1}{\mu} \text{ terms,}$$

for we have to associate every composition of λ with every composition of μ.

In general we may write

$$D_\lambda D_\mu D_\nu \dots (\phi_1\phi_2 \dots \phi_m)$$
$$= \Sigma\Sigma\Sigma \dots D_{\lambda_1}D_{\mu_1}D_{\nu_1} \dots \phi_1 . D_{\lambda_2}D_{\mu_2}D_{\nu_2} \dots \phi_2 . \dots . D_{\lambda_m}D_{\mu_m}D_{\nu_m} \dots \phi_m$$

and the number of terms on the dexter is

$$\binom{m+\lambda-1}{\lambda}\binom{m+\mu-1}{\mu}\binom{m+\nu-1}{\nu} \dots .$$

31. So far we have been concerned with a system of operators for use when symmetric functions are expressed in terms of the elementary symmetric functions $a_1, a_2, a_3, \dots$; we must now find another system for use when the functions are expressed in terms of the homogeneous product sums $h_1, h_2, h_3, \dots .$

We may evidently write

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \dots$$
$$= x^n - h_1x^{n-1} + (h_1^2 - h_2)x^{n-2} - (h_1^3 - 2h_1h_2 + h_3)x^{n-3} + \dots$$

since we know the formula for expressing any a in terms of the quantities h.

The introduction of a new factor $(x-\kappa)$ on the left-hand side causes h_w to become

$$h_w + h_{w-1}\kappa + h_{w-2}\kappa^2 + \ldots + \kappa^w,$$

and thence any function ϕ of $h_1, h_2, h_3, \ldots$ becomes

$$\phi(h_1+\kappa,\quad h_2+h_1\kappa+\kappa^2,\quad h_3+h_2\kappa+h_1\kappa^2+\kappa^3,\ \ldots)$$

or $$\phi+(\kappa\delta_1+\kappa^2\delta_2+\kappa^3\delta_3+\ldots)\phi+\frac{1}{2!}(\kappa\delta_1+\kappa^2\delta_2+\kappa^3\delta_3+\ldots)^2\phi+\ldots$$

where $$\delta_s=\frac{d}{dh_s}+h_1\frac{d}{dh_{s+1}}+h_2\frac{d}{dh_{s+2}}+\ldots$$

and the multiplications of operators are algebraic as in Taylor's Theorem.

This expression must be written in the form

$$\phi+\kappa\Delta_1\phi+\kappa^2\Delta_2\phi+\kappa^3\Delta_3\phi+\ldots;$$

where

$$\Delta_1=\delta_1,$$
$$2!\,\Delta_2=(\delta_1^2)+2\delta_2,$$
$$3!\,\Delta_3=(\delta_1^3)+6(\delta_1\delta_2)+6\delta_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s!\,\Delta_s=\Sigma\frac{s!}{l!\,m!\,n!\ldots}(\delta_\lambda^l\delta_\mu^m\delta_\nu^n\ldots),$$

and $$l\lambda+m\mu+n\nu+\ldots=s.$$

From these relations we deduce

$$\delta_1=\Delta_1,$$
$$\delta_2=-(\Delta_1^2-2\Delta_2),$$
$$\delta_3=\Delta_1^3-3\Delta_1\Delta_2+3\Delta_3,$$
$$\delta_4=-(\Delta_1^4-4\Delta_1^2\Delta_2+2\Delta_2^2+4\Delta_1\Delta_3-4\Delta_4),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\delta_s=\{s\},$$

where $\{s\}$ is formed from $\Delta_1, \Delta_2, \Delta_3, \ldots$ in the same way as (s) from $h_1, h_2, h_3, \ldots$.

The introduction of the quantity κ causes

$$(\lambda^l\mu^m\nu^n\ldots)$$

to become

$$(\lambda^l\mu^m\nu^n\ldots)+\kappa^\lambda(\lambda^{l-1}\mu^m\nu^n\ldots)+\kappa^\mu(\lambda^l\mu^{m-1}\nu^n\ldots)+\kappa^\nu(\lambda^l\mu^m\nu^{n-1}\ldots)+\ldots;$$

and if $$\phi=(\lambda^l\mu^m\nu^n\ldots)$$

this is also $$(1+\kappa\Delta_1+\kappa^2\Delta_2+\kappa^3\Delta_3+\ldots)(\lambda^l\mu^m\nu^n\ldots);$$

yielding the results

$$\Delta_\lambda(\lambda^l\mu^m\nu^n\ldots)=(\lambda^{l-1}\mu^m\nu^n\ldots),$$
$$\Delta_\mu(\lambda^l\mu^m\nu^n\ldots)=(\lambda^l\mu^{m-1}\nu^n\ldots),$$
$$\Delta_\lambda^l\Delta_\mu^m\Delta_\nu^n\ldots(\lambda^l\mu^m\nu^n\ldots)=1;$$

and if s does not occur among the parts $\lambda, \mu, \nu, \ldots$

$$\Delta_s (\lambda^l \mu^m \nu^n \ldots) = 0.$$

It will thus be seen that Δ_s when operating upon a function, symbolized by a partition, is precisely equivalent to D_s.

The further conclusion is that there exists in regard to the h functions a theory exactly parallel to that of Hammond in regard to the a functions. Thus if

$$(\lambda^l \mu^m \nu^n \ldots) = \phi (h_1, h_2, h_3, \ldots),$$

we have the corresponding operator relation

$$(\delta_\lambda^l \delta_\mu^m \delta_\nu^n \ldots) \div l!\, m!\, n! \ldots = \phi (\Delta_1, \Delta_2, \Delta_3, \ldots).$$

Ex. gr. Since $(21) = -3h_3 + 5h_2h_1 - 2h_1^3,$

we have $(\delta_2\delta_1) = -3\Delta_3 + 5\Delta_2\Delta_1 - 2\Delta_1^3.$

CHAPTER V

APPLICATIONS OF THE OPERATORS d AND D

32. It will have been observed that every symmetric function formula which involves the monomial functions $(pqr\ldots)$, the elementary functions $a_1, a_2, a_3, \ldots$ and the power sums (otherwise one-part functions) $s_1, s_2, s_3, \ldots$ gives a relation between the operators and *vice versâ*. Thus

$$(2^m) = a_m^2 - 2a_{m-1}a_{m+1} + 2a_{m-2}a_{m+2} - \ldots (-)^m 2a_{2m}$$

gives $$(d_2^m) \div m! = D_m^2 - 2D_{m-1}D_{m+1} + 2D_{m-2}D_{m+2} - \ldots (-)^m 2D_{2m}.$$

The operators may be applied to the calculation of the expression of the function $(\lambda^l \mu^m \nu^n \ldots)$ in terms of the elementary functions.

It must first be noted that if $s > m$

$$D_s a_{p_1} a_{p_2} \ldots a_{p_m} = D_s (1^{p_1})(1^{p_2}) \ldots (1^{p_m}) = 0.$$

Also if $s = m$

$$D_m a_{p_1} a_{p_2} \ldots a_{p_m} = D_m (1^{p_1})(1^{p_2}) \ldots (1^{p_m}) = a_{p_1 - 1} a_{p_2 - 1} \ldots a_{p_m - 1}.$$

It follows from this that the function $(\lambda^l \mu^m \nu^n \ldots)$ is of degree λ when expressed in terms of $a_1, a_2, a_3, \ldots$, for clearly $D_s (\lambda^l \mu^m \nu^n \ldots) = 0$ if $s > \lambda$.

We may therefore write

$$(\lambda^l \mu^m \nu^n \ldots) = \Sigma P a_{p_1} a_{p_2} \ldots a_{p_\lambda} + \text{terms of lower degree.}$$

Thence operating with D_λ we find

$$(\lambda^{l-1} \mu^m \nu^n \ldots) = \Sigma P a_{p_1 - 1} a_{p_2 - 1} \ldots a_{p_\lambda - 1},$$

or, in words, a unit diminution of suffix in the terms of highest degree in $(\lambda^l \mu^m \nu^n \ldots)$ gives immediately the complete expression of $(\lambda^{l-1} \mu^m \nu^n \ldots)$.

Moreover suppose that we are given the expression of $(\lambda^{l-1} \mu^m \nu^n \ldots)$ and that we make it homogeneous of degree λ by introducing into the various terms the proper power of a_0, we see that a unit increase of suffix throughout the expression gives at once the whole of the terms of highest degree in the expression of $(\lambda^l \mu^m \nu^n \ldots)$.

Next suppose that

$$(\kappa \lambda^l \mu^m \nu^n \ldots) = \Sigma P a_{p_1} a_{p_2} \ldots a_{p_\kappa} + \text{terms of lower degree,}$$

the operation of D_κ produces

$$a_0^{\kappa-\lambda}(\lambda^l\mu^m\nu^n\ldots)=\Sigma P a_{p_1-1}a_{p_2-1}\ldots a_{p_\kappa-1},$$

indicating that we produce $(\lambda^l\mu^m\nu^n\ldots)$ by making a unit decrease of suffix in $(\kappa\lambda^l\mu^m\nu^n\ldots)$ and subsequent division by $a_0^{\kappa-\lambda}$.

Moreover starting with the expression of $(\lambda^l\mu^m\nu^n\ldots)$ we have merely to make it homogeneous of degree κ by introducing the proper power of a_0 into every term and then make a unit increase of suffix to obtain the whole of the terms of highest degree in the expression of $(\kappa\lambda^l\mu^m\nu^n\ldots)$.

Ex. gr. Since

$$(3)=a_1^3-3a_0a_1a_2+3a_0^2a_3$$

we find $(53)=a_1^2(a_2^3-3a_1a_2a_3+3a_1^2a_4)+$ terms of lower degree.

33. If in the equation

$$(\lambda^l\mu^m\nu^n\ldots)=f(a_1,\ a_2,\ a_3,\ \ldots)$$

we operate on the right-hand side with d_1, d_2, d_3, etc. ... and on the left-hand side with their equivalents

$$D_1,\ \ D_1^2-2D_2,\ \ D_1^3-3D_1D_2+3D_3,\ \ldots$$

we obtain a set of differential equations of the first order all satisfied by f. If κ be an integer, less than $\Sigma l\lambda$, none of whose partitions are contained in

$$(\lambda^l\mu^m\nu^n\ldots),$$

we have a differential equation of the form

$$d_\kappa f=0;$$

and thus, frequently, we are able to calculate f without reference to known values of symmetric functions of lower weight. Thus if $f=(3^2)$

$$d_1f=0,\ \ d_2f=0,\ \ d_4f=0,\ \ d_5f=0,$$

and, without using $d_1f=0$, we find

$$(3^2)=a_2^3-3a_1a_2a_3+3a_1^2a_4+3a_3^2-3a_2a_4-3a_1a_5+3a_6.$$

The remaining equations may be employed when Tables of functions of lower weight are available.

The operation of $d_1\equiv D_1$ alone naturally derives a Table of weight $w-1$ from a given Table of weight w.

The operator d_n when performed on a function of weight n, or on a function which does not involve an a with a suffix greater than n, reduces to $\dfrac{d}{da_n}$; and similarly the non-linear operator $(d_\lambda d_\mu d_\nu\ldots)$ reduces to

$$\frac{d}{da_\lambda}\cdot\frac{d}{da_\mu}\cdot\frac{d}{da_\nu}\cdots$$

when the operand has a weight $\lambda+\mu+\nu+\ldots$.

Hammond's Proof of the Second Law of Symmetry.

34. It has been shewn that there is a complete correspondence between symmetric functions and operational formulas. Thus suppose that the two symmetric functions $(pqr\ldots)$, $(p_1q_1r_1\ldots)$ of like weight are developed so as to give

$$(pqr\ldots) = \ldots + Aa_{p_1}a_{q_1}a_{r_1}\ldots + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(a),$$

$$(p_1q_1r_1\ldots) = \ldots + A'a_pa_qa_r\ldots + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(a').$$

We have to shew that $A = A'$.

The first equation (a) leads to the operator formula

$$(d_pd_qd_r\ldots) = \ldots + AD_{p_1}D_{q_1}D_{r_1}\ldots + \ldots$$

and using each side of this on the opposite side of equation (a') we find

$$(\ldots + AD_{p_1}D_{q_1}D_{r_1}\ldots + \ldots)(p_1q_1r_1\ldots) = (d_pd_qd_r\ldots)(\ldots + A'a_pa_qa_r\ldots + \ldots).$$

Now the left-hand side is equal to A since

$$(D_{p_1}D_{q_1}D_{r_1}\ldots)(p_1q_1r_1\ldots) = 1$$

and the remaining operations cause $(p_1q_1r_1\ldots)$ to vanish. Also on the right-hand

$$d_pd_qd_r\ldots \equiv \frac{d}{da_p}\cdot\frac{d}{da_q}\cdot\frac{d}{da_r}\cdots$$

since the operand is of the same weight $p+q+r+\ldots$; hence the right-hand side is equal to A', the remaining terms of the operand vanishing under the operation. Accordingly

$$A = A',$$

establishing the theorem.

The requisite modifications can be made in the proof when there are any equalities between either the parts $p, q, r, \ldots$ or the parts $p_1, q_1, r_1, \ldots$.

35. To prove the reciprocal theorem, write

$$a_pa_qa_r\ldots = \ldots + B(p_1q_1r_1\ldots) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(b),$$

$$a_{p_1}a_{q_1}a_{r_1}\ldots = \ldots + B'(pqr\ldots) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(b').$$

From equation (b) we derive

$$D_pD_qD_r\ldots = \ldots + B(d_{p_1}d_{q_1}d_{r_1}\ldots) + \ldots$$

and operating as before on the relation (b') we find

$$B = B',$$

establishing the reciprocal theorem.

36. We may now generalize these results.

Suppose that, of the same weight, we have two results

$$\Sigma A_{\lambda^l\mu^m\nu^n\ldots}(\lambda^l\mu^m\nu^n\ldots) = \Sigma A'_{\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots}\, a_{\lambda'}^{l'} a_{\mu'}^{m'} a_{\nu'}^{n'}\ldots,$$

$$\Sigma B'_{\lambda^l\mu^m\nu^n\ldots}\, a_\lambda^l a_\mu^m a_\nu^n\ldots = \Sigma B_{\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots}(\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots).$$

Equivalent to the first of these results we have

$$\Sigma A_{\lambda^l\mu^m\nu^n\ldots}\frac{(d_\lambda^l d_\mu^m d_\nu^n\ldots)}{l!\,m!\,n!\ldots} = \Sigma A'_{\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots} D_{\lambda'}^{l'} D_{\mu'}^{m'} D_{\nu'}^{n'}\ldots;$$

and since $$\frac{(d_\lambda^l d_\mu^m d_\nu^n\ldots)}{l!\,m!\,n!\ldots} \equiv \frac{1}{l!\,m!\,n!\ldots}\left(\frac{d}{da_\lambda}\right)^l\left(\frac{d}{da_\mu}\right)^m\left(\frac{d}{da_\nu}\right)^n\ldots,$$

when the operand is $a_\lambda^l a_\mu^m a_\nu^n\ldots$, a term of the same weight as the operator,

$$\frac{(d_\lambda^l d_\mu^m d_\nu^n\ldots)}{l!\,m!\,n!\ldots} a_\lambda^l a_\mu^m a_\nu^n\ldots = 1;$$

and moreover $$D_{\lambda'}^{l'} D_{\mu'}^{m'} D_{\nu'}^{n'}\ldots(\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots) = 1;$$

we find by operation of the operator identity upon the two sides of the second of the two results in symmetric functions

$$\Sigma A_{\lambda^l\mu^m\nu^n\ldots} B'_{\lambda^l\mu^m\nu^n\ldots} = \Sigma A'_{\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots} B_{\lambda'^{l'}\mu'^{m'}\nu'^{n'}\ldots}.$$

As an example of this elegant theorem take from Tables the two formulas

$$(4)+(31)+(2^2)+(21^2)+(1^4) = -a_4 + 2a_3a_1 + a_2^2 - 3a_2a_1^2 + a_1^4,$$

$$a_4 + a_3a_1 + a_2^2 + a_2a_1^2 + a_1^4 = (4) + 5(31) + 9(2^2) + 20(21^2) + 47(1^4).$$

Now multiply each coefficient in the first relation by the coefficient immediately below it in the second relation so as to obtain the relation

$$1\times1+1\times1+1\times1+1\times1+1\times1 = -1\times1+2\times5+1\times9-3\times20+1\times47$$

or $$5 = -1+10+9-60+47,$$

which is true.

Similarly since $$(21) = a_1a_2 - 3a_3,$$

$$a_1^3 = 3(21)+(3)+6(1^3),$$

$$0 = 3-3 = 0.$$

Applications of the Operators δ and Δ.

37. If the student has grasped the method of employing the operators d and D he will have little difficulty with those now under discussion.

The following results are easily verified:

$$\Delta_s h_m = h_{m-s}, \quad \text{if } s < m;$$

$$\Delta_s h_s = 1,$$

$$\Delta_w h_\lambda^l h_\mu^m h_\nu^n\ldots = 1, \quad \text{if } l\lambda + m\mu + n\nu + \ldots = w.$$

It follows from this last result that if the symmetric function (w) be expressed in terms of $h_1, h_2, h_3, \ldots$ the sum of the coefficients in such expression is unity. But that if any other symmetric function, symbolized by a single partition, be so expressed the sum of the coefficients is zero.

If in the equation

$$(\lambda^l \mu^m \nu^n \ldots) = \phi(h_1, h_2, h_3, \ldots),$$

we operate on the right-hand side with $\delta_1, \delta_2, \delta_3, \ldots$ and on the left-hand side with the equivalents

$$\Delta_1, \quad -(\Delta_1^2 - 2\Delta_2), \quad \Delta_1^3 - 3\Delta_1\Delta_2 + 3\Delta_3, \ldots$$

we obtain a set of differential equations of the first order all satisfied by ϕ, and thus it is just as easy to calculate symmetric functions in terms of the functions h as in terms of the functions a. But it will be found that in *every* case an h product presents itself in correspondence with every partition of the weight; there are no cells empty in the square table and every symmetric function in brackets ( ) is of degree 6 when expressed in $h_1, h_2, h_3, \ldots$.

New Proof of the First Law of Symmetry.

Taking the two relations

$$(pqr\ldots) = \ldots + A h_{p_1} h_{q_1} h_{r_1} \ldots,$$
$$(p_1 q_1 r_1 \ldots) = \ldots + A' h_p h_q h_r \ldots,$$

the first gives rise to $(\delta_p \delta_q \delta_r \ldots) = \ldots + A \Delta_{p_1} \Delta_{q_1} \Delta_{r_1} \ldots$

and performing the sides of this upon the opposite sides of the second relation we find

$$A = A'$$

which establishes the direct theorem. For the reciprocal theorem write

$$h_p h_q h_r \ldots = \ldots + B(p_1 q_1 r_1 \ldots) + \ldots,$$
$$h_{p_1} h_{q_1} h_{r_1} \ldots = \ldots + B'(pqr\ldots) + \ldots;$$

from the first relation

$$\Delta_p \Delta_q \Delta_r \ldots = \ldots + B(\delta_{p_1} \delta_{q_1} \delta_{r_1} \ldots) + \ldots$$

and this, combined with the second relation, gives

$$B = B'.$$

If there be repetitional numbers in the partition we have merely to introduce the appropriate factorials and the proof proceeds in a similar manner.

General Law of Symmetry.

38. Suppose that we have any two formulas of the same weight

$$\Sigma A \lambda^l \mu^m \nu^n \ldots (\lambda^l \mu^m \nu^n \ldots) = \Sigma A' \lambda'^{l'} \mu'^{m'} \nu'^{n'} \ldots (h_\lambda^{l'} h_{\mu'}^{m'} h_{\nu'}^{n'} \ldots),$$
$$\Sigma B' \lambda^l \mu^m \nu^n \ldots h_\lambda^l h_\mu^m h_\nu^n \ldots = \Sigma B \lambda'^{l'} \mu'^{m'} \nu'^{n'} \ldots (\lambda'^{l'} \mu'^{m'} \nu'^{n'} \ldots).$$

The proof proceeds precisely as in the case of the functions $a_1, a_2, a_3, \ldots$ and we find

$$\Sigma A_{\lambda^l \mu^m \nu^n \ldots} B'_{\lambda^l \mu^m \nu^n \ldots} = \Sigma A'_{\lambda'^{l'} \mu'^{m'} \nu'^{n'} \ldots} B_{\lambda'^{l'} \mu'^{m'} \nu'^{n'} \ldots}.$$

Ex. gr. Take any two formulas from the Tables such as

$$h_4 + h_3 h_1 + h_2^2 + h_2 h_1^2 + h_1^4 = 5(4) + 12(31) + 16(2^2) + 27(21^2) + 47(1^4),$$
$$(31) \qquad = -4h_4 + 7h_3 h_1 + 2h_2^2 - 7h_2 h_1^2 + 2h_1^4,$$

multiplying together the coefficients of corresponding terms we have

$$1 = -20 + 84 + 32 - 189 + 94$$

which is right.

Operation of D or Δ upon a product.

39. Example 1.

$$D_6 (43)(2)(1) = \{D_{(42)} + D_{(321)}\}(43)(2)(1) = (3)(1) + (4).$$

Example 2. $$D_\lambda (1)^n = D_{(1^\lambda)} (1)^n = \binom{n}{\lambda} (1)^{n-\lambda}.$$

Hence at once the multinomial theorem, for if

$$(1)^n = \Sigma M (\lambda^l \mu^m \nu^n \ldots),$$

we have $$D_\lambda^l D_\mu^m D_\nu^n \ldots (1)^n = M$$

and $$\frac{n!}{(\lambda!)^l (n - l\lambda)!} D_\mu^m D_\nu^n \ldots (1)^{n-l\lambda} = M;$$

and thence $$M = \frac{n!}{(\lambda!)^l (\mu!)^m (\nu!)^n \ldots};$$

or $$(1)^n = \Sigma \frac{n!}{(\lambda!)^l (\mu!)^m (\nu!)^n \ldots} (\lambda^l \mu^m \nu^n \ldots),$$

the symmetric function expression of the multinomial theorem of algebra.

Example 3. The operators may be freely used for the purpose of multiplying symmetric functions together. Thus suppose we require the coefficient of (543) in the development of (321)(21)(21); writing

$$(321)(21)(21) = \ldots + A(543) + \ldots$$

we find $$A = D_5 D_4 D_3 (321)(21)(21);$$

now write $$D_5 = D_{(32)} + D_{(311)} + D_{(221)},$$

$$\therefore D_5 (321)(21)(21) = (\bullet 21)(\bullet 1)(21) + (\bullet 21)(21)(\bullet 1) + (\bullet 21)(2\bullet)(2\bullet) + (3\bullet 1)(\bullet 1)(2\bullet) + (3\bullet 1)(2\bullet)(\bullet 1) + (32\bullet)(\bullet 1)(\bullet 1)$$

(the dot indicating the part picked out)

$$= 2(21)(21)(1) + (21)(2)(2) + 2(31)(2)(1) + (32)(1)(1);$$

and since $$D_4 = D_{(22)} + D_{(211)},$$

$$\begin{aligned} D_5 D_4 (321)(21)(21) &= 2(\bullet 1)(\bullet 1)(1) + (21)(\bullet)(\bullet) + (\bullet 1)(\bullet)(2) + (\bullet 1)(2)(\bullet) \\ &\quad + 2(\bullet 1)(2\bullet)(\bullet) + 2(2\bullet)(\bullet 1)(\bullet) + 2(3\bullet)(\bullet)(\bullet) + (3\bullet)(\bullet)(\bullet) \\ &= 3(3) + 6(2)(1) + 2(1)(1)(1) + (21); \end{aligned}$$

and since $$D_3 = D_{(3)} + D_{(21)} + D_{(111)},$$

$$D_5 D_4 D_3 (321)(21)(21) = 3(\bullet) + 6(\bullet)(\bullet) + 2(\bullet)(\bullet)(\bullet) = 11.$$

Hence on multiplying out (321)(21)(21) we get one term 11 (543).

In like manner the other terms can be obtained.

The most important application of these operators is postponed to Section V.

SECTION II

THEORY OF SEPARATIONS

CHAPTER I

THE ALGEBRAIC THEORY

40. In the foregoing Section the Theory of Symmetric Functions and the Theory of Distributions have been developed together up to an interesting point. The theories have been mutually helpful but a new idea will now lead to more interesting and refined developments in both theories. The notion is to associate the Theory of Symmetric Functions not with a number but with the partition of a number and to shew that the theory of Section I is that particular part, of a more general theory, which is associated with that partition of a number which is composed wholly of units. This leads to a great advance in the algebra and the Theory of Distributions proceeds *pari passu*—at one time being urged along by the algebra, at another time pulling along the algebra after it. The relation of the theory of Section I to the theory of this Section will become clear, and attention will be drawn to those theorems which are quite general for every partition and to those which are peculiar to certain partitions.

The Separation of a Partition.

41. A partition of any number is "separated" into "separates" by writing down a set of partitions, each partition in its own brackets, from left to right so that when all of the parts of these partitions are assembled in a single bracket, the partition separated is reproduced. We thus obtain what is called a "Separation" of the partition.

Thus of a partition $(p_1p_2p_3p_4p_5)$,

$$(p_1p_2)(p_3p_4)(p_5),\quad (p_1p_2p_3)(p_4p_5),$$

are separations. The first and second of these are composed of separates (p_1p_2), (p_3p_4), (p_5); $(p_1p_2p_3)$, (p_4p_5) respectively.

Just as we speak of a number, quâ its partitions, as the partible number, so we speak of a partition, quâ its separations, as the separable partition.

42. It is convenient to arrange the separates of a separation in descending order of magnitude of weight or content from left to right. If these successive weights be $w_1, w_2, w_3, \ldots$ the separation is said to have a "Specification"

$$(w_1 w_2 w_3 \ldots),$$

and it must be noticed that both the separable partition and the specification of its separation are "partitions of the same number."

The sum of the highest part of the separates of a separation is called the "degree" of the separation. Thus the separation

$$(p_1 p_2)(p_3 p_4)(p_5)$$

has a specification $(p_1 + p_2, p_3 + p_4, p_5)$ and a degree $p_1 + p_3 + p_5$.

43. If the separates are some of them repeated in a separation we have the notion of "multiplicity." Thus if a separation be

$$(p_1 \ldots)^{j_1} (p_2 \ldots)^{j_2} (p_3 \ldots)^{j_3} \ldots$$

the multiplicity is defined by the succession of indices

$$j_1, j_2, j_3, \ldots.$$

Admitting unity as a possible value of a j, it is clear that every separation has a "multiplicity"; the sum of the multiplicity numbers is equal to the number of separates which appear in the separation.

Every separation has the characteristics (i) weight, (ii) separable partition, (iii) specification, (iv) degree, (v) multiplicity. There is also a sixth characteristic which will be reached and considered presently. At present the inquiry must be into the number of separations possessed by a given partition. Let us have before us the whole of the separations of the partition

$$(\lambda^2 \mu^2);$$

they are

$$(\lambda^2 \mu^2),$$
$$(\lambda^2 \mu)(\mu), \quad (\lambda \mu^2)(\lambda), \quad (\lambda^2)(\mu^2), \quad (\lambda \mu)^2,$$
$$(\lambda^2)(\mu)^2, \quad (\lambda)^2 (\mu^2), \quad (\lambda \mu)(\lambda)(\mu),$$
$$(\lambda)^2 (\mu)^2.$$

We observe that the number of the separations does not depend upon the absolute values of λ and μ but merely upon the repetitional exponents 2 and 2 which present themselves in the separable partition. The next remark is that separations correspond to every partition of the "multipartite number" (22). These partitions are

(22),	one part,
(21)(01), (12)(10), (20)(02), $(11)^2$,	two parts,
(20)$(01)^2$, $(10)^2(02)$, (11)(10)(01),	three ,,
$(10)^2(01)^2$,	four ,,

nine in number. In fact we see from Section I that these bipartite partitions correspond to the distributions of objects of type (22) into one, two, three and four similar parcels respectively. The notion of multiplicity arises naturally. It will be remembered that in Section I it was found that the consideration of a symmetric function

$$\Sigma\alpha^{\lambda}\beta^{\mu}\gamma^{\nu}\dots$$

could be dealt with for most purposes by merely considering the numbers $\lambda, \mu, \nu, \dots$. So for some purposes a partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots)$$

can be dealt with by restricting the attention to the repetitional numbers

$$\pi_1, \pi_2, \pi_3, \dots;$$

and we may say that the multiplicity of the partition is

$$|\,\pi_1\pi_2\pi_3\dots\,|.$$

In the same notation we say that the multiplicity of a separation

$$(p_1\dots)^{j_1}(p_2\dots)^{j_2}(p_3\dots)^{j_3}\dots \text{ is } |\,j_1j_2j_3\dots\,|.$$

We find instances where it is only necessary to consider the multiplicity of a separation. It will be observed that quâ partition $(\lambda^2\mu^2)$ has a multiplicity $|\,22\,|$ but quâ separation a multiplicity $|\,1\,|$.

It can now be asserted that the separable partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots)$$

of multiplicity $|\,\pi_1\pi_2\pi_3\dots\,|$ has separations which are in one-to-one correspondence with the partitions of the multipartite number

$$(\pi_1\pi_2\pi_3\dots),$$

and with the distributions of objects of type $(\pi_1\pi_2\pi_3\dots)$ into parcels of type (m) when m may have all integer values from 1 to $\Sigma\pi$.

In the particular case of the separable partition (p^{π}) the number of separations is equal to the number of partitions of the multiplicity π.

Groups of Separations.

44. The separations of the partition $(\lambda^2\mu^2)$ were arranged above according to the number of separates that made up the separations. There is a much more important arrangement. If in the separations we strike out μ altogether we are left either with (λ^2) or with $(\lambda)(\lambda)$; similarly if we strike out λ we are left either with (μ^2) or with $(\mu)(\mu)$. In fact in any separation λ considered alone occurs in one of the separations of (λ^2) and μ considered apart from λ occurs in one of the separations of (μ^2).

Those separations which involve both λ and μ in identical separations of (λ^2), (μ^2) respectively form a group. There are thus four groups associated with

$$(\lambda^2),\quad (\mu^2),$$
$$(\lambda)^2,\quad (\mu^2),$$
$$(\lambda^2),\quad (\mu)^2,$$
$$(\lambda)^2,\quad (\mu)^2,$$

or, attending only to exponents, we may write these

$$Gr\{(2),(2)\},\quad Gr\{(1^2),(2)\},\quad Gr\{(2),(1^2)\},\quad Gr\{(1^2),(1^2)\};$$

and we find for

$Gr\{(2),(2)\}$ $\quad (\lambda^2\mu^2),\quad (\lambda^2)(\mu^2),$

$Gr\{(1^2),(2)\}$ $\quad (\lambda\mu^2)(\lambda),\quad (\lambda)^2(\mu^2),$

$Gr\{(2),(1^2)\}$ $\quad (\lambda^2\mu)(\mu),\quad (\lambda^2)(\mu)^2,$

$Gr\{(1^2),(1^2)\}$ $\quad (\lambda\mu)^2,\quad (\lambda\mu)(\lambda)(\mu),\quad (\lambda)^2(\mu)^2.$

In the case of the separable partition (p^π) there are as many groups as there are separations; each group contains one separation only.

It is clear, in regard to a separable partition

$$(p_1^{\pi_1}p_2^{\pi_2}),$$

that there will be a group corresponding to every way possible of associating a partition of π_1 with a partition of π_2, and in general if P_π denote the number of partitions of π, the number of groups of separations appertaining to the separable partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$$

is

$$P_{\pi_1}.P_{\pi_2}.P_{\pi_3}.\ldots.$$

The sixth characteristic of a separation is thus the group of separations to which it belongs.

The usual Tables of Symmetric Functions comprise results such as

$$(3)=(1)^3-3(1)(1^2)+3(1^3)$$

and the reader will observe that on the dexter, to numerical coefficients *près*, each term is a separation of the separable partition (1^3). In general a Table of weight w gives the expressions of all symmetric functions symbolized by a single partition of w as linear functions of separations of the partition (1^w). The theory of Section I is in fact based on the particular partition (1^w) of the number w and is thus a unitary theory. We are about to see that there is a theory based upon any partition of w selected at pleasure and that there is also a parallel Theory of Distributions.

Generalization of the Theory of Section I.

45. In Section I we considered the distribution of n objects into n parcels and found that the Distribution Function of n objects into parcels of type $(p_1 q_1 r_1 \ldots)$, where $p_1 + q_1 + r_1 + \ldots = n$, is

$$h_{p_1} h_{q_1} h_{r_1} \ldots .$$

We analyse this result in the following manner:

Write
$$c_1 = (1)\, b_1,$$
$$c_2 = (2)\, b_2 + (1^2)\, b_1^2,$$
$$c_3 = (3)\, b_3 + (21)\, b_2 b_1 + (1^3)\, b_1^3,$$
$$c_4 = (4)\, b_4 + (31)\, b_3 b_1 + (2^2)\, b_2^2 + (21^2)\, b_2 b_1^2 + (1^4)\, b_1^4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and in general

$$c_s = \Sigma\, (\lambda \mu \nu \ldots)\, b_\lambda b_\mu b_\nu \ldots,$$

the summation being for every partition of the number s.

Observe that, if $b_1 = b_2 = b_3 = \ldots = 1$, then $c_s = h_s$.

Consider the result of multiplication

$$c_{p_1} c_{q_1} c_{r_1} \ldots = \Sigma P b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} b_{s_3}^{\sigma_3} \ldots ;$$

in the product $c_{p_1} c_{q_1} c_{r_1} \ldots$ every symmetric function product that presents itself is a separation (of some partition of weight $p_1 + q_1 + r_1 + \ldots$) of specification $(p_1 q_1 r_1 \ldots)$; the partition of which it is a separation will become manifest on inspection of the b-product to which it is attached. Thus if the attached b-product be

$$b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} b_{s_3}^{\sigma_3} \ldots$$

it is clear that it is a separation of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots);$$

hence P must be a linear function of separations of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots)$$

each of which has a specification $(p_1 q_1 r_1 \ldots)$.

It can now be shewn that the linear function P is also the Distribution Function of n objects into parcels

$$(p_1 q_1 r_1 \ldots)$$

when the distributions are subject to certain restrictions.

Recalling Section I, Art. 10 it will be seen that we considered the particular case of distribution of objects of type (321^2) into parcels of type (43);

we wrote down h_4 and h_3 at length in terms of the quantities $\alpha, \beta, \gamma, \delta$ and found that *term* of the symmetric function (321^2) which arose from the multiplication $\alpha^3\beta \times \beta\gamma\delta$; we supposed the letters $\alpha\alpha\alpha\beta$ to be four objects placed in similar parcels (one object in each parcel) of one kind and the letters $\beta\gamma\delta$ to be three objects placed in three similar parcels of a second kind. In the present theory the product $\alpha^3\beta$ being a term of the symmetric function (31) would have the literal product b_3b_1 attached to it, and the product $\beta\gamma\delta$ being a term of the symmetric function (1^3) would have the product b_1^3 attached to it. The meaning of b_3 here is clearly that we have in the distribution three similar objects, viz. $\alpha\alpha\alpha$ in similar parcels; and the meaning of b_1 which multiplies b_3 is that one object β of a certain kind is found in a parcel; so also the product b_1^3 attached to $\beta\gamma\delta$ indicates that objects of three different kinds are found in parcels; had the two objects β, β been found in *similar* parcels b_2 would have found a place in the b-product.

From these considerations we see that the whole b-product attached to the term $\alpha^3\beta^2\gamma\delta$, which has been derived from the multiplication

$$\alpha^3\beta \times \beta\gamma\delta,$$

indicates that the corresponding distribution is such that if we agree to write down a factor b_s whenever we observe *s similar objects in s similar parcels* we shall obtain the product

$$b_3b_1^4,$$

where (31^4) and (321^2) are necessarily partitions of the same number.

In this particular case therefore we see that the product $b_3b_1^4$ will appear whenever a distribution is of the type (31^4). (See Sec. I, Art. 2.)

In general if in any distribution of n objects into n parcels (one object in one parcel) we agree to write down a factor

$$b_\xi$$

whenever we observe ξ similar objects in ξ similar parcels, we of necessity write down a b-product

$$b_{\xi_1}b_{\xi_2}b_{\xi_3}\ldots,$$

where

$$(\xi_1\xi_2\xi_3\ldots)$$

is some partition of n. We then say that the distribution is of type

$$(\xi_1\xi_2\xi_3\ldots).$$

Thence it follows that P, the linear function of symmetric function products which is the coefficient of

$$b_{s_1}^{\sigma_1}b_{s_2}^{\sigma_2}b_{s_3}^{\sigma_3}\ldots,$$

is the Distribution Function of n objects into parcels $(p_1q_1r_1\ldots)$, the distributions being all of them of type

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots).$$

46. Suppose now that P is expanded so as to be a linear function of monomial symmetric functions, viz.:

$$P = \Sigma\theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots);$$

we will write $(p_1q_1r_1\ldots)$ in the more general form

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),$$

and we then have

$$c_{p_1}^{\pi_1}c_{p_2}^{\pi_2}c_{p_3}^{\pi_3}\ldots = \ldots + \{\Sigma\theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)\}\,b_{s_1}^{\sigma_1}b_{s_2}^{\sigma_2}b_{s_3}^{\sigma_3}\ldots + \ldots,$$

indicating that there are precisely θ ways of distributing objects of type

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)$$

into parcels of type $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$,

the distributions being all of them of type

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots).$$

The reader will have little difficulty in grasping the nature of the type in any particular case of distribution. Thus realizing a distribution as in Section I such as

A	A	A	A	A	B	B	B	C	C
a	a	a	b	b	b	b	c	c	c
3			2		2		1	2	

the type is seen to be (32^21).

It is seen intuitively that interchanging capital and small letters does *not* alter the restricting partition; hence there are also θ ways of distributing objects of type

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$$

into parcels of type

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots),$$

the distributions being all of them of type

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots).$$

The effect of this reciprocity on the algebra is immediate, for we see that if

$$c_{p_1}^{\pi_1}c_{p_2}^{\pi_2}c_{p_3}^{\pi_3}\ldots = \ldots\ldots + \theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)\,b_{s_1}^{\sigma_1}b_{s_2}^{\sigma_2}b_{s_3}^{\sigma_3}\ldots + \ldots$$

we are at liberty to interchange the partitions

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots),\quad (\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots),$$

without altering the b-product, so that also

$$c_{\lambda_1}^{l_1}c_{\lambda_2}^{l_2}c_{\lambda_3}^{l_3}\ldots = \ldots\ldots + \theta\,(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)\,b_{s_1}^{\sigma_1}b_{s_2}^{\sigma_2}b_{s_3}^{\sigma_3}\ldots + \ldots,$$

indicating a law of symmetry in symmetric algebra of great generality and

interest. The meaning in general is that if we take that linear function of the separations of the symmetric function

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots),$$

each of which has necessarily the specification

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

which arises from the product

$$c_{p_1}^{\pi_1} c_{p_2}^{\pi_2} c_{p_3}^{\pi_3} \ldots,$$

and then express such linear function of separations as a linear function of single partition symmetric functions, the function

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots)$$

will appear with a coefficient which will remain unchanged when in the process the partitions

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots), \quad (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots)$$

are interchanged.

We obtain special cases by giving the quantities $b_1, b_2, b_3, \ldots$ the values zero or unity. If we put each b equal to unity we have the First Law of Symmetry of Section I, there being no restricting partition. If we put $b_2 = b_3 = \ldots = 0$, merely retaining b_1, we have the Second Law and so on.

The reader must notice that the theorem is in regard to any three partitions of the same number and has to do with the separations of one of those partitions; it is further concerned with the separations of a certain specification denoted by a second of those partitions; these separations are linearly connected in such wise that each separation has attached to it a coefficient which indicates the number of ways of permuting the separates of the separation, the only permutations allowable being those amongst *separates of the same content*; this is clear because such coefficients only arise in the powers

$$c_{p_1}^{\pi_1}, \quad c_{p_2}^{\pi_2}, \quad c_{p_3}^{\pi_3}, \ldots$$

which make up the c-product.

Let us now fix the attention on the partition whose separations are under consideration; these separations are of various specifications

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$$

and they occur in a number of linear functions P, each of which has one of the said specifications, and moreover when the functions P are developed they give rise to a number of monomial functions

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots);$$

the theorem of symmetry or reciprocity derived from the distribution theory now shews that the various monomials

$$(\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots)$$

are, in some order, identical with the various specifications

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots);$$

and that writing the expressions P in any order vertically and the corresponding specifications in the same order from left to right we are able to express the linear functions of separations P in terms of monomial symmetric functions by means of a Table possessing row and column symmetry.

Theorem. "Of a separable partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots)$$

the linear functions of separations P are expressible by means of monomial symmetric functions whose partitions are identical with the specifications of the separations, and a symmetrical Table may be thus formed.

Conversely by solving a number of linear equations or otherwise the monomial functions are expressible as linear functions of the functions P and the Table thus formed must, by elementary determinant theory, possess row and column symmetry."

A theorem of expressibility now arises, a generalization of the fact that all monomial symmetric functions are expressible in terms of the elementary functions, or as we may say by means of symmetric function products each of which is a separation of the partition (1^w). The symmetric functions which are expressible by means of products which are symbolized by separations of a given partition are denoted by partitions which are identical with the specifications of the separations of the given partition; moreover if $(pqr\ldots)$ be a specification of a separation of a given partition it is possible to arrive at that partition by substituting

for p some partition of p
„ q „ „ q
„ r „ „ r
.....

47. Hence the following

Theorem. "If a symmetric function be symbolized by

$$(pqr\ldots)$$

and

$(p_1 p_2 p_3 \ldots)$ be any partition of p
$(q_1 q_2 q_3 \ldots)$ „ „ q
$(r_1 r_2 r_3 \ldots)$ „ „ r
.....

the symmetric function $(pqr\ldots)$ is expressible by means of symmetric

function products each of which is symbolized by a separation of the partition

$$(p_1p_2p_3\ldots q_1q_2q_3\ldots r_1r_2r_3\ldots).''$$

Ex. gr. The symmetric function (321) *is* expressible by means of separations of (1^6), (21^4), (2^21^2), (31^3), or (321) but *is not* expressible by means of separations of (2^3), (3^2), (41^2), (42), (51) or (6).

This is an important theorem in regard to algebraic expressibility. We have now before us the complete separation Tables, Direct and Inverse, as far as weight 6 inclusive. From each row of the Direct Tables the reader will be able to enunciate a theorem of distribution.

CHAPTER II

GENERALIZATION OF GIRARD'S FORMULA

48. Turning back to the formula of Girard given in Section I, Art. 5, we will give it a different form by writing $S(1^n)$ in place of s_n, the notation indicating that the sum of the nth powers of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ or of $\alpha, \beta, \gamma, \ldots$ is expressed by symmetric function products symbolized by separations of (1^n). We then write the formula

$$(-)^n \frac{(n-1)!}{n!} S(1^n) = \Sigma (-)^{\Sigma j} \frac{(\Sigma j - 1)!}{j_1! j_2! j_3! \ldots} (1)^{j_1} (1^2)^{j_2} (1^3)^{j_3} \ldots .$$

Now, if we denote by $S(\lambda^l \mu^m \ldots)$ the expression of s_n by means of symmetric function products, each of which is symbolized by a separation of the partition

$$(\lambda^l \mu^m \ldots)$$

of the number n, we can establish the elegant result

$$(-)^{l+m+\ldots} \frac{(l+m+\ldots-1)!}{l!\, m! \ldots} S(\lambda^l \mu^m \ldots)$$

$$= \Sigma (-)^{\Sigma j} \frac{(\Sigma j - 1)!}{j_1! j_2! j_3! \ldots} (J_1)^{j_1} (J_2)^{j_2} (J_3)^{j_3} \ldots,$$

where $$(J_1)^{j_1} (J_2)^{j_2} (J_3)^{j_3} \ldots$$

is a separation of the partition $(\lambda^l \mu^m \ldots)$ and the summation is for every such separation.

We recur to the system of relations

$$\begin{aligned} c_1 &= (1)\, b_1, \\ c_2 &= (2)\, b_2 + (1^2)\, b_1^2, \\ c_3 &= (3)\, b_3 + (21)\, b_2 b_1 + (1^3)\, b_1^3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

which we may look upon, from one point of view, as a transformation by means of which the quantities c are transformed into the quantities b; and we may enquire if there exists a system of invariants of this transformation; whether in fact we can form a system of functions of $c_1, c_2, c_3, \ldots$, which, to symmetric function multipliers *près*, are equal to the like functions of

$b_1, b_2, b_3, \ldots$. A complete system of such functions does exist and they are of fundamental importance.

In the first place the reader will notice that c_1 is such a function. The complete system is found as follows:

Granted that the symmetric functions on the dexter sides of the relations refer to quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$, we observe that the expression

$$1 + c_1 + c_2 + c_3 + \ldots$$

may be broken up into factors of the form

$$1 + \alpha_s b_1 + \alpha_s^2 b_2 + \alpha_s^3 b_3 + \ldots,$$

so that we have the identity

$$1 + c_1 + c_2 + c_3 + \ldots = \prod_{\alpha} (1 + \alpha b_1 + \alpha^2 b_2 + \alpha^3 b_3 + \ldots),$$

a factor appearing for each of the quantities α which may be supposed to be infinite in number.

We may also introduce an arbitrary quantity μ and write

$$1 + \mu c_1 + \mu^2 c_2 + \mu^3 c_3 + \ldots = \prod_{\alpha} (1 + \mu\alpha b_1 + \mu^2\alpha^2 b_2 + \mu^3\alpha^3 b_3 + \ldots).$$

Taking logarithms

$$\log (1 + \mu c_1 + \mu^2 c_2 + \mu^3 c_3 + \ldots) = \sum_{\alpha} \log (1 + \mu\alpha b_1 + \mu^2\alpha^2 b_2 + \mu^3\alpha^3 b_3 + \ldots);$$

and expanding

$$\mu c_1 + \mu^2 (c_2 - \tfrac{1}{2} c_1^2) + \mu^3 (c_3 - c_2 c_1 + \tfrac{1}{3} c_1^3) + \ldots$$
$$= \mu (1) b_1 + \mu^2 (2) (b_2 - \tfrac{1}{2} b_1^2) + \mu^3 (3) (b_3 - b_2 b_1 + \tfrac{1}{3} b_1^3) + \ldots;$$

and by equating coefficients of like powers of μ we obtain the complete system of functions we are searching for. Thus for μ^l

$$\Sigma (-)^{\Sigma l - 1} \frac{(\Sigma l - 1)!}{l_1!\, l_2! \ldots} c_{\lambda_1}^{l_1} c_{\lambda_2}^{l_2} \ldots = (l) \Sigma (-)^{\Sigma l - 1} \frac{(\Sigma l - 1)!}{l_1!\, l_2! \ldots} b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots,$$

where $l = \Sigma l\lambda$.

Hence in general

$$\Sigma (-)^{\Sigma l - 1} \frac{(\Sigma l - 1)!}{l_1!\, l_2! \ldots} c_{\lambda_1}^{l_1} c_{\lambda_2}^{l_2} \ldots$$

is an invariant function of the transformation for all integer values of l.

If $c_1, c_2, c_3, \ldots$ be elementary symmetric functions quâ elements $\gamma_1, \gamma_2, \gamma_3, \ldots$ the invariant function, when multiplied by $\Sigma l\lambda$ and taken with negative sign when $\Sigma l\lambda$ is even, is the sum of the lth powers of $\gamma_1, \gamma_2, \gamma_3, \ldots$; moreover if $b_1, b_2, b_3, \ldots$ be elementary symmetric functions of quantities $\beta_1, \beta_2, \beta_3, \ldots$ and we write the symmetric functions as partitions $(\ )_\alpha, (\ )_\beta, (\ )_\gamma$, according as they refer to the quantities α, β or γ we have

$$(l)_\gamma = (l)_\alpha (l)_\beta,$$

a simple relation which shews that in the original set of relations we are at liberty to interchange the quantities α with the quantities β; in fact they may be written

$$c_1 = (1)_\beta a_1,$$
$$c_2 = (2)_\beta a_2 + (1^2)_\beta a_1^2,$$
$$c_3 = (3)_\beta a_3 + (21)_\beta a_2 a_1 + (1^3)_\beta a_1^3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

To proceed, we have the identity

$$\Sigma (-)^{\Sigma l - 1} \frac{(\Sigma l - 1)!}{l_1! \, l_2! \ldots} c_{\lambda_1}^{l_1} c_{\lambda_2}^{l_2} \ldots = (l) \, \Sigma (-)^{\Sigma l - 1} \frac{(\Sigma l - 1)!}{l_1! \, l_2! \ldots} b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots,$$

and we have to express the left-hand side in terms of $b_1, b_2, b_3, \ldots$ by means of the relations connecting the functions c with the functions b so as to pick out the term which involves

$$b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots;$$

suppose that a term on the left-hand side involves the c product

$$c_{\mu_1}^{m_1} c_{\mu_2}^{m_2} \ldots;$$

in this the cofactor of $b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots$ consists of products of symmetric functions, each product being a separation of the function

$$(\lambda_1^{l_1} \lambda_2^{l_2} \ldots);$$

if such separation be

$$(J_1)^{j_1} (J_2)^{j_2} \ldots,$$

we see, by applying the multinomial theorem separately to $c_{\mu_1}^{m_1}, c_{\mu_2}^{m_2} \ldots$, that the whole cofactor of $b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots$ is

$$\Sigma \frac{m_1! \, m_2! \ldots}{j_1! \, j_2! \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots,$$

the summation being for all the separations.

Hence the complete term

$$(-)^{\Sigma m - 1} \frac{(\Sigma m - 1)!}{m_1! \, m_2! \ldots} c_{\mu_1}^{m_1} c_{\mu_2}^{m_2} \ldots$$

$$= \ldots + (-)^{\Sigma j - 1} \frac{(\Sigma j - 1)!}{j_1! \, j_2! \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots + \ldots,$$

because $\Sigma m = \Sigma j$.

Hence substituting in the identity and equating cofactors of $b_{\lambda_1}^{l_1} b_{\lambda_2}^{l_2} \ldots$ we have

$$(-)^{\Sigma l} \frac{(\Sigma l - 1)!}{l_1! \, l_2! \ldots} \cdot (l) = \Sigma (-)^{\Sigma j} \frac{(\Sigma j - 1)!}{j_1! \, j_2! \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots,$$

or

$$(-)^{\Sigma l} \frac{(\Sigma l - 1)!}{l_1! \, l_2! \ldots} S (\lambda_1^{l_1} \lambda_2^{l_2} \ldots) = \Sigma (-)^{\Sigma j} \frac{(\Sigma j - 1)!}{j_1! \, j_2! \ldots} (J_1)^{j_1} (J_2)^{j_2} \ldots,$$

where $(J_1)^{j_1}(J_2)^{j_2}\ldots$

is a separation of $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$,

as was to be shewn.

This remarkable result enables one to write down at once the expression of s_n in terms of separations of any partition of n.

Ex. gr. $s_6 = S(2^2 1^2)$,

and
$$\tfrac{3}{2}S(2^2 1^2) = \tfrac{3}{2}(2)^2(1)^2 - 2(21)(2)(1) - (2^2)(1)^2 - (1^2)(2)^2 + (2^2)(1^2) + \tfrac{1}{2}(21)^2 + (21^2)(2) + (2^2 1)(1) - (2^2 1^2).$$

Properties of the Coefficients in the Expressions for s_n.

49. We recur to the definition above given Art. 44 of a "Group of Separations" and we will establish that if s_n be expressed as a linear function of separations of a partition of n, and provided that such partition does not consist merely of repetitions of a single part, the algebraic sum of the coefficients appertaining to each group of separations is zero.

Ex. gr. Arranged by groups of separations

	$\tfrac{3}{2}S(2^2 1^2)$				
	$= \tfrac{3}{2}(2)^2(1)^2$	$-(2^2)(1)^2$	$-(1^2)(2)^2$	$+(2^2)(1^2)$	
	$-2(21)(2)(1) + (2^2 1)(1) + (21^2)(2) - (2^2 1^2)$				
	$+\tfrac{1}{2}(21)^2$				
Sum of coefficients	$=$ 0	0	0	0	
Group is	$\{(1^2), (1^2)\}$	$\{(2), (1^2)\}$	$\{(1^2), (2)\}$	$\{(2), (2)\}$	

It will be noted that the partition $(2^2 1^2)$ satisfies the condition that there must be at least two *different* parts.

To establish the theorem we put the group in evidence by writing

$$c_1 = (1)_a A_1,$$
$$c_2 = (2)_a B_1 + (1^2)_a A_2,$$
$$c_3 = (3)_a C_1 + (21)_a A_1 B_1 + (1^3)_a A_3,$$
$$c_4 = (4)_a D_1 + (31)_a A_1 C_1 + (2^2)_a B_2 + (21^2)_a A_2 B_1 + (1^4)_a A_4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$c_l = \Sigma(1^{k_1}2^{k_2}3^{k_3}\ldots)_a A_{k_1} B_{k_2} C_{k_3}\ldots,$$

where the summation is for all partitions $(1^{k_1}2^{k_2}3^{k_3}\ldots)$ of l.

If we form any product

$$c_f c_g c_h \ldots,$$

we obtain on the right-hand side a number of terms of type

$$(1^{f_1}2^{f_2}3^{f_3}\ldots)_a\,(1^{g_1}2^{g_2}3^{g_3}\ldots)_a\,(1^{h_1}2^{h_2}3^{h_3}\ldots)_a\ldots$$
$$\ldots A_{f_1}B_{f_2}C_{f_3}\ldots A_{g_1}B_{g_2}C_{g_3}\ldots A_{h_1}B_{h_2}C_{h_3}\ldots\ldots,$$

where the portion $A_{f_1}B_{f_2}C_{f_3}\ldots A_{g_1}B_{g_2}C_{g_3}\ldots A_{h_1}B_{h_2}C_{h_3}\ldots\ldots$

indicates that the separation

$$(1^{f_1}2^{f_2}3^{f_3}\ldots)_a\,(1^{g_1}2^{g_2}3^{g_3}\ldots)_a\,(1^{h_1}2^{h_2}3^{h_3}\ldots)_a\ldots$$

belongs to the group of separations

$$G\,\{(f_1g_1h_1\ldots),\ (f_2g_2h_2\ldots),\ (f_3g_3h_3\ldots),\ \ldots\}.$$

When we were dealing with the relations

$$c_1=(1)_a b_1,$$
$$c_2=(2)_a b_2+(1^2)_a b_1^2,$$
$$\text{etc.}$$

we obtained a result which as a particular case is

$$c_1^4-4c_1^2c_2+2c_2^2+4c_1c_3-4c_4=(4)_a\,(b_1^4-4b_1^2b_2+2b_2^2+4b_1b_3-4b_4).$$

For the relations now under view we find

$$c_1^4-4c_1^2c_2+2c_2^2+4c_1c_3-4c_4$$
$$=\{(1)_a^4A_1^4-4\,(1)_a^2\,(1^2)_aA_1^2A_2+2\,(1^2)_a^2A_2^2+4\,(1)_a\,(1^3)_aA_1A_3-4\,(1^4)_aA_4\}$$
$$-4\,\{(1)_a^2\,(2)_aA_1^2B_1-(1)_a\,(21)_aA_1^2B_1-(2)_a\,(1^2)_aA_2B_1+(21^2)_aA_2B_1\}$$
$$+2\,\{(2)_a^2B_1^2-2\,(2^2)_aB_2\}$$
$$+4\,\{(1)_a\,(3)_aA_1C_1-(31)_aA_1C_1\}$$
$$-4\,\{(4)_aD_1\}$$

and it is observed that each separation has attached to it a literal product which indicates the group of separations to which it belongs.

This latter result becomes the former by writing

$$(A_k,\ B_k,\ C_k,\ \ldots)=(b_1^k,\ b_2^k,\ b_3^k,\ \ldots)\ \text{respectively,}$$

so that if we put therein the suffixed letters A, B, C, D, ... each equal to unity, the portions of the five lines in brackets $\{\ \}$ are respectively equal to s_4 or $(4)_a$ expressed in terms of separations of (1^4), (21^2), (2^2), (31), (4).

If the theorem to be established is true the second and fourth lines should vanish when all of the symmetric function factors of the separations are put equal to unity. They visibly do vanish shewing that in the formula

$$s_4=(1)^2\,(2)-(1)\,(21)-(2)\,(1^2)+(21^2)$$

the sum of the coefficients vanishes in each of the two groups and in the formula

$$s_4=(1)\,(3)-(31)$$

in the single group.

It will suffice therefore to shew that, when s_n is formed for the elementary functions $c_1, c_2, c_3, \ldots$ and on the right-hand side every symmetric function $(\ )_a$ is put equal to unity, only literal products of the forms

$$A_{f_1}A_{g_1}A_{h_1}\ldots$$
$$B_{f_2}B_{g_2}B_{h_2}\ldots$$
$$C_{f_3}C_{g_3}C_{h_3}\ldots$$
$$\ldots\ldots\ldots\ldots$$

survive—in fact that no product containing more than one of the different symbols $A, B, C, \ldots$ survives.

For this purpose put $c_1' = A_1,$
$c_2' = B_1 + A_2,$
$c_3' = C_1 + A_1B_1 + A_3,$
$\ldots\ldots\ldots\ldots\ldots\ldots$

so that

$$1 + c_1' + c_2' + c_3' + \ldots$$
$$= (1 + A_1 + A_2 + A_3 + \ldots)(1 + B_1 + B_2 + B_3 + \ldots)(1 + C_1 + C_2 + C_3 + \ldots)\ldots.$$

Taking logarithms of each side and expanding we find

$$\begin{aligned} & c_1' - \tfrac{1}{2}(c_1'^2 - 2c_2') + \tfrac{1}{3}(c_1'^3 - 3c_1'c_2' + 3c_3') - \ldots \\ &= A_1 - \tfrac{1}{2}(A_1^2 - 2A_2) + \tfrac{1}{3}(A_1^3 - 3A_1A_2 + 3A_3) - \ldots \\ &+ B_1 - \tfrac{1}{2}(B_1^2 - 2B_2) + \tfrac{1}{3}(B_1^3 - 3B_1B_2 + 3B_3) - \ldots \\ &+ C_1 - \tfrac{1}{2}(C_1^2 - 2C_2) + \tfrac{1}{3}(C_1^3 - 3C_1C_2 + 3C_3) - \ldots \\ &+ \ldots\ldots. \end{aligned}$$

This relation proves the theorem because the right-hand side visibly contains no product which involves more than one of the different symbols $A, B, C, \ldots$.

Theorem. "If s_n be expressed as a linear function of separations of a partition of n and if such partition contain at least two different parts the algebraic sum of the coefficients appertaining to each group of separations is zero."

It follows, as a necessary corollary, that under the same circumstances the sum of the whole of the coefficients in the expression of s_n is zero. This theory of the group can be extended to other symmetric functions. Consider for a moment the expression of (3^2) by means of separations of (21^4); we can reach it by writing

$$2(3^2) = s_3^2 - s_6 = S(21)\,S(1^3) - S(21^4).$$

It is clear that the sum of the whole of the coefficients is zero because both $S(21)$ and $S(21^4)$ possess that property. If on the right-hand side there had been such a product as $S(1^4)\,S(2)$ this would not have been the case because the algebraic sum of the coefficients is not zero either in $S(1^4)$

or in $S(2)$. This product does not present itself because the partition (3^2) possesses no separation of specification (42). The reasoning is general and we are led to the

Theorem. "In the expression of the symmetric function

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$$

by means of separations of $(s_1^{\sigma_1}s_2^{\sigma_2}\ldots)$

the algebraic sum of the coefficients will be zero provided that

(i) the partition $(s_1^{\sigma_1}s_2^{\sigma_2}\ldots)$ contains at least two different parts;

(ii) the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$ possesses no separation of specification $(\sigma_1 s_1, \sigma_2 s_2, \ldots)$."

Similarly if we express $S(21)$, $S(1^3)$ and $S(21^4)$ as sums of groups of separations we can express $2(3^2)$ as a sum of groups and each such group would arise as the sum of two terms, one of which would be a product of a group of $S(21)$ by a group of $S(1^3)$ and the other would be a group of $S(21^4)$. Since each group of $S(21)$ and $S(21^4)$ has zero for the sum of its coefficients it follows that each group in the expression of $2(3^2)$ has zero for the sum of its coefficients.

Hence in the case of any monomial symmetric function expressed in terms of separations of a given symmetric function the sum of the coefficients of each group vanishes under the same circumstances as the whole of the coefficients.

Theorem. "In the expression of the symmetric function

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$$

by means of separations of $(s_1^{\sigma_1}s_2^{\sigma_2}\ldots)$

the algebraic sum of the coefficients of each group of separations will be zero provided that

(i) the partition $(s_1^{\sigma_1}s_2^{\sigma_2}\ldots)$ contains at least two different parts;

(ii) the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$ possesses no separation of specification $(\sigma_1 s_1, \sigma_2 s_2, \ldots)$."

These laws may be verified in the Tables of Symmetric Functions.

CHAPTER III

THE DIFFERENTIAL OPERATORS OF THE THEORY OF SEPARATIONS

50. In Section I we expressed the obliterating operators $D_1, D_2, D_3, \ldots$ in a form suitable to the separation theory by writing

$$D_1 = D_{(1)},$$
$$D_2 = D_{(2)} + D_{(12)},$$
$$D_3 = D_{(3)} + D_{(21)} + D_{(13)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D_\lambda = \Sigma D_{(\lambda_1 \lambda_2 \ldots \lambda_s)},$$

the summation being for every partition of λ.

We also established the law of operation of

$$D_{(\lambda_1 \lambda_2 \ldots \lambda_s)}$$

upon a symmetric function product.

We now require a similar extension of the operators $d_1, d_2, d_3, \ldots$. Consider the separable partition

$$(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)$$

and note that, if it be separated, any combination of its parts may occur as a separate; the number of distinct separates that may occur depends only upon the multiplicity of the partition separated and is equal to

$$(1 + l_1)(1 + l_2) \ldots - 1.$$

In order to adapt the operators $d_1, d_2, d_3, \ldots$ to an operand composed of separations of a partition it is necessary to consider all of the separates as independent variables.

Let (v) be any such separate, then by a well-known theorem of the differential calculus we have

$$d_s = \Sigma d_s(v) \frac{d}{d(v)},$$

the summation being in regard to every separate.

Since by Section I, Art. 27

$$\frac{(-)^s}{s} d_s = \Sigma \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} D_1^{\pi_1} D_2^{\pi_2}\ldots$$

where $$\pi_1 + 2\pi_2 + 3\pi_3 + \ldots = s;$$

if $$(v) = (1^{p_1+\pi_1} 2^{p_2+\pi_2}\ldots),$$

$$\frac{(-)^s}{s} d_s(v) = \Sigma\,(-)^{\Sigma\pi} \frac{(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} (1^{p_1} 2^{p_2}\ldots),$$

the summation being for every partition

$$(1^{\pi_1} 2^{\pi_2}\ldots)$$

of s.

Hence $$d_s \equiv \Sigma\Sigma\,(-)^{\Sigma\pi+s} \frac{(\Sigma\pi - 1)!\,s}{\pi_1!\,\pi_2!\ldots} (1^{p_1} 2^{p_2}\ldots) \frac{d}{d\,(1^{p_1+\pi_1} 2^{p_2+\pi_2}\ldots)},$$

where on the right-hand side the operand is a linear function of separations of a certain partition and the summation is in regard to

(i) every separate $(1^{p_1+\pi_1} 2^{p_2+\pi_2}\ldots)$,

(ii) every partition $(1^{\pi_1} 2^{\pi_2}\ldots)$ of s.

Let us now put

$$\Sigma\,(1^{p_1} 2^{p_2}\ldots) \frac{d}{d\,(1^{p_1+\pi_1} 2^{p_2+\pi_2}\ldots)} = d_{(1^{\pi_1} 2^{\pi_2}\ldots)},$$

wherein the indices $\pi_1, \pi_2, \ldots$ are constant, and the summation is in regard to every separate of the form

$$(1^{p_1+\pi_1} 2^{p_2+\pi_2}\ldots);$$

then $$d_s = \Sigma\,(-)^{\Sigma\pi+s} \frac{(\Sigma\pi - 1)!\,s}{\pi_1!\,\pi_2!\ldots} d_{(1^{\pi_1} 2^{\pi_2}\ldots)}.$$

We call $$d_{(1^{\pi_1} 2^{\pi_2}\ldots)}$$

a linear partition operator, and of course

$$(1^{\pi_1} 2^{\pi_2}\ldots)$$

is a partition of s and we have expressed the number or weight operator d_s as a linear function of linear partition operators by a law similar to that of Girard for expressing the one-part symmetric functions in terms of the elementary functions.

In particular
$$d_1 = d_{(1)},$$
$$d_2 = d_{(1^2)} - 2d_{(2)},$$
$$d_3 = d_{(1^3)} - 3d_{(21)} + 3d_{(3)},$$
$$d_4 = d_{(1^4)} - 4d_{(21^2)} + 2d_{(2^2)} + 4d_{(31)} - 4d_{(4)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Ex. gr. Suppose the partition separated to be

$$(2^2 1^2):$$

this has separations involving eight separates, viz.:

$$(2^2 1^2),\quad (2^2 1),\quad (2^2),\quad (21^2),\quad (21),\quad (2),\quad (1^2),\quad (1)$$

and therefore

$$d_{(1)} = \frac{d}{d(1)} + (1)\frac{d}{d(1^2)} + (2)\frac{d}{d(21)} + (21)\frac{d}{d(21^2)} + (2^2)\frac{d}{d(2^2 1)} + (2^2 1)\frac{d}{d(2^2 1^2)};$$

$$d_{(1^2)} = \frac{d}{d(1^2)} + (2)\frac{d}{d(21^2)} + (2^2)\frac{d}{d(2^2 1^2)};$$

$$d_{(2)} = \frac{d}{d(2)} + (1)\frac{d}{d(21)} + (1^2)\frac{d}{d(21^2)} + (2)\frac{d}{d(2^2)} + (21)\frac{d}{d(2^2 1)} + (21^2)\frac{d}{d(2^2 1^2)};$$

$$d_{(21)} = \frac{d}{d(21)} + (1)\frac{d}{d(21^2)} + (2)\frac{d}{d(2^2 1)} + (21)\frac{d}{d(2^2 1^2)};$$

$$d_{(21^2)} = \frac{d}{d(21^2)} + (2)\frac{d}{d(2^2 1^2)};$$

$$d_{(2^2)} = \frac{d}{d(2^2)} + (1)\frac{d}{d(2^2 1)} + (1^2)\frac{d}{d(2^2 1^2)};$$

$$d_{(2^2 1)} = \frac{d}{d(2^2 1)} + (1)\frac{d}{d(2^2 1^2)};$$

$$d_{(2^2 1^2)} = \frac{d}{d(2^2 1^2)};$$

and no other partition operators need be considered. The number of potent operators is clearly always equal to the number of separates of the given partition. We thus see that any weight operation d_s may be carried out, when the operand consists of separations of a given partition, by a number of differentiations in regard to the separates which compose the separation.

Now let us have before us the identity

$$3(51) = 2(2^2 1^2) + (2^2 1)(1) - 2(21^2)(2) - 2(2^2)(1^2) - (21)^2 - (2^2)(1)^2 + (21)(2)(1) + 2(2)^2(1^2).$$

Observing that $d_2 = D_1^2 - 2D_2$ it is clear that the left-hand side vanishes under the operation of d_2; therefore the right-hand side also vanishes under d_2 and therefore under

$$d_{(1^2)} - 2d_{(2)}.$$

If we operate with $d_{(1^2)}$ we obtain a number of separations of (2^2), and, if with $d_{(2)}$, a number of separations of (21^2); hence the terms which result being separations of (2^2) and (21^2) respectively must separately vanish and we see therefore that the right-hand side must vanish when *either* of the operations $d_{(1^2)}$, $d_{(2)}$ is performed.

In fact $d_{(1^2)}$ yields

$$2(2^2) - 2(2)^2 - 2(2^2) + 2(2)^2 = 0,$$

and $d_{(2)}$ gives

$$-2(21^2)+(21)(1)+4(2)(1^2)-2(1)(21)+(2)(1)^2-2(1^2)(2)$$
$$-2(2)(1^2)-(2)(1)^2+(21)(1)+2(21^2)=0.$$

We thus draw the important general conclusion that if a linear function of separations of a partition vanishes by the performance of any weight operator it must also vanish by the performance of every partition operator of that weight. This fact renders the calculation of Tables of Separations a very easy matter in many cases.

Ex. gr. Suppose we require to express (41) by means of separations of (21^3).

Assume

$$(41)=A(21^3)+B(21^2)(1)+C(21)(1^2)+C(2)(1^3)+D(21)(1)^2+E(2)(1^2)(1),$$

where observe that $(21)(1^2)$ and $(2)(1^3)$, being separations having the same specification, have necessarily the same coefficient; and that the separation $(2)(1)^3$ being the only one of degree 5 must be absent.

The operations of $d_{(12)}$, $d_{(2)}$, $d_{(1^3)}$, $d_{(21)}$ must cause the expression to vanish; from $d_{(12)}$ we get $B+C+E=0$, $A+C=0$, and from $d_{(2)}$, $D=0$; and thence, from $d_{(21)}$, $B=0$; thus

$$(41)=A\{(21^3)-(21)(1^2)-(2)(1^3)+(2)(1^2)(1)\},$$

and the operation of D_4 shews that A is unity.

In regard to weight operators we have

$$(d_\lambda)(d_\mu)=(d_\lambda d_\mu)+d_{\lambda+\mu};$$

we require the corresponding result for partition operators.

Since

$$d_{(1^{\pi_1}2^{\pi_2}\ldots)}=\Sigma(1^{p_1}2^{p_2}\ldots)\frac{d}{d(1^{p_1+\pi_1}2^{p_2+\pi_2}\ldots)},$$

and

$$d_{(1^{\rho_1}2^{\rho_2}\ldots)}=\Sigma(1^{p_1+\pi_1}2^{p_2+\pi_2}\ldots)\frac{d}{d(1^{p_1+\pi_1+\rho_1}2^{p_2+\pi_2+\rho_2}\ldots)},$$

it follows that

$$d_{(1^{\pi_1}2^{\pi_2}\ldots)}\dagger d_{(1^{\rho_1}2^{\rho_2}\ldots)}=d_{(1^{\pi_1+\rho_1}2^{\pi_2+\rho_2}\ldots)};$$

see Art. 27; and thence

$$(d_{(1^{\pi_1}2^{\pi_2}\ldots)})(d_{(1^{\rho_1}2^{\rho_2}\ldots)})=(d_{(1^{\pi_1}2^{\pi_2}\ldots)}d_{(1^{\rho_1}2^{\rho_2}\ldots)})+d_{(1^{\pi_1+\rho_1}2^{\pi_2+\rho_2}\ldots)}.$$

This result leads to the conclusion that any linear partition or weight operator is commutative with every other linear partition or weight operator. Consider the solutions of the linear partial differential equation

$$\Delta=0,$$

where Δ is any linear partition or weight operator. If ϕ be one solution so that identically

$$\Delta\phi = 0,$$

and Δ' be any other linear partition or weight operator,

$$\Delta\Delta'\phi = \Delta'\Delta\phi = 0,$$

so that $\Delta'\phi$ is also a solution of the same equation.

Relations between the Partition d and D Operators.

51. It will be gathered from Art. 50 of this Section and Art. 27 of Section I that

$$\Sigma \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} d_{(1^{\pi_1}2^{\pi_2}\ldots)} = \Sigma \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\pi_1!\,\pi_2!\ldots} D_1^{\pi_1} D_2^{\pi_2}\ldots,$$

the summations being for every partition

$$(1^{\pi_1}2^{\pi_2}\ldots)$$

of a number n.

We thence obtain the series of relations

$$d_{(1)} = D_1,$$

$$d_{(1^2)} - 2d_{(2)} = D_1^2 - 2D_2,$$

$$d_{(1^3)} - 3d_{(21)} + 3d_{(3)} = D_1^3 - 3D_2D_1 + 3D_3,$$

$$d_{(1^4)} - 4d_{(21^2)} + 2d_{(2^2)} + 4d_{(31)} - 4d_{(4)} = D_1^4 - 4D_2D_1^2 + 2D_2^2 + 4D_3D_1 - 4D_4,$$

.....

and thence expressing the weight operators D in terms of partition operators, the series

$$d_{(1)} = D_{(1)},$$

$$d_{(1^2)} - 2d_{(2)} = \{D_{(1)}^2 - 2D_{(1^2)}\} - 2D_{(2)},$$

$$d_{(1^3)} - 3d_{(21)} + 3d_{(3)} = \{D_{(1)}^3 - 3D_{(1^2)}D_{(1)} + 3D_{(1^3)}\} - 3\{D_{(2)}D_{(1)} - D_{(21)}\} + 3D_{(3)},$$

$$d_{(1^4)} - 4d_{(21^2)} + 2d_{(2^2)} + 4d_{(31)} - 4d_{(4)} = \{D_{(1)}^4 - 4D_{(1^2)}D_{(1)}^2 + 2D_{(1^2)}^2 + 4D_{(1^3)}D_{(1)} - 4D_{(1^4)}\}$$
$$- 4\{D_{(2)}D_{(1)}^2 - D_{(2)}D_{(1^2)} - D_{(21)}D_{(1)} + D_{(21^2)}\} + 2\{D_{(2)}^2 - 2D_{(2^2)}\}$$
$$+ 4\{D_{(3)}D_{(1)} - D_{(31)}\} - 4D_{(4)},$$

.....

Consider in particular the second of these, viz.:

$$d_{(1^2)} - 2d_{(2)} = D_{(1)}^2 - 2D_{(1^2)} - 2D_{(2)};$$

the two sides of this relation must produce with any operand the same result identically and those functions which are separations of the same function must be equal amongst themselves; hence $d_{(1^2)}$ must produce the same as $D_{(1)}^2 - 2D_{(1^2)}$ and $d_{(2)}$ the same as $D_{(2)}$; this argument leads to the important series of relations

$$
\begin{cases} d_{(1)} = D_{(1)}, \end{cases}
$$

$$
\begin{cases} d_{(1^2)} = D_{(1)}^2 - 2D_{(1^2)}, \\ d_{(2)} = D_{(2)}, \end{cases}
$$

$$
\begin{cases} d_{(1^3)} = D_{(1)}^3 - 3D_{(1^2)}D_{(1)} + 3D_{(1^3)}, \\ d_{(21)} = D_{(2)}D_{(1)} - D_{(21)}, \\ d_{(3)} = D_{(3)}, \end{cases}
$$

$$
\begin{cases} d_{(1^4)} = D_{(1)}^4 - 4D_{(1^2)}D_{(1)}^2 + 2D_{(1^2)}^2 + 4D_{(1^3)}D_{(1)} - 4D_{(1^4)}, \\ d_{(21^2)} = D_{(2)}D_{(1)}^2 - D_{(2)}D_{(1^2)} - D_{(21)}D_{(1)} + D_{(21^2)}, \\ d_{(2^2)} = D_{(2)}^2 - 2D_{(2^2)}, \\ d_{(31)} = D_{(3)}D_{(1)} - D_{(31)}, \\ d_{(4)} = D_{(4)}, \end{cases}
$$

etc.;

in general

$$
(-)^{\Sigma\pi}\frac{(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\dots}\,d_{(p_1^{\pi_1}p_2^{\pi_2}\dots)} = \Sigma\,(-)^{\Sigma j}\frac{(\Sigma j-1)!}{j_1!\,j_2!\dots}\,D_{(J_1)}^{j_1}D_{(J_2)}^{j_2}\dots,
$$

the summation being for all separations

$$
(J_1)^{j_1}(J_2)^{j_2}\dots \text{ of the partition } (p_1^{\pi_1}p_2^{\pi_2}\dots).
$$

The mode of operation of $D_{(J)}$ upon a product of monomial symmetric functions has been explained in Art. 29 of Section I; that of $d_{(J)}$, where (J) may be any partition, is from Art. 50 clearly such that the parts of (J) are to be subtracted in all possible ways from the symmetric function factors, *the whole of the parts always from a single factor*; thus

$$
d_{(J)}\,(JJ\lambda\mu\nu)\,(J\lambda'\mu'\nu')
$$
$$
= (J\lambda\mu\nu)\,(J\lambda'\mu'\nu') + (JJ\lambda\mu\nu)\,(\lambda'\mu'\nu').
$$

In particular

$$
d_{(31)}\,(4321)\,(3^21^2)\,(31)
$$
$$
= (42)\,(3^21^2)\,(31) + (4321)\,(31)\,(31) + (4321)\,(3^21^2),
$$
$$
d_{(31)}\,(31) = 1,
$$
$$
d_{(J)}\,(J) = 1.
$$

To verify the relation

$$
d_{(21)} = D_{(2)}D_{(1)} - D_{(21)},
$$

take as operand $(321^2)\,(21^3)$, then

$$
d_{(21)}\,(321^2)\,(21^3) = (31)\,(21^3) + (321^2)\,(1^2),
$$

whilst

$$
D_{(1)}\,(321^2)\,(21^3) = (321)\,(21^3) + (321^2)\,(21^2),
$$
$$
D_{(2)}D_{(1)}\,(321^2)\,(21^3) = (31)\,(21^3) + (321)\,(1^3) + (31^2)\,(21^2) + (321^2)\,(1^2),
$$
$$
D_{(21)}\,(321^2)\,(21^3) = (31^2)\,(21^2) + (321)\,(1^3),
$$

a verification.

We must now reverse the system and it is easy to establish that

$$D_{(1)} = d_{(1)},$$

$$\begin{cases} 2D_{(1^2)} = d_{(1)}^2 - d_{(1^2)}, \\ D_{(2)} = d_{(2)}, \end{cases}$$

$$\begin{cases} 6D_{(1^3)} = d_{(1)}^3 - 3d_{(1^2)}d_{(1)} + 2d_{(1^3)}, \\ D_{(21)} = d_{(2)}d_{(1)} - d_{(21)}, \\ D_{(3)} = d_{(3)}, \end{cases}$$

$$\begin{cases} 24D_{(1^4)} = d_{(1)}^4 - 6d_{(1^2)}d_{(1)}^2 + 3d_{(1^2)}^2 + 8d_{(1^3)}d_{(1)} - 6d_{(1^4)}, \\ 2D_{(21^2)} = d_{(2)}d_{(1)}^2 - 2d_{(21)}d_{(1)} - d_{(2)}d_{(1^2)} + 2d_{(21^2)}, \\ 2D_{(2^2)} = d_{(2)}^2 - d_{(2^2)}, \\ D_{(31)} = d_{(3)}d_{(1)} - d_{(31)}, \\ D_{(4)} = d_{(4)}, \end{cases}$$

etc.

In both of these systems of relations the multiplications of operations have denoted successive operations. We may transform the system last written so that the multiplication of the d operators may be algebraic or symbolic.

Thus since

$$d_{(1)}^2 = (d_{(1)}^2) + d_{(1^2)},$$

where the brackets ( ) denote symbolic multiplication of the factors which they contain, we have

$$2!\, D_{(1^2)} = (d_{(1)}^2),$$

and

$$6D_{(1^3)} = (d_{(1)}^3) + 3\,(d_{(1^2)}d_{(1)}) + d_{(1^3)} - 3\,\{(d_{(1^2)}d_{(1)}) + d_{(1^3)}\} + 2d_{(1^3)} = (d_{(1)}^3),$$

also

$$2D_{(21^2)} = d_{(2)}\,\{(d_{(1)}^2) + d_{(1^2)}\} - 2\,(d_{(21)}d_{(1)}) - 2d_{(21^2)} - (d_{(2)}d_{(1^2)}) - d_{(21^2)} + 2d_{(21^2)} = (d_{(2)}d_{(1)}^2),$$

and the reader will have little difficulty in establishing the general result

$$\pi_1!\,\pi_2!\ldots D_{(p_1^{\pi_1}p_2^{\pi_2}\ldots)} = (d_{(p_1)}^{\pi_1}d_{(p_2)}^{\pi_2}\ldots),$$

which is very remarkable and suggestive.

52. There is yet another important set of relations which have an important application in Section V of this work. It will be remembered that the mode of operation of D_s upon a product of symmetric functions was given in Art. 29. This was important because D_s picked out all partitions of s in all possible ways from the product, one part only being taken from each factor of the product. We are now to see that there is an analogous operator

$$L_{(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)},$$

which picks out from a product of symmetric functions all *separations* of the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\dots)$ in all possible ways from the product, one *separate* only being taken from each factor.

The L operators are defined as follows:

$$L_{(\lambda)} = d_{(\lambda)} = D_{(\lambda)},$$

$$\begin{cases} L_{(\lambda\mu)} = (d_{(\lambda)}d_{(\mu)}) + d_{(\lambda\mu)} = d_{(\lambda)}d_{(\mu)} = D_{(\lambda)}D_{(\mu)}, \\ 2L_{(\lambda^2)} = (d^2_{(\lambda)}) + 2d_{(\lambda^2)} = d^2_{(\lambda)} + d_{(\lambda^2)} = 2\{D^2_{(\lambda)} - D_{(\lambda^2)}\}, \end{cases}$$

$$\begin{cases} L_{(\lambda\mu\nu)} = (d_{(\lambda)}d_{(\mu)}d_{(\nu)}) + (d_{(\lambda\mu)}d_{(\nu)}) + (d_{(\lambda\nu)}d_{(\mu)}) + (d_{(\mu\nu)}d_{(\lambda)}) + d_{(\lambda\mu\nu)} = d_{(\lambda)}d_{(\mu)}d_{(\nu)} \\ \qquad\qquad = D_{(\lambda)}D_{(\mu)}D_{(\nu)}, \\ 2L_{(\lambda^2\mu)} = (d^2_{(\lambda)}d_{(\mu)}) + 2(d_{(\lambda^2)}d_{(\mu)}) + 2(d_{(\lambda\mu)}d_{(\lambda)}) + 2d_{(\lambda^2\mu)} \\ \qquad = \{d^2_{(\lambda)} + d_{(\lambda^2)}\}d_{(\mu)} = 2\{D^2_{(\lambda)} - D_{(\lambda^2)}\}D_{(\mu)}, \\ 6L_{(\lambda^3)} = (d^3_{(\lambda)}) + 6(d_{(\lambda^2)}d_{(\lambda)}) + 6d_{(\lambda^3)} = d^3_{(\lambda)} + 3d_{(\lambda^2)}d_{(\lambda)} + 2d_{(\lambda^3)} \\ \qquad = 6\{D^3_{(\lambda)} - 2D_{(\lambda^2)}D_{(\lambda)} + D_{(\lambda^3)}\}, \end{cases}$$

while in general

$$L_{(\lambda^l)} = \Sigma \frac{(d^{\pi_1}_{(\lambda)}d^{\pi_2}_{(\lambda^2)}d^{\pi_3}_{(\lambda^3)}\dots)}{\pi_1!\,\pi_2!\,\pi_3!\dots}$$

$$= \Sigma \frac{d^{\pi_1}_{(\lambda)}d^{\pi_2}_{(\lambda^2)}d^{\pi_3}_{(\lambda^3)}\dots}{\pi_1!\,\pi_2!\,\pi_3!\dots 1^{\pi_1}\cdot 2^{\pi_2}\cdot 3^{\pi_3}\cdot\dots}$$

$$= \Sigma(-)^{l+\Sigma\pi}\frac{(\Sigma\pi)!}{\pi_1!\,\pi_2!\,\pi_3!\dots}D^{\pi_1}_{(\lambda)}D^{\pi_2}_{(\lambda^2)}D^{\pi_3}_{(\lambda^3)}\dots,$$

---

* Let

$$\alpha + 2\beta + 3\gamma + \dots = n,$$
$$\alpha + \beta + \gamma + \dots = r.$$

Then

$$(t_1 + t_2 + t_3 + \dots)^r = \Sigma \frac{r!}{\alpha!\,\beta!\,\gamma!\dots} t_1^{\alpha} t_2^{\beta} t_3^{\gamma}\dots.$$

Put $t_s = \frac{t^s}{s}$ so that

$$\left(\frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{3} + \dots\right)^r = \Sigma \frac{r!}{\alpha!\,\beta!\,\gamma!\dots 1^{\alpha}\cdot 2^{\beta}\cdot 3^{\gamma}\dots} t^{\alpha+2\beta+3\gamma+\dots}.$$

Writing $\frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{3} + \dots = T$, we have

$$e^T = 1 + \frac{T}{1!} + \frac{T^2}{2!} + \dots = \Sigma\Sigma_r \frac{t^n}{\alpha!\,\beta!\,\gamma!\dots 1^{\alpha}\cdot 2^{\beta}\cdot 3^{\gamma}\dots}.$$

Now

$$T = \log\frac{1}{1-t},$$

and if ρ be an arbitrary numerical magnitude

$$e^{\rho T} = e^{\rho\log\frac{1}{1-t}} = \Sigma\Sigma_r \frac{\rho^r t^n}{\alpha!\,\beta!\,\gamma!\dots 1^{\alpha}\cdot 2^{\beta}\cdot 3^{\gamma}\dots};$$

and $e^{\rho T} = (1-t)^{-\rho}$ and comparing coefficients of t^n on each side we find that

$$\Sigma\Sigma_r \frac{\rho^r}{\alpha!\,\beta!\,\gamma!\dots 1^{\alpha}\cdot 2^{\beta}\cdot 3^{\gamma}\dots}$$

is equal to the coefficient of t^n in the expansion of $(1-t)^{-\rho}$.

the summations being for all solutions of the equation in integers

$$\pi_1 + 2\pi_2 + 3\pi_3 + \ldots = l,$$

and

$$L_{(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)} = L_{(\lambda_1^{l_1})}\, L_{(\lambda_2^{l_2})}\, L_{(\lambda_3^{l_3})} \cdots,$$

the operations on the right-hand side being performed successively.

From these we observe that

$$L_{(\lambda\mu)} = L_{(\lambda)} L_{(\mu)},$$
$$L_{(\lambda\mu\nu)} = L_{(\lambda)} L_{(\mu)} L_{(\nu)},$$
$$L_{(\lambda^2\mu)} = L_{(\lambda^2)} L_{(\mu)},$$

and we shall find that in general

$$L_{(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)} = L_{(\lambda_1^{l_1})}\, L_{(\lambda_2^{l_2})}\, L_{(\lambda_3^{l_3})} \cdots,$$

where

$$L_{(\lambda)} = d_{(\lambda)} = D_{(\lambda)},$$

$$L_{(\lambda^2)} = \frac{1}{2!}\{d^2_{(\lambda)} + d_{(\lambda^2)}\} = D^2_{(\lambda)} - D_{(\lambda^2)},$$

$$L_{(\lambda^3)} = \frac{1}{3!}\{d^3_{(\lambda)} + 3d_{(\lambda^2)}d_{(\lambda)} + 2d_{(\lambda^3)}\} = D^3_{(\lambda)} - 2D_{(\lambda^2)}D_{(\lambda)} + D_{(\lambda^3)},$$

and in general

$$L_{(\lambda^l)}$$

has the expressions above given.

It will be noticed that in the first summation the law is that of the expression of the elementary functions a in terms of the sums of powers,

---

It thence follows that

(i) when $\rho = 1$ we have Cauchy's Theorem

$$\Sigma \frac{1}{\alpha!\beta!\gamma!\ldots 1^\alpha . 2^\beta . 3^\gamma \ldots} = 1,$$

where n is a positive integer and the summation is for all solutions of the equation in integers,

$$\alpha + 2\beta + 3\gamma + \ldots = n;$$

(ii) when $\rho = -1$ we have Cayley's Theorem

$$\Sigma \frac{(-)^{\alpha+\beta+\gamma+\ldots}}{\alpha!\beta!\gamma!\ldots 1^\alpha . 2^\beta . 3^\gamma \ldots} = 0,$$

where n is a positive integer greater than unity.

The Theorem as it stands is due to Sylvester.

Other interesting particular cases are obtained by putting ρ equal to $-\frac{1}{2}$, $+\frac{1}{2}$ and $+2$.

Also an easy corollary is that

$$\Sigma \frac{n!}{\alpha!\beta!\gamma!\ldots 1^\alpha . 2^\beta . 3^\gamma \ldots}$$
$$= \text{Coeff. of } \rho^{r-1} \text{ in } (\rho+1)(\rho+2)\ldots(\rho+n-1),$$

a result which is also due to Sylvester.

but here the signs are all positive; whilst in the second the law is that of the expression of the homogeneous product sums h in terms of the elementary functions a.

In the summations multiplications of operators denote successive multiplication.

We must now establish the law by which $L_{(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)}$ operates upon a product of symmetric functions. The expression of L that must be considered primarily is that which is a linear function of operators each of which is a symbolic product of linear operators. Thus we have

$$L_{(\lambda\mu)} = (d_{(\lambda)}\, d_{(\mu)}) + d_{(\lambda\mu)},$$

where in connexion with the right-hand side

$$(\lambda)\,(\mu), \quad (\lambda\mu)$$

are the two separations of $(\lambda\mu)$ which occurs on the left-hand side.

The operation of $d_{(\lambda\mu)}$ upon a symmetric function product picks out from such product the separate $(\lambda\mu)$ from each factor in succession and adds the results together; thus

$$d_{(\lambda\mu)}\,(\lambda^2\mu^2)\,(\lambda\mu 1) = (\lambda\mu)\,(\lambda\mu 1) + (\lambda^2\mu^2)\,(1).$$

Next consider the operator $(d_{(\lambda)}\, d_{(\mu)})$ of the second order.

It is, if J_1, J_2 denote any assemblages of integers in descending order of magnitude so that (J_1), (J_2) denote any partitions of any numbers,

$$\Sigma\Sigma\,(J_1)\,(J_2)\,\frac{d^2}{d_{(J_1\lambda)}\, d_{(J_2\mu)}},$$

the summation being for all partitions of all numbers, separately in respect of J_1 and J_2. The particular term of the operator that has been specified under the summation sign may act upon an operand

$$(J_1\lambda)\,(J_2\mu)\,(\ldots)\,(\ldots)\ldots,$$

and the result of the operation would be

$$(J_1)\,(J_2)\,(\ldots)\,(\ldots)\ldots,$$

shewing that the operator has deleted the separation $(\lambda)\,(\mu)$ from the product, one separate only from one factor. Similarly if the operand be

$$(J_1\lambda)^l\,(J_2\mu)^m\,(J_3\lambda\mu)^n\,(\ldots),$$

wherein as before only the factors which involve the numbers λ, μ are specified, the operator $(d_{(\lambda)}\, d_{(\mu)})$ produces

$$\begin{aligned} &lm\,(J_1)\,(J_2)\,(J_1\lambda)^{l-1}\,(J_2\mu)^{m-1}\,(J_3\lambda\mu)^n\,(\ldots) \\ &+ ln\,(J_1)\,(J_3\lambda)\,(J_1\lambda)^{l-1}\,(J_2\mu)^m\,(J_3\lambda\mu)^{n-1}\,(\ldots) \\ &+ mn\,(J_2)\,(J_3\mu)\,(J_1\lambda)^l\,(J_2\mu)^{m-1}\,(J_3\lambda\mu)^{n-1}\,(\ldots); \end{aligned}$$

and it will be observed that the separation $(\lambda)(\mu)$ has been deleted in all possible ways from the operand, one separate only from each factor.

Hence $L_{(\lambda\mu)}$ when performed upon an operand deletes a separation of $(\lambda\mu)$ from the operand in all possible ways so that one separate only is taken from a single factor. Thus for example

$$L_{(32)}(3^2 2^2)(32) = (32)(32) + (3^2 2^2) + (32^2)(3) + (3^2 2)(2).$$

Next consider the operation of

$$\frac{1}{2!}(d^2_{(\lambda)}),$$

which is $\frac{1}{2!}\Sigma\left\{(J_1)^2\frac{d^2}{d^2_{(J_1\lambda)}} + 2(J_1)(J_2)\frac{d^2}{d_{(J_1\lambda)}d_{(J_2\mu)}} + (J_2)^2\frac{d^2}{d^2_{(J_2\mu)}}\right\}$.

Taking for operand $(J_1\lambda)^{l_1}(J_2\lambda)^{l_2}$,

the result is

$$\binom{l_1}{2}(J_1)^2(J_1\lambda)^{l_1-2}(J_2\lambda)^{l_2} + l_1 l_2 (J_1)(J_2)(J_1\lambda)^{l_1-1}(J_2\lambda)^{l_2-1} + \binom{l_2}{2}(J_2)^2(J_1\lambda)^{l_1}(J_2\lambda)^{l_2-2},$$

which is precisely the result of deleting the separation $(\lambda)(\lambda)$ in all possible ways from the operand, one separate only from a single factor. It is clear that J_1 or J_2 also involving λ any number of times makes no difference.

In general $$L_{(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)} = \Sigma\frac{(d^{j_1}_{(J_1)}d^{j_2}_{(J_2)}\ldots)}{j_1!\,j_2!\ldots},$$

the summation being for every separation

$$(J_1)^{j_1}(J_2)^{j_2}(J_3)^{j_3}\ldots \text{ of the partition } (\lambda_1^{l_1}\lambda_2^{l_2}\ldots);$$

and the argument used above shews that the particular term

$$\frac{(d^{j_1}_{(J_1)}d^{j_2}_{(J_2)}\ldots)}{j_1!\,j_2!\ldots}$$

of the operation is performed by deletion of the separation

$$(J_1)^{j_1}(J_2)^{j_2}\ldots$$

from the operand in all possible ways, one separate only from any one factor of the operand. Hence the complete operation

$$L_{(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)}$$

is performed by deletion of all separations

$$(J_1)^{j_1}(J_2)^{j_2}\ldots$$

of the partition $$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$$

in all possible ways from the operand, one separate only from any one factor of the operand; the separate results being of course all added together.

The operator $L_{(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)}$ is thus clearly a most important instrument in the theory of distributions. It will be freely used at a later stage of this work.

It should be remarked that, in the relation

$$L_{(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots)} = L_{(\lambda_1^{l_1})} L_{(\lambda_2^{l_2})} L_{(\lambda_3^{l_3})} \cdots,$$

the operations $L_{(\lambda_1^{l_1})}$, $L_{(\lambda_2^{l_2})}$, $L_{(\lambda_3^{l_3})}$, ... are to be performed *successively.*

CHAPTER IV

A CALCULUS OF BINOMIAL COEFFICIENTS

53. Of the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ consider the symmetric function

$$\Sigma\alpha_1^0\alpha_2^0\alpha_3^0\ldots\alpha_s^0;$$

this is a mere number, is equal to the number of combinations of the n quantities s together and has the value $\binom{n}{s}$; in the partition notation this is the symmetric function

$$(0^s)$$

which we may write $(0^s)_n$ if we wish to keep the number n in evidence.

Thus $$(1+x)^n = 1 + (0)_n x + (0^2)_n x^2 + \ldots + (0^n)_n x^n,$$

and if we introduce a new quantity α_{n+1}

$$(1+x)^{n+1} = 1 + (0)_{n+1} x + (0^2)_{n+1} x^2 + \ldots + (0^n)_{n+1} x^n + (0^{n+1})_{n+1} x^{n+1};$$

whence $$(0^s)_{n+1} = (0^s)_n + (0^{s-1})_n \quad \text{if} \quad s < n+1.$$

If then $$(0^s)_n = \phi(n),$$

$$(0^s)_n + (0^{s-1})_n = \phi(n+1) = \left(1 + \frac{d}{dn} + \frac{1}{2!}\frac{d^2}{dn^2} + \ldots\right)\phi(n) = e^{\frac{d}{dn}}\phi(n).$$

Write $$\frac{d}{dn} + \frac{1}{2!}\frac{d^2}{dn^2} + \ldots = D_0,$$

so that $$\phi(n+1) = (1+D_0)\,\phi(n) \quad \text{and} \quad D_0 = e^{\frac{d}{dn}} - 1,$$

and $$D_0(0^s)_n = (0^{s-1})_n,$$

where observe that the effect of D_0 is to delete a part zero from the partition.

Now suppose $$\psi(n) = (0^p)(0^q)(0^r),$$

$$\psi(n+1) = \{(0^p) + (0^{p-1})\}\{(0^q) + (0^{q-1})\}\{(0^r) + (0^{r-1})\},$$

and then

$$\begin{aligned} D_0\psi(n) = {} & (0^{p-1})(0^q)(0^r) + (0^p)(0^{q-1})(0^r) + (0^p)(0^q)(0^{r-1}) \\ & + (0^{p-1})(0^{q-1})(0^r) + (0^{p-1})(0^q)(0^{r-1}) + (0^p)(0^{q-1})(0^{r-1}) \\ & + (0^{p-1})(0^{q-1})(0^{r-1}); \end{aligned}$$

from which it appears that in the first line on the right a zero is deleted in succession from each partition; in the second two zeros are deleted from two partitions, one zero only from a partition in all possible ways; in the third three zeros are deleted, one zero from each partition in all possible ways. In fact D_0 operates through the partitions (0), (0^2), (0^3). ... of zero, just as in a previous case D_n operated upon a product of symmetric functions through the partitions of n. Briefly stated, D_0 picks out one, two, three, etc. zeros in all possible ways from the product.

This fact supplies us not only with a calculus of binomial coefficients, but also with a new instrument for use in the Theory of Distributions. The results in that theory will be found to have no direct reference to binomial coefficients at all.

We may break up the operation D_0 into a number of distinct operations and write

$$D_0 = D_{(0)} + D_{(0^2)} + D_{(0^3)} + \ldots,$$

and since

$$e^{\frac{d}{dn}} = 1 + D_0,$$

we have

$$\frac{d}{dn} = \log(1 + D_0) = D_0 - \tfrac{1}{2}D_0^2 + \tfrac{1}{3}D_0^3 - \ldots.$$

If we write

$$X_0 = (0)\,x_0 + (0^2)\,x_0^2 + (0^3)\,x_0^3 + \ldots,$$

then

$$X_0^p = \Sigma \frac{(p_1 + p_2 + p_3 + \ldots)!}{p_1!\,p_2!\,p_3!\ldots}(0)^{p_1}(0^2)^{p_2}(0^3)^{p_3}\ldots x_0^{p_1+2p_2+3p_3+\ldots};$$

where

$$p_1 + p_2 + p_3 + \ldots = p.$$

The term $(0)^{p_1}(0^2)^{p_2}(0^3)^{p_3}\ldots$ which occurs on the right is a separation of

$$(0^{p_1+2p_2+3p_3+\ldots})$$

and has a specification (0^p); it has attached to it a coefficient which is equal to the number of permutations of its separates.

If we regard only that portion of the right-hand side which is the coefficient of the particular power of x_0, viz. $x_0^{p_1+2p_2+3p_3+\ldots}$, we have

$$P x_0^{p_1+2p_2+3p_3+\ldots}$$

where P is a linear function of the separations of $(0^{p_1+2p_2+3p_3+\ldots})$, each separation having the specification (0^p). This function P may be expanded by multiplication so as to be a linear function of the binomial coefficients

$$(0^s),\ (0^{s+1}),\ \ldots\ (0^{p_1+2p_2+3p_3+\ldots}),$$

s being determined by the consideration that every term in P contains a separate (0^{p_1}) in which p_1 is at least as large as s. In other words a partition of $p_1 + 2p_2 + 3p_3 + \ldots$ into p parts must contain a part at least as large as s.

We therefore write

$$P = \Sigma A_\sigma (0^\sigma),$$

and $$X_0^p = \Sigma\Sigma\, A_\sigma (0^\sigma)\, x_0^{p_1+2p_2+3p_3+\dots},$$

$$\Sigma \frac{(p_1+p_2+p_3+\dots)!}{p_1!\,p_2!\,p_3!\dots}(0)^{p_1}(0^2)^{p_2}(0^3)^{p_3}\dots = \overset{w}{\Sigma} A_\sigma (0^\sigma),$$

where $$p_1+2p_2+3p_3+\dots = w, \quad p_1+p_2+p_3+\dots = p.$$

54. We can now shew that if

$$\Sigma \frac{(\sigma_1+\sigma_2+\sigma_3+\dots)!}{\sigma_1!\,\sigma_2!\,\sigma_3!\dots}(0)^{\sigma_1}(0^2)^{\sigma_2}(0^3)^{\sigma_3}\dots = \overset{w}{\Sigma} B_p (0^p),$$

where $$\sigma_1+2\sigma_2+3\sigma_3+\dots = w, \quad \sigma_1+\sigma_2+\sigma_3+\dots = \sigma,$$

then $$B_p = A_\sigma.$$

Take the particular case

$$(0^2)^2 + 2\,(0^3)\,(0) = \dots + 12\,(0^3) + \dots,$$
$$3\,(0^2)\,(0)^2 = \dots + 12\,(0^2) + \dots,$$

the left-hand side of the first of these relations is

$$(\Sigma\alpha^0\beta^0)^2 + 2\,(\Sigma\alpha^0\beta^0\gamma^0)\,(\Sigma\alpha^0),$$

while on the right-hand side a term of (0^3) may arise from the development in either of the forms $\alpha^0\alpha^0\beta^0\gamma^0$, $\alpha^0\beta^0\beta^0\gamma^0$, $\alpha^0\beta^0\gamma^0\gamma^0$.

The multiplications which give these forms are

$$\alpha^0\beta^0 . \alpha^0\gamma^0; \quad \alpha^0\gamma^0 . \alpha^0\beta^0; \quad \alpha^0\beta^0\gamma^0 . \alpha^0; \quad \alpha^0 . \alpha^0\beta^0\gamma^0;$$
$$\alpha^0\beta^0 . \beta^0\gamma^0; \quad \beta^0\gamma^0 . \alpha^0\beta^0; \quad \alpha^0\beta^0\gamma^0 . \beta^0; \quad \beta^0 . \alpha^0\beta^0\gamma^0;$$
$$\alpha^0\gamma^0 . \beta^0\gamma^0; \quad \beta^0\gamma^0 . \alpha^0\gamma^0; \quad \alpha^0\beta^0\gamma^0 . \gamma^0; \quad \gamma^0 . \alpha^0\beta^0\gamma^0;$$

twelve in all yielding the desired coefficient 12 in the first relation.

To find the corresponding distribution theorem we note that the sets of parcels involved are of the three types (2^2), (31), (13) and we must, in co-relation, consider the sets of objects of types (211), (121), (112); there is also the restriction on the distribution that two or more similar parcels must not contain more than one object of the same kind. We realise the distributions in the following manner:

A A B B A A B B	A A B B A A B B	A A B B A A B B
a b a c a c a b	b a b c b c b a	c a c b c b c a

A A A B	A A A B	A A A B
a b c a	a b c b	a b c c
A B B B	A B B B	A B B B
a a b c	b a b c	c a b c

twelve in number and there are no more. In the above the capital and small letters denote parcels and objects respectively and it will be noted that

by reason of the restriction on the distribution in each distribution no two columns of letters are identical.

If we now interchange capital and small letters and rearrange we find

$A\ A\ B\ C$	$A\ A\ B\ C$	$A\ A\ B\ C$	$A\ A\ B\ C$
$a\ b\ a\ b$	$a\ b\ b\ a$	$a\ b\ a\ a$	$a\ b\ b\ b$
$A\ B\ B\ C$	$A\ B\ B\ C$	$A\ B\ B\ C$	$A\ B\ B\ C$
$a\ a\ b\ b$	$b\ a\ b\ a$	$a\ a\ b\ a$	$b\ a\ b\ b$
$A\ B\ C\ C$	$A\ B\ C\ C$	$A\ B\ C\ C$	$A\ B\ C\ C$
$a\ b\ a\ b$	$b\ a\ a\ b$	$a\ a\ a\ b$	$b\ b\ a\ b$

and this is the scheme of distribution we should arrive at from a consideration of the second of the relations; for the left-hand side is

$$3\,(\Sigma\alpha^0\beta^0)\,(\Sigma\alpha^0)\,(\Sigma\alpha^0),$$

while on the right-hand side a term of (0^2) may arise from the development in either of the forms $\alpha^0\alpha^0\beta^0\beta^0$, $\alpha^0\alpha^0\alpha^0\beta^0$, $\alpha^0\beta^0\beta^0\beta^0$ and the multiplications which give these terms are

$$\begin{array}{llll}
\alpha^0\beta^0.\alpha^0.\beta^0; & \alpha^0\beta^0.\beta^0.\alpha^0; & \alpha^0\beta^0.\alpha^0.\alpha^0; & \alpha^0\beta^0.\beta^0.\beta^0; \\
\alpha^0.\alpha^0\beta^0.\beta^0; & \beta^0.\alpha^0\beta^0.\alpha^0; & \alpha^0.\alpha^0\beta^0.\alpha^0; & \beta^0.\alpha^0\beta^0.\beta^0; \\
\alpha^0.\beta^0.\alpha^0\beta^0; & \beta^0.\alpha^0.\alpha^0\beta^0; & \alpha^0.\alpha^0.\alpha^0\beta^0; & \beta^0.\beta^0.\alpha^0\beta^0;
\end{array}$$

where as the reader will see we have put

$$3\,(\Sigma\alpha^0\beta^0)\,(\Sigma\alpha^0)\,(\Sigma\alpha^0)$$

in the form

$$(\Sigma\alpha^0\beta^0)\,(\Sigma\alpha^0)\,(\Sigma\alpha^0)+(\Sigma\alpha^0)\,(\Sigma\alpha^0\beta^0)\,(\Sigma\alpha^0)+(\Sigma\alpha^0)\,(\Sigma\alpha^0)\,(\Sigma\alpha^0\beta^0),$$

and we are now concerned with the distributions of objects of types (2^2), (31), (13) into parcels of types (211), (121), (112) and the distributions are restricted as before. We have thus established the truth of the coefficient 12 which appears in the second relation.

The reasoning employed is of general application and leads to the conclusion that $B_p = A_\sigma$ in the two general relations above written.

We are thus enabled to construct Tables of Binomial Coefficients which possess row and column symmetry. The inverse Tables will also be symmetrical and we have calculated them as far as the order 6. See Tables.

55. We will now associate D_0 not with the partitions of zero, (0), (0^2), ... (0^m) (when operating upon a symmetric function $\phi_1\phi_2 \ldots \phi_m$ wherein each factor may involve zero parts) but with the compositions of zero into zeros of the nature 0 and into zeros of the nature ω (see Article 30), the zero ω being non-effective and having the effect only of exhibiting the

law of operation. Taking $m = 3$, we have the compositions of zero into exactly three parts, the zero ω being admissible as a part:

$$\begin{array}{ccc} 0 & \omega & \omega \\ \omega & 0 & \omega \\ \omega & \omega & 0 \\ 0 & 0 & \omega \\ 0 & \omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

and the result

$$\begin{aligned} D_0\phi_1\phi_2\phi_3 = {} & D_0\phi_1 \,.\, \phi_2\phi_3 + \phi_1 \,.\, D_0\phi_2 \,.\, \phi_3 + \phi_1\phi_2 \,.\, D_0\phi_3 \\ & + D_0\phi_1 \,.\, D_0\phi_2 \,.\, \phi_3 + D_0\phi_1 \,.\, \phi_2 \,.\, D_0\phi_3 + \phi_1 \,.\, D_0\phi_2 \,.\, D_0\phi_3 \\ & + D_0\phi_1 \,.\, D_0\phi_2 \,.\, D_0\phi_3, \end{aligned}$$

and in general $D_0\phi_1\phi_2 \dots \phi_m$ is split up into $2^m - 1$ operations.

The advantage of utilizing compositions is that the order of the factors of the operand is preserved throughout.

Generally

$$D_0\phi_1\phi_2 \dots \phi_m = \Sigma\, D_{\lambda_1}\phi_1 \,.\, D_{\lambda_2}\phi_2 \,.\, \dots\, D_{\lambda_m}\phi_m,$$

where $\lambda_1, \lambda_2, \dots \lambda_m$ are, each of them either 0 or ω and D_ω is put equal to unity; and the summation is in regard to every composition of zero into *exactly* m parts. We can now operate with D_0 upon each of the $2^m - 1$ terms on the dexter, and following Section I, Art. 39 arrive at the result

$$\begin{aligned} & D_0D_0D_0 \dots \phi_1\phi_2 \dots \phi_m \\ = {} & \Sigma\Sigma\Sigma \dots D_{\lambda_1}D_{\mu_1}D_{\nu_1} \dots \phi_1 \,.\, D_{\lambda_2}D_{\mu_2}D_{\nu_2} \dots \phi_2 \,.\, \dots\, D_{\lambda_m}D_{\mu_m}D_{\nu_m} \dots \phi_m, \end{aligned}$$

where the summation is in regard to the compositions

$$(\lambda_1\lambda_2 \dots \lambda_m), \quad (\mu_1\mu_2 \dots \mu_m), \quad (\nu_1\nu_2 \dots \nu_m), \dots$$

of zero into exactly m parts as above explained. The dexter therefore involves

$$(2^m - 1)(2^m - 1)(2^m - 1) \dots \text{ terms},$$

since we find involved every combination of compositions, repetitions allowed.

CHAPTER V

THEORY OF THREE IDENTITIES

56. We recall the series of relations

$$c_1 = (1)\, b_1,$$
$$c_2 = (2)\, b_2 + (1^2)\, b_1^2,$$
$$c_3 = (3)\, b_3 + (21)\, b_2 b_1 + (1^3) b_1^3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

with reference to Art. 45 to observe that they may be regarded as arising from the three identities

$$1 + a_1 x + a_2 x^2 + \ldots = (1 + \alpha_1 x)(1 + \alpha_2 x) \ldots,$$
$$1 + b_1 x + b_2 x^2 + \ldots = (1 + \beta_1 x)(1 + \beta_2 x) \ldots,$$
$$1 + c_1 x + c_2 x^2 + \ldots = (1 + \gamma_1 x)(1 + \gamma_2 x) \ldots,$$

and the connecting relation, y being arbitrary,

$$1 + c_1 y + c_2 y^2 + \ldots = \prod_s (1 + \alpha_s b_1 y + \alpha_s^2 b_2 y^2 + \ldots);$$

the series of relations being obtained by multiplying out the right-hand side and then equating corresponding powers of y.

If we write $(1 + \alpha_1 x)(1 + \alpha_2 x) \ldots$ in the form

$$\exp\{(1)\, x - \tfrac{1}{2}(2)\, x^2 + \tfrac{1}{3}(3)\, x^3 - \ldots\},$$

the three identities yield the operator relations

$$1 + {}_\alpha D_1 x + {}_\alpha D_2 x^2 + \ldots = \exp\{{}_\alpha d_1 x - \tfrac{1}{2}\, {}_\alpha d_2 x^2 + \tfrac{1}{3}\, {}_\alpha d_3 x^3 - \ldots\},$$
$$1 + {}_\beta D_1 x + {}_\beta D_2 x^2 + \ldots = \exp\{{}_\beta d_1 x - \tfrac{1}{2}\, {}_\beta d_2 x^2 + \tfrac{1}{3}\, {}_\beta d_3 x^3 - \ldots\},$$
$$1 + {}_\gamma D_1 x + {}_\gamma D_2 x^2 + \ldots = \exp\{{}_\gamma d_1 x - \tfrac{1}{2}\, {}_\gamma d_2 x^2 + \tfrac{1}{3}\, {}_\gamma d_3 x^3 - \ldots\},$$

wherein ${}_\alpha D_s$, ${}_\beta D_s$, ${}_\gamma D_s$ is a notation indicating that the operators have reference to the quantities α, β, γ, respectively and, similarly, the notation ${}_\alpha d_s$, ${}_\beta d_s$, ${}_\gamma d_s$.

Writing the connecting relation in the form

$$U = u_{a_1} u_{a_2} u_{a_3} \ldots,$$

we have

$$ {}_\beta d_l U = ({}_\beta d_l u_{a_1})\, u_{a_2} u_{a_3} \ldots + u_{a_1} ({}_\beta d_l u_{a_2})\, u_{a_3} \ldots + u_{a_1} u_{a_2} ({}_\beta d_l u_{a_3}) \ldots + \ldots, $$

and

$$ {}_\beta d_l u_{a_1} = \left(\frac{d}{db_l} + b_1 \frac{d}{db_{l+1}} + \ldots \right) (1 + \alpha_1 b_1 y + \alpha_1^2 b_2 y^2 + \ldots) $$

$$ = \alpha_1^l y^l + b_1 \alpha_1^{l+1} y^{l+1} + b_2 \alpha_1^{l+2} y^{l+2} + \ldots = \alpha_1^l y^l u_{a_1}; $$

hence

$$ {}_\beta d_l U = (l)_a y^l U, $$

leading to

$$ {}_\beta d_l c_s = c_{s-l}\, (l)_a, $$

where $s \geqslant l$, and $c_0 = 1$.

Thence

$$ {}_\beta d_l = ({}_\beta d_l c_l) \frac{d}{dc_l} + ({}_\beta d_l c_{l+1}) \frac{d}{dc_{l+1}} + \ldots $$

$$ = (l)_a \left\{ \frac{d}{dc_l} + c_1 \frac{d}{dc_{l+1}} + \ldots \right\}, $$

or

$$ {}_\beta d_l = (l)_{a\ \gamma} d_l. $$

This means that regarding the series of relations as defining a transformation of the functions $c_1, c_2, c_3, \ldots$ into functions of $b_1, b_2, b_3, \ldots$ the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ entering as constants of the transformation, the operation

$$ {}_\gamma d_l $$

is an invariant.

The relation ${}_\beta d_l = (l)_{a\ \gamma} d_l$ enables us to write

$$ {}_\beta d_1 y - \tfrac{1}{2}\, {}_\beta d_2 y^2 + \tfrac{1}{3}\, {}_\beta d_3 y^3 - \ldots = (1)_{a\ \gamma} d_1 y - \tfrac{1}{2} (2)_{a\ \gamma} d_2 y^2 + \tfrac{1}{3} (3)_{a\ \gamma} d_3 y^3 - \ldots, $$

and thence we are led to the relation

$$ 1 + {}_\beta D_1 y + {}_\beta D_2 y^2 + \ldots = \prod_s (1 + \alpha_s\, {}_\gamma D_1 y + \alpha_s^2\, {}_\gamma D_2 y^2 + \ldots), $$

and now comparison with the relation connecting the three identities

$$ 1 + c_1 y + c_2 y^2 + \ldots = \prod_s (1 + \alpha_s b_1 y + \alpha_s^2 b_2 y^2 + \ldots), $$

enables us to assert that

"In any relation connecting the quantities

$$ c_1, c_2, c_3, \ldots, $$

with the quantities

$$ b_1, b_2, b_3, \ldots, $$

we are at liberty to substitute

$$ {}_\beta D_\kappa \text{ for } c_\kappa, $$

and

$$ {}_\gamma D_\kappa \text{ for } b_\kappa, $$

and we so obtain a relation between operators."

57. We can apply this theorem at once.

From the series of relations we can derive

$$c_{p_1}^{\pi_1} c_{p_2}^{\pi_2} \ldots = \ldots + L b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

$$c_{\lambda_1}^{l_1} c_{\lambda_2}^{l_2} \ldots = \ldots + M b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + A\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + B\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta + \ldots. \quad \ldots\text{(III)}$$

The relation (I) yields the operator relation

$$_\beta(D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} \ldots) = \ldots + L\,_\gamma(D_{s_1}^{\sigma_1} D_{s_2}^{\sigma_2} \ldots) + \ldots,$$

the occurrence of β and γ as suffixes before the brackets denoting that the attached operations have reference to the symmetric functions of the series β, γ, respectively.

Performing each side of the operator relation upon the opposite side of the relation (III) we obtain

$$L\,_\gamma(D_{s_1}^{\sigma_1} D_{s_2}^{\sigma_2} \ldots)\,(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = A\,_\beta(D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} \ldots)\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta,$$

no other terms surviving the operation.

Hence $L = A$ and similarly $M = B$.

We thus have

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + M\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta + \ldots.$$

Now we have seen that the right-hand side of this relation is symmetrical in regard to the quantities α and β so that we may also write

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots) = \ldots + J\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + J\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\beta\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha + \ldots,$$

and we have

$$L = \ldots + J\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha + \ldots,$$

$$M = \ldots + J\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha + \ldots,$$

shewing that the relations (I) and (II) may be written

$$c_{p_1}^{\pi_1} c_{p_2}^{\pi_2} \ldots = \ldots + J\,(\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha\, b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots,$$

$$c_{\lambda_1}^{l_1} c_{\lambda_2}^{l_2} \ldots = \ldots + J\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha\, b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots,$$

putting in evidence a law of symmetry which has already been established by means of a theory of distributions.

58. The above investigation yields also the results

$$c_{p_1}^{\pi_1} c_{p_2}^{\pi_2} \ldots = \ldots + L\, b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots,$$

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + L\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta + \ldots,$$

which involve a law of symmetry. The reader will have no difficulty in seeing that L is a linear function of the tabular separations of the symmetric function

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\alpha,$$

of specification $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, and it thus becomes easy to form the expression of

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma,$$

in terms of the symmetric functions of the quantities α and β.

59. Again suppose two results to be

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + P b_{p_1}^{\pi_1} b_{p_2}^{\pi_2} \ldots + \ldots,$$

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\gamma = \ldots + Q b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots,$$

there is no difficulty in proving that $P = Q$; for we derive the operator relation

$$\frac{1}{\pi_1 ! \, \pi_2 ! \ldots}{}_\beta(d_{p_1}^{\pi_1} d_{p_2}^{\pi_2} \ldots) = \ldots + Q_\gamma(D_{s_1}^{\sigma_1} D_{s_2}^{\sigma_2} \ldots) + \ldots,$$

and thence

$$Q_\gamma(D_{s_1}^{\sigma_1} D_{s_2}^{\sigma_2} \ldots)(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \frac{1}{\pi_1 ! \, \pi_2 ! \ldots}{}_\beta(d_{p_1}^{\pi_1} d_{p_2}^{\pi_2} \ldots) P b_{p_1}^{\pi_1} b_{p_2}^{\pi_2} \ldots,$$

or

$$Q = P.$$

Hence if $(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)_\gamma = \ldots + A (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha b_{p_1}^{\pi_1} b_{p_2}^{\pi_2} \ldots + \ldots,$

then $(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\gamma = \ldots + A (\lambda_1^{l_1} \lambda_2^{l_2} \ldots)_\alpha b_{s_1}^{\sigma_1} b_{s_2}^{\sigma_2} \ldots + \ldots,$

a law of symmetry the interpretation of which is very interesting.

The complete coefficient of $b_{p_1}^{\pi_1} b_{p_2}^{\pi_2} \ldots$ is a linear function of separations of the function $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ formed according to a law determined by the function $(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)$, and we have shewn that this is equal to the linear function of separations of the function $(s_1^{\sigma_1} s_2^{\sigma_2} \ldots)$ formed according to a law determined by the function $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$.

Actually forming a table of weight four which reads from left to right:

	b_4	$b_3 b_1$	b_2^2
$(4)_\gamma$	$-4\,(4)$	$4\,(3)\,(1) - 4\,(31)$	$2\,(2)^2 - 4\,(2^2)$
$(31)_\gamma$	$4\,(4)$	$-(3)\,(1) + 4\,(31)$	$-2\,(2)^2 + 4\,(2^2)$
$(2^2)_\gamma$	$2\,(4)$	$-2\,(3)\,(1) + 2\,(31)$	$(2)^2 + 2\,(2^2)$
$(21^2)_\gamma$	$-4\,(4)$	$(3)\,(1) - 4\,(31)$	$-4\,(2^2)$
$(1^4)_\gamma$	(4)	(31)	(2^2)

	$b_2 b_1^2$	b_1^4
$(4)_\gamma$	$-4\,(2)\,(1)^2 + 4\,(2)\,(1^2) + 4\,(1)\,(21) - 4\,(21^2)$	$(1)^4 - 4\,(1^2)\,(1)^2 + 2\,(1^2)^2 + 4\,(1^3)\,(1) - 4\,(1^4)$
$(31)_\gamma$	$(2)\,(1)^2 - 4\,(2)\,(1^2) - (1)\,(21) + 4\,(21^2)$	$(1^2)\,(1)^2 - 2\,(1^2)^2 - (1)\,(1^3) + 4\,(1^4)$
$(2^2)_\gamma$	$2\,(2)\,(1^2) - 2\,(1)\,(21) + 2\,(21^2)$	$(1^2)^2 - 2\,(1)\,(1^3) + 2\,(1^4)$
$(21^2)_\gamma$	$(1)\,(21) - 4\,(21^2)$	$(1)\,(1^3) - 4\,(1^4)$
$(1^4)_\gamma$	(21^2)	(1^4)

Further Theory of Distributions.

60. In Section I, Chapter II the number of distributions of objects of type $(p_1p_2p_3\ldots)$ into parcels of type $(q_1q_2q_3\ldots)$ was found to be the coefficient of the symmetric function $(q_1q_2q_3\ldots)$ in the development of the product $h_{p_1}h_{p_2}h_{p_3}\ldots$. It was later noted that the idea of the parcel is not necessary in the particular case when one object is in one parcel, and that we may consider the distribution as appertaining to two sets of objects of types $(p_1p_2p_3\ldots)$, $(q_1q_2q_3\ldots)$ respectively—the objects specified by $(p_1p_2p_3\ldots)$ being different in *kind* from those specified by $(q_1q_2q_3\ldots)$. If objects of a certain kind are included in the set defined by $(p_1p_2p_3\ldots)$, no objects of the same kind are to occur in the set specified by $(q_1q_2q_3\ldots)$. The two sets of objects are distributed so as to form n pairs, each pair involving one object from each set. We have then a set of n two-fold objects which may be also specified as to type by a partition of the number n. In Section II, Chapter II it was proposed to find the number of distributions of given type $(s_1s_2s_3\ldots)$ resulting from the distribution of objects of type $(p_1p_2p_3\ldots)$ with objects of type $(q_1q_2q_3\ldots)$, and the solution was found by writing

$$c_s = \Sigma\,(pqr\ldots)\,b_pb_qb_r\ldots, \qquad s = 1, 2, 3, \ldots,$$

the summation being for all partitions of s, and then taking the coefficient of the term $(q_1q_2q_3\ldots)\,b_{s_1}b_{s_2}b_{s_3}\ldots$ in the development of the product $c_{p_1}c_{p_2}c_{p_3}\ldots$.

In order to extend this theory so as to involve three or more sets of objects of distinct and given types we may in the first place consider four identities

$$1 + a_1x + a_2x^2 + \ldots = (1+\alpha_1x)(1+\alpha_2x)\ldots,$$
$$1 + b_1x + b_2x^2 + \ldots = (1+\beta_1x)(1+\beta_2x)\ldots,$$
$$1 + c_1x + c_2x^2 + \ldots = (1+\gamma_1x)(1+\gamma_2x)\ldots,$$
$$1 + d_1x + d_2x^2 + \ldots = (1+\delta_1x)(1+\delta_2x)\ldots^*,$$

and therewith an auxiliary identity

$$1 + k_1x + k_2x^2 + \ldots = (1+\kappa_1x)(1+\kappa_2x)\ldots.$$

Assume the quantities involved to be connected by the two relations

$$1 + k_1y + k_2y^2 + \ldots = \prod_s (1 + \alpha_sb_1y + \alpha_s^2b_2y^2 + \ldots),$$
$$1 + d_1y + d_2y^2 + \ldots = \prod_s (1 + \gamma_sk_1y + \gamma_s^2k_2y^2 + \ldots),$$

which lead, as shewn in Art. 48, to the relations

$$(l)_\kappa = (l)_\alpha(l)_\beta,$$
$$(l)_\delta = (l)_\gamma(l)_\kappa,$$

* The symbols d_1, d_2, ... employed in this article are algebraic, appertaining to the series $a, b, c, \ldots$, and are not to be confounded with operational symbols previously employed.

so that elimination of $(l)_\kappa$ gives

$$(l)_\delta = (l)_\alpha (l)_\beta (l)_\gamma.$$

Thus the expression $(l)_\delta$ remains unchanged for any permutation of the sets of quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$; $\beta_1, \beta_2, \beta_3, \ldots$; $\gamma_1, \gamma_2, \gamma_3, \ldots$; and every symmetric function of the quantities $\delta_1, \delta_2, \delta_3, \ldots$ will enjoy the same property. Thus we have

$$(p_1p_2p_3\ldots)_\delta = \ldots + J\{(q_1q_2q_3\ldots)_\alpha (r_1r_2r_3\ldots)_\beta (s_1s_2s_3\ldots)_\gamma + 5 \text{ similar expressions obtained by permuting } \alpha, \beta, \gamma\}.$$

Following Art. 56 there is no difficulty in establishing the operator relations

$$(l)_{\alpha\ \alpha} d_l = (l)_{\beta\ \beta} d_l = (l)_{\gamma\ \gamma} d_l = (l)_{\delta\ \delta} d_l.$$

The two relations which connect the four identities and the auxiliary identity give rise by comparison of the coefficients of like powers of y to the two series of relations

$$\begin{aligned}
k_1 &= (1)_\alpha b_1, \\
k_2 &= (2)_\alpha b_2 + (1^2)_\alpha b_1^2, \\
k_3 &= (3)_\alpha b_3 + (21)_\alpha b_2 b_1 + (1^3)_\alpha b_1^3, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
k_s &= \Sigma (pqr\ldots)_\alpha b_p b_q b_r \ldots;
\end{aligned}$$

$$\begin{aligned}
d_1 &= (1)_\gamma k_1, \\
d_2 &= (2)_\gamma k_2 + (1^2)_\gamma k_1^2, \\
d_3 &= (3)_\gamma k_3 + (21)_\gamma k_2 k_1 + (1^3)_\gamma k_1^3, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
d_s &= \Sigma (pqr\ldots)_\gamma k_p k_q k_r \ldots.
\end{aligned}$$

From the second series the development

$$d_{p_1} d_{p_2} d_{p_3} \ldots = \ldots + \theta (q_1q_2q_3\ldots)_\gamma k_{\lambda_1} k_{\lambda_2} k_{\lambda_3} \ldots + \ldots$$

shews that there are θ ways of distributing objects of type $(p_1p_2p_3\ldots)$ with objects of type $(q_1q_2q_3\ldots)$ so that the distribution is of type $(\lambda_1\lambda_2\lambda_3\ldots)$. We have now sets of two-fold objects of type $(\lambda_1\lambda_2\lambda_3\ldots)$ which we may distribute with objects of type $(r_1r_2r_3\ldots)$, and since from the first series of relations

$$k_{\lambda_1} k_{\lambda_2} k_{\lambda_3} \ldots = \ldots + \theta' (r_1r_2r_3\ldots)_\alpha b_{\mu_1} b_{\mu_2} b_{\mu_3} \ldots + \ldots,$$

we see that the set of two-fold objects of type $(\lambda_1\lambda_2\lambda_3\ldots)$ may be distributed with a set of objects of type $(r_1r_2r_3\ldots)$ so that the result is a set of three-fold objects of type $(\mu_1\mu_2\mu_3\ldots)$ in θ' ways. Hence from the three sets of objects of types $(p_1p_2p_3\ldots)$, $(q_1q_2q_3\ldots)$, $(r_1r_2r_3\ldots)$ respectively, we can form a set of three-fold objects of type $(\mu_1\mu_2\mu_3\ldots)$, the first two sets forming a two-fold

set of type $(\lambda_1\lambda_2\lambda_3 \ldots)$ in $\theta\theta'$ ways. We can sum these distributions for all of the intermediate two-fold types $(\lambda_1\lambda_2\lambda_3 \ldots)$ by simply eliminating the quantities $k_1, k_2, k_3, \ldots$ so as to obtain the formula

$$d_{p_1}d_{p_2}d_{p_3}\ldots = \ldots + \theta''(q_1q_2q_3\ldots)_\gamma(r_1r_2r_3\ldots)_\alpha b_{\mu_1}b_{\mu_2}b_{\mu_3}\ldots + \ldots,$$

where θ'' enumerates the number of sets of three-fold objects which are of type $(\mu_1\mu_2\mu_3\ldots)$ that can be obtained by distributing together the sets of objects of type $(p_1p_2p_3\ldots)$, $(q_1q_2q_3\ldots)$, $(r_1r_2r_3\ldots)$ respectively.

To obtain the whole number of such distributions of all types we must sum the result just reached for all possible types $(\mu_1\mu_2\mu_3\ldots)$. This is simply done by writing $b_1 = b_2 = b_3 = \ldots = 1$, when k_s becomes h_s, viz.: the homogeneous product sum of order s of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$. We have then

$$\begin{aligned}
d_1 &= (1)_\gamma h_1,\\
d_2 &= (2)_\gamma h_2 + (1^2)_\gamma h_1^2,\\
d_3 &= (3)_\gamma h_3 + (21)_\gamma h_2h_1 + (1^3)_\gamma h_1^3,\\
d_4 &= (4)_\gamma h_4 + (31)_\gamma h_3h_1 + (2^2)_\gamma h_2^2 + (21^2)_\gamma h_2h_1^2 + (1^4)_\gamma h_1^4,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and if

$$d_{p_1}d_{p_2}d_{p_3}\ldots = \ldots + \theta'''(q_1q_2q_3\ldots)_\gamma(r_1r_2r_3\ldots)_\alpha + \ldots,$$

θ''' enumerates the distributions under consideration.

Let D_s, Δ_s be obliterating operators having reference to symmetric functions of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$; $\gamma_1, \gamma_2, \gamma_3, \ldots$ respectively.

Then

$$\Delta_{q_1}\Delta_{q_2}\Delta_{q_3}\ldots D_{r_1}D_{r_2}D_{r_3}\ldots(d_{p_1}d_{p_2}d_{p_3}\ldots) = \theta''',$$

constituting the analytical solution of the problem.

The easily established results of operations simplify the calculations, viz.: these are

$$\Delta_s d_p = h_s d_{p-s}, \quad D_s d_p = h_s' d_{p-s},$$

where h_s' refers to the series $\gamma_1, \gamma_2, \gamma_3, \ldots$.

Thus suppose $(p_1p_2p_3\ldots)$, $(q_1q_2q_3\ldots)$, $(r_1r_2r_3\ldots) = (211)$, (22), (211) respectively,

$$\begin{aligned}
\theta''' &= \Delta_2^2 D_2 D_1^2 . d_2 d_1^2 = D_2 D_1^2 . \Delta_2 \{(h_2 + 2h_1^2)\, d_1^2 + h_1^2 d_2\}\\
&= D_2 D_1^2 (2h_1^2 h_2 + 2h_1^4) = D_2 D_1 (2h_1^3 + 4h_1h_2 + 8h_1^3)\\
&= D_2 (30h_1^2 + 4h_1^2 + 4h_2) = 38.
\end{aligned}$$

To verify this enumeration take the objects to be

$$a_1a_1b_1c_1, \quad a_2a_2b_2b_2, \quad a_3a_3b_3c_3 \text{ respectively.}$$

The distributions are

$a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$
$a_2a_2b_2b_2$ $a_2a_2b_2b_2$ $a_2a_2b_2b_2$ $a_2a_2b_2b_2$ $a_2a_2b_2b_2$ $a_2a_2b_2b_2$ $a_2a_2b_2b_2$
$a_3a_3b_3c_3$ $a_3a_3c_3b_3$ $a_3b_3a_3c_3$ $a_3b_3c_3a_3$ $b_3c_3a_3a_3$ $a_3c_3a_3b_3$ $a_3c_3b_3a_3$

$a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$
$b_2b_2a_2a_2$ $b_2b_2a_2a_2$ $b_2b_2a_2a_2$ $b_2b_2a_2a_2$ $b_2b_2a_2a_2$ $b_2b_2a_2a_2$ $b_2b_2a_2a_2$
$a_3a_3b_3c_3$ $a_3a_3c_3b_3$ $a_3b_3a_3c_3$ $a_3b_3c_3a_3$ $b_3c_3a_3a_3$ $a_3c_3a_3b_3$ $a_3c_3b_3a_3$

$a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$
$a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$
$a_3a_3b_3c_3$ $a_3a_3c_3b_3$ $a_3b_3a_3c_3$ $a_3c_3a_3b_3$ $a_3b_3c_3a_3$ $a_3c_3b_3a_3$

$a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$
$a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$ $a_2b_2a_2b_2$
$b_3a_3a_3c_3$ $c_3a_3a_3b_3$ $b_3a_3c_3a_3$ $c_3a_3b_3a_3$ $b_3c_3a_3a_3$ $c_3b_3a_3a_3$

$a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$
$a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$
$a_3a_3b_3c_3$ $a_3a_3c_3b_3$ $a_3b_3a_3c_3$ $a_3c_3a_3b_3$ $a_3b_3c_3a_3$ $a_3c_3b_3a_3$

$a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$ $a_1a_1b_1c_1$
$a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$ $a_2b_2b_2a_2$
$b_3a_3a_3c_3$ $c_3a_3a_3b_3$ $b_3a_3c_3a_3$ $c_3a_3b_3a_3$ $b_3c_3a_3a_3$ $c_3b_3a_3a_3$

The first written of these is to be understood to mean a set of four three-fold objects $a_1a_2a_3$, $a_1a_2a_3$, $b_1b_2b_3$, $c_1b_2c_3$, and it will be noted that it is of type (211); in fact if in the calculation we do not put $b_1 = b_2 = b_3 = \ldots = 1$, we find

$$d_{p_1}d_{p_2}d_{p_3}\ldots = \ldots + (q_1q_2q_3\ldots)_\gamma\,(r_1r_2r_3\ldots)_\alpha\,(4b_2b_1^2 + 34b_1^4) + \ldots,$$

indicating that 4 and 34 of the distributions are of types (21^2), (1^4) respectively.

The four of type (21^2) consist of that above written and also the second in the first row and the first and second in the second row.

SECTION III

PERMUTATIONS

CHAPTER I

THE ENUMERATION OF PERMUTATIONS

61. We commence the study of permutations to which this Section is devoted by applying the general Theory of Distributions of Section I to the special case of permutations. We first of all study the combinations involved. We are given any assemblage of letters specified by

$$\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_s^{p_s},$$

where $p_1+p_2+\dots+p_s=n$, and we require the number of different combinations of the letters therein occurring which involve m letters, alike or not alike. In fact we require the number of combinations of the letters m together.

Conceive m exactly similar parcels of one kind and $n-m$ similar parcels of a second kind. Suppose that all the letters of the given assemblage (n in number) to be distributed into these n parcels in such wise that each parcel contains one letter. The number of ways in which this can be done is equal to the number of combinations m at a time of the letters of the assemblage. The number in question is by Section I equal to the coefficient of the symmetric function

$$(p_1p_2\dots p_s),$$

in the development of the homogeneous product-sum product

$$h_mh_{n-m}.$$

The function h_mh_{n-m} is the generating function for the enumeration of combinations m at a time of all possible assemblages of n letters.

Taking $p_1, p_2, p_3, \dots$ to be in descending order of magnitude there is of course a one-to-one correspondence between the assemblages which involve n letters and the partitions of n.

From the symmetry of the function h_mh_{n-m} we gather that the combinations $n-m$ at a time are equi-numerous with the combinations m at a time. This is also intuitively evident *à priori*.

As an example of the theorem, since

$$h_4h_2 = (6) + 2\,(51) + 3\,(42) + 4\,(41^2) + 3\,(3^2) + 5\,(321) + 7\,(31^3) + 6\,(2^3) + 8\,(2^21^2) + 11\,(21^4) + 15\,(1^6),$$

we reach the solution, when $m = 4$ or 2, for any possible assemblage of letters. Thus for the assemblage

$$\alpha\alpha\alpha\beta\gamma\delta,$$

since one term in the development of h_4h_2 is $7\,(31^3)$ we find that 7 combinations, 2 (or 4) at a time, can be derived from the letters of the assemblage. These are of course (for 2 at a time)

$$\alpha\alpha,\ \alpha\beta,\ \alpha\gamma,\ \alpha\delta,\ \beta\gamma,\ \beta\delta,\ \gamma\delta.$$

Employing Hammond's differential operators which have been defined and explained in Section I we find that, since

$$D_{p_1}D_{p_2}\dots D_{p_s}(p_1p_2\dots p_s) = 1,$$

the sought number is

$$D_{p_1}D_{p_2}\dots D_{p_s}h_mh_{n-m}.$$

The number in simple cases is readily calculable because

$$D_sh_m = h_{m-s},$$

$$D_sh_ph_q = h_{p-s}h_q + h_{p-s+1}h_{q-1} + h_{p-s+2}h_{q-2} + \dots + h_ph_{q-s}.$$

The reader will observe that here the operator D_s is operative through the 2-part compositions of s,

$$s,\ 0;\quad s-1,\ 1;\quad s-2,\ 2;\quad \dots\ 0,\ s.$$

In the above example the calculation proceeds as follows:

$$D_3D_1^3h_4h_2 = D_1^3(h_1h_2 + h_2h_1 + h_3) = D_1^2(2h_1^2 + 2h_2 + h_2)$$
$$= D_1(4h_1 + 3h_1) = 7.$$

In Section I a theorem of reciprocity was established which yields the relation

$$D_{p_1}D_{p_2}\dots D_{p_s}h_mh_{n-m} = D_mD_{n-m}\,h_{p_1}h_{p_2}\dots h_{p_s},$$

which may be sometimes used to simplify the calculation because the right-hand side operation may be more easily carried out than that on the left-hand side.

Moreover we observe the identity

$$D_pD_{n-p}h_mh_{n-m} = D_mD_{n-m}h_ph_{n-p},$$

which at once gives us a theorem in combinations, for it establishes that the combinations p at a time drawn from the assemblage

$$\alpha^m\beta^{n-m}$$

are equi-numerous with the combinations m at a time drawn from the assemblage

$$\alpha^p\beta^{n-p}.$$

The reader will verify this otherwise without difficulty.

If we take as a particular case the assemblage

$$\alpha_1 \alpha_2 \dots \alpha_n,$$

the number of combinations m at a time is

$$D_1^n h_m h_{n-m} \equiv D_m D_{n-m} h_1^n \equiv D_m D_{n-m} (1)^n,$$

where (1) is the symmetric function $\Sigma\alpha$.

Without difficulty the number is found to be $\binom{n}{m}$.

In the above we have not restricted the combinations in any way. Suppose however that we make the condition that no combination is to involve more than k letters which are identical, we must construct the product

$$k_m h_{n-m},$$

wherein k_m denotes the product-sum h_m from which all terms involving repetitional exponents exceeding k have been deleted.

The number of combinations is now

$$D_{p_1} D_{p_2} \dots D_{p_s} k_m h_{n-m}.$$

In this formula the combinations which are left after withdrawal of the combinations m at a time have not been restricted.

If the two sets of combinations are to be simultaneously restricted the enumerating number is

$$D_{p_1} D_{p_2} \dots D_{p_s} k_m k_{n-m}.$$

The reader will observe that, with the restriction before us, the combinations m and $n-m$ at a time are not equi-numerous. The generating functions are respectively $k_m h_{n-m}$ and $h_m k_{n-m}$.

In particular consider the case $k=1$. The generating function is

$$a_m h_{n-m},$$

wherein a_m is the elementary symmetric function $\Sigma\alpha_1 \alpha_2 \dots \alpha_n$.

Since $$D_s a_p h_q = a_{p-1} h_{q-s+1} + a_p h_{q-s},$$

because D_s operates through the compositions $1, s-1$; $0, s$ of s, the calculation is easy.

Thus for the assemblage $\alpha\alpha\alpha\beta\gamma$ and $m=2$

$$D_3 D_2 D_1 a_2 h_4 = D_2 D_1 (a_1 h_2 + a_2 h_1) = D_1 (h_1 + a_1 + a_1) = 3.$$

The combinations are evidently $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$.

For $m=4$, it will be found that

$$D_3 D_2 D_1 a_4 h_2 = 0,$$

as should be the case because there is no combination of 4 letters, drawn from the given assemblage, which does not contain some repetitions of letters.

In general the calculation is performed by the aid of the theorem

$$D_s k_p h_q = k_p h_{q-s} + k_{p-1} h_{q-s+1} + k_{p-2} h_{q-s+2} + \dots,$$

wherein the last term is either $k_{p-k} h_{q-s+k}$ or h_{q-s+p}.

62. The enumeration of permutations is treated in a similar manner.

Instead of the product $h_m h_{n-m}$ we must now take $h_1^m h_{n-m}$ because the number of permutations of combinations of m letters drawn from the assemblage

$$\alpha_1^{p_1} \alpha_2^{p_2} \dots \alpha_s^{p_s} \qquad (\Sigma p = n)$$

is equal to the number of distributions of the letters into n parcels, one letter in each parcel, of which m are different from one another and the remaining $n - m$ similar but different from any included in the former.

Thus

$$h_1^m h_{n-m}$$

is the generating function which on development shews by the coefficient of the symmetric function

$$(p_1 p_2 \dots p_s),$$

the number we are considering.

Hence the number is

$$D_{p_1} D_{p_2} \dots D_{p_s} h_1^m h_{n-m},$$

or its equivalent

$$D_1^m D_{n-m} h_{p_1} h_{p_2} \dots h_{p_s}.$$

Thus since

$$h_4 h_1^2 = (6) + 3\,(51) + 4\,(42) + 7\,(41^2) + 4\,(3^2) + 8\,(321) + 13\,(31^3) + 9\,(2^3) + 14\,(2^2 1^2) + 21\,(21^4) + 30\,(1^6),$$

we obtain at once the solution for $m = 2$, in the case of every possible assemblage of 6 letters.

For the assemblage $\alpha\alpha\beta\beta\gamma\delta$ we see by the coefficient of $(2^2 1^2)$ that the number is 14. These permutations are

$\alpha\alpha$	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$
$\beta\beta$	$\beta\alpha$	$\gamma\alpha$	$\delta\alpha$	$\gamma\beta$	$\delta\beta$	$\delta\gamma$.

In particular if the assemblage be

$$\alpha_1^{n-p} \alpha_2 \alpha_3 \dots \alpha_{p+1},$$

the number sought is

$$D_1^p D_{n-p} h_1^m h_{n-m},$$

or its equivalent

$$D_1^m D_{n-m} h_1^p h_{n-p}.$$

This equivalence establishes that the permutations of combinations of m letters drawn from the assemblage

$$\alpha_1^{n-p} \alpha_2 \alpha_3 \dots \alpha_{p+1}$$

are equi-numerous with the permutations of combinations of p letters drawn from the assemblage

$$\alpha_1^{n-m}\alpha_2\alpha_3 \dots \alpha_{m+1}.$$

Moreover since

$$D_1^p D_{n-p} h_m h_{n-m} = D_m D_{n-m} h_1^p h_{n-p},$$

we can assert that the *combinations* of m letters drawn from the assemblage

$$\alpha_1^{n-p}\alpha_2\alpha_3 \dots \alpha_{p+1}$$

are equi-numerous with the *permutations of combinations* of p letters drawn from the assemblage

$$\alpha_1^m \alpha_2^{n-m}.$$

As an example of this result take $n = 6$, $m = 2$, $p = 3$.

The combinations of 2 letters drawn from $\alpha\alpha\alpha\beta\gamma\delta$ are

$$\alpha\alpha, \quad \alpha\beta, \quad \alpha\gamma, \quad \alpha\delta, \quad \beta\gamma, \quad \beta\delta, \quad \gamma\delta,$$

whilst the permutations of combinations of 3 letters drawn from $\alpha\alpha\beta\beta\beta\beta$ are

$$\alpha\alpha\beta, \quad \alpha\beta\alpha, \quad \beta\alpha\alpha, \quad \alpha\beta\beta, \quad \beta\alpha\beta, \quad \beta\beta\alpha, \quad \beta\beta\beta,$$

the number in each case being 7.

Calculation proceeds according to the formula

$$D_p h_1^m h_{n-m} = h_1^m h_{n-m-p} + \binom{m}{1} h_1^{m-1} h_{n-m-p+1} + \binom{m}{2} h_1^{m-2} h_{n-m-p+2} + \dots + \binom{m}{p} h_1^{m-p} h_{n-m}.$$

Thus the evaluation of $D_2^2 D_1^2 h_1^2 h_4$ is

$$\begin{aligned} D_2^2 D_1^2 h_1^2 h_4 &= D_2 D_1^2 (h_1^2 h_2 + 2h_1 h_3 + h_4) \\ &= D_1^2 (h_1^2 + 2h_1^2 + h_2 + 2h_1^2 + 2h_2 + h_2) \\ &= D_1 (10h_1 + 4h_1) = 14. \end{aligned}$$

We may restrict in any manner the number of identical letters that may appear in the permutations. If not more than k identical letters are to appear in the permutation we take h_1^m which is

$$\Sigma \frac{m!}{p_1!\, p_2! \dots p_t!} (p_1 p_2 \dots p_t),$$

where

$$p_1 + p_2 + \dots + p_t = m,$$

and delete therefrom all symmetric functions which involve an exponent greater than k (or from the partitions, which denote the symmetric functions, all partitions which involve a part greater than k).

This is particularly simple when $k = 1$, for then the only part of h_1^m which survives is $m!\, a_m$ and the generating function is

$$\cdot\ m!\, a_m h_{n-m}.$$

In particular for $n = 6$, $m = 2$, we find

$$2!\, a_2 h_4 = 2(51) + 2(42) + 6(41^2) + 2(3^2) + 6(321) + 12(31^3) + 6(2^3) + 12(2^2 1^2) + 20(21^4) + 30(1^6).$$

Thus with regard to the assemblage $\alpha\alpha\alpha\beta\gamma\delta$, we find 12 permutations of combinations of 2 letters such that no letter is repeated, viz.:

$$\begin{array}{cccccc} \alpha\beta & \alpha\gamma & \alpha\delta & \beta\gamma & \beta\delta & \gamma\delta \\ \beta\alpha & \gamma\alpha & \delta\alpha & \gamma\beta & \delta\beta & \delta\gamma. \end{array}$$

In fact we can see *à priori* that the permutations in question are derived from the corresponding combinations by multiplication by $m!$, so that the generating function $a_m h_{n-m}$ already found above for the combinations necessarily leads to the generating function $m!\, a_m h_{n-m}$ for the permutations.

CHAPTER II

THE THEORY OF PERMUTATIONS

63. We will now consider in some interesting details the distribution of n objects of given type into n parcels of type (1^n); no two of the parcels are alike and if we view the parcels as being also a set of objects the association results in sets of two-fold objects, one set of objects being such that no two are alike. Further we may regard the theory as connected with the distribution of objects of type (1^n) into n parcels of given type.

It was shewn in Section I that the number of permutations is given by the multinomial expansion

$$(1)^n = \Sigma \frac{n!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}(p_3!)^{\pi_3}\ldots}(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots);$$

$(1)^n$ is the distribution function, so that the number of permutations of objects of type $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$ is

$$\frac{n!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}(p_3!)^{\pi_3}\ldots}.$$

Consider $X_1, X_2, \ldots X_n$ to be linear functions of quantities $x_1, x_2, \ldots x_n$ given by the matricular relation

$$(X_1, X_2, \ldots X_n) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} (x_1, x_2, \ldots x_n),$$

so that $$X_s = a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n.$$

If $X = x_1 + x_2 + \ldots + x_n$ the number of permutations of the factors of the product $x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$ is the coefficient of the like term in

$$X^{\xi_1+\xi_2+\ldots+\xi_n}.$$

To further examine the matter we now consider the expansion of the function

$$X_1^{\xi_1} X_2^{\xi_2} \dots X_n^{\xi_n},$$

and we require an important theorem.

Consider the algebraic fraction

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)\dots(1-s_nX_n)};$$

the general term in its expansion is

$$s_1^{\xi_1} s_2^{\xi_2} \dots s_n^{\xi_n} X_1^{\xi_1} X_2^{\xi_2} \dots X_n^{\xi_n},$$

and, if we are merely looking for the coefficient of $x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$ in the development of

$$X_1^{\xi_1} X_2^{\xi_2} \dots X_n^{\xi_n},$$

we require that portion of

$$s_1^{\xi_1} s_2^{\xi_2} \dots s_n^{\xi_n} X_1^{\xi_1} X_2^{\xi_2} \dots X_n^{\xi_n}$$

which is a function of the products

$$s_1x_1, \quad s_2x_2, \quad s_3x_3, \quad \dots \quad s_nx_n$$

only; so in general we only require *that* portion of the expansion of the fraction

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)\dots(1-s_nX_n)}.$$

For the sake of simplicity we will take $n=3$ and write the matricular relation

$$(X_1, X_2, X_3) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} (x_1, x_2, x_3)$$

and we will now shew that that portion of the fraction

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)(1-s_3X_3)}$$

which is a function of s_1x_1, s_2x_2, s_3x_3 only is equal to

$$\frac{1}{|(1-a_1s_1x_1)(1-b_2s_2x_2)(1-c_3s_3x_3)|},$$

where the denominator is in symbolic form and is such that after multiplication the a, b, c products are to be written in determinant brackets; thus

$$1 - a_1s_1x_1 - b_2s_2x_2 - c_3s_3x_3 + |a_1b_2|s_1s_2x_1x_2 + |a_1c_3|s_1s_3x_1x_3 + |b_2c_3|s_2s_3x_2x_3 - |a_1b_2c_3|s_1s_2s_3x_1x_2x_3,$$

the determinants denoting the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and its co-axial minors. $|a_1b_2c_3|$ is the determinant itself; $|a_1c_3|$ the minor $a_1c_3 - a_3c_1$ and so forth.

64. To establish this, consider the fraction

$$\frac{|(1-a_1s_1x_1)(1-b_2s_2x_2)(1-c_3s_3x_3)|}{(1-s_1X_1)(1-s_2X_2)(1-s_3X_3)},$$

which may be written

$$\frac{|\{1-s_1X_1+s_1(X_1-a_1x_1)\}\{1-s_2X_2+s_2(X_2-b_2x_2)\}\{1-s_3X_3+s_3(X_3-c_3x_3)\}|}{(1-s_1X_1)(1-s_2X_2)(1-s_3X_3)},$$

and now carrying out the multiplication of the numerator factors this is

$$1+\frac{s_1(X_1-a_1x_1)}{1-s_1X_1}+\frac{s_2(X_2-b_2x_2)}{1-s_2X_2}+\frac{s_3(X_3-c_3x_3)}{1-s_3X_3}$$
$$+\frac{s_2s_3|(X_2-b_2x_2)(X_3-c_3x_3)|}{(1-s_2X_2)(1-s_3X_3)}+\frac{s_1s_3|(X_1-a_1x_1)(X_3-c_3x_3)|}{(1-s_1X_1)(1-s_3X_3)}$$
$$+\frac{s_1s_2|(X_1-a_1x_1)(X_2-b_2x_2)|}{(1-s_1X_1)(1-s_2X_2)},$$

the last term vanishing because, as will be seen presently, the expression

$$|(X_1-a_1x_1)(X_2-b_2x_2)(X_3-c_3x_3)|$$

vanishes identically.

It will be shewn that the terms on the right-hand side which follow unity contain on expansion no terms which are functions of

$$s_1x_1,\ s_2x_2,\ s_3x_3 \text{ only.}$$

For consider, in respect of the n quantities $x_1, x_2, \ldots x_n$, the expression

$$\frac{s_1s_2\ldots s_t|(X_1-a_1x_1)(X_2-b_2x_2)\ldots(X_t-t_tx_t)|}{(1-s_1X_1)(1-s_2X_2)\ldots(1-s_tX_t)},$$

where t is an integer not greater than n; this fraction is specified by the first t natural numbers, but what follows can be readily modified to meet the case of a fraction specified by any selection of t natural numbers, which are unequal and not superior to n.

To shew that this fraction contains on expansion no terms which are functions of $s_1x_1, s_2x_2, \ldots s_nx_n$ only, it is merely necessary to shew that every term in the development of the expression

$$|(X_1-a_1x_1)(X_2-b_2x_2)\ldots(X_t-t_tx_t)|$$

contains at least one of the quantities $x_{t+1}, x_{t+2}, \ldots x_n$, for visibly the fraction under examination contains neither $s_{t+1}, s_{t+2}, \ldots$ nor s_n. For this to be the case the expression should vanish by putting

$$x_{t+1} = x_{t+2} = \ldots = x_n = 0.$$

The expression, in determinant form, is

$$\begin{vmatrix} X_1 - a_1x_1, & -a_2x_1 & \ldots & -a_tx_1 \\ -b_1x_2, & X_2 - b_2x_2 & \ldots & -b_tx_2 \\ . & . & \ldots & . \\ . & . & \ldots & . \\ -t_1x_t, & -t_2x_t & \ldots & X_t - t_tx_t \end{vmatrix},$$

for it is at once evident that the co-factor of any X product

$$X_pX_qX_r \ldots$$

is the same in the expression and in the determinant.

In the determinant putting

$$x_{t+1} = x_{t+2} = \ldots = x_n = 0,$$

and then adding x_1 times the first column, x_2 times the second, ... x_t times the tth, we obtain a column of zeros. Hence on the supposition the determinant vanishes and also

$$|(X_1 - a_1x_1)(X_2 - b_2x_2) \ldots (X_n - n_nx_n)|$$

vanishes identically. Every term in the development of the expression under examination contains as factor at least one of the quantities $x_{t+1}, x_{t+2}, \ldots x_n$, and thus the fraction under examination has on expansion no term which is a function of $s_1x_1, s_2x_2, \ldots s_nx_n$ only.

It has now been shewn, dividing each side of the identity above by

$$|(1 - a_1s_1x_1)(1 - b_2s_2x_2)(1 - c_3s_3x_3)|,$$

that

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)(1 - s_3X_3)}$$

is the product of

$$\frac{1}{|(1 - a_1s_1x_1)(1 - b_2s_2x_2)(1 - c_3s_3x_3)|}$$

and the series

$$1 + \frac{s_1(X_1 - a_1x_1)}{1 - s_1X_1} + \frac{s_2(X_2 - b_2x_2)}{1 - s_2X_2} + \frac{s_3(X_3 - c_3x_3)}{1 - s_3X_3}$$
$$+ \frac{s_1s_2|(X_1 - a_1x_1)(X_2 - b_2x_2)|}{(1 - s_1X_1)(1 - s_2X_2)} + \frac{s_1s_3|(X_1 - a_1x_1)(X_3 - c_3x_3)|}{(1 - s_1X_1)(1 - s_3X_3)}$$
$$+ \frac{s_2s_3|(X_2 - b_2x_2)(X_3 - c_3x_3)|}{(1 - s_2X_2)(1 - s_3X_3)},$$

and from what has been established above it is clear that that portion of the expanded fraction

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)(1-s_3X_3)}$$

which is a function of s_1x_1, s_2x_2, s_3x_3 only is represented by

$$\frac{1}{|(1-a_1s_1x_1)(1-b_2s_2x_2)(1-c_3s_3x_3)|},$$

and it is equally clear now that that portion of the expanded fraction

$$\frac{1}{(1-s_1X_1)(1-s_2X_2)\dots(1-s_nX_n)}$$

which is a function of s_1x_1, s_2x_2, $\dots$ s_nx_n only is represented by

$$\frac{1}{|(1-a_1s_1x_1)(1-b_2s_2x_2)\dots(1-n_ns_nx_n)|}.$$

65. From this theorem it at once follows that the coefficient of the term

$$x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n},$$

in the expansion of the product

$$X_1^{\xi_1}X_2^{\xi_2}X_3^{\xi_3}\dots X_n^{\xi_n},$$

is equal to the coefficient of the same term in the expansion of the fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|};$$

and it will be noted that this fraction does not involve the numbers ξ_1, ξ_2, $\dots$ ξ_n explicitly.

66. *Theorem.* "If X_1, X_2, $\dots$ X_n be given by the matricular relation

$$(X_1, X_2, \dots X_n) = \begin{pmatrix} a_{11}, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22}, & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1}, & a_{n2}, & \dots & a_{nn} \end{pmatrix} (x_1, x_2, \dots x_n)$$

the coefficient of the term

$$x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n},$$

in the development of the product

$$X_1^{\xi_1}X_2^{\xi_2}\dots X_n^{\xi_n},$$

is equal to the coefficient of the term

$$x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n},$$

in the expansion of the fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|},$$

wherein the denominator is in symbolic form in such wise that on multiplication the factors a_1b_2, $a_1b_2c_3$, ... are to be placed in determinant brackets $|a_1b_2|$, $|a_1b_2c_3|$, ... and denote the co-axial minors of the determinant

$$|a_1b_2 \dots n_n|,$$

which appertains to the matricular relation."

This is a master theorem in the Theory of Permutations.

We will write

$$V_n$$

for the expression

$$|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|,$$

and the reader will be able to verify that V_n has also the expression

$$(-)^n x_1x_2\dots x_n \begin{vmatrix} a_1-1/x_1 & a_2 & \dots & a_n \\ b_1 & b_2-1/x_2 & \dots & b_n \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ n_1 & n_2 & \dots & n_n-1/x_n \end{vmatrix}.$$

Applications of the Theorem.

67. We suppose the product

$$X_1^{\xi_1}X_2^{\xi_2}\dots X_n^{\xi_n}$$

to be written out so that the $\xi_1+\xi_2+\dots+\xi_n$ factors extend from left to right; to perform the multiplication we imagine one term to be selected from each factor in such wise that when the terms are all multiplied together there results a term which is $x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n}$ together with a coefficient which is a monomial composed of symbols a, b, c, ... n with certain suffixes. According to the mode of selection we have, in correspondence, a certain permutation of the factors of $x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n}$ and the number of different modes of selection is equal to the whole number of such permutations. If the result of any particular selection be

$$a_1^{\alpha_1}b_1^{\beta_1}\dots a_2^{\alpha_2}b_2^{\beta_2}\dots\dots x_1^{\xi_1}x_2^{\xi_2}\dots x_n^{\xi_n},$$

it will indicate that the corresponding permutation is such that

x_1 occurs α_1 times in places originally occupied by an x_1,
" " β_1 " " " x_2,
. . . . . .
x_2 " α_2 " " " x_1,
" " β_2 " " " x_2,
etc.

The proper generating function for the enumeration of the permutations possessing this property is $\frac{1}{V_n}$.

CHAPTER III

THE THEORY OF DISPLACEMENTS

68. We will apply the master theorem to the problem of determining the number of permutations of the quantities in

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n},$$

which are such that every quantity is displaced; in other words no quantity x_s is to be in a place originally occupied by an x_s.

The particular case when each of the exponents ξ is equal to unity leads to the determination of the permutations of

$$x_1 x_2 \dots x_n$$

which have the property under examination. This is often spoken of as the "Problème des rencontres" and has been much studied. The results that we shall obtain will be of a general character and include this particular case. If this case were alone under consideration it would be possible to adopt a simpler method of investigation, but such a procedure would not be so suggestive as the one here adopted.

If in the product

$$(a_1x_1 + a_2x_2 + \dots + a_nx_n)^{\xi_1} (b_1x_1 + b_2x_2 + \dots + b_nx_n)^{\xi_2} \dots (n_1x_1 + n_2x_2 + \dots + n_nx_n)^{\xi_n}$$

we put $a_1 = 0$, it is clearly impossible to select x_1 from the first ξ_1 factors: so also if we put $b_2 = 0$ we cannot select x_2 from either of the next ξ_2 factors. It follows that in forming a permutation by selection of one of the quantities $x_1, x_2, \dots x_n$ from each of the factors in succession, the result of putting

$$a_1 = b_2 = c_3 = \dots = n_n = 0$$

is to ensure that such permutation has every letter or quantity displaced. The number of displacements is therefore the coefficient of $x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$ in the product

$$(x_2 + x_3 + \dots + x_n)^{\xi_1} (x_1 + x_3 + \dots + x_n)^{\xi_2} \dots (x_1 + x_2 + \dots + x_{n-1})^{\xi_n},$$

for we may put the remaining symbols $a, b, c, \dots n$ equal to unity as not

being required to be put in evidence. This then is the redundant generating function, and through the guiding determinant

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & . & . \\ 1 & 0 & 1 & 1 & 1 & . & . \\ 1 & 1 & 0 & 1 & 1 & . & . \\ 1 & 1 & 1 & 0 & 1 & . & . \\ 1 & 1 & 1 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix},$$

of the order n, we are led to the condensed or true generating function

$$\frac{1}{1-\Sigma x_1x_2-2\Sigma x_1x_2x_3-3\Sigma x_1x_2x_3x_4-\ldots-(n-1)\,x_1x_2\ldots x_n},$$

since it is easily proved that every co-axial minor of order s has the value

$$(-)^{s-1}(s-1).$$

We write

$$(x-x_1)(x-x_2)\ldots(x-x_n)=x^n-p_1x^{n-1}+p_2x^{n-2}-\ldots,$$

so that the generating function is

$$\frac{1}{1-p_2-2p_3-\ldots-(n-1)\,p_n}.$$

69. If $\xi_1=\xi_2=\ldots=\xi_n=1$, some properties of the numbers are obtained in a simple manner from the redundant generating function.

Thus it is easy to find the coefficient of $x_1x_2\ldots x_n$ in the development of

$$(x_2+x_3+\ldots+x_n)(x_1+x_3+\ldots+x_n)\ldots(x_1+x_2+\ldots+x_{n-1});$$

for this expression may be written

$$(p_1-x_1)(p_1-x_2)\ldots(p_1-x_n),$$

which is

$$p_1^n-p_1^{n-1}\Sigma x_1+p_1^{n-2}\Sigma x_1x_2-\ldots+(-)^n\,x_1x_2\ldots x_n,$$

or

$$p_1^{n-2}\,p_2-p_1^{n-3}\,p_3+\ldots+(-)^n\,p_n.$$

Observing that

$$p_1^{n-s}\,p_s=\ldots+\frac{n\,!}{s\,!}\,p_n+\ldots,$$

where $(1)^{n-s}(1^s)$ is multiplied out so as to be a linear function of monomial symmetric functions, the coefficient of $x_1x_2\ldots x_n$ or of p_n is seen to be

$$\frac{n\,!}{2\,!}-\frac{n\,!}{3\,!}+\ldots+(-)^n\frac{n\,!}{n\,!},$$

which may be written

$$n\,!\left\{1-\frac{1}{1\,!}+\frac{1}{2\,!}-\frac{1}{3\,!}+\ldots+(-)^n\frac{1}{n\,!}\right\},$$

the well-known value of the coefficient under examination, which will be denoted by

$$\{0\,;\ 1^n\},$$

in correspondence with a notation which expresses by

$$\{m\,;\ \xi_1\xi_2 \dots \xi_n\}$$

the number of permutations such that exactly m of the quantities which compose the product

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$$

are in the places they originally occupied.

The reader will observe that this notation involves an extension of the problem of displacements as originally enunciated and also of its particular case, the "Problème des rencontres." The utility of this extension will appear in the next following investigation.

70. If we are considering the question of the permutations which are such that exactly m of the quantities are in places they originally occupied the redundant generating function is

$$(ax_1 + x_2 + x_3 + \dots + x_n)^{\xi_1} (x_1 + ax_2 + x_3 + \dots + x_n)^{\xi_2} \dots (x_1 + x_2 + \dots + ax_n)^{\xi_n},$$

in which we seek the coefficient of

$$a^m x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}.$$

Putting $1 - a = b$, and $\xi_1 = \xi_2 = \dots = 1$, we have the product

$$(p_1 - bx_1)(p_1 - bx_2) \dots (p_1 - bx_n),$$

which is $$p_1^n - bp_1^n + b^2 p_1^{n-2} p_2 - \dots + (-)^n b^n p_n,$$

and developing so as to obtain the coefficient of p_n we find

$$n! \left\{1 - b + \frac{b^2}{2!} - \dots + (-)^n \frac{b^n}{n!}\right\} p_n;$$

and now expressing this in terms of a we find

$$\{0\,;\ 1^n\} + \binom{n}{1} a\,\{0\,;\ 1^{n-1}\} + \binom{n}{2} a^2\,\{0\,;\ 1^{n-2}\} + \dots + \binom{n}{n} a^n\,\{0\,;\ 1^0\}.$$

Hence

$$\{m\,;\ 1^n\} = \binom{n}{m}\{0\,;\ 1^{n-m}\},$$

shewing that the number of permutations of $x_1 x_2 \dots x_n$ which are such that exactly m of the letters are in the places originally occupied is

$$\binom{n}{m}\{0\,;\ 1^{n-m}\}.$$

Moreover it is easy to see *à priori* that this result is correct.

For we can select the m letters which are to remain undisplaced in $\binom{n}{m}$ ways and for each of these selections the remaining letters can be so permuted as to be all displaced in $\{0\,;\,1^{n-m}\}$ ways. Thence the result found is clearly correct.

The whole of the permutations can be found by summing $\{m\,;\,1^n\}$ from $m=0$ to $m=n$. Hence the well-known formula

$$\{0\,;\,1^n\}+\binom{n}{1}\{0\,;\,1^{n-1}\}+\binom{n}{2}\{0\,;\,1^{n-2}\}+\ldots+\binom{n}{n}=n\,!.$$

The values of $\{0\,;\,1^n\}$ for $n=0,\,1,\,2,\,3,\,4,\,5,\,6,\,7,\,8,\,\ldots$ are

$$1,\ 0,\ 1,\ 2,\ 9,\ 44,\ 265,\ 1854,\ 14833,\ \ldots,$$

and from the results already reached it is easy to shew that

$$\{0\,;\,1^n\}=(n-1)\,[\{0\,;\,1^{n-1}\}+\{0\,;\,1^{n-2}\}],$$

$$\{0\,;\,1^n\}=n\,\{0\,;\,1^{n-1}\}+(-1)^n,$$

both of them well-known relations.

71. We will now turn to the condensed generating function

$$\frac{1}{1-p_2-2p_3-3p_4-\ldots-(n-1)\,p_n},$$

and consider its expanded form which has been shewn to be

$$\sum_{\xi}\{0\,;\,\xi_1\xi_2\ldots\xi_n\}\,.\,(\xi_1\xi_2\ldots\xi_n);$$

for the theory of symmetric functions shews us that we can deal with the function $\Sigma x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$ instead of with the single term $x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$.

We may regard $\xi_1\xi_2\ldots\xi_n$ as being numbers in descending order of magnitude and n as indefinitely great. $\xi_1,\ \xi_2\ldots\xi_n$ may be any integers, zero not excluded, but there are certain symmetric functions that *à priori* must be absent; for clearly

$$\{0\,;\,\xi_1\xi_2\ldots\xi_n\}=0,$$

if $\xi_1>\xi_2+\xi_3+\ldots+\xi_n$. For example such functions as (21), (421), ... do not present themselves in the development.

In the first instance we will restrict ourselves to the numbers $\{0\,;\,1^s\}$ and for convenience put $\{0\,;\,1^s\}=P_s$. We write

$$\frac{1}{1-p_2-2p_3-3p_4-\ldots}=1+P_1p_1+P_2p_2+P_3p_3+\ldots+P_np_n+\ldots+\text{other terms},$$

which in the present process do not affect the numbers P_s.

Multiplying up we get

$$1=(1-p_2-2p_3-\ldots)\,(1+P_1p_1+P_2p_2+\ldots+P_np_n+\ldots)+\text{other terms}.$$

On the right-hand side, to find the coefficient of p_n, the relevant terms are

$$P_n p_n - P_{n-2}p_2 p_{n-2} - 2P_{n-3}p_3 p_{n-3} - 3P_{n-4}p_4 p_{n-4} - \ldots - (n-3)\,P_2 p_{n-2} p_2 \\ -(n-2)\,P_1 p_{n-1} p_1 - (n-1)\,p_n,$$

and if $n > 0$ the coefficient of p_n must be zero; hence

$$P_n = \binom{n}{2} P_{n-2} + 2\binom{n}{3} P_{n-3} + 3\binom{n}{4} P_{n-4} + \ldots + (n-1)\binom{n}{n},$$

or

$$\{0\,;\,1^n\} = \binom{n}{2}\{0\,;\,1^{n-2}\} + 2\binom{n}{3}\{0\,;\,1^{n-3}\} + 3\binom{n}{4}\{0\,;\,1^{n-4}\} + \ldots + (n-1)\binom{n}{n},$$

a new relation, the verification of which for $n = 6$ being

$$265 = 1.15.9 + 2.20.2 + 3.15.1 + 4.6.0 + 5.1.1.$$

The law that has been established is better exhibited by putting

$$(n-s-1)\,P_s = Q_{s,n},$$

so that $Q_{n,n} = -P_n$ and $Q_{n-1,n} = 0$; then

$$0 = Q_{n,n} + \binom{n}{1} Q_{n-1,n} + \binom{n}{2} Q_{n-2,n} + \ldots + \binom{n}{n} Q_{0,n},$$

or, if we write symbolically $Q_{s,n} = Q^s_{0,n}$,

$$(Q_{0,n} + 1)^n = 0.$$

72. Next consider the expansion of

$$\frac{1}{1 - p_2 - 2p_3 - 3p_4 - \ldots}$$

in ascending powers of $(p_2 + 2p_3 + 3p_4 + \ldots)$; we have

$$(p_2 + 2p_3 + 3p_4 + \ldots)^s = \Sigma \frac{s!}{s_2!\,s_3!\,s_4!\ldots} 1^{s_2}.2^{s_3}.3^{s_4}.\ldots.p_2^{s_2}p_3^{s_3}p_4^{s_4}\ldots,$$

where $s_2 + s_3 + s_4 + \ldots = s$, and in the first instance we are concerned with the coefficient of p_n or (1^n) which arises when the products $p_2^{s_2}p_3^{s_3}p_4^{s_4}\ldots$ are multiplied out.

The coefficient of (1^n) in the development of

$$(1^2)^{s_2}(1^3)^{s_3}(1^4)^{s_4}\ldots$$

is, by a theorem of symmetry established in Section I, equal to the coefficient of

$$(2^{s_2}3^{s_3}4^{s_4}\ldots)$$

in the expansion of $(1)^n$, and this by the multinomial theorem is

$$\frac{n!}{(2!)^{s_2}(3!)^{s_3}(4!)^{s_4}\ldots}.$$

Hence the portion of the right-hand side that we require is

$$\Sigma \frac{s!\,n!}{s_2!\,s_3!\,s_4!\ldots(2.0!)^{s_2}(3.1!)^{s_3}(4.2!)^{s_4}\ldots}(1^n),$$

and summing for s we find that

$$\{0;\ 1^n\} = \Sigma\Sigma_s \frac{s!\,n!}{s_2!\,s_3!\,s_4!\ldots(2.0!)^{s_2}(3.1!)^{s_3}(4.2!)^{s_4}\ldots},$$

where s may have all suitable values and

$$2s_2 + 3s_3 + 4s_4 + \ldots = n,$$

$$s_2 + s_3 + s_4 + \ldots = s.$$

Ex. gr. for $n = 8$, we have to consider the non-unitary partitions of 8, viz.: 8, 62, 53, 44, 422, 332, 2222, and from the scheme

s_2	s_3	s_4	s_5	s_6	s_7	s_8
0	0	0	0	0	0	1
1	0	0	0	1	0	0
0	1	0	1	0	0	0
0	0	2	0	0	0	0
2	0	1	0	0	0	0
1	2	0	0	0	0	0
4	0	0	0	0	0	0

we find

$$7 + 280 + 896 + 630 + 3780 + 6720 + 2520 = 14833,$$

which is right.

73. This method of procedure however is only appropriate to reach results over a limited range of the expanded function. We require theorems of a more general character and we proceed to shew that the symmetric function operators of Section II are competent to produce them. Writing

$$\frac{1}{1 - p_2 - 2p_3 - 3p_4 - \ldots} = \Sigma\{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}.(1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

we put $p_2 + 2p_3 + 3p_4 + \ldots = B$, so that the function to be examined is

$$\frac{1}{1 - B}.$$

The operators available are

$$d_s = \frac{d}{dp_s} + p_1\frac{d}{dp_{s+1}} + p_2\frac{d}{dp_{s+2}} + \ldots,$$

$$D_s = \frac{1}{s!}(d_1^s).$$

It will be found that, for the particular operand $(1-B)^{-1}$, these operators are connected by special relations and that every such relation is of significance in the theory of the generating function.

A special object of the following investigation is the determination of operators which have the effect of leaving the special operand unaltered.

We have

$$d_s B = (s-1) + s(p_1 + p_2 + p_3 + \ldots) + B,$$

so that

$$(d_s - d_t) B = (s-t)(1 + p_1 + p_2 + p_3 + \ldots),$$

$$(d_u - d_v) B = (u-v)(1 + p_1 + p_2 + p_3 + \ldots);$$

giving us

$$(u-v)(d_s - d_t) B = (s-t)(d_u - d_v) B,$$

shewing that, for an operand B,

$$(u-v)(d_s - d_t) \quad \text{and} \quad (s-t)(d_u - d_v)$$

are equivalent operations. Since these operations are linear this is also the case for an operand which is any power of B and therefore also for an operand

$$\frac{1}{1-B},$$

u, v, s, t being any positive integers, zero being excluded.

74. This result has been arrived at by elimination of $(p_1 + p_2 + p_3 + \ldots)$ and B between four equations of the type

$$d_s B = s - 1 + s(p_1 + p_2 + p_3 + \ldots) + B.$$

Only three equations are *necessary* and then we should reach the particular case

$$(s-v)(d_s - d_t) \equiv (s-t)(d_s - d_v).$$

If from two relations

$$d_s B = (s-1) + s(p_1 + p_2 + p_3 + \ldots) + B,$$

$$d_t B = (t-1) + t(p_1 + p_2 + p_3 + \ldots) + B,$$

we eliminate $(p_1 + p_2 + p_3 + \ldots)$ we find

$$(td_s - sd_t) B = (s-t)(1-B);$$

leading to the important result

$$\frac{td_s - sd_t}{s-t} \frac{1}{1-B} = \frac{1}{1-B},$$

establishing that the operation

$$\frac{td_s - sd_t}{s-t}$$

leaves the operand $(1-B)^{-1}$ entirely unaltered.

75. The two results that have now been established lead to a vast number of relations between the operators which may be applied forthwith to the study of the properties of the coefficients which arise in the development of the operand $\frac{1}{1-B}$. The difficulty is in selecting the relations which best exhibit those properties. Generally in applying the operators to the expanded form of the function we first express the operators d_s in terms of the operators D_s in order to take advantage of the great facility with which the latter operations are performed upon symmetric functions which are denoted by partitions.

We will in the first place consider the equivalence

$$(u-v)(d_s-d_t) \equiv (s-t)(d_u-d_v),$$

and commence by applying the simplest particular case

$$d_3 \equiv 2d_2 - d_1;$$

since $$d_3 = D_1^3 - 3D_2D_1 + 3D_3, \quad d_2 = D_1^2 - 2D_2, \quad d_1 = D_1,$$

the operation

$$(D_1^3 - 3D_2D_1 + 3D_3) - 2(D_1^2 - 2D_2) + D_1$$

must reduce the function $(1-B)^{-1}$ to zero. Writing this function in the form

$$\Sigma \{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\} \cdot (1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

we find that, after the operation, the symmetric function

$$(1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)$$

must appear with a zero coefficient. Hence

$$\{0;\ 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots\} - 3\{0;\ 1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\ldots\} + 3\{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots\}$$
$$- 2[\{0;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\} - 2\{0;\ 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\}] + \{0;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\} = 0.$$

We thus obtain a linear relation between certain groups of coefficients which are found throughout the whole extent of the expansion of the function; for the numbers $\pi_1, \pi_2, \pi_3, \ldots$ are at our disposal absolutely. The way in which the specification of the numbers $\{\ \}$ is connected with the formula which expresses the sum of the powers of quantities in terms of the elementary symmetric functions will be noted. In fact the above relation might be denoted by

$$(3)_{\{\}} - 2(2)_{\{\}} + (1)_{\{\}} = 0,$$

in mathematical shorthand.

As a simple example put $\pi_1 = 1$, $\pi_2 = \pi_3 = \ldots = 0$; then

$$\{0;\ 1^4\} - 3\{0;\ 21^2\} + 3\{0;\ 31\} - 2\{0;\ 1^3\} + 4\{0;\ 21\} + \{0;\ 1^2\} = 0.$$

Since $\{0;31\}$ and $\{0;21\}$ are both zero the relation gives us the expression of $\{0;21^2\}$ in terms of well-known numbers $\{0;1^s\}$; thus

$$3\{0;21^2\} = \{0;1^4\} - 2\{0;1^3\} + \{0;1^2\} = 9 - 4 + 1 = 6,$$

so that $\{0;21^2\} = 2$ which is obviously correct.

By giving $\pi_1, \pi_2, \pi_3, \ldots$ other values progressively we can obtain many results of a similar character and in like manner we can proceed with the general formula

$$(u - v)(d_s - d_t) \equiv (s - t)(d_u - d_v),$$

which when applied to the function yields the relation

$$(u - v)[(s)_{\{\}} - (t)_{\{\}}] = (s - t)[(u)_{\{\}} - (v)_{\{\}}].$$

76. Passing now to the equivalence

$$td_s - sd_t \equiv s - t,$$

it will be found that there are several ways of dealing with it.

We will first consider the particular case

$$td_1 - d_t \equiv -(t - 1);$$

putting $t = 2$, we deduce

$$2!\,D_2 = D_1^2 - 2D_1 - 1;$$

putting $t = 3$ and reducing by means of the relation just found there results

$$3!\,D_3 = D_1^3 - 6D_1^2 + 3D_1 + 4,$$

and thence, similarly,

$$4!\,D_4 = D_1^4 - 12D_1^3 + 30D_1^2 + 4D_1 - 15,$$

$$5!\,D_5 = D_1^5 - 20D_1^4 + 110D_1^3 - 140D_1^2 - 95D_1 + 56,$$

etc.

and it is clear we can express D_s in terms of D_1.

To calculate these relations we observe that the algebraic equivalent of the relation $td_1 - d_t \equiv -(t - 1)$ is

$$tp_1 - (t) = -(t - 1).$$

Since p_s corresponds to D_s we have to express p_s in terms of p_1 being given that

$$tp_1 - s_t = -(t - 1),$$

where we have replaced (t) by the more convenient symbol s_t; thus since

$$2!\,p_2 = s_1^2 - s_2,$$

$$3!\,p_3 = s_1^3 - 3s_2s_1 + 2s_3,$$

$$4!\,p_4 = s_1^4 - 6s_2s_1^2 + 3s_2^2 + 8s_3s_1 - 6s_4,$$

etc.

we find

$$2!\,p_2 = p_1^2 - (2p_1+1),$$
$$3!\,p_3 = p_1^3 - 3(2p_1+1)p_1 + 2(3p_1+2),$$
$$4!\,p_4 = p_1^4 - 6(2p_1+1)p_1^2 + 3(2p_1+1)^2 + 8(3p_1+2)p_1 - 6(4p_1+3),$$
$$\text{etc.},$$

where the general formula is known by Art. 6, and now we have merely to write D_s for p_s to arrive at the relations before us.

Applying these relations to the generating function, we first obtain

$$2!\{0; 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\} = \{0; 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\} - 2\{0; 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\} - \{0; 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\},$$

and a particular case is, putting $\{0; 1^s\} = P_s$,

$$2!\{0; 21^s\} = P_{s+2} - 2P_{s+1} - P_s,$$

a convenient formula for $\{0; 21^s\}$. From it we derive the values

$$s = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots,$$
$$\{0; 21^s\} = 0 \quad 0 \quad 2 \quad 12 \quad 84 \quad 640 \quad \ldots,$$

and from the known properties of P_s we deduce

$$2\{0; 21^s\} = (s^2+s-1)P_s + (-)^{s-1}(s-1).$$

We next obtain

$$3!\{0; 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots\} = \{0; 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots\} - 6\{0; 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\} + 3\{0; 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\} + 4\{0; 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\},$$

and thence

$$6\{0; 31^s\} = P_{s+3} - 6P_{s+2} + 3P_{s+1} + 4P_s,$$

and we derive for

$$s = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots,$$
$$\{0; 31^s\} = 0 \quad 0 \quad 0 \quad 6 \quad 72 \quad 780 \quad \ldots,$$

and

$$6\{0; 31^s\} = \{(s+1)(s+2)(s+3) - 6(s+1)(s+2) + 3(s+1) + 4\}P_s + (-)^{s+1}(s-1)^2.$$

Similarly we obtain

$$24\{0; 41^s\} = P_{s+4} - 12P_{s+3} + 30P_{s+2} + 4P_{s+1} - 15P_s,$$

and we derive for

$$s = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots,$$
$$\{0; 41^s\} = 0 \quad 0 \quad 0 \quad 0 \quad 24 \quad 480 \quad \ldots,$$

and

$$24\{0; 41^s\} = \left\{\frac{(s+4)!}{s!} - 12\frac{(s+3)!}{s!} + 30\frac{(s+2)!}{s!} + 4(s+1) - 15\right\}P_s + (-)^{s+1}(s-1)(s^2-3s-1).$$

There is no difficulty in obtaining a general formula of this nature.

77. From the general relation between the coefficients derived from the relation

$$2\,!\,D_2 = D_1^2 - 2D_1 - 1,$$

we obtain as another particular case

$$4\,\{0\,;\,2^2 1^s\} = 2\,\{0\,;\,21^{s+2}\} - 4\,\{0\,;\,21^{s+1}\} - 2\,\{0\,;\,21^s\},$$

and thence

$$4\,\{0\,;\,2^2 1^s\} = P_{s+4} - 4P_{s+3} + 2P_{s+2} + 4P_{s+1} + P_s\,;$$

and

$$4\,\{0\,;\,2^2 1^s\} = \left\{\frac{(s+4)\,!}{s\,!} - 4\,\frac{(s+3)\,!}{s\,!} + 2\,\frac{(s+2)\,!}{s\,!} + 4\,(s+1) + 1\right\} P_s + (-)^{s+1}(s^3 + 4s^2 + 6s + 5),$$

two forms of result which can no doubt be reached in the case of the number

$$\{0\,;\,1^s 2^{\pi_2} 3^{\pi_3} \ldots\}.$$

It will be observed that one formula of reduction is

$$2^r\,\{0\,;\,2^r 1^s\} = 2^{r-1}\,\{0\,;\,2^{r-1}1^{s+2}\} - 2^r\,\{0\,;\,2^{r-1}1^{s+1}\} - 2^{r-1}\,\{0\,;\,2^{r-1}1^s\}.$$

78. The generating function

$$\frac{1}{1 - ap_1 + (a-1)(a+1)p_2 - (a-1)^2(a+2)p_3 + \ldots}$$

arises similarly from the redundant product

$$(ax_1 + x_2 + x_3 + \ldots)^{\xi_1}(x_1 + ax_2 + x_3 + \ldots)^{\xi_2}(x_1 + x_2 + ax_3 + \ldots)^{\xi_3} \ldots (x_1 + x_2 + \ldots + ax_n)^{\xi_n}.$$

This, when expanded, is

$$\Sigma\,(m\,;\,1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)\,a^m\,(1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

and may be similarly dealt with. For write it $(1 - C)^{-1}$ where

$$C = ap_1 - (a-1)(a+1)p_2 + (a-1)^2(a+2)p_3 - \ldots\,;$$

then

$$d_s C = (-)^{s+1}(a-1)^s(1-C) + (-)^{s+1}s(a-1)^{s-1}E,$$

where

$$E = 1 - (a-1)p_1 + (a-1)^2 p_2 - \ldots\,;$$

whence

$$\{(-)^t d_s + (-)^{s+1}(a-1)^{s-t}d_t\}\,C = (-)^{s+t-1}(a-1)^{s-1}(s-t)\,E\,;$$

or putting, as in Art. 68, $1 - a = b$,

$$(b^{1-s}d_s - b^{1-t}d_t)\,C = (s-t)\,E;$$

shewing that, for an operand C,

$$(u-v)(b^{1-s}d_s - b^{1-t}d_t) \equiv (s-t)(b^{1-u}d_u - b^{1-v}d_v)$$

are equivalent operations. This equivalence reduces to

$$(u-v)(d_s - d_t) \equiv (s-t)(d_u - d_v),$$

when $a = 0$, or $b = 1$, as it should.

79. If from the original expression of d_s we eliminate E first instead of $1-C$ we find

$$(tb^t d_s - sb^s d_t)\, C = (s-t)\, b^{s+t}(1-C),$$

whence

$$\frac{tb^{-s}d_s - sb^{-t}d_t}{s-t}\,\frac{1}{1-C} = \frac{1}{1-C},$$

and we have established that the operation

$$\frac{tb^{-s}d_s - sb^{-t}d_t}{s-t}$$

leaves the generating function unchanged—or, in other form, the operation

$$tb^{-s}d_s - sb^{-t}d_t - s + t$$

causes the function to vanish.

It is just as easy to deal with the function $\frac{1}{1-C}$ as with $\frac{1}{1-B}$.

From the relation

$$(u-v)(b^{1-s}d_s - b^{1-t}d_t) \equiv (s-t)(b^{1-u}d_u - b^{1-v}d_v),$$

an equivalence which holds in general for any power of C and therefore for $(1-C)^{-1}$, we deduce as the simplest possible case

$$d_3 \equiv 2bd_2 - b^2 d_1,$$

and comparing this with the result when a is zero, viz.:

$$d_3 \equiv 2d_2 - d_1,$$

we see that regarding b as being of weight unity we have merely to render each term of the same weight by the introduction of the proper power of b as a factor in each term in order to pass from the latter to the former.

Applying the relation to

$$\Sigma\,(m\,;\, 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)\, a^m\, (1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

we first of all express the operators d_3, d_2, d_1 in terms of the operators D_1, D_2, D_3, and then we find

$$\sum_{m,\pi}(m;\, 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)a^m\{(1^{\pi_1-3}2^{\pi_2}3^{\pi_3}\ldots)-3(1^{\pi_1-1}2^{\pi_2-1}3^{\pi_3}\ldots)+3(1^{\pi_1}2^{\pi_2}3^{\pi_3-1}\ldots)\}$$

$$-2\sum_{m,\pi}(m\,;\, 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)(a^m - a^{m+1})\,\{(1^{\pi_1-2}2^{\pi_2}3^{\pi_3}\ldots) - 2\,(1^{\pi_1}2^{\pi_2-1}3^{\pi_3}\ldots)\}$$

$$+\sum_{m,\pi}(m\,;\, 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)(a^m - 2a^{m+1} + a^{m+2})(1^{\pi_1-1}2^{\pi_2}3^{\pi_3}\ldots) = 0.$$

Herein selecting the coefficient of $(1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)$ we have

$$\sum_{m}\{(m;\, 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots) - 3\,(m;\, 1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\ldots) + 3\,(m;\, 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots)\}\, a^m$$

$$-2\sum_{m}\{(m;\, 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots) - 2\,(m;\, 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots)\}\,(a^m - a^{m+1})$$

$$+\sum_{m}(m\,;\, 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots)(a^m - 2a^{m+1} + a^{m+2}) = 0,$$

and herein selecting the coefficient of a^m

$$(m\,;\,1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots)-3\,(m\,;\,1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\ldots)+3\,(m\,;\,1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots)$$
$$-2\,\{(m\,;\,1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots)-2\,(m\,;\,1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots)\}$$
$$+2\,\{(m-1\,;\,1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots)-2\,(m-1)\,;\,1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\}$$
$$+\;(m\,;\,1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots)-2\,(m-1\,;\,1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots)+(m-2\,;\,1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots)=0,$$

a relation connecting ten of the coefficients.

In applying this formula it must be noted that

$$(m\,;\,1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

denoting as it does the number of permutations in which exactly m of the letters are not displaced, must be zero

(i) when m is negative,

(ii) when $m > \Sigma s\pi_s$, i.e. greater than the number of letters in the permutation.

Also it is obvious that

$$(\Sigma s\pi_s\,;\,1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)=(\Sigma s\pi_s-1\,;\,1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

since if all but one of the letters are in original positions then all must be so.

Bearing these facts in mind there is no difficulty in verifying the formula in some simple cases.

80. Passing now to the relation

$$tb^{-s}d_s-sb^{-t}d_t-s+t=0,$$

and putting $s=1$, $t=2$, we find

$$2\,!\,D_2=D_1^2-2bD_1-b^2\,;$$

and without difficulty we reach the further results

$$3\,!\,D_3=D_1^3-6bD_1^2+3b^2D_1+4b^3,$$
$$4\,!\,D_4=D_1^4-12bD_1^3+30b^2D_1^2+4b^3D_1-15b^4,$$
$$5\,!\,D_5=D_1^5-20bD_1^4+110b^2D_1^3-140b^3D_1^2-95b^4D_1+56b^5,$$

etc.

which can be written down from those given, by the case $a=0$, by simply introducing the proper power of b in each term.

81. Application of the relation

$$2\,!\,D_2=D_1^2-2bD_1-b^2,$$

gives

$$2\,(m\,;\,1^{\pi_1}2^{\pi_2+1}\ldots)=(m\,;\,1^{\pi_1+2}2^{\pi_2}\ldots)-2\,(m\,;\,1^{\pi_1+1}2^{\pi_2}\ldots)-(m\,;\,1^{\pi_1}2^{\pi_2}\ldots)$$
$$+2\,(m-1\,;\,1^{\pi_1+1}2^{\pi_2}\ldots)+2\,(m-1\,;\,1^{\pi_1}2^{\pi_2}\ldots)$$
$$-(m-2\,;\,1^{\pi_1}2^{\pi_2}\ldots);$$

and thence

$$2(m; 21^s) = (m; 1^{s+2}) - 2(m; 1^{s+1}) - (m; 1^s) + 2(m-1; 1^{s+1}) + 2(m-1; 1^s) - (m-2; 1^s);$$

and, since

$$(m; 1^s) = \binom{s}{m}(0; 1^{s-m}) = \binom{s}{m} P_{s-m},$$

we have

$$2(m; 21^s) = \left\{\binom{s+2}{m} + 2\binom{s+1}{m-1} - \binom{s}{m-2}\right\} P_{s-m+2} - 2\left\{\binom{s+1}{m} - \binom{s}{m-1}\right\} P_{s-m+1} - \binom{s}{m} P_{s-m}.$$

This relation shews us that the number $(m; 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)$ is ultimately expressible as a linear function of the numbers P_{s-m}, P_{s-m+1}, ... and thence also in the form

$$f(s, m) P_{s-m} + \phi(s, m).$$

82. The number $\{0; 1^n\}$ or P_n may also be studied by means of the elementary notions of the Theory of Substitutions.

A substitution is the *operation* by which one permutation of the symbols

$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n$$

is connected with another.

If we start with the particular permutation just written and then remove α_1 from the left and place it to the right of α_n we obtain the new permutation

$$\alpha_2\alpha_3\alpha_4 \ldots \alpha_n\alpha_1,$$

and what we have done is to substitute α_2 for α_1, α_3 for α_2, ... α_n for α_{n-1}, α_1 for α_n. Such an operation of substitution is said to be *circular* and is denoted by

$$(\alpha_1\alpha_2\alpha_3 \ldots \alpha_n),$$

the meaning being that for each letter occurring in the brackets we have to substitute the next succeeding letter and that when we come to the last letter we have to substitute the first letter just as if the row of letters were bent round so as to form a closed chain or circle of letters. Such a substitution may equally well be written so as to commence with any letter—for example in the form

$$(\alpha_3\alpha_4 \ldots \alpha_n\alpha_1\alpha_2).$$

Such a circular substitution which involves the whole of the n letters is said to be of *order* n. It is clear that it displaces each of the letters and from this circumstance arises its effectiveness in the discussion of the problem of displacements. To obtain a circular substitution of order n we may write the letters in brackets in $n!$ different ways, but these do not all express

different substitutions since each substitution is expressible in n different ways by circulation of the symbols in the bracket. Hence we see that there are

$$(n-1)!$$

different circular substitutions of order n.

A substitution may displace each of the letters without being a circular substitution of order n. It may be made up of circular substitutions of lower orders. Thus the substitution which converts the permutation $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ into $\alpha_2\alpha_1\alpha_4\alpha_5\alpha_6\alpha_3$ is composed of the two circular substitutions

$$(\alpha_1\alpha_2)\,(\alpha_3\alpha_4\alpha_5\alpha_6)$$

of orders 2 and 4 respectively.

It is clear that the circular substitution of order one—for example (α_4)—merely indicates that the symbol α_4 is not displaced. If we include circular substitutions of order one, the whole of the permutations of the n symbols may be represented as circular substitutions. If however we are merely concerned with those permutations in which each letter is displaced we are restricted to circular substitutions composed of such substitutions whose order is not less than two.

We may call these non-unitary circular substitutions and we observe that we have such substitutions appertaining to every non-unitary partition of n. It is also an easy observation to say that the number of permutations in which all of the letters of a given permutation are displaced is the same as the number of non-unitary circular substitutions. Let us verify this statement in the case of four letters; we have to consider the non-unitary partitions of 4 which are (4) and (22). As before remarked we have 3! substitutions of order 4, viz. these are $(abcd)$, $(abdc)$, $(acbd)$, $(acdb)$, $(adbc)$, $(adcb)$; also we can form three substitutions each of which involves a pair of substitutions of the order 2, viz.:

$$(ab)\,(cd), \quad (ac)\,(bd), \quad (ad)\,(bc),$$

and

$$3!+3=9=\{0\,;\,1^4\}=P_4,$$

as we know.

Clearly we have to determine the number of circular substitutions which are in correspondence with the non-unitary partition

$$(2^{\pi_2}3^{\pi_3}4^{\pi_4}\ldots)$$

of the number n.

If we distribute the n letters in any manner into $\pi_2+\pi_3+\pi_4+\ldots$ parcels so that π_s parcels contain each s letters where s has the values 2, 3, 4, ... we obtain a definite circular substitution corresponding to any assigned order of the letters in the parcels. Now a parcel which contains s letters may have the letters permuted in $(s-1)!$ different ways so as to give $(s-1)!$ different

circular substitutions of order s; so that if N be the number of ways of distributing n different letters into $\pi_2 + \pi_3 + \pi_4 + \ldots$ parcels so that $\pi_2, \pi_3, \pi_4, \ldots$ parcels contain 2, 3, 4, ... letters respectively, the number of substitutions thence derivable will be

$$N \,.\, (1\,!)^{\pi_2} (2\,!)^{\pi_3} (3\,!)^{\pi_4} \ldots.$$

We have therefore to determine N and from Section I we find that N is the coefficient of (1^n) in the development of

$$\frac{p_2^{\pi_2} p_3^{\pi_3} p_4^{\pi_4} \ldots}{\pi_2\,!\,\pi_3\,!\,\pi_4\,!\ldots},$$

and this is

$$\frac{n\,!}{(2\,!)^{\pi_2}(3\,!)^{\pi_3}(4\,!)^{\pi_4}\ldots\,\pi_2\,!\,\pi_3\,!\,\pi_4\,!\ldots},$$

and

$$N \,.\, (1\,!)^{\pi_2} (2\,!)^{\pi_3} (3\,!)^{\pi_4} \ldots = \frac{n\,!}{2^{\pi_2} \,.\, 3^{\pi_3} \,.\, 4^{\pi_4} \,.\ldots\, \pi_2\,!\,\pi_3\,!\,\pi_4\,!\ldots}.$$

This is the number of circular substitutions which are in correspondence with the partition $(2^{\pi_2} 3^{\pi_3} 4^{\pi_4} \ldots)$ of the number n.

Hence the total number of circular substitutions of n letters or of displacements is

$$\{0\,;\, 1^n\} = \sum_{\pi} \frac{n\,!}{2^{\pi_2} \,.\, 3^{\pi_3} \,.\, 4^{\pi_4} \,.\ldots\, \pi_2\,!\,\pi_3\,!\,\pi_4\,!\ldots},$$

the summation being for every non-unitary partition $(2^{\pi_2} 3^{\pi_3} 4^{\pi_4} \ldots)$ of n.

83. The above expression should be compared with the expression found in Art. 72, viz.:

$$\{0\,;\, 1^n\} = \Sigma \frac{n\,!}{2^{\pi_2} \,.\, 3^{\pi_3} \,.\, 4^{\pi_4} \,.\ldots\, \pi_2\,!\,\pi_3\,!\,\pi_4\,!\ldots} \,.\, \frac{(\pi_2 + \pi_3 + \pi_4 + \ldots)\,!}{(2\,!)^{\pi_2}(3\,!)^{\pi_3}\ldots}.$$

The contrast is remarkable.

The new expression gives, for $n = 6$,

$$\frac{6\,!}{6} + \frac{6\,!}{2\,.\,4} + \frac{6\,!}{9\,.\,2} + \frac{6\,!}{8\,.\,3\,!} = 120 + 90 + 40 + 15 = 265.$$

The old expression gives

$$120 \,.\, \frac{1}{24} + 90 \,.\, \frac{2}{2} + 40 \,.\, \frac{2}{1} + 15 \,.\, 6 = 265,$$

the four terms on the left corresponding in each case to the partitions (6), (42), (3^2), (2^3) respectively.

The identity which appears and is of the form

$$\sum_{\pi} A_{\pi_2 \pi_3 \pi_4 \ldots} = \sum_{\pi} A_{\pi_2 \pi_3 \pi_4 \ldots} B_{\pi_2 \pi_3 \pi_4 \ldots}$$

is very remarkable.

CHAPTER IV

OTHER APPLICATIONS OF THE MASTER THEOREM

84. If between the exponents ξ there is any equality such as $\xi_s = \xi_t$ it is clear that we may interchange X_s and X_t and thus arrive at an alternative form of generating function V_n^{-1}; for the interchange of X_s and X_t means the interchange of the sth and tth rows of the matricular determinant. In general we may permute in any manner those linear functions X which have identical exponents, and when in particular the exponents are all equal we may carry out $n!$ permutations and obtain $n!$ different forms of generating function.

Thus suppose $n = 3$ and we require the number of permutations of the symbols in

$$x_1^\xi x_2^\xi x_3^\xi,$$

which are such that no symbol x_s ($s = 1$, 2, or 3) is in a place originally occupied by x_s, we have first of all the product

$$(x_2 + x_3)^\xi (x_1 + x_3)^\xi (x_1 + x_2)^\xi,$$

giving the matricular determinant

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix},$$

and

$$V_n = 1 - x_1x_2 - x_1x_3 - x_2x_3 - 2x_1x_2x_3;$$

but by permutations of X_1, X_2, X_3 we obtain five other matricular determinants

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix},$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}.$$

which yield the generating functions

$$\frac{1}{1-x_2-x_3-x_1x_2-x_1x_3+x_2x_3+2x_1x_2x_3}, \quad \frac{1}{1-x_1-x_2+x_1x_2-x_1x_3-x_2x_3+2x_1x_2x_3},$$

$$\frac{1}{1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-2x_1x_2x_3}, \quad \frac{1}{1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-2x_1x_2x_3},$$

$$\frac{1}{1-x_1-x_3-x_1x_2+x_1x_3-x_2x_3+2x_1x_2x_3},$$

in each of which the coefficient of $x_1^\xi x_2^\xi x_3^\xi$ is the desired number; two of these five are identical, viz.:

$$\frac{1}{1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-2x_1x_2x_3}.$$

This form arrests the attention, for it may be written as

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)-x_1x_2x_3},$$

or, putting $x_1x_2x_3=u$, as

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)}+\frac{u}{(1-x_1)^2(1-x_2)^2(1-x_3)^2}+\frac{u^2}{(1-x_1)^3(1-x_2)^3(1-x_3)^3}+\ldots;$$

the part of this which is a function of u only is

$$\begin{aligned}
&1+u+u^2+u^3+\ldots\\
&+u\left\{1+\binom{2}{1}^3u+\binom{3}{2}^3u^2+\binom{4}{3}^3u^3+\ldots\right\}\\
&+u^2\left\{1+\binom{3}{1}^3u+\binom{4}{2}^3u^2+\binom{5}{3}^3u^3+\ldots\right\}\\
&+u^3\left\{1+\binom{4}{1}^3u+\binom{5}{2}^3u^2+\binom{6}{3}^3u^3+\ldots\right\}\\
&+\ldots,
\end{aligned}$$

wherein the coefficient of u^ξ is

$$1+\binom{\xi}{1}^3+\binom{\xi}{2}^3+\binom{\xi}{3}^3+\ldots+\binom{\xi}{\xi-1}^3+\binom{\xi}{\xi}^3,$$

viz.: the sum of the cubes of the coefficients in the expansion of

$$(1+x)^\xi,$$

and this number therefore enumerates the permutations which are derangements of the product

$$x_1^\xi x_2^\xi x_3^\xi;$$

and since the two determinants utilized correspond to the products

$$(x_1+x_2)^\xi(x_1+x_2)^\xi(x_2+x_3)^\xi,$$
$$(x_1+x_2)^\xi(x_2+x_3)^\xi(x_1+x_3)^\xi,$$

and the four not utilized to four other products, we can give four other interpretations to the coefficient in the theory of the permutations.

85. This particular theorem is easily generalized because the determinant of order n

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & . & . \\ 0 & 1 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 1 & 0 & . & . \\ 0 & 0 & 0 & 1 & 1 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}$$

derived from the product

$$(x_1 + x_2)^\xi (x_2 + x_3)^\xi (x_3 + x_4)^\xi \dots (x_n + x_1)^\xi,$$

gives the generating function

$$\frac{1}{(1 - x_1)(1 - x_2) \dots (1 - x_n) - x_1 x_2 \dots x_n},$$

and herein the coefficient of $x_1^\xi x_2^\xi \dots x_n^\xi$ is

$$1 + \binom{\xi}{1}^n + \binom{\xi}{2}^n + \dots + \binom{\xi}{\xi - 1}^n + \binom{\xi}{\xi}^n,$$

and this enumerates the permutations of

$$x_1^\xi x_2^\xi \dots x_n^\xi,$$

which are such that in the places originally occupied by x_s there are found either x_s or x_{s+1}; where when $s = n$, x_1 takes the place of x_{n+1}.

Other interpretations arise from the permutations of the linear functions X.

86. Divide the places occupied by the quantities $x_1, x_2, x_3, \dots$ into compartments $A_1, A_2, A_3, \dots$ so that the first ξ_1 places are in A_1, the next ξ_2 in A_2 and so on, and let us find the number of the permutations which possess the property that no quantity with an even suffix is in a compartment with an even suffix and no quantity with an uneven suffix is in a compartment with an uneven suffix. In the X product we have merely to put

$$a_1 = a_3 = a_5 = \dots = 0,$$
$$b_2 = b_4 = b_6 = \dots = 0,$$
$$c_1 = c_3 = c_5 = \dots = 0,$$
$$\dots\dots\dots\dots\dots\dots$$

and the remaining letters $a, b, c, \dots$ equal to unity. The guiding determinant has a chess-board pattern and is of course of order n:

$$\begin{vmatrix} 0 & 1 & 0 & 1 & . & . \\ 1 & 0 & 1 & 0 & . & . \\ 0 & 1 & 0 & 1 & . & . \\ 1 & 0 & 1 & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{vmatrix}.$$

All the co-axial minors of order one are zero; one of order two has either the value zero or negative unity; if the minor be formed from the pth and qth rows and the pth and qth columns the value will be zero if $q-p \equiv 0 \bmod 2$ and will be negative unity in all other cases; co-axial minors of order >2 as well as the whole determinant vanish because in every case two rows are found to be identical.

Hence the generating function is

$$\frac{1}{1 - x_1(x_2 + x_4 + \ldots) - x_2(x_3 + x_5 + \ldots) - x_3(x_4 + x_6 + \ldots) - \ldots - x_{n-1}x_n}.$$

Examples of this nature may be multiplied indefinitely.

We proceed to consider some of a different character.

Self-Conjugate Permutations.

87. A permutation of

$$x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n},$$

supposed separated as above into compartments $A_1, A_2, A_3, \ldots A_n$, is said to be self-conjugate when if x_s be in compartment A_t, x_t is in compartment A_s for all values of s and t. Thus of

$$x_1^2 \mid x_2 \mid x_3,$$

$$x_2 x_3 \mid x_1 \mid x_1$$

is a self-conjugate permutation.

If $\xi_1 = \xi_2 = \ldots = \xi_n = 1$, Rothe (A.D. 1800) shewed that if U_n be the number sought

$$U_n = U_{n-1} + (n-1)\, U_{n-2},$$

with

$$U_2 = 2U_1 = 2.$$

In the more general case before us, first let $n=3$; we then have to find the coefficient of

$$\lambda^0 \mu^0 \nu^0 x_1^{\xi_1} x_2^{\xi_2} x_3^{\xi_3}$$

in

$$(x_1 + \lambda x_2 + \mu x_3)^{\xi_1} \left(\frac{x_1}{\lambda} + x_2 + \nu x_3\right)^{\xi_2} \left(\frac{x_1}{\mu} + \frac{x_2}{\nu} + x_3\right)^{\xi_3},$$

for if λ occurs to the power zero it shews that x_2 has occurred just as often in A_1 as x_1 has in A_2 and so on. The guiding determinant is

$$\begin{vmatrix} 1 & \lambda & \mu \\ \frac{1}{\lambda} & 1 & \nu \\ \frac{1}{\mu} & \frac{1}{\nu} & 1 \end{vmatrix},$$

so that we are led, putting $\frac{\lambda\nu}{\mu} = p$, to the generating function

$$\frac{1}{1 - x_1 - x_2 - x_3 - \left(p + \frac{1}{p} - 2\right) x_1 x_2 x_3};$$

and the reader will have no difficulty in proving that the portion of this function which is free from p is

$$\{(1 - x_1 - x_2 - x_3)(1 - x_1 - x_2 - x_3 + 4x_1 x_2 x_3)\}^{-\frac{1}{2}}.$$

It is useful to observe that the generating function which involves p would arise also from the determinant

$$\begin{vmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ \frac{1}{p} & 1 & 1 \end{vmatrix},$$

and from the X product

$$(x_1 + x_2 + px_3)^{\xi_1} (x_1 + x_2 + x_3)^{\xi_2} \left(\frac{x_1}{p} + x_2 + x_3\right)^{\xi_3};$$

indicating that the conditions of the problem before us are redundant; it suffices to say that x_3 shall occur as often in A_3 as x_3 does in A_1.

This is another instance of the power of the Master Theorem.

88. On proceeding to consider the self-conjugate permutations of

$$x_1^{\xi_1} x_2^{\xi_2} x_3^{\xi_3} x_4^{\xi_4},$$

we find similarly that the true X product involves three and not six auxiliary coefficients and has the form

$$(x_1 + x_2 + px_3 + qx_4)^{\xi_1} (x_1 + x_2 + x_3 + rx_4)^{\xi_2} \times \left(\frac{x_1}{p} + x_2 + x_3 + x_4\right)^{\xi_3} \left(\frac{x_1}{q} + \frac{x_2}{r} + x_3 + x_4\right)^{\xi_4}.$$

The generating function may now be written down, but the isolation from it of that portion which is free from p, q and r is not an easy matter.

Generally for the order n it would be necessary to isolate from the generating function that portion which is free from

$$\binom{n-1}{2}$$

auxiliary coefficients.

89. In Art. 85 we have seen a connexion between the sum of the nth powers of the binomial coefficients, which arise from any binomial expansion,

and the Theory of Permutations. The Master Theorem is well adapted to display the properties of these numbers. Thus since

$$1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \dots$$

is the coefficient of $x^p y^q$ in $(x+y)^p (x+y)^q$ we are led through the guiding determinant

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

to the generating function $\frac{1}{1-x-y}$, in which the coefficient of $x^p y^q$ is

$$\binom{p+q}{p},$$

that is

$$1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \dots = \binom{p+q}{p}.$$

The coefficient of $x^p y^p$ in $(ax+by)^p (bx+ay)^p$ is

$$a^{2p} + \binom{p}{1}^2 a^{2p-2} b^2 + \binom{p}{2}^2 a^{2p-4} b^4 + \dots,$$

while the generating function derived from the matrix

$$\begin{vmatrix} a & b \\ b & a \end{vmatrix}$$

is either $\frac{1}{1-a(x+y)-(b^2-a^2)xy}$ or $\frac{1}{1-b(x+y)-(a^2-b^2)xy}$,

and in the first of these the coefficient of $x^p y^p$ is

$$(b^2-a^2)^p + \binom{p+1}{2}\binom{2}{1} a^2 (b^2-a^2)^{p-1} + \binom{p+2}{4}\binom{4}{2} a^4 (b^2-a^2)^{p-2} + \dots;$$

and now putting $a^2 = \alpha$, $b^2 = \beta$ we have the identity

$$\alpha^p + \binom{p}{1}^2 \alpha^{p-1}\beta + \binom{p}{2}^2 \alpha^{p-2}\beta^2 + \dots$$

$$= (\beta-\alpha)^p + \binom{p+1}{2}\binom{2}{1}\alpha(\beta-\alpha)^{p-1} + \binom{p+2}{4}\binom{4}{2}\alpha^2(\beta-\alpha)^{p-2} + \dots + \binom{2p}{p}\alpha^p$$

$$= (\alpha-\beta)^p + \binom{p+1}{2}\binom{2}{1}\beta(\alpha-\beta)^{p-1} + \binom{p+2}{4}\binom{4}{2}\beta^2(\alpha-\beta)^{p-2} + \dots + \binom{2p}{p}\beta^p$$

since the left-hand side shews that α and β may be interchanged.

This curious result gives an expression for the same linear function of the squares of the binomial coefficients as the binomial theorem does in respect of the first powers of the coefficients.

90. The sum

$$1-\binom{p}{1}^3+\binom{p}{2}^3-\binom{p}{3}^3+\ldots,$$

which was considered by Dixon*, is the coefficient of $x^p y^p z^p$ in

$$(y-z)^p(z-x)^p(x-y)^p;$$

the guiding matrix is

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix},$$

and gives us the generating function

$$\frac{1}{\sqrt{1+yz+zx+xy}},$$

a striking result which gives immediately much information concerning the sum under consideration. The term $x^p y^p z^p$ can only occur in the term

$$(-)^{\frac{3p}{2}}(yz+zx+xy)^{\frac{3p}{2}},$$

and hence the sum vanishes unless p be even and is positive or negative according as p is evenly or unevenly even. To evaluate the coefficient put $yz=u^2$, $zx=v^2$, $xy=w^2$ so that $xyz=uvw$, and then the coefficient of $u^p v^p w^p$ in

$$\frac{1}{\sqrt{1+u^2+v^2+w^2}}$$

is

$$(-)^{\frac{3p}{2}}\frac{\left(\frac{3p}{2}\right)!}{\left(\left(\frac{p}{2}\right)!\right)^3},$$

yielding the result

$$1-\binom{2p}{1}^3+\binom{2p}{2}^3-\ldots=(-)^p\frac{(3p)!}{(p!)^3}.$$

Another form of generating function, obtained by permuting the rows of the guiding matrix, is

$$\frac{1}{\sqrt{1-x-y-z+yz+zx+xy}}.$$

91. We may consider the coefficient of $x^p y^p z^p$ in

$$(ay+bz)^p(bx+az)^p(ax+by)^p,$$

which is

$$a^{3p}+\binom{p}{1}^3 a^{3p-3}b^3+\binom{p}{2}^3 a^{3p-6}b^6+\ldots,$$

* *Messenger of Mathematics*, Vol. xx. p. 79.

and we are led through the guiding matrix

$$\begin{vmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{vmatrix},$$

and the generating function

$$\frac{1}{1 - ab(yz + zx + xy) - (a^3 + b^3)xyz},$$

to the identity (putting $a^3 = \alpha$, $b^3 = \beta$)

$$\alpha^p + \binom{p}{1}^3 \alpha^{p-1}\beta + \binom{p}{2}^3 \alpha^{p-2}\beta^2 + \ldots$$

$$= (\alpha + \beta)^p + \frac{(p+1)!}{(1!)^3 (p-2)!}\alpha\beta(\alpha + \beta)^{p-2} + \frac{(p+2)!}{(2!)^3 (p-4)!}\alpha^2\beta^2(\alpha + \beta)^{p-4} + \ldots.$$

Another interesting form of the generating function for the same sum is

$$\frac{1}{1 - b(x + y + z) + b^2(yz + zx + xy) - (a^3 + b^3)xyz}.$$

92. It has been shewn above that for special values of the exponents ξ the generating function V_n^{-1} may have more than one form. The general form of V_n is such that the equation $V_n = 0$ gives each quantity x_s as a homographic function of the remaining $n - 1$ quantities, and it is interesting to inquire whether assuming the coefficients of V_n arbitrarily it is possible to pass back to an X product. The coefficients of V_n must be the co-axial minors of some determinant, and it was shewn* that the coefficients assumed arbitrarily must satisfy $2^n - n^2 + n - 2$ conditions and that assuming the satisfaction of these conditions an X product can be constructed which involves $n - 1$ undetermined quantities. From any particular X product we can thus pass to an equivalent X product which involves $n - 1$ undetermined quantities. Assuming these quantities at pleasure we obtain a number of different X products each of which may have its own meaning in the theory of permutations, and thus the number of arithmetical correspondences obtainable is subject to no finite limit.

The Master Theorem is also valid when the quantities $a_1, a_2, a_3, \ldots$ $b_1, b_2, b_3, \ldots$ are no longer real quantities but umbral symbols such that

$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^{\xi_1}$$

is a symbolic form of the general homogeneous quantic of degree ξ_1 in n variables. The theorem then becomes one which considers the coefficient of $x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$ in the product of n quantics of degrees $\xi_1, \xi_2, \ldots \xi_n$ respectively in n variables. The theorem in fact expresses such terms for all quantics of

* *Phil. Trans. Roy. Soc. of London*, 1894, A, "On a certain class of generating functions in the Theory of Numbers."

all degrees. In the umbral notation we say that the coefficient of $x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$ in the product

$$a_x^{\xi_1} b_x^{\xi_2} \dots n_x^{\xi_n}$$

is equal to the coefficient of the same term in the expansion of the fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|}.$$

Ex. gr. Consider $a_x^{\xi_1} b_x^{\xi_2}$ where $a_x = a_1x_1 + a_2x_2$, $b_x = b_1x_1 + b_2x_2$; herein the coefficient of $x_1^{\xi_1} x_2^{\xi_2}$ is

$$C_{\xi_1,\,\xi_2} = a_1^{\xi_1} b_2^{\xi_2} + \binom{\xi_1}{1}\binom{\xi_2}{1} a_1^{\xi_1-1} a_2 b_1 b_2^{\xi_2-1} + \binom{\xi_1}{2}\binom{\xi_2}{2} a_1^{\xi_1-2} a_2^2 b_1^2 b_2^{\xi_2-2} + \dots$$

and

$$\Sigma\Sigma C_{\xi_1,\,\xi_2} x_1^{\xi_1} x_2^{\xi_2} = \frac{1}{(1-a_1x_1)(1-b_2x_2)} + \frac{a_2b_1x_1x_2}{(1-a_1x_1)^2(1-b_2x_2)^2} + \frac{a_2^2b_1^2x_1^2x_2^2}{(1-a_1x_1)^3(1-b_2x_2)^3} + \dots$$

$$= \frac{1}{1-a_1x_1-b_2x_2+(ab)x_1x_2},$$

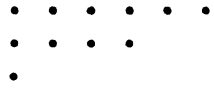
where $(ab) = a_1b_2 - a_2b_1$.

CHAPTER V

LATTICE PERMUTATIONS

93. Consider an assemblage of letters $\alpha^p\beta^q\gamma^r\ldots$ in which the numbers $p, q, r, \ldots$ are in descending order of magnitude. This particular permutation of the assemblage can be denoted by a regular graph consisting of rows of nodes. The successive rows will have $p, q, r, \ldots$ nodes respectively and the graph is the same as serves to denote the partition $(pqr\ldots)$ of the number $p+q+r+\ldots$.

Such a graph may be

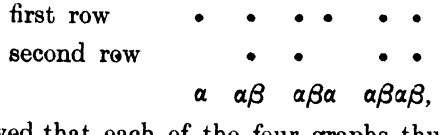


for $p=6, q=4, r=1$.

The successive rows correspond to the letters $\alpha, \beta, \gamma, \ldots$ respectively.

If we take *any* permutation of $\alpha^p\beta^q\gamma^r\ldots$ we shall arrive finally at the same graph by proceeding from left to right of the permutation and placing a node in the first row, or in the second, or in the third according as we reach a letter α or β or γ, etc.

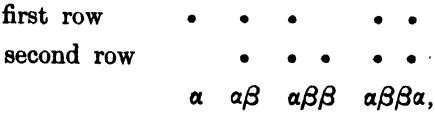
Thus if we take the permutation $\alpha\beta\alpha\beta$ of the assemblage we obtain successively in this manner



and it will be observed that each of the four graphs thus reached is regular and is in fact the graph of a partition of a number.

Since the permutation possesses this property of yielding a succession of regular graphs it is termed a "lattice permutation."

On the other hand if we treat the permutation $\alpha\beta\beta\alpha$ in the same way we reach the graphs



and since the third of these graphs is irregular we have not had before us a "lattice permutation."

In general for a permutation of $\alpha^p\beta^q\gamma^r\ldots$ if the successive graphs are all regular it is a "lattice permutation."

In other words if a dividing line be drawn between *any two* letters of the permutation and the assemblage of letters to the left of the line is found to be $\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\ldots$, where $p_1 \geqslant q_1 \geqslant r_1 \geqslant \ldots$, the permutation is said to be a lattice permutation.

Ex. gr. of the assemblage $\alpha^2\beta^2$, there are only two lattice permutations, viz. $\alpha\alpha\beta\beta$ and $\alpha\beta\alpha\beta$.

These special permutations are of much use in the Theory of Partitions taken up in Volume II of this work, but they also have a special interest of their own. For instance in the Theory of Probabilities:

Suppose that there are $p+q+r+\ldots$ electors at an election and that $p, q, r, \ldots$ electors vote for candidates $\alpha, \beta, \gamma, \ldots$ by handing in tickets marked $\alpha, \beta, \gamma, \ldots$ respectively. The electors may present themselves in any order and if such gives a lattice permutation it is clear that if the flow of electors be stopped at any time and the votes be counted, the count will give a result which is not inconsistent with the final result. The enumeration of the lattice permutations leads therefore to the probability of such non-inconsistency obtaining.

94. We will first set forth certain properties possessed by the permutations.

Every permutation necessarily commences with α.

Consider any lattice permutation, say $\alpha\beta\alpha\gamma\beta$, of the assemblage $\alpha^2\beta^2\gamma$.

Write underneath the letters the first five numbers in descending order, viz.:

$$\begin{array}{ccccc} \alpha & \beta & \alpha & \gamma & \beta \\ 5 & 4 & 3 & 2 & 1 \end{array}$$

Starting from the left, place each number in the first, second or third row of a graph according as it stands beneath an α, a β or a γ. Thus:

$$\begin{array}{cc} 5 & 3 \\ 4 & 1 \\ 2 & \end{array}$$

This graph, since it has been formed from a lattice permutation, has the property that the numbers are in descending order of magnitude in each row read from left to right and in each column read from top to bottom. There is a one-to-one correspondence between the two-dimensional array of numbers,

formed upon the graph of the partition (221) of the number 5, which possess this property and the lattice permutations of the assemblage $\alpha^2\beta^2\gamma$. For we can pass uniquely from any such array to the corresponding permutation. Thus from the array

$$\begin{array}{ll} 5 & 4 \\ 3 & 1 \\ 2 & \end{array}$$

we consider the numbers in descending order of magnitude and write down from left to right an α, a β or a γ according as the number considered is in the first, second or third row. We thus reach the permutation

$$\alpha\alpha\beta\gamma\beta.$$

In general we see that there exists a one-to-one correspondence between the lattice permutations of the assemblage $\alpha^p\beta^q\gamma^r\ldots$ and the two-dimensional arrays of the first $p+q+r+\ldots$ numbers at the nodes of the graph of the partition $(pqr\ldots)$ of the number $p+q+r+\ldots$, which are such that there is a descending order of magnitude alike in each row and in each column of the graph.

95. We can now transform these arrays so as to establish and exhibit an important property of lattice permutations. For suppose that we take the array

$$\begin{array}{ll} 5 & 3 \\ 4 & 1 \\ 2 & \end{array}$$

and write the rows as columns thus:

$$\begin{array}{lll} 5 & 4 & 2 \\ 3 & 1 & \end{array}$$

we obtain an array at the nodes of the graph of the partition (32) which is the partition conjugate to the partition (221) appertaining to the former graph. The transformed array leads to the lattice permutation

$$\alpha\alpha\beta\alpha\beta \text{ of the assemblage } \alpha^3\beta^2,$$

and in consequence there must be a one-to-one correspondence between the lattice permutations of the two assemblages

$$\alpha^2\beta^2\gamma, \quad \alpha^3\beta^2.$$

The lattice permutations of these two assemblages are therefore equinumerous. In general we may say that there is a one-to-one correspondence between the lattice permutations of the two assemblages

$$\alpha^p\beta^q\gamma^r\ldots, \quad \alpha^{p'}\beta^{q'}\gamma^{r'}\ldots,$$

if $(pqr\ldots)$, $(p'q'r'\ldots)$ are conjugate partitions.

96. There is another correspondence which it is useful to note.

If we take the graph

$$\begin{matrix} 5 & 3 \\ 4 & 1 \\ 2 & \end{matrix}$$

we observe descending orders of magnitude five times, viz. in the three rows and in the two columns. The descending orders are 53, 41, 2, 542, 31.

We can arrange these five numbers in a row so that the five descending orders are in evidence. To do this we proceed from the lattice permutations

$\alpha\alpha\beta\beta\gamma$	$\alpha\alpha\beta\gamma\beta$	$\alpha\beta\alpha\gamma\beta$	$\alpha\beta\alpha\beta\gamma$	$\alpha\beta\gamma\alpha\beta$
5 3 4 1 2	5 3 4 2 1	5 4 3 2 1	5 4 3 1 2	5 4 2 3 1

and write 5, 3 in order underneath the letters α; 4, 1 in order underneath the letters β and 2 underneath the γ.

We have thus a line-arrangement exhibiting the five descending orders in correspondence with each lattice permutation, and the descending orders have been derived from one (any one) of the associated two-dimensional arrays.

In general we determine a one-to-one correspondence between the lattice permutations of the assemblage $\alpha^p\beta^q\gamma^r\ldots$ and the permutations of the first $p+q+r\ldots$ natural numbers which exhibit the descending orders which are derived from the rows and columns of *any one* of the two-dimensional arrays associated with the lattice permutations.

97. We will now be concerned with the enumeration of the lattice permutations. First let us take the assemblage $\alpha^p\beta^q$ and denote by $(pq;)$ the number of the lattice permutations. If $q=p$, the last letter of a lattice permutation must be β, and if we delete this β we shall get every lattice permutation of the assemblage $\alpha^p\beta^{p-1}$. Hence

$$(pp;) = (p, p-1;).$$

If $p>q$, the last letter may be α or β; if this last letter be deleted we obtain all the lattice permutations of $\alpha^{p-1}\beta^q$ and of $\alpha^p\beta^{q-1}$; hence

$$(pq;) = (p-1, q;) + (p, q-1;).$$

Of this difference equation $\dfrac{(p+q)!}{(p+s)!\,(q-s)!}$ is a solution, and the particular solution required is

$$(pq;) = \frac{(p+q)!}{p!\,q!} - \frac{(p+q)!}{(p+1)!\,(q-1)!} = \frac{(p+q)!}{(p+1)!\,q!}(p-q+1).$$

Also

$$(pp;) = \frac{(2p)!}{(p+1)!\,p!}.$$

It follows from the expression obtained for $(pq;)$ that if two candidates A, B at an election can command p, q voters respectively $(p > q)$, the probability that at no time during the balloting A will have fewer votes than B is

$$\frac{p-q+1}{p+1}.$$

It is seen without difficulty that $(pq;)$ is equal to the coefficient of $x^p y^q$ in the expansion of the function

$$\frac{1-\frac{y}{x}}{1-x-y},$$

for $(pq;)$ has been shewn to be the difference of $\frac{(p+q)!}{p!\,q!}$ and $\frac{(p+q)!}{(p+1)!(q-1)!}$.

This is a redundant generating function because it contains many terms which are not applicable to the question under examination.

98. The exact generating function is obtained in the following manner.

Consider

$$\sum_p (pp;)(xy)^p = 1 + xy + 2x^2y^2 + 5x^3y^3 + 14x^4y^4 + \ldots = u_{xy}.$$

The general term in u_{xy} is

$$\frac{(2p)!}{(p+1)!\,p!}x^p y^p,$$

and since

$$\sqrt{(1-4xy)} = 1 - 2xy - 2x^2y^2 - \ldots - \frac{2(2p)!}{(p+1)!\,p!}(xy)^{p+1} - \ldots,$$

we find that $$2xy\,u_{xy} = 1 - \sqrt{(1-4xy)}.$$

Thus $$u_{xy} = \frac{1}{2xy}\{1 - \sqrt{(1-4xy)}\},$$

and thence a relation that will be useful

$$u_{xy} = \frac{1}{1 - xy\,u_{xy}}.$$

99. There is another way of establishing this result which is valuable for the purpose in hand. If we examine the various lattice permutations of the assemblage $\alpha^p\beta^p$ we find that they are of two kinds, viz. prime and composite. Composite arrangements are those which are decomposable into shorter lattice permutations appertaining to assemblages $\alpha^q\beta^q$, where $q < p$. The prime lattice permutations are those which are not so decomposable.

Thus the assemblage $\alpha^2\beta^2$ has the two lattice permutations

$$\alpha\alpha\beta\beta, \quad \alpha\beta\,|\,\alpha\beta;$$

the first is prime; the second is composite, because it is decomposable into two shorter lattice permutations, each of which is $\alpha\beta$.

Similarly the assemblage $\alpha^3\beta^3$ has two prime arrangements, viz. $\alpha\alpha\alpha\beta\beta\beta$, $\alpha\alpha\beta\alpha\beta\beta$, and three which are composite, viz.

$$\alpha\alpha\beta\beta \mid \alpha\beta, \quad \alpha\beta \mid \alpha\alpha\beta\beta, \quad \alpha\beta \mid \alpha\beta \mid \alpha\beta.$$

The theory of prime lattice permutations becomes very simple directly the observation is made, that from *every* lattice permutation of the assemblage $\alpha^{p-1}\beta^{p-1}$ a prime lattice permutation of the assemblage $\alpha^p\beta^p$ is derivable by simply prefixing the letter α and affixing the letter β, and that in this way the whole of the lattice permutations are obtained.

Thus we have

Lattice Permutations	Prime Lattice Permutations
1	$\alpha\beta$
$\alpha\beta$	$\alpha\alpha\beta\beta$
$\alpha\alpha\beta\beta$	$\alpha\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\beta$
$\alpha\alpha\alpha\beta\beta\beta$	$\alpha\alpha\alpha\alpha\beta\beta\beta\beta$
$\alpha\alpha\beta\alpha\beta\beta$	$\alpha\alpha\alpha\beta\alpha\beta\beta\beta$
$\alpha\alpha\beta\beta\alpha\beta$	$\alpha\alpha\alpha\beta\beta\alpha\beta\beta$
$\alpha\beta\alpha\alpha\beta\beta$	$\alpha\alpha\beta\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\alpha\beta\beta$
⋮	⋮

Lattice permutations are either prime or decomposable into primes.

This fact will lead us to the generating functions. For it is clear that the enumerating generating function of the prime lattice permutations is

$$xyu_{xy}.$$

Moreover these permutations may be combined *in all ways* to produce lattice permutations. Hence the relation

$$u_{xy} = \frac{1}{1 - xyu_{xy}}.$$

The relation shews simply the derivation of all lattice permutations from prime lattice permutations.

The same principle will now be applied to determine the enumerating generating function of lattice permutations of the assemblage $\alpha^p\beta^q$ where $p \geqslant q$. When $p > q$ there is no prime lattice permutation except in the particular case $p = 1$, $q = 0$, when the form is

$$\alpha.$$

Ex. gr. for the assemblage $\alpha^3\beta^2$ the arrangements are all composite, viz.

$$\alpha \mid \alpha\alpha\beta\beta, \quad \alpha \mid \alpha\beta \mid \alpha\beta, \quad \alpha\alpha\beta\beta \mid \alpha,$$
$$\alpha\beta \mid \alpha \mid \alpha\beta, \quad \alpha\beta \mid \alpha\beta \mid \alpha.$$

The generating function for the prime lattice permutations is now

$$x + xy\,u_{xy},$$

and if we write $\sum_{p,q} (pq;)\,x^p y^q = v_{x,y}$, the fact that the arrangements are either prime or composed of primes leads at once to the relation

$$v_{x,y} = \frac{1}{1 - x - xy\,u_{xy}};$$

and this, using the relation above satisfied by u_{xy}, may be written

$$v_{x,y} = \frac{u_{xy}}{1 - x u_{xy}},$$

where

$$u_{xy} = \frac{1}{2xy}\{1 - \sqrt{(1 - 4xy)}\}.$$

This is the exact form of generating function to which we have been led by the notion of prime lattice permutations.

If s be a given integer we deduce from the above result that

$$\sum_q (q + s,\ q;)\,(xy)^q = u_{xy}^{s+1},$$

a relation which leads to the expression of $(q + s,\ q;)$ in terms of $(11;)$, $(22;)$, $(33;)$, etc.

Thus

$$(s + 1,\ 1;) = \binom{s+1}{1}(11;),$$

$$(s + 2,\ 2;) = \binom{s+1}{1}(22;) + \binom{s+1}{2}(11;)^2,$$

etc.

100. Reverting to the difference equation, we find

$$\begin{aligned}(pp;) &= (p,\ p-1;)\\ &= (p,\ p-2;) + (p-1,\ p-1;)\\ &= (p,\ p-3;) + 2\,(p-1,\ p-2;)\\ &= (p,\ p-4;) + 3\,(p-1,\ p-3;) + 2\,(p-2,\ p-2;),\end{aligned}$$

and we notice that the last result may be written

$$(pp;) = (40;)\,(p,\ p-4;) + (31;)\,(p-1,\ p-3) + (22;)\,(p-2,\ p-2;).$$

In view of this it is natural to suspect the law

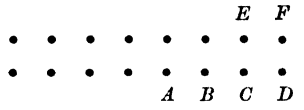
$$(pp;) = \sum (st;)\,(p-t,\ p-s;),$$

where

$$s + t = \text{constant}.$$

This may be established by utilizing the correspondence between lattice permutations and the arrangements of different numbers at the nodes of the graph of the partition (pp) of the number $2p$.

For consider the graph



The four lowest numbers may be placed

(i) at the points A, B, C, D,

(ii) „ B, C, D, F,

(iii) „ C, D, E, F.

Taking the case (ii) the numbers may be

$$\begin{matrix} 2 & 3 & 4 \\ 431, & 421, & 321. \end{matrix}$$

Subtracting each of these numbers from the number 5, the arrangements become

$$\begin{matrix} 3 & 2 & 1 \\ 124, & 134, & 234, \end{matrix}$$

and it is clear that they are enumerated by the number $(31;)$.

Similarly the arrangements at the nodes A, B, C, D and C, D, E, F are enumerated by the numbers $(40;)$, $(22;)$ respectively, and we are led to the relation

$$(pp;) = (40;)(p, p-4;) + (31;)(p-1, p-3;) + (22;)(p-2, p-2;).$$

Similarly it is shewn that

$$(pp;) = \Sigma (st;)(p-t, p-s;), \text{ where } s+t = \text{constant}.$$

101. Putting herein $s+t=p$, we find

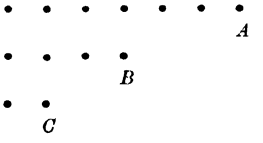
$(pp;) = (p, 0;)^2 + (p-1, 1;)^2 + (p-2, 2;)^2 + \ldots$ to $\frac{1}{2}(p+1)$ or $\frac{1}{2}(p+2)$ terms, according as p is uneven or even.

Hence the identity

$$\frac{(2p)!}{(p+1)!\,p!} = 1^2 + (p-1)^2 + \{\tfrac{1}{2}p(p-3)\}^2 + \left\{\frac{1}{3!}p(p-1)(p-5)\right\}^2 + \ldots$$

to $\frac{1}{2}(p+1)$ or $\frac{1}{2}(p+2)$ terms.

102. Taking up the lattice permutations of the assemblage $\alpha^p\beta^q\gamma^r$ and considering the associated graph



with different numbers placed at the nodes in such wise that descending order of magnitude is visible alike in each row and in each column, we find that we may detach a node without destroying the regularity of the graph in three ways. The nodes are marked A, B, C, ...

But if q were equal to p we could not detach A; if r were equal to q we could not detach B; if q and r were both equal to p we could not detach B or A.

Hence the difference equations

$$(pqr;) = (p-1, q, r;) + (p, q-1, r;) + (p, q, r-1;),$$
$$(ppq;) = (p, p-1, q;) + (p, p, q-1;),$$
$$(pqq;) = (p-1, q, q;) + (p, q, q-1;),$$
$$(ppp;) = (p, p, p-1;).$$

A solution of the first of these equations is

$$\frac{(p+q+r)!}{(p+p')!(q+q')!(r+r')!},$$

where p', q', r' are positive or negative (including zero) integers, such that

$$p' + q' + r' = 0.$$

This may be at once established by applying to it the difference equation.

The particular solution which corresponds to the present problem is

$$\frac{(p+q+r)!}{p!q!r!} - \frac{(p+q+r)!}{(p+1)!(q-1)!r!} - \frac{(p+q+r)!}{p!(q+1)!(r-1)!}$$
$$+ \frac{(p+q+r)!}{(p+2)!(q-1)!(r-1)!} + \frac{(p+q+r)!}{(p+1)!(q+1)!(r-2)!} - \frac{(p+q+r)!}{(p+2)!q!(r-2)!}.$$

This may be simplified so as to exhibit the result

$$(pqr;) = \frac{(p+q+r)!}{p!q!r!}\left(1 - \frac{q}{p+1}\right)\left(1 - \frac{r}{q+1}\right)\left(1 - \frac{r}{p+2}\right),$$

which is true whatever equalities subsist between p, q, and r.

The unsimplified form shews that $(pqr;)$ is equal to the coefficient of $x^p y^q z^r$ in the expansion of the redundant generating function

$$\frac{\left(1 - \frac{y}{x}\right)\left(1 - \frac{z}{y}\right)\left(1 - \frac{z}{x}\right)}{1 - x - y - z},$$

for the expanded numerator is

$$1 - \frac{y}{x} - \frac{z}{y} + \frac{yz}{x^2} + \frac{z^2}{xy} - \frac{z^2}{x^2},$$

and its six terms yield respectively the six terms of the unsimplified expression for $(pqr;)$.

The reader should also make note of another form of $(pqr;)$, viz.

$$\frac{(p+q+r)!}{(p+2)!\,(q+1)!\,r!}(p-q+1)(q-r+1)(p-r+2).$$

The Theory of the Prime Lattice Permutations of the assemblage $\alpha^p\beta^q\gamma^r$ awaits investigation. It seems to present a certain amount of difficulty. Until this has been surmounted we cannot pass to the real generating function from the redundant form above given.

103. We now pass to the general case, viz. the number

$$(p_1p_2p_3\dots p_n;).$$

The difference equation to be satisfied is

$$(p_1p_2p_3\dots p_n;)$$
$$=(p_1-1,\,p_2p_3\dots p_n;)+(p_1,\,p_2-1,\,p_3\dots p_n;)+\dots+(p_1p_2p_3\dots p_n-1;).$$

We are led to the redundant generating function

$$\frac{\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}\left(1-\frac{x_s}{x_t}\right)}{1-(x_1+x_2+x_3+\dots+x_n)},$$

and to the two forms of result

$$(p_1p_2p_3\dots p_n;)=\frac{(p_1+p_2+p_3+\dots+p_n)!}{p_1!\,p_2!\,p_3!\dots p_n!}\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}\left(1-\frac{p_s}{p_t+s-t}\right),$$

$$(p_1p_2p_3\dots p_n;)$$
$$=\frac{(p_1+p_2+p_3+\dots+p_n)!}{(p_1+n-1)!\,(p_2+n-2)!\,(p_3+n-3)!\dots p_n!}\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}(p_t-p_s+s-t).$$

The question in probability that is here solved may be stated as follows:

If n candidates at an election have

$$p_1,\,p_2,\,p_3,\,\dots\,p_n$$

voters in their favour respectively, where

$$p_1\geqslant p_2\geqslant p_3\dots\geqslant p_n,$$

and if at any instant $P_1,\,P_2,\,P_3,\,\dots\,P_n$ voters have recorded their votes in favour of the several candidates respectively, the probability that

$$P_1\geqslant P_2\geqslant P_3\dots\geqslant P_n$$

always is

$$\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}\left(1-\frac{p_s}{p_t+s-t}\right).$$

It will be remarked also that the number $(p_1p_2p_3\dots p_n;)$ enumerates the arrangements of $p_1+p_2+p_3+\dots+p_n$ different numbers at the nodes of a

lattice which has $p_1, p_2, p_3, \ldots p_n$ nodes (respectively) in the successive rows where the arrangements are such that there is in evidence a descending order of magnitude alike in each row and in each column.

The circumstance that this enumeration is not altered by interchanging all the rows and all the columns establishes the fact that if

$$(p_1 p_2 p_3 \ldots), \ (q_1 q_2 q_3 \ldots)$$

be conjugate partitions

$$(p_1 p_2 p_3 \ldots;) = (q_1 q_2 q_3 \ldots;).$$

In view of the results above set forth this involves a remarkable property of numbers.

CHAPTER VI

THE INDICES OF PERMUTATIONS

104. The assemblage of objects is taken to be

$$\alpha^i\beta^j\gamma^k\ldots \quad \text{or} \quad \alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\ldots$$

indifferently.

A contact in a permutation of these letters which is (say) $\alpha_s\alpha_t$ is called a major, equal, or minor contact according as $s >, =,$ or $< t$.

The first definitions have reference to the major contacts in a permutation.

The Greater Index of a Permutation. Of $\alpha^1\beta^2\gamma^3$ let any permutation be (say)

$$\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma.$$

Whenever a letter is the left-hand member of a major contact we write under it a number which shews how many places it is from the left of the permutation. Thus if the sth letter stands immediately before a letter prior to it in alphabetical order we place under it the number s. In this we obtain

$$\begin{array}{l}\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma. \\ 1 \quad\quad 67\end{array}$$

We add up the numbers so placed and obtain $1+6+7=14$, a number which is called the "greater index" of the permutation.

Similarly if in any permutation of any assemblage of letters or of ordered objects the p_1th, p_2th, p_3th ... letters are the left-hand members of major contacts, we have the definition

$$\text{Greater Index} = p_1 + p_2 + p_3 + \ldots = p.$$

If, in so forming the number p, we have to add m numbers, or in other words if the permutation possesses m major contacts, the permutation is said to be of Class m *quâ* major contacts.

The Equal Index of a Permutation. If, in any permutation, the q_1th, q_2th, q_3th ... letters immediately precede letters identical with themselves so that they are the left-hand members of equal contacts, we make the definition

$$\text{Equal Index} = q_1 + q_2 + q_3 + \ldots = q.$$

Thus from $\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma$
$23\ 5$

we obtain the equal index $2 + 3 + 5 = 10$.

If, in so forming the number q, we have to add m numbers or if, in other words, the permutation possesses m equal contacts, we speak of the permutation as being of Class m *quâ* equal contacts.

The Lesser Index of a Permutation. If, in any permutation, the r_1th, r_2th, r_3th ... letters immediately precede letters which are later in alphabetical order we make the definition

$$\text{Lesser Index} = r_1 + r_2 + r_3 + \ldots = r.$$

If, in so forming the number r, we have to add m numbers, or if, in other words, the permutation possesses m minor contacts, we speak of the permutation as being of Class m *quâ* minor contacts.

The Greater and Equal Index. This refers to letters which immediately precede letters which are *not later* in alphabetical order. It is equal to the sum of the greater and equal indices or to $p + q$.

In the same manner we may have to consider other combinations of the indices p, q, r.

105. If p be the greater index of a permutation and we denote by Σx^p a summation in respect of the whole of the permutations of a given assemblage, it will be established in Volume II in connexion with the Theory of Partitions, wherein the notion of the index plays a very important part, that

$$\Sigma x^p = \frac{(1-x)(1-x^2)\ldots(1-x^{i+j+k+\ldots})}{(1-x)(1-x^2)\ldots(1-x^i)(1-x)(1-x^2)\ldots(1-x^j)(1-x)(1-x^2)\ldots(1-x^k)\ldots}.$$

This beautiful theorem will be taken for granted here and various interesting consequences will be deduced.

The late Professor Cayley, who did much work in regard to expressions of this form, was in the habit of writing

$$(1 - x^s) = (\mathbf{s})$$

for the sake of brevity and of general convenience. We do the same here so that the theorem under consideration is written

$$\Sigma x^p = \frac{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i}+\mathbf{j}+\mathbf{k}+\ldots)}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i})(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j})(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})\ldots}.$$

106. *Important property of* Σx^p. From the symmetry of the expression just written we gather the information that Σx^p is unaltered by any permutation of the numbers i, j, k, ... Ex. gr. the sum has the same expression

for each of the assemblages $\alpha^3\beta^2\gamma$, $\alpha^3\beta\gamma^2$, $\alpha^2\beta^3\gamma$, $\alpha^2\beta\gamma^3$, $\alpha\beta^3\gamma^2$, $\alpha\beta^2\gamma^3$. For each of these assemblages the same number of permutations have the greater index equal to a given integer. Thus, developing the algebraic fraction for these assemblages, we find a term $+6x^3$, indicating that 6 permutations have a greater index equal to 3. For $\alpha\alpha\alpha\beta\beta\gamma$ the six permutations are

$$\alpha\alpha\beta\alpha\beta\gamma, \quad \alpha\beta\beta\alpha\alpha\gamma, \quad \gamma\beta\alpha\alpha\alpha\beta,$$
$$\alpha\alpha\gamma\alpha\beta\beta, \quad \alpha\beta\gamma\alpha\alpha\beta, \quad \beta\beta\gamma\alpha\alpha\alpha.$$

For $\alpha\alpha\beta\gamma\gamma\gamma$ the six are

$$\alpha\alpha\gamma\beta\gamma\gamma, \quad \gamma\beta\alpha\alpha\gamma\gamma, \quad \alpha\beta\gamma\alpha\gamma\gamma,$$
$$\alpha\gamma\gamma\alpha\beta\gamma, \quad \beta\gamma\gamma\alpha\alpha\gamma, \quad \gamma\gamma\gamma\alpha\alpha\beta.$$

107. *Proof that for any assemblage* $\Sigma x^r = \Sigma x^p$. This follows from the last article. For consider any permutation of any assemblage, viz.

$$\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma,$$

and transform it by the substitution

$$\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix}$$

(equivalent to changing the assemblage $\alpha^4\beta^2\gamma^3$ into $\alpha^3\beta^2\gamma^4$). The permutation becomes

$$\beta\gamma\gamma\gamma\alpha\alpha\beta\gamma\alpha.$$

Now the major contacts $\beta\alpha$, $\gamma\beta$, $\gamma\alpha$ in the former necessarily become the minor contacts $\beta\gamma$, $\alpha\beta$, $\alpha\gamma$ in the latter, so that the minor index of the latter is equal to the major index of the former. Hence Σx^p for the former is equal to Σx^r for the latter. Moreover it was shewn that Σx^p remains unchanged by the transformation. Hence for every assemblage

$$\Sigma x^r = \Sigma x^p,$$

for we have merely to make the substitution

$$\begin{pmatrix}\alpha_1\alpha_2 & \dots\alpha_s\\ \alpha_s\alpha_{s-1}\dots & \alpha_1\end{pmatrix}$$

and employ the same reasoning.

108. *The expression of* Σx^{p+q}, *where* p, q *are the greater and equal indices of the same permutation.*

Since every contact in a permutation is either major, equal or minor, it follows that in every permutation

$$p+q+r = 1+2+\dots+(\Sigma i-1) = \binom{\Sigma i}{2},$$

where of course $\binom{\Sigma i}{2}$ denotes $\frac{1}{2}\Sigma i(\Sigma i-1)$.

Hence

$$\Sigma x^{p+q} = x^{p+q+r}\,\Sigma x^{-r}$$

$$= x^{\binom{\Sigma i}{2}} \frac{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\cdots\left(1-\frac{1}{x^{i+j+\cdots}}\right)}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\cdots\left(1-\frac{1}{x^i}\right)\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\cdots\left(1-\frac{1}{x^j}\right)\cdots},$$

which on simplification gives the result

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\,\Sigma x^p,$$

a noteworthy formula the truth of which can be seen *à priori* when regarded, as will be the case in Volume II, from the point of view of the Theory of Partitions.

109. *The assemblage $\alpha^i\beta^j$ in particular.* It will be shewn that the number $p-r$ has the value $-i$ or $+j$ according as the permutation terminates with β or α.

Assume the law to obtain for $\alpha^i\beta^j$ and denote by $(\alpha^i\beta^j)_\alpha$, $(\alpha^i\beta^j)_\beta$ any permutations terminating with α, β respectively.

Observe the results

for $(\alpha^i\beta^j)_\alpha\alpha$ $\quad p-r = +j,$

$(\alpha^i\beta^j)_\alpha\beta$ $\quad$ „ $= +j-(i+j) = -i,$

$(\alpha^i\beta^j)_\beta\alpha$ $\quad$ „ $= -i+i+j = +j,$

$(\alpha^i\beta^j)_\beta\beta$ $\quad$ „ $= -i,$

$(\alpha^i\beta^j)_\beta\beta\alpha$ $\quad$ „ $= -i+(i+j+1) = +j+1,$

$(\alpha^i\beta^j)_\beta\alpha\beta$ $\quad$ „ $= -i+(i+j)-(i+j+1) = -i-1,$

$(\alpha^i\beta^j)_\alpha\beta\alpha$ $\quad$ „ $= +j-(i+j)+(i+j+1) = +j+1,$

$(\alpha^i\beta^j)_\alpha\alpha\beta$ $\quad$ „ $= +j-(i+j+1) = -i-1,$

which, combined with observation in simple cases, establish the law.

We are now in a position to find the expression of Σx^q for the assemblage $\alpha^i\beta^j$. For there are

$$\binom{i+j-1}{i}$$

permutations ending with β, so that *for those permutations*

$$\Sigma x^{p-r} = \binom{i+j-1}{i} x^{-i}.$$

Similarly *for the permutations ending with* α

$$\Sigma x^{p-r} = \binom{i+j-1}{j} x^j$$

For the whole of the permutations

$$\Sigma x^{p-r} = \binom{i+j-1}{i} x^{-i} + \binom{i+j-1}{j} x^{j}.$$

Moreover (see Art. 108) for the whole of the permutations

$$\Sigma x^{p+q+r} = \binom{i+j}{i} x^{\binom{i+j}{2}}.$$

Hence by simple multiplication of the last but one written formula by x^{p+q+r} on one side and by $x^{\binom{i+j}{2}}$ on the other,

$$\Sigma x^{2p+q} = \binom{i+j-1}{i} x^{\binom{i+j}{2}-i} + \binom{i+j-1}{j} x^{\binom{i+j}{2}+j}.$$

Writing Σx^p for the whole of the permutations equal to $F_{i,j}(x)$ we see that for those ending with β,

$$\Sigma x^p = F_{i,j-1}(x),$$

for those ending with α, $\quad \Sigma x^p = x^j F_{i-1,j}(x)$.

Hence

$$\Sigma x^q = \Sigma x^{2p+q-2p} = x^{\binom{i+j}{2}-i} F_{i,j-1}\left(\frac{1}{x^2}\right) + x^{\binom{i+j}{2}+j-2j} F_{i-1,j}\left(\frac{1}{x^2}\right),$$

easily reducing to the formula

$$\Sigma x^q = x^{\binom{i-j+1}{2}} F_{i,j-1}(x^2) + x^{\binom{j-i+1}{2}} F_{i-1,j}(x^2).$$

It will be noted that the index q is always even when $i-j \equiv 0 \bmod 4$, and always uneven when $i-j \equiv 2 \bmod 4$.

We can find the expression of Σx^{p+r}. For this is

$$\Sigma x^{p-r+2r};$$

and since Σx^r is $x^i F_{i,j-1}(x)$ or $F_{i-1,j}(x)$ according as the permutation ends with β or α,

$$\begin{aligned}\Sigma x^{p+r} &= x^{-i} \cdot x^{2i} F_{i,j-1}(x^2) + x^j F_{i-1,j}(x^2) \\ &= x^i F_{i,j-1}(x^2) + x^j F_{i-1,j}(x^2).\end{aligned}$$

Resuming consideration of the assemblage $\alpha^i \beta^j \gamma^k \ldots$ it is to be remarked that $p-r$ has no longer (the number of different letters exceeding 2) any simple value for different classes of permutations. It is not possible to obtain expressions for Σx^q and Σx^{p+r} in the foregoing manner.

110. *The Parity of the Greater Index.* The permutations of $\alpha^i \beta^j \gamma^k \ldots$ are such that the number with an even index is equal to the number with an uneven index whenever more than one of the numbers $i, j, k, \ldots$ are uneven.

Consider the function

$$\frac{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i}+\mathbf{j}+\mathbf{k}+\ldots)}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i})(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j})(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})\ldots};$$

if we put, therein, $x = -1$ and the function happens to vanish it must be because the permutations with even index are equal in number to those with uneven index. This is the case because the function is equal to $\Sigma C_p x^p$ where there are C_p permutations having the index equal to p.

Under what conditions does $x = -1$ cause the function to vanish?

The factor (m) becomes 2 for $x = -1$ whenever m is uneven. Eliminating these factors we are left with a function which may be of the form $\frac{0}{0}$ when $x = -1$ and can be evaluated. This happens when numerator and denominator contain the same number of factors (m), wherein m is even. If this be not the case the numerator must contain more of such factors than the denominator and the function must vanish for $x = -1$. Observe that the numerator cannot involve fewer of such factors than the denominator, for in that case the function would become infinite for $x = -1$, which from the nature of the function is absurd. We have, then, to find out the circumstances under which the numerator of the *complete* function involves more factors (m) than the denominator, m being even. If we consider the product

$$(1)(2)\dots(s)$$

and denote by $E\sigma$ the greatest integer in σ, it will be seen that the number of factors (m), m even, in the product, is $E\frac{1}{2}s$. Hence the condition

$$E\tfrac{1}{2}(i+j+k+\dots) - E\tfrac{1}{2}i - E\tfrac{1}{2}j - E\tfrac{1}{2}k - \dots > 0,$$

which is obviously satisfied when more than one of the numbers $i, j, k, \dots$ is uneven. Under these circumstances the number of permutations with an even index is equal to the number with an uneven index.

111. There is an analogous result in the case of every prime modulus μ. For putting $x = \omega$, where ω is an imaginary μth root of unity, if the number of permutations which have indices $\equiv \sigma \bmod \mu$ does not vary by variation of σ, the expanded form of the function will reduce to a numerical multiple of $1 + \omega + \omega^2 + \dots + \omega^{\mu-1}$ and will therefore vanish. If in the unexpanded fraction we put $x = \omega$ we obtain a number of factors $1 - \omega^t$, wherein t is not $\equiv 0 \bmod \mu$. Besides these we have a number of factors $1 - x^s$, where $x = \omega$ and s is a multiple of μ. If the number of such factors in the numerator is greater than the number in the denominator the fraction vanishes.

In the complete fraction we must find the condition that the number of factors (m), where $m \equiv 0 \bmod \mu$, in the numerator exceeds the number in the denominator. We are led to the condition

$$E\frac{1}{\mu}(i+j+k+\dots) - E\frac{1}{\mu}i - E\frac{1}{\mu}j - E\frac{1}{\mu}k - \dots > 0.$$

This happens if

$$i \equiv i' \bmod \mu, \quad j \equiv j' \bmod \mu, \quad k \equiv k' \bmod \mu, \dots$$

whenever

$$i' + j' + k' + \dots > \mu - 1.$$

Under these circumstances the number of permutations which have indices $\equiv \sigma \bmod \mu$ does not vary by variation of σ.

Ex. gr. Take the assemblage $\alpha^2\beta\gamma$,

	$p=$		$p=$
$\alpha\alpha\beta\gamma$	0	$\beta\alpha\alpha\gamma$	1
$\alpha\alpha\gamma\beta$	3	$\gamma\alpha\alpha\beta$	1
$\alpha\beta\alpha\gamma$	2	$\beta\alpha\gamma\alpha$	4
$\alpha\gamma\alpha\beta$	2	$\gamma\alpha\beta\alpha$	4
$\alpha\beta\gamma\alpha$	3	$\beta\gamma\alpha\alpha$	2
$\alpha\gamma\beta\alpha$	5	$\gamma\beta\alpha\alpha$	3

The reader will verify that since more than one of the numbers 2, 1, 1 are uneven, the even and uneven indices are equi-numerous.

Also since 2, 1, 1 are $\equiv 2, 1, 1 \bmod 3$ respectively and $2+1+1>2$, the indices $\equiv 0, 1, 2 \bmod 3$ are equi-numerous.

112. *Average Values of the Indices.* The highest power of x which presents itself in the expression for Σx^p is

$$\binom{i+j+k+\ldots+1}{2}-\binom{i+1}{2}-\binom{j+1}{2}-\binom{k+1}{2}-\ldots$$

or
$$ij+ik+jk+\ldots,$$

viz. the sum of the products two and two together of the numbers $i, j, k, \ldots$. Hence it is clear that *one* permutation of the assemblage possesses an index $ij+ik+jk+\ldots$. Moreover it is easily proved (it is a well-known property of such an algebraic fraction) that the coefficients of x^p and of $x^{ij+ik+jk+\ldots-p}$ in the development of the fraction are equal. This shews that the permutations having indices p and $ij+ik+jk+\ldots-p$ are equal in number. From these facts it is a natural inference that the *average* value of the index p is

$$\tfrac{1}{2}(ij+ik+jk+\ldots).$$

Another procedure will establish this and lead on to other results.

Write Σx^p in the form $\Sigma C_p x^p$ so that C_p permutations have p for index. The average value of p is the quotient of $\Sigma C_p p$ by

$$\frac{(i+j+k+\ldots)!}{i!\,j!\,k!\ldots}.$$

To find $\Sigma C_p p$ we differentiate $\Sigma C_p x^p$ with regard to x and then put x equal to unity.

To evaluate
$$\left\{\partial_x \frac{(1)(2)\ldots(i+j)}{(1)(2)\ldots(i)(1)(2)\ldots(j)}\right\}_{x=1},$$

we require to evaluate
$$\left\{\partial_x \frac{(i+\sigma)}{(\sigma)}\right\}_{x=1}.$$

This is the limit when $x = 1$ of

$$\frac{-(i+\sigma)x^{i+\sigma-1}(1-x^{\sigma})+\sigma x^{\sigma-1}(1-x^{i+\sigma})}{(1-x^{\sigma})^2}.$$

Differentiating numerator and denominator this becomes

$$\frac{-(i+\sigma)(i+\sigma-1)x^{i+\sigma-2}(1-x^{\sigma})+\sigma(i+\sigma)x^{i+2\sigma-2}+\sigma(\sigma-1)x^{\sigma-2}(1-x^{i+\sigma})-\sigma(i+\sigma)x^{i+2\sigma-2}}{-2\sigma x^{\sigma-1}(1-x^{\sigma})}$$

$$= \frac{i(i+\sigma)}{2\sigma} \text{ when } x=1.$$

Hence

$$\left\{\partial_x \frac{(\mathbf{i}+\mathbf{1})\dots(\mathbf{i}+\mathbf{j})}{(\mathbf{1})\dots(\mathbf{j})}\right\}_{x=1}$$

$$= \frac{(i+j)!}{i!\,j!} \sum_{\sigma=1}^{\sigma=j} \frac{2\sigma}{\dfrac{i+\sigma}{\sigma}} = \frac{(i+j)!}{i!\,j!} \cdot \tfrac{1}{2} ij,$$

establishing that the average value of the greater (or lesser) index of the permutations of $\alpha^i\beta^j$ is $\frac{1}{2}ij$.

Thence

$$\left\{\partial_x \frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i}+\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i})(\mathbf{1})(\mathbf{2})\dots(\mathbf{j})(\mathbf{1})(\mathbf{2})\dots(\mathbf{k})}\right\}_{x=1}$$

$$= \left\{\frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i}+\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i})(\mathbf{1})(\mathbf{2})\dots(\mathbf{j}+\mathbf{k})}\right\}_{x=1} \cdot \left\{\partial_x \frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{j})(\mathbf{1})(\mathbf{2})\dots(\mathbf{k})}\right\}_{x=1}$$

$$+ \left\{\frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{j})(\mathbf{1})(\mathbf{2})\dots(\mathbf{k})}\right\}_{x=1} \cdot \left\{\partial_x \frac{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i}+\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2})\dots(\mathbf{i})(\mathbf{1})(\mathbf{2})\dots(\mathbf{j}+\mathbf{k})}\right\}_{x=1}$$

$$= \frac{(i+j+k)!}{i!(j+k)!} \cdot \frac{(j+k)!}{j!\,k!} \cdot \tfrac{1}{2}jk + \frac{(j+k)!}{j!\,k!} \cdot \frac{(i+j+k)!}{i!(j+k)!} \cdot \tfrac{1}{2} i(j+k)$$

$$= \frac{(i+j+k)!}{i!\,j!\,k!} \cdot \tfrac{1}{2}(ij+ik+jk),$$

establishing that the average value of the greater (or lesser) index of the permutations of $\alpha^i\beta^j\gamma^k$ is

$$\tfrac{1}{2}(ij+ik+jk).$$

From the above it is evident that for the permutations of $\alpha^i\beta^j\gamma^k\dots$ the average value of the greater (or lesser) index is

$\frac{1}{2}$(sum of the products two and two together of the numbers $i, j, k, \dots$).

113. *Average Value of the Equal Index.* From the formula

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\dots} \Sigma x^{p},$$

we deduce that the average value of the equal index is

$$\binom{i}{2} + \binom{j}{2} + \binom{k}{2} + \dots.$$

Otherwise, since the average value of $p+r$ in a permutation is

$$ij+ik+jk+\ldots,$$

and the sum $p+q+r$ is invariably

$$\binom{i+j+k+\ldots}{2},$$

the average value of q is seen to be

$$\binom{i+j+k+\ldots}{2}-(ij+ik+jk+\ldots),$$

which is

$$\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\ldots.$$

114. *Average Values of the Square and Higher Powers of the Greater Index.* Writing the expression for Σx^p in the form $F_{i,j,k,\ldots}(x)$, the average value of the square is derived from the average value of $p(p-1)$, which is the quotient of

$$\partial_x^2 F_{i,j,k,\ldots}(x),\ x \text{ being put equal to unity,}$$

by

$$\frac{(i+j+k+\ldots)!}{i!\,j!\,k!\ldots}.$$

We have merely to add the average value of p. Or we may say that the average value of p^2 is equal to the quotient of

$$\partial_x x \partial_x F_{i,j,k,\ldots}(x),\ x \text{ being put equal to unity,}$$

by the number of permutations.

In any case the evaluation requires the value of

$$\left\{\partial_x^2 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}.$$

Theoretically there is no difficulty in determining the average value of any given positive integral power of the index. It depends upon finding the value of

$$\left\{\partial_x^s \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}.$$

The simplest cases corresponding to $s=1, 2, 3$, may be written

$$\left\{\partial_x \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1} = \mu \frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right),$$

$$\left\{\partial_x^2 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1} = \mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\} - \mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

$$\left\{\partial_x^3 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1} = \mu^3\left\{\frac{1}{4}\frac{\lambda^2}{\mu^2}\left(\frac{\lambda}{\mu}-1\right)^2\right\} - 3\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}$$
$$+ 2\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\}.$$

The general result depends upon the sums of the powers of the natural numbers and therefore ultimately upon the numbers of Bernoulli*.

The above formulas may be often simplified. Thus we find

$$\left\{\partial_x^2 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1} = \frac{1}{6}\frac{\lambda}{\mu}(\lambda-\mu)(2\lambda-\mu-3),$$

$$\left\{\partial_x^3 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1} = \frac{1}{4}\frac{\lambda}{\mu}(\lambda-\mu)(\lambda-\mu-2)(\lambda-2).$$

On applying them to the case in hand it is found that for the assemblage $\alpha^i\beta^j$

$$\text{Average value of } p^2 = \tfrac{1}{12}ij(3ij+i+j+1),$$

$$\text{,,} \qquad p^3 = \tfrac{1}{8}i^2j^2(i+1)(j+1),$$

and for the assemblage $\alpha^i\beta^j\gamma^k$

$$\text{Average value of } p^2 = \tfrac{1}{4}(ij+ik+jk)^2 + \tfrac{1}{12}(i^2j+i^2k+j^2k+ij^2+ik^2+jk^2+2ijk+ij+ik+jk),$$

$$\text{,,} \qquad p^3 = \tfrac{1}{8}(\Sigma i^3j^3 + 3\Sigma i^3j^2k + 6i^2j^2k^2 + \Sigma i^3j^2 + 2\Sigma i^3jk + 4\Sigma i^2j^2k + 2\Sigma i^2jk + \Sigma i^2j^2).$$

115. *Average Values from another point of view.* For the assemblage

$$\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_s^{i_s},$$

we will enquire into the effect which the particular major contact $\alpha_\mu\alpha_\lambda\,(\mu>\lambda)$ has upon the sum of the greater indices Σp. When the contact has σ letters to its left, where σ may be zero or any integer not greater than $\Sigma i-2$, it adds $\sigma+1$ to the index of the permutation.

To find the number of permutations in which this will happen, we observe that the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_\lambda^{i_\lambda-1}\dots\alpha_\mu^{i_\mu-1}\dots\alpha_s^{i_s}$ may be placed in any permutation in the $\Sigma i-2$ places not occupied by the major contact.

Hence the number of permutations in which the contact $\alpha_\mu\alpha_\lambda$ adds $\sigma+1$ to the index is

$$\frac{(\Sigma i-2)!}{i_1!\,i_2!\dots i_s!}i_\lambda i_\mu,$$

so that altogether the effect of the contact $\alpha_\mu\alpha_\lambda$ is to add to Σp the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2}\frac{(\Sigma i-2)!}{i_1!\,i_2!\dots i_s!}i_\lambda i_\mu(\sigma+1)$$

or

$$\frac{(\Sigma i)!}{i_1!\,i_2!\dots i_s!}\tfrac{1}{2}i_\lambda i_\mu.$$

* *American Journal of Mathematics*, Vol. xxxv. No. 3, p. 296 *et seq.*

From this we see that on the average the major contact $\alpha_\mu\alpha_\lambda$ adds the number $\frac{1}{2}i_\lambda i_\mu$ to the greater index of a permutation, and thence we find that the average value of p is

$$\tfrac{1}{2}\Sigma i_\lambda i_\mu,$$

verifying the conclusion of Art. 112.

116. The number p is obtained as the sum of m numbers $p_1, p_2, \ldots p_m$, where m is the class of the permutation *quâ* the greater index. These m numbers are termed the m components of the index.

We will consider the sum

$$\Sigma\,(p_1^\nu + p_2^\nu + \ldots + p_m^\nu),$$

the summation being in respect to every permutation of the assemblage.

We enquire into the effect which the particular major contact $\alpha_\mu\alpha_\lambda$ has upon such sum. From the previous Article we see at once that it has the effect of adding the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2} \frac{(\Sigma i-2)!}{i_1!\,i_2!\ldots i_s!}\, i_\lambda i_\mu (\sigma+1)^\nu,$$

which may be written

$$\frac{(\Sigma i)!}{i_1!\,i_2!\ldots i_s!}\cdot\frac{1^\nu + 2^\nu + \ldots + (\Sigma i-1)^\nu}{\Sigma i\,(\Sigma i-1)}\, i_\lambda i_\mu.$$

Hence the contact $\alpha_\mu\alpha_\lambda$ on the average adds to the sum

$$p_1^\nu + p_2^\nu + \ldots + p_m^\nu,$$

with regard to a permutation, the number

$$\frac{1^\nu + 2^\nu + \ldots + (\Sigma i-1)^\nu}{\Sigma i\,(\Sigma i-1)}\, i_\lambda i_\mu;$$

leading to the result that the average value of

$$p_1^\nu + p_2^\nu + \ldots + p_m^\nu$$

is

$$\frac{1^\nu + 2^\nu + \ldots + (\Sigma i-1)^\nu}{\Sigma i\,(\Sigma i-1)}\,\Sigma i_\lambda i_\mu.$$

Ex. gr. The average value of $p_1^2 + p_2^2 + \ldots + p_m^2$, m of course not being constant, is

$$\tfrac{1}{6}\,(2\Sigma i_1 - 1)\,\Sigma i_1 i_2,$$

and of $p_1^3 + p_2^3 + \ldots + p_m^3$ is

$$\tfrac{1}{4}\Sigma i_1\,(\Sigma i_1 - 1)\,\Sigma i_1 i_2$$

by making use of known results in regard to the sums of powers of the natural numbers.

To verify these results, take the assemblage $\alpha^2\beta\gamma$:

	p_1	p_2	$p_1^2+p_2^2$	$p_1^3+p_2^3$		p_1	p_2	$p_1^2+p_2^2$	$p_1^3+p_2^3$
$\alpha\alpha\beta\gamma$	0	0	0	0	$\beta\alpha\alpha\gamma$	1	0	1	1
$\alpha\alpha\gamma\beta$	3	0	9	27	$\gamma\alpha\alpha\beta$	1	0	1	1
$\alpha\beta\alpha\gamma$	2	0	4	8	$\beta\alpha\gamma\alpha$	1	3	10	28
$\alpha\gamma\alpha\beta$	2	0	4	8	$\gamma\alpha\beta\alpha$	1	3	10	28
$\alpha\beta\gamma\alpha$	3	0	9	27	$\beta\gamma\alpha\alpha$	2	0	4	8
$\alpha\gamma\beta\alpha$	2	3	13	35	$\gamma\beta\alpha\alpha$	1	2	5	9
			39	105				31	75

shewing that
$$\Sigma(p_1^2+p_2^2) = 39+31 = 70,$$
$$\Sigma(p_1^3+p_2^3) = 105+75 = 180,$$
giving the averages of $p_1^2+p_2^2$ and $p_1^3+p_2^3$ equal to $\frac{35}{6}$ and 15 respectively.

Here $\Sigma i_1 = 4$, $\Sigma i_1 i_2 = 5$, so that from the formula
$$\tfrac{1}{6}(2\Sigma i_1 - 1)\Sigma i_1 i_2 = \tfrac{1}{6}(8-1).5 = \tfrac{35}{6},$$
$$\tfrac{1}{4}\Sigma i_1(\Sigma i_1 - 1)\Sigma i_1 i_2 = \tfrac{1}{4}\,4.3.5 \quad = 15,$$
a verification.

117. *Average Value of the Class of a Permutation* quâ *the greater index*

If in the foregoing results we put $\nu = 0$,
$$p_1^0 + p_2^0 + \ldots + p_m^0 = m,$$
the class of the permutation.

Hence the average value of the class is
$$\frac{(\Sigma i - 1)}{\Sigma i(\Sigma i - 1)}\Sigma i_1 i_2 \quad \text{or} \quad \frac{\Sigma i_1 i_2}{\Sigma i_1}.$$

In other words permutations on the average possess
$$\frac{\Sigma i_1 i_2}{\Sigma i_1}$$
major contacts.

In the above example this number is $\frac{5}{4}$, and we see that one permutation is of class 0, seven of class 1 and four of class 2, so the average is
$$\frac{0+7+8}{12} = \frac{5}{4}.$$

118. If we divide the average value of $p_1^\nu + p_2^\nu + \ldots + p_m^\nu$ by the average value of the class we obtain the average value of the νth power of a component of the index. We thus find that the average value of p_θ^ν is
$$\frac{1}{\Sigma i - 1}\{1^\nu + 2^\nu + \ldots + (\Sigma i - 1)^\nu\},$$
p_θ being not equal to zero.

Putting $\nu = 1$, we find that the average value of a component is

$$\tfrac{1}{2}\Sigma i;$$

in other words, the average situation of a major contact in a permutation is $\frac{1}{2}\Sigma i$ places from the beginning. The situation of course refers to the left-hand letter of the contact.

What has been said about major contacts refers equally to minor contacts, so that we are at liberty to say that a permutation possesses on the average

$$2\frac{\Sigma i_1 i_2}{\Sigma i_1}$$

contacts which are either major or minor.

Since moreover the whole number of contacts is $\Sigma i_1 - 1$, it follows that on the average a permutation possesses

$$\Sigma i_1 - 1 - 2\frac{\Sigma i_1 i_2}{\Sigma i_1} \quad \text{or} \quad 2\frac{\Sigma \binom{i_1}{2}}{\Sigma i_1}$$

equal contacts.

This last result will be obtained independently.

119. *Average Values connected with the Equal Index.* We determine the effect which the particular equal contact $\alpha_\lambda \alpha_\lambda$ has upon the sum Σq, where q is the equal index and the summation is for the whole of the permutations. When such a contact has σ letters to its left it adds $\sigma + 1$ to the equal index. This happens in the case of

$$\frac{(\Sigma i - 2)!}{i_1!\, i_2! \ldots i_s!}\, 2 \binom{i_\lambda}{2}$$

permutations; so that on the whole the contact adds to Σq the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2} \frac{(\Sigma i - 2)!}{i_1!\, i_2! \ldots i_s!}\, 2 \binom{i_\lambda}{2} (\sigma + 1)$$

or

$$\frac{(\Sigma i)!}{i_1!\, i_2! \ldots i_s!} \binom{i_\lambda}{2}.$$

Hence the contact $\alpha_\lambda \alpha_\lambda$ on the average adds the number

$$\binom{i_\lambda}{2}$$

to the equal index of a permutation. Thence it appears that the average value of the equal index is

$$\Sigma \binom{i_1}{2}.$$

Moreover the contact $\alpha_\lambda \alpha_\lambda$ on the average adds to the sum

$$q_1^\nu + q_2^\nu + \ldots + q_m^\nu$$

($q_1, q_2, \ldots q_m$ being the components of the index q) the number

$$2\,\frac{1^\nu + 2^\nu + \ldots + (\Sigma i - 1)^\nu}{\Sigma i\,(\Sigma i - 1)} \binom{i_\lambda}{2},$$

and thence we gather that the average value of

$$q_1^\nu + q_2^\nu + \ldots + q_m^\nu$$

is

$$2\,\frac{1^\nu + 2^\nu + \ldots + (\Sigma i - 1)^\nu}{\Sigma i\,(\Sigma i - 1)} \Sigma \binom{i_1}{2}.$$

The average values of $q_1^2 + q_2^2 + \ldots + q_m^2$ and $q_1^3 + q_2^3 + \ldots + q_m^3$ respectively are

$$\tfrac{1}{3}\,(2\Sigma i_1 - 1)\,\Sigma \binom{i_1}{2},$$

$$\tfrac{1}{2}\Sigma i\,(\Sigma i - 1)\,\Sigma \binom{i_1}{2}.$$

Moreover putting $\nu = 0$ we find for the average class *quâ* the equal index

$$\frac{2}{\Sigma i}\,\Sigma \binom{i_1}{2},$$

agreeing with that obtained above in another manner.

120. *Properties of the Components of the Indices.* It has been shewn that the number of permutations in which the contact $\alpha_\mu\alpha_\lambda$ adds $\sigma + 1$ to the index is

$$\frac{(\Sigma i - 2)!}{i_1!\,i_2!\ldots i_s!}\,i_\lambda i_\mu.$$

It thence follows that the number of permutations in which some contact or another adds $\sigma + 1$ to the index is

$$\frac{(\Sigma i - 2)!}{i_1!\,i_2!\ldots i_s!}\,\Sigma i_1 i_2.$$

Since this number is independent of σ it follows that, viewing the whole of the permutations, each of the components

$$1, 2, 3, \ldots \Sigma i - 1$$

occurs precisely

$$\frac{(\Sigma i_1 - 2)!}{i_1!\,i_2!\ldots i_s!}\,\Sigma i_1 i_2 \text{ times.}$$

In the example of the assemblage $\alpha\alpha\beta\gamma$ given above it will be observed as a verification that each of the components 1, 2, 3 occurs five times.

Similarly the same reasoning shews that each of the components of the equal index occurs precisely

$$\frac{(\Sigma i_1 - 2)!}{i_1!\,i_2!\ldots i_s!}\,2\Sigma \binom{i_1}{2} \text{ times.}$$

It is interesting to apply some of the above theorems to a pack of ordinary playing cards whose specification may be taken to be

$$\alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\alpha_5^4\alpha_6^4\alpha_7^4\alpha_8^4\alpha_9^4\alpha_{10}^4\alpha_{11}^4\alpha_{12}^4\alpha_{13}^4.$$

Here $$\Sigma i_1 = 52, \quad \Sigma i_1 i_2 = 1248, \quad \Sigma \binom{i_1}{2} = 78,$$

giving $$\frac{\Sigma i_1 i_2}{\Sigma i_1} = 24, \qquad \frac{2}{\Sigma i_1} \Sigma \binom{i_1}{2} = 3.$$

Thus, in a random dealing of the cards, we may expect on the average

24 major contacts, 3 equal contacts and 24 minor contacts.

If we deal out m packs the average number of equal contacts will be $4m - 1$.

SECTION IV

THEORY OF THE COMPOSITIONS OF NUMBERS

CHAPTER I

UNIPARTITE NUMBERS

121. This Section deals with that part of the general theory of distributions which is concerned with the distributions of objects into parcels of type (1^m); that is to say, the parcels are such that no two are alike. If the objects distributed are of type (n) the theory is that of the compositions of the (unipartite) number n into m parts; where the objects are of type $(pqr\ldots)$ we have to deal with the compositions into m parts of the multipartite number $pqr\ldots$.

122. Compositions of a number as understood in the theory of distributions are merely partitions of a number in which the order of occurrence of the parts is essential; thus while the partitions of the number 3 are (3), (21), (1^3), the compositions are (3), (21), (12), (1^3).

In Section I a general formula is given for the enumeration of the compositions of multipartite numbers into exactly m parts. In *this* particular case it is best to arrive at results from elementary considerations and not to follow the general formula.

The number of compositions of n into exactly m parts is given by the coefficient of x^n in the expansion of

$$(x + x^2 + x^3 + \ldots)^m;$$

for writing the function as a product of m factors and performing the multiplication by picking out the terms $x^{p_1}, x^{p_2}, \ldots x^{p_m}$, where

$$p_1 + p_2 + \ldots + p_m = n,$$

in succession, we obtain for this particular selection the term $x^{p_1+p_2+\ldots+p_m}$ of the product, where $(p_1 p_2 \ldots p_m)$ is one composition of n into exactly m parts.

The number is therefore $\binom{n-1}{m-1}$.

123. The generating function of the total number of compositions is

$$\sum_{1}^{\infty}(x+x^2+x^3+\ldots)^m=\frac{x}{1-2x};$$

hence the number is 2^{n-1}.

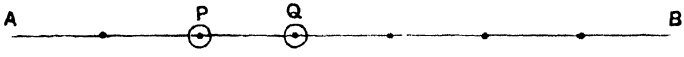
124. If the parts of the composition are limited not to exceed s in magnitude the generating function of the compositions into exactly m parts is

$$x^m\left(\frac{1-x^s}{1-x}\right)^m.$$

If the number of parts be unrestricted the generating function is

$$\sum_{m}x^m\left(\frac{1-x^s}{1-x}\right)^m=\frac{x(1-x^s)}{1-2x+x^{s+1}}.$$

125. There are two methods of representing the compositions of numbers graphically. The number n is denoted by a straight line divided at $n-1$ points into n equal segments. The graph of a composition is obtained by placing nodes at certain of these $n-1$ points of division.



AB being the graph of 7, for the composition (214) nodes are placed at the points P, Q so that in moving from A to B by steps proceeding from node to node, 2, 1 and 4 segments of the line are passed over in succession.

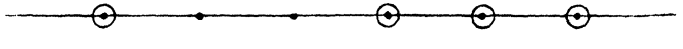
The number of parts of the composition exceeds by unity the number of nodes on the graph. For a composition into m parts we can place nodes at any $m-1$ of the $n-1$ points of division, so that the number of the compositions is at once seen to be

$$\binom{n-1}{m-1}.$$

Since each of the $n-1$ points of division is or is not the position of a node the total number of compositions is 2^{n-1}.

126. Associated with any one graph there is another obtained by obliterating the nodes and placing nodes at the points not previously occupied.

These graphs are said to be conjugate, and the associated compositions are said to be conjugate.



(13111) is conjugate to (214).

If a graph denotes a composition of n into m parts, the conjugate graph denotes a composition of n into $n-m+1$ parts. These compositions are equi-numerous. This fact also appears from the consideration that $\binom{n-1}{m-1}$ remains unaltered by the change of m into $n-m+1$.

127. Two compositions are said to be inverse, the one of the other, when the parts of the one read from left to right are identical with those of the other when read from right to left.

A composition may therefore be self-inverse.

In the graph of a self-inverse composition, the nodes must be symmetrically placed with respect to the centre of the graph. If the number n be even, the number of segments of the graph is even, and the two central nodes, or nodes nearest to the centre of the graph, may be coincident or may include any even number of segments. A self-inverse composition of an even number $2n'$ into $2m'$ parts can only occur when the two central nodes are coincident, and attending to one side of this node we find that such compositions are just as numerous as the compositions of n' into m' parts. The enumerating number is therefore $\binom{n'-1}{m'-1}$.

In the case of self-inverse compositions of $2n'$ into $2m'-1$ parts, the two central nodes must be distinct and may include any number 2κ of segments; the corresponding number of self-inverse compositions is equal to the number of compositions of $n'-\kappa$ into $m'-1$ parts; hence giving κ all values from 1 to $n'-m'+1$ the enumerating number is

$$\binom{n'-2}{m'-2}+\binom{n'-3}{m'-2}+\ldots+\binom{m'-2}{m'-2},$$

which has the value $\binom{n'-1}{m'-1}$.

Self-inverse compositions of uneven numbers occur only when the number of parts is uneven, and it is readily shewn that those of $2n'-1$ into $2m'-1$ parts are enumerated by

$$\binom{n'-1}{m'-1}.$$

Without restriction of the number of parts the self-inverse compositions of $2n'$ are enumerated by

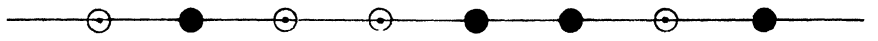
$$2\left\{\binom{n'-1}{0}+\binom{n'-1}{1}+\ldots+\binom{n'-1}{n'-1}\right\}=2^{n'},$$

and those of $2n'+1$ by

$$\binom{n'}{0}+\binom{n'}{1}+\ldots+\binom{n'}{n'}=2^{n'}.$$

128. Two compositions which are at once conjugate and inverse are termed "inverse conjugates." A composition whose conjugate is its own inverse is said to be "inversely conjugate." Inversely conjugate compositions of n which have m parts can only occur when $m = n - m + 1$ or $n = 2m - 1$ an uneven number. The compositions of this nature of $2m + 1$ are composed of $m + 1$ parts.

Consider a graph in which white and black nodes have reference to the two inverse conjugates respectively.

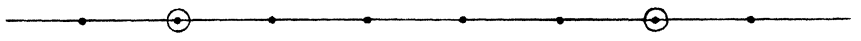


The black nodes are placed to the right and left of the centre in a manner similar to the white nodes to the left and right. Of the graph of the number $2m + 1$ there are m points to the right of the centre at which white and black nodes can be placed in 2^m distinct ways. Hence it follows that the number $2m + 1$ possesses 2^m inversely conjugate compositions.

It will be observed that the number $2m + 1$ has precisely the same number 2^m of self-inverse compositions and we can readily establish a one-to-one correspondence between the compositions which are inversely conjugate and those which are self-inverse. For in the graph of the number 9 above depicted, the reading according to the black nodes yields the inversely conjugate composition

(23121).

We now obliterate the black nodes to the right of the centre and also the white nodes to the left of the centre; substituting white nodes for the black nodes remaining, we have the graph



of the self-inverse composition (252).

If we interchange black and white nodes in the above process, we find the correspondence between the inverse-conjugate (12132) and the self-inverse (1211121).

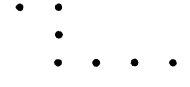
Every composition of the number n gives rise to two compositions of the number $n + 1$

(i) by prefixing the part unity,

(ii) by increasing the magnitude of the first part by unity.

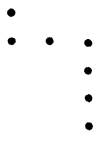
All of the compositions thus reached are necessarily distinct.

129. The second graphical method of representing compositions will be clear from a few examples. The composition (214) is denoted by



the several rows of nodes contain numbers of nodes corresponding to the successive parts of the composition. The left-hand node in each row is placed beneath the right-hand node in the row above it.

Similarly (13111) is given by



and it will be observed that a composition and its conjugate can both be read from the same graph. Whereas the composition is read from left to right in successive rows from top to bottom, the conjugate is read from top to bottom in successive columns from left to right.

The inverses of each of these can also be read, so that a single graph can in general be read in four ways; if the composition be self-inverse so also is the conjugate, and only two different compositions are obtainable from the four readings.

These are called zig-zag graphs, and as will be seen later they form one connecting link between the theories of compositions and partitions.

It suffices to say here that the Ferrers-Sylvester graph of a partition is completely bounded (i) by an axis of x, (ii) by an axis of y, (iii) by a zig-zag graph.

CHAPTER II

MULTIPARTITE NUMBERS

130. It was shewn in Section I that the number of distributions of objects of type $(pqr\ldots)$ into m parcels of type (1^m) is given by the coefficient of the symmetric function $(pqr\ldots)$ in the expansion of

$$(h_1 + h_2 + h_3 + \ldots)^m$$

into a series of monomial symmetric functions; h_s denoting the sum of the homogeneous products of degree s of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$. An expression was found for this number, so that the enumeration is analytically complete.

The generating function of the whole number of compositions is

$$\sum_1^\infty (h_1 + h_2 + h_3 + \ldots)^m$$

$$= \frac{h_1 + h_2 + h_3 + \ldots}{1 - h_1 - h_2 - h_3 - \ldots}.$$

Now, if $a_1, a_2, a_3, \ldots$ be the elementary symmetric functions,

$$h_1 + h_2 + h_3 + \ldots = \frac{a_1 - a_2 + a_3 - \ldots}{1 - a_1 + a_2 - a_3 + \ldots};$$

and thence the generating function is

$$\frac{a_1 - a_2 + a_3 - \ldots}{1 - 2(a_1 - a_2 + a_3 - \ldots)}.$$

We now take the important step of adding the fraction $\frac{1}{2}$ to the function, equivalent to the assumption that the number of compositions of multipartite zero is $\frac{1}{2}$. We find

$$\frac{1}{2} + \frac{a_1 - a_2 + a_3 - \ldots}{1 - 2(a_1 - a_2 + a_3 - \ldots)}$$

$$= \frac{1}{2} \frac{1}{1 - 2(a_1 - a_2 + a_3 - \ldots)}.$$

The whole number of compositions of the multipartite number

$$(pqr\ldots)$$

is equal to the coefficient of the symmetric function

$$(pqr\ldots)$$

in the development of the algebraic fraction

$$\frac{1}{2}\frac{1}{1-2(a_1-a_2+a_3-\ldots)}.$$

This may be written

$$\frac{1}{2}\frac{1}{1-2\Sigma a_1+2\Sigma a_1a_2-2\Sigma a_1a_2a_3+\ldots+(-)^m 2\Sigma a_1a_2\ldots a_m},$$

wherein m may be considered to be as large as may be necessary.

131. The reader who has made himself familiar with the master theorem of Section III will at once recognize the possibility of the form of generating function now reached being of the form V_m^{-1} and being thus connected with a redundant X product

$$\tfrac{1}{2}X_1^{\xi_1}X_2^{\xi_2}\ldots X_m^{\xi_m},$$

where each X is a certain linear function of the quantities

$$a_1,\ a_2,\ a_3,\ \ldots\ a_m.$$

Consider the coefficient of $a_1^{\xi_1}a_2^{\xi_2}a_3^{\xi_3}$ in the product

$$(2a_1+a_2+a_3)^{\xi_1}(2a_1+2a_2+a_3)^{\xi_2}(2a_1+2a_2+2a_3)^{\xi_3};$$

the guiding matrix being

$$\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{vmatrix}.$$

We arrive at the generating function

$$\frac{1}{1-2\Sigma a_1+2\Sigma a_1a_2-2a_1a_2a_3},$$

and in general, when the guiding matrix is

$$\begin{vmatrix} 2 & 1 & 1 & 1 & 1 & . & . \\ 2 & 2 & 1 & 1 & 1 & . & . \\ 2 & 2 & 2 & 1 & 1 & . & . \\ 2 & 2 & 2 & 2 & 1 & . & . \\ 2 & 2 & 2 & 2 & 2 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix},$$

of order m, it is easy to establish that every co-axial minor not excluding the determinant itself is equal to 2. Hence the X product

$$\begin{aligned}&(2\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_m)^{p_1}\\ &\times (2\alpha_1 + 2\alpha_2 + \alpha_3 + \ldots + \alpha_m)^{p_2}\\ &\times (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + \alpha_m)^{p_3}\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\times (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_m)^{p_m}\end{aligned}$$

leads to the generating function

$$\frac{1}{1 - 2\Sigma\alpha_1 + 2\Sigma\alpha_1\alpha_2 - 2\Sigma\alpha_1\alpha_2\alpha_3 + \ldots + (-)^m 2\alpha_1\alpha_2 \ldots \alpha_m};$$

and thence we are able to conclude that the number of compositions of the multipartite number

$$(p_1 p_2 p_3 \ldots p_m)$$

is equal to the coefficient of

$$\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} \ldots \alpha_m^{p_m}$$

in the product

$$\tfrac{1}{2}(2\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_m)^{p_1}(2\alpha_1 + 2\alpha_2 + \alpha_3 + \ldots + \alpha_m)^{p_2}(2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + \alpha_m)^{p_3} \ldots (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_m)^{p_m}.$$

This is the particular redundant generating function. The general redundant generating function is

$$\frac{1}{2}\cdot\frac{1}{\{1 - s_1(2\alpha_1 + \alpha_2 + \ldots + \alpha_m)\}\{1 - s_2(2\alpha_1 + 2\alpha_2 + \ldots + \alpha_m)\}\ldots\{1 - s_m(2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_m)\}}.$$

132. We notice that we have here a connecting link between the Theory of Compositions and the Theory of Permutations. In the particular redundant generating function the term $\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_m^{p_m}$, attached to some numerical coefficient, arises in connexion with every permutation of the letters in the term. One factor of such coefficient is obviously

$$2^{p_1 - 1};$$

if the letter α_2 occurs q_2 times in the last $p_2 + p_3 + \ldots + p_m$ places of the permutation there is another factor

$$2^{q_2};$$

and if α_s, in the same permutation, occur q_s times in the last $p_s + p_{s+1} + \ldots + p_m$ places there will be a factor

$$2^{q_s}.$$

Hence the particular permutation considered occurs with a coefficient

$$2^{p_1 - 1 + q_2 + q_3 + \ldots + q_m},$$

and the number of compositions must therefore be

$$2^{p_1 - 1}\Sigma 2^{q_2 + q_3 + \ldots + q_m},$$

the summation being in regard to every permutation of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_m^{p_m}.$$

Ex. gr. To find in this manner the number of compositions of the bipartite number (22), we have the following scheme:

			$q_2 =$
$p_1 = 2$	$\alpha_1\alpha_1$	$\alpha_2\alpha_2$	2
	$\alpha_1\alpha_2$	$\alpha_1\alpha_2$	1
	$\alpha_1\alpha_2$	$\alpha_2\alpha_1$	1
	$\alpha_2\alpha_1$	$\alpha_1\alpha_2$	1
	$\alpha_2\alpha_1$	$\alpha_2\alpha_1$	1
	$\alpha_2\alpha_2$	$\alpha_1\alpha_1$	0

hence the number is $2(2^2 + 2 + 2 + 2 + 2 + 2^0) = 26.$

133. We proved in Section I that the generating function of the number of distributions of objects of any specified type into m different parcels when no parcel is allowed to contain more than one object of any given kind is

$$(a_1 + a_2 + a_3 + \dots + a_m)^m,$$

where $a_1, a_2, a_3, \dots$ are the elementary symmetric functions; in other words this function generates the number of compositions into m parts of all multipartite numbers when the compositions are restricted so as to contain no figure greater than unity in their expression.

Hence the whole number of such compositions is enumerated by

$$\frac{a_1 + a_2 + a_3 + \dots + a_m}{1 - a_1 - a_2 - a_3 - \dots - a_m};$$

adding unity to this it becomes

$$\frac{1}{1 - a_1 - a_2 - a_3 - \dots - a_m},$$

and we may now seek the corresponding particular redundant generating function by applying the master theorem of Section III.

Slight reflexion shews that the guiding matrix must be

$$\begin{vmatrix} 1 & 2 & 2 & 2 & 2 & . & . & . \\ 1 & 1 & 2 & 2 & 2 & . & . & . \\ 1 & 1 & 1 & 2 & 2 & . & . & . \\ 1 & 1 & 1 & 1 & 2 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{vmatrix}$$

of m rows. For each co-axial minor of order one has the value unity; of the order two the value -1; of the order three $+1$ and so on; every co-axial

minor of uneven order has the value $+1$ and of even order the value -1. Thus this matrix leads from the particular redundant generating function

$$(\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_m)^{p_1}(\alpha_1 + \alpha_2 + 2\alpha_3 + \ldots + 2\alpha_m)^{p_2} \ldots (\alpha_1 + \alpha_2 + \ldots + \alpha_m)^{p_m}$$

to the true condensed generating function

$$\frac{1}{1 - a_1 - a_2 - a_3 - \ldots - a_m}.$$

The general redundant generating function is

$$\frac{1}{\{1 - s_1(\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_m)\}\{1 - s_2(\alpha_1 + \alpha_2 + 2\alpha_3 + \ldots + 2\alpha_m)\} \ldots \{1 - s_m(\alpha_1 + \alpha_2 + \ldots + \alpha_m)\}}.$$

134. We have before us another connecting link with the Theory of Permutations. In the particular redundant generating function a particular permutation of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_m^{p_m}$$

occurs in the multiplication with a factor

$$2^{q_2}$$

if α_2 appears q_2 times in the first p_1 places of the permutation, and with a factor

$$2^{q_s}$$

if α_s appears q_s times in the first $p_1 + p_2 + \ldots + p_{s-1}$ places of the permutation.

The number of compositions, of the nature under view, of the multipartite number $(p_1 p_2 \ldots p_m)$ is thus

$$\Sigma 2^{q_2 + q_3 + \ldots + q_m},$$

where q_s is the number of times that α_s appears in the first $p_1 + p_2 + \ldots + p_{s-1}$ places of the permutation of $\alpha_1^{p_1}\alpha_2^{p_2} \ldots \alpha_m^{p_m}$ and the summation is in respect of the whole of the permutations.

Ex. gr. To find in this way the number of compositions of (22), in which no figure greater than unity appears, we have the scheme:

		$q_2 =$
$\alpha_1\alpha_1$	$\alpha_2\alpha_2$	0
$\alpha_1\alpha_2$	$\alpha_1\alpha_2$	1
$\alpha_1\alpha_2$	$\alpha_2\alpha_1$	1
$\alpha_2\alpha_1$	$\alpha_1\alpha_2$	1
$\alpha_2\alpha_1$	$\alpha_2\alpha_1$	1
$\alpha_2\alpha_2$	$\alpha_1\alpha_1$	2

so that the number sought is $2^0 + 2 + 2 + 2 + 2 + 2^2$ or 13.

The compositions referred to are in fact

(10.10.01.01)	(10.01.10.01)	(10.01.01.10)
(01.01.10.10)	(01.10.01.10)	(01.10.10.01)
	(11.11)	
(11.10.01)	(11.01.10)	(10.11.01)
(10.01.11)	(01.11.10)	(01.10.11).

135. Let $F(p_1p_2p_3\ldots)$ and $f(p_1p_2p_3\ldots; r)$ denote respectively the total number of compositions and the number comprised of r parts of the multipartite number $(p_1p_2p_3\ldots)$; so that

$$F(p_1p_2p_3\ldots) = \sum_r f(p_1p_2p_3\ldots; r).$$

From Section I, Art. 16, we have if

$$h_1 + h_2 + h_3 + \ldots = H,$$

$$H^r = \Sigma f(p_1p_2p_3\ldots; r).(p_1p_2p_3\ldots),$$

$$\frac{H}{1-H} = \Sigma F(p_1p_2p_3\ldots).(p_1p_2p_3\ldots);$$

where $(p_1p_2p_3\ldots)$ denotes the symmetric function $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}\ldots$.

Recalling the operators

$$d_r = \frac{d}{da_r} + a_1\frac{d}{da_{r+1}} + a_2\frac{d}{da_{r+2}} + \ldots,$$

$$D_r = \frac{1}{r!}(d_1^r),$$

we have $$D_\mu H^r = \Sigma f(p_1p_2p_3\ldots\mu; r).(p_1p_2p_3\ldots).$$

In order to evaluate $D_\mu H^r$ we first of all find that

$$d_\mu h_s = (-)^{\mu+1}h_{s-\mu},$$

$$d_\mu H = (-)^{\mu+1}(1+H).$$

Hence, when operating upon a function of H, d_μ is equivalent to d_1 when μ is uneven and to $-d_1$ when μ is even. This leads us to the formula

$$D_\mu = \frac{1}{\mu!}d_1(d_1+1)(d_1+2)\ldots(d_1+\mu-1),$$

the products on the right denoting successive operations.

To see how this is we recur to the analogy between quantities and operations that was broached at some length in Section I. For the operator relation

$$d_\mu \equiv (-)^{\mu+1}d_1$$

corresponds with a quantity relation $s_\mu = (-)^{\mu+1}s_1$; shewing that the formula

$$1 - a_1x + a_2x^2 - \ldots = \exp\{-(s_1x + \tfrac{1}{2}s_2x^2 - \ldots)\}$$

becomes $$1 - a_1x + a_2x^2 - \ldots = (1+x)^{-s_1},$$

leading to $$a_\mu = \frac{1}{\mu!}s_1(s_1+1)(s_1+2)\ldots(s_1+\mu-1),$$

and thence to the desired operator relation

$$D_\mu = \frac{1}{\mu!}d_1(d_1+1)(d_1+2)\ldots(d_1+\mu-1).$$

To find the effect of D_μ upon H^r, operate successively with

$$d_1,\quad \tfrac{1}{2}(d_1+1),\quad \tfrac{1}{3}(d_1+2),\quad \ldots$$

and we find $$D_1H^r = rH^{r-1} + rH^r,$$

$$D_2H^r = \binom{r}{2}H^{r-2} + \binom{r}{1}^2 H^{r-1} + \binom{r+1}{2}H^r.$$

Suppose that, in general,

$$D_\mu H^r = \sum_{s=0}^{s=\mu}\binom{r+s-1}{s}\binom{r}{\mu-s}H^{r-\mu+s};$$

operation with $\dfrac{1}{\mu+1}(d_1+\mu)$ gives

$$D_{\mu+1}H^r = \sum_{s=0}^{s=\mu}\frac{1}{\mu+1}\binom{r+s-1}{s}\binom{r}{\mu-s}\{(r-\mu+s)H^{r-\mu+s-1}(1+H) + \mu H^{r-\mu+s}\}$$

$$= \sum_{s=0}^{s=\mu+1}\binom{r+s-1}{s}\binom{r}{\mu+1-s}H^{r-(\mu+1)+s},$$

so that the assumed law is established inductively.

Substituting in an identity above

$$\sum_{s=0}^{s=\mu}\binom{r+s-1}{s}\binom{r}{\mu-s}H^{r-\mu+s} = \Sigma f(p_1p_2p_3\ldots\mu;\ r).(p_1p_2p_3\ldots)$$

or $$\sum_{s=0}^{s=\mu}\binom{r+s-1}{s}\binom{r}{\mu-s}\Sigma f(p_1p_2p_3\ldots;\ r-\mu+s).(p_1p_2p_3\ldots)$$

$$= \Sigma f(p_1p_2p_3\ldots\mu;\ r).(p_1p_2p_3\ldots).$$

This is an absolute identity, so that equating coefficients, after putting

$$\binom{r+s-1}{s}\binom{r}{\mu-s} = \phi(r, s),$$

we obtain

$$f(p_1p_2p_3\ldots\mu;\ r)$$
$$= \phi(r,\mu)f(p_1p_2p_3\ldots;\ r) + \phi(r,\mu-1)f(p_1p_2p_3\ldots;\ r-1) + \ldots$$
$$+ \phi(r,\mu-r+1)f(p_1p_2p_3\ldots;\ 1),$$

a formula which enables the calculation of the number $f(p_1p_2p_3\ldots\mu;\ r)$ from the successive numbers

$$f(p_1p_2p_3\ldots;\ r),\quad f(p_1p_2p_3\ldots;\ r-1),\quad \ldots f(p_1p_2p_3\ldots;\ 1).$$

136. A more useful result is obtained by summing each side of this identity for values of r from 1 to $\Sigma p + \mu$. This is, after reduction,

$$F(p_1p_2p_3\dots\mu) = \sum_r 2^{\mu-1}\binom{r+\mu-1}{r}\binom{2r+\mu}{\mu} f(p_1p_2p_3\dots; r).$$

Ex. gr. We can apply this formula to find a series for $F(p_1p_2)$; for since

$$f(p_1; r) = \binom{p_1-1}{r-1},$$

we have

$$F(p_1p_2) = 2^{p_2-1}\left\{\binom{p_2}{1}\frac{p_2+2}{p_2}\binom{p_1-1}{1} + \binom{p_2+1}{2}\frac{p_2+4}{p_2}\binom{p_1-1}{2}\right.$$
$$\left. + \binom{p_2+2}{3}\frac{p_2+6}{p_2}\binom{p_1-1}{3} + \dots\right\}.$$

137. It has been shewn in Art. 135 that $(-)^{\mu+1}d_\mu$ is equivalent to d_1 when the operand is a function of H only. Now

$$\frac{(-)^{\mu+1}}{\mu} d_\mu = \Sigma (-)^{p+q+r+\dots-1}\frac{(p+q+r+\dots-1)!}{p!\,q!\,r!\dots} D_1^p D_2^q D_3^r \dots,$$

the summation being for all positive integer solutions of

$$p + 2q + 3r + \dots = \mu.$$

Whence operating on the expression

$$\Sigma f(p_1p_2p_3\dots; r)\,.\,(p_1p_2p_3\dots)$$

with $(-)^{\mu+1}d_\mu$ for successive positive integer values of μ, and then equating the coefficients of the symmetric function $(p_1p_2p_3\dots)$, we find

$$f(p_1p_2p_3\dots 1; r)$$
$$= -\{f(p_1p_2p_3\dots 1^2; r) - 2f(p_1p_2p_3\dots 2; r)\},$$
$$= +\{f(p_1p_2p_3\dots 1^3; r) - 3f(p_1p_2p_3\dots 21; r) + 3f(p_1p_2p_3\dots 3; r)\},$$
$$= \dots$$
$$= (-)^{s+1}\Sigma(-)^{p+q+\dots-1}\frac{(p+q+\dots-1)!\,s}{p!\,q!\dots} f(p_1p_2p_3\dots\dots 2^q 1^p; r);$$

wherein

$$p + 2q + \dots = s.$$

138. Summing with respect to r we reach the set of identities

$$F(p_1p_2p_3\dots 1)$$
$$= -\{F(p_1p_2p_3\dots 1^2) - 2F(p_1p_2p_3\dots 2)\},$$
$$= +\{F(p_1p_2p_3\dots 1^3) - 3F(p_1p_2p_3\dots 21) + 3F(p_1p_2p_3\dots 3)\},$$
$$= \dots$$
$$= (-)^{s+1}\Sigma(-)^{p+q+\dots-1}\frac{(p+q+\dots-1)!\,s}{p!\,q!\dots} F(p_1p_2p_3\dots\dots 2^q 1^p).$$

In the particular case $(p_1p_2p_3 \dots 1) = (1)$, we have by reference to the Tables, the verification

$$\begin{aligned} & F(1) \\ &= -\{F(1^2) - 2F(2)\}, \\ &= +\{F(1^3) - 3F(21) + 3F(3)\}, \\ &= -\{F(1^4) - 4F(21^2) + 2F(2^2) + 4F(31) - 4F(4)\}, \\ &= \dots \end{aligned}$$

verified by

$$\begin{aligned} & 1 \\ &= -\ 3 + 2.2, \\ &= \ 13 - 3.8 + 3.4, \\ &= -75 + 4.44 - 2.26 - 4.20 + 4.8, \\ &= \dots \end{aligned}$$

139. Another very useful result is reached in the following manner:

Since the supposition

$$(-)^{\mu+1} s_\mu = s_1$$

leads to the formula

$$h_\mu = \frac{1}{\mu!} s_1 (s_1 - 1) \dots (s_1 - \mu + 1);$$

and

$$h_\mu = \Sigma (-)^{\mu+p+q+\dots} \frac{(p+q+\dots)!}{p!\,q!\dots} a_1^p a_2^q \dots,$$

we reach the operator relation

$$\frac{1}{\mu!} d_1 (d_1 - 1) \dots (d_1 - \mu + 1) = \Sigma (-)^{\mu+p+q+\dots} \frac{(p+q+\dots)!}{p!\,q!\dots} D_1^p D_2^q \dots,$$

whenever as in the present case the operand is such that

$$(-)^{\mu+1} d_\mu = d_1.$$

Since

$$d_1 H^r = \binom{r}{1} H^{r-1} (1 + H),$$

and

$$\frac{1}{2!} d_1 (d_1 - 1) H^r = \binom{r}{2} H^{r-2} (1 + H)^2,$$

assume

$$\frac{1}{\mu!} d_1 (d_1 - 1) \dots (d_1 - \mu + 1) H^r = \binom{r}{\mu} H^{r-\mu} (1 + H)^\mu;$$

and we readily find that this leads to the relation

$$\frac{1}{(\mu+1)!} d_1 (d_1 - 1) \dots (d_1 - \mu) H^r = \binom{r}{\mu+1} H^{r-\mu-1} (1 + H)^{\mu+1},$$

verifying the assumption.

Now operation upon the relation

$$H^r = \Sigma f(p_1p_2p_3\dots\,;\, r)\,.\,(p_1p_2p_3\dots)$$

yields

$$\binom{r}{\mu} H^{r-\mu}(1+H)^\mu$$

$$= \Sigma(-)^{\mu+p+q+\dots}\,\frac{(p+q+\dots)!}{p!\,q!\dots}\,D_1^p D_2^q \dots \Sigma f(p_1p_2p_3\dots\,;\, r)\,.\,(p_1p_2p_3\dots);$$

whence expanding both sides, equating coefficients of $(p_1p_2p_3\dots)$ and inverting,

$$f(p_1p_2p_3\dots 1^\mu\,;\, r) - \binom{\mu-1}{1} f(p_1p_2p_3\dots 21^{\mu-2};\, r) + \binom{\mu-2}{1} f(p_1p_2p_3\dots 31^{\mu-3};\, r)$$
$$+ \binom{\mu-2}{2} f(p_1p_2p_3\dots 2^2 1^{\mu-4};\, r) + \dots$$
$$= \binom{r}{\mu}\left\{ f(p_1p_2p_3\dots\,;\, r-\mu) + \binom{\mu}{1} f(p_1p_2p_3\dots\,;\, r-\mu+1) \right.$$
$$\left. + \binom{\mu}{2} f(p_1p_2p_3\dots\,;\, r-\mu+2) + \dots \right\}.$$

140. Summing in regard to r,

$$F(p_1p_2p_3\dots 1^\mu) - \binom{\mu-1}{1} F(p_1p_2p_3\dots 21^{\mu-2}) + \dots$$
$$+ (-)^{\mu+p+q+\dots}\,\frac{(p+q+\dots)!}{p!\,q!\dots}\,F(p_1p_2p_3\dots 2^q 1^p) + \dots$$
$$= \Sigma\theta_{\mu,s} f(p_1p_2p_3\dots\,;\, s);$$

where $$\theta_{\mu,s} = \binom{\mu}{s} + \binom{\mu+1}{1}\binom{\mu}{s-1} + \binom{\mu+2}{2}\binom{\mu}{s-2} + \dots + \binom{\mu+s}{s}$$

or $$\Sigma\theta_{\mu,s}x^s = (1+x)^\mu(1-x)^{-\mu-1}.$$

Ex. gr. if $r<\mu$ in the formula of Art. 139 the dexter vanishes, and putting

$$r=2,\quad \mu=3,\quad p_1p_2p_3\dots = 2,$$

we get $$f(21^3;\, 2) - 2f(2^21;\, 2) + f(32;\, 2) = 0;$$

verified by $$22 - 2\,.\,16 + 10 = 0.$$

Again if $r=\mu=3$,

$$f(21^3;\, 3) - 2f(2^21;\, 3) + f(32;\, 3) = f(2;\, 0) + 3f(2;\, 1) + 3f(2;\, 2),$$

verified by $$93 - 2\,.\,57 + 27 = 0 + 3\,.\,1 + 3\,.\,1.$$

In the summation formula of the present article, if we put

$$(p_1p_2p_3\dots 1^\mu) = (1^\mu),$$

the dexter vanishes and we obtain the series of identities

$$F(1)$$
$$= F(1^2) - F(2),$$
$$= F(1^3) - 2F(21) + F(3),$$
$$= F(1^4) - 3F(21^2) + F(2^2) + 2F(31) - F(4),$$
$$= \dots$$

(the law being that of the expression of the functions h in terms of the functions a), verified by

$$\begin{aligned} &1 \\ &= 3 - 2, \\ &= 13 - 2.8 + 4, \\ &= 75 - 3.44 + 26 + 2.20 - 8, \\ &= \ldots. \end{aligned}$$

These relations should be compared with those of Art. 138 in which the coefficients followed the law of the expression of the sums of powers in terms of the functions h.

As another example put $(p_1 p_2 p_3 \ldots) = (4)$, $\mu = 3$, giving

$$F(41^3) - 2F(421) + F(43)$$

$$= f(4;\ 0) + (3+4.1) f(4;\ 1) + (3+4.3+10.1) f(4;\ 2)$$

$$+ (1+4.3+10.3+20.1) f(4;\ 3) + (4.1+10.3+20.3+35.1) f(4;\ 4),$$

$$= f(4;\ 0) + 7f(4;\ 1) + 25f(4;\ 2) + 63f(4;\ 3) + 129f(4;\ 4);$$

verified by $3408 - 3776 + 768$

$$= 0 + 7.1 + 25.3 + 63.3 + 129.1 = 400.$$

The Graphical Representation of the Compositions of Bipartite Numbers.

141. The first graphical method that has been employed in the case of the compositions of unipartite numbers can be extended so as to meet the cases of bipartite, tripartite and multipartite numbers in general. For the present the bipartite case is under consideration.

The graph of the bipartite number (pq) is derived directly from the graphs of the unipartite numbers (p), (q).

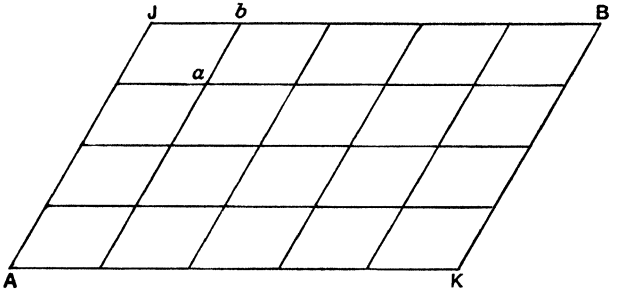
Take $q+1$ exactly similar graphs of the number p and place them parallel to one another at equal distances apart and so that their left-hand extremities lie upon a straight line; corresponding points of the $q+1$ graphs can then be joined by straight lines and a reticulation or lattice will be formed which is the graph of the bipartite number (pq).

We have AK a graph of the number p and $q+1$ such graphs parallel to one another; and AJ a graph of the number q and $p+1$ such graphs parallel to one another.

The points A, B are the initial and final points of the graph.

The remaining intersections are termed the "points" of the graph.

The lines of the graph have either the direction AK or the direction AJ. These will be called the α and β directions respectively. Through each point of the graph pass lines in each of these directions. Each line is made up of segments, and we speak of α and β segments to indicate that the lines, on which lie the segments, are in the α and β directions.



Suppose a traveller to proceed from A to B by successive steps. A step is taken by moving over a certain number of α segments and subsequently moving over a certain number of β segments. A step is thus made up of two figures—say an α figure and a β figure. The number of segments moved over may be zero in either but not in both of these two figures of the step.

A step may be taken from A to any point a of the graph; a second step may be taken from a to any point of the graph aB, which has a and B for its initial and final points; subsequent steps are taken on a similar principle and the last step terminates at the point B and completes the procession from the point A to the point B.

A step which takes x α-segments followed by y β-segments is taken to be a representation of a bipartite part (xy). A procession from the initial to the final point of the graph thus represents a sequence of bipartite parts which constitutes a composition of the bipartite number (pq). To every procession from A to B corresponds a composition of the bipartite number (pq) and the enumeration of the different processions is identical with the enumeration of the whole number of different compositions.

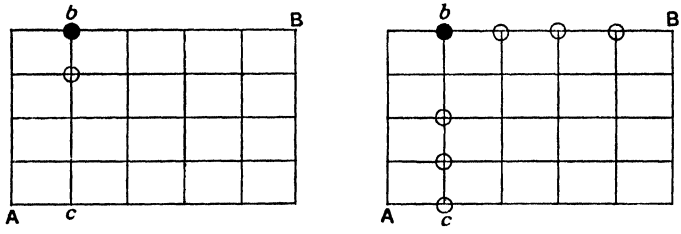
The steps of a procession are marked out by nodes placed at the points of the reticulation which terminate the first, second, third, etc. and penultimate steps. When nodes are thus placed we have the graph of a composition. If nodes be placed at the points a, b of the diagram we obtain the graph of the composition (13, 01, 40) of the bipartite (54). The number of parts in a composition is one greater than the number of nodes in its graph.

Having under view the number of compositions of (pq) which have two parts, it is clear that we may place the single graph node at any of

$(p+1)(q+1)-2$ points of the lattice. Hence the number of two-part compositions of (pq) is

$$p+q+pq.$$

142. The graph of a composition traced from A to B passes over certain segments and may be said to follow a certain "line of route" through the lattice. Other compositions may follow the same line of route; they all have their defining nodes upon the line of route and in general a *certain number* of defining nodes will be common to them all. Consider the path $AcbB$ in the graph given below. All compositions whose graphs follow this line of route must have the node b; b is an "essential node" along this line of route. Essential nodes occur on a line of route at all points where the line changes from the β to the α direction. We may regard a line of route as specified by these essential nodes, since the line of route is completely determined by these nodes.



The number of different lines of route that can be traced upon the graph of the bipartite number (pq) is equal to the number of permutations of p symbols α and q symbols β, for this is the number of ways in which the p α-segments and the q β-segments which compose a line of route can form a succession.

Hence the number of different lines of route is

$$\binom{p+q}{p}.$$

This result is evident from another point of view; for it will be observed that the line of route $AcbB$ is with reference to the points upon it the zig-zag graph of the composition 151111 of the unipartite number 10 (ten). In general every line of route through the lattice of the bipartite number (pq) is the zig-zag graph of a composition, possessing $p+1$ parts, of the unipartite number $p+q+1$; the number of such compositions therefore is the same as the number of lines of route and is therefore by Art. 125 of this Section

$$\binom{p+q}{p}.$$

This fact constitutes an interesting bond of union between the compositions of bipartite and unipartite numbers.

143. Every graph of a composition involves nodes of two kinds, essential and non-essential. We will now discuss the compositions appertaining to a given line of route. These naturally arrange themselves in pairs. Associated with any one graph is another obtained by obliterating those nodes which are non-essential and then inserting nodes at those points upon the line of route which were not previously occupied by nodes. These two graphs are said to be conjugate. Also the compositions represented by these graphs are said to be conjugate.

Conjugate graphs are shewn in the diagrams, the essential node b being black, the non-essential nodes white. The conjugate compositions are

$$(13, 01, 40), \quad (10, 01, 01, 02, 10, 10, 10, 10)$$

of the bipartite (54).

If the graph of a composition of (pq), having r parts, has s essential nodes, it is clear that the conjugate composition has $p+q-r+s+1$ parts; thus of compositions of (pq), which have graphs with s essential nodes, there is a one-to-one correspondence between those of r parts and those of $p+q-r+s+1$ parts.

Corresponding to an essential node in the graph of a composition there exist in the composition itself adjacent parts $(\ldots p_1q_1, p_2q_2, \ldots)$ such that q_1 and p_2 are both greater than zero. Thus from inspection of a composition we are enabled to determine the number of essential nodes in its graph.

It is useful to recognize four species of contact between adjacent parts of a composition

in $(\ldots p_1q_1, p_2q_2, \ldots)$

if q_1	is	zero	and p_2	zero	there	is a zero-zero	contact
„	„	„		positive	„	zero-positive	„
„		positive	„	zero	„	positive-zero	„
„	„	„		positive	„	positive-positive	„

In this nomenclature we may say that the graph of a composition possesses as many essential nodes as the composition itself possesses positive-positive contacts. The theorem reached may be stated as follows:

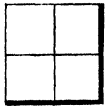
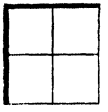
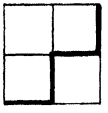
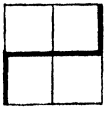
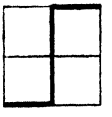
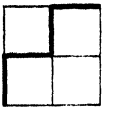
"Of compositions of the bipartite number (pq) possessing s positive-positive contacts there is a one-to-one correspondence between those of $s+t+1$ parts and those of $p+q-t$ parts."

From any line of route may be derived a composition whose graph exhibits essential nodes and no others; this is termed the "principal composition" along the line of route; it will have $s+1$ parts and each of the s contacts of its parts will be positive-positive.

The associated composition of the unipartite number $p+q+1$ which has $p+1$ parts is such that if the last or right-hand part be deleted, there will remain s parts whose magnitude is superior to unity.

The associated permutation of the letters in $\alpha^p\beta^q$ clearly possesses s $\beta\alpha$-contacts.

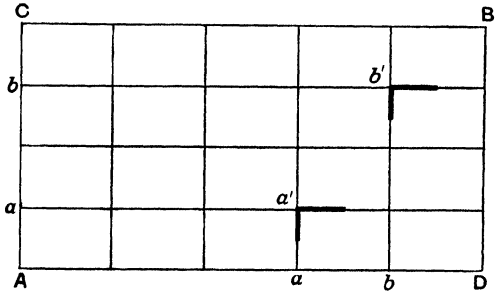
Ex. gr. for the bipartite number (22) the correspondence is

Lines of route						
Bipartite compositions	(22)	(02, 20)	(11, 11)	(01, 21)	(12, 10)	(01,11,10)
Unipartite compositions	(113)	(311)	(122)	(212)	(131)	(221)
Permutations	$\alpha\alpha\beta\beta$	$\beta\beta\alpha\alpha$	$\alpha\beta\alpha\beta$	$\beta\alpha\alpha\beta$	$\alpha\beta\beta\alpha$	$\beta\alpha\beta\alpha$

144. We now enquire into the number of lines of route through the lattice which possess exactly s essential nodes; it will be shewn that the number we seek is

$$\binom{p}{s}\binom{q}{s}.$$

First proof.



In each of the adjacent sides AD, AC of the graph of (pq) select any s points (see definition of "point") $a, b, c, \ldots$ in order from the point A. The two points a are seen to determine an essential node a'; the two points b one b' and so on, and a line of route necessarily exists which possesses these s essential nodes and no others.

The points along AD, AC from which s points can be selected are in number p, q respectively and the selection can take place in $\binom{p}{s}$, $\binom{q}{s}$ ways respectively; hence the number of lines of route having exactly s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

145. *Second proof.* We determine the number of permutations of $\alpha^p\beta^q$ which possess s $\beta\alpha$-contacts. Write down the s $\beta\alpha$-sequences

$$\ldots\beta\alpha\ldots\beta\alpha\ldots\beta\alpha\ldots\beta\alpha\ldots$$

and the $s+1$ intervals and end spaces. In these we have to distribute the letters in

$$\alpha^{p-s}\beta^{q-s}$$

in such manner as to introduce no fresh $\beta\alpha$-contacts. For each of these $p+q-2s$ letters there is a choice of $s+1$ spaces. The number of ways of so distributing the $p-s$ letters α is equal to the coefficient of x^{p-s} in the development of

$$(1+x+x^2+\ldots)^{s+1}$$

or of

$$(1-x)^{-s-1},$$

and this is $\binom{p}{s}$; similarly the $q-s$ letters β may be distributed in $\binom{q}{s}$ ways, and as each of the distributions of the letters α may occur with each of those of the letters β, the total number of permutations which possess exactly s $\beta\alpha$-contacts is

$$\binom{p}{s}\binom{q}{s}.$$

146. *Third proof.* This proof will be found to be of fundamental importance. We consider, as in the second proof, the permutations of the $p+q$ letters $\alpha^p\beta^q$ which exhibit exactly s $\beta\alpha$-contacts. The proof depends upon establishing that there exists a one-to-one correspondence between these permutations and those in which the letter β occurs exactly s times in the first p places counted from the left.

Suppose a permutation with s $\beta\alpha$-contacts to be

$$\alpha^{x_1}\beta^{y_1}\beta\alpha\,\alpha^{x_2}\beta^{y_2}\beta\alpha\,\alpha^{x_3}\beta^{y_3}\beta\alpha\,\alpha^{x_4}\beta^{y_4},$$

where for convenience s has the special value 3.

Any of the indices x, y may be zero and

$$\Sigma x+3=p,\quad \Sigma y+3=q.$$

Now obliterate the letters β which do not occur in $\beta\alpha$-contacts and the letters α which do so occur; there remains a succession

$$\alpha^{x_1}\beta\alpha^{x_2}\beta\alpha^{x_3}\beta\alpha^{x_4},$$

in which there are p letters and β occurs s times.

Next obliterate in the original permutation the letters α which do not occur in $\beta\alpha$-contacts and the letters β which do so occur; there remains a succession

$$\beta^{y_1}\alpha\beta^{y_2}\alpha\beta^{y_3}\alpha\beta^{y_4},$$

in which there are q letters and α occurs s times.

Take these two successions for the left- and right-hand portions of a new permutation, viz.:

$$\alpha^{x_1}\beta\alpha^{x_2}\beta\alpha^{x_3}\beta\alpha^{x_4}\,.\,\beta^{y_1}\alpha\beta^{y_2}\alpha\beta^{y_3}\alpha\beta^{y_4},$$

and it will be observed that we have made a perfectly definite transformation of a permutation involving exactly s $\beta\alpha$-contacts into another possessing the property that the letter β occurs exactly s times in the first p places.

It will be also remarked that if k of the indices ($2s$ in number)

$$x_2,\ x_3,\ x_4;\quad y_1,\ y_2,\ y_3$$

be zero, the new permutation possesses $2s-k$ $\beta\alpha$-contacts.

Ex. gr. To transform

$$\beta^4\alpha^2\beta^3\alpha$$

write it

$$\alpha^0\beta^3\beta\alpha\ \alpha\beta^2\beta\alpha$$

from which we obtain the successions $\alpha^0\beta\alpha\beta$, $\beta^3\alpha\beta^2\alpha$ and the permutation

$$\beta\alpha\beta^4\alpha\beta^2\alpha,$$

in which β presents itself twice in the first three places.

These transformed permutations are very easily enumerated, because the number is evidently the coefficient of $\lambda^s\alpha^p\beta^q$ in the development of

$$(\alpha+\lambda\beta)^p(\alpha+\beta)^q$$

or, using the master theorem of Section III, in the development of

$$\frac{1}{1-\alpha-\beta+(1-\lambda)\alpha\beta},$$

which may be written

$$\frac{1}{(1-\alpha)(1-\beta)}+\frac{\lambda\alpha\beta}{(1-\alpha)^2(1-\beta)^2}+\frac{\lambda^2\alpha^2\beta^2}{(1-\alpha)^3(1-\beta)^3}+\ldots;$$

wherein the coefficient of $\alpha^p\beta^q$ is readily found to be

$$1+\binom{p}{1}\binom{q}{1}\lambda+\binom{p}{2}\binom{q}{2}\lambda^2+\ldots+\binom{p}{s}\binom{q}{s}\lambda^s+\ldots.$$

The required result is thus established by a method which will be found later to be of great utility.

147. Upon each of the $\binom{p}{s}\binom{q}{s}$ lines of route which have s essential nodes may be represented

$$2^{p+q-s-1}$$

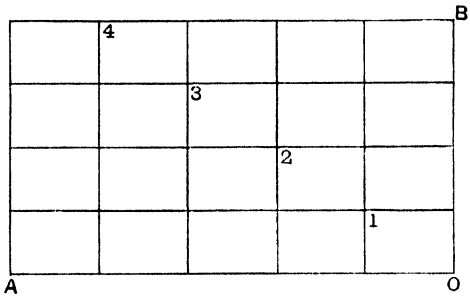
compositions because the $p+q-s-1$ non-essential nodes upon each line of route may be selected as composition nodes in this number of ways. Hence the total number of compositions is

$$F(pq)=\sum_s\binom{p}{s}\binom{q}{s}2^{p+q-s-1}.$$

Analytically this result is equivalent to the algebraic expansion

$$\frac{1}{1-2(x+y-xy)}$$
$$= \frac{1}{(1-2x)(1-2y)} + \frac{2xy}{(1-2x)^2(1-2y)^2} + \frac{2^2x^2y^2}{(1-2x)^3(1-2y)^3} + \ldots.$$

Viewed graphically that portion of a line of route traced by the initial p segments terminates in one of the points 0, 1, 2, 3, 4, ... in the diagram.



All of these points lie upon a straight line through the right-hand lower corner. If one of these points be marked s, s of the p segments are β segments and every line of route passing through the point s has the property that s symbols β present themselves in the first p places of the corresponding permutation of $\alpha^p\beta^q$. Hence there is a one-to-one correspondence between the lines of route passing through the points numbered 0, 1, 2, ... s, ... and the permutations of $\alpha^p\beta^q$ which are such that β occurs 0, 1, 2, ... s, ... times in the first p places.

The number of lines of route in the graph of which A, s are the initial and final points is $\binom{p}{s}$, and in the graph of which s, B are the initial and final points, $\binom{q}{s}$, so that again we have the enumerating number

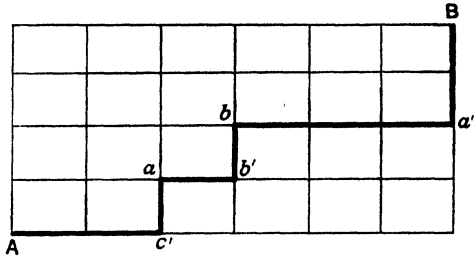
$$\binom{p}{s}\binom{q}{s}.$$

Inverse Bipartite Compositions.

148. A line of route being marked out upon a lattice from A to B the inverse line of route is obtained by rotating the lattice through two right angles and interchanging the letters A, B.

Consider a line of route from A to B having essential nodes a, b. On the inverse line of route a', b', c' are the essential nodes.

The principal compositions along these lines of route are



(21, 11, 32) from A to B.

(02, 31, 11, 20) from B to A.

All of the contacts in these compositions are necessarily positive-positive; hence the leading and ending parts are the only ones that can involve a zero element. Every part of a principal composition which does not possess a zero element necessitates an essential node upon the graph of the principal composition upon the inverse line of route. Thus if the one principal composition has s nodes upon its graph, $s+1$ parts and t parts without a zero element, the other has t nodes upon its graph, $t+1$ parts and s parts without a zero element. These may be called inverse principal compositions.

The number of pairs of inverse principal compositions is equal to half the number of distinct lines of route through the lattice.

We may say that in regard to compositions, all of whose contacts are positive, there is a one-to-one correspondence between those having $s+1$ parts and t parts without a zero element and those having $t+1$ parts and s parts without a zero element.

The number $s+1-t$ or the number $t+1-s$ is either 0, 1 or 2.

Ex. gr. for the bipartite (22) the correspondence is

$s=0,\ t=1$	$s=1,\ t=0$
(22)	(02, 20)
$s=1,\ t=1$	
(01, 21)	(12, 10)
$s=1,\ t=2$	$s=2,\ t=1$
(11, 11)	(01, 11, 10)

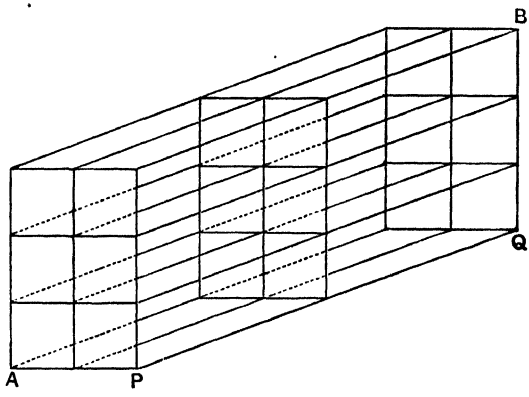
As regards the permutations of $\alpha^p\beta^q$ the correspondence is between those which have s $\beta\alpha$-contacts and t $\alpha\beta$-contacts and those which have t $\beta\alpha$-contacts and s $\alpha\beta$-contacts, the one permutation being obviously the other read backwards.

Suppose a line of route traced on a lattice and thereon to be marked the s essential nodes and the t essential nodes on the inverse line. These s and t nodes are distinct and we may place an additional node at an unoccupied point and interpret the new compositions as read before and after the rotations of the lattice. They have each acquired an additional part which has either introduced a zero-positive contact into each or a positive-zero contact into each according as the added node is on an α or on a β line between adjacent nodes. Altogether there are $p+q-1-s-t$ points at disposal which may be selected as positions for nodes in $2^{p+q-1-s-t}$ ways. We have thus pairs of inverse compositions and we may also have self-inverse compositions. Such can only occur upon self-inverse lines of route and one at least of the elements of the bipartite numbers must be even.

CHAPTER III

THE GRAPHICAL REPRESENTATION OF THE COMPOSITIONS OF TRIPARTITE AND MULTIPARTITE NUMBERS

149. The graph of a tripartite number may be in either two or three dimensions. We derive it from a bipartite graph just as the bipartite has been derived from the unipartite graph. For the tripartite number (pqr) we take $r+1$ exactly similar graphs of the bipartite number (pq) and place them similarly with corresponding lines parallel and similar points lying upon straight lines; when these straight lines are drawn the graph is complete. The $r+1$ bipartite graphs may be in the same plane or in parallel planes according as the tripartite graph is required to be in two or three dimensions.



The figure depicts the graph of the tripartite (233). The points of the lattice are identical with the points of the $r+1$ lattices of the $r+1$ bipartites (23). Other than the initial and final points A, B, there are

$$(p+1)(q+1)(r+1)-2$$

points. The graph involves lines in three different directions; say an α, a β and a γ direction; these are parallel respectively to AP, PQ and QB.

Through each point of the graph lines pass in all three directions and a segment joining two adjacent points is called an α segment when it is in the α direction. A line of route proceeds from A to B, from point to point of the lattice, its direction being either that of $A \rightarrow P$, or $P \rightarrow Q$, or $Q \rightarrow B$.

The number of lines of route is the same as the number of permutations of the symbols in $\alpha^p\beta^q\gamma^r$ and is therefore

$$\binom{p+q+r}{p,\ q}.$$

A step along a line of route traverses in succession any number of α, β, γ segments and in any one step the segments must be taken in the order α, β, γ. The number of segments traversed in a step may be zero in one or in two but not in three of these directions. A step which involves p_1 α-, q_1 β- and r_1 γ-segments is represented by a tripartite number $(p_1q_1r_1)$ and a succession of steps, the first starting at A and the last terminating at B, is represented by a succession of tripartite numbers which constitutes a composition of the tripartite number (pqr).

The graph of a composition is obtained by placing nodes at the points which terminate the first, second, ... penultimate steps.

Essential nodes occur upon lines of route whenever the direction at a point changes from β to α, from γ to α or from γ to β. These will be alluded to briefly as $\beta\alpha$, $\gamma\alpha$, $\gamma\beta$ essential nodes. To these essential nodes on the graph of a composition correspond

(i) $\beta\alpha$, $\gamma\alpha$, $\gamma\beta$ contacts in the associated permutation of $\alpha^p\beta^q\gamma^r$; these will be called "major" contacts.

(ii) Zero-positive, positive-positive, and positive-zero contacts in the composition itself. These will be called "essential" contacts.

The theory of conjugate composition exists as in the bipartite theory.

On every line of route there are $p+q+r-1$ points which are possible points on the graph of a composition, and if there be s essential nodes $2^{p+q+r-1-s}$ distinct composition graphs can be delineated along the line of route. Each of these has s essential contacts, and as in the bipartite case we establish a one-to-one correspondence between the compositions having s essential contacts and t non-essential contacts and those having s essential contacts and $p+q+r-1-s-t$ non-essential contacts; in other words the correspondence is between those with s essential contacts and $s+t+1$ parts and those with s essential contacts and $p+q+r-t$ parts.

There is also a theory of inverse compositions.

150. For the multipartite number

$$(p_1p_2\cdots p_{n-1}p_n),$$

we take p_n+1 exactly similar graphs of the multipartite numbers

$$(p_1p_2\cdots p_{n-1})$$

and place them similarly with corresponding lines parallel and corresponding points lying on straight lines. When these straight lines are drawn the

graph is complete. Other than the initial and final points there are $(p_1+1)(p_2+1)\dots(p_n+1)-2$ points in the graph. There are n different directions passing through or meeting at each point; we call these the $\alpha_1, \alpha_2, \dots \alpha_n$ directions. A step through the lattice traverses in succession any number of $\alpha_1, \alpha_2, \dots \alpha_n$ segments, but in any one step the segments must be taken in the order $\alpha_1, \alpha_2, \dots \alpha_n$. In a step the number of segments traversed may be zero in any one, any two, etc. ..., any $n-1$ of the n given directions. A step that involves p_1' segments in the α_1 direction, p_2' segments in the α_2 direction, and so on, may be represented by the multipartite number

$$(p_1' p_2' \dots p_n').$$

A succession of steps, the first starting at A and the last terminating at B, is represented by a succession of multipartite numbers, constituting a composition of the multipartite

$$(p_1 p_2 \dots p_n).$$

Any composition follows a certain line of route through the lattice.

The number of lines of route is the same as the number of permutations of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n},$$

and is therefore

$$\binom{p_1+p_2+\dots+p_n}{p_1, p_2, \dots p_{n-1}}$$

(the extension of a previous notation is obvious).

The graph of a composition is obtained by placing nodes at the points which terminate the first, second, etc. ... penultimate steps.

Essential nodes occur upon a line of route whenever the direction at a point changes from α_u to α_t where $u>t$. There are $\binom{n}{2}$ pairs of numbers u, t which satisfy this condition so that there are $\binom{n}{2}$ varieties of essential nodes. At such a point the contact between the adjacent parts of the composition is such that a part terminating with $n-u$ zero elements precedes a part commencing with $t-1$ zero elements. We speak of this as a contact of $n-u$ zeros with $t-1$ zeros, and the number of zeros in contact is

$$n-1-(u-t),$$

and this may be any number from 0 to $n-2$ according to the magnitude of $u-t$.

Corresponding to a line of route with s essential nodes there is a permutation of $\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$ which possesses s major contacts.

Upon every line of route there are $p_1+p_2+\dots+p_n-1$ points, and if the line has s essential nodes,

$$2^{p_1+p_2+\dots+p_n-1-s}$$

distinct composition graphs can be delineated along the line of route.

Included in these will be the principal composition graph along which there are only essential nodes. The principal composition thence derived is such that all contacts between adjacent parts are of the nature

$$\dots 0^{n-u}, \quad 0^{t-1} \dots,$$

where $u > t$ and t may have all values from 1 to $n-1$.

We establish a one-to-one correspondence between the compositions which have s essential contacts and $s+t+1$ parts and those which have s essential contacts and $p_1 + p_2 + \dots + p_n - t$ parts.

151. We now enquire, in respect of the tripartite lattice, into the number of lines of route which possess exactly

s_{21} $\beta\alpha$ essential nodes
s_{32} $\gamma\beta$,,
s_{31} $\gamma\alpha$,,

This is the same thing as the investigation of the number of permutations of the symbols in

$$\alpha^p \beta^q \gamma^r,$$

which involve s_{21}, $\beta\alpha$; s_{32}, $\gamma\beta$; s_{31}, $\gamma\alpha$ contacts, where we suppose that the number of major contacts is

$$s_{21} + s_{32} + s_{31} = s.$$

It will be shewn that the number is equal to the number of permutations in which

β occurs s_{21} times in the first p places
γ ,, s_{31} ,, ,,
γ ,, s_{32} ,, in the q places following the first p.

A permutation of the former kind involves successions of letters

$$\beta\alpha, \ \gamma\beta, \ \gamma\alpha, \ \gamma\beta\alpha,$$

and as regards these successions and attending only to them the permutation exhibits some permutation of the terms in

$$(\beta\alpha)^{s_{21}-\sigma} (\gamma\beta)^{s_{32}-\sigma} (\gamma\alpha)^{s_{31}} (\gamma\beta\alpha)^{\sigma};$$

the number of these terms is $s_{21} + s_{32} + s_{31} - \sigma$, and they admit of

$$\frac{(s_{21} + s_{32} + s_{31} - \sigma)!}{(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,s_{31}!\,\sigma!}$$

permutations. Selecting any one of these there are with reference to the terms $s_{21} + s_{32} + s_{31} - \sigma + 1$ positions or intervals available for the insertion of other letters.

It is clear that α cannot be placed after a term $(\gamma\beta)$ or γ before a term $(\beta\alpha)$, for these placings would lead to additional major contacts; the letter α

may be placed before any of the terms $(\beta\alpha)$, $(\gamma\beta)$, $(\gamma\alpha)$, $(\gamma\beta\alpha)$, and after any of the terms $(\beta\alpha)$, $(\gamma\alpha)$, $(\gamma\beta\alpha)$; hence $s_{21}+s_{31}+1$ out of $s_{21}+s_{32}+s_{31}-\sigma+1$ positions may be occupied by α; similarly γ may occupy only $s_{32}+s_{31}+1$ different positions; the letter β may be placed anywhere without the introduction of additional major contacts.

Besides the letters occurring in the terms

$$(\beta\alpha)^{s_{21}-\sigma}(\gamma\beta)^{s_{32}-\sigma}(\gamma\alpha)^{s_{31}}(\gamma\beta\alpha)^{\sigma},$$

α occurs $p-s_{21}-s_{31}$ times

β ,, $q-s_{21}-s_{32}+\sigma$,,

γ ,, $r-s_{32}-s_{31}$,,

The $p-s_{21}-s_{31}$ letters α, $q-s_{21}-s_{32}+\sigma$ letters β, $r-s_{32}-s_{31}$ letters γ can be distributed in $s_{21}+s_{31}+1$, $s_{21}+s_{32}+s_{31}-\sigma+1$, $s_{32}+s_{31}+1$ positions in

$$\binom{p}{s_{21}+s_{31}};\quad \binom{q+s_{31}}{s_{21}+s_{32}+s_{31}-\sigma};\quad \binom{r}{s_{32}+s_{31}}$$

ways respectively.

This is the case because the number of distributions of A letters α in P positions is given by the coefficient of x^A in the expansion of $\left(\frac{1}{1-x}\right)^P$ and is thus equal to $\binom{A+P-1}{P-1}$ or to $\binom{A+P-1}{A}$.

Hence for any given permutation of the terms in

$$(\beta\alpha)^{s_{21}-\sigma}(\gamma\beta)^{s_{32}-\sigma}(\gamma\alpha)^{s_{31}}(\gamma\beta\alpha)^{\sigma},$$

there are

$$\binom{p}{s_{21}+s_{31}}\binom{q+s_{31}}{s_{21}+s_{32}+s_{31}-\sigma}\binom{r}{s_{32}+s_{31}}$$

arrangements of the remaining letters which do not introduce fresh major contacts. Hence for a given value of σ there are

$$\binom{p}{s_{21}+s_{21}}\binom{q+s_{31}}{s_{21}+s_{32}+s_{31}-\sigma}\binom{r}{s_{32}+s_{31}}\frac{(s_{21}+s_{32}+s_{31}-\sigma)!}{(s_{21}-\sigma)!\,(s_{32}-\sigma)!\,s_{31}!\,\sigma!}$$

permutations.

To complete the enumeration we have to sum this expression in regard to σ. We find

$$\begin{aligned}
&\Sigma\binom{q+s_{31}}{s_{21}+s_{32}+s_{31}-\sigma}\frac{(s_{21}+s_{32}+s_{31}-\sigma)!}{(s_{21}-\sigma)!\,(s_{32}-\sigma)!\,s_{31}!\,\sigma!}\\
&=\frac{(q+s_{31})!}{s_{32}!\,s_{31}!\,(q-s_{32})!}\Sigma\binom{s_{32}}{\sigma}\binom{q-s_{32}}{s_{21}-\sigma}\\
&=\frac{(q+s_{31})!}{s_{32}!\,s_{31}!\,(q-s_{32})!}\binom{q}{s_{21}}\\
&=\binom{s_{21}+s_{31}}{s_{21}}\binom{q}{s_{32}}\binom{q+s_{32}}{s_{21}+s_{31}}.
\end{aligned}$$

Hence the number of permutations sought is

$$\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{32}+s_{31}}.$$

152. By direct expansion it can be shewn that this number is the coefficient of $\lambda_{21}^{s_{21}}\lambda_{32}^{s_{32}}\lambda_{31}^{s_{31}}\alpha^p\beta^q\gamma^r$ in the expansion of

$$(\alpha+\lambda_{21}\beta+\lambda_{31}\gamma)^p(\alpha+\beta+\lambda_{32}\gamma)^q(\alpha+\beta+\gamma)^r,$$

and is thus equal to the number of permutations of $\alpha^p\beta^q\gamma^r$ which are such that β and γ occur s_{21} and s_{31} times respectively in the first p places and γ occurs s_{32} times in the q places succeeding the first p. Hence by the master theorem of Section III the true generating function is

$$\frac{1}{1-\alpha-\beta-\gamma+(1-\lambda_{21})\alpha\beta+(1-\lambda_{32})\beta\gamma+(1-\lambda_{31})\alpha\gamma-(1-\lambda_{21})(1-\lambda_{32})\alpha\beta\gamma},$$

and that for the lines of route which have s essential nodes is

$$\frac{1}{1-\alpha-\beta-\gamma+(1-\lambda)(\alpha\beta+\beta\gamma+\alpha\gamma)-(1-\lambda)^2\alpha\beta\gamma}.$$

In the expansion of this function the coefficient of $\alpha^p\beta^q\gamma^r$ is

$$C_0+C_1\lambda+C_2\lambda^2+\ldots+C_s\lambda^s+\ldots,$$

where $$C_s=\Sigma\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{32}+s_{31}},$$

the summation being controlled by $s_{21}+s_{31}+s_{32}=s$.

Denoting the whole number of compositions of (pqr) by $F(pqr)$, we have

$$F(pqr)=\Sigma C_s 2^{p+q+r-s-1}=2^{p+q+r-1}\sum_s C_s\left(\tfrac{1}{2}\right)^s,$$

and thence we verify the true generating function previously obtained in Art. 131, for the redundant function

$$2^{p+q+r-1}(\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma)^p(\alpha+\beta+\tfrac{1}{2}\gamma)^q(\alpha+\beta+\gamma)^r,$$

otherwise written

$$\tfrac{1}{2}(2\alpha+\beta+\gamma)^p(2\alpha+2\beta+\gamma)^q(2\alpha+2\beta+2\gamma)^r,$$

leads to the generating function

$$\frac{1}{2}\cdot\frac{1}{1-2(\alpha+\beta+\gamma-\beta\gamma-\gamma\alpha-\alpha\beta+\alpha\beta\gamma)}.$$

153. The idea of composition is capable of enlargement from a particular point of view.

In regard to unipartite numbers consider p units placed in a row

1 1 1 1 1 1 .

there are $p-1$ spaces between them which may be occupied by algebraic symbols at pleasure. We may choose k different symbols and choose any one to occupy any one of the $p-1$ spaces; we thus arrive at k^{p-1} different expressions involving the p units. We may take as one of these symbols the simple unoccupied blank space, since such space left between any two numbers, quantities or expressions of any kind has, in every case, a definite signification, not always however the same, in mathematical notation. This blank space *might* be denoted by a definite symbol such as O.

If we restrict ourselves to a single symbol only one expression involving the units is possible; if the symbol be +, indicating addition, we merely get

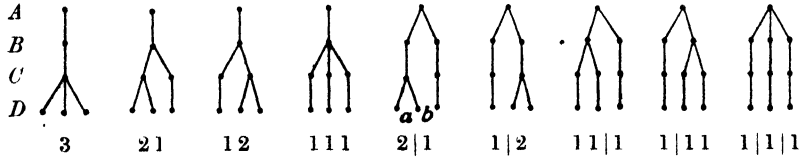
$$1+1+1+1+1+1+\ldots,$$

which is p or the number which enumerates the units; a blank space would yield a succession of p units; the symbol of multiplication unity and so forth. All of these modes of obtaining expressions from the p units are termed "combinations of the first order" in respect of the p units.

Passing to the case of two different symbols we may choose to employ the sign of addition and the blank space; we thus obtain 2^{p-1} different expressions, which are the several compositions of the unipartite number p. These are "combinations of the second order."

In general, expressions obtained from any k different symbols may be called "combinations of order k" in respect of the p units.

There is a correspondence between these combinations and the "Trees" which have an altitude k and p terminal knots. To form a "Tree" we start from a single point or knot, and from it draw any number σ_1 of branches each terminating in a knot; this would be a tree of altitude one and having σ_1 terminal knots; from each of these knots again we can draw branches in a similar manner and arrive at trees of altitude two and having σ_2 terminal knots and so on. As an example take $k=p=3$, and further take as symbols the sign of addition, the blank space and the symbol | of unspecified signification. The correspondence between the nine trees and the nine "combinations of order three" is shewn below:



The process consists in writing down a unit for each terminal knot. The $p-1$ intervals between the units correspond to the $p-1$ inter-terminal knot spaces. If a space leads to a bifurcation in row C the symbol is +; if in row B it is a blank space; if in row A it is |.

Thus for the fifth tree above $1+1 \mid 1$ or $2 \mid 1$ because the space a leads to a bifurcation in row C and the space b to one in row A.

On this principle a tree can be drawn to represent any combination of order k in respect of p units.

In the case of multipartite numbers there is also an extension of the idea of composition. Let us take tripartite numbers as being representative.

Arrange a row of tripartite units of the three kinds, viz.

$$100\ 100\ 100\ \ldots\ 010\ 010\ 010\ \ldots\ 001\ 001\ 001\ \ldots,$$

in number p_1, p_2, p_3 respectively. We assume that these units are added together in the manner

$$100 + 100 + 010 + 001 = 211.$$

By using the two symbols, the sign of addition and the blank space, we obtain a number of different compositions of the tripartite $(p_1 p_2 p_3)$; we may also permute the above succession of units in

$$\binom{p_1+p_2+p_3}{p_1,\ p_2}$$

ways and using the same two symbols obtain other compositions, the whole number *including* every composition of $(p_1 p_2 p_3)$. For a given succession of the units and a given placing of the two symbols in the $p_1+p_2+p_3-1$ spaces, it is clear that we may permute the units which lie between adjacent blank space symbols without altering the composition arrived at. Thence we see that the number of different expressions (in this case compositions) arrived at is not

$$2^{p_1+p_2+p_3-1}\binom{p_1+p_2+p_3}{p_1,\ p_2},$$

but some lesser number. This arises from the circumstance that the symbol $+$ is commutative with respect to the units, whereas the blank space symbol is not commutative in regard to units of different kinds.

If we were dealing with k different non-commutative symbols the number of combinations of order k would be

$$k^{p_1+p_2+p_3-1}\binom{p_1+p_2+p_3}{p_1,\ p_2},$$

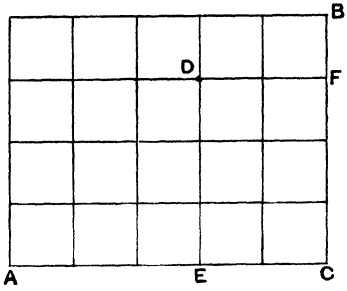
and the generating function might be written

$$\frac{1}{k}\frac{1}{1-k(\alpha_1+\alpha_2+\ldots+\alpha_n)},$$

but the number is less if one or more of the symbols be commutative.

154. The important case for the present purpose is that in which one symbol, the blank space, is non-commutative, and the remaining $k-1$

symbols commutative as between the units of different kinds. Consider the lattice of a bipartite number



A combination of order k (of the nature under examination) is regarded as having m parts when $m-1$ blank space symbols occur in the combination.

In regard to any line of route a blank space node may be essential or non-essential, but an essential node *must* be a blank space node.

We will enquire into the number of combinations which possess only a single part; the line of route must not have an essential node, and therefore can only be ACB; the graphs of the one-part combinations must, all of them, be along this line of route. Similarly in the lattice of the multipartite $(p_1 p_2 \dots p_n)$ the graphs of the one-part combinations must, all of them, be along the line of route traced out in order by the p_1 segments α_1, the p_2 segments α_2, etc., the p_n segments α_n. The $k-1$ different symbols at our disposal may be placed at pleasure at the $p_1+p_2+\dots+p_n-1$ points along this line of route. Hence

$$(k-1)^{p_1+p_2+\dots+p_n-1}$$

different one-part combinations are obtainable. This number depends only upon the sum of the integers $p_1, p_2, \dots p_n$, which define the multipartite number, and therefore

$$\Sigma\,(k-1)^{p_1+p_2+\dots+p_n-1}\,\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$$

is the generating function; this is

$$h_1+(k-1)\,h_2+(k-1)^2 h_3+\dots \textit{ ad inf.},$$

h_s denoting the homogeneous product sum, degree s, of the quantities $\alpha_1, \alpha_2, \dots \alpha_n$.

Next as to the combinations which have two parts. At any point D of the lattice place a blank space node. This point *may* be on the line of route ACB, and then and then only will it not be an essential node. All two-part combinations whose graphs pass through D must follow the line of route $AEDFB$, for otherwise an additional essential (and blank space) node would be introduced. The whole combination may be split up into a one-part combination along the line of route AED followed by a blank space and

a one-part combination along the line of route *DFB*. We must associate every one-part combination in the lattice AD with every one-part combination in the lattice DB, and thus we see that the whole number of two-part combinations (of order k) of the multipartite number $(p_1 p_2 \dots p_n)$ is

$$\Sigma (k-1)^{p_1'+p_2'+\dots-1} (k-1)^{p_1''+p_2''+\dots-1},$$

where $(p_1' p_2' \dots, p_1'' p_2'' \dots)$ is a bipartite composition (where composition denotes a combination of order 2) of $(p_1 p_2 \dots)$. Hence denoting as before by

$$f(p_1 p_2 \dots ; 2)$$

the number of bipartite compositions of the multipartite $(p_1 p_2 \dots)$, we find that the number of two-part combinations of the multipartite $(p_1 p_2 \dots)$ is

$$f(p_1 p_2 \dots ; 2)(k-1)^{p_1+p_2+\dots-2},$$

and the symmetric function generating function will be

$$\Sigma f(p_1 p_2 \dots ; 2)(k-1)^{p_1+p_2+\dots-2}(p_1 p_2 \dots).$$

Moreover D being any point in the reticulation and remaining unspecified we have to associate one-part combinations in the lattice AD with one-part combinations in the lattice DB, and since *each of these* has the symmetric function generating function

$$\Sigma (k-1)^{p_1+p_2+\dots-1}(p_1 p_2 \dots),$$

it is clear that the two-part combinations must have the generating function

$$\{\Sigma (k-1)^{p_1+p_2+\dots-1}(p_1 p_2 \dots)\}^2$$

or

$$\{h_1 + (k-1)h_2 + (k-1)^2 h_3 + \dots\}^2.$$

This is equivalent to

$$\Sigma f(p_1 p_2 \dots ; 2)(k-1)^{p_1+p_2+\dots-2}(p_1 p_2 \dots),$$

the summation being for all partitions of all numbers, but is in a handier and more tractable form.

By precisely similar reasoning it is established that the generating function of the m-part combinations of the multipartite $(p_1 p_2 \dots)$ is

$$\{h_1 + (k-1)h_2 + (k-1)^2 h_3 + \dots\}^m.$$

As a verification it is observed that putting $k=2$ we reach the generating function of the m-part *compositions* of the multipartite.

155. The complete generating function for the enumeration of the combinations of order k, one symbol only being non-commutative, is consequently

$$\frac{h_1 + (k-1)h_2 + (k-1)^2 h_3 + \dots}{1 - h_1 - (k-1)h_2 - (k-1)^2 h_3 - \dots};$$

but $$h_1 - (k-1)h_2 + (k-1)^2 h_3 - \dots = \frac{a_1 - (k-1)a_2 + (k-1)^2 a_3 - \dots}{1 - (k-1)a_1 + (k-1)^2 a_2 - \dots},$$

so that transforming it and adding the fractional number $\frac{1}{k}$ it may be expressed as

$$\frac{1}{k}\,\frac{1}{1-k\Sigma\alpha_1+k(k-1)\Sigma\alpha_1\alpha_2-k(k-1)^2\Sigma\alpha_1\alpha_2\alpha_3+\ldots},$$

and if we restrict ourselves to a finite value of n the denominator may be considered as terminating with the term

$$(-)^n k(k-1)^{n-1}\alpha_1\alpha_2\ldots\alpha_n.$$

The master theorem of Section III shews that this expression is derived through the guiding determinant

$$\frac{1}{k}\begin{vmatrix} k & k & k & k & \ldots \\ 1 & k & k & k & \ldots \\ 1 & 1 & k & k & \ldots \\ 1 & 1 & 1 & k & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad \text{or} \quad \frac{1}{k}\begin{vmatrix} k & 1 & 1 & 1 & \ldots \\ k & k & 1 & 1 & \ldots \\ k & k & k & 1 & \ldots \\ k & k & k & k & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

of order n; and thence a redundant generating function is

$$\frac{1}{k}(k\alpha_1+k\alpha_2+\ldots+k\alpha_n)^{p_1}(\alpha_1+k\alpha_2+\ldots+k\alpha_n)^{p_2}\ldots(\alpha_1+\alpha_2+\ldots+\alpha_{n-1}+k\alpha_n)^{p_n}$$

or $$\frac{1}{k}(k\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_1}(k\alpha_1+k\alpha_2+\ldots+\alpha_n)^{p_2}\ldots(k\alpha_1+k\alpha_2+\ldots+k\alpha_n)^{p_n}.$$

The latter form may be written

$$k^{p_1+p_2+\ldots+p_{n-1}}\left\{\alpha_1+\frac{1}{k}(\alpha_2+\ldots+\alpha_n)\right\}^{p_1}\left\{\alpha_1+\alpha_2+\frac{1}{k}(\alpha_3+\ldots+\alpha_n)\right\}^{p_2}\ldots(\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_n}$$

and herein the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$ is

$$\Sigma_s C_s k^{\Sigma p-s-1},$$

where C_s is the coefficient of $\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$ in the development of

$$\{\alpha_1+\lambda(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\ldots+\alpha_n)\}^{p_2}\ldots(\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_n}.$$

The number of combinations of order k, one symbol being non-commutative, of the multipartite number $(p_1p_2\ldots p_n)$ is therefore

$$\Sigma_s C_s k^{\Sigma p-s-1},$$

but if in the lattice of the multipartite number there be D_s lines of route which possess exactly s essential nodes,

$$D_s k^{\Sigma p-s-1}$$

combinations of order k may be represented upon these lines, and hence the whole number of combinations of order k, one symbol being non-commutative, is also

$$\Sigma D_s k^{\Sigma p-s-1},$$

which leads us to the relation

$$\sum_s C_s k^{\Sigma p - s - 1} = \sum_s D_s k^{\Sigma p - s - 1},$$

which, being true for all integral values of k, shews that

$$C_s = D_s.$$

In other words the number of permutations of $\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$ which exhibit precisely s major contacts is equal to the coefficient of $\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$ in the development of the product

$$\{\alpha_1 + \lambda(\alpha_2 + \dots + \alpha_n)\}^{p_1}\{\alpha_1 + \alpha_2 + \lambda(\alpha_3 + \dots + \alpha_n)\}^{p_2}\dots(\alpha_1 + \alpha_2 + \dots + \alpha_n)^{p_n},$$

or in the expansion of the algebraic fraction

$$\frac{1}{1 - \Sigma\alpha_1 + (1-\lambda)\Sigma\alpha_1\alpha_2 - (1-\lambda)^2\Sigma\alpha_1\alpha_2\alpha_3 + \dots},$$

an important theorem in permutations. Interpreting the redundant product we have the following:

Theorem. "The number of permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_n^{p_n}$$

which possess exactly s major contacts is equal to the number of permutations for which

$$r_2 + r_3 + \dots + r_n = s,$$

r_t denoting the number of times that the letter α_t occurs in the first

$$p_1 + p_2 + \dots + p_{t-1}$$

places of the permutation."

CHAPTER IV

SIMON NEWCOMB'S PROBLEM

156. The question which we now take up involves much that is interesting in the Theory of Distributions and its discussion is a convenient way of introducing new methods and new points of view. The problem was suggested to the late Professor Newcomb by a game of "patience" played with ordinary playing cards which he found to be a recreation in the few hours that he could spare from astronomical work. It may be stated as follows:

"A pack of cards of any specification is taken—say there are p cards marked 1, q cards 2, r cards 3 and so on—and being shuffled is dealt out on a table; so long as the cards that appear have numbers that are in ascending order of magnitude, equality of number counting as ascending order, they are placed together in one pack, but directly the ascending order is broken a fresh pack is commenced and so on until all the cards have been dealt. The probability that there will result exactly m packs or at most m packs is required."

The result of the deal will be m packs containing in order $a, b, c, \ldots$ cards respectively, where, n being the number of cards in the whole pack,

$$(abc\ldots)$$

is some composition, of the number n, involving m parts.

We have then for discussion:

(i) the number of ways of arranging the cards so as to yield a given composition $(abc\ldots)$;

(ii) the number of arrangements which lead to a distribution into exactly m packs.

The solution of these questions depends only upon the numbers $p, q, r, \ldots$, that is to say upon the specification of the pack and not at all upon the actual numbers marked upon the cards. We will commence by considering the case where the pack has a specification (1^n)—that is to say the cards are all different—and it is convenient to suppose that the n different numbers are in fact the first n natural numbers.

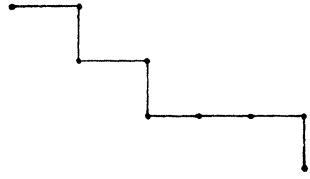
Consider the permutations of the first n integers and for simplicity let $n = 9$. Writing down a permutation at random

$$8|67|29|5|4|13,$$

we can draw lines separating the numbers into compartments in such wise that in each compartment the numbers are in ascending order of magnitude. We can then write down a succession of numbers which specify the sizes of the compartments proceeding from left to right, and arrive at a composition 122112 of the number 9; we say that the permutation under examination has an ascending specification (122112) or $(12^2 1^2 2)$. Similarly from the descending character

$$86|72|9541|3,$$

we say that the descending specification is (2241) or $(2^2 41)$ and it will be remarked that $(12^2 1^2 2)$ and $(2^2 41)$ are conjugate compositions which the zig-zag graph



puts in evidence. If we invert the permutation the ascending specification is the inverse of the conjugate composition; thus

$$3|1459|27|68,$$

and if we substitute for the number p in the permutations the number $n - p + 1$ we obtain

$$24|38|1569|7,$$

$$79|6|5|18|34|2,$$

which have specifications (2241) and (211221), and thus from a single permutation we derive three other permutations, and the four associated compositions are those derivable from a single zig-zag graph by the four modes of reading. If then we formulate the question: of the permutations of the first n natural numbers how many have an ascending specification denoted by a given composition of the number n? it is clear that whatever the answer, the same answer must in general be given for three other compositions, viz. the three others derivable from the zig-zag graph. In two cases the four compositions shrink to two, viz.

(i) when the composition is self-inverse,

(ii) when the conjugate and the inverse are identical.

The compositions therefore may be grouped in lots of either 4 or 2, and the reader will have no difficulty in establishing from the results reached in

a previous chapter that, if n is an even number $2n'$, the number of lots is $2^{n'-2}(2^{n'-1}+1)$ and that, if it is an uneven number $2n'+1$, the number of lots is $2^{n'-1}(2^{n'-1}+1)$. For $n=2, 3, 4, 5, 6, \ldots$ the numbers of lots are $1, 2, 3, 6, 10, \ldots$ respectively.

When the pack has a specification $(pqr\ldots)$ we denote by

$$N(abc\ldots)_{(pqr\ldots)}$$

the number of permutations of the cards which have the specification $(abc\ldots)$. At present we are concerned with the numbers

$$N(abc\ldots)_{(1^n)}$$

where $(abc\ldots)$ is some composition of the number n, for the partition (1^n) is the specification of n cards no two of which are alike, and therefore also of the first n natural numbers. When there is no risk of misunderstanding we shall write simply $N(abc\ldots)$, the partition subscript being understood.

157. Obviously $N(n)=N(a)=1$, for it is only when the numbers are in ascending order that the associated composition is (a).

To determine $N(ab)$, where $a+b=n$, we separate the n integers into two groups, a left-hand group of a numbers chosen at random and a right-hand group comprising the remaining b numbers; this can be done in $\binom{n}{a}$ different ways. We next arrange the numbers in each group in ascending orders for each of the $\binom{n}{a}$ separations and thus obtain each of the permutations enumerated by $N(a, b)$ and also the one permutation enumerated by $N(a+b)$.

Hence

$$N(a, b)+N(a+b)=\binom{n}{a}$$

or

$$N(ab)=\binom{n}{a}-N(a+b)=\binom{n}{a}-1.$$

Again to find $N(abc)$ where $a+b+c=n$, we separate the n integers into three groups containing, in order from left to right, a, b and c integers respectively; this can be done in

$$\frac{n!}{a!\,b!\,c!}$$

different ways. Placing the numbers in each group in ascending order we obtain the permutations enumerated by

$$N(abc),\quad N(a+b, c),\quad N(a, b+c),\quad N(a+b+c),$$

because the right-hand number in a group is or is not greater than the left-hand number of the group standing to its right. Hence

$$N(abc)+N(a+b, c)+N(a, b+c)+N(a+b+c)=\frac{n!}{a!\,b!\,c!},$$

leading to

$$N(abc) = \frac{n!}{a!\,b!\,c!} - \frac{n!}{(a+b)!\,c!} - \frac{n!}{a!\,(b+c)!} + \frac{n!}{(a+b+c)!},$$

wherein now we retain $\frac{n!}{(a+b+c)!}$ instead of its value unity, the better to shew the law of formation.

Similarly we find

$$N(abcd) + N(a+b,\, c,\, d) + N(a,\, b+c,\, d) + N(a,\, b,\, c+d)$$
$$+ N(a+b,\, c+d) + N(a+b+c,\, d) + N(a,\, b+c+d) + N(a+b+c+d) = \frac{n!}{a!\,b!\,c!\,d!}$$

leading to

$$N(abcd) = \frac{n!}{a!\,b!\,c!\,d!} - \frac{n!}{(a+b)!\,c!\,d!} - \frac{n!}{a!\,(b+c)!\,d!} - \frac{n!}{a!\,b!\,(c+d)!}$$
$$+ \frac{n!}{(a+b)!\,(c+d)!} + \frac{n!}{(a+b+c)!\,d!} + \frac{n!}{a!\,(b+c+d)!} - \frac{n!}{(a+b+c+d)!}\,*.$$

The general laws of the sinister of the first relation and of the dexter of the second are quite clear. Additions 2, 3, 4, ... at a time occur in all possible ways between the numbers $a, b, c, d, \ldots$, *provided that* the numbers are always in the order $a, b, c, d, \ldots$; the combinations are essentially between adjacent numbers, and in the second relation the sign of a term is determined by the number of factorial factors in its denominator. It is convenient to have before us the simplest results so that formulæ may be verified. Thus

$N(2) = N(1^2)$	$= 1$	total	$2 = 2!$
$N(3) = N(1^3)$	$= 1$	„	2
$N(21) = N(12)$	$= 2$	„	4
			$6 = 3!$
$N(4) = N(1^4)$	$= 1$	„	2
$N(31) = N(13) = N(21^2) = N(1^2 2)$	$= 3$	„	12
$N(22) = N(121)$	$= 5$	„	10
			$24 = 4!$

* Observe that in general $N(abc\ldots)$ may be given the determinant expression

$$n! \begin{vmatrix} \frac{1}{a!} & \frac{1}{(a+b)!} & \frac{1}{(a+b+c)!} & \cdots \\ 1 & \frac{1}{b!} & \frac{1}{(b+c)!} & \cdots \\ 0 & 1 & \frac{1}{c!} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix}.$$

$$
\begin{array}{rrrrrr}
N(5) & = N(1^5) & = 1 & \text{total} & 2 \\
N(41) = N(14) = N(21^3) & = N(1^32) & = 4 & \text{,,} & 16 \\
N(32) = N(23) = N(121^2) & = N(1^21) & = 9 & \text{,,} & 36 \\
N(31^2) & = N(1^23) & = 6 & \text{,,} & 12 \\
N(2^21) & = N(12^2) & = 16 & \text{,,} & 32 \\
N(131) & = N(212) & = 11 & \text{,,} & 22 \\
 & & & & \underline{120} = 5!
\end{array}
$$

158. Some simple summations are obtainable from elementary considerations.

Let us find the number of arrangements which give a cards in the first pack. We have to find the sum of the numbers $N(abc\ldots)$ in which the first part of the composition is a. Take any $a+1$ of the numbers 1, 2, 3, ... n; this can be done in $\binom{n}{a+1}$ ways. Now take the highest of these numbers and place any one of the others to its right and the remaining $a-1$ numbers in ascending order to its left; this can be done in a ways, so that the whole operation can be carried out in $a\binom{n}{a+1}$ ways. It is thus certain that, these numbers forming the left-hand part of the permutation, the first part in the composition will be a. Now the remaining $n-a-1$ numbers can be placed to the right of these to complete the permutation in $(n-a-1)!$ ways.

Hence

$$\Sigma N(a\ldots) = a\binom{n}{a+1}(n-a-1)! = a\,\frac{n!}{(a+1)!}.$$

Ex. gr. for $n=6$,

$$N(21^4)+N(24)+N(21^22)+N(2^21^2)+N(213)+N(231)+N(2121)+N(2^3)$$
$$= 5 + 14 + 19 + 35 + 26 + 40 + 40 + 61$$
$$= 240 = 2\cdot\frac{6!}{3!}.$$

The conjugate of the composition $(a\ldots)$ is of the form $(1^{a-1}\ldots)$; hence

$$\Sigma N(1^{a-1}\ldots) = a\,\frac{n!}{(a+1)!},$$

the number following 1^{a-1} being of course >1.

Putting $a=1$, we thus find that

$$\Sigma N(a\ldots) = \frac{n!}{2!},$$

the summation being for all values of a greater than unity. Thus exactly half of the arrangements of the n cards yield a first pack containing but a single card.

Putting $a = 2$ we find

$$\Sigma N(1\ldots) = \tfrac{1}{3}n!$$

the number following the unit being > 1. Thus in $\frac{1}{3}$ of the whole of the arrangements the first pack has one card and the second pack more than one card.

A fundamental property of the numbers $N(\ldots)$ will now be established. In a subsequent chapter it will be generalized.

The Multiplication Theorem.

159. Let $N(a_1 a_2 \ldots a_s)$ be derived from the permutations of p different integers and $N(a_{s+1} a_{s+2} \ldots a_{s+t})$ from the permutations of $n - p$ different integers. It is to be proved that

$$\binom{n}{a_1 + a_2 + \ldots + a_s} N(a_1 a_2 \ldots a_s)\, N(a_{s+1} a_{s+2} \ldots a_{s+t})$$
$$= N(a_1 a_2 \ldots a_{s-1},\ a_s + a_{s+1},\ a_{s+2},\ \ldots a_{s+t}) + N(a_1 a_2 \ldots a_{s+t}),$$

where, on the dexter, the derivation is from n different integers.

Out of the n numbers 1, 2, $\ldots n$, we can select $a_1 + a_2 + \ldots + a_s$ numbers in

$$\binom{n}{a_1 + a_2 + \ldots + a_s} \text{ ways,}$$

and we can arrange each selection, so as to have an ascending specification

$$(a_1 a_2 \ldots a_s),$$

in $N(a_1 a_2 \ldots a_s)$ ways;

the remaining numbers can then be arranged, so as to have an ascending specification

$$(a_{s+1} a_{s+2} \ldots a_{s+t}),$$

in $N(a_{s+1} a_{s+2} \ldots a_{s+t})$ ways;

placing the latter to the right of the former there appear

$$\binom{n}{a_1 + a_2 + \ldots + a_s} N(a_1 a_2 \ldots a_s)\, N(a_{s+1} a_{s+2} \ldots a_{s+t}) \text{ arrangements.}$$

Now combining the two groups of numbers we find that there is not or there is a break in the ascending order between a_s and a_{s+1}; hence the number of arrangements is also

$$N(a_1 a_2 \ldots a_{s-1},\ a_s + a_{s+1},\ a_{s+2}, \ldots a_{s+t}) + N(a_1 a_2 \ldots a_{s+t}). \qquad \text{Q.E.D.}$$

Regarded as a numerical theorem the multiplication is commutative, but in regard to form this is not the case. Thus the multiplication

$$N(a_{s+1} a_{s+2} \ldots a_{s+t})\, N(a_1 a_2 \ldots a_s)$$

gives in form a different result and establishes the linear relation

$$N(a_1a_2\dots a_{s-1},\ a_s+a_{s+1},\ a_{s+2}\dots a_{s+t})+N(a_1a_2\dots a_{s+t})$$
$$=N(a_{s+1}a_{s+2}\dots a_{s+t-1},\ a_{s+t}+a_1,\ a_2\dots a_s)+N(a_{s+1}\dots a_{s+t}a_1\dots a_s).$$

Ex. gr.

$$\binom{5}{3}N(12)\,N(11)=N(131)+N(1211)=N(122)+N(1112)$$
$$10\ .\ 2\ .\ 1\ =\ 9\ +\ 11\ =\ 4\ +\ 16$$
$$N(123)+N(15)=N(312)+N(42)$$
$$35\ +\ 5\ =\ 26\ +\ 14.$$

The fact that the multiplication is not formally commutative is of great service in the theory of these numbers.

160. Extending the theorem to the product of three numbers

$$N(a_1a_2\dots a_s),\quad N(b_1b_2\dots b_t),\quad N(c_1c_2\dots c_u),$$

we find $$\frac{n!}{(\Sigma a)!\,(\Sigma b)!\,(\Sigma c)!}N(a_1a_2\dots a_s)\,N(b_1b_2\dots b_t)\,N(c_1c_2\dots c_u)$$
$$=N(a_1a_2\dots a_{s-1},\ a_s+b_1,\ b_2\dots b_{t-1},\ b_t+c_1,\ c_2\dots c_u)$$
$$+N(a_1a_2\dots a_{s-1},\ a_s+b_1,\ b_2\dots b_tc_1\dots c_u)$$
$$+N(a_1a_2\dots a_sb_1b_2\dots b_{t-1},\ b_t+c_1,\ c_2\dots c_u)+N(a_1a_2\dots a_sb_1b_2\dots b_tc_1c_2\dots c_u),$$

and we may give the dexter 3! different forms corresponding to the 3! permutations of the three numbers.

If we take the product of m numbers to form the dexter we combine the last integer of a number $N(\dots)$ with the first integer of the next following number $N(\dots)$ s times in $\binom{m-1}{s}$ ways; hence

$$\sum_{s=0}^{s=m-1}\binom{m-1}{s}=2^{m-1}$$

numbers present themselves on the dexter. The dexter can be given as many different forms as there are permutations of the m numbers, and counting reversals of order of the compositions this number is further multiplied by 2^m, subject to a diminution when one or more of the numbers is self-inverse.

Applications of the Multiplication Theorem.

161. The theorems arrived at above are particular cases of multiplication.

Thus the formulæ of which

$$N(abc)+N(a+b,\ c)+N(a,\ b+c)+N(a+b+c)=\frac{n!}{a!\,b!\,c!}$$

is a type are equivalent to results of which

$$\frac{n!}{a!\,b!\,c!}N(a)\,N(b)\,N(c) = N(abc) + N(a+b,\,c) + N(a,\,b+c) + N(a+b+c)$$

is representative, since $N(a) = N(b) = N(c) = 1$.

Observe that

$$2!\,\{N(1)\}^2 = N(2) + N(1^2),$$
$$3!\,\{N(1)\}^3 = N(3) + N(21) + N(12) + N(1^3),$$

and that in general

$$n!\,\{N(1)\}^n = \Sigma N(abc\ldots),$$

the summation being for every composition of n. Since $N(1) = 1$, this merely states that the sum of the numbers $N(\ldots)$ in respect of a pack of specification (1^n) is $n!$.

162. Suppose that we require the sum of all numbers $N(\ldots)$ of given weight which are such that each associated composition commences with a given series of numbers $a_1 a_2 \ldots a_m$; or in other words suppose that we wish to evaluate

$$\Sigma N(a_1 a_2 \ldots a_m \ldots);$$

the solution is at once given by

$$\frac{n!}{(\Sigma a + 1)!}N(a_1 a_2 \ldots a_m 1)\,\{N(1)\}^{n-\Sigma a-1} = \Sigma N(a_1 a_2 \ldots a_m \ldots),$$

for by the multiplication process the unit which terminates the composition in $N(a_1 a_2 \ldots a_m 1)$ combined with

$$\{N(1)\}^{n-\Sigma a-1}$$

gives every composition of the number $n - \Sigma a$. Hence since $N(1) = 1$,

$$\Sigma N(a_1 a_2 \ldots a_m \ldots) = \frac{n!}{(\Sigma a + 1)!}N(a_1 a_2 \ldots a_m 1),$$

and from this we verify the result of Art. 158 for $N(a1) = a$ and thus

$$\Sigma N(a\ldots) = a\,\frac{n!}{(a+1)!}.$$

163. Again

$$\frac{n!}{(\Sigma a + 2)!}\{N(1)\}^{p-1}N(1\,a_1 a_2 \ldots a_m 1)\,\{N(1)\}^{n-\Sigma a-p-1} = \Sigma N(\ldots a_1 a_2 \ldots a_m \ldots),$$

where on the dexter the summation is for every composition of p before a_1 and for every composition of $n - \Sigma a - p$ after a_m. Thus

$$\Sigma N(\ldots a_1 a_2 \ldots a_m \ldots) = \frac{n!}{(\Sigma a + 2)!}N(1\,a_1 a_2 \ldots a_m 1),$$

and it is remarkable that the sum in question is independent of p, p being at least equal to unity.

Ex. gr. take $n=6$, $m=1$, $a_1=2$, $p=2$. We have

$$N(2^3)+N(221^2)+N(1^2 22)+N(1^2 21^2)=\frac{6!}{4!}N(121),$$

verified by $61 + 35 + 35 + 19 = 30 . 5.$

Also when $n=6$, $m=0$, $p=3$, we have to take every composition of 3 with every composition of 3 and so

$$\begin{aligned}N(33)+N(321)+N(312)+N(31^3)+N(213)+N(2121)+N(2112)+N(21^4)\\ +N(123)+N(1221)+N(1212)+N(121^3)+N(1^3 3)+N(1^3 21)\\ +N(1^4 2)+N(1^6)\\ =\frac{6!}{2!}N(11)=360,\end{aligned}$$

verified by

$$\begin{aligned}&19+35+26+10+26+40+19+5\\ &+35+61+40+14+10+14+5+1\\ &=360.\end{aligned}$$

164. Again if $n=ra$,

$$\frac{(ra)!}{(a!)^r}\{N(a)\}^r=\Sigma N(m_1 a, m_2 a, \ldots),$$

where the summation is for every composition of ra into parts each of which is a multiple of a. Thus

$$\Sigma N(m_1 a, m_2 a, \ldots)=\frac{(ra)!}{(a!)^r}.$$

Ex. gr., for $a=2$, $r=3$,

$$N(6)+N(42)+N(24)+N(2^3)=\frac{6!}{(2!)^3}=90;$$

verified by $1 + 14 + 14 + 61 = 90.$

165. As another example of the power of the theorem let

$$\Sigma N(a_1 a_2 \ldots a_p \ldots b_1 b_2 \ldots b_q)$$

denote a summation for all compositions of $n-\Sigma a-\Sigma b$ placed between the parts a_p and b_1; we find

$$\begin{aligned}&\Sigma N(a_1 a_2 \ldots a_p \ldots b_1 b_2 \ldots b_q),\\ &=\frac{n!}{(\Sigma a+1)!(\Sigma b+1)!}N(a_1 a_2 \ldots a_p 1)\{N(1)\}^{n-\Sigma a-\Sigma b-2}N(1 b_1 b_2 \ldots b_q),\\ &=\frac{n!}{(\Sigma a+1)!(\Sigma b+1)!}N(a_1 a_2 \ldots a_p 1)N(1 b_1 b_2 \ldots b_q),\\ &=\frac{n!}{(\Sigma a+\Sigma b+2)!}\{N(a_1 a_2 \ldots a_p 2 b_1 b_2 \ldots b_q)+N(a_1 a_2 \ldots a_p 1^2 b_1 b_2 \ldots b_q)\}.\end{aligned}$$

166. Consider now the multiplication

$$\frac{n!}{(s_1+2)!\,(s_2+2)!}\{N(1)\}^{w_1-1}N(1^{s_1+2})\{N(1)\}^{w_2-2}N(1^{s_2+2})\{N(1)\}^{w_3-1},$$

wherein $\Sigma w+\Sigma s=n$, $w_1 \nleq 1$, $w_2 \nleq 2$, $w_3 \nleq 1$ and s_1, s_2 are each of them zero or any positive integer.

The result of the multiplication consists of numbers $N(\ldots)$ such that there is

(i) a composition of w_1 followed by s_1 units, succeeded by

(ii) a composition of w_2 followed by s_2 units, succeeded by

(iii) a composition of w_3.

Denoting this sum by

$$\Sigma N(w_1 1^{s_1} w_2 1^{s_2} w_3),$$

we find that its value is $\dfrac{n!}{(s_1+2)!\,(s_2+2)!}$;

and in general

$$\Sigma N(w_1 1^{s_1} w_2 1^{s_2}\ldots 1^{s_p} w_{p+1}) = \frac{n!}{(s_1+2)!\,(s_2+2)!\ldots(s_p+2)!}, \quad \Sigma w+\Sigma s=n,$$

establishing that the sum depends merely upon the numbers $s_1, s_2, \ldots s_p$ and not at all upon the numbers $w_1, w_2, \ldots w_{p+1}$. Observe that w_1 and w_{p+1} must not be less than unity; $w_2, w_3, \ldots w_p$ must not be less than 2; and that no restriction is placed upon $s_1, s_2, \ldots s_p$.

If $s_1=s_2=\ldots=s_p=0$,

$$\Sigma N(w_1 w_2\ldots w_{p+1}) = \frac{n!}{(2!)^p}.$$

Ex. gr. take $n=6$, $p=2$, $w_1=1$, $w_2=2$, $w_3=3$, leading to

$$N(123)+N(1^33)+N(1221)+N(11121)$$
$$+N(1212)+N(1^42)+N(121^3)+N(1^6)=\frac{6!}{(2!)^2};$$

verified by

$$35+10+61+14$$
$$+40+\ 5+14+\ 1=180.$$

CHAPTER V

GENERALIZATION OF THE FOREGOING THEORY

167. In the preceding pages we have had under view the permutations of n different numbers. For the discussion of the general case it is convenient to slightly alter the point of view.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ denote numbers in ascending order of magnitude and supposing the number α_s to occur p_s times we consider the assemblage of numbers which form the ascending product

$$\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}\ldots;$$

this assemblage is specified by the composition $(p_1p_2p_3\ldots)$ of n. As equalities may occur between the numbers $p_1, p_2, p_3, \ldots$ we take for greater generality the specifying composition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots).$$

It will be seen later that the order of occurrence of the parts of this composition is immaterial, so that we may regard the numbers $p_1, p_2, p_3, \ldots$ as being in descending order of magnitude and the specification of the numbers to be denoted by a partition.

Ex. gr. we obtain the same results for each of the six assemblages

$$\alpha\alpha\alpha\beta\beta\gamma, \quad \alpha\alpha\alpha\beta\gamma\gamma, \quad \alpha\alpha\beta\beta\beta\gamma,$$
$$\alpha\alpha\beta\gamma\gamma\gamma, \quad \alpha\beta\beta\beta\gamma\gamma, \quad \alpha\beta\beta\gamma\gamma\gamma,$$

the specification of each being (321).

Every permutation has an ascending specification; ex. gr.

$$\alpha\beta|\alpha\alpha\gamma|\beta$$

has the ascending specification (231).

In the case already considered the assemblage of numbers had the specification (1^n) and the numbers $N(\ldots)_{(1^n)}$ were expressed in terms of coefficients obtained from the multinomial expansion

$$(\alpha_1 + \alpha_2 + \ldots + \ldots)^n = h_1^n.$$

In the present general case it will appear that the auxiliary coefficients are derived from the auxiliary generating function

$$h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots,$$

h_s being the homogeneous product, of order s, of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$.

It will be remembered that the number of ways of distributing numbers specified by $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$ into *different* compartments specified by $(abc\ldots)$ is the coefficient of the symmetric function $(abc\ldots)$ in the development of the symmetric function

$$h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} h_{p_3}^{\pi_3} \ldots;$$

let this coefficient be denoted by $C(abc\ldots)$ and let the number of arrangements of the numbers, which have an ascending specification $(abc\ldots)$, be denoted by $N(abc\ldots)_{(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)}$ or by $N(abc\ldots)$ when there is no danger of a misunderstanding.

By precisely the same train of reasoning as before we arrive at the relations connecting the numbers $N(\ldots)$ with the numbers $C(\ldots)$, viz.:

$$\begin{aligned}
N(a) &= C(a),\\
N(ab) &= C(ab) - C(a+b),\\
N(abc) &= C(abc) - C(a+b, c) - C(a, b+c) + C(a+b+c),\\
N(abcd) &= C(abcd) - C(a+b, c, d) - C(a, b+c, d) - C(ab, c+d)\\
&\quad + C(a+b, c+d) + C(a+b+c, d) + C(a, b+c+d)\\
&\quad - C(a+b+c+d),
\end{aligned}$$

etc.,

the numbers $N(\ldots)$ being all expressible in terms of the coefficients of the auxiliary generating function.

Ex. gr. Take numbers $\alpha_1\alpha_1\alpha_1\alpha_2\alpha_2\alpha_3$ of specification (321); since

$$\begin{aligned}
h_3 h_2 h_1 = {} & (6) + 3(51) + 5(42) + 8(41^2) + 6(3^2) + 12(321)\\
& + 19(31^3) + 15(2^3) + 24(2^2 1^2) + 38(21^4) + 60(1^6),
\end{aligned}$$

we calculate from the above formulæ

$$N(6) = 1; \quad N(51) = 3 - 1 = 2; \quad N(3^2) = 6 - 1 = 5;$$

$$N(321) = 12 - 3 - 6 + 1 = 4$$

and so on.

Ex. gr. the five arrangements enumerated by $N(3^2)$ are

$$\alpha_1\alpha_1\alpha_2 \,|\, \alpha_1\alpha_2\alpha_3, \quad \alpha_1\alpha_1\alpha_3 \,|\, \alpha_1\alpha_2\alpha_2, \quad \alpha_1\alpha_2\alpha_3 \,|\, \alpha_1\alpha_1\alpha_2,$$

$$\alpha_1\alpha_2\alpha_2 \,|\, \alpha_1\alpha_1\alpha_3, \quad \alpha_2\alpha_2\alpha_3 \,|\, \alpha_1\alpha_1\alpha_1.$$

The complete results for numbers specified by (321) are

$N(31^3)$,	$N(1^33)$,	$N(21^4)$,	$N(1^42)$,	$N(2112)$	each 0 ...	0
	$N(121^3)$,	$N(1^321)$,	$N(1^221^2)$,	$N(1^6)$		
$N(6)$,	$N(41^2)$,	$N(1^24)$,	$N(131^2)$,	$N(1^231)$	each 1 ...	9
	$N(2^21^2)$,	$N(1^22^2)$,	$N(2121)$,	$N(1212)$		
	$N(51)$,	$N(15)$,	$N(312)$,	$N(213)$	each 2 ...	8
			$N(141)$,	$N(1221)$	each 3 ...	6
	$N(42)$,	$N(24)$,	$N(321)$,	$N(123)$	each 4 ...	16
		$N(3^2)$,	$N(231)$,	$N(132)$	each 5 ...	15
				$N(2^3)$	6 ...	6
						60

$60 = \dfrac{6!}{3!\,2!\,1!}$ being the whole number of permutations of the numbers.

The formulæ themselves establish that the number $N(\ldots)$ is unchanged by reversal of the order of the numbers in the composition. Also the fact that we are dealing with symmetric functions shews that the results are only dependent upon the magnitudes of the parts in the specification of the assemblages and not upon the order of their occurrence. It will be observed that conjugate compositions do not give the same result but they are connected, as will be shewn later.

Investigation of a Generating Function.

168. We have seen how the value of $N(abc\ldots)_{(pqr\ldots)}$ can be obtained from the coefficients that appear when $h_p h_q h_r \ldots$ is expanded into a linear function of monomial symmetric functions. We are able to utilize the differential operators of Section II for the further elucidation of the question. Denoting by

$$C(abc\ldots)_{(pqr\ldots)}$$

the coefficient of $(abc\ldots)$ in the development of $h_p h_q h_r \ldots$ we have

$$C(abc\ldots)_{(pqr\ldots)} = D_a D_b D_c \ldots h_p h_q h_r \ldots.$$

Now in particular

$$N(abc) = C(abc) - C(a+b,\, c) - C(a,\, b+c) + C(a+b+c),$$

where the subscript $(pqr\ldots)$ is omitted for convenience. Thence

$$N(abc) = (D_a D_b D_c - D_{a+b} D_c - D_a D_{b+c} + D_{a+b+c})\, h_p h_q h_r \ldots,$$

and since from Section II

$$D_{p_1} D_{q_1} D_{r_1} \ldots h_p h_q h_r \ldots = D_p D_q D_r \ldots h_{p_1} h_{q_1} h_{r_1} \ldots,$$

we derive the result

$$N(abc) = D_p D_q D_r \ldots (h_a h_b h_c - h_{a+b} h_c - h_a h_{b+c} + h_{a+b+c}).$$

In words we may say that $N(abc)_{(pqr...)}$ is equal to the coefficient of the symmetric function $(pqr...)$ in the development of

$$h_a h_b h_c - h_{a+b} h_c - h_a h_{b+c} + h_{a+b+c},$$

or the expression last written is the true generating function of the numbers $N(abc)_{(pqr...)}$. This expression follows the law by which the dexter of the relation

$$N(abc) = C(abc) - C(a+b, c) - C(a, b+c) + C(a+b+c)$$

is formed.

We now write $h_a h_b h_c - h_{a+b} h_c - h_a h_{b+c} + h_{a+b+c} = h_{abc}$,

and in general

$$\begin{aligned} & h_a h_b h_c h_d \ldots \\ & - h_{a+b} h_c h_d \ldots - h_a h_{b+c} h_d \ldots - h_a h_b h_{c+d} \ldots - \ldots \\ & + h_{a+b+c} h_d \ldots + h_a h_{b+c+d} \ldots + \ldots \\ & + h_{a+b} h_{c+d} \ldots + h_{a+b} h_c h_{d+e} \ldots + \ldots \\ & - \ldots \end{aligned}$$

(the *law* being similar to that which obtains in the expression of $N(abcd...)_{(1^n)}$ in terms of factorials) is written $h_{abcd...}$.

$h_{abcd...}$ is then the true generating function of the numbers

$$N(abcd...)_{(pqr...)}.$$

We have $$h_{abcd...} = \Sigma N(abcd...)_{(pqr...)} . (pqr...),$$

and the expansion of $h_{abcd..}$ as a linear function of monomial symmetric functions yields a complete account of the numbers $N(abcd...)_{(pqr..)}$.

The new function can be expressed elegantly in determinant form.

For
$$h_{ab} = \begin{vmatrix} h_a & h_{a+b} \\ 1 & h_b \end{vmatrix},$$

$$h_{abc} = \begin{vmatrix} h_a & h_{a+b} & h_{a+b+c} \\ 1 & h_b & h_{b+c} \\ 0 & 1 & h_c \end{vmatrix},$$

and in general

$$h_{abcd..} = \begin{vmatrix} h_a & h_{a+b} & h_{a+b+c} & h_{a+b+c+d} & \ldots \\ 1 & h_b & h_{b+c} & h_{b+c+d} & \ldots \\ 0 & 1 & h_c & h_{c+d} & \ldots \\ 0 & 0 & 1 & h_d & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

and there is no difficulty in shewing that, if we write the determinants in brackets $\{\ \}$ to denote that on development multiplications $h_s h_t h_u \ldots$ are to be changed to $h_{stu..}$,

$$h_a h_b = \begin{Bmatrix} h_a & h_{a+b} \\ \bar{1} & h_b \end{Bmatrix}^*,$$

$$h_a h_b h_c = \begin{Bmatrix} h_a & h_{a+b} & h_{a+b+c} \\ \bar{1} & h_b & h_{b+c} \\ 0 & \bar{1} & h_c \end{Bmatrix},$$

$$h_a h_b h_c h_d \ldots = \begin{Bmatrix} h_a & h_{a+b} & h_{a+b+c} & h_{a+b+c+d} & \ldots \\ \bar{1} & h_b & h_{b+c} & h_{b+c+d} & \ldots \\ 0 & \bar{1} & h_c & h_{c+d} & \ldots \\ 0 & 0 & \bar{1} & h_d & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{Bmatrix}.$$

Thus ex. gr. the second of these symbolical determinants gives

$$h_a h_b h_c = h_{abc} + h_{a,b+c} + h_{a+b,c} + h_{a+b+c}.$$

Before proceeding to an examination of the important symmetric function $h_{abc\ldots}$ that has now presented itself, it is well to have the simpler results before us, as verifications are thereby greatly assisted.

$n=2$

	(2)	(1^2)	= specification
$N(2)$	1	1	
$N(1^2)$	—	1	

$n=3$

	(3)	(21)	(1^3)
$N(3)$	1	1	1
$N(21)$	—	1	2
$N(1^3)$	—	—	1

$n=4$

	(4)	(31)	(2^2)	(21^2)	(1^4)
$N(4)$	1	1	1	1	1
$N(31)$	—	1	1	2	3
$N(2^2)$	—	1	2	3	5
$N(121)$	—	—	1	2	5
$N(21^2)$	—	—	—	1	3
$N(1^4)$	—	—	—	—	1

* $\bar{1}$ is written for negative unity.

$n=5$

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
$N(5)$	1	1	1	1	1	1	1
$N(41)$	—	1	1	2	2	3	4
$N(32)$	—	1	2	3	4	6	9
$N(131)$	—	—	1	2	3	6	11
$N(2^21)$	—	—	1	2	4	8	16
$N(31^2)$	—	—	—	1	1	3	6
$N(212)$	—	—	—	1	2	5	11
$N(121^2)$	—	—	—	—	1	3	9
$N(21^3)$	—	—	—	—	—	1	4
$N(1^5)$	—	—	—	—	—	—	1

$n=6$

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
$N(6)$	1	1	1	1	1	1	1	1	1	1	1
$N(51)$	—	1	1	1	2	2	2	3	3	4	5
$N(42)$	—	1	2	2	3	4	5	6	7	10	14
$N(3^2)$	—	1	2	3	3	5	6	7	9	13	19
$N(141)$	—	—	1	1	2	3	4	6	7	12	19
$N(231)$	—	—	1	2	2	5	7	9	13	23	40
$N(312)$	—	—	—	—	1	2	3	5	7	14	26
$N(321)$	—	—	1	1	2	4	6	8	11	20	35
$N(2^3)$	—	—	1	2	2	6	10	11	18	33	61
$N(41^2)$	—	—	—	—	1	1	1	3	3	6	10
$N(31^3)$	—	—	—	—	—	—	—	1	1	4	10
$N(12^21)$	—	—	—	1	—	3	6	6	13	28	61
$N(2^21^2)$	—	—	—	—	—	1	2	3	6	15	35
$N(131^2)$	—	—	—	—	—	1	2	3	5	12	26
$N(2121)$	—	—	—	—	—	1	3	3	7	17	40
$N(21^22)$	—	—	—	—	—	—	—	1	2	7	19
$N(1^22^21^2)$	—	—	—	—	—	—	1	—	2	6	19
$N(121^3)$	—	—	—	—	—	—	—	—	1	4	14
$N(21^4)$	—	—	—	—	—	—	—	—	—	1	5
$N(1^6)$	—	—	—	—	—	—	—	—	—	—	1

To explain, it will be found from row 4 of the Table, for $n=5$,

$$h_{131} = (32) + 2\,(31^2) + 3\,(2^21) + 6\,(21^3) + 11\,(1^5).$$

The Symmetric Functions $h_{p_1p_2p_3\ldots}$, $a_{p_1p_2p_3\ldots}$.

169. The quantities a_1, a_2, a_3, ... denote the elementary symmetric functions.

It has been proved in Section I that the two series of elements h and a are connected in such wise that in any relation between the elements the symbols a and h may be interchanged.

In the new function $h_{p_1p_2p_3\ldots}$, $(p_1p_2p_3\ldots)$ is any composition of the number n; of the given weight there are 2^{n-1} such functions, one of which is h_n. The complete definition is given by the multiplication law

$$h_{p_1p_2\ldots p_s}h_{q_1q_2\ldots q_t} = h_{p_1p_2\ldots p_{s-1},\, p_s+q_1,\, q_2q_3\ldots q_t} + h_{p_1p_2\ldots p_sq_1q_2\ldots q_t},$$

where the functions $h_{p_1p_2\ldots p_s}$, $h_{q_1q_2\ldots q_t}$ are, or are not, of the same weight.

170. A second new function $a_{p_1p_2p_3\ldots}$ is similarly defined by the same law; viz.

$$a_{p_1p_2\ldots p_s}a_{q_1q_2\ldots q_t} = a_{p_1p_2\ldots p_{s-1},\, p_s+q_1,\, q_2q_3\ldots q_t} + a_{p_1p_2\ldots p_sq_1q_2\ldots q_t}.$$

What follows applies generally to both of the new functions.

171. Since the multiplication is commutative we have the first property, viz.

$$h_{p_1p_2\ldots p_{s-1},\, p_s+q_1,\, q_2q_3\ldots q_t} + h_{p_1p_2\ldots p_sq_1q_2\ldots q_t}$$
$$= h_{q_1q_2\ldots q_{t-1}q_t+p_1p_2p_3\ldots p_s} + h_{q_1q_2\ldots q_tp_1p_2\ldots p_s}.$$

Every product of the functions h is expressible in terms of the new functions, ex. gr.

$$h_ph_q = h_{pq} + h_{p+q},$$

$$h_ph_qh_r = h_{pqr} + h_{p+q,\, r} + h_{p,\, q+r} + h_{p+q+r},$$

and in general, employing a symbol θ to indicate the law,

$$h_{p_1}h_{p_2}\ldots h_{p_s} = \theta_h\{(p_1)(p_2)\ldots(p_s)\}.$$

These relations shew that $h_{p_1p_2\ldots p_s} = h_{p_sp_{s-1}\ldots p_1}$.

Similarly $$h_{pq} = h_ph_q - h_{p+q},$$

$$h_{pqr} = h_ph_qh_r - h_{p+q}h_r - h_ph_{q+r} + h_{p+q+r},$$

and in general, employing a symbol ϕ,

$$h_{p_1p_2\ldots p_s} = \phi_h\{(p_1)(p_2)\ldots(p_s)\}.$$

If moreover we define the symbol θ to mean in general that

$$\theta_h\{(p_1\ldots p_s)(q_1\ldots q_t)(r_1\ldots r_u)\ldots\}$$
$$= h_{p_1\ldots p_sq_1\ldots q_tr_1\ldots r_u\ldots} + h_{p_1\ldots p_{s-1},\, p_s+q_1,\, q_2\ldots q_tr_1\ldots r_u\ldots} + \ldots,$$

addition between elements in *contact* being taken 0, 1, 2, ... at a time, we have

$$h_{p_1 \dots p_s} h_{q_1 \dots q_t} h_{r_1 \dots r_u} \dots = \theta_h \{(p_1 \dots p_s)(q_1 \dots q_t)(r_1 \dots r_u) \dots\};$$

and, similarly, defining the symbol ϕ to mean that

$$\phi_h \{(p_1 p_2 \dots p_s)(q_1 \dots q_t)(r_1 \dots r_u) \dots\}$$
$$= h_{p_1 \dots p_s} h_{q_1 \dots q_t} h_{r_1 \dots r_u} \dots - h_{p_1 \dots p_{s-1} p_s + q_1,\ q_2 \dots q_t} h_{r_1 \dots r_u} \dots - \dots,$$

addition between elements in contact being taken 0, 1, 2, ... at a time, but the sign changing as the number of factors changes, we have

$$h_{p_1 \dots p_s q_1 \dots q_t r_1 \dots r_u} \dots = \phi_h \{(p_1 p_2 \dots p_s)(q_1 \dots q_t)(r_1 \dots r_u) \dots\}.$$

It is noteworthy that in regard to the symbol ϕ the elements $p, q, r, \dots$ in the bracket $\{\ \}$ may be bracketed in any manner. Thus the reader will verify that

$$\phi_h \{(p)(q)(r)\}, \quad \phi_h \{(pq)(r)\}, \quad \phi_h \{(p)(qr)\} \quad \text{and} \quad \phi_h (pqr)$$

have each the same value

$$h_p h_q h_r - h_{p+q} h_r - h_p h_{q+r} + h_{p+q+r},$$

which we denote by h_{pqr}; the important point to notice is that however the bracketing be effected the law of symbols in contact obtains. This is not the case with the symbol θ, but here we find that following that symbol the bracketed factors can be permuted in any manner; thus

$$\theta_h \{(p)(q)(r)\} = \theta_h \{(q)(p)(r)\} = \dots = h_p h_q h_r.$$

172. Important relations connect the new h and a functions.

We observe that

$$a_{11} = a_1^2 - a_2 = h_2,$$
$$a_{111} = a_1^3 - 2a_1 a_2 + a_3 = h_3,$$
$$a_{1111} = a_1^4 - 3a_1^2 a_2 + a_2^2 + 2a_1 a_3 - a_4 = h_4,$$

and, in general, the law of symbols in contact applied to the present case shews that the numerical value of the coefficient of $a_1^{p_1} a_2^{p_2} a_3^{p_3} \dots$ in the expression of a_{1^n} is equal to the number of permutations of which that term is susceptible.

Hence $$a_{1^n} = h_n,$$

and by the known reciprocity derived from Section 1,

$$h_{1^n} = a_n.$$

173. This is a very concise and illuminating expression of h_n in terms of the elementary functions and *vice versâ*. It will be noticed moreover that

n and 1^n are zig-zag conjugate compositions of n, and it will now be proved that if $(pqr\ldots)$ and $(pqr\ldots)'$ be zig-zag conjugate compositions

$$h_{(pqr\ldots)} = a_{(pqr\ldots)'}.$$

From the relation $$a_{pq} = a_p a_q - a_{p+q},$$

is now deduced $$a_{pq} = h_{1^p} h_{1^q} - h_{1^{p+q}} = h_{1^{p-1}21^{q-1}},$$

because clearly $$h_{1^p} h_{1^q} = h_{1^{p-1}21^{q-1}} + h_{1^{p+q}}.$$

It will be noted that (pq) and $(1^{p-1}21^{q-1})$ are zig-zag conjugates.

Hence writing $$(1^{p-1}21^{q-1}) = (pq)',$$

we obtain $$a_{pq} = h_{(pq)'}.$$

To prove the general theorem observe that

$$a_{p_1 p_2 \ldots p_s} = a_{p_1 p_2 \ldots p_{s-1}} a_{p_s} - a_{p_1 p_2 \ldots p_{s-2},\, p_{s-1}+p_s}.$$

Assume that the law has been proved when the composition subscript of a has at most $s-1$ numbers; then

$$a_{p_1 p_2 \ldots p_s} = h_{(p_1 p_2 \ldots p_{s-1})'} h_{(p_s)'} - h_{(p_1 p_2 \ldots p_{s-2},\, p_{s-1}+p_s)'},$$

and we have to shew that the dexter is equal to

$$h_{(p_1 p_2 \ldots p_s)'}.$$

It is best to write the composition in a more general form

$$p_1 1^{\pi_1} p_2 1^{\pi_2} \ldots p_{s-1} 1^{\pi_{s-1}} p_s 1^{\pi_s},$$

where the numbers $p_1, p_2, \ldots p_{s-1}, p_s$ are, each of them, greater than unity and the numbers $\pi_1, \pi_2, \ldots \pi_{s-1}, \pi_s$ may, each of them, be zero or any positive integer.

First suppose $\pi_s = 0$.

Then

$$(p_1 1^{\pi_1} p_2 1^{\pi_2} \ldots p_{s-1} 1^{\pi_{s-1}} p_s)' = 1^{p_1-1}.\pi_1 + 2.1^{p_2-2}.\pi_2 + 2.\ldots.1^{p_{s-1}-2}.\pi_{s-1} + 2.1^{p_s-1},$$

$$(p_1 1^{\pi_1} p_2 1^{\pi_2} \ldots p_{s-1} 1^{\pi_{s-1}})' = 1^{p_1-1}.\pi_1 + 2.1^{p_2-2}.\pi_2 + 2.\ldots.1^{p_{s-1}-2}.\pi_{s-2} + 2.1^{p_{s-1}-2}.\pi_{s-1} + 1,$$

$$(p_s)' = 1^{p_s},$$

$$(p_1 1^{\pi_1} p_2 1^{\pi_2} \ldots p_{s-1} 1^{\pi_{s-1}-1}.1 + p_s)' = 1^{p_1-1}.\pi_1 + 2.1^{p_2-2}.\pi_2 + 2.\ldots.1^{p_{s-1}-2}.\pi_{s-1} + 1.1^{p_s},$$

as will be seen by actually forming the zig-zag graphs.

Now by the contact law

$$h_{1^{p_1-1}.\pi_1 + 2.\ \pi_{s-2} + 2.1^{p_{s-1}-2}.\pi_{s-1} + 1} h_{1^{p_s}}$$

$$= h_{1^{p_1-1}.\pi_1 + 2.\ .\pi_{s-2} + 2.1^{p_{s-1}-2}.\pi_{s-1} + 2.1^{p_s-1}}$$

$$+ h_{1^{p_1-1}.\pi_1 + 2.\ldots.\pi_{s-2} + 2.1^{p_{s-1}-2}.\pi_{s-1} + 1.1^{p_s}}.$$

Hence

$$h_{(p_1 1^{\pi_1} p_2 1^{\pi_2} \dots 1^{\pi_{s-1}} p_s)'} = h_{(p_1 1^{\pi_1} p_2 1^{\pi_2} \dots p_{s-1} 1^{\pi_{s-1}})'} h_{(p_s)'} - h_{(p_1 1^{\pi_1} p_2 1^{\pi_2} \dots p_{s-1} 1^{\pi_{s-1}-1} . 1 + p_s)'},$$

or on the assumption made

$$a_{p_1 1^{\pi_1} p_2 1^{\pi_2} \dots 1^{\pi_{s-1}} p_s} = h_{(p_1 1^{\pi_1} p_2 1^{\pi_2} \dots 1^{\pi_{s-1}} p_s)'}.$$

The proof when $\pi_s > 0$ proceeds in a similar manner and the general theorem is thus established.

This theorem is interesting because it makes easy the expression of functions of quantities h, which frequently are those which naturally present themselves, in terms of the quantities a which are more easy to handle algebraically.

There is a function h_{pqr} of given weight for every composition $(pqr\dots)$ of the weight, but since this function is not altered when the composition is inverted, the number of different functions is equal to the number of self-inverse compositions added to half the number of those which are not self-inverse. A previous chapter shews that this number is $2^{m-1} + 2^{2m-2}$ when $n = 2m$ and $2^{m-1} + 2^{2m-1}$ when $n = 2m+1$.

174. The reader will have little difficulty in proving that, s_n being the one-part symmetric function of weight n,

$$s_n = a_{1^n} - a_{21^{n-2}} + a_{31^{n-3}} - \dots (-)^{n+1} a_n.$$

The generating function of $N(abc\dots)_{(pqr\ )}$ is either $h_{(abc\dots)}$ or $a_{(abc\ )'}$.

We can determine the first symmetric function in dictionary order of the parts that occurs in the expression of $a_{(abc\ )}$; we have merely to express $(abc\dots)$ as a partition and then take the conjugate; this will specify the symmetric function.

Ex. gr. $h_{2121} = a_{132}$; the partition (321) is self-conjugate so that the symmetric function (321) is the first that presents itself in the expression of h_{2121} (see Tables).

The differential operators of Section II can be used with effect upon these functions; thus it is easy to prove that

$$D_1 h_{ab} = h_{a-1,b} + h_{a,b-1} + h_{a+b-1};$$

and thence

$$N(ab)_{pqr\ 1} = N(a-1, b)_{pqr\ } + N(a, b-1)_{pqr\ } + N(a+b-1)_{pqr\ };$$

and, in particular,

$$N(42)_{2^2 1^2} = N(32)_{221} + N(41)_{221} + N(5)_{221};$$

verified by $7 = 4 + 2 + 1.$

Similarly it can be shewn that

$$D_1 h_{abc} = h_{a-1,b,c} + h_{a,b-1,c} + h_{a,b,c-1} + h_{a+b-1,c} + h_{a,b+c-1},$$

leading to

$$N(abc)_{pqr\ 1} = N(a-1,\ b,\ c)_{pqr} + N(a,\ b-1,\ c)_{pqr} + N(a,\ b,\ c-1)_{pqr} + N(a+b-1,\ c)_{pqr} + N(a,\ b+c-1)_{pqr}.$$

In these and in similar formulæ a modification is necessary when any of the numbers a, b, c, ... are equal to unity. This fact arises from the circumstance that the formula $h_{ab} = h_a h_b - h_{a+b}$ is not applicable when either a or b is zero*.

---

* The reader will have no difficulty in establishing three fundamental formulæ connected with these functions. Thus of the order 8 we have

(A)
$$a_{s_1s_2s_3}a_{t_1t_2}a_{u_1u_2}a_v$$
$$= a_{s_1s_2s_3t_1t_2u_1u_2v}$$
$$+ a_{s_1s_2,\ s_3+t_1,\ t_2u_1u_2v} + a_{s_1s_2s_3t_1,\ t_2+u_1,\ u_2v} + a_{s_1s_2s_3t_1t_2u_1,\ u_2+v}$$
$$+ a_{s_1s_2,\ s_3+t_1,\ t_2+u_1,\ u_2v} + a_{s_1s_2,\ s_3+t_1,\ t_2u_1,\ u_2+v} + a_{s_1s_2s_3t_1,\ t_2+u_1,\ u_2+v}$$
$$+ a_{s_1s_2,\ s_3+t_1,\ t_2+u_1,\ u_2+v};$$

(B)
$$a_{s_1s_2s_3t_1t_2u_1u_2v}$$
$$= a_{s_1s_2s_3}a_{t_1t_2}a_{u_1u_2}a_v$$
$$- a_{s_1s_2,\ s_3+t_1,\ t_2}a_{u_1u_2}a_v - a_{s_1s_2s_3}a_{t_1,\ t_2+u_1,\ u_2}a_v - a_{s_1s_2s_3}a_{t_1t_2}a_{u_1,\ u_2+v}$$
$$+ a_{s_1s_2,\ s_3+t_1,\ t_2+u_1,\ u_2}a_v + a_{s_1s_2,\ s_3+t_1,\ t_2}a_{u_1,\ u_2+v} + a_{s_1s_2s_3}a_{t_1,\ t_2+u_1,\ u_2+v}$$
$$- a_{s_1s_2,\ s_3+t_1,\ t_2+u_1,\ u_2+v};$$

(C)
$$a_{s_1s_2,\ s_3+t_1,\ t_2+u_1,\ u_2+v}$$
$$= a_{s_1s_2s_3}a_{t_1t_2}a_{u_1u_2}a_v$$
$$- a_{s_1s_2s_3t_1t_2}a_{u_1u_2}a_v - a_{s_1s_2s_3}a_{t_1t_2u_1u_2}a_v - a_{s_1s_2s_3}a_{t_1t_2}a_{u_1u_2v}$$
$$+ a_{s_1s_2s_3t_1t_2u_1u_2}a_v + a_{s_1s_2s_3t_1t_2}a_{u_1u_2v} + a_{s_1s_2s_3}a_{t_1t_2u_1u_2v}$$
$$- a_{s_1s_2s_3t_1t_2u_1u_2v}.$$

These formulæ have to do with the ordered succession of integers

$$s_1,\ s_2,\ s_3,\ t_1,\ t_2,\ u_1,\ u_2,\ v$$

and they may be divided into sets, the order being preserved, in any manner. In the example given the division is into the sets $s_1s_2s_3$, t_1t_2, u_1u_2, v.

There is a division corresponding to every composition of n if the number of letters be n. There are thus 2^{n-1} different divisions possible.

In the formulæ we may write h for a throughout or we may write $h_{c'}$ for a_c where c, c' are conjugate compositions.

As particular cases of the order 4 we have

(A) $a_1^4 = a_{1111} + a_{211} + a_{121} + a_{112} + a_{31} + a_{22} + a_{13} + a_4$,

(B) $a_{1111} = a_1^4 - 3a_2a_1^2 + a_2^2 + 2a_3a_1 - a_4$,

(C) $a_4 = a_1^4 - 3a_{11}a_1^2 + a_{11}^2 + 2a_{111}a_1 - a_{1111}$.

Herein (B) may be also written

$$a_{11.1.1} = a_{11}a_1^2 - a_{12}a_1 - a_{11}a_2 + a_{13},$$
$$a_{1.11.1} = a_{11}a_1^2 - a_{21}a_1 - a_1a_{12} + a_{22},$$
$$a_{11.11} = a_{11}^2 - a_{121},$$
$$a_{111.1} = a_{111}a_1 - a_{112},$$

and (C) may be also written

$$a_{2+1+1} = a_2a_1^2 - a_{21}a_1 - a_2a_{11} + a_{211},$$
$$a_{1+2+1} = a_2a_1^2 - a_{12}a_1 - a_1a_{21} + a_{121},$$
$$a_{2+2} = a_2^2 - a_{22},$$
$$a_{3+1} = a_3a_1 - a_{31}.$$

The Conjugate Law.

175. It has been seen in Chapter IV, Art. 156, of this Section that $(ab\ldots)$, $(ab\ldots)'$ being conjugate compositions,

$$N(ab\ldots)_{(1^n)} = N(ab\ldots)'_{(1^n)};$$

and we may enquire into the existence of an analogous theorem when the numbers permuted have any other specification.

Consider the expression

$$h_{(ab\ )} - h_{(ab\ )'},$$

which is the generating function for the difference between $N(ab\ldots)$ and $N(ab\ldots)'$ for all specifications of the numbers permuted. The generating function may be written

$$h_{ab\ } - a_{ab}$$

according to a theorem already proved (Art. 173).

The differential operator D_1 has the equivalent forms

$$\frac{d}{da_1} + a_1\frac{d}{da_2} + a_2\frac{d}{da_3} + \ldots; \quad \frac{d}{dh_1} + h_1\frac{d}{dh_2} + h_2\frac{d}{dh_3} + \ldots,$$

so that $D_1^\mu h_{ab}$ is the same function of h_1, h_2, h_3, ... that $D_1^\mu a_{ab}$ is of a_1, a_2, a_3, ...; it follows at once that $D_1^n(h_{ab\ } - a_{ab\ }) = 0$ or

$$N(ab\ldots)_{1^n} = N(ab\ldots)'_{1^n},$$

the result established in Art. 156.

Next, considering the function

$$h_{(ab\ )} + h_{(ab\ )'} \quad \text{or} \quad h_{ab\ } + a_{ab\ },$$

it is clear that $D_1^{n-2}(h_{ab\ } + a_{ab\ })$ must be of the form

$$Ah_2 + Bh_1^2 + Aa_2 + Ba_1^2,$$

where A and B are numbers that it is not necessary to know; now this expression is

$$(A + 2B)\{(2) + 2(1^2)\};$$

and, remembering from Art. 26 the manner of operation of D_1 and D_2, it is plain that

$$D_1^{n-2}D_2(h_{ab\ } + a_{ab\ }) = \tfrac{1}{2}D_1^n(h_{ab\ } + a_{ab\ }) = D_1^n h_{ab\ };$$

that is to say, $N(ab\ldots)_{21^{n-2}} + N(ab\ldots)'_{21^{n-2}} = N(ab\ldots)_{1^n}$.

Ex. gr., from the Table for $n = 6$,

$$N(33)_{21^4} + N(1^221^2)_{21^4} = N(33)_{1^6},$$

verified by $\quad 13 \quad + \quad 6 \quad = \quad 19.$

176. Again, operating with D_1^{n-3} upon

$$h_{ab\ .} - a_{ab\ ,}$$

we obtain a result of the form

$$A\,(h_3 - a_3) + B\,(h_2 h_1 - a_2 a_1),$$

or
$$(A + B)\,\{(3) + (21)\};$$

whence
$$D_1^{n-3} D_3\,(h_{ab\ .} - a_{ab\ .}) = D_1^{n-2} D_2\,(h_{ab\ .} - a_{ab\ ,}),$$

equivalent to

$$N\,(ab\ldots)_{31^{n-3}} - N\,(ab\ldots)'_{31^{n-3}} = N\,(ab\ldots)_{21^{n-2}} - N\,(ab\ldots)'_{21^{n-2}}.$$

Ex. gr. $N\,(321)_{313} - N\,(2^2 1^2)_{313} = N\,(321)_{21^4} - N\,(2^2 1^2)_{21^4}$,

verified by $8 - 3 = 20 - 15$.

177. Again, operating with D_1^{n-4} upon $h_{ab\ .} - a_{ab\ .}$ and upon $h_{ab\ .} + a_{ab\ .}$, we obtain the new results

$$N\,(ab\ldots)_{2^2 1^{n-4}} - N\,(ab\ldots)'_{2^2 1^{n-4}} = N\,(ab\ldots)_{31^{n-3}} - N\,(ab\ldots)'_{31^{n-3}};$$

$$2N\,(ab\ldots)_{41^{n-4}} + 2N\,(ab\ldots)'_{41^{n-4}} + N\,(ab\ldots)_{21^{n-2}} + N\,(ab\ldots)'_{21^{n-2}}$$
$$= 2N\,(ab\ldots)_{31^{n-3}} + 2N\,(ab\ldots)'_{31^{n-3}} + N\,(ab\ldots)_{2^2 1^{n-4}} + N\,(ab\ldots)'_{2^2 1^{n-4}}.$$

We might also make use of the operators

$$\frac{d}{da_2} + a_1 \frac{d}{da_3} + a_2 \frac{d}{da_4} + \ldots \equiv -\frac{d}{dh_2} - h_1 \frac{d}{dh_3} - h_2 \frac{d}{dh_4} - \ldots \equiv D_1^2 - 2D_2,$$

$$\frac{d}{da_3} + a_1 \frac{d}{da_4} + a_2 \frac{d}{da_5} + \ldots \equiv \frac{d}{dh_3} + h_1 \frac{d}{dh_4} + h_2 \frac{d}{dh_5} + \ldots \equiv D_1^3 - 3D_2 D_1 + D_3,$$

etc.,

but the results obtainable are more complicated and less interesting.

Enough has been said to shew the nature of the results that can be reached.

Theory of the Compositions containing a given Number of Parts.

178. So far we have been considering Newcomb's card problem so as to determine the probability that a distribution of the cards will give a number of packs specified by a given composition of the number of cards. We have found that the mathematical problem is concerned with the ascending (or descending) specification of the permutations of objects of given type. We have now to study the probability that a given number of packs will result from the distribution. Mathematically we have to find the number of permutations of objects of given type such that the ascending specification is denoted by a composition containing a given number m of parts. Let this number be denoted by

$$N_{m,pqr\ldots}.$$

We have shewn in Art. 17 that the number of ways of distributing objects of type $(pqr\ldots)$ into m different parcels is

$$F_m = \binom{m+p-1}{p}\binom{m+q-1}{q}\binom{m+r-1}{r}\ldots$$
$$-\binom{m}{1}\binom{m+p-2}{p}\binom{m+q-2}{q}\binom{m+r-2}{r}\ldots$$
$$+\binom{m}{2}\binom{m+p-3}{p}\binom{m+q-3}{q}\binom{m+r-3}{r}\ldots$$
$$-\ldots;$$

or say for brevity

$$F_m = G_m - \binom{m}{1} G_{m-1} + \binom{m}{2} G_{m-2} - \ldots.$$

Consider on the one hand the arrangements enumerated by F_m and on the other those enumerated by N_{m-s}. In the F_m arrangements place the numbers in each parcel in ascending order and place the parcels in a row from left to right. It may be possible to merge two or more adjacent parcels or compartments into a single parcel or compartment and still preserve the ascending order in the new compartments. We observe $m-1$ dividing lines between compartments that it may be possible to obliterate and still preserve the ascending order. In the arrangements enumerated by N_{m-s} there are $m-s-1$ dividing lines between compartments and in each compartment there is ascending order, while it is not possible to obliterate any dividing line without breaking the ascending order. There are moreover

$$(n-1)-(m-s-1)$$

pairs of numbers between which it is possible to insert dividing lines so as to obtain arrangements enumerated by $F_{m-s+1}, F_{m-s+2}, \ldots F_m$. If out of these $n-m+s$ pairs of numbers we choose at random s, we can insert s additional dividing lines and so obtain an F_m arrangement. The choice can be made in

$$\binom{n-m+s}{s} \text{ ways,}$$

so that a particular N_{m-s} arrangement can be obtained, by obliteration of s lines from $\binom{n-m+s}{s}$ different F_m arrangements. Hence the forms F_m include the forms N_{m-s} each $\binom{n-m+s}{s}$ times. Therefore we have the relation

$$F_m = N_m + \binom{n-m+1}{1} N_{m-1} + \binom{n-m+2}{2} N_{m-2} + \ldots + \binom{n-1}{m-1} N_1;$$

leading, with little trouble, to the next relation

$$N_m = F_m - \binom{n-m+1}{1} F_{m-1} + \binom{n-m+2}{2} F_{m-2} - \ldots (-)^{m+1}\binom{n-1}{m-1} F_1.$$

Substituting for F_m, F_{m-1}, ... their expressions in terms of G_m, G_{m-1}, ..., we find

$$N_m = \sum_{s=0}^{s=m-1} (-)^s \left\{ \binom{m}{s} + \binom{m-1}{m-s}\binom{n-m+1}{1} + \binom{m-2}{m-s}\binom{n-m+2}{2} + \dots + \binom{n-m+s}{s} \right\} G_{m-s};$$

and the term in { } is seen to be the coefficient of x^s in

$$(1-x)^{-m+s-1} \times (1-x)^{-n+m-1} \quad \text{or} \quad (1-x)^{-n+s-2},$$

which is $$\binom{n+1}{s}.$$

Hence

$$N_m = G_m - \binom{n+1}{1} G_{m-1} + \binom{n+1}{2} G_{m-2} - \dots + (-)^{m+1} \binom{n+1}{m-1} G_1.$$

This formula was used to calculate the subjoined Tables of N_m.

Summing each side of the relation, from $m=1$ to $m=m$, we have

$$N_m + N_{m-1} + \dots + N_1$$

$$= G_m - \binom{n}{1} G_{m-1} + \binom{n}{2} G_{m-2} - \dots + (-)^{m+1} \binom{n}{m-1} G_1;$$

and, repeating the summation θ times,

$$N_m + \binom{\theta+1}{1} N_{m-1} + \binom{\theta+2}{2} N_{m-2} + \dots + \binom{\theta+m-1}{m-1} N_1$$

$$= G_m - \binom{n-\theta}{1} G_{m-1} + \binom{n-\theta}{2} G_{m-2} - \dots + (-)^{m+1} \binom{n-\theta}{m-1} G_1;$$

so that, when $\theta = n$,

$$N_m + \binom{n+1}{1} N_{m-1} + \binom{n+2}{2} N_{m-2} + \dots + \binom{n+m-1}{m-1} N_1 = G_m.$$

Again, taking differences,

$$N_m - N_{m-1} = G_m - \binom{n+2}{1} G_{m-1} + \binom{n+2}{2} G_{m-2} - \dots + (-)^{m+1} \binom{n+2}{m-1} G_1;$$

and, in general,

$$N_m - \binom{p}{1} N_{m-1} + \binom{p}{2} N_{m-2} - \dots + (-)^p N_{m-p}$$

$$= G_m - \binom{n+p+1}{1} G_{m-1} + \binom{n+p+1}{2} G_{m-2} - \dots + (-)^{m+1} \binom{n+p+1}{m-1} G_1.$$

These results are given by the two formulæ, viz. the one just written and

$$N_m + \binom{p}{1} N_{m-1} + \binom{p+1}{2} N_{m-2} + \ldots + \binom{p+m-2}{m-1} N_1$$

$$= G_m - \binom{n-p+1}{1} G_{m-1} + \binom{n-p+1}{2} G_{m-2} - \ldots + (-)^{m+1} \binom{n-p+1}{m-1} G_1;$$

and these become the same when $p = 0$.

The Generating Function of the Number $N_{m,(pqr\ldots)}$.

179. The theory of these numbers is identical with that of the permutations of the symbols in $\alpha_1^p \alpha_2^q \alpha_3^r \ldots$ which possess exactly $m-1$ major contacts. See Art. 155.

For taking any permutation, α_1, α_2, α_3, ... being symbols in ascending order of numerical magnitude,

$$\ldots \alpha_2 \alpha_1 \ldots \alpha_3 \alpha_1 \ldots \alpha_3 \alpha_2 \ldots \alpha_3 \alpha_2 \alpha_1 \ldots,$$

which exhibits all of the major contacts and the vacant spaces are supposed to be occupied by symbols in ascending order of numerical magnitude, it is clear that we can separate the permutations into compartments by lines drawn between every pair of symbols that are in major contact so that in each compartment the symbols are in ascending order and that order is broken between adjacent compartments

$$\ldots \alpha_2 \,|\, \alpha_1 \ldots \alpha_3 \,|\, \alpha_1 \ldots \alpha_3 \,|\, \alpha_2 \ldots \alpha_3 \,|\, \alpha_2 \,|\, \alpha_1 \ldots.$$

The number of such compartments is *one greater* than the number of major contacts. Hence $N_{m,(p_1 p_2 \ldots p_k)}$ is the coefficient of

$$\lambda^{m-1} \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_k^{p_k}$$

in the development of

$$(\alpha_1 + \lambda\alpha_2 + \lambda\alpha_3 + \ldots + \lambda\alpha_k)^{p_1} (\alpha_1 + \alpha_2 + \lambda\alpha_3 + \ldots + \lambda\alpha_k)^{p_2} \ldots (\alpha_1 + \alpha_2 + \ldots + \alpha_k)^{p_k},$$

or of

$$\frac{1}{1 - \Sigma\alpha_1 + (1-\lambda)\Sigma\alpha_1\alpha_2 - (1-\lambda)^2 \Sigma\alpha_1\alpha_2\alpha_3 + \ldots + (-)^k (1-\lambda)^{k-1} \alpha_1\alpha_2\ldots\alpha_k}.$$

Introducing the elementary functions a_1, a_2, a_3, ..., putting $1 - \lambda = b$ and supposing, for the present, $k = \infty$, this may be written

$$\frac{1}{1 - a_1 + ba_2 - b^2 a_3 + b^3 a_4 - \ldots}.$$

Since the generating function is symmetrical in regard to α_1, α_2, α_3, ... the value of N_m is not affected by any permutation of the letters α_1, α_2, α_3, ...

180. We can shew that

$$N_{m,(p^k)} = N_{(k-1)p-m+2,(p^k)};$$

for the coefficient of $\lambda^{m-1}(\alpha_1\alpha_2\ldots\alpha_k)^p$ in

$$\{(\alpha_1+\lambda\alpha_2+\ldots+\lambda\alpha_k)(\alpha_1+\alpha_2+\lambda\alpha_3+\ldots+\lambda\alpha_k)\ldots(\alpha_1+\alpha_2+\ldots+\alpha_k)\}^p$$

is, by writing $\frac{1}{\lambda}$ for λ and $\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \ldots$ for $\alpha_1, \alpha_2, \alpha_3, \ldots$, equal to the coefficient of $\lambda^{kp-m+1}(\alpha_1\alpha_2\ldots\alpha_k)^p$ in

$$\{(\lambda\alpha_1+\alpha_2+\ldots+\alpha_k)(\lambda\alpha_1+\lambda\alpha_2+\alpha_3+\ldots+\alpha_k)\ldots(\lambda\alpha_1+\lambda\alpha_2+\ldots+\lambda\alpha_k)\}^p,$$

or of $\lambda^{(k-1)p-m+1}(\alpha_1\alpha_2\ldots\alpha_k)^p$ in

$$\{(\lambda\alpha_1+\alpha_2+\ldots+\alpha_k)(\lambda\alpha_1+\lambda\alpha_2+\alpha_3+\ldots+\alpha_k)\ldots(\alpha_1+\alpha_2+\ldots+\alpha_k)\}^p,$$

or, since we may permute the symbols $\alpha_1, \alpha_2, \ldots\alpha_k$ at pleasure, in

$$\{(\alpha_1+\lambda\alpha_2+\ldots+\lambda\alpha_k)(\alpha_1+\alpha_2+\lambda\alpha_3+\ldots+\lambda\alpha_k)\ldots(\alpha_1+\alpha_2+\ldots+\alpha_k)\}^p.$$

Hence $$N_{m,(p^k)} = N_{(k-1)p-m+2,(p^k)}.$$

The numbers N_m range from N_1 to $N_{(k-1)p+1}$, and $(k-1)p+1$ is the maximum number of parts in an ascending specification.

181. We must now examine the generating function

$$\frac{1}{1-A},$$

where $$A = a_1 - ba_2 + b^2a_3 - b^3a_4 + \ldots.$$

Taking as in Section II the symmetric function operators

$$d_s = \frac{d}{da_s} + a_1\frac{d}{da_{s+1}} + a_2\frac{d}{da_{s+2}} + \ldots,$$

$$D_s = \frac{1}{s!}\left(\frac{d}{da_1} + a_1\frac{d}{da_2} + a_2\frac{d}{da_3} + \ldots\right)^s = \frac{1}{s!}(d_1^s),$$

we find that, with the operand A, they are connected by special relations

For $$d_sA = (-)^{s-1}b^{s-1}(1-bA) = (-)^{s-1}b^{s-1}d_1A,$$

so that $$d_s \equiv (-b)^{s-1}d_1,$$

and thence, from the relations

$$\begin{aligned} d_2 &= D_1^2 - 2D_2 &&= (2)_D,\\ d_3 &= D_1^3 - 3D_2D_1 + 3D_3 &&= (3)_D,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ d_s &= D_1^s - \ldots &&= (s)_D, \end{aligned}$$

we derive the special relations

$$\begin{aligned} 2!D_2 &\equiv D_1(D_1+b),\\ 3!D_3 &\equiv D_1(D_1+b)(D_1+2b),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ s!D_s &\equiv D_1(D_1+b)\ldots\{D_1+(s-1)b\}; \end{aligned}$$

and, if $s > t$,

$$s!D_s \equiv t!D_t(D_1+tb)\dots\{D_1+(s-1)b\}.$$

Using the above we find

$$(p)_D = d_p = (-b)^{p-1}D_1;$$

$$(pq)_D = (d_pd_q) = (d_p)(d_q) - d_{p+q},$$

$$\equiv (-b)^{p+q-2}D_1^2 - (-b)^{p+q-1}D_1 \equiv (-b)^{p+q-2}D_1(D_1+b),$$

$$\equiv (-b)^{p+q-2}2!D_2;$$

$$(pqr)_D = (d_pd_qd_r) = (d_p)(d_q)(d_r) - (d_{p+q})(d_r) - (d_{p+r})(d_q) - (d_{q+r})(d_p) + 2d_{p+q+r},$$

$$\equiv (-b)^{p+q+r-3}D_1^3 - 3(-b)^{p+q+r-2}D_1^2 + 2(-b)^{p+q+r-1}D_1,$$

$$\equiv (-b)^{p+q+r-3}3!D_3;$$

and in general $$(p_1p_2\dots p_k)_D \equiv (-b)^{\Sigma p-k}k!D_k.$$

Moreover since $$(p_1^{\pi_1}p_2^{\pi_2}\dots)_D = \frac{1}{\pi_1!\pi_2!\dots}(d_{p_1}^{\pi_1}d_{p_2}^{\pi_2}\dots),$$

we have, still more generally,

$$(p_1^{\pi_1}p_2^{\pi_2}\dots)_D \equiv (-b)^{\Sigma\pi p-\Sigma\pi}\frac{(\Sigma\pi)!}{\pi_1!\pi_2!\dots}D_{\Sigma\pi};$$

or if $\Sigma\pi p = n$, $\Sigma\pi = i$,

$$(p_1^{\pi_1}p_2^{\pi_2}\dots)_D \equiv (-b)^{n-i}\frac{i!}{\pi_1!\pi_2!\dots}D_i.$$

Hence if $(q_1^{\chi_1}q_2^{\chi_2}\dots)$ be of the same weight and degree as $(p_1^{\pi_1}p_2^{\pi_2}\dots)$, it follows that

$$\pi_1!\pi_2!\dots(p_1^{\pi_1}p_2^{\pi_2}\dots)_D = \chi_1!\chi_2!\dots(q_1^{\chi_1}q_2^{\chi_2}\dots)_D,$$

a relation which is free from b.

$n=3$

	(3)	(21)	(1^3)
N_1	1	1	1
N_2	—	2	4
N_3	—	—	1

$n=4$

	(4)	(31)	(2^2)	(21^2)	(1^4)
N_1	1	1	1	1	1
N_2	—	3	4	7	11
N_3	—	—	1	4	11
N_4	—	—	—	—	1

$n=5$

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
N_1	1	1	1	1	1	1	1
N_2	—	4	6	10	12	18	26
N_3	—	—	3	9	15	33	66
N_4	—	—	—	—	2	8	26
N_5	—	—	—	—	—	—	1

$n=6$

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
N_1	1	1	1	1	1	1	1	1	1	1	1
N_2	—	5	8	13	9	17	25	20	29	41	57
N_3	—	—	6	16	9	33	67	48	93	171	302
N_4	—	—	—	—	1	9	27	20	53	131	302
N_5	—	—	—	—	—	—	—	1	4	16	57
N_6	—	—	—	—	—	—	—	—	—	—	1

To explain, observe that the number at the intersection of the row N_3 and the column (2^21^2) shews that

$$N_{3,(2^21^2)}=93.$$

Applications of the Foregoing to the Generating Function.

182. It has been established that

$$\frac{1}{1-a_1+(1-\lambda)a_2-(1-\lambda)^2a_3+\dots}$$
$$=\Sigma N_{m,(p_1p_2\dots p_k)}\lambda^{m-1}(p_1p_2\dots p_k)=\Sigma N_{m,(p_1p_2\dots p_k)}(1-b)^{m-1}(p_1p_2\dots p_k)$$
$$=\Sigma N_{m,(1^{c_1}2^{c_2}3^{c_3}\dots)}(1-b)^{m-1}(1^{c_1}2^{c_2}3^{c_3}\dots).$$

We will, first of all, examine what results from the equivalence of operators

$$2!D_2=D_1^2+(1-\lambda)D_1=D_1^2+bD_1.$$

Recalling the mode of operation of D_s we clearly have

$$2!N_{m,(1^{c_1}2^{c_2+1}3^{c_3}\dots)}=N_{m,(1^{c_1+2}2^{c_2}3^{c_3}\dots)}+(N_m-N_{m-1})_{(1^{c_1+1}2^{c_2}3^{c_3}\dots)},$$

an interesting relation between these numbers. It will be observed that we have above written

$$N_{m,(\dots)}-N_{m-1,(\dots)}$$

in the abbreviated form $(N_m-N_{m-1})_{(\dots)}$.

Ex. gr. put $m=3$, $c_1=0$, $c_2=2$, $c_3=c_4=\dots=0$, and we obtain

$$2!N_{3,(2^2)}=N_{3,(1^22^2)}+(N_3-N_2)_{(1^22)},$$

verified by $2.48 = 93 + 15 - 12.$

Again, in the same formula, put $c_1 = n-2$, $c_2 = c_3 = \ldots = 0$, and we get

$$2N_{m,\,(21^{n-2})} = N_{m,\,(1^n)} + (N_m - N_{m-1})_{(1^{n-1})}.$$

In this write $n-m+1$ for m, obtaining

$$2N_{n-m+1,\,(21^{n-2})} = N_{n-m+1,\,(1^n)} + (N_{n-m+1} - N_{n-m})_{(1^{n-1})};$$

from these, since

$$N_{m,\,(1^n)} = N_{n-m+1,\,(1^n)}; \quad N_{m,\,(1^{n-1})} = N_{n-m,\,(1^{n-1})}; \quad N_{m-1,\,(1^{n-1})} = N_{n-m+1,\,(1^{n-1})},$$

we find by addition and subtraction

$$N_{m,\,(21^{n-2})} + N_{n-m+1,\,(21^{n-2})} = N_{m,\,(1^n)},$$
$$N_{m,\,(21^{n-2})} - N_{n-m+1,\,(21^{n-2})} = N_{m,\,(1^{n-1})} - N_{n-m+1,\,(1^{n-1})}.$$

These are the relations connecting N_m and N_{n-m+1} for the subscript (21^{n-2}) analogous to that connecting the same numbers for the subscript (1^n).

Since $b^\sigma = (1-\lambda)^\sigma = 1 - \binom{\sigma}{1}\lambda + \binom{\sigma}{2}\lambda^2 - \ldots$ we can immediately, from any operator relation, form a relation between the numbers N_m by substituting for

$$b^\sigma D_s^p D_t^q \ldots$$

the expression

$$\left\{N_m - \binom{\sigma}{1} N_{m-1} + \binom{\sigma}{2} N_{m-2} - \ldots\right\}_{(1^{c_1} 2^{c_2} \ldots s^{c_s+p} t^{c_t+q} \ldots)};$$

and this it is sometimes convenient to denote by

$$N^{(\sigma)}_{m,\,(1^{c_1} 2^{c_2} \ldots s^{c_s+p} t^{c_t+q} \ldots)}.$$

Thus corresponding to the relation

$$6D_3 = D_1^3 + 3bD_1^2 + 2b^2 D_1,$$

we obtain

$$\begin{aligned} 6N_{m,\,(1^{c_1} 2^{c_2} 3^{c_3+1} \ldots)} &= N_{m,\,(1^{c_1+3} 2^{c_2} 3^{c_3} \ldots)} + 3N^{(1)}_{m,\,(1^{c_1+2} 2^{c_2} 3^{c_3} \ldots)} + 2N^{(2)}_{m,\,(1^{c_1+1} 2^{c_2} 3^{c_3} \ldots)} \\ &= N_{m,\,(1^{c_1+3} 2^{c_2} 3^{c_3} \ldots)} \\ &\quad + 3\,(N_m - N_{m-1})_{(1^{c_1+2} 2^{c_2} 3^{c_3} \ldots)} \\ &\quad + 2\,(N_m - 2N_{m-1} + N_{m-2})_{(1^{c_1+1} 2^{c_2} 3^{c_3} \ldots)}. \end{aligned}$$

From the symmetric function relation

$$(1^2)(1^2) = (2^2) + 2(21^2) + 6(1^4)$$

we find

$$D_2^2 = b^2 D_2 - 6bD_3 + 6D_4,$$

leading to

$$\begin{aligned} N_{m,\,(1^{c_1} 2^{c_2+2} 3^{c_3} 4^{c_4} \ldots)} &= (N_m - 2N_{m-1} + N_{m-2})_{(1^{c_1} 2^{c_2+1} 3^{c_3} 4^{c_4} \ldots)} \\ &\quad - 6\,(N_m - N_{m-1})_{(1^{c_1} 2^{c_2} 3^{c_3+1} 4^{c_4} \ldots)} \\ &\quad + 6N_{m,\,(1^{c_1} 2^{c_2} 3^{c_3} 4^{c_4+1} \ldots)}, \end{aligned}$$

a relation which, for

$$m = 4, \quad c_1 = 2, \quad c_2 = c_3 = c_4 = \ldots = 0,$$

can be verified from the Tables.

SECTION V

DISTRIBUTIONS UPON A CHESS BOARD, TO WHICH IS PREFIXED A CHAPTER ON PERFECT PARTITIONS

CHAPTER I

THEORY OF PERFECT PARTITIONS OF NUMBERS

183. Before discussing the general subject of Partitions of Numbers it is appropriate to consider certain special partitions which have a notable and remarkable application to the theory of the distributions of a certain nature. The application will be found in the next chapter which deals with arrangements on a chess board of given dimensions.

A "perfect" partition of a number is one which contains one and only one partition of every lesser number.

Ex. gr. (41^3) is a perfect partition of 7 because by using the parts we can build up one and only one partition of each of the numbers 1, 2, 3, 4, 5, 6; thus (1), (1^2), (1^3), (4), (41), (41^2) are the partitions referred to.

Other partitions possess a very similar property. These are termed "subperfect" and may be defined as follows:

A "subperfect" partition of a number is one which contains one and only one partition of every lesser number if it be permissible to regard the several parts as affected with *either* the positive or negative sign.

These theories have an application to the theories of weights and measures; for perfect partitions solve the problem:

(1) to partition a weight of u lbs. so as to be able to weigh, in only one manner, any weight of an integral number of lbs. from 1 to u inclusive, it being only permissible to place the weights in one scale-pan.

Moreover subperfect partitions solve the problem:

(2) to solve the same problem as in (1) with this difference, that the weights may be placed in either the goods pan or the weight pan.

Ex. gr. (31) is a subperfect partition of 4 because there are the partitions of the numbers 1, 2, 3, viz. (1), (3$\bar{1}$), (3), where observe that $\bar{1}$ denotes the part -1.

184. Before proceeding to the general case of enumeration we will study the problem of perfect partitions in the case of numbers of the form

$$p^\alpha - 1,$$

where p is a prime number. It will become manifest that the enumeration of the perfect partitions is identical with the enumeration of the compositions of the number α.

All numbers have clearly one perfect partition composed wholly of unit parts. This fact is exhibited, in the case of the numbers before us, by the relation

$$\frac{1-x^{p^\alpha}}{1-x} = 1 + x + x^2 + \ldots + x^{p^\alpha - 1};$$

for it shews that the partition $(1^{p^\alpha - 1})$ is such that every number from 1 to $p^\alpha - 1$ can be made up by selecting, from the partition, 1, 2, ... $p^\alpha - 1$ units. This fact depends upon the circumstance that $1 - x^{p^\alpha}$ is divisible by $1 - x$.

It may happen that the fraction

$$\frac{1-x^{p^\alpha}}{1-x}$$

may be put into the form

$$\frac{1-x^{p^\alpha}}{1-x^q} \cdot \frac{1-x^q}{1-x^r} \cdot \frac{1-x^r}{1-x^s} \cdot \cdots \frac{1-x^v}{1-x},$$

wherein $q, r, s, \ldots 1$ are factors of p^α, $q, r, \ldots v$ respectively. To each such exhibition corresponds a perfect partition of the number $p^\alpha - 1$. For consider the particular case

$$\frac{1-x^{p^4}}{1-x^p} \cdot \frac{1-x^p}{1-x},$$

or $$(1 + x^p + x^{2p} + \ldots + x^{p^4 - p})(1 + x + x^2 + \ldots + x^{p-1});$$

the identity

$$(1 + x^p + x^{2p} + \ldots + x^{p^4 - p})(1 + x + x^2 + \ldots + x^{p-1}) = 1 + x + x^2 + \ldots + x^{p^4 - 1}$$

shews, by multiplying out the left-hand side, that each of the numbers $1, 2, \ldots p^4 - 1$ can be made up, *in one way only*, from the parts of the partition

$$(p^{p^3 - 1} 1^{p-1}) \text{ of the number } p^4 - 1.$$

Let us denote the fraction

$$\frac{1-x^{p^\sigma}}{1-x^{p^\tau}} \text{ by } \sigma - \tau,$$

which, observe, is the difference of the exponents σ and τ of p in the numerator and denominator; then we may denote

$$\frac{1-x^{p^4}}{1-x^p}\cdot\frac{1-x^p}{1-x} \text{ by } 3, 1,$$

where (31) is necessarily a composition of 4. There is also an exhibition represented by

$$\frac{1-x^{p^4}}{1-x^{p^3}}\cdot\frac{1-x^{p^3}}{1-x} = 1, 3,$$

where the same numbers 3, 1 occur but in different order.

Whatever instance of factorization be exhibited we inevitably reach a succession of numbers which constitutes a composition of 4.

The whole of the factorizations are shewn with the perfect partitions which flow from them.

Form	Composition	Perfect partition
$\frac{1-x^{p^4}}{1-x}$	(4)	(1^{p^4-1})
$\frac{1-x^{p^4}}{1-x^p}\cdot\frac{1-x^p}{1-x}$	(31)	$(p^{p^3-1}1^{p-1})$
$\frac{1-x^{p^4}}{1-x^{p^3}}\cdot\frac{1-x^{p^3}}{1-x}$	(13)	$(\overline{p^3}^{p-1}1^{p^3-1})$
$\frac{1-x^{p^4}}{1-x^{p^2}}\cdot\frac{1-x^{p^2}}{1-x}$	(22)	$(\overline{p^2}^{p^2-1}1^{p^2-1})$
$\frac{1-x^{p^4}}{1-x^{p^2}}\cdot\frac{1-x^{p^2}}{1-x^p}\cdot\frac{1-x^p}{1-x}$	(211)	$(\overline{p^2}^{p^2-1}p^{p-1}1^{p-1})$
$\frac{1-x^{p^4}}{1-x^{p^3}}\cdot\frac{1-x^{p^3}}{1-x^p}\cdot\frac{1-x^p}{1-x}$	(121)	$(\overline{p^3}^{p-1}p^{p^2-1}1^{p-1})$
$\frac{1-x^{p^4}}{1-x^{p^3}}\cdot\frac{1-x^{p^3}}{1-x^{p^2}}\cdot\frac{1-x^{p^2}}{1-x}$	(112)	$(\overline{p^3}^{p-1}\overline{p^2}^{p-1}1^{p^2-1})$
$\frac{1-x^{p^4}}{1-x^{p^3}}\cdot\frac{1-x^{p^3}}{1-x^{p^2}}\cdot\frac{1-x^{p^2}}{1-x^p}\cdot\frac{1-x^p}{1-x}$	(1111)	$(\overline{p^3}^{p-1}\overline{p^2}^{p-1}p^{p-1}1^{p-1})$

We have an instance of factorization and therefore of perfect partition corresponding to every composition of the number 4.

Returning to the more general case of the factorization of

$$\frac{1-x^{p^\alpha}}{1-x},$$

it is clear that

$$\frac{1-x^{p^\alpha}}{1-x^{p^{\alpha_1}}}\cdot\frac{1-x^{p^{\alpha_1}}}{1-x^{p^{\alpha_2}}}\cdot\frac{1-x^{p^{\alpha_2}}}{1-x^{p^{\alpha_3}}}\cdot\cdots\frac{1-x^{p^{\alpha_s}}}{1-x}$$

will yield the composition $\alpha-\alpha_1, \alpha_1-\alpha_2, \alpha_2-\alpha_3, \ldots \alpha_s$ of the number α.

Moreover any composition $(\theta_1\theta_2\ldots\theta_{s+1})$ of the number can be put into the form

$$\Big(\alpha-(\alpha-\theta_1),\ \alpha-\theta_1-(\alpha-\theta_1-\theta_2),\ \alpha-\theta_1-\theta_2-(\alpha-\theta_1-\theta_2-\theta_3),\ \ldots,$$
$$\alpha-\theta_1-\theta_2-\ldots-\theta_s-(\alpha-\theta_1-\theta_2-\ldots-\theta_{s+1})\Big),$$

and will correspond to a factorization

$$\frac{1-x^{p^\alpha}}{1-x^{p^{\alpha-\theta_1}}}\cdot\frac{1-x^{p^{\alpha-\theta_1}}}{1-x^{p^{\alpha-\theta_1-\theta_2}}}\cdot\frac{1-x^{p^{\alpha-\theta_1-\theta_2}}}{1-x^{p^{\alpha-\theta_1-\theta_2-\theta_3}}}\cdot\ldots\frac{1-x^{p^{\alpha-\theta_1-\theta_2-\ldots-\theta_s}}}{1-x}.$$

Hence there is a one-to-one correspondence between the factorizations of $(1-x^{p^\alpha})/(1-x)$ and the compositions of the number α. Hence the special problem is solved in the form:

"A number of the form $p^\alpha-1$, where p is a prime, possesses the same number of perfect partitions as the number α has compositions and therefore the number of such is $2^{\alpha-1}$."

The number of *different* parts which appear in the partition is equal to the number of parts in the composition.

Corresponding to the composition $(\theta_1\theta_2\ldots\theta_{s+1})$ the perfect partition has, altogether,

$$p^{\theta_1}+p^{\theta_2}+\ldots+p^{\theta_{s+1}}-s-1 \text{ parts.}$$

The smallest value of this number is $\alpha(p-1)$.

185. Every number is of the form $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\ldots-1$ where $p_1, p_2, p_3, \ldots$ are primes.

We will now shew that the number of perfect partitions possessed by this number is equal to the number of compositions of the multipartite number $(\alpha_1\alpha_2\alpha_3\ldots)$.

The number of perfect partitions depends upon the factorization of

$$\frac{1-x^{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\ldots}}{1-x}$$

into factors of the form $(1-x^q)/(1-x^p)$ where p is a divisor of q.

Suppose such to be

$$\frac{1-x^{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\ldots}}{1-x^{p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}\ldots}}\cdot\frac{1-x^{p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}\ldots}}{1-x^{p_1^{\gamma_1}p_2^{\gamma_2}p_3^{\gamma_3}\ldots}}\cdot\ldots\frac{1-x^{p_1^{\nu_1}p_2^{\nu_2}p_3^{\nu_3}\ldots}}{1-x},$$

which, in analogy with what has gone before, we denote by

$$(\alpha_1-\beta_1,\ \alpha_2-\beta_2,\ \alpha_3-\beta_3\ldots;\ \beta_1-\gamma_1,\ \beta_2-\gamma_2,\ \beta_3-\gamma_3\ldots;\ \ldots\ldots\nu_1,\ \nu_2,\ \nu_3\ldots;).$$

This may be regarded as a composition of the multipartite number

$$(\alpha_1\alpha_2\alpha_3\ldots).$$

As before we see that there is a one-to-one correspondence between the factorizations and the compositions, and we have the theorem:

"The number $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots - 1$ possesses as many perfect partitions as the multipartite number $(\alpha_1 \alpha_2 \alpha_3 \ldots)$ has compositions."

The number of parts in a perfect partition has the least value

$$\alpha_1 (p_1 - 1) + \alpha_2 (p_2 - 1) + \alpha_3 (p_3 - 1) + \ldots$$

in correspondence with the composition

$$(100\ldots^{\alpha_1} 010\ldots^{\alpha_2} 001\ldots^{\alpha_3} \ldots).$$

In correspondence with the factorization above written the perfect partition has

$p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2} p_3^{\alpha_3 - \beta_3} \ldots - 1$ parts each equal to $p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \ldots$,

$p_1^{\beta_1 - \gamma_1} p_2^{\beta_2 - \gamma_2} p_3^{\beta_3 - \gamma_3} \ldots - 1$ „ „ $p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3} \ldots$,

.....

$p_1^{\nu_1} p_2^{\nu_2} p_3^{\nu_3} \ldots - 1$ „ „ 1.

186. The following examples of perfect partitions are given:

Number $15 = 2^4 - 1$

Unipartite number 4

Group	Composition	Perfect partition
(4)	(4)	(1^{15})
(31)	(31)	$(2^7 1)$
	(13)	$(8 1^7)$
(22)	(22)	$(4^3 1^3)$
(211)	(211)	$(4^3 21)$
	(121)	$(8 2^3 1)$
	(112)	$(8 4 1^3)$
(1111)	(1111)	(8421)

Number $244 = 7^2 . 5 - 1$

Multipartite number (21)

Group	Composition	Perfect partition
(21)	(21)	(1^{244})
(20, 01)	(20, 01)	$(5^{48} . 1^4)$
	(01, 20)	$(49^4 . 1^{48})$
(11, 10)	(11, 10)	$(7^{34} . 1^6)$
	(10, 11)	$(35^6 . 1^{34})$
(10, 10, 01)	(10, 10, 01)	$(35^6 . 5^6 . 1^4)$
	(10, 01, 10)	$(35^6 . 7^4 . 1^6)$
	(01, 10, 10)	$(49^4 . 7^6 . 1^6)$

It will be noted that, 244 being equal to $7^2.5-1$, corresponding to a part (xy) in a composition, the perfect partition involves a repetitional number $7^x.5^y-1$.

187. The general form of a perfect partition is

$$1^A.(1+A)^B\{(1+A)(1+B)\}^C\{(1+A)(1+B)(1+C)\}^D\ldots.$$

It is moreover clear that every perfect partition of every number gives rise directly to an unlimited number of perfect partitions of higher numbers. For this purpose the repetitional index of the highest part may be increased to any extent, or a part equal to the sum of all the parts with unity added. From this point of view let A_p denote the number of perfect partitions of the number p so that a generating function (when expanded) is

$$A_0+A_1x+A_2x^2+\ldots+A_px^p+\ldots.$$

The perfect partitions having a highest part equal to $p+1$ are derivable from the perfect partitions of the number p by prefixing to each of the latter the part $p+1$ any number of times repeated. Hence the generating function of such is

$$\frac{A_px^{2p+1}}{1-x^{p+1}},$$

and we are led to the algebraic identity

$$\frac{A_0x}{1-x}+\frac{A_1x^3}{1-x^2}+\frac{A_2x^5}{1-x^3}+\ldots+\frac{A_px^{2p+1}}{1-x^{p+1}}+\ldots$$

$$=A_1x+A_2x^2+A_3x^3+\ldots+A_px^p+\ldots+A_{2p+1}x^{2p+1}+\ldots.$$

Herein equating coefficients of like powers of x we find

$$A_p=\sum_s A_s,$$

the summation being for values of s which make $s+1$ a divisor less than $p+1$ of $p+1$.

Thus $$A_{11}=A_0+A_1+A_2+A_3+A_5,$$

since 12 has the divisors 1, 2, 3, 4, 6.

188. It can be shewn that there is a one-to-one correspondence between the perfect partitions (all numbers being in evidence) containing s parts and the compositions of the number s. For write down the perfect partitions having less than 5 parts as follows:

			(1)				
		(1^2)				(21)	
	(1^3)			(31^2)			(421)
(1^4)	(41^3)	(3^21^2)	(631^2)		(2^31)	(62^31)	(4^221)	(8421)

It will be observed that in any line each perfect partition gives birth to two perfect partitions in the line below

(a) by prefixing a part equal to the existing highest part,

(b) by prefixing a part which is greater by unity than the sum of all the existing parts.

Write down next instead of the partitions themselves the repetitional numbers of the parts of those partitions. The scheme follows:

						1									
			2							11					
		3			12				21				111		
4		13		22		112		31		121		211		1111	

where the successive lines involve the compositions of the numbers 1, 2, 3, 4. Each composition in a line gives rise to two compositions in the line below by rules derived from (a) and (b) above. These are:

(a') increase the first part by unity,

(b') prefix a part unity.

There are therefore 2^{s-1} perfect partitions involving s parts.

189. The theory of the subperfect partitions is derived at once from what has been established above; for if we put, for the number u,

$$\phi_{u,1} = \sum_{k=-u}^{k=u} x^k = \frac{1 - x^{2u+1}}{x^u (1 - x)},$$

we have to consider the divisors of $2u + 1$ instead of those of $u + 1$, and the reader will have no difficulty in proving that if

$$2u + 1 = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots,$$

the number of subperfect partitions of u is equal to the number of compositions of the multipartite number $(\alpha_1 \alpha_2 \alpha_3 \ldots)$.

CHAPTER II

ARRANGEMENTS ON A CHESS BOARD

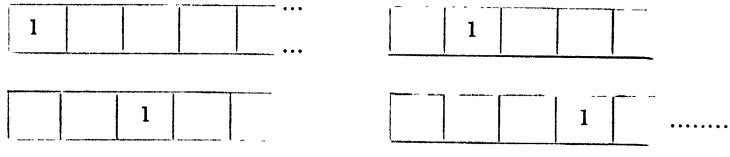
190. In this section we have for discussion arrangements of objects or numbers in the compartments of square or rectangular lattices. In previous chapters the distributions have been essentially on a line or in one dimension. We now come to two dimensions The numbers or objects themselves will as usual be specified as to type while the distributions will be subject to conditions depending upon the rows and columns of the lattice. The method employed depends upon the dissection of certain operations (usually differential operations) upon certain functions. A very simple example will suffice to lead up to the general idea. Consider the operation of $\frac{d}{dx}$ upon x^n where n is a positive integer. If we write x^n as the product of n factors each equal to x

$$xxxx \ldots$$

it is clear that we can pick out an x in n different ways; for example we can substitute unity for a factor x in n different ways; if we do so and then add the results together we arrive at nx^{n-1}, viz. the result of performing the operation $\frac{d}{dx}$ upon the operand x^n. Thus

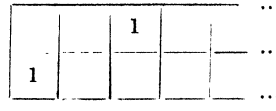
$$\frac{d}{dx}x^n = 1\,.\,xxx \ldots + x1xx \ldots + xx1x \ldots + xxx1 \ldots + \ldots = nx^{n-1}.$$

Observe that here we have dissected the operation of $\frac{d}{dx}$ into n separate operations of substitution. Moreover we can represent these n minor operations by means of n diagrams

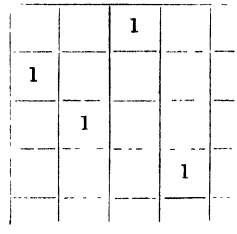


the unit indicating that particular x which is subjected to substitution.

If upon each of the terms $1 . xxx \ldots x1xx \ldots$, etc., we operate again in a similar manner, by substituting unity for x in all possible ways, we arrive at $n(n-1)$ terms and, adding these, we obtain $n(n-1)\,x^{n-2}$ which is the result of the operation $\dfrac{d^2}{dx^2}$ upon x^n. In correspondence therewith we have $n(n-1)$ diagrams each consisting of two rows of a lattice of order n. One of these diagrams is



Proceeding in this manner we find that the operation of $\dfrac{d^n}{dx^n}$ upon x^n can be dissected into $n!$ minor operations or processes which are in correspondence with $n!$ diagrams, each of which involves a distribution of n units in the compartments of a square lattice of order n of the nature



The diagrams enumerated are such that there is a distribution of n units in the cells of a square lattice of order n subject to the conditions that one single unit appears in each row and in each column. The problem of enumeration is solved by

$$\frac{d^n}{dx^n}\,x^n,$$

n being, by definition, an integer.

191. This result is quite as suggestive as it is trivial. We have on the one hand an operation and on the other a function such that the result of the operation on the function is an integer ν; we find it possible to exhibit the operation as the result of ν distinct processes and to represent each process by a two-dimensional diagram. These diagrams in their construction obey certain laws or satisfy certain conditions which enable us to define or specify the diagrams and they are enumerated by the number ν; that is to say the enumeration is given by the conjunction of the operator and function which leads naturally to the diagrams.

If we are given a system of diagrams, with compartments numbered or lettered according to given laws and conditions, are we able to assign an operator and a function the conjunction of which will enumerate the diagrams?

Two methods of research are clearly open to us. We may be given the diagrams formed according to certain laws and then seek to design the operator and the function *or* we may commence by designing operators and functions with the object of discovering the laws of formation of the systems of diagrams to which they naturally lead. Both methods are useful and will be freely employed in what follows.

We shall frequently write $\frac{d}{dx}$ in the notation ∂_x.

Let us adopt the second method and take

$$\text{as operator } (\partial_{x_1}\partial_{x_2}\partial_{x_3})^3, \quad \text{as function } (x_1x_2x_3)^3.$$

Leibnitz' theorem in the differential calculus gives us

$$\begin{aligned}(\partial_{x_1}\partial_{x_2}\partial_{x_3})(x_1x_2x_3)^3 = {} & (\partial_{x_1}\partial_{x_2}\partial_{x_3})(x_1x_2x_3).(x_1x_2x_3).(x_1x_2x_3) + 2 \text{ similar terms} \\ & + (\partial_{x_1}\partial_{x_2})(x_1x_2x_3).\partial_{x_3}(x_1x_2x_3).(x_1x_2x_3) + 5 \quad ,, \quad ,, \\ & + (\partial_{x_2}\partial_{x_3})(x_1x_2x_3).\partial_{x_1}(x_1x_2x_3).(x_1x_2x_3) + 5 \quad ,, \quad ,, \\ & + (\partial_{x_3}\partial_{x_1})(x_1x_2x_3).\partial_{x_2}(x_1x_2x_3).(x_1x_2x_3) + 5 \quad ,, \quad ,, \\ & + \partial_{x_1}(x_1x_2x_3).\partial_{x_2}(x_1x_2x_3).\partial_{x_3}(x_1x_2x_3) + 5 \quad ,, \quad ,,\end{aligned}$$

On the right there are 3^3 terms corresponding to the 27 permuted partitions of $x_1x_2x_3$ into *exactly* 3 parts, zero being reckoned as a part. Selecting any one of the 27 terms, say

$$\partial_{x_1}\partial_{x_2}(x_1x_2x_3).\partial_{x_3}(x_1x_2x_3).(x_1x_2x_3),$$

we have before us one of 27 minor operations into which the operation of $(\partial_{x_1}\partial_{x_2}\partial_{x_3})$ upon $(x_1x_2x_3)^3$ can be broken up or dissected. We denote this diagrammatically by

12	3	

The selected term is equal to

$$(\ \bullet\ \bullet\ x_3)\,(x_1x_2\ \bullet\ )\,(x_1x_2x_3),$$

and the diagram either denotes that x_1x_2 has been erased from the first factor and x_3 from the second *or* that $\partial_{x_1}\partial_{x_2}$ has been performed upon the first factor and ∂_{x_3} upon the second.

Keeping to the selected term we now again operate with $\partial_{x_1}\partial_{x_2}\partial_{x_3}$. Of the whole number of 27 minor operations which we again have before us only a certain number will be effective in producing a term which is not zero; for clearly the first operator factor must not contain ∂_{x_1} or ∂_{x_2} and the second must not contain ∂_{x_3}. However some of the minor operations survive and one of them is

$$(\ \bullet\ \bullet\ x_3).(\partial_{x_1}\partial_{x_2})(x_1x_2\ \bullet\ ).\partial_{x_3}(x_1x_2x_3),$$

resulting in

$$(\ \bullet\ \bullet\ x_3)(\ \bullet\ \bullet\ \bullet\ )(x_1x_2\ \bullet\ ).$$

The two successive operations we have now performed are diagrammatically

12	3	
	12	3

Again, operating with $\partial_{x_1}\partial_{x_2}\partial_{x_3}$, only one of the 27 minor operations is effective; this is

$$\partial_{x_3}(\ \bullet\ \bullet\ x_3)(\ \bullet\ \bullet\ \bullet\ )(\partial_{x_1}\partial_{x_2})(x_1x_2\ \bullet\ ),$$

resulting in unity.

The three successive minor operations have now produced the diagram

12	3	
	12	3
3		12

and it is clear that we can select three successive minor operations in $(\partial_{x_1}\partial_{x_2}\partial_{x_3})^3\,(x_1x_2x_3)^3$ ways since for each such selection unity is the result, and the result for all combinations of three selections must be $(\partial_{x_1}\partial_{x_2}\partial_{x_3})^3\,(x_1x_2x_3)^3$. We obtain therefore in this manner

$$(\partial_{x_1}\partial_{x_2}\partial_{x_3})^3\,(x_1x_2x_3)^3$$

diagrams, and to obtain a theorem we have to

(1) Define the diagrams,

(2) Evaluate the expression.

The diagrams are easily defined. It is clear that in each row the numbers 1, 2, 3 must appear and that they must also appear in each column, but there is *no restriction* as to how many numbers may appear in each cell of the lattice.

The enumerating expression has the value $(3\,!)^3$.

Generally for the order n the enumeration of the diagrams is given by

$$(\partial_{x_1}\partial_{x_2}\dots\partial_{x_n})^n\,(x_1x_2\dots x_n)^n = (n\,!)^n;$$

the numbers 1, 2, 3, ... n present themselves in each row and in each column, and none, some or all *may* appear in any cell of the lattice. For $n = 2$, the four arrangements are

1	2
2	1

2	1
1	2

12	
	12

	12
12	

192. Next consider

as operator $(\partial_{x_1}\partial_{x_2}\dots\partial_{x_n})^m$, as function $(x_1x_2\dots x_n)^n$,

where $m \neq n$.

We are clearly led to lattices of m rows and m columns; the number of rows is m because we consider the operator in m successive operations; the number of columns is m because we consider the function or operand as a product of m factors. The diagrams are such that in the square of m^2 compartments the numbers 1, 2, 3, ... n each present themselves once in each row and in each column while the number of them that *may* appear in each cell is unrestricted. The enumerating integer is

$$(\partial_{x_1}\partial_{x_2}\dots\partial_{x_n})^m (x_1x_2\dots x_n)^m \quad \text{or} \quad (m\,!)^n.$$

193. It is of great importance to notice that the nature of the diagrams produced depends not only upon the operator and operand but also upon the *modus operandi*. Consider for a moment the operator $(\partial_{x_1}\partial_{x_2}\partial_{x_3})^2$ and the operand $(x_1x_2x_3)^2$. In the example above given the operator and operand when factorized are shewn by the method in which they were written; but we might also have written them in the forms

$$(\partial_{x_1}\partial_{x_2})(\partial_{x_1}\partial_{x_3})(\partial_{x_2}\partial_{x_3}) \quad \text{and} \quad (x_2x_3)(x_1x_3)(x_1x_2),$$

the operator factors being performed in the order $(\partial_{x_2}\partial_{x_3})$, $(\partial_{x_1}\partial_{x_3})$, $(\partial_{x_1}\partial_{x_2})$.

Here it is seen that the performance of $(\partial_{x_2}\partial_{x_3})$ does not give the number 1 in the first row of a diagram of 3 rows and 3 columns; for we associate the number 1 with the deletion of x_1 from a factor of the operand; further the subsequent operation of $(\partial_{x_1}\partial_{x_3})$ cannot lead to the number 2 in the second row, and lastly the final operation of $(\partial_{x_1}\partial_{x_2})$ cannot lead to a number 3 in the third or last row. Moreover from the way in which the operand has been factorized the numbers 1, 2, 3 cannot find places in the first, second and third columns of a diagram respectively. Hence the diagrams possess the properties:

(i) the first row and first column each contain the two numbers 2, 3,
(ii) the second „ second „ „ „ „ 3, 1,
(iii) the third „ third „ „ „ „ 1, 2.

The system of eight diagrams to which we are led is:

	3	2
3	1	
2		1

	3	2
3		1
2	1	

3		2
	3	1
2	1	

2	3	
3		1
	1	2

32		
	3	1
	1	2

3		2
	31	
2		1

2	3	
3	1	
		21

32		
	31	
		21

equi-numerous with those derived from the *modus operandi*

$$(\partial_{x_1}\partial_{x_2}\partial_{x_3})^2\,(x_1x_2x_3)^2,$$

which possess the property that in a diagram of two rows and two columns the numbers 1, 2, 3 appear once in each row and once in each column without any restriction. These eight diagrams are:

21	3
3	21

3	21
21	3

31	2
2	31

2	31
31	2

32	1
1	32

1	32
32	1

321	
	321

	321
321	

The fact that the two sets comprise the same number of diagrams is at once evident from the method employed; it might be otherwise troublesome to establish.

194. Again keeping to the same operator and operand we may take the *modus operandi* to be

$$(\partial_{x_1}\partial_{x_2})\,(\partial_{x_3}\partial_{x_1})\,(\partial_{x_2}\partial_{x_3})\,(x_1x_2x_3)^2,$$

and be led to the same number of diagrams, viz. 8. These diagrams possess the properties:

(i) they have 3 rows and 2 columns,

(ii) each column exhibits the numbers 1, 2, 3,

(iii) the sth row does not exhibit the number s where s has the values 1, 2, 3.

The eight diagrams are:

2	3
3	1
1	2

3	2
1	3
2	1

2	3
31	
	21

3	2
	31
21	

32	
1	3
	21

32	
	31
1	2

	32
3	1
21	

	32
31	
2	1

If the *modus operandi* had been

$$(\partial_{x_1}\partial_{x_2}\partial_{x_3})^2\,(x_2x_3)\,(x_1x_3)\,(x_1x_2)$$

the diagrams would have been those last written rotated counter-clockwise through a right angle.

This does not exhaust the sets of diagrams obtainable from this operator and operand which are enumerated by the number 8.

It will be evident that a very large number of theorems concerning distributions in the cells of rectangular lattices are obtainable from a consideration of the operator and operand

$$(\partial_{x_1}\partial_{x_2}\dots\partial_{x_n})^m \quad \text{and} \quad (x_1x_2\dots x_n)^m$$

and the innumerable *modi operandi*. These theorems all refer to equinumerous sets of diagrams.

195. We next introduce the differential operators d_s, D_s which we studied in Section II. Suitable operands will clearly be the associated symmetric functions expressed in the notation of partitions. If a_1, a_2, a_3, ... be the elementary symmetric functions of the quantities α_1, α_2, α_3, ... α_n,

$$d_1 = \partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \dots,$$

$$d_s = \partial_{a_s} + a_1\partial_{a_{s+1}} + a_2\partial_{a_{s+2}} + \dots,$$

$$D_s = \frac{1}{s!}(d_1^s),$$

where (d_1^s) is an operator of order s obtained by symbolical multiplication.

Taking as operator D_1^n and as operand $(1)^n$ and remembering the law of operation of D_s upon a product of symmetric functions expressed in the partition notation we see that the effect of D_1 upon $(1)^n$ or upon

$$(1)(1)(1)(1)\dots$$

is to obliterate one of the factors in all possible ways and by summation to obtain $n(1)^{n-1}$. This shews that we proceed to the same system of diagrams as in the case of the operator ∂_x^n taken with the operand x^n.

This is an example of what one frequently finds in this subject; that many pairs of operator and operand may be designed which lead to the same set of diagrams. As a set-off against this however we have already met with the remarkable circumstance that one pair, operator and operand, may lead to a large number of different sets of diagrams.

196. We will now suppose that we have to enumerate the permutations of the quantities in $\alpha_1^{\pi_1}\alpha_2^{\pi_2}\dots\alpha_n^{\pi_n}$. We are at once led to design

$$\text{as operator } D_{\pi_1}D_{\pi_2}\dots D_{\pi_n} \text{ and as operand } (1)^n,$$

where $\Sigma\pi = n$.

Since
$$(1)^n = \ldots + \frac{n!}{\pi_1!\,\pi_2!\ldots\pi_n!}\Sigma\alpha_1^{\pi_1}\alpha_2^{\pi_2}\ldots\alpha_n^{\pi_n} + \ldots$$
$$= \ldots + \frac{n!}{\pi_1!\,\pi_2!\ldots\pi_n!}(\pi_1\pi_2\ldots\pi_n) + \ldots$$

and
$$D_{\pi_1}D_{\pi_2}\ldots D_{\pi_n}(\pi_1\pi_2\ldots\pi_n) = 1,$$

it is clear that
$$D_{\pi_1}D_{\pi_2}\ldots D_{\pi_n}(1)^n = \frac{n!}{\pi_1!\,\pi_2!\ldots\pi_n!},$$

giving the number of diagrams to which we must be led.

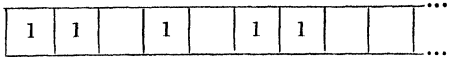
Suppose for convenience that the operators D are performed in the order $D_{\pi_1}, D_{\pi_2}, \ldots D_{\pi_n}$. The operation of D_{π_1} upon $(1)^n$ is performed by striking out π_1 factors in all possible ways from the product (of n factors)

$$(1)(1)(1)\ldots$$

and then adding. We thus get

$$\binom{n}{\pi_1}(1)^{n-\pi_1},$$

and in obtaining this we have broken up the operation of D_{π_1} into $\binom{n}{\pi_1}$ minor operations. These may be represented by the first rows of diagrams and we thus obtain $\binom{n}{\pi_1}$ different first rows, one of which may be



(number of units $= \pi_1$).

The term considered is now

$$(\bullet)(\bullet)(1)(\bullet)(1)(\bullet)(\bullet)(1)(1)\ldots,$$

as it appears when the π_1 factors have been obliterated in the particular manner under view.

The operation of D_{π_2} is performed upon each of the $\binom{n}{\pi_1}$ terms, each of which has $n-\pi_1$ factors, by striking out π_2 factors in all the ways possible. This can be done in each case in $\binom{n-\pi_1}{\pi_2}$ ways and therefore in each case can be broken up into $\binom{n-\pi_1}{\pi_2}$ minor operations. Altogether $D_{\pi_1}D_{\pi_2}$ has been broken up into $\binom{n}{\pi_1}\binom{n-\pi_1}{\pi_2}$ or $\frac{n!}{\pi_1!\,\pi_2!(n-\pi_1-\pi_2)!}$ minor operations. Suppose that one of the $\binom{n-\pi_1}{\pi_2}$ minor operations associated with D_{π_2} is performed upon the particular term

$$(\bullet)(\bullet)(1)(\bullet)(1)(\bullet)(\bullet)(1)(1)\ldots,$$

the result may be the striking out of the factors in the third, eighth, ninth and other places (π_2 in number). This would yield a diagram of two rows

1	1		1		1	1			...	(number of units $= \pi_1$),
		1	.				1	1	...	(number of units $= \pi_2$),

which is one of the $\dfrac{n!}{\pi_1!\,\pi_2!\,(n-\pi_1-\pi_2)!}$ diagrams of two rows which correspond to the like number of minor operations into which the double operation $D_{\pi_1} D_{\pi_2}$ has been broken up.

Proceeding in this manner we finally arrive at a diagram of n rows and n columns such that there is one and only one unit in each column while the numbers of units in the 1st, 2nd, ... nth rows are $\pi_1, \pi_2, \pi_3, \ldots \pi_n$ respectively. We thus obtain

$$\frac{n!}{\pi_1!\,\pi_2! \ldots \pi_n!}$$

diagrams of this nature corresponding to the like number of permutations of the quantities in $\alpha_1^{\pi_1} \alpha_2^{\pi_2} \ldots \alpha_n^{\pi_n}$.

197. Let us now consider the problem of placing units in the compartments of a lattice of m rows and l columns, not more than one unit in each, in such wise that we can count $\mu_1, \mu_2, \ldots \mu_m$ units in the successive rows and $\lambda_1, \lambda_2, \ldots \lambda_l$ units in the successive columns.

Clearly $\Sigma\mu = \Sigma l$. We take

$$\text{as operator } D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m}, \quad \text{as operand } (1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l}).$$

It will be gathered from Section II that if

$$(1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l}) = \ldots + A\,(\mu_1 \mu_2 \ldots \mu_m) + \ldots,$$

then

$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m} (1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l}) = A,$$

and we shall shew that A enumerates the lattices under consideration.

Since the operand involves only units

$$D_{\mu_s} \equiv D_{(1^{\mu_s})},$$

that is to say when D_{μ_s} operates upon the operand it does so by selecting μ_s units in all possible ways from the operand, taking care to select *one unit only* from each factor of the operand. We may assume $\mu_1, \mu_2, \ldots$ to be in descending order of magnitude and then μ_1 must be equal to or less than l; otherwise there will be no diagrams, as A will be zero.

Commencing with D_{μ_1} we pick out the μ_1 units in each of the $\binom{l}{\mu_1}$ possible ways and select one of them for the first row of a diagram; we

follow with D_{μ_2} and operate with it on the selected term, picking out another μ_2 units, and so on until finally we have completed one diagram which naturally possesses the desired properties. Each of the A diagrams can be similarly formed and thus the diagrams in question are enumerated by

$$D_{\mu_1} D_{\mu_2} \dots D_{\mu_m} (1^{\lambda_1}) (1^{\lambda_2}) \dots (1^{\lambda_l}).$$

Ex. gr. Take

$$\lambda_1 = 3, \ \lambda_2 = 2, \ \lambda_3 = 1, \ \mu_1 = 2, \ \mu_2 = 2, \ \mu_3 = 1, \ \mu_4 = 1,$$

finding $$(1^3)(1^2)(1) = a_3 a_2 a_1 = \dots + 8\,(2211) + \dots$$

because $$D_2^2 D_1^2 a_3 a_2 a_1 = 8.$$

The eight diagrams are

1	1	
1	1	
1		
		1

1	1	
1	1	
		1
1		

	1	1
1	1	
1		
1		

1	1	
	1	1
1		
1		

1		1
1	1	
1		
	1	

1		1
1	1	
	1	
1		

1	1	
1		1
1		
	1	

1	1	
1		1
	1	
1		

There are 2, 2, 1, 1 units in the successive rows and 3, 2, 1 units in the successive columns. No others possess the desired property.

198. We can now apply the method so as to be an instrument of reciprocation in algebra. If we rotate the diagrams through a right angle so that they read by rows as they formerly did by columns, the effect is to interchange the set of numbers $\lambda_1, \lambda_2, \dots \lambda_l$ with the set $\mu_1, \mu_2, \dots \mu_m$ and the number of diagrams is not altered. Hence the reciprocal theorem:

"If $$(1^{\lambda_1})(1^{\lambda_2}) \dots (1^{\lambda_l}) = \dots + A\,(\mu_1 \mu_2 \dots \mu_m) + \dots,$$

then $$(1^{\mu_1})(1^{\mu_2}) \dots (1^{\mu_m}) = \dots + A\,(\lambda_1 \lambda_2 \dots \lambda_l) + \dots,$$"

a well-known theorem of symmetry which has been established in a previous section; but notice that when we first met with it the number A was associated with the number of distributions of objects of type $(\lambda_1 \lambda_2 \dots \lambda_l)$ into parcels of type $(\mu_1 \mu_2 \dots \mu_m)$, no two objects of a kind being in any one parcel. Here it is associated with the distribution of units in the cells of a given lattice under prescribed conditions. The easy intuitive nature of this proof of the theorem is quite remarkable.

199. In the above the magnitude of the numbers, appearing in the cells of the lattice, has been restricted so as not to exceed unity.

This restriction is removed in the following manner:

Consider the homogeneous product sums of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$, viz. $h_1, h_2, h_3, \ldots$ and recall the result of Section III, Art. 61,

$$D_s h_\lambda = h_{\lambda - s},$$

and also
$$D_s h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l} = \Sigma h_{\lambda_1 - \sigma_1} h_{\lambda_2 - \sigma_2} \ldots h_{\lambda_l - \sigma_l},$$

where $(\sigma_1 \sigma_2 \ldots \sigma_l)$ is a partition of s and the sum is taken for all such partitions and for a particular partition for all ways of operating upon the suffixes with the parts of the partition.

We take

$$\text{as operator } D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m}, \quad \text{as operand } h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l},$$

and commencing by operating with D_{μ_1} we operate with every partition of μ_1 upon the suffixes of the operand as above explained. We thus obtain the first row of a diagram

σ_1	σ_2	σ_3	σ_4	...	σ_l

$(\Sigma\sigma = \mu_1)$.

The particular term associated with this first row is then the subject of operation by D_{μ_2} and, selecting one of the resulting terms, we derive a second row of the diagram

σ_1	σ_2	σ_3	σ_4	...	σ_l
τ_1	τ_2	τ_3	τ_4	...	τ_l

$(\Sigma\sigma = \mu_1)$,

$(\Sigma\tau = \mu_2)$.

Proceeding similarly till we have completed a lattice of m rows and l columns we reach one of the diagrams possessing the property that the sums of the numbers in the successive rows are $\mu_1, \mu_2, \ldots \mu_m$ and in the successive columns $\lambda_1, \lambda_2, \ldots \lambda_l$, no restriction being placed upon the magnitude of the numbers.

Now if
$$h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l} = \ldots + B(\mu_1 \mu_2 \ldots \mu_m) + \ldots,$$

we have
$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m} (h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l}) = B.$$

B enumerates the diagrams in question.

We can of course rotate the diagram and establish that

$$h_{\mu_1} h_{\mu_2} \ldots h_{\mu_m} = \ldots + B(\lambda_1 \lambda_2 \ldots \lambda_l) + \ldots,$$

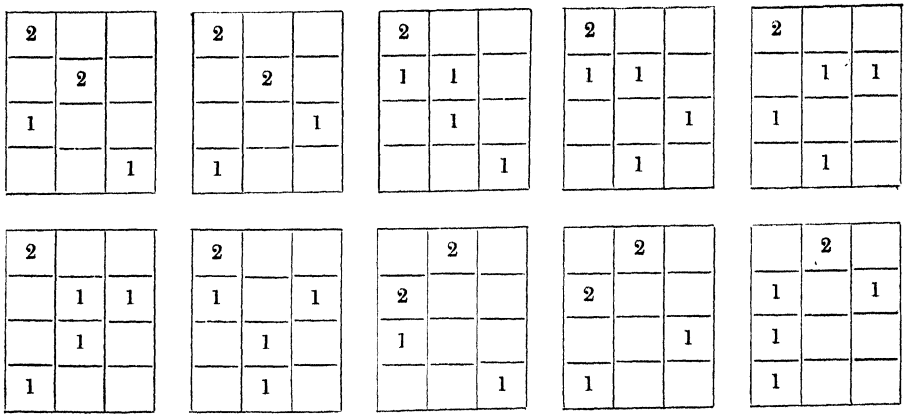
another law of symmetry being evolved. This has been previously established in another manner in Section II.

Ex. gr. We have $D_2^2 D_1^2 h_3 h_2 h_1 = 18,$

$$h_3 h_2 h_1 = \ldots + 18\,(2^2 1^2) + \ldots,$$

$$h_2^2 h_1^2 = \ldots + 18\,(321) + \ldots,$$

we have therefore 18 diagrams; eight of these in which the compartment numbers do not exceed unity have been depicted above; the remaining ten are



200. We may consider the same problem with a restriction k upon the magnitude of the compartment numbers; k is to be an upper limit.

Let k_s denote the homogeneous product sum of order s in which none of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ is raised to a higher power than k.

Ex. gr. if $k=2$; $k_3 = (21) + (1^3)$ and not $(3) + (21) + (1^3)$.

We have

$D_\lambda k_s = k_{s-\lambda}$ where λ is equal to or less than both k and s,

$D_\lambda k_s = 0$ if $\lambda > k$ or $> s$.

Take

$$\text{as operator } D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m}, \quad \text{as operand } k_{\lambda_1} k_{\lambda_2} \ldots k_{\lambda_l}.$$

We shall obtain a number of diagrams of m rows and l columns which possess the property that the sums of the numbers in the successive rows are $\mu_1, \mu_2, \ldots \mu_m$ and in the successive columns $\lambda_1, \lambda_2, \ldots \lambda_l$, the magnitude of the compartment numbers not exceeding k.

The number of such diagrams is

$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m} k_{\lambda_1} k_{\lambda_2} \ldots k_{\lambda_l} = C,$$

where $$k_{\lambda_1} k_{\lambda_2} \ldots k_{\lambda_l} = \ldots + C\,(\mu_1 \mu_2 \ldots \mu_m) + \ldots.$$

By rotating the lattices we thence find that

$$k_{\mu_1} k_{\mu_2} \ldots k_{\mu_m} = \ldots + C\,(\lambda_1 \lambda_2 \ldots \lambda_l) + \ldots,$$

establishing another law of symmetry in symmetrical algebra.

201. It is clear that any specification of restrictions in regard to the magnitude of the compartment numbers must similarly lead to a law of algebraic symmetry, provided that the operator is suitably adapted to the operand. Thus suppose that unity is not to be present. We may take b_s equal to the sum of the monomial symmetric functions whose partitions are free from unity and then of course we must omit D_1 from the operator, and taking

$$\text{as operator } D_{\mu_1} D_{\mu_2} \dots D_{\mu_m}, \quad \text{as operand } b_{\lambda_1} b_{\lambda_2} \dots b_{\lambda_l},$$

we readily find the law of symmetry shewn by the relations

$$b_{\mu_1} b_{\mu_2} \dots b_{\mu_m} = \dots + E(\lambda_1 \lambda_2 \dots \lambda_l) + \dots,$$
$$b_{\lambda_1} b_{\lambda_2} \dots b_{\lambda_l} = \dots + E(\mu_1 \mu_2 \dots \mu_m) + \dots,$$

wherein unity does not appear in the partitions.

202. It may be gathered from what has been said that every case of symmetric function multiplication is connected with a theory of distribution in the cells of a lattice.

In some cases we are led to but a single diagram. For example take

$$\text{as operator } D_3 D_2^2 D_1, \quad \text{as operand } (1^4)(1^3)(1),$$

(32^21) and (431) being conjugate partitions.

Operating with D_3 by picking out unity from each factor of the operand we obtain $(1^3)(1^2)$ with a first row of a diagram, viz.:

1	1	1

The operation of D_2 now gives $(1^2)(1)$ and a second row

1	1	

and proceeding we finally obtain the single diagram

1	1	1
1	1	
1	1	
1		

which is none other than the graph of the partition (32^21) or of (431) according as it is read by rows or by columns. We might also have operated with $D_4 D_3 D_1$ upon $(1^3)(1^2)^2(1)$; and in general if $(\pi_1 \pi_2 \pi_3 \dots)$, $(\rho_1 \rho_2 \rho_3 \dots)$ be conjugate partitions we obtain their graphs either by operating with $D_{\pi_1} D_{\pi_2} D_{\pi_3} \dots$ upon $(1^{\rho_1})(1^{\rho_2})(1^{\rho_3}) \dots$ or with $D_{\rho_1} D_{\rho_2} D_{\rho_3} \dots$ upon $(1^{\pi_1})(1^{\pi_2})(1^{\pi_3}) \dots$.

203. We will now consider a general question. Take

$$\text{as operator } D_p D_q D_r \ldots,$$

$$\text{as operand } (\lambda_1 \mu_1 \nu_1 \ldots)(\lambda_2 \mu_2 \nu_2 \ldots)(\lambda_3 \mu_3 \nu_3 \ldots)\ldots(\lambda_s \mu_s \nu_s \ldots).$$

where $$p + q + r + \ldots = \Sigma\lambda + \Sigma\mu + \Sigma\nu + \ldots.$$

We have

$$D_p D_q D_r \ldots (\lambda_1 \mu_1 \nu_1 \ldots)(\lambda_2 \mu_2 \nu_2 \ldots)\ldots(\lambda_s \mu_s \nu_s \ldots) = A,$$

where

$$(\lambda_1 \mu_1 \nu_1 \ldots)(\lambda_2 \mu_2 \nu_2 \ldots)\ldots(\lambda_s \mu_s \nu_s \ldots) = \ldots + A\,(pqr\ldots) + \ldots.$$

If the partition $(pqr\ldots)$ contains t parts we are led to lattices of s columns and t rows. The operator D_p acts through its various partitions upon the product of monomial functions and any mode of picking out a partition of p from the factors, one part from each factor, constitutes a minor operation, of the major operation D_p, which yields the row of a diagram. D_q is similarly responsible for all the second rows of the diagrams and so forth; finally every diagram which results possesses properties which are defined in the following manner:

(i) the numbers in the diagram are the whole assemblage

$$\lambda_1, \mu_1, \nu_1, \ldots \lambda_2, \mu_2, \nu_2, \ldots \lambda_s, \mu_s, \nu_s, \ldots;$$

(ii) the numbers in the successive rows are compositions of the numbers $p, q, r, \ldots$ respectively;

(iii) the numbers in the successive columns are

$$\lambda_1, \mu_1, \nu_1, \ldots;\ \lambda_2, \mu_2, \nu_2, \ldots;\ \ldots \lambda_s, \mu_s, \nu_s, \ldots;$$

respectively.

Ex. gr. Take

$$\text{as operator } D_{13} D_{15} D_{17}, \quad \text{as operand } (987)(654)(321),$$

where $$(987)(654)(321) = \ldots + 6\,(17 \cdot 15 \cdot 13) + \ldots.$$

The six diagrams are

7	5	1
8	4	3
9	6	2

7	5	1
9	4	2
8	6	3

7	4	2
8	6	1
9	5	3

7	4	2
9	5	1
8	6	3

8	4	1
7	6	2
9	5	3

8	4	1
7	5	3
9	6	2

In the first, second and third rows there are partitions of 13, 15 and 17 respectively, while in the successive columns are numbers 7, 8, 9; 4, 5, 6;

1, 2, 3 respectively. The number 6 may be determined algebraically by expressing both operator and operand in terms of the elementary functions and actually performing the differentiations.

The reader must now notice that a theorem of symmetry does not result from the mere rotation of the lattices, because such rotation does not exhibit the same collections of numbers in the respective columns. For example after rotation in the first column we find three collections, viz. 7, 5, 1; 7, 4, 2; 8, 4, 1.

There is however a beautiful symmetrical theorem which a close examination of a particular case will lead us to.

Write

$$\lambda_1 + \mu_1 + \nu_1 + \ldots = \lambda,$$
$$\lambda_2 + \mu_2 + \nu_2 + \ldots = \mu,$$
$$\lambda_3 + \mu_3 + \nu_3 + \ldots = \nu,$$
$$\ldots\ldots\ldots\ldots$$

and consider diagrams satisfying the conditions:

(i) the numbers in the compartments are the assemblage

$$\lambda_1, \mu_1, \nu_1, \ldots;\ \lambda_2, \mu_2, \nu_2, \ldots;\ \ldots \lambda_s, \mu_s, \nu_s, \ldots;$$

(ii) the numbers $\lambda, \mu, \nu, \ldots$ appertaining to the columns and the numbers $p, q, r, \ldots$ appertaining to the rows are unaltered.

These conditions do not define the lattices under examination above because other lattices comply with them, viz. those in which the whole assemblage of compartment numbers remaining unchanged the column numbers while satisfying the condition (ii) are other than

$$\lambda_1, \mu_1, \nu_1, \ldots;\ \lambda_2, \mu_2, \nu_2, \ldots;\ \ldots \lambda_s, \mu_s, \nu_s, \ldots.$$

We may have other collections of numbers drawn from the given assemblage such that

$$\lambda_1' + \mu_1' + \nu_1' + \ldots = \lambda,$$
$$\lambda_2' + \mu_2' + \nu_2' + \ldots = \mu,$$
$$\lambda_3' + \mu_3' + \nu_3' + \ldots = \nu,$$
$$\ldots\ldots\ldots\ldots$$

the assemblage of dashed letters being in some order identical with the assemblage of undashed letters.

The new conditions include lattices enumerated by

$$D_p D_q D_r \ldots (\lambda_1'\mu_1'\nu_1'\ldots)(\lambda_2'\mu_2'\nu_2'\ldots)\ldots(\lambda_s'\mu_s'\nu_s'\ldots),$$

and the totality of lattices implied by them is enumerated by

$$D_p D_q D_r \ldots \Sigma\,(\lambda_1'\mu_1'\nu_1'\ldots)(\lambda_2'\mu_2'\nu_2'\ldots)\ldots(\lambda_s'\mu_s'\nu_s'\ldots),$$

the summation being for every separation of the assemblage of numbers

$$\lambda_1\mu_1\nu_1\ldots\lambda_2\mu_2\nu_2\ldots\ \ldots\lambda_s\mu_s\nu_s\ldots$$

into partitions $(\lambda_1'\mu_1'\nu_1'\ldots), (\lambda_2'\mu_2'\nu_2'\ldots), \ldots (\lambda_s'\mu_s'\nu_s'\ldots),$

such that

$$\lambda_1' + \mu_1' + \nu_1' + \ldots = \lambda,$$
$$\lambda_2' + \mu_2' + \nu_2' + \ldots = \mu,$$
$$\lambda_3' + \mu_3' + \nu_3' + \ldots = \nu,$$
$$\ldots\ldots\ldots\ldots$$

or, as it is convenient to say and moreover consistent with the previous nomenclature, for every separation of the given assemblage of numbers which has the specification $(\lambda\mu\nu\ldots)$.

We may now assert that the successive row numbers have a specification $(pqr\ldots)$ and the successive column numbers a specification $(\lambda\mu\nu\ldots)$ and we may now define lattices as follows:

(i) The numbers occurring in the compartments of a lattice of s columns and t rows are those of the assemblage

$$\lambda_1, \mu_1, \nu_1, \ldots \lambda_2, \mu_2, \nu_2, \ldots\ \ldots\lambda_s, \mu_s, \nu_s, \ldots.$$

(ii) The successive row and successive column numbers have the sums $p, q, r, \ldots$; $\lambda, \mu, \nu, \ldots$ respectively.

We will now consider a particular case in order to find the condition for a symmetrical theorem due to rotation of lattices.

Let the assemblage of numbers be 2, 2, 1, 1 and consider the two results

$$(2)^2(1)^2 = \ldots + 6(2^3) + \ldots; \quad (2)^2(1^2) = \ldots + 2(2^2 1^2) + \ldots$$

derived from $D_2^3(2)^2(1)^2 = 6$, $D_2^2 D_1^2 (2)^2(1^2) = 2$ respectively.

In the first case the row and column specifications are (222), (2211), and in the second case (2211), (222) respectively.

The first yields the six diagrams

2			
	2		
		1	1

	2		
2			
		1	1

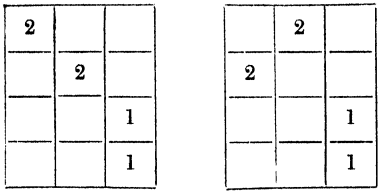
2			
		1	1
	2		

	2		
		1	1
2			

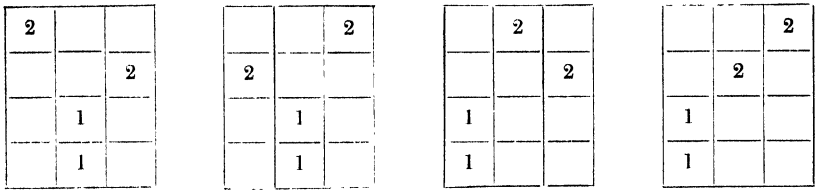
		1	1
	2		
2			

		1	1
2			
	2		

and the second the two diagrams



If we rotate the six lattices we obtain four in addition to these two, viz.



The first pair of these would be derived from $D_2^2 D_1^2 (2)(1^2)(2)$ and the second pair from $D_2^2 D_1^2 (1)^2 (2)(2)$. Hence it appears that to obtain identity of enumeration we must multiply $(2)^2(1^2)$ by a number equal to the number of ways of permuting its factors; in the present case the whole of the permutations are involved, but reflexion shews that we have only to consider the permutations amongst factors which have the same specification (compare also Section II, Art. 46). Thus for a symmetrical theorem we must use $(2)^2(1)^2$ and $(2)^2(1^2)$ with the multipliers 1 and 3 respectively.

Let then an operand be

$$(L_1)^{l_1}(L_2)^{l_2}\ldots(M_1)^{m_1}(M_2)^{m_2}\ldots\ \ldots,$$

$(L_1), (L_2), \ldots$ on the one hand and $(M_1), (M_2), \ldots$ on the other denoting partitions of the same weight. We attach a coefficient

$$\frac{(l_1+l_2+\ldots)!}{l_1!\,l_2!\ldots}\cdot\frac{(m_1+m_2+\ldots)!}{m_1!\,m_2!\ldots}\ldots.$$

Let any operand so multiplied be denoted by the operand with the prefix Co. The law of symmetry is then:

"From a given finite assemblage of numbers construct all the products

$$(\lambda_1\mu_1\nu_1\ldots)(\lambda_2\mu_2\nu_2\ldots)(\lambda_3\mu_3\nu_3\ldots)\ldots$$

which have a given specification $(\lambda\mu\nu\ldots)$, and further all the products

$$(p_1q_1r_1\ldots)(p_2q_2r_2\ldots)(p_3q_3r_3\ldots)\ldots$$

which have a given specification $(pqr\ldots)$; then if

$$\Sigma\,\mathrm{Co}\,(\lambda_1\mu_1\nu_1\ldots)(\lambda_2\mu_2\nu_2\ldots)(\lambda_3\mu_3\nu_3\ldots)\ldots = \ldots + A\,(pqr\ldots) + \ldots$$

we have also

$$\Sigma \operatorname{Co}(p_1q_1r_1\ldots)(p_2q_2r_2\ldots)(p_3q_3r_3\ldots)\ldots = \ldots + A(\lambda\mu\nu\ldots) + \ldots,$$

and the number A is represented by a set of A diagrams constructed according to definite laws."

204. So far the operations have been those of the infinitesimal calculus, and the numbers involved in the partitions of the functions have been positive integers, excluding zero. If we admit zero as a part in the partitions we have to do with the operations of the calculus of finite differences.

At the commencement of this section it was shewn that ∂_x is a selective symbol of operation, for when the operand is a power of x, the said power being positive and integral, it has the effect of summing the results obtained by substituting unity for x in all possible ways in the product of x's. The corresponding operator of the calculus of finite differences, viz. Δ, has the effect of substituting unity for one x, two x's, three x's, etc. in all possible ways and then summing the results. Thus

$$\Delta x^3 = 1.x.x + x.1.x + x.x.1 + 1.1.x + 1.x.1 + x.1.1 + 1.1.1$$
$$= 3x^2 + 3x + 1.$$

This simple fact indicates that we may expect a corresponding theory of diagrams whose compartments are lettered or numbered according to definite laws. Moreover directly we introduce zero as a part into the partitions of our symmetric functions we find that our differential operators become effectively equivalent to those of the calculus of finite differences; for remark the result derived from Section II, Art. 53,

$$D_0(0)^3 = (\bullet)(0)(0) + (0)(\bullet)(0) + (0)(0)(\bullet) + (\bullet)(\bullet)(0) + (\bullet)(0)(\bullet)$$
$$+ (0)(\bullet)(\bullet) + (\bullet)(\bullet)(\bullet)$$
$$= 3(0)^2 + 3(0) + 1,$$

and compare it with

$$\Delta x^3 = 3x^2 + 3x + 1.$$

In fact we recall that if the symmetric functions are formed from n quantities,

$$d_0 = \partial_n, \quad D_0 = e^{\partial_n} - 1,$$

so that the operations $\quad D_0, \quad d_0, \quad 1 + D_0$

of Section II correspond to the operations

$$\Delta, \quad \partial_n, \quad E$$

of the calculus of finite differences.

If we consider partitions which *only* involve zero parts, we have only to deal with finite difference operations; if we have other integers, we have to do with mixed operations drawn both from the finite and from the infinitesimal

calculus. The important point to notice is that the importation into symmetric function theory of the zero part and of the corresponding zero operator relieves us from the necessity of making any *direct* reference to the calculus of finite differences or of making use of its special operative symbols.

The reader should make a few comparisons like the following:

$$n^{(m)} = n(n-1)(n-2)\dots(n-m+1) = m!\,(0^m)_n,$$

$$n^{(m-1)} = (m-1)!\,(0^{m-1})_n,$$

$$D_0 n^{(m)} = D_0 m!\,(0^m)_n = m!\,(0^{m-1})_n = m n^{(m-1)},$$

to be compared with the well-known formula

$$\Delta n^{(m)} = m n^{(m-1)},$$

and also satisfy himself of the equivalence of the relations

$$d_0 = \log(1 + D_0),$$

$$\partial_n = \log(1 + \Delta).$$

205. Consider the lattice theory reached by taking

$$\text{as operator } D_0^s, \quad \text{as operand } (0^\lambda)(0^\mu)(0^\nu)\dots.$$

The operand when multiplied out will appear in the form of a linear function of (0^λ), $(0^{\lambda+1})$, $\dots$ $(0^{\lambda+\mu+\nu+\dots})$, the coefficients being all positive integers (cf. Section II, Art. 53); the numbers $\lambda, \mu, \nu, \dots$ are supposed to be in descending order of magnitude and s clearly is limited to be one of the numbers

$$\lambda,\ \lambda+1,\ \dots\ \lambda+\mu+\nu+\dots.$$

To find therein the coefficient of (0^s) we operate with D_0^s and the number in question is the resulting numerical term. If the factors (0^λ), (0^μ), (0^ν), $\dots$ be t in number we are concerned with lattices of s rows and t columns. The first operation of D_0 results in a first row whose compartments contain t or fewer zeros placed in any manner, so that not more than one zero is in each compartment. In fact the first row is specified by a composition of zero into exactly t parts, a zero denoted by ω being permissible as a part (cf. Section II, Art. 55).

The first row of t compartments will therefore be of the form

...	ω	0	0	ω	0	0	...

the zero 0 appearing *at least* once, and ($\omega 00 \omega 00 \dots$) being a composition of zero into *exactly* t parts as defined. The number of such compositions is $2^t - 1$ (cf. Section II, Art. 55).

Each succeeding row will involve a composition of zero similarly defined and the complete diagram will be subject to the condition that the successive

columns must involve $\lambda, \mu, \nu, \ldots$ zeros 0 respectively. The number of such diagrams must be

$$\{D_0^s(0^\lambda)(0^\mu)(0^\nu)\ldots\}_{n=0},$$

or

$$\left\{(e^{\partial_n}-1)^s\binom{n}{\lambda}\binom{n}{\mu}\binom{n}{\nu}\ldots\right\}_{n=0},$$

or symbolically

$$(e^{\partial_0}-1)^s\binom{0}{\lambda}\binom{0}{\mu}\binom{0}{\nu}\ldots,$$

n of course being the number of quantities appertaining to the symmetric functions.

Ex. gr. $$(0^2)(0)^3 = \ldots + 8(0^2) + \ldots,$$

and the eight diagrams are

0	ω	ω	ω
0	0	0	0

0	0	ω	ω
0	ω	0	0

0	ω	0	ω
0	0	ω	0

0	ω	ω	0
0	0	0	ω

0	0	0	ω
0	ω	ω	0

0	0	ω	0
0	ω	0	ω

0	ω	0	0
0	0	ω	ω

0	0	0	0
0	ω	ω	ω

As a theorem of enumeration the symbols 0, ω may have any significance; the only essential is that they shall be different. Thus we may put 0, ω equal to 1, 0 respectively and then we see that the first of the diagrams may be represented by the composition (1000, 1111) of the multipartite number (2111). The other seven diagrams will be denoted by (1100, 1011), (1010, 1101), (1001, 1110), (1110, 1001), (1101, 1010), (1011, 1100), (1111, 1000) respectively, and these constitute the complete set of compositions of the multipartite number (2111) into exactly two parts, the magnitudes of the constituents of the parts being limited so as not to exceed unity. We may call such compositions "*unitary*," and we clearly are led to the general theorem:

"The coefficient of (0^s) in the development of the product

$$(0^\lambda)(0^\mu)(0^\nu)\ldots$$

is equal to the number of unitary compositions of the multipartite number

$$(\lambda\mu\nu\ldots)$$

into *exactly* s parts!"

It will be noted that the word "exactly" appears in the theorem because in the transformed diagrams the number unity must appear at least once in each row. The theorem itself is a direct consequence of the "composition" definition of the operation of D_0 upon a product.

206. It is very interesting to contrast the result just reached with that which arises from the process

$$D_1^s (1^\lambda)(1^\mu)(1^\nu)\ldots,$$

the numbers $\lambda, \mu, \nu, \ldots$ being t in number, and

$$\lambda + \mu + \nu + \ldots = s.$$

Recalling *ante* Art. 196 it will be remembered that we had to do with lattices of s rows and t columns; that the diagrams were such that one unit only appeared in each row, while there were $\lambda, \mu, \nu, \ldots$ units in the successive columns. Again the process

$$D_{\pi_1} D_{\pi_2} D_{\pi_3} \ldots (1^\lambda)(1^\mu)(1^\nu)\ldots,$$

the numbers $\pi_1, \pi_2, \pi_3, \ldots$ being s in number and the numbers $\lambda, \mu, \nu, \ldots$ t in number, led to lattices of s rows and t columns such that of identical symbols the successive rows involved $\pi_1, \pi_2, \pi_3, \ldots$ respectively and the successive columns $\lambda, \mu, \nu, \ldots$ respectively.

In regard to the first of these it is clear that we may denote the diagrams by the compositions of the multipartite number $(\lambda\mu\nu\ldots)$ into exactly s parts, each part containing but a single unit; we may call such compositions single-unitary, and we have the theorem that the number of single-unitary compositions of the multipartite number $(\lambda\mu\nu\ldots)$ into exactly s parts is given by the coefficient of (1^s) in the development of $(1^\lambda)(1^\mu)(1^\nu)\ldots$.

Ex. gr. $$(1^3)(1^2) = \ldots + 10\,(1^5),$$

and it is clear that derived from the unitary partition (10, 10, 10, 01, 01) of the multipartite (32) there are precisely ten compositions corresponding to the ten permutations of the parts of the partition.

As to the second we are conducted to an interesting relationship between the coefficients in the developments of $(1^\lambda)(1^\mu)(1^\nu)\ldots$ and of $(0^\lambda)(0^\mu)(0^\nu)\ldots$ respectively. For, as we have seen, in the diagrams connected with the multiplication $(0^\lambda)(0^\mu)(0^\nu)\ldots$ we may take the symbols to be 1 and 0, the conditions being for the coefficient of (0^s):

(i) that each row is to contain at least one unit,

(ii) that the successive columns are to contain $\lambda, \mu, \nu, \ldots$ units respectively.

Also for the multiplication $(1^\lambda)(1^\mu)(1^\nu)\ldots$ we may take the diagram symbols to be 1 and 0, the conditions being for the coefficient of $(\pi_1\pi_2\pi_3\ldots)$:

(i) that the successive rows are to involve $\pi_1, \pi_2, \pi_3, \ldots$ units respectively,

(ii) that the successive columns are to involve $\lambda, \mu, \nu, \ldots$ units respectively.

In both cases the diagrams have s rows and t columns, and of course there are s numbers $\pi_1, \pi_2, \pi_3, \ldots$.

If we take any partition of $\lambda+\mu+\nu+\ldots$ into exactly s parts, viz. $(\pi_1\pi_2\pi_3\ldots)$, and from it derive every composition of $\lambda+\mu+\nu+\ldots$ which involves the parts $\pi_1, \pi_2, \pi_3, \ldots$, we can proceed to a diagram of the zero multiplication from every such composition; for the rows of the unitary diagrams can be permuted without going outside of the zero diagrams, and this will be the case for every partition of $\lambda+\mu+\nu+\ldots$ into exactly s parts.

Hence, if $$(0^\lambda)(0^\mu)(0^\nu)\ldots = \ldots + B_s(0^s) + \ldots,$$

$$(1^\lambda)(1^\mu)(1^\nu)\ldots = \ldots + A_{\pi_1\pi_2\ldots}(\pi_1\pi_2\ldots) + A_{\pi_1'\pi_2'\ldots}(\pi_1'\pi_2'\ldots) + \ldots,$$

where $(\pi_1\pi_2\ldots)$, $(\pi_1'\pi_2'\ldots)$, ... are all partitions containing exactly s parts, we must have

$$B_s = P_{\pi_1\pi_2\ldots}A_{\pi_1\pi_2\ldots} + P_{\pi_1'\pi_2'\ldots}A_{\pi_1'\pi_2'\ldots} + \ldots,$$

where $P_{\pi_1\pi_2\ldots}$ denotes the number of compositions which involve the numbers $\pi_1, \pi_2, \ldots$, or in other words the number of ways of permuting those numbers.

Ex. gr. we have the formula

$$(1^2)(1)(1) = (31) + 2(2^2) + 5(21^2) + 12(1^4),$$

and we can immediately deduce the corresponding formula for $(0^2)(0)(0)$, for write

$$(0^2)(0)(0) = B_2(0^2) + B_3(0^3) + B_4(0^4).$$

To find B_2 we take the terms (31), $2(2^2)$, multiply each coefficient by the number of ways of permuting the numbers in the partition to which each is attached and then add; therefore

$$B_2 = 2\times 1 + 1\times 2 = 4,$$

because (31) has two permutations and (2^2) one.

Also $B_3 = 3\times 5$ because (21^2) has three permutations,

and $B_4 = 12$ because (1^4) has one permutation.

Hence $$(0^2)(0)(0) = 4(0^2) + 15(0^3) + 12(0^4)$$

CHAPTER III

THE THEORY OF THE LATIN SQUARE

207. The applications of the method, which has been explained, to problems of a chess-board character which will now be made are of a special kind and possess considerable historical interest.

Euler in 1782 in *Verhandelingen uitgegeven door het Zeeuwsch Genootschap der Wetenschappen te Vlissingen*, vol. 9, has a paper entitled "Recherches sur une nouvelle espèce de Quarrés Magiques." He commences as follows:

"Une question fort curieuse qui a exercé pendant quelque temps la sagacité de bien du monde, m'a engagé à faire les recherches suivantes, qui semblent ouvrir une nouvelle carrière dans l'Analyse et en particulier dans la doctrine des combinaisons. Cette question rouloit sur une assemblée de 36 officiers de six différens grades et tirés de six Régimens différens, qu'il s'agissoit de ranger dans un quarré, de manière que sur chaque ligne tant horizontale que verticale il se trouva six officiers tant de différens caractères que de Régimens différens. Or après toutes les peines qu'on s'est donné pour résoudre ce Problème on a été obligé de reconnoître qu'un tel arrangement est absolument impossible, quoiqu'on ne puisse pas en donner de démonstration rigoureuse."

He denotes the six regiments by the Latin letters a, b, c, d, e, f and the six ranks or grades by the Greek letters $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ and remarks that the character of an officer is determined by two letters, the one Latin and the other Greek, and that the problem consists in arranging the 36 combinations

$$\begin{array}{cccccc} a\alpha & a\beta & a\gamma & a\delta & a\epsilon & a\zeta \\ b\alpha & b\beta & b\gamma & b\delta & b\epsilon & b\zeta \\ c\alpha & c\beta & c\gamma & c\delta & c\epsilon & c\zeta \\ d\alpha & d\beta & d\gamma & d\delta & d\epsilon & d\zeta \\ e\alpha & e\beta & e\gamma & e\delta & e\epsilon & e\zeta \\ f\alpha & f\beta & f\gamma & f\delta & f\epsilon & f\zeta \end{array}$$

in a square in such manner that every row and every column contains the six Latin as well as the six Greek letters.

He finds no solution of this particular problem and gives his opinion that none is obtainable when the order of the square is of the form 2 mod 4. He observes that the first step is to arrange the Latin letters in a square so that no letter is missing either from any row or from any column. Such a square of the order 4 is for instance

$$\begin{matrix} a & b & c & d \\ b & d & a & c \\ c & a & d & b \\ d & c & b & a \end{matrix}$$

the four different letters appearing in each row and in each column.

He and subsequent writers have called such squares "Latin Squares," and we have the question as to their enumeration for a given order. In regard to this matter he writes:

"J'observe encore à cette occasion que le parfait dénombrement de tous les cas possibles de variations semblables seroit un objet digne de l'attention des Géomètres, d'autant plus que tous les principes connus dans la doctrine des combinaisons n'y sçauroient prêter le moindre secours."

And again:

"J'avois observé ci-dessus, qu'un parfait dénombrement de toutes les variations possibles des quarrés latins seroit une question très importante, mais qui me paroissoit extrêmement difficile et presque impossible dès que le nombre n surpassoit 5. Pour approcher de cette énumération il faudroit commencer par cette question: En combien de manières différentes, la première bande horizontale étant donnée, peut-on varier la seconde bande horizontale pour chaque nombre proposé n?"

Euler in fact gives a solution of the question proposed in the last part of the extract quoted. It is the well-known problem of derangements, known in France as the "Problème des rencontres"; it was first proposed by Montmort, and many solutions have been given since Euler's time. The complete solution of a problem in which it is included as a particular case is given in Section III of this work. Yet another solution is given in the present chapter.

Cayley in 1890 reviewed what had been written on the subject of the Latin Square and remarked upon the difficulty, given two rows of a square, of finding the number of arrangements possible for the third row and adds "the difficulty of course increases for the next following lines."

The "Problème des ménages" involves to a limited degree the consideration of the second and third lines of the square (see *post* Art. 214).

The method of operators with which we were concerned in the preceding chapter is peculiarly adapted to the discussion of these and similar questions.

Of the problem of the Latin Square two distinct solutions will be found, and there will be no difficulty in determining the number of arrangements of the $(s+1)$th row of the square when the first s rows are given.

It will be found possible to generalize the problem and also to give a useful discussion of the far more difficult question of enumeration connected with the Græco-Latin Square of Euler.

208. From any given Latin Square we can form $n!(n-1)!$ others by permuting the columns in $n!$ ways and the last $n-1$ rows in $(n-1)!$ ways. The whole number of Latin Squares will therefore be a multiple of $n!(n-1)!$. We may write it $n!(n-1)!R_n$, where R_n is the number of "reduced Latin Squares" of the order n, and we may take these to be those squares in which the first row is $a, b, c, d, \ldots$, the letters being in alphabetical order and in which the first column also involves the letters in alphabetical order.

Taking $n=4$, for simplicity, we construct a symmetric function operand

$$(abcd)\,(abcd)\,(abcd)\,(abcd),$$

where of course $(abcd) = \Sigma\alpha_1^a\alpha_2^b\alpha_3^c\alpha_4^d$, and we must design an operator which will have the effect of picking out the numbers a, b, c, d, one from each factor of the operand. It is clear that $D_{a+b+c+d}$ will be such *if* $(abcd)$ is the only partition of $a+b+c+d$ into 4 or fewer parts, which involves the parts a, b, c, d repeated or not. This would not be the case for example if there existed the relation $a=b+c$ for then, since $a+b+c+d=2b+2c+d$, $D_{a+b+c+d}$ would operate not only through the medium of the partition $(abcd)$ but also through the partition $(b+c,\ b+c,\ d)$ or $(a,\ a,\ d)$.

In order that $D_{a+b+c+d}$ may be effective for the purpose in hand we must give a, b, c, d such numerical values that the coefficient of $x^{a+b+c+d}$ in the expansion of $\frac{1}{4!}(x^a+x^b+x^c+x^d)^4$ may be equal to unity.

We are naturally brought to the consideration of the perfect partitions of numbers and we observe that if $(abcd)=(8421)$, the number 15 can be composed in only one way with the parts 8, 4, 2, 1 repeated or not, when the number of parts is restricted to be 4 or less. (8421) is a perfect partition of the number 15 and is the simplest solution of the problem. Many other sets of numbers would do—for example 19, 8, 2, 1, since the number 30 can be partitioned in only one way, into 4 or fewer parts, so as to involve only the numbers 19, 8, 2, 1 repeated or not.

We will suppose then that $(abcd)$ is such a partition of $a+b+c+d$ and then

$$D_{a+b+c+d}\,(abcd)\,(abcd)\,(abcd)\,(abcd)$$
$$= (\bullet bcd)\,(a\bullet cd)\,(ab\bullet d)\,(abc\bullet)$$
$$+ 23 \text{ other terms},$$

since we can arrange *abcd* in 24 permutations and pick out the first letter from the first factor, the second from the second, the third from the third and the fourth from the fourth factor.

We have therefore broken up the operation $D_{a+b+c+d}$ into 4! minor operations and we may take for the first row of our diagram any one of the 4! permutations of *abcd*. We will select the minor operation which gives the first row

c	a	d	b

the corresponding portion of the operand being

$$(ab \bullet d)\,(\bullet bcd)\,(abc \bullet)\,(a \bullet cd).$$

Operating upon this portion with $D_{a+b+c+d}$ we can pick out the parts a, b, c, d in a number of ways; one of these gives a second row a, b, c, d and a diagram of two rows

c	a	d	b
a	b	c	d

the corresponding portion of the operand being now

$$(\bullet b \bullet d)\,(\bullet\bullet cd)\,(ab\bullet\bullet)\,(a \bullet c \bullet).$$

Operating again we reach a third row b, d, a, c, leaving finally a fourth row d, c, b, a and the complete diagram

c	a	d	b
a	b	c	d
b	d	a	c
d	c	b	a

which is a Latin Square.

It is thus clear that if the numbers a, b, c, d be suitably selected, and

$$D^4_{a+b+c+d}\,(abcd)^4 = I_4,$$

I_4 enumerates the whole number of Latin Squares of the fourth order. It is clear in fact from the construction of the operand that each column must contain each of the letters a, b, c, d, and from the relation of the operand to the operator due to a suitable selection of the numbers represented by the letters that each row also must involve each of the letters.

This discussion is of general application and shews that if the n letters $a, b, c, \ldots$ be suitably selected and

$$D^n_{a+b+c+\ldots}\,(abc\ldots)^n = I_n,$$

I_n enumerates the whole of the Latin Squares of order n.

The number of reduced Latin Squares is $\dfrac{I_n}{n!\,(n-1)!}$.

209. The simplest selection it is possible to make for the numbers $a, b, c, d, \ldots$ is the series of powers of two, 1, 2, 4, 8, 16, $\ldots 2^{n-1}$, and the theorem then states that the whole number of Latin Squares of order n is

$$D^n_{2^n-1}(1248\ldots 2^{n-1})^n = I_n.$$

For the first few orders

$$I_2 = 1,$$
$$I_3 = 12 = 3!\,2!,$$
$$I_4 = (4!)^2,$$
$$I_5 = 52 \cdot 5!\,4!,$$

and if we denote by R_n the number of reduced Latin Squares of order n so that

$$n!\,(n-1)!\,R_n = I_n,$$

we find $R_2 = 1, \quad R_3 = 1, \quad R_4 = 4, \quad R_5 = 52.$

For the third order the form is

$$\begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix}$$

and for the fourth order the forms are

$$\begin{matrix} a\ b\ c\ d \\ b\ a\ d\ c \\ c\ d\ b\ a \\ d\ c\ a\ b \end{matrix} \qquad \begin{matrix} a\ b\ c\ d \\ b\ a\ d\ c \\ c\ d\ a\ b \\ d\ c\ b\ a \end{matrix} \qquad \begin{matrix} a\ b\ c\ d \\ b\ c\ d\ a \\ c\ d\ a\ b \\ d\ a\ b\ c \end{matrix} \qquad \begin{matrix} a\ b\ c\ d \\ b\ d\ a\ c \\ c\ a\ d\ b \\ d\ c\ b\ a \end{matrix}$$

it will be observed that all of these forms are self-conjugate in that they read the same by columns as by rows. When we come to the order 5 we find that only six of the forms are self-conjugate, the remainder consisting of twenty-three conjugate pairs. The six forms are

$$\begin{matrix} a\ b\ c\ d\ e \\ b\ c\ d\ e\ a \\ c\ d\ e\ a\ b \\ d\ e\ a\ b\ c \\ e\ a\ b\ c\ d \end{matrix} \qquad \begin{matrix} a\ b\ c\ d\ e \\ b\ c\ e\ a\ d \\ c\ e\ d\ b\ a \\ d\ a\ b\ e\ c \\ e\ d\ a\ c\ b \end{matrix} \qquad \begin{matrix} a\ b\ c\ d\ e \\ b\ e\ a\ c\ d \\ c\ a\ d\ e\ b \\ d\ c\ e\ b\ a \\ e\ d\ b\ a\ c \end{matrix}$$

$$\begin{matrix} a\ b\ c\ d\ e \\ b\ e\ d\ a\ c \\ c\ d\ b\ e\ a \\ d\ a\ e\ c\ b \\ e\ c\ a\ b\ d \end{matrix} \qquad \begin{matrix} a\ b\ c\ d\ e \\ b\ d\ a\ e\ c \\ c\ a\ e\ b\ d \\ d\ e\ b\ c\ a \\ e\ c\ d\ a\ b \end{matrix} \qquad \begin{matrix} a\ b\ c\ d\ e \\ b\ d\ e\ c\ a \\ c\ e\ b\ a\ d \\ d\ c\ a\ e\ b \\ e\ a\ d\ b\ c \end{matrix}$$

210. To find the number I_n we may express both operator and operand in terms of the elementary symmetric functions and actually carry out the operation.

Some simplifications may be made. For suppose, for the third order,

$$(421)^3 = \ldots + I_3(777) + \ldots;$$

we may restrict the number of quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ to which the symmetric functions refer to three and thus reduce the relation to

$$(31)^3 = \ldots + I_3(444) + \ldots,$$

and

$$D_4(31)^3 = 6\,(1)\,(3)\,(31),$$

$$D_4^2(31)^3 = 6\,(31) + 12\,(3)\,(1),$$

$$D_4^3(31)^3 = 12,$$

shewing that

$$I_3 = 12.$$

Similarly we reduce the relation

$$(8421)^4 = \ldots + I_4(15, 15, 15, 15) + \ldots$$

to

$$(731)^4 = \ldots + I_4(11, 11, 11, 11) + \ldots,$$

and thence, without introducing the elementary functions but relying merely upon the known properties of the operator, establish the result

$$I_4 = (4\,!)^2.$$

The calculations for higher orders become impracticable, but we shall find later in this work that there is a simpler solution of the question depending upon the symmetric functions of several systems of quantities.

211. We may in general consider Latin Rectangles. A Latin Rectangle is merely the first s rows of a Latin Square and would then have s rows and n columns. Latin Rectangles may be *reduced* as regards columns by always arranging the first row in alphabetical order. This reduced number is obtained by dividing by $n\,!$. We *may* also reduce them as regards rows by only considering those which have a first column arranged in alphabetical order. To obtain this enumeration we must *further* divide by

$$\frac{(n-1)\,!}{(n-s)\,!}.$$

Consider the Latin Rectangles of 2 rows and n columns which obviously represent the solution of the "Problème des rencontres" or the "derangements" of a permutation. Denote the total number by

$$n\,!\,(n-1)\,n_2,$$

so that $(n-1)\,n_2$, corresponding to rectangles reduced as to columns but unreduced as regards rows, is the number required by the "Problème des rencontres."

If $a, b, c, d, \ldots$ be n suitably selected numbers,

$$D_{a+b+c+\ldots}(abc\ldots)^n = n!(\bullet bcd\ldots)(a\bullet cd\ldots)(ab\bullet d)(\quad)\ldots$$

and $$D_{a+b+c+\ldots}(\bullet bcd\ldots)(a\bullet cd\ldots)(ab\bullet d)(\quad)\ldots$$

clearly results in a number of terms equal to $(n-1)n_2$, for in forming the second row, the first being in alphabetical order, the letters $a, b, c, \ldots$ clearly cannot appear in the first, second, third, etc. columns respectively.

Hence

$n!(n-1)n_2 =$ number of terms resulting from the operation $D^2_{a+b+c+\ldots}(abc\ldots)^n$.

Each term resulting from the operation is of the form

$$(cde\ldots)(bde\ldots)(ade\ldots)(\quad)\ldots,$$

being a product of n functions each of which is denoted by a partition involving $n-2$ different parts; and since when we put

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = 1,$$

each term becomes equal to $$\left(\frac{n!}{2!}\right)^n,$$

it follows that the number of terms which result from the operation is

$$\frac{[D^2_{a+b+c+\ldots}(abc\ldots)^n]_{\alpha_1=\alpha_2=\ldots=1}}{\left(\frac{n!}{2!}\right)^n},$$

and thence we find

$$(n-1)n_2 = \frac{[D^2_{a+b+c+\ldots}(abc\ldots)^n]_{\alpha=1}}{n!\left(\frac{n!}{2!}\right)^n},$$

where for brevity $\alpha = 1$ is put for $\alpha_1 = \alpha_2 = \ldots = 1$.

This yields another solution of the problem considered.

The method and reasoning are quite general and lead to the result for s rows:

$$\frac{(n-1)!}{(n-s)!}n_s = \frac{[D^s_{a+b+c+\ldots}(abc\ldots)^n]_{\alpha=1}}{n!\left(\frac{n!}{s!}\right)^n},$$

where $$\frac{(n-1)!}{(n-s)!}n_s$$

enumerates the solution of the "Problème des rencontres" generalized to s rows, and n_s enumerates the number of ways of forming the first s rows of a Latin Square reduced both as to columns and rows.

212. The reader will observe that n_n is precisely what we have denoted above by I_n, viz. the number of reduced Latin Squares.

A Latin Square is exactly determined by its first $n-1$ rows since $n-1$ rows determine the nth row. Hence we must have $n_{n-1} = n_n$ and to prove that this does in fact result from the formula, note that

$$n_{n-1} = \frac{[D^{n-1}_{a+b+c+\ldots}(abc\ldots)^n]_{a=1}}{n!\,(n-1)!\,n^n},$$

$$n_n = \frac{D^n_{a+b+c+\ldots}(abc\ldots)^n}{n!\,(n-1)!},$$

and that in the expression of n_n it is superfluous to put $\alpha = 1$ because the numerator is a mere number and does not involve the quantities α.

Now if $n_{n-1} = n_n$ we should have

$$[D^{n-1}_{a+b+c+\ldots}(abc\ldots)^n]_{a=1} = n^n D^n_{a+b+c+\ldots}(abc\ldots)^n.$$

The operation $D^{n-1}_{a+b+c+\ldots}(abc\ldots)^n$ results in a number of terms of the form

$$(c_1)(c_2)\ldots(c_n).$$

This number is clearly

$$D^n_{a+b+c+\ldots}(abc\ldots)^n,$$

and since, on putting $\alpha = 1$, the term

$$(c_1)(c_2)\ldots(c_n) \text{ becomes } n^n,$$

it follows that the left-hand side is equal to

$$n^n D^n_{a+b+c+\ldots}(abc\ldots)^n.$$

213. The general solution of the Latin Square may be written

$$(\Sigma\alpha_1\alpha_2^2\alpha_3^4\ldots\alpha_n^{2^{n-1}})^n = \ldots + n!\,(n-1)!\,n_n\Sigma(\alpha_1\alpha_2\alpha_3\ldots\alpha_n)^{2^n-1} + \ldots.$$

Another solution will appear in Vol. II.

214. We will now consider the "Problème des ménages."

Lucas in his *Théorie des Nombres* states the question in this manner:

"Des femmes, en nombre n, sont rangées autour d'une table, dans un ordre déterminé; on demande quel est le nombre des manières de placer leurs maris respectifs, de telle sorte qu'un homme soit placé entre deux femmes, sans se trouver à côté de la sienne?"

He then remarks that it is necessary to determine the number of "permutations discordantes" with the two permutations

$$1, 2, 3, 4, \ldots n-1, n,$$

$$2, 3, 4, 5, \ldots n, 1.$$

He says "Nous ne connaissons aucune solution simple de cette question, dont l'énoncé donne lieu à l'étude du nombre des permutations discordantes de deux permutations déjà discordantes et plus généralement, du nombre des permutations discordantes de deux permutations quelconques."

Solutions have been given by M. Laisant and M. C. Moreau, of which the most convenient is represented by the difference equation

$$(n-1)\,\lambda_{n+1} = (n^2-1)\,\lambda_n + (n+1)\,\lambda_{n-1} + 4\,(-)^n$$

with the initial values $(\lambda_3, \lambda_4) = (1, 2)$.

215. The reader who has mastered the foregoing solution of the problem of the Latin Square will have no difficulty in applying the same method here.

Construct for the operand the symmetric function of n factors

$$(cde\ldots)\,(ade\ldots)\,(abe\ldots)\ldots(bcd\ldots a_{n-1}),$$

where the sth factor from the left is deprived of the sth and $(s+1)$th letters of the alphabet, while the last factor is deprived of the nth and 1st letters.

Take as operator $D_{a+b+c+\ldots}$ and choose $a, b, c, \ldots$ to be appropriate numbers—most simply 1, 2, 4, 8, ... 2^{n-1}.

The operator must clearly pick out permutations of $a, b, c, \ldots$ from the factors which are discordant in Lucas' meaning with each of the permutations

$$a, b, c, d, \ldots a_{n-1}, a_n,$$

$$b, c, d, e, \ldots a_n, a.$$

In fact $D_{a+b+c+\ldots}(cde\ldots)\,(ade\ldots)\,(abe\ldots)\ldots(bcd\ldots a_{n-1})$

$$= \Sigma\, D_{v_1}(cde\ldots)\, D_{v_2}(ade\ldots)\, D_{v_3}(abe\ldots)\ldots D_{v_n}(bcd\ldots a_{n-1}),$$

the summation being for every permutation

$$v_1, v_2, v_3, \ldots v_n$$

of the letters $a, b, c, \ldots a_n$.

The number of products that survive is precisely the number of *ménages* denoted, by Lucas, by the symbol λ_n.

Each factor of each product is denoted by a partition containing $n-3$ literal symbols and for $\alpha_1 = \alpha_2 = \ldots = 1$ has the value

$$\frac{n!}{3!}.$$

Hence

$$[D_{a+b+c+\ldots}(cde\ldots)\,(ade\ldots)\,(abe\ldots)\ldots]_{a=1} = \lambda_n \left(\frac{n!}{3!}\right)^n$$

and

$$\lambda_n = \left(\frac{n!}{3!}\right)^{-n} [D_{a+b+c+\ldots}(cde\ldots)\,(ade\ldots)\,(abe\ldots)\ldots]_{a=1}.$$

For the calculation we substitute the selected numbers for the letters.

216. On a similar principle we enumerate the number of permutations which are discordant with *any number* of given permutations.

Suppose that the m given permutations are

$$\begin{array}{l} v_1', \quad v_2', \quad v_3', \quad \ldots \quad v_n', \\ v_1'', \quad v_2'', \quad v_3'', \quad \ldots \quad v_n'', \\ v_1''', \quad v_2''', \quad v_3''', \quad \ldots \quad v_n''', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_1^{(m)}, \quad v_2^{(m)}, \quad v_3^{(m)}, \quad \ldots \quad v_n^{(m)}, \end{array}$$

which may be *either* mutually discordant or not.

Of the m letters $v_s', v_s'', v_s''', \ldots v_s^{(m)}$ let the *different* ones be

$$u_s', u_s'', \ldots u_s^{(k_s)}, \ k_s \text{ in number.}$$

Take for the sth factor of an operand

$$D_{u_s'} D_{u_s''} \ldots D_{u_s^{(k_s)}} (abcd \ldots a_n),$$

or in other words the factor $(abcd\ldots)$ after it has been deprived of the letters $u_s', u_s'', \ldots u_s^{(k_s)}$.

Denote by P the operand thus formed; then

$$D_{a+b+c+\ldots} P$$

must result in a number of terms exactly equal to the number of permutations which are discordant with the given permutations.

Proceeding as before the number in question is

$$\left\{\frac{n!}{(k_1+1)!}\right\}^{-j_1} \left\{\frac{n!}{(k_2+1)!}\right\}^{-j_2} \ldots \left\{\frac{n!}{(k_s+1)!}\right\}^{-j_s} \ldots \times [D_{a+b+c+\ldots} P]_{a=1},$$

where $j_1, j_2, \ldots j_s, \ldots$ are numbers ascertainable from the given permutations and $\Sigma j = n$.

217. A more direct generalization of the "Problème des ménages" is obtained by imposing the condition that no husband is to have less than $2m$ persons between himself and his wife.

In the problem considered above $m = 1$. If $m = 2$ we must enumerate the permutations discordant with

$$\begin{array}{llllllll} a_1, & a_2, & a_3, & a_4, & \ldots & a_{n-3}, & a_{n-2}, & a_{n-1}, & a_n, \\ a_2, & a_3, & a_4, & a_5, & \ldots & a_{n-2}, & a_{n-1}, & a_n, & a_1, \\ a_3, & a_4, & a_5, & a_6, & \ldots & a_{n-1}, & a_n, & a_1, & a_2, \\ a_4, & a_5, & a_6, & a_7, & \ldots & a_n, & a_1, & a_2, & a_3, \end{array}$$

and we form the product

$$P_4 = (a_5 \ldots a_n)(a_1 a_6 \ldots a_n)(a_1 a_2 a_7 \ldots a_n) \ldots$$
$$(a_1 \ldots a_{n-4})(a_2 \ldots a_{n-3})(a_3 \ldots a_{n-2})(a_4 \ldots a_{n-1}),$$

the solution being given by

$$\text{number} = \left(\frac{n!}{5!}\right)^{-n} [D_{\Sigma a} P_4]_{a=1},$$

and in general

$$\text{number} = \left\{\frac{n!}{(2m+1)!}\right\}^n [D_{\Sigma a} P_{2m}]_{a=1}.$$

218. All questions which depend upon the enumeration of the permutations which are discordant with a given number of permutations can be better solved by having recourse to the Master Theorem of Section III.

Thus for the "Problème des ménages" for $n=3$ we are led through the determinant

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

to the generating function

$$\frac{1}{1 - x_1 x_2 x_3};$$

and for $n=4$ through the determinant

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix}$$

to the generating function

$$\frac{1}{1 - x_1x_3 - x_2x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4},$$

in which we have to evaluate the coefficients of $x_1x_2x_3$ and $x_1x_2x_3x_4$ respectively; we thus obtain the numbers 1 and 2 corresponding to the arrangements of tables

$$\begin{array}{ccc} & H_3 & \\ W_1 & & W_2 \\ H_2 & & H_1 \\ & W_3 & \end{array}$$

$$\begin{array}{ccccc} & & H_3 & & \\ & W_1 & & W_2 & \\ H_2 & & & & H_4 \\ & W_4 & & W_3 & \\ & & H_1 & & \end{array} \qquad \begin{array}{ccccc} & & H_4 & & \\ & W_1 & & W_2 & \\ H_3 & & & & H_1 \\ & W_4 & & W_3 & \\ & & H_2 & & \end{array}$$

Although the Master Theorem is of much more general application the particular numbers connected with the "Problème des ménages" are much more easily obtained from the difference equation quoted. The succession of numbers is

λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	...
1	2	13	80	579	4738	...

219. The notion of a Latin Rectangle may be generalized. Instead of n different letters we may have s of one kind, t of another kind and so on; so that the letters are specified by the partition $(st\ldots)$ of the number n.

Let the letters be $a^s b^t c^u \ldots$.

To obtain a Latin Rectangle of k rows we take k permutations of the letters such that in no column does a occur more than s times, b more than t times, c more than u times and so on.

The reduced rectangles have the first row and first column in the same assigned order and evidently we can obtain their number by dividing the number of unreduced rectangles by

$$\frac{n!}{s!\,t!\,u!\ldots} \cdot \frac{(n-1)!}{(s-1)!\,t!\,u!\ldots}$$

in the case when the rectangle is a square and by factors of similar forms in the other cases.

Examples of such Latin Squares are

$a\ a\ b\ c$	$a\ a\ b\ b$	$a\ a\ a\ b$
$a\ b\ c\ a$	$a\ b\ a\ b$	$a\ a\ b\ a$
$b\ c\ a\ a$	$b\ b\ a\ a$	$a\ b\ a\ a$
$c\ a\ a\ b$	$b\ a\ b\ a$	$b\ a\ a\ a$

220. We have Latin Squares and Rectangles associated with every partition of n; the three, given above, correspond to the partitions (21^2), (2^2), (31) of 4; we have already considered the case *abcd* corresponding to (1^4) and there remains the trivial case *aaaa* which corresponds to the partition (4).

First take the simple case $a^{n-1}b$ where $(n-1)a+b=w$. The numbers a, b are to be selected in such wise that $(a^{n-1}b)$ is the *only* partition of w into n or fewer parts drawn from the symbols a, b each any number of times repeated. For, that being so, by the law of operation of D_w,

$$D_w (a^{n-1}b)^n = n\,(a^{n-2}b)^{n-1}(a^{n-1}).$$

The appearance of the coefficient n indicates that for unreduced rectangles and squares there are n possible first rows; these are obviously the n permutations of $a^{n-1}b$.

Also $$D_w^2 (a^{n-1}b)^n = n\,(n-1)\,(a^{n-3}b)^{n-2}(a^{n-2})^2,$$

the coefficient $n(n-1)$ indicating that there are $n(n-1)$ possible pairs of two first rows in unreduced rectangles.

Further $$D_w^k (a^{n-1}b)^n = \frac{n!}{(n-k)!}(a^{n-k-1}b)^{n-k}(a^{n-k})^k,$$

indicating $\dfrac{n!}{(n-k)!}$ unreduced rectangles of k rows.

Hence
$$D_w^{n-1}(a^{n-1}b)^n = n!(a)^{n-1}(b),$$
$$D_w^n(a^{n-1}b)^n = n!,$$

intimating to us (as is otherwise immediately evident) that the number of unreduced rectangles of $n-1$ rows *or* of squares is $n!$.

221. We can enumerate the reduced rectangles of k rows because an examination of the square, based upon *aaab* figured above, shews that in Row 1 we have in the reduced rectangle one place for b instead of n places in the unreduced rectangle; in Row 2, $n-2$ places instead of $n-1$; etc. and in Row k, $n-k$ places instead of $n-k+1$.

Therefore the unreduced rectangles are

$$\frac{n}{1}\cdot\frac{n-1}{n-2}\cdot\frac{n-2}{n-3}\cdots\frac{n-k+1}{n-k} \text{ times}$$

as numerous as the reduced rectangles. This number is

$$\frac{n!}{(n-k)!}\cdot\frac{(n-k-1)!}{(n-2)!};$$

and so we find that the number of reduced rectangles is

$$\frac{n!}{(n-k)!}\div\frac{n!}{(n-k)!}\cdot\frac{(n-k-1)!}{(n-2)!},$$

or
$$\frac{(n-2)!}{(n-k-1)!} \text{ where } k<n.$$

If $k=n-1$ we find that the number of reduced rectangles of $n-1$ rows *or* of squares is $(n-2)!$.

In the formula we may not put $k=n$ because, in that case, the last fractional factor $\frac{n-k+1}{n-k}$ of the divisor must be omitted and the divisor, in that case, is $n(n-1)$ and this leads to the number

$$n!\div n(n-1),$$

or
$$(n-2)!.$$

It is in fact sufficient to consider $n-1$ rows because in that case the square is completely determined.

222. If we next proceed to enumerate the Latin Rectangles based upon

$$a^{n-2}b^2$$

we find that the enumeration and the formation of the rectangle depend upon

$$D_w^n(a^{n-2}b^2)^n,$$

where $w=(n-2)a+2b$.

It will be found that

$$D_w(a^{n-2}b^2)^n = \binom{n}{2}(a^{n-3}b^2)^{n-2}(a^{n-2}b)^2,$$

$$D_w^2(a^{n-2}b^2)^n = \binom{n}{2}\left\{(a^{n-4}b^2)^{n-2}(a^{n-2})^2 + 2\binom{n-2}{1}(a^{n-4}b^2)^{n-3}(a^{n-3}b)^2(a^{n-2}) + \binom{n-2}{2}(a^{n-4}b^2)^{n-4}(a^{n-3}b)^4\right\},$$

and it will be noticed that the right-hand side of the last written relation, since it contains terms of three different types, is not simply evaluated for unit values of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$; moreover the terms will not have a single type until we reach D_w^{n-1} which is the case of the square.

The number of unreduced squares is

$$D_w^n(a^{n-2}b^2)^n,$$

and, dividing this number by $\binom{n}{2}\binom{n-1}{2}$, we obtain the number of reduced squares. For the calculation the numbers a, b must be appropriately chosen or the process will fail. It is necessary for the number w to be composable in only one way with the numbers a, b repeated or not; the simplest solution is to so choose a and b that $(a^{n-2}b^2)$ is a perfect partition of $(n-2)a+2b$. We may therefore give a, b the values 3, 1 respectively, for then $(3^{n-2}1^2)$ is a perfect partition of $3(n-2)+2$.

223. We obtain a similar result for the enumeration of the Latin Squares derived from the letters

$$a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k} \text{ where } \Sigma sa = w,\ \Sigma s = n.$$

The number of reduced squares is

$$D_w^n(a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k})^n \div \frac{n!}{s_1!\,s_2!\ldots s_k!}\cdot\frac{(n-1)!}{(s_1-1)!\,s_2!\ldots s_k!},$$

and we must choose $a_1, a_2, \ldots a_k$ so that $(a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k})$ is a perfect partition of w. Reference to Art. 186 shews that we may take

$$a_1 = 1,\ a_2 = s_1+1,\ a_3 = (s_1+1)(s_2+1),\ \ldots a_k = (s_1+1)(s_2+1)\ldots(s_{k-1}+1).$$

In these squares each row and each column contains the whole of the letters $a_1^{s_1}, a_2^{s_2}, \ldots a_k^{s_k}$.

Ex. gr. If we evaluate $D_8^4(3^21^2)^4$ we readily find the number 90 for the

enumeration of the Latin Squares based upon a^2b^2 and division by $\binom{4}{2}\binom{3}{1}$ or 18 gives 5 for the number of reduced squares. These are

$a\ a\ b\ b$	$a\ a\ b\ b$	$a\ a\ b\ b$	$a\ a\ b\ b$	$a\ a\ b\ b$
$a\ a\ b\ b$	$a\ b\ a\ b$	$a\ b\ b\ a$	$a\ b\ a\ b$	$a\ b\ b\ a$
$b\ b\ a\ a$	$b\ a\ b\ a$	$b\ b\ a\ a$	$b\ b\ a\ a$	$b\ a\ a\ b$
$b\ b\ a\ a$	$b\ b\ a\ a$	$b\ a\ a\ b$	$b\ a\ b\ a$	$b\ b\ a\ a$

224. The Latin Squares that have been considered involve the property that one and only one letter appears in one cell of the lattice. We may have under view squares which have the property that the letters all occur in each row and in each column but which are divorced from the restriction as to the number of letters that may appear in a single cell or compartment.

Take

$$\text{as operator } L^n_{(a_1 a_2 \ldots a_n)}, \quad \text{as operand } (a_1 a_2 \ldots a_n)^n$$

where the operator $L_{(a_1 a_2 \ldots a_n)}$ is that studied in Section II, Chapter III which operates through the medium of the separations of the partition $(a_1 a_2 \ldots a_n)$.

For simplicity consider

$$L_{(abcd)}\,(abcd)\,(abcd)\,(abcd)\,(abcd).$$

$L_{(abcd)}$ picks out from the operand every separation of $(abcd)$, one separate only from each factor, in all possible ways. These separations are

$(abcd)$	$(ab)\,(c)\,(d)$
$(a)\,(bcd)$	$(ac)\,(b)\,(d)$
$(b)\,(acd)$	$(ad)\,(b)\,(c)$
$(c)\,(abd)$	$(bc)\,(a)\,(d)$
$(d)\,(abc)$	$(bd)\,(a)\,(c)$
$(ab)\,(cd)$	$(cd)\,(a)\,(b)$
$(ac)\,(bd)$	$(a)\,(b)\,(c)\,(d)$
$(ad)\,(bc)$	

fifteen in number. If we consider these separations in the permutations of their factors we arrive at the notion of the compositions of a partition which are in the same relation to the separations of a partition as the compositions of numbers are to the partitions of numbers.

The above fifteen separations of the partition $(abcd)$ yield 75 compositions of the partition $(abcd)$. If however we include the zero separate (ω) and regard the separations as all being composed of exactly 4 separates, zero being included as a separate, the number of compositions to be considered will be 4^4. Hence the diagrams will have first rows of 4^4 different kinds. It is easy to see *à priori* that this is so because the letters a, b, c, d must occur once each in the first row, and as there is a choice of 4 compartments for each of the 4 letters the number of first rows must be 4^4. Now consider the

second rows and remember that the letters a, b, c, d must only occur once in each column. Take a particular first row and observe that for each letter there is now only a choice of three compartments. This shews us that associated with a particular first row there can be 3^4 second rows. Further for a particular "first two rows" there can be 2^4 third rows. Finally the whole number of diagrams is seen to be $4^4 . 3^4 . 2^4 . 1^4$ or

$$(4\,!)^4.$$

In general the diagrams associated with

$$\text{the operator } L^n_{(abcd\ldots)}, \quad \text{the operand } (abcd\ldots)^n,$$

the number of letters being equal to n, are $(n\,!)^n$ in number.

The enumeration of this particular generalization of the Latin Square presents no difficulty whatever and the operator and operand are not necessary for the discussion. It will be observed that the diagrams are the same as those associated with the operator $(\partial_{x_1}\partial_{x_2}\ldots\partial_{x_n})^n$ and the operand $(x_1x_2\ldots x_n)^n$ (*vide* Art. 191), and the enumerating number is at once evident from the latter process.

The mode of operation of

$$L^n_{(abcd\ldots)}$$

is however worth a few words.

The number of separations of the partition $(abcd\ldots)$ is at once seen to be equal to the number of partitions of the multipartite number (1^n). One is convinced of this directly one writes down the partitions of (1^n), for a one-to-one correspondence with the separations under view is evident.

It follows at once that the number of compositions of the partition $(abcd\ldots)$ is equal to the number of compositions of the multipartite number (1^n). The special properties of these numbers are considered in other parts of this book.

Next consider

$$\text{as operator } L^m_{(abcd\ldots)}, \quad \text{as operand } (abcd\ldots)^m$$

and it will be seen that we get the same $(m\,!)^n$ diagrams that in Art. 191 were enumerated by

$$(\partial_{x_1}\partial_{x_2}\ldots\partial_{x_n})^m\,(x_1x_2\ldots x_n)^m.$$

We may now pass on to consider

$$\text{as operator } L^n_{(a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k})}, \quad \text{as operand } (a_1^{s_1}a_2^{s_2}\ldots a_k^{s_k})^n,$$

a generalization of the case considered in the last Article.

We take $\Sigma s = n$ and the diagram will have n rows and n columns.

The partition $(a_1^{s_1}a_2^{s_2}\dots a_k^{s_k})$ has just as many separations and compositions as the multipartite number $(s_1 s_2 \dots s_k)$ has partitions and compositions. We can find an expression for the number of diagrams without making use of the operator and operand. Consider the letters $a_1, a_2, \dots a_k$ separately. The letters $a_1^{s_1}$ must be distributed in the n compartments of the first row. The problem is that of the distribution of s_1 similar objects into n different parcels—in other words objects of type (s_1) into parcels of type (1^n). Suppose that the letters $a_1^{s_1}$ occupy exactly i compartments of the row and that in these i compartments we notice the combinations

$$a_1^{\sigma_1},\ a_1^{\sigma_2},\ \dots\ a_1^{\sigma_i},$$

reading the row from left to right. Then $\Sigma\sigma = s_1$ and $(\sigma_1\sigma_2\dots\sigma_i)$ is a composition of the unipartite number s_1 into exactly i parts. The number of such compositions is $\binom{s_1-1}{i-1}$ and the number of ways of selecting i of the n compartments is $\binom{n}{i}$. Hence for a given value of i there are

$$\binom{s_1-1}{i-1}\binom{n}{i} \text{ first rows}$$

and altogether the number of first rows must be

$$\sum_{i=1}^{i=s_1}\binom{s_1-1}{i-1}\binom{n}{i}$$

$$=\binom{n}{1}+\binom{s_1-1}{1}\binom{n}{2}+\binom{s_1-1}{2}\binom{n}{3}+\dots+\binom{s_1-1}{s_1-1}\binom{n}{s_1}.$$

Treating $a_2^{s_2}, a_3^{s_3}, \dots a_k^{s_k}$ similarly and then combining the distributions we see that the operation

$$L_{(a_1^{s_1}a_2^{s_2}\dots a_k^{s_k})}\,(a_1^{s_1}a_2^{s_2}\dots a_k^{s_k})$$

leads to

$$\left\{\sum_{i=1}^{i=s_1}\binom{s_1-1}{i-1}\binom{n}{i}\right\}\left\{\sum_{i=1}^{i=s_2}\binom{s_2-1}{i-1}\binom{n}{i}\right\}\dots\left\{\sum_{i=1}^{i=s_k}\binom{s_k-1}{i-1}\binom{n}{i}\right\}$$

one-row diagrams.

It is not easy similarly to obtain the number of two-row diagrams, but there is no difficulty in enumerating the complete diagrams of n rows and n columns. For having under view merely the letter a_1 and the aggregate $a_1^{s_1}$ we can enter the s_1 letters into the diagrams by remarking that the exponents that appear in the compartments must be such that in each row and in each column the sum of the exponents must be s_1. Hence reference to Art. 199 shews that the number of such diagrams is

$$D^n_{s_1}h^n_{s_1},$$

and thus the number of diagrams associated with

$$L^n_{(a_1^{s_1}a_2^{s_2}\dots a_k^{s_k})}\,(a_1^{s_1}a_2^{s_2}\dots a_k^{s_k})^n$$

is $$D^n_{s_1} h^n_{s_1} . D^n_{s_2} h^n_{s_2} . \ldots . D^n_{s_k} h^n_{s_k},$$

for we are at liberty to combine all the diagrams appertaining to the letters $a_1, a_2, \ldots a_k$ separately. These diagrams possess the property that the letters $a_1^{s_1} a_2^{s_2} \ldots a_k^{s_k}$ occur in each row and in each column without restriction upon the number of them that may appear in any one compartment. As a verification observe that if $s_1 = s_2 = \ldots = s_k = 1$, $k = n$ the enumeration is given by

$$(D^n_1 h^n_1)^n \quad \text{or} \quad (n\,!)^n.$$

We can clearly take as operand the product of n functions whose partitions involve n given combinations of the letters $a_1, a_2, \ldots a_k$, say

$$(B_1), (B_2), \ldots (B_n),$$

and as operator n separation operators

$$L_{(A_1)} L_{(A_2)} \ldots L_{(A_n)},$$

where $A_1, A_2, \ldots A_n$ are also given combinations and

$$A_1 A_2 \ldots A_n = B_1 B_2 \ldots B_n.$$

The diagrams then reached are those in which the rows contain the several combinations $A_1, A_2, \ldots A_n$ and the columns the several combinations $B_1, B_2, \ldots B_n$, and there is no reduction in the number of letters that may appear in any one compartment.

If the successive row combinations be

$$a^{\alpha_1} b^{\beta_1} c^{\gamma_1} \ldots, \quad a^{\alpha_2} b^{\beta_2} c^{\gamma_2} \ldots, \quad a^{\alpha_3} b^{\beta_3} c^{\gamma_3} \ldots,$$

and the successive column combinations be

$$a^{\alpha_1'} b^{\beta_1'} c^{\gamma_1'} \ldots, \quad a^{\alpha_2'} b^{\beta_2'} c^{\gamma_2'} \ldots, \quad a^{\alpha_3'} b^{\beta_3'} c^{\gamma_3'} \ldots,$$

the reasoning employed establishes a number of diagrams given by

$$(D_{\alpha_1'} D_{\alpha_2'} D_{\alpha_3'} \ldots h_{\alpha_1} h_{\alpha_2} h_{\alpha_3} \ldots)\,(D_{\beta_1'} D_{\beta_2'} D_{\beta_3'} \ldots h_{\beta_1} h_{\beta_2} h_{\beta_3} \ldots)$$
$$(D_{\gamma_1'} D_{\gamma_2'} D_{\gamma_3'} \ldots h_{\gamma_1} h_{\gamma_2} h_{\gamma_3} \ldots) \ldots,$$

and this can be evaluated by means of tables giving the h products in terms of the single partition symmetric functions.

It will be gathered from Art. 200 that any desired restriction can be placed upon the magnitudes of the exponents of the letters which appear in the diagrams.

SECTION VI

THE ENUMERATION OF THE PARTITIONS OF MULTIPARTITE NUMBERS

CHAPTER I

ENUMERATION OF THE PARTITIONS OF BIPARTITE NUMBERS

225. The enumeration of the partitions of multipartite numbers which is sometimes called "compound denumeration" may be studied by a direct application of the Theory of Distributions which was developed in Section I. It will be shewn that the enumeration may be made to depend upon the symmetric function of one or more *single* systems of quantities. One such system being

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

we write as usual

$$(1-\alpha_1 x)(1-\alpha_2 x)\ldots = 1 - a_1 x + a_2 x^2 - \ldots = \frac{1}{1 + h_1 x + h_2 x^2 + \ldots},$$

and employ the system of D operators defined by the relations

$$d_s = \partial_{a_s} + a_1 \partial_{a_{s+1}} + a_2 \partial_{a_{s+2}} + \ldots,$$

$$s!\, D_s = (d_1^s),$$

the outer bracket ( ) in which d_1^s is enclosed denoting as usual algebraic multiplication which yields an operator of order s and *not* s successive operations of the linear operator, which would be denoted by $(d_1)^s$.

We first consider the partitions of a bipartite number (pq) and recall that the partitions are separable into groups which depend upon the partitions of the unipartite numbers p, q respectively. Thus the partitions of the bipartite number (22), nine in number, are separated into four groups:

$Gr\{(2), (2)\}$	$Gr\{(2), (1^2)\}$	$Gr\{(1^2), (2)\}$	$Gr\{(1^2), (1^2)\}$
(22)	(21, 01)	(12, 10)	(11, 11)
(20, 02)	(20, 01, 01)	(10, 10, 02)	(11, 10, 01)
			(10, 10, 01, 01)

in fact if the numbers p, q possess P and Q partitions respectively there will be PQ groups, for every partition of P can be associated with every partition of Q.

We now study the enumeration of the partitions which appertain to a given group

$$Gr\,\{(p_1^{\pi_1}p_2^{\pi_2}\ldots),\ (q_1^{\chi_1}q_2^{\chi_2}\ldots)\},$$

the two partitions which define the group being partitions of the constituent numbers p, q of the bipartite (pq).

In such a partition the first element of a bipart may be p_s or 0, the second element may be q_s or 0. Hence the biparts have one of three forms

$$(p_sq_t),\quad (p_s0),\quad (0q_t).$$

If the first of these forms does not occur the number of biparts will be

$$\Sigma\pi + \Sigma\chi.$$

On the other hand if $\Sigma\pi \geqslant \Sigma\chi$ and the form $(0q_t)$ does not present itself the number of biparts will be $\Sigma\pi$; while if $\Sigma\pi < \Sigma\chi$ and the form (p_s0) is absent there will be $\Sigma\chi$ biparts. Thus the maximum number of parts is

$$\Sigma\pi + \Sigma\chi,$$

and the minimum number the greatest of the integers $\Sigma\pi$, $\Sigma\chi$.

Suppose p_1, p_2, ... to denote different objects and q_1, q_2, ... different parcels; then the collection of objects

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots)$$

may be spoken of as objects of type

$$(\pi_1\pi_2\ldots),$$

and the collection of parcels $(q_1^{\chi_1}q_2^{\chi_2}\ldots)$

as parcels of type $(\chi_1\chi_2\ldots)$.

If $\Sigma\pi$ were equal to $\Sigma\chi$, we could distribute the objects into the parcels, one object into each parcel, in a number of ways which is given by the expression

$$D_{\pi_1}D_{\pi_2}\ldots h_{\chi_1}h_{\chi_2}\ldots,$$

or by the expression $D_{\chi_1}D_{\chi_2}\ldots h_{\pi_1}h_{\pi_2}\ldots,$

where the symbol h has reference to the homogeneous product-sums of a single system of quantities.

This is the distribution which evidently yields the partitions of the group which involve exactly $\Sigma\pi\,(=\Sigma\chi)$ parts. The enumeration of the distributions gives also the enumeration of those special partitions appertaining to the group.

Here however $\Sigma\pi$ is not in general equal to $\Sigma\chi$ and the parts of the partitions vary in number between the greatest of the integers $\Sigma\pi$, $\Sigma\chi$ and the integer $\Sigma\pi + \Sigma\chi$.

Suppose that we add, to the objects $\Sigma\chi$, objects of species r and, to the parcels $\Sigma\pi$, parcels of species s; so that the objects and parcels are

$$(p_1^{\pi_1} p_2^{\pi_2} \dots r^{\Sigma\chi}), \quad (q_1^{\chi_1} q_2^{\chi_2} \dots s^{\Sigma\pi})$$

respectively. They are now of types

$$(\pi_1 \pi_2 \dots \Sigma\chi), \quad (\chi_1 \chi_2 \dots \Sigma\pi),$$

and the number of objects is equal to the number of parcels.

Observe that the types are denoted by two partitions of the number

$$\Sigma\pi + \Sigma\chi.$$

We will now distribute the objects into the parcels, one object into each parcel. In this case the notion of the "parcel" is not essential. We may consider two sets of objects of specifications

$$(\pi_1 \pi_2 \dots \Sigma\chi), \quad (\chi_1 \chi_2 \dots \Sigma\pi),$$

and the problem is the enumeration of the sets of duads of objects that can be formed by making $\Sigma\pi + \Sigma\chi$ pairs of objects, each pair consisting of an object taken from each set of objects.

This problem is precisely the same as that of determining the number of partitions of the bipartite number (pq) which appertain to the group

$$\{(p_1^{\pi_1} p_2^{\pi_2} \dots), \ (q_1^{\chi_1} q_2^{\chi_2} \dots)\}.$$

To explain this consider the partitions of the bipartite number (33) which appertain to the group

$$\{(21), \ (1^3)\}.$$

Here $\pi_1 = 1, \quad \pi_2 = 1, \quad \Sigma\pi = 2, \quad \chi_1 = 3, \quad \Sigma\chi = 3.$

We take $p_1 = 2, \quad p_2 = 1, \quad r = 0; \quad q_1 = 1, \quad s = 0,$

so that the two sets of objects are

$$2, \ 1, \ 0, \ 0, \ 0,$$

$$1, \ 1, \ 1, \ 0, \ 0,$$

respectively.

The first set of objects supplies the left-hand elements of the biparts and the second set the right-hand elements. To exhibit this we write

the first set $2*, \ 1*, \ 0*, \ 0*, \ 0*$

and the second set $*1, \ *1, \ *1, \ *0, \ *0.$

We have to associate the two sets in pairs in all possible ways, and since however this is carried out we can always arrange the pairs so that the first set of objects is in the order

$$2,\ 1,\ 0,\ 0,\ 0,$$

we will keep this order fixed and permute the objects of the second set so as to give all possible sets of pairs.

We find that the possible sets are:

$$21,\ 11,\ 01,\ 00,\ 00,$$
$$21,\ 10,\ 01,\ 01,\ 00,$$
$$20,\ 11,\ 01,\ 01,\ 00,$$
$$20,\ 10,\ 01,\ 01,\ 01,$$

from which deleting the zero parts we are left with the four partitions

$$(21,\ 11,\ 01),$$
$$(21,\ 10,\ 01,\ 01),$$
$$(20,\ 11,\ 01,\ 01),$$
$$(20,\ 10,\ 01,\ 01,\ 01)$$

of the bipartite (33) which appertain to the group

$$\{(21),\ (1^3)\}.$$

Before deletion of the zero parts we have all the partitions into exactly five parts, the bipartite zero being admissible as a part.

In general it is clear that we obtain all the partitions into exactly $\Sigma\pi + \Sigma\chi$ parts, the bipartite zero being admissible as a part; and after deletion of the zero parts we obtain the partitions appertaining to the group, zero not being admissible as a part in a partition.

The Theory of Distributions shews that the number we seek is the coefficient of the symmetric function

$$(\pi_1 \pi_2 \ldots \Sigma\chi)$$

in the development of the symmetric function product

$$h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi},$$

and equal to $$D_{\pi_1} D_{\pi_2} \ldots D_{\Sigma\chi} h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi},$$

or to $$D_{\chi_1} D_{\chi_2} \ldots D_{\Sigma\pi} h_{\pi_1} h_{\pi_2} \ldots h_{\Sigma\chi}.$$

This is a very elegant solution of the problem, for it enables us to enumerate the partitions in each group by a mere reference to the Tables which express products of h functions as linear functions of the monomial symmetric functions of the same weight.

226. As an example the method is applied to the partitions of the bipartite (33).

We have

Group	π_1	π_2	$\Sigma\chi$	χ_1	χ_2	$\Sigma\pi$	Number of Partitions	
(3, 3)	1	.	1	1	.	1	$D_1^2h_1^2=2,$	2
(3, 21)	1	.	2	1	1	1	$D_2D_1h_1^3=3,$	3
(3, 1^3)	1	.	3	3	.	1	$D_3D_1h_3h_1=2,$	2
(21, 3)	1	1	1	1	.	2	$D_1^3h_2h_1=3,$	3
(21, 21)	1	1	2	1	1	2	$D_2D_1^2h_2h_1^2=7,$	7
(21, 1^3)	1	1	3	3	.	2	$D_3D_1^2h_3h_2=4,$	4
(1^3, 3)	3	.	1	1	.	3	$D_3D_1h_3h_1=2,$	2
(1^3, 21)	3	.	2	1	1	3	$D_3D_2h_3h_1^2=4,$	4
(1^3, 1^3)	3	.	3	3	.	3	$D_3^2h_3^2=4,$	4
							Total	31

Hence the total number of partitions of the bipartite (33) is 31.

To explain the above calculation notice that for the group (21, 21) we have

$$D_2D_1^2h_2h_1^2=7,$$

because the Tables shew that

$$h_2h_1^2=\ldots+7\,(21^2)+\ldots,$$

and thence

$$D_2D_1^2h_2h_1^2=7.$$

But, *à priori*, the calculation is rapid, for

$$D_2h_2h_1^2=h_1^2+h_2+2h_1^2=3h_1^2+h_2,$$

$$D_2D_1h_2h_1^2=D_1(3h_1^2+h_2)=6h_1+h_1=7h_1,$$

$$D_2D_1^2h_2h_1^2=D_1.7h_1=7.$$

The formula for the whole of the partitions is

$$D_1^2h_1^2+D_2D_1h_1^3+D_3D_1h_3h_1+D_1^3h_2h_1+D_2D_1^2h_2h_1^2$$
$$+D_3D_1^2h_3h_2+D_3D_1h_3h_1+D_3D_2h_3h_1^2+D_3^2h_3^2,$$

and this expression, if we for the moment regard D_1, D_2, D_3 as symbols of numerical magnitude, may be written

$$(D_1h_1+D_2h_1^2+D_3h_3)(D_1h_1+D_1^2h_2+D_3h_3).$$

In the general case the formula for the whole of the partitions is

$$\Sigma\Sigma D_{\pi_1}D_{\pi_2}\ldots D_{\Sigma\chi}h_{\chi_1}h_{\chi_2}\ldots h_{\Sigma\pi},$$

the double summation being in regard to the whole of the partitions of both p and q.

We at once see that this expression may be algebraically factorized and be written

$$(\Sigma D_{\pi_1} D_{\pi_2} \ldots h_{\Sigma\pi})(\Sigma D_{\Sigma\chi} h_{\chi_1} h_{\chi_2} \ldots).$$

This is the generalization of the particular factorization met with in the case of the bipartite (33).

The reader will remark particularly that this factorization is symbolic in character and must be multiplied out and arranged in the original form before calculation.

By the above method the following numbers have been calculated:

Number partitioned	(11)	(21)	(31)	(22)	(41)	(32)	(51)	(42)	(33)
Number of partitions	2	4	7	9	12	16	19	29	31

Number partitioned	(52)	(43)	(53)	(44)	(54)	(55)
Number of partitions	47	57	97	109	189	336

These numbers agree with those obtained by expanding the generating function

$$\frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2)(1-x^3)(1-x^2y)(1-xy^2)(1-y^3)\ldots}.$$

227. The distribution of $\Sigma\pi + \Sigma\chi$ objects into $\Sigma\pi + \Sigma\chi$ parcels has necessarily resulted in our obtaining the whole of the partitions of the group under view; because the maximum number of parts that such a partition may have is precisely $\Sigma\pi + \Sigma\chi$.

If $\Sigma\pi$ is by hypothesis $\nless \Sigma\chi$ we may if we please make a distribution of $\Sigma\pi + s$ objects into $\Sigma\pi + s$ parcels, where s is any number included in the series $0, 1, 2, \ldots \Sigma\chi$. The partitions in correspondence are those which contain $\Sigma\pi + s$ or fewer parts. These are enumerated by

$$D_{\pi_1} D_{\pi_2} \ldots D_s h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi - \Sigma\chi + s},$$

a number which also denotes the number of ways of distributing objects of type $(\pi_1 \pi_2 \ldots s)$ into parcels of type $(\chi_1 \chi_2 \ldots \Sigma\pi - \Sigma\chi + s)$.

From this we learn that the number of partitions of the group

$$Gr\,\{(p_1^{\pi_1} p_2^{\pi_2} \ldots), (q_1^{\chi_1} q_2^{\chi_2} \ldots)\}$$

which contain exactly $\Sigma\pi + s$ parts is given by

$$D_{\pi_1} D_{\pi_2} \ldots D_s h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi - \Sigma\chi + s} - D_{\pi_1} D_{\pi_2} \ldots D_{s-1} h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi - \Sigma\chi + s - 1},$$

or as it may be written

$$D_{\pi_1} D_{\pi_2} \ldots \{D_s h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi - \Sigma\chi + s} - D_{s-1} h_{\chi_1} h_{\chi_2} \ldots h_{\Sigma\pi - \Sigma\chi + s - 1}\}.$$

Ex. gr. Consider the partitions of the bipartite (44) which appertain to the group $\{(211), (211)\}$. Here $\pi_1 = 1$, $\pi_2 = 2$, $\chi_1 = 1$, $\chi_2 = 2$, $\Sigma\pi = \Sigma\chi = 3$.

The whole of the partitions are enumerated by

$$D_3 D_2 D_1 h_3 h_2 h_1,$$

which from the Tables or otherwise in rapid fashion is found to have the value twelve.

The twelve partitions are

$$\left.\begin{matrix}(22,\,11,\,11)\\(21,\,12,\,11)\end{matrix}\right\}\text{ three parts,}$$

$$\left.\begin{matrix}(22,\,11,\,10,\,01)\\(21,\,12,\,10,\,01)\\(21,\,11,\,10,\,02)\\(20,\,12,\,11,\,01)\\(20,\,11.\,11,\,02)\end{matrix}\right\}\text{ four parts,}$$

$$\left.\begin{matrix}(22,\,10,\,10,\,01,\,01)\\(21,\,10,\,10,\,02,\,01)\\(20,\,12,\,10,\,01,\,01)\\(20,\,11,\,10,\,02,\,01)\end{matrix}\right\}\text{ five parts,}$$

$$(20,\,10,\,10,\,02,\,01,\,01)\quad\text{six parts.}$$

If we put $s=2$, we find that the partitions containing five or fewer parts are given by

$$D_2^2D_1h_2^2h_1=11.$$

For four or fewer parts we put $s=1$ and then

$$D_2D_1^2h_2h_1^2=7,$$

while for three parts, and there cannot be fewer, we put $s=0$, giving

$$D_2D_1h_2h_1=2.$$

These numbers are consequently verified.

For the whole of the partitions of the bipartite number (44) we enumerate the partitions involving at most one part by

$$D_1h_1=1\,;$$

at most two parts by

$$(2D_1^2+D_2)(2h_1^2+h_2)=13\,;$$

at most three parts by

$$(D_1^3+3D_2D_1)(h_1^3+3h_2h_1)=42\,;$$

at most four parts by

$$(2D_2D_1^2+D_2^2+D_3D_1+D_4)(2h_2h_1^2+h_2^2+h_3h_1+h_4)=74\,;$$

at most five parts by

$$(D_2^2D_1+D_3D_1^2+D_3D_2+2D_4D_1)(h_2^2h_1+h_3h_1^2+h_3h_2+2h_4h_1)=95\,;$$

at most six parts by

$$(D_3D_2D_1+D_4D_1^2+2D_4D_2+D_5D_1)(h_3h_2h_1+h_4h_1^2+2h_4h_2+h_5h_1)=105\,;$$

at most seven parts by

$$(D_4D_2D_1+D_4D_3+D_5D_1^2+D_5D_2+D_6D_1)(h_4h_2h_1+h_4h_3+h_5h_1^2+h_5h_2+h_6h_1)=108\,;$$

at most eight parts by

$$(D_4^2+D_5D_2D_1+D_6D_1^2+D_6D_2+D_7D_1)(h_4^2+h_5h_2h_1+h_6h_1^2+h_6h_2+h_7h_1)=109;$$

so that the whole number of partitions is 109 and the enumerations of those containing exactly 1, 2, 3, 4, 5, 6, 7, 8 parts respectively are given by the numbers

1, 12, 29, 32, 21, 10, 3, 1.

228. It will be noted that the above expressions break up in every case into an operator and an operand. This is invariably the case, but the symmetry between operator and operand is due to the equality of the numbers which compose the bipart under consideration.

For the bipart (32) the expressions are

$$\begin{aligned} D_1h_1 &= 1,\\ 2D_1^2(h_1^2+h_2) &= 6,\\ 2(D_1^3+D_2D_1+D_3)h_2h_1 &= 12,\\ (D_2D_1^2+2D_3D_1)(h_2^2+h_3h_1) &= 15,\\ (D_3D_1^2+D_3D_2+D_4D_1)(h_3h_2+h_4h_1) &= 16, \end{aligned}$$

wherein the symmetry does not exist.

That the operator can always be separated from the operand can be seen by studying the expression

$$D_{\pi_1}D_{\pi_2}\dots D_s h_{\chi_1}h_{\chi_2}\dots h_{\Sigma\pi-\Sigma\chi+s};$$

for if we consider $\Sigma\pi+s$ parts, the two groups

$$(p_1^{\pi_1}p_2^{\pi_2}\dots,\ q_1^{\chi_1}q_2^{\chi_2}\dots),\quad (p_1^{\pi_1}p_2^{\pi_2}\dots,\ q_1^{\chi_1'}q_2^{\chi_2'}\dots),$$

yield respectively

$$D_{\pi_1}D_{\pi_2}\dots D_s h_{\chi_1}h_{\chi_2}\dots h_{\Sigma\pi-\Sigma\chi+s},$$
$$D_{\pi_1}D_{\pi_2}\dots D_s h_{\chi_1'}h_{\chi_2'}\dots h_{\Sigma\pi-\Sigma\chi'+s},$$

which added together become

$$D_{\pi_1}D_{\pi_2}\dots D_s(h_{\chi_1}h_{\chi_2}\dots h_{\Sigma\pi-\Sigma\chi+s}+h_{\chi_1'}h_{\chi_2'}\dots h_{\Sigma\pi-\Sigma\chi'+s}).$$

Also the two groups

$$(p_1^{\pi_1'}p_2^{\pi_2'}\dots,\ q_1^{\chi_1}q_2^{\chi_2}\dots),\quad (p_1^{\pi_1'}p_2^{\pi_2'}\dots,\ q_1^{\chi_1'}q_2^{\chi_2'}\dots),$$

yield respectively

$$D_{\pi_1'}D_{\pi_2'}\dots D_{\Sigma\pi-\Sigma\pi'+s}h_{\chi_1}h_{\chi_2}\dots h_{\Sigma\pi-\Sigma\chi+s},$$
$$D_{\pi_1'}D_{\pi_2'}\dots D_{\Sigma\pi-\Sigma\pi'+s}h_{\chi_1'}h_{\chi_2'}\dots h_{\Sigma\pi-\Sigma\chi'+s},$$

which added together become

$$D_{\pi_1'}D_{\pi_2'}\dots D_{\Sigma\pi-\Sigma\pi'+s}(h_{\chi_1}h_{\chi_2}\dots h_{\Sigma\pi-\Sigma\chi+s}+h_{\chi_1'}h_{\chi_2'}\dots h_{\Sigma\pi-\Sigma\chi'+s}).$$

Combining this result with the former we find for the four groups a sum of two operators separated from a sum of two h functions; and evidently the reasoning is general.

229. In the next place let us examine the effect of employing the elementary functions $a_1, a_2, a_3, \ldots$ instead of the homogeneous product-sums $h_1, h_2, h_3, \ldots$.

The distributions enumerated by

$$D_{\pi_1} D_{\pi_2} \ldots D_{\Sigma\chi} a_{\chi_1} a_{\chi_2} \ldots a_{\Sigma\pi},$$

are those of objects of type

$$(\pi_1 \pi_2 \ldots \Sigma\chi),$$

into parcels of type

$$(\chi_1 \chi_2 \ldots \Sigma\pi), \text{ one object in each parcel,}$$

subject to the restriction that no two similar objects are to be placed in similar parcels.

The corresponding partitions of the bipartite number (pq) are those which appertain to the group

$$\{(p_1^{\pi_1} p_2^{\pi_2} \ldots),\ (q_1^{\chi_1} q_2^{\chi_2} \ldots)\}.$$

In these partitions the zero bipart (00) is admissible and no identical biparts occur. The number of parts in the partitions is $\Sigma\pi + \Sigma\chi$.

Taking as an illustration the bipartite number (33) the calculation may proceed as follows*:

Group	Enumerating Expression	Number of Partitions	Number of Parts
$\{(3), (3)\}$	$D_1^2 a_1^2$	2	2
$\{(3), (21)\}$	$D_1 D_2 a_1^3$	3	3
$\{(3), (1^3)\}$	$D_1 D_3 a_1 a_3$	0	—
$\{(21), (3)\}$	$D_1^3 a_1 a_2$	3	3
$\{(21), (21)\}$	$D_1^2 D_2 a_1^2 a_2$	5	4
$\{(21), (1^3)\}$	$D_1^2 D_3 a_2 a_3$	0	—
$\{(1^3), (3)\}$	$D_1 D_3 a_1 a_3$	0	—
$\{(1^3), (21)\}$	$D_2 D_3 a_1^2 a_3$	0	—
$\{(1^3), (1^3)\}$	$D_3^2 a_3^2$	0	—

The partitions enumerated by the numbers 2, 3, 3, 5 respectively are

(33, 00)	(32, 01, 00)	(23, 10, 00)	(21, 10, 02, 00)
(30, 03)	(31, 02, 00)	(13, 20, 00)	(22, 10, 01, 00)
	(30, 02, 01)	(03, 20, 10)	(20, 02, 10, 01)
			(12, 01, 20, 00)
			(20, 11, 02, 00)

* Observe, in the calculation, that D_σ operating upon a product of less than σ elementary symmetric functions necessarily produces zero.

The complete enumerating expression may be placed in the symbolic form

$$(D_1a_1 + D_2a_1^2 + D_3a_3)\,(D_1a_1 + D_1^2a_2 + D_3a_3),$$

and has the value 13.

In general the symbolic form of

$$\Sigma D_{\pi_1} D_{\pi_2} \dots D_{\Sigma\chi} a_{\chi_1} a_{\chi_2} \dots a_{\Sigma\pi}$$

is

$$(\Sigma D_{\pi_1} D_{\pi_2} \dots a_{\Sigma\pi}) \times (\Sigma D_{\Sigma\chi} a_{\chi_1} a_{\chi_2} \dots).$$

230. Again the distributions, enumerated by the expression

$$D_{\pi_1} D_{\pi_2} \dots D_s a_{\chi_1} a_{\chi_2} \dots a_{\Sigma\pi - \Sigma\chi + s},$$

are those of objects of type

$$(\pi_1 \pi_2 \dots s),$$

into parcels of type

$$(\chi_1 \chi_2 \dots \Sigma\pi - \Sigma\chi + s), \text{ one object in each parcel,}$$

subject to the restriction that no two similar objects are to be placed in similar parcels.

The corresponding partitions of the bipartite pq are those which appertain to the group

$$\{(p_1^{\pi_1} p_2^{\pi_2} \dots),\ (q_1^{\chi_1} q_2^{\chi_2} \dots)\},$$

and contain exactly $\Sigma\pi + s$ parts, the zero bipart 00 not being excluded as a possible part, and no particular part (including the part 00) occurs more than once. In the case of the bipartite number (33) the calculation is as follows:

Group			Partition
		One Part.	
{(3), (3)}	$s = 0$	$D_1a_1 = 1$	(33)
		Two Parts.	
{(3), (3)}	$s = 1$	$D_1^2a_1^2 = 2$	(33, 00)
			(30, 03)
{(3), (21)}	$s = 1$	$D_1^2a_1^2 = 2$	(32, 01)
			(31, 02)
{(21), (3)}	$s = 0$	$D_1^2a_1^2 = 2$	(23, 10)
			(13, 20)
{(21), (21)}	$s = 0$	$D_1^2a_1^2 = 2$	(21, 12)
			(22, 11)

Three Parts.

Group			Partition
$\{(3), (3)\}$	$s=2$	$D_1D_3a_1a_2=1$	(30, 03, 00)
$\{(3), (21)\}$	$s=2$	$D_1D_3a_1^3=3$	(32, 01, 00)
			(31, 02, 00)
			(30, 02, 01)
$\{(21), (3)\}$	$s=1$	$D_1^3a_1a_2=3$	(23, 10, 00)
			(13, 20, 00)
			(03, 20, 10)
$\{(21), (21)\}$	$s=1$	$D_1^3a_1^3=6$	(21, 12, 00)
			(22, 11, 00)
			(21, 10, 02)
			(22, 10, 01)
			(20, 11, 02)
			(20, 12, 01)
$\{(21), (1^3)\}$	$s=1$	$D_1^3a_3=1$	(21, 11, 01)
$\{(1^3), (21)\}$	$s=0$	$D_3a_1^3=1$	(12, 11, 10)

Four Parts.

$\{(3), (21)\}$	$s=3$	$D_1D_3a_1^2a_2=1$	(30, 02, 01, 00)
$\{(21), (3)\}$	$s=2$	$D_1^2D_2a_1a_3=1$	(03, 20, 10, 00)
$\{(21), (21)\}$	$s=2$	$D_1^2D_2a_1^2a_2=5$	(21, 10, 02, 00)
			(22, 10, 01, 00)
			(20, 11, 02, 00)
			(20, 12, 01, 00)
			(20, 10, 01, 02)
$\{(21), (1^3)\}$	$s=2$	$D_1^2D_2a_1a_3=1$	(21, 11, 01, 00)
$\{(1^3), (21)\}$	$s=1$	$D_1D_3a_1^2a_2=1$	(12, 11, 10, 00)

Five Parts.

$\{(21), (21)\}$	$s=3$	$D_1^2D_3a_1^2a_3=1$	(20, 10, 01, 02, 00)

It will be seen that for

	one	two	three	four	five	parts,
we have	1	8	15	9	1	partitions.

The partitions possessing r parts, one of which is a zero part, are derivable from those possessing $r-1$ parts no one of which is a zero part.

Bearing this in mind we see that the partitions into different parts, no one of which is zero, are enumerated in the manner:

one part,	number is		1,
two parts,	„	$8-1=7$,	
three „	„	$15-7=8$,	
four „	„	$9-8=1$,	
five „	„	$1-1=0$,	

so that altogether there are $1+7+8+1=17$ such partitions.

Otherwise, the matter may be viewed in the following manner.

Denote by $F_r(a)$ the enumerating expression in regard to r parts so that in the present case

$$F_1(a)=D_1a_1;\ F_2(a)=4D_1^2a_1^2;\ F_3(a)=D_1^3(a_1^3+a_1a_2+a_3)+D_1D_2(a_1a_2+a_1^3)+D_3a_1^3;$$
$$F_4(a)=D_1^2D_2(a_1^3a_2+2a_1a_3)+2D_1D_3a_1^2a_2;\ F_5(a)=D_1^2D_3a_1^2a_3.$$

The enumerating expressions for the numbers of partitions which involve 1, 2, 3, 4, 5 different parts, the zero part not being admissible as a part, are

$$F_1(a),$$
$$F_2(a)-F_1(a),$$
$$F_3(a)-F_2(a)+F_1(a),$$
$$F_4(a)-F_3(a)+F_2(a)-F_1(a),$$
$$F_5(a)-F_4(a)+F_3(a)-F_2(a)+F_1(a),$$

respectively; so that the expression which enumerates the whole of such partitions is by addition

$$F_1(a)+F_3(a)+F_5(a).$$

In fact we verify in respect of the numbers

$$1,\ 8,\ 15,\ 9,\ 1$$

that the number is $1+15+1=17$.

But it is clear also that the expression

$$F_6(a)-F_5(a)+F_4(a)-F_3(a)+F_2(a)-F_1(a)$$

is zero. Observe that *both* $F_6(a)$ and $F_5(a)-F_4(a)+F_3(a)-F_2(a)+F_1(a)$ are zero.

Hence also adding this expression we find that the required number is

$$F_2(a)+F_4(a).$$

We thus see in general that if we have obtained the numbers

$$F_1(a),\ F_2(a),\ F_3(a),\ F_4(a),\ \text{etc. ...},$$

the partitions into different parts, the zero not admissible, are enumerated by *either* of the expressions

$$F_1(a) + F_3(a) + F_5(a) + \ldots + F_{2r+1}(a) + \ldots,$$

$$F_2(a) + F_4(a) + \ldots + F_{2r}(a) + \ldots.$$

The number is in fact exactly half of that which enumerates the partitions in which the zero part is admissible.

We can verify this result *à priori* because every partition without a zero part has corresponding to it a partition obtained by adding a zero part to it. The number of them without a zero part must therefore be equal to the number with a zero part.

Instead of employing the functions h and a we might use functions derived from the functions h by deleting all partitions which contain numbers higher than any given integer. The reader will have no difficulty, recollecting Section I, in using such functions to obtain further information concerning the partitions appertaining to any given group.

CHAPTER II

ENUMERATION OF THE PARTITIONS OF TRIPARTITE AND OTHER MULTIPARTITE NUMBERS

231. In the case of the tripartite number pqr we examine the partitions appertaining to the group

$$\{(p_1^{\pi_1}p_2^{\pi_2}\ldots),\ (q_1^{\chi_1}q_2^{\chi_2}\ldots),\ (r_1^{\rho_1}r_2^{\rho_2}\ldots)\}.$$

The partitions involve parts which cannot be less in number than the greatest of the integers $\Sigma\pi$, $\Sigma\chi$, $\Sigma\rho$ nor greater in number than $\Sigma\pi+\Sigma\chi+\Sigma\rho$. Here we are concerned with three assemblages of objects of types

$$(\pi_1\pi_2\ldots),\ (\chi_1\chi_2\ldots),\ (\rho_1\rho_2\ldots),\ \text{respectively.}$$

In order to arrive, from these, at assemblages which contain equal numbers of objects we proceed to the assemblages of types

$$(\pi_1\pi_2\ldots\Sigma\chi+\Sigma\rho),\ (\chi_1\chi_2\ldots\Sigma\pi+\Sigma\rho),\ (\rho_1\rho_2\ldots\Sigma\pi+\Sigma\chi)$$

and, reasoning as in the bipartite case, we can assert that the partitions under examination are equi-numerous with the different sets of $\Sigma\pi+\Sigma\chi+\Sigma\rho$ triads of objects that can be formed by taking one object from each assemblage to form a triad until the objects are exhausted.

In the bipartite case we employed an auxiliary set of letters $\alpha_1, \alpha_2, \alpha_3, \ldots$ and connected therewith the homogeneous product sums $h_1, h_2, h_3, \ldots$ and the elementary functions $a_1, a_2, a_3, \ldots$. Here we require two auxiliary sets of letters $\alpha_1, \alpha_2, \alpha_3, \ldots$; $\beta_1, \beta_2, \beta_3, \ldots$ and therewith homogeneous product sums

$$h_{1,\alpha},\ h_{2,\alpha},\ h_{3,\alpha},\ \ldots;\ h_{1,\beta},\ h_{2,\beta},\ h_{3,\beta},\ \ldots$$

and elementary functions

$$a_{1,\alpha},\ a_{2,\alpha},\ a_{3,\alpha},\ \ldots;\ a_{1,\beta},\ a_{2,\beta},\ a_{3,\beta},\ \ldots$$

associated with the sets of letters respectively.

Partitions in brackets $(\ )_\alpha$, $(\ )_\beta$ will refer to the sets respectively.

Section I shews that from the relations

$$h_{1,\alpha}=(1)_\alpha, \qquad h_{1,\beta}=(1)_\beta,$$
$$h_{2,\alpha}=(2)_\alpha+(1^2)_\alpha, \qquad h_{2,\beta}=(2)_\beta+(1^2)_\beta,$$
$$h_{3,\alpha}=(3)_\alpha+(21)_\alpha+(1^3)_\alpha, \qquad h_{3,\beta}=(3)_\beta+(21)_\beta+(1^3)_\beta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$

we define other functions in the manner

$$h_{1,\alpha\beta} = (1)_\beta h_{1,\alpha},$$
$$h_{2,\alpha\beta} = (2)_\beta h_{2,\alpha} + (1^2)_\beta h^2_{1,\alpha},$$
$$h_{3,\alpha\beta} = (3)_\beta h_{3,\alpha} + (21)_\beta h_{2,\alpha} h_{1,\alpha} + (1^3)_\beta h^3_{1,\alpha},$$
$$h_{4,\alpha\beta} = (4)_\beta h_{4,\alpha} + (31)_\beta h_{3,\alpha} h_{1,\alpha} + (2^2)_\beta h^2_{2,\alpha} + (21^2)_\beta h_{2,\alpha} h^2_{1,\alpha} + (1^4)_\beta h^4_{1,\alpha},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_{p,\alpha\beta} = \Sigma\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta\, h^{\pi_1}_{p_1,\alpha} h^{\pi_2}_{p_2,\alpha} \ldots.$$

It will be gathered from Sections I and II that the right-hand sides of these relations are such that in them the symbols α, β may be interchanged. Therefore we may also write

$$h_{1,\alpha\beta} = (1)_\alpha h_{1,\beta},$$
$$h_{2,\alpha\beta} = (2)_\alpha h_{2,\beta} + (1^2)_\alpha h^2_{1,\beta},$$
$$h_{3,\alpha\beta} = (3)_\alpha h_{3,\beta} + (21)_\alpha h_{2,\beta} h_{1,\beta} + (1^3)_\alpha h^3_{1,\beta},$$
$$h_{4,\alpha\beta} = (4)_\alpha h_{4,\beta} + (31)_\alpha h_{3,\beta} h_{1,\beta} + (2^2)_\alpha h^2_{2,\beta} + (21^2)_\alpha h_{2,\beta} h^2_{1,\beta} + (1^4)_\alpha h^4_{1,\beta},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_{p,\alpha\beta} = \Sigma\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha\, h^{\pi_1}_{p_1,\beta} h^{\pi_2}_{p_2,\beta} \ldots.$$

We now form the product

$$h_{\rho_1,\alpha\beta}\, h_{\rho_2,\alpha\beta} \ldots h_{\Sigma\pi+\Sigma\chi,\alpha\beta}$$

and develop it by means of the relations above written until it is entirely composed of terms of the form

$$K(\quad)_\alpha(\quad)_\beta,$$

where K is a numerical coefficient.

Amongst these terms will be one

$$C(\pi_1\pi_2\ldots\Sigma\chi + \Sigma\rho)_\alpha(\chi_1\chi_2\ldots\Sigma\pi + \Sigma\rho)_\beta.$$

That being so, the Theory of Distributions of Section I asserts that the different sets of triads and therefore also the partitions under examination are enumerated by the number C.

Let $D_{m,\alpha}$, $D_{m,\beta}$ be obliterating operators in respect of symmetric functions of the sets of letters α, β respectively.

Then writing

$$h_{\rho_1,\alpha\beta}\, h_{\rho_2,\alpha\beta} \ldots h_{\Sigma\pi+\Sigma\chi,\alpha\beta} = \ldots + C(\pi_1\pi_2\ldots\Sigma\chi + \Sigma\rho)_\alpha(\chi_1\chi_2\ldots\Sigma\pi + \Sigma\rho)_\beta + \ldots,$$

we see from the known properties of the operators that

$$D_{\pi_1,\alpha} D_{\pi_2,\alpha} \ldots D_{\Sigma\chi+\Sigma\rho,\alpha} D_{\chi_1,\beta} D_{\chi_2,\beta} \ldots D_{\Sigma\pi+\Sigma\rho,\beta}\, h_{\rho_1,\alpha\beta}\, h_{\rho_2,\alpha\beta} \ldots h_{\Sigma\pi+\Sigma\chi,\alpha\beta} = C.$$

This is the analytical solution of the problem.

It must be noted that in this result the symbols π, χ, ρ may be permuted in any manner so that there are six (in general $n!$) forms of solution. This

multiplicity of solution supplies an important means for the verification of results.

We proceed to consider the evaluation of the expression.

We must find out how to operate upon

$$h_{m,\alpha\beta}$$

with $D_{s,\alpha}$ and $D_{s,\beta}$.

Since $$h_{p,\alpha\beta} = \Sigma\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha\, h_{p_1,\beta}^{\pi_1} h_{p_2,\beta}^{\pi_2} \ldots,$$

$$D_{p_1,\alpha}\, h_{p,\alpha\beta} = \{\Sigma\,(p_1^{\pi_1 - 1} p_2^{\pi_2} \ldots)_\alpha\, h_{p_1,\beta}^{\pi_1 - 1} h_{p_2,\beta}^{\pi_2} \ldots\}\, h_{p_1,\beta} = h_{p_1,\beta}\, h_{p-p_1,\alpha\beta}.$$

Hence $$D_{s,\alpha}\, h_{m,\alpha\beta} = h_{s,\beta}\, h_{m-s,\alpha\beta}.$$

Again since $$h_{p,\alpha\beta} = \Sigma\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta\, h_{p_1,\alpha}^{\pi_1} h_{p_2,\alpha}^{\pi_2} \ldots,$$

$$D_{p_1,\beta}\, h_{p,\alpha\beta} = \{\Sigma\,(p_1^{\pi_1 - 1} p_2^{\pi_2} \ldots)_\beta\, h_{p_1,\alpha}^{\pi_1 - 1} h_{p_2,\alpha}^{\pi_2} \ldots\}\, h_{p_1,\alpha} = h_{p_1,\alpha}\, h_{p-p_1,\alpha\beta}.$$

Hence $$D_{s,\beta}\, h_{m,\alpha\beta} = h_{s,\alpha}\, h_{m-s,\alpha\beta}.$$

These important relations are a direct consequence of the law

$$D_{p_\mu}(p_1^{\pi_1} p_2^{\pi_2} \ldots p_\mu^{\pi_\mu} \ldots) = (p_1^{\pi_1} p_2^{\pi_2} \ldots p_\mu^{\pi_\mu - 1} \ldots).$$

Before exhibiting particular examples it will be useful to further consider the operations of $D_{s,\alpha}$, $D_{s,\beta}$ upon a product

$$h_{m_1,\alpha\beta}\, h_{m_2,\alpha\beta}\, h_{m_3,\alpha\beta} \ldots.$$

Referring back to Art. 29 of Section I it will be seen that $D_{s,\alpha}$ must operate upon such a product in a manner which depends upon the compositions of the number s. Thus suppose that we have to operate with $D_{3,\alpha}$ upon

$$h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}.$$

The compositions of 3 are 3, 21, 12, 111. The parts of these compositions must be associated with the product in the ways following:

$$\begin{array}{cccc} h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}, & h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}, & h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}, & h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}, \\ 3 & 2\quad 1 & 1\quad 2 & 1\quad\quad 2 \end{array}$$

$$\begin{array}{cc} h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}, & h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta} \\ 1\quad 2 & 1\quad 1\quad 1 \end{array}$$

and then $$D_{3,\alpha}\, h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{4,\alpha\beta}$$

$$= h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, (D_{3,\alpha}\, h_{4,\alpha\beta}) + h_{1,\alpha\beta}\, (D_{2,\alpha}\, h_{2,\alpha\beta})\, (D_{1,\alpha}\, h_{4,\alpha\beta}) + (D_{1,\alpha}\, h_{1,\alpha\beta})\, (D_{2,\alpha}\, h_{2,\alpha\beta})\, h_{4,\alpha\beta}$$
$$+ (D_{1,\alpha}\, h_{1,\alpha\beta})\, h_{2,\alpha\beta}\, (D_{2,\alpha}\, h_{4,\alpha\beta}) + h_{1,\alpha\beta}\, (D_{1,\alpha}\, h_{2,\alpha\beta})\, (D_{2,\alpha}\, h_{4,\alpha\beta})$$
$$+ (D_{1,\alpha}\, h_{1,\alpha\beta})\, (D_{1,\alpha}\, h_{2,\alpha\beta})\, (D_{1,\alpha}\, h_{4,\alpha\beta}),$$

$$= h_{1,\alpha\beta}\, h_{2,\alpha\beta}\, h_{3,\beta}\, h_{1,\alpha\beta} + h_{1,\alpha\beta}\, h_{2,\beta}\, h_{1,\beta}\, h_{3,\alpha\beta} + h_{1,\beta}\, h_{2,\beta}\, h_{4,\alpha\beta}$$
$$+ h_{1,\beta}\, h_{2,\alpha\beta}\, h_{2,\beta}\, h_{2,\alpha\beta} + h_{1,\alpha\beta}\, h_{1,\beta}\, h_{1,\alpha\beta}\, h_{2,\beta}\, h_{2,\alpha\beta} + h_{1,\beta}\, h_{1,\beta}\, h_{1,\alpha\beta}\, h_{1,\beta}\, h_{3,\alpha\beta},$$

$$= (h_{3,\beta} + h_{1,\beta}\, h_{2,\beta})\, h_{1,\alpha\beta}^2\, h_{2,\alpha\beta} + h_{1,\beta}\, h_{2,\beta}\, h_{2,\alpha\beta}^2$$
$$+ (h_{1,\beta}^3 + h_{1,\beta}\, h_{2,\beta})\, h_{1,\alpha\beta}\, h_{3,\alpha\beta} + h_{1,\beta}\, h_{2,\beta}\, h_{4,\alpha\beta}.$$

If, instead, we are operating with $D_{3,\beta}$ we have merely to interchange α and β in the result just reached, such interchange of course leaving $h_{m,\alpha\beta}$ unaltered.

In this way the operators gradually reduce the product until it is a function of functions $h_{s,\alpha}$, $h_{s,\beta}$ only. These functions are then similarly dealt with according to the laws

$$D_{s,\alpha}\, h_{m,\alpha} = h_{m-s,\alpha}, \quad D_{s,\beta}\, h_{m,\beta} = h_{m-s,\beta};$$

and as above through the compositions of s upon a product of functions.

232. We will now take as an example the tripartite number (322).

We have

Group	π_1	π_2	$\Sigma\pi$	χ_1	$\Sigma\chi$	ρ_1	$\Sigma\rho$	Number of Partitions
$\{(3), (2), (2)\}$	1	0	1	1	1	1	1	$D_{1,\alpha} D_{2,\alpha} D_{1,\beta} D_{2,\beta} h_{1,\alpha\beta} h_{2,\alpha\beta}$,
$\{(3), (2), (1^2)\}$	1	0	1	1	1	2	2	$D_{1,\alpha} D_{3,\alpha} D_{1,\beta} D_{3,\beta} h^2_{2,\alpha\beta}$
$\{(3), (1^2), (2)\}$	1	0	1	2	2	1	1	$D_{1,\alpha} D_{3,\alpha} D^2_{2,\beta} \quad h_{1,\alpha\beta} h_{3,\alpha\beta}$ },
$\{(3), (1^2), (1^2)\}$	1	0	1	2	2	2	2	$D_{1,\alpha} D_{4,\alpha} D_{2,\beta} D_{3,\beta} h_{2,\alpha\beta} h_{3,\alpha\beta} = 9$,
$\{(21), (2), (2)\}$	1	1	2	1	1	1	1	$D^2_{1,\alpha} D_{2,\alpha} D_{1,\beta} D_{3,\beta} h_{1,\alpha\beta} h_{3,\alpha\beta}$,
$\{(21), (2), (1^2)\}$	1	1	2	1	1	2	2	$D^2_{1,\alpha} D_{3,\alpha} D_{1,\beta} D_{4,\beta} h_{2,\alpha\beta} h_{3,\alpha\beta}$
$\{(21), (1^2), (2)\}$	1	1	2	2	2	1	1	$D^2_{1,\alpha} D_{3,\alpha} D_{2,\beta} D_{3,\beta} h_{1,\alpha\beta} h_{4,\alpha\beta}$ }
$\{(21), (1^2), (1^2)\}$	1	1	2	2	2	2	2	$D^2_{1,\alpha} D_{4,\alpha} D_{2,\beta} D_{4,\beta} h_{2,\alpha\beta} h_{4,\alpha\beta}$,
$\{(1^3), (2), (2)\}$	3	0	3	1	1	1	1	$D_{3,\alpha} D_{2,\alpha} D_{1,\beta} D_{4,\beta} h_{1,\alpha\beta} h_{4,\alpha\beta}$,
$\{(1^3), (2), (1^2)\}$	3	0	3	1	1	2	2	$D^2_{3,\alpha} \quad D_{1,\beta} D_{5,\beta} h_{2,\alpha\beta} h_{4,\alpha\beta}$ }
$\{(1^3), (1^2), (2)\}$	3	0	3	2	2	1	1	$D^2_{3,\alpha} \quad D_{2,\beta} D_{4,\beta} h_{1,\alpha\beta} h_{5,\alpha\beta}$ },
$\{(1^3), (1^2), (1^2)\}$	3	0	3	2	2	2	2	$D_{3,\alpha} D_{4,\alpha} D_{2,\beta} D_{5,\beta} h_{2,\alpha\beta} h_{5,\alpha\beta} = 19$.

It must be noted that the bracketed expressions have necessarily the same value. Inspection of the corresponding groups proves this. It is otherwise evident because from the theory of symmetric function symmetry the expression

$$D_{\pi_1,\alpha} D_{\pi_2,\alpha} \dots D_{\Sigma\chi+\Sigma\rho,\alpha} D_{\chi_1,\beta} D_{\chi_2,\beta} \dots D_{\Sigma\pi+\Sigma\rho,\beta}\, h_{\rho_1,\alpha\beta}\, h_{\rho_2,\alpha\beta} \dots h_{\Sigma\pi+\Sigma\chi,\alpha\beta}$$

admits of all permutations of the symbols π, χ, ρ.

The calculation is given for the case of the group $\{(3), (1^2), (1^2)\}$:

$$\begin{aligned}
& D_{1,\alpha} D_{4,\alpha} D_{2,\beta} D_{3,\beta} h_{2,\alpha\beta} h_{3,\alpha\beta} \\
&= D_{1,\alpha} D_{4,\alpha} D_{2,\beta} \{h_{2,\alpha\beta} h_{3,\alpha} + h_{2,\alpha} h_{1,\alpha} h_{2,\alpha\beta} + h_{1,\alpha\beta} h_{1,\alpha} h_{1,\alpha\beta} h_{2,\alpha}\}, \\
&= D_{1,\alpha} D_{4,\alpha} D_{2,\beta} \{(h_{3,\alpha} + h_{2,\alpha} h_{1,\alpha}) h_{2,\alpha\beta} + h_{2,\alpha} h_{1,\alpha} h^2_{1,\alpha\beta}\}, \\
&= D_{1,\alpha} D_{4,\alpha} \{(h_{3,\alpha} + h_{2,\alpha} h_{1,\alpha}) h_{2,\alpha} + h_{2,\alpha} h_{1,\alpha} h^2_{1,\alpha}\}, \\
&= D_{1,\alpha} D_{4,\alpha} \{h_{3,\alpha} h_{2,\alpha} + h^2_{2,\alpha} h_{1,\alpha} + h_{2,\alpha} h^3_{1,\alpha}\}, \\
&= D_{4,\alpha} (h^2_{2,\alpha} + h_{3,\alpha} h_{1,\alpha} + 2h_{2,\alpha} h^2_{1,\alpha} + h^2_{2,\alpha} + h^4_{1,\alpha} + 3h_{2,\alpha} h^2_{1,\alpha}), \\
&= 9,
\end{aligned}$$

for the final operation by $D_{4,\alpha}$ consists merely in adding the numerical coefficients.

The nine partitions that have been enumerated are found to be

$$\begin{cases} (311,\ 011) \\ (311,\ 010,\ 001) \\ (310,\ 011,\ 001) \\ (301,\ 011,\ 010) \\ (300,\ 011,\ 011) \\ (310,\ 010,\ 001,\ 001) \\ (301,\ 010,\ 010,\ 001) \\ (300,\ 011,\ 010,\ 001) \\ (300,\ 010,\ 010,\ 001,\ 001). \end{cases}$$

Similarly the reader may ascertain by calculation that the number of partitions appertaining to the group $\{(1^3), (1^2), (1^2)\}$ is nineteen and verify that this is the correct number.

It is not possible to split up the complete enumerating expression for the whole of the partitions into a symbolic product. Such expression is a special feature of the bipartite theory.

233. We now deal with the partitions into a definite number of parts. This number cannot be less than the greatest of the integers $\Sigma\pi$, $\Sigma\chi$, $\Sigma\rho$, and may be as large as $\Sigma\pi + \Sigma\chi + \Sigma\rho$.

Previous reasoning shews us that for partitions into $\Sigma\pi + s$ or fewer parts, where

$$0 \leqslant s \leqslant \Sigma\chi + \Sigma\rho,$$

we have to deal with three assemblages of objects of types

$$(\pi_1\pi_2 \dots s), \quad (\chi_1\chi_2 \dots \Sigma\pi - \Sigma\chi + s), \quad (\rho_1\rho_2 \dots \Sigma\pi - \Sigma\rho + s),$$

where s can only have such values as make

$$\Sigma\pi - \Sigma\chi + s \text{ and } \Sigma\pi - \Sigma\rho + s$$

non-negative integers.

We have to consider the number of ways of forming $\Sigma\pi + s$ triads of objects by taking one object from each assemblage to form a triad. The Theory of Distributions tells us that this number is

$$D_{\pi_1,\alpha} D_{\pi_2,\alpha} \dots D_{s,\alpha} D_{\chi_1,\beta} D_{\chi_2,\beta} \dots D_{\Sigma\pi-\Sigma\chi+s,\beta} h_{\rho_1,\alpha\beta} h_{\rho_2,\alpha\beta} \dots h_{\Sigma\pi-\Sigma\rho+s,\alpha\beta}.$$

Taking as an example the group $\{(3), (1^2); (1^2)\}$ of the tripartite (322), the calculation may be carried out in the following manner:

Group	s	No. of Parts or fewer	π_1	χ_1	ρ_1	Number of Partitions
$\{(3), (1^2), (1^2)\}$	1	2	1	2	2	$D_{1,\alpha}^2 D_{2,\beta}$ $h_{2,\alpha\beta}$
,,	2	3	1	2	2	$D_{1,\alpha} D_{2,\alpha} D_{2,\beta} D_{1,\beta} h_{2,\alpha\beta} h_{1,\alpha\beta}$
,,	3	4	1	2	2	$D_{1,\alpha} D_{3,\alpha} D_{2,\beta}^2$ $h_{2,\alpha\beta}^2$
,,	4	5	1	2	2	$D_{1,\alpha} D_{4,\alpha} D_{2,\beta} D_{3,\beta} h_{2,\alpha\beta} h_{3,\alpha\beta}$

$$D_{1,\alpha}^2 D_{2,\beta} h_{2,\alpha\beta} = D_{1,\alpha}^2 h_{2,\alpha} = 1.$$

$$D_{1,\alpha} D_{2,\alpha} D_{2,\beta} D_{1,\beta} h_{2,\alpha\beta} h_{1,\alpha\beta}$$

$$= D_{1,\alpha} D_{2,\alpha} D_{1,\beta} (h_{2,\alpha} h_{1,\alpha\beta} + h_{1,\alpha}^2 h_{1,\alpha\beta}),$$

$$= D_{1,\alpha} D_{2,\alpha} (h_{2,\alpha} h_{1,\alpha} + h_{1,\alpha}^3), \quad = D_{2,\alpha} (h_{1,\alpha}^2 + h_{2,\alpha} + 3h_{1,\alpha}^2), \quad = 5.$$

$$D_{1,\alpha} D_{3,\alpha} D_{2,\beta}^2 h_{2,\alpha\beta}^2$$

$$= D_{1,\alpha} D_{3,\alpha} D_{2,\beta} (h_{2,\alpha} h_{2,\alpha\beta} + h_{1,\alpha} h_{1,\alpha\beta} h_{1,\alpha} h_{1,\alpha\beta} + h_{2,\alpha\beta} h_{2,\alpha}),$$

$$= D_{1,\alpha} D_{3,\alpha} D_{2,\beta} (2h_{2,\alpha} h_{2,\alpha\beta} + h_{1,\alpha}^2 h_{1,\alpha\beta}^2),$$

$$= D_{1,\alpha} D_{3,\alpha} (2h_{2,\alpha}^2 + h_{1,\alpha}^4), \quad = D_{3,\alpha} (4h_{1,\alpha} h_{2,\alpha} + 4h_{1,\alpha}^3), \quad = 8.$$

$$D_{1,\alpha} D_{4,\alpha} D_{2,\beta} D_{3,\beta} h_{2,\alpha\beta} h_{3,\alpha\beta}$$

$$= D_{1,\alpha} D_{4,\alpha} D_{2,\beta} (h_{2,\alpha} h_{1,\alpha} h_{2,\alpha\beta} + h_{1,\alpha} h_{1,\alpha\beta} h_{2,\alpha} h_{1,\alpha\beta} + h_{2,\alpha\beta} h_{3,\alpha}),$$

$$= D_{1,\alpha} D_{4,\alpha} (h_{2,\alpha} h_{1,\alpha} h_{2,\alpha} + h_{1,\alpha} h_{2,\alpha} h_{1,\alpha}^2 + h_{2,\alpha} h_{3,\alpha}),$$

$$= D_{4,\alpha} (h_{1,\alpha}^2 h_{2,\alpha} + h_{2,\alpha}^2 + h_{2,\alpha} h_{1,\alpha}^2 + 3h_{2,\alpha} h_{1,\alpha}^2 + h_{1,\alpha}^4 + h_{1,\alpha} h_{3,\alpha} + h_{2,\alpha}^2), \quad = 9.$$

We thus obtain the numbers 1, 5, 8, 9 and derive from them the numbers

$$1,$$
$$5 - 1 = 4,$$
$$8 - 5 = 3,$$
$$9 - 8 = 1,$$

which as shewn in the last Article enumerate the partitions of the group into exactly 2, 3, 4, 5 parts respectively.

234. When the parts of the partitions are required to be different from one another we adopt the notation and definitions:

$$a_{1,\alpha} = (1)_\alpha, \quad a_{2,\alpha} = (1^2)_\alpha, \quad \ldots \; a_{s,\alpha} = (1^s)_\alpha, \quad \ldots,$$

and obtain

$$a_{1,\alpha\beta} = a_{1,\alpha}\, a_{1,\beta},$$

$$a_{2,\alpha\beta} = (2)_\alpha\, a_{2,\beta} + (1^2)_\alpha\, a_{1,\beta}^2 = (2)_\beta\, a_{2,\alpha} + (1^2)_\beta\, a_{1,\alpha}^2,$$

$$a_{3,\alpha\beta} = (3)_\alpha\, a_{3,\beta} + (21)_\alpha\, a_{2,\beta}\, a_{1,\beta} + (1^3)_\alpha\, a_{1,\beta}^3 = (3)_\beta\, a_{3,\alpha} + (21)_\beta\, a_{2,\alpha}\, a_{1,\alpha} + (1^3)_\beta\, a_{1,\alpha}^3,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{p,\alpha\beta} = \Sigma (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\alpha\, a_{p_1,\beta}^{\pi_1} a_{p_2,\beta}^{\pi_2} \ldots = \Sigma (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\beta\, a_{p_1,\alpha}^{\pi_1} a_{p_2,\alpha}^{\pi_2} \ldots.$$

The substitution of the symbol a for h ensures that in the distributions considered no two triads of objects can be identical; this in turn ensures that no two parts of the corresponding partitions can be identical.

Previous reasoning shews that the number of these special partitions, which appertain to the group

$$\{(p_1^{\pi_1} p_2^{\pi_2} \ldots),\ (q_1^{\chi_1} q_2^{\chi_2} \ldots),\ (r_1^{\rho_1} r_2^{\rho_2} \ldots)\},$$

is
$$D_{\pi_1, a} D_{\pi_2, a} \ldots D_{\Sigma\chi+\Sigma\rho, a} \cdot D_{\chi_1, \beta} D_{\chi_2, \beta} \ldots D_{\Sigma\pi+\Sigma\rho, \beta}\, a_{\rho_1, a\beta}\, a_{\rho_2, a\beta} \ldots.$$

To evaluate this expression we note that

$$D_{s, a}\, a_{m, a\beta} = a_{s, \beta}\, a_{m-s, a\beta},$$

$$D_{s, \beta}\, a_{m, a\beta} = a_{s, a}\, a_{m-s, a\beta},$$

results that enable us to reduce the operand to a function of the functions

$$a_{s, a}, \quad a_{s, \beta}.$$

$D_{s, a}$ and $D_{s, \beta}$ then merely operate through the partition of s which is composed wholly of units.

Ex. gr. $D_{3, a}\, a_{1, a}\, a_{2, a}\, a_{3, a}\, a_{4, a} = D_{3, a}\, (1)_a\, (1^2)_a\, (1^3)_a\, (1^4)_a,$

$= \bullet\, (1)_a\, (1^2)_a\, (1^4)_a + \bullet\, (1)_a\, (1^3)_a\, (1^3)_a + \bullet\, (1^2)_a\, (1^2)_a\, (1^3)_a + (1)_a\, (1)_a\, (1^2)_a\, (1^3)_a,$

$= a_{1, a}\, a_{2, a}\, a_{4, a} + a_{1, a}\, a_{3, a}^2 + a_{2, a}^2\, a_{3, a} + a_{1, a}^2\, a_{2, a}\, a_{3, a},$

the parts 1, 1, 1 of the partition (111) of the number 3 (the subscript of $D_{3, a}$) being picked out in all possible ways from the factors of the operand.

In the partitions which are enumerated by the expression above written it must be carefully noted that the bipartite zero 000 can only occur once as a part.

Take again the particular case of the tripartite number (322).

Group	π_1	π_2	$\Sigma\pi$	χ_1	$\Sigma\chi$	ρ_1	$\Sigma\rho$	Number of Partitions
$\{(3), (2), (2)\}$	1	0	1	1	1	1	1	$D_{1, a}\, D_{2, a}\, D_{1, \beta}\, D_{2, \beta}\, a_{1, a\beta}\, a_{2, a\beta} = 4,$
$\{(3), (2), (1^2)\}$	1	0	1	1	1	2	2	$D_{1, a}\, D_{3, a}\, D_{1, \beta}\, D_{3, \beta}\, a_{2, a\beta}^2,$
$\{(3), (1^2), (2)\}$	1	0	1	2	2	1	1	$D_{1, a}\, D_{3, a}\, D_{2, \beta}^2 \quad a_{1, a\beta}\, a_{3, a\beta},$
$\{(3), (1^2), (1^2)\}$	1	0	1	2	2	2	2	$D_{1, a}\, D_{4, a}\, D_{2, \beta}\, D_{3, \beta}\, a_{2, a\beta}\, a_{3, a\beta},$
$\{(21), (2), (2)\}$	1	1	2	1	1	1	1	$D_{1, a}^2\, D_{2, a}\, D_{1, \beta}\, D_{3, \beta}\, a_{1, a\beta}\, a_{3, a\beta},$
$\{(21), (2), (1^2)\}$	1	1	2	1	1	2	2	$D_{1, a}^2\, D_{3, a}\, D_{1, \beta}\, D_{4, \beta}\, a_{2, a\beta}\, a_{3, a\beta},$
$\{(21), (1^2), (2)\}$	1	1	2	2	2	1	1	$D_{1, a}^2\, D_{3, a}\, D_{2, \beta}\, D_{3, \beta}\, a_{1, a\beta}\, a_{4, a\beta},$
$\{(21), (1^2), (1^2)\}$	1	1	2	2	2	2	2	$D_{1, a}^2\, D_{4, a}\, D_{2, \beta}\, D_{4, \beta}\, a_{2, a\beta}\, a_{4, a\beta},$
$\{(1^3), (2), (2)\}$	3	0	3	1	1	1	1	$D_{3, a}\, D_{2, a}\, D_{1, \beta}\, D_{4, \beta}\, a_{1, a\beta}\, a_{4, a\beta},$
$\{(1^3), (2), (1^2)\}$	3	0	3	1	1	2	2	$D_{3, a}^2 \quad D_{1, \beta}\, D_{5, \beta}\, a_{2, a\beta}\, a_{4, a\beta},$
$\{(1^3), (1^2), (2)\}$	3	0	3	2	2	1	1	$D_{3, a}^2 \quad D_{2, \beta}\, D_{4, \beta}\, a_{1, a\beta}\, a_{5, a\beta},$
$\{(1^3), (1^3), (1^2)\}$	3	0	3	2	2	2	2	$D_{3, a}\, D_{4, a}\, D_{2, \beta}\, D_{5, \beta}\, a_{2, a\beta}\, a_{5, a\beta} = 0.$

In each group the number of parts in the partitions is $\Sigma\pi + \Sigma\chi + \Sigma\rho$.

Thus in the case of the group $\{(3), (2), (2)\}$ the partitions have three parts. The number of them is found by calculation to be four. They are

$$(320, 002, 000),$$
$$(302, 020, 000),$$
$$(300, 022, 000),$$
$$(300, 020, 002).$$

On the other hand for the group $\{(1^3), (1^2), (1^2)\}$ the partitions would have seven parts, but calculation shews that none exist.

235. The subject is better elucidated by considering, for each group, the partitions which have a given definite number of parts.

Following the path of the bipartite case we find that the partitions into $\Sigma\pi + s$ parts are given by the expression

$$D_{\pi_1,\alpha} D_{\pi_2,\alpha} \dots D_{s,\alpha} D_{\chi_1,\beta} D_{\chi_2,\beta} \dots D_{\Sigma\pi-\Sigma\chi+s,\beta}\, a_{\rho_1,\alpha\beta}\, a_{\rho_2,\alpha\beta} \dots a_{\Sigma\pi-\Sigma\rho+s,\alpha\beta},$$

and the scheme for a group is as follows:

Group	s	No. of Parts	No. of Partitions	Evaluation
$\{(3), (2), (2)\}$	0	1	$D_{1,\alpha} D_{1,\beta}\, a_{1,\alpha\beta}$	1
„	1	2	$D^2_{1,\alpha} D^2_{1,\beta}\, a^2_{1,\alpha\beta}$	4
„	2	3	$D_{1,\alpha} D_{2,\alpha} D_{1,\beta} D_{2,\beta}\, a_{1,\alpha\beta}\, a_{2,\alpha\beta}$	3
„	3	4	$D_{1,\alpha} D_{3,\alpha} D_{1,\beta} D_{3,\beta}\, a_{1,\alpha\beta}\, a_{3,\alpha\beta}$	0

The partitions enumerated by the numbers 1, 4, 3 are

$$(322),$$
$$(322, 000),\quad (320, 002),\quad (302, 020),\quad (300, 022),$$
$$(320\ 002, 000),\quad (302, 020, 000),\quad (300, 022, 000).$$

If the number of the partitions which involve s parts be denoted by P_s, we have

$$P_1 = 1,\quad P_2 = 4,\quad P_3 = 3,\quad P_4 = 0.$$

The number P_s evidently enumerates

(i) the partitions involving s different parts, the part 000 being excluded;

(ii) the partitions involving $s-1$ different parts, the part 000 being excluded, together with one additional part 000.

Hence if Q_s denotes the number of the partitions which do not involve one part 000, we have

$$P_s = Q_s + Q_{s-1}.$$

Thus
$$Q_1 = P_1,$$
$$Q_2 = P_2 - P_1,$$
$$Q_3 = P_3 - P_2 + P_1,$$
$$\ldots\ldots\ldots\ldots$$

and $$Q_1 + Q_2 + Q_3 + \ldots = P_1 + P_3 + P_5 + \ldots = P_2 + P_4 + P_6 + \ldots.$$

In the above example
$$Q_1 = 1,$$
$$Q_2 = 3,$$
$$Q_3 = 0,$$
$$Q_1 + Q_2 = 4 = P_1 + P_3 = P_2 + P_4 = \tfrac{1}{2}(P_1 + P_2 + P_3 + P_4).$$

In general $$\Sigma Q = \tfrac{1}{2}\Sigma P.$$

Take as another example the group $\{(1^3), (1^2), (1^2)\}$.

Group	s	No. of Parts	No. of Partitions	Evaluation
$\{(1^3), (1^2), (1^2)\}$	0	3	$D_{3,\alpha} D_{2,\beta} D_{1,\beta} a_{2,\alpha\beta} a_{1,\alpha\beta}$	1
,,	1	4	$D_{3,\alpha} D_{1,\alpha} D^2_{2,\beta} a^2_{2,\alpha\beta}$	4
,,	2	5	$D_{3,\alpha} D_{2,\alpha} D_{2,\beta} D_{3,\beta} a_{2,\alpha\beta} a_{3,\alpha\beta}$	4
,,	3	6	$D^2_{3,\alpha} D_{2,\beta} D_{4,\beta} \quad a_{2,\alpha\beta} a_{4,\alpha\beta}$	1

Here $$P_3 = 1, \quad P_4 = 4, \quad P_5 = 4, \quad P_6 = 1,$$
$$Q_3 = 1, \quad Q_4 = 3, \quad Q_5 = 1.$$

The partitions enumerated by Q_3, Q_4 and Q_5 are

(111, 110, 101),

(111, 110, 100, 001), (111, 101, 100, 010), (110, 101, 011, 100),

(110, 101, 100, 010, 001).

236. The theory in respect of multipartite numbers in general is now clear. We add to the foregoing defining relations further sets. The first one is

$$h_{1,\alpha\beta\gamma} = (1)_\gamma h_{1,\alpha\beta},$$
$$h_{2,\alpha\beta\gamma} = (2)_\gamma h_{2,\alpha\beta} + (1^2)_\gamma h^2_{1,\alpha\beta},$$
$$h_{3,\alpha\beta\gamma} = (3)_\gamma h_{3,\alpha\beta} + (21)_\gamma h_{2,\alpha\beta} h_{1,\alpha\beta} + (1^3)_\gamma h^3_{1,\alpha\beta},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_{p,\alpha\beta\gamma} = \Sigma (p_1^{\pi_1} p_2^{\pi_2} \ldots)_\gamma h^{\pi_1}_{p_1,\alpha\beta} h^{\pi_2}_{p_2,\alpha\beta} \ldots,$$

and we note that in these relations the symbols α, β, γ may be permuted in any manner.

Also as before
$$D_{s,\alpha} h_{m,\alpha\beta\gamma} = h_{s,\beta\gamma} h_{m-s,\alpha\beta\gamma},$$
$$D_{s,\beta} h_{m,\alpha\beta\gamma} = h_{s,\alpha\gamma} h_{m-s,\alpha\beta\gamma},$$
$$D_{s,\gamma} h_{m,\alpha\beta\gamma} = h_{s,\alpha\beta} h_{m-s,\alpha\beta\gamma}.$$

We add further sets

$$h_{p,\alpha\beta\gamma\delta} = \Sigma' (p_1^{\pi_1} p_2^{\pi_2} \dots)_\delta h_{p_1,\alpha\beta\gamma}^{\pi_1} h_{p_2,\alpha\beta\gamma}^{\pi_2} \dots,$$

.....

$$h_{p,\alpha\beta\gamma\dots\nu} = \Sigma' (p_1^{\pi_1} p_2^{\pi_2} \dots)_\nu h_{p_1,\alpha\beta\dots\mu}^{\pi_1} h_{p_2,\alpha\beta\dots\mu}^{\pi_2} \dots,$$

and observe the formulæ of operation

$$D_{s,\alpha} h_{m,\alpha\beta\gamma\dots\nu} = h_{s,\beta\gamma\dots\nu} h_{m-s,\alpha\beta\gamma\dots\nu},$$

etc.

We have then for the group

$$\{(p_1^{\pi_1} p_2^{\pi_2} \dots), (q_1^{\chi_1} q_2^{\chi_2} \dots), \dots (z_1^{\zeta_1} z_2^{\zeta_2} \dots)\}$$

of the multipartite number $pq \dots z$, partitions enumerated by

$$D_{\pi_1,\alpha} D_{\pi_2,\alpha} \dots D_{\Sigma\chi+\dots+\Sigma\zeta,\alpha} D_{\chi_1,\beta} D_{\chi_2,\beta} \dots D_{\Sigma\pi+\Sigma\rho+\dots} \dots h_{\zeta_1,\alpha\beta\gamma\dots} h_{\zeta_2,\alpha\beta\gamma\dots} \dots h_{\Sigma\pi+\dots+\Sigma\eta}.$$

The enumeration of the partitions with a definite number of parts follows in the general case the procedure that has been set out at length for bipartite and tripartite numbers.

237. The functions $h_{m,\alpha\beta}$, $h_{m,\alpha\beta\gamma}$, ...

which present themselves in the investigation possess some elegant properties.

The relations

$$h_{1,\alpha\beta} = (1)_\beta h_{1,\alpha},$$

$$h_{2,\alpha\beta} = (2)_\beta h_{2,\alpha} + (1^2)_\alpha h_{1,\alpha}^2,$$

etc.

are such that the series

$$1 + h_{1,\alpha\beta} + h_{2,\alpha\beta} + \dots$$

can be broken up into factors of the form

$$1 + \beta_s h_{1,\alpha} + \beta_s^2 h_{2,\alpha} + \beta_s^3 h_{3,\alpha} + \dots,$$

where $\beta_1, \beta_2, \beta_3, \dots$ is the set of numerical quantities of which the partitions in brackets $(\ )_\beta$ are symmetric functions. In fact introducing an arbitrary quantity μ we may write

$$1 + \mu h_{1,\alpha\beta} + \mu^2 h_{2,\alpha\beta} + \mu^3 h_{3,\alpha\beta} + \dots$$

$$= (1 + \mu\beta_1 h_{1,\alpha} + \mu^2 \beta_1^2 h_{2,\alpha} + \mu^3 \beta_1^3 h_{3,\alpha} + \dots)$$

$$\times (1 + \mu\beta_2 h_{1,\alpha} + \mu^2 \beta_2^2 h_{2,\alpha} + \mu^3 \beta_2^3 h_{3,\alpha} + \dots)(\ ) \dots.$$

Take logarithms of both sides, expand and equate coefficients of like powers of μ and we obtain the relations

$$h_{1,\alpha\beta} = (1)_\beta h_{1,\alpha},$$

$$h_{1,\alpha\beta}^2 - 2h_{2,\alpha\beta} = (2)_\beta (h_{1,\alpha}^2 - 2h_{2,\alpha}),$$

$$h_{1,\alpha\beta}^3 - 3h_{2,\alpha\beta} h_{1,\alpha\beta} + 3h_{3,\alpha\beta} = (3)_\beta (h_{1,\alpha}^3 - 3h_{2,\alpha} h_{1,\alpha} + 3h_{3,\alpha}),$$

.....

relations which may be written:

$$h_{1,\alpha\beta} = + h_{1,\beta} h_{1,\alpha},$$

$$h^2_{1,\alpha\beta} - 2h_{2,\alpha\beta} = - (h^2_{1,\beta} - 2h_{2,\beta})(h^2_{1,\alpha} - 2h_{2,\alpha}),$$

$$h^3_{1,\alpha\beta} - 3h_{2,\alpha\beta} h_{1,\alpha\beta} + 3h_{3,\alpha\beta} = + (h^3_{1,\beta} - 3h_{2,\beta} h_{1,\beta} + 3h_{3,\beta})(h^3_{1,\alpha} - 3h_{2,\alpha} h_{1,\alpha} + 3h_{3,\alpha}),$$

etc.,

the signs on the dexter being alternately positive and negative and the coefficients in the factors following the law which appears when the sums of the powers of a set of numerical quantities are expressed in terms of the elementary symmetric functions.

Similarly
$$h_{1,\alpha\beta\gamma} = h_{1,\gamma} h_{1,\alpha\beta},$$

$$h^2_{1,\alpha\beta\gamma} - 2h_{2,\alpha\beta\gamma} = - (h^2_{1,\gamma} - 2h_{2,\gamma})(h^2_{1,\alpha\beta} - 2h_{2,\alpha\beta}),$$

$$h^3_{1,\alpha\beta\gamma} - 3h_{2,\alpha\beta\gamma} h_{1,\alpha\beta\gamma} + 3h_{3,\alpha\beta\gamma}$$
$$= + (h^3_{1,\gamma} - 3h_{2,\gamma} h_{1,\gamma} + 3h_{3,\gamma})(h^3_{1,\alpha\beta} - 3h_{2,\alpha\beta} h_{1,\alpha\beta} + 3h_{3,\alpha\beta}),$$

etc.

and similarly also for the functions $h_{m,\alpha\beta\gamma\delta}$, $h_{m,\alpha\beta\gamma\delta\epsilon}$, ...

Combining results we find that

$$h_{1,\alpha\beta\gamma\ldots\nu} = h_{1,\alpha} h_{1,\beta} h_{1,\gamma} \ldots h_{1,\nu},$$

$$h^2_{1,\alpha\beta\gamma\ldots\nu} - 2h_{2,\alpha\beta\gamma\ldots\nu} = (-)^{\nu+1}(h^2_{1,\alpha} - 2h_{2,\alpha})(h^2_{1,\beta} - 2h_{2,\beta})(h^2_{1,\gamma} - 2h_{2,\gamma}) \ldots (h^2_{1,\nu} - 2h_{2,\nu}),$$

$$h^3_{1,\alpha\beta\ldots\nu} - 3h_{2,\alpha\beta\ldots\nu} h_{1,\alpha\beta\ldots\nu} + 3h_{3,\alpha\beta\ldots\nu}$$
$$= (h^3_{1,\alpha} - 3h_{2,\alpha} h_{1,\alpha} + 3h_{3,\alpha})(h^3_{1,\beta} - 3h_{2,\beta} h_{1,\beta} + 3h_{3,\beta}) \ldots (h^3_{1,\nu} - 3h_{2,\nu} h_{1,\nu} + 3h_{3,\nu}),$$

etc.,

the signs being alternately positive and $(-)^{\nu+1}$.

Partitions into dissimilar parts are treated by the function

$$a_{s,\alpha\beta\ldots\nu}$$

and in the above formulæ for computation it is merely necessary to substitute the symbol a for the symbol h.

TABLES

Products of Homogeneous Product Sums in terms of Monomial Symmetric Functions.

	(1)
h_1	1

	(2)	(1^2)
h_2	1	1
h_1^2	1	2

	(3)	(21)	(1^3)
h_3	1	1	1
h_2h_1	1	2	3
h_1^3	1	3	6

	(4)	(31)	(2^2)	(21^2)	(1^4)
h_4	1	1	1	1	1
h_3h_1	1	2	2	3	4
h_2^2	1	2	3	4	6
$h_2h_1^2$	1	3	4	7	12
h_1^4	1	4	6	12	24

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
h_5	1	1	1	1	1	1	1
h_4h_1	1	2	2	3	3	4	5
h_3h_2	1	2	3	4	5	7	10
$h_3h_1^2$	1	3	4	7	8	13	20
$h_2^2h_1$	1	3	5	8	11	18	30
$h_2h_1^3$	1	4	7	13	18	33	60
h_1^5	1	5	10	20	30	60	120

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
h_6	1	1	1	1	1	1	1	1	1	1	1
h_5h_1	1	2	2	3	2	3	4	3	4	5	6
h_4h_2	1	2	3	4	3	5	7	6	8	11	15
$h_4h_1^2$	1	3	4	7	4	8	13	9	14	21	30
h_3^2	1	2	3	4	4	6	8	7	10	14	20
$h_3h_2h_1$	1	3	5	8	6	12	19	15	24	38	60
$h_3h_1^3$	1	4	7	13	8	19	34	24	42	72	120
h_2^3	1	3	6	9	7	15	24	21	33	54	90
$h_2^3h_1^2$	1	4	8	14	10	24	42	33	58	102	180
$h_2h_1^4$	1	5	11	21	14	38	72	54	102	192	360
h_1^6	1	6	15	30	20	60	120	90	180	360	720

Thus $h_2h_1 = (3) + 2\,(21) + 3\,(1^3)$.

Products of Elementary Symmetric Functions in terms of Monomial Symmetric Functions.

	(1)
a_1	1

	(2)	(1^2)
a_2		1
a_1^2	1	2

	(3)	(21)	(1^3)
a_3			1
a_{21}		1	3
a_1^3	1	3	6

	(4)	(31)	(2^2)	(21^2)	(1^4)
a_4					1
a_3a_1				1	4
a_2^2			1	2	6
$a_2a_1^2$		1	2	5	12
a_1^4	1	4	6	12	24

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
a_5							1
a_4a_1						1	5
a_3a_2					1	3	10
$a_3a_1^2$				1	2	7	20
$a_2^2a_1$			1	2	5	12	30
$a_2a_1^3$		1	3	7	12	27	60
a_1^5	1	5	10	20	30	60	120

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
a_6											1
a_5a_1										1	6
a_4a_2									1	4	15
$a_4a_1^2$							1		2	9	30
a_3^2								1	2	6	20
$a_3a_2a_1$						1	3	3	8	22	60
$a_3a_1^3$				1		3	10	6	18	48	120
a_2^3					1	3	6	6	15	36	90
$a_2^2a_1^2$			1	2	2	8	18	15	34	78	180
$a_2a_1^4$		1	4	9	6	22	48	36	78	168	360
a_1^6	1	6	15	30	20	60	120	90	180	360	720

Thus $$a_4a_1^2 = (31^3) + 2(2^21^2) + 9(21^4) + 30(1^6).$$

Monomial Symmetric Functions in terms of Products of Elementary Symmetric Functions.

	a_1
(1)	1

	a_2	a_1^2
(2)	$\bar{2}$	1
(1^2)	1	

	a_3	a_2a_1	a_1^3
(3)	3	$\bar{3}$	1
(21)	$\bar{3}$	1	
(1^3)	1		

	a_4	a_3a_1	a_2^2	$a_2a_1^2$	a_1^4
(4)	$\bar{4}$	4	2	$\bar{4}$	1
(31)	4	1	$\bar{2}$	1	
(2^2)	2	2	1		
(21^2)	$\bar{4}$	1			
(1^4)	1				

	a_5	a_4a_1	a_3a_2	$a_3a_1^2$	$a_2^2a_1$	$a_2a_1^3$	a_1^5
(5)	5	$\bar{5}$	$\bar{5}$	5	5	$\bar{5}$	1
(41)	$\bar{5}$	1	5	$\bar{1}$	$\bar{3}$	1	
(32)	$\bar{5}$	5	$\bar{1}$	$\bar{2}$	1		
(31^2)	5	$\bar{1}$	$\bar{2}$	1			
(2^21)	5	$\bar{3}$	1				
(21^3)	$\bar{5}$	1					
(1^5)	1						

	a_6	a_5a_1	a_4a_2	$a_4a_1^2$	a_3^2	$a_3a_2a_1$	$a_3a_1^3$	a_2^3	$a_2^2a_1^2$	$a_2a_1^4$	a_1^6
(6)	$\bar{6}$	6	6	$\bar{6}$	3	$\bar{1}\bar{2}$	6	$\bar{2}$	9	$\bar{6}$	1
(51)	6	$\bar{1}$	$\bar{6}$	1	$\bar{3}$	7	$\bar{1}$	2	$\bar{4}$	1	
(42)	6	$\bar{6}$	2	2	$\bar{3}$	4	$\bar{2}$	2	1		
(41^2)	$\bar{6}$	1	2	1	3	$\bar{3}$	1				
(3^2)	3	$\bar{3}$	$\bar{3}$	3	3	$\bar{3}$		1			
(321)	$\bar{1}\bar{2}$	7	4	$\bar{3}$	$\bar{3}$	1					
(31^3)	6	$\bar{1}$	$\bar{2}$	1							
(2^3)	$\bar{2}$	2	$\bar{2}$		1						
(2^21^2)	9	$\bar{4}$	1								
(21^4)	$\bar{6}$	1									
(1^6)	1										

Thus $$(2^21^2) = 9a_6 - 4a_5a_1 + a_4a_2.$$

The simplest cases of Distribution Functions.

Number of Objects	Number of Parcels	Type of Parcels	Distribution Function
1	1	(1)	h_1
2	1	(1)	h_2
,,	2	(2)	h_2
,,	2	(1^2)	h_1^2
3	1	(1)	h_3
,,	2	(2)	$h_2 h_1$
,,	2	(1^2)	$2h_2 h_1$
,,	3	(3)	h_3
,,	3	(21)	$h_2 h_1$
,,	3	(1^3)	h_1^3
4	1	(1)	h_4
,,	2	(2)	$h_4 + h_2^2$
,,	2	(1^2)	$2h_3 h_1 + h_2^2$
,,	3	(3)	h_2^2
,,	3	(21)	$h_2^2 + h_2 h_1^2$
,,	3	(1^3)	$3h_2 h_1^2$
,,	4	(4)	h_4
,,	4	(31)	$h_3 h_1$
,,	4	(2^2)	h_2^2
,,	4	(21^2)	$h_2 h_1^2$
,,	4	(1^4)	h_1^4
5	1	(1)	h_5
,,	2	(2)	$h_4 h_1 + h_3 h_2$
,,	2	(1^2)	$2h_4 h_1 + 2h_3 h_2$
,,	3	(3)	$h_4 h_1 + h_3 h_2 - h_3 h_1^2 + h_2^2 h_1$
,,	3	(21)	$h_4 h_1 + h_3 h_2 + 2h_2^2 h_1$
,,	3	(1^3)	$3h_3 h_1^2 + 3h_2^2 h_1$
,,	4	(4)	$h_3 h_2$
,,	4	(31)	$h_3 h_2 + h_2^2 h_1$
,,	4	(2^2)	$2h_2^2 h_1$
,,	4	(21^2)	$2h_2^2 h_1 + h_2 h_1^3$
,,	4	(1^4)	$4h_2 h_1^3$
6	1	(1)	h_6
,,	2	(2)	$h_6 + 2h_4 h_2$
,,	2	(1^2)	$2h_5 h_1 + 2h_4 h_2 + h_3^2$
,,	3	(3)	$h_6 - h_5 h_1 + h_4 h_2 + h_4 h_1^2 + h_3^2 - h_3 h_2 h_1 + h_2^3$
,,	3	(21)	$2h_4 h_2 + h_4 h_1^2 + 2h_3 h_2 h_1 + h_2^3$
,,	3	(1^3)	$3h_4 h_1^2 + 6h_3 h_2 h_1 + h_2^3$
,,	4	(4)	$h_4 h_2 + h_3^2 - h_3 h_2 h_1 + h_2^3$
,,	4	(31)	$h_4 h_1^2 + h_3^2 + h_3 h_2 h_1 - h_3 h_1^3 + h_2^3 + h_2^2 h_1^2$
,,	4	(2^2)	$2h_4 h_2 + 2h_3^2 + h_2^3 h_1^2$
,,	4	(21^2)	$h_4 h_1^2 + 2h_3 h_2 h_1 + h_2^3 + 3h_2^2 h_1^2$
,,	4	(1^4)	$4h_3 h_1^3 + 6h_2^2 h_1^2$
,,	5	(5)	$h_4 h_2$
,,	5	(41)	$h_4 h_2 + h_3 h_2 h_1$
,,	5	(32)	$h_3 h_2 h_1 + h_2^3$
,,	5	(31^2)	$2h_3 h_2 h_1 + h_2^3 h_1^2$
,,	5	$(2^2 1)$	$h_2^3 + 2h_2^3 h_1^2$
,,	5	(21^3)	$3h_2^2 h_1^2 + h_2 h_1^4$
,,	5	(1^5)	$5h_2 h_1^4$

Symmetrical Tables of Binomial Coefficients. $(0^s) = \binom{n}{s}$.

	(0)	(0^2)
$(0)^2$	1	2
(0^2)		1

	(0)	(0^2)	(0^3)
$(0)^3$	1	6	6
$2\,(0^2)\,(0)$		4	6
(0^3)			1

	(0)	(0^2)	(0^3)	(0^4)
$(0)^4$	1	14	36	24
$3\,(0^2)\,(0)^2$		12	45	36
$(0^2)^2+2\,(0^3)\,(0)$		1	12	14
(0^4)				1

	$(0)^4$	$3\,(0^2)\,(0)^2$	$(0^2)^2+2\,(0^3)\,(0)$	(0^4)
(0)	1	$-\frac{4}{3}$	2	-4
(0^2)		$\frac{4}{33}$	$-\frac{5}{11}$	2
(0^3)		$-\frac{1}{99}$	$\frac{4}{33}$	$-\frac{4}{3}$
(0^4)				1

	(0)	(0^2)	(0^3)	(0^4)	(0^5)
$(0)^5$	1	30	150	240	120
$4\,(0^2)\,(0)^3$		32	228	432	240
$3\,(0^2)^2\,(0)+3\,(0^3)\,(0)^2$		6	90	228	150
$2\,(0^4)\,(0)+2\,(0^3)\,(0^2)$			6	32	30
(0^5)					1

	(0)	(0^2)	(0^3)	(0^4)	(0^5)	(0^6)
$(0)^6$	1	62	540	1560	1800	720
$5\,(0^2)\,(0)^4$		80	975	3300	4200	1800
$6\,(0^2)^2\,(0)^2+4\,(0^3)\,(0)^3$		24	522	2248	3300	1560
$(0^2)^3+6\,(0^3)\,(0^2)\,(0)+3\,(0^4)\,(0)^2$		1	78	522	975	540
$(0^3)^2+2\,(0^4)\,(0^2)+2\,(0^5)\,(0)$			1	24	80	62
(0^6)						1

These tables may be reversed. A specimen is given above for the order 4.

Thus $2\,(0^2)\,(0) = 4\,(0^2)+6\,(0^3)$

or $\binom{n}{2}\binom{n}{1} = 2\binom{n}{2}+3\binom{n}{3}$.

Enumeration of Compositions of Multipartite Numbers.

No.	1	part	Total
1	1		1

No.	1	2	parts	Total
2	1	1		2
$1\bar{1}$	1	2		3

No.	1	2	3	parts	Total
3	1	2	1		4
$\bar{2}\bar{1}$	1	4	3		8
$\bar{1}\bar{1}\bar{1}$	1	6	6		13

No.	1	2	3	4	parts	Total
4	1	3	3	1		8
$3\bar{1}$	1	6	9	4		20
$\bar{2}\bar{2}$	1	7	12	6		26
$\bar{2}\bar{1}\bar{1}$	1	10	21	12		44
$\bar{1}\bar{1}\bar{1}\bar{1}$	1	14	36	24		75

No.	1	2	3	4	5	parts	Total
5	1	4	6	4	1		16
$\bar{4}\bar{1}$	1	8	18	16	5		48
$\bar{3}\bar{2}$	1	10	27	28	10		76
$\bar{3}\bar{1}\bar{1}$	1	14	45	52	20		132
$\bar{2}\bar{2}\bar{1}$	1	16	57	72	30		176
$\bar{2}\bar{1}\bar{1}\bar{1}$	1	22	93	132	60		308
$\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}$	1	30	150	240	120		541

Thus the multipartite number $\bar{2}\bar{2}$ possesses 7 compositions into two parts, viz.

(20, 02), (02, 20), (21, 01), (01; 21), (12, 10), (10, 12), (11, 11).

No.	1	2	3	4	5	6	parts	Total
6	1	5	10	10	5	1		32
$\overline{51}$	1	10	30	40	25	6		112
$\overline{42}$	1	13	48	76	55	15		208
$\overline{411}$	1	18	78	136	105	30		368
$\overline{33}$	1	14	55	92	70	20		252
$\overline{321}$	1	22	111	220	190	60		604
$\overline{3111}$	1	30	177	388	360	120		1076
$\overline{222}$	1	25	138	294	270	90		818
$\overline{2211}$	1	34	219	516	510	180		1460
$\overline{21111}$	1	46	345	900	960	360		2612
$\overline{111111}$	1	62	540	1560	1800	720		4683

No.	1	2	3	4	5	6	7	parts	Total
7	1	6	15	20	15	6	1		64
$\overline{61}$	1	12	45	80	75	36	7		256
$\overline{52}$	1	16	75	160	175	96	21		544
$\overline{511}$	1	22	120	280	325	186	42		976
$\overline{43}$	1	18	93	216	255	150	35		768
$\overline{421}$	1	28	183	496	655	420	105		1888
$\overline{4111}$	1	38	288	856	1205	810	210		3408
$\overline{331}$	1	30	207	588	810	540	140		2316
$\overline{322}$	1	34	255	772	1120	780	210		3172
$\overline{3211}$	1	46	399	1324	2050	1500	420		5740
$\overline{31111}$	1	62	621	2260	3740	2880	840		10404
$\overline{2221}$	1	52	489	1728	2820	2160	630		7880
$\overline{22111}$	1	70	759	2940	5130	4140	1260		14300
$\overline{211111}$	1	94	1173	4980	9300	7920	2520		25988
$\overline{1111111}$	1	126	1806	8400	16800	15120	5040		47293

Tables of the values of $a_{pqr\ldots}$ *and* $h_{pqr\ldots}$. (N.B. $a_{pqr\ldots} = a_{\ldots rqp}$; $h_{pqr\ldots} = h_{\ldots rqp}$.)

	(2)	(1^2)
a_{11} or h_2	1	1
a_2 or h_{11}		1

	(3)	(21)	(1^3)
a_{111} or h_3	1	1	1
a_{21} or h_{21}		1	2
a_3 or h_{111}			1

	(4)	(31)	(2^2)	(21^2)	(1^4)
a_{1111} or h_4	1	1	1	1	1
a_{211} or h_{31}		1	1	2	3
a_{121} or h_{22}		1	2	3	5
a_{22} or h_{121}			1	2	5
a_{31} or h_{211}				1	3
a_4 or h_{1111}					1

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
a_{11111} or h_5	1	1	1	1	1	1	1
a_{2111} or h_{41}		1	1	2	2	3	4
a_{1211} or h_{32}		1	2	3	4	6	9
a_{212} or h_{131}			1	2	3	6	11
a_{221} or h_{221}			1	2	4	8	16
a_{311} or h_{311}				1	1	3	6
a_{131} or h_{212}				1	2	5	11
a_{32} or h_{1211}					1	3	9
a_{41} or h_{2111}						1	4
a_5 or h_{11111}							1

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
a_{111111} or h_6	1	1	1	1	1	1	1	1	1	1	1
a_{21111} or h_{51}		1	1	1	2	2	2	3	3	4	5
a_{12111} or h_{42}		1	2	2	3	4	5	6	7	10	14
a_{11211} or h_{33}		1	2	3	3	5	6	7	9	13	19
a_{2112} or h_{141}			1	1	2	3	4	6	7	12	19
a_{2121} or h_{231}			1	2	2	5	7	9	13	23	40
a_{1311} or h_{312}					1	2	3	5	7	14	26
a_{2211} or h_{321}			1	1	2	4	6	8	11	20	35
a_{1221} or h_{222}			1	2	2	6	10	11	18	33	61
a_{3111} or h_{411}					1	1	1	3	3	6	10
a_{411} or h_{3111}								1	1	4	10
a_{222} or h_{1221}				1		3	6	6	13	28	61
a_{321} or h_{2211}						1	2	3	6	15	35
a_{312} or h_{1311}						1	2	3	5	12	26
a_{231} or h_{2121}						1	3	3	7	17	40
a_{141} or h_{2112}								1	2	7	19
a_{33} or h_{11211}							1		2	6	19
a_{42} or h_{12111}									1	4	14
a_{51} or h_{21111}										1	5
a_6 or h_{111111}											1

Thus $a_{211} = a_{112} = h_{31} = h_{13} = (31) + (2^2) + 2(21^2) + 3(1^4)$.

Examples of Symmetric Tables associated with every Partition of every Number.

Weight 3. Partition (21).

	(3)	(21)
(21)		1
(2) (1)	1	1

	(21)	(2)(1)
(3)	$\bar{1}$	1
(21)	1	

Weight 4. Partition (21^2).

	(4)	(31)	$2(2^2)$	(21^2)
(21^2)				1
(21) (1)		1	1	2
(2) (1^2)		1		1
(2) $(1)^2$	1	2	1	2

	(21^2)	(21) (1)	(2) (1^2)	(2) $(1)^2$
(4)	1	$\bar{1}$	$\bar{1}$	1
(31)	1		1	
$2(2^2)$	$\bar{1}$	1	$\bar{1}$	
(21^2)	1			

Weight 4. Partition (2^2).

	(4)	(2^2)
(2^2)		1
$(2)^2$	1	2

	(2^2)	$(2)^2$
(4)	$\bar{2}$	1
(2^2)	1	

Weight 4. Partition (31).

	(4)	(31)
(31)		1
(3) (1)	1	1

	(31)	(3)(1)
(4)	$\bar{1}$	1
(31)	1	

Weight 5. Partition (21^3).

	(5)	(41)	(32)	(31^2)	$2(2^21)$	(21^3)
(21^3)						1
(21^2) (1)				1	1	3
(21) (1^2) + (1^3) (2)			1	3	1	4
(21) $(1)^2$		1	3	4	3	6
(2) (1^2) (1)		1	1	3	1	3
(2) $(1)^3$	1	3	4	6	3	6

	(21^3)	(21^2) (1)	(21) (1^2) + (1^3) (2)	(21) $(1)^2$	(2) (1^2) (1)	(2) $(1)^3$
(5)	$\bar{5}$	5	5	$\bar{5}$	$\bar{10}$	5
(41)	5		5		5	
(32)	5	$\bar{5}$	1	2	$\bar{2}$	
(31^2)	$\bar{5}$		2	$\bar{1}$	1	
$2(2^21)$	$\bar{10}$	5	$\bar{2}$	1	$\bar{1}$	
(21^3)	5					

Divide coefficients by 5.

Weight 5. Partition (2^21).

	(5)	(41)	(32)	(2^21)
(2^21)				1
(2^2) (1)			1	2
(21) (2)		1	1	2
$(2)^2$ (1)	1	1	2	2

	(2^21)	(2^2) (1)	(21) (2)	$(2)^2$ (1)
(5)	1	$\bar{1}$	$\bar{1}$	1
(41)	$\bar{1}$	$\bar{1}$	1	
(32)	$\bar{1}$	1		
(2^21)	1			

Thus $(31^2) = \frac{1}{5}\{-5(21^3) + 2(21)(1^2) + 2(1^3)(2) - (21)(1)^2 + (2)(1^2)(1)\}$.

Weight 6. Partition (21^4).

	(6)	(51)	(42)	$2\,(3^2)$	(41^2)	(321)	(31^3)	$3\,(2^3)$	$2\,(2^21^2)$	(21^4)
(21^4)										1
$(21^3)\,(1)$							1		1	4
$(21^2)\,(1^2)$ $+(1^4)\,(2)$						1	4	1	2	7
$(21)\,(1^3)$						1	3		1	4
$(21^2)\,(1)^2$					1	3	6	2	5	12
$(21)\,(1^2)\,(1)$ $+(1^3)\,(2)\,(1)$			1	1	3	6	13	2	6	16
$(21)\,(1)^3$		1	4	3	6	13	18	6	12	24
$(2)\,(1^2)^2$			1		2	2	6		2	6
$(2)\,(1^2)\,(1)^2$		1	2	1	5	6	12	2	5	12
$(2)\,(1)^4$	1	4	7	4	12	16	24	6	12	24

	(21^4)	$(21^3)\,(1)$	$(21^2)\,(1^2)$ $+(1^4)\,(2)$	$(21)\,(1^3)$	$(21^2)\,(1)^2$	$(21)\,(1^2)\,(1)$ $+(1^3)\,(2)\,(1)$	$(21)\,(1)^3$	$(2)\,(1^2)^2$	$(2)\,(1^2)\,(1)^2$	$(2)\,(1)^4$
(6)	30	$\bar{3}\bar{0}$	$\bar{3}\bar{0}$	$\bar{3}\bar{0}$	30	60	$\bar{3}\bar{0}$	30	$\bar{9}\bar{0}$	30
(51)	$\bar{3}\bar{0}$		30	30		$\bar{3}\bar{0}$		$\bar{3}\bar{0}$	30	
(42)	$\bar{3}\bar{0}$	30	$\bar{1}\bar{0}$	30	$\bar{1}\bar{0}$	$\bar{2}\bar{0}$	10	30	$\bar{1}\bar{0}$	
$2\,(3^2)$	$\bar{3}\bar{0}$	30	30	$\bar{6}\bar{0}$	$\bar{3}\bar{0}$	30		$\bar{3}\bar{0}$		
(41^2)	30		$\bar{1}\bar{0}$	$\bar{3}\bar{0}$	5	10	$\bar{5}$		5	
(321)	60	$\bar{3}\bar{0}$	$\bar{2}\bar{0}$	30	10	$\bar{4}$	2		$\bar{2}$	
(31^3)	$\bar{3}\bar{0}$		10		$\bar{5}$	2	$\bar{1}$		1	
$3\,(2^3)$	30	$\bar{3}\bar{0}$	30	$\bar{3}\bar{0}$						
$2\,(2^21^2)$	$\bar{9}\bar{0}$	30	$\bar{1}\bar{0}$		5	$\bar{2}$	1		$\bar{1}$	
(21^4)	30									

Divide coefficients by 30.

Weight 6. Partition $(2^2 1^2)$.

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	$3(2^3)$	$(2^2 1^2)$
$(2^2 1^2)$								1
$(2^2 1)\,(1)$						1	1	2
$(21^2)\,(2)$ $+(2^2)\,(1^2)$				1	1	2		3
$(21)^2$			1	2	2	2	2	4
$(2^2)\,(1)^2$			1	2		2	1	2
$(21)\,(2)\,(1)$		1	2	2	2	3	2	4
$(2)^2\,(1^2)$		1		2	1	2		2
$(2)^2\,(1)^2$	1	2	3	4	2	4	2	4

	$(2^2 1^2)$	$(2^2 1)\,(1)$	$(21^2)\,(2)$ $+(2^2)\,(1^2)$	$(21)^2$	$(2^2)\,(1)^2$	$(21)\,(2)\,(1)$	$(2)^2\,(1^2)$	$(2)^2\,(1)^2$
(6)	$\bar{1}\bar{0}$	10	10	5	$\bar{1}\bar{0}$	$\bar{2}\bar{0}$	$\bar{1}\bar{0}$	15
(51)	10	5	$\bar{1}\bar{0}$	$\bar{5}$	$\bar{5}$	5	10	
(42)	10	$\bar{1}\bar{0}$	2	$\bar{5}$	4	8	8	
(3^2)	5	$\bar{5}$	$\bar{5}$	5	5	$\bar{5}$	5	
(41^2)	$\bar{1}\bar{0}$	$\bar{5}$	4	5	$\bar{7}$	1	$\bar{1}$	
(321)	$\bar{2}\bar{0}$	5	8	$\bar{5}$	1	2	$\bar{2}$	
$3\,(2^3)$	$\bar{1}\bar{0}$	10	$\bar{8}$	5	$\bar{1}$	$\bar{2}$	2	
$(2^2 1^2)$	15							

Divide coefficients by 15.

Weight 6. Partition (2^3).

	(6)	(42)	(2^3)
(2^3)			1
$(2^2)\,(2)$		1	3
$(2)^3$	1	3	6

	(2^3)	$(2^2)(2)$	$(2)^3$
(6)	3	$\bar{3}$	1
(42)	$\bar{3}$	1	
(2^3)	1		

Weight 6. Partition (31^3).

	(6)	(51)	$2\,(3^2)$	(42)	(41^2)	(321)	(31^3)
(31^3)							1
$(31^2)\,(1)$					1	1	3
$(3)\,(1^3)$					1		1
$(31)\,(1^2)$				1	2	1	3
$(31)\,(1)^2$		1	1	2	4	3	6
$(3)\,(1^2)\,(1)$		1		1	3	1	3
$(3)\,(1)^3$	1	3	1	3	6	3	6

	(31^3)	$(31^2)\,(1)$	$(3)\,(1^3)$	$(31)\,(1^2)$	$(31)\,(1)^2$	$(3)\,(1^2)\,(1)$	$(3)\,(1)^3$
(6)	$\bar{1}$	1	1	1	$\bar{1}$	$\bar{2}$	1
(51)	1		$\bar{1}$	$\bar{1}$		1	
$2\,(3^2)$	1	$\bar{1}$	2	1	1	$\bar{1}$	
(42)	1	$\bar{1}$	$\bar{1}$	1			
(41^2)	$\bar{1}$		1				
(321)	2	1	$\bar{1}$				
(31^3)	1						

Weight 6. Partition (321).

	(6)	(51)	(42)	$2\,(3^2)$	(321)
(321)					1
$(32)\,(1)$			1	1	1
$(31)\,(2)$		1		1	1
$(3)\,(21)$		1	1		1
$(3)\,(2)\,(1)$	1	1	1	1	1

	(321)	$(32)\,(1)$	$(31)\,(2)$	$(3)\,(21)$	$(3)\,(2)\,(1)$
(6)	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	2
(51)	1	$\bar{1}$	1	1	
(42)	$\bar{1}$	1	$\bar{1}$	1	
$2\,(3^2)$	$\bar{1}$	1	1	$\bar{1}$	
(321)	2				

Divide coefficients by 2.

Weight 6. Partition (3^2).

	(6)	(3^2)
(3^2)		1
$(3)^2$	1	2

	(3^2)	$(3)^2$
(6)	$\bar{2}$	1
(3^2)	1	

Weight 6. Partition (41^2).

	(6)	(51)	(42)	(41^2)
(41^2)				1
(41) (1)		1	1	2
(4) (1^2)		1		1
(4) $(1)^2$	1	2	1	2

	(41^2)	(41) (1)	(4) (1^2)	(4) $(1)^2$
(6)	1	1	1	1
(51)	1		1	
(42)	$\bar{1}$	1	$\bar{1}$	
(41^2)	1			

Weight 6. Partition (42).

	(6)	(42)
(42)		1
(4) (2)	1	1

	(42)	(4)(2)
(6)	$\bar{1}$	1
(42)	1	

Weight 6. Partition (51).

	(6)	(51)
(51)		1
(5) (1)	1	1

	(51)	(5)(1)
(6)	$\bar{1}$	1
(51)	1	

CAMBRIDGE: PRINTED BY JOHN CLAY, M.A. AT THE UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
C. F. CLAY, MANAGER
London: FETTER LANE, E.C.
Edinburgh: 100 PRINCES STREET

New York: G. P. PUTNAM'S SONS
Bombay and Calcutta: MACMILLAN AND CO., LTD.
Toronto: J. M. DENT AND SONS, LTD.
Tokyo: THE MARUZEN-KABUSHIKI-KAISHA