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THE HIGHER TRIGONOMETRY.

SUPERRATIONALS OF SECOND ORDER.

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THE HIGHER TRIGONOMETRY.

SUPERRATIONALS OF SECOND ORDER.

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THE HIGHER TRIGONOMETRY.

1. To make this title intelligible, we must interpret the epithet *Lower* as applied to Trigonometry. Every one who enters the Differential Calculus learns the definitions and Differentiation of the Trigonometrical Functions; and from $d \cdot \sin x = \cos x dx$, with $d \cdot \cos x = -\sin x dx$, quickly learns

$$\int_0 \cos x dx = \sin x, \text{ and } \int_0 \sin x dx = 1 - \cos x,$$

which by repetition, give most directly the series for $\sin x$ and $\cos x$ in powers of x. We have only to *begin* from the hypothesis of x very small. The reader is supposed to be acquainted with these series, and with the inverse series for $\sin^{-1}(x)$ and $\tan^{-1}(x)$. To be explicit, the series are

$$\left.\begin{aligned} \sin \omega &= \omega - \frac{\omega^3}{2.3} + \frac{\omega^5}{2.3.4.5} - \frac{\omega^7}{2.3\ldots7} + \&c. \\ \cos \omega &= 1 - \frac{\omega^2}{2} + \frac{\omega^4}{2.3.4} - \frac{\omega^6}{2.3\ldots6} + \&c. \end{aligned}\right\},$$

$$\left.\begin{aligned} \sin^{-1} x &= x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \&c. \\ \tan^{-1} x &= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c. \end{aligned}\right\},$$

but the reader is *not* supposed to have dealt with $\sqrt{-1}$ as a multiple of an exponent. Here I define $A^{x\sqrt{-1}}$ as follows: "Expand A^y in rising powers of y, and in the series resulting, write

$$x\sqrt{-1} \text{ for } y, \ (x\sqrt{-1})^n \text{ for } y^n."$$

From this I begin.

2. $\epsilon^{x\sqrt{-1}}$ defined from the series which alone can explain it, is equivalent to $\cos x + \sqrt{-1} . \sin x$. This has nothing paradoxical; but is clearly understood by geometry. No other *proof* is needed, but to put $x\sqrt{-1} = u$, and to develop ϵ^u into its algebraic equivalent

$$1 + u + \frac{u^2}{2} + \frac{u^3}{2.3} + \frac{u^4}{2.3.4} + \&c.$$

Then by restoring to u the $(x\sqrt{-1})$ for which it stands and grouping the terms separately in which $\sqrt{-1}$ vanishes, we find their sum to be exactly $\cos x$. The rest, in which the exponent of u was *odd*, have each of them $\sqrt{-1}$ as a factor; and the sum of the coefficients of $\sqrt{-1}$ is visibly $\sin x$. Hence $\epsilon^{x\sqrt{-1}}$ when thus developed is visibly

$$\cos x + \sqrt{-1} \sin x.$$

3. Some students, justly cautious, will ask, When $x\sqrt{-1}$ cannot be defined as an exponent, ought we to assume that it obeys the law of exponents? The reply is not far to seek. Let $\phi(x)$ denote the series intended by $\epsilon^{x\sqrt{-1}}$, which we see is equivalent to

$$\cos x + \sqrt{-1} \sin x.$$

Then also $\phi(y)$ for $\epsilon^{y\sqrt{-1}}$ means $\cos y + \sqrt{-1} \sin y$. Then the product

$$\phi(x) . \phi(y) = (\cos x + \sqrt{-1} \sin x) . (\cos y + \sqrt{-1} \sin y),$$

which by the known values of $\cos(x+y)$ and $\sin(x+y)$ shows that the last product is equivalent to

$$\cos(x+y) + \sqrt{-1} \sin(x+y),$$

or which is what we here mean by $\phi(x+y)$. Thus

$$\phi x . \phi y = \phi(x+y).$$

But this is the cardinal law of A^x, making $A^x . A^y = A^{x+y}$. Indeed Cauchy in his *Elementary Algebra* proposed the Problem, "to find the nature of the function which for all *real* values of x and y fulfils the equation $\phi x . \phi y = \phi(x+y)$," and by the simplest reasoning shows that it *must* have the form A^x, in which A is unchangeable, but may have any intelligible value. Here, we may say

$$A = \epsilon^{\sqrt{-1}} \text{ or } (\cos . 1 + \sqrt{-1} \sin 1);$$

but at any rate we find that our $\epsilon^{x\sqrt{-1}}$ is subservient to the laws of exponents.

The secondary property $(A^x)^u = A^{xu}$, follows from the other when u is numerical.

4. COR. 1. From $\epsilon^{x\sqrt{-1}}$, *identical* with $\cos x + \sqrt{-1}\sin x$, we get by changing x into $-x$, $\epsilon^{-x\sqrt{-1}}$ identical with $\cos x - \sqrt{-1}\sin x$.

Hence by addition

$$\left.\begin{aligned} \epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}} &= 2\cos x, \\ \epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}} &= 2\sqrt{-1}\,.\sin x \end{aligned}\right\}.$$

by subtraction

Thus when for convenience we write u for $\epsilon^{x\sqrt{-1}}$, we have

$$u + u^{-1} = 2\cos x.$$

COR. 2. If in these identities we write nx for x, $\epsilon^{x\sqrt{-1}}$ changes to $\epsilon^{nx\sqrt{-1}}$ or u^n, thus we have simultaneously

$$u + u^{-1} = 2\cos x \text{ and } u^n + u^{-n} = 2\cos nx.$$

Evidently of the two last equations each implies the other.

5. $(\cos x)^n$, when n is a positive integer, is hence expressible in a finite series of *linear* cosines. For by the Binomial Theorem we can expand $(u + u^{-1})^n$ into $n + 1$ terms;

$$\therefore\ (2\cos x)^n = (u + u^{-1})^n = u^n + \frac{n}{1}u^{n-1}\,.\,u^{-1} + \frac{n}{1}\,.\,\frac{n-1}{2}\,.\,u^{n-2}\,.\,u^{-2} + \&\text{c}.$$

$$= u^n + \frac{n}{1}\,.\,u^{n-2} + \frac{n}{1}\,.\,\frac{n-1}{2}\,u^{n-4} + \dots$$

$$+ \frac{n}{1}\,.\,\frac{n-1}{2}\,.\,u^{-n+4} + \frac{n}{1}\,.\,u^{-n+2} + u^{-n}.$$

When n is *odd*, $\overline{n+1}$ is even, and the series has no term free from u. But when n is *even*, there is a middle term N free from u. The general term being

$$\frac{n\,.\,(n-1)\dots(n-m+1)}{1\,.\,2\dots m},$$

in the case of n even, the middle term N is

$$\frac{n(n-1)\dots(n-\frac{1}{2}n+1)}{1\,.\,2\dots\frac{1}{2}n},$$

or

$$N = \frac{n}{1}\,.\,\frac{n-1}{2}\dots\frac{\frac{1}{2}n+1}{\frac{1}{2}n}.$$

Introvert the series, then

$$(2\cos x)^n = (u^n + u^{-n}) + \frac{n}{1}(u^{n-2} + u^{-n+2}) + \frac{n}{1} \cdot \frac{n-1}{2}(u^{n-4} + u^{-n+4}) + \&c.$$

$$= 2\cos nx + \frac{n}{1} \cdot 2\cos(n-2)x$$

$$+ \frac{n}{1} \cdot \frac{n-1}{2} \cdot 2\cos(n-4)x + \&c.,$$

which demonstrates our Theorem

COR. If we wish to deal similarly with $(2\sin x)^n$ the simplest way is to transform it by assuming

$$x = \tfrac{1}{2}\pi - y,$$

$$\therefore \sin x = \cos y.$$

6. PROBLEM. To attain the value of π, exact to many decimal places.

The inverse equation of $\sin^{-1}x$ in series gives the earliest elementary method. For if in it we make $x = \frac{1}{2}$ which $= \sin 30^\circ$ or $\sin\frac{\pi}{6}$, we have

$$\sin^{-1}x = \frac{\pi}{6},$$

so that we get $\frac{\pi}{6}$ in series, in fact

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{2^{-3}}{3} + \frac{1.3}{2.4} \cdot \frac{2^{-5}}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{2^{-7}}{7} + \&c.$$

Represent the double of this series by

$$\tfrac{1}{3}\pi = A_0 + \tfrac{1}{3}A_1 + \tfrac{1}{5}A_2 + \tfrac{1}{7}A_3 + \&c.,$$

which implies

$$A_n = \left(1 - \frac{1}{2n}\right) \cdot \frac{A_{n-1}}{4},$$

then starting from $\qquad A_0 = 1,$

we easily compute the series A in succession, and without much trouble can find

$$\pi = 3{\cdot}141592653591,$$

which is correct to 11 decimals. But a more powerful method is easily deducible.

7. We had $\tan^{-1} x$ in powers of x; change x to v^{-1}, then $\tan^{-1} x$ becomes $\tan^{-1}.(v^{-1})$. But if v^{-1} is the tangent of an arc, v must be its cotangent. Hence

$$\cot^{-1}(v) = v^{-1} - \tfrac{1}{3}v^{-3} + \tfrac{1}{5}v^{-5} - \tfrac{1}{7}v^{-7} + \&c.,$$

which is a new identity, if only v is > 1, since x was < 1.

Begin anew, from

$$\cot x = u,\ \cot y = v,\ \cot(x+y) = z,$$

then by the Elements of Trigonometry

$$z = \frac{uv-1}{u+v},$$

or $$uv - (u+v)z = 1,$$

and $$(u-z)(v-z) = 1 + z^2.$$

Take for z such a whole number as shall make $1 + z^2$ the product of the two factors s, t. Nothing then forbids our assuming $u - z = s$, which makes

$$v - z = t, \text{ or } u = s + z,\ v = t + z.$$

But $$x = \cot^{-1} u,\ y = \cot^{-1} v,$$

also $$x + y = \cot^{-1} z,$$

$$\therefore \cot^{-1} z = \cot^{-1}(s+z) + \cot^{-1}(t+z)$$

under the condition that $st = z^2 + 1$.

In particular let $t = 1$, $\therefore\ s = z^2 + 1$,

and $$\cot^{-1} z = \cot^{-1}(1 + z^2 + z) + \cot^{-1}(1+z),$$

without conditions. If $z = 1$,

$$\cot^{-1}(1) \text{ or } \tfrac{1}{4}\pi = \cot^{-1}(3) + \cot^{-1}(2).$$

This is Euler's formula, and very elegant.

8. But if $z = 2$, $\cot^{-1}(2) = \cot^{-1}(7) + \cot^{-1}(3)$.

From these two results eliminate $\cot^{-1}(2)$ which has the worst convergence,

$$\therefore \tfrac{1}{4}\pi = 2\cot^{-1}(3) + \cot^{-1}(7) = 2a + b,$$

if $$a = 3^{-1} - \tfrac{1}{3}.3^{-3} + \tfrac{1}{5}.3^{-5} - \tfrac{1}{7}.3^{-7} + \&c.$$

and $$b = 7^{-1} - \tfrac{1}{3}.7^{-3} + \tfrac{1}{5}.7^{-5} - \tfrac{1}{7}.7^{-7} + \&c.$$

It is hard to surpass this at once in simplicity and power. Observe, that we may also write

$$a = \tfrac{1}{3}\{1 - \tfrac{1}{3} . 9^{-1} + \tfrac{1}{5} . 9^{-2} - \tfrac{1}{7} . 9^{-3} + \&c.\}.$$

Again, if z is odd, $1 + z^2$ is even, and we may take 2 as one factor. Indeed, if we write $z = 2r - 1$,

$$1 + z^2 = 2 - 4r + 4r^2,$$

let $t = 2$, then
$$s = 1 - 2r + 2r^2,$$
$$u = 3 + z = 2r^2,$$
$$v = t + z = 2r + 1.$$

Hence $\cot^{-1}(2r - 1) = \cot^{-1}(2r^2) + \cot^{-1}(2r + 1)$, an identity.

9. But far more celebrated and important is *Machin's* formula for π. He assumes $\cot x = 5$. Then

$$\cot 2x = \frac{(\cot x)^2 - 1}{2 \cot x} = \frac{5^2 - 1}{10} = \frac{12}{5}.$$

Further
$$\cot . 4x = \frac{(\cot 2x)^2 - 1}{2 \cot 2x} = \frac{12^2 - 5^2}{2 . 12 . 5} = \frac{119}{120}$$

very near to 1. Then $4x$ must be very near to $\tfrac{1}{4}\pi$. Let

$$4x = \tfrac{1}{4}\pi + \omega,$$

$$\therefore \quad \cot \omega = \cot (4x - \tfrac{1}{4}\pi) = \frac{1 + \cot 4x}{1 - \cot 4x} = \frac{120 + 119}{120 - 119} = 239,$$

and
$$\omega = \cot^{-1}(239).$$

Now
$$\tfrac{1}{4}\pi = 4x - \omega = 4 \cot^{-1}(5) - \cot^{-1}(239).$$

This result is powerful, but 239 is a clumsy divisor. An unexpected relief is attained by taking $z = 70$,

$$\therefore \quad 1 + z^2 = 4901 = 169 . 29.$$

Take $s = 169$, $t = 29$, which yield

$$u = z + s = 239,$$
$$v = z + t = 99,$$

then
$$\cot^{-1} . 70 = \cot^{-1} . (239) + \cot^{-1}(99).$$

Here 70 and 99 are highly convenient divisors, and *eliminating* $\cot^{-1}(239)$ we have remaining

$$\tfrac{1}{4}\pi = 4 \cot^{-1}(5) + \cot^{-1}(99) - \cot^{-1}(70).$$

By this formula π can be found accurately to a very unreasonable number of decimal places. For 20 decimals Mathematicians acquiesce in $\pi = 3{\cdot}14159\ 26535\ 89793\ 23846$. So much of π.

Factors of $\sin x$.

10. The late Professor Jarrett introduced the notation $\underline{|n}$ for $(1 . 2 . 3 . 4 \ldots n)$, and it has been accepted at Cambridge. Thus

$$\sin x = x - \frac{x^3}{1 . 2 . 3} + \ldots \pm \frac{x^n}{\underline{|n}} \mp \&c.$$

However great x may be, the denominator $\underline{|n}$ at length becomes so vast as to overwhelm the numerator x^n, and the sum of the terms that follow becomes always at last insignificant. Thus in the Elements there is a suggestion that by making n vast, yet finite, we may treat $(\sin x = 0)$ as an equation of the nth degree. Indeed, making $x^2 = y$,

$$\frac{\sin x}{x} = 1 - \frac{y}{1 . 2 . 3} + \frac{y^2}{\underline{|5}} - \frac{y^3}{\underline{|y}} + \ldots \pm \frac{y^n}{\underline{|2n+1}},$$

which seems simpler still. But the argument is not rigid enough; we must go deeper for proof of a Theorem, on which so very much depends.

In the Elements it is easily shown that

$$\sin 3x,\ \sin 5x,\ \sin 7x \ldots$$

and in general $\qquad \sin (2r - 1) x$

can be expressed as odd integer functions of $\sin x$. This justifies our here assuming, if $\sin x = u$,

$$\sin . nx = nu - N_3 u^3 + N_5 u^5 - \&c. \ldots \pm N_n u^n = \Sigma (A_m u^m) \text{ when } n \text{ is } \textit{odd}.$$

Now by making $\sin nx = 0$, we find for nx the values $0, \pi, 2\pi, 3\pi, m\pi, \ldots$ and for x the values $0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots \frac{m\pi}{n} \ldots$ which give only n values to u, viz.

$$0, \sin \frac{\pi}{n}, \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n}.$$

Also $\sin \frac{m\pi}{n}$ and $\sin \frac{(n-m)\pi}{n}$ or $\sin \left(\pi - \frac{m\pi}{n}\right)$ are equal. Thus, after $u = 0$, every other root has its pair. Put $n = 2r + 1$, then by the theory of integer functions $\sin nx$, the equivalent of

$$\Sigma (A_m u^m) = nu \,.\left(1 - \frac{u^2}{u_1^{\,2}}\right)\left(1 - \frac{u^2}{u_2^{\,2}}\right) \dots \left(1 - \frac{u^2}{u_r^{\,2}}\right),$$

if $u_1, u_2, u_3 \dots u_r$ denote the roots

$$\sin\frac{\pi}{n},\ \sin\frac{2\pi}{n},\ \sin\frac{3\pi}{n},\ \dots \sin\frac{r\pi}{n}.$$

As an identity then,

$$\sin (nx) = n \sin x \,.\left(1 - \frac{\sin^2 x}{\sin^2 . n^{-1}\pi}\right)\left(1 - \frac{\sin^2 x}{\sin^2 . 2n^{-1}\pi}\right)\left(1 - \frac{\sin^2 x}{\sin^2 . 3n^{-1}\pi}\right)\dots$$
$$\left(1 - \frac{\sin^2 x}{\sin^2 . rn^{-1}\pi}\right):$$

the ease with which this great result is obtained from

$$\sin (nx) = \Sigma (A_m x^m)$$

is very striking. From it we are to deduce the factors of $\sin x$ of the form $(1 - a^2x^2)$.

11. Our first step is in the last *identity* to write $\frac{x}{n}$ for x; the next is to increase x beyond all limit. First we have

$$\sin x = \left(n \sin\frac{x}{n}\right)$$

multiplied by r factors, each of the form

$$1 - \frac{\sin^2 . (n^{-1} x)}{\sin^2 . (m \,. n^{-1}\pi)}.$$

With n infinite, $n \sin (n^{-1} x)$ evidently $= x \,. \frac{\sin (n^{-1}x)}{(n^{-1}x)}$ and converges to mere x. The general factor converges to

$$1 - \left(\frac{n^{-1}x}{mn^{-1}\pi}\right)^2 \text{ or } 1 - \left(\frac{x}{m\pi}\right)^2;$$

so that if we can be sure that *the conditions of convergence in the product of the factor are fulfilled* when n is made infinite, we deduce universally

$$\sin x = x \,.\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{4^2\pi^2}\right) \dots \text{\&c.} \quad \dots(4).$$

But just as $a_m + a_{m+1} + a_{m+2} +$ &c. *ad infinitum* may be a finite quantity, though every term vanishes when m is infinite, so may an infinite product $a_m a_{m+1} . a_{m+2} \dots$ be other than 1, though every factor separately converge to 1 when m is infinite. Therefore to investigate the convergence remains as a problem to us.

12. Cauchy has treated this question of convergence in the IXth supplementary Note of his *Cours d'Analyse*, Paris, 1821, and after much detail proves our point only when x is less than $\frac{1}{2}\pi$; after which his argument is embarrassed by the logarithm of a negative quantity.

Serret has dealt with the topic (for our particular case) rigorously; but his proof is not easy to follow, and is still harder to remember. This encourages another method. In equation (4) write ξ for $\frac{x^2}{\pi^2}$. The product

$$x . (1 - \xi)(1 - 2^{-2}\xi)(1 - 3^{-2}\xi)(1 - 4^{-2}\xi) \dots \&c.$$

will converge, *if its logarithm* converges. The general term of the logarithm is $\log(1 - m^{-2}\xi)$ and its differential is

$$-\frac{m^{-2} d\xi}{1 - m^{-2}\xi};$$

of which the denominator converges to 1 when m increases indefinitely. Thus after a certain high value of m, the later terms converge to

$$-d\xi . \{m^{-2} + (m + 1)^{-2} + (m + 2)^{-2} + (m + 3)^{-2} + \dots\}.$$

The infinite series in parenthesis is known to be finite, since

$$S_2 = 1^{-1} + 2^{-2} + 3^{-2} + \&c.$$

is finite. Then stepping back, from the differential to the integral, the sum of the logarithm, whose general term was $\log(1 - m^{-2}\xi)$, is finite. Consequently the series of products above marked (4) converges to a finite value, and its definite equality to $\sin x$ is beyond objection.

The differential of $\log \sin x$ (divided by dx) is

$$x \cot x = 1 - \Sigma \frac{2x^2}{m^2\pi^2 - x^2};$$

where m means 1, 2, 3, 4,... The value of this result is great.

13. COR. 1. Since $\cos x = \frac{\sin 2x}{2 \sin x}$, it gives $\cos x$ also in algebraic factors.

COR. 2. Having obtained

$$\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{2^2\pi^2 - x^2} - \frac{2x}{3^2\pi^2 - x^2} - \&c$$

we further deduce tan $(\frac{1}{2}x)$ by the known formula

$$\tfrac{1}{2}\tan(\tfrac{1}{2}x) = \tfrac{1}{2}\cot(\tfrac{1}{2}x) - \cot x,$$

whence $$\tfrac{1}{2}\tan(\tfrac{1}{2}x) = \frac{2x}{\pi^2 - x^2} + \frac{2x}{3^2\pi^2 - x^2} + \frac{2x}{5^2\pi^2 - x^2} + \&c.$$

COR. 3. Again, since $\operatorname{cosec} x = \frac{1}{2}\cot(\frac{1}{2}x) + \frac{1}{2}\tan(\frac{1}{2}x)$, we infer

$$\frac{1}{\sin x} = \frac{1}{x} + \frac{2x}{\pi^2 - x^2} - \frac{2x}{2^2\pi^2 - x^2} + \frac{2x}{3^2\pi^2 - x^2} - \frac{2x}{4^2\pi^2 - x^2} + \&c.$$

From the last we take a step further, to obtain (virtually) $\dfrac{1}{\cos x}$.

First since $\dfrac{2x}{m^2\pi^2 - x^2} = \dfrac{1}{m\pi - x} - \dfrac{1}{m\pi + x}$, we have also

$$\frac{1}{\sin x} = \frac{1}{x} + \left(\frac{1}{\pi - x} - \frac{1}{\pi + x}\right) - \left(\frac{1}{2\pi - x} - \frac{1}{2\pi + x}\right)$$
$$+ \left(\frac{1}{3\pi - x} - \frac{1}{3\pi - x}\right) - \&c.$$

In this, let $x = \frac{1}{2}\pi - y$, or $\sin x = \cos y$. Also write p for $\frac{1}{2}\pi$, $x = p - y$.

$$\therefore \frac{1}{\cos y} = \frac{1}{p - y} + \left(\frac{1}{p + y} - \frac{1}{3p - y}\right) - \left(\frac{1}{3p + y} - \frac{1}{5p - y}\right)$$
$$+ \left(\frac{1}{5p + y} - \frac{1}{7p - y}\right) - \&c.,$$

which being a new identity, permits us to change at pleasure the y into x.

Also, coupling the terms differently, you have

$$\frac{1}{\cos y} = \frac{2p}{p^2 - y^2} - \frac{6p}{3^2p^2 - y^2} + \frac{10p}{5^2p^2 - y^2} - \&c.$$

But this convergence may be improved. Observe that

$$\frac{2mp}{m^2p^2 - y^2} = \frac{2p}{mp^2}\left\{1 + \frac{y^2}{m^2p^2 - y^2}\right\},$$

where m may successively be 1, 3, 5, 7, 9,....

Collect $\Sigma \frac{2}{\pm mp}$ or $\frac{2}{p}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.\right)$ into P,

then $$\frac{1}{\cos y} = P + \frac{2}{p}\left\{\frac{y^2}{p^2 - y^2} - \frac{\frac{1}{3}y^2}{3^2p^2 - y^2} + \frac{\frac{1}{5}y^2}{5^2p^2 - y^2} - \&c.\right\}.$$

Let $y = 0$, then since we know that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. = \frac{\pi}{4} = \frac{p}{2},\ P = \frac{2}{p}\cdot\frac{p}{2} = 1.$$

Finally then, changing y to x, we attain

$$\frac{1}{\cos x} = 1 + \frac{2}{p}\left\{\frac{x^2}{p^2 - x^2} - \frac{\frac{1}{3}x^2}{3^2p^2 - x^2} + \frac{\frac{1}{5}x^2}{5^2p^2 - x^2} - \&c.\right\}.$$

14. We can obtain $x \cot x$ in powers of x by an elementary process. To save space, write a_m for the reciprocal of $1.2.3\ldots m$; then

$$x \cot x \text{ or } \frac{\cos x}{\left(\frac{\sin x}{x}\right)} = \frac{1 - a_2x^2 + a_4x^4 - a_6x^6 + \&c.}{1 - a_3x^2 + a_5x^4 - a_7x^6 + \&c.},$$

a fraction which algebraically admits the form

$$1 - 2H_1x^2 - 2H_2x^4 - 2H_3x^6 - \&c.$$

This determines the series $H_1, H_2, H_3\ldots$ on which Legendre bestowed much care, but I believe they are called "Euler's H series." It converges notably, and seems to supersede the hideous numbers of Bernoulli. Supposing the H series fitly determined, convergence is not affected nor therefore identity disturbed by changing $-x^2$ to $+x^2$. Then

$$(1 + a_2x^2 + a_4x^4 + a_6x^6 + \&c.)$$
$$= (1 + a_3x^2 + a_5x^4 + \&c.).(1 + 2H_1x^2 - 2H_2x^4 + 2H_3x^6 - \&c.).$$

Multiplying out on the right, equate the coefficients of x^{2n}, and you easily get, observing that $a_m - a_{m+1} = m.a_{m+1}$ and dividing by 2,

$$H_1.a_{2n-1} - H_2.a_{2n-2} + H_3.a_{2n-3} - \&c.\ldots \pm H_n = a_{2n+1};$$

a general relation, from which $H_1, H_2, H_3\ldots$ are successively determined by taking n as 1, 2, 3, 4...

First $$H_1 = a_3.$$

Next $$H_1a_3 - H_2 = 2a_5.$$

Further $$H_1a_5 - H_2a_3 + H_3 = 3a_7;\ \text{and so on.}$$

The five first ought to be remembered; viz.

$$H_1 = \tfrac{1}{6}; \qquad H_2 = \tfrac{1}{90};$$

$$H_3 = \frac{1}{3 \,.\, 5 \,.\, 7 \,.\, 9};$$

$$H_4 = \tfrac{1}{10} \,.\, H_3; \qquad H_5 = \tfrac{1}{9} \,.\, \tfrac{1}{11} \,.\, H_3.$$

Beyond, I calculate them as follows,—to 16 decimals:

n	H_n
6	·0000 0108 2202 1406
7	....10 9629 7392
8	.... 1 1107 3044
9	. ... 1125 3919
10	.... 114 0257
11	....11 5532
12	. ... 1 1872
13	 1206
14	 122
15	12
16	 1

Here it is apparent, that any H barely exceeds *one-tenth* of the preceding. We presently learn that $\frac{1}{10}$ is the limit to which the ratio tends.

This H series is founded on the expansion of $\cot x$ in powers of x, but the Trigonometry which deduced $\tan(\frac{1}{2}x)$ from $\cot x$, will equally deduce it in *powers* of x, so will it deduce $\operatorname{cosec} x$. But it fails when applied to $\sec x$. From

$$\int \cot x dx$$

we step back to $\log \sin x$ in powers of x, and from

$$\log \cos x = \log(\sin 2x) - \log(2 \sin x),$$

we obtain $\log \cos x$ in powers of x by the same H series.—(Legendre most ingeniously exhibits H_n in finite terms as a function of n; but though the law is clear, the formula has no practical use.)

15. From the equation

$$(x \cot x) = 1 - 2H_1x^2 - 2H_2x^4 - 2H_3x^6 - 2H_4x^8 - \&c.$$

by aid of the identity

$$\tan \tfrac{1}{2}x = \cot \tfrac{1}{2}x - 2 \cot x,$$

or $$\tfrac{1}{2}x \,.\, \tan \tfrac{1}{2}x = \tfrac{1}{2}x \,.\, \cot \tfrac{1}{2}x - x \cot x,$$

we deduce

$$\frac{x}{2}\tan\frac{1}{2}x = \left(1 - 2H_1\,.\,\overline{\frac{x}{2}}\Big|^2 - 2H_2\,.\,\frac{x}{2}\Big|^4 - 2H_3\,.\,\overline{\frac{x}{2}}\Big|^6 - \&c.\right)$$

$$- (1 - 2H_1\,.\,x^2 - 2H_2\,.\,x^4 - 2H_3\,.\,x^6 - \&c.)$$

$$= (2 - 2^{-1})\,H_1x^2 + (2 - 2^{-3})\,H_2x^4 + (2 - 2^{-5})\,H_3x^6 + \&c.$$

Write S_n for $1^{-n} + 2^{-n} + 3^{-n} + 4^{-n} + \&c.$, and let $x = \omega\pi$,

$$\therefore\ \omega\pi\,.\cot(\omega\pi) = 1 - 2H_1\omega^2\pi^2 - 2H_2\omega^4\pi^4 - 2H_3\omega^6\pi^6 - \&c.,$$

which also $$= 1 - \frac{2\omega^2}{1 - \omega^2} - \frac{2\omega^2}{2^2 - \omega^2} - \frac{2\omega^2}{3^2 - \omega^2} - \&c.$$

Expand each of these fractions, then combine all the multipliers that affect the same power of ω, and the series is transformed to

$$1 - 2S_2\omega^2 - 2S_4\omega^4 - 2S_6\omega^6 - \&c.$$

Comparison with the first series shews that

$$S_2 = H_1\pi^2,\ S_4 = H_2\pi^4,\ S_6 = H_3\pi^6,$$

and generally $$S_n = H_n\pi^{2n}.$$

This further yields $$\frac{H_n\pi^{2n}}{H_{n-1}\pi^{2n-2}} = \frac{S_{2n}}{S_{2n-2}},$$

or $$\frac{H_n}{H_{n-1}} = \pi^{-2}\,.\,\frac{S_{2n}}{S_{2n-2}}.$$

But π^{-2} is not much short of $\frac{1}{10}$, and the last fraction converges towards 1 as n increases. This displays the convergence of the H series.

If next we write $T_n = 1^{-n} + 3^{-n} + 5^{-n} + 7^{-n} + \&c.$,

it gives $T_n = (1 - 2^{-n})\,S_n$;

likewise if $U_n = 1^{-n} - 2^{-n} + 3^{-n} - 4^{-n} + \&c.$,

then $U_n = (1 - 2^{1-n})\,S_n$, .

and by developing the fractions in $\tan(\frac{1}{2}x)$ and $\frac{x}{\sin x}$, with $x = \omega\pi$,

we obtain $$\tfrac{1}{4}\pi\,.\tan(\tfrac{1}{2}\omega\pi) = T_2\omega + T_4\omega^3 + T_6\omega^5 + \&c.$$

and $$\frac{\omega\pi}{\sin(\omega\pi)} = 1 + 2U_2\omega^2 + 2U_4\omega^4 + 2U_6\omega^6 + \&c.$$

Evidently S_n, T_n and U_n tend to 1, as n increases. Therefore we generally increase convergence by working with $S_n - 1$, $T_n - 1$, $1 - U_n$, instead of by S_n, T_n, U_n. Thus if for a Trigonometrical table we desire to find log sin and log cos by a direct formula, put first

$$-\log\frac{\sin\omega\pi}{\omega\pi} = S_2\omega^2 + \tfrac{1}{2}S_4\omega^4 + \tfrac{1}{3}S_6\omega^6 + \&c.;$$

then subtract from it

$$-\log(1-\omega^2) = \omega^2 + \tfrac{1}{2}\omega^4 + \tfrac{1}{3}\omega^6 + \&c.;$$

whence

$$-\log\frac{\sin\omega\pi}{\omega\pi} + \log(1-\omega^2) = (S_2-1)\,\omega^2 + \tfrac{1}{2}(S_4-1)\,\omega^4 + \tfrac{1}{3}(S_6-1)\omega^6 + \&c.$$

Again, since

$$-\log\cos(\tfrac{1}{2}\omega\pi) = T_2\omega^2 + \tfrac{1}{2}T_4\omega^4 + \tfrac{1}{3}T_6\omega^6 + \&c.,$$

subtract as before, then

$$-\log(\cos\tfrac{1}{2}\omega\pi) + \log(1-\omega^2) = (T_2-1)\,\omega^2$$
$$+\tfrac{1}{2}(T_4-1)\,\omega^4 + \tfrac{1}{3}(T_6-1)\,\omega^6 + \&c.$$

To avail ourselves of these formulae, we need tables of $S_n - 1$, $T_n - 1$ and $1 - U_n$. Legendre has given S_n to 16 decimals. A fourth series is needed, namely

$$V_n = 1^{-n} - 3^{-n} + 5^{-n} - 7^{-n} + \&c.,$$

which is connected with the development of sec x. Above we found

$$\frac{1}{\cos x} = 1 + \frac{2}{p}\left\{\frac{x^2}{p^2-x^2} - \frac{\frac{1}{3}x^2}{3^2p^2-x^2} + \frac{\frac{1}{5}x^2}{5^2p^2-x^2} - \&c.\right\}.$$

Expand every fraction, and write for the result

$$\sec x = 1 + K_1x^2 + K_2x^4 + K_3x^6 + \&c.,$$

then you arrive at

$$\tfrac{1}{2}K_1 = p^{-3}\{1 - 3^{-3} + 5^{-3} - 7^{-3} + \&c.\} = V_3\,.\,p^{-3},$$

and generally

$$\tfrac{1}{2}K_n = p^{-m}\{1 - 3^{-m} + 5^{-m} - \&c.\} = V_m\,.\,p^{-m},$$

if $$m = 2n + 1.$$

The ratio $(K_{n+1} : K_n)$ converges towards

$$(p^2V_{2n+3} : V_{2n+1}),$$

or $(1 : p^2)$ when 3^{-n} is evanescent. That is, it never is so small as $4 : \pi^2$, say $4 : 10$.

16. The series V_n is of far less use than S_n. I have never seen a professed table of it, but Gudermann's coefficients imply that he had calculated it. These four slowly converging series will be further treated below.

It was observed above, in treating of

$$x^{-1}\frac{\cos x}{\sin x} = 1 - 2H_1x^2 - 2H_2x^4 - \&c.$$

that the formula being an identity when the H series is rightly assigned, nothing forbids the change of $-x^2$ to $+x^2$, which gives

$$x \,.\, \frac{\epsilon^x + \epsilon^{-x}}{\epsilon^x - \epsilon^{-x}} = 1 + 2H_1x^2 - 2H_2x^4 + 2H_3x^6 - \&c.,$$

which also then $$= 1 + \Sigma\,\frac{2x^2}{m^2\pi^2 + x^2};$$

where m means 1, 2, 3, 4 .... Hence, working back, we find

$$\tfrac{1}{2}\,(\epsilon^x - \epsilon^{-x}) = x,$$

multiplied by factors $$\left(1 + \frac{x^2}{m^2\pi^2}\right),$$

where m as before, means 1, 2, 3, 4, &c. This appearance of π in a purely exponential function is very remarkable.

Since $$\frac{\epsilon^x + \epsilon^{-x}}{\epsilon^x - \epsilon^{-x}} = \frac{2\epsilon^x}{\epsilon^x - \epsilon^{-x}} - 1 = \frac{2\epsilon^{2x}}{\epsilon^{2x} - 1} - 1,$$

substitute this in the penultimate equation. Then

$$\frac{2x \,.\, \epsilon^{2x}}{\epsilon^{2x} - 1} - x = 1 + 2H_1x^2 - 2H_2x^4 + 2H_3x^6 - \&c.,$$

a new identity. Change x to $\tfrac{1}{2}x$. Then

$$\frac{x\epsilon^x}{\epsilon^x - 1} = \frac{x}{2} + 1 + 2^{-1}H_1x^2 - 2^{-3}H_2x^4 + 2^{-5}H_3x^6 - \&c.,$$

a formula of cardinal value, as will appear.

The left member is also $\dfrac{x}{1 - \epsilon^{-x}}$, which we find to be identical with

$$1 + \tfrac{1}{2}x + 2^{-1}H_1x^2 - 2^{-3}H_2x^4 + \&c.$$

17. PROBLEM. To distribute $\frac{\phi(\mu x)}{\phi(x)}$ into simpler fractions, when the simple factors of $\phi(x)$ are known, and μ is a numerical fraction, less than 1.

First, assume $\phi(x)$ to mean

$$\left(1-\frac{x^2}{a_1^2}\right)\left(1-\frac{x^2}{a_2^2}\right)\left(1-\frac{x^2}{a_3^2}\right)\cdots\left(1-\frac{x^2}{a_n^2}\right),$$

where n is a finite integer, and the series $a_1, a_2, a_3 \ldots a_n$ are constants that increase from first to last. Then mere substitution gives on reversing the sign in every numerator and denominator,

$$\frac{\phi(\mu x)}{\phi(x)}=\frac{\mu^2x^2-a_1^2}{x^2-a_1^2}\cdot\frac{\mu^2x^2-a_2^2}{x^2-a_2^2}\cdots\frac{\mu^2x^2-a_n^2}{x^2-a_n^2};$$

call the product of the denominators $F(x)$, and by it divide the product of the numerators. The only *quotient* will be μ^{2n}. Let the *remainder* be called $f(x)$ which will be integer in x^2, but only of the degree $2n-2$. Then

$$\frac{\phi(\mu x)}{\phi(x)}=\mu^{2n}+\frac{f(x)}{F(x)}.$$

The last fraction then admits the form

$$\Sigma\frac{A_r}{x^2-a_r^2},$$

in which a_r is the general term of the a series, and r means $1, 2, 3 \ldots n$. In the course of our argument we shall suppose n to increase indefinitely. This will make the term μ^{2n} vanish, since by hypothesis μ is <1. Nothing now forbids us to assume $a_r=r\pi$, then $x\phi(x)$ will approximate to $\sin x$, when n has unlimited increase.

Thus $$\mu^{2n}+\frac{fx}{Fx}=\frac{\phi(\mu x)}{\phi(x)},$$

and for huge values of n we may neglect μ^{2n}. When $a_r=r\pi$, the two equations

$$\phi(x)=\frac{\sin x}{x} \text{ and } \mu^{2n}=0$$

are more and more nearly true as n becomes greater. Then $\frac{\phi(\mu x)}{\phi(x)}$

approaches to $\frac{\sin(\mu x)}{\mu\sin x}$. We had as equivalent $\Sigma\frac{A_r}{x^2-a_r^2}$, but we may conveniently write $2a_r.B_r$ for A_r. Then

$$\frac{A_r}{x^2-a_r^2}=\frac{B_r}{x-a_r}-\frac{B_r}{x+a_r}.$$

By the received rule in breaking up a rational fraction of this order, designated as

$$\frac{\left(\frac{\sin\mu x}{\mu x}\right)}{\left(\frac{\sin x}{x}\right)},\text{ we have } B_r=\frac{\left(\frac{\sin\mu a_r}{\mu a_r}\right)}{\frac{d}{dx}\left(\frac{\sin x}{x}\right)},$$

if we make $x=a_r$ after the differentiation. We must also remember that we made $a_r=r\pi$. Now

$$\frac{d}{dx}.\left(\frac{\sin x}{x}\right)=\frac{\cos x}{x}-\frac{\sin x}{x^2}=(\text{here})\frac{\cos r\pi}{r\pi}-\frac{\sin r\pi}{(r\pi)^2}.$$

But $\sin r\pi=0$, $\cos r\pi=(-1)^r$. Thus the denominator to B_r is found to be $\frac{(-1)^r}{r\pi}$, which yields

$$B_r=\frac{r\pi}{(-1)^r}.\frac{\sin(\mu r\pi)}{\mu r\pi}=(-1)^r.\frac{\sin(\mu r\pi)}{\mu}.$$

Then $$A_r=2a_rB_r=(-1)^r.\frac{2r\pi}{\mu}.\sin(\mu r\pi).$$

Hence $\frac{\sin(\mu x)}{\mu\sin x}$ is an identity with $\Sigma\frac{A_r}{x^2-r^2\pi^2}$, if n becomes infinite.

Multiply by μ,

$$\therefore\ \frac{\sin(\mu x)}{\sin x}=\Sigma\frac{(-1)^{r+1}.2r\pi\sin(\mu r\pi)}{r^2\pi^2-x^2};$$

or at full,

$$\frac{\sin(\mu x)}{\sin x}=2\left\{\frac{\pi\sin\mu\pi}{\pi^2-x^2}-\frac{2\pi\sin 2\mu\pi}{2^2\pi^2-x^2}+\frac{3\pi\sin 3\mu\pi}{3^2\pi^2-x^2}-\&c.\right\}$$

when μ is <1. But we can improve the convergence, since

$$\frac{r^2\pi^2}{r^2\pi^2-x^2}=1+\frac{x^2}{r^2\pi^2-x^2},$$

so that $$\frac{r\pi}{r^2\pi^2-x^2}=\frac{1}{r\pi}\left\{1+\frac{x^2}{r^2\pi^2-x^2}\right\}.$$

Collect all the portions thus independent of x into $2E$,

$$\therefore\ E\pi = \sin\mu\pi - \tfrac{1}{2}\sin 2\mu\pi + \tfrac{1}{3}\sin 3\mu\pi - \&c.,$$

and $$\frac{\sin(\mu x)}{\sin x} = 2E + \frac{2x^2}{\pi}\left\{\frac{\sin\mu\pi}{\pi^2 - x^2} - \frac{2^{-1}\sin 2\mu\pi}{2^2\pi^2 - x^2} + \frac{3^{-1}\sin 3\mu\pi}{3^2\pi^2 - x^2} - \&c.\right\}.$$

Make $$x = 0,$$

$$\therefore\ \mu = 2E;$$

or $$\tfrac{1}{2}\mu\pi = \sin\mu\pi - \tfrac{1}{2}\sin 2\mu\pi + \tfrac{1}{3}\sin 3\mu\pi - \&c.,$$

which we perhaps knew already.

Finally we find

$$\frac{\sin\mu x}{\sin x} = \mu + \frac{2x^2}{\pi}\cdot\left\{\frac{\sin\mu\pi}{\pi^2 - x^2} - \frac{2^{-1}\sin 2\mu\pi}{2^2\pi^2 - x^2} + \frac{3^{-1}\sin 3\mu\pi}{3^2\pi^2 - x^2} - \&c.\right\}$$

an identity if μ is < 1.

If we differentiate the last, with x constant and μ variable, the result is, after dividing by x;

$$\frac{\cos\mu x}{\sin x} = \frac{1}{x} + 2x\left\{\frac{\cos\mu\pi}{\pi^2 - x^2} - \frac{\cos 2\mu\pi}{2^2\pi^2 - x^2} + \frac{\cos 3\mu\pi}{3^2\pi^2 - x^2} - \&c.\right\}.$$

Simpler perhaps these two series become, if we assume $x = a\pi$, $\mu\pi = \omega$, whence $\mu x = a\omega$. Then because μ is < 1, ω is $< \pi$. The pair is now

$$\frac{\sin a\omega}{\sin a\pi} = \frac{\omega}{\pi} + \frac{2a^2}{\pi}\left\{\frac{\sin\omega}{1^2 - a^2} - \frac{2^{-1}\sin 2\omega}{2^2 - a^2} + \frac{3^{-1}\sin 3\omega}{3^2 - a^2} - \&c.\right\}\ldots(1),$$

$$\frac{\cos a\omega}{\sin a\pi} = \frac{1}{a\pi} + \frac{2a}{\pi}\cdot\left\{\frac{\cos\omega}{1^2 - a^2} - \frac{\cos 2\omega}{2^2 - a^2} + \frac{\cos 3\omega}{3^2 - a^2} - \&c\right\}\ \ldots\ldots(2),$$

if ω be $< \pi$.

18. We may proceed similarly for $\dfrac{\phi(\mu x)}{\phi(x)}$, first by supposing the number of factors large, yet finite, but $\mu < 1$. In details this is somewhat easier. Put p for $\frac{1}{2}\pi$; and it leads to

$$\frac{\cos\mu x}{\cos x} = \frac{2p\cos\mu p}{p^2 - x^2} - \frac{6p\cos 3\mu p}{3^2p^2 - x^2} + \frac{10p\cos 5\mu p}{5^2p^2 - x^2} - \&c.$$

But $$\frac{2rp}{r^2p^2 - x^2} = \frac{2}{rp}\cdot\frac{2x^2}{rp\,(r^2p^2 - x^2)}.$$

Substitute this in each fraction and put

$$F = 2\mu^{-1} . (\cos \mu p - \tfrac{1}{3} \cos 3\mu\pi + \tfrac{1}{5} \cos 5\mu\pi - \&c.),$$

then $$\frac{\cos \mu x}{\cos x} = F + \frac{2x^2}{p}\left\{\frac{\cos \mu p}{p^2 - x^2} - \frac{\frac{1}{3}\cos 3\mu\pi}{3^2p^2 - x^2} + \frac{\frac{1}{5}\cos 5\mu\pi}{5^2p^2 - x^2} - \&c.\right\}.$$

Make $x = 0$; $\therefore$ $F = 1$; which gives

$$\frac{p}{2} \text{ or } \frac{\pi}{4} = \cos \mu p - \tfrac{1}{3}\cos 3\mu p + \tfrac{1}{5}\cos 5\mu p - \&c.$$

$$= \cos v - \tfrac{1}{3}\cos 3v + \tfrac{1}{5}\cos 5v - \&c.,$$

so long as v is less than p or $\frac{1}{2}\pi$.

Differentiate the last, with μ sole variable; then

$$\frac{\sin \mu x}{\cos x} = 2x\left\{\frac{\sin \mu p}{p^2 - x^2} - \frac{\sin 3\mu p}{3^2p^2 - x^2} + \frac{\sin 5\mu p}{5^2p^2 - x^2} - \&c.\right\}.$$

Again, change x to ap, μp to ω, *which requires* ω *now less than* $\frac{1}{2}\pi$, then

$$\frac{\cos a\omega}{\cos \frac{1}{2}a\pi} = 1 + \frac{4a^2}{\pi}\left\{\frac{\cos \omega}{1^2 - a^2} - \frac{3^{-1}\cos 3\omega}{3^2 - a^2} + \frac{5^{-1}\cos 5\omega}{5^2 - a^2} - \&c.\right\} \ldots(3),$$

and

$$\frac{\sin a\omega}{\cos \frac{1}{2}a\pi} = \frac{4a}{\pi}\left\{\frac{\sin \omega}{1^2 - a^2} - \frac{\sin 3\omega}{3^2 - a^2} + \frac{\sin 5\omega}{5^2 - a^2} - \&c.\right\} \ldots\ldots(4).$$

The four equations are identities while ω abides under the prescribed limits; and each has two independent elements, a and ω.

19. Special Problem. To find $A = \int_0^\infty \frac{\sin \omega}{\omega} . d\omega$. First assume

$$A_n = \int_0^{n\pi} \sin \omega . \frac{d\omega}{\omega}$$

which is evidently finite, since, when

$$\omega = 0, \ \frac{\sin \omega}{\omega} = 1.$$

Observe that $\int_{r\pi}^{(r+1)\pi} \sin \omega . \frac{d\omega}{\omega}$ is identical with

$$\int_0^\pi \cos r\pi . \sin \theta . \frac{d\theta}{r\pi + \theta},$$

if $\omega = r\pi + \theta$. But the last integral vanishes, if r is infinite. *A fortiori* $\int_{r\pi}^{r\pi+a} \sin\omega \,.\, \frac{d\omega}{\omega}$ vanishes if a be less than π. Break up

$$\int_0^{n\pi} \text{ into } \int_0^{\pi} + \int_{\pi}^{2\pi} + \int_{2\pi}^{3\pi} + \ldots\ldots + \int_{(n-1)\pi}^{n\pi}$$

and in these put $\omega_1, \omega_2 \ldots \omega_n$ successively for ω. Next assume

$$\omega_1 = \pi - \theta,\ \omega_2 = \pi + \theta,\ \omega_3 = 3\pi - \theta,\ \omega_4 = 3\pi + \theta,$$

and so on,

$$\therefore\ A_n = \int_0^{\pi} \left\{\frac{\sin\theta\, d\theta}{\pi - \theta} - \frac{\sin\theta\, d\theta}{\pi + \theta} + \frac{\sin\theta\, d\theta}{3\pi - \theta} + \frac{\sin\theta\, d\theta}{3\pi + \theta} - \text{\&c. to } n \text{ terms}\right\},$$

or if $\quad \Theta = \frac{1}{\pi - \theta} - \frac{1}{\pi + \theta} + \frac{1}{3\pi - \theta} - \frac{1}{3\pi + \theta} - \text{\&c. to } n \text{ terms},$

$$A_n = \int_0^{\pi} \Theta \,.\, \sin\theta\, d\theta.$$

Then we obtain A by making n infinite.

But above we found [Art. 13]

$$\frac{\tan\frac{1}{2}\theta}{2} = \frac{2\theta}{\pi^2 - \theta^2} + \frac{2\theta}{3^2\pi^2 - \theta^2} + \frac{2\theta}{5^2\pi^2 - \theta^2} + \text{\&c.}$$

which is separable into

$$\frac{1}{\pi - \theta} - \frac{1}{\pi + \theta} + \frac{1}{3\pi - \theta} - \frac{1}{3\pi + \theta} + \text{\&c.}$$

so that with n infinite $\Theta = \frac{1}{2}\tan\frac{1}{2}\theta$, whence

$$A = \int_0^{\pi} \tfrac{1}{2}\tan\tfrac{1}{2}\theta \,.\, \sin\theta\, d\theta.$$

Now $$\tan\tfrac{1}{2}\theta = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta},$$

$$\sin\theta = 2\sin\tfrac{1}{2}\theta \,.\, \cos\tfrac{1}{2}\theta,$$

their product is $2(\sin\frac{1}{2}\theta)^2$ or $1 - \cos\theta$,

$$\therefore\ \int_0 \tfrac{1}{2}\tan\tfrac{1}{2}\theta \,.\, \sin\theta\, d\theta = \int \tfrac{1}{2}(1 - \cos\theta)\, d\theta = \tfrac{1}{2}(\theta - \sin\theta).$$

Put $\theta = \pi$ at the upper limit, $\sin\theta = 0$,

then $$A = \tfrac{1}{2}\pi \text{ or } \int_0^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \tfrac{1}{2}\pi.$$

Observe; if we write $\omega = \mu x$, and make μ *positive*, the limits of x are the same as of ω, therefore

$$\int_0^\infty \frac{\sin(\mu x)\,dx}{x} = \tfrac{1}{2}\pi,$$

for *all positive* values of x; but to make μ negative changes the sign of the result. Such a phenomenon is new.

20. PROBLEM. To expand $(\sin^{-1} u)^2$ in powers of u. That this is possible, is clear from the fact that $\sin^{-1} u$ is thus expressible, when if $u = \sin x$, x is less than $\frac{1}{2}\pi$.

Assume $\quad V = (\sin^{-1} u)^2 = A_2 u^2 + A_4 u^4 + A_6 u^6 + \&c.$

Differentiate; then

$$\frac{dV}{du} = 2\sin^{-1} u \,.\, \frac{d\sin^{-1} u}{du} = \frac{2\sin^{-1} u}{\sqrt{(1-u^2)}}.$$

Hence $$(1-u^2)\,.\left(\frac{dV}{du}\right)^2 = (2\sin^{-1} u)^2 = 4V$$

Differentiate anew:

$$\therefore\ -2u\,.\left(\frac{dV}{du}\right)^2 + (1-u^2)\,.\,2\,.\,\frac{dV}{du}\,.\,\frac{d^2V}{du^2} = 4\frac{dV}{du}.$$

Divide by $2\,.\,\frac{dV}{du}$; then $2 + u\,.\,\frac{dV}{du} + (u^2-1)\,\frac{d^2V}{du^2} = 0.$

$$\left.\begin{aligned} &\text{But} && \frac{dV}{du} = 2A_2 u + 4A_4 u^3 + 6A_6 u^5 + \&c. \\ &\text{also} && \frac{d^2V}{du^2} = 1\,.\,2A_2 + 3\,.\,4A_4 u^2 + 5\,.\,6A_6 u^4 + \&c. \end{aligned}\right\}$$

Introduce these two series into the antepenultimate; collect the coefficient of every power u^m, and it must vanish separately. This gives *binomial* equations:

$$A_2 = 1;\quad A_4 = \frac{2^2}{3\,.\,4}A_2;\quad A_6 = \frac{4^2}{5\,.\,6}A_4;$$

and the law is evident. Hence

$$(\sin^{-1} u)^2 = u^2 + \frac{2^2}{3\,.\,4}u^4 + \frac{2^2}{3\,.\,4}\cdot\frac{4^2}{5\,.\,6}u^6 + \frac{2^2}{3\,.\,4}\cdot\frac{4^2}{5\,.\,6}\cdot\frac{6^2}{7\,.\,8}u^8 + \&c.$$

or $$x^2 = V = u^2 + \frac{2}{3}\cdot\frac{u^4}{2} + \frac{2\,.\,4}{3\,.\,5}\cdot\frac{u^6}{3} + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\cdot\frac{u^8}{4} + \&c.$$

COR. Differentiate the last,

$$\therefore \frac{x\,dx}{du} = u + \frac{2}{3}u^3 + \frac{2 \cdot 4}{3 \cdot 5}u^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}u^7 + \&c.$$

21. PROBLEM. To expand $(\sin^{-1}u)^n$ whether n be positive or negative, if x is as before.

Take new values (unknown) for A_1, A_2, A_3... then in algebraic form x^n,

or $$(\sin^{-1}u)^n = u^n(1 + A_1u^2 + A_2u^4 + A_3u^6 + \&c.).$$

Let $$U = x^n, \ \therefore \ \frac{dU}{U} = n \cdot \frac{dx}{x} \text{ or } \frac{dU}{dx} = n \cdot \frac{U}{x}.$$

and $$dx = \frac{du}{\sqrt{(1-u^2)}} \text{ or } \frac{dx}{du} = \frac{\sqrt{(1-u^2)}}{1-u^2}.$$

But identically $$\frac{dU}{du} = \frac{dU}{dx} \cdot \frac{dx}{du}.$$

Eliminate $\frac{dU}{dx}$, then

$$\frac{dU}{du} = \left(n \cdot \frac{U}{x}\right) \cdot \frac{\sqrt{(1-u^2)}}{1-u^2}$$

or $$(1-u^2)\frac{dU}{du} \cdot x = nU\sqrt{(1-u^2)}.$$

Multiply the last by $$\frac{dx}{du} = \frac{1}{\sqrt{(1-u^2)}},$$

then $$(1-u^2)\frac{dU}{du} \cdot \frac{x\,dx}{du} = nU.$$

In the Corollary to the preceding we know $\frac{x\,dx}{du}$ in series of powers of u.

Put $$Y = (1-u^2)\frac{x\,dx}{du} = \left.\begin{array}{l} u + \frac{2}{3}u^3 + \frac{2 \cdot 4}{3 \cdot 5}u^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}u^7 + \&c. \\ \qquad - u^3 - \frac{2}{3}u^5 - \frac{2 \cdot 4}{3 \cdot 5}u^7 - \&c. \end{array}\right\}$$

$$= u - \frac{u^3}{3} - \frac{2}{3} \cdot \frac{u^5}{5} - \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{u^7}{7} - \&c.$$

with the law still clear.

We have now $Y.\dfrac{dU}{du}=nU$; in which on the left two series of powers of u are to be multiplied together. It is a somewhat elaborate, though mere algebraical process to compare the result with nU or nx^n or

$$n(u^n + A_1u^{n+2} + A_2u^{n+4} + \&c.),$$

also $$\frac{dU}{du} = nu^{n-1} + (n+2)A_1u^{n+1} + (n+4)A_2u^{n+3} + \&c.$$

First we have $nA_1 = (n+2)A_1 - n.\frac{1}{3}$; or $2A_1 = n.\frac{1}{3}$.

Next $$nA_2 = (n+4)A_2 - (n+2)\frac{1}{3}.A_1 - n.\frac{2}{3.5},$$

or more simply

$$4A_2 = (n+2)\frac{1}{3}A_1 - n.\frac{2}{3.5}.$$

In general for A_m we easily find the law; for

$$nA_m = (n+2m)A_m$$
$$+ (n+2m-2)\frac{1}{3}A_{m-1} - (n+2m-4)\frac{2}{3.5}A_{m-2} - \&c.$$

down to the last term

$$-n.\frac{2.4.6\ldots(2m-2)}{3.5.7\ldots(2m-1)}.\frac{1}{2m+1}.$$

Blot out nA_m from each side of the equation and for conciseness write $n+2m=r$, then

$$2mA_m = \frac{r-2}{3}A_{m-1} + \frac{2}{3}.\frac{r-4}{5}A_{m-2} + \frac{2.4}{3.5}.\frac{r-6}{7}A_{m-3}$$
$$+ \frac{2.4.6}{3.5.7}.\frac{r-8}{9}.A_{m-4} + \ldots + \frac{2.4.6\ldots 2m-2}{3.5.7\ldots 2m-1}.\frac{r-2m}{2m-1}.A_0,$$

where $A_0 = 1$, $r-2m=n$, and the law is evident.

It is of great interest that n may be *negative*, or indeed fractional. The cases of $n=-1$ and $n=-2$, serve Legendre excellently in elliptic interpolation. He deduces U^{-1} and U^{-2} by direct reciprocating of U and U^2.

We can verify our general result by trying it on $n=2$. The reduction to our known series is somewhat elaborate; but thereby yields the more satisfactory test.

22. To interpret $\log(A + B\sqrt{-1})$. Let

$$A = \rho\cos\omega,\ B = \rho\sin\omega,\ \therefore\ A^2 + B^2 = \rho^2,\ \frac{B}{A} = \tan\omega.$$

Inversely then, $\rho = \sqrt{(A^2 + B^2)},\ \omega = \tan^{-1}\frac{B}{A}$.

Thus while A and B are real and finite, so are ρ and ω. Then

$$\log(A + B\sqrt{-1}) = \log\{\rho(\cos\omega + \sqrt{-1}\sin\omega)\}$$
$$= \log\{\rho\,.\,\epsilon^{\omega\sqrt{-1}}\} = \log\rho + \log(\epsilon^{\omega\sqrt{-1}}).$$

The last term can only mean $\omega\sqrt{-1}$. Thus we obtain

$$\log(A + B\sqrt{-1}) = \log\rho + \sqrt{-1}\,.\,\omega$$
$$= \tfrac{1}{2}\log(A^2 + B^2) + \sqrt{-1}\,.\tan^{-1}\frac{B}{A}.$$

23. Now let x be numerically less than 1 and $u = x\,.\,\epsilon^{\omega\sqrt{-1}}$, then

$$u^r = x^r\,.\,(\epsilon^{r\omega\sqrt{-1}}) \text{ or } u_r = x^r(\cos r\omega + \sqrt{-1}\,.\sin r\omega).$$

The series $1 + u + u^2 + u^3 +$ &c. will converge when x is less than 1. For it is equivalent to $M + N\sqrt{-1}$, when

$$M = 1 + x\cos\omega + x^2\cos 2\omega + x^3\cos 3\omega + \text{\&c.} \ldots\ldots\ldots(a),$$
$$N = \quad x\sin\omega + x^2\sin 2\omega + x^3\cos 3\omega + \text{\&c.} \ldots\ldots\ldots(b).$$

Every cosine and sine being less than 1, each series must converge when simple $1 + x + x^2 + x^3 +$ &c. converges; that is, when x is < 1. In the same case $M + N\sqrt{-1}$ and its equivalent $1 + u + u^2 + u^3 +$ &c. converges.

Further

$$\frac{1}{1-u} = \frac{1}{1 - x(\cos\omega + \sqrt{-1}\sin\omega)} = \frac{1}{(1 - x\cos\omega) - \sqrt{-1}\,.\,x\sin\omega}$$

Multiply numerator and denominator of the last fraction by

$$(1 - x\cos\omega) + \sqrt{-1}\,.\,x\sin\omega,$$

then
$$\frac{1}{1-u} = \frac{(1 - x\cos\omega) + \sqrt{-1}\,.\,x\sin\omega}{(1 - x\cos\omega)^2 + (x\sin^2\omega)^2}.$$

But
$$\frac{1}{1-u} = 1 + u + u^2 + \text{\&c.}$$

which $= M + N\sqrt{-1}$.

Hence $$M = \frac{1 - x\cos\omega}{X},\quad N = \frac{x\sin\omega}{X} \quad\ldots\ldots\ldots\ldots\ldots (c),$$
if X stands for $(1 - x\cos\omega)^2 + (x\sin^2\omega)^2$ or $1 - 2x\cos\omega + x^2$.

Thus we obtain in finite terms the infinite series M and N.

Cor. Since $2M - 1 = \frac{2 - 2x\cos\omega}{X} - 1 = \frac{1 - x^2}{X}$,
we obtain
$$\frac{1 - x^2}{X} = 1 + 2x\cos\omega + 2x^2\cos 2\omega + 2x^3\cos 3\omega + \&c.$$

24. Integrate $-Nd\omega$ with x constant, observing that
$$-\sin\omega\, d\omega = d\cos\omega,$$
then $-\frac{1}{2}\log X = x\cos\omega + \frac{1}{2}x^2\cos 2\omega + \frac{1}{3}x^3\cos 3\omega + \&c. + \phi(x)$.
In precaution we add an *arbitrary of integration* $\phi(x)$, which cannot involve ω.

Put $\omega = 0$, $\therefore X = (1 - x)^2$, the series becomes
$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \&c. \text{ or } -\log(1 - x),$$
$$\therefore -\log(1 - x) = -\log(1 - x) + \phi(x);$$
whence $$\phi(x) = 0.$$

Having now attained, while $x^2 < 1$,
$$-\tfrac{1}{2}\log(1 - 2x\cos\omega + x^2) = x\cos\omega + \tfrac{1}{2}x^2\cos 2\omega + \tfrac{1}{3}\cos 3\omega + \&c.$$
we may push x to its extreme, $x = 1$ since the series still converge. The left-hand then becomes
$$-\tfrac{1}{2}\log(2 - 2\cos\omega) \text{ or } -\log\sin\tfrac{1}{2}\omega;$$
hence changing ω to 2ω, we find
$$-\log\sin\omega = \cos 2\omega + \tfrac{1}{2}\cos 4\omega + \tfrac{1}{3}\cos 6\omega + \&c.,$$
a curious result, in some rare cases good to know.

25. Again, integrate $N\frac{dx}{x}$ with ω constant, from $x = 0$
$$\therefore \tan^{-1}\frac{x\sin\omega}{1 - x\cos\omega} = x\sin\omega + \frac{1}{2}x^2\sin 2\omega + \frac{1}{3}x^3 3\omega + \&c.$$
Change x to $-x$, and
$$\tan^{-1}\frac{x\sin\omega}{1 + x\cos\omega} = x\sin\omega - \frac{1}{2}x^2\sin 2\omega + \frac{1}{3}x^3 3\omega - \&c.$$

Call these two $\tan^{-1}h$ and $\tan^{-1}k$. Form the sum

$$\tan^{-1}\frac{h+k}{1-hk},$$

where

$$h+k=\frac{2x\sin\omega}{1-x^2\cos^2\omega};\quad hk=\frac{x^2\sin^2\omega}{1-x^2\cos^2\omega};\quad 1-hk=\frac{1-x^2}{1-x^2\cos^2\omega}.$$

Halve the sum, then

$$\frac{1}{2}\tan^{-1}\frac{2x\sin\omega}{1-x^2}=x\sin\omega+\frac{1}{3}x^3\sin 3\omega+\frac{1}{5}x^5\sin 5\omega+\&c.$$

COR. In general, x is by hypothesis <1; but if we push to the extreme $x=1$, the last series is still convergent. Its equivalent is then $\frac{1}{2}\tan^{-1}\left(\frac{2\sin\omega}{1-1}\right)$, that is, *while ω is between* 0 and π,

$$\tfrac{1}{4}\pi=\sin\omega+\tfrac{1}{3}\sin 3\omega+\tfrac{1}{5}\sin 5\omega+\&c.$$

Series of this class can generally be tested by first assuming $U=$ the finite series to n terms, then differentiate, sum the series to n terms, integrate back, and make n infinite.

26. Above, we had

$$\frac{1-x^2}{X}=1+2x\cos\omega+2x^2\cos 2\omega+2x^3\cos 3\omega+\&c.$$

Multiply by $\cos n\omega\,.\,d\omega$, suppose x constant, and integrate. The general term of the new series is

$$\int x^m\,.\,2\cos m\omega\cos n\omega d\omega \quad\text{or}\quad x^m\int_0(\cos\overline{m-n}\omega+\cos\overline{m+n}\omega)\,d\omega,$$

where m has the successive values 0, 1, 2, 3,... When m differs from n, the result is

$$x^m\left\{\frac{\sin(m-n)\omega}{m-n}+\frac{\sin(m+n)\omega}{m+n}\right\},$$

which vanishes at the upper limit $\omega=\pi$. Only when $m=n$, the term is finite, giving

$$(1-x^2)\int_0^\pi\frac{\cos n\omega\,.\,d\omega}{X}=\int_0^\pi x^n(1+\cos 2n\omega)\,d\omega.$$

But $$\int_0(1+\cos 2n\omega)\,d\omega=\omega+\frac{\sin 2n\omega}{2n};$$

and the last term vanishes when $\omega = \pi$, making our integral $x^n . \pi$. Dividing both sides by $(1 - x^2)$, there remains

$$\int_0^\pi \frac{\cos n\omega \,.\, d\omega}{1 - 2x\cos\omega + x^2} = \frac{\pi x^n}{1 - x^2}.$$

27. In the original M change x to $-x$ and M to M', X to X', and take $\frac{1}{2}(M - M')$; observe that

$$X \,.\, X' = 1 - 2x^2 \cos 2\omega + x^4$$

by the Elements of Trigonometry, then

$$\frac{(x - x^3)\cos\omega}{1 - 2x^2\cos 2\omega + x^4} = x\cos\omega + x^3\cos 3\omega + x^5 \cos 5\omega + \&c.$$

Divide by x, then write everywhere x for x^2, then

$$\frac{(1 - x)\cos\omega}{1 - 2x\cos 2\omega + x^2} = \cos\omega + x\cos 3\omega + x^2\cos 5\omega + x^3 \cos 7\omega + \&c.$$

Not to dwell on special results when the assumption $x = 1$ is permissible, suppose

$$x = \tan\tfrac{1}{2}\theta, \quad \therefore \tfrac{1}{2}\theta < 45°, \ \theta < \tfrac{1}{2}\pi.$$

Then $\quad X = (1 + x^2) - 2x\cos\omega = (1 + x^2)\left\{1 - \frac{2x}{1 + x^2} \,.\, \cos\omega\right\}.$

But $\quad 1 + x^2 = (\sec\tfrac{1}{2}\theta)^2,$

and $\quad \dfrac{2x}{1 + x^2} = \sin\theta,$

$$\therefore X = \sec^2\tfrac{1}{2}\theta\,(1 - \sin\theta\cos\omega).$$

But we had $-\frac{1}{2}\log X$ in series of powers of x^2, whence

$$-\tfrac{1}{2}\log(1 - \sin\theta\cos\omega) = \log\sec\tfrac{1}{2}\theta + x\cos\omega + \tfrac{1}{2}x^2\cos 2\omega + \tfrac{1}{3}x^3\cos 3\omega + \&c.$$

Reverse the signs of x and θ, which does not affect $\sec(\frac{1}{2}\theta)$,

$$\therefore -\tfrac{1}{2}\log(1 + \sin\theta\cos\omega) = \log\sec\tfrac{1}{2}\theta - x\cos\omega + \tfrac{1}{2}x^2\cos 2\omega - \tfrac{1}{3}x^3\cos 3\omega + \&c.$$

Halve the sum of the two last, then

$$-\tfrac{1}{4}\log(1 - \sin^2\theta\cos^2\omega) = \log\sec\tfrac{1}{2}\theta + \tfrac{1}{2}x^2\cos 2\omega + \tfrac{1}{4}x^4\cos 4\omega + \&c.$$

In the last we may change

$$\omega \text{ to } (\tfrac{1}{2}\pi - \omega),\ 2n\omega \text{ to } n\pi - 2n\omega,\ \cos 2n\omega \text{ to } \cos n\pi \,.\, \cos 2n\omega,$$

then $-\frac{1}{4}\log(1 - \sin^2\theta \sin^2\omega)$

$$= \log \sec \tfrac{1}{2}\theta - \tfrac{1}{2}x^2 \cos 2\omega + \tfrac{1}{4}x^4 \cos 4\omega - \&c.$$

28. A more arduous transformation is possible. For x write $\epsilon^{-y\sqrt{-1}}$, which being $= \cos y - \sqrt{-1} \sin y$ will not damage convergence in the series. Now $\cos\theta = \dfrac{1 - x^2}{1 + x\epsilon}$. Hence

$$\cos\theta = \frac{1 - \epsilon^{-2y\sqrt{-1}}}{1 + \epsilon^{-2y\sqrt{-1}}} = \frac{\epsilon^{y\sqrt{-1}} - \epsilon^{-y\sqrt{-1}}}{\epsilon^{y\sqrt{-1}} + \epsilon^{-y\sqrt{-1}}} = \frac{\sin y}{\cos y} \,.\, \sqrt{-1},$$

or

$$\cos\theta = \sqrt{-1}\,\tan y,$$

also

$$\sin\theta = \frac{2x}{1 + x^2} = \frac{2\epsilon^{-y\sqrt{-1}}}{1 + y^{-2y\sqrt{-1}}} = \frac{2}{\epsilon^{y\sqrt{-1}} + \epsilon^{-y\sqrt{-1}}} = \sec y.$$

Then

$$1 - \sin^2\theta \sin^2\omega = 1 - \sec^2 y \sin^2\omega.$$

The coefficient of $\sin^2\omega$ was previously *less* than 1. We now make it to exceed 1.

Also

$$x^n = \epsilon^{-ny\sqrt{-1}} = \cos ny - \sqrt{-1} \sin ny.$$

Thus the series splits in two. We have likewise

$$\sec^2 \tfrac{1}{2}\theta = 1 + x^2 = 1 + \cos 2y - \sqrt{-1} \sin 2y.$$

Put

$$A = 1 + \cos 2y;\ B = \sin 2y,$$

$$\log(A + B\sqrt{-1}) = \frac{1}{2}\log(A^2 + B^2) + \sqrt{-1}\,\tan^{-1}\frac{B}{A},$$

where

$$A^2 + B^2 = 1 + 2\cos 2y + 1,$$

$$\frac{B}{A} = \frac{\sin 2y}{1 + \cos 2y} = \tan y,$$

and if y is $< \frac{1}{2}\pi$,

$$\tan^{-1}\tan y = y.$$

Observe that

$$\log \sec \tfrac{1}{2}\theta = \tfrac{1}{2}\log(\sec^2 \tfrac{1}{2}\theta).$$

These give

$$\log \sec^2 \tfrac{1}{2}\theta = \tfrac{1}{2}\log(2 + 2\cos 2y) + \sqrt{-1}\,y;$$

and

$$2(1 + \cos 2y) = 4\cos^2 y.$$

From the real terms of our general equation we now find

$-\frac{1}{4}\log(1 - \sec^2 y \sin^2\omega)$

$$= \tfrac{1}{2}\log(2\cos y) - \tfrac{1}{2}\cos 2y \cos 2\omega + \tfrac{1}{4}\cos 4y \cos 4\omega - \tfrac{1}{6}, \&c.,$$

and more simply, by transposing $\frac{1}{2}\log\cos y = \frac{1}{4}\log\cos^2 y$,

$$-\tfrac{1}{4}\log(\cos^2 y - \sin^2\omega)$$
$$= \tfrac{1}{2}\log 2 - \tfrac{1}{2}\cos 2y\cos 2\omega + \tfrac{1}{4}\cos 4y\cos 4\omega - \&c.$$

From the imaginary terms,

$$\tfrac{1}{2}y = \tfrac{1}{2}\cos 2\omega\sin 2y - \tfrac{1}{4}\cos 4\omega\sin 4y + \tfrac{1}{6}\cos 6\omega\sin 6y - \&c.,$$

a series well known when ω vanishes, but new and surprising here, where ω is arbitrary.

We have supposed y less than $\frac{1}{2}\pi$. Change 2ω, $2y$ to ω and η, then η is less than π. Then the last series doubled gives

$$\tfrac{1}{2}\eta = \cos\omega\sin\eta - \tfrac{1}{2}\cos 2\omega\sin 2\eta + \tfrac{1}{3}\cos 3\omega\sin 3\eta - \&c.$$

Also the previous one admits improvement, since

$$2\cos^2\eta - 2\sin^2\omega = (1+\cos 2\eta) - (1-\cos 2\omega) = \cos 2\eta + \cos 2\omega,$$

so that by the same changes

$$\tfrac{1}{2}\log(2\cos\eta + 2\cos\omega)$$
$$= \cos\eta\cos\omega - \tfrac{1}{2}\cos 2\eta\cos 2\omega + \tfrac{1}{3}\cos 3\omega\cos 3\eta - \&c.$$

The symmetry indicates then ω, like η, must be less than π, though it has not appeared in what stage that limitation first came in.

29. It is worth while to verify these two series by an entirely different process.

Put

$$Z = u\cos\omega - \frac{1}{2}u^2\cos 2\omega + \frac{1}{3}u^3\cos 3\omega - \&c.\ldots - \frac{1}{2n}\,.\,u^{2n}\cos 2n\omega;$$

a series of $2n$ terms. Suppose u variable and ω constant, also

$$u = \epsilon^{\theta\sqrt{-1}},\quad v = \epsilon^{\omega\sqrt{-1}}.$$

Differentiate;

$$\therefore\ \frac{dZ}{du} = \cos\omega - u\cos 2\omega + u^2\cos 3\omega - \&c.\ \ldots - u^{2n-1}\cos 2n\omega.$$

For the cosines substitute from $2\cos\omega = v + v^{-1}$, and generally $2\cos r\omega = v^r + v^{-r}$,

$$\therefore\ \frac{2dZ}{du} = \left.\begin{aligned} &v - uv^2 + u^2v^3 - \&c.\ \ldots - u^{2n-1}v^{2n} \\ &+ v^{-1} - uv^{-2} + u^2v^{-3} - \&c.\ \ldots - u^{2n-1}v^{-n} \end{aligned}\right\}.$$

These two series can be summed separately, making

$$\frac{v(1-\overline{uv}|^{2n})}{1+vu}+\frac{v^{-1}(1-u^{2n}v^{-2n})}{1+v^{-1}u},$$

and integrating back

$$2Z=\int_0\frac{vdu}{1+vu}+\int_0\frac{v^{-1}du}{1+v^{-1}u}-v^{2n+1}\int_0\frac{u^{2n}du}{1+vu}-v^{-2n-1}\int_0\frac{u^{2n}du}{1+v^{-1}u}.$$

The formula

$$\int\frac{u^{2n}du}{x}=\frac{u^{2n+1}}{(2n+1)x}-\frac{1}{2n+1}\int_0 u^{2n+1}d\left(\frac{1}{x}\right)$$

applies to both the last integrals, if we make $x=1+v^{\pm 1}u$. Observe that

$$u^{2n+1}=\cos(2n+1)\theta+\sqrt{-1}\sin(2n+1)\theta,$$

varying through narrow limits. If then the denominators $1+vu$ and $1+v^{-1}u$ never vanish, the hypothesis of n infinite expunges both of the uncertain integrals, and leaves accurately

$$2Z=\int_0\frac{vdu}{1+vu}+\int_0\frac{v^{-1}du}{1+v^{-1}u}.$$

Now $1+vu=1+\epsilon^{\omega\sqrt{-1}}\epsilon^{\theta\sqrt{-1}}=1+\epsilon^{(\omega+\theta)\sqrt{-1}}$

$$=1+\cos(\omega+\theta)+\sqrt{-1}\sin(\omega+\theta),$$

which cannot vanish unless the real and imaginary parts vanish, each separately. Now $\sin(\omega+\theta)$ vanishes when $\omega+\theta$ is *any* multiple of π. Can $1+\cos(\omega+\theta)$ then also vanish? Only if $(\omega+\theta)$ be an *odd* multiple of π. Exactly in the same way it appears that

$$1+uv^{-1}=1+\cos(\theta-\omega)+\sqrt{-1}\sin(\theta-\omega),$$

and vanishes only when $\theta-\omega$ is an odd multiple of π. If either $(\theta+\omega)$ or $(\theta-\omega)$ be such, our process fails to prove anything. In integrating back, we regain the series for $2Z$; but in a new integration

$$2Z=C+\log(1+vu)+\log(1+v^{-1}u),$$

where C is zero, because Z in the original series vanishes with u. Suppose ω less than $\frac{1}{2}\pi$, and θ beginning from zero, never to become so great that $\theta+\omega=\pi$, this secures the needful conditions.

We have now

$$\begin{aligned}2Z&=\log\{\overline{1+vu}\,.\,\overline{1+v^{-1}u}\}=\log(1+\overline{v+v^{-1}}\,.\,u+u^2)\\&=\log\{1+2\cos\omega(\cos\theta+\sqrt{-1}\sin\theta)\\&\qquad+(\cos 2\theta+\sqrt{-1}\sin 2\theta)\};\end{aligned}$$

call this $\log(A + \sqrt{-1}B)$,

$$\therefore\ A = 1 + 2\cos\omega\cos\theta + \cos 2\theta = 2\cos^2\theta + 2\cos\omega\cos\theta$$
$$= 2\cos\theta\,.\,(\cos\omega + \cos\theta),$$

and $B = 2\cos\omega\sin\theta + \sin 2\theta = 2\cos\omega\sin\theta + 2\sin\theta\cos\theta$

$$= 2\sin\theta\,(\cos\omega + \cos\theta),$$

whence $$\frac{B}{A} = \frac{\sin\theta}{\cos\theta} = \tan\theta,$$

and $$\tan^{-1}\frac{B}{A} = \tan^{-1}\tan\theta,$$

or here $= \theta$, also

$$A^2 + B^2 = 4\,(\cos^2\theta + \sin^{-2}\theta)\,(\cos\omega + \cos\theta)^2,$$
$$\sqrt{(A^2 + B^2)} = 2\,(\cos\omega + \cos\theta).$$

Thus $$2Z = \log(2\cos\omega + 2\cos\theta) + \sqrt{-1}\,.\,\theta.$$

At the same time Z *now*

$$= u\cos\omega - \tfrac{1}{2}u^2\cos 2\omega + \tfrac{1}{3}u^3\cos 3\omega - \tfrac{1}{4}, \text{\&c.} \ldots \textit{ad infin.}$$

or $$\left.\begin{array}{l} = \cos\theta\cos\omega - \frac{1}{2}\cos 2\theta\cos 2\omega + \frac{1}{3}\cos 3\theta\cos 3\omega - \text{\&c.} \\ + \sqrt{-1}\,\{\sin\theta\cos\omega - \frac{1}{2}\sin 2\theta\cos 2\omega + \frac{1}{3}\sin 3\theta\cos 3\omega - \text{\&c.}\}\end{array}\right\}.$$

Equating then separately the real and imaginary parts, the penultimate series $= \frac{1}{2}\log(2\cos\omega + 2\cos\theta)$, and (removing $\sqrt{-1}$), the last $= \frac{1}{2}\theta$.

This exactly agrees with the series above, which involved η and ω. But we now attain the strict condition that neither $\theta + \omega$ nor $\theta - \omega$ must be (within the limits here concerned) anywhere an odd multiple of π.

Plunge into the Calculus of Symbols.

30. The Trigonometry rising out of the Factors of $\sin x$ carries us unexpectedly farther.

We found various identities in which x appeared quite arbitrary, and among these

$$\frac{x\,.\,\epsilon^x}{\epsilon^x - 1} \text{ or } \frac{x}{1 - \epsilon^{-x}} = 1 + \frac{x}{2} + 2^{-1}.H_1.x^2 - 2^{-3}.H_2.x^4 + 2^{-5}.H_3.x^6 - \text{\&c.},$$

or again, if we change x to $-y$, it yields

$$\frac{y}{\epsilon^y - 1} = 1 - \frac{y}{2} + 2^{-1}H_1y^2 - 2^{-3}H_2y^4 + 2^{-5}H_3y^6 - \text{\&c.} \ldots\ldots\ldots\ldots \text{(A)}.$$

But we have no right to treat this equation as an identity, unless the series converges, and if we substitute a mere *symbol* as Δ or Σ for y, *we have no test of convergence;* therefore the result becomes not a demonstrated equation, but a something to be *inquired into,* a probability in fact; and this in strict Mathematics! This is paradoxical at first, but the thought underlies all interpolation. Maclaurin first was bold enough to start the application to equation (A), and the student will quickly see that our conviction of truth here springs, not from the cogency of the proof, but from the harmony of results.

In Taylor's Theorem, write D for $\frac{d}{dx}$; then

$$\phi(x+h) = \phi(x) + \frac{h}{1} D\phi x + \frac{h^2}{1.2} D^2\phi x + \&c.,$$

which may be shortened into

$$(1+\Delta)\phi x = \left\{1 + \frac{h}{1} D + \frac{h^2}{1.2} . D^2 + \frac{h^3}{1.2.3} D^3 + \&c.\right\} \phi x.$$

The series of operations implied on the right, may be compressed into the symbol ϵ^{hD}; after which we discern from this equation that the complex of operations denoted by ϵ^{hd} is equivalent to the simple $1+\Delta$.

Now if it be proposed to sum the series

$$\phi(x) + \phi(x+1) + \phi(x+2) + \&c. \textit{ ad infin.}$$

in which $\phi(x+n)$ vanishes when n is infinite, and the sum is presumed to be finite, we may call it $F(x)$. Then we see that $F(x+1)$ is *less* than $F(x)$ by the first term $\phi(x)$, so that

$$\Delta . F(x) = -\phi(x),$$

of which we might represent the integral by

$$F(x) = -\Sigma(\phi x),$$

if we include a possible arbitrary constant. But instead of Σ it is lawful to write the reverse operation Δ^{-1}, then

$$Fx = -\Delta^{-1} . \phi(x).$$

Superadd the *differential* process by the operation $\frac{d}{dx} = D$, and observe that the order of operation $D . \Delta^{-1}$ is equivalent to $\Delta^{-1} . D$, so that each may be denoted by $\frac{D}{\Delta}$, then our equivalence of operations is

$$D \,.\, F(x) = -\frac{D}{\Delta} \,.\, \phi(x).$$

Above, assume $h = 1$, to adapt it to our series in which x increases by 1, then

$$1 + \Delta = \epsilon^D,$$

and

$$D \,.\, F(x) = \left(\frac{-D}{\epsilon^D - 1}\right) \phi(x).$$

In equation (A) above, introduce the symbol D instead of the numerical y; then $D \,.\, F(x)$ is *probably* equivalent to

$$\{-1 + \tfrac{1}{2}D - 2^{-1}H_1 D^2 + 2^{-3}H_2 D^4 - \&c.\}\ \phi(x).$$

Multiply by dx and integrate; which lowers D^n to D^{n-1}, D into D^0 or 1;

$$\therefore\ F(x) = \int - \phi x dx$$

$$+ \left(\frac{1}{2} - 2^{-1}H_1 D + 2^{-3} H_2 D^3 - 2^{-5} H_3 D^5 + \&c.\right) \phi(x) \ \ldots\ldots.(B).$$

Such is Maclaurin's celebrated "Sum-Formula."

31. Its first application is to

$$S_n = 1^{-n} + 2^{-n} + 3^{-n} + 4^{-n} + \&c.,$$

where we may by actual addition of the $(m-1)$ first terms, leave only a residue

$$R = m^{-n} + (m+1)^{-n} + (m+2)^{-n} + \ldots$$

of infinite length. Here m being at our free choice, we may write x for it; and thus have

$$R(x) = x^{-n} + (x+1)^{-n} + (x+2)^{-n} + \&c. \ldots$$

$\therefore\ \phi(x)$ now means x^{-n}. Our application of it to S_n supposes x integer; but we need not so *limit* it in $R(x)$. Making x vary, we find

$$\int - \phi(x)\, dx = \int - x^{-n} dx = \frac{x^{-n+1}}{n-1},$$

which vanishes when x is infinite. The same we shall find true of every other term, therefore no constant of integration is required.

Further $D\phi(x) = -nx^{-n-1}$;

$$D^3\phi(x) = -n.n+1.n+2.x^{-n-3};$$

$$D^5\phi(x) = -n.n+1.n+2.n+3.n+4.x^{-n-5};$$

and the law is clear.

Hence

$$\left.\begin{aligned} R(x) = \frac{x^{-n+1}}{n-1} + \frac{1}{2}.x^{-n} + 2^{-1}H_1.\frac{n}{x^{n+1}} - 2^{-3}H_2.\frac{n.n+1.n+2}{x^{n+3}} \\ + 2^{-5}H_3.\frac{n.n+1.n+2.n+3.n+4}{x^{n+5}} - \&c. \end{aligned}\right\} \ldots(C).$$

After a certain number of terms, the numerator wins on the denominator, and the series ceases to converge. *There* our calculation must cease. The divergency which follows has signs alternately + and −, and is to us unmeaning. Legendre calls such series half-converging.

32. Apply (C) first to calculate S_2 by assuming $n = 2$. This gives for (C) from known values of the H series;

$$R_2(x) = x^{-1} + \frac{1}{2}x^{-2} + \frac{1}{6}x^{-3} - \frac{x^{-5}}{30} + \frac{x^{-7}}{6.7} - \frac{x^{-9}}{30} + \frac{10x^{-11}}{11.12} - \&c.$$

if we may stop at the term containing H_5. Call the term containing H_r simply M_r, then

$$M_5 = \text{here } \frac{10x^{-11}}{11.12},$$

whence $$\frac{M_6}{M_5} = \frac{H_6}{H_5}.\frac{n+9.n+10}{2^2.x^2}$$

in general, and *here* $$\frac{H_6}{H_5}.\frac{11.12}{4x^2},$$

whence $$M_6 = \frac{H_6}{H_5}.\frac{10}{4}.x^{-13},$$

or nearly $\frac{1}{4}x^{-13}$, since H_6 very little exceeds $\frac{1}{10}H_5$. If then x is taken as large as 10, the terms written down seem likely to give us a result accurate to 12 decimals. But try $x = 15$, and then to check it, try also $x = 16$. Then we have as a test of accuracy,

$$R_2(15) = 15^{-2} + R_2(16).$$

When $x = 15$, we obtain for $R_2(15)$

15^{-1}	·0666 6666 6666 6667
$\frac{1}{2}15^{-2}$	+ 22 2222 2222 2222
$\frac{1}{6}15^{-3}$	... + 4938 2716 0494
$-M_2$	...... − 4 3895 7475
$+M_3$	.... + 139 3518
$-M_4$	.... − 3670
$+M_5$	.... + 88
$-M_6$ slightly exceeds $\frac{1}{4}.15^{-13}$ or	− 1, 3
$R_2(15) =$	·0689 3822 7847 6836

Next with $x = 16$

16^{-1}	·0625
$\frac{1}{2}16^{-2}$	+ 19 5312 5
M_1	... + 4069 0104 1667
$-M_2$	...... − 3 1789 1439
M_3	.... + 88 6973
$-M_4$	.... − 4051
M_5	.... + 43
$-M_6$	.... − 0, 5
$R_2(16) =$	·0644 9378 3403 2393
Add $15^{-2} =$	44 4444 4444 4444
therefore $R_2(15) =$	·0689 3822 7847 6837

The two results differ only by a unit in the 16th place. Apparently if we had carried the work to 18 decimals, we should have found agreement in the 16th. See the enormous chances against such agreement, if the equation (C), *which has not been absolutely demonstrated,* were not trustworthy, when rightly managed. Again, try $x = 20$, also $x = 21$; work with 17 decimals.

For $x = 20$

	·05
	+ ·0012 5
	+ 2083 3333 33333
	− 1 0416 66666
	+ 18 60119
	− 6510
	+ 37
$R_2(20) =$	·0512 7082 2935 20313

For $x = 21$

$$\begin{array}{rl} & \cdot 0476\ 1904\ 7619\ 04762 \\ & +\ 11\ 3378\ 6848\ 07256 \\ & +\ 1799\ 6616\ 63607 \\ & -\ 8161\ 73090 \\ & +\ 13\ 21952 \\ & -\ 4196 \\ & +\ 21 \\ \hline R_2(21) = & \cdot 0487\ 7082\ 2935\ 20312 \\ \text{Add} & \cdot 0025 = 20^{-2} \\ \hline \therefore\ R_2(20) = & \cdot 0512\ 7082\ 2935\ 20312 \end{array}$$

We now have agreement all but a unit in the 17th decimal. Above, Art. 14, we gave decimal computation of H_6, H_7, H_8 ... where $H_6 = \cdot 000001082...$ or barely $\frac{9\,.\,12}{10^8}$, by which we can on occasion find M_6, or learn that it is insignificant.

33. Proceed to $n = 3$,

$$\therefore R_3(x) = \tfrac{1}{2}x^{-2} + \tfrac{1}{2}x^{-3} + \tfrac{1}{4}x^{-4} - \tfrac{1}{12}(x^{-6} - x^{-8}) - \tfrac{3}{20}x^{-10} + \tfrac{10}{24}x^{-12} - \&c.$$

When $n = 4$,

$$R_4(x) = \tfrac{1}{3}x^{-3} + \tfrac{1}{2}x^{-4} + \tfrac{1}{3}x^{-5} - \tfrac{1}{6}x^{-7} + \tfrac{2}{9}x^{-9} - \tfrac{1}{2}x^{-11} + \tfrac{10}{6}x^{-13} + \&c.$$

When $n = 5$,

$$R_5(x) = \tfrac{1}{4}x^{-4} + \tfrac{1}{2}x^{-5} + \tfrac{5}{12}x^{-6} - \tfrac{1}{3}\,.\,\tfrac{7}{8}x^{-8} + \tfrac{1}{2}x^{-10} - \tfrac{11}{8}x^{-12} + 5\left(1 + \tfrac{1}{12}\right)x^{-14} - \&c.$$

When $n = 6$,

$$R_6(x) = \tfrac{1}{5}x^{-5} + \tfrac{1}{2}x^{-6} + \tfrac{1}{2}x^{-7} - \tfrac{7}{15}x^{-9} + x^{-11} - 3\,.\,\tfrac{11}{10}x^{-13} + 13\,.\,\tfrac{7}{6}x^{-15} - \&c.$$

so $$R_7(x) = \tfrac{1}{6}x^{-6} + \tfrac{1}{2}x^{-7} + \tfrac{7}{12}x^{-8} - \tfrac{7}{10}x^{-10} + \tfrac{11}{6}x^{-12} - \frac{11\,.\,13}{20}x^{-14} + \frac{5\,.\,7\,.\,13}{3\,.\,4}x^{-16} - \&c.$$

$$R_8(x) = \tfrac{1}{7}x^{-7} + \tfrac{1}{2}x^{-8} + \tfrac{2}{3}x^{-9} - x^{-11} + \tfrac{22}{7}x^{-13} - \frac{11\,.\,13}{10}x^{-15} + \&c.$$

$$R_9(x) = \tfrac{1}{8}x^{-8} + \tfrac{1}{2}x^{-9} + \tfrac{5}{4}x^{-10} - \tfrac{11}{8}x^{-12} + \&c.,$$

and so on; which explains the organic process for calculating S_n as far as we choose, and the ease of verifying.

In each case we have to prefix the initial terms

$$1^{-n} + 2^{-n} + 3^{-n} + \ldots + (m-1)^n,$$

which we omitted in Art. 31.

34. But we cannot define S_1, because

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c.\ldots \text{ is infinite.}$$

Nevertheless, assuming $f(n)$ for n terms of this series, observe that

$$n = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1}.$$

Take the logarithm of the last from the former, then

$$f(n) - \log n = 1 - \log\frac{2}{1} + \frac{1}{2} - \log\frac{3}{2} + \frac{1}{3} - \&c.\ldots - \log\frac{n}{n-1} + \frac{1}{n};$$

LEMMA: $\frac{1}{m}$ lies between $\log\frac{m+1}{m}$ and $\log\frac{m}{m-1}$.

For first $\frac{1}{m}$ is *greater* than

$$\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \&c.$$

or $$> \log\left(1 + \frac{1}{m}\right),$$

that is, $$\log\frac{m+1}{m}.$$

Next $\frac{1}{m}$ is *less* than

$$\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^2} + \&c.$$

or than $$-\log\left(1 - \frac{1}{m}\right),$$

or $$\log\frac{m}{m-1}.$$

This being the case, the terms in the series of alternate sign, equivalent to $f(n) - \log n$, successively diminish, and if n be perpetually increased, the total converges to a finite limit

$$\gamma = 1 - \log\tfrac{2}{1} + \tfrac{1}{2} - \log\tfrac{3}{2} + \tfrac{1}{3} - \&c.\ldots \text{ ad infin.}$$

This γ is called Euler's Integral. Professor J. C. Adams has computed and verified it up to 263 decimals. To 20 figures it is

$$\gamma = \cdot 57721\ 56649\ 01532\ 86060.$$

This constant arises out of Euler's Function called Γ (Gamma) by Legendre. It is in some sense a substitute for S_1.

The γ can be computed from $f(n) - \log(n)$ by Maclaurin's Sum Formula, with n infinite, but more elegantly by several series of the *Gamma* Function especially

$$\gamma + \log 2 = 1 + \tfrac{1}{2}(S_2 - 1) - \tfrac{1}{3}(S_3 - 1) + \tfrac{1}{4}(S_4 - 1) - \&c.,$$

and with greater convergence,

$$1 - \gamma = \log\frac{3}{2} + \frac{2^{-2}}{3}(S_3 - 1) + \frac{2^{-4}}{5}(S_5 - 1) + \frac{2^{-6}}{7}(S_7 - 1) + \&c.$$

The values of S_n are supposed to be known, and no idea of *half-*convergence now enters. A transcendal equivalent of γ is in

$$-\gamma = \int_0^\infty \epsilon^{-x} . \log x . dx,$$

which rather needs than gives light.

35. Legendre's table of S_n to 16 decimals may be conveniently presented as follows;

n	$S_n - 1$	n	$S_n - 1$
2	·6449 3406 6848 2264	19	190 8212 7166
3	·2020 5690 3159 5943	20	95 3962 0339
4	823 2323 3711 1382	21	47 6932 9868
5	369 2775 5143 3700	22	23 8450 5027
6	173 4306 1984 4491	23	11 9219 9260
7	83 4927 7381 9227	24	5 9608 1891
8	40 7735 6197 9443	25	2 9803 5035
9	20 0839 2826 0822	26	1 4901 5548
10	9 9457 5127 8180	27	...... 7450 7118
11	4 9418 8604 1194	28	...... 3725 3340
12	2 4608 6553 3080	29	...... 1862 6597
13	1 2271 3347 5785	30	...... 931 3274
14	...... 6124 8135 0587	31	...... 465 6629
15	...... 3058 8236 3070	32	...... 232 8312
16	...... 1528 2259 4086	33	...... 116 4155
17	...... 763 7197 6379	34	...... 58 2077
18	..... 381 7293 2650	35	...... 29 1038

When n exceeds 34, no term after the first affects the 16th decimal. Hence $S_n - 1$ confounds itself with 2^{-n}, when we neglect decimals beyond.

I write T_n for $1^{-n} + 3^{-n} + 5^{-n} + 7^{-n} + \&c. = (1 - 2^{-n}) S_n$.

Then from Legendre's $S_n - 1$ I deduce

n	$T_n - 1$	n	$T_n - 1$
2	·2337 0055 0136 1698	13	62 8055 6218
3	517 9979 0204 6450	14	20 9240 5193
4	146 7803 1604 1920	15	6 9724 7031
5	45 2376 2795 1397	16	2 3237 1573
6	14 4707 6640 9421	17	...... 7744 8400
7	4 7154 8652 3765	18	...... 2581 4375
8	1 5517 9025 2961	19	...... 860 4442
9	...... 5134 5183 8438	20	...... 286 8077
10	...... 1704 1363 0447	21	...... 95 6012
11	...... 566 6051 0901	22	...... 31 8668
12	...... 188 5848 5831	23	...... 10 6222

When x exceeds 23, 5^{-n} gives only a unit to 17th decimal;

$$\therefore\ T_n - 1 = 3^{-n}.$$

Further, from S_n we deduce U_n, or rather

$$1 - U_n = 2^{-n} - 3^{-n} + 4^{-n} - \&c.$$

n	$1 - U_n$	n	$1 - U_n$
2	·1775 3296 6575 8866	19	190 6491 8289
3	·0984 5932 2630 3043	20	95 3388 4184
4	·0529 6717 0502 7541	21	47 6741 7845
5	·0278 8022 9553 0906	22	23 8386 7692
6	·0144 4890 8702 5649	23	11 9198 6815
7	74 0618 0077 1698	24	5 9601 1076
8	37 6699 8147 3521	25	2 9801 1431
9	19 0570 2458 3947	26	1 4900 7680
10	9 6049 2401 7284	27	...... 7450 4495
11	4 8324 7126 9390	28	...... 3725 2466
12	2 4231 4856 1418	29	...... 1862 6306
13	1 2145 7236 7349	30	...... 931 3177
14	...... 6082 9654 0203	31	...... 465 6597
15	...... 3044 8786 9008	32	..... 232 8301
16	...... 1523 5785 0939	33	.. ... 116 4151
17	...... 762 1707 9599	34	...... 58 2076
18	...... 381 2130 3899	35	$= 2^{-n}$

From Gudermann (p. 72) I infer the following, correcting only an error of a unit in 7th place of $\frac{1}{12}\pi^2$.

n	$1 - U_n$
2	·1775 3296 6575 8867 8177
4	·0529 6717 0502 7640 8242
6	·0144 4890 8702 5648 9591
8	37 6699 8147 3521 0076
10	9 6049 2401 7283 8435
12	2 4231 4856 1418 0912
14	...... 6082 9654 0202 8188
16	...... 1523 5785 0938 9360
18	...... 381 2130 3898 8649

Gudermann's value of $L\left(\frac{1}{2}\pi - v\pi\right)$ represents the series $\int \frac{-\pi dv}{\sin \pi v}$ (see Art. 4, above) or $\log \frac{1}{v} + c - \Sigma \frac{1}{m} . U_{2m} . v^{2m}$. He gives the coefficients $m^{-1}.U_{2m}$ in figures. Given $m^{-1}.U_{2m}=a$, I infer $U_{2m}=ma$, and $1-U_{2m}=1-ma$ from which I have made out the last table.

36. *On* $V_n = 1 - 3^{-n} + 5^{-n} - 7^{-n} + \&c.$

We may treat this as the difference of two series

$$P_n = 3^{-n} + 7^{-n} + 11^{-n} + \&c.;\quad Q_n = 5^{-n} + 9^{-n} + 13^{-n} + \&c.$$

If each be calculated by Maclaurin's method, $P_n + Q_n = T_n - 1$, which last is known already, and gives a severe test,

then $$P_n - Q_n = 1 - V_n.$$

This process is elaborate, but very safe.

To adapt P and Q to the form of

$$\phi(x) + \phi(x+1) + \phi(x+2) + \&c.$$

in which the variable increases by 1 at each step, we must first change x to u or v, then make $u = 4x \mp 1$. It replaces du or dv by $4dx$, so that the symbol D each time doubles the result. Thus our $R_n(x)$ becomes

$$\int -\phi(u)\,dx + \frac{1}{2}u^{-n} + 2^1.H_1 n u^{-n-1}$$
$$-2^3.H_2 n.(n+1)(n+2)u^{-n-3} + 2^5 H.n.(n+1)\ldots(n+4)u^{-n-4}.$$

But, to begin from $u=2$, and $u = 4x \pm 1$,

$$\int -\phi(u)\,dx = \frac{1}{4}u^{-1},$$

$$M_1 = \frac{2}{3}u^{-3};\quad M_2 = 2^3.\frac{2.3.4}{90}u^{-5} = \frac{2^6}{30}u^{-5}.$$

$$M_3 = 2^5.\frac{2.3.4.5.6}{3.5.7.9}u^{-7} = \frac{2^9}{3.7}u^{-7};$$

and so on. *First*, take $u=45$, for $5^{-2}+9^{-2}+13^{-2}+\&c.$ to find the residue $R_2(45)$ to 18 decimals.

$$
\begin{array}{rr}
\tfrac{1}{4}u^{-1} = & 00555\ 55555\ 55555\ 555 \\
\tfrac{1}{2}u^{-2} = & 24\ 69135\ 30246\ 913 \\
M_1 = & 73159\ 57933\ 232 \\
-M_2 = & -115\ 61019\ 944 \\
M_3 = & 65247\ 379 \\
-M_4 = & -721\ 757 \\
M_5 = & 12\ 943 \\
\hline
\therefore\ R_2(45) = & 00580\ 97735\ 97254\ 034
\end{array}
$$

Observe that $R_2(45) = 45^{-2} + R_2(47)$.

Next with $u = 49$

$$
\begin{array}{rr}
 & 00510\ 20408\ 16326\ 530 \\
 & \ldots\ 20\ 82465\ 62931\ 695 \\
 & \ldots\ldots\ 56665\ 73168\ 209 \\
 & \ldots\ldots\ldots\ -75\ 52284\ 106 \\
 & \ldots\ldots\ldots\ldots\ 35948\ 278 \\
 & \ldots\ldots\ldots\ldots\ -335\ 377 \\
 & \ldots\ldots\ldots\ldots\ldots\ 5\ 079 \\
\hline
 & \cdot 00531\ 59464\ 36760\ 308 = R_2(49) \\
\text{Add } 45^{-2} & 49\ 38271\ 60493\ 827 \\
\text{whence } R_2(45) = & \cdot 00580\ 97735\ 97254\ 135
\end{array}
$$

Our error is only a unit in 16th decimal, though we have not taken account of M_6.

By direct addition find

$$A = 5^{-2} + 9^{-2} + 13^{-2} + \ldots 41^{-2},$$

or

$$
\begin{array}{rr}
A = & \cdot 06902\ 32985\ 59440\ 393 \\
\text{Add } R_2(45) = & 580\ 97735\ 97254\ 034 \\
\hline
Q_2 = & 07483\ 30721\ 56694\ 427
\end{array}
$$

Proceed to $u = 4x - 1$, and compute $R_2(47)$ side by side with $R_2(51)$.

For $R_2(47)$

$$
\begin{array}{rr}
 & 00531\ 91489\ 33170\ 212 \\
 & 22\ 63467\ 63241\ 285 \\
M_1 & 64211\ 84772\ 802 \\
-M_2 & -93\ 01852\ 091 \\
M_3 & 48124\ 437 \\
-M_4 & -487\ 997 \\
M_5 & 1\ 944 \\
\hline
 & 00555\ 19076\ 29976\ 592
\end{array}
$$

For $R_2(51)$

$$\begin{array}{r} 00490\ 19607\ 84313\ 725 \\ 19\ 22337\ 56247\ 597 \\ 50257\ 19117\ 584 \\ -\ 61\ 83122\ 328 \\ 27168\ 110 \\ -\ 233\ 974 \\ 3\ 177 \\ \hline \cdot 00509\ 92141\ 03493\ 891 \\ 45\ 26935\ 26482\ 571 \\ \hline \cdot 00555\ 19076\ 29976\ 462 \end{array} \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ = R_2(51) \\ = 47^{-2} \\ = R_2(47) \end{array}$$

The two results again differ by a unit in the 16th decimal place, the former being the greater. But in neither is M_6 estimated; a negative term, greater in the former. With direct calculus of $R_2(47)$ M_6 (roughly) gives 206 for the three last figures, thus reducing 592 to 386: and with $R_2(51)$ I make M_6 (roughly) $= 077$, reducing the other result 462 to 385, with a difference of only a unit in the 18th decimal place. Thus we may trust as secure

$$R_2(47) = \cdot 00555\ 19076\ 29976\ 38|6.$$

It only remains to add the previous terms

$$3^{-2} + 7^{-2} + \ldots + 43^{-2}.$$

Now $\qquad 3^{-2} + 7^{-2} + \ldots + 43^{-2} = B$

$= \cdot 15531\ 55703\ 49498\ 914$

Add $\qquad 555\ 19076\ 29976\ 386 = R_2(47)$,

$\therefore\ \cdot 15886\ 74779\ 79475\ 300 = 3^{-2} + 7^{-2} + \&c.$ ad infin. P_2.

Add $\qquad \cdot 07483\ 30721\ 56694\ \ 39 = 5^{-2} + 9^{-2} + \&c.$ ad infin. Q_2,

$\therefore\ \cdot 23370\ 05501\ 36169\ \ 69 = 3^{-2} + 5^{-2} + 7^{-2} + \&c.$

The last agrees in 15 decimal places with $T_2 - 1$ in the table above, deduced from Legendre's S_n. In the 16th figure, that has 8, but in estimating T_2 as $= (1 - \frac{1}{4}) S_2$, it was as legitimate to write 75 for 80 as 16th and 17th figures; and from 69 we of course write 7 for the 16th. Thus we may fairly claim agreement up to 16 places.

We likewise now by taking $(P_2 - Q_2)$ obtain

$1 - V_2$ or $3^{-2} - 5^{-2} + 7^{-2} - 9^{-2} + \&c. = \cdot 08403\ 44058\ 22780\ 91.$

I have confirmed this result (to 13 decimals) by a different process, which must now be explained.

On series with alternate signs.

37. Suppose the series which we desire to sum has alternate signs, as $F(x) = \phi x - \phi(x+1) + \phi(x+2) - \&c.,$

which by the notation $E = 1 + \Delta$, becomes

$$F(x) = \phi x - E \, . \, \phi x + E^2 \, . \, \phi x - \&c. = (1 - E + E^2 - \&c.)\, \phi x;$$

or
$$F = \frac{1}{1+E} \, . \, \phi;$$

or since $E = e^{hD}$, and here h or $\Delta x = 1$,

$$F = \frac{1}{1+e^D} \, . \, \phi.$$

But in the identity

$$\frac{1}{1+e^v} = \frac{1}{2} - (2 - 2^{-1}) H_1 v + (2 - 2^{-3}) H_2 v^3 - \&c. \text{ (given above)},$$

we may write D for v;

$$\therefore \; F = \{\tfrac{1}{2} - (2 - 2^{-1}) H_1 D + (2 - 2^{-3}) H_2 D^3 - \&c.\ldots\} \, \phi,$$

whence $F(x) = \frac{1}{2}\phi x - (2 - 2^{-1}) H_1 \, . \, D\phi x + (2 - 2^{-3}) H_2 D^3 \phi x - \&c.,$

which differs remarkably from Maclaurin's Sum-Formula in involving no integration. But it does not converge so rapidly as Maclaurin's. It evidently applies to the series

$$1^{-n} - 2^{-n} + 3^{-n} - 4^{-n} + \&c.\ldots$$

and to
$$1^{-n} - 3^{-n} + 5^{-n} - 7^{-n} + \&c.\ldots$$

but the former is deducible from S_n.

Put
$$\phi x = x^{-4},$$
$$\phi' x = -4x^{-5}, \&c.$$

$$R_4(x) = \frac{1}{2} x^{-4} + \frac{2^2 - 1}{2} \, . \, \frac{4}{6} x^{-5} - \frac{2^4 - 1}{2^3} \, . \, \frac{4\,.\,5\,.\,6}{90} x^{-7}$$

$$+ \frac{2^6 - 1}{2^5} \, . \, \frac{4\,.\,5\,.\,6\,.\,7\,.\,8}{3\,.\,5\,.\,7\,.\,9} x^{-9} - \frac{2^8 - 1}{2^7} \, . \, \frac{4\,.\,5 \ldots 9\,.\,10}{3\,.\,5\,.\,7\,.\,9\,.\,10} x^{-11}$$

$$+ \frac{2^{10} - 1}{2^9} \, . \, \frac{4\,.\,5 \ldots 11\,.\,12}{3\,.\,5\,.\,7\,.\,9\,.\,9\,.\,11} x^{-13} - \&c.$$

To get fair convergence, I try $x = 33$. Reducing, we find

$$R_4(x) = \frac{1}{2}x^{-4} + x^{-5} - \frac{5}{2}x^{-7} + 14x^{-9} + \frac{3 \cdot 9 \cdot 13}{2}x^{-11}$$

$$+ (2 + 2^{-5}) \cdot 7 \cdot 10 \cdot 12x^{-11} - \&c.$$

Applying the formula to V_n, we have $\phi(x) = (2x-1)^{-n}$, and each differentiation multiplies by 2. Observe that

$$(2^2 - 1) H_1 = \tfrac{1}{2},\ (2^4 - 1) H_2 = \tfrac{1}{6};$$

$$(2^6 - 1) H_3 = \frac{1}{3 \cdot 5};$$

$$(2^8 - 1) H_4 = \tfrac{1}{7} \cdot \tfrac{1}{10} \cdot (2 - \tfrac{1}{9});$$

$$(2^{10} - 1) H_5 = \frac{31}{5 \cdot 7 \cdot 9 \cdot 9}.$$

Also $$\phi' x = -2n \cdot (2x-1)^{-n-1};$$

$$\phi''' x = -2^3 \cdot n \cdot \overline{n+1} \cdot \overline{n+2} \cdot (2x-1)^{-n-3}, \&c.,$$

Let $$2x - 1 = u.$$

Put $$F_n(x) = \tfrac{1}{2}u^{-n} + \tfrac{1}{2}nu^{-n-1} - M_2 u^{-n-3} + M_3 u^{-n-5}$$

$$- M_4 u^{-n-7} + M_5 u^{-n-9} - \&c.$$

then $$M_2 = \tfrac{1}{6}n \cdot \overline{n+1} \cdot \overline{n+2};$$

$$M_3 = \frac{n \cdot \overline{n+1} \ldots \overline{n+4}}{3 \cdot 5};$$

$$M_4 = \frac{2 - \frac{1}{9}}{7 \cdot 10} \cdot n \cdot \overline{n+1} \ldots \overline{n+6};$$

$$M_5 = \frac{31}{5 \cdot 7 \cdot 9 \cdot 9} \cdot n \cdot \overline{n+1} \ldots \overline{n+8}.$$

I find $u = 33$ convenient. Then

$$1 - 3^{-n} + 5^{-n} - \&c. \ldots - 31^{n}$$

must be computed by elementary methods. By this formula I have made out the following values of $1 - V_n$, to 13 decimals.

n	$1 - V_n$ (13 decimals)	n	$1 - V_n$	n	$1 - V_n$
1	$1 - \frac{1}{2}\pi$	8	15 0009 7532	15	0069 6591
2	·08403 4405 8228	9	5 0315 8128	16	23 2240
3	·03105 3853 7406	10	6835 9738	17	7 7422
4	·01105 5488 2589	11	...... 5625 0262	18	2 5809
5	384 2171 9229	12	...... 1877 6494	19	 8604
6	131 4777 7816	13	...... 0626 4169	20	$(=3^{-20})$ 2868
7	44 5492 1095	14	...... 0208 9127	21	$(=3^{-21})$

Having no check on trivial error, contingent on a tired brain, I have to speak diffidently of this little table. But I find it to agree, so far as it can be compared with Gudermann's calculation of

$$\frac{2}{r}(1 - V_r)$$

with r odd only,—except as to V_{19}, where I think him wrong and inconsistent with himself. He has

r	$2r^{-1}.(1 - V_r)$
1	·42920 36732 05103 38076 86783
3	2070 25691 60420 41301 09101
5	153 68687 69164 77439 74722
7	12 72834 59845 74014 39010
9	1 11812 91728 86892 85874
11	...... 10227 32032 05469 6540
13	...... 963 71727 40906 13
15	...... 92 87887 65025
17	...... 9 10849 178

Computing even to 20 decimals, we easily find

$$1 - V_{19} = 3^{-19} - 5^{-19} + 7^{-19} - 9^{-19}.$$

Indeed 11^{-19} gives a unit in last figure. Hence against Gudermann

$$1 - V_{19} = \ldots\ldots 8\ 60339\ 25544.$$

Bolder computers may carry my table of $1 - V_n$ to 16 decimals if there is an adequate object.

F. W. N.
Aug. 26, 1891.

(END)

SUPERRATIONALS OF SECOND ORDER.

SUPERRATIONALS OF SECOND ORDER.

PREFACE.

MY excuse for this little treatise, is, that its topic is not at all new to me. Allured into it by a cursory perusal of Spence's Essay on his Superlogarithm, I essayed to analyze all Logarithmic Integrals of the Second Order, and at length in 1847 some discussion of the three from my pen appeared in the *Cambridge and Dublin Mathematical Journal.* Then first I learned that Kummer had treated the subject in Crelle, indeed had pushed on into a Third Order. It does not appear that he was aware of Spence's Essay, or of Hill's dealing with my Third Integral, for which I sought in vain in the British Museum.

As these Integrals seem not to have a place in the ordinary Calculus, a treatise not too elaborate may be acceptable to some, though it cannot pretend to any real novelty or, I believe, necessity. In correcting the press in 1847 I learned many small lessons.

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CHAPTER II.

SUPERRATIONALS OF SECOND ORDER. THE SECOND INTEGRAL OR *supersine.*

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CHAPTER I.

THE FIRST SUPERLOGARITHM. SPENCE'S INTEGRAL.

Kummer's General Treatment.

1. WE learn in the Elements of the Integral Calculus, to integrate $\int X\,dx$, when X is a rational fraction of x. If the numerator of X involve x to a higher degree than its denominator, mere algebraic division reduces X to $M+\frac{N}{P}$, where MNP are integer functions of x, and N is of lower degree than P. Next $\frac{N}{P}$ is reducible to $\Sigma\frac{A}{x-a}$, where A, a are constants, i.e. independent of x, and a is either real, or has the form $m+n\sqrt{-1}$. Hence

$$\int X\,dx=\int M dx+\Sigma.\int\frac{A\,dx}{x-a}.$$

We know familiarly, that $\int M dx$ is an integer function; also when a is real, $\Sigma\int\frac{A\,dx}{x-a}$ is a series of logarithms, but if any of the values of a has the form $m+n\sqrt{-1}$, then with it a circular function (tan^{-1}) is involved, as well as a logarithm. Thus from integrating a Rational Function there arises, besides a new Integer form, *two novel forms* of Functions, Logarithmic and Cyclic; both which, as here discovered or generated, we may fitly call *Superrational of the First Order.* Let one be $P_1(x)$.

2. Kummer embraced a higher set in the formula $\int P_1 \,.\, X dx$, where X is, as before, a rational fraction. The new form has for equivalent

$$\int P_1 \left[M\,dx + \Sigma \frac{A\,dx}{x-a} \right].$$

In $\int P_1 M dx$ there cannot be anything new. For we may write $M dx = dP$, where P is integer in x. Then

$$\int P_1 \,.\, M dx = \int P_1 \,.\, dP = P_1 \,.\, P - \int P \,.\, dP_1.$$

But dP_1 has the form

$$N dx + \Sigma \frac{B dx}{x-b};$$

and there is nothing new in

$$P_1 \,.\, P = \int P \left(N dx + \Sigma \frac{B dx}{x-b} \right).$$

Thus it only remains to consider

$$\int P_1 \,.\, \Sigma \frac{A dx}{x-a}.$$

Again for P_1 we may substitute,

$$\int \left\{ N dx + \Sigma \frac{B dx}{x-b} \right\},$$

and we have as result

$$\iint \left\{ N dx + E \frac{B dx}{x-b} \right\} \Sigma \frac{A dx}{x-a},$$

or $$\int \left(\int N dx \right) \Sigma \frac{A dx}{x-b} + \iint \left\{ \Sigma \frac{B dx}{x-b} \right\} \Sigma \frac{A dx}{x-a}.$$

But since N is integer, we may put $N dx = dQ$, then

$$\int Q \,.\, \Sigma \frac{B dx}{x-b}$$

is Superrational of First Order, and nothing remains new but a series of terms, each with the form

$$\int . \int \frac{dx}{x-b} \,.\, \frac{dx}{x-a}.$$

In these both a and b may have the form $m + n\sqrt{-1}$. Every such integral is entitled *Superrational of the Second Order*.

Simplest Integral in $\iint \frac{dx}{x-b} \cdot \frac{dx}{x-a}$.

3. Evidently our simplest case is when a and b are both real. On that type it becomes

$$\int \frac{[\log (x-b)]\, dx}{x-a},$$

or indeed

$$\int \log (x-b)\, d \log (x-a),$$

which alternately is also

$$\log (x-b) \log (x-a) - \int \log (x-a)\, d \log (x-b).$$

Therefore we may deal with b and a in alternate order at pleasure.

In these pages $\log u$ can only arise out of $\int \frac{du}{u}$; and since then

$$\frac{du}{u} = \frac{-du}{-u},$$

we are free to change $\log u$ into $\log (-u)$, if we are careful to remember that at worst it only adds $\log (-1)$ in the equation resulting.

4. To solve $X = \int \frac{\log (x-b)\, dx}{x-a} = \iint \frac{dx}{x-b} \cdot \frac{dx}{x-a}$,

put $x - b = cy$, leaving the constant c at our disposal.

First let $b - a$ be positive, and $c = b - a$; $x - a = cy + c$; then

$$X = \iint \frac{cdy}{cy} \cdot \frac{cdy}{cy+c},$$

or

$$X = \int \frac{\log y \,.\, dy}{y+1}$$

$$= \int_0 \frac{\log y dy}{y+1}.$$

The last integral is *Kummer's first.* It may sometimes be convenient to denote it by $K(y)$. But if $(b-a)$ be negative, put $c = a - b$, and we alight upon

$$\int_0 \log y \frac{dy}{y-1},$$

which is virtually *Spence's* integral $L(y)$. It gives

$$L(1+y) = \int \log(1+y)\frac{dy}{y},$$

or, when y^2 is less than 1,

$$= y - \frac{y^2}{2^2} + \frac{y^3}{3^2} + \frac{y^4}{4^2} + \&c. \text{ with } L(1) = 0.$$

No doubt Kummer worked by this very same series.

It is $$\int_0 \frac{\log y dx}{y+1} = \log y \,.\, \log(1+y) - \int \log(1+y)\frac{dy}{y}.$$

5. Spence not quite logically proposed his series as his *definition;* but proceeded to give his variable an unlimited range; which shews that he really was dealing with

$$\int_0 \log x \,.\, d \log(x-1).$$

He sets 2 over his L, which here is needless, since we do not use L for simple logarithm. I have wished to write Sp. for Spence's integral, as more distinctive. Here

$$K(x) = \log x \,.\, \log(x+1) - L(1+x).$$

Whence $$K(1) = -L2.$$

Spence's integral is discontinuous where $1 + y = 0$, because $\log(1+y)$ is then infinite. This calls for attention to our constant of integration.

6. In Trigonometry we learn that

$$1^{-2} + 2^{-2} + 3^{-2} + 4^{-2} + \&c. \dots \text{ or } S_2 = \frac{\pi^2}{6}.$$

Also $$1^{-2} - 2^{-2} + 3^{-2} - 4^{-2} + \&c. \dots = \frac{\pi^2}{12}.$$

Since now $$L(1+x) = x - 2^{-2}x^2 + 3^{-2}x^3 - 4^{-2}x^4 + \&c. \dots\dots\dots\dots (A).$$

Put $x = 1$; then

$$L(2) = \frac{\pi^2}{12} = \cdot 8224\ 6703\ 3424\ 1132 \dots\dots\dots\dots (B).$$

Put $x = -1$; then

$$L(0) = -\frac{\pi^2}{6} = -2 \,.\, L(2).$$

We have also by hypothesis, $L(1) = 0$. Such are our 3 constants.

7. *Complementary Equation.* Let x be < 1. For conciseness write l for *log*. Since

$$d\,(lx\,.\,l\overline{1-x}) = lx\,.\,\frac{-dx}{(1-x)} + l\,(1-x)\,\frac{dx}{x};$$

integrate both sides; then

$$l\,(x)\,.\,l\,(1-x) = Lx + L\,(1-x) + c.$$

The left hand vanishes when $x = 1$; indeed also when $x = 0$;

$$\therefore\ 0 = L\,(0) + c;$$

hence $$c = -L\,(0) = 2L2;$$

and $$Lx + L\,(1-x) + 2L\,(2) = lx\,.\,l\,(1-x) \ldots\ldots\ldots\ldots\ldots (\text{C});$$

COR. When $$x = \tfrac{1}{2},\ 1 - x = \tfrac{1}{2};$$

$$\therefore\ 2\,.\,L\,(\tfrac{1}{2}) + 2L\,(2) = (l\,.\,2)^2,$$

by which $L\,(\frac{1}{2})$ is known $= \frac{1}{2}\,(l\,.\,2)^2 - L\,(2)$.

8. *Reciprocal Equation.* Let $xu = 1$,

$$\therefore\ L\,\frac{1}{x} = Lu = \int \log u\,.\,\frac{du}{u-1}.$$

But $\log u = -l\ x$, and

$$\frac{d\,.\,x^{-1}}{x^{-1}-1} = \frac{-dx}{x\,(1-x)}.$$

Thus $$L\,\frac{1}{x} = \int l\,(x)\,.\,\frac{dx}{x\,(1-x)} = \int l(x)\left\{\frac{dx}{x} + \frac{dx}{1-x}\right\}$$

$$= \tfrac{1}{2}\,(lx)^2 + \int lx\,.\,\frac{dx}{1-x} = \tfrac{1}{2}\,(lx)^2 - Lx + C.$$

We suppose x positive: let it be 1, $\therefore\ C = 0$.

Finally $$Lx + Lx^{-1} = \tfrac{1}{2}\,(lx)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{D}).$$

We deduce the COR. of the last, by making $x = \frac{1}{2}$ or $x = 2$.

9. Observe the difference of

$$K\,(x) = \int_0 \frac{lx\,.\,dx}{1+x}.$$

Here $$K\,(x^{-1}) = \int \frac{lx^{-1}\,.\,dx^{-1}}{1+x^{-1}},$$

which further $$= \int \frac{lx\,.\,x^{-2}\,dx}{1+x^{-1}} = \int \frac{lx\,.\,dx}{x\,(x+1)} = \int lx\left\{\frac{dx}{x} - \frac{dx}{1+x}\right\}$$

$$= \tfrac{1}{2}\,(lx)^2 - K\,(lx) + C'.$$

Let $x = 1$, $\therefore\ 2K(1) = C.$

But $K(1) = -L(2).$

Hence $Kx + Kx^{-1} = \frac{1}{2}(lx)^2 - 2L2$..................(D$_2$).

10. *Quasi-Reciprocal Equation.*

$$L(1 + x^{-1}) \text{ means } \int l(1+x^{-1}).\frac{dx^{-1}}{x^{-1}},$$

which $$= \int l\frac{x+1}{x}.\left(\frac{-dx}{x}\right) = \int \{l(x+1) - lx\}\left(\frac{-dx}{x}\right)$$

$$= -L(1+x) + \frac{1}{2}(lx)^2 + c.$$

Let $x = 1$, $\therefore\ 2L2 = c,$

or $L(1+x^{-1}) + L(1+x) = \frac{1}{2}(l\,.\,x)^2 + 2L2$ ....(E).

This reduces Ly, however large y, to $L(1+x)$ with a small x.

In fact, let $1 + x^{-1} = 1 + \epsilon^{+\rho},$

$\therefore\ l\,.\,x = -\rho,\ (l\,.\,x)^2 = \rho^2.$

Develop $L(1+x)$, put $x^n = \epsilon^{-n\rho}$.

$\therefore\ L(1+\epsilon^{\rho}) = 2L2 + \frac{1}{2}\rho^2 - \{\epsilon^{-\rho} - 2^{-2}.\epsilon^{-2\rho} + 3^{-2}.\epsilon^{-3\rho} - \&c.\}$

This convergence is good, if ρ exceed 3, or say $1 + x^{-1}$ exceeds 21, and the terms are easy to compute from a table of ϵ^{-x}.

11. Let x be > 1, so that $1 - x^{-1}$ is positive.

Then $L(1 - x^{-1})$ means

$$\int l(1-x^{-1}).\frac{d.x^{-1}}{x-1} = \int l.\frac{x-1}{x}.\left(\frac{-dx}{x}\right),$$

or $$\int (lx - l\overline{x-1})\frac{dx}{x} = \frac{1}{2}(lx)^2 - lxl(x-1) + \int lx\frac{dx}{x-1},$$

or $$Lx - lx(l\overline{x-1} - \tfrac{1}{2}lx) + C.$$

12. Let x recede to its smallest value 1,

$\therefore\ C = L(0) = -2L2.$

Finally $Lx - L(1-x^{-1}) = lx.(l\,.\,\overline{x-1} - \frac{1}{2}lx) + 2L2$........(F).

13. GENERAL PROBLEM. To reduce $L(u)$ to $L(1 \pm x)$, in which x shall never exceed $\frac{1}{2}$.

I. If u be between 0 and $\frac{1}{2}$, write $x = 1 - u$, then by equation (C)

$$Lx + Lu + 2L2 = lx \,.\, lu,$$

which reduces Lu to Lx or $L(1-u)$, in which u at utmost $= \frac{1}{2}$. This writing u for x, is the Problem in hand.

II. Let u be between $\frac{1}{2}$ and $\frac{3}{2}$. If between $\frac{1}{2}$ and 1, put $u = 1 - x$; but if between 1 and $\frac{3}{2}$, put $u = 1 + x$. In the two cases

$$Lu = L(1-x) \text{ and } x < \tfrac{1}{2}.$$

III. Let u be between $\frac{3}{2}$ and 2. Put $u(1-x) = 1$. Then by Reciprocal Equation (D)

$$L(1-x) + Lu = \tfrac{1}{2}(l \,.\, u)^2,$$

and $1 - x$ is between $\frac{2}{3}$ and $\frac{1}{2}$. This shows x between $\frac{1}{3}$ and $\frac{1}{2}$; or never exceeding $\frac{1}{2}$.

IV. Finally, let u have 3 for its least value: then by equation (3) we obtain Lu with greater ease, the larger is x; for with $u = 1 + x^{-1}$, the x diminishes as u increases, and Lu is always reduced to $L(1+x)$.

This concludes the Problem.

No doubt, both Kummer and Spence substantially solved this whole Problem; but it does not seem to have occurred to them in how narrow a compass it could be presented in tabular form.

14. It is now proved, that if we have registered in two Tables, *first*, the values of $L(1+x)$ from $x = 0$ to $x = \frac{1}{2}$ as the upper limit, and *next* the values of $-L(1-x)$ with the same upper limits, we can, by aid of natural *logs*, and our constant π^2, reduce Ly to finite terms, whatever the value of y, positive or negative. Our further Problem is, to calculate $L(1+x)$ and $-L(1-x)$; and the process is easy.

Put
$$\phi(x) = x + 3^{-2}x^3 + 5^{-2}x^5 + 7^{-2}x^7 + \&c.$$
$$\psi(x) = 2^{-2}x^2 + 4^{-2}x^4 + 6^{-2}x^6 + \&c.$$
$$= \tfrac{1}{4}\{x^2 + 2^{-2}x^4 + 3^{-2}x^6 + \&c.\}.$$

$$\begin{array}{ll}\text{Evidently then} & L(1+x)=\phi(x)-\psi(x) \\ \text{and} & -L(1-x)=\phi(x)+\psi(x)\end{array}\Bigg\}.$$

It remains chiefly to compute the two series $\phi(x)$ and $\psi(x)$.

This computation was performed by myself as I was best able; but, wishing to get my results tested by *differencing*, I begged access to a trustworthy arithmetician from the local knowledge of the eminent astronomer, Prof. J. C. Adams. But to my surprise, one whose time is so valuable for his own purposes, performed the whole calculation independently and corrected all my errors. He kindly sent me his tables and their verification. I do not know how duly to return thanks for a favour so unsolicited, and so hard to estimate: but the public has the advantage. Nothing remained for me but to add and subtract in $\phi(x) \pm \psi(x)$. My previous errors had mostly been in the last figures of 12 decimals.

15. In my ambition to present so many decimal places as 12, I had overlooked how rare is the possession of logarithms to this extent. I have never possessed or used so full a table, and I suppose very few learners are likely to have access to natural logarithms of 12 figures. This has one remedy, if the use is general of Weddell's leaflets, which enable any logarithm *required* to be quickly calculated to 16 decimals.

Meanwhile, I venture to print a short table; showing the logarithms of primes up to 97, which I computed years ago without thought of publishing. I carried it to 13 decimals, and think it must be accurate to the 12th. It will at least give a certain completeness *to illustrate* the Calculus of L. Weddell's leaflet is short to reprint, yet this might be a violation of Copyright.

In my own little Table I computed every logarithm by at least *two* independent methods, and by their accurate agreement I am satisfied. I briefly exhibit my double procedure, to give the reader confidence in the Table.

The first column shows the number whose logarithm is sought, *in order of smallness*, assuming those from 2 to 10 as notorious to 16 decimals. The 2nd column shows the equations by which I worked.

11	$100 - 1 = 9 \cdot 11$ $120 + 1 = 11^2$	41	$41 = 40 + 1$ $11 \cdot 41 = 450 + 1$	71	$71 = 70 + 1$ $69 \cdot 71 = 70^2 - 1$ $9 \cdot 71 = 640 - 1$
13	$90 + 1 = 7 \cdot 13$ $\frac{13}{11} = \frac{1 + 12^{-1}}{1 - 12^{-1}}$	43	$7 \cdot 43 = 300 + 1$ $\frac{43}{41} = \frac{1+42^{-1}}{1-42^{-1}}$	73	$3 \cdot 73 = 320 - 1$ $7 \cdot 73 = 510 + 1$
17	$50 + 1 = 3 \cdot 17$ $120 - 1 = 7 \cdot 17$	47	$17 \cdot 47 = 800 + 1$ $23 \cdot 47 = 1080 + 1$	79	$19 \cdot 79 = 1500 + 1$ $79 \cdot 81 = 80^2 - 1$
19	$360 + 1 = 19^2$ $\frac{21}{19} = \frac{1+20^{-1}}{1-20^{-1}}$	53	$17 \cdot 53 = 900 + 1$ $3 \cdot 53 = 160 - 1$	83	$3 \cdot 83 = 250 - 1$ $47 \cdot 83 = 3900 + 1$
23	$7 \cdot 23 = 160 + 1$ $13 \cdot 23 = 300 - 1$	59	$59 = 60 - 1$ $19 \cdot 59 = 1120 + 1$	89	$89 = 90 - 1$ $9 \cdot 89 = 800 + 1$ $\frac{91}{89} = \frac{1 + 90^{-1}}{1 - 90^{-1}}$ $89 \cdot 91 = 90^2 - 1$
31	$31^2 = 960 + 1$ $29 \cdot 31 = 30^2 - 1$ $\frac{31}{29} = \frac{1+30^{-1}}{1-30^{-1}}$	61	$61 = 60 + 1$ $9 \cdot 61 = 550 - 1$ $59 \cdot 61 = 60^2 - 1$		
37	$3 \cdot 37 = 110 + 1$ $13 \cdot 37 = 480 + 1$	67	$3 \cdot 67 = 200 + 1$ $43 \cdot 67 = 2880 + 1$	97	$3 \cdot 97 = 290 + 1$ $23 \cdot 97 = 3200 + 1$ $97 = 96 + 1$

As we ascend from 100 to 10,000 the ease of computing the logarithms of any given primes generally increases. It must be odd.

Short Table with x a Prime. (Natural *logs.*)

x	$\log x$	x	$\log x$
11	2·3978 9527 27983	53	3·9702 9191 35521
13	2·5649 4935 74615	59	4·0775 3744 39059
17	2·8332 1334 40562	61	4·1108 7386 41733
19	2·9444 3897 91661	67	4·2046 9261 93908
23	3·1354 9421 59294	71	4·2626 7987 70483
29	3·3672 9582 99863	73	4·2904 5944 11483
31	3·4339 8720 44851	79	4·3694 4785 24671
37	3·6109 1791 26443	83	4·4188 4060 77967
41	3·7135 7206 67042	89	4·4886 3636 97321
43	3·7612 0011 56934	97	4·5747 1097 85032
47	3·8501 4760 17098		

x	$L(1+x)$	x	$L(1+x)$
·01	·0099 7511 0490	·26	·2448 0759 1657
·02	·0199 0087 9015	·27	·2536 78[illegible]5 4093
·03	·0297 7795 0327	·28	·2625 1267 6614
·04	·0396 0695 5096	·29	·2713 1124 3880
·05	·0493 8851 0345	·30	·2800 7433 3760
·06	·0591 2321 9862	·31	·2888 0231 7509
·07	·0688 1167 4606	·32	·2974 9555 9926
·08	·0784 5445 3083	·33	·3061 5441 9513
·09	·0880 5212 1722	·34	·3147 7924 8625
·10	·0976 0523 5229	·35	·3233 7039 3626
·11	·1071 1433 6926	·36	·3319 2819 5021
·12	·1165 7995 9083	·37	·3404 5298 0605
·13	·1260 0262 3229	·38	·3489 4510 0590
·14	·1353 8284 0467	·39	·3574 0485 7745
·15	·1447 2111 1757	·40	·3658 3257 7513
·16	·1540 1792 8204	·41	·3742 2861 3137
·17	·1632 7377 1334	·42	·3825 9315 2783
·18	·1724 8911 3350	·43	·3909 2661 9655
·19	·1816 6441 7392	·44	·3992 2927 2051
·20	·1908 0013 7778	·45	·4075 0140 3725
·21	·1998 9672 0242	·46	·4157 4330 3503
·22	·2089 5460 2162	·47	·4239 5525 5868
·23	·2179 7421 2783	·48	·4321 3754 0822
·24	·2269 5597 3423	·49	·4402 9043 4033
·25	·2359 0029 7686	·50	·4484 1420 6923

$L(1+x)$ means $1^{-2}x - 2^{-2}x^2 + 3^{-2}x^3 - 4^{-2}x^4 + \&c.$

x	$-L(1-x)$	x	$-L(1-x)$
·01	·0100 2511 1740	·26	·2791 9665 5857
·02	·0201 0089 9019	·27	·2908 1500 4677
·03	·0302 2805 1617	·28	·3025 0901 1592
·04	·0404 0727 5324	·29	·3142 8004 3520
·05	·0506 3929 2465	·30	·3261 2951 0076
·06	·0609 2484 2460	·31	·3380 5886 5503
·07	·0712 6468 2410	·32	·3500 6961 0724
·08	·0816 5958 7699	·33	·3621 6329 5513
·09	·0921 1035 2632	·34	·3743 4152 0809
·10	·1026 1779 1099	·35	·3866 0594 1160
·11	·1131 8273 7272	·36	·3989 5826 7343
·12	·1238 0604 6417	·37	·4114 0026 2133
·13	·1344 8859 5213	·38	·4239 3377 8266
·14	·1452 3128 3445	·39	·4365 6069 1597
·15	·1560 3503 3945	·40	·4492 8297 4471
·16	·1669 0079 3918	·41	·4601 0270 4323
·17	·1778 2953 5770	·42	·4750 2187 4533
·18	·1888 2225 8086	·43	·4880 4679 8559
·19	1998 7998 6634	·44	·5011 6771 4411
·20	·2110 0377 5440	·45	·5043 9898 9154
·21	·2221 9470 7914	·46	·5277 3908 4519
·22	·2334 5389 8030	·47	·5411 9056 1906
·23	·2447 8249 1573	·48	·5547 5608 8554
·24	·2561 8166 745·	·49	·5684 3844 3897
·25	·2676 5263 9082	·50	·5822 4052 6465

$-L(1-x)$ means $1^{-2}x + 2^{-2}x^2 + 3^{-2}x^3 + \&c.$

PROBLEM. To sum the series

$$S = 1^{-2}x + 3^{-2}x^3 + 5^{-2}x^5 + \&c$$

when $x < 1$. Here we have

$$\frac{dS}{dx}x = 1^{-1}x + 3^{-1}x^3 + 5^{-1}x^5 + \&c. = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

$$\therefore\quad dS = \tfrac{1}{2}\log(1+x)\,d\log x - \tfrac{1}{2}\log(1-x)\,d\log x,$$

or

$$S = \tfrac{1}{2}L(1+x) - \tfrac{1}{2}L(1-x);$$

no constant of integration needed. Otherwise:

$$\left.\begin{aligned}S = x + 2^{-2}x^2 + 3^{-2}x^3 + 4^{-2}x^4 + 5^{-2}x^5 + 6^{-2}x^6 + \&c.\\ - 2^{-2}x^2 \qquad - 4^{-2}x^4 \qquad - 6^{-2}x^6 \quad \&c.\end{aligned}\right\}$$

$$= - L(1-x) + 2^{-1} . L(1-x^2);$$

a result identical with the former.

16. Picture of Curve $y = Lx$.

We mean x and y to be Rectangular Coordinates.

We have $L(1) = 0$, $L(2) = \frac{1}{12}\pi^2$

Take OAX opposite to BOY, but $OB = \frac{\pi^2}{6}$.

When x exceeds 1, $\frac{dy}{dx} = \frac{\log x}{x-1}$, which is positive.

Also when $x = 1$, $\frac{dy}{dx} = \frac{0}{0} = \frac{d \log x}{d(x-1)} = \frac{1}{x} = 1$.

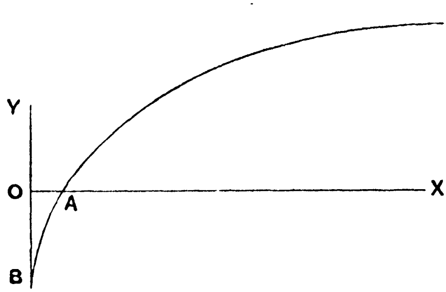
With the increase of x, $\frac{\log x}{x-1}$ lessens therefore after x passes 1, the curve is concave to the axis OX, indeed rapidly tends to parallelism. But when x is < 1,

$$\frac{dy}{dx} = \frac{\log x^{-1}}{1-x},$$

which is still positive, though y is now negative. The curve is continuous, until it reaches YOB in B. There

$$\frac{dy}{dx} = \frac{\log x^{-1}}{1-x},$$

which then (since $x = 0$) becomes $\log \frac{1}{x}$ or $\log \infty$. Thus the curve touches YB at the point B.



When XO is continued along negative x, if we change x to $-x'$, we have $y = L(x')$ or

$$\int lx \,.\, \frac{dx}{x-1} = \int l(-x) \,.\, \frac{-dx}{-x-1},$$

$$\text{or } \int (lx + c)\, \frac{dx}{x+1},$$

since $l(-x)$ only means

$$\int \frac{-dx}{-x} + c.$$

17. Spence showed that by putting

$$u = \frac{1-ax}{1-x}$$

with a constant and working on $L(u)$ we can deduce an equation with five integrals of the type L, with the term

$$\frac{1}{2} \,.\, l^2 \,.\, \frac{1-x}{1-a}.$$

I prefer a method originated in Elliptic Integrals by Mr Fox Talbot. It is here not more fruitful, but fully as much, and more elegant.

Assume $U = L(x) + L(y)$, and seek a *symmetrical* relation between x and y which will make dU integrable.

Try the relation

$$(1-u)^2 = v(u-a) \quad\text{.........................}(A)$$

with a constant, and let x, y be the *two roots* of u in this equation, while v varies.

Arranging for u, we have

$$u^2 - (2+v)\,u + (1+av) = 0.$$

Then, because x, y are the two roots, we find

$$x + y = 2 + v;$$

and $\quad xy = 1 + av$ and $lx + ly = l(1-av)$.

From these eliminate v, then

$$a(x+y) - xy = 2a - 1.$$

Otherwise $\quad (x-a)(y-a) = (a-1)^2 \quad\text{....................}(B).$

Also $\quad dlv = dl(av) = dl(xy-1) \quad\text{..............}(B').$

Now by definition of L,

$$Lx = \int lx \,.\, dl\,(x-1),$$

and
$$Ly = \int ly \,.\, dl\,(y-1).$$

$$\therefore\ 2U = 2Lx + 2Ly = \int lx \,.\, dl \,.\, \overline{x-1}|^2 + \int ly \,.\, dl \,.\, \overline{y-1}|^2.$$

But
$$(x-1)^2 = v\,(x-a),$$
and
$$(y-1)^2 = v\,(y-a),$$
by making alternately $u = x$ and $u = y$ in the original equation (A).

Hence $$2U = \int lxdl\,[v\,.\,\overline{x-a}] + \int lydl\,[v\,.\,\overline{y-a}]$$

$$= \int (lx + ly)\,dlv + \int lxdl\,(x-a) + \int lydl\,(y-a).$$

Of these three, the first $= \int l\,(1+av)\,dl\,(av)$ else

$$= \int l\,(xy)\,dl\,(xy-1) = L\,(xy). \quad \text{See (B}'\text{)}.$$

In the second, make $x = at$, and it becomes

$$\int [la + lt]\,dl\,(t-1) = la\,l\,(x-a) + Lt = lxl\,(x-a) + l\,\frac{x}{a}.$$

Similarly the third $= la\ l\,(y-a) + L\,\dfrac{y}{a}$.

Hence $$2U = L\,(xy) + la\,l\,(\overline{x-a\,.\,y-a}) + L\,\frac{x}{a} + L\,\frac{y}{a}.$$

Now by (B) $$(x-a)\,(y-a) = (a-1)^2;$$
so that we may include this term in the constant of integration. The assumption, $x = 1$, $y = 1$, fulfils the last equation, and shows that on the left hand we need $2L(a^{-1})$ as the constant of integration. Finally, we may change a^{-1} to z, then

$$2Lx + 2Ly + 2Lz = L\,(xy) + L\,(xz) + L\,(yz),$$

and our equation of condition

$$a\,(x+y) - xy = 2a - 1$$

becomes
$$x + y + z = xyz + 2.$$

Since x, y, z are connected by only one equation of condition, any two of them are arbitrary.

The number of such equations here seems unlimited, and the results in detail obtainable are endless, but *always less useful than curious.*

From the equations of 7—12, marked C D E F G, many relations between *only two* integrals of the type L result; and I alighted on the following, which is not likely to be quite unique:

$$\tfrac{1}{6} \cdot L9 = L3 - L2.$$

18. It is easy to produce a formula on the steps of Taylor's Theorem: namely, if x is much smaller than a, assume

$$L(a+x) = L(a) + A_1 x - A_2 \cdot \frac{x^2}{2} + A_3 \cdot \frac{x^3}{3} - A_4 \cdot \frac{x^4}{4} + \&c.$$

then, to find $A_1\, A_2\, A_3\, \ldots$ as functions of a, assume a constant, x variable, and differentiate: whence

$$\frac{\log(a+x)}{(a+x)-1} = A_1 - A_2 x + A_3 x^2 - A_4 x^3 + \&c.$$

Multiply by $(a-1)+x$, and develop the log; whence

$$\log a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \&c. = \left.\begin{matrix} A_1(a-1) - A_2(a-1)x + A_3(a-1)x^2 - \&c. \\ +A_1 x \qquad -A_2 x^2 \qquad + \&c. \end{matrix}\right\}.$$

Equate like coefficients: then

$$A_1 = \frac{\log a}{a-1}; \qquad A_2 = \frac{A_1 - a^{-1}}{a-1};$$

$$A_3 = \frac{A_2 - \frac{1}{2}a^{-2}}{a-1}; \quad A_4 = \frac{A_3 - \frac{1}{3}a^{-3}}{a-1}; \ \&c.$$

where the law of succession is clear.

The convergence is only that of $(a-1)^{-1}$, which is the same as that in equation (E) of Art. 10. But our x in this Article may be ever so small; which essentially alters the question.

CHAPTER II.

CLAUSEN'S INTEGRAL;—*the Supersine?*

(1) IN $\iint \frac{dx}{x-a} \cdot \frac{dx}{x-b}$ we have hitherto supposed a and b both real. Next suppose only one real. It does not signify which. For

$$\int \log(x-a) \, . \, \frac{dx}{x-b}$$

yields $\quad \log(x-a) \, . \, \log(x-b) - \int \log(x-b) \, d \log(x-a).$

Suppose then a to be real, and let

$$x = a + y,$$

$$\therefore \; x - b = y + (a-b).$$

For $(a-b)$, which is imaginary, we may write now

$$r(\cos\mu + \sqrt{-1}\sin\mu),$$

which changes our integral to

$$\int \frac{\log y dy}{(y + r\cos\mu) + \sqrt{-1} \, . \, r \sin\mu},$$

or $$\int \frac{(y + r\cos\mu) - \sqrt{-1} \, . \, r\sin\mu}{(y + r\cos\mu)^2 + r^2\sin^2\mu} \log y dy.$$

Next, put $$y = rz,$$

$$\log y = \log r + \log z.$$

That part of the integral which is then multiplied by $\log r$, is only of the earlier order. Write Z^2 for

$$z^2 + 2z\cos\mu + 1;$$

and the residue becomes

$$\int \frac{z + \cos\mu}{Z^2} \log z dz - \sqrt{-1} \int \frac{\sin\mu}{Z^2} \log z dz,$$

since $$Z \, . \, dZ = (Z + \cos\mu) \, dz.$$

The former part is reducible to

$\int \log z \, . \, d \log Z$, Hill's integral (the Dilogarithm ?)

involving the constant angle μ. In the second part, write $\tan\omega$ for

$$\frac{z\sin\mu}{1+z\cos\mu};$$

which gives $$\sin\omega\,(1+z\cos\mu)=\cos\omega\,.\,z\sin\mu,$$

whence $$z=\frac{\sin\omega}{\sin\mu\cos\omega-\cos\mu\sin\omega},$$

or $$z=\frac{\sin\omega}{\sin(\mu-\omega)},$$

$$\log z=\log 2\sin\omega-\log 2\sin(\mu-\omega).$$

Again: differentiate

$$\tan\omega=\frac{z\sin\mu}{1+z\cos\mu},$$

whence $$\sec^2\omega\,.\,d\omega=\frac{\sin\mu\,dz}{(1+z\cos\mu)^2}.$$

But $$\sec^2\omega=1+\tan^2\omega=\frac{Z^2}{(1+z\cos\mu)^2};$$

whence $$d\omega=\frac{\sin\mu\,dz}{Z^2}.$$

We deduce $$\int\frac{\sin\mu}{Z^2}\log z\,dz=\int\log z\,d\omega$$

$$=\int\log(2\sin\omega)\,d\omega-\int\log 2\sin(\mu-\omega)\,d\omega.$$

Put $\mu-\omega=v$, and the residual integral is

$$\int\log(2\sin\omega)\,d\omega+\int\log(2\sin v)\,dv,$$

where each part is of the same form, and involves only one arbitrary. This is Clausen's integral. Let Cl. (θ) denote

$$\int_0-\log_e(\text{chord }\theta)\,d\theta,$$

or $$-\int\log_e(2\sin\tfrac{1}{2}\theta)\,d\theta;$$

the symbol Cl. alluding to Clausen. Then we have

$$\int_0\log z\,\frac{\sin\mu\,dz}{Z^2}=const.-\tfrac{1}{2}\,\text{Cl.}\,(2\omega)-\tfrac{1}{2}\,\text{Cl.}\,(2\mu-2\omega).$$

Change (for convenience) z to $-u$ and ω to $-\psi$;

$$\therefore\ \tan\psi=\frac{u\sin\mu}{1-u\cos\mu},$$

and observe that log z meaning only $\int \frac{dz}{z}$ is exchangeable to $\int \frac{du}{u}$ or log u, and multiply by -2;

$$\therefore \int_0 2 \log u \frac{\sin \mu . du}{u^2 - 2u \cos \mu + 1} = \text{Cl.} (2\mu + 2\psi) - \text{Cl.} (2\psi) - \text{Cl.} (2\mu)$$

which reduces this portion to Clausen's Integral; the sign $-$ after *log* being here unimportant.

(2) We thus have found, that if

$$X^2 = 1 - 2x \cos \mu + x^2$$

and
$$\tan \omega = \frac{x \sin \mu}{1 - x \cos \mu},$$

which entails
$$x = \frac{\sin \omega}{\sin (\omega + \mu)},$$

and
$$d\omega = \frac{\sin \mu \, dx}{X^2},$$

we have
$$\int_0 2 \log x \, d\omega = \text{Cl.} (2\omega + 2\mu) - \text{Cl.} (2\omega) - \text{Cl.} (2\mu).$$

But in Trigonometry we learn that ω or

$$\tan^{-1} \left(\frac{x \sin \mu}{1 - x \cos \mu} \right) = x \sin \mu + \tfrac{1}{2} x^2 \sin 2\mu + \tfrac{1}{3} x^3 \sin 3\mu + \&\text{c}.$$

Now
$$\int 2 \log x \, d\omega = 2\omega \log x - \int 2\omega \frac{dx}{x},$$

therefore (since μ is constant)

$$= 2\omega \log x + 1^{-2} x \sin \mu + 2^{-2} x^2 \sin 2\mu + 3^{-2} x^3 \sin 3\mu + \&\text{c}.$$

Let $\mu = \frac{1}{2}\pi$, then the terms with x^2, x^4, x^6... vanish, and $\tan \omega = x$, making

$$1^{-2} x - 3^{-2} x^3 + 5^{-2} x^5 - \&\text{c}. = \text{Cl.} (2\omega + \pi) - \text{Cl.} (2\omega) - \text{Cl.} (\pi) - 2\omega \log x.$$

To this series Spence devoted a careful discussion, but we see it is solved at once by Clausen's integral; as is the wider series with μ arbitrary.

[I have also in Ch. I. before Art. 14,

$$1^{-2} x + 3^{-2} x^3 + 5^{-2} x^5 + \&\text{c}. = \tfrac{1}{2} L (1 + x) - \tfrac{1}{2} L (1 - x).]$$

(3) It is known also that

$$-\log(2\sin\tfrac{1}{2}\theta) = \cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta + \&c....$$

whence $\text{Cl.}(\theta) = 1^{-2}\sin\theta + 2^{-2}\sin 2\theta + 3^{-2}\sin 3\theta + \&c.$ ....(1),

of which the sum vanishes when $2\theta = n\pi$, (n an integer). We lose this advantage, if we remove the factor 2 after *log*.

Such is the cardinal series, which Clausen has tabulated, by processes of his own. The pictorial form of his function suggests *Supersine* as a name for it. If x and y' are rectilinear coordinates of this curve, by means of $x = \cos\omega$, $y' = \text{Cl.}(\omega)$, and x, y are the coordinates of a circle, by $x = \cos\omega$, and $y = \sin\omega$,

$$\frac{dy'}{d\omega} = -\log(2\sin\tfrac{1}{2}\omega) = -\log \text{chord}\ \omega = -\log\sqrt{2(1-x)}.$$

This is positive while ω is less than 60°. When $\omega = 60°$,

$$\tfrac{1}{2}\omega = 30°,\quad x = \tfrac{1}{2},\quad 2(1-x) = 1,$$

and

$$\frac{dy'}{d\omega} = 0.$$

When ω passes 60°, $\frac{dy'}{d\omega}$ becomes negative, and y' diminishes until $\omega = 180°$, at which y' becomes zero, then $\frac{dy'}{dx}$ is *infinite*. For

$$\frac{dy'}{d\omega}\cdot\frac{d\omega}{dx} = -\log\sqrt{2(1-x)}\cdot\frac{d\omega}{(-\sin\omega\, d\omega)} = \log\sqrt{2(1-x)}\cdot\frac{1}{\sin\omega},$$

which when $x = 180°$, becomes

$$\frac{\log\sqrt{2\cdot 2}}{\sin 180°} \text{ or } \frac{\log 2}{0}.$$

Thus the curve has the form of a *kettle-drum*, very flat where ω is small, tapering but still rounded when ω approaches 180°.

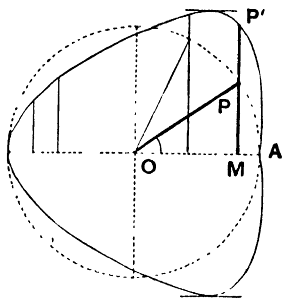
(4) Let S_n as with Legendre mean $1^{-n} + 2^{-n} + 3^{-n} + \&c.$, a known series: a few other series may in certain cases avail us in computing $\text{Cl.}(\theta)$. First observe that the series (1) shows

$$\text{Cl.}(-\theta) = -\text{Cl.}(\theta);$$
$$\text{Cl.}(0) = 0 = \text{Cl.}(\pi);$$
$$\text{Cl.}(2\pi - \theta) = \text{Cl.}(\theta);$$
$$\text{Cl.}(\pi + \omega) = -\text{Cl.}(\pi - \omega),$$

and when n is integer, $\text{Cl.}(2n\pi + \theta) = \text{Cl.}(\theta)$.

Thus if Cl. (θ) is known from $\theta = 0$ to $\theta = \pi$, it is known universally.

Curve of Supersine or Kettle Drum?



$$\left.\begin{array}{l} y' = \text{Cl.}\ (\theta) \\ x = \cos\theta \end{array}\right\} \quad \begin{array}{l} \angle AOP = \theta \\ OM = x,\ MP' = y'. \end{array}$$

In fact, by simple equations, we can deduce the values from $\theta = 60^\circ$ to $\theta = 180^\circ$ from the earlier values, as will soon appear.

Also $\qquad \text{Cl.}\ (90^\circ) = 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \&\text{c.},$

which is known to be $= \frac{1}{4}\pi$, or

$$\cdot 9159\ 6559\ 4177\ 2190.$$

Let $\qquad \sin\frac{1}{2}\theta = u,$

$$\therefore \int - \log(2u)\, d\theta = -\theta\log(2u) + \int \theta \,.\, \frac{du}{u}\,.$$

But $\theta = 2\sin^{-1} u$, of which the development in powers of u is known:

$$\therefore \text{Cl.}\ (\theta) = -\theta\log(2u) + \int 2\left\{u + \frac{1}{2}\,.\,\frac{u^3}{3} + \frac{1\,.\,3}{2\,.\,4}\,.\,\frac{u^5}{5} + \&\text{c.}\right\}.\ \frac{du}{u}$$

$$= -\theta\log(2u) + 2\left\{\frac{u}{1^2} + \frac{1}{2}\,.\,\frac{u^3}{3^2} + \frac{1\,.\,3}{2\,.\,4}\,.\,\frac{u^5}{5^2} + \&\text{c.}\right\}\ldots(2),$$

which, when u is a convenient fraction, yields a value. But it suffices to make $\theta = 60^\circ$, $u = \frac{1}{2}$, $\log(2u) = 0$, whence Cl. (60°) is found

$$= 1{\cdot}0149\ 4160\ 6409\ 6536.$$

(5) We cannot compute Cl. (θ) from series (1). Series (2) avails us only when u is a simple small fraction; and this does not yield an exact number of degrees.

The scope of this Chapter is, *first*, to give series by which it is possible to get values when θ is *less than* 60°, and economize labour; also series by which second computations may be made, affording verification: *next*, it will be demonstrated, that,—Given a table up to 60°, the entries up to 180° can be deduced by easy linear equations, without new infinite series or logarithms. Lastly, some uses of this new integral will be pointed out.

(6) As a general series for actual computation, the following seems to claim chief attention: $\frac{1}{2}$ Cl. $(2x)$, or

$$-\int_0 \log(2\sin x)\,dx = -x\log(2\sin x) + \int_0 x\,d\log(2\sin x)$$

$$= -x\log(2\sin x) + \int_0 x\cot x\,dx.$$

But $\qquad \cot x = \frac{1}{x} - \Sigma\frac{2x}{r^2\pi^2 - x^2}$; if r mean 1, 2, 3, 4 ...

$$\therefore \int_0 x\cot x\,dx = x - \Sigma\int_0 \frac{2x^2\,dx}{r^2\pi^2 - x^2}$$

$$= x + \Sigma\int_0 \left\{\frac{r^2\pi^2}{r^2\pi^2 - x^2}\right\} 2dx$$

$$= x + \Sigma\left\{2x - r\pi\log\frac{r\pi + x}{r\pi - x}\right\};$$

in which we have only to make $x = \frac{1}{2}\theta$, $dx = \frac{1}{2}d\theta$, and double the equation.

$$\therefore \text{Cl.}(\theta) = -\theta\log(2\sin\tfrac{1}{2}\theta) + \theta + \Sigma\left\{2\theta - 2r\pi\log\frac{2r\pi + \theta}{2r\pi - \theta}\right\}.$$

Call the last summation Θ, which is an infinite series. Putting 1, 2, 3 ... for r, the convergence is slow. Take θ a submultiple of 2π, or $\theta = 2\pi v$, and develop in powers of v. The general term is then

$$2\theta - 2r\pi\log\frac{r+v}{r-v} = 4\pi v - 2r\pi\,.\,2\left\{\frac{v}{r} + \frac{v^3}{3r^3} + \frac{v^5}{5r^5} + \&c.\right\},$$

or simply

$$-4\pi\left\{\frac{v^3}{3r^2} + \frac{v^5}{5r^4} + \frac{v^7}{7r^6} + \&c.\right\}.$$

(7) Collect $-4\pi A_{2n+1}$ the *total* coefficient of v^{2n+1} by making $r = 1, 2, 3, 4\ldots$ and you find

$$A_{n+1} = \frac{1}{2n+1}\{1^{-2n} + 2^{-2n} + 3^{-2n} + \&c.\},$$

or

$$\frac{1}{2n+1} \cdot S_{2n}.$$

Thus $$\Theta = -4\pi\{\tfrac{1}{3}S_2 v^3 - \tfrac{1}{5}S_4 v^5 + \tfrac{1}{7}S_6 v^7 + \&c.\}.$$

But $$2\pi \log \frac{1+v}{1-v} = 4\pi\{v + \tfrac{1}{3}v^3 + \tfrac{1}{5}v^5 + \tfrac{1}{7}v^7 + \&c.\}.$$

$$\therefore\ \Theta + 2\pi \log \frac{1+v}{1-v}$$

$$= +4\pi v - \{\tfrac{1}{3}(S_2 - 1)v^3 + \tfrac{1}{5}(S_4 - 1)v^5 + \&c.\}\,4\pi \ldots\ldots(3),$$

which improves the convergence, and shows that we ought *not* to have developed the *first* term

$$\left\{2\theta - 2\pi \log \frac{2\pi + \theta}{2\pi - \theta}\right\}.$$

The highest value for θ which we need is $\theta = 60^\circ$, which is gained by $v = \frac{1}{6}$: and with v *less* than $\frac{1}{6}$, we have good convergence.

Take as values of v^{-1},

6, 8, 9, 10, 12, 15, 18, 20, $\frac{90}{4}$, 24, 30

and the values of θ will be

60°, 45°, 40°, 36°, 30°, 24°, 20°, 18°, 16°, 15°, 12°

36,	40,	45,	60,	72,	90,	120,	240,	360.
10°,	9°,	8°,	6°,	5°,	4°,	3°,	2°,	1°.

Thus 20 values of Cl. (θ) are obtainable directly from equation (3), while in every case, except $v = \frac{4}{90}$, v^{-1} is an integer.

By what process Clausen obtained his results so far as to 16 decimals, I have to guess.

(8) When θ is near to 180°, Cl. θ can be computed (as will presently be shown) from the earliest entries; but a direct method will give verification. Let $\theta = 180^\circ - \omega$, and ω be small.

Begin anew : $\sin \frac{1}{2}\theta = \cos \frac{1}{2}\omega,$

$$\text{Cl. } \theta = +\int_0 \log(\cos \tfrac{1}{2}\omega)\, d\omega = C + \omega \log\cos(\tfrac{1}{2}\omega)$$
$$+\int_0 \tfrac{1}{2}\tan(\tfrac{1}{2}\omega)\, \omega d\omega.$$

When $\omega = 0$, $\text{Cl. } \theta = \text{Cl. } 180^\circ = 0.$

$$\therefore\ C = 0$$

Put $\omega = u\pi$, u being a fraction. Develop $\tan(\frac{1}{2}\omega)$ in powers of u.

$$\tfrac{1}{4}\pi \tan(\tfrac{1}{2}\pi u) = T_2 u + T_4 u^3 + T_6 u^5 + \&\text{c.}$$

if T_n mean $1^{-n} + 3^{-n} + 5^{-n} + \&\text{c.}$

Multiply by udu and integrate; then, since

$$\omega d\omega = \pi^2 . udu,$$

$$\int_0 \tfrac{1}{2}\tan(\tfrac{1}{2}\omega)\,\omega d\omega = 2\pi\left\{T_2 . \frac{u^3}{3} + T_4 . \frac{u^5}{5} + T_6 . \frac{u^7}{7} + \&\text{c.}\right\};$$

and if we have extended tables of natural logarithms, we may increase convergence by subtracting

$$\pi \log\frac{1+u}{1-u} = 2\pi\left\{u + \frac{u^3}{3} + \frac{u^5}{5} + \frac{u^7}{7} + \&\text{c.}\right\}.$$

Thus Cl. (θ) is obtained, T_n being known to 16 decimals.

(9) For small values of θ, simplest of all in appearance is the use of Euler's coefficients H_n, which result algebraically from

$$x\cot x = 1 - 2H_1 x^2 - 2H_2 x^4 - 2H_3 x^6 - \&\text{c.}$$

The five first are easy to remember and convenient to use, viz.

$$H_1 = \tfrac{1}{6};\quad H_2 = \tfrac{1}{90};$$
$$H_3 = \frac{1}{3.5.7.9};$$
$$H_4 = \tfrac{1}{10} . H_3 . \quad H_5 = \tfrac{1}{9} . \tfrac{1}{11} . H_3.$$

It is known that

$$\log\sin x = \log x - H_1 x^2 - \tfrac{1}{2}H_2 x^4 - \tfrac{1}{3}H_3 x^6 - \&\text{c.}$$

Add log 2 to both sides, and make $x = \frac{1}{2}\theta$,

$$\therefore\ -\log(2\sin\tfrac{1}{2}\theta) = -\log\theta + 2^{-2}H_1\theta^2 + 2^{-4}H_2\frac{\theta^4}{2} + 2^{-6}H_3\frac{\theta^6}{3} + \&\text{c.}$$

Multiply by $d\theta$, and integrate from $\theta = 0$, observing that

$$\int_0 - \log\theta . d\theta = \theta(1 - \log\theta).$$

Finally

$$\text{Cl.}(\theta) = \theta(1 - \log\theta) + 4^{-1}H_1\frac{\theta^3}{1.3} + 4^{-2}H_2\frac{\theta^5}{2.5}$$

$$+ 4^{-3}H_3\frac{\theta^7}{3.7} + \&c. \quad \ldots\ldots\ldots(4).$$

Algebraically, this is good; but θ being an arc, not an exact number of degrees, we are troubled by its *powers*, even when it is small. If we make

$$\theta = 2\pi\iota, \quad H_n = \pi^{-2n}S_{2n},$$

$$\text{Cl.}(\theta) = \theta(1 - \log\theta) + 4\pi v\left\{\frac{v^2S_2}{2.3} + \frac{v^4S_4}{4.5} + \frac{v^6S_6}{6.7} + \&c.\right\}.$$

For the series in v write $4\pi V$ when every S is replaced by 1, or

$$V = \frac{v^3}{2.3} + \frac{v^5}{4.5} + \frac{v^7}{6.7} + \&c.$$

Now $$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n.\overline{n+1}},$$

whence $$V = \left(\frac{v^3}{2} + \frac{v^5}{4} + \frac{v^7}{6} + \&c.\right) - \left(\frac{v^3}{3} + \frac{v^5}{5} + \frac{v^7}{7} + \&c.\right)$$

$$= -\tfrac{1}{2}v\log(1 - v^2) + v - \tfrac{1}{2}\log\frac{1+v}{1-v},$$

then

$$\text{Cl.}(2\pi v) = 2\pi v(1 - \log 2\pi v) + 4\pi V + 4\pi\left\{\frac{v^3}{2.3}(S_2 - 1)\right.$$

$$\left. + \frac{v^5}{4.5}(S_4 - 1) + \frac{v^7}{6.7}(S_6 - 1) + \&c.\right\} \quad \ldots\ldots.(5)$$

perhaps better than equation (3).

We buy better convergence at the price of a complex V.

(10) How to obtain with accuracy 16 decimals (from ignorance what tables are available) I cannot explain: but for liberal knowledge of these integrals the following may suffice.

All tables furnish us with log sin x, and the *differences* of log sin x when x proceeds by single degrees are small,—the smaller as x

increases. If then we begin by series (3) or (4) while θ is small, we soon proceed safely by addition of Δ Cl. (θ), calculated from

$$\Delta^n \log \sin (\tfrac{1}{2}\theta)$$

in some method of Mechanical Quadrature. Very simple is that founded on the presumed identity of

$$\frac{\Delta}{\log(1+\Delta)} \text{ with } 1 + m_1\Delta - m_2\Delta^2 + m_3\Delta^3 - \&c....$$

where $m_1 = \frac{1}{2}$, $\quad m_2 = \frac{1}{12}$, $m_3 = \frac{1}{24}$,

$$m_4 = \frac{19}{60\,.\,12} = +\frac{1}{6^2}(1 - \tfrac{1}{20}).$$

Make $$-Fx = \log \sin x\,;$$

then since $$\frac{h\Delta}{\log(1+\Delta)} = \Delta D^{-1} = \Delta\int......dx,$$

we have

$$\Delta\int - \log\sin x dx = h\,[Fx + m_1\Delta Fx + m_2\Delta^2 Fx + m_3\Delta^3 Fx + \&c.]...(6),$$

which ought to agree with the equivalent series

$$h\,[F(x+h) - m_1\Delta Fx + m_2\Delta^2 F(x-h) - m_3\Delta^3 F(x-2h) + \&c.]$$

The two afford a reasonable check on error; but the latter is less trustworthy here.

To obtain Δ Cl. (θ) we have to double the last (changing dx to $2dx$) and then write θ for $2x$. (Thus x ought to advance by half degrees, if θ is to advance by degrees.) But we must first subtract

$$\Delta\int \log 2dx,$$

to insert the factor 2 before $\sin x$, and from the right-hand take

$$\Delta(\log 2\,.\,x) \text{ or } h\,.\log 2.$$

If the values of Cl. up to $\theta = 10^\circ$ have been obtained by series (3). and in (5) we make $x = 5^\circ$, $h = \frac{1}{2}$ a degree or $30'$, we obtain Δ Cl. 10° which added to Cl. 10° yields to us Cl. 11°, hitherto unknown. By a second step make $x = 5^\circ\,30'$, $h = 30'$, and you can get Δ Cl. 12° which was previously known by series (3). Thus at once a test is obtained of the validity of our method. Carry it further we have new tests, wherever Cl. was already otherwise known. Nothing is here gained, rather we lose, by *descending* to

$$F(x-h)\,.\,F(x-2h)\,\&c.$$

(11) Nevertheless for the arduous task of 16 decimals a quicker converging series, proceeding by $1\ \Delta^2\Delta^4\Delta^6\ldots$ may be desired. Perhaps one of those by which Legendre computed his Elliptic Table deserves attention. Its theory may be here expounded with slight variation from his point of view.

First, given $\dfrac{x^2}{1+x}=4u^2$, we may develop $\log(1+x)$ in rising powers of u^2.

Solve for x; then $\qquad x = 2u^2 + 2\sqrt{(u^2+u^4)}$.

Differentiate:

$$\therefore\ \frac{dx}{2u\,du} = 2 + \frac{1+2u^2}{\sqrt{(u^2+u^4)}} = \frac{1+x}{\sqrt{(u^2+u^4)}};$$

whence
$$\frac{dx}{2(1+x)} = \frac{du}{\sqrt{(1+u^2)}}.$$

Develop the right member into series, and integrate:

$$\therefore\ \tfrac{1}{2}\log(1+x) = u - \tfrac{1}{2}.\frac{u^3}{3} + \frac{1.3}{2.4}\cdot\frac{u^5}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{u^7}{7} + \&c.$$

The coefficients are familiar in the series for $\sin^{-1}x$; call them $N_1\,N_2\,N_3\ldots$ whence

$$\frac{\log(1+x)}{2u} = 1 - N_1u^2 + N_2u^4 - N_3u^6 + \&c.$$

Reciprocate this equation,

$$\therefore\ \frac{2u}{\log(1+x)} = 1 + M_1u^2 + M_2u^4 + M_3u^6 + \&c. \ \ldots\ldots. \ (7),$$

and you compute $M_1\,M_2\,M_3\ldots$ from $N_1\,N_2\,N_3\ldots$, namely

$$M_1 = N_1 = \tfrac{1}{6};$$
$$M_2 = M_1N_1 - N_2;$$
$$M_3 = M_2N_1 - M_1N_2 + N_3;\ \&c.$$

Treat the equation (7) as an *identity* while convergence is sure; and assume *symbols* for quantities; viz. put Δ for x, $\therefore (2u)^2$ becomes $\dfrac{\Delta^2}{1+\Delta}$, or, say, $\dfrac{\Delta^2}{E}$, and u^2 becomes $4^{-1}.\Delta^2E^{-1}$. In (6) *multiply* (so to speak) by $\sqrt{E}$ or $E^{\frac{1}{2}}$.

$$\therefore \frac{\Delta}{\log(1+\Delta)} = E^{+\frac{1}{2}} + E^{+\frac{1}{2}}(4^{-1}\Delta^2 E^{-1}) M_1 + E^{+\frac{1}{2}}(4^{-1}\Delta^2 E^{-1})^2 M_2 + \&c.,$$

or

$$= E^{+\frac{1}{2}} + 4^{-1}M_1\Delta^2 E^{+\frac{3}{2}} + 4^{-2}M_2\Delta^4 E^{+\frac{5}{2}} + \&c.$$

Apply this to $F(x)$, and as before change $\frac{h\Delta}{\log(1+\Delta)}$ into ΔD^{-1}, then

$$\Delta\int Fx\,dx = h\,\{E^{+\frac{1}{2}}Fx + 4^{-1}M_1\Delta^2 E^{-\frac{1}{2}}Fx + 4^{-2}M_2\Delta^4 E^{-\frac{3}{2}}Fx + \&c.\},$$

which means

$$h\,\{F(x+\tfrac{1}{2}h) + 4^{-1}M_1\Delta^2 F(x-\tfrac{1}{2}h) + 4^{-2}M_2\Delta^4 F(x-\tfrac{3}{2}h) + 4^{-3}M_3\Delta^6 F(x-\tfrac{5}{2}h) + \&c.\},$$

and it remains to assume, as before

$$Fx = -\log\sin x,\ \ 2x=\theta,\ \ \&c.$$

Continuation of a table of Cl. (θ) *beyond* $\theta = 60°$.

(12) We know in Trigonometry that

$$2\sin nx = 2\sin x . 2\sin\left(x+\frac{\pi}{n}\right)......2\sin\left(x+\frac{n-1}{n}\pi\right);$$

where n is any positive integer. Let $x=\frac{1}{2}\theta$; take the logarithm on each side, multiply by $d\theta$, and integrate; then

$$\frac{1}{n}\,\text{Cl.}\,(n\theta) + \alpha = \text{Cl.}\,(\theta) + \text{Cl.}\left(\theta+\frac{2\pi}{n}\right) + \text{Cl.}\left(\theta+\frac{4\pi}{n}\right) + \ldots + \text{Cl.}\left(\theta+\frac{2n-2}{n}\pi\right) \ldots (8).$$

To find α, make $\theta = 0$,

$$\therefore \alpha = \text{Cl.}\,\frac{2\pi}{n} + \text{Cl.}\,\frac{4\pi}{n} + \ldots + \text{Cl.}\left(\frac{2n-2}{n}\pi\right);$$

in reverse order,

$$\alpha = \text{Cl.}\left(\frac{2n-2}{n}\pi\right) + \text{Cl.}\left(\frac{2n-4}{n}\pi\right) + \ldots + \text{Cl.}\,\frac{2\pi}{n}.$$

Add each term to that above, remembering that

$$\text{Cl.}\,(\theta) + \text{Cl.}\,(2\pi-\theta) = 0;$$

$$\therefore\ 2\alpha = 0.$$

In the general equation (8) first make

$$n = 2, \ \theta = x;$$

and observe that

$$\text{Cl.}\left(x + \frac{2\pi}{2}\right) = -\text{Cl.}(\pi - x),$$

or

$$\tfrac{1}{2}\,\text{Cl.}(2x) = \text{Cl.}(x) = \text{Cl.}(180^\circ - x) \ \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Next, for θ write y, with $n = 3$. Note that

$$\text{Cl.}\left(y + \frac{4\pi}{3}\right) = -\text{Cl.}(120^\circ - y),$$

$$\therefore\ \tfrac{1}{3}\,\text{Cl.}(3y) = \text{Cl.}(y) + \text{Cl.}(120^\circ + y) - \text{Cl.}(120^\circ - y) \ \ldots(10).$$

By taking $n = 5$, we have a formula of verification,

$$\tfrac{1}{5}\,\text{Cl.}(5\theta) = \text{Cl.}(\theta) + \text{Cl.}(\theta + 72^\circ) + \text{Cl.}(\theta + 144^\circ)$$
$$- \text{Cl.}(144^\circ - \theta) - \text{Cl.}(72^\circ - \theta).$$

For a moment suppose that only 90° and 60° are known as by Art. 4.

In (9) make $x = 60^\circ$ and in (10) make $y = 30^\circ$,

$$\therefore\ \tfrac{3}{2}\,\text{Cl.}\ 120^\circ = \text{Cl.}\ 60^\circ,$$

known by Art. 4. Hence Cl. 120° is known, and

$$\tfrac{4}{3}\,\text{Cl.}\ 90^\circ = \text{Cl.}\ 30^\circ + \text{Cl.}\ 150^\circ.$$

Thus Cl. 150° is known.

Again, in (9) make $x = 30^\circ$,

$$\therefore\ \text{Cl.}\ 150^\circ = \text{Cl.}\ 30^\circ - \tfrac{1}{2}\,\text{Cl.}\ 60^\circ.$$

This is a check and test of accuracy.

(13) Further, in (9) make $x = 36^\circ$ and in (10) make $y = 48^\circ$.

Then

$$\tfrac{1}{2}\,\text{Cl.}\ 72^\circ = \text{Cl.}\ 36^\circ - \text{Cl.}\ 144^\circ,$$

$$\tfrac{1}{3}\,\text{Cl.}\ 144^\circ = \text{Cl.}\ 48^\circ + \text{Cl.}\ 168^\circ - \text{Cl.}\ 72^\circ,$$

which are of avail to find Cl. 72° and Cl. 144°, *after* Cl. is calculated up to $\theta = 60^\circ$. For then by (9)

$$\text{Cl.}\ 168^\circ = \text{Cl.}\ 12^\circ - \tfrac{1}{2}\,\text{Cl.}\ 24^\circ,$$

and if $p = \text{Cl.}\ 36^\circ$ and $q = \text{Cl.}\ 48^\circ + \text{Cl.}\ 168^\circ$, both p and q become known, and in the two equations only Cl. 72° and Cl. 144° are unknown. Call them v and u,

$$\therefore\ v = \frac{6q - 2p}{5}, \qquad u = \frac{6p - 3q}{5},$$

if $\qquad p = \text{Cl. } 36^\circ$ and $q = \text{Cl. } 48^\circ - \tfrac{1}{2}\,\text{Cl. } 24^\circ + \text{Cl. } 12^\circ$.

By a persevering use of equations (9) and (10) *every* value of Cl. θ from $\theta = 61^\circ$ to $\theta = 179^\circ$ can now be deduced from the values which have θ less than 60°. Obviously from (9) alone we obtain the values from $\theta = 179^\circ$ to $\theta = 150^\circ$. The successive steps are less obvious in other parts of the table; but only when θ is less than 60° is the method of independent series *indispensable*.

Clausen's Integral.

Again, if $\quad x = 74^\circ,\quad \text{Cl. } 106^\circ = \text{Cl. } 74^\circ - \tfrac{1}{2}\,\text{Cl. } 148^\circ,$

$x = 76^\circ,\quad \text{Cl. } 104^\circ = \text{Cl. } 76^\circ - \tfrac{1}{2}\,\text{Cl. } 152^\circ,$

$x = 80^\circ,\quad \text{Cl. } 100^\circ = \text{Cl. } 80^\circ - \tfrac{1}{2}\,\text{Cl. } 160^\circ,$

$x = 82^\circ,\quad \text{Cl. } 98^\circ = \text{Cl. } 82^\circ - \tfrac{1}{2}\,\text{Cl. } 164^\circ,$

$x = 86^\circ,\quad \text{Cl. } 94^\circ = \text{Cl. } 86^\circ - \tfrac{1}{2}\,\text{Cl. } 172^\circ,$

$x = 88^\circ,\quad \text{Cl. } 92^\circ = \text{Cl. } 88^\circ - \tfrac{1}{2}\,\text{Cl. } 176^\circ.$

Also from $\quad x = 46^\circ,\quad \text{Cl. } 134^\circ = \text{Cl. } 46^\circ - \tfrac{1}{2}\,\text{Cl. } 92^\circ,$

$x = 47^\circ,\quad \text{Cl. } 133^\circ = \text{Cl. } 47^\circ - \tfrac{1}{2}\,\text{Cl. } 94^\circ,$

$x = 49^\circ,\quad \text{Cl. } 131^\circ = \text{Cl. } 49^\circ - \tfrac{1}{2}\,\text{Cl. } 98^\circ,$

$x = 50^\circ,\quad \text{Cl. } 130^\circ = \text{Cl. } 50^\circ - \tfrac{1}{2}\,\text{Cl. } 100^\circ,$

$x = 52^\circ,\quad \text{Cl. } 128^\circ = \text{Cl. } 52^\circ - \tfrac{1}{2}\,\text{Cl. } 104^\circ,$

$x = 53^\circ,\quad \text{Cl. } 127^\circ = \text{Cl. } 53^\circ - \tfrac{1}{2}\,\text{Cl. } 106^\circ.$

Next, let $\quad y = 39^\circ,\ \therefore\ \text{Cl. } 81^\circ = \text{Cl. } 39^\circ + \text{Cl. } 159^\circ - \tfrac{1}{3}\,\text{Cl. } 117^\circ,$

$y = 37^\circ,\quad \text{Cl. } 83^\circ = \text{Cl. } 37^\circ + \text{Cl. } 157^\circ - \tfrac{1}{3}\,\text{Cl. } 111^\circ,$

$y = 35^\circ,\quad \text{Cl. } 85^\circ = \text{Cl. } 35^\circ + \text{Cl. } 155^\circ - \tfrac{1}{3}\,\text{Cl. } 105^\circ,$

$x = 81^\circ,\quad \text{Cl. } 99^\circ = \text{Cl. } 81^\circ - \tfrac{1}{2}\,\text{Cl. } 182^\circ,$

$x = 83^\circ,\quad \text{Cl. } 97^\circ = \text{Cl. } 83^\circ - \tfrac{1}{2}\,\text{Cl. } 186^\circ,$

$$x = 85^\circ, \qquad \text{Cl. } 95^\circ = \text{Cl. } 85^\circ - \tfrac{1}{2}\,\text{Cl. } 170^\circ,$$

$$y = 41^\circ, \qquad \text{Cl. } 79^\circ = \text{Cl. } 41^\circ + \text{Cl. } 161^\circ - \tfrac{1}{3}\,\text{Cl. } 123^\circ,$$

$$y = 43^\circ, \qquad \text{Cl. } 77^\circ = \text{Cl. } 43^\circ + \text{Cl. } 163^\circ - \tfrac{1}{3}\,\text{Cl. } 129^\circ,$$

$$x = 77^\circ, \qquad \text{Cl. } 103^\circ = \text{Cl. } 77^\circ - \tfrac{1}{2}\,\text{Cl. } 154^\circ,$$

$$y = 33^\circ, \qquad \text{Cl. } 87^\circ = \text{Cl. } 33^\circ + \text{Cl. } 153^\circ - \tfrac{1}{3}\,\text{Cl. } 99^\circ,$$

$$y = 27^\circ, \qquad \text{Cl. } 93^\circ = \text{Cl. } 27^\circ + \text{Cl. } 147^\circ - \tfrac{1}{3}\,\text{Cl. } 81^\circ,$$

$$y = 31^\circ, \qquad \text{Cl. } 89^\circ = \text{Cl. } 31^\circ + \text{Cl. } 151^\circ - \tfrac{1}{3}\,\text{Cl. } 93^\circ,$$

$$x = 89^\circ, \qquad \text{Cl. } 91^\circ = \text{Cl. } 89^\circ - \tfrac{1}{2}\,\text{Cl. } 178^\circ,$$

$$x = 79^\circ, \qquad \text{Cl. } 101^\circ = \text{Cl. } 79^\circ - \tfrac{1}{2}\,\text{Cl. } 158^\circ,$$

$$y = 13^\circ, \qquad \text{Cl. } 170^\circ = \text{Cl. } 13^\circ + \text{Cl. } 133^\circ - \tfrac{1}{3}\,\text{Cl. } 39^\circ,$$

$$x = 71^\circ, \qquad \text{Cl. } 109^\circ = \text{Cl. } 71^\circ - \tfrac{1}{2}\,\text{Cl. } 142^\circ$$

[might have been earlier],

$$x = 70^\circ, \qquad \text{Cl. } 110^\circ = \text{Cl. } 70^\circ - \tfrac{1}{2}\,\text{Cl. } 140^\circ,$$

$$x = 68^\circ, \qquad \text{Cl. } 112^\circ = \text{Cl. } 68^\circ - \tfrac{1}{2}\,\text{Cl. } 136^\circ,$$

$$x = 67^\circ, \qquad \text{Cl. } 113^\circ = \text{Cl. } 67^\circ - \tfrac{1}{2}\,\text{Cl. } 134^\circ,$$

$$x = 65^\circ, \qquad \text{Cl. } 115^\circ = \text{Cl. } 65^\circ - \tfrac{1}{2}\,\text{Cl. } 130^\circ,$$

$$x = 64^\circ, \qquad \text{Cl. } 116^\circ = \text{Cl. } 64^\circ - \tfrac{1}{2}\,\text{Cl. } 128^\circ,$$

$$x = 56^\circ, \qquad \text{Cl. } 124^\circ = \text{Cl. } 56^\circ - \tfrac{1}{2}\,\text{Cl. } 112^\circ,$$

$$x = 62^\circ, \qquad \text{Cl. } 118^\circ = \text{Cl. } 62^\circ - \tfrac{1}{2}\,\text{Cl. } 124^\circ,$$

$$x = 58^\circ, \qquad \text{Cl. } 122^\circ = \text{Cl. } 58^\circ - \tfrac{1}{2}\,\text{Cl. } 116^\circ,$$

$$x = 61^\circ, \qquad \text{Cl. } 119^\circ = \text{Cl. } 61^\circ - \tfrac{1}{2}\,\text{Cl. } 122^\circ,$$

$$x = 59^\circ, \qquad \text{Cl. } 121^\circ = \text{Cl. } 59^\circ - \tfrac{1}{2}\,\text{Cl. } 118^\circ.$$

Finally, $x = 55^\circ, \qquad \text{Cl. } 125^\circ = \text{Cl. } 55^\circ - \tfrac{1}{2}\,\text{Cl. } 110^\circ,$

which completes our problem of this article. We can construct many verifications by taking new values of y or x.

I proceed to copy out Clausen's table of

$$z = -\int_0 \log_e ch\,(\omega)\,d\omega, \text{ to 16 decimals.}$$

(14) TABLE IV.

Copied out from Crelle.

ω	Cl. (ω)	ω	Cl. (ω)
1°	·0881 0825 5769 3816	31°	·8755 9079 8996 7808
2°	·1520 2155 3586 8417	32°	·8862 5316 1826 4641
3°	·2068 0333 4607 0624	33°	·8963 8427 2291 5068
4°	·2556 5584 9659 5340	34°	·9068 0111 6233 6120
5°	·3001 0018 6767 3956	35°	9151 1970 6837 0285
6°	·3410 3242 9322 0966	36°	·9237 5516 8100 5353
7°	·3790 4482 9624 8711	37°	·9319 2180 9030 1292
8°	·4145 5845 3838 3392	38°	·9396 3318 9807 3681
9°	·4478 8824 4813 3546	39°	·9469 0218 0990 0054
10°	·4792 7876 0075 2437	40°	·9537 4101 6638 8513
11°	·5089 2545 8247 8208	41°	·9601 6134 2130 6023
12°	·5369 8817 7258 7339	42°	·9661 7425 7305 1169
13°	·5636 0008 7633 3457	43°	·9[illegible]17 9035 5502 9060
14°	·5888 7387 4464 0590	44°	·9770 1975 8971 0157
15°	·6129 0614 4155 6218	45°	·9818 7215 1050 2034
16°	·6357 8065 0927 6729	46°	·9863 5680 5501 2737
17°	·6575 7071 3646 7311	47°	·9904 8261 3281 6730
18°	·6783 4106 [illegible]111 0970	48°	·9942 5810 7043 7516
19°	·6901 4927 1851 7745	49°	·9976 9148 3592 1363
20°	·7170 4689 6085 3112	50°	1·0007 9062 4508 5728
21°	·7350 8037 0749 8140	51°	1·0035 6311 5127 5813
22°	·7522 91[illegible]4 6792 7012	52°	1·0060 1626 2024 6885
23°	·7687 1928 9063 6796	53°	1·0081 5710 9160 5273
24°	·7843 9797 0873 9228	54°	1·0099 9245 2806 2885
25°	·7993 5988 6000 1699	55°	1·0115 2885 5367 5050
26°	·8136 3459 4674 3617	56°	1·0127 7265 8199 6113
27°	·8272 4941 6271 8459	57°	1·0137 2999 3512 1934
28°	·8402 2967 8604 2658	58°	1 0144 0679 5437 3551
29°	·8525 9893 1517 4475	59°	1·0148 0881 0336 6062
30°	·8643 7913 1853 8927	60°	1·0149 ·4160 6409 6536

TABLE IV. (*continued*).

ω	Cl. (ω)	ω	Cl. (ω)
61°	1·0148 1058 2663 8056	91°	·9098 4103 0699 7436
62°	1·0144 2097 7296 8533	92°	·9035 6679 0135 4674
63°	1·0137 7787 5538 9387	93°	·8971 4546 3151 8095
64°	1·0128 8621 6999 4148	94°	·8905 7959 7112 5609
65°	1·0117 5080 2554 3869	95°	·8838 7169 7533 1242
66°	1·0103 7630 0812 0751	96°	·8770 2422 9475 7791
67°	1·0087 6725 4187 4170	97°	·8700 3961 8888 1217
68°	1·0069 2808 4615 1755	98°	·8629 2025 3887 6263
69°	1·0048 6309 8928 1536	99°	·8556 6848 5995 1023
70°	1·0025 7649 3924 8056	100°	·8482 8663 1319 6489
71°	1·0000 7236 1148 4501	101°	·8407 7697 1697 5556
72°	·9973 5469 1398 4148	102°	·8331 4175 5787 4517
73°	·9944 2757 8091 7486	103°	·8253 8320 0123 8715
74°	·9912 9422 5792 6051	104°	·8175 0349 0131 2800
75°	·9879 5894 5025 0142	105°	·8095 0478 1100 4812
76°	·9844 2516 4883 5026	106°	·8013 8919 9129 2267
77°	·9806 9643 1954 8802	107°	·7931 5884 2028 7364
78°	·9767 7621 4463 4752	108°	·7848 1578 0197 7508
79°	·9726 6790 5351 1523	109°	·7763 6205 7465 6417
80°	·9683 7482 5202 5917	110°	·7677 9969 1906 0278
81°	·9639 0022 5025 5169	111°	·7591 3067 6622 2615
82°	·9592 4728 8894 8467	112°	·7503 5698 0506 0800
83°	·9544 1913 6469 0848	113°	·7414 8054 8970 6471
84°	·9494 1882 5386 6654	114°	·7325 0330 4659 1441
85°	·9442 4935 3549 4182	115°	·7234 2714 8130 0127
86°	·9389 1366 1299 8123	116°	·7142 5395 8519 8906
87°	·9334 1463 3498 1743	117°	7049 8559 4185 2321
88°	·9277 5510 1505 6496	118°	·6956 2389 3323 5527
89°	·9219 3784 5078 2809	119°	·6861 7067 4575 1907
90°	·9159 6559 4177 2190	120°	·6766 2773 7606 4358

TABLE IV. (*continued*).

ω	Cl. (ω)	ω	Cl. (ω)
121°	·6669 9686 3674 8299	151°	·3453 9553 3798 7699
122°	·6572 7981 6177 4098	152°	·3338 4334 9504 4452
123°	·6474 7834 1182 6213	153°	·3222 5318 9871 7016
124°	·6375 9416 7946 6013	154°	·3106 2646 3662 0175
125°	·6276 2900 9414 4910	155°	·2989 6457 3745 8835
126°	·6175 8456 2707 4131	156°	·2872 6891 7352 0470
127°	·6074 6250 9595 7139	157°	·2755 4088 6313 0428
128°	·5972 6451 6959 0485	158°	·2637 8186 7307 1933
129°	·5869 9223 7233 8555	159°	·2519 9324 2097 2555
130°	·5766 4730 8848 7484	160°	·2401 7638 7765 8856
131°	·5662 3135 6648 3232	161°	·2283 3267 6948 0904
132°	·5557 4599 2305 8620	162°	·2164 6347 8060 8294
133°	·5451 9281 4725 3925	163°	·[illegible]045 7015 5529 9251
134°	·5345 7341 0433 5400	164°	·1926 5407 0014 4409
135°	·5238 8935 3961 5938	165°	·1807 1657 8628 6755
136°	·5131 4220 8218 1909	166°	·1687 5903 5161 9261
137°	·5023 3352 4853 0008	167°	·1567 8279 0296 1649
138°	·4914 6484 4611 7842	168°	·1447 8919 1821 7725
139°	·4805 3769 7683 1790	169°	·1327 7958 4851 4702
140°	·4695 5360 4037 5555	170°	·1207 5531 2032 5880
141°	·4585 1407 3758 2678	171°	·1087 1771 3757 8061
142°	·4474 2060 7365 6168	172°	·0966 6812 8374 5027
143°	·4362 7469 6133 8266	173°	·0846 0789 2392 8416
144°	·4250 7782 2401 3279	174°	·0725 3834 0692 7297
145°	·4138 3145 9879 6257	175°	·0604 6080 6729 7737
146°	·4025 3707 3926 0213	176°	·0483 7662 2740 3644
147°	·3911 9612 1885 4692	177°	·0362 8711 9946 [illegible]140
148°	·3798 1005 3326 7567	178°	·0241 9362 8757 0748
149°	·3683 8031 0348 3542	179°	·0120 9747 8975 9607
150°	·3569 0832 7849 0659	180°	*zero*

Some Uses of Clausen's Integral.

(15) (*a*) To find

$$\Theta = -\int_0 \log(\sin^2\theta - \sin^2\alpha)\,d\theta,$$

observe that the quantity under *log*

$$= \sin(\theta+\alpha).\sin(\theta-\alpha).$$

Hereby Θ is reducible to Cl. $(2\theta \pm 2\alpha)$.

(*b*)
$$\Theta = -\int_0 \log(\cos\theta - \cos\alpha)\,d\theta.$$

But
$$\cos\theta - \cos\alpha = 2\sin^2\tfrac{1}{2}\alpha - 2\sin^2\tfrac{1}{2}\theta,$$

which admits a reduction as the preceding.

(*c*)
$$\Theta = \int_0 \log\left(1 + \frac{\cos\theta}{\cos\alpha}\right)d\theta.$$

The *log* being equivalent to

$$\log(\cos\alpha + \cos\theta) - \log\cos\alpha,$$

the second term involves nothing new. Put $\alpha = \pi - \beta$, and for the former we regain the last case.

(*d*)
$$\Theta = \int_0 \log(1 + \sin\alpha\cos\theta)\,d\theta.$$

This is less obvious. Put

$$\tan\eta = \frac{\sin\theta}{\tan\frac{1}{2}\alpha + \cos\theta},$$

and write
$$m = \tan\tfrac{1}{2}\alpha;$$

then
$$m = \frac{\sin(\theta-\eta)}{\sin\eta}.$$

But
$$\sin\alpha = \frac{2m}{1+m^2},$$

$$1 + \sin\alpha\cos\theta = \frac{1 + 2m\cos\theta + m^2}{1+m^2}.$$

Now $$1 + 2m\cos\theta + m^2 = \sin^2\theta + (\cos\theta + m)^2,$$

and $$m + \cos\theta = \frac{\sin(\theta-\eta)}{\sin\eta} + \cos\theta = \frac{\sin\theta\cos\eta}{\sin\eta},$$

whence we find $$1 + 2m\cos\theta + m^2 = \frac{\sin^2\theta}{\sin^2\eta},$$

$$1 + m^2 = \sec^2\tfrac{1}{2}\alpha;$$

$$\therefore\ 1 + \sin\alpha\cos\theta = \frac{\sin^2\theta}{\sin^2\eta}\cot^2\tfrac{1}{2}\alpha,$$

of which the *log* is

$$2\log\sin\theta - 2\log\sin\eta + 2\log\cos\tfrac{1}{2}\alpha.$$

Add $$2\log 2 - 2\log 2 = 0.$$

$$\therefore\ \Theta = -2\theta\log(\sec\tfrac{1}{2}) - \text{Cl.}(2\theta) - \int\log 2\sin\eta\, 2d\theta.$$

Now $$d\theta = d\eta + d(\theta-\eta),$$

whence the last integral

$$= \int\log 2\sin\eta\, 2d\eta + \int\log 2\sin\eta\, 2d(\theta-\eta),$$

or since $$\sin\eta = \frac{\sin(\theta-\eta)}{m},$$

we get $$-\text{Cl.}(2\eta) + \int\log\left(\frac{2\sin(\theta-\eta)}{m}\right)2a(\theta-\eta).$$

But $$\log\frac{2\sin(\theta-\eta)}{m} = \log 2\sin(\theta-\eta) - \log m.$$

Further, we have

$$-\text{Cl.}(2\eta) - \text{Cl.}(2\theta-2\eta) - 2\log m(\theta-\eta).$$

Since η vanishes with θ,

$$\Theta = -2\theta\log\sec\tfrac{1}{2}\alpha - \text{Ci.}(2\theta) + \text{Cl.}(2\eta)$$
$$+ \text{Cl.}(2\theta-2\eta) + 2\log m(\theta-\eta),$$

where restoring to m its value

$$\log m - \log\sec\tfrac{1}{2}\alpha = \log(\tan\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha) = \log\sin\tfrac{1}{2}\alpha,$$

and the algebraic terms make

$$2\theta\log\sin\tfrac{1}{2}\alpha - 2\eta\log\sec\tfrac{1}{2}\alpha.$$

Generally then

$$\int \log (A + B\cos\theta)\, d\theta,$$

is integrable by Cl., whatever the relative magnitude of A and B.

(*e*) When $F(x)$ is rational,

$$\int \log F(x) \frac{dx}{\sqrt{(p^2 - x^2)}}$$

is reducible to the forms

$$\int \log (a + bx) \frac{dx}{\sqrt{(p^2 - x^2)}},$$

and

$$\int \log X \frac{dx}{\sqrt{(p^2 - x^2)}},$$

of which each is solved by Cl. So is

$$\int \frac{x^m \log x\, dx}{(a + bx)^{n+\frac{1}{2}}},$$

when m and n are integers.

(*f*) So indeed is

$$\int_0 \log (1 - 2r\cos\mu \sin\omega + r^2 \sin^2\omega)\, d\omega.$$

CHAPTER III.

HILL'S INTEGRAL;—*The Dilogarithm?*

1. IN the Reduction of

$$\int\left[\int\frac{dx}{x-a}\right]\cdot\frac{ax}{x-b}$$

when of a and b one is real, we found in Ch. II. two new Integrals, Clausen's, which is of single entry, and Hill's, which being of the type

$$\int \log x\, d \log \sqrt{(x^2 - 2x\cos\mu + 1)},$$

involves a constant arc μ. The treatment of the latter was deferred[1].

I have now to add that Kummer elicits only these two Integrals from the general formula, even when both a and b in the above are of the form $m + n\sqrt{-1}$. First reducing to

$$\int\frac{\log x\, dx}{x+1}$$

which makes x imaginary, he assumes

$$x = (\cos\mu + \sqrt{-1}\sin\mu)\, y,$$

as a general representation of an imaginary variable and makes his μ constant with his y proportionate to x. This is not with me in itself obviously admissible; for the y here has superseded a more complex form

$$y = \left\{\frac{x}{m + n\sqrt{-1}} + (p + q\sqrt{-1})\right\}$$

and y does not vanish with x. I cannot doubt that Kummer can make his case good, also Professor Cayley by an *aliter* method of his own justified it. But I cannot trust myself to expound the argument. Years ago I essayed an after process by presuming a *perfect table of double* entry from which I argued as though the distinction of constant and variable were abandoned. I thought this satisfactory; but I have lost my paper and fail to recover the outline. I can only *announce* how Kummer brings out this result.

Let X^2 stand for $x^2 - 2x\cos\alpha + 1$. We may find it convenient also to use Y^2 for $y^2 - 2y\cos\alpha + 1$ otherwise with a variation of the constant, Y^2 for $y^2 - 2y\cos\beta + 1$. So too with z and Z.

Supplemental Integrals.

2. Let $\Lambda(x, \alpha)$ stand for $\int_0 \log x \, d\log X$,

and $\lambda(x, \alpha)$ stand for $\int_0 \log X \, d\log x$,

which are in some sense supplemental, since evidently their sum

$$= \log x \log X.$$

COR. $\lambda(x, 0) = \int_0 \log\sqrt{(1 - 2x + x^2)}\, d\log x$

$$= \int_0 l(1-x)\frac{dx}{x} = L(1-x).$$

So $$\Lambda(x, 0) = \int_0 lx \, dl(1-x) = Lx + 2L2.$$

Each integral is known from the other, but λ has a distinct development in powers of x. For $-\frac{1}{2}\log X$ is known

$$= x\cos\alpha + \tfrac{1}{2}x^2\cos 2\alpha + \tfrac{1}{3}x^2\cos 3\alpha + \&c.,$$

and $$d\log x = \frac{dx}{x},$$

whence

$$\lambda(x, \alpha) = 1^{-2}x\cos\alpha + 2^{-2}x^2\cos 2\alpha + 3^{-2}x^3\cos 3\alpha + \&c.\ldots(1).$$

The simplicity of this series commends it, yet even with $x = \frac{1}{3}$, the convergence is poor.

3. What then is our present aim? To attain *power* of evaluation, must be our first object, and therefore to attain, so far as may be, such interchanges as lead to easier estimates. But besides this, we inquire under what values of the constant α, Λ and λ are reducible to L; and whether at definite values for x or any relation of x to α, the Λ and λ become reducible to L or indeed to a function of single entry.

4. The constant arc α involved in the integral may at our pleasure vary independently. Then we find

$$\frac{d}{d\alpha}\lambda(x,\alpha)=\frac{d}{d\alpha}\int_0 lX\,dlx=\int_0 \tfrac{1}{2}\frac{d}{d\alpha}lX^2\,dlx$$

$$=\int_0 \frac{x\sin\alpha}{X^2}\cdot\frac{dx}{x}=\int_0 \frac{\sin\alpha\,dx}{X^2}$$

But in Ch. II. we found that when $\tan\omega$ is taken

$$=\frac{x\sin\alpha}{1-x\cos\alpha},$$

it is equivalent to $\qquad x=\dfrac{\sin\omega}{\sin(\omega+\alpha)},$

and yields $\qquad X=\dfrac{\sin\alpha}{\sin(\omega+\alpha)};$

also $\qquad \dfrac{d\omega}{dx}=\dfrac{\sin\alpha}{X^2};$

from $\dfrac{d}{d\alpha}\lambda(x,\alpha)=\displaystyle\int_0 \frac{d\omega}{dx}dx$ or ω, since ω vanishes with x.

Now universally $d\lambda$, when both x and α vary,

$$=\left(\frac{d}{dx}\lambda\right)dx+\left(\frac{d}{d\alpha}\lambda\right)d\alpha,$$

or *total* $\qquad d\lambda(x,\alpha)=lXdlx+\omega d\alpha,$

whence also $\qquad d\Lambda(x,\alpha)=lxdlX-\omega d\alpha$ ....(2),

since $lXlx$ is the sum of both.

In these last formulae x and α varying independently, we may at pleasure assign any relation between them, making them *vary together*.

New form of Integral.

5. Further, since $\qquad \dfrac{d}{d\alpha}\lambda(x,\alpha)=\omega,$

when x is constant, $\qquad \therefore\ \lambda(x,\alpha)=\displaystyle\int_0 \omega d\alpha.$

We may better remember that x is constant, by changing x to c, for a moment, which gives

$$\tan\omega=\frac{c\sin\alpha}{1-c\cos\alpha},\ \text{and}\ c=\frac{\sin\omega}{\sin(\omega+\alpha)}.$$

Put $\omega + \alpha = \theta,\ \alpha = \theta - \omega,\ \omega d\alpha = \omega\,(d\theta - d\omega),$

and $\sin\omega = c\sin\theta,$

$$\therefore\ \lambda\,(c,\,\alpha) = \int \omega\,d\theta - \int \omega\,d\omega = \int \omega\,d\theta - \tfrac{1}{2}\omega^2.$$

But $\omega = \sin^{-1}(c\sin\theta),$

finally $$\lambda\,(c,\,\alpha) = \int_0 \sin^{-1}(c\sin\theta)\,d\theta - \tfrac{1}{2}\omega^2 + C,$$

when $\alpha = 0,\ c = 1,\ \omega = 0 = \theta,\ \lambda\,(c.\ 0) = L\,(1-c),\ \sin^{-1}(c\sin\theta) = \theta,$

$$\therefore\ L\,(1-c) = \int_0 \theta\,d\theta - \tfrac{1}{2}\omega^2 + C \text{ or } C = L\,(1-c).$$

Finally $$\lambda\,(c,\,\alpha) = L\,(1-c) - \tfrac{1}{2}\omega^2 + \int_0 \sin^{-1}(c\sin\theta)\,d\theta \ \ldots\ldots(3).$$

Here, if x, (that is c) be *less than* 1, the last integral is continuous from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, and so on with $\sin\theta$ retrograding to zero when $\theta = \pi$. From $\theta = \pi$ to $\theta = 2\pi$, the integral is negative, and destroys itself.

It is remarkable how the Logarithmic integral is here changed into Cyclic.

Special Cases of x in Dilogarithm.

6. Since $lxlX$ vanishes when $x = 1$, and

$$\Lambda\,(x,\,\alpha) + \lambda\,(x,\,\alpha) = lxlX,\ \therefore\ \Lambda\,(1,\,\alpha) + \lambda\,(1,\,\alpha) = 0.$$

But we found $d\lambda\,(1,\,\alpha) = \omega d\alpha,$

where $\tan\omega$ or $\dfrac{x\sin\alpha}{1 - x\cos\alpha}$ becomes $\dfrac{\sin\alpha}{1-\cos\alpha}$,

or $\cot\frac{1}{2}\alpha$, or $\tan(\frac{1}{2}\pi - \frac{1}{2}\alpha)$; that is $\omega = \frac{1}{2}(\pi - \alpha)$. Hence

$$\lambda\,(1-\alpha) = \int \tfrac{1}{2}(\pi - \alpha)\,d\alpha.$$

Integrating, $\therefore\ \lambda\,(1,\,\alpha) = C - \frac{1}{4}(\pi - \alpha)^2.$

To find C, make $\alpha = 0$; $\lambda\,(x,\,0)$ was $L\,(1-x)$, by Cor. in Art. 2;

$$\therefore\ \lambda\,(1,\,0) = L\,(0) = -\frac{\pi^2}{6},\ \therefore\ -\frac{\pi^2}{6} = C - \tfrac{1}{4}\pi^2,$$

and $C = \frac{1}{12}\pi^2$, or

$$\lambda\,(1,\,\alpha) = \tfrac{1}{12}\pi^2 - \tfrac{1}{4}(\pi-\alpha)^2,\quad \Lambda\,(1,\,\alpha) = \tfrac{1}{4}(\pi-\alpha)^2 - \tfrac{1}{12}\pi^2,$$

or again, $\Lambda\,(1,\,\alpha) = \frac{1}{4}(\pi - \alpha)^2 - L2.$

7. To find $\lambda(x, \alpha)$, when $x = 2\cos\alpha$. Proceed by Art. 4, in which x and α varying together, always fulfil this equation. Then

$$X^2 = 1 - x \cdot x + x^2 = 1, \quad lX = 0,$$

then $$\tan\omega = \frac{x\sin\alpha}{1 - x\cos\alpha} = \frac{2\cos\alpha\sin\alpha}{1 - 2\cos^2\alpha} = \frac{\sin 2\alpha}{-\cos 2\alpha}$$

$$= -\tan 2\alpha = \tan(\pi - 2\alpha),$$

or $\omega = \pi - 2\alpha$. In general

$$d\lambda(x, \alpha) = lX dlx + \omega d\alpha,$$

which here $= \omega d\alpha$, (since $lX = 0$) which further $= (\pi - 2\alpha) d\alpha$, making

$$\lambda(2\cos\alpha, \alpha) = C + \pi\alpha - \alpha^2.$$

To find C, make $\alpha = \frac{1}{2}\pi$, $x = 0$; then

$$\lambda(0, \tfrac{1}{2}\pi) = C + \tfrac{1}{2}\pi^2 - \tfrac{1}{4}\pi^2, \text{ or } C = -\tfrac{1}{4}\pi^2.$$

Finally $$\left.\begin{aligned}\lambda(2\cos\alpha, \alpha) &= -(\tfrac{1}{2}\pi - \alpha)^2 \\ \Lambda(2\cos\alpha, \alpha) &= +(\tfrac{1}{2}\pi - \alpha)^2\end{aligned}\right\}$$
also

The sum of these vanishes, because lX here $=$ zero.

8. When $x = \cos\alpha$. Then

$$X^2 = 1 - 2x \cdot x + x^2 = 1 - x^2, \quad lX = \tfrac{1}{2}l(1 - x^2).$$

Then

$$\tan\omega = \frac{\cos\alpha\sin\alpha}{1 - \cos^2\alpha} = \frac{\cos\alpha}{\sin\alpha} = \cot\alpha = \tan(\tfrac{1}{2}\pi - \alpha),$$

and $\omega = \frac{1}{2}\pi - \alpha$. Here

$$d\lambda(x, \alpha) = lX dlx + \omega d\alpha = \tfrac{1}{2}l(1 - x^2)\frac{dx}{x} + (\tfrac{1}{2}\pi - \alpha) d\alpha,$$

whence $$\lambda(x, \alpha) = \tfrac{1}{4}L(1 - x^2) - \tfrac{1}{2}(\tfrac{1}{2}\pi - \alpha)^2 + C.$$

Make $x = 0$, $\alpha = \frac{1}{2}\pi$, $\lambda(x, \alpha) = 0$, $L(1 - x^2) = 0$, $\therefore$ $C = 0$.

Finally $$\lambda(\cos\alpha, \alpha) = \tfrac{1}{4}L\sin^2\alpha - \tfrac{1}{2}(\tfrac{1}{2}\pi - \alpha)^2.$$

Similarly treated

$$\Lambda(\cos\alpha, \alpha) = \tfrac{1}{4}L\cos^2\alpha + \tfrac{1}{2}(\tfrac{1}{2}\pi - \alpha)^2 + \tfrac{1}{2}L2.$$

9. When $x = \sec \alpha$. Here $x \cos \alpha = 1$, $X^2 = x^2 - 2 + 1 = x^2 - 1$;

$$\tan \omega = \frac{x \sin \alpha}{1 - 1} = \infty, \quad \omega = \tfrac{1}{2}\pi,$$

$$\lambda(x, \alpha) = \tfrac{1}{2} \int_0 l(x^2 - 1)\, dlx + \int \tfrac{1}{2}\pi d\alpha,$$

which by integration is

$$\tfrac{1}{4} K(x^2 - 1) + \tfrac{1}{2}\pi\alpha + C,$$

or indeed

$$\tfrac{1}{4} l(x^2 - 1)\, l(x^2) - \tfrac{1}{4} \int l(x^2)\, dl(x^2 - 1) + \tfrac{1}{2}\pi\alpha + C,$$

where the $\int$ means $L(x^2)$. To find C in the last, make $\alpha = 0$, $x = 1$,

$$\lambda(x, \alpha) = \lambda(1, 0) = -2L2, \quad \therefore \; -2L2 = C,$$

and $$\lambda(\sec \alpha, \alpha) = \tfrac{1}{4}\, l(\tan^2 \alpha)\, l(\sec^2 \alpha) - \tfrac{1}{4} L(\sec^2 \alpha) + \tfrac{1}{2}\pi\alpha - \frac{\pi^2}{6}$$

So too $$\Lambda(\sec \alpha, \alpha) = \tfrac{1}{4} L(x^2) - \tfrac{1}{2}\pi\alpha + C',$$

and $$C' = \Lambda(1, 0) = 2L2 = \tfrac{1}{4} L(\sec^2 \alpha) - \tfrac{1}{2}\pi\alpha + 2L2.$$

10. When $x = \tfrac{1}{2} \sec \alpha$.

Now $$\tan \omega = \frac{\tfrac{1}{2} \sec \alpha \sin \alpha}{1 - \tfrac{1}{2}} = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha;$$

whence $\omega = \alpha$, unless indeed $\omega = \pi - \alpha$. But since

$$x = \frac{\sin \omega}{\sin(\omega + \alpha)} \text{ and } x = \frac{\sec \alpha}{2},$$

$$\therefore \; 2 \sin \omega \cos \alpha = \sin(\omega + \alpha);$$

this gives preference to $\omega = \alpha$. Also

$$X^2 = 1 - \sec \alpha \cos a + x^2 = x^2.$$

Thence $$d\lambda(x, \alpha) = lx\,dlx + \alpha d\alpha,$$

and $$\lambda(x, \alpha) = \tfrac{1}{2}(lx)^2 + \tfrac{1}{2}\alpha^2 + C.$$

To find C, it is well to make $\alpha = \tfrac{1}{3}\pi$, then

$$\cos \alpha = \tfrac{1}{2}, \quad \sec \alpha = 2, \quad x \text{ or } \tfrac{1}{2} \sec \alpha = 1.$$

$$\therefore \; \lambda(1, \tfrac{1}{3}\pi) = \tfrac{1}{18}\pi^2 + C.$$

[But we must return from Art. 12, IV, for the equivalent

$$\therefore\ 2\lambda(1, \tfrac{1}{3}\pi) = \tfrac{1}{3}L2 - L2 = -\tfrac{2}{3}L2;$$

$$\therefore\ C = -\tfrac{1}{3}L2 - \tfrac{2}{3}L2 = -L2.$$

Also $\Lambda(x, \alpha) + \lambda(x, \alpha) = lx\,lY$ — here $lx\,lx = (lx)^2$.]

11. Lastly, if

$$x = \frac{1-\sin\alpha}{\cos\alpha} = \sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} = \tan\frac{\frac{1}{2}\pi - \alpha}{2};\ \text{say} = \tan\mu;$$

then, $$\tan\omega = \frac{x\sin\alpha}{1 - x\cos\alpha}.$$

But $$x\cos\alpha = 1 - \sin\alpha,\quad 1 - x\cos\alpha = \sin\alpha.$$

$$\therefore\ \tan\omega = \frac{x\sin\alpha}{\sin\alpha} = x.$$

But $$x = \tan\mu;\quad \therefore\ \tan\omega = \tan\mu,$$

and $\omega = \mu$ not $\omega = \pi - \mu$; $\sin\omega$ increases with x, and $\omega = \frac{1}{2}(\pi - \alpha)$. Making all elements vary,

$$d\lambda(x, \alpha) = lX\,dlx + \omega d\alpha.$$

Now $\cos\alpha$ in terms of x $$= \frac{2x}{1+x^2},$$

since $$\tfrac{1}{2}\pi - \alpha = 2\mu,$$

$$\cos\alpha = \sin 2\mu = \frac{2\tan\mu}{1+\tan^2\mu} = \frac{2\tan\omega}{1+\tan^2\omega} = \frac{2x}{1+x^2}.$$

Hence $$X^2 = 1 - 2x\frac{2x}{1+x^2} + x^2 = \frac{1-2x^2+x^4}{1+x^2}.$$

Hence if $\alpha < 90^\circ$, x is < 1 and positive. We need not care for x negative. Only for

$$X = \frac{1-x^2}{\sqrt{(1+x^2)}}.$$

Thus $$d\lambda(x, \alpha) = [l(1-x^2) - \tfrac{1}{2}l(1+x^2)]\,dlx + \mu d\alpha;$$

and $$dlx = \frac{dx}{x} = \tfrac{1}{2}dl(x^2)\ \lambda(x, \alpha) = \tfrac{1}{2}L(1-x^2) - \tfrac{1}{4}L(1+x^2)$$

$$= \tfrac{1}{4}(\tfrac{1}{2}\pi - \alpha)^2 + C.$$

But when $x = 0$, $\alpha = \frac{1}{2}\pi$; so that *without* any C, every term vanishes.

This completes the solution.

Special cases of the constant α.

12. I. We have already found in the COR. to ART. 2

$$\lambda(x, 0) = L(1 - x);$$

and
$$\Lambda(x, 0) = Lx + 2L2.$$

II. Take now $\alpha = \pi$. Then

$$\lambda(x, \pi) = \int_0 l(1 + x)\,dlx = L(1 + x),$$

and
$$\Lambda(x, \pi) = \int_0 lx\,dl(1 + x) = lx\,l(1 + x) - L(1 + x + C).$$

Make $x = 1$, $\therefore \Lambda(1, \pi) = C$, which is not yet known.

But since universally

$$\Lambda(x, \alpha) + \lambda(x, \alpha) = lx\,lX;$$

and here
$$\alpha = \pi, \quad X = 1 + x.$$

$$\therefore \Lambda(x, \pi) = lx\,l(1 + x) - L(1 + x),$$

without new integration.

COR. Since our last C is thus found needless,

$$\therefore \Lambda(1, \pi) = 0.$$

III. Take $\alpha = \frac{1}{2}\pi$,

$$\lambda(x, \tfrac{1}{2}\pi) = \int_0 \tfrac{1}{2}l(1 + x^2)\,dlx = \tfrac{1}{4}L(1 + x^2),$$

whence
$$\Lambda(x, \tfrac{1}{2}\pi) = \tfrac{1}{2}l(1 + x^2)\,lx - \tfrac{1}{4}L(1 + x^2).$$

IV. Take $\alpha = \frac{1}{3}\pi$. Falling back on Trigonometry, write for a moment $f(x, \alpha)$ for X^2, then

$$f(x^3, 3\alpha) = f(x, \alpha)\,f(x, \alpha + \tfrac{2}{3}\pi)\,f(x, \alpha + \tfrac{4}{3}\pi).$$

Take log and multiply $\frac{1}{6}\log x^3 = \frac{1}{2}\log x$. Then integrate;

$$\tfrac{1}{3}\Lambda(x^3, 3\alpha) = \Lambda(x, \alpha) + \Lambda(x, \alpha + \tfrac{2}{3}\pi) + \Lambda(x, \alpha + \tfrac{4}{3}\pi).$$

Let
$$\alpha = \tfrac{1}{3}\pi;$$

$$\therefore \tfrac{1}{3}\Lambda(x^3, \pi) = \Lambda(x, \tfrac{1}{3}\pi) + \Lambda(x, \pi) + \Lambda(x, \tfrac{5}{3}\pi).$$

Now $\Lambda(x^3, \pi)$ is known by writing x^3 for x in II. above, where $\Lambda(x, \pi)$ is found. Also

$$\cos \tfrac{5}{3}\pi = \cos(2\pi - \tfrac{1}{3}\pi) = \cos \tfrac{1}{3}\pi.$$

Hence $$2\Lambda(x, \tfrac{1}{3}\pi) = \tfrac{1}{3}\Lambda(x^3, \pi) - \Lambda(x, \pi),$$
known by L.

Somewhat simpler,

$$2\lambda(x, \tfrac{1}{3}\pi) = \tfrac{1}{3}\lambda(x^3, \pi) - \lambda(x, \pi) = \tfrac{1}{3}L(1 + x^3) - L(1 + x).$$

13. The *Reciprocal* of x. Assume $xy = 1$,

$$Y^2 = 1 - 2y\cos\alpha + y^2,$$

then $$Y^2 = x^{-2}(x^2 - 2x\cos\alpha + 1) = x^{-2}X^2;$$

$$Y = x^{-1}X;\ lY = lX - lx,$$

also $$ly = -lx,$$

$$\therefore\ d\Lambda(y, \alpha) = lydlY = -lx(dlX - dlx) = -d\Lambda(x, \alpha) + lxdlx;$$

whence $$\Lambda(y, \alpha) + \Lambda(x, \alpha) = \tfrac{1}{2}(lx)^2 + C.$$

Make $$x = 1 = y;$$

then $$C = 2\Lambda(1, \alpha),$$

which was found in Art. 6, so that

$$\Lambda(x, \alpha) + \Lambda(x^{-1}, \alpha) = \tfrac{1}{2}(lx)^2 + \tfrac{1}{4}(\pi - \alpha)^2 - L2 \ldots\ldots\ldots(4).$$

By similar steps, we have only to change Λ to λ, and C' to $2\lambda(1, \alpha)$ and we know that

$$\Lambda(1, \alpha) + \lambda(1, \alpha) = 0 \text{ or } C' = -C.$$

Thus an x greater than 1 is at once reduced to x^{-1} less than 1; which by slight effort is an enormous lessening of our problem.

14. A *Complement* of x, viz. from $x + y = 2\cos\alpha$.

Here $$Y^2 = 1 - (x + y)y + y^2 = 1 - xy,$$

by symmetry $= X^2$,

$$\therefore\ \lambda x + \lambda y = \int_0 \tfrac{1}{2}l(1 - xy)\,d(lx + ly)$$

$$= \tfrac{1}{2}\int_0 l(1 - xy)\,dl(xy) = \tfrac{1}{2}L(1 - xy) + C.$$

To find C, make $$x = 2\cos\alpha, \quad y = 0,$$

$\therefore\ C = \lambda(2\cos\alpha, \alpha)$, known in Art. 7, or make $x = y = \cos\alpha$;

$\therefore\ C = 2\lambda(\cos x, \alpha) = \tfrac{1}{2}L(\sin^2\alpha) + C$, less simply.

Hence $$\lambda(x, \alpha) + \lambda(y, \alpha) = \tfrac{1}{2}L(1 - xy) - (\tfrac{1}{2}\pi - \alpha)^2 \ldots\ldots\ldots(5);$$

When we work similarly for Λ, we get

$$\Lambda(x, \alpha) + \Lambda(y, \alpha) = \tfrac{1}{2} \int l(xy)\, dl\,(1 - xy).$$

Though $$dl\,(1 - xy) = dl\,(xy - 1),$$
yet xy is at its maximum when $x = y = \cos\alpha$,

$$\Lambda(x, \alpha) + \lambda(y, \alpha) = \tfrac{1}{2} l(xy)\, l\,(1 - xy) - \tfrac{1}{2} \int l(1 - xy)\, dl\,(xy)$$

$$= \tfrac{1}{2} l\,(xy)\, l\,(1 - xy) - \tfrac{1}{2} L\,(1 - xy) + C' \ldots (5b).$$

Let $$x = y = \cos\alpha,$$

$$\therefore\ 2\Lambda(\cos\alpha, \alpha) = \tfrac{1}{2} l\,(\cos\alpha)^{2}\, l\,(\sin^2\alpha) - \tfrac{1}{2} L \sin^2\alpha + C'.$$

But in Art. 8,

$$2\Lambda(\cos\alpha, \alpha) = \tfrac{1}{2} L \cos^2\alpha + (\tfrac{1}{2}\pi - \alpha)^2 + L2,$$

$$\therefore\ C' = \tfrac{1}{2}\,(L \sin^2\alpha + L \cos^2\alpha) + (\tfrac{1}{2}\pi - \alpha)^2 + L2 - 2l \cos\alpha\, l \sin\alpha.$$

Now, because $$\sin^2\alpha + \cos^2\alpha = 1,$$

$$\therefore\ L \sin^2\alpha + L \cos^2\alpha = l \cos^2\alpha\, l \sin^2\alpha - 2L2,$$

whence $$C' = (\tfrac{1}{2}\pi - \alpha)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5c).$$

COR. Only when $(2\cos\alpha - x)$ is small, does this relieve the case of x almost $= 1$.

15. A *Decrement* of x; viz. $x - y = 2\cos\alpha$; $2\cos\alpha$ positive and less than x.

Suppose also $\alpha = \pi - \beta$, $Y^2 = 1 - 2y\cos\beta + y^2$,

$$Y^2 = 1 + 2y\cos\alpha + y^2 = 1 + (x - y)\,y + y^2 = 1 + xy,$$

$$X^2 = 1 - 2x\cos\alpha + y^2 = 1 - (x - y)\,x + x^2 = 1 + yx.$$

$$\therefore\ \lambda(x, \alpha) + \lambda(y, \beta) = \int \tfrac{1}{2} l(1 + xy)\, dl\,(xy) = \tfrac{1}{2} L\,(1 + xy) + C.$$

Put $y = 0$, $\quad \therefore\ \lambda(2\cos\alpha, \alpha) = C = -(\tfrac{1}{2}\pi - \alpha)^2$,

or $$\lambda(x, \alpha) + \lambda(y, \pi - \alpha) = \tfrac{1}{2} L\,(1 + xy) - (\tfrac{1}{2}\pi - \alpha)^2 \ldots\ldots\ldots (6),$$

when $$x - y = 2\cos\alpha.$$

Here if x be somewhat too large for the service of equation (1) Art. 2, y may *possibly* be small enough to relieve us.

COR. In our worst case, x a little exceeds 1 or a little falls short, y does not help us, unless $2\cos\alpha$ is in like case with x, i.e. unless $x - 2\cos\alpha$ is small.

Table of $\operatorname{Tan} x = \dfrac{1 - \epsilon^{-2x}}{1 + \epsilon^{-2x}}$,

from $x = \cdot 01$ to $x = 1$.

For $\displaystyle\int_0 \frac{d\theta}{\cos\theta}$.

December, 1884, recopied from original papers.

Corrected by Professor J. C. Adams.

x	Tan x	x	Tan x
·01	·0099 9966 6680	·26	·2542 9553 2627
·02	·0199 9733 3760	·27	·2636 2483 5472
·03	·0299 9100 3239	·28	·2729 0508 0563
·04	·0399 7868 0318	·29	·2821 3481 2670
·05	·0499 5837 4958	·30	·2913 1261 2452
·06	·0599 2810 3529	·31	·3004 3709 7147
·07	·0698 8589 0316	·32	·3095 0692 1213
·08	·0798 2976 9111	·33	·3185 2077 6903
·09	·0897 5778 4747	·34	·3274 7739 4808
·10	·0996 6799 4625	·35	·3363 7554 4337
·11	·1095 5847 0215	·36	·3452 1403 4136
·12	·1194 2729 8535	·37	·3539 9171 2477
·13	·1292 7258 3606	·38	·3627 0746 7578
·14	·1390 9244 7878	·39	·3713 6022 7877
·15	·1488 8503 3623	·40	·3799 4896 2255
·16	·1586 4850 4297	·41	·3884 7268 0216
·17	·1683 8104 5870	·42	·3969 3043 2005
·18	·1780 8086 8117	·43	·4053 2130 8689
·19	·1877 4620 5869	·44	·4136 4444 2187
·20	·1973 7532 0225	·45	·4218 9900 5251
·21	·2069 6649 9730	·46	·4300 8421 1403
·22	·2165 1806 1493	·47	·4381 9931 4833
·23	·2260 2835 2279	·48	·4462 4361 0249
·24	·2354 9574 9539	·49	·4542 1643 2683
·25	·2449 1866 2403	·50	·4621 1715 7260

x	Tan x	x	Tan x
·51	·4699 4519 8933	·76	·6410 7696 1186
·52	·4777 0001 2168	·77	·6469 2945 0442
·53	·4853 8109 0606	·78	·6527 0670 5962
·54	·4929 8796 0675	·79	·6584 0903 5955
·55	·5005 2021 1190	·80	·6640 3677 0267
·56	·5079 7743 2898	·81	·6695 9025 9620
·57	·5153 5927 8008	·82	·6750 6987 4836
·58	·5226 6542 9685	·83	·6804 7600 6113
·59	·5298 9560 7528	·84	·6858 0906 2230
·60	·5370 4956 6998	·85	·6910 6946 9833
·61	·5441 2709 8854	·86	·6962 5767 2687
·62	·5511 2802 8538	·87	·7013 7413 0938
·63	·5580 5[illegible]21 5559	·88	·7064 1932 0[illegible]97
·64	·5648 9955 2846	·89	·7113 9373 1818
·65	·5716 6996 6085	·90	·7162 9787 0200
·66	·5783 6341 3044	·91	·7211 3225 4077
·67	·5849 7988 2881	·92	·7258 9741 4849
·68	·5915 1939 5433	·93	·7305 9389 6096
·69	·5979 8200 0499	·94	·7352 2225 2916
·70	·6043 6777 7117	·95	·7397 8305 1273
·71	·6106 7683 2817	·96	·7442 7685 7362
·72	·6169 0930 2877	·97	·7487 0428 6969
·73	·6230 6534 9572	·98	·7530 6590 4870
·74	·6291 4516 1414	·99	·7573 6232 4216
·75	·6351 4895 2388	1·00	·7615 9415 5955

To use $\sin\theta = \tan x$, when x is given, this Table may aid.

Table of $\phi(x) = x + 3^{-2}x^3 + 5^{-2}x^5 + 7^{-2}x^7 + \&c.$

Corrected by Professor Adams.

For Spence's Integral. December, 1884.

x	$\phi(x)$	x	$\phi(x)$
·01	·0100 0011 1115	·26	·2620 0212 3757
·02	·0200 0088 9017	·27	·2722 4662 9385
·03	·0300 0300 0972	·28	·2825 1084 4103
·04	·0400 0711 5210	·29	·2927 9564 3700
·05	·0500 1390 1405	·30	·3031 0192 1918
·06	·0600 2403 1161	·31	·3134 3059 1506
·07	·0700 3817 8508	·32	·3237 8258 5325
·08	·0800 5702 0391	·33	·3341 5885 7513
·09	·0900 8123 7177	·34	·3445 6038 4717
·10	·1001 1151 3164	·35	·3549 8816 7393
·11	·1101 4853 7099	·36	·3654 4323 1182
·12	·1201 9300 2705	·37	·3759 2662 8369
·13	·1302 4560 9221	·38	·3864 3943 9428
·14	·1403 0706 1956	·39	·3969 8277 4671
·15	·1503 7807 2851	·40	·4075 5777 5992
·16	·1604 5936 1061	·41	·4181 6561 8730
·17	·1705 5165 3552	·42	·4288 0751 2658
·18	·1806 5568 5718	·43	·4394 8470 9107
·19	·1907 7220 2[illegible]13	·44	·4501 9849 3231
·20	·2009 0195 6609	·45	·4609 5019 6440
·21	·2110 457[illegible] 4078	·46	·4717 4119 4011
·22	·2212 0425 0096	·47	·4825 7290 8887
·23	·2313 7835 2178	·48	·4934 4681 4688
·24	·2415 6882 0437	·49	·5043 6443 8965
·25	·2517 7646 8384	·50	·5153 2736 6694

Table of $\psi(x) = 2^{-2}x^2 + 4^{-2}x^4 + 6^{-2}x^6 + \&c.$

x	$\psi(x)$	x	$\psi(x)$
·01	·0000 2500 0625	·26	·0171 9453 2100
·02	·0001 0001 0002	·27	·0185 6837 5292
·03	·0002 2505 0645	·28	·0199 9816 7489
·04	·0004 0016 0114	·29	·0214 8439 9820
·05	0006 2539 1060	·30	·0230 2758 8158
·06	·0009 0081 1299	·31	·0246 2827 3997
·07	·0012 2650 3902	·32	·0262 8702 5399
·08	·0016 0256 7208	·33	·0280 0443 8000
·09	·0020 2911 5455	·34	·0297 8113 6092
·10	·0025 0627 7935	·35	·0316 1777 3767
·11	·0030 3420 0173	·36	·0335 1503 6161
·12	·0036 1204 3622	·37	·0354 7364 0754
·13	·0042 4298 5992	·38	·0374 9433 8838
·14	·0049 2422 1489	·39	·0395 7791 6926
·15	·0056 5696 1094	·40	·0417 2519 8479
·16	·0064 4143 2857	·41	·0439 3704 5593
·17	·0072 7788 2218	·42	·0462 1436 0875
·18	·0081 6657 2368	·43	·0485 5808 9452
·19	·0091 0778 4621	·44	·0509 6922 1130
·20	·0101 0181 8831	·45	·0534 4879 2714
·21	·0111 4899 3836	·46	·0559 9789 0508
·22	·0122 4964 7934	·47	·0586 1765 3019
·23	·0134 0413 9395	·48	·06[illegible] 0927 3866
·24	·0146 1284 7014	·49	·0640 [illegible] 4932
·25	·0158 7617 0698	·50	·0669 1315 9771

16. In the *Cambridge and Dublin Mathematical Journal* I obtained besides two *independent integrations*,

1st of $$V = \lambda\,(x,\ \alpha) - \lambda\,(y,\ \beta),$$

when $$x \cos \alpha + y \cos \beta = 1, \text{ and } \alpha + \beta = \tfrac{1}{2}\pi.$$

Here of $$x = \cos \alpha, \quad y = \cos \beta.$$

2nd of $$V = \lambda\,(x,\ \alpha) - 2\lambda\,(y,\ \beta),$$

when $$1 - x = 2y \cos \beta, \text{ with } \alpha + 2\beta = \pi,$$

which again allows $$x = \cos \alpha \text{ with } y = \cos \beta,$$

but in integrating back the constant of integration is too complex to be of any service in our only troublesome case. Therefore it is not worth while to add the investigation here.

17. *Commutative System.* With

$$\tan \omega = \frac{x \sin \omega}{1 - x \cos \omega},$$

we had $$X = \frac{\sin \alpha}{\sin(\omega + \alpha)} \text{ and } x = \frac{\sin \omega}{\sin(\omega + \alpha)};$$

so that to exchange ω with α exchanges x with X.

Now we had

$$d\lambda\,(x,\ \alpha) = lX\,dlx + \omega d\alpha,$$

when x and α vary together.

Hence $$d\lambda\,(X,\ \omega) = lx\,dlX + \alpha d\omega.$$

Take the sum and integrate; then

$$\lambda\,(x,\ \alpha) + \lambda\,(X,\ \omega) = lX\ lx + \omega\alpha + C,$$

in which C is numerical. When

$$x = 0,\ \omega = 0,\ X = 1,\ lX = 0,\ \therefore\ \lambda\,(1,\ 0) = C, \text{ or } C = -2\,L2.$$

Thus $$\lambda\,(x,\ \alpha) + \lambda\,(X,\ \omega) = lX\ lx + \omega\alpha - 2\,L2.$$

But our difficulty as to convergence is, when x is a little less or greater than 1; but X is as liable to that inconvenience as x. To pass to it is no remedy.

18. *The Obverse System.* Take β as arc in Y^2, and θ related to $y\beta$ as ω to $x\alpha$. But assume

$$\alpha + \beta = \pi - \omega, \text{ and } \omega = \psi.$$

Then x, α and y, β are symmetrical.

Since now $\omega+\alpha=\pi-\beta$, and $\omega+\beta=\pi-\alpha$,

$$x=\frac{\sin\omega}{\sin\beta}.\quad X=\frac{\sin\alpha}{\sin\beta},\quad y=\frac{\sin\omega}{\sin\alpha},\quad Y=\frac{\sin\beta}{\sin\alpha};$$

whence $y=\frac{x}{X}$, $Y=X^{-1}$, $ly=lx-lX$, $lY=-lX$.

Thus

$$d\lambda(x,\ \alpha)=lX\,dlx+\omega d\alpha;\quad d\lambda(y,\ \beta)=lY\,dly+\omega d\beta$$
$$=-lX\,d(lx-lX)+\omega d\beta.$$

Sum of the two last equations is

$$lX\,dlX+\omega d(\alpha+\beta),$$

or $\lambda(x,\ \alpha)+\lambda(y,\ \beta)=\frac{1}{2}(lX)^2+\int\omega\,d(\pi-\omega)=\frac{1}{2}(lX)^2-\frac{1}{2}\omega^2+C.$

But $C=0$ since every term vanishes with ω.

Still $y=\frac{x}{X}$ is not more favourable to convergence than x.

Case of x near to 1.

19. If x is less than $\frac{1}{4}$, perhaps no series for $\lambda(x)$ can compete with that of Art. 2 which I call Cardinal: nay, even when x does not exceed $\frac{1}{3}$. If x is >3, the Reciprocal equation enables us to pass to $x'<\frac{1}{3}$. Between $x=\frac{1}{2}$ and $x=2$, is our main difficulty. When x is a little less than 1, every series hitherto disappoints us.

Dissatisfied with these, I proposed in two ways to use the assumption $y=\frac{1-x}{1+x}$, and think my methods, with slight modification, worthy of reprint.

If $\qquad x=\frac{1}{3},\quad y=\frac{1}{2},\quad y^2=\frac{1}{4},$

put $\qquad \cos\alpha=\frac{1-m^2}{1+m^2};$

then $\qquad m=\tan\frac{1}{2}\alpha.$

If $\qquad \alpha$ is $<\frac{1}{2}\pi$, m is <1;

but if $\qquad \alpha$ is $>\frac{1}{2}\pi$, $-\cos\alpha$ being $=\frac{m^2-1}{m^2+1}$;

then m is >1, and may be enormous. This uncertain value of m must be kept in mind. Now

$$(1+m^2)X^2=(1+m^2)\{1-2x\cos\alpha+x^2\},$$

or eliminating $\cos\alpha$,

$$(1+m^2)X^2 = (1+m^2) - 2x(1-m^2) + (1+m^2)x^2$$
$$= (1-x)^2 + m^2(1+x)^2.$$

Assume

$$y = \frac{1-x}{1+x}, \quad \therefore\ (1+m^2)X^2 = (1+x)^2\{y^2+m^2\},$$

or $$2\log X = 2\log(1+x) + \log(y^2+m^2) - \log(1+m^2),$$

$$dlX = dl(1+x) + \tfrac{1}{2}dl(y^2+m^2);$$

whence $$\Lambda(x,\ \alpha) = K(x) + C + \frac{1}{2}\int_0 l(x)\frac{2y\,dy}{y^2+m^2}.$$

To find C, make $y=0, \quad x=1,$

$$\therefore\quad C = \Lambda(1,\ \alpha) - K(1) = \{\tfrac{1}{4}(\pi-\alpha)^2 - L2\} + L2,$$

or $$C = \tfrac{1}{4}(\pi-\alpha)^2.$$

Put $$Q = \int_0 lx\,\frac{y\,dy}{y^2+m^2};$$

or $$\Lambda(x,\ \alpha) = \tfrac{1}{4}(\pi-\alpha)^2 + K(x) + Q.$$

Our remaining Problem is to evaluate Q. The relative magnitude of y^2 and m^2 being uncertain, to develop $(y^2+m^2)^{-1}$ is inexpedient. But

$$lx = l\,\frac{1-y}{1+y} = -2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \&c.\right);$$

$$\therefore\quad Q = -2\left\{\int_0\frac{y^2dy}{y^2+m^2} + \frac{1}{3}\int_0\frac{y^4dy}{y^2+m^2} + \frac{1}{5}\int_0\frac{y^6dy}{y^2+m^2} + \&c.\right\},$$

add $$R = -2\left\{\int_0\frac{m^2dy}{y^2+m^2} - \frac{1}{3}\int_0\frac{m^4dy}{y^2+m^2} + \frac{1}{5}\int_0\frac{m^6dy}{y^2+m^2} - \&c.\right\},$$

at least when m is <1; for in that case the series

$$m^2 - \tfrac{1}{3}m^4 + \tfrac{1}{5}m^6 + \tfrac{1}{7}m^8 + \&c. \text{ converges, and } = m\tan^{-1}m,$$

so that R is a finite quantity. We must afterwards inquire as to the case of $m>1$.

When m is <1,

$$R = -2\int_0\frac{m\tan^{-1}m\,dy}{y^2+m^2} = -2\tan^{-1}m\tan^{-1}\frac{y}{m} = -\alpha\tan^{-1}\frac{y}{m};$$

$$\therefore\quad -\frac{1}{2}(Q+R)$$

$$= \int_0 dy - \frac{1}{3}\int_0\frac{m^4-y^4}{m^2+y^2}dy + \frac{1}{5}\int_0\frac{m^6+y^6}{m^2+y^2}dy - \frac{1}{7}\int_0\frac{m^8-y^8}{m^2+y^2}dy + \&c.$$

Every numerator is now divisible by the denominator, and becomes integrable.

We may write

$$-\tfrac{1}{2}(Q+R) = y - \tfrac{1}{3}Y_3 + \tfrac{1}{5}Y_5 - \tfrac{1}{7}Y_7 + \&c.$$

where

$$Y_3 = \int_0 (m^2 - y^2)\,dy;\quad Y_5 = \int_0 (m^4 - m^2y^2 + y^4)\,dy;$$

$$Y_7 = \int_0 (m^6 - m^4y^2 + m^2y^4 - y^6)\,dy, \ \&c.$$

Arrange these terms so as to place all the partial coefficients of y^{2n-1} in the same vertical line, after effecting the integrations.

Then $-\tfrac{1}{2}(Q+R)$

$$\begin{aligned} &= y - \tfrac{1}{3}(m^2y - \tfrac{1}{3}y^3) \\ &\quad + \tfrac{1}{5}(m^4y - \tfrac{1}{3}m^2y^3 + \tfrac{1}{5}y^5) \\ &\quad - \tfrac{1}{7}(m^6y - \tfrac{1}{3}m^4y^3 + \tfrac{1}{5}m^2y^5 - \tfrac{1}{7}y^7) \\ &\quad + \&c. \qquad \&c. \end{aligned}$$

Call the total

$$M_1y + \tfrac{1}{3}M_3y^3 + \tfrac{1}{5}M_5y^5 + \&c.;$$

$$\therefore\quad Q = a\tan^{-1}\frac{y}{m} - 2(M_1y + \tfrac{1}{3}M_3y^3 + \&c.).$$

We deduce

$$M_1 = 1 - \tfrac{1}{3}m^2 + \tfrac{1}{5}m^4 - \tfrac{1}{7}m^6 + \&c. = \frac{\tan^{-1}m}{m};$$

if m is < 1

$$M_3 = \tfrac{1}{3} - \tfrac{1}{5}m^2 + \tfrac{1}{7}m^4 - \tfrac{1}{9}m^6 + \&c. \ \text{or} = \frac{1 - M_1}{m^2}.$$

$$M_5 = \tfrac{1}{5} - \tfrac{1}{7}m^2 + \tfrac{1}{9}m^4 - \tfrac{1}{11}m^6 + \&c. \ \text{or} = \frac{\tfrac{1}{3} - M_3}{m^2}.$$

The law is manifest. We have

$$M_7 = \frac{\tfrac{1}{5} - M_5}{m^2};\quad M_9 = \frac{\tfrac{1}{7} - M_7}{m^2};\ \text{and so on.}$$

The less is m, the slower is the convergence. The worst case is that of $m =$ zero, which gives

$$Q = -2\int_0 \{1 + \tfrac{1}{3}y^2 + \tfrac{1}{5}y^4 + \tfrac{1}{7}y^6 + \&c.\}\,dy$$

$$= -2\{y + 3^{-2}y^3 + 5^{-2}y^5 + 7^{-2}y^7 + \&c.\},$$

such is our slowest convergence. The larger is the divisor m, the more rapidly do $M_1 M_3 M_5 M_7 \dots$ diminish, and we attain

$$\left(\text{since } R = -\alpha \tan^{-1} \frac{y}{m}\right),$$

$$\Lambda(x, \alpha) = \tfrac{1}{4}(\pi - \alpha)^2 + K(x) + \alpha \tan^{-1} \frac{y}{m}$$
$$- 2\{M_1 y + \tfrac{1}{3} M_3 y^3 + \tfrac{1}{5} M_5 y^5 + \&c.\} \dots (A),$$

where $m = \tan(\tfrac{1}{2}\alpha)$, and $y = \frac{1-x}{1+x}$;

$M_1 M_3 M_5 \dots$ are deduced by Binomial operations.

This equation has been established beyond question, so long as m is < 1; thus it is an *identity* continuous, between $\alpha = 0$, and $\alpha = \frac{1}{2}\pi$. $M_1 = \frac{\tan^{-1} m}{m}$ is always intelligible, and always less than 1. Continuous identity between two separate finite values, while there is no discontinuity of form, establishes identity beyond, so long as the series remain convergent. But here, to make m greater and greater, does but improve convergence. We may conclude that the last series (A) is really general, and that when m exceeds 1, we may confidently *deduce* the series $M_1 M_2 M_3 M_4 \dots$ each from the preceding, without caring for the series which in *this case* diverge.

The lessening of the series M gives our new process a vast advantage over series (1).

Besides, we have here only the *odd* powers of y, while in series (1) *all* the powers of x have to be computed. Moreover the case which before was hardest (when x was a little smaller than 1), is now peculiarly favourable.

20. Nevertheless when m is > 1, put

$$\alpha = \pi - 2\beta, \text{ and } p = \tan\beta.$$

Since $m = \tan \tfrac{1}{2}\alpha = \tan(\tfrac{1}{2}\pi - \beta) = \cot\beta$, $p = m^{-1}$,

and
$$\frac{m^2 + y^2}{m^2 + 1} = \frac{1 + p^2 y^2}{1 + p^2}.$$

This is favourable to development in powers of y, p and y being now *both* small.

Hence it becomes simpler to integrate λ than Λ.

$$dlx = dl\,\frac{1-y}{1+y} = \frac{-2dy}{1-y^2}.$$

We had

$$2 \log X = 2 \log (1+x) + \log \frac{y^2+m^2}{1+m^2}$$

$$= 2 \log (1+x) + \log \frac{p^2y^2+1}{p^2+1};$$

$$\therefore \quad \lambda(x, \alpha) = \int_0 lX\, dlx$$

$$= L(1+x) + \int_0 \tfrac{1}{2} l \frac{1+p^2}{1+p^2y^2} \frac{2dy}{1-y^2} + C.$$

When $\qquad y = 0, \quad x = 1,$

$$\therefore \quad C = \lambda(1, \alpha) - L2 = -\tfrac{1}{4}(\pi - \alpha)^2 = -\beta^2.$$

Differentiate under $\int_0$ for β and p variable;

$$\therefore \quad \frac{d\lambda}{dp} = \int_0 \frac{d}{dp}(l\,.\,\overline{1+p^2} - l\,.\,\overline{1+p^2y^2}) \frac{dy}{1-y^2} - \frac{d}{dp}\beta^2.$$

Now

$$\frac{d}{dp}\{l\,.\,\overline{1+p^2} - l\,.\,\overline{1+p^2y^2}\} = \frac{2p}{1+p^2} - \frac{2py^2}{1+p^2y^2} = \frac{2p(1-y^2)}{(1+p^2)(1+p^2y^2)},$$

so that $$\frac{d\lambda}{dp} = \int_0 \frac{2pdy}{(1+p^2)(1+p^2y^2)} - 2\beta\frac{d\beta}{dp}$$

$$= \frac{2\tan^{-1}(py)}{1+p^2} - 2\beta\frac{d\beta}{dp}.$$

When p vanishes, so does β, and $\lambda(x, \alpha)$ becomes $L(1+x)$. Integrating for p variable, you get

$$\lambda(x, \alpha) = L(1+x) - \beta^2 + 2\int_0 \tan^{-1}(py)\frac{dp}{1+p^2}.$$

But $$\frac{dp}{1+p^2} = d\beta,$$

and $$\tan^{-1}(py) = py - \tfrac{1}{3}p^3y^3 + \tfrac{1}{5}p^5y^5 - \&c.,$$

so that development in powers of y becomes very simple. Let

$$B_n \text{ stand for } \int_0 p^n d\beta.$$

Then $\lambda(x, \pi - 2\beta)$

$$= L(1+x) - \beta^2 + 2\{B_1y - \tfrac{1}{3}B_3y^3 + \tfrac{1}{5}B_5y^5 - \&c.\} \ldots\ldots(B).$$

The series B_n is easily computed in succession. First

$$B_1 = \int_0 \tan\beta\, d\beta = \log\sec\beta.$$

Next, generally $B_{n-1} + B_{n+1}$

$$= \int_0 \tan^{n-1}\beta\,(1+\tan^2\beta)\,d\beta = \int_0 \tan^{n-1}\beta\; d\tan\beta = \frac{\tan^n\beta}{n}.$$

Thus each B_{n+1} in succession is known from B_{n-1} preceding.

Likewise

$$B_n = \frac{p^{n+1}}{n+1} - B_{n+2} = \frac{p^{n+1}}{n+1} - \frac{p^{n+3}}{n+3} + B_{n+4}, \text{ and so on.}$$

Since p is < 1, B_n vanishes when $n = \infty$; so that

$$B_n = \frac{p^{n+1}}{n+1} - \frac{p^{n+3}}{n+3} + \frac{p^{n+5}}{n+5} - \&c.,$$

is a converging series, and B_n is always *less* than $\dfrac{p^{n+1}}{n+1}$, but *greater* than

$$\frac{p^{n+1}}{n+1} - \frac{p^{n+3}}{n+3}.$$

Equations (A) and (B) both convince; yet are hard to identify.

The convergence of (A) is at its worst when α is very small, yet if x be very near to 1, the smallness of y makes up. It is hard to get any advantage from the smallness of α, that shall compare to the advantage gained from y.

21. New Problem. *To find λ with advantage, when α is near to $90°$.*

We know that if

$$\alpha = \tfrac{1}{2}\pi, \quad \lambda(x, \alpha) \text{ becomes } \tfrac{1}{4}L(1+x^2).$$

Let
$$P = \tfrac{1}{4}L(1+x^2) - \lambda(x, \alpha);$$

$$\therefore\ P = \frac{1}{2}\int_0 \log\left(\frac{1+x^2}{1+x^2-2x\cos\alpha}\right) dlx$$

$$= -\frac{1}{2}\int_0 \log\left(1 - \frac{2x\cos\alpha}{1+x^2}\right) dlx.$$

Let
$$x = \tan\tfrac{1}{2}\theta,$$

$$\therefore\ \frac{2x}{1+x^2} = \sin\theta, \quad dlx = \frac{d\theta}{\sin\theta};$$

$$P = -\frac{1}{2}\int_0 \log(1 - \sin\theta\cos\alpha)\frac{d\theta}{\sin\theta}.$$

Now if $\cos\alpha$ is extremely small, possibly nothing is better than to expand in series of its powers; which yields directly

$$2P = \theta\cos\alpha + \Sigma\frac{1}{n}(\cos\alpha)^n\int_0 (\sin\theta)^{n-1}\,d\theta\,;$$

where $$n = 2,\ 3,\ 4\ \dots\,.$$

It is easy to work out $\int_0 (\sin\theta)^{n-1}\,d\theta$ into linear sines or cosines, and retain the series in its existing order. If

$$\Theta_n = \int_0 (\sin\theta)^n\,d\theta,$$

evidently

$$\Theta_n = \int \sec\theta\,(\sin\theta)^n\,d\sin\theta = \sec\theta\,\frac{(\sin\theta)^{n+1}}{n+1} - \int\frac{(\sin\theta)^{n+1}}{n+1}\,d\sec\theta.$$

The last being positive, while θ is $< 90^\circ$, shews that Θ_n is within this limit *less* than

$$\sec\theta\,\frac{(\sin\theta)^{n+1}}{n+1},$$

that is. than $$\frac{(\sin\theta)^n}{n+1}\tan\theta.$$

Besides, $$\frac{(\cos\alpha)^n}{n}\ \text{is small.}$$

But otherwise; we may differentiate P with a variable under $\int$, and θ constant.

Then

$$\frac{dP}{d\alpha} = -\frac{1}{2}\int_0 \frac{\sin\theta\sin\alpha}{1-\sin\theta\cos\alpha}\,\frac{d\theta}{\sin\theta} = -\frac{1}{2}\int_0 \frac{\sin\alpha\,d\theta}{1-\sin\theta\cos\alpha}.$$

Let $$\theta = 90^\circ - \psi,\ \text{and}\ \ \alpha = 90^\circ - 2\beta.$$

By hypothesis, β is small;

$$d\alpha = -2d\beta;\quad d\theta = -d\psi;$$

$$\therefore\ \frac{dP}{d\beta} = \int\frac{-\cos 2\beta\,d\psi}{1-\cos\psi\sin 2\beta}$$

$$= -\int[1 + 2p\cos\psi + 2p^2\cos 2\psi + 2p^3\cos 3\psi + \&c.]\,d\psi$$

by known Trigonometry, if $p = \tan\beta$, and p is < 1. Integrate for

ψ variable:

$$\frac{dP}{d\beta} = C - [\psi + 2p \sin \psi + 2p^2 \tfrac{1}{2} \sin 2\psi + 2p^2 \tfrac{1}{3} \sin 3\psi + \&c.].$$

To find C, make

$$\psi = \tfrac{1}{2}\pi, \quad \theta = 0, \quad P = 0,$$

$$\therefore \quad C = \tfrac{1}{2}\pi + 2(p - \tfrac{1}{3}p^3 + \tfrac{1}{5}p^5 - \&c.),$$

or

$$C = \tfrac{1}{2}\pi + 2 \tan^{-1} p = \tfrac{1}{2}\pi + 2\beta.$$

Integrate next for β variable;

$$\therefore \; P = \int (\tfrac{1}{2}\pi + 2\beta)\, d\beta - \psi\beta - 2 \left(\sin \psi \int_0 p d\beta + \tfrac{1}{2} \sin 2\psi \int_0 p^2 d\beta \right.$$

$$\left. + \tfrac{1}{3} \sin 3\psi \int_0 p^3 d\beta + \tfrac{1}{4} \sin 4\psi \int_0 p^4 d\beta + \&c.\right).$$

P vanishes with $\cos\alpha$, that is, when $\alpha = 90^\circ$. Then also $\beta = 0$; hence

$$\int (\tfrac{1}{2}\pi + 2\beta)\, d\beta \text{ must be interpreted } \tfrac{1}{2}\pi\beta + \beta^2,$$

and

$$\tfrac{1}{2}\pi\beta - \psi\beta = \theta\beta.$$

Finally

$$\lambda(x, \alpha) = \tfrac{1}{4} L(1 + x^2) - \beta^2 - \theta\beta$$

$$+ 2\{B_1 \sin \psi + \tfrac{1}{2} B_2 \sin 2\psi + \tfrac{1}{3} B_3 \sin 3\psi + \&c.\},$$

if B_n means

$$\int_0 (\tan \beta)^n\, d\beta; \quad \psi = 90^\circ - \theta; \quad \tan \tfrac{1}{2}\theta = x; \quad \alpha = 90^\circ - 2\beta.$$

Since $(\cos\alpha)^n$ or $(\sin 2\beta)^n$ is increasingly greater than

$$\int_0 (\tan \beta)^n\, d\beta,$$

we may seem to have improved the convergence by this last step. Yet

$$\int_0 (\sin \theta)^n\, d\theta$$

may chance to be less than

$$\int_0 (\tan \beta)^n\, d\beta;$$

as an equivalent form, each may have its service.

General equivalent of λ in linear cosines.

22. We may also adapt the method of Art. 21 to attain a series in linear cosines applicable for all values of x, instead of the series in

$$y\, y^3\, y^5 \dots \&c.$$

It gives a new equivalent of λ. Let

$$Q = -\frac{1}{2}\int_0 \log \frac{X^2}{(1+x)^2}\, dlx;$$

or

$$Q = L(1+x) - \lambda(x, \alpha).$$

Put $\qquad 4\beta = \pi - \alpha, \quad p = \tan\beta, \quad x = (\tan \tfrac{1}{2}\omega)^2,$

or

$$\frac{2\sqrt{x}}{1+x} = \sin\omega\,;\ dlx = \frac{2d\omega}{\sin\omega},$$

$$\sqrt{x} = \tan \tfrac{1}{2}\omega.$$

Hence

$$Q = -\frac{1}{2}\int_0 \log\left\{1 - \frac{4x}{(1+x)^2}\cos^2\frac{\alpha}{2}\right\} dlx$$

$$= -\int_0 \log\{1 - \sin^2\omega\,(\sin 2\beta)^2\}\,\frac{d\omega}{\sin\omega},$$

or by known Trigonometry,

$$\tfrac{1}{4}Q = \int_0 \{\log\sec\beta - \tfrac{1}{2}p^2\cos 2\omega + \tfrac{1}{4}p^4\cos 4\omega - \tfrac{1}{6}p^6\cos 6\omega + \&c.\}\,\frac{d\omega}{\sin\omega}.$$

This guides to assume

$$\tfrac{1}{4}Q = A_0 - 2A_1\cos\omega + \tfrac{2}{3}A_3\cos 3\omega - \tfrac{2}{5}A_5\cos 5\omega + \&c.,$$

whence by differentiation,

$$\frac{1}{4}\frac{dQ}{d\omega} = 2A_1\sin\omega - 2A_3\sin 3\omega + 2A_5\sin 5\omega - \&c.$$

But

$$\frac{1}{4}\frac{dQ}{d\omega}\sin\omega = \log\sec\beta - \tfrac{1}{2}p^2\cos 2\omega + \tfrac{1}{4}p^4\cos 4\omega - \&c.,$$

which is therefore identical with

$$A_1(2\sin\omega\sin\omega) - A_3(2\sin 3\omega\sin\omega) + A_5(2\sin 5\omega\sin\omega) - \&c.,$$

or $\quad A_1(1-\cos 2\omega) - A_3(\cos 2\omega - \cos 4\omega)$

$$+ A_5(\cos 4\omega - \cos 6\omega) - \&c.,$$

or $\quad A_1 - (A_1 + A_3)\cos 2\omega + (A_3 + A_5)\cos 4\omega$

$$-(A_5 + A_7)\cos 6\omega + \&c.$$

This gives us equivalents

$$A_1 = \log\sec\beta;\quad A_1 + A_3 = \frac{p^2}{2};$$

$$A_3 + A_5 = \frac{p^4}{4};\quad A_5 + A_7 = \frac{p^6}{6},\quad \&c.$$

and, as $$p = \tan\beta,\ \text{ and }\ \log\sec\beta = \int_0 p\,d\beta,$$

these equations establish universally

$$A_n = \int_0 p^n d\beta.$$

But we have not yet found the value of A_0 in $\frac{1}{4}Q$. Q must vanish when $x = 0$, in which case $\omega = 0$, which makes all the cosines in that series to become $+1$;

$$\therefore\quad 0 = A_0 - 2A_1 + \tfrac{2}{3}A_3 - \tfrac{2}{5}A_5 + \&c.,$$

or $$\tfrac{1}{2}A_0 = A_1 - \tfrac{1}{3}A_3 + \tfrac{1}{5}A_5 - \&c. = \int_0 \{p - \tfrac{1}{3}p^3 + \tfrac{1}{5}p^5 - \&c.\}\,d\beta$$

$$= \int_0 \tan^{-1} p\ d\beta = \int_0 \beta d\beta;\quad \therefore\ A_0 = \beta^2.$$

Finally $\lambda(x, \alpha)$

$$= L(1+x) - 4\beta^2 + 8\{A_1 \cos\omega - \tfrac{1}{3}A_3 \cos 3\omega + \tfrac{1}{5}A_5 \cos 5\omega - \&c.\},$$

where $$\beta = \tfrac{1}{4}(\pi - \alpha),\quad A_n = \int_0 (\tan\beta)^n\,d\beta,\quad x = (\tan\tfrac{1}{2}\omega)^2.$$

The Symmetrical system.

23. As in Spence's $L(x)$, so here Fox Talbot's principle of Symmetry yields an integration,—of whatever worth.

Let x and y be the two roots of u in

$$u^2 - 2u\cos\alpha + 1 = (m - u)\,v,$$

or $$u^2 - (2\cos\alpha - v)\,u + (1 - mv) = 0,$$

in which α, m are constants, but v varies with the two roots x, y, of which it is a symmetrical function.

By the property of quadratic equations

$$x + y = 2\cos\alpha - v,\ \text{ and }\ xy = 1 - mv,$$

$$X^2 = (m - x)\,v,\quad Y^2 = (m - y)\,v.$$

We suppose α and m positive,

$$M^2 = 1 - 2m\cos\alpha + m^2.$$

Then we easily obtain

$$2\Lambda x + 2\Lambda y - C = L(xy) + L\frac{x}{m} + L\frac{y}{m},$$

by differentiating $2\Lambda x + 2\Lambda y$ and integrating back after certain changes.

To find C, make $y = 0$, and $x = p$,

$$\therefore \quad v = m^{-1},$$

and since

$$x + y = 2\cos\alpha - v, \quad p = 2\cos\alpha - m^{-1}, \quad 2\Lambda p - C = 2L0 + L\frac{p}{m}.$$

Then observing that $2L(0) = -4L(2)$, you find

$$2\Lambda x + 2\Lambda y - 2\Lambda p = L\frac{x}{m} + L\frac{y}{m} - L\frac{p}{m} + 4L(2),$$

where $p = 2\cos\alpha - m^{-1}$.

We easily find also

$$\frac{dx}{X^2} + \frac{dy}{Y^2} = 0, \quad d\omega + d\theta = 0, \quad \omega + \theta = \text{constant}.$$

Though the formulae are elegant, they do not aid us.

If we eliminate m,

$$2\Lambda x + 2\Lambda y - 2\Lambda p$$
$$= L(x\,.\,\overline{2\cos\alpha - p}) + L(y\,.\,\overline{2\cos\alpha - p}) - L(p\,.\,\overline{p - 2\cos\alpha}) + 4L(2).$$

24. On the complete integral of Art. 5. Write

$$\phi(\theta) = \int_0 \sin^{-1}(c\sin\theta)\,d\theta.$$

The c here represents our original x, which we suppose to have been so reduced as to be less than 1 *at its upper limit;* there represented by c.

Problem. *To find $\phi(\theta)$ when $\theta = \frac{1}{2}\pi$.*

We had universally

$$\phi(\theta) = \lambda(c, \alpha) + \tfrac{1}{2}\omega^2 - L(1 - c),$$

where $\theta = \omega + \alpha, \quad \sin\omega = c\sin\theta.$

When $$\theta = \tfrac{1}{2}\pi, \quad \sin\omega = c.$$

Also $$\omega = \theta - \alpha = \tfrac{1}{2}\pi - \alpha,$$

so that $$c = \cos\alpha.$$

By Art. 8 above, we know that when $x = c\cos\alpha$

$$\lambda(x, \alpha) = \tfrac{1}{4}L(\sin^2\alpha) - \tfrac{1}{2}(\tfrac{1}{2}\pi - \alpha)^2,$$

and this x is our present c.

Therefore now $$\lambda(c, \alpha) = \tfrac{1}{4}L(\sin^2\alpha) - \tfrac{1}{2}\omega^2,$$

and $$\sin^2\alpha = 1 - c^2.$$

Thus $$\phi(\tfrac{1}{2}\pi) = \tfrac{1}{4}L(1 - c^2) - L(1 - c).$$

But in Spence's integral,

$$\tfrac{1}{4}L(1 - c^2) = \tfrac{1}{2}L(1 + c) + \tfrac{1}{2}L(1 - c);$$

therefore also $$\phi(\tfrac{1}{2}\pi)$$

or $$\int_0^{\frac{1}{2}\pi} \sin^{-1}(c\sin\theta)\,d\theta = \tfrac{1}{2}L(1 + c) - \tfrac{1}{2}L(1 - c).$$

Some Integrating machine seems hopeful for $\phi(\theta)$, which will supersede calculation of Λ and λ.

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