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SIMILAR FIGURES, SPACE, AND SOLIDS

*BEING A NEW GEOMETRY OF THE SUBJECT-MATTER
OF EUCLID, BOOKS V., VI., AND XI.*

BY

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PREFACE.

THE present work is a continuation of *The New Matriculation Geometry* by the same authors, and the characteristic features of that book have been maintained in the present volume, *The New Matriculation Geometry* is a modern course of Elementary Geometry covering the subject matter of Euclid, Books I.-IV., and including additional chapters on Similar Figures. This book is a modern treatment of the subject matter of Euclid, Books V., VI., and XI., including also theorems relating to the surfaces and volumes of simple solids, and those extensions of Euclidian Geometry—Polyhedra, Solids of Revolution, Harmonic Section, Centres of Similitude, Inversion, Radical Axes, Theorems of Ceva and Menelaus, Nine Points Circle, Poles and Polars, Complete Quadrilateral, and the Tangency Problems—usually taught in schools.

When the ties which bound the schools to Euclidian Geometry were severed many advocates of the change rushed to the other extreme and confusion followed, From the experience gained, however, certain principles may be regarded as definitely established: *The New Matriculation Geometry* and *Similar Figures, Space, and Solids* have been written in accordance with these principles.

Special attention has been paid to accurate definitions and to the logical sequence of the leading propositions, the proofs of which are clearly stated with a view to logical accuracy and training in deductive reasoning. Thus is retained one of the recognised and distinctive features of Euclid's Elements, leading to the inculcation of the true spirit of Geometry.

Exceptional care has been devoted to the arrangement of the propositions and their proofs and to the use of various types. Each proposition, with only a few exceptions, is presented complete on a single page or on two facing pages, whilst explanatory notes are set in a type distinct from that of the text to which they refer. These devices appeal to the eye, assist the memory, and play an important rôle in the acquisition by the student of a thorough and efficient training in the subject.

A simple system of reference numbers to propositions has been adopted which avoids large numbers by the classification of related propositions into groups, is easily remembered, and retains the merits of the Euclidian system modified to meet the new treatment.

In accordance with the accepted modern practice, the theory of Proportion and Similar Figures has been based on the arithmetical definition of Ratio as the Euclidian definition is by no means easy to comprehend and apply, and is scarcely suitable for use in schools. This treatment, of course, necessitates the assumption that any two magnitudes of the same kind may be regarded as commensurable.

The difficulty experienced in teaching Euclid's Book XI. in schools is largely due to the fact that Euclid's treatment is cursory, and omits certain important concepts and theorems. The student is hurried on too rapidly, without becoming sufficiently experienced in the properties of solid space. For example, the idea of a line parallel to a plane is ignored by Euclid, though there are a large number of important propositions which depend upon this relation. Criticism may also be advanced of Euclid's order of proof. Thus the properties of parallel lines and planes in space do not depend upon the properties of lines perpendicular to planes, and are easily deduced without them. It is hoped that the fuller treatment and different order of proof given in this work will render the subject much clearer to the average student.

In the preparation of this book use has been made of the well known *Intermediate Geometry* by W. P. Workman, M.A., B.Sc., and A. G. Cracknell, M.A., B.Sc. In the present work, however, the matter has been entirely re-written, proofs, exercises, riders, etc., re-arranged, new explanatory matter and many new proofs and diagrams substituted, and the various types used and the type arrangements completely changed. The book is, in fact, a new work brought up to modern teaching requirements, and embodying those distinctive features of arrangement of matter and use of type which mean so much to the learner and teacher.

A. G. C.
G. F. P.

CONTENTS.

CHAPTER	PAGE
I. RATIO AND PROPORTION	1
II. SIMILAR FIGURES	31
III. PROPORTION THEOREMS ON AREAS	51
IV. RECTANGLE THEOREMS	65
V. PROBLEMS ON RATIO AND PROPORTION AND SIMILAR FIGURES	76
VI. STRAIGHT LINES AND PLANES	87
VII. PARALLELS AND PARALLELEPIPEDS	98
VIII. LINES PERPENDICULAR TO PLANES	116
IX. DIHEDRAL, TRIHEDRAL, AND POLYHEDRAL ANGLES	131
X. POLYHEDRA	145
XI. SOLIDS OF REVOLUTION—THE SPHERE, CYLINDER, AND CONE	169
XII. SURFACE AREAS AND VOLUMES OF SIMPLE SOLIDS	177
XIII. MISCELLANEOUS PROPOSITIONS	195
LIST OF PROPOSITIONS	241
ANSWERS	258
INDEX TO DEFINITIONS AND TERMS	260

LIST OF SYMBOLS.

$\because$ because	$\parallel$ is similar to
$\therefore$ therefore	$\angle$ angle
$=$ is (or are) equal to	R a right angle
$>$ is (or are) greater than	$\perp$ perpendicular ; is (or are) perpendicular to
$<$ is (or are) less than	$\parallel$ parallel ; is (or are) parallel to
$\sim$ the difference between	Δ triangle
Σ the sum of	
$\equiv$ is congruent to	

The letter *s* may be added to a symbol to form the plural, *e.g.*, Δs may be used for triangles.

REFERENCES TO PROPOSITIONS.

D.—The propositions in Chapters I.-V. (inclusive) are designated by the group letter D : thus :—

Theorem 1D, Theorem 2D, etc.

Problem 1D, Problem 2D, etc.

These include propositions on Ratio and Proportion, and Similar Figures (covering the subject-matter of Euclid, V. and VI.).

E.—The propositions in Chapters VI.-XII. (inclusive) are designated by the group letter E : thus :—

Theorem 1E, Theorem 2E, etc.

Problem 1E, Problem 2E, etc.

These include propositions on Straight Lines and Planes, Parallels and Parallelepipeds, Lines Perpendicular to Planes, Dihedral, Trihedral, and Polyhedral Angles, Polyhedra and Solids of Revolution (covering Euclid, XI., with additional matter).

F.—The propositions in Chapter XIII. are designated by the group letter F : thus :—

Theorem 1F, Theorem 2F, etc.

Problem 1F.

This chapter contains miscellaneous propositions on Harmonic Section, Centres of Similitude, Inversion, the Radical Axis, the Theorems of Ceva and Menelaus, the Nine-Points Circle, Poles and Polars, the Complete Quadrilateral, and the Tangency Problems. A special system of reference letters is used for the Tangency Problems ; this is explained in the text.

On pp. 241-257 will be found a list of the propositions including those given in *The New Matriculation Geometry* under the reference letters A, B, and C.

In the Exercises, examples which are marked with an asterisk are somewhat difficult, and the student is advised not to attempt these on a first reading of the book.

CHAPTER I.

RATIO AND PROPORTION.

1. Introduction.—In the *Elements of Geometry*, Book V., Euclid treats of Ratio and Proportion, and in Book VI. he applies the theory of Proportion to establish certain properties of geometrical figures. The theory of Ratio and Proportion is essentially numerical and is fully treated in books on Arithmetic and Algebra. The student is, therefore, advised to read the treatment in some good text-book, say on Algebra, before proceeding further.

In this chapter the essential definitions and theorems on Ratio and Proportion are briefly dealt with, this work being followed in the next four chapters by a treatment, entirely on modern lines, of the subject-matter of Euclid, Book VI. In the last chapter of the book such other important theorems in modern Geometry as are usually associated with this part of the subject are dealt with.

2. Ratio.—If **A** and **B** are two quantities of the same kind, *e.g.* two lengths or two areas, it is known from Arithmetic that the one can be expressed as a fraction of the other. **The ratio of two quantities of the same kind is expressed by the fraction which gives their relative magnitude.**

The same idea may be explained thus:—**A** and **B** are two quantities of the same kind which, therefore, can be measured in the same units. Suppose **A** contains p such units and **B** contains q such units: then the ratio of **A** to **B** is expressed by the fraction $\frac{\text{measure of A}}{\text{measure of B}}$, *i.e.* by the fraction $\frac{p}{q}$.

The notation employed in writing the ratio of a quantity **A** to a quantity **B** of the same kind is

A : **B** (read **A** is to **B**), and sometimes $\frac{\mathbf{A}}{\mathbf{B}}$.

In the ratio **A** : **B**, **A** and **B** are called the **terms** of the ratio, **A** being called the **antecedent** and **B** the **consequent**.

Examples.—(1) The ratio 3 inches : 3 feet = $\frac{3 \text{ inches}}{3 \text{ feet}} = \frac{3 \text{ inches}}{36 \text{ inches}}$
 $= \frac{1}{12}$.

(2) The ratio 4 sq. inches : 1 sq. yard = $\frac{4 \text{ sq. inches}}{1 \text{ sq. yard}} = \frac{4 \text{ sq. inches}}{1296 \text{ sq. inches}}$
 $= \frac{1}{324}$.

(3) The ratio 9d. : 2s. = $\frac{9\text{d.}}{2\text{s.}} = \frac{9\text{d.}}{24\text{d.}} = \frac{3}{8}$.

Note the importance of the order : thus the ratio of 2s. to 9d. is $\frac{8}{3}$.

Warning.—It should also be noted that whilst the numerator and denominator of a fraction *must be numbers*, the antecedent and consequent of a ratio *may be either two numbers or two concrete quantities of the same kind*. Thus 3 yards : 10 feet or $\frac{3 \text{ yards}}{10 \text{ feet}}$ is a ratio (for both quantities are lengths) but it must be remembered that we ought not to perform any arithmetical operation with this ratio *until we have reduced it to the equivalent fraction by expressing both antecedent and consequent in terms of the same unit* (in this case, taking the foot as the unit, the fraction is $\frac{9}{10}$). On the other hand, the expression, say $\frac{3}{10}$, where 3 and 10 are simple numbers, may be regarded either as a ratio or as a fraction, and has exactly the same meaning in either case.

3. Commensurable and Incommensurable Quantities.

—In the examples of Art. 2 we have seen that to reduce a ratio to the equivalent fraction it is necessary to express both antecedent and consequent in terms of the same unit. In the three simple cases there considered, the measure of the two quantities in terms of the unit was always an integer. Other cases occur in which the measure of one or both of the quantities is not an integer but involves a fraction. (In this connection it should be noted that “fraction” means either a vulgar or a decimal fraction, for both a finite and a recurring decimal can be reduced to an equivalent vulgar fraction.)

For example, if two lines **A** and **B** were 1·6 inches and 2·6 inches long respectively their ratio would be 1·6 : 2·6 or $\frac{1·6}{2·6}$ expressed in terms of the inch as unit. It is easy to see,

however, that if $\frac{1}{10}$ inch instead of one inch were taken as the unit of length, the ratio of the two lines **A** and **B** would be expressed as $16 : 26$ or $\frac{16}{26}$. Similarly, if two lines **C** and **D** were 1.47 and 3.73 inches long respectively their ratio would be expressed as $1.47 : 3.73$ or $\frac{1.47}{3.73}$ if one inch were regarded as the unit of length, or as $147 : 373$ or $\frac{147}{373}$ if $\frac{1}{100}$ inch were regarded as the unit of length. It is, therefore, possible in all the above cases to find a third quantity which is contained an exact number of times in each of the given quantities. In such cases the two given quantities are said to be **commensurable**.

Two quantities are said to be commensurable if it is possible to find a third quantity which is contained in each of them an exact number of times. It follows that the ratio of two commensurable quantities can always be expressed as the ratio of two whole numbers.

The question next arises whether all quantities of the same kind are commensurable, and a little consideration will show that quantities are sometimes met with which are not commensurable. For example, the ratio of the length of the side of a square to the length of its diagonal is $1 : \sqrt{2}$ or $1 : 1.41421 \dots$, and there is no third quantity which is contained an exact number of times in each of these, or, in other words, the ratio of the two quantities cannot be exactly expressed as the ratio of two whole numbers.

Two quantities which are not commensurable are called incommensurable quantities. A single numerical quantity is said to be incommensurable if it is incommensurable with unity.

Incommensurable quantities introduce considerable difficulty into the logical theory of Geometry, but a consideration of this question is beyond the scope of the present book, and in the theorems which follow *we shall therefore assume that all the magnitudes involved are commensurable.*

Euclid, in his Fifth Book—a masterpiece of mathematical ingenuity—gives a treatment of Ratio and Proportion which is logically complete and yet avoids the necessity of distinguish-

ing between commensurable and incommensurable quantities. This method is, however, difficult, and is rarely employed at the present day.

It should be noted that, although incommensurable quantities introduce difficulties into geometrical theory, they do not cause any difficulty in practical work, for we can always replace incommensurable quantities by commensurable quantities which are practically equal to them—in fact we can always obtain commensurable quantities which are equal to given incommensurable quantities to *any required degree of accuracy* (short of absolute theoretical accuracy).

Thus we can find $\sqrt{2}$ as a decimal correct to any required decimal place. Correct to the 7th decimal place the result is 1.4142135. Hence (in any calculation), instead of the incommensurable quantities 1 and $\sqrt{2}$, we may use the commensurable quantities 1 and 1.4, or 1 and 1.41, or 1 and 1.414, etc., according to the degree of accuracy required in the result.

It may also be mentioned in passing that probably *no* two lengths occurring in nature are really commensurable.

4. Further Definitions in connection with Ratio.—

The ratio **compounded** of two or more given ratios is the fraction obtained by multiplying *the corresponding fractions*. Thus, if **A, B, C, D, E, F** are *numbers*, then the ratio compounded of the three ratios **A : B, C : D, E : F** is the ratio **ACE : BDF**, for the fraction obtained by multiplying the fractions

$$\frac{A}{B}, \frac{C}{D}, \text{ and } \frac{E}{F} \text{ is } \frac{ACE}{BDF}.$$

Example.—Find the ratio compounded of the ratios 1 inch : 1 foot, 1 penny : 1 shilling, and 1 hour : 1 minute.

$$\frac{1 \text{ inch}}{1 \text{ foot}} = \frac{1 \text{ inch}}{12 \text{ inches}} = \frac{1}{12}; \quad \frac{1 \text{ penny}}{1 \text{ shilling}} = \frac{1 \text{d.}}{12 \text{d.}} = \frac{1}{12};$$

$$\frac{1 \text{ hour}}{1 \text{ minute}} = \frac{60 \text{ minutes}}{1 \text{ minute}} = \frac{60}{1},$$

$$\therefore \text{the ratio compounded of the three ratios is } \frac{1}{12} \times \frac{1}{12} \times \frac{60}{1} = \frac{5}{12}.$$

Again, if **A** and **B** are *numbers*, then:—

A² : B² is called the **duplicate ratio** of **A : B**;

A³ : B³ is called the **triplicate ratio** of **A : B**;

$\sqrt{A} : \sqrt{B}$ is called the **sub-duplicate ratio** of **A : B**;

$\sqrt[3]{A} : \sqrt[3]{B}$ is called the **sub-triplicate ratio** of **A : B**.

A **ratio of equality** is a ratio in which the antecedent is equal to the consequent (the fraction is equal to unity): a **ratio of greater inequality** is a ratio in which the antecedent is greater than the consequent (the fraction is greater than unity): a **ratio of less inequality** is a ratio in which the antecedent is less than the consequent (the fraction is less than unity).

5. Proportion.—If the ratio of **A** to **B** is equal to the ratio of **C** to **D**, the four quantities **A**, **B**, **C**, and **D** are called **proportionals** or are said to be **in proportion**. Thus a **proportion is an equality of ratios**. A proportion is usually expressed thus:—

$$A : B :: C : D,$$

which is read “**A** is to **B** as **C** is to **D**”: it is also written

$$A : B = C : D,$$

which may be read as above or “the ratio of **A** to **B** is equal to the ratio of **C** to **D**”: it is also written

$$\frac{A}{B} = \frac{C}{D}.$$

In the proportion $A : B = C : D$, the quantities **A** and **D** are called the **extremes** and the quantities **B** and **C** the **means**: the quantity **D** is called a **fourth proportional** to **A**, **B**, and **C**. The terms **A** and **C**, which are both antecedents of the ratios, are called **corresponding terms**: **B** and **D**, which are both consequents, are also corresponding terms.

In any proportion the *first two* quantities must be of the same kind (for otherwise they do not constitute a ratio) and the *last two* quantities must be of the same kind, but the first two need not be of the same kind as the last two. Thus it is correct to write 2 feet : 1 yard = 16d. : 2s., for 2 feet is $\frac{2}{3}$ of 1 yard and 16d. is $\frac{2}{3}$ of 2s., and accordingly the two ratios are equal. But it would be wrong to write 4 feet : 6 sq. yards = 10 feet : 15 sq. yards, for there is no such thing as the ratio 4 feet : 6 sq. yards, or 10 feet : 15 sq. yards.

If four quantities **A**, **B**, **C**, and **D** are such that $A : B = D : C$ then **C** and **D** are said to be **inversely proportional** to **A** and **B**. Similarly, **A** and **B** are inversely proportional to **C** and **D**.

Three magnitudes of the same kind are said to be **proportionals** if the ratio of the first to the second is equal to the ratio of the second to the third: thus **A**, **B**, and **C** are proportionals if

$$A : B = B : C.$$

In this case **B** is called a **mean proportional** to **A** and **C** and **C** is called a **third proportional** to **A** and **B**.

If $A : B = B : C = C : D = \text{etc.}$, the magnitudes **A**, **B**, **C**, **D**, etc., are said to be in **continued proportion**.

6. Harmonic Ranges.—A line **AB** is said to be divided internally (or externally) in a given ratio at **C** when the point **C** is taken in the line (or in the line produced) so that the ratio **AC : BC** is equal to the given ratio.

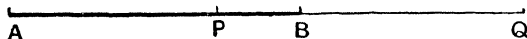


Fig. 1.

A straight line **AB** is said to be divided **harmonically** when it is divided internally and externally in the same ratio. Thus the line **AB** (Fig. 1) is divided harmonically at **P** and **Q** if $AP : PB = AQ : QB$. The points **A**, **P**, **B**, **Q** are called a **harmonic range** and the points **P** and **Q** are called **harmonic conjugates** with respect to **A** and **B**. It is easy to show that **A** and **B** also divide the line **PQ** harmonically, so that **A** and **B** are also harmonic conjugates with respect to **P** and **Q**.

Example.—Suppose **AB** = 2·1 inches and **P** and **Q** are taken so that **AP** = 1·5 inches and **AQ** = 3·5 inches: then **PB** = ·6 inches and **QB** = 1·4 inches.

$$\therefore AP : PB = 1·5 : ·6 = 5 : 2,$$

$$AQ : QB = 3·5 : 1·4 = 5 : 2.$$

Thus each of the ratios **AP : PB** and **AQ : QB** is equal to the ratio 5 : 2. The line **AB** is said to be divided *internally* at **P** and *externally* at **Q** in the ratio 5 : 2, or **P** and **Q** are said to divide **AB** *harmonically* and the points **A**, **P**, **B**, **Q** are a *harmonic range*.

Further, note that **A** and **B** divide the line **PQ** harmonically, for **QB : BP** = 1·4 : ·6 = 7 : 3, and **QA : AP** = 3·5 : 1·5 = 7 : 3: thus **A** and **B** divide **PQ** internally and externally in the ratio 7 : 3.

7. Extreme and Mean Ratio.—A straight line is said to be divided in extreme and mean ratio when the ratio of the whole line to the greater segment is equal to the ratio of the greater segment to the lesser segment. Thus PQ (Fig. 2) is divided in extreme and mean ratio at R if $PQ : PR = PR : RQ$. A line divided in extreme and mean ratio is also said to be divided in **medial section**. The division may be either internal or external. (After reading Theorem 6D, page 13, the student should read again the note on “medial section” in Problem 4C and compare with the definition of extreme and mean ratio above.)

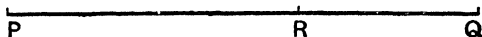


Fig. 2.

8. Warning regarding Concrete Quantities in Proportion.—The proportion $A : B = C : D$ is equivalent to the statement that the two ratios $\frac{A}{B}$ and $\frac{C}{D}$ work out to the same integer or fraction, say n . Thus $\frac{A}{B} = n$, and $\frac{C}{D} = n$, or $A = nB$ and $C = nD$.

Conversely, given that $A = nB$ and $C = nD$, it follows at once that $A : B = C : D$.

Most theorems in proportion are easily proved by supposing the ratios reduced in this way to their equivalent fractions; but in working through such proofs the student must be careful to remember the following points:—

(1) The terms of a proportion are generally concrete quantities and not mere numbers.

(2) Two concrete quantities must not be added or subtracted unless they are of the same kind, *e.g.* both lengths or both sums of money.

(3) Two concrete quantities must not be multiplied together. The definition of multiplication implies that the multiplier is necessarily an abstract number.

(4) One concrete quantity must not be divided by another unless they are of the same kind, this restriction being implied in the definition of division, and when they are of the same kind the process of division is equivalent to reducing the ratio of the two quantities to a fraction.

THEOREM 1D.

A ratio of greater inequality is diminished and one of lesser inequality is increased by adding the same quantity to each term.

PART I.

Given that In the ratio $A : B$ the term A is greater than the term B .

A quantity X is added to each term, A and B .

Required to prove that The original ratio $A : B$ is greater than the new ratio $A + X : B + X$.

Proof. If $\frac{A}{B}$ is greater than $\frac{A + X}{B + X}$,
 then $\frac{A}{B} \times \frac{B}{A}$ is greater than $\frac{A + X}{B + X} \times \frac{B}{A}$,
 that is, 1 is greater than $\frac{B(A + X)}{A(B + X)}$.
 $\therefore A(B + X)$ is greater than $B(A + X)$,
 that is, AX is greater than BX .
 $\therefore A$ is greater than B .

But A is greater than B . (Hyp.)

Hence the ratio $A : B$ is greater than the ratio $A + X : B + X$.

PART II.

Given that In the ratio $A : B$ the term A is less than the term B .

A quantity X is added to each term, A and B .

Required to prove that The original ratio $A : B$ is less than the new ratio $A + X : B + X$.

Proof. If $\frac{A}{B}$ is less than $\frac{A + X}{B + X}$,
 then $\frac{A}{B} \times \frac{B}{A}$ is less than $\frac{A + X}{B + X} \times \frac{B}{A}$,

that is, 1 is less than $\frac{B(A+X)}{A(B+X)}$.

$\therefore A(B+X)$ is less than $B(A+X)$,

that is, AX is less than BX .

$\therefore A$ is less than B .

But A is less than B . (Hyp.)

Hence the ratio $A : B$ is less than the ratio $A + X : B + X$.

COROLLARY TO THEOREM 1D.

A ratio of greater inequality is increased and a ratio of lesser inequality is diminished by subtracting the same quantity from each term.

THEOREM 2D.

If the ratio $A : B$ is equal to the ratio $C : D$, then shall the ratio $B : A$ be equal to the ratio $D : C$.

Given that $A : B = C : D$.

Required to prove that $B : A = D : C$.

Proof. Because $\frac{B}{A} = 1 \div \frac{A}{B}$, and $\frac{A}{B} = \frac{C}{D}$,

$$\therefore \frac{B}{A} = 1 \div \frac{C}{D}.$$

$$\text{But } \frac{D}{C} = 1 \div \frac{C}{D}.$$

$$\text{Hence } \frac{B}{A} = \frac{D}{C}.$$

Notes.—(1) Theorem 2D may also be stated thus:—"If four magnitudes are proportionals, they are also proportionals when taken inversely."

(2) The theorem is sometimes called **Invertendo**.

THEOREM 3D.

If four quantities **A, B, C, D** are all of the same kind, and if the ratio **A : B** is equal to the ratio **C : D**, then shall the ratio **A : C** be equal to the ratio **B : D**.

Given that **A, B, C, D** are quantities of the same kind, and
 $A : B = C : D$.

Required to prove that $A : C = B : D$.

Proof. Let the ratio **A : B** and the ratio **C : D** be each equal to n .

$$\text{Because } \frac{A}{B} = n, \therefore A = nB.$$

$$\text{Because } \frac{C}{D} = n, \therefore C = nD.$$

$$\therefore \frac{A}{C} = \frac{nB}{nD}.$$

$$\text{Hence } \frac{A}{C} = \frac{B}{D}.$$

Notes.—(1) Theorem 3D may also be stated thus:—“If four magnitudes of the same kind are proportionals they are also proportionals when taken alternately.”

(2) Since, by the hypothesis, we are dealing with the ratio of the two quantities **A** and **B**, and with the ratio of the two quantities **C** and **D**, it follows that **A** and **B** must be quantities of the same kind, and **C** and **D** must be of the same kind. Similarly the conclusion necessitates **A** and **C** being of the same kind, and **B** and **D** being of the same kind. Hence all four quantities must be of the same kind (as stated).

(3) The theorem is sometimes called **Alternando**.

THEOREM 4D.

If the ratio $A : B$ is equal to the ratio $C : D$, then shall the ratio $A + B : B$ be equal to the ratio $C + D : D$ and the ratio $A - B : B$ be equal to the ratio $C - D : D$.

Given that $A : B = C : D$.

Required to prove that (1) $A + B : B = C + D : D$.

(2) $A - B : B = C - D : D$.

Proof. Let the ratios $A : B$ and $C : D$ be each equal to n .

$$\therefore \frac{A}{B} = n, \therefore A = nB; \text{ also, } \therefore \frac{C}{D} = n, \therefore C = nD.$$

$$(1) \therefore \frac{A + B}{B} = \frac{nB + B}{B} = \frac{B(n + 1)}{B} = n + 1,$$

$$\text{and } \frac{C + D}{D} = \frac{nD + D}{D} = \frac{D(n + 1)}{D} = n + 1.$$

$$\text{Hence } \frac{A + B}{B} = \frac{C + D}{D}.$$

$$(2) \text{ Similarly, } \frac{A - B}{B} = \frac{B(n - 1)}{B} = n - 1,$$

$$\text{and } \frac{C - D}{D} = \frac{D(n - 1)}{D} = n - 1.$$

$$\text{Hence } \frac{A - B}{B} = \frac{C - D}{D}.$$

COROLLARY TO THEOREM 4D.

If $A : B = C : D$ then $A + B : A - B = C + D : C - D$.

Notes.—(1) The first part may also be stated thus :—“If four magnitudes are proportionals the sum of the first antecedent and its consequent is to that consequent as the sum of the second antecedent and its consequent is to that consequent.” This part is sometimes called **Componendo**.

(2) The second part may be stated thus :—“If four magnitudes are proportionals the difference between the first antecedent and its consequent is to that consequent as the difference between the second antecedent and its consequent is to that consequent.” This part is sometimes called **Dividendo**.

(3) The Corollary is sometimes called **Componendo and Dividendo**.

THEOREM 5D.

If the quantities A, B, C, D, E, F are all of the same kind, and if $A : B = C : D = E : F$, then shall each of these ratios be equal to the ratio $A + C + E : B + D + F$.

Given that A, B, C, D, E, F are quantities of the same kind, and $A : B = C : D = E : F$.

Required to prove that $\frac{A + C + E}{B + D + F} = \frac{A}{B} = \frac{C}{D} = \frac{E}{F}$.

Proof. Let each of the ratios $A : B, C : D, E : F$ be equal to n .

Because $\frac{A}{B} = n$, $\therefore A = nB$.

Similarly, $C = nD$ and $E = nF$.

$$\therefore A + C + E = nB + nD + nF \\ = n(B + D + F),$$

$$i.e. \frac{A + C + E}{B + D + F} = n.$$

$$\text{Hence } \frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{A + C + E}{B + D + F}.$$

Notes.—(1) Theorem 5D is true for any number of equal ratios: hence it may also be stated thus:—"If any number of ratios are equal, all the magnitudes being of the same kind, then as any one of the antecedents is to its consequent so is the sum of all the antecedents to the sum of all the consequents."

(2) The theorem is sometimes called **Addendo**.

(3) The student should also prove that each of the above ratios, $\frac{A}{B}, \frac{C}{D}$, etc., is equal to $\frac{lA + mC + nE}{lB + mD + nF}$, where l, m, n are any quantities, positive or negative.

THEOREM 6D.

If four numbers are proportionals, the product of the extremes is equal to the product of the means; and conversely, if the product of one pair of numbers is equal to the product of another pair, the four numbers shall be proportionals if taken in such an order that one pair form the means and the other pair the extremes.

PART I.

Given that A, B, C, and D represent four abstract numbers, and $A : B = C : D$.

Required to prove that $AD = BC$.

Proof. Because $\frac{A}{B} = \frac{C}{D}$, $\therefore \frac{A}{B} \times BD = \frac{C}{D} \times BD$.

Hence $AD = BC$.

PART II.

Given that $AD = BC$.

Required to prove that $A : B = C : D$.

Proof. Because $AD = BC$, $\therefore \frac{AD}{BD} = \frac{BC}{BD}$.

Hence $\frac{A}{B} = \frac{C}{D}$.

COROLLARIES TO THEOREM 6D.

COROLLARY 1.—If three numbers are proportionals the product of the extremes is equal to the square of the mean, and conversely if the product of two quantities is equal to the square of a third, the three quantities are proportionals, the third quantity being the mean.

COROLLARY 2.—If four straight lines are proportionals the rectangle contained by the extremes is equal to the rectangle contained by the means (see Theorem 33D).

COROLLARY 3.—If three straight lines are proportionals the rectangle contained by the extremes is equal to the square on the mean (see Theorem 33D, Cor.)

THEOREM 7D.

If three numbers are proportionals the ratio of the first to the third shall be equal to the duplicate ratio of the first to the second.

Given that A , B , and C are three numbers such that

$$A : B = B : C.$$

Required to prove that $A : C = A^2 : B^2$.

Proof. Because $\frac{A}{B} = \frac{B}{C}$, $\therefore \frac{A}{B} \times \frac{A}{B} = \frac{B}{C} \times \frac{A}{B}$.

$$\text{Hence } \frac{A}{C} = \frac{A^2}{B^2}.$$

THEOREM 8D.

If the ratio of two numbers is equal to the ratio of two other numbers then shall their duplicate, triplicate, . . . ratios be equal.

Given that $A : B = C : D$.

Required to prove that $A^2 : B^2 = C^2 : D^2$,

$$A^3 : B^3 = C^3 : D^3, \text{ etc.}$$

Proof. Because $\frac{A}{B} = \frac{C}{D}$, $\therefore \frac{A}{B} \times \frac{A}{B} = \frac{C}{D} \times \frac{A}{B} = \frac{C}{D} \times \frac{C}{D}$.

$$\text{Hence } \frac{A^2}{B^2} = \frac{C^2}{D^2}.$$

In the same way it can be shown that $A^3 : B^3 = C^3 : D^3$, etc.

COROLLARY TO THEOREM 8D.

If the duplicate ratio of two numbers is equal to the duplicate ratio of two other numbers then shall the ratio of the first pair of numbers be equal to the ratio of the second pair.

THEOREM 9D.

If the ratio $A : B$ is equal to the ratio $P : Q$, and the ratio $B : C$ is equal to the ratio $Q : R$, then shall the ratio $A : C$ be equal to the ratio $P : R$.

Given that $A : B = P : Q$
and $B : C = Q : R$.

Required to prove that $A : C = P : R$.

Proof. Because $\frac{A}{B} = \frac{P}{Q}$ and $\frac{B}{C} = \frac{Q}{R}$,
 $\therefore \frac{A}{B} \times \frac{B}{C} = \frac{P}{Q} \times \frac{B}{C} = \frac{P}{Q} \times \frac{Q}{R}$.
 Hence $\frac{A}{C} = \frac{P}{R}$.

Notes.—(1) Theorem 9D may also be stated thus:—“*Ratios compounded of equal ratios are equal.*”

(2) The theorem is sometimes called **Ex aequali**.

THEOREM 10D.

If the ratio $A : B$ is equal to the ratio $C : D$, and the ratio $E : B$ is equal to the ratio $F : D$, then shall the ratio $A + E : B$ be equal to the ratio $C + F : D$.

Given that $A : B = C : D$
and $E : B = F : D$.

Required to prove that $A + E : B = C + F : D$.

Proof. Because $\frac{A}{B} = \frac{C}{D}$ and $\frac{E}{B} = \frac{F}{D}$,
 therefore, adding equals to equals,

$$\frac{A}{B} + \frac{E}{B} = \frac{C}{D} + \frac{F}{D}.$$

$$\text{Hence } \frac{A + E}{B} = \frac{C + F}{D}.$$

THEOREM 11D.

If a series of parallel lines divide each of two transversals into segments, then any two segments on the one line shall be proportional to the corresponding two segments on the other.

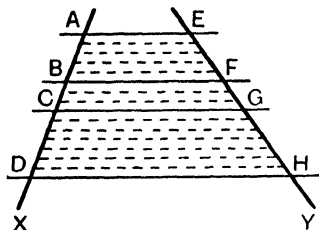


Fig. 3.

Given that The lines **AE, BF, CG, DH** are all parallel
and cut the transversals **X** and **Y** in the points
A, B, C, D, and **E, F, G, H.**

Required to prove that Any two segments on the transversal
X are proportional to the corresponding segments on
the transversal **Y**, *e.g.*

$$\mathbf{AB : CD = EF : GH,}$$

$$\mathbf{AD : BC = EH : FG,}$$

and so on.

Construction. Assuming that the lengths **AB, BC,** and **CD**
are commensurable,* there is some length **P** which is
contained an exact number of times in each of them.

Divide **AB, BC,** and **CD** into equal parts each of length **P.**

Let **AB** contain *r*, **BC** contain *s*, and **CD** contain *t* of
these parts.

* See Art. 3.

Through the points of division draw lines parallel to **AE**,
and therefore parallel to **BF**, **CG**, **DH**, and to one another.
(Theorem 8A.)

Proof. Because these parallels divide **AB**, **BC**, and **CD** into
equal parts,

$\therefore$ they divide **EF**, **FG**, and **GH** into the same number
of equal parts each of length, say, **Q**;
(Theorem 31A.)

that is, **EF** contains r , **FG** contains s , and **GH** contains
 t of these parts.

Because **AB** = rP and **CD** = tP ,

$$\therefore \mathbf{AB} : \mathbf{CD} = \frac{\mathbf{AB}}{\mathbf{CD}} = \frac{rP}{tP} = \frac{r}{t}.$$

Because **EF** = rQ and **GH** = tQ ,

$$\therefore \mathbf{EF} : \mathbf{GH} = \frac{\mathbf{EF}}{\mathbf{GH}} = \frac{rQ}{tQ} = \frac{r}{t}.$$

Hence **AB** : **CD** = **EF** : **GH**.

Again, **AD** : **BC**

$$= \frac{\mathbf{AD}}{\mathbf{BC}} = \frac{\mathbf{AB} + \mathbf{BC} + \mathbf{CD}}{\mathbf{BC}} = \frac{(r + s + t)P}{sP},$$

$$\text{that is, } \mathbf{AD} : \mathbf{BC} = \frac{r + s + t}{s}.$$

Further **EH** : **FG**

$$= \frac{\mathbf{EH}}{\mathbf{FG}} = \frac{\mathbf{EF} + \mathbf{FG} + \mathbf{GH}}{\mathbf{FG}} = \frac{(r + s + t)Q}{sQ},$$

$$\text{that is, } \mathbf{EH} : \mathbf{FG} = \frac{r + s + t}{s}.$$

Hence **AD** : **BC** = **EH** : **FG**,

and so on.

THEOREM 12D.

If a straight line is drawn parallel to one side of a triangle it shall divide the other sides (internally or externally) in the same ratio.

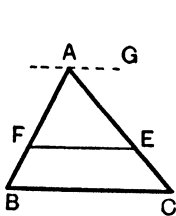


Fig. 4.

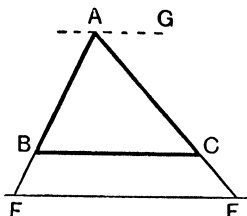


Fig 5.

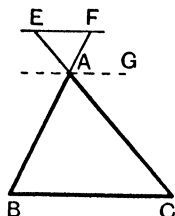


Fig 6.

Given that The straight line **FE** (Fig. 4) is parallel to the side **BC** of $\triangle ABC$ and divides the sides **AB** and **AC** at the points **F** and **E**.

Required to prove that $AF : FB = AE : EC$.

Construction. Through **A** draw **AG** parallel to **BC**.

Proof. Because **AG**, **FE**, and **BC** are parallel straight lines and **AB** and **AC** two transversals,
 $\therefore$ any two segments of **AB** are proportional to the corresponding segments of **AC**. (*Theorem 11D.*)

Hence $AF : FB = AE : EC$.

In the same way it can be shown (by Theorem 11D) that if the straight line **FE** cuts the sides **AB** and **AC** produced either way (*i.e.* divides them externally), as indicated in Figs. 5 and 6, the same relation holds, *i.e.*,
 $AF : FB = AE : EC$.

Note.—It should be noted that in each of the three figures, the segments **AF** and **FB** correspond to the segments **AE** and **EC**.

THEOREM 13D.

If two sides of a triangle are divided (internally or externally) in the same ratio, then the line joining the points of section shall be parallel to the third side.

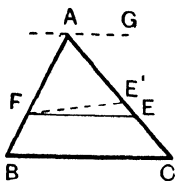


Fig. 7.

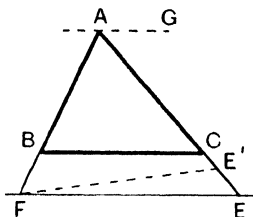


Fig. 8.

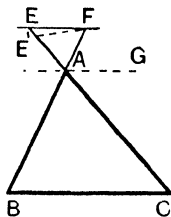


Fig. 9.

Given that

$$AF : FB = AE : EC.$$

Required to prove that FE is parallel to BC .

Construction. Through A draw AG parallel to BC , and through F draw FE' parallel to BC , meeting AC at E' .

Proof.

In all three cases,

$$AF : FB = AE : EC$$

(Hyp.)

$$\therefore \text{(Fig. 7)} \quad AF : AF + FB = AE : AE + EC,$$

(Theorem 4D.)

$$\text{or (Fig. 8)} \quad AF : AF - FB = AE : AE - EC,$$

(Theorem 4D.)

$$\text{or (Fig. 9)} \quad AF : FB - AF = AE : EC - AE,$$

(Theorem 4D.)

that is, in all three cases $AF : AB = AE : AC$.

Again, because the parallels AG , FE' , and BC meet the transversals AB and AC ,

$$\therefore AF : AB = AE' : AC.$$

(Theorem 11D.)

$$\text{But } AF : AB = AE : AC.$$

(Proved.)

$$\therefore AE : AC = AE' : AC,$$

whence $AE = AE'$,

that is, E and E' coincide.

But FE' is parallel to BC .

Hence FE is parallel to BC .

THEOREM 14D.

The line which bisects the interior vertical angle of a triangle divides the base internally in the ratio of the sides ; and conversely.

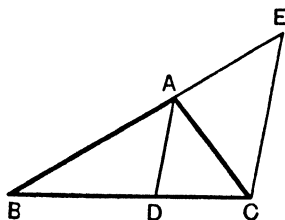


Fig. 10.

PART I.

Given that In $\triangle ABC$, the straight line AD bisects $\angle BAC$,
i.e. $\angle BAD = \angle DAC$.

Required to prove that $BD : DC = BA : AC$.

Construction. Draw the straight line CE parallel to DA ,
meeting BA produced at the point E .

Proof. Because AD is parallel to EC ,
 $\therefore$ using BE as transversal, $\angle BAD = \angle AEC$,
(Theorem 7A.)

and using AC as transversal, $\angle DAC = \angle ACE$.
(Theorem 6A.)

But $\angle BAD = \angle DAC$, (Hyp.)

$\therefore \angle AEC = \angle ACE$;

hence $AC = AE$. (Theorem 14A.)

Again, in $\triangle BCE$, because AD is parallel to EC ,

$\therefore BD : DC = BA : AE$. (Theorem 12D.)

But $AE = AC$. (Proved.)

Hence $BD : DC = BA : AC$.

PART II.

Given that In $\triangle ABC$, the straight line **AD** drawn from the vertex to the base is such that

$$\mathbf{BD : DC = BA : AC.}$$

Required to prove that

$$\mathbf{AD \text{ bisects } \angle BAC, \text{ i.e. } \angle BAD = \angle DAC.}$$

Construction. Draw the straight line **CE** parallel to **DA**, meeting **BA** produced at the point **E**.

Proof. In $\triangle BCE$, because **AD** is parallel to **EC**,

$$\therefore \mathbf{BD : DC = BA : AE.} \quad (\text{Theorem 12D.})$$

$$\text{But } \mathbf{BD : DC = BA : AC,} \quad (\text{Hyp.})$$

$$\therefore \mathbf{BA : AC = BA : AE.}$$

$$\text{Hence } \mathbf{AC = AE,}$$

$$\therefore \mathbf{\angle AEC = \angle ACE.} \quad (\text{Theorem 13A.})$$

Again, because **AD** is parallel to **EC**,

$$\therefore \text{using } \mathbf{BE} \text{ as transversal, } \mathbf{\angle AEC = \angle BAD,} \quad (\text{Theorem 7A.})$$

$$\text{and using } \mathbf{AC} \text{ as transversal, } \mathbf{\angle ACE = \angle DAC.} \quad (\text{Theorem 6A.})$$

$$\text{But } \mathbf{\angle AEC = \angle ACE,} \quad (\text{Proved.})$$

$$\therefore \mathbf{\angle BAD = \angle DAC}$$

$$\mathbf{Hence AD \text{ bisects } \angle BAC.}$$

Note.—As an exercise the student should use Theorem 14D to prove that the bisectors of the angles of a triangle are concurrent.

THEOREM 15D.

The line which bisects the exterior vertical angle of a triangle divides the base externally in the ratio of the sides; and conversely.

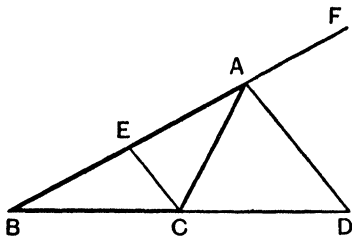


Fig. 11.

PART I.

Given that In $\triangle ABC$, the straight line AD bisects the exterior $\angle CAF$, i.e. $\angle FAD = \angle DAC$.

Required to prove that $BD : DC = BA : AC$.

Construction. Draw the straight line CE parallel to DA meeting BA at the point E .

Proof. Because AD is parallel to EC ,

$\therefore$ using BF as transversal, $\angle FAD = \angle AEC$,
(Theorem 7A.)

and using AC as transversal, $\angle DAC = \angle ACE$.
(Theorem 6A.)

But $\angle FAD = \angle DAC$, (Hyp.)

$\therefore \angle AEC = \angle ACE$;

hence $AC = AE$. (Theorem 14A.)

Again, in $\triangle BCE$, the sides BC and BE are divided externally at D and A by the straight line DA parallel to the side CE ,

$\therefore BD : DC = BA : AE$. (Theorem 12D.)

But $AE = AC$.

Hence $BD : DC = BA : AC$.

PART II.

Given that In $\triangle ABC$, the straight line AD is drawn so that
 $BD : DC = BA : AC$.

Required to prove that

AD bisects exterior $\angle FAC$, i.e. $\angle FAD = \angle DAC$.

Construction. Draw CE parallel to DA meeting BA at E .

Proof. Because in $\triangle BCE$ the sides BC and BE are divided
 externally at D and A by the line DA
 parallel to CE ,

$$\therefore BD : DC = BA : AE. \quad (\text{Theorem 12D.})$$

$$\text{But } BD : DC = BA : AC, \quad (\text{Hyp.})$$

$$\therefore AC = AE;$$

$$\text{hence } \angle AEC = \angle ACE. \quad (\text{Theorem 13A.})$$

Again, because AD is parallel to EC ,

$$\therefore \text{using } BF \text{ as transversal, } \angle AEC = \angle FAD, \quad (\text{Theorem 7A.})$$

$$\text{and using } AC \text{ as transversal, } \angle ACE = \angle DAC. \quad (\text{Theorem 6A.})$$

$$\text{But } \angle AEC = \angle ACE, \quad (\text{Proved.})$$

$$\therefore \angle FAD = \angle DAC.$$

Hence AD bisects $\angle FAC$.

Notes.—(1) The above proof is usually accepted as sufficient. It assumes, however, the two following facts:—(1) The same line bisects each of the two exterior vertical angles; (2) If in Fig. 11, $AB > AC$, then the bisector of $\angle CAF$ meets BC produced, but if $AC > AB$, then this bisector meets CB produced. The student should prove these two statements as riders. Does the theorem still hold if $AB = AC$?

(2) It follows from Theorems 14D and 15D that “the base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle” for in each case the base is divided in the ratio of the other two sides, the division being in one case internal and in the other external (See Art. 6, page 6).

THEOREM 16D.

Given the base of a triangle and the ratio of the sides, the locus of the vertex is a circle; or:—

The locus of a point which moves so that its distances from two fixed points are in a given ratio is a circle.

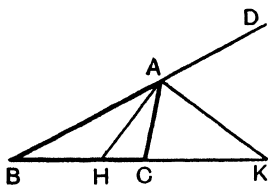


Fig. 12.

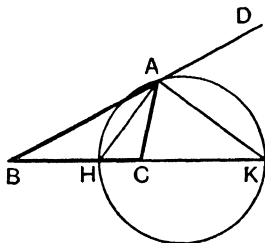


Fig. 13.

Given that In $\triangle ABC$, the base **BC** is fixed and the vertex **A** moves so that the ratio **AB : AC** is constant.

Required to prove that The locus of **A** is a circle.

Construction. Let the ratio **AB : AC** be equal to the constant ratio $l : m$.

Let **H** divide **BC** internally so that **BH : HC** = $l : m$.

Let **K** divide **BC** externally so that **BK : KC** = $l : m$.

Join **AH** and **AK** and produce **BA** to **D**.

Proof.

In the triangle **ABC**,

$$\mathbf{BH : HC = l : m.} \quad (\text{Cons.})$$

$$\text{But } \mathbf{BA : AC = l : m,} \quad (\text{Hyp.})$$

$$\therefore \mathbf{BH : HC = BA : AC.}$$

Hence **AH** bisects $\angle \mathbf{BAC}$,

that is, $\angle \mathbf{HAB} = \angle \mathbf{CAH.}$ (Theorem 14D.)

Again, in the triangle **ABC**,

$$\mathbf{BK : KC = l : m,} \quad (\text{Cons.})$$

$$\therefore \mathbf{BK : KC = BA : AC.}$$

Hence **AK** bisects exterior $\angle \mathbf{DAC}$,

$$\text{that is, } \angle \mathbf{DAK} = \angle \mathbf{KAC.} \quad (\text{Theorem 15D.})$$

$$\text{Because } \angle \mathbf{CAH} = \angle \mathbf{HAB} = \frac{1}{2} \angle \mathbf{CAB},$$

$$\text{and } \angle \mathbf{KAC} = \angle \mathbf{DAK} = \frac{1}{2} \angle \mathbf{DAC},$$

$$\begin{aligned} \therefore \angle \mathbf{CAH} + \angle \mathbf{KAC} &= \frac{1}{2} (\angle \mathbf{CAB} + \angle \mathbf{DAC}) \\ &= \text{a right angle;} \end{aligned}$$

$$\text{that is, } \angle \mathbf{HAK} = \text{a right angle.}$$

Because **BC** is a fixed line,

$$\therefore \mathbf{HK} \text{ is a fixed line,}$$

$\therefore$ the fixed line **HK** subtends a right angle at **A**.

But the angle in a semi-circle is a right angle.

Hence the locus of A is a circle with diameter HK.

COROLLARY TO THEOREM 16D.

*If **B, H, C** are three points in a straight line, then the locus of a point at which **BH, HC** subtend equal angles is a circle.*

$$\text{For if } \angle \mathbf{BAH} = \angle \mathbf{CAH},$$

$$\text{then } \mathbf{BA : AC = BH : HC.} \quad (\text{Theorem 14D.})$$

That is, **A** moves so that the ratio of the sides of

$$\triangle \mathbf{ABC} \text{ is constant.}$$

Hence the locus of **A** is a circle.

Notes.—(1) In the figure it is assumed that l is greater than m . If l is less than m , **K** lies in **CB** produced.

(2) The circle is called the **Circle of Apollonius**.

THEOREM 17D.

In equal circles the arcs of sectors are proportional to their angles.

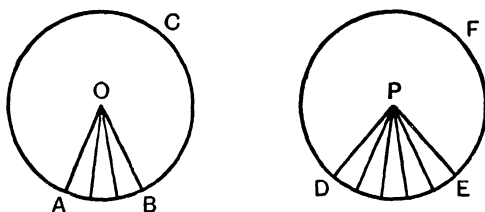


Fig. 14.

Given that **ABC** and **DEF** are two equal circles with centres **O** and **P** respectively, and **AOB** and **DPE** are any two sectors.

Required to prove that $\text{Arc } \mathbf{AB} : \text{arc } \mathbf{DE} = \angle \mathbf{AOB} : \angle \mathbf{DPE}$.

Construction. Assuming that the arcs **AB**, **DE** are commensurable there is some arc **L** which is contained an exact number of times in each of them.

Suppose **AB** and **DE** divided respectively into m and n parts of length **L**. Join **O** to the points of division of **AB**, and **P** to the points of division of **DE**.

Then $\angle \mathbf{AOB}$ is divided into m angles, and $\angle \mathbf{DPE}$ into n angles.

Proof. In the equal circles **ABC** and **DEF**,

$\therefore$ the arcs into which **AB** and **DE** are divided are all equal,

$\therefore$ the angles into which $\angle \mathbf{AOB}$ and $\angle \mathbf{DPE}$ are divided are all equal. (Theorem 20B.)

Let each of these angles be θ .

$\therefore \angle \mathbf{AOB} = m\theta$, and $\angle \mathbf{DPE} = n\theta$,

$$\therefore \frac{\angle \mathbf{AOB}}{\angle \mathbf{DPE}} = \frac{m}{n}.$$

Because arc $AB = mL$, and arc $DE = nL$,

$$\therefore \frac{\text{arc } AB}{\text{arc } DE} = \frac{m}{n}.$$

$$\text{Hence } \frac{\text{arc } AB}{\text{arc } DE} = \frac{\angle AOB}{\angle DPE}.$$

COROLLARY TO THEOREM 17D.

In equal circles angles at the circumference have the same ratio as the arcs upon which they stand.

For the angles at the centre have the same ratio as the arcs, and the angles at the centre are each double of the angles at the circumference: hence the angles at the circumference have the same ratio as the arcs.

Note.—An angle called a *radian* is used in Trigonometry: it is the angle at the centre of a circle subtended by an arc equal to the radius. Thus if $\angle AOB = \alpha$ radians and radius $OA = r$ —

$$\frac{\text{arc } AB}{r} = \alpha, \quad \therefore \text{arc } AB = r\alpha,$$

an expression frequently employed in Mathematics.

The relation between a radian and a right angle can easily be deduced. By the definition above, the angle at the centre of a circle subtended by an arc equal to the radius = 1 radian; also the angle at the centre of a circle subtended by half the circumference = 2 right angles.

Hence, by Theorem 17D,

$$\begin{aligned} \frac{2 \text{ right angles}}{1 \text{ radian}} &= \frac{\frac{1}{2} \text{ circumference}}{\text{radius}} \\ &= \frac{\frac{1}{2}(2\pi r)}{r} = \frac{\pi r}{r} = \pi. \\ \therefore 2 \text{ right angles} &= \pi \text{ radians} \\ \text{or } 1 \text{ radian} &= \frac{2 \text{ right angles}}{\pi} \\ &= \frac{180^\circ}{3.1416} \\ &= 57.3^\circ \text{ approx.} \end{aligned}$$

It is evident that the radian bears a constant ratio to a right angle which does not depend on the size of the circle used in the construction; in other words, the radian is an invariable angle.

Exercises I.

CALCULATIONS.

1. Reduce the following ratios to their equivalent vulgar fractions :—
 (i) 5 ft. : 2 yds. ; (ii) 300 sq. metres : 4 sq. decametres ; (iii) 10 right angles : 1000° ; (iv) 20 ft. : 10 sec.

2. Reduce the following ratios to their equivalent decimal fractions, correct to three significant figures :—(i) 1 metre : 273 cm. ; (ii) 9 kilom. : 13 kilom. ; (iii) 132 ft. : 10 sq. yds. ; (iv) 21 R : 10,000°.

3. Express as vulgar fractions :—(i) the duplicate ratio of 4 ft. : 2 yds. ; (ii) the sub-duplicate ratio of 8 ft. : 6 yds. ; (iii) the triplicate ratio of 10 yds. : 40 ft. ; (iv) the sub-triplicate ratio of 1 cub. in. : 1 cub. ft.

4. Express in decimals correct to three significant figures the ratio compounded of the ratios :—(i) 2 : 5 and 25 : 8 ; (ii) 3 yds. : 2 ft. and 5 ft. : 2 yds. ; (iii) 10 sq. ft. : 2 sq. yds. and 10 in. : 2 ft. ; (iv) 1 hour : 1 day, 1 cwt. : 1 ton, and 1 metre. : 1 cm.

5. Find the unknown quantity x in each of the following proportions :—(i) $x : 1 \text{ ft.} = 2 : 3$; (ii) $x : 1 \text{ sq. yd.} = 20 \text{ ft.} : 15 \text{ yds.}$; (iii) $4\text{R} : 200^\circ = x : 10 \text{ sq. ft.}$; (iv) 100 miles : 1000 furlongs = $x : 50 \text{ sq. kilom.}$

6. ABCDE are five points in order on a straight line. If
 $AB : BC : CD : DE = 2 : 3 : 1 : 5$,
 evaluate the ratios AC : CE, AD : BE, and BD : AE.

RIDERS.

7. A, B, C, D, E, F are six points in order on a straight line. Given that $AB : BC = DE : EF$, prove that $AB : AC = DE : DF$, and that $AC : DF = BC : EF$.

8. A, B, C, D are four points in order on a straight line, and a, b, c, d are four points in order on another. Given that $ab = \frac{2}{3}AB$, $bc = \frac{2}{3}BC$, $cd = \frac{2}{3}CD$, prove that $ad : bc = AD : BC$, and that $ac : AC = bd : BD$.

9. If a median of a triangle bisects the corresponding angle, the triangle is isosceles.

10. ABC is a triangle of which AD is a median. DK, DL bisect the angles ADC, ADB respectively, and meet AC, AB in K, L respectively. Prove that KL is parallel to BC.

11. ABC is any triangle, and O is any given point. Join OA , OB , and OC , and on OA , OB , and OC take the points P , Q , and R such that QR is parallel to BC and RP to CA . Prove that PQ is parallel to AB .

12. If the line joining the points of intersection of two sides of a triangle with the bisectors of the opposite angles is parallel to the base, the triangle is isosceles.

13. Two circles touch one another internally at O , and a chord PQ of the outer circle touches the inner circle at R . Show that

$$OP : OQ = PR : RQ.$$

14. If O is the in-centre of ABC , and AO meets BC in D , prove that

$$AO : OD = AB + AC : BC.$$

15. If in a quadrilateral the bisectors of one pair of opposite angles meet on one diagonal, then the bisectors of the other pair of opposite angles meet on the other diagonal.

16. Two intersecting chords in a circle are such that the segments of one are in the same ratio as the segments of the other. Prove that the line bisecting the angle formed by corresponding segments passes through the centre of the circle.

17. AB is a diameter of a circle and CD is a chord at right angles to it. E is any point on CD and AE , BE meet the circle in F , G . Prove that any two adjacent sides of the quadrilateral $CFDG$ are in the same ratio as the other two.

18. $ABCD$ is a quadrilateral and P is any point. Prove that the centroids of the triangles PAB , PBC , PCD , PDA form a parallelogram.

19. OA , OB are any two straight lines intersecting in O , and R and S are any two points on OB , OA respectively; BN parallel to AR meets OA in N , and AM parallel to BS meets OB in M . Prove that MN is parallel to RS .

20. P is a point outside a circle, and chords PAB , PCD are drawn to meet the circle in A , B , C , D . PEF bisects $\angle APC$, and meets the chord AC in E , and the chord BD in F ; prove that $BF : FD = CE : EA$.

21. The external bisector of the angle A of a triangle ABC meets BC produced in D . E is the middle point of BC , and a line through C parallel to AE meets BA produced in F . If EG is drawn parallel to DF to meet AB in G , prove that $AG = AC$.

- 22.** Trisect a given straight line by means of Theorem 14D.
- 23.** Draw through a point between two given convergent straight lines a straight line terminated by these lines and divided at the point in a given ratio.
- 24.** Find a point on a given line (or circle) whose distances from two given points are in a given ratio.
- 25.** Determine a point whose distances from three given points are in given ratios.
- 26.** Construct a triangle given one side, an angle adjacent to it, and the ratio of the other two sides.
- 27.** Construct a triangle given one side, the angle opposite to it, and the ratio of the other two sides.
- 28.** Construct a triangle given one side, the ratio of the other two sides, and the area.
- 29.** Construct a triangle given the base, the altitude, and the ratio of the other two sides.

CHAPTER II.

SIMILAR FIGURES.

9. Definitions.—Two rectilinear figures (triangles, quadrilaterals, polygons) are said to be **equiangular** to one another when the angles of the first taken in order are equal respectively to the angles of the second taken in order.

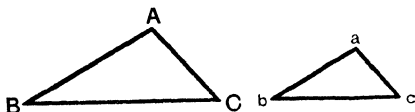


Fig. 15.

Two rectilinear figures are said to be similar if they are equiangular to one another and if also the sides joining corresponding vertices are in a constant ratio. This constant ratio is called the ratio of similitude of the two figures.

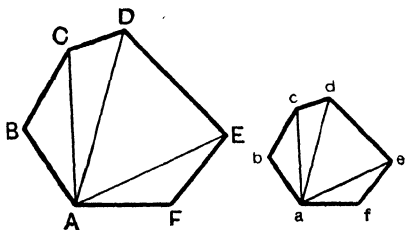


Fig. 16.

Thus in Fig. 15 the angles **A, B, C** of the triangle **ABC** are respectively equal to the angles *a, b, c* of the triangle *abc*, and each side of the triangle *abc* is two thirds of the corresponding side of the triangle **ABC**: the two triangles are therefore similar, the ratio of similitude being 3:2. Again, in Fig. 16 corresponding angles are equal and each side of *abcdef* is three fifths of the corresponding side of

ABCDEF: the two polygons are therefore similar, the ratio of similitude being 5 : 3.

It should be particularly noted that the *two necessary conditions* for similarity are (1) the figures must have their corresponding angles equal, and (2) corresponding sides must be in a constant ratio.

In the case of triangles these two conditions are not independent: thus it can be shown that if two triangles have their corresponding angles equal their corresponding sides must be in a constant ratio, and it can also be shown that if the corresponding sides of two triangles are in a constant ratio their corresponding angles must be equal.

This, however, is not the case with quadrilaterals and polygons. Thus the quadrilaterals **ABCD** and **EBFG** (Fig. 17) have corresponding angles equal, but corresponding sides are certainly not in a constant ratio and the quadrilaterals are not similar. Again, in Fig. 18 corresponding sides of the

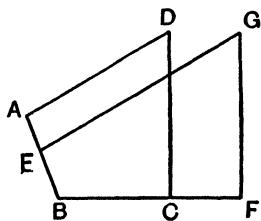


Fig. 17.

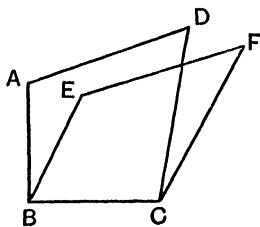


Fig. 18.

two quadrilaterals are in a constant ratio but corresponding angles are certainly not equal and the quadrilaterals are not similar.

Two triangles which are equiangular, and therefore similar, are said to be of the same **species**.

If two unequal similar figures are placed so that their corresponding sides are parallel, the lines joining their corresponding points can be shown to be concurrent: this point of concurrence is called a **centre of similitude** or **centre of similitude** of the two figures. Thus in Figs. 19 and 20 the point **O** is the centre of similitude of the two similar polygons there shown. It should be noted that in

Fig. 19 the two similar figures have corresponding sides parallel and *in the same sense*, whilst in Fig. 20 they have corresponding sides parallel but *in opposite senses*.

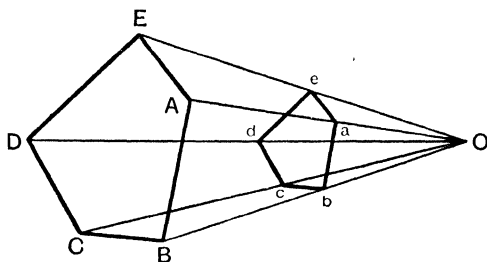


Fig. 19.

If two similar figures have their corresponding sides parallel and drawn *in the same sense* they are said to be **similarly situated**: thus the polygons in Fig. 19 are similarly situated.

Two rectilinear figures which have their sides about an angle in each proportional, but in such a manner that a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first, are said to have their sides about these angles **reciprocally proportional**.

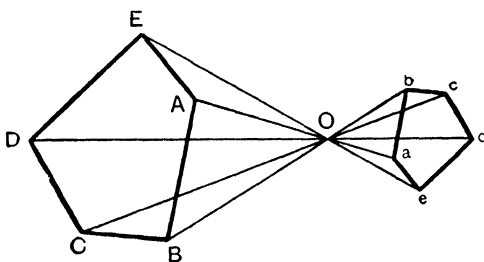


Fig. 20.

The symbol $\parallel$ is sometimes used to denote that two figures are similar: thus " $\triangle ABC \parallel \triangle DEF$ " means "triangle **ABC** is similar to triangle **DEF**."

THEOREM 18D.

If two triangles are equiangular they are similar.

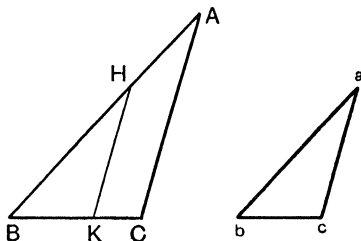


Fig. 21.

Given that In the triangles **ABC** and **abc**,
 $\angle BAC = \angle bac$, $\angle ABC = \angle abc$, and $\angle ACB = \angle acb$.

Required to prove that $\frac{AB}{ab} = \frac{BC}{bc} = \frac{CA}{ca}$,

that is, that *both* conditions for similarity hold and the triangles are therefore similar.

Construction. From **BA** mark off **BH** = **ba**, and from **BC** mark off **BK** = **bc**. Join **HK**.

Proof. In the triangles **HBK** and **abc**,
 $\therefore BH = ba$, $BK = bc$, $\angle ABC = \angle abc$,
 $\therefore \angle BHK = \angle bac$. (Theorem 11A.)

But $\angle BAC = \angle bac$, $\therefore \angle BHK = \angle BAC$.

Hence (using **BA** as transversal) **HK** is parallel to **AC**.

$\therefore \frac{AB}{HB} = \frac{BC}{BK}$. But **HB** = **ab** and **BK** = **bc**,

$$\therefore \frac{AB}{ab} = \frac{BC}{bc}.$$

Similarly, by marking off lengths along **CA** and **CB** equal to **ca** and **cb** respectively it can be proved that

$$\frac{BC}{bc} = \frac{CA}{ca}.$$

$$\therefore \frac{AB}{ab} = \frac{BC}{bc} = \frac{CA}{ca}.$$

Hence the triangles **ABC and **abc** are similar.**

COROLLARIES TO THEOREM 18D.

COROLLARY 1.—*If two parallels are cut by two intersecting transversals the two triangles so formed are similar.*

Thus in Figs. 30, 31, **PQ**, *pq* are the parallels and **AO**, **BO** are the intersecting transversals.

Then it is easily proved that $\triangle$ s **OAB**, **Oab** are equiangular and therefore similar.

COROLLARY 2.—*If the three sides of one triangle are respectively parallel (or respectively perpendicular) to the three sides of another then the triangles are similar.*

Suppose that the sides **AB**, **BC**, **CA** of $\triangle$ **ABC** are respectively parallel (or respectively perpendicular) to the sides *ab*, *bc*, *ca* of $\triangle$ *abc*.

It is easy to prove that, since the arms of $\angle$ **A** are respectively parallel (or respectively perpendicular) to the arms of $\angle$ *a*, therefore $\angle$ s **A** and *a* are either equal or supplementary.

Similarly for $\angle$ s **B** and *b*, and for $\angle$ s **C** and *c*.

Now $\angle$ **A** + $\angle$ *a* + $\angle$ **B** + $\angle$ *b* + $\angle$ **C** + $\angle$ *c* = four right angles.
(Theorem 10A.)

(i) It is not possible for $\angle$ s **A**, **B**, **C** to be respectively supplementary to $\angle$ s *a*, *b*, *c*, for this would give $\angle$ **A** + $\angle$ *a* + $\angle$ **B** + $\angle$ *b* + $\angle$ **C** + $\angle$ *c* = six right angles.

(ii) It is not possible for two of the angles **A**, **B**, **C** to be respectively supplementary to two of the angles *a*, *b*, *c*, and the remaining angles to be equal, for then $\angle$ **A** + $\angle$ *a* + $\angle$ **B** + $\angle$ *b* + $\angle$ **C** + $\angle$ *c* would necessarily be greater than four right angles.

(iii) Hence at least two of the angles **A**, **B**, **C** must be respectively equal to two of the angles *a*, *b*, *c*.

But in this case the third angles are also equal.

Hence the two triangles are equiangular, and therefore similar.

Note.—Theorem 18D may obviously be stated as follows:—*If two triangles have two angles in the one triangle respectively equal to two angles in the other triangle, then the two triangles shall be similar.*

THEOREM 19D.

If in two triangles two sides of the one are proportional to two sides of the other, and if also the angles included between these proportional sides are equal, then the triangles shall be similar.

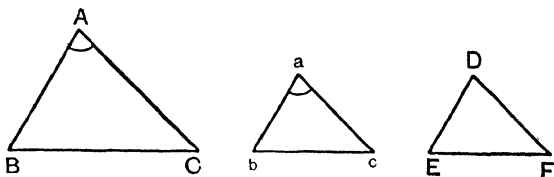


Fig. 22.

Given that In the triangles **ABC** and *abc*,

$$\mathbf{AB : ab = AC : ac, \text{ and } \angle BAC = \angle bac.}$$

Required to prove that The triangles **ABC** and *abc* are similar.

Construction. Draw the triangle **DEF**, making **DE = ab**,
 $\angle \mathbf{EDF} = \angle \mathbf{BAC}$, and $\angle \mathbf{DEF} = \angle \mathbf{ABC}$.

Then also $\angle \mathbf{DFE} = \angle \mathbf{ACB}$. (*Theorem 10A.*)

Proof. Because $\triangle$ s **ABC** and **DEF** are equiangular,

$$\therefore \frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}}, \text{ that is, } \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{AC}}{\mathbf{DF}}. \text{ (Theorem 18D.)}$$

$$\text{But } \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{AC}}{\mathbf{ac}}, \therefore \mathbf{DF} = \mathbf{ac}.$$

Again, $\angle \mathbf{EDF} = \angle \mathbf{BAC}$ and $\angle \mathbf{bac} = \angle \mathbf{BAC}$,

$$\therefore \angle \mathbf{EDF} = \angle \mathbf{bac}.$$

Again, in the triangles *abc* and **DEF**,

$$\therefore \mathbf{ab} = \mathbf{DE}, \mathbf{ac} = \mathbf{DF}, \text{ and } \angle \mathbf{bac} = \angle \mathbf{EDF},$$

$$\therefore \triangle \mathbf{abc} \text{ is equiangular to } \triangle \mathbf{DEF}. \text{ (Theorem 11A.)}$$

But $\triangle \mathbf{ABC}$ is equiangular to $\triangle \mathbf{DEF}$,

$$\therefore \triangle \mathbf{ABC} \text{ is equiangular to } \triangle \mathbf{abc}.$$

Hence $\triangle$ s **ABC and *abc* are similar.**

Note.—Compare this with Theorem 11A. Also note the use of Theorem 11A in the proof.

THEOREM 20D.

If the three sides of one triangle are respectively proportional to the three sides of another, the two triangles shall be similar.

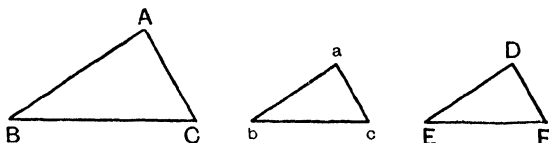


Fig. 23.

Given that In the triangles **ABC** and *abc*,

$$\mathbf{AB : ab = BC : bc = CA : ca.}$$

Required to prove that $\triangle$ s **ABC** and *abc* are similar.

Construction. Draw the triangle **DEF**, making

$$\mathbf{DE = ab, \angle EDF = \angle BAC, \text{ and } \angle DEF = \angle ABC.}$$

Then also $\angle DFE = \angle ACB$. (*Theorem 10A.*)

Proof. Because $\triangle$ s **ABC**, **DEF** are equiangular,

$$\therefore \frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{BC}}{\mathbf{EF}} = \frac{\mathbf{CA}}{\mathbf{FD}}. \quad (\textit{Theorem 18D.})$$

But $\mathbf{DE = ab}$, (*Cons.*)

$$\therefore \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{BC}}{\mathbf{EF}} = \frac{\mathbf{CA}}{\mathbf{FD}}.$$

$$\text{But } \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{BC}}{\mathbf{bc}} = \frac{\mathbf{CA}}{\mathbf{ca}}. \quad (\textit{Hyp.})$$

$$\therefore \mathbf{EF = bc \text{ and } FD = ca.}$$

Thus $\triangle abc \equiv \triangle DEF$, (*Theorem 15A.*)

that is, $\triangle abc$ is equiangular to $\triangle DEF$.

But $\triangle \mathbf{ABC}$ is equiangular to $\triangle DEF$,

$\therefore \triangle \mathbf{ABC}$ is equiangular to $\triangle abc$.

Hence $\triangle$ s **ABC and *abc* are similar.**

Notes.—(1) Note that Theorem 20D is the converse of Theorem 18D.

(2) Compare this with Theorem 15A. Also note the use of Theorem 15A in the proof.

THEOREM 21D.

If the hypotenuse and one side of one right-angled triangle are respectively proportional to the hypotenuse and one side of another right-angled triangle, then the two triangles shall be similar.

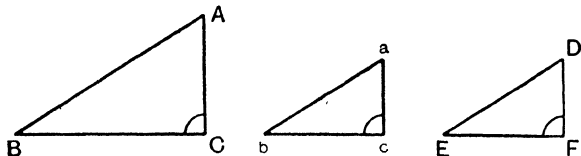


Fig. 24.

Given that In the triangles **ABC**, **abc**, $\mathbf{AB} : \mathbf{ab} = \mathbf{AC} : \mathbf{ac}$,
and $\angle \mathbf{BCA} = \angle \mathbf{bca} = \text{a right angle}$.

Required to prove that The triangles **ABC**, **abc** are similar.

Construction. Draw the triangle **DEF**, making
 $\mathbf{DE} = \mathbf{ab}$, $\angle \mathbf{EDF} = \angle \mathbf{BAC}$, and $\angle \mathbf{DEF} = \angle \mathbf{ABC}$.

Then $\angle \mathbf{DFE} = \angle \mathbf{ACB} = \text{a right angle}$. (*Theorem 10A.*)

Proof. Because $\triangle$ s **ABC**, **DEF** are equiangular,

$$\therefore \frac{\mathbf{AB}}{\mathbf{DE}} = \frac{\mathbf{AC}}{\mathbf{DF}}. \quad (\text{Theorem 18D.})$$

$$\text{But } \mathbf{DE} = \mathbf{ab}, \therefore \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{AC}}{\mathbf{DF}}.$$

$$\text{But } \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{AC}}{\mathbf{ac}}, \therefore \mathbf{DF} = \mathbf{ac}.$$

In the triangles **abc**, **DEF**,

$$\therefore \mathbf{ab} = \mathbf{DE}, \mathbf{ac} = \mathbf{DF}, \text{ and } \angle \mathbf{acb} = \angle \mathbf{DFE} = \mathbf{R},$$

$$\therefore \triangle \mathbf{abc} \equiv \triangle \mathbf{DEF}, \quad (\text{Theorem 16A.})$$

that is, $\triangle \mathbf{abc}$ is equiangular to $\triangle \mathbf{DEF}$.

But $\triangle \mathbf{ABC}$ is equiangular to $\triangle \mathbf{DEF}$,

$\therefore \triangle \mathbf{ABC}$ is equiangular to $\triangle \mathbf{abc}$.

Hence $\triangle$ s **ABC and **abc** are similar.**

Note.—Compare this with Theorem 16A. Also note the use of Theorem 16A in the proof.

THEOREM 22D.

If two sides of one triangle are proportional to two sides of another, and if the angles opposite to one pair of corresponding sides are equal, then shall the angles opposite to the other pair of corresponding sides be either equal or supplementary.

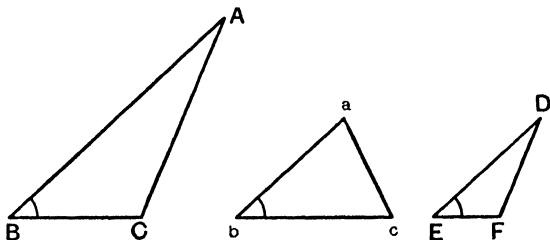


Fig. 25.

Given that In the triangles **ABC**, **abc**,
 $AB : ab = AC : ac$, and $\angle ABC = \angle abc$.

Required to prove that $\angle ACB$ and $\angle acb$ are either equal or supplementary.

Construction. Draw the triangle **DEF**, making
 $DE = ab$, $\angle EDF = \angle BAC$, and $\angle DEF = \angle ABC$.
 Then also $\angle DFE = \angle ACB$. (*Theorem 10A.*)

Proof. $\because \triangle s$ **ABC**, **DEF** are equiangular, $\therefore \frac{AB}{DE} = \frac{AC}{DF}$.

Now $DE = ab$, $\therefore \frac{AB}{ab} = \frac{AC}{DF}$. But $\frac{AB}{ab} = \frac{AC}{ac}$, $\therefore ac = DF$.

Also $\angle abc = \angle ABC = \angle DEF$.

In $\triangle s$ **abc**, **DEF**, $\because ab = DE$, $ac = DF$, and $\angle abc = \angle DEF$,
 $\therefore \angle acb$ and $\angle DFE$ are either equal or supplementary.
 (*Theorem 17A.*)

But $\angle DFE = \angle ACB$.

Hence $\angle ACB$ and $\angle acb$ are either equal or supplementary.

Note.—Compare this with Theorem 17A. Also note the use of Theorem 17A in the proof.

THEOREM 23D.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.

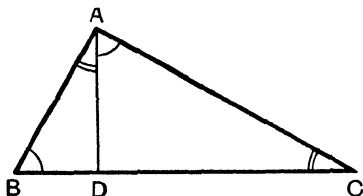


Fig. 26.

Given that In the triangle **ABC**, $\angle BAC$ is a right angle,
and **AD** is perpendicular to **BC**.

Required to prove that The triangles **ABC**, **DBA**, **DAC**
are similar.

Proof. In $\triangle$ s **ABC**, **DBA**, $\angle ABC = \angle DBA$,
and $\angle BAC = \text{a right angle} = \angle BDA$.

Therefore $\triangle$ s **ABC**, **DBA** are equiangular. (*Theorem 10A.*)

In $\triangle$ s **ABC**, **DAC**, $\angle ACB = \angle DCA$,
and $\angle BAC = \text{a right angle} = \angle ADC$.

Therefore $\triangle$ s **ABC**, **DAC** are equiangular. (*Theorem 10A.*)

$\therefore$ the triangles **ABC**, **DBA**, **DAC** are equiangular.

Hence the triangles ABC, DBA, DAC are similar.

COROLLARIES TO THEOREM 23D.

COROLLARY 1.—**DA** is the mean proportional between **DB**
and **DC**, since, by Theorem 18D, $DB : DA = DA : DC$.

COROLLARY 2.—**BA** is the mean proportional between **BD**
and **BC**, since, by Theorem 18D, $BD : BA = BA : BC$.

COROLLARY 3.—**CA** is the mean proportional between **CD**
and **CB**, since, by Theorem 18D, $CD : CA = CA : CB$.

Note.—These three results are easy to remember if it is noted that the three lines mentioned in Corollary 1 have a common extremity at **D**, the three mentioned in Corollary 2 have a common extremity at **B**, and those in Corollary 3 have a common extremity at **C**.

THEOREM 24D.

Two similar polygons can always be divided into similar triangles.

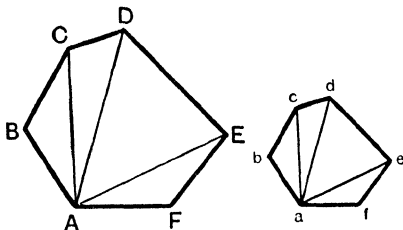


Fig. 27.

Given that **ABCDEF . . .** and **abcdef . . .** are similar polygons whose ratio of similitude is k , and in which the vertices **A, B, C, . . .** correspond respectively to the vertices **a, b, c, . . .**

Required to prove that The two polygons can be divided into similar triangles.

Construction. Join **AC, AD, . . .**, **ac, ad, . . .**

Proof. In the triangles **ABC, abc**, $\angle \mathbf{ABC} = \angle \mathbf{abc}$,

$$\text{and } \frac{\mathbf{AB}}{\mathbf{ab}} = \frac{\mathbf{BC}}{\mathbf{bc}} = k. \quad (\text{Hyp.})$$

$\therefore \triangle \mathbf{s} \mathbf{ABC}$ and **abc** are similar, (*Theorem 19D.*)

whence $\angle \mathbf{BCA} = \angle \mathbf{bca}$, and $\frac{\mathbf{AC}}{\mathbf{ac}} = k$.

Because $\angle \mathbf{BCD} = \angle \mathbf{bcd}$, and $\angle \mathbf{BCA} = \angle \mathbf{bca}$,

$$\therefore \angle \mathbf{ACD} = \angle \mathbf{acd}.$$

$$\text{Also } \frac{\mathbf{AC}}{\mathbf{ac}} = k = \frac{\mathbf{CD}}{\mathbf{cd}}.$$

$\therefore \triangle \mathbf{s} \mathbf{ACD}$ and **acd** are similar, (*Theorem 19D*)

whence $\angle \mathbf{CDA} = \angle \mathbf{cda}$, and $\frac{\mathbf{AD}}{\mathbf{ad}} = k$.

In the same way $\triangle \mathbf{s} \mathbf{ADE}$ and **ade** may be proved similar, and so on.

Hence the two polygons ABCDEF . . ., abcdef . . ., have been divided into similar triangles.

THEOREM 25D.

Two similar polygons can always be placed so as to have a centre of similitude.

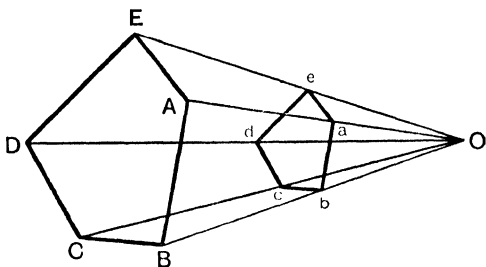


Fig. 28.

Given that **ABCDE** and **abcde** are two similar polygons
in which the vertices **A, B, C, . . .** correspond
respectively to the vertices **a, b, c, . . .**

Required to prove that The two polygons can be placed
so as to have a centre of similitude.

Construction. Place the two polygons so that corresponding sides **AB** and **ab** are parallel; and so that **BC** and **bc** are on the *same* sides of **AB** and **ab** respectively if **AB** and **ab** are drawn in the *same* sense (Fig. 28); or so that **BC** and **bc** are on *opposite* sides of **AB** and **ab** respectively, if **AB** and **ab** are drawn in *opposite* senses (Fig. 29).

Let **Aa** and **Bb** meet in **O**, and let **OC** meet **bc** at **c'**.

Proof. In either figure, $\angle ABC = \angle abc$, (*Hyp.*)
and $\angle ABO = \angle abO$; (*Theorem 7A or 6A.*)

$$\therefore \angle OBC = \angle Obc,$$

whence **BC** is parallel to **bc**.

(*Theorem 5A or 4A.*)

Because **AB** is parallel to *ab*, (Cons.)
 $\therefore$ the $\triangle$ s **OAB** and **Oab** are equiangular,
 and they are therefore similar. (*Theorem 18D, Cor. 1.*)
 Also, because **BC** is parallel to *bc'*, (Proved.)
 $\therefore$ the $\triangle$ s **OBC** and **Obc'** are equiangular,
 and they are therefore similar. (*Theorem 18D, Cor. 1.*)

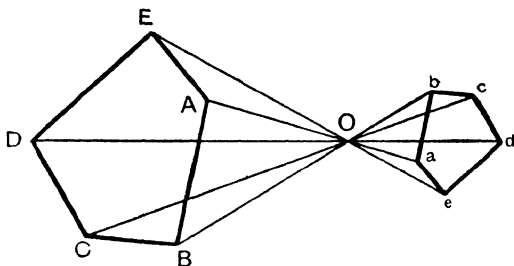


Fig. 29.

$$\text{Hence } \frac{AB}{ab} = \frac{OB}{Ob} = \frac{BC}{bc'}.$$

$$\text{But } \frac{AB}{ab} = \frac{BC}{bc}, \quad (\text{Hyp.})$$

whence $bc' = bc$.

Hence c' coincides with c ; that is to say, **OC**
 passes through c .

Similarly, it can be proved that **CD** is parallel to *cd*,
 and **OD** passes through d ; and so on.

**Hence O is a centre of similitude for the polygons
 ABCDE, abcde.**

COROLLARY TO THEOREM 25D.

$$\frac{OA}{Oa} = \frac{OB}{Ob} = \frac{OC}{Oc} = \dots = \frac{AB}{ab} = \text{ratio of similitude.}$$

THEOREM 26D.

If two parallel straight lines are cut by a series of transversals which all pass through one point, then the intercepts formed on the two parallel straight lines are proportional.

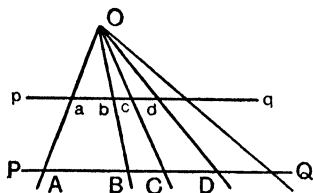


Fig. 30.

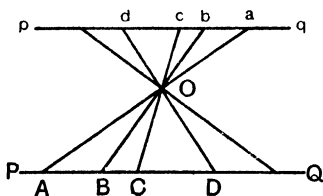


Fig. 31.

Given that PQ and pq are two parallel straight lines, and that $OA, OB, OC, \dots$ are a number of transversals cutting PQ in $A, B, C, \dots$ and pq in $a, b, c, \dots$

Required to prove that $AB : ab = BC : bc = CD : cd = \dots$

Proof.

Because PQ is parallel to pq ,

$$\therefore \angle OAB = \angle Oab \text{ and } \angle OBA = \angle Oba.$$

Hence $\triangle$ s OAB and Oab are similar,

$$\text{and } \frac{AB}{ab} = \frac{OB}{Ob}. \quad (\text{Theorem 18D.})$$

Similarly, $\angle OBC = \angle Obc$, and $\angle OCB = \angle Ocb$.

Hence $\triangle$ s OBC and Obc are similar,

$$\text{and } \frac{BC}{bc} = \frac{OB}{Ob}. \quad (\text{Theorem 18D.})$$

$$\text{Hence } \frac{AB}{ab} = \frac{BC}{bc}.$$

$$\text{Similarly } \frac{BC}{bc} = \frac{OC}{Oc}, \text{ and } \frac{CD}{cd} = \frac{OC}{Oc}.$$

$$\text{Hence } \frac{BC}{bc} = \frac{CD}{cd}, \text{ and so on.}$$

Hence the intercepts made on PQ are proportional to the intercepts made on pq .

THEOREM 27D.

The locus of a point within a given angle whose distances from the arms of the angle are in a constant ratio is a straight line through the vertex of the angle.

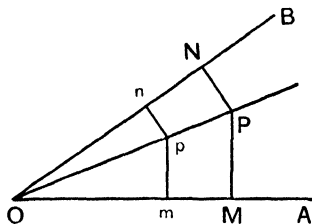


Fig. 32.

Given that The point **P** moves within the angle **AOB** in such a way that its distances from **OA** and **OB** are in a constant ratio k .

Required to prove that The locus of **P** is a straight line through **O**.

Construction. Let p be any one position of the moving point.

Take any other point **P** on **Op**, or **Op** produced.

Let pm , **PM**, pn , **PN** be the perpendicular distances of p and **P** from **OA**, **OB**.

Then $pm : pn = k$.

Proof. In the triangles pOm and **POM**,

$\angle pOm = \angle POM$,

and $\angle pmO = \text{a right angle} = \angle PMO$;

$\therefore$ the triangles pOm and **POM** are similar,

and $\frac{pm}{PM} = \frac{pO}{PO}$. (Theorem 18D.)

Similarly, $\frac{pn}{PN} = \frac{pO}{PO}$. (Theorem 18D.)

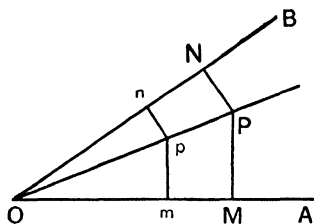


Fig. 32.

$$\therefore \frac{pm}{PM} = \frac{pn}{PN},$$

whence $\frac{PM}{PN} = \frac{pm}{pn} = k$. (Theorem 3D.)

Thus any point on Op satisfies the required condition.

Hence Op is the required locus.

COROLLARY TO THEOREM 27D.

If $PM = k \cdot PN + l$, where k and l are constants, then the locus of P is still a straight line.

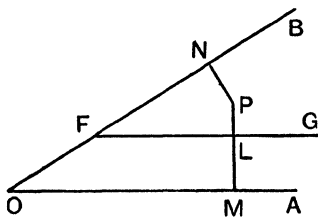


Fig. 33.

Let FG be a fixed line drawn parallel to OA at a distance l ; also let FG meet PM at L .

Then $LM = l$.

Thus, if $PM = k \cdot PN + l$, $PL = PM - l = k \cdot PN$.

Hence the locus of P is a straight line through F , where F is at a distance l from OA .

10. Similar Figures other than Triangles, Quadrilaterals, and Polygons.—In the preceding portion of this chapter similar triangles, quadrilaterals, and polygons have been dealt with. To complete the theory of similar figures, however, reference should be made to any two similar aggregations of points, whether the points are isolated or form continuous lines, straight or curved. This would include, for example, two maps of the same country drawn to different scales, as shown in Fig. 34.

Two geometrical figures are similar if to any point on the one figure there corresponds one point on the other, and if the ratio of the join of any two points on the one figure to the join of the corresponding two points on the other figure is constant. This constant ratio is the **ratio of similitude**. Thus, in Fig. 34, the points *A, B, C, D, E . . .* on the one map correspond respectively to the points *a, b, c, d, e . . .* on the other, and

$$\frac{AB}{ab} = \frac{AC}{ac} = \frac{BE}{be} = \frac{CD}{cd} = \dots = 1.6.$$

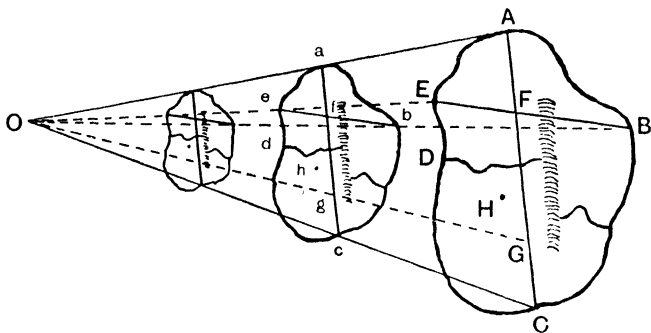


Fig. 34.

Hence the two maps are similar, and the ratio of similitude is 1.6. Similarly, each of the two larger maps is similar to the smallest map.

In similar figures corresponding lines are paths traced by corresponding points. Further, corresponding lines intersect at corresponding points and corresponding angles are equal.

Exercises II.

1. Two isosceles triangles whose vertical angles are equal have their altitudes proportional to their bases.

2. A quadrilateral $ABCD$ is inscribed in a circle, and the two sides BA and CD are produced to meet in O ; prove that the triangles OAD and OBC are similar.

3. If two straight lines AOD , BOC cut one another internally or externally in O , so that $OA : OB = OC : OD$, prove that A, B, C, D are concyclic.

4. ABC is any triangle, and the lines AD, BE, CF are drawn making the angles BAD, CBE, ACF all equal to one another; prove that the lines either meet in a point or else form a triangle similar to the triangle ABC .

5. If two circles touch each other either externally or internally, any two straight lines drawn through the point of contact will be cut proportionally by the circumferences.

6. If AD, BE be drawn from the vertices A and B of the triangle ABC perpendicular to the opposite sides, and if the perpendiculars cut each other in O , prove that $BD : DO = AD : DC$.

7. BC is any chord parallel to the tangent at A to a circle, and a line AD meets BC in E and the circle in D ; prove that

$$AE : AB = AB : AD.$$

8. $ABCD$ is a parallelogram, and P and Q are any two points in AD and CD . PM, QN are any two parallel lines meeting AB in M , BC in N respectively; prove that $AM \cdot CN$ is constant.

9. Prove the following construction for inscribing a square in a triangle ABC . On the side BC remote from A describe a square $BCDE$. Join AD, AE meeting BC in F, G . The square on FG is the square required.

10. A common tangent to two circles cuts the join of the centres externally or internally in the ratio of the radii.

11. Two circles intersect in A and B . Any chord CBD cuts the circles in C and D . Prove that the ratio $AC : AD$ is constant for all positions of the chord.

12. ABC is a triangle right-angled at A ; a point D is taken in the hypotenuse BC such that $CB : BA = BA : BD$. Prove that AD is perpendicular to BC .

13. The line AB being divided at the points C and D , so that $AB : AC = AC : AD$, and any line AE (equal to AC) being drawn through A , prove that EC bisects $\angle BED$.

14. ABC is a straight line, and BD, CE are any two parallel lines on the same side of ABC . Points D and E are taken such that $BD : CE = AB : AC$; prove that ADE is a straight line.

15. Show how to divide an arc of a circle into two parts whose chords are in a given ratio.

16. From a given point on the circumference of a circle draw two chords which are in a given ratio and contain a given angle.

17. ABC is a triangle which is not obtuse-angled. If the perpendicular AD from A to BC is a mean proportional between BD and DC , prove that $\angle BAC$ is a right angle.

18. Find the locus of points at which two given circles subtend equal angles.

19. Find a point at which three given circles subtend the same angle.

20. $ABCD$ is a parallelogram, L , M , and P are points in AD , CD , BD respectively, LP and MP meet BC , BA respectively in N and O . Prove that LM is parallel to ON .

21. $ABCD$ is a quadrilateral. On the side of AB remote from C , $\angle BAE$ is made equal to $\angle CAD$, $\angle ABE$ equal to $\angle ADC$, and EC is joined. Prove that $\angle ECA = \angle BDA$.

22. $ABCD$ is a parallelogram, and P and Q are any two points such that PQ is parallel to AB , PA and QB meet at R , and PD and QC meet at S ; show that RS is parallel to AD .

23. Any line parallel to the base of a triangle is bisected by the median.

24. The diagonals of a trapezium cut each other in the same ratio.

25. ABC is a triangle, having the angle BAC a right angle, and CD and CE the internal and external bisectors of the angle at C , cutting AB in D and E respectively. Show that AC is a mean proportional between AD and AE .

26. In a right-angled triangle the perpendicular on the hypotenuse is a fourth proportional to the hypotenuse and the other two sides.

27. The straight lines which join corresponding angles of two similar triangles whose corresponding sides are parallel will meet in a point.

28. If a square $DEFG$ is inscribed in a right-angled triangle ABC so that a side DE of the square lies along the hypotenuse BC , prove that DE is a mean proportional to BD and EC .

29. Two circles with centres A and B touch externally at C . P is a point such that PC bisects $\angle APB$. If PD , PE are tangents to the circles, prove that PC bisects $\angle DPE$.

30. Determine a point whose perpendicular distances from the three sides of a triangle are in the ratio $1 : 2 : 3$.

31. A square is described with one side always along a given line and one corner always on another; find the locus of the vertex which lies on neither.

32. If two triangles ABC and DEF are on equal bases BC and EF , and between the same parallels AD and BF , any straight line drawn parallel to AD will cut off equal areas from ABC and DEF .

33. If two circles touch one another externally their common tangent is a mean proportional to their diameters.

34. If one of the sides of a right-angled triangle is double the other, prove that the perpendicular from the right angle to the hypotenuse divides it in the ratio 4 : 1.

35. The radius of a circle is a mean proportional between the segments of any tangent made by its point of contact and a pair of parallel tangents.

36. BC is a diameter of a circle and A is any point on the circumference. A point D is taken in BC and a line DE is drawn perpendicular to BC meeting BA , the circle, and CA in E , F , G respectively. Prove that $DF^2 = DE \cdot DG$.

37. Given the base and the vertical angle of a triangle, find the locus of the point of intersection of the medians.

38. Construct a right-angled triangle such that one side shall be a mean proportional between the hypotenuse and the other side.

39. A , B , C , D are collinear points and on AC and BD two similar triangles APC and BQD are described, so that the corresponding sides AP and BQ , and again CP and DQ , are parallel. If QP and DA intersect in O , prove that $OA : OB = OC : OD$.

40. The line joining any point P on a given circle to a fixed point Q is divided in a given ratio at R . Prove that the locus of R is a circle.

41. $ABCD$ and $EFGH$ are two equiangular quadrilaterals, and $AB : EF = CD : GH$. Prove that the quadrilaterals are similar. Is there any exception to this?

42. $ABCD$ is a quadrilateral, and P , Q , R , S are points on AB , BC , CD , DA such that $AP : AB = CQ : CB = CR : CD = AS : AD$. Prove that $PQRS$ is a parallelogram.

CHAPTER III.

PROPORTION THEOREMS ON AREAS.

11. On Algebraical Operations in Geometry.—It is often convenient in geometrical argument to allow the symbol **AB** to denote either the *line AB* or the *measure* of the line **AB** in terms of some unit of length, and similarly to allow the symbol $\triangle \mathbf{ABC}$ to denote either the *triangle* or its *measure* in terms of some unit of area. The value of this practice lies in the fact that any mathematical operation (such as multiplication or the extraction of a root) may be performed with the *measures*, as these are abstract numbers.

This practice does not lead to confusion of ideas or to incorrect argument. It may be used in either of the following cases:—

(1) *In equations involving only ratios.*—

For the symbols $\frac{\mathbf{AB}}{\mathbf{CD}}$ and $\frac{\triangle \mathbf{ABH}}{\triangle \mathbf{CDK}}$ have the same values whether they refer to the ratios of these lengths and areas, or whether they refer to the ratios of the *measures* of these lengths and areas.

(2) *In equations involving mensuration formulae.*—

For the fact that the measure of the area of a parallelogram is the product of the measures of its base and altitude is correctly expressed by the equation

$$\text{rect. } \mathbf{BADC} = \mathbf{AB} \times \mathbf{AD},$$

provided that the measures of these quantities are expressed in corresponding units of length and area; also all other mensuration formulae in areas are derived from this formula.

THEOREM 28D.

Triangles of equal altitude are proportional to their bases; and triangles on equal bases are proportional to their altitudes.

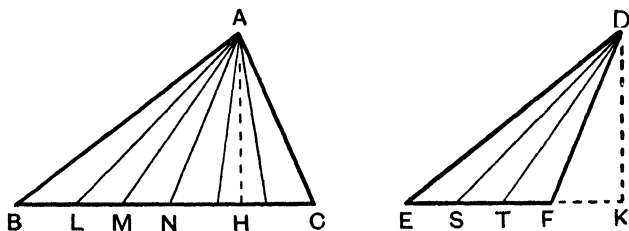


Fig. 35.

PART I.

Given that In $\triangle$ s **ABC**, **DEF**, altitude **AH** = altitude **DK**.

Required to prove that

$$\triangle \text{ABC} : \triangle \text{DEF} = \text{base BC} : \text{base EF}.$$

Construction. Assuming that **BC** and **EF** are commensurable, there is some length **P** which is contained an exact number of times in each of them.

Suppose **BC** and **EF** divided respectively into r and s parts each equal to **P**.

In each triangle join the points of section of the base to the vertex.

Proof. $\triangle$ s **ABL**, **ALM**, **AMN**, ..., **DES**, **DST**, ...

are on equal bases and of equal altitude,

hence they are all of equal area. (*Theorem 39A.*)

Thus $\triangle \text{ABC} = r \cdot \triangle \text{ABL}$, and $\triangle \text{DEF} = s \cdot \triangle \text{ABL}$

$$\therefore \frac{\triangle \text{ABC}}{\triangle \text{DEF}} = \frac{r}{s}.$$

Again, $\because \text{BC} = r \text{ P}$ and $\text{EF} = s \text{ P}$, $\therefore \frac{\text{BC}}{\text{EF}} = \frac{r}{s}.$

$$\text{Hence } \frac{\triangle \text{ABC}}{\triangle \text{DEF}} = \frac{\text{BC}}{\text{EF}}.$$

PART II.

Given that In $\triangle s$ ABC , DEF , base $BC =$ base EF .

Required to prove that $\triangle ABC : \triangle DEF =$ altitude $AH :$
altitude DK .

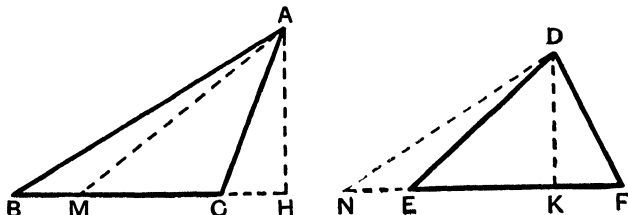


Fig. 36.

Construction. Let $HM = KN = BC = EF$. Join AM , DN .

Proof. $\because MH = BC, \therefore \triangle AMH = \triangle ABC$. (Theorem 39A.)

$\because NK = EF, \therefore \triangle DNK = \triangle DEF$. (Theorem 39A.)

$\because \triangle s$ AMH , DNK have equal altitudes MH , NK ,

$\therefore$ they are proportional to their bases AH , DK , (Proved.)

i.e. $\triangle AMH : \triangle DNK = AH : DK$.

Hence $\triangle ABC : \triangle DEF = AH : DK$.

 COROLLARY TO THEOREM 28D.

Parallelograms of equal altitude are proportional to their bases; and parallelograms on equal bases are proportional to their altitudes.

For the area of a parallelogram is twice the area of a triangle having the same base and altitude.

Note.—The following is an alternative proof of Theorem 28D:—

Let b_1, b_2 be the bases of the two triangles; h_1, h_2 their respective altitudes; and Δ_1, Δ_2 their respective areas.

Then, by Theorem 40A, Cor., $\Delta_1 = \frac{1}{2} b_1 h_1$, $\Delta_2 = \frac{1}{2} b_2 h_2$.

$$\text{Thus } \frac{\Delta_1}{\Delta_2} = \frac{b_1 h_1}{b_2 h_2}.$$

Thus if $h_1 = h_2$, then $\frac{\Delta_1}{\Delta_2} = \frac{b_1}{b_2}$; or if $b_1 = b_2$, then $\frac{\Delta_1}{\Delta_2} = \frac{h_1}{h_2}$.

A similar proof obviously holds in the case of parallelograms.

THEOREM 29D.

If two triangles have one angle of the one equal to one angle of the other, their areas are in the ratio of the products of the sides containing the equal angles.

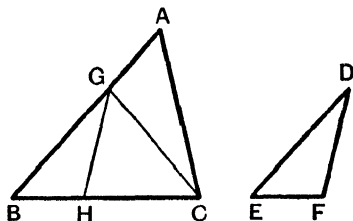


Fig. 37.

Given that In the triangles **ABC** and **DEF**,
 $\angle ABC = \angle DEF$.

Required to prove that $\triangle ABC : \triangle DEF$
 $= AB \cdot BC : DE \cdot EF$.

Construction. Let $\triangle DEF$ be placed so that the arms of the angle **E** lie along the arms of the equal angle **B**, and let **GBH** represent $\triangle DEF$ in its new position.

Join **GC**.

Proof. Because $\triangle$ s **ABC**, **GBC** on the bases **AB**, **GB**, have the same altitude,

$$\therefore \frac{\triangle ABC}{\triangle GBC} = \frac{AB}{GB}. \quad (\text{Theorem 28D.})$$

Because $\triangle$ s **GBC**, **GBH** on the bases **BC**, **BH**, have the same altitude,

$$\therefore \frac{\triangle GBC}{\triangle GBH} = \frac{BC}{BH}. \quad (\text{Theorem 28D.})$$

$$\therefore \frac{\triangle ABC}{\triangle GBC} \cdot \frac{\triangle GBC}{\triangle GBH} = \frac{AB}{GB} \cdot \frac{BC}{BH},$$

$$\text{i.e. } \frac{\triangle ABC}{\triangle GBH} = \frac{AB \cdot BC}{GB \cdot BH}.$$

But $GB = DE$, and $BH = EF$.

$$\text{Hence } \frac{\triangle ABC}{\triangle GBH} = \frac{AB \cdot BC}{DE \cdot EF}.$$

COROLLARIES TO THEOREM 29D.

COROLLARY 1.—*If two parallelograms have one angle of the one equal to one angle of the other, their areas are in the ratio of the products of the sides containing the equal angles.*

This follows at once from the fact that either diagonal of a parallelogram bisects the parallelogram.

COROLLARY 2.—*Triangles (or parallelograms) which are equal in area and which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional.*

Taking two triangles ABC and DEF , of equal area, in which $\angle ABC = \angle DEF$,

$$\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot BC}{DE \cdot EF}.$$

But the triangles are equal in area,

hence $AB \cdot BC = DE \cdot EF$,

$$\therefore \frac{AB}{DE} = \frac{EF}{BC} \text{ or } \frac{AB}{EF} = \frac{DE}{BC}.$$

Hence the sides about the equal angles are reciprocally proportional.

COROLLARY 3.—*Triangles (or parallelograms) which have an angle of the one equal to an angle of the other and the sides about these angles reciprocally proportional are equal in area.*

Note.—In Theorem 29D we have used the notation explained justified in Art. 11.

THEOREM 30D.

Similar triangles are proportional to the squares on corresponding sides.

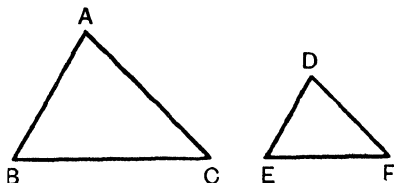


Fig. 38.

Given that $\triangle ABC$ is similar to $\triangle DEF$.

Required to prove that $\triangle ABC : \triangle DEF = BC^2 : EF^2$.

Proof. Because $\triangle ABC$ is similar to $\triangle DEF$,

$$\therefore \frac{AB}{DE} = \frac{BC}{EF}.$$

Because $\angle ABC = \angle DEF$,

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot BC}{DE \cdot EF} \quad (\text{Theorem 29D.})$$

$$\text{or } \frac{\triangle ABC}{\triangle DEF} = \frac{AB}{DE} \cdot \frac{BC}{EF}.$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF} \cdot \frac{BC}{EF}.$$

$$\text{Hence } \frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}.$$

Note.—Euclid worded this theorem as follows: “Similar triangles are to one another in the duplicate ratio of their homologous sides” (i.e. corresponding sides). It will be remembered that the duplicate ratio of BC to EF is the same as the ratio of BC^2 to EF^2 .

THEOREM 31D.

Similar polygons are proportional to the squares on corresponding sides.

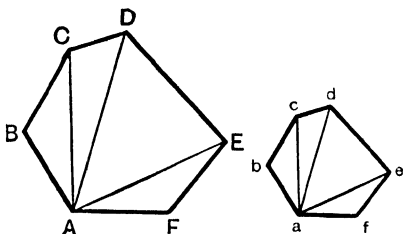


Fig. 39.

Given that **ABCDEF**, *abcdef* are similar polygons.

Required to prove that **ABCDEF** : *abcdef* = **AB**² : *ab*².

Proof. Since the polygons are similar they can be divided into similar triangles **AFE**, *afe*, etc. (*Theorem 24D.*)

$\therefore \triangle \mathbf{AFE}$ is similar to $\triangle afe$,

$\therefore \triangle \mathbf{AFE} : \triangle afe = \mathbf{AF}^2 : af^2$. (*Theorem 30D.*)

But **AF** : *af* = **AB** : *ab* ;

$\therefore \mathbf{AF}^2 : af^2 = \mathbf{AB}^2 : ab^2$. (*Theorem 8D.*)

Hence $\triangle \mathbf{AFE} : \triangle afe = \mathbf{AB}^2 : ab^2$.

Similarly $\triangle \mathbf{AED} : \triangle aed = \mathbf{AB}^2 : ab^2$,

and $\triangle \mathbf{ADC} : \triangle adc = \mathbf{AB}^2 : ab^2$,

and $\triangle \mathbf{ACB} : \triangle acb = \mathbf{AB}^2 : ab^2$.

Hence **ABCDEF** : *abcdef* = **AB**² : *ab*². (*Theorem 5D.*)

COROLLARIES TO THEOREM 31D.

COROLLARY 1.—*If a pair of similar rectilinear figures are described on two given straight lines, and if another pair of similar rectilinear figures are described on two other straight lines, then (i) if the four lines are proportional, the figures shall be proportional; and, conversely, (ii) if the figures are proportional, the lines shall be proportional.*

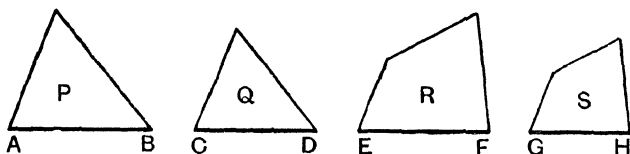


Fig. 40.

For in Fig. 40, $P : Q = AB^2 : CD^2$

and $R : S = EF^2 : GH^2$.

Hence (i) *Given that* $AB : CD = EF : GH$;

then, by Theorem 8D, $AB^2 : CD^2 = EF^2 : GH^2$,

$\therefore P : Q = R : S$;

and (ii) *Given that* $P : Q = R : S$;

then $AB^2 : CD^2 = EF^2 : GH^2$,

$\therefore AB : CD = EF : GH$,

as can be easily proved by the method of exhaustion.

COROLLARY 2.—*If three straight lines are proportional the first is to the third as any polygon described on the first is to a similar and similarly situated polygon described on the second.*

Let M be a third proportional to AB and ab

so that $AB : ab = ab : M$.

Hence $\frac{AB}{M} = \frac{AB^2}{ab^2}$. (Theorem 7D.)

But $\frac{AB}{M} = \frac{\text{first line}}{\text{third line}}$

and $\frac{AB^2}{ab^2} = \frac{\text{polygon on first line}}{\text{similar polygon on second line}}$.

THEOREM 32D.

In a right-angled triangle, any rectilinear figure described on the hypotenuse is equal to the sum of the two similar rectilinear figures described on the sides containing the right angle.

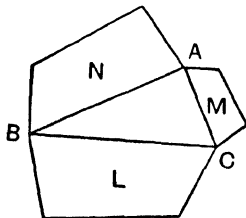


Fig. 41.

Given that The triangle **ABC** is right-angled at **A**, and the figures **L**, **N**, **M** are similar.

Required to prove that Area **L** = area **N** + area **M**.

Proof. $N : L = AB^2 : BC^2$, (Theorem 31D.)

and $M : L = AC^2 : BC^2$ (Theorem 31D.)

$\therefore N + M : L = AB^2 + AC^2 : BC^2$. (Theorem 10D.)

But $BC^2 = AB^2 + AC^2$. (Theorem 42A.)

Hence $L = N + M$.

Notes.—(1) Theorem 32D is obviously an extension and generalisation of the Theorem of Pythagoras (Theorem 42A). Theorem 42A is used in the proof.

(2) Note that the theorem applies to circles: "The area of a circle drawn on the hypotenuse of a right-angled triangle as diameter is equal to the sum of the areas of the circles similarly drawn on the other two sides." Of course areas of circles are proportional to the squares on their diameters.

THEOREM 33D.

If four lines are proportional the rectangle contained by the extremes is equal to the rectangle contained by the means; and conversely.

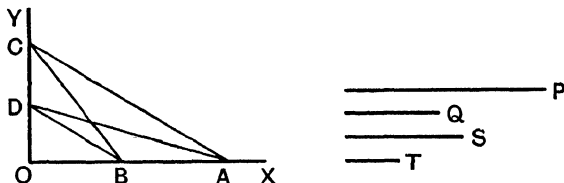


Fig. 42.

PART I.

Given that P, Q, S, T are four straight lines such that
 $P : Q = S : T$.

Required to prove that $\text{Rect. } (P, T) = \text{rect. } (Q, S)$.

Construction. Draw two perpendicular lines OX, OY .

Along OX mark off $OA = P$ and $OB = Q$.

Along OY mark off $OC = S$ and $OD = T$.

Join AC, BD, AD, BC .

Proof.

$$P : Q = S : T;$$

$$\text{i.e. } OA : OB = OC : OD;$$

$$\text{whence } AB : OB = CD : OD. \quad (\text{Theorem 4D.})$$

$$\text{Thus } BD \text{ is parallel to } AC. \quad (\text{Theorem 13D.})$$

$$\therefore \triangle BDA = \triangle BDC. \quad (\text{Theorem 38A.})$$

To each add $\triangle BOD$; thus $\triangle AOD = \triangle BOC$.

$$\therefore 2 \triangle AOD = 2 \triangle BOC,$$

$$\therefore \text{rect. } (AO, OD) = \text{rect. } (BO, OC).$$

$$\text{Hence rect. } (P, T) = \text{rect. } (Q, S).$$

PART II.

Given that P, Q, S, T are four straight lines such that
 rect. $(P, T) = \text{rect. } (Q, S)$.

Required to prove that $P : Q = S : T$.

Construction. As in Part I.

Proof. Rect. $(P, T) = \text{rect. } (Q, S)$,

$\therefore \text{rect. } (AO, OD) = \text{rect. } (BO, OC)$,

i.e. $2 \triangle AOD = 2 \triangle BOC$; (Theorem 26A.)

$\therefore \triangle AOD = \triangle BOC$.

From each take $\triangle BOD$;

thus $\triangle BDA = \triangle BDC$.

Hence BD is parallel to AC . (Theorem 41A, Cor. 1.)

$\therefore AB : OB = CD : OD$, (Theorem 12D.)

$\therefore OA : OB = OC : OD$. (Theorem 4D.)

Hence $P : Q = S : T$.

COROLLARY TO THEOREM 33D.

If three lines are proportional the rectangle contained by the extremes is equal to the square on the mean.

Notes.—(1) It will be remembered that Theorem 33D and the Corollary were indicated in Theorem 6D; for if $A : B = C : D$, A, B, C , and D being numbers, we have $AD = BC$, and if $A : B = B : C$ we have $AC = B^2$.

(2) The student should also prove the following statements :—(a) If two rectangles are equal in area their sides are inversely proportional. (b) If four lines are proportionals either the two extremes or the two means may be interchanged. (c) If the square on a line is equal to the rectangle contained by two other lines the first line is the mean proportional between the other two. (d) If the areas of four polygons are proportionals either the two extremes or the two means may be interchanged.

THEOREM 34D.

In equal circles the areas of sectors are proportional to their angles and also to their arcs.

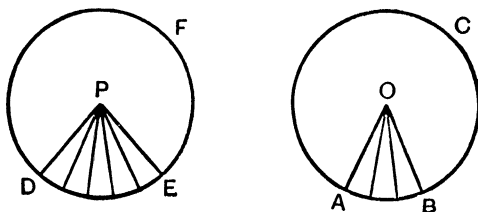


Fig. 43.

Given that **ABC** and **DEF** are two equal circles of which **O** and **P** are the respective centres.

Required to prove that Area of sector **AOB** : area of sector **DPE** = $\angle$ **AOB** : $\angle$ **DPE** = arc **AB** : arc **DE**.

Construction. Assuming that the angles **AOB**, **DPE** are commensurable, there is some angle θ which is contained an exact number of times in each of them.

Suppose these angles divided respectively into m and n parts each of magnitude θ .

Proof. The areas **AOB** and **DPE** are divided into m and n sectors respectively.

By superposition these sectors are all congruent. Let the area of each of these congruent sectors be **H**.

$$\therefore \text{area } \mathbf{AOB} = m\mathbf{H}, \text{ and area } \mathbf{DPE} = n\mathbf{H},$$

$$\therefore \frac{\text{area } \mathbf{AOB}}{\text{area } \mathbf{DPE}} = \frac{m}{n}.$$

$$\therefore \angle \mathbf{AOB} = m\theta, \text{ and } \angle \mathbf{DPE} = n\theta,$$

$$\therefore \frac{\angle \mathbf{AOB}}{\angle \mathbf{DPE}} = \frac{m}{n}.$$

$$\text{Also } \frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}} = \frac{\angle \mathbf{AOB}}{\angle \mathbf{DPE}} = \frac{m}{n}. \quad (\text{Theorem 17D.})$$

$$\text{Hence } \frac{\text{area } \mathbf{AOB}}{\text{area } \mathbf{DPE}} = \frac{\angle \mathbf{AOB}}{\angle \mathbf{DPE}} = \frac{\text{arc } \mathbf{AB}}{\text{arc } \mathbf{DE}}.$$

Exercises III.

In these and subsequent Exercises, examples marked with an asterisk are somewhat difficult, and should not be attempted on a first reading of the book.

1. If straight lines **AO**, **BO**, **CO** be drawn from the vertices of a triangle **ABC** to any point **O** within the triangle, and if **AO** produced cuts **BC** in **L**, prove that

$$\triangle AOB : \triangle AOC = BL : LC.$$

2. **ABCD** is a quadrilateral, and its diagonals **AC** and **BD** intersect in **O**. Show that

$$\triangle AOB : \triangle BOC = \triangle AOD : \triangle DOC.$$

3. **ABC** is a triangle. Find a point **P** in **ABC** such that the triangles **PAB**, **PBC**, and **PCA** are all equal.

4. A quadrilateral is divided into four triangles by its diagonals; show that if two of these triangles are equal the remaining two are either equal or similar.

5. In Figs. 4, 5, and 6 join **BE**, **CF**; then prove theorems 12D and 13D by means of Theorem 28D.

6. If the areas of the triangles **ABC**, **DEF** are in the ratio of the rectangles **AB.AC**, **DE.DF** the angles at **A** and **D** are either equal or supplementary.

7. Prove Theorem 29D by the Corollary to Theorem 40A.

8. If two triangles are drawn on opposite sides of the same base, their areas are proportional to the segments into which the base divides the join of their vertices.

9. **ABCD** is a quadrilateral. Prove that if it is possible to find a point **P** inside it such that the triangles **PAB**, **PBC**, **PCD**, **PDA** are all equal, then one diagonal bisects the other.

10. **ABC** is a triangle and **P** is a point inside it. **AP**, **BP**, **CP** meet **BC**, **CA**, **AB** in **D**, **E**, **F** respectively. Prove that $\frac{AP}{AD} + \frac{BP}{BE} + \frac{CP}{CF} = 2$.

11. Triangles, which have one angle of the one supplementary to one angle of the other and the products of the sides about the supplementary angles equal, are equal in area.

12. A trapezium **ABCD** has its sides **AB** and **CD** parallel, and **AB** is three times **CD**. If the diagonals intersect in **O**, show that the triangle **AOB** is nine times the triangle **COD**.

13. Prove that the areas of two similar cyclic figures are proportional to the squares of the radii of their circumcircles.

14. The rectangle contained by two straight lines is a mean proportional between the squares on the lines.

15. ABC is any triangle, and BE and CF are perpendiculars to AC and AB . Prove that (i) $\triangle ABE : \triangle ACF = \text{sq. on } AB : \text{sq. on } AC$;
(ii) $\triangle AEF : \triangle ABC = \text{sq. on } EF : \text{sq. on } BC$.

16. AOC and BOD are two chords of a circle which intersect in O . Prove that $\triangle AOB : \triangle COD = \text{sq. on } OA : \text{sq. on } OD$.

17. Similar triangles are to one another in the ratio of the squares of (1) corresponding medians, (2) the radii of their inscribed circles, (3) the radii of their circumscribed circles.

18. Prove that in similar figures corresponding areas are proportional to the squares of corresponding lengths.

19. Draw a straight line parallel to the base of a triangle to bisect the triangle.

20. Draw a straight line parallel to the base of a triangle, to divide its area into two parts which are in a given ratio.

21. Draw an isosceles triangle equal in area to a given triangle and having the same vertical angle.

22. ABC is a right-angled triangle, and CD is drawn perpendicular to the hypotenuse AB . Prove that
 $AD : DB = AC^2 : CB^2$.

23. ABC is a triangle having the side BC produced to D , and CD is equal to BC ; also AB is bisected in E . Show that the line ED will cut AC in a point P such that

$$EP : PD = CP : PA.$$

24. On AB and AC , two sides of the triangle ABC , squares are described externally, viz. $ABPQ$ and $ACRS$. Prove that the triangle PAS is equal to the triangle QAR in area.

25. The ratio of twice the area of a triangle to the rectangle contained by two of the sides is equal to the ratio of the third side to the diameter of the circumscribed circle of the triangle.

26.* ABC is a triangle, and equal lines AX , BY , CZ are drawn to meet the opposite sides of ABC . If, from any point P within ABC , PL , PM , PN are drawn parallel to AX , BY , CZ , to meet the corresponding sides of ABC , then
 $AX = PL + PM + PN$.

27.* Find the greatest triangle which can be inscribed in a given triangle having one side parallel to the base.

28.* If X and Y are the points of intersection of a variable circle, drawn through two fixed points A and B , with a given circle, then the ratio $AX \cdot AY : BX \cdot BY$ is constant.

CHAPTER IV.

RECTANGLE THEOREMS.

12. Introductory Note.—Before proceeding with the new propositions of this chapter the student should note the following alternative proofs (using the principles of ratio and proportion and similar figures) of Theorems 8C, 10C, and 11C, also given in Chapter XVI. of *The New Matriculation Geometry*.

ALTERNATIVE PROOF OF THEOREM 8C.

If two chords of a circle intersect at a point within the circumference, then the rectangle contained by the segments of the one chord is equal to the rectangle contained by the segments of the other.

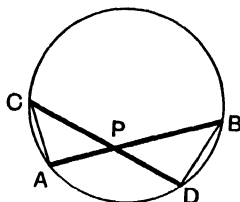


Fig. 44.

Given that The chords **AB** and **CD** of the circle **CAB** intersect at the point **P**.

Required to prove that **$AP \cdot PB = CP \cdot PD$** .

Construction. Join **AC**, **BD**.

Proof. In the segment **ACBD**, $\angle ACD = \angle DBA$.

In the segment **CADB**, $\angle CAB = \angle BDC$.

In the triangles **PCA**, **PBD**,

$\therefore \angle ACP = \angle DBP$ and $\angle CAP = \angle BDP$,

$\therefore \triangle PCA$ is similar to $\triangle PBD$, (*Theorem 18D.*)

and **$AP : PD = CP : PB$** .

Hence $AP \cdot PB = CP \cdot PD$.

ALTERNATIVE PROOF OF THEOREM 11C.

If a chord of a circle is divided externally by a tangent, the rectangle contained by the segments of the chord is equal to the square on the tangent.

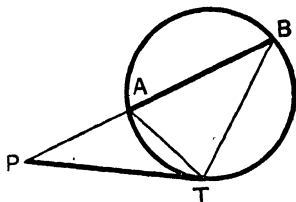


Fig. 46.

Given that The tangent **PT** intersects the chord **AB** of the circle **ATB** at **P**.

Required to prove that $PA \cdot PB = PT^2$.

Construction. Join **AT**.

Proof. $\because$ **PT** is a tangent and **AT** a chord,
 $\therefore \angle PTA = \angle ABT$. (Theorem 17B.)
 In the triangles **PAT**, **PBT**,
 $\therefore \angle PTA = \angle PBT$,
 and $\angle APT = \angle BPT$,
 $\therefore \triangle PAT$ is similar to $\triangle PBT$, (Theorem 18D.)
 and $PA : PT = PT : PB$.
Hence $PA \cdot PB = PT^2$.

Notes.—(1) The preceding three theorems might be included in a single enunciation as follows:—*If any two chords of a circle cut one another internally or externally the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.* The definition of a tangent as the limit of a secant (or chord) when the two points of intersection with the circumference have come to coincide, must, of course, be borne in mind.

(2) The student should work out for himself the proofs of the converse Theorems 9C, 12C, and 13C.

THEOREM 35D.

If a quadrilateral is inscribed in a circle, the sum of the rectangles contained by the two pairs of opposite sides is equal to the rectangle contained by its diagonals.

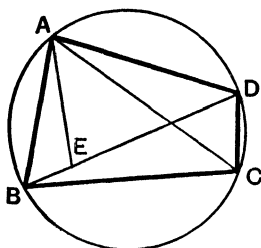


Fig. 47.

Given that **ABCD** is a cyclic quadrilateral.

Required to prove that The rectangle contained by **AB** and **CD** together with the rectangle contained by **BC** and **AD** is equal to the rectangle contained by **AC** and **BD**, *i.e.*

$$\mathbf{AB \cdot CD + BC \cdot AD = AC \cdot BD.}$$

Construction. Draw **AE** making $\angle \mathbf{BAE} = \angle \mathbf{DAC}$ and meeting **BD** at **E**.

Proof. Because $\angle \mathbf{BAE} = \angle \mathbf{DAC}$,
 $\therefore \angle \mathbf{BAE} + \angle \mathbf{EAC} = \angle \mathbf{DAC} + \angle \mathbf{EAC}$,
 that is, $\angle \mathbf{BAC} = \angle \mathbf{EAD}$.

Again, in the segment **ABCD**, $\angle \mathbf{ABD} = \angle \mathbf{ACD}$,
(*Theorem 12B.*)

and in the segment **ADCB**, $\angle \mathbf{ACB} = \angle \mathbf{ADB}$.
(*Theorem 12B.*)

Now in the triangles **BAE** and **CAD**,

$$\therefore \angle \mathbf{BAE} = \angle \mathbf{CAD} \text{ and } \angle \mathbf{ABE} = \angle \mathbf{ACD},$$

$$\therefore \angle \mathbf{BEA} = \angle \mathbf{CDA}.$$

Hence $\triangle$ s **BAE** and **CAD** are equiangular, and therefore similar,

$$\therefore \mathbf{AB : AC = BE : CD}, \quad (\text{Theorem 18D.})$$

that is, $\mathbf{AB \cdot CD = AC \cdot BE}$.

Again, in the triangles **BAC** and **EAD**,

$$\therefore \angle \mathbf{BAC} = \angle \mathbf{EAD} \text{ and } \angle \mathbf{ACB} = \angle \mathbf{ADE},$$

$$\therefore \angle \mathbf{ABC} = \angle \mathbf{AED}.$$

Hence $\triangle$ s **BAC** and **EAD** are equiangular, and therefore similar,

$$\therefore \mathbf{BC : ED = AC : AD}, \quad (\text{Theorem 18D.})$$

that is, $\mathbf{BC \cdot AD = AC \cdot ED}$.

$$\text{But } \mathbf{AB \cdot CD = AC \cdot BE}, \quad (\text{Proved.})$$

$$\therefore \mathbf{AB \cdot CD + BC \cdot AD = AC \cdot BE + AC \cdot ED}.$$

$$\text{Also } \mathbf{AC \cdot BE + AC \cdot ED = AC (BE + ED) = AC \cdot BD}.$$

$$\text{Hence } \mathbf{AB \cdot CD + BC \cdot AD = AC \cdot BD}.$$

Notes.—(1) Theorem 35D is known as **Ptolemy's Theorem**.

(2) In the case of a quadrilateral which is *not cyclic* it can be shown that *the sum of the rectangles contained by the two pairs of opposite sides is greater than the rectangle contained by the diagonals*.

(3) As further exercises on the cyclic quadrilateral the student should prove that—(a) If the diagonals of the quadrilateral of Fig. 47 are at right angles the sum of the rectangles contained by the opposite sides is twice the area of the figure. (b) The rectangle contained by two adjacent sides of the quadrilateral is to the rectangle contained by the other two sides in the ratio of the segments into which one of the diagonals is divided by the other.

THEOREM 36D.

If from the vertical angle of a triangle a perpendicular is drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by this perpendicular and the diameter of the circle described about the triangle.

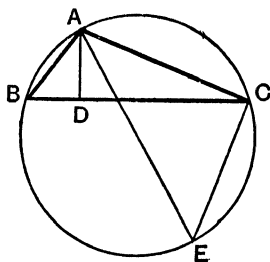


Fig. 48.

Given that **ABC** is a triangle and **AD** is perpendicular to **BC**.
AE is a diameter of the circum-circle **ABEC**.

Required to prove that The rectangle contained by **AB** and **AC** is equal to the rectangle contained by **AD** and **AE**, i.e.
AB . AC = AD . AE.

Construction.

Join **EC**.

Proof. In the semi-circle **ACE**, $\angle ACE = \text{a right angle}$,
 and in the segment **ABEC**, $\angle ABC = \angle AEC$.

Now in the triangles **BAD** and **EAC**,

$$\therefore \angle ADB = \angle ACE \text{ and } \angle ABD = \angle AEC,$$

$$\therefore \angle BAD = \angle EAC.$$

Hence $\triangle$ s **BAD** and **EAC** are equiangular, and
 therefore similar, (Theorem 18D.)

$$\therefore AB : AE = AD : AC.$$

$$\text{Hence } AB . AC = AD . AE.$$

THEOREM 37D.

If the vertical angle of a triangle is bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base *plus* the square on the line which bisects the vertical angle.

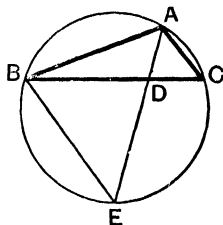


Fig. 49.

Given that **ABC** is a triangle and **AD** bisects the vertical angle **BAC** and cuts the base at **D**.

Required to prove that The rectangle contained by **BA** and **AC** is equal to the rectangle contained by **BD** and **DC** together with the square on **AD**, i.e.

$$\mathbf{BA \cdot AC = BD \cdot DC + AD^2.}$$

Construction. Describe a circle about the triangle **ABC**, produce **AD** to meet the circumference at **E**, and join **BE**.

Proof. In the segment **BECA**, $\angle \mathbf{BEA} = \angle \mathbf{BCA}$.

Now in the triangles **BAE** and **DAC**,

$$\therefore \angle \mathbf{BAE} = \angle \mathbf{DAC} \text{ and } \angle \mathbf{BEA} = \angle \mathbf{DCA},$$

$$\therefore \angle \mathbf{EBA} = \angle \mathbf{CDA}.$$

Hence $\triangle$ s **BAE** and **DAC** are similar,

$$\therefore \mathbf{BA : AD = EA : AC}, \text{ that is, } \mathbf{BA \cdot AC = AD \cdot EA}.$$

$$\text{But } \mathbf{AD \cdot EA = AD (ED + AD) = AD \cdot ED + AD^2},$$

$$\therefore \mathbf{BA \cdot AC = AD \cdot ED + AD^2}.$$

$$\text{But } \mathbf{AD \cdot ED = BD \cdot DC.} \quad (\text{Theorem 8c.})$$

$$\mathbf{Hence \quad BA \cdot AC = BD \cdot DC + AD^2.}$$

THEOREM 38D.

If the vertical angle of a triangle is bisected externally by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base *minus* the square on the line which bisects the vertical angle externally.

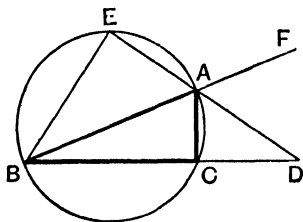


Fig. 50.

Given that $\triangle ABC$ is a triangle and AD bisects the angle FAC and cuts the base at D .

Required to prove that The rectangle contained by BA and AC is equal to the rectangle contained by DC and DB diminished by the square on DA , i.e.

$$BA \cdot AC = DC \cdot DB - DA^2.$$

Construction. Describe a circle about the triangle ABC , produce DA to meet the circumference at E , and join BE .

Proof. In the cyclic quadrilateral $ACBE$, $\angle ACD = \angle BEA$.
 (Theorem 14B, Cor.)

Again, $\angle CAD = \angle DAF = \angle EAB$.

Now in the triangles ACD and AEB ,

$\therefore \angle ACD = \angle BEA$ and $\angle CAD = \angle EAB$,

$\therefore \angle CDA = \angle EBA$.

Hence $\triangle s$ ACD and AEB are equiangular, and therefore similar. (Theorem 18D.)

$$\therefore BA : DA = EA : AC,$$

that is, $BA \cdot AC = DA \cdot EA$.

$$\text{But } DA \cdot EA = DA (DE - DA) = DA \cdot DE - DA^2,$$

$$\therefore BA \cdot AC = DA \cdot DE - DA^2.$$

Again, because the chords EA and BC intersect at D ,

$$\therefore DA \cdot DE = DC \cdot DB. \quad (\text{Theorem 10c.})$$

$$\text{Hence } BA \cdot AC = DC \cdot DB - DA^2.$$

13. Radius of the Circum-circle of a Triangle.—

Referring to Fig. 48, page 70, let the lengths of the sides of the triangle ABC which are opposite the angles A , B , and C be denoted by a , b , and c respectively, let p be the length of the perpendicular AD , and R the radius of the circum-circle. By Theorem 36D:—

$$2R \cdot p = bc,$$

$$\therefore R = \frac{bc}{2p}.$$

$$\text{But } pa = \text{twice area of triangle } ABC = 2\Delta,$$

$$\therefore p = \frac{2\Delta}{a}.$$

$$\text{Hence } R = \frac{abc}{4\Delta}.$$

Exercises IV.

1. The diagonals of a cyclic quadrilateral intersect at right angles. Show that the area of the sum of the rectangles of the opposite sides is double the area of the quadrilateral.

2. $ABCD$ is a cyclic quadrilateral, and the diagonal BD bisects AC . Prove that the rectangle $AB \cdot AD$ is equal to the rectangle $CB \cdot CD$.

3. A triangle ABC is inscribed in a circle, and BD is drawn parallel to the tangent at A to meet AC , produced if necessary, in D . Show that the rectangle $AB \cdot BC$ is equal to the rectangle $AC \cdot BD$.

4. ABC is a triangle inscribed in a circle, and the exterior angle at A is bisected by a straight line which meets the circle in E and BC in D . Show that the rectangle $AB \cdot AC$ is equal to the rectangle $EA \cdot AD$.

5. If the sides of a right-angled triangle are in continued proportion, then the perpendicular from the right angle divides the hypotenuse in medial section.

6. O, A, B, C, D are collinear points such that the rectangle $OA \cdot OD$ is equal to the rectangle $OB \cdot OC$, and a circle is described with centre O and the square of whose radius is equal to $OA \cdot OD$ or $OB \cdot OC$. Show that the lines AB and CD subtend equal angles at any point on the circumference of this circle.

7. ABC is an equilateral triangle inscribed in a circle, and P is any point in the arc AB . Join PA, PB , and PC ; and prove that PC is equal to the sum of PA and PB .

8. If a regular hexagon be inscribed in a circle, and P be any point on the circumference, prove that the sum of the distances of P from the four nearest vertices of the hexagon is equal to the sum of its distances from the two most remote.

9. ABC is an isosceles triangle, right-angled at A ; find the locus of a point P which moves so that $PA \cdot \sqrt{2} = (PB + PC)$.

10. $ABCD$ is a square inscribed in a circle, and P is any point in the arc AD . Prove

$$(1) \frac{PA - PD}{PB - PC} = \text{const.}$$

$$(2) PB (PB + PD) = PC (PC + PA).$$

11. Construct a triangle given the base, the altitude, and the rectangle contained by the sides.

12. $ABCD$ is a cyclic quadrilateral whose diagonals intersect in E . Prove that $AB \cdot BC : AD \cdot DC = BE : ED$.

13. The product of the perpendicular distances of any point on a circle from two opposite sides of an inscribed quadrilateral is equal to the product of the distances from the other two sides.

14. Given the base and the vertical angle of a triangle, construct it so that the rectangle contained by the sides should be a maximum.

15. From the vertex of a triangle draw a line to the base such that it is a mean proportional between the segments of the base.

16.* Two circles with centres **A** and **B** touch externally at **C**. **P** is any point at which **AC**, **BC** subtend equal angles. **PQ**, **PR** are tangents to the circles. Prove that $PQ \cdot PR = PC^2$. [Prove $PQ^2 : PC^2 = AC : BC$.]

17.* **ABCDE** is a regular pentagon inscribed in a circle, and **P** is any point on the arc **AE**. Prove that $PA + PC + PE = PB + PD$.

18.* **ABCD** is a parallelogram, and a circle through **A** cuts **AB**, **AC**, **AD** in **P**, **Q**, **R** respectively. Prove that $AB \cdot AP + AD \cdot AR = AC \cdot AQ$.

19.* **ABCD** is a cyclic quadrilateral. Prove that
 $AB \cdot AD + CB \cdot CD : BA \cdot BC + DA \cdot DC = AC : BD$.

20.* **ABC** is an isosceles triangle. Show that the locus of a point **P**, which moves so that the rectangle under its distances from the equal sides **AB** and **AC** is equal to the square on its distance from the base **BC**, is the circle which touches **AB** and **AC** at **B** and **C**.

CHAPTER V.

PROBLEMS ON RATIO AND PROPORTION AND SIMILAR FIGURES.

14. Introductory Note.—In this chapter various Problems are considered, the solutions of which depend upon Propositions proved in the earlier chapters of the book.

PROBLEM 1D.

To divide a straight line proportionally to a given divided line.

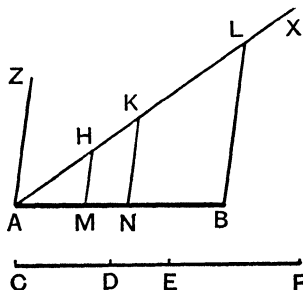


Fig. 51.

Given A straight line **AB**, and a divided straight line **CDEF**.

Required To divide the line **AB** proportionally to the divided line **CDEF**.

Construction. From **A** draw a line **AX** at a convenient angle with **AB**.

From **AX** mark off **AH, HK, KL** equal to **CD, DE, EF** respectively.

Join **BL**.

Draw **HM** and **KN** parallel to **LB** and meeting **AB** at **M** and **N**.

Then AB shall be divided proportionally to CF.

Proof. Through **A** draw **AZ** parallel to **BL**.

Because the parallels **AZ**, **MH**, **NK**, **BL** meet the transversals **AB**, **AL**,

$$\therefore \mathbf{AM : MN : NB = AH : HK : KL.}$$

(Theorem 11b.)

But **AH** = **CD**, **HK** = **DE** and **KL** = **EF**.

Hence **AM : MN : NB = CD : DE : EF**.

COROLLARIES TO PROBLEM 1D.

COROLLARY 1.—To divide a given line **AN** internally in a given ratio, say $l : m$.

Draw **AX** at any convenient angle, and (choosing any convenient units) mark off from it **AH** = l units and **HK** = m units (Fig. 51).

Draw **HM** parallel to **KN** to meet **AN** at **M**.

Then **AM : MN** = **AH : HK** = $l : m$.

COROLLARY 2.—To divide a given line **AB** externally in a given ratio, say $l : m$.

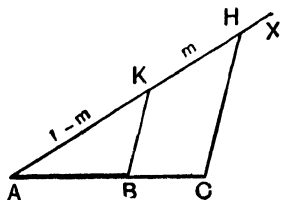


Fig. 52.

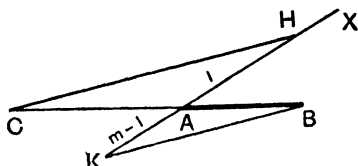


Fig. 53.

Draw **AX** at any convenient angle, and (choosing any convenient units) mark off from it **AH** = l units and **HK** = m units, drawing **AH** and **HK** in opposite senses.

(In Fig. 52 $l > m$, and in Fig. 53 $l < m$).

Join **KB**. Draw **HC** parallel to **KB** to meet **AB** at **C**.

Then **AC : CB** = **AH : HK** = $l : m$.

PROBLEM 2D.

To find a fourth proportional to three given lines.

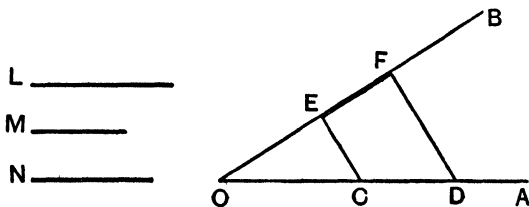


Fig. 54.

Given Three straight lines, **L**, **M**, and **N**.

Required To find a fourth proportional to **L**, **M**, and **N**.

Construction. Draw **OA** and **OB** making an angle **AOB**.

From **OA** cut off **OC = L**, and **CD = M**;

from **OB** cut off **OE = N**. Join **CE**.

Through **D** draw **DF** parallel to **CE**, meeting **OB** in **F**.

Then EF shall be the fourth proportional to L, M, and N.

Proof. In $\triangle ODF$, $\because CE$ is parallel to DF ,
 $\therefore OC : CD = OE : EF$. (Theorem 12D.)
Hence $L : M = N : EF$. (Cons.)

COROLLARIES TO PROBLEM 2D.

COROLLARY 1.—To find the third proportional to two given lines.

To find the third proportional to **L** and **M** (Fig. 54).

Make **OC = L**, **CD = M**, **OE = M**, and draw

DF parallel to **CE**. Then **$L : M = M : EF$** .

COROLLARY 2.—To find a line which shall be in a given ratio to a given line.

To find a line which shall be to **L** in the ratio $h : k$ (Fig. 54).

Make **OE = L**; also using any convenient units make

OC = k units, **CD = h** units. Draw **DF** parallel to **CE**.

Then **$EF : L = h : k$** .

PROBLEM 3D.

To find the mean proportional to two given lines.

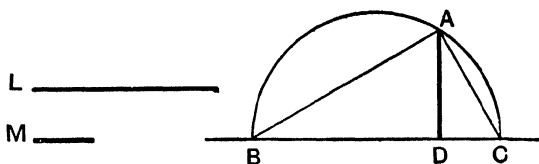


Fig. 55.

Given Two straight lines **L** and **M**.

Required To find the mean proportional between **L** and **M**.

Construction. Take any straight line, and in it cut off
BD = L and **DC = M**.

On **BC** as diameter describe the semicircle **BAC**.

From **D** draw **DA** perpendicular to **BC**, meeting the
circumference in **A**.

**Then DA shall be the mean proportional between
L and M.**

Proof. Join **BA** and **AC**.

Because **BAC** is a semicircle,

$\therefore \angle \text{BAC}$ is a right angle. (*Theorem 16B.*)

Also **AD** is drawn from the right angle perpendicular to **BC**.

Hence $\triangle \text{BDA}$ is similar to $\triangle \text{ADC}$,

and **BD : DA = DA : DC**. (*Theorem 23D.*)

Hence L : DA = DA : M. (*Cons.*)

Note.—Since **DC** is the third proportional to **BD**, **AD**, and **BC** is the third proportional to **BD**, **BA**, this figure provides also two distinct methods for finding the third proportional to two given lines. The student should work out both of these suggestions.

PROBLEM 4D.

(FIRST METHOD.)

On a given straight line as base to describe a polygon similar to a given polygon.

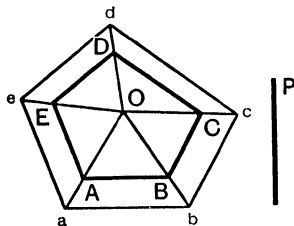


Fig. 56.

Given A straight line **P** and a polygon **ABCDE**.

Required To describe a polygon similar to the polygon **ABCDE**, having a base of length **P**.

Construction. Draw a line *ab*, in any convenient position, parallel to **AB** and equal to **P**.

Join *aA* and *bB*, intersecting at some point **O**.

Draw *bc*, *cd*, *de* respectively parallel to **BC**, **CD**, **DE**, meeting **OC**, **OD**, **OE** in *c*, *d*, *e* respectively.

Join *ae*.

Then *abcde* shall be the required polygon.

Proof. In $\triangle$ s **Oab**, **OAB**, $\because ab$ is parallel to **AB**,
 $\therefore \angle Oab = \angle OAB$, and $\angle Oba = \angle OBA$.

Similarly, in the $\triangle$ s **Obc**, **OBC**,
 $\angle Obc = \angle OBC$ and $\angle Ocb = \angle OCB$.

Hence $\angle Oba + \angle Obc = \angle OBA + \angle OBC$,
 that is, $\angle abc = \angle ABC$.

Also, $\triangle Oab \parallel \triangle OAB$, and $\triangle Obc \parallel \triangle OBC$,

$$\text{and } \frac{ab}{AB} = \frac{Ob}{OB} = \frac{bc}{BC}. \quad (\text{Theorem 18D.})$$

Similarly, it can be proved that each angle of polygon *abcde* is equal to the corresponding angle of polygon **ABCDE**, and that corresponding sides of the two polygons are in a constant ratio.

Hence polygon *abcde* is similar to polygon **ABCDE.**

PROBLEM 4D.

(SECOND METHOD.)

On a given straight line as base to describe a polygon similar to a given polygon.

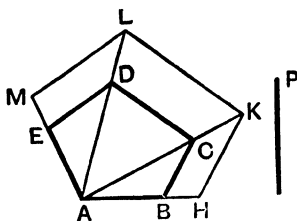


Fig. 57.

Given A straight line **P** and a polygon **ABCDE**.

Required To describe a polygon similar to the polygon **ABCDE**, having a base of length **P**.

Construction. Along the base **AB** mark off **AH = P**.

Draw **HK** parallel to **BC**, meeting **AC** at **K**.

Draw **KL** parallel to **CD**, meeting **AD** at **L**.

Draw **LM** parallel to **DE**, meeting **AE** at **M**.

Then AHKLM shall be the required polygon.

Proof. $\because$ the two arms of any angle in the polygon **AHKLM** are respectively parallel to and drawn in the same sense as the two arms of the corresponding angle in the polygon **ABCDE**,

$\therefore$ the two polygons are equiangular.

$\because$ **BC** $\parallel$ **HK**, **CD** $\parallel$ **KL**, **DE** $\parallel$ **LM**,

$\therefore \triangle AHK \parallel \triangle ABC$, $\triangle AKL \parallel \triangle ACD$,

$\triangle ALM \parallel \triangle ADE$.

(Theorem 18D.)

$\therefore \triangle AHK$ is similar to $\triangle ABC$,

$$\therefore \frac{AH}{AB} = \frac{HK}{BC} = \frac{AK}{AC}.$$

$\therefore \triangle AKL$ is similar to $\triangle ACD$,

$$\therefore \frac{AK}{AC} = \frac{KL}{CD} = \frac{AL}{AD}.$$

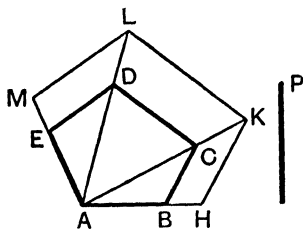


Fig. 57.

$\therefore \triangle ALM$ is similar to $\triangle ADE$,

$$\therefore \frac{AL}{AD} = \frac{LM}{DE} = \frac{MA}{EA}.$$

Hence $\frac{AH}{AB} = \frac{HK}{BC} = \frac{KL}{CD} = \frac{LM}{DE} = \frac{MA}{EA}.$

Hence polygon AHKLM is similar to polygon ABCDE.

COROLLARY TO PROBLEM 4D.

To describe a polygon similar to a given polygon, and equal to a given fraction of it in area.

Suppose a polygon is required similar to **AHKLM** (Fig. 57) and equal to one-half of it in area.

Let **AB** be the mean proportional between **AH** and a line **AF** (not shown) equal to half **AH**.

On **AB** describe a polygon **ABCDE** similar to **AHKLM**.

$$\begin{aligned} \text{Then } \frac{\text{polygon } ABCDE}{\text{polygon } AHKLM} &= \frac{AB^2}{AH^2} = \frac{AF \cdot AH}{AH^2} \\ &= \frac{AF}{AH} = \frac{1}{2}. \end{aligned}$$

PROBLEM 5D.

To construct a polygon similar to one polygon and equal to another.

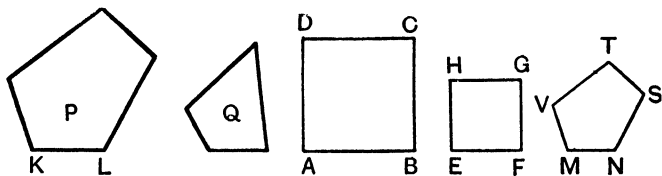


Fig. 58.

Given Two polygons **P** and **Q**.

Required To construct a polygon similar to **P** and equal to **Q**.

Construction. Construct triangles equal to the polygons **P** and **Q**. (Problem 18A.)

Construct rectangles equal to these triangles. (In each case draw a rectangle on half the base of the triangle and of equal altitude.)

Construct squares **ABCD**, **EFGH**, equal to these rectangles. (Problem 1c.)

Find **MN**, the fourth proportional to **AB**, **EF**, **KL**.

On **MN** construct a polygon **MNSTV** similar to **P**.

Then polygon MNSTV shall be equal to Q.

Proof. $\therefore AB : EF = KL : MN$, (Cons.)

$\therefore \text{sq. on } AB : \text{sq. on } EF = \text{sq. on } KL : \text{sq. on } MN$.

But **P** : **MNSTV** = sq. on **KL** : sq. on **MN** ;

hence **P** : **MNSTV** = sq. on **AB** : sq. on **EF**.

But **P** = sq. on **AB**. (Cons.)

Hence MNSTV = sq. on EF = Q.

Exercises V.

PRACTICAL GEOMETRY.

1. Draw a straight line 2 in. long, and divide it in the ratio of the side of a square to its diagonal. Measure the longer segment.

2. Draw a straight line 2 cm. long, and divide it externally in the ratio of the side of an equilateral triangle to its altitude. Measure the longer segment.

3. Draw an equilateral triangle of side 1·2 in. Construct and measure the third proportional to its altitude and base.

4. Draw an equilateral triangle **ABC** of side 8 cm. Bisect **BC** at **D**. Construct a line **L**, such that $BD : AD = L : AB$. Measure **L**.

5. Draw a line **AB** of length 6 cm. Divide it externally at **C** in the ratio $1 : \sqrt{7}$. Measure **AC**.

6. Show how to find geometrically the value of $a \times b \div c$, where a , b , and c are given numbers.

Hence evaluate geometrically (i) $29 \times 17 \div 19$; (ii) $2\cdot1 \times 3\cdot2$; (iii) $4\cdot5 \div 1\cdot7$; (iv) $4\cdot2 \div 7\cdot3$.

7. Find the mean proportional to the side and altitude of an equilateral triangle whose altitude is 4 cm.

8. Show how to find geometrically the value of $\sqrt[4]{abcd}$, where a , b , c , and d are given numbers.

Hence evaluate geometrically the fourth roots of 12, 30, and 12,000.

9. Evaluate by geometrical constructions (i) $\sqrt{(2\sqrt{2})}$, (ii) $\sqrt{(10\sqrt{7})}$, (iii) $\sqrt{(84\sqrt{69})}$.

10. Construct a quadrilateral **ABCD** in which **BC** = 5 cm., $\angle ABC = 80^\circ$, $\angle DBC = 35^\circ$, $\angle DCB = 70^\circ$, $\angle ACB = 40^\circ$. Construct a similar quadrilateral in which the side corresponding to **AD** measures 5 cm. Measure the side corresponding to **BC**.

11. Construct a quadrilateral similar to those in question 10, and equal in area to their sum. Measure the line corresponding to **AC**.

12. Copy Fig. 59 (page 85) to the given dimensions. *Without performing any calculations* draw a figure similar to your copy and of half the area. Measure the line which corresponds to **AD**.

13. Construct a figure similar to your copy of Fig. 59 and of area equal to 3 sq. inches. Measure the line which corresponds to **BD** in Fig. 59.

RIDERS.

- 14.** Use Theorem 8C to find a fourth (or third) proportional.
- 15.** Use Theorem 10C to find a fourth (or third) proportional.
- 16.** Use Theorem 11C to find a mean proportional.
- 17.** Divide a given straight line into two parts such that their mean proportional shall be equal to a given straight line.
- 18.** The Geometric Mean between any two quantities is less than the Arithmetic Mean, but approaches to it in value as the two quantities become more nearly equal.

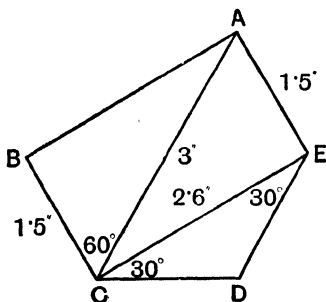


Fig. 59.

- 19.** On a given straight line construct a rectangle equal to a given rectangle.
- 20.** From a point P outside a circle draw a line PQR meeting the circle and such that QR is a mean proportional between PQ and PR .
- 21.** OQ , OR are two intersecting straight lines and P is any point in the angle which they form. Draw through P a straight line QPR such that OP is a mean proportional between PQ and PR .
- 22.** OP and OQ are two perpendicular radii of a circle. Draw a chord of the circle which is trisected by OP and OQ .
- 23.** From a given parallelogram cut off a similar parallelogram by a straight line parallel to one of its sides.
- 24.** Through a point of intersection of two circles draw a line such that the chords intercepted by the two circles are in a given ratio.
- 25.** The line OC lies within the angle AOB . From any point P in OA draw a line PQ to meet OB in Q , such that PQ is divided in a given ratio by OC .

26. A given straight line **AD** is divided into any three parts **AB**, **BC**, and **CD**. Show how to find a point **P** in **AD** produced such that

$$PA : PB = PC : PD.$$

27. Inscribe in a triangle a triangle similar to a given triangle and having one side parallel to a given line.

28. Inscribe in a triangle a triangle whose sides are parallel to three given lines.

29. Construct a triangle given any angle, one side, and the ratio of the other two sides.

30. Construct a triangle given the angles and the length of any line connected with the triangle (*e.g.* the bisector of an angle).

31. Construct a triangle given its angles and its area.

32. Find a point **P** within a triangle **ABC** such that the areas **BPC**, **CPA**, **APB** are in given ratios.

33. In a given triangle inscribe a square.

34. In a given triangle inscribe a parallelogram similar to a given parallelogram.

35. Describe a rectangle equal to a given square, and having its sides in a given ratio.

36. Draw a straight line such that the perpendiculars on it from three given points shall be in given ratios. How many solutions are there?

37. Bisect a triangle by a straight line perpendicular to one of its sides.

38. Draw a circle touching two circles, one of them at a given point.

39.* **OL**, **OM**, **ON** are three concurrent straight lines. Prove that the locus of a point **P** which moves so that its perpendicular distance from **OM** is a mean proportional between its perpendicular distances from **OL** and **ON** is a pair of straight lines.

40.* Through a given point outside or inside a circle show how to draw a chord which is divided at the point in a given ratio.

41.* Construct a triangle given the three perpendiculars.

(If the perpendiculars are given the ratios of the sides are known.)

CHAPTER VI.

STRAIGHT LINES AND PLANES.

15. The Plane.—The earlier portions of this book have been devoted almost entirely to **Plane Geometry**, *i.e.* to the consideration of geometrical figures lying entirely in one plane. The remainder of the work will be devoted to a modern treatment of the elements of **Solid Geometry**, *i.e.* to the consideration of geometrical figures which do not lie in one plane. Solid Geometry forms the subject-matter of Euclid, Books XI. and XII.

A plane surface (or plane) is a surface such that the straight line which joins any two points upon it lies entirely on the surface. Examples of plane surfaces are the face of a wall, the flat top of a table, the surface of a garden lawn, etc. Fig. 60 represents two plane surfaces, say the surface of a rectangular lawn, **A**, and the face of a rectangular wall, **B**, along one side of the lawn: thus **A** represents a horizontal (*i.e.* level) plane, and **B** represents a vertical (*i.e.* upright) plane.

A complete straight line is infinite in length, but in a figure it is represented by any convenient portion of itself, which portion may be produced if desired. In the same way **a complete plane is infinite in length and breadth**, but in a figure it is represented by any convenient portion of itself (usually a rectangle), which portion may be produced in length or breadth or both, if desired. For example, in Fig. 60 the complete plane represented by the rectangle **B** extends in all directions above, below, to the right of, and to the left of the rectangle **B**.

Notice that though **A** and **B** in Fig. 60 *represent* rectangles they are not actually *drawn* as rectangles. They are irregular quadrilaterals—approximately parallelograms. If a geometrical figure lies in one plane it can be drawn in correct shape on a flat piece of paper; but if a geometrical

figure does not lie in one plane it can only be represented on paper by a "picture" of the figure, *i.e.* by a diagram drawn in *perspective*.

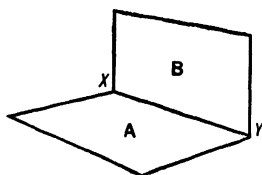


Fig. 60.

Thus in the diagrams of Solid Geometry a *complete plane* is represented by some convenient portion of itself (usually a rectangle) drawn in *perspective*.

A straight line is said to be parallel to a plane if the complete straight line and the complete plane have no point in common. For example, any straight

line drawn on the flat top of a table standing in a room is parallel to the plane of the floor and also to that of the ceiling, or any straight line drawn on one wall of a room is parallel to the opposite wall.

Two planes are said to be parallel if when completed they have no point in common. Thus two opposite walls in a room are parallel planes, and the flat top of a table standing in a room is parallel to the floor.

It might also be noted in passing that if lines are drawn on a plane (or if a plane can be made to contain them) they are called **co-planar lines**, whilst if a plane cannot be made to contain them, they are called **skew lines**.

16. Axioms used in Solid Geometry.—The following axioms, additional to those used in Plane Geometry, are required in Solid Geometry:—

(1) *Any point on a complete straight line divides it into two parts.*

(2) *Any complete straight line on a complete plane divides it into two parts.*

(3) *Any complete plane divides space into two parts.*

Thus in Fig. 60, the plane A, if produced indefinitely in all directions, divides space into two parts, or **regions**, one above the plane and one below it.

(4) *A half-plane, rotated about its bounding line, sweeps through all space.*

Thus in Fig. 60, the complete line XY divides the complete plane A into two halves. If either half makes one revolution about XY it sweeps through every point in space.

(5) *Two intersecting surfaces meet (in general) in a line (or lines), not in a point or isolated points.*

Thus in Fig. 61, the two planes **AA**, **BB** meet in the line **XY**.

(6) *Two intersecting complete planes divide space into four regions.*

Thus in Fig. 61 the two complete planes **AA**, **BB** divide space into the four regions **P**, **Q**, **R**, **S**. Note that, similarly, two intersecting complete straight lines divide a plane into four regions.

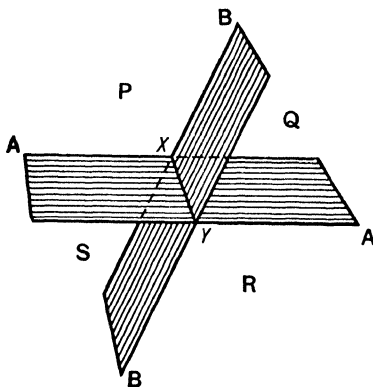


Fig. 61.

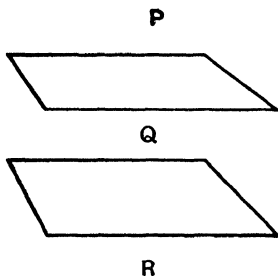


Fig. 62.

(7) *Two parallel complete planes divide space into three regions.*

See, for example, the regions **P**, **Q**, **R** in Fig. 62. Note that, similarly, two parallel complete lines divide a plane into three regions.

(8) *Through a given straight line or through two given points in space an infinite number of planes may pass.*

Thus one plane can be imagined containing a line **AB** or two points **A** and **B**, and then this plane can be rotated about the line **AB**, thus occupying in succession an infinite number of positions.

It follows from Axiom 8 that *it is possible to pass a plane through any three given points in space*. For one plane, **X**, can be passed through two of the given points (**A**, **B**). Moreover,

by the definition of a plane, X contains the complete straight line AB . Hence, by Axiom 4 above, if X does not contain the third point C , it can be rotated about AB until it does.

17. Intersections and Parallels.—*If two straight lines are co-planar they may either intersect or be parallel, and if they intersect they cannot do so in more than one point. If two straight lines are not co-planar (i.e. if they are skew) they neither intersect nor are parallel.*

Again, if two straight lines intersect, a plane can always be imagined which contains both, i.e. the two lines are co-planar: moreover, the definition of parallel lines involves the fact that two parallel lines are co-planar.

If a line be drawn on the table pointing north and another line on the floor pointing north-east the two lines cannot intersect, and they are not parallel: further, no plane can be imagined which contains both, i.e. the lines are not co-planar but skew. Of course, if both lines be drawn pointing north (or both north-east) the two lines will be parallel, and a plane can be imagined which contains both.

A straight line and a plane cannot intersect at more than one point. If they do not intersect they are parallel (by Definition).

Two planes can only intersect in a straight line. If they do not intersect they are parallel (by Definition). (See Theorem 2E.)

18. Diagrams of Solid Figures.—Much of the difficulty of Solid Geometry lies in the fact that figures which do not lie entirely in one plane cannot be accurately represented by a diagram drawn on a flat sheet of paper. This difficulty can always be overcome by constructing models of the solid figures with wood, cardboard, wire, string, or other suitable materials; occasionally this is well worth the trouble, but for general purposes it is far too tedious a process.

The most satisfactory way to solve the difficulty is to learn to draw reasonably good "pictures" (i.e. perspective drawings) of the solid figures—sufficiently good to present to the mind an accurate conception of the actual positions of the different points, lines, and planes represented. Examples of such drawings appear in this and subsequent chapters.

THEOREM 1E.

Only one plane can be drawn through three given points which are not collinear.

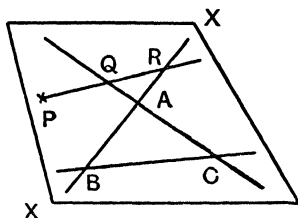


Fig. 63.

Given that **A**, **B**, and **C** are three points which are not collinear.

Required to prove that Only one plane can be drawn through the three points **A**, **B**, and **C**.

Construction. If possible let **XX** and **YY** be two different planes, each of which passes through the three points **A**, **B**, and **C** (Fig. 63; **YY** is not represented in the figure).

Let **P** be any point in the plane **XX**.

Through **P** in the plane **XX** draw any straight line which does not pass through **A**, **B**, or **C**.

This straight line must intersect at least two sides of $\triangle ABC$, for it cannot be parallel to more than one side of the triangle.

Also, since the straight line does not pass through an angular point of the triangle, it must intersect the two sides in different points.

Let the line meet **CA**, **BA** in **Q**, **R** respectively.

Proof. Since the points **C**, **A** lie in the plane **YY**, (*Hyp.*)
 $\therefore$ complete line **CA** lies in plane **YY**, (*Def. of plane.*)
 $\therefore$ point **Q** lies in plane **YY**.

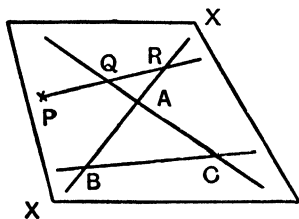


Fig. 63.

Since the points **A**, **B** lie in the plane **YY**,
 $\therefore$ complete line **AB** lies in plane **YY**,
 $\therefore$ point **R** lies in plane **YY**.

Since the points **Q**, **R** lie in the plane **YY**, (*Proved.*)
 $\therefore$ complete line **QR** lies in plane **YY**,
 $\therefore$ point **P** lies in plane **YY**.

But point **P** lies in plane **XX**, (Const.)
 $\therefore$ the point **P** lies in both planes **XX** and **YY**.

Similarly, any point in plane **XX** can be shown to lie
in plane **YY**,
and also any point in plane **YY** can be shown to lie
in plane **XX**.
 $\therefore$ the two planes are coincident.

Hence only one plane can be drawn through the points A, B, and C.

COROLLARIES TO THEOREM 1E.

COROLLARY 1. *One and only one plane can be drawn through (1) two given intersecting lines, or (2) two given parallel lines, or (3) a given line and a given point outside it.*

(1) If the two given intersecting lines are **AB, AC**, then the points **A, B, C** determine a plane which contains the complete lines **AB, AC**.

(2) If the two parallel lines are **DE, FG**, then by the definition of parallel lines there is one plane which contains them both. Also there cannot be more than one plane through the three points **E, F, G**.

(3) If the line is **HK** and the point **L**, then the three points **H, K, L** determine a plane which contains the complete line **HK**.

COROLLARY 2. *Any three straight lines, of which each pair cut one another, must be co-planar, i.e. any three straight lines which form a triangle lie in a plane.*

Notes.—(1) Theorem 1E is frequently assumed without proof.

(2) It follows from Theorem 1E that the position of a plane is definitely fixed

- (a) if it passes through three points not in a straight line ;
- (b) if it contains two given intersecting lines ;
- (c) if it contains two given parallel lines ;

and (d) if it contains a given straight line and a given point outside that line.

(3) It further follows that a plane may be **generated by**

- (a) a straight line which turns about a fixed point and slides over a fixed straight line ;
- (b) a straight line which slides over two straight lines (either parallel or intersecting) which lie in the same plane ;

or (c) a straight line which moves parallel to itself and slides over a fixed straight line.

(4) Whilst the sides of every triangle must lie in one plane, the sides of a quadrilateral (or polygon) need not do so. This may be verified, for example, by folding a quadrilateral about a diagonal, in which case two adjacent sides will be in one plane, the other two in another. Such a quadrilateral is called a **skew** or **gauche** quadrilateral.

THEOREM 2E.

If two planes meet their intersection is a straight line.

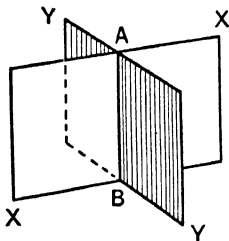


Fig. 64.

Given that **XX** and **YY** are two planes, and **A** and **B** are any two points on their line of intersection.

Required to prove that The intersection of the planes **XX** and **YY** is a straight line.

Proof. Because the two points **A**, **B** lie in the plane **XX**,
 $\therefore$ the straight line **AB** lies in the plane **XX**. (*Def.*)

Similarly, the straight line **AB** lies in the plane **YY**.

Again, the two planes cannot meet in any point **C** which is not in **AB**, for only one plane can pass through the three points **A**, **B**, **C**. (*Theorem 1E.*)

Hence the intersection of the two planes is the straight line AB.

COROLLARY TO THEOREM 2E.

If a point lies in each of two planes it must lie on their line of intersection.

THEOREM 3E.

If three planes intersect in pairs their three lines of intersection are coincident, concurrent, or parallel.

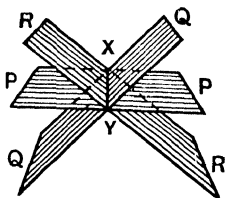


Fig. 65.

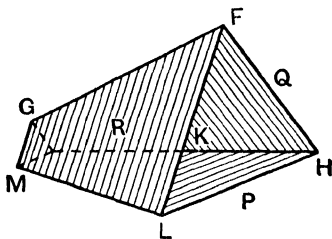


Fig. 66.

Given that **P**, **Q**, and **R** are three planes each of which meets the other two.

Required to prove that The three lines of intersection, viz. where **Q** meets **R**, where **R** meets **P**, and where **P** meets **Q**, are coincident, concurrent, or parallel.

CASE I.—IF **P** INTERSECTS **Q** AND **R** IN THE SAME LINE **XY** (Fig. 65).

Since the line **XY** lies in each of the planes **Q** and **R**,
 $\therefore$ it is the intersection of **Q** and **R**.

Hence the three lines of intersection are coincident.

CASE II.—IF **P** INTERSECTS **Q** AND **R** IN TWO INTERSECTING LINES **HK** AND **LM** RESPECTIVELY (Fig. 66),

Proof. Let **Q** and **R** intersect each other in the line **FG**.
 Let **HK** and **LM** intersect at a point **I**, (not shown in the figure).

$\therefore$ **I** lies in **HK** and **HK** lies in the plane **Q**,

$\therefore$ **I** lies in the plane **Q**.

$\therefore$ **I** lies in **LM** and **LM** lies in the plane **R**,

$\therefore$ **I** lies in the plane **R**.

$\therefore I$ lies in both the planes Q and R ,
 $\therefore$ it lies in their line of intersection FG .

(Theorem 2E, Cor.)

**Hence the three lines FG , HK , LM are concurrent,
 meeting at the one point I .**

CASE III.—If P intersects Q and R in two parallel
 lines HK and LM respectively (Fig. 67).

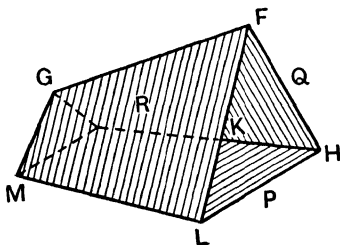


Fig. 67.

Proof. If possible let FG meet LM at a point I (not shown
 in the figure).

Then, arguing as in Case II., it can be proved that I lies in
 both the planes P and Q , and therefore lies in
 their intersection HK .

Thus HK and LM meet at I .

But this is impossible since they are parallel.

Hence FG is parallel to LM .

Similarly it can be proved that FG is parallel to HK .

**Hence the three lines of intersection FG , HK , LM
 are all parallel.**

Exercises VI.

1. Of three straight lines, not in the same plane, each meets the other two. How is this possible?

2. Into how many regions is space divided (*a*) by two parallel planes and a third plane intersecting both? (*b*) by two pairs of parallel planes?

3. Three faces of a rectangular block meet at one corner; into how many regions is space divided by the three complete planes containing these three faces?

4. Into how many regions is space divided by two intersecting planes?

5. Into how many regions is space divided by three planes which meet in one straight line?

6. Into how many regions is space divided by three planes which meet in three parallel straight lines?

7. Into how many regions is space divided by three planes which meet in three concurrent straight lines? [Work this by starting as in Question 5.]

CHAPTER VII.

PARALLELS AND PARALLELEPIPEDS.

19. Parallelism in Solid Geometry.—In Solid Geometry there are three cases of parallelism, viz. (1) straight lines parallel to straight lines, (2) planes parallel to planes, and (3) straight lines parallel to planes.

It is always possible to draw a straight line through a given point **A** parallel to a given straight line **BC**. To do this pass a plane through the three points **A**, **B**, **C**, and then use Plane Geometry.

There is no need to *assume* the possibility of two planes being parallel: this will be demonstrated in Theorem 6E (Cor.), where it will be shown that it is always possible to draw a plane through a given point parallel to a given plane.

20. Definitions.—A solid or solid figure is any portion of space bounded by one or more surfaces, plane or curved. The surfaces are referred to as the **faces** of the solid, the intersections of adjacent faces are called **edges**, and the points where three or more edges meet are called **vertices**, or **corners**, or **angular points** of the solid figure.

The name **polyhedron** is a general name used to denote all solid figures bounded by plane faces. Thus:—

A *tetrahedron* is a polyhedron with four faces.

A *pentahedron* „ „ „ five „

A *hexahedron* „ „ „ six „

An *octahedron* „ „ „ eight „

A *dodecahedron* „ „ „ twelve „

An *icosahedron* „ „ „ twenty „

It is clear that a polyhedron must have at least four faces.

If the faces of a polyhedron are equal in all respects and the number of edges meeting at each vertex is the same it is called a **regular polyhedron**. It will be shown later (Chapter X.) that there are five, and only five, *regular polyhedra*, sometimes spoken of as the *five regular solids*, and known to the Greeks as the *Platonic bodies*. (It may be mentioned in passing that these are the *regular tetrahedron*, *regular hexahedron (cube)*, *regular octahedron*, *regular dodecahedron*, and *regular icosahedron*).

A **parallelepiped** is a solid figure bounded by three pairs of parallel plane faces: it is a special type of polyhedron with six faces, opposite faces being parallel. Thus Fig. 68 represents a parallelepiped bounded by the following three pairs of parallel planes:—

ABFE, DCGH; ABCD, EFGH; BFGC, AEHD.

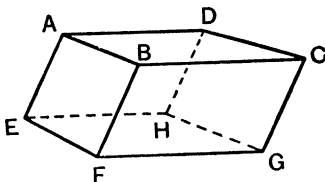


Fig. 68.

A brick is an instance of a parallelepiped in which each face is a rectangle. Such a parallelepiped is called a **cuboid** or **rectangular solid**. If the faces are all squares the parallelepiped is called a **cube**. From the definition it follows that a cube is a *regular parallelepiped*, or *regular hexahedron*.

The line joining two opposite corners of a parallelepiped is called a **diagonal**. There are eight corners in any parallelepiped; and therefore four diagonals, viz. **AG, BH, CE, DF** (Fig. 68).

It will be proved in Theorem 8E that:—

- (1) each face of a parallelepiped is a parallelogram;
 - (2) opposite faces are congruent;
 - (3) any two diagonals bisect each other;
- and (4) the four diagonals are concurrent.

THEOREM 4E.

If two intersecting planes pass respectively through two parallel lines, their intersection is parallel to each of these lines.

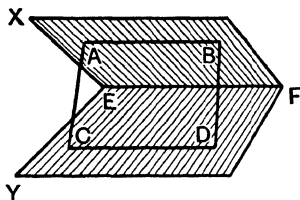


Fig. 69.

Given that The straight line **AB** is parallel to the straight line **CD**, and the planes **X** and **Y**, which pass through **AB** and **CD** respectively, intersect in the line **EF**.

Required to prove that **EF** is parallel to **AB** and **CD**.

Proof. Because **AB** is parallel to **CD**,
 $\therefore$ **ABDC** is a plane.

The three planes **X**, **Y**, **ABDC** intersect in the three lines **AB**, **CD**, **EF**,

hence the three lines **AB**, **CD**, **EF** are coincident, concurrent, or parallel. (Theorem 3E.)

But **AB** and **CD** are parallel.

Hence EF is parallel to AB and CD.

Note.—As an illustration of Theorem 4E, note that it is not possible to draw two parallel lines, one on the wall and one on the floor, unless each is parallel to the line of intersection of the wall and the floor.

THEOREM 5E.

If two straight lines are each parallel to a third they are parallel to each other.

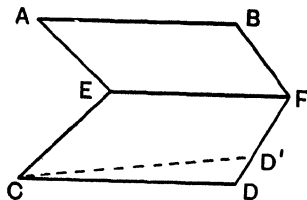


Fig. 70.

Given that The straight lines **AB** and **CD** are both parallel to **EF**.

Required to prove that **CD** is parallel to **AB**.

CASE I.—IF THE THREE LINES LIE IN ONE PLANE.

This case has already been proved in Theorem 8A.

CASE II.—IF THE THREE LINES DO NOT LIE IN ONE PLANE.

Proof. Because **AB** $\parallel$ **EF**, $\therefore$ **ABFE** is a plane.

Similarly, **CDFE** is a plane.

Suppose the plane **ABC** intersects the plane **CDFE** in the line **CD'**.

Then the three lines **AB**, **EF**, **CD'** are the intersections of the three planes **ABFE**, **CDFE**, **ABC**.

Hence these three lines are coincident, concurrent, or parallel.

But **AB** and **EF** are $\parallel$, hence also **CD'** $\parallel$ **AB** and $\parallel$ **EF**.

But **CD** $\parallel$ **EF**, $\therefore$ **CD'** coincides with **CD**.

Hence CD is parallel to AB.

COROLLARY TO THEOREM 5E.

It is possible to have a set of lines in space such that any two of them are parallel.

For example, suppose that in Fig. 70 a set of points **P**, **Q**, **R**, **S**, . . . is taken in plane **DCEF**, and suppose that through each of these points a line is drawn parallel to **EA**. Then any two of these lines are parallel.

THEOREM 6E.

If two intersecting lines in one plane are respectively parallel to two intersecting lines in another, then (i) the two planes are parallel, and (ii) the two pairs of lines contain equal angles.

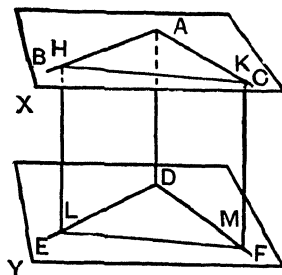


Fig. 71.

PART I.

Given that The lines **AB** and **AC** in the plane **X** are respectively parallel to the lines **DE** and **DF** in the plane **Y**.

Required to prove that The plane **X** is parallel to the plane **Y**.

Proof. Because the planes **X** and **Y** pass through the two parallel lines **AB** and **DE**,

$\therefore$ they intersect (if at all) in a line parallel to **AB**.
(Theorem 4B.)

Similarly, these planes intersect (if at all) in a line parallel to **AC**.
(Theorem 4E.)

But there cannot be a line parallel to both **AB** and **AC**.

Hence the planes X and Y cannot intersect, i.e. they are parallel.

PART II.

Given that The lines **AB** and **AC** in the plane **X** are respectively parallel to the lines **DE** and **DF** in the plane **Y**.

Required to prove that $\angle \text{BAC} = \angle \text{EDF}$.

Construction. On **AB**, **DE** mark off equal lengths **AH**, **DL** in the same sense.

On **AC**, **DF** mark off equal lengths, **AK**, **DM** in the same sense.

Join **AD**, **HL**, **KM**.

Proof. Because **AH** is equal and parallel to **DL**,
 $\therefore$ **HL** is equal and parallel to **AD**.
 (Theorem 28A.)

Because **AK** is equal and parallel to **DM**,
 $\therefore$ **KM** is equal and parallel to **AD**.
 (Theorem 28A.)

$\therefore$ **HL** is equal and parallel to **KM**.
 (Theorem 5E.)

Hence **HK** is equal and parallel to **LM**.
 (Theorem 28A.)

In the triangles **AHK** and **DLM**,

because **AH** = **DL**, (Cons.)

and **AK** = **DM**, (Cons.)

and **HK** = **LM**, (Proved.)

$\therefore \triangle \text{AHK} \equiv \triangle \text{DLM}$ (Theorem 15A.)

and $\angle \text{HAK} = \angle \text{LDM}$.

Hence $\angle \text{BAC} = \angle \text{EDF}$.

COROLLARY TO THEOREM 6E.

Through a given point A to draw a plane parallel to a given plane Y.

In the plane **Y** take any two intersecting lines **DE**, **DF**.
 Through **A** draw **AB** parallel to **DE**, and **AC** parallel to **DF**.
 Then plane **ABC** is parallel to plane **Y**.

THEOREM 7E.

If a plane intersects two parallel planes, the intersections are parallel lines.

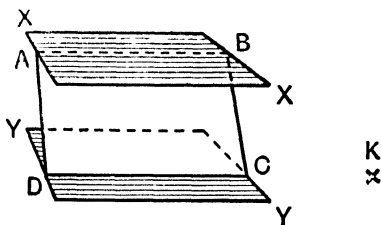


Fig. 72.

Given that **XX** and **YY** are two parallel planes, and the plane **ABCD** intersects them in the lines **AB** and **DC**.

Required to prove that **AB** is parallel to **DC**.

Proof. If possible, suppose that **AB** is not parallel to **DC**.
Then, since they are in the same plane **ABCD**, they meet
at some point **K**.

$\therefore$ **K** lies in **AB**, and **AB** lies in the plane **XX**,
 $\therefore$ **K** lies in the plane **XX**.

Similarly, **K** lies in **CD**, and therefore in the plane **YY**.
Hence the planes **XX** and **YY** have at least one point **K**
in common.

But this is not the case, since **XX** is parallel to **YY**.

Hence AB must be parallel to DC.

THEOREM 8E.

In any parallelepiped (i) each face is a parallelogram, (ii) opposite faces are congruent, and (iii) any two diagonals bisect each other.

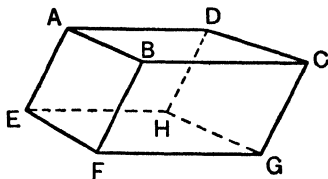


Fig. 73.

PART I.

Given that **FGDA** is a parallelepiped.

Required to prove that Each face of **FGDA** is a parallelogram.

Proof. $\because$ plane **ABFE** is parallel to plane **DCGH**, and plane **ABCD** meets these parallel planes in the lines **AB**, **DC**,

$\therefore$ **AB** is parallel to **DC**. (Theorem 7E.)

$\because$ plane **AEHD** is parallel to plane **BFGC**, and plane **ABCD** meets these parallel planes in the lines **AD**, **BC**,

$\therefore$ **AD** is parallel to **BC**. (Theorem 7E.)

Hence **ABCD** is a parallelogram.

Similarly, it can be proved that any other face of **FGDA** is a parallelogram.

Hence each face of the parallelepiped FGDA is a parallelogram.

PART II.

Given that **FGDA** is a parallelepiped.

Required to prove that Opposite faces of **FGDA** are congruent.

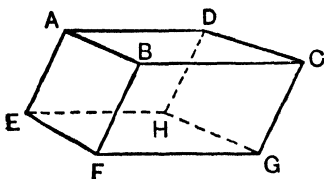


Fig. 73.

Proof.

In the parallelogram **ABFE**,
BA is parallel to **FE**, and **BA = FE**.

(Theorem 26A.)

In the parallelogram **CBFG**,
BC is parallel to **FG**, and **BC** = **FG**.

(Theorem 26A.)

In the planes **ABC**, **EFG**,

$\therefore$ **BA** is parallel to **FE**, and **BC** is parallel to **FG**,

$$\therefore \angle ABC = \angle EFG. \quad (\text{Theorem 6E.})$$

Thus in the parallelograms **ABCD**, **EFGH**,

BA = FE,

and $\mathbf{BC} = \mathbf{FG}$.

and $\angle ABC = \angle EFG$,

$$\therefore \text{parallelogram } \mathbf{ABCD} \equiv \text{parallelogram } \mathbf{EFGH}.$$

(Theorem 30A.)

Similarly, any other pair of opposite faces may be proved congruent.

Hence opposite faces of the parallelepiped FGDA are congruent.

PART III.

Given that **FGDA** is a parallelepiped.

Required to prove that Any two diagonals of **FGDA** bisect each other.

Proof. In the parallelogram **ABCD**,
AD = BC, and **AD** is parallel to **BC**.
(Theorem 26A.)
 In the parallelogram **BFGC**,
FG = BC, and **FG** is parallel to **BC**,
 $\therefore$ **AD = FG**, and **AD** is parallel to **FG**,
 whence **AFGD** is a parallelogram. (Theorem 28A.)
 But **AG**, **DF** are the diagonals of this parallelogram;
 whence **AG**, **DF** bisect one another.
(Theorem 29A.)

A similar proof holds for any other pair of diagonals.

**Hence any two diagonals of the parallelepiped
 FGDA bisect each other.**

COROLLARY TO THEOREM 8E.

In any parallelepiped the four diagonals are concurrent.

Any two diagonals bisect each other. Hence the mid-point of one diagonal coincides with the mid-point of each of the others. Thus the four diagonals are concurrent.

Note.—It will be seen later (Art. 24) that a parallelepiped is a particular type of the solid figure known as a **prism**. A cube or a cuboid (rectangular parallelepiped) is a particular type of a *right prism*, while any other parallelepiped is an oblique prism. The determination of the surface area and volume of a parallelepiped is dealt with in Art. 29.

THEOREM 9E.

If two planes are parallel, any straight line in one plane is parallel to the other plane.

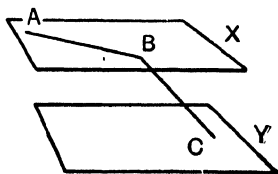


Fig. 74.

Given that **X** and **Y** are two parallel planes, and **AB** is a straight line in **X**.

Required to prove that **AB** is parallel to the plane **Y**.

Proof. If possible, suppose that **AB** is not parallel to the plane **Y**.

Then **AB** will meet the plane **Y** in some point **C**.

Because **C** lies in **AB**, and **AB** lies in the plane **X**,

$\therefore$ **C** lies in the plane **X**.

But **C** lies in the plane **Y**.

Hence the planes **X** and **Y** have at least one point **C** in common.

But this is impossible since the plane **X** is parallel to the plane **Y**.

Hence AB must be parallel to the plane Y.

Note.—As illustrations of Theorem 9E note that any straight line drawn on the ceiling of a room is parallel to the floor, and that any straight line drawn on one face of a parallelepiped is parallel to the opposite face.

THEOREM 10E.

If two lines are parallel then either of them is parallel to any plane through the other.

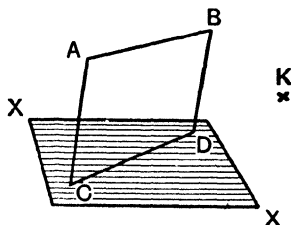


Fig. 75.

Given that Line **AB** is parallel to line **CD**, and **CD** lies in the plane **XX**.

Required to prove that **AB** is parallel to the plane **XX**.

Proof. If possible, let **AB** meet the plane **XX** at some point **K**.

$\therefore$ **AB** is parallel to **CD**,

$\therefore$ these two lines lie in one plane **ABDC**.

$\therefore$ **K**, which lies in **AB**, also lies in the plane **ABDC**.

But **K** lies in the plane **XX**. (*Hyp.*)

Hence **K** lies in the intersection of the planes

ABDC and **XX**;

i.e. **K** lies in the line **CD**.

But this is impossible, since **AB** is parallel to **CD**. (*Hyp.*)

Hence AB is parallel to the plane XX.

Note.—As illustrations of Theorem 10E the student should note that if a rod is held parallel to the hinge line of a door, it is parallel to the plane of the door however the door may swing. An edge of a parallelepiped is parallel to any plane through the opposite edge.

THEOREM 11E.

If a given straight line is parallel to a given plane, and if a plane through the given line intersects the given plane, then the line of intersection is parallel to the given line.

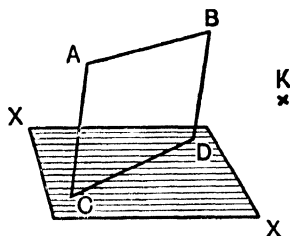


Fig. 76.

Given that **AB** is parallel to the plane **XX**, and the plane **ABDC** intersects the plane **XX** in the line **CD**.

Required to prove that **CD** is parallel to **AB**.

Proof. If possible, suppose that **CD** is not parallel to **AB**.

Then **CD** and **AB** will meet in some point **K**.

Because **K** lies in **CD**, and **CD** lies in the plane **X**,

$\therefore$ **K** lies in the plane **X**.

But **K** lies in the line **AB**.

Hence the line **AB** cuts the plane **X** in **K**.

But this is impossible, since **AB** is parallel to the plane **X**.

Hence **CD** and **AB** do not meet.

But **CD** and **AB** lie in the same plane **ABDC**.

Hence CD and AB must be parallel.

THEOREM 12E.

If a straight line is parallel to each of two intersecting planes it is parallel to their line of intersection.

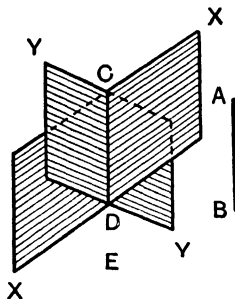


Fig. 77.

Given that Line **AB** is parallel to the plane **X** and to the plane **Y**.

Required to prove that **AB** is parallel to the line of intersection of **X** and **Y**.

Proof. Let **C** be any point in the line of intersection of the planes **X** and **Y**.

Consider the plane **ABC**.

Because **AB** is parallel to plane **X**,
 $\therefore$ plane **ABC** meets **X** in some line **CD** which is parallel to **AB**. (Theorem 11E.)

Because **AB** is parallel to **Y**,
 $\therefore$ plane **ABC** meets **Y** in some line **CE** which is parallel to **AB**. (Theorem 11E.)

Since **CD** and **CE** are both parallel to **AB** they must coincide;

i.e. **CD** lies in both **X** and **Y** and is therefore their line of intersection.

Hence **AB is parallel to the line of intersection of the planes **X** and **Y**.**

Note.—Theorem 12E may be illustrated thus :—Let the two planes be represented by a wall and a half-open door whose hinge line lies in the wall. If a rod is held parallel to both these planes, it will be found to be parallel to the hinge line. See also illustrations in Fig. 73.

THEOREM 13E.

If two planes are each parallel to a third plane they are parallel to one another.

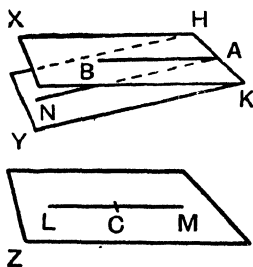


Fig. 78.

Given that The plane **X** is parallel to the plane **Z**, and
the plane **Y** is parallel to the plane **Z**.

Required to prove that The plane **X** is parallel to the plane **Y**.

Proof. If possible, suppose that the planes **X** and **Y**
are not parallel; then they meet in some line **HK**.

Take three points **A**, **B**, and **C** such that

A lies on **HK**,

B lies on the plane **X**, but not on **HK**,

C lies on the plane **Z**.

Let **W** be the plane which contains these three points.

Then **W** meets **X** in the line **AB**; it meets **Y** in some
different line **AN**, through the point **A**; and it meets
Z in some line **LM**.

Because the plane **X** is parallel to the plane **Z**,

$\therefore$ **BA** is parallel to **LM**. (Theorem 7E.)

Because the plane **Y** is parallel to the plane **Z**,

$\therefore$ **NA** is parallel to **LM**. (Theorem 7E.)

Thus we have two intersecting lines **BA** and **NA** parallel
to the same line **LM**; which is impossible.

Hence X and Y must be parallel planes.

THEOREM 14E.

If two straight lines are cut by three parallel planes, they are cut proportionally.

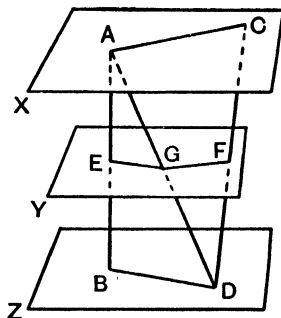


Fig. 79.

Given that The straight lines **AB, CD**, are cut by the three parallel planes **X, Y, Z** at **A, E, B** and **C, F, D** respectively.

Required to prove that **$AE : EB = CF : FD$** .

Construction. Join **AC, BD, AD**.

Let **AD** meet the plane **Y** in **G**.

Join **EG, FG**.

Proof. Because the two parallel planes **Y** and **Z** are cut by the third plane **ABD**,

$\therefore$ their common sections **EG** and **BD** are parallel. (Theorem 7E.)

Because the two parallel planes **X** and **Y** are cut by the third plane **ADC**,

$\therefore$ their common sections **AC** and **GF** are parallel. (Theorem 7E.)

In $\triangle ABD$, $\therefore EG$ is parallel to **BD**,

$\therefore AE : EB = AG : GD$; (Theorem 12D.)

and in $\triangle ACD$, $\therefore GF$ is parallel to **AC**,

$\therefore AG : GD = CF : FD$. (Theorem 12D.)

Hence $AE : EB = CF : FD$.

Exercises VII.

1. If two intersecting lines in one plane are each parallel to another plane, then the two planes are parallel.

2. If two sides of a triangle are parallel to a given plane so also is the third.

3. Two parallel planes cut two intersecting planes in pairs of lines which contain equal angles.

4. Draw through a given point, **A**, a straight line to intersect each of two straight lines, **BC**, **DE**, which are neither parallel nor concurrent.

5. Three lines are parallel but not in one plane. If two planes make equal intercepts on these three lines prove that they are parallel.

6. Show that an infinite number of straight lines may be drawn to intersect each of three given straight lines, no two of which intersect or are parallel.

7. How many lines can be drawn through a given point parallel to a given plane?

8. A given straight line is parallel to a given plane. Prove that any straight line, drawn through a point in the given plane, and parallel to the given line, will lie entirely in the given plane.

9. If two planes are parallel any straight line which is parallel to one of them is also parallel to the other.

10. Two straight lines are neither parallel nor concurrent. Show that it is always possible to find one and only one plane which passes through one of them and is parallel to the other.

11. Two straight lines are neither parallel nor concurrent. Show how to find two parallel planes which pass through these two lines.

12. Given two straight lines which are neither parallel nor concurrent, and a point which does not lie on either, show that it is always possible to find a plane through the given point parallel to both of the given lines.

13. Any section of a parallelepiped by a plane which cuts two pairs of opposite faces only is a parallelogram.

14. A solid figure is bounded by six quadrilateral faces. If two opposite faces are congruent parallelograms having corresponding sides parallel, prove that the figure is a parallelepiped.

15. Two parallelograms are placed in different planes and intersect in a line which divides either parallelogram into two congruent parallelograms. Prove that the eight vertices of these two parallelograms are the corners of a parallelepiped.

16. If through a variable point in a plane a line is drawn in a fixed direction and of constant length, find the locus of its extremity.

17. If P and Q are points one on each of two lines which are neither parallel nor concurrent, the locus of the middle point of PQ is a plane.

18. A system of planes drawn through one line meets any other plane in a system of lines which are either concurrent or parallel.

19. The intercepts formed on a system of lines by two parallel planes are divided in a constant ratio. Find the locus of the point of section.

20. A pyramid is described on a given polygon as base. Show that the section of the pyramid made by any plane parallel to the base is a polygon similar to the base.

21. A pyramid is described on a given triangle as base, and the three edges which meet at the vertex are divided in the same ratio. Show that the three points of section determine a plane which is parallel to the base.

22. A pyramid is described on a given polygon as base, and the edges which meet at the vertex are all divided in the same ratio. Show that the points of section all lie in one plane which is parallel to the base.

23. A series of planes are drawn through a point, each containing one of a set of parallel lines. Prove that any plane which does not pass through the given point intersects these planes in a series of parallel or concurrent lines.

24. Three planes intersect in a single line ; prove that they divide any two parallel lines proportionately.

CHAPTER VIII.

LINES PERPENDICULAR TO PLANES.

21. Definitions.—The projection of a line on a plane is the locus of the feet of the perpendiculars from all points of the line to the plane. Thus in Figs. 80 and 81 the projection of the line AB on the plane X is the line ab . It is clear that the projection of a straight line on a plane is itself a straight line.

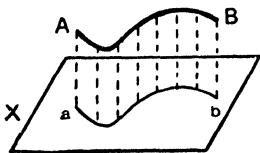


Fig. 80.

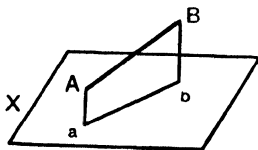


Fig. 81.

The inclination of a given straight line to a given plane is measured by the acute angle between the given straight line and its projection on the given plane.

A straight line is said to be perpendicular to a plane if it is perpendicular to every line in the plane which meets it (see Fig. 82). A straight line which is perpendicular to a plane is called a **normal** to the plane.

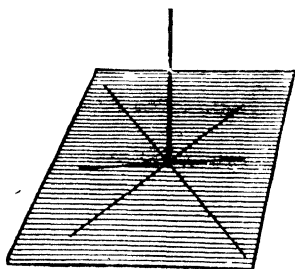


Fig. 82.

For example, a flagstaff standing on level ground should be perpendicular to any straight line on the ground drawn from its foot, *i.e.* the flagstaff should be perpendicular to the plane of the ground; in machinery the axle of a wheel should be perpendicular to every spoke of the

wheel, and therefore to the plane of the wheel.

THEOREM 15E.

If a given straight line is perpendicular to two straight lines in a given plane which meet it, it shall be perpendicular to the given plane.

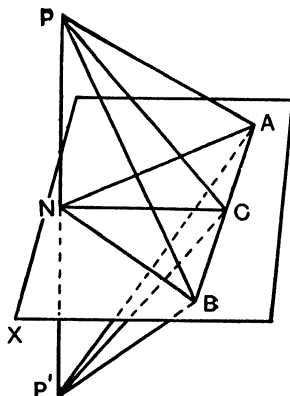


Fig. 83.

Given that **PN** meets the plane **X** in **N**, and is perpendicular to **NA** and **NB**, both of which lie in the plane **N**.

Required to prove that **PN** is perpendicular to any other straight line **NC** drawn through **N** in the plane **X**, and hence is perpendicular to the plane **X**.

Construction. Produce **PN** through the plane **X** to **P'**, making **P'N = PN**.

In the plane **X** draw any straight line meeting the three lines **NA**, **NB**, **NC** in **A**, **B**, **C** respectively.

Join **P** and **P'** to each of the points **A**, **B**, **C**.

Proof. Because **NA** bisects **PP'** at right angles,

$$\therefore PA = P'A. \quad (\text{Theorem 34A.})$$

Because **NB** bisects **PP'** at right angles,

$$\therefore PB = P'B. \quad (\text{Theorem 34A.})$$

In $\triangle$ s **PAB** and **P'AB**, $\therefore PA = P'A$, (Proved.)
 and **PB** = **P'B**, (Proved.)
 and **AB** is common to both triangles,

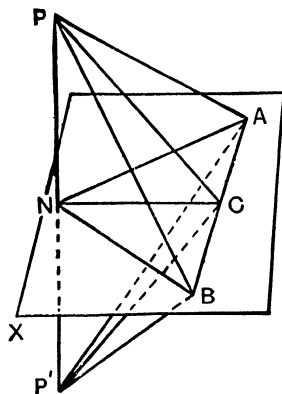


Fig. 84.

$\therefore \triangle PAB \equiv \triangle P'AB$, (Theorem 15A.)
 and $\angle PAC = \angle P'AC$.

In $\triangle$ s **PAC** and **P'AC**, $\therefore PA = P'A$, (Proved.)
 and **AC** is common to both triangles,
 and $\angle PAC = \angle P'AC$, (Proved.)
 $\therefore \triangle PAC \equiv \triangle P'AC$, (Theorem 11A.)
 and **PC** = **P'C**.

In $\triangle$ s **PNC** and **P'NC**, $\therefore PN = P'N$, (Cons.)
 and **PC** = **P'C**, (Proved.)
 and **NC** is common to both triangles,
 $\therefore \triangle PNC \equiv \triangle P'NC$, (Theorem 15A.)
 and $\angle PNC = \angle P'NC$.
 $\therefore PNP'$ is perpendicular to **NC**.

Hence PNP' is perpendicular to the plane X.

THEOREM 16E.

If a straight line is perpendicular to each of three concurrent straight lines at their point of intersection, these three straight lines shall lie in one plane.

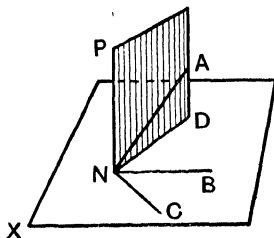


Fig. 85.

Given that PN is perpendicular to each of the three straight lines NA , NB , NC .

Required to prove that NA , NB , NC lie in one plane.

Construction. Let X be the plane which passes through NB and NC .

Let a plane pass through NP and NA ; and let ND be the intersection of this plane with the plane X .

Proof. NB , NC , ND lie in the same plane X ; (Cons.)
also PN is perpendicular to NB , and PN is perpendicular to NC ,

$\therefore PN$ is perpendicular to ND . (Theorem 15E.)

But PN is perpendicular to NA ,
and PN , NA , ND are in one plane.

Thus NA and ND coincide.

Hence NA lies in the plane X .

COROLLARIES TO THEOREM 16E.

COROLLARY 1.—If from a given point in a given straight line a series of lines are drawn perpendicular to the given straight line, the locus of these lines is a plane perpendicular to the given straight line.

COROLLARY 2.—Through a given point in a given straight line it is possible to draw one and only one plane perpendicular to the given straight line.

THEOREM 17E.

Through a given point in a given plane it is always possible to draw one and only one straight line perpendicular to the given plane.

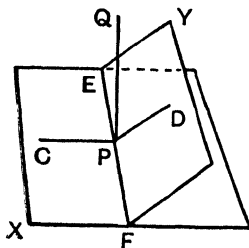


Fig. 86.

Given that **P** is any point in the plane **X**.

Required to prove that One and only one straight line can be drawn through **P** perpendicular to the plane **X**.

Construction. In the plane **X** through the point **P** draw any two perpendicular lines **EF**, **PC**.

Take any other plane **Y** through **EF**, and in it draw **PD** perpendicular to **EF**.

In the plane **CPD** draw **PQ** perpendicular to **PC**.

Then **PQ** is the required perpendicular.

Proof. (1) **To prove that PQ is perpendicular to the plane X.**

FP is perpendicular to **PC** and to **PD**, (Cons.)

$\therefore$ **FP** is perpendicular to the plane **CPD**.

(Theorem 15E.)

But **PQ** lies in the plane **CPD**, (Cons.)

$\therefore$ **FP** is perpendicular to **PQ**.

Because **PQ** is perpendicular to **FP**, (Proved.)

and **PQ** is perpendicular to **PC**, (Cons.)

$\therefore$ **PQ** is perpendicular to the plane **FPC**,

(Theorem 15E.)

that is, PQ is perpendicular to the plane X.

- (2) To prove that no other line can be drawn through P perpendicular to the plane X .

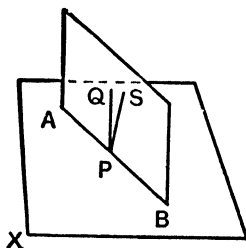


Fig. 87.

Suppose that another line PS is drawn through P perpendicular to the plane X .

Let the plane PQS meet the plane X in the line AB .

Then, since PQ and PS are both perpendicular to the plane X , and since AB lies in the plane X ,

$\therefore QPB$ and SPB are both right angles.

But these are angles in the same plane, and so cannot be equal unless PS and PQ coincide.

Hence PQ is the only line through P which is perpendicular to the plane X .

Note.—A construction for drawing a straight line perpendicular to a plane from a given point *outside* it will be given in Problem 1E. The student should note that by a proof similar to that given in the second part of Theorem 17E it can be shown that only one perpendicular can be drawn to the plane when the given point is outside the plane. It is thus clear that it is always possible to draw one perpendicular, and only one, from a given point to a given plane, whether the given point lies in or outside the plane.

THEOREM 18E.

Planes to which the same straight line is perpendicular are parallel to one another.

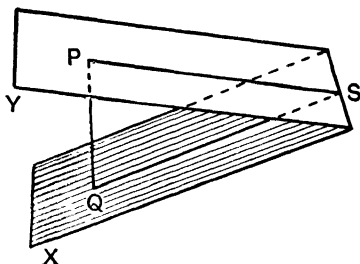


Fig. 88.

Given that **PQ** is perpendicular to each of the planes **X**, **Y**.

Required to prove that The plane **X** is parallel to the plane **Y**.

Construction. If the planes **X** and **Y** be not parallel, they will meet if produced.

Let **S** be any point in their intersection.

Join **PS**, **QS**.

Proof. Because **PQ** is perpendicular to the plane **X**,

$\therefore \angle PQS$ is a right angle.

Because **PQ** is perpendicular to the plane **Y**,

$\therefore \angle QPS$ is a right angle.

Thus in $\triangle PQS$, $\angle PQS + \angle QPS = 2$ right angles.

But this is impossible. (*Theorem 10A.*)

Hence the planes X and Y must be parallel.

THEOREM 19E.

If two planes are parallel any line which is perpendicular to one is also perpendicular to the other.

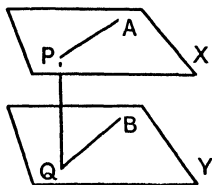


Fig. 89.

Given that The plane **X** is parallel to the plane **Y**, and
PQ is perpendicular to the plane **X**.

Required to prove that **PQ** is perpendicular to the plane **Y**.

Construction. Let **QB** be any straight line through **Q**
 in the plane **Y**,

and let the plane **PQB** meet the plane **X** in the line **PA**.

Proof. Because **PA**, **QB** are the intersections of the
 plane **PQB** with the parallel planes **X**, **Y**,
 $\therefore$ **PA** is parallel to **QB**. (Theorem 7E.)

Hence $\angle \text{APQ} + \angle \text{PQB} = 2 \text{ right angles}$.

Because **PQ** is perpendicular to the plane **X**,

$\therefore \angle \text{APQ}$ is a right angle.

Hence $\angle \text{PQB}$ is a right angle,

i.e. **PQ** is perpendicular to **QB**.

Similarly, **PQ** is perpendicular to any other line in the
 plane **Y**.

Hence PQ is perpendicular to the plane Y.

THEOREM 20E.

If two straight lines are parallel, and one is perpendicular to a plane, the other is perpendicular to the same plane.

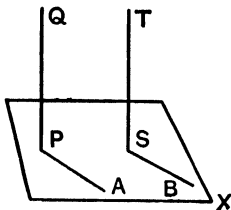


Fig. 90.

Given that **PQ** is parallel to **ST**, and **PQ** is perpendicular to the plane **X**.

Required to prove that **ST** is perpendicular to the plane **X**.

Construction. In the plane **X** draw any straight line **SB** through **S**, and a parallel line **PA** through **P**.

Proof. Because **PQ** is perpendicular to the plane **X**,

$\therefore \angle QPA$ is a right angle.

In the planes **QPA**, **TSB**,

because **PQ** is parallel to **ST** and **PA** is parallel to **SB**,

$\therefore \angle QPA = \angle TSB$. (Theorem 6E.)

Hence $\angle TSB$ is a right angle,

i.e. **ST** is perpendicular to **SB**.

Similarly, **ST** is perpendicular to any other straight line drawn through **S** in the plane **X**.

Hence ST is perpendicular to the plane X.

THEOREM 21E.

If two lines are perpendicular to the same plane they are parallel to each other.

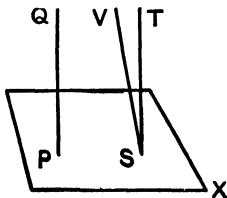


Fig. 91.

Given that **PQ** and **ST** are both perpendicular to the plane **X**.

Required to prove that **ST** is parallel to **PQ**.

Construction. Through **S** draw **SV** parallel to **PQ**.

Proof. Because **PQ** is perpendicular to the plane **X**, and **SV** is parallel to **PQ**,

$\therefore$ **SV** is perpendicular to the plane **X**.

(Theorem 20E.)

Also **ST** is perpendicular to the plane **X**. (Hyp.)

But there cannot be two different straight lines through

S perpendicular to the plane **X**, (Theorem 17E.)

$\therefore$ **ST** and **SV** coincide.

But **SV** is parallel to **PQ**.

Hence ST is parallel to PQ.



THEOREM 22E.

If a straight line is perpendicular to two intersecting lines which it does not meet, it is perpendicular to the plane containing them.

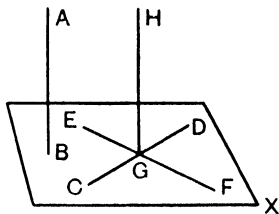


Fig. 92.

Given that The straight line **AB** is perpendicular to the two intersecting lines **CD** and **EF**.

Required to prove that **AB** is perpendicular to the plane **X** which contains **CD** and **EF**.

Construction. Through **G**, the point of intersection of **CD** and **EF**, draw **GH** parallel to **BA**.

Proof. Because **HG** is parallel to **AB**, and **AB** is perpendicular to **CD**,
 $\therefore$ **HG** is perpendicular to **CD**.

Similarly, **HG** is perpendicular to **EF**.
 $\therefore$ **HG** is perpendicular to **CD** and **EF**,
 whence **HG** is perpendicular to the plane **X**.
(Theorem 15E.)

But **AB** is parallel to **HG**,
 $\therefore$ **AB** is perpendicular to the plane **X**.
(Theorem 20E.)

Hence AB is perpendicular to the plane containing CD and EF.

COROLLARY TO THEOREM 22E.

If a straight line is perpendicular to two sides of a triangle it is also perpendicular to the third.

PROBLEM 1E.

To draw a straight line perpendicular to a given plane from a given point without it.

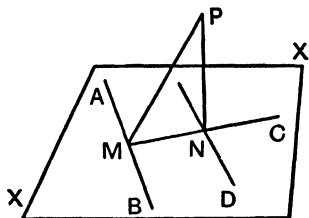


Fig. 93.

Given A plane **X** and a point **P** outside it.

Required To draw from **P** a straight line perpendicular to the plane **X**.

Construction. Draw any straight line **AB** in the plane **X**.

From **P** draw **PM** perpendicular to **AB**.

Then, if **PM** is perpendicular to the plane **X**, what was required is done.

If not, from **M** draw **MC** in the plane **X** perpendicular to **AB**.

From **P** draw **PN** perpendicular to **MC**.

Then PN shall be perpendicular to the plane X.

Proof. Let **ND** be drawn through **N** in the plane **X** parallel to **AB**.

Because **AB** is perpendicular to **MP** and to **MC**,

$\therefore$ **AB** is perpendicular to the plane **PMC**.

(Theorem 15E.)

But **DN** is parallel to **AB**,

$\therefore$ **DN** is perpendicular to the plane **PMC**.

(Theorem 20E.)

Hence $\angle$ **PND** is a right angle.

But $\angle$ **PNM** is a right angle.

(Cons.)

$\therefore$ **PN** is perpendicular to the plane **MND**.

(Theorem 15E.)

Hence PN has been drawn perpendicular to the plane X.

PROBLEM 2E.

Through two given straight lines, which do not meet and are not parallel, to draw a straight line which is perpendicular to both.

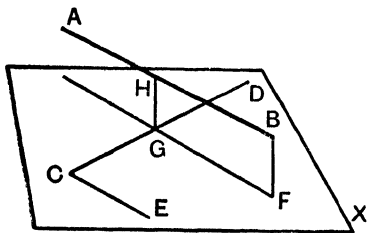


Fig. 94.

Given Two straight lines **AB** and **CD**, which do not meet and are not parallel.

Required To draw a straight line perpendicular to both **AB** and **CD**.

Construction. From any point **C** in **CD** draw **CE** parallel to **AB**.

Pass a plane **X** through **CD** and **CE**.

From any point **B** in **AB** draw **BF** perpendicular to the plane **X**.

In the plane **X**, from **F** draw **FG** parallel to **EC**, meeting **CD** in **G**.

From **G** draw **GH** perpendicular to **AB**.

Then GH shall be perpendicular to AB and CD.

Proof. Because **FG** is parallel to **CE** and **AB** is parallel to **CE**,

$\therefore$ **FG** is parallel to **AB**. (Theorem 5E.)

Thus **FG** and **AB** lie in one plane, viz. the plane **ABFG**.

Because **BF** is perpendicular to the plane **X**,

$\therefore$ **BF** is perpendicular to **FG**.

Also **FG** is parallel to **AB**,

$\therefore$ **BF** is perpendicular to **AB**.

Also **GH** is perpendicular to **AB**, and lies in the same plane as **BF**,

$\therefore$ **GH** is parallel to **BF**.

Since **GH** is parallel to **BF**, and **BF** is perpendicular to the plane **X**,

$\therefore$ **GH** is perpendicular to the plane **X**.

(Theorem 20E.)

But **CD** lies in the plane **X**,

$\therefore$ **GH** is perpendicular to **CD**.

But **GH** is also perpendicular to **AB**. (Cons.)

Hence GH has been drawn perpendicular to both AB and CD.

Note.—The figure of Problem 2E is the more easily understood if it be supposed that **AB** is a line drawn on the ceiling, and **CD** a line drawn on the floor. The plane **X** will then coincide with the floor, and **FG** will be the line in the floor which is vertically below the line **AB**.

Exercises VIII.

1. **ABC** is a triangle, and equal lines **AP**, **BQ**, **CR** are drawn perpendicular to the plane **ABC** and on the same side of it. Prove that triangles **ABC**, **PQR** are congruent.

2. If a straight line drawn through the centre of a circle at right angles to its plane is called the axis of the circle, prove that any point on the axis is equidistant from all points on the circumference of the circle.

3. Prove that the locus of all points in space which are equidistant from two given points is a plane.

4. The perpendicular is the least straight line that can be drawn from an external point to a plane.

5. From a given point without a given plane one and only one straight line can be drawn perpendicular to the given plane.

6. From a given point without a given straight line one and only one plane can be drawn perpendicular to the given straight line.

7. If perpendiculars be drawn from a given point to a plane and to any straight line in the plane, show that the line joining the feet of the perpendiculars is at right angles to the straight line in the plane.

8. PN is perpendicular to a plane, and NA is drawn in the plane perpendicular to a straight line BC in the plane. Prove that PA is also perpendicular to BC .

9. One and only one plane can be drawn through a given point parallel to a given plane.

10. OA, OB, OC are any three lines which meet at O . Show how to draw a line OP making equal angles with all three lines.

11. OA, OB, OC are any three lines in a plane which meet at O . Show that the only line through O making equal angles with all three of them is the perpendicular at O to the plane.

12. Find the locus of points equally distant from three given points.

13. If O is the orthocentre of a triangle ABC , and OP is drawn at right angles to the plane of ABC , show that PA is perpendicular to the straight line through A parallel to BC .

14. Straight lines are drawn from two given points A, B without a given plane, and on the same side of it, to a point P within the plane. Find when $AP + BP$ is a minimum.

15. Given two points (i) on adjacent walls, (ii) on opposite walls of a room, find the shortest path from one to the other along the walls.

16. AB and PQ are two straight lines, PQ being perpendicular to a plane passing through AB . Prove that all the perpendiculars to AB from different points in PQ pass through a fixed point.

17. The shortest distance between two straight lines which do not lie in one plane is perpendicular to each of them.

18. There is only one straight line which intersects at right angles each of two straight lines which do not lie in one plane.

19. GH is the shortest distance between two straight lines AHB, CGD , and $AH = HB, CG = GD$. Prove that $AC = BD$.

20. A and B are two points on opposite sides of a plane. Find a point P in the plane such that $PA \sim PB$ is a maximum.

21. Four points A, B, C, D are so situated that AB is perpendicular to the plane BCD , and $AD^2 = AB^2 + BC^2 + CD^2$. Prove that CD is perpendicular to the plane ABC .

22. If the perpendicular from P to the plane of a triangle ABC meets it in the orthocentre of the triangle, show that, if one of the three angles APC, CPB, APB is a right angle, then the others are also right angles. [Use Theorem 42A.]

CHAPTER IX.

DIHEDRAL, TRIHEDRAL, AND POLYHEDRAL ANGLES.

22. Dihedral Angles.—Two half planes drawn from the same line are said to form a dihedral angle (see Fig. 95). It follows, therefore, that two intersecting complete planes form four dihedral angles (see Fig. 61), just as two intersecting straight lines form four angles.

A dihedral angle is measured by the angle formed by two

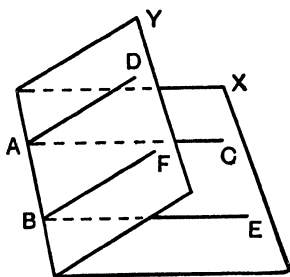


Fig. 95.

lines, one in each half plane, drawn from any point in the line of intersection of the half planes and perpendicular to that line of intersection. Thus in Fig. 95 **BE** is drawn in the half plane **X** perpendicular to the line of intersection **AB**, and from the same point **B**, **BF** is drawn in the half plane **Y** perpendicular to the line of intersection **AB**: the dihedral angle is measured by the angle **EBF**.

The dihedral angle is also measured by the angle **CAD**, or any similar angle. That all such angles are equal may be readily proved thus:—

In the plane **X**, **AC** is perpendicular to **AB**
and **BE** is perpendicular to **AB**,
∴ **AC** is parallel to **BE**.

Similarly, in the plane **Y**, **AD** is parallel to **BF**.

Now in the planes **DAC** and **FBE**,
because **AC** is parallel to **BE** and **AD** is parallel to **BF**,
∴ $\angle CAD = \angle EBF$.

Two intersecting planes are said to be *perpendicular* if any one of the four dihedral angles which they form is a right

angle. It is evident, of course, that if one of the four is a right angle, each of the others is also a right angle.

23. Trihedral and Polyhedral Angles.—If three intersecting planes meet at a point they are said to form a **trihedral angle**. Thus in Fig. 96, the three planes **OAC**, **OBC**, and **OAB** form a trihedral angle at **O**.

If more than three intersecting planes meet at a point they are said to form a **polyhedral angle**. Thus in Fig. 97, the five planes **OAE**, **OED**, **ODC**, **OBC**, and **OAB** form a polyhedral angle at **O**.

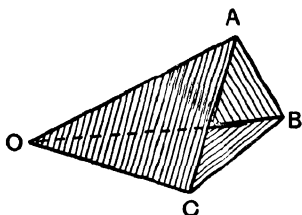


Fig. 96.

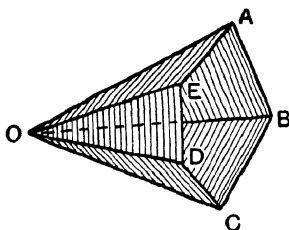


Fig. 97.

A polyhedral angle is said to be **convex** if a section of it made by any plane which intersects all its faces is a convex polygon.

Trihedral and polyhedral angles are sometimes included under the general name of **solid angles**. Thus if three or more intersecting planes meet at a point they form a solid angle: the point is called the **vertex** of the solid angle: the lines of intersection of consecutive planes are called the **edges** of the solid angle: the angles between consecutive planes are its **dihedral angles**: the angles formed by consecutive edges are called its **face angles**. The three face angles and the three dihedral angles of a trihedral angle are called its **six parts**.

The solid angle at **O** (Fig. 97) is referred to as the angle (**OABCDE**) or simply as the angle at **O**. It may be mentioned that the solid angle at a point **O** is measured by the area which it cuts off from a sphere drawn with centre **O** and unit radius. (Compare the circular measure of a plane angle in Trigonometry.)

THEOREM 23E.

If a straight line is perpendicular to a given plane then any plane passing through this straight line is also perpendicular to the given plane.

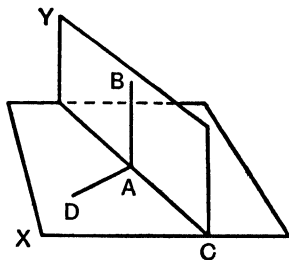


Fig. 98.

Given that The straight line **AB** is perpendicular to the plane **X**.

Required to prove that Any plane **Y** passing through **AB** is perpendicular to the plane **X**.

Construction. Let **AC** be the intersection of the planes **X** and **Y**.

Draw **AD** in the plane **X** perpendicular to **AC**.

Proof. Because **AB** is perpendicular to the plane **X**,
 $\therefore \angle \mathbf{BAD} = \angle \mathbf{BAC} = \text{a right angle.}$

Because **AB** and **AD** are each perpendicular to **AC**,
 $\therefore$ the dihedral angle is measured by $\angle \mathbf{BAD}$.

But $\angle \mathbf{BAD} = \text{a right angle.}$

Hence the plane Y is perpendicular to the plane X.

THEOREM 24E.

If two planes are perpendicular any line in one plane perpendicular to their intersection is perpendicular to the other plane.

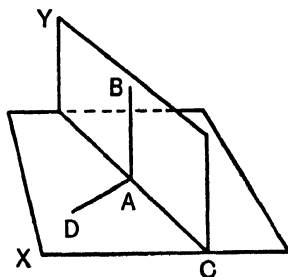


Fig. 99.

Given that The plane **Y** is perpendicular to the plane **X**,
 and **AB** in the plane **Y** is perpendicular to **AC**,
 the line of intersection of the two planes.

Required to prove that **AB** is perpendicular to the plane **X**.

Construction. In the plane **X** draw **AD** perpendicular to **AC**.

Proof. Because the plane **Y** is perpendicular to the plane **X**
 and **AB** and **AD** are each perpendicular to **AC**,

$$\therefore \angle BAD = \text{a right angle.}$$

$$\text{Also } \angle BAC = \text{a right angle.} \quad (\text{Hyp.})$$

Hence AB is perpendicular to the plane X.

THEOREM 25E.

If two intersecting planes are each perpendicular to a third, then the line of intersection of these two planes is perpendicular to the third.

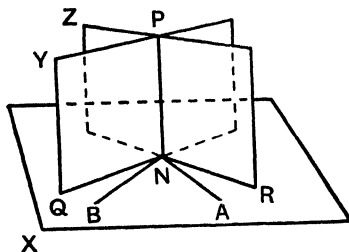


Fig. 100.

Given that The planes **Y** and **Z** are each perpendicular to the plane **X**.

Required to prove that **NP**, the line of intersection of the planes **Y** and **Z**, is perpendicular to the plane **X**.

Construction. Let **NQ** and **NR** be the intersections of the planes **Y** and **Z** with the plane **X**.

In the plane **X** draw **NA** perpendicular to **NQ** and **NB** perpendicular to **NR**.

Proof. Because the plane **X** is perpendicular to the plane **Y**, and because **NA** (in plane **X**) is perpendicular to the intersection **NQ**,

$\therefore$ **NA** is perpendicular to the plane **Y**.

(Theorem 24E.)

Thus **NA** is perpendicular to **NP**.

Similarly, **NB** is perpendicular to the plane **Z**, whence **NB** is perpendicular to **NP**.

Thus **NP** is perpendicular to **NA** and **NB**, and **NA** and **NB** are in the plane **X**.

Hence NP is perpendicular to the plane X.

PROBLEM 3E.

Through any given straight line to draw a plane perpendicular to a given plane.

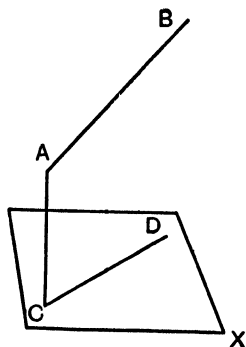


Fig. 101.

Given The straight line **AB** and the plane **X**.

Required To draw a plane through the straight line **AB** perpendicular to the plane **X**.

CASE I.—GIVEN STRAIGHT LINE PERPENDICULAR TO GIVEN PLANE.

If the given straight line **AB** is perpendicular to the given plane **X**, then *any* plane through **AB** will be perpendicular to the plane **X**. (*Theorem 23E.*)

CASE II.—GIVEN STRAIGHT LINE NOT PERPENDICULAR TO GIVEN PLANE.

Construction. From the point **A** in the line **AB** draw **AC** perpendicular to the plane **X**.

Then shall BAC be the required plane perpendicular to the plane X.

Proof. Because **AC** is in the plane **BAC**,
and **AC** is perpendicular to the plane **X**,

$\therefore$ the plane **BAC** is perpendicular to the plane **X**.
 (Theorem 23E.)

Hence BAC is the plane required.

COROLLARY TO PROBLEM 3E.

The projection of a given straight line on a given plane is a straight line, unless the given straight line is perpendicular to the plane.

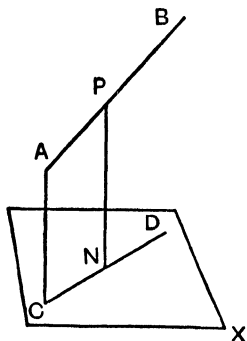


Fig. 102.

In Fig. 102, the projection of **AB** on **X** is the line **CD**, which is the intersection of the planes **BAC** and **X**.

Let **P** be any point on **AB**. Draw **PN** perpendicular to **CD**.

Because the plane **BACD** is perpendicular to the plane **X**, and **PN** is perpendicular to **CD**,

$\therefore$ **PN** is perpendicular to the plane **X**.

(Theorem 24E.)

Thus the projection of **P** on the plane **X** is a point **N** on **CD**.

Similarly for any other point on **AB**.

Hence the projection of the straight line **AB** on the plane **X** is the straight line **CD**.

It is clear that if **AB** is perpendicular to **X** the line **CD** disappears and the projection of **AB** on **X** becomes a point.

THEOREM 26E.

If a straight line intersects a plane, then the acute angle which the straight line makes with its projection on the plane is less than the angle which it makes with any other straight line in the plane which meets it.

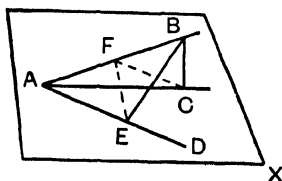


Fig. 103.

Given that The straight line **AB** meets the plane **X** at **A**;
BC is perpendicular to the plane **X**, and therefore **AC** is
the projection of **AB** on **X**;

AD is any other line drawn from **A** in the plane **X**.

Required to prove that $\angle \text{BAD}$ is greater than $\angle \text{BAC}$.

CASE I.—If $\angle \text{BAD}$ IS OBTUSE OR A RIGHT ANGLE.

In this case $\angle \text{BAD}$ is necessarily greater than $\angle \text{BAC}$.

CASE II.—If $\angle \text{BAD}$ IS AN ACUTE ANGLE.

Construction. Draw **BE** perpendicular to **AD**.

Bisect **AB** at **F**. Join **CF** and **EF**.

Proof. Because **BC** is perpendicular to the plane **X**,
 $\therefore \angle \text{BCE} = \text{a right angle}$.

Hence **BE** is greater than **BC**. (*Theorem 19A.*)

Because $\angle \text{AEB} = \angle \text{ACB} = \text{a right angle}$,

$\therefore \text{F}$ is the centre of the two circles **AEB** and **ACB**.

Hence **FE** = **FB** = **FC**,

also $\angle \text{BAE} = \frac{1}{2} \angle \text{BFE}$, and $\angle \text{BAC} = \frac{1}{2} \angle \text{BFC}$.

Now in $\triangle \text{s}$ **BFE** and **BFC**, $\therefore \text{FE} = \text{FC}$,

and **FB** is common to both triangles,

and **BE** is greater than **BC**,

$\therefore \angle \text{BFE}$ is greater than $\angle \text{BFC}$. (*Theorem 23A.*)

But $\angle \text{BAE}$ (*i.e.* $\angle \text{BAD}$) = $\frac{1}{2} \angle \text{BFE}$ and $\angle \text{BAC} = \frac{1}{2} \angle \text{BFC}$.

Hence $\angle \text{BAD}$ is greater than $\angle \text{BAC}$.

Proof. In the triangles **AVC** and **AVD**,
because **VC = VD**, (Cons.)
and **AV** is common to both triangles,
and $\angle \mathbf{AVC} = \angle \mathbf{AVD}$, (Cons.)
 $\therefore \triangle \mathbf{AVC} \equiv \triangle \mathbf{AVD}$ (Theorem 11A.)
and **AC = AD**.
In the triangle **ABC**,
AC + CB > AD + DB. (Theorem 20A.)

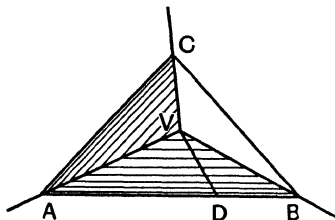


Fig 104.

But **AC = AD**,
 $\therefore \mathbf{CB} > \mathbf{DB}$.
In the triangles **BVC** and **BVD**,
because **VC = VD**, (Cons.)
and **VB** is common to both triangles,
and **CB > DB**,
 $\therefore \angle \mathbf{BVC} > \angle \mathbf{BVD}$. (Theorem 23A.)
But $\angle \mathbf{AVC} = \angle \mathbf{AVD}$, (Cons.)
 $\therefore \angle \mathbf{AVC} + \angle \mathbf{BVC} > \angle \mathbf{AVD} + \angle \mathbf{BVD}$.
Hence $\angle \mathbf{AVC} + \angle \mathbf{BVC} > \angle \mathbf{AVB}$.

THEOREM 28E.

In any trihedral or convex polyhedral angle the sum of the face angles is less than four right angles.

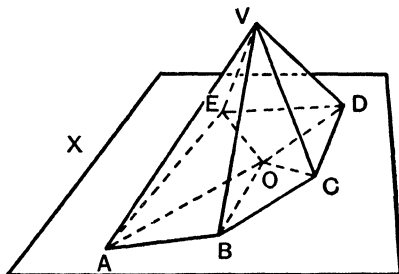


Fig. 105.

Given that The convex solid angle at **V** is contained by the plane angles **AVB, BVC, CVD, . . .**

Required to prove that The sum of these angles is less than four right angles.

Construction. Let a plane **X** intersect the planes of all the face angles on the same side of the vertex **V**, and let **AB, BC, CD, . . .** be the common sections of the plane **X** with the planes of the face angles.

Take any point **O** within the polygon **ABC . . .** and join **OA, OB, OC, . . .**

Proof. Because the solid angle at **A** is contained by the three plane angles **VAE, VAB, EAB**,

$$\therefore \angle VAE + \angle VAB > \angle EAB \quad (\text{Theorem 27E}).$$

Similarly $\angle VBA + \angle VBC > \angle ABC$,

$\angle VCB + \angle VCD > \angle BCD$,

and so on.

Hence, by addition,

the sum of the base angles of the triangles whose vertices are **V** is greater than the sum of the base angles of the triangles whose vertices are **O**.

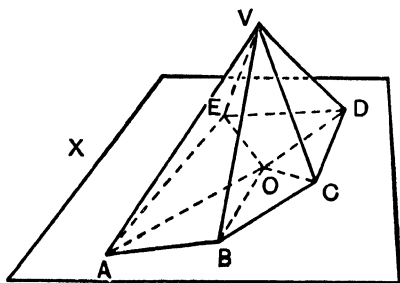


Fig. 105.

But these two sets of triangles are equal in number ;

$\therefore$ the sum of all the angles of the triangles with vertices

V is equal to the sum of all the angles of the

triangles with vertices **O**.

$\therefore$ the sum of all the vertical angles at **V** is less than the

sum of all the vertical angles at **O**.

But the sum of the angles at **O** is four right angles.

Hence the sum of all the angles at the vertex V is less than four right angles.

Exercises IX.

1. Show that if two dihedral angles are congruent, their measures are equal ; and conversely.

2. If two planes intersect prove that two vertically opposite dihedral angles are equal and two adjacent dihedral angles are supplementary.

3. If a plane intersects two parallel planes prove that any two of the dihedral angles so formed are either equal or supplementary.

4. If three planes intersect in three parallel lines show that the sum of the three interior dihedral angles is equal to two right angles.

5. Show how to bisect a given dihedral angle.

6. Equal straight lines drawn from the same point to a given plane are equally inclined to the plane.

7. Prove Theorem 25E indirectly by drawing lines from **N** in planes **Y** and **Z** perpendicular to their intersections with plane **X**.

8. Through a given point draw a straight line whose direction is perpendicular to the directions of two given lines which are neither parallel nor concurrent.

9. The angle of inclination of two planes is equal to an angle between any two intersecting lines perpendicular to them.

10. A straight line is equally inclined to two parallel planes.

11. Given two parallel straight lines and one plane, prove that the straight lines are equally inclined to the plane.

12. The projections of a straight line on parallel planes are both equal and parallel.

13. The projections of equal and parallel straight lines on the same plane are equal and parallel.

14. From a given point **P** two perpendiculars **PA**, **PB** are drawn, one to each of two intersecting planes. Prove that the common section of the planes is also perpendicular to the plane containing **PA**, **PB**.

15. In an acute dihedral angle the angle which any line in one plane makes with the other plane is not greater than the measure of the dihedral angle.

16. Find the locus of a point which moves so that it is always equidistant from two given intersecting straight lines.

17. Find the locus of a line which meets two given lines at their point of intersection and is equally inclined to them.

18. A, B are two fixed points in space. Find the locus of a point P if $PA^2 - PB^2$ is constant.

19. Given any three lines, no two of which lie in the same plane, show how to draw through a given point a straight line which shall be equally inclined to all three. How many solutions are there?

20. Two straight lines neither intersect nor are parallel. Show how to find planes on which their projections will be (a) parallel, (b) perpendicular, (c) inclined at any given angle.

21. If two straight lines in one plane are equally inclined to a second plane, then they are equally inclined to the common section of the two planes, and conversely.

22. The straight line OX lies within the trihedral angle $O(ABC)$. Prove that

$$\angle XO B + \angle XO C < \angle AO B + \angle AO C.$$

23. In the figure of Question 22, prove that

$$\angle XO A + \angle XO B + \angle XO C < \angle BO C + \angle CO A + \angle AO B.$$

24. If the straight line OX lies within or without the trihedral angle $O(ABC)$, prove that

$$\angle XO A + \angle XO B + \angle XO C > \frac{1}{2}(\angle BO C + \angle CO A + \angle AO B).$$

CHAPTER X.

POLYHEDRA.

24. Definitions.—Before beginning this Chapter the student should revise Art. 20. The further definitions given in the present Article are also required at this stage.

A **diagonal** of a polyhedron is a straight line joining *any two vertices not lying in the same face*.

Two polyhedra are said to be **similar polyhedra** if they have the same number of faces, similar and similarly placed, and have their corresponding polyhedral angles equal.

A **prism** is a polyhedron of which two faces, called the **ends**, are parallel and equal in every respect (*i.e.* are congruent), and the others, called **side faces**, are **parallelograms** (Figs. 106, 107). The ends of a prism may be triangles, quadrilaterals, or polygons, the prism being spoken of as a triangular prism, quadrilateral prism, and so on. The side edges are all equal and parallel to each other and to the **axis**, which is the line joining the centres of the two ends.

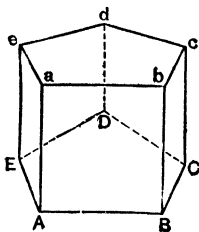


Fig. 106.

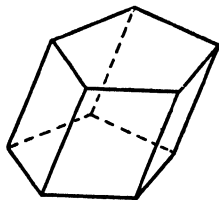


Fig. 107.

If the ends of a prism are at right angles to the axis the prism is called a **right prism** (Fig. 106) : if the ends are not at right angles to the axis it is called an **oblique prism** (Fig. 107). It is clear that a parallelepiped is a particular

type of prism, and that a rectangular block and cube are particular types of right prisms. A **truncated prism** is the portion of a prism included between the base (end) and a plane section inclined to the base.

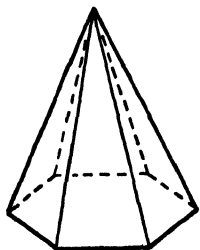


Fig. 108.

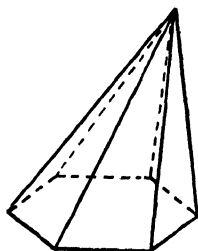


Fig. 109.

A **pyramid** is a polyhedron of which one face, called the **base**, is any rectilinear figure, and the others are **triangles meeting at a point**, called the **vertex**, which is outside the plane of the base (Figs. 108, 109). A pyramid whose base is a regular polygon is said to be a **right pyramid** when the vertex lies in the straight line drawn perpendicular to the base from its central point (*i.e.* the centre of the inscribed or circumscribed circle) as shown in Fig. 108: in other cases it is **oblique** (Fig. 109).

A **truncated pyramid** is a portion of a pyramid included between the base and a plane section inclined to the base.

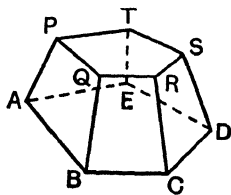


Fig. 110.

A **frustum of a pyramid** is a portion of a pyramid included between the base and a plane section parallel to the base or it is a portion cut off between two planes parallel to the base (Fig. 110).

A **tetrahedron** is a pyramid on a **triangular base**: it is thus contained by four triangular faces.

A **regular tetrahedron** is a tetrahedron all of whose **edges are equal**: it is thus contained by four equilateral triangular faces (Fig. 111).

There are analogies between the properties of the tetrahedron and the triangle: *e.g.* inscribed, circumscribed, and escribed spheres of a tetrahedron correspond to the inscribed, circum-

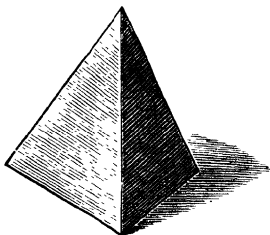


Fig. 111.

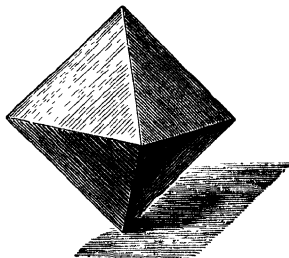


Fig. 112.

scribed, and escribed circles of a triangle. In the triangle the three lines drawn from the vertices perpendicular to the opposite sides are *always* concurrent, the point where they meet being the orthocentre. In the tetrahedron, however, the four lines drawn from the vertices perpendicular to the opposite faces are *not generally* concurrent: if they *are* concurrent the figure is called an **orthocentric tetrahedron**.

A regular octahedron is a polyhedron contained by eight equilateral triangles (Fig. 112).

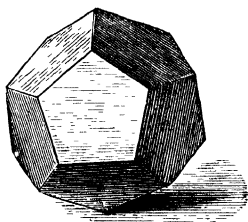


Fig. 113.

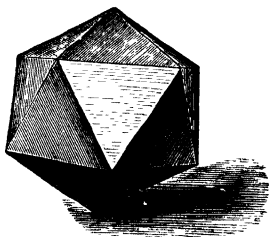


Fig. 114.

A regular dodecahedron is a polyhedron contained by twelve regular pentagons (Fig. 113).

A regular icosahedron is a polyhedron contained by twenty equilateral triangles (Fig. 114).

THEOREM 29E.

There cannot be more than five different types of regular polyhedra.

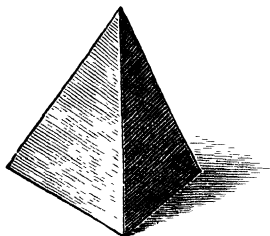


Fig. 115.

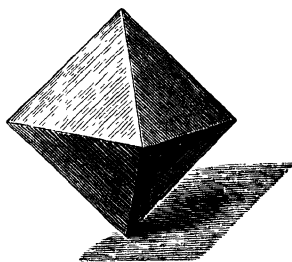


Fig. 116.

Proof. All the plane angles which bound a convex solid angle are together less than four right angles.

(Theorem 28E.)

Also a solid angle is bounded by three or more plane angles.

Therefore in a regular polyhedron, where all the plane angles are equal, each must be less than four-thirds of a right angle.

Again, the only regular polygons having angles each less than four-thirds of a right angle are equilateral triangles, squares, and regular pentagons.

Thus the faces of a regular polyhedron can only be
(i) equilateral triangles, (ii) squares, or
(iii) regular pentagons.

CASE I.—FACES EQUILATERAL TRIANGLES.

Each angle in an equilateral triangle is two-thirds of a right angle.

Therefore only three, four, or five of such angles can form a solid angle, for six or more of these angles are equal to four right angles or more.

Hence there may be regular polyhedra having three, four, or five angles of equilateral triangles to form each solid angle, but not more.

This gives three possible types of regular polyhedra whose faces are equilateral triangles, viz. :—

- (1) *The regular tetrahedron* (Fig. 115) *bounded by four equilateral triangles* (4 faces, 4 vertices, 6 edges).
- (2) *The regular octahedron* (Fig. 116) *bounded by eight equilateral triangles* (8 faces, 6 vertices, 12 edges).
- (3) *The regular icosahedron* (Fig. 117) *bounded by twenty equilateral triangles* (20 faces, 12 vertices, 30 edges).

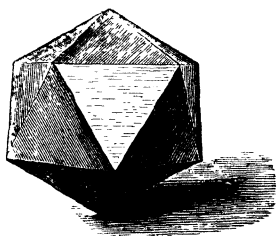


Fig. 117.

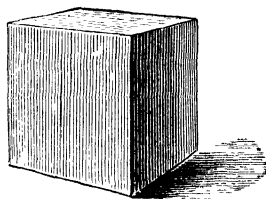


Fig. 118.

CASE II.—FACES SQUARES.

Each angle in a square is a right angle, therefore only three such angles can form a solid angle.

Hence there may be regular polyhedra having three right angles to form each solid angle, but not more.

This gives one possible type of regular polyhedron whose faces are squares, viz. :—

- (4) *The cube* (Fig. 118) *bounded by six squares* (6 faces, 8 vertices, 12 edges).

CASE III.—FACES REGULAR PENTAGONS.

Each angle in a regular pentagon is six-fifths of a right angle,

therefore only three such angles can form a solid angle, for four such angles are greater than four right angles.

Hence there may be regular polyhedra having three angles of regular pentagons to form each solid angle, but not more.

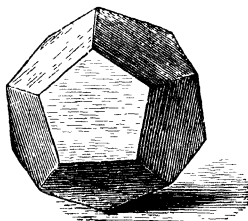


Fig. 119.

This gives one possible type of regular polyhedron, whose faces are regular pentagons, viz.:—

- (5) *The regular dodecahedron* (Fig. 119) *bounded by twelve regular pentagons* (12 faces, 20 vertices, 30 edges).

Hence there are five and only five different types of regular polyhedra.

Note.—If the faces of the above regular polyhedra are imagined to be unfolded and laid out flat we obtain plane figures made up of equilateral triangles, squares, and pentagons, which are called the **nets** of the various regular polyhedra. Thus:—

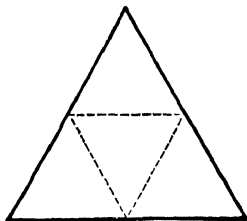


Fig. 120.

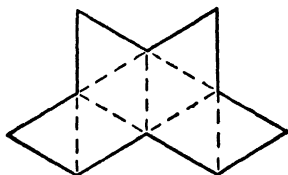


Fig. 121.

(a) Fig. 120 is the net of the regular tetrahedron: by cutting this out in cardboard and folding at the dotted lines a model of the solid is obtained.

(b) Similarly, Fig. 121 is the net of the regular octahedron. Again by folding at the dotted lines a model of the figure is obtained.

(c) A model of the regular icosahedron is obtained in the same way from the net shown in Fig. 122.

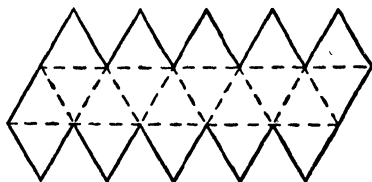


Fig. 122.

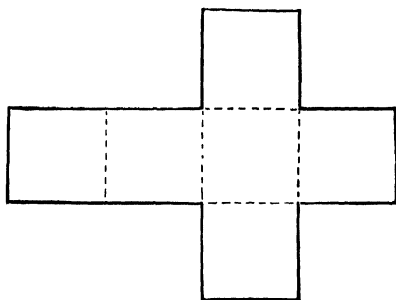


Fig. 123.

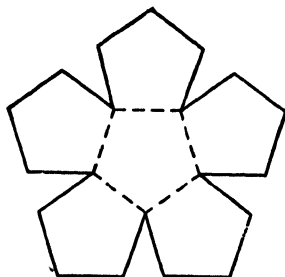


Fig. 124.

(d) The cube can be constructed from the net shown in Fig. 123.

(e) Finally, a model of the dodecahedron can be made by cutting out *two* figures like Fig. 124 and fitting them together.

THEOREM 30E.

(Euler's Theorem.)

If F , E , and C are the numbers of faces, edges, and corners respectively of any convex polyhedron, then

$$E + 2 = F + C.$$

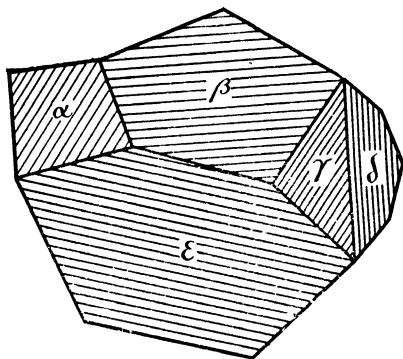


Fig. 125.

Proof. Suppose that the polyhedron is built up by starting with one face and adding new faces one by one, in such a way that at any stage of the process the faces form a *continuous* surface (which however is not a *closed* surface until the last face has been added).

In this process each new face will introduce a certain number of *new* edges and *new* corners. For example, suppose that in Fig. 125 we have the faces α , β , γ , δ ; then when the new face ϵ is added this introduces 3 new edges and 2 new corners.

Also it is obvious that when any new face is added (except the last face which *closes* the figure) **the number of new edges introduced is always one more than the number of new corners.**

Now when there is only one face, the number of edges is equal to the number of corners; *i.e.* $E = C$.

Also $F = 1$.

Hence in this case $E + 1 = F + C \dots\dots\dots (i)$

Again, each time a new face is added (except the last face) the value of E is increased by one more than

the value of C , (*Proved.*)

and the value of F is increased by 1.

Hence equation (i) is still true.

When the last face is added the values of E and C are unchanged, but the value of F is increased by 1.

Hence for the complete polyhedron

$$E + 2 = F + C.$$

Note.—As simple exercises on the two preceding propositions, the student should work the following at this stage:—

(1) A tetrahedron is being built up, face by face. State the number of edges and corners in the partially constructed figure for each value of F from 1 to 4.

(2) No polyhedron can have less than four faces and four corners.

(3) No polyhedron can have less than six edges and with this exception none can have less than eight edges.

(4) If the faces of a polyhedron be triangles then $F = 2C - 4$, where C is the number of corners.

THEOREM 31E.

The two lines which join any two vertices of a tetrahedron to the centroids of the opposite faces divide one another in the ratio 3 : 1.

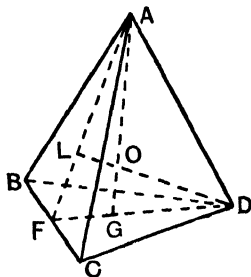


Fig. 126.

Given that In the tetrahedron $ABCD$, AG joins the vertex A to G , the centroid of the face BCD ; and DL joins the vertex D to L , the centroid of the face ABC .

Required to prove that AG and DL divide one another in the ratio 3 : 1.

Construction. Bisect BC at F , and join FA and FD .

Proof. $\because G$ is the centroid of $\triangle BCD$, and F bisects BC ,
 $\therefore G$ lies in FD , and $FD = 3FG$.

Similarly, L lies in FA , and $FA = 3FL$.

$\therefore AG$ and DL both lie in the plane of $\triangle ADF$,

$\therefore AG$ and DL meet at some point O .

$\because FA = 3FL$, and $FD = 3FG$, $\therefore \triangle FAD \parallel \triangle FLG$.

Hence $AD = 3LG$ and LG is parallel to AD .

$\therefore LG$ is parallel to AD , $\therefore \triangle LGO \parallel \triangle DAO$.

Hence $AO : OG = DO : OL = AD : GL = 3 : 1$.

COROLLARY TO THEOREM 31E.

The four lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.

For each of the lines joining B and C to the centroids of the opposite faces divides AG in the ratio 3 : 1, and therefore passes through O .

THEOREM 32E.

In a regular tetrahedron the perpendicular from any angular point to the opposite face meets that face at its centroid.

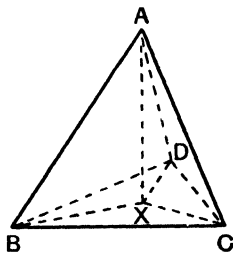


Fig. 127.

Given that **ABCD** is a regular tetrahedron, and **AX** is the perpendicular from **A** to the face **BCD**.

Required to prove that **X** is the centroid of the face **BCD**.

Construction. Join **XB**, **XC**, and **XD**.

Proof. Because **AX** is perpendicular to the plane **BCD**,

$$\therefore \angle AXB = \angle AXC = \angle AXD = \text{a right angle.}$$

$$\therefore BX^2 = AB^2 - AX^2.$$

$$\text{But } AB = AC,$$

$$\therefore BX^2 = AC^2 - AX^2.$$

$$\text{But } CX^2 = AC^2 - AX^2.$$

$$\text{Thus } BX = CX.$$

Similarly, it can be proved that **BX = DX**.

$$\text{Hence } BX = CX = DX.$$

$$\therefore \mathbf{X} \text{ is the circum-centre of } \triangle BCD.$$

But $\triangle BCD$ is an equilateral triangle, and therefore its circum-centre is its centroid.

Hence the perpendicular AX meets the opposite face at its centroid.

THEOREM 33E.

In any tetrahedron any two lines which join the middle points of opposite edges bisect one another.

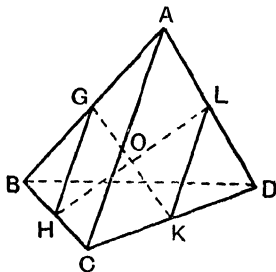


Fig. 128.

Given that In the tetrahedron **ABCD**, **GK** joins the mid-points of edges **AB**, **CD**, and **HL** joins the mid-points of edges **BC**, **AD**.

Required to prove that **GK** and **HL** bisect each other.

Construction. Join **GH**, **KL**.

Proof. In $\triangle ABC$, $\therefore$ **G** bisects **AB**, and **H** bisects **BC**,
 $\therefore$ $GH = \frac{1}{2}AC$ and **GH** is parallel to **AC**.

Similarly, in $\triangle ADC$, $LK = \frac{1}{2}AC$, and **LK** is parallel to **AC**.

Hence $GH = LK$ and **GH** is parallel to **LK**.

Thus **GH**, **LK** are in one plane and form a parallelogram.

$\therefore$ **GK**, **LH** are the diagonals of a parallelogram.

Hence GK, LH bisect each other.

COROLLARY TO THEOREM 33E.

In any tetrahedron the three lines which join the middle points of opposite edges are concurrent.

For if **M**, **N** are the mid-points of edges **AC**, **BD**, then **MN** bisects **GK** and therefore passes through **O**.

PROBLEM 4E.

About any tetrahedron to circumscribe a parallelepiped such that opposite edges of the tetrahedron are diagonals of opposite faces of the parallelepiped.

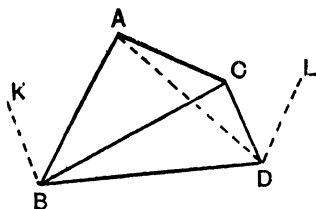


Fig. 129.

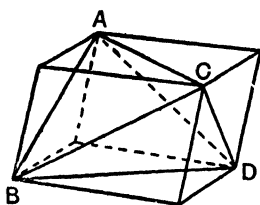


Fig. 130.

Given A tetrahedron **ABCD**.

Required To circumscribe about **ABCD** a parallelepiped such that opposite edges of **ABCD** are diagonals of opposite faces of the parallelepiped.

Construction. Take one pair of opposite edges of the tetrahedron, viz. **AB**, **CD**.

From **B** draw **BK** parallel to **DC**, and from **D** draw **DL** parallel to **BA**.

Then the planes **KBA**, **CDL** are parallel planes through **AB** and **CD** respectively. (*Theorem 6E.*)

Draw a second pair of parallel planes through the opposite edges **AC**, **BD**; and a third pair through the opposite edges **AD**, **BC**.

Proof. These three pairs of parallel planes form a parallelepiped. (*Def.*)

Again, three faces of the parallelepiped meet at **A**, viz. those drawn through the lines **AB**, **AC**, **AD**.

Hence **A** is a corner of the parallelepiped.

Similarly, each corner of the tetrahedron is a corner of the parallelepiped.

Also any two opposite edges of the tetrahedron lie in opposite faces of the parallelepiped. (*Cons.*)

It remains to show that each edge of the tetrahedron is a *diagonal* of the corresponding face of the parallelepiped (not an *edge* of the parallelepiped) :—

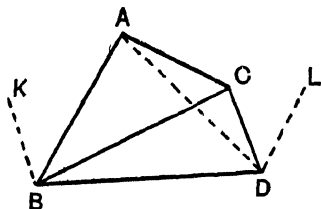


Fig. 129.

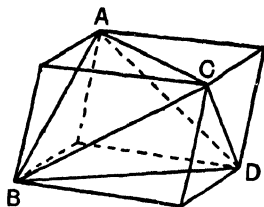


Fig. 130.

Consider the line **AB**.

This is a diagonal of the face in which it lies (*viz.* the plane through **AB**), because the other two faces which meet at **A** are respectively parallel to the other two faces which meet at **B**, *viz.*

the planes through **AC** and **AD** are respectively parallel to the planes

through **BD** and **BC**.

(*Cons.*)

Hence the parallelepiped fulfils the required conditions.

Note.—It is a useful exercise to prove Theorem 33E by Fig. 130, which shows the circumscribing parallelepiped.

THEOREM 34E.

If in a tetrahedron two pairs of opposite edges consist of perpendicular lines, then the third pair of opposite edges consists of perpendicular lines.

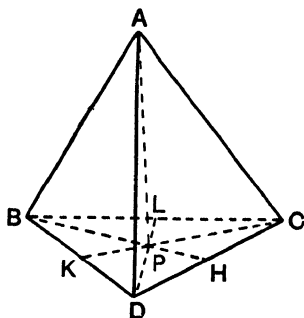


Fig. 131.

Given that $ABCD$ is a tetrahedron in which AB is perpendicular to CD , and AC is perpendicular to BD .

Required to prove that BC is perpendicular to AD .

Construction. Let P be the orthocentre of $\triangle BCD$, so that BH is perpendicular to CD , CK is perpendicular to DB , DL is perpendicular to BC .

Join AP .

Proof. In $\triangle APC$, BD is perpendicular to AC , (Hyp.)
and BD is perpendicular to PC ; (Cons.)

$\therefore BD$ is perpendicular to AP . (Theorem 22E, Cor.)

In $\triangle APB$, CD is perpendicular to AB , (Hyp.)

and CD is perpendicular to PB ; (Cons.)

$\therefore CD$ is perpendicular to AP . (Theorem 22E, Cor.)

In $\triangle BCD$, AP is perpendicular to BD and AP is perpendicular to CD ; (Proved.)

$\therefore AP$ is perpendicular to BC . (Theorem 22E, Cor.)

In $\triangle APD$, BC is perpendicular to AP , (Proved.)

and BC is perpendicular to PD . (Cons.)

Hence BC is perpendicular to AD .

Note.—This property is also easily proved by using Problem 4E.

THEOREM 35E.

If each edge of a tetrahedron is perpendicular to the opposite edge then the tetrahedron is orthocentric.

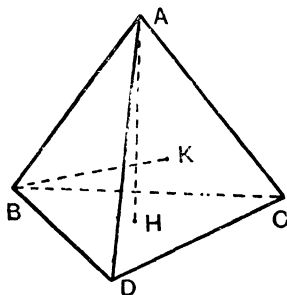


Fig. 132.

Given that In the tetrahedron **ABCD**, **AB** is perpendicular to **CD**, **AC** is perpendicular to **BD**, **AD** is perpendicular to **BC**.

Required to prove that The tetrahedron **ABCD** is orthocentric.

Construction. Let **AH**, **BK**, **CL**, **DM** be the four lines drawn from the vertices perpendicular to the opposite faces. (**CL**, **DM** are not shown in the figure.)

Proof. Because **AH** is perpendicular to $\triangle BCD$, (Cons.)
 $\therefore$ **AH** is perpendicular to **CD**.

Because **BK** is perpendicular to $\triangle ACD$, (Cons.)
 $\therefore$ **BK** is perpendicular to **CD**.

Because **CD** is perpendicular to **AB**, (Hyp.)
 and **CD** is perpendicular to **AH**, (Proved.)

$\therefore$ **CD** is perpendicular to the plane **BAH**.

(Theorem 22E.)

Because **CD** is perpendicular to **AB**, (Hyp.)
 and **CD** is perpendicular to **BK**, (Proved.)
 $\therefore$ **CD** is perpendicular to the plane **ABK**.
 (Theorem 22E.)

Because the planes **BAH** and **ABK** are both
 perpendicular to **CD**,
 $\therefore$ these planes are either parallel or coincident.
 (Theorem 18E.)

But they have the two points **A** and **B** in common, and are
 therefore coincident.

Hence **AH** and **BK** lie in the same plane.

Now **AH** and **BK** are not parallel, for if they were the
 faces to which they are perpendicular would be
 either parallel or coincident.

$\therefore$ **AH** and **BK** intersect.

Similarly, it can be proved that any two of the lines
AH, BK, CL, DM intersect.

Now since any two of the three lines **BK, CL, DM** intersect,
 these three lines either lie in one plane (which would
 be the plane **BCD**), or else are concurrent.

But the line **AH**, which does *not* lie in the plane **BCD**,
 meets each of the lines **BK, CL, DM**. This is not
 possible unless all four lines are concurrent.

Hence the tetrahedron is orthocentric.

THEOREM 36E.

Any plane section of a pyramid taken parallel to the base is similar to the base, and the area of such a section varies as the square of its distance from the vertex.

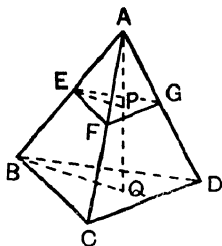


Fig. 133.

Given that **ABCD** is a pyramid, and **EFG** a plane section parallel to the base **BCD**.

Required to prove that The figure **EFG** is similar to the base **BCD**, and its area varies as the square of its distance from the vertex.

Construction. Draw **APQ** perpendicular to the base meeting the section at **P**.

Join **EP** and **BQ**.

Proof.

PART I.

Because the planes **EFG** and **BCD** are parallel and they are cut by the plane **BCFE**,

$\therefore$ **EF** is parallel to **BC**. (Theorem 7E.)

Similarly, **FG** is parallel to **CD**.

$\therefore \angle \text{EFG} = \angle \text{BCD}$. (Theorem 6E.)

In the same way it can be shown that the angles at **E** and **G** are equal to the angles at **B** and **D** respectively.

*Hence the figures **EFG** and **BCD** are equiangular.*

Again, because **EF** is parallel to **BC**,

$$\therefore \triangle \text{s } \mathbf{AEF}, \mathbf{ABC} \text{ are similar, i.e. } \frac{\mathbf{EF}}{\mathbf{BC}} = \frac{\mathbf{AF}}{\mathbf{AC}}.$$

$$\text{Similarly, } \triangle \text{s } \mathbf{AFG}, \mathbf{ACD} \text{ are similar, i.e. } \frac{\mathbf{FG}}{\mathbf{CD}} = \frac{\mathbf{AF}}{\mathbf{AC}}.$$

$$\therefore \frac{\mathbf{EF}}{\mathbf{BC}} = \frac{\mathbf{FG}}{\mathbf{CD}}.$$

In the same way it can be shown that

$$\frac{\mathbf{EF}}{\mathbf{BC}} = \frac{\mathbf{EG}}{\mathbf{BD}},$$

$$\therefore \frac{\mathbf{EF}}{\mathbf{BC}} = \frac{\mathbf{FG}}{\mathbf{CD}} = \frac{\mathbf{EG}}{\mathbf{BD}}.$$

*Hence the ratios of corresponding sides of the figures **EFG** and **BCD** are equal,*

and the figures have been proved equiangular.

Hence the figures **EFG and **BCD** are similar.**

PART II.

Because the parallel planes **EFG**, **BCD** are cut by the plane **BQPE**,

$\therefore$ **EP** is parallel to **BQ**. (Theorem 7E.)

$\therefore$ $\triangle$ s **AEP**, **ABQ** are similar.

$$\text{Thus } \frac{\text{Area } \mathbf{EFG}}{\text{Area } \mathbf{BCD}} = \frac{\mathbf{EF}^2}{\mathbf{BC}^2} = \frac{\mathbf{AE}^2}{\mathbf{AB}^2} = \frac{\mathbf{AP}^2}{\mathbf{AQ}^2}.$$

Hence the areas vary as the squares of their distances from the vertex.

COROLLARY TO THEOREM 36E.

If two pyramids are on equal bases and are of equal height the sections made by planes parallel to and at equal distances from the bases are equal.

Notes.—(1) The *cone* (see Chapter XI.) is a pyramid on a circular base, and the above theorem applies to it. The proof can be worked out by inscribing a regular polygon in the circular base and then imagining the number of sides to be increased indefinitely.

(2) It will be seen later that the *volume* of the pyramid $\mathbf{AEFG}$: the *volume* of the whole pyramid $\mathbf{ABCD} = \mathbf{AP^3 : AQ^3}$, and that this ratio is also equal to the ratio of the *cubes of corresponding edges*. The two pyramids are in fact similar, and the general statement is "*the volumes of similar polyhedra are to one another as the cubes of the lengths of corresponding edges.*"

Exercises X.

1. Verify Euler's Theorem in the case of a pyramid on an irregular hexagonal base. Also in the case of the solid remaining if the upper portion of this pyramid is cut away by a plane cut.

2. Work out the number of edges, faces, and corners in each type of regular polyhedron, and verify Euler's Theorem in each case.

3. Each edge of a tetrahedron is equal to the opposite edge. Show that all four faces are congruent.

4. Prove that the four middle points of two pairs of opposite edges of a tetrahedron are coplanar.

5. The six planes each passing through one edge of a tetrahedron and bisecting the opposite edge meet in a point.

6. $\mathbf{ABCD}$ is a regular tetrahedron, and $\mathbf{AP}$ is drawn perpendicular to the plane $\mathbf{BCD}$. Prove that (i) $\mathbf{AP}$ is equally inclined to $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{AD}$, (ii) $\mathbf{P}$ is the circum-centre of $\triangle \mathbf{ABC}$, and (iii) $\mathbf{CD}$ is perpendicular to the plane $\mathbf{APB}$.

7. A tetrahedron $ABCD$ has each edge 2 inches long. Calculate the length of the perpendicular AP from A to the opposite face.

8. In the previous question calculate the inclination of the edge AB to the face BCD , and the dihedral angle between the two faces ACD , BCD . (Use Mathematical Tables.)

9. In Question 7, calculate the distance between the middle points of opposite edges of the tetrahedron.

10. In the tetrahedron $ABCD$, $AB = CD = 4$ inches, $AC = BD = 5$ inches, $AD = BC = 6$ inches. Calculate the edges of the circumscribing parallelepiped, and the lengths of the three lines joining middle points of opposite sides of the tetrahedron.

11. Prove that every section of a tetrahedron made by a plane parallel to two opposite edges is a parallelogram. When is this parallelogram (a) a rectangle, (b) a square?

12. A plane section of a tetrahedron is a parallelogram. Prove that the plane of section is parallel to two opposite edges of the tetrahedron.

13. In a tetrahedron the line joining the middle points of two opposite edges is perpendicular to both. Prove that of the other four edges any one is equal in length to the opposite one.

14. A parallelepiped is circumscribed to a tetrahedron (see Problem 4E). Prove that

(i) If each edge of the tetrahedron is equal to the opposite edge the parallelepiped is rectangular.

(ii) If the tetrahedron is regular the parallelepiped is a cube.

(iii) If the tetrahedron is orthocentric the parallelepiped has all its edges equal.

15. Can the orthocentre of an orthocentric tetrahedron lie (i) at a vertex, (ii) on an edge but not at a vertex, (iii) on a face but not on an edge?

16. Each edge of a tetrahedron is equal to the opposite edge. Prove that the sum of the squares on the edges is equal to four times the sum of the squares on the three lines joining the middle points of opposite edges.

17. In the tetrahedron $ABCD$, given that

$$AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2,$$

prove that each edge is perpendicular to the opposite edge.

CHAPTER XI.

SOLIDS OF REVOLUTION: THE SPHERE, CYLINDER, AND CONE.

25. Definitions.—A sphere is a solid figure bounded by one curved surface which is such that all points on this surface are at equal distances from a certain point within the solid called the **centre** of the sphere: any line from the centre to the surface is called a **radius**.

We may also say that a sphere is a solid figure generated by the revolution of a semicircle about the diameter: thus, if the semicircle **ABC** revolves about the diameter **AC** it will describe the surface of a sphere, all points of which will be the same distance from the centre **O**, this constant distance being the radius of the sphere. Any line through **O** terminated both ways by the surface is a **diameter**: clearly all diameters of a sphere are equal.

As is proved in Theorem 37E, page 169, every plane section of a sphere is a circle. Thus in Fig. 134, the plane section **PQRS** is a circle. The diameter **AC** of the sphere, which is perpendicular to the section **PQRS**, is called the **axis of this section**, and the ends **A** and **C** are called its **poles**.

If a plane passes through the centre of a sphere, then the line of section of this plane and the surface of the sphere is called a **great circle**: the lines of section of all other planes and the surface of the sphere are called **small circles**. Thus **XYZ** (Fig. 134) is a great circle, and **PQRS** is a small circle. It follows that the radius of a great circle is equal

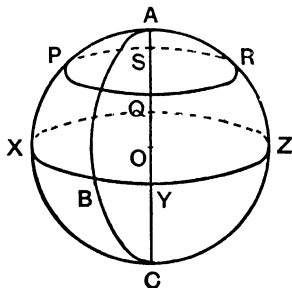


Fig. 134.

to the radius of the sphere, while the radius of a small circle is less than that of the sphere.

Any plane which meets the surface of a sphere at one point only is called a **tangent plane**: it is perpendicular to the radius of the sphere at that point (see Theorem 38E). Any straight line in this plane which passes through the point of contact is a **tangent line**: clearly there is an infinite number of tangent lines at any point on a sphere, and all are perpendicular to the radius at that point.

A **frustum of a sphere** is a part included between two parallel planes (Fig. 135). The curved surface of a frustum is called a **zone**.

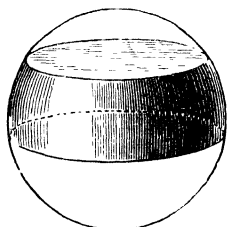


Fig. 135.

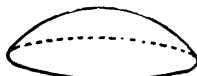


Fig. 136.



Fig. 137.

A **segment of a sphere** is a part cut off by a single plane (Fig. 136).

A **sector of a sphere** is a part contained by a segment and the conical part swept out by a radius of the sphere moving round the edge of the segment (Fig. 137).

A **right circular cylinder** or **right cylinder** is a solid figure generated by the revolution of a rectangle about one of its sides: thus if the rectangle $ABCD$ (Fig. 138) revolves about AD it generates the solid shown in the figure. AD is called the **axis** of the cylinder. Clearly the plane ends of this cylinder are circles.

An **oblique circular cylinder** is one whose ends are parallel circles to which the generating lines are oblique (Fig. 139).

A **truncated cylinder** is the portion of a cylinder included between an end and a plane section inclined at an angle to the end (Fig. 140).

A **right circular cone** or **right cone** is a solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle: thus if the right-angled triangle ADB (Fig. 141) revolves about AD it generates the solid shown in the figure. AD is called the **axis** of the cone, and A the **vertex**. The base is a circle.

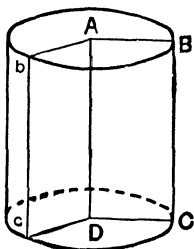


Fig. 138.

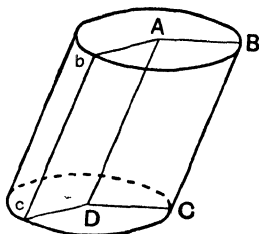


Fig. 139.

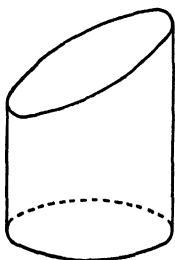


Fig. 140.

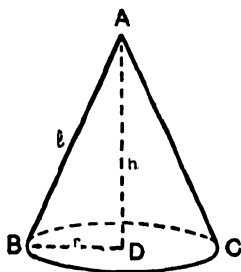


Fig. 141.

An **oblique circular cone** is one in which the line joining the centre of the base to the vertex (*i.e.* the axis) is oblique to the plane of the base.

A **truncated cone** is the portion of a cone included between the base and a plane section inclined to the base.

A **frustum of a cone** is the portion included between the base and a plane parallel to the base, or it is the portion cut off between two planes parallel to the base.

THEOREM 37E.

Any plane section of a sphere is a circle.

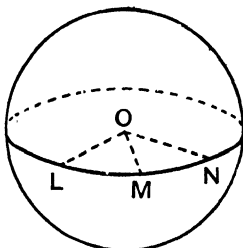


Fig. 142.

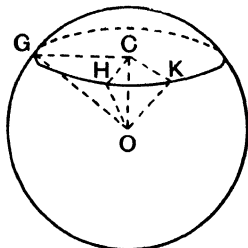


Fig. 143.

CASE I.—WHEN THE PLANE SECTION PASSES THROUGH THE CENTRE O OF THE SPHERE (Fig. 142).

Proof. Let $L, M, N, \dots$ be any points on the section.
 Thus $OL, OM, ON, \dots$ are radii of the sphere,
 $\therefore OL, OM, ON, \dots$ are all equal.

Hence the section is a circle whose centre is O , and whose radius is equal to that of the sphere.

CASE II.—WHEN THE PLANE SECTION DOES NOT PASS THROUGH THE CENTRE O OF THE SPHERE (Fig. 143).

Construction. Draw OC perpendicular to the given plane.
 Let $G, H, K, \dots$ be any points on the section.
 Join O and C to $G, H, K, \dots$

Proof. Because OC is perpendicular to the plane GHK ,
 $\therefore \angle OCG = \angle OCH = \angle OCK = \dots = \text{a right angle.}$

In $\triangle s$ $OCG, OCH, OCK, \dots$

Because $OG = OH = OK = \dots$

and OC is common to all the triangles,

and $\angle OCG = \angle OCH = \angle OCK = \dots = \text{a right angle,}$

$\therefore \triangle OCG \equiv \triangle OCH \equiv \triangle OCK \equiv \dots$

and $CG = CH = CK = \dots$ (*Theorem 16A.*)

Hence the section GHK is a circle whose centre is C .

THEOREM 38E.

The plane drawn through a point on the surface of a sphere perpendicular to the radius at that point is a tangent plane.

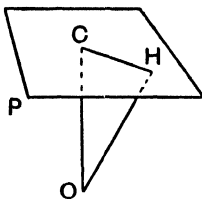


Fig. 144.

Given that O is the centre of a sphere, C a point on its surface, and P a plane through C perpendicular to the radius OC of the sphere.

Required to prove that The plane P is the tangent plane at C , that is, it meets the sphere at no other point.

Construction. Take any other point H on the plane P .
Join HC and HO .

Proof. Because OC is perpendicular to the plane P ,
 $\therefore \angle OCH = \text{a right angle.}$

In $\triangle HOC$, because $\angle OCH$ is a right angle,
 $\therefore \angle OHC$ is less than a right angle.

(Theorem 10A.)

$\therefore OH$ is greater than OC . (Theorem 19A.)

But OC is the radius of the sphere.

Thus OH is greater than the radius, and therefore H lies outside the sphere.

Similarly, it can be shown that all points on P except C lie outside the sphere.

Hence the plane P is the tangent plane at the point C .

COROLLARIES TO THEOREM 38E.

COROLLARY 1. *A tangent line at any point of a sphere is perpendicular to the radius at that point.*

COROLLARY 2. *An infinite number of tangent lines may be drawn at any point of a sphere, and these all lie in the tangent plane at that point.*

PROBLEM 5E.

To find the locus of a point which is equidistant from three given points.

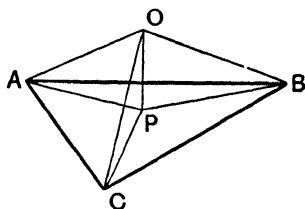


Fig. 145.

Given Three fixed points, **A**, **B**, and **C**.

Required To find the locus of a point which moves so that its distances from **A**, **B**, and **C** are equal.

Construction. Let **O** be any point on the required locus, so that $OA = OB = OC$.

Draw **OP** perpendicular to the plane **ABC**.
Join **AP**, **BP**, **CP**.

Proof. In $\triangle$ s **OPA**, **OPB**, **OPC**,
because **OP** is common,
and $OA = OB = OC$,
and $\angle OPA = \angle OPB = \angle OPC =$ a right angle,
 $\therefore \triangle OPA \equiv \triangle OPB \equiv \triangle OPC$,
and $PA = PB = PC$, (Theorem 16A.)
that is, **P** is the centre of the circum-circle of $\triangle ABC$.

Hence PO is the required locus, that is, the locus is the line through the centre of the circum-circle of the triangle ABC perpendicular to the plane of the triangle.

Note.—The above is often met with in the form of a theorem thus :—
If a straight line be drawn through the circum-centre of a triangle perpendicular to its plane, every point in this line is equidistant from the angular points of the triangle.

PROBLEM 6E.

To draw a sphere through four given points which do not lie in one plane.

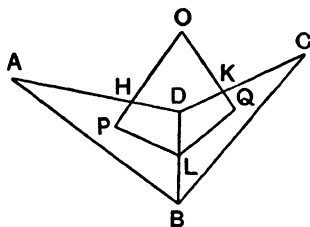


Fig. 146.

Given Four points, **A**, **B**, **C**, and **D**, which do not lie in one plane.

Required To draw a sphere passing through **A**, **B**, **C**, and **D**.

Construction. Join **AB**, **BC**, **CD**, **DA**, and **DB**.

Find **P** the circum-centre of $\triangle ABD$ and **Q** the circum-centre of $\triangle CBD$.

Draw **PH** perpendicular to the plane **ABD**, and **QK** perpendicular to the plane **CBD**.

Bisect **BD** at **L**.

Join **PL** and **QL**.

Proof. (1) **To prove that PH and QK must intersect.**

Because **P** is the centre of the circum-circle of $\triangle ABD$,

$\therefore$ **PL** is perpendicular to **BD**.

Similarly, **QL** is perpendicular to **BD**.

Hence **BD** is perpendicular to the plane **PLQ**.

(Theorem 15E.)

Thus the plane **PLQ** is perpendicular to the plane **ABD**.

(Theorem 23E.)

Again, **HP** is perpendicular to the plane **ABD**.

Thus the plane **HPL** is perpendicular to the plane **ABD**.

(Theorem 23E.)

Because the planes **HPL** and **PLQ** are both perpendicular to the plane **ABD**,

$\therefore$ the planes **HPL** and **PLQ** coincide.

$\therefore$ **PH** lies in the plane **PLQ**.

Similarly, it can be proved that the planes **KQL** and **PLQ** coincide,

$\therefore$ **QK** lies in the plane **PLQ**.

Thus **PH** and **QK** lie in the same plane,

$\therefore$ **PH** and **QK** either intersect or are parallel.

But if **PH** and **QK** are parallel, planes which are perpendicular to them will also be parallel ;

(Theorems 18E, 20E.)

that is, planes **ABD** and **BCD** will be parallel.

But this is not the case.

Hence **PH** and **QK** intersect at some point **O**.

(2) **To prove that O is equidistant from A, B, C, and D.**

In $\triangle$ s **OPA**, **OPB**, **OPD**,

because **OP** is common,

and **PA = PB = PD**,

and $\angle \text{OPA} = \angle \text{OPB} = \angle \text{OPD} = \text{a right angle}$,

$\therefore \triangle \text{OPA} \equiv \triangle \text{OPB} \equiv \triangle \text{OPD}$,

and **OA = OB = OD**. (Theorem 16A.)

Similarly, it can be proved that **OC = OB = OD**.

$\therefore$ **O** is equidistant from the four points **A, B, C, D**.

Hence a sphere with centre O and radius OA will pass through the four points and be the sphere required.

Notes.—(1) The preceding might have been enunciated:—“*To draw the circumscribed sphere of a tetrahedron, i.e. the sphere passing through the four angular points of the tetrahedron.*”

(2) The student should, as an exercise, prove that one and only one sphere can be drawn through the four points **A, B, C, D** of Fig. 146.

(3) It is easy to see that an infinite number of spheres can be drawn through *two* given points, and that their centres all lie in a fixed plane. For if **A** and **B** are the given points, the locus of points equidistant from **A** and **B** is the plane bisecting **AB** at right angles. Thus taking *any* point on this plane as a centre a sphere can be drawn through **A** and **B**.

(4) A little consideration will also show that an infinite number of spheres can be drawn through *three* given points, and that their centres all lie on a fixed line.

PROBLEM 7E.

To find the locus of a point which is equidistant from two given intersecting planes.

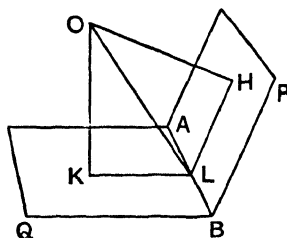


Fig. 147.

Given Two planes **P** and **Q**, intersecting in **AB**.

Required To find the locus of a point which moves so that its distances from **P** and **Q** are equal.

Proof. Let **O** be any point on the locus, so that $OH = OK$, where **OH**, **OK** are perpendicular to **P** and **Q** respectively.

Let the plane **OHK** meet **AB** in the point **L**.

Because **OH** is perpendicular to the plane **P**,

$\therefore$ the plane **OHK** is perpendicular to the plane **P**.

Similarly, the plane **OHK** is perpendicular to the plane **Q**.

$\therefore$ plane **OHK** is perpendicular to intersection of **P** and **Q**.

That is, the plane **OHK** is perpendicular to **AB**;

hence **AB** is perpendicular to **LH**, **LK**, and **LO**,
and $\angle HLK$ measures the dihedral angle between **P** and **Q**.

Again, because in the plane **OHK**,

OH is perpendicular to **LH** and **OK** is perpendicular to **LK**,
and $OH = OK$, $\therefore$ line **OL** bisects $\angle HLK$.

Hence the plane OLA bisects the dihedral angle between P and Q, and this plane is the locus required.

Note.—The two complete planes **P** and **Q** form two pairs of vertically opposite dihedral angles. Thus the complete locus consists of the two complete planes bisecting these two pairs of dihedral angles. Compare Theorem 35A.

PROBLEM 8E.

To inscribe a sphere in a given tetrahedron.

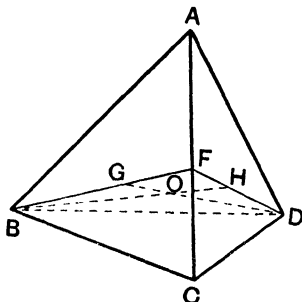


Fig. 148.

Given A tetrahedron **ABCD**.

Required To inscribe a sphere in the given tetrahedron.

Proof. The locus of points which are equidistant from the two faces **ABD**, **CBD** is (by Problem 7E) the plane bisecting the dihedral angle between these two faces—say the plane **BFD**, where **F** lies between **A** and **C**.

The locus of points which are equidistant from the two faces **ABC**, **DBC** is the plane bisecting the dihedral angle between these two faces.

This plane intersects plane **BFD** in some line **BH**, where **H** lies between **F** and **D**.

The locus of points which are equidistant from the faces **ACD**, **BCD** is the plane bisecting the dihedral angle between these two faces.

This plane intersects plane **BFD** in some line **DG**, where **G** lies between **F** and **B**.

The lines **BH** and **DG** are obviously not parallel, but meet at some point **O** within $\triangle BFD$.

This point **O** is the intersection of the three locus planes, and is therefore equidistant from all the faces of the tetrahedron.

Hence O is the centre of the required inscribed sphere.

Exercises XI.

1. Prove that, if two spheres intersect, their line of intersection is a circle.

2. If two spheres touch internally or externally, the join of their centres passes through the point of contact.

3. If two spheres meet at a point on the join of their centres they touch one another at that point.

4. Two planes touch a sphere at points **A**, **B**, and intersect in a line **LM**. If **O** is the centre of the sphere prove that **AB** is bisected at right angles by the plane **OLM**.

5. **A**, **B** are two fixed points in space. Find the locus of a point **P** if $PA^2 + PB^2$ is constant.

6. A tetrahedron has each edge 2 inches long. Calculate the radii of the circumscribed and inscribed spheres.

7. In the tetrahedron **ABCD**, $AB = AC = AD = 2'$, $BC = CD = DB = 3'$. Calculate the radius of the circumscribing sphere.

8. Prove that a sphere can be drawn through any two circles, not in the same plane, which cut one another in two points.

If the planes of the two circles are at right angles, if their radii are α , β , and the length of their common chord $2c$, show that the square of the radius of the sphere is $\alpha^2 + \beta^2 - c^2$.

9. Three equal spheres each 1 inch in diameter rest in contact with each other in a spherical bowl whose diameter is 6 inches. Find the distance of the plane passing through the centres of the spheres from the centre of the bowl.

10. Through a fixed point **O** on a given sphere a straight line is drawn cutting the sphere at **P**, and on **OP** a point **Q** is taken such that the rectangle **OP**, **OQ** is equal to a given square. Show that **Q** lies on a fixed plane whose distance from **O** is a third proportional to the diameter of the sphere and the side of the given square.

CHAPTER XII.

SURFACE AREAS AND VOLUMES OF SIMPLE SOLIDS.

26. Rectangular Solid and Cube.—Let p, q, r be the lengths of the three edges AB, AD, AE respectively (Fig. 149). It is clear, as is shown in all books on simple mensuration, that :—

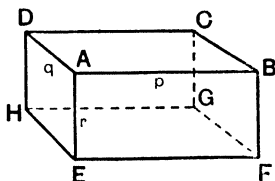


Fig. 149.

Surface area of solid = sum of the areas of the six faces
 $= 2(pq + pr + rq) \dots \dots \dots (1)$

Volume of solid = area of base $\times$ height
 $= p \times q \times r = pqr \dots \dots \dots (2)$

while for any one of the four diagonals, such as EC , we have :—

$$\begin{aligned} EC^2 &= EG^2 + GC^2 \\ &= (EF^2 + FG^2) + GC^2 \\ &= p^2 + q^2 + r^2, \end{aligned}$$

$$\therefore \text{Diagonal of solid} = \sqrt{p^2 + q^2 + r^2} \dots \dots \dots (3)$$

If in the above $p = q = r$, the rectangular block becomes a cube of side p , say, and :—

$$\text{Surface area of cube} = 6p^2 \dots \dots \dots (4)$$

$$\text{Volume of cube} = p^3 \dots \dots \dots (5)$$

$$\text{Diagonal of cube} = \sqrt{3}p^2 = p\sqrt{3} \dots \dots (6)$$

27. Right Prism.—Consider the triangular prism **ABCD FE** (Fig. 150), and let **EG** be the perpendicular from **E** to the edge **BC**. The sides of the prism are rectangles. Hence if **BC** = *a*, **CE** = *b*, **EB** = *c*, **EG** = *p*, and the height **AB** = *h*, it follows that:—

$$\begin{aligned}\text{Lateral surface area of prism} &= ah + bh + ch \\ &= (a + b + c)h \\ &= \text{perimeter of base} \\ &\quad \times \text{height} \dots\dots\dots (7)\end{aligned}$$

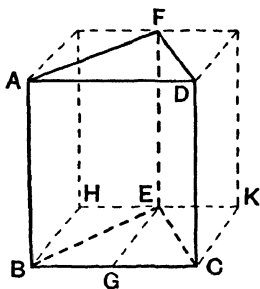


Fig. 150.

$$\begin{aligned}\text{Total surface area of prism} &= \text{lateral surface} + \text{area of ends} \\ &= (a + b + c)h + 2 (\text{area of base}) \\ &= (a + b + c)h + 2 \left(\frac{ap}{2} \right) \\ &= (a + b + c)h + ap \dots\dots\dots (8)\end{aligned}$$

To determine the volume, complete the rectangular block shown by the dotted lines, where **BH** is parallel to **GE**. From the figure it is evident that:—

$$\begin{aligned}\text{Volume of prism} &= \frac{1}{2} (\text{volume of rectangular block}) \\ &= \frac{1}{2} (\text{area of } \mathbf{HBCK} \times \text{height}) \\ &= \text{area of base } \mathbf{EBC} \times \text{height}.\end{aligned}$$

Further, if the base be a polygon, it can be suitably divided into triangles, and the same proof holds. Hence for any right prism:—

$$\text{Volume of right prism} = \text{area of base} \times \text{height} \dots (9)$$

28. Oblique Prism.—In Fig. 151 **ABCD** and **EFGH** are the parallel ends of the oblique prism there shown. Clearly **AE = BF = CG = DH** = the *length* of the prism.

Through the point **F** draw the plane **KFLM** (indicated by the dotted lines) at *right angles* to the edges **AE, BF**, etc.

Imagine the top portion **EKFLMH** cut off and placed on the other end of the prism, the point **E** being put on **A**, **F** on **B**, **G** on **C**, and **H** on **D**.

The oblique prism is now made into a right prism of the same volume, the base being the right section **KFLM**, and the height being **BF**, the *length* of the original prism. Hence from Art. 27:—

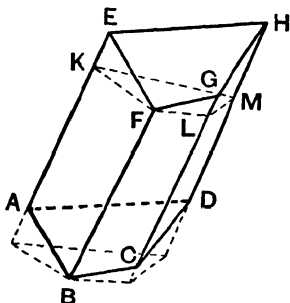


Fig. 151.

Volume of oblique prism

= volume of equivalent right prism

= base area of right prism $\times$ height of right prism

= (area of right section of oblique prism) $\times$ (length of oblique prism) ... (10)

= area **KFLM** $\times$ length **AE**.

It is matter of simple proof that the volume of an oblique prism is also given by:—

Volume of oblique prism

= area of end $\times$ perpendicular height (11)

The student should prove that the two expressions on the right-hand sides of (10) and (11) are equal. (See Art. 29.) To summarise briefly:—

Volume of oblique prism = right section $\times$ length

= base $\times$ perpendicular height.

The *lateral surface area* of an oblique prism is the sum of the areas of its sides. Referring to Fig. 151, area of

$ABFE = AE \times KF$. Similarly, area of $FBCG = BF \times FL$, area of $GCDH = CG \times LM$, and area of $DHEA = DH \times MK$.

Lateral surface area of oblique prism

$$= AE \times KF + BF \times FL + CG \times LM + DH \times MK \\ = AE (KF + FL + LM + MK),$$

since $AE = BF = CG = DH$. The part in brackets is the perimeter of a right section of the pyramid: hence:—

Lateral surface area of oblique prism

$$= \text{lateral edge} \times \text{perimeter of right section} \dots\dots(12)$$

The *total surface area* is, of course, the lateral area + the area of the ends.

29. Parallelepiped.—The parallelepiped is simply an oblique prism, and its *surface area* and *volume* are calculated as indicated in Art. 28. Thus (Fig. 152):—

Volume of parallelepiped

$$= \text{area of right section} \times \text{length} \dots\dots\dots(13)$$

$$= \text{area of } EFGH \times AB.$$

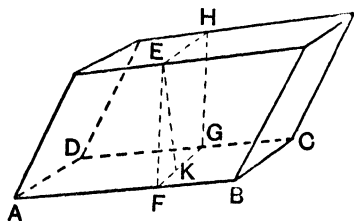


Fig. 152.

Let EK be drawn perpendicular to FG , so that, regarding $ABCD$ as the base, EK is the perpendicular height or altitude. Now area of $EFGH = EK \times FG$. Hence:—

$$\text{Volume of parallelepiped} \\ = EK \times FG \times AB,$$

$$\text{and } FG \times AB = \text{area of base } ABCD.$$

$$\therefore \text{Volume of parallelepiped} = \text{area of base} \times \text{altitude} \dots\dots(14)$$

The *total surface area* is the sum of the areas of the six faces calculated in the usual way.

30. Pyramid.—Consider first a triangular pyramid $ABCD$ (Fig. 153). Divide AC into a number of equal parts, and through the points of division draw planes parallel to the base BCD .

Also draw planes $CDFE$, etc., parallel to AB as shown in the figure, thus making a step-like construction of prismatic slabs.

Now the volume of the pyramid is less than the sum of the volumes of these prisms by the sum of the pieces **CDFEGH**, etc.

But a little consideration and examination of the figure will show that the sum of these pieces is less than the bottom slab. Further, this can be made as small as we please by dividing **AC** into a sufficiently large number of parts.

Hence the volume of the pyramid is equal to the limit of the sum of the volumes of the prisms.

Imagine now a second pyramid on an equal base and of equal altitude to be divided into the same number of slabs.

Then, since planes at equal distances from the bases of pyramids on equal bases and of equal heights make equal sections (Theorem 36E, Corollary), the section **KGH**, etc., of one is equal to the corresponding section of the other, and, therefore, the sum of the volumes of the prisms of one is equal to the sum of the volumes of the prisms of the other.

But the volume of each pyramid is the limit of the sum of the volumes of the prisms when their number is indefinitely increased (and consequently their thickness is indefinitely diminished).

Hence the volumes of triangular pyramids on equal bases and of equal heights are equal, and the same applies to pyramids other than triangular pyramids.

Now consider the triangular pyramid **PABC** (Fig. 154).

Through **P** draw the plane **PQR** parallel to the base **ABC**, and through **BC** draw the plane **BCRQ** parallel to the edge **PA**, thus making with the planes **PAB** and **PAC** produced a prism on the same base and of the same height as the pyramid. Join **BR**.

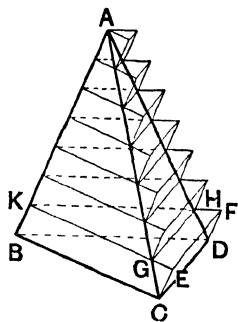


Fig. 153.

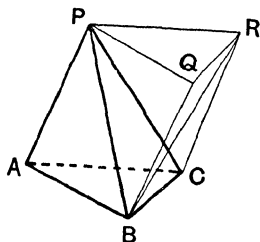


Fig. 154.

The prism is divided into three triangular pyramids **PABC**, **BPQR**, and **BCRP**.

Now the pyramids **PABC** and **BPQR** are equal in volume, for they are on equal bases **ABC** and **PQR**, and are equal in height.

Further, the pyramids **PABC** and **BCRP** are equal in volume, for they are on equal bases **PAC** and **CRP**, and are equal in height.

Thus the volumes of the three pyramids are equal, and each is, therefore, one-third the volume of the prism.

But the volume of the prism is equal to the area of the base multiplied by the altitude. Hence:—

$$\text{Volume of pyramid} = \frac{1}{3} (\text{area of base} \times \text{altitude}) \dots (15)$$

If the base be a polygon, the pyramid can be divided into a number of triangular pyramids, and it will be seen that the same expression holds. The rule applies to both right and oblique pyramids.

The above formula is proved in Euclid, Book XII., Proposition 7.

The proposition was first solved by Eudoxus in the time of Plato.

The *lateral surface area* of a pyramid is the sum of the areas of the triangular faces, and in general each must be separately calculated.

In the case, however, of a *right* pyramid whose base is a regular polygon of **N** sides each of length *a*, the area of each triangular face will be $\frac{1}{2}al$, where *l* is the slant height: hence:—

Lateral surface area of right pyramid

$$\begin{aligned} &= \frac{1}{2}al \times \mathbf{N} \\ &= \frac{1}{2}a\mathbf{N} \times l \\ &= \frac{1}{2}(\text{perimeter of base} \times \text{slant height}) \dots\dots (16) \end{aligned}$$

The *total surface area* of a pyramid will be the lateral surface area + the area of the base.

31. Frustum of a Pyramid.—Let **P** and **P'** be the ends of the frustum. Draw the complete pyramid as indicated in Fig. 155. From **O** draw **OS**, **S** perpendicular to the ends. Let **SS**₁ = *h*, the height of the frustum. Let **OS**₁ = *x*, and

denote the areas of the ends **P** and **P'** by **A** and **A'** respectively. Now:—

$$\frac{A}{A'} = \frac{OS^2}{OS_1^2} = \frac{(x+h)^2}{x^2}$$

$$\therefore \frac{x+h}{x} = \frac{\sqrt{A}}{\sqrt{A'}}$$

$$\therefore x = \frac{\sqrt{A'}}{\sqrt{A} - \sqrt{A'}} h$$

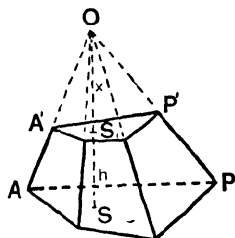


Fig. 155.

Volume of frustum of pyramid

= vol. of whole pyramid – vol. of top pyramid

$$= \frac{1}{3} [A(x+h) - A'x]$$

$$= \frac{1}{3} [Ah + x(A - A')]$$

$$= \frac{h}{3} \left[A + \frac{\sqrt{A'}}{\sqrt{A} - \sqrt{A'}} (A - A') \right]$$

$$= \frac{h}{3} [A + \sqrt{AA'} + A'] \dots\dots\dots (17)$$

As an exercise the student should prove that *for a right frustum*

$$\left(\begin{array}{c} \text{Lateral surface} \\ \text{of frustum} \end{array} \right) = \frac{1}{2} \left(\begin{array}{c} \text{sum of perimeters} \\ \text{of ends} \end{array} \right) \times \begin{array}{c} \text{slant} \\ \text{height} \end{array} \dots (18)$$

32. Cylinder.—If h be the height of a *right* cylinder and r the radius of the base it follows from the preceding expressions for the prism that:—

$$\begin{aligned} \text{Curved surface of cylinder} &= \text{circumference of base} \times \text{height} \\ &= 2\pi rh \dots\dots\dots (19) \end{aligned}$$

$$\begin{aligned} \text{Total surface of cylinder} &= \text{curved surface} + \text{end areas} \\ &= 2\pi rh + 2\pi r^2 \\ &= 2\pi r(h + r) \dots\dots\dots (20) \end{aligned}$$

$$\begin{aligned} \text{Volume of cylinder} &= \text{area of base} \times \text{height} \\ &= \pi r^2 h \dots\dots\dots (21) \end{aligned}$$

The student who has carefully studied the preceding sections will have no difficulty in seeing that:—

(a) The area of the curved surface of an *oblique* cylinder is measured by the product of the slant height and the perimeter of a right section.

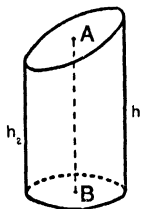


Fig. 156.

(b) The volume of an *oblique* cylinder is measured by the product of the slant height and the area of a right section or by the product of the base and the perpendicular height.

In the case of the *truncated* cylinder, shown in Fig. 156, it is easily seen that:—

$$\text{Curved surface} = 2\pi r \times \mathbf{AB} = 2\pi r \times \frac{h_1 + h_2}{2},$$

$$\text{Volume} = \pi r^2 \times \mathbf{AB} = \pi r^2 \times \frac{h_1 + h_2}{2},$$

where h_1 and h_2 = the greatest and least heights of the truncated cylinder respectively.

33. Cone.—Imagine the curved surface of the *right* cone (Fig. 157) to be divided into an infinite number of triangles having all their vertices at the vertex of the cone and whose bases added together make up the circumference of the base of the cone. In the limit the areas of all these triangles added together give, of course, the curved surface area of the cone.

If l be the slant height of the cone then l is the altitude of each of these triangles: hence:—

Curved surface of right cone

= sum of the areas of the triangles

= $\Sigma \frac{1}{2} bl$, where b = base of triangles

= $\frac{1}{2} l \Sigma b$

= $\frac{1}{2} l \times$ circumference of base

= $\frac{1}{2} 2\pi r l = \pi r l$ (22)

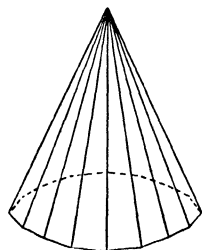


Fig. 157.

where r is the radius of the base. The *total surface area* = $\pi r l$ + area of base = $\pi r l + \pi r^2 = \pi r (l + r)$.

Again, since the volume of a pyramid is one third that of a prism of the same base and height, the volume of the right

cone is one third that of the corresponding cylinder; hence if h be the perpendicular height:—

$$\begin{aligned}\text{Volume of right cone} &= \frac{1}{3} (\text{area of base} \times \text{perpendicular height}) \\ &= \frac{1}{3} \pi r^2 h \dots\dots\dots (23)\end{aligned}$$

Just as in the case of the oblique pyramid it should be noted that the volume of an oblique cone is also equal to $\frac{1}{3}$ the product of the area of base and the perpendicular height.

34. Frustum of Cone.—Let **ABCD** (Fig. 158) be the frustum of a *right* cone, and **FG** the section midway between and parallel to the ends **AB** and **DC**.

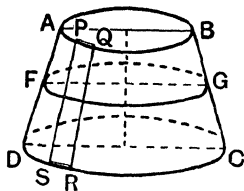


Fig. 158.

Let r and R be the radii of the ends **AB** and **CD** respectively, and let **AD** = l = the slant height.

Let **PQ** be the side of a regular polygon inscribed in the circular end **AB**, and **SR** the corresponding side of a similar and similarly situated regular polygon inscribed in the end **CD**. Then **PQRS** is a trapezium.

Now the lateral surface of the frustum may be regarded as the limit of the sum of the areas of the trapeziums formed by joining the corresponding vertices of the two polygons inscribed in **AB** and **CD**.

The area of the trapezium **PQRS** = $\frac{1}{2}PS(PQ + SR)$, and the limits of the perimeters of the polygons inscribed in **AB** and **CD** are the circumferences of the circles **APB** and **DSC** respectively. Hence:—

$$\begin{aligned}\text{Lateral surface of right frustum} &= \frac{1}{2}PS(2\pi r + 2\pi R) \\ &= PS \times 2\pi \times \frac{1}{2}(r + R) \\ &= \pi l(r + R) \dots\dots\dots (24)\end{aligned}$$

Now since the section **FG** is half-way between and parallel to the ends, the radius of the section **FG** will be the arithmetic mean of the radii of the ends, viz. $\frac{1}{2}(r + R)$. Thus $2\pi \times \frac{1}{2}(r + R)$ is the circumference of the section **FG**, and **PS** $\times 2\pi \times \frac{1}{2}(r + R)$ is the lateral surface of a cylinder whose height is **PS**, the slant side of the frustum, and whose base is the mean section **FG** of the frustum. Thus:—

$$\begin{aligned} & \text{Lateral surface of right frustum} \\ &= (\text{circumference of mid section}) \times (\text{slant height}) \dots\dots (25) \end{aligned}$$

The *total surface area* of the frustum will be the lateral surface area + the area of the two ends.

The volume of the frustum of *any* cone can be found in the same way as the volume of the frustum of the pyramid in Art. 31. Substituting in the formula of that section we have:—

$$\begin{aligned} \text{Volume of frustum} &= \frac{h}{3} [A + \sqrt{AA'} + A'] \\ &= \frac{h}{3} [\pi R^2 + \sqrt{\pi R^2 \cdot \pi r^2} + \pi r^2] \\ &= \frac{\pi h}{3} [R^2 + Rr + r^2] \dots\dots\dots (26) \end{aligned}$$

where h is the perpendicular height of the frustum.

35. The Prismoidal Formula.—Let **M** be the area of the mid section of the preceding frustum. Then:—

$$\begin{aligned} \text{Volume of frustum} &= \frac{\pi h}{3} (R^2 + Rr + r^2) \\ &= \frac{\pi h}{6} (2R^2 + 2Rr + 2r^2) \\ &= \frac{\pi h}{6} (R^2 + (R + r)^2 + r^2) \\ &= \frac{\pi h}{6} \left(R^2 + 4\left(\frac{R + r}{2}\right)^2 + r^2 \right) \\ &= \frac{h}{6} (A + 4M + A') \dots\dots\dots (27) \end{aligned}$$

The last formula is known as the *prismoidal formula*. It applies to *any* solid whose ends are parallel and have the

same number of sides, corresponding sides being parallel; the frusta of *any* pyramid or *any* cone are specific examples. Such a solid is frequently spoken of as a *prismoid*.

36. Sphere.—Imagine a cylinder to be circumscribed to a sphere as shown in Fig. 159. It is comparatively easy to prove that :—

(a) *The surface area of the sphere is equal to the area of the curved surface of the cylinder: thus if r be the radius of the sphere, the circumference of the cylinder is $2\pi r$ and the length $2r$, so that the area of the curved surface of the cylinder is $4\pi r^2$, and this is the surface area of the sphere.*

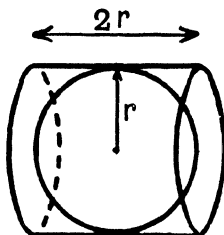


Fig. 159.

(b) *The surface area of any belt of the cylinder between two planes perpendicular to the axis is equal to the surface of the belt of the sphere between the same two planes.*

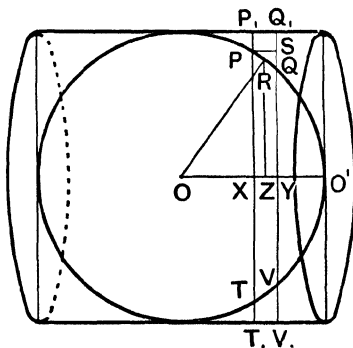


Fig. 160.

In Fig. 160, **PQVT** and **P₁Q₁V₁T₁** are corresponding belts on the sphere and on the circumscribing cylinder contained between two planes at right angles to the axis **OO'**. The planes are supposed to be near together. They cut **OO'** in **X** and **Y**. **RZ** is drawn from the middle of **PQ** perpendicular to **OO'**.

The belt **PQVT** of the sphere is generated by the revolution of **PQ** about

OO'. As **PQ** is short we may assume it to be a straight line, and the surface generated by **PQ** in revolving about

OO' is the *curved surface of the frustum of a cone*. Hence from Art. 34:—

$$\begin{aligned}\text{Surface of belt of sphere} &= \text{curved surface of frustum of cone} \\ &= 2\pi \cdot \text{RZ} \cdot \text{PQ}\end{aligned}$$

Draw **PS** perpendicular to **Q₁Q** and join **RO**. Then since **OR** is perpendicular to **PQ**, the triangles **PSQ** and **RZO** are similar: hence:—

$$\begin{aligned}\frac{\text{OR}}{\text{RZ}} &= \frac{\text{PQ}}{\text{PS}} = \frac{\text{PQ}}{\text{P}_1\text{Q}_1} \\ \therefore \text{RZ} \cdot \text{PQ} &= \text{OR} \cdot \text{P}_1\text{Q}_1\end{aligned}$$

$$\begin{aligned}\therefore \text{Surface of belt of sphere} \\ &= 2\pi \text{OR} \cdot \text{P}_1\text{Q}_1 \\ &= 2\pi \cdot \text{P}_1\text{X} \cdot \text{P}_1\text{Q}_1 \quad (\text{Since } \text{OR} = \text{P}_1\text{X}) \\ &= \text{surface generated by } \text{P}_1\text{Q}_1 \text{ revolving about } \text{OO}' \\ \text{i.e. Surface of belt of sphere} \\ &= \text{surface of corresponding belt of cylinder.}\end{aligned}$$

Total Surface of Sphere.—By dividing the sphere and the circumscribing cylinder into a large number of corresponding belts, for each of which the above applies, it will be clear that:—

$$\begin{aligned}\text{Total surface area of sphere} \\ &= \text{total curved surface of cylinder} \\ &= 2\pi r \times 2r \\ &= 4\pi r^2 \dots\dots\dots (28)\end{aligned}$$

Volume of Sphere.—The volume of a sphere may be determined as follows:—Imagine the surface divided into an infinite number of infinitely small polygons. By connecting these to the centre we get an infinite number of pyramids, each of height equal to the radius. The volumes of all these added together give the volume of the sphere. Hence:—

$$\begin{aligned}\text{Volume of sphere} &= \text{sum of volumes of pyramids} \\ &= \sum \frac{1}{3} (\text{areas of bases} \times \text{heights}) \\ &= \frac{1}{3} r \sum \text{areas of bases} \\ &= \frac{1}{3} r \times \text{area of sphere} = \frac{1}{3} r \times 4\pi r^2 \\ &= \frac{4}{3} \pi r^3 \dots\dots\dots (29)\end{aligned}$$

for the sum of the areas of the bases of the polygons ultimately equals the surface area of the sphere.

37. Sector, Segment, and Frustum of Sphere.—The surface areas and volumes of these are readily determined.

(a) *Curved Surface (or Cap) of Segment.*—If h be the height of the segment (Fig. 161), and r the radius of the sphere:—

Curved surface of segment = corresponding belt of cylinder
 $= 2\pi rh$ (30)

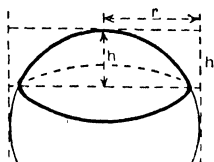


Fig. 161.

(b) *Curved Surface of Frustum (Zone).*—If h be the height of the belt of the sphere of radius r , i.e. if h be the perpendicular distance between the parallel planes of the frustum, it follows as before that

Curved surface of frustum = corresponding belt of cylinder
 $= 2\pi rh$ (31)

In both (a) and (b) the total surface will be the curved surface + the area of the flat circular end (or ends).

(c) *Surface of Sector.*—The total surface area will be the sum of the curved surface of the segment and the lateral surface of the cone which together make up the sector.

(d) *Volume of Sector.*—By the same method as in the case of the whole sphere, it can be shown that the volume of a sector is given by:—

$$\text{Volume of sector} = \frac{1}{3}Ar \text{ (32)}$$

where A denotes the curved surface or segmented cap, and r is the radius of the sphere.

(e) *Volume of Segment*.—Let r_1 = radius of base of segment, and h = height of segment (Fig. 162). Then :—

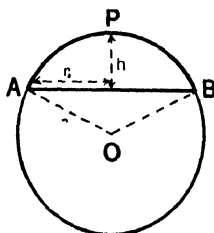


Fig. 162.

$$\begin{aligned}
 & \left\{ \begin{array}{c} \text{Volume of segment} \\ \text{ABP} \end{array} \right\} \\
 &= \left\{ \begin{array}{c} \text{volume of sector} \\ \text{OAPB} \end{array} \right\} - \left\{ \begin{array}{c} \text{volume of cone} \\ \text{OAB} \end{array} \right\} \\
 &= \frac{1}{3} 2\pi r h \cdot r - \frac{1}{3} \pi r_1^2 (r - h) \\
 &= \frac{\pi}{3} \left\{ 2r^2 h - r_1^2 (r - h) \right\} \dots\dots\dots (33)
 \end{aligned}$$

Now $(2r - h)h = r_1^2$: hence, if $(2r - h)h$ be substituted for r_1^2 in the preceding, the volume of the segment is obtained in terms of r and h .

(f) *Volume of Frustum of Sphere*.—If the distance between the parallel faces of the frustum be l , the volume of the frustum can be found by writing down the difference between the volumes of two segments, one of height h_1 and the other of height h_2 , where $h_1 - h_2 = l$. The volume is given by the expression :—

$$\text{Volume of frustum} = \frac{\pi}{6} l (3r_1^2 + 3r_2^2 + l^2) \dots (34)$$

r_1 and r_2 being the radii of the circular ends. The working out of this formula is left as an exercise to the student.

THEOREM 39E.

If a section of a pyramid be made by a plane parallel to its base the volume of the pyramid thus cut off is to the volume of the whole pyramid as the ratio of the cubes of corresponding edges, and also as the ratio of the cubes of their altitudes.

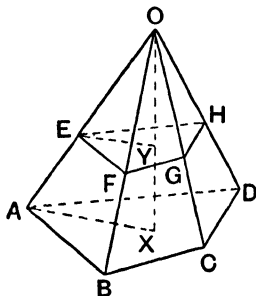


Fig. 163.

Given that **OABCD** is a pyramid, and **EFGH** a plane parallel to the base. **OYX** is perpendicular to the base and the parallel plane.

Required to prove that

$$\frac{\text{Vol. of pyramid } \mathbf{OEF GH}}{\text{Vol. of pyramid } \mathbf{OABCD}} = \text{say } \frac{\mathbf{OE^3}}{\mathbf{OA^3}} = \frac{\mathbf{OY^3}}{\mathbf{OX^3}}.$$

Proof.

$$\begin{aligned} \frac{\text{Vol. of } \mathbf{OEF GH}}{\text{Vol. of } \mathbf{OABCD}} &= \frac{\frac{1}{3} (\text{Area of } \mathbf{EFGH} \times \mathbf{OY})}{\frac{1}{3} (\text{Area of } \mathbf{ABCD} \times \mathbf{OX})} \\ &= \frac{\mathbf{OY^2}}{\mathbf{OX^2}} \times \frac{\mathbf{OY}}{\mathbf{OX}}. \quad (\text{Theorem 36E.}) \end{aligned}$$

$$\text{Hence } \frac{\text{Volume of } \mathbf{OEF GH}}{\text{Volume of } \mathbf{OABCD}} = \frac{\mathbf{OY^3}}{\mathbf{OX^3}}.$$

Again, $\triangle$ s OEY, OAX are similar,

$$\therefore \frac{OY}{OX} = \frac{OE}{OA}.$$

$$\text{Hence } \frac{\text{Volume of OEFGH}}{\text{Volume of OABCD}} = \frac{OE^3}{OA^3}.$$

COROLLARIES TO THEOREM 39E.

COROLLARY 1. *The volumes of similar pyramids are to one another in the ratio of the cubes of corresponding edges and also in the ratio of the cubes of their altitudes.*

COROLLARY 2. *The volumes of similar polyhedra are to one another as the cubes of their corresponding edges.*

For by taking two points similarly situated, one inside each, they can be divided into equal numbers of similar pyramids.

Now for each pair of similar pyramids the volumes are
as the cubes of corresponding sides of their bases.

Hence the sums of these similar pyramids in each are in
the same ratio.

From this it follows that the volumes of similar polyhedra
are as the cubes of corresponding edges.

Exercises XII.

1. Find the surface-areas of the rectangular parallelepipeds whose edges are—

- | | |
|-------------------------------------|------------------------------------|
| (i) 10 inches, 15 inches, 6 inches. | (iii) $a+b$, $b+c$, $c+a$. |
| (ii) 1 foot, 14 inches, 15 inches. | (iv) $a+b-c$, $b+c-a$, $c+a-b$. |

2. Find the volumes of the rectangular parallelepipeds whose edges are—

- | | |
|------------------------------------|------------------------------------|
| (i) 10 inches, 4 inches, 7 inches. | (iv) 30 cm., 12 cm., 15 cm. |
| (ii) 1 foot, 15 inches, 6 inches. | (v) 1 metre, 80 cm., 50 cm. |
| (iii) 2 yd., 2 ft. 6 in., 4 ft. | (vi) $a+b-c$, $b+c-a$, $c+a-b$. |

3. Find the surface-area of a cube whose edge is 10 inches.
4. Find the lateral surface-area of a right prism whose length is 10 inches, the base of the prism being a regular pentagon whose side is 2 inches.
5. What would be the cost of gilding, at 2d. per square inch, the entire surface of a right prism, its length being 32 inches, and its ends triangles whose sides are 3 inches, 4 inches, and 5 inches respectively.
6. Find the volume of a column in the shape of a right prism whose length is 24 feet, the base being an equilateral triangle with a side of 12 inches.
7. The area of the base of a pyramid is 20 square feet. Find its height, in order that its volume may be the same as that of a cube whose edge is 5 feet.
8. Find, to the nearest cubic inch, the volume of a pyramid, the base being a regular hexagon whose side is 10 feet, and the slant height from the middle point of a side of the base being also 10 feet.
9. Find the total surface-area of a pyramid on an equilateral triangular base whose side is 8 inches, the lateral heights of the faces of the pyramid being 10 inches, 12 inches, and 10 inches respectively.
10. A frustum of a pyramid 6 inches high stands on a square base whose sides are 5 inches, and its top is a square whose sides are 3 inches. Find the volume of the frustum in cubic inches.
11. Find the area of the sloping surface of a frustum of a pyramid on a square base whose side is 6 inches, the perpendicular height of the pyramid being 6 inches, and the side of the square top 1 inch.
12. The total surface of a cylinder, whose radius is 18 inches, is 120 square feet; find the length of the cylinder.
13. Determine, in tons, the quantity of iron plate required in the construction of a cylindrical gas-holder (including the *flat* top) 140 feet in diameter and 120 feet high, the iron plate weighing 10·2 lb. to the square foot.
14. Determine the volume, in cubic feet, of a cylindrical gas-holder, 140 feet in diameter and 120 feet high.
15. Find the whole surface of a right cone, the radius of whose base is 4 inches, and whose slant height is 8 inches.
16. The circumference of the base of a cone is 44 feet, and its slant height is $8\frac{1}{2}$ feet. Find the volume of the cone.
17. The altitude of a right circular cone is equal to the circumference of the base. Calculate the volume and the area of the whole surface of the cone, the radius of the base being given.

18. Find the volume, in cubic feet, of the frustum of a cone, the radii of whose ends are 1 foot and 4 feet respectively, the slant height of the frustum being 5 feet.

19. Find the area of the Earth's surface to the nearest 1,000 square miles, taking its diameter as 8,000 miles. ($\pi = 3.141592$.)

20. The diameter of the Earth being taken as 8,000 miles, find its volume to within a million cubic miles. ($\pi = 3.141592$.)

21. Compare the volumes of a sphere whose radius is 10 inches and a cylinder whose length is 15 inches and the radius of whose base is 4 inches.

22. Find the area of the curved surface of the zone of a sphere of radius 12 inches intercepted by two planes whose distance apart is 5 inches.

CHAPTER XIII.

MISCELLANEOUS PROPOSITIONS.

I.—HARMONIC SECTION.

38. Definitions.—Before proceeding with this section the student should again read Art. 6 of Chapter I.

Three quantities are in **harmonic progression** if the ratio of the first to the third is equal to the ratio of the difference between the first and second to the difference between the second and third. Thus if a , b , and c are in harmonic progression

$$\frac{a}{c} = \frac{a-b}{b-c}$$

and b is said to be a **harmonic mean** between a and c . Clearly it follows that

$$b = \left\{ \begin{array}{l} \text{harmonic mean} \\ \text{between } a \text{ and } c \end{array} \right\} = \frac{2ac}{a+c}.$$

A **range** is a series of points in a straight line. A **pencil** is a series of straight lines passing through one point called the **vertex**. A **ray** is one of the straight lines forming a pencil.

As explained in Art. 6, when a straight line AB (Fig. 165) is divided internally (at P) and externally (at Q) in the same ratio (*i.e.* so that $AP:PB$

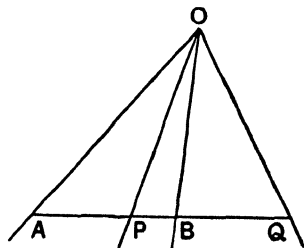


Fig. 164.

$= AQ:QB$), it is said to be cut harmonically, and the four points A , P , B , Q thus obtained are said to be a **harmonic range**. The points P and Q are called **harmonic conjugates** with respect to A and B .

If the four points **A, P, B, Q** (Fig. 164) of a harmonic range are joined to a point **O** outside the range, the pencil formed by the four lines **OA, OP, OB, OQ** is called a **harmonic pencil**: thus a **harmonic pencil is a pencil of rays which passes through the four points of a harmonic range**.

A pencil whose vertex is **O** and which is composed of, say four rays, passing through the points **A, B, C, D**, is denoted thus:—(**O . ABCD**) or thus:—**O(ABCD)**.

The notation $\{ABCD\} = -1$ is often used to denote that the four points **A, B, C, D** form a harmonic range.

THEOREM 1F.

If the points **P** and **Q** divide the line **AB** harmonically, then (1) the points **A** and **B** divide the line **PQ** harmonically, (2) the lengths **AQ, AB**, and **AP** are in harmonical progression, and the lengths **QB, QP**, and **QA** are in harmonical progression.



Fig. 165.

Proof. (1) Because $AP : PB = AQ : QB$

$$\therefore AP \cdot QB = PB \cdot AQ.$$

$$\text{Thus } \frac{AP \cdot QB}{AP \cdot BP} = \frac{PB \cdot AQ}{AP \cdot BP}.$$

$$\therefore \frac{QB}{BP} = \frac{AQ}{AP},$$

$$\text{i.e. } QB : BP = QA : AP.$$

Hence **A** and **B** divide the line **PQ** harmonically.

$$(2) \text{ Since } \frac{AQ}{AP} = \frac{QB}{BP}, \quad (\text{Proved.})$$

$$\therefore \frac{AQ}{AP} = \frac{AQ - AB}{AB - AP}.$$

Hence **AQ, AB** and **AP** are in harmonical progression.

Similarly, **QB, QP** and **QA** are in harmonical progression.

THEOREM 2F.

If a straight line AB is bisected at O, and if P and Q are harmonic conjugates with respect to A and B, then $OP \cdot OQ = OB^2$, and conversely if $OP \cdot OQ = OB^2$, then P and Q are harmonic conjugates with respect to A and B.

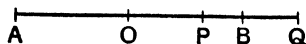


Fig. 166.

Proof

Because O bisects AB,

$$\therefore AP = OB + OP, \quad PB = OB - OP,$$

$$AQ = OQ + OB, \quad QB = OQ - OB.$$

$$\text{If } AP : PB = AQ : QB,$$

$$\therefore AP + PB : AP - PB = AQ + QB : AQ - QB,$$

$$\text{i.e. } 2OB : 2OP = 2OQ : 2OB.$$

$$\therefore OB : OP = OQ : OB.$$

$$\text{Hence } OP \cdot OQ = OB^2.$$

Again, if $OP \cdot OQ = OB^2$,

$$\therefore OB : OP = OQ : OB.$$

$$\text{i.e. } OB + OP : OB - OP = OQ + OB : OQ - OB,$$

$$\therefore AP : PB = AQ : QB.$$

Hence P and Q are harmonic conjugates with respect to A and B.

THEOREM 3F.

If $(O.APBQ)$ is a harmonic pencil, then the rays OP , OQ cut off equal segments on a line through B parallel to OA ; and conversely.

Given that $APBQ$ is a harmonic range and $(O.APBQ)$ a harmonic pencil, and that HBK is parallel to OA .

Required to prove that $HB = BK$.

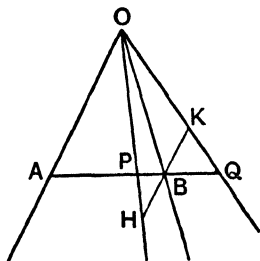


Fig. 167.

Proof Because $\triangle POA$ is similar to $\triangle PHB$,

$$\therefore AP : PB = AO : BH;$$

Because $\triangle QOA$ is similar to $\triangle QKB$,

$$\therefore AQ : QB = AO : BK.$$

$$\text{But } AP : PB = AQ : QB,$$

(Hyp.)

$$\therefore AO : BH = AO : BK.$$

$$\text{Hence } BH = BK.$$

The converse theorem can obviously be proved by reversing the argument.

THEOREM 4F.

Any transversal cuts a harmonic pencil in a harmonic range.

Given that $APBQ$ (Fig. 168) is a harmonic range, and hence $(O.APBQ)$ a harmonic pencil.

Required to prove that $apbq$ *is a harmonic range.*

Construction. Draw HBK and hbk each parallel to OA .

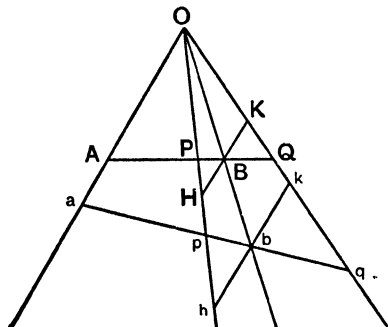


Fig. 168.

Proof. Because $APBQ$ is a harmonic range,

$$\therefore HB = BK.$$

Because the parallel lines HK and hbk are cut by three transversals which meet at O ,

$$\therefore HB : BK = hb : bk. \quad (\text{Theorem 11D.})$$

$$\text{Thus } hb = bk.$$

Hence $apbp$ is a harmonic range.

Note.—The range $apbq$ is called a projection of the range $APBQ$, and O is called the vertex of projection. Hence we may express this theorem in the form:—*The projection of a harmonic range on any straight line is another harmonic range.* [This is a **conical** projection, and must not be confused with **orthogonal** projection : see Art. 21].

II.—CENTRES OF SIMILITUDE.

39. Definitions.—Before proceeding with this section the student should again read Art. 9 of Chapter II.

The centres of similitude of two circles are the two points which divide the line joining their centres in-

ternally and externally in the ratio of their radii. It is evident that the centres of similitude are harmonically conjugate with respect to the two circle centres.

It will be seen later that the *direct* common tangent to the two circles passes through one centre of similitude (the *external* dividing point), and the *transverse* common tangent passes through the other (the *internal* dividing point). The former is called the **centre of direct similitude**, and the latter the **centre of transverse similitude**: sometimes they are referred to as the external and internal centres of similitude respectively.

It will also be seen later that the use of the term "centre of similitude," as given above, is quite in harmony with the use as given in Chapter II.

THEOREM 5F.

If in two circles two parallel radii are drawn, then the line joining their outer extremities meets the line joining their centres in a centre of similitude.

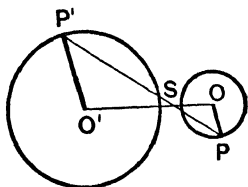


Fig. 169.

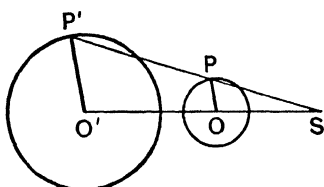


Fig. 170.

Given that OP and $O'P'$ are parallel radii, and $P'P$ meets the join of the centres O, O' at S .

Required to prove that S is a centre of similitude of the two circles.

Proof. S divides OO' *internally* if OP and $O'P'$ are drawn in different senses (Fig. 169), and *externally* if OP and $O'P'$ are drawn in the same sense (Fig. 170).

In both cases, because OP is parallel to $O'P'$,

$\therefore \triangle SOP$ is similar to $\triangle SO'P'$,

$\therefore OS : SO' = OP : O'P'$.

Thus S in Fig. 169 divides OO' *internally* in the ratio of the radii, and S in Fig. 170 divides OO' *externally* in the ratio of the radii.

Hence S in each figure is one of the centres of similitude.

Notes.—(1) It follows that **the direct and transverse common tangents pass through the centres of similitude**, for the radii to the points of contact of the common tangents are parallel (Fig. 171); hence the statement that “The transverse and direct common tangents to two circles intersect the join of the centres at points which divide that line harmonically.”

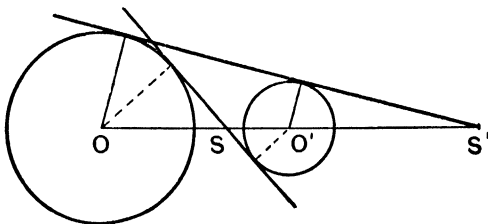


Fig. 171.

(2) In both Fig. 169 and Fig. 170 $SP : SP' = OP : O'P' = \text{a constant}$. This shows that the present use of the term “centre of similitude” is in harmony with the use of the term in Chapter II.

THEOREM 6F.

If a variable line through a centre of similitude, S , of two circles meets the circles in P, Q, P', Q' , where P corresponds to P' and Q to Q' , then $SP \cdot SQ' = SP' \cdot SQ = \text{a constant}$.

Proof.

Since **S** is a centre of similitude,
 $\therefore$ radius **OP** is parallel to radius **O'P'**,
 and radius **OQ** is parallel to radius **O'Q'**.

Therefore, by similar triangles,
 $SP : SP' = SO : SO' = SQ : SQ'$.

Hence $SP \cdot SQ' = SP' \cdot SQ$.

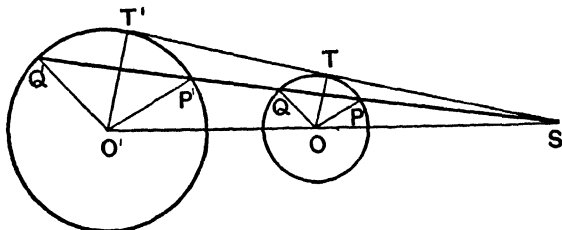


Fig. 172.

Again, if **STT'** is a common tangent through **S**, then by similar triangles,

$$ST : ST' = SO : SO' = SQ : SQ',$$

$$\therefore ST : SQ = ST' : SQ'.$$

$$\text{Now } SP \cdot SQ = ST^2,$$

$$\text{i.e. } SP : ST = ST : SQ = ST' : SQ'.$$

$$\therefore SP \cdot SQ' = ST \cdot ST'.$$

But $ST \cdot ST' = \text{a constant}$.

Hence $SP \cdot SQ' = SP' \cdot SQ = \text{a constant}$.

THEOREM 7F.

If a variable circle touches two fixed circles the line joining their points of contact passes through a centre of similitude.

Given that **FGH** and **KLM** are fixed circles, and **HVK** is a variable circle touching them at **H** and **K** respectively.

Required to prove that **KH** passes through a centre of similitude of the circles **FGH**, **KLM**.

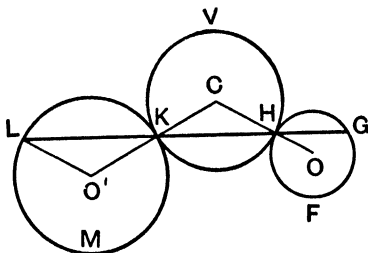


Fig. 173.

Proof. **OC** passes through **H** and **O'C** through **K**.

$$\angle OHG = \angle KHC = \angle HKC = \angle O'KL = \angle O'LK.$$

Thus, using **GL** as transversal, **OH** and **O'L** are parallel radii.

Hence LH (i.e. KH) passes through a centre of similitude.

THEOREM 8F.

If the line joining a fixed point *S* to a variable point *P* on a given circle is divided in a given ratio at *P'*, the locus of *P'* is a circle, and *S* is a centre of similitude of the two circles.

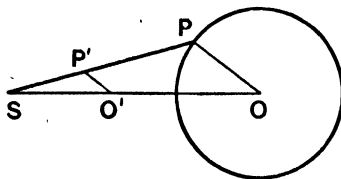


Fig. 174.

Construction. Let **O'** divide **SO** in the given ratio, then **O'** is a fixed point.

Proof. Because $SP' : P'P = SO' : O'O$,

$\therefore O'P'$ is parallel to OP .

Thus, by similar triangles,

$O'P' : OP = SO' : SO = \text{a constant ratio.}$

But OP is of constant length,

$\therefore O'P'$ is of constant length.

Hence the locus of P' is a circle whose centre is O' .

Also OP and $O'P'$ are parallel radii.

Hence S is a centre of similitude of the two circles.

III.—INVERSION.

40. Definitions.—If O (Fig. 175) is any fixed point, P a point which travels along some path AB , and Q a point in OP or OP produced, such that the rectangle $OP \cdot OQ$ remains constant and equal, say, to k^2 , then:—

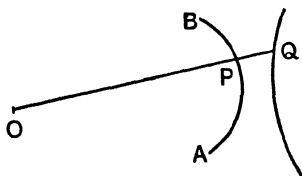


Fig. 175.

(1) The locus of Q is called an **inverse of the curve AB** , i.e. the locus of Q is the inverse of the locus of P .

(2) The point O is called the **centre of inversion** or **origin of inversion**.

(3) The length k is called the **radius of inversion**, and k^2 is sometimes called the **constant of inversion**.

(4) The points **P** and **Q** are called **inverse points**.

From the definition it is clear that *the inverse of a straight line through the centre of inversion is the same straight line*, for if **P** (Fig. 175) travels along a straight line through **O**, then **Q** travels along the same straight line.

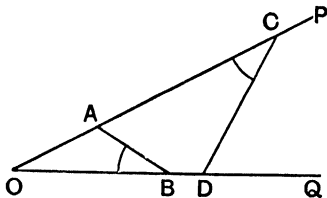


Fig. 176.

Two lines **AB** and **CD** (Fig. 176) are said to be **anti-parallel** with respect to the angle **POQ** if they are not parallel, and yet form equiangular triangles with the arms of the angle **POQ**,—in other words, if the angle **OBA** be equal to the angle **OCD**. It is obvious that in this case the points **A, B, C, D** are concyclic (Theorem 15B).

THEOREM 9F.

The inverse of a circle which passes through the centre of inversion is a straight line perpendicular to the diameter through the centre of inversion, and conversely, the inverse of a straight line (not passing through the centre of inversion) is a circle through the centre of inversion such that its diameter through the centre of inversion is perpendicular to the given line.

PART I.

Given that **OPA** is a circle which passes through **O**, the centre of inversion, and k is the radius of inversion.

Required to prove that The inverse of the circle is a straight line perpendicular to the diameter through **O**.

Construction. Let **P** be any point on the circle, and
Q the inverse point: then $OP \cdot OQ = k^2$.
 Draw the diameter **OA**, and on it find a point **C**
 such that $OC \cdot OA = k^2$.
 Join **QC**.

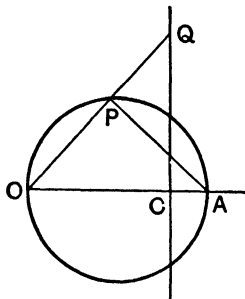


Fig. 177.

Proof. Because $OP \cdot OQ = OC \cdot OA$,
 $\therefore$ the points **A, C, P, Q** are concyclic. (Theorem 12C.)
 Hence $\angle ACQ = \angle APQ$. (Theorem 12B.)
 But $\angle APO =$ a right angle,
 $\therefore \angle APQ =$ a right angle.
 Hence $\angle ACQ =$ a right angle.
 Now **C** is a fixed point, and it has just been proved that
 the line joining **C** to **Q** is perpendicular to **OA**.
Hence the locus of Q (i.e. the inverse of the circle)
is the straight line CQ through C
perpendicular to OA.

PART II.

Given that **QC** is a straight line not passing through **O**,
 the centre of inversion.
k is the radius of inversion.

Required to prove that The inverse of **QC** is a circle passing through **O** whose diameter through **O** is perpendicular to **QC**.

Construction. Let **Q** be any point on the given line, and **P** the inverse point: then $OP \cdot OQ = k^2$.

Draw **OC** perpendicular to **QC**, and on it find a point **A** such that $OC \cdot OA = k^2$.

Join **PA**.

Proof. As in Part I., the points **A**, **C**, **P**, **Q** are concyclic.

Hence $\angle APQ = \angle ACQ =$ a right angle.

$\therefore \angle OPA =$ a right angle.

Hence the locus of P (i.e. the inverse of the line CQ) is the circle OPA on the diameter OA perpendicular to CQ.

THEOREM 10F.

The inverse of a circle which does not pass through the centre of inversion is another circle: also the centre of inversion is a centre of similitude.

Given that **ABE** (Fig. 178) is the circle, **O** the centre of inversion, and k the radius of inversion.

Required to prove that The inverse of the circle **ABE** is another circle, and **O** is a centre of similitude of the two circles.

Construction. Let **P** be any point on the circle, and **Q** its inverse point: then $OP \cdot OQ = k^2$.

Through **O** draw the diameter **OAB**, and find **C** and **D**, the inverse points of **A** and **B** respectively; then $OA \cdot OC = k^2$, and $OB \cdot OD = k^2$.

Join **PA**, **PB**, **QC**, **QD**.

Proof.

Because $OA \cdot OC = OP \cdot OQ$,
 $\therefore APQC$ is a cyclic quadrilateral. (Theorem 12c.)

Thus $\angle PQC = \angle PAO$. (Theorem 14B, Cor.)
 Similarly, $BPQD$ is a cyclic quadrilateral,
 $\therefore \angle PQD = \angle PBA$.

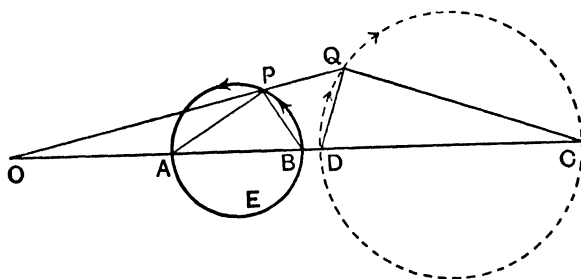


Fig. 178.

Thus $\angle PQC - \angle PQD = \angle PAO - \angle PBA$,
 $\therefore \angle DQC = \angle APB$.

But $\angle APB$ = a right angle, since AB is a diameter,
(Theorem 16B.)

$\therefore \angle DQC$ = a right angle.

Also D and C are fixed points.

Hence the locus of Q (i.e. the inverse of the circle ABE) is a circle DQC on the diameter DC .

Again, if OP were a tangent to the given circle, it would meet it in one point only, and would therefore meet the inverse curve in one point only.

Thus OPQ would now be a common tangent.

Hence O is a centre of similitude.

THEOREM 11F.

If A and B are the inverse points of C and D with respect to O, then AB and CD are antiparallels with reference to the angle COD.

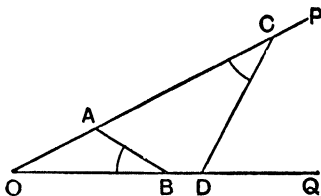


Fig. 179.

Proof. $OA \cdot OC = k^2$ and $OB \cdot OD = k^2$,

$$\therefore OA \cdot OC = OB \cdot OD.$$

Thus A, B, D, C are concyclic points,

(Theorem 12c.)

$$\therefore \angle OBA = \angle OCD. \quad (\text{Theorem 14B, Cor.})$$

Hence AB and CD are antiparallels with reference to the angle COD.

Note.—As a corollary to the above it is easy to prove that “The tangents to a curve and its inverse at corresponding points are equally inclined to the join of these points.” For, in Fig. 179, if C, D are consecutive points on the original curve, and A, B the corresponding points on the inverse curve, then AB and CD become the tangents at A and C. Also these tangents are equally inclined to OP and OQ, which are now coincident lines.

IV.—THE RADICAL AXIS.

41. Definitions.—The radical axis of two given circles is the locus of a point from which the tangents to the two circles are of equal length. Thus if P (Fig. 180) moves so that the tangents PC and PD are always equal, then the locus of P is the radical axis of the two circles. It will be seen later that the radical axis is a straight

line perpendicular to the line joining the centre of the two circles: thus in Fig. 180 PX is the radical axis.

It will also be seen later that the radical axes of *three circles taken in pairs* are concurrent. Thus in Fig. 181 OD is the radical axis of the circles B and C , OE is the radical axis of the circles A and C , and OF is the radical axis of the circles A and B : these three meet at the point O . **The radical centre of three circles is the point of intersection of their three radical axes**: thus in Fig. 181 O is the radical centre.

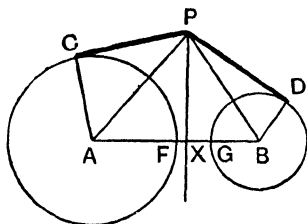


Fig. 180.

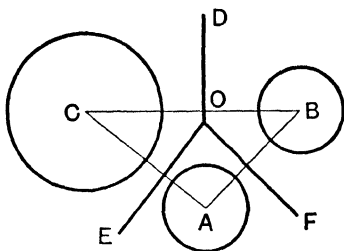


Fig. 181.

A system of circles is said to be **coaxial** if the radical axis of any two circles in the system is always the same line.

THEOREM 12F.

The radical axis of two circles is a straight line perpendicular to the join of their centres.

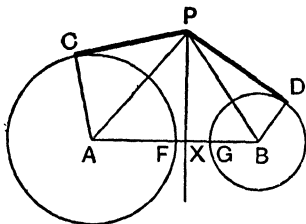


Fig. 182.

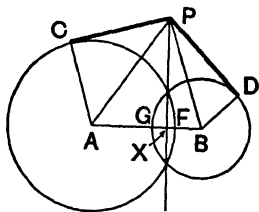


Fig. 183.

Given that **CF** and **GD** are two circles with centres **A** and **E** respectively.

PC and **PD** are tangents from **P** to these circles.

$$\mathbf{PC} = \mathbf{PD}.$$

Required to prove that The locus of **P** is a straight line perpendicular to **AB**.

Construction. Join **AP** and **BP**.

Draw **PX** perpendicular to **AB**.

Proof. Because **PC** = **PD**,

$$\therefore \mathbf{PC}^2 = \mathbf{PD}^2.$$

But $\angle \mathbf{PCA} = \angle \mathbf{PDB} = \text{a right angle},$

$$\therefore \mathbf{PA}^2 - \mathbf{AC}^2 = \mathbf{PB}^2 - \mathbf{BD}^2,$$

$$\text{i.e. } \mathbf{PA}^2 - \mathbf{PB}^2 = \mathbf{AC}^2 - \mathbf{BD}^2 = \mathbf{AF}^2 - \mathbf{BG}^2.$$

Thus $\mathbf{PA}^2 - \mathbf{PB}^2$ is constant, being equal to the difference of the squares on the radii.

But $\angle \mathbf{AXP} = \angle \mathbf{BXP} = \text{a right angle},$

$$\begin{aligned} \therefore \mathbf{PA}^2 - \mathbf{PB}^2 &= (\mathbf{AX}^2 + \mathbf{PX}^2) - (\mathbf{BX}^2 + \mathbf{PX}^2) = \mathbf{AX}^2 - \mathbf{BX}^2. \\ &= (\mathbf{AX} + \mathbf{BX})(\mathbf{AX} - \mathbf{BX}). \end{aligned}$$

Hence $(\mathbf{AX} + \mathbf{BX})(\mathbf{AX} - \mathbf{BX})$ is constant.

But $\mathbf{AX} + \mathbf{BX} = \mathbf{AB}$ and is constant,

$$\therefore \mathbf{AX} - \mathbf{BX} \text{ is constant,}$$

i.e. **X** is a fixed point in **AB**.

And **X** is on the perpendicular from **P** to **AB**.

Hence the locus of P (i.e. the radical axis of the two circles) is a straight line perpendicular to AB.

Note.—If the circles cut each other the two tangents from the part of **XP** exterior to the circles are equal (Fig. 183).

PROBLEM 1F.

To construct the radical axis of two given circles.

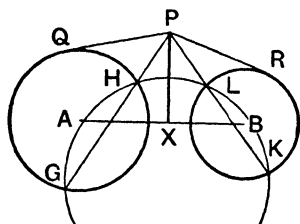


Fig. 184.

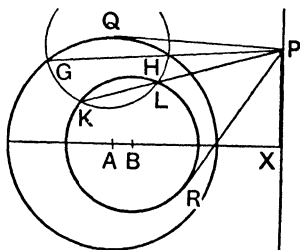


Fig. 185.

Given The two circles **GQH** and **KLR**.

Required To construct the radical axis of the two circles.

Construction. With any convenient centre and radius describe a circle **GKLH** cutting the two given circles at **G, H**, and **K, L** respectively.

Let the chords **GH** and **KL** intersect at **P**.

Through **P** draw **PX** perpendicular to the join of the centres **A, B**.

Then PX shall be the required radical axis.

Proof. Let **PQ, PR** be tangents to the two given circles.

In the circle **GQH**, $PQ^2 = PG \cdot PH$.

(Theorem 11c.)

In the circle **GKL**, $PG \cdot PH = PK \cdot PL$.

(Theorem 10c.)

In the circle **KRL**, $PK \cdot PL = PR^2$.

(Theorem 11c.)

Thus $PQ^2 = PR^2$,

whence $PQ = PR$.

$\therefore P$ is a point on the radical axis.

But the radical axis is a straight line perpendicular to **AB**.

Hence PX is the required radical axis.

Note.—If the circles touch, the radical axis is the common tangent at the point of contact. If the circles intersect, the radical axis is the common chord produced.

THEOREM 13F.

The radical axes of three circles taken in pairs are concurrent.

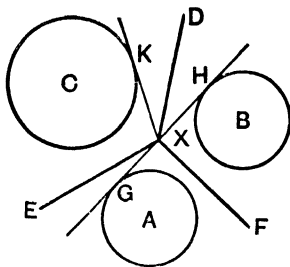


Fig. 186.

Given that **EX** is the radical axis of the circles **A** and **C**,
and **DX** the radical axis of the circles **B** and **C**.

X is their point of intersection.

Required to prove that The radical axis of the circles **A**
and **B** passes through **X**.

*It will be found that the point X is either without
or within all the circles.*

CASE I.—WHEN **X** IS WITHOUT ALL THE CIRCLES.

Construction. Draw the tangents **XG**, **XH**, **XK**.

Proof. Because **X** is on the radical axis of the circles **A** and **C**,
 $\therefore \mathbf{XG} = \mathbf{XK}$.

Because **X** is on the radical axis of the circles **B** and **C**,
 $\therefore \mathbf{XH} = \mathbf{XK}$.

Hence $\mathbf{XG} = \mathbf{XH}$.

$\therefore$ **X** is on the radical axis (**XF**) of the circles **A** and **B**.

Hence the three radical axes are concurrent.

CASE II.—WHEN **X** IS WITHIN ALL THE CIRCLES.

In this case the radical axes are the common chords of the three circles taken in pairs, and it is easy to prove that these common chords are concurrent.

The proof is left as an exercise for the student.

THEOREM 14F.

The difference between the squares on the tangents from a given point to two given circles is double of the rectangle contained by the join of the centres and the distance of the given point from the radical axis.

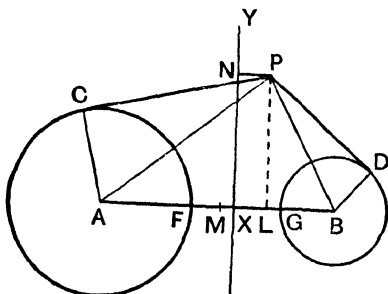


Fig 187.

Given that **XY** is the radical axis of the two circles **CF** and **GD**.

PC and **PD** are tangents from **P** to the two circles, and **PN** is perpendicular to **XY**.

Required to prove that $PC^2 - PD^2 = 2PN \cdot AB$.

Construction. Draw **PL** perpendicular to **AB**.

Bisect **AB** at **M**.

Join **AP**, **BP**, **AC**, **BD**.

Proof. Because $\angle ACP = \angle BDP =$ a right angle,

$$\therefore PC^2 - PD^2 = (PA^2 - AC^2) - (PB^2 - BD^2), \quad (\text{Theorem 42A.})$$

$$\text{whence } PC^2 - PD^2 = (PA^2 - PB^2) - (AC^2 - BD^2). \quad (1)$$

In $\triangle APB$, **PL** is perpendicular to **AB** and **AB** is bisected at **M**.

$$\begin{aligned}
 \therefore PA^2 - PB^2 &= AL^2 + PL^2 - (BL^2 + PL^2) \\
 &= AL^2 - BL^2 \\
 &= (AM + ML)^2 - (AM - ML)^2 \\
 &= 4AM \cdot ML \\
 &= 2AB \cdot ML \dots\dots\dots (2)
 \end{aligned}$$

Because X is on the radical axis,

$$\therefore AX^2 - XB^2 = AC^2 - BD^2 \dots\dots\dots (3)$$

Also $AX^2 - XB^2 = (AX + XB)(AX - XB)$, (*Theorem 5c.*)

$$\therefore AX^2 - XB^2 = AB \cdot 2MX \dots\dots\dots (4)$$

Thus from (3) and (4) $AC^2 - BD^2 = 2AB \cdot MX \dots (5)$

Therefore from (1), (2), (5) it follows that:—

$$PC^2 - PD^2 = 2AB \cdot ML - 2AB \cdot MX = 2AB (ML - MX),$$

$$\text{i.e. } PC^2 - PD^2 = 2AB \cdot LX$$

$$\text{and } LX = PN.$$

$$\text{Hence } PC^2 - PD^2 = 2AB \cdot PN.$$

V.—THE THEOREMS OF CEVA AND MENELAUS.

42. Introduction.—In these two theorems the three sides (BC , CA , AB) of a triangle are divided *internally or externally* by three points (X , Y , Z). (See Figs. 188, 189, 190.) The theorems are concerned with the ratios of the two segments of each side, *i.e.* with the ratios.

$$BX : XC,$$

$$CY : YA,$$

$$\text{and } AZ : ZB.$$

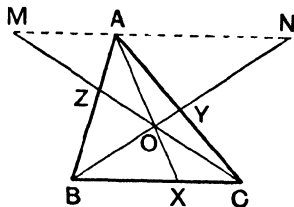


Fig. 188.

The student should note the order of the segments in each ratio, which is such that if each side is divided *internally* the six segments are taken in order round the triangle.

Further, the ratio of the two segments is reckoned *positive* if the side is divided internally (as the two segments point in the *same* direction), and *negative* if the side is divided externally (as the two segments then point in opposite directions).

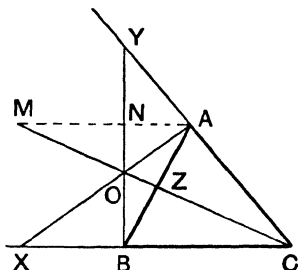


Fig. 189.

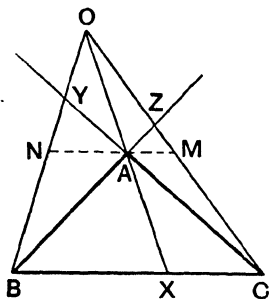


Fig. 190.

Ceva's Theorem proves that if AX , BY , CZ are concurrent, the product of these three ratios is equal to $+1$; and conversely.

Menelaus' Theorem proves that if X , Y , Z are collinear, the product of these three ratios is equal to -1 ; and conversely.

THEOREM 15F. (CEVA'S THEOREM.)

If three lines drawn from the vertices of a triangle are concurrent, then the product of the ratios of the segments into which these lines divide the opposite sides is equal to $+1$: and conversely.

Given The construction shown in Figs. 188, 189, 190, where MN drawn through A is parallel to BC .

Required to prove that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1$.

Proof. (1) **To prove that the product of the three ratios is positive.**

If O lies within the triangle (Fig. 188) each ratio is obviously positive and the product therefore positive.

If **O** lies without the triangle (Figs. 189 and 190) two of the ratios are negative and the third positive, and the product therefore positive.

(2) To prove that the product is numerically equal to 1.

In each of the three figures:—

Because the three lines through **O** make intercepts on the parallels **BC** and **MN**,

$$\therefore \frac{BX}{XC} = \frac{NA}{AM}. \quad (\text{Theorem 26D.})$$

Again, because $\triangle$ s **YBC**, **YNA** are equiangular,

$$\therefore \frac{CY}{YA} = \frac{BC}{NA}. \quad (\text{Theorem 18D.})$$

And because $\triangle$ s **ZAM**, **ZBC** are equiangular,

$$\therefore \frac{AZ}{ZB} = \frac{AM}{BC}. \quad (\text{Theorem 18D.})$$

$$\text{Thus } \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{NA}{AM} \cdot \frac{BC}{NA} \cdot \frac{AM}{BC}.$$

$$\text{Hence } \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

The converse is left as an exercise for the student.

THEOREM 16F. (MENELAUS' THEOREM.)

If three points taken on the three sides of a triangle are collinear, then the product of the ratios of the segments into which they divide the sides is -1 : and conversely.

Given The construction shown in Figs. 191 and 192, where **AL**, **BM**, **CN** are each perpendicular to **XYZ**.

Required to prove that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$

Proof. (1) **To prove that the product of the three ratios is negative.**

If the line **XYZ** passes through the triangle (Fig. 191), two of the ratios are positive and the third is negative, and the product therefore negative.

If the line **XYZ** lies entirely without the triangle (Fig. 192), all three ratios are negative, and the product therefore negative.

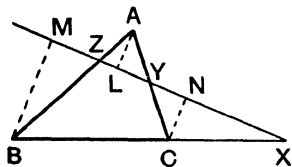


Fig. 191.

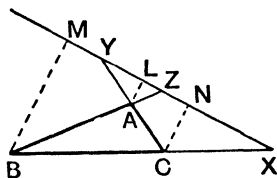


Fig. 192.

(2) **To prove that the product is numerically equal to 1.**

The triangles **BMX**, **CNX** are similar, for they have right angles at **M** and **N** and equal angles at **X**.

$$\text{Hence } \frac{BX}{XC} = \frac{BM}{CN}.$$

$$\text{Similarly } \frac{CY}{YA} = \frac{CN}{AL},$$

$$\text{and } \frac{AZ}{ZB} = \frac{AL}{BM}.$$

$$\text{Thus } \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{BM}{CN} \cdot \frac{CN}{AL} \cdot \frac{AL}{BM}.$$

Hence $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$ (numerically), and the product is negative as already proved.

The converse is left as an exercise for the student.

VI.—THE NINE-POINTS CIRCLE.

43. Introduction.—Theorem 17F has already been given on page 273 of *The New Matriculation Geometry*, but it is repeated here so that the theorems in this section may be complete. In itself Theorem 17F is an interesting example of a rider on concyclic points.

THEOREM 17F.

In any triangle the following nine points are concyclic:—the middle points of the sides, the middle points of the lines joining the orthocentre to the vertices, and the feet of the perpendiculars from the orthocentre to the sides.

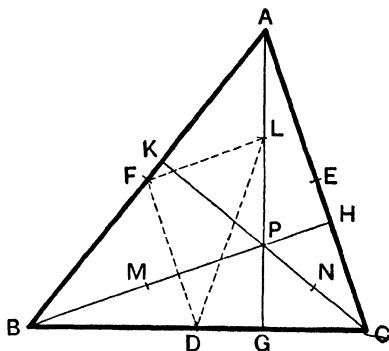


Fig. 193.

Given that P is the orthocentre of $\triangle ABC$; D, E, F bisect the sides; L, M, N bisect PA, PB, PC.

Required to prove that The circle DEF passes through the points G, H, K, L, M, N.

Proof. (1) Consider the circle on diameter LD.

$\therefore$ F, D bisect two sides of $\triangle ABC$, $\therefore$ FD is parallel to HA.

$\therefore$ F, L bisect two sides of $\triangle ABP$, $\therefore$ FL is parallel to HB.

(Theorem 32A.)

Hence $\angle \text{LFD} = \angle \text{AHB} = \text{a right angle.}$
(Theorem 9A.)

Similarly $\angle \text{LED} = \angle \text{AKC} = \text{a right angle.}$

Also $\angle \text{LGD} = \text{a right angle.}$

Hence the circle on diameter **LD** passes through **F, E, G.**

But only one circle can be drawn through the three
 points **D, E, F.**

Hence the circle **DEF** passes through **L** and **G.**

(2) Similarly, by considering the circle on diameter **ME**, it can be proved that circle **DEF** passes through **M** and **H**; and by considering the circle on diameter **NF**, that circle **DEF** passes through **N** and **K.**

Hence the circle through D, E, F also passes through G, H, K, L, M, N.

From this property, the circle which passes through the middle points of the sides of a triangle is called the **Nine-Points Circle.**

Notes.—(1) As a corollary the student should prove that **LD, ME, NF** are concurrent and equal and bisect each other. (*Hint*: They are all diameters of the nine-points circle.)

(2) A remarkable theorem concerning the nine-points circle was given by Feuerbach, namely, that it touches the inscribed and the three escribed circles of a triangle. The proof is somewhat difficult.

THEOREM 18F.

In any triangle the following four points are collinear:—the orthocentre, the centre of the nine-points circle, the intersection of medians, and the circum-centre.

Given that A point **P** is the orthocentre, **Q** the centre of the nine-points circle, **V** the intersection of the medians, and **O** the circum-centre.

Required to prove that **P, Q, V, O** are collinear points.

Proof. (1) To prove that $OD = \frac{1}{2}AP$ (Fig. 194).

Because OD is perpendicular to BC , and AP is perpendicular to BC ,

$\therefore OD$ is parallel to AP . (Theorem 5A.)

Because F bisects BA and M bisects BP ,

$\therefore FM$ is parallel to AP and equal to $\frac{1}{2}AP$.

(Theorem 32A.)

Hence OD is parallel to FM .

Similarly OF , PC , and DM are parallel,

whence $ODMF$ is a parallelogram.

$\therefore OD = FM = \frac{1}{2}AP$. (Theorem 26A.)

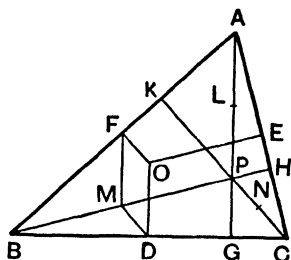


Fig. 194.

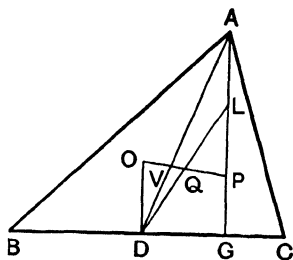


Fig. 195.

(2) To prove that the intersection of medians and the centre of the nine-points circle lie in OP (Fig. 195).

Let AD meet OP in V , and LD meet OP in Q .

Because OD is parallel to AP ,

$\therefore \triangle OVD$ is similar to $\triangle PVA$. (Theorem 18D.)

But $OD = \frac{1}{2}AP$,

$\therefore DV = \frac{1}{2}VA$.

Thus V is the intersection of the medians.

Again, because $LP = \frac{1}{2}AP$,

$\therefore OD$ is equal and parallel to LP ,

$\therefore ODPL$ is a parallelogram and Q bisects LD .

(Theorems 28A, 29A.)

But LD is a diameter of the nine-points circle.

Thus Q is the centre of the nine-points circle.

Hence P , Q , V , and O are collinear.

Note.—As a corollary the student should prove that $OV = \frac{1}{3}OP$ and $OQ = \frac{2}{3}OP$: the proof is simple, and the facts are used in the next theorem.

THEOREM 19F.

In any triangle the radius of the nine-points circle is half the radius of the circum-circle; also the orthocentre and the intersection of medians are the centres of similitude of the circum-circle and the nine-points circle.

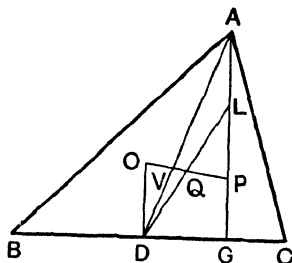


Fig. 196.

Proof. Because OD is equal and parallel to AL ,
(Theorem 18E.)

$\therefore OA$ is equal and parallel to DL ,

i.e. radius of circum-circle = diameter of nine-points circle.

$\therefore$ radius of circum-circle : radius of nine-points circle = 2 : 1.

i.e. radius of nine-points circle is half radius of circum-circle

$\therefore OV = \frac{1}{3}OP$ and $OQ = \frac{1}{2}OP$, $\therefore OV = \frac{2}{3}OQ$.

Hence $OV : VQ = 2 : 1$,

i.e. V divides the join of the centres *internally* in the ratio of the radii.

Again, because $OQ = QP$,

$\therefore OP : PQ = 2 : 1$,

i.e. P divides the join of the centres *externally* in the ratio of the radii.

Thus V and P are the two centres of similitude.

Hence radius of nine-points circle is half the radius of the circum-circle, also orthocentre and intersection of medians are centres of similitude of these two circles.

VII.—POLES AND POLARS.

44. Introduction.—Before proceeding with this section, the student should again read the remarks on Inversion in Art. 40.

The polar of any point P (Fig. 197) with respect to any circle of centre O and radius OA is the straight line QN , which is perpendicular to OP and meets it at a point N such that $OP \cdot ON = OA^2$. P is called the pole of the line QN .

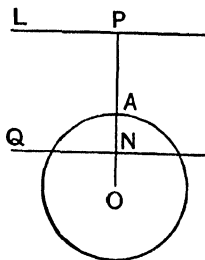


Fig. 197.

Similarly, in Fig. 197 LP is the polar of the point N , and N is the pole of LP .

Further, since P and N are inverse points (OA being the radius of inversion), the polar of a given point with respect to a given circle may be defined as the straight line drawn through the inverse of the given point at right angles to the line which joins the given point to the centre.

Obviously, (1) if the pole P lies outside the circle, N lies inside it and the polar of P intersects the circle. (2) If the pole P lies within the circle, N lies outside it, and the polar of P lies outside the circle. (3) If the pole P lies on the circumference, N coincides with P , and the polar of P becomes the tangent at P . (4) By taking P inside the circle and moving it nearer and nearer to O , N (which is outside) moves further and further from the circle: thus if P falls at O , N and the polar of P are at infinity.

THEOREM 21F.

If a point lies outside a circle its polar is the chord of contact of tangents drawn from the given point to the circle.

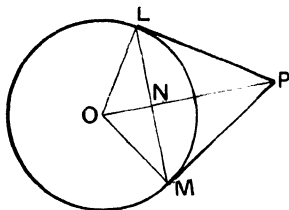


Fig. 201.

Given that **PL** and **PM** are tangents to a circle.

Required to prove that **LM**, the chord of contact, is the polar of **P**.

Construction. Join **OL**, **OP**, **OM**.

Proof. $\triangle ONL$ is similar to $\triangle OLP$,

$$\therefore ON : OL = OL : OP,$$

$$\therefore ON \cdot OP = OL^2.$$

Also **LM** is perpendicular to **OP**. (*Theorem 8B.*)

Hence LM is the polar of P.

THEOREM 22F.

If a variable chord passes through a fixed point, then the locus of the intersection of the tangents at the extremities of the chord is the polar of the point.

Given that **LM** is a variable chord through a fixed point **P**.
LQ and **MQ** are tangents at **L** and **M** meeting at **Q**.

Required to prove that The locus of **Q** is the polar of **P**.

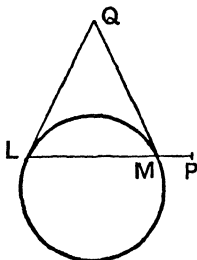


Fig. 202.

Proof. **LM** is the polar of **Q**, (Theorem 21F.)
 $\therefore$ **P** lies on the polar of **Q**.
 $\therefore$ **Q** lies on the polar of **P**. (Theorem 20F.)
Hence the locus of Q is the polar of P.

THEOREM 23F. (SALMON'S THEOREM.)

The distances of any two points from the centre of a circle are in the same ratio as the two distances of each from the polar of the other.

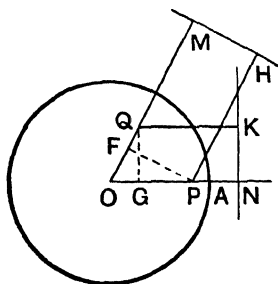


Fig. 203.

Given that **P** and **Q** are two points whose distances from the centre are **OP** and **OQ**.

KN and **HM** are the polars of **P** and **Q** respectively.

Required to prove that $OP : OQ = PH : QK$.

Construction. Draw PF perpendicular to OQ and QG perpendicular to OP .

Proof. Because $\angle PFQ = \angle PGQ =$ a right angle,
 $\therefore$ points P, G, F, Q are concyclic,
 $\therefore OP \cdot OG = OQ \cdot OF$. (Theorem 10c.)

Hence $OP : OQ = OF : OG$.

Again, because $OP \cdot ON = OA^2 = OQ \cdot OM$,

$\therefore OP : OQ = OM : ON$.

Hence $OP : OQ = OM - OF : ON - OG$,

i.e. $OP : OQ = FM : GN$.

But $FM = PH$ and $GN = QK$.

Hence $OP : OQ = PH : QK$.

THEOREM 24F.

Any chord of a circle drawn through a given point is cut harmonically by the point and its polar.

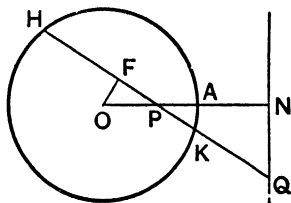


Fig. 204.

Given that HK is a chord passing through a given point P , and QN is the polar of P .

Required to prove that HK is divided harmonically at P and Q .

Construction. Draw OF perpendicular to HK .

Proof. Because $\angle OFP = \angle PNQ =$ a right angle,
 $\therefore$ points **O, F, N, Q** are concyclic;

$$\therefore FP \cdot QP = OP \cdot PN. \quad (\text{Theorem 8c.})$$

$$\text{Now } FK^2 = OK^2 - OF^2 = OP \cdot ON - OF^2,$$

$$\text{i.e. } FK^2 = OP \cdot PN + OP^2 - OF^2 = FP \cdot PQ + FP^2.$$

$$\text{Thus } FK^2 = FP \cdot FQ. \quad (\text{Theorem 2c.})$$

But **F** bisects **HK**.

Hence HPKQ is a harmonic range, i.e. HK is divided harmonically at P and Q.

VIII.—THE COMPLETE QUADRILATERAL.

45. Definition.—If **ABCD** (Fig. 205) is a quadrilateral, and the pairs of opposite sides **AB, DC** and **AD, BC** when produced meet at **L** and **M** respectively, the figure so formed is called a **complete quadrilateral**, and **LM** is called the **third diagonal** (**BD** and **AC** being the other two diagonals).

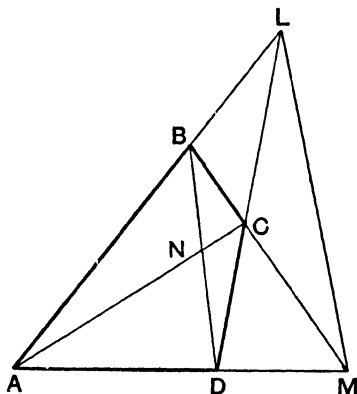


Fig. 205.

Another definition is as follows:—A complete quadrilateral is formed by four points, the **vertices A, B, C, D**, joined by six lines, **AB, BC, CD, DA, AC, BD**, which intersect in seven points, namely, the vertices and the three **harmonic points L, M, N**.

THEOREM 25F.

The mid-points of the three diagonals of a complete quadrilateral are collinear.

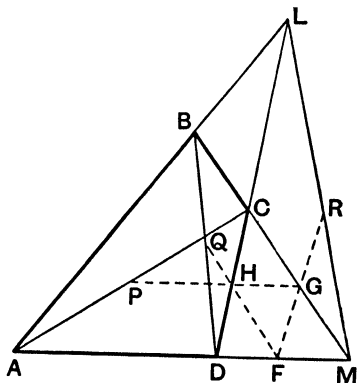


Fig. 206.

Given that **P, Q, R** are the mid-points of the three diagonals of a complete quadrilateral.

Required to prove that **P, Q, R** are collinear.

Construction. Bisect the sides of the triangle **CDM** at **F, G, H**.

Proof. **GH** bisects the diagonal **AC**, **FH** bisects the diagonal **BD**, and **FG** bisects the diagonal **LM**.

Also $HP = \frac{1}{2}DA$, $GR = \frac{1}{2}CL$, etc.

Because the points **A, B, L** (lying on sides of $\triangle MCD$) are collinear,

$$\therefore \text{(by Menelaus' Theorem)} \quad \frac{DA}{AM} \cdot \frac{MB}{BC} \cdot \frac{CL}{LD} = -1.$$

But this still holds if each segment is replaced by a segment of half its length.

$$\text{Thus } \frac{HP}{PG} \cdot \frac{FQ}{QH} \cdot \frac{GR}{RF} = -1.$$

Hence (by the converse of Menelaus' Theorem) the points PQR (lying on sides of $\triangle FGH$) are collinear.

Note.—As an exercise on Ceva's and Menelaus' Theorems, the student should prove that in Fig. 207 the ranges **ASDM**, **APBL**, **SNQL**, **DRCL**, **PNRM**, **BQCM**, **ANCT** are harmonic ranges. For example :—

The lines **AC**, **DB**, **LS** (through vertices of $\triangle ADL$) are concurrent.

$\therefore$ by Ceva's Theorem

$$\frac{AS}{SD} \cdot \frac{DC}{CL} \cdot \frac{LB}{BA} = 1.$$

Again, the points **M**, **C**, **B** (lying on sides of $\triangle ADL$) are collinear.

$\therefore$ by Menelaus' Theorem

$$\frac{AM}{MD} \cdot \frac{DC}{CL} \cdot \frac{LB}{BA} = -1.$$

$$\therefore \frac{AS}{SD} \div \frac{AM}{MD} = -1.$$

Hence ASDM form a harmonic range,

and so with the others.

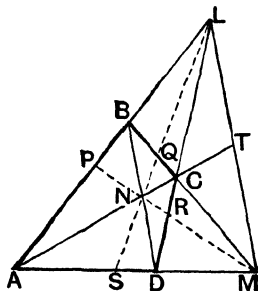


Fig. 207.

IX.—THE TANGENCY PROBLEMS.

46. Introduction.—In general a straight line can be made to satisfy *two* conditions, *e.g.* to pass through two points, or to pass through one point and be parallel to a given line.

Similarly, a circle can usually be made to satisfy *three* conditions, *e.g.* to pass through three points, or to have a given radius and to pass through two points.

The name of **Tangency Problems** is usually applied to an interesting group of problems on constructing circles where the conditions are of the three following types :—

- (1) that the required circle is to pass through a given point,
- (2) that the required circle is to touch a given line,
- (3) that the required circle is to touch a given circle.

If these three types of conditions are represented respectively by the three letters **P**, **L**, **C** (point, line, circle), then the different types of problem can be represented by various combinations of these letters: for example, the symbol **PCC** represents the problem of describing a circle to pass through one given point, and to touch two given circles; and the symbol **LLL** represents the problem of describing a circle to touch three given lines.

We know already that the latter problem has *four solutions*: that is to say, in general four different circles can be drawn, each of which touches the three given lines, viz. the inscribed circle and the three escribed circles of the triangle formed by the three lines.

The following is a complete list of the various types of problem, with the number of solutions to be expected in each type:—

One solution: **PPP**.

Two solutions: **PPL**, **PLL**, **PPC**.

Four solutions: **LLL**, **PLC**, **PCC**.

Eight solutions: **LLC**, **LCC**, **CCC**.

47. Solutions.—The methods required in the various cases are as follows:—

PPP. See Problem 6B.

LLL. See Problems 7B, 8B.

PPL.—**To draw a circle passing through two given points and touching a given straight line.**

Let **A**, **B** be the given points and **XY** the given line.

Let **AB** meet **XY** at **C**.

Draw any circle through **A** and **B**.

From **C** draw a tangent **CT** to this circle.

On **XY** mark off **CD = CT**.

Draw a circle through **A**, **B**, **D**.

There are **two solutions** because **CD** may be marked off on either side of **C**.

Prove the construction.

PLL.—To draw a circle passing through a given point and touching two given straight lines.

Let **OB**, **OC** be the given lines, and let **A**, the given point, be within the angle **BOC**.

Bisect $\angle BOC$ by the straight line **OD**.

With any convenient point **F** on **OD** as centre, draw a circle touching **OB** and **OC**.

Let **OA** meet this circle at **G** and **H**.

Through **A** draw lines parallel to **FG** and **FH**, meeting **OD** at **I** and **K**.

Then **I** and **K** are the centres of the required circles.

There are thus **two solutions**.

Prove the construction.

PPC.—To draw a circle passing through two points and touching a given circle.

Let **A** and **B** be the given points.

Draw any circle through **A** and **B** meeting the given circle in **P** and **Q**.

Let the straight lines **AB**, **PQ** meet in **C**.

Draw a tangent **CT** to the given circle.

Draw a circle through **A**, **B**, **T**.

There are **two solutions** because either of the tangents from **C** to the given circle may be used.

Prove the construction.

PCC.—To draw a circle passing through a given point and touching two given circles.

In Fig. 208 let **R** be the given point, and **HQK**, **LPM** the given circles.

Analysis.—If the required circle touches the two given circles at **P**, **Q**, then (by Theorem 7F), **QP** passes through a centre of similitude **S**. If **SR** meets the required circle again at **R'**, then $SR \cdot SR' = SQ \cdot SP = SK \cdot SL$ (Theorem 6F). Thus **K**, **R**, **R'**, **L** are concyclic; whence we obtain the following construction :—

Solution. Find a centre of similitude **S**.

Through **S** draw any secant **SMLKH**. (In practice the most convenient secant is the join of the centres.)

Draw the circle **KLR**, cutting **SR** again in **R'**.

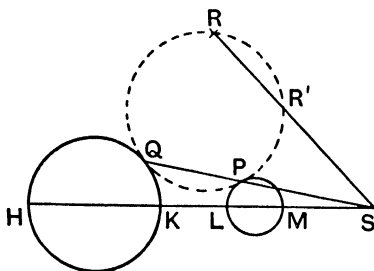


Fig. 208.

Draw a circle through **R** and **R'** to touch the circle **HQK** ;
this will also touch the circle **LPM**.

There are **four solutions** : for (1) we may use either of
the two centres of similitude, and (2) we may use
either of the two circles through **RR'** which
touch the circle **HQK**.
Prove the construction.

PLC.—To draw a circle passing through a given point
and touching a given line and a given circle.

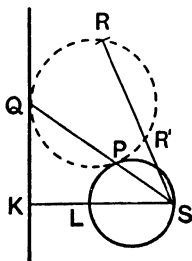


Fig. 209.

In Fig. 209, let **R** be the given point, **QK** the given
straight line, and **LPS** the given circle.

Analysis.—Fig. 209 may be regarded as a limiting case of Fig. 208; the line **QK** in Fig. 209 corresponds to the circle **HQK** in Fig. 208, for the line **QK** in Fig. 209 may be regarded as a portion of a circle of an infinitely large radius. In this case it is easily seen that the two centres of similitude of the circle **LPS** and the line-circle **QK** are the extremities of that diameter of **LPS** which is perpendicular to **QK**.

Solution. The solution can now be inferred by making the suitable alterations in the solution of **PCC**.

There are **four solutions**.

The student should prove his construction.

LLC.—To draw a circle touching two given lines and a given circle.

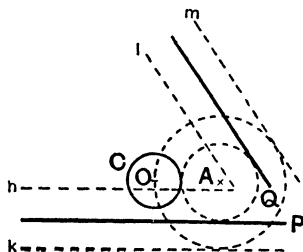


Fig 210.

Let **C** be the given circle, and **P, Q** the given lines.

On each side of **P, Q** draw lines parallel to them at a distance equal to the radius of **C**, viz. the lines **h, k, l, m**.

Analysis.—If **A** is the centre of a circle which touches **P** and **Q**, and touches **C** *externally* (Fig. 210), it is easily proved that **A** is also the centre of a circle which touches **k** and **m**, and passes through **O**. Hence **A** can be found by the construction for **PLL**.

Similarly, if **B** is the centre of a circle which touches **P** and **Q**, and touches **C** *internally*, it is easily proved that **B** is also the centre of a circle which touches **h** and **l** and passes through **O**. Hence again **B** can be found by the construction for **PLL**.

Solution. The case **LLC** reduces to the case **PLL**.

There are in general **eight solutions**, but the eight circles will all be real only if both the lines **P, Q** cut the circle **C**.

The student should prove his construction.

CCC.—To draw a circle touching three given circles.

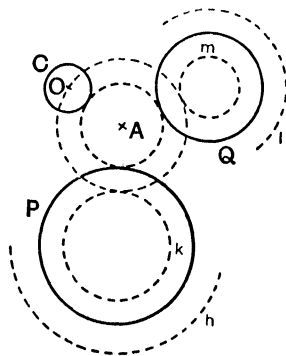


Fig. 211.

Let **P, Q, C** be the three circles, **C** being the smallest.

*Draw two circles **h, k** concentric with **P**, at a distance from **P** equal to the radius of **C**.*

*Draw two circles **l, m** concentric with **Q**, and at a distance from **Q** equal to the radius of **C**.*

Analysis.—If **A** is the centre of a circle which touches the circles **P, Q**, and **C** all *externally* (Fig. 211), it is easily proved that **A** is also the centre of a circle which passes through **O** and touches the circles **k** and **m** *externally*. Hence **A** can be found by the construction for **PCC**.

Similarly, if **B** is the centre of a circle which touches the circle **P** *internally* and the circles **Q** and **C** *externally*, it is easily proved that **B** is also the centre of a circle which passes through **O**, touches the circle **h** *internally*, and touches the circle **m** *externally*. Hence **B** can be found by the construction for **PCC**.

Solution. The case **CCC** reduces to the case **PCC**.

There remain six other solutions; *e.g.* a circle can be drawn touching **P** and **C** *internally* and **Q** *externally*.

These are all treated as above.

Thus there are **eight solutions**.

The student should prove his construction.

LCC.—To draw a circle touching a given line and two given circles.

Let **P** be the given line and **Q** and **C** the given circles.

Let **C** be the smaller circle.

Draw two lines parallel to **P** at a distance equal to the radius of **C**.

Draw two circles concentric with **Q** at a distance from **Q** equal to the radius of **C**.

Compare the cases **LLC** and **CCC**, and it will be seen that **LCC** reduces to **PLC**.

There are **eight solutions**.

The student should prove his construction.

Exercises XIII.

ON HARMONIC SECTION.

1. If **A, P, B, Q** be a harmonic range, the circles described on **AB** and **PQ** as diameters will cut each other orthogonally.

2. If one pair of rays of a harmonic pencil are at right angles to each other, then these rays are the internal and external bisectors of the angle between the other pair of rays.

3. If one ray of a harmonic pencil bisects the angle between two other rays, then this ray will be perpendicular to its conjugate ray.

4. If **ABCD** and **Abcd** be two harmonic ranges having the common point **A**, then **Bb, Cc**, and **Dd** will be concurrent.

5. Find a fourth harmonic point to three given points.

6. Draw a fourth harmonic ray to three given rays.

7. A diameter of a circle is divided harmonically by any tangent and the orthogonal projection on the diameter of the point of contact.

ON CENTRES OF SIMILITUDE.

8. Where are the two centres of similitude of two circles :—

- (a) If one circle cuts the other ?
- (b) If one circle lies entirely within the other ?
- (c) If one circle touches the other ?
- (d) If the two circles are equal ?
- (e) If one circle is replaced by a straight line ?

9. The internal centre of similitude of any two escribed circles of a triangle falls at a vertex of the triangle.

10. The external centre of similitude of the inscribed circle of a triangle and one of the escribed circles falls at a vertex of the triangle.

11. Prove Theorem 6F when **S** is the internal centre of similitude.

12. Prove Theorem 7F when the variable circle touches the other two circles internally.

13. Prove Theorem 7F when the variable circle touches one circle externally and one internally.

14. The internal centre of similitude of the inscribed circle of a triangle and one of the escribed circles divides one side of the triangle internally in the ratio of the other two sides.

15. The external centre of similitude of two of the escribed circles of a triangle divides one side of the triangle externally in the ratio of the other two sides.

16. Given the base and the vertical angle of a triangle, find the locus of the centroid.

17. Given the circum-circle of a triangle, the vertex, and the length of the base, find the locus of the centroid.

ON INVERSION.

18. If **O, A, B, C** are collinear points and **A', B', C'** the inverses of **A, B, C** with respect to **O**, prove that if **OA, OB, OC** are in arithmetical progression, then **OA', OB', OC'** are in harmonical progression.

19. If **O, X, Y, Z** are collinear points and **X', Y', Z'** the inverses of **X, Y, Z** with respect to **O**, prove that if **OX, OY, OZ** are in geometrical progression, then **OX', OY', OZ'** are also in geometrical progression.

20. Find the inverse of the circum-circle of an isosceles triangle with respect to the vertex of the triangle as origin.

21. Show that if a straight line cuts a circle each may be inverted into the other by suitable selection of the origin and constant of inversion.

22. PQ is a chord of a circle bisected at O . Show that, with O as origin and OP as radius of inversion, the circle inverts into itself.

ON THE RADICAL AXIS.

23. The centres of two circles are A , B , and AB cuts the circumferences at H , K respectively; also their radical axis meets AB at X . If radius $AH >$ radius BK , prove that $AX^2 - BX^2 = AH^2 - BK^2$.

24. Find the locus of the centre of a circle which cuts two given circles orthogonally.

25. Show how to draw a circle to cut each of three given circles orthogonally.

26. Find the locus of a point P , if the difference of the squares on the tangents from P to two given circles is constant.

27. The common chords to a fixed circle and to all circles which can be drawn through two given points pass through a fixed point.

28. Find the locus of a point P which moves so that its distance from a fixed point O is equal to the length of the tangent from P to a fixed circle.

ON THE THEOREMS OF CEVA AND MENELAUS.

29. Deduce from Ceva's Theorem that the following sets of lines connected with a triangle are concurrent :—

(a) The three medians.

(b) The three perpendiculars.

(c) The three internal bisectors of the angles.

(d) The lines joining the vertices to the points of contact of the in-circle.

(e) The lines joining the vertices to the points of contact of an ex-circle.

30. ABC is a triangle and EF is parallel to BC ; BE and CF intersect in K . Prove that AK bisects BC .

31. Prove Menelaus' Theorem by drawing AK parallel to BC , meeting the transversal in K , and considering the similar triangles thus produced.

32. Prove Menelaus' Theorem by drawing AK parallel to the transversal and considering the similar triangles thus produced.

33. Extend Menelaus' Theorem to the case of a quadrilateral. [N.B.—The converse is not true.]

34. The in-circle of a triangle **ABC** touches the sides in **D, E, F**. **FE** produced meets **BC** produced in **L**. Show that $\{BDCL\} = -1$.

35. The in-circle of a triangle **ABC** touches the sides in **D, E, F**. **EF, FD, DE** produced meet **BC, CA, AB** produced, respectively, in **L, M, N**. Prove that **LMN** is a straight line.

36. **AD, BE, CF** are any three concurrent lines within the triangle **ABC**; **EF, FD, DE** produced meet **BC, CA, AB** produced in **L, M, N**. Prove that **LMN** is a straight line.

37. Prove Ceva's Theorem by applying Menelaus' Theorem first to the triangle **ABX** cut by the transversal **COZ**, then to the triangle **AXC** cut by the transversal **BOY** (Fig. 188).

38. **ABC, DEF** are two triangles. **BC, EF** meet at **P**; **CA, FD** meet at **Q**; **AB, DE** meet at **R**. Given that **AD, BE, CF** are parallel, prove that **P, Q, R** are collinear.

39.* **E** is a point in the side **AC** of a triangle **ABC**, and **F** is taken in **BA** produced such that $BF : AF = CE : EA$; also **BE** and **CF** intersect in **O**. Prove that **AO** is parallel to **BC**.

40.* The lines joining the middle point of each side of a triangle to the middle point of the perpendicular upon it from the opposite angle meet in a point. [Apply Ceva's Theorem to the triangle formed by the middle points of the sides.]

41.* The lines joining the middle point of each side of a triangle to the middle point of the bisector of the opposite angle meet in a point.

42.* **AD, BE, CF** are lines drawn to meet in a point through the vertices of a triangle **ABC**; the circum-circle of **DEF** meets the sides again in **D', E', F'**. Prove that **AD', BE', CF'** meet in a point.

ON THE NINE-POINTS CIRCLE.

43.* Given any two of the four points **O, V, Q, P** (Fig. 195) and one angular point, construct the triangle.

44.* Show that the nine-points circle of $\triangle AEF, BFD, CDE$ touches the nine-points circle of $\triangle DEF$ at the middle points of **EF, FD, DE**.

45.* Construct a triangle given one side and the nine-points centre.

46.* Given one side and the opposite angle, find the locus of the nine-points centre.

ON POLES AND POLARS.

47. Show that the angle between two given lines is equal to the angle subtended at the centre by the join of their poles with respect to any given circle.

48. Show that the pole of a diameter is the point at infinity in the direction perpendicular to the diameter.

49. Draw tangents to a circle from a given point using the ruler only.

ON THE COMPLETE QUADRILATERAL.

50. In a complete quadrilateral prove that any diagonal is divided harmonically by the other two.

ON THE TANGENCY PROBLEMS.

51. Draw a circle touching two parallel straight lines and a circle which intersects them.

52.* Given three equal circles, draw a circle

- (i) to touch two of them externally and one internally,
- (ii) to touch one of them externally and two internally.

LIST OF PROPOSITIONS.

The chief object of this list of Propositions is to serve for an Index to the reference symbols used in *The New Matriculation Geometry* and in *Similar Figures, Space, and Solids*. References to the Propositions in *The New Matriculation Geometry* are made where necessary, under these symbols, in the present book. The list will also be found useful as a summary of the course of theoretical Geometry covered by the two books.

ANGLES AT A POINT: PARALLEL STRAIGHT LINES.

Theorem 1A. The two angles which one straight line makes with another straight line on one side of it are together equal to two right angles.

Theorem 2A. If at a point in a straight line, two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in the same straight line.

Theorem 3A. Each of the angles formed by two intersecting straight lines is equal to the vertically opposite angle.

Theorem 4A. If a straight line meeting two other straight lines in the same plane makes two alternate angles equal to one another, these two straight lines shall be parallel.

Theorem 5A. If a straight line, meeting two other straight lines, makes an exterior angle equal to the interior and opposite angle on the same side of the line; or if it makes the two interior angles on the same side together equal to two right angles; then the two straight lines shall be parallel.

Theorem 6A. If a straight line meets two parallel straight lines, it shall make the alternate angles equal to one another.

Theorem 7A. If a straight line meets two parallel straight lines, it shall make each exterior angle equal to the interior and opposite angle on the same side of the line, and each pair of interior angles on the same side of the line together equal to two right angles.

Theorem 8A. If two straight lines are parallel to the same straight line they are parallel to one another.

Theorem 9A. If the two arms of one angle are respectively parallel to the two arms of another, and are drawn in the same senses from the angular points, then the angles are equal.

TRIANGLES : CONGRUENCE OF TRIANGLES.

Theorem 10A. If any side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior and opposite angles : also, the three angles of a triangle are together equal to two right angles.

Theorem 11A. If two triangles have two sides and the included angle in the one respectively equal to two sides and the included angle in the other, then shall these triangles be congruent.

Theorem 12A. If two triangles have two angles and one side in the one triangle respectively equal to two angles and the corresponding side in the other triangle, then the two triangles shall be congruent.

Theorem 13A. If two sides of a triangle are equal then shall the angles opposite to these sides be equal, *i.e.*

The angles at the base of an isosceles triangle are equal.

Theorem 14A. If two angles of a triangle are equal, then the sides opposite to these angles are also equal.

Theorem 15A. If the three sides of one triangle are respectively equal to the three sides of another triangle, the two triangles shall be congruent.

Theorem 16A. If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles shall be congruent.

Theorem 17A. If two sides of one triangle are respectively equal to two sides of another, and if the angles opposite to one pair of equal sides are equal, then shall the angles opposite to the other pair of equal sides be either equal or supplementary.

Theorem 18A. If one side of a triangle is greater than another, then the angle opposite to the greater side shall be greater than the angle opposite to the less.

Theorem 19A. If one angle of a triangle is greater than another, then the side opposite to the greater angle shall be greater than the side opposite to the less.

Theorem 20A. Any two sides of a triangle are together greater than the third.

Theorem 21A. The shortest line which can be drawn from a given point to a given line is the line which is perpendicular to the given line ; and of any other two lines drawn from the given point to the given line on one side of the perpendicular, that which is nearer to the perpendicular is less than that which is more remote.

Theorem 22A. If the two sides of one triangle are respectively equal to the two sides of another, and if the vertical angle of the first triangle is greater than the vertical angle of the second triangle ; then the base of the first triangle shall be greater than the base of the second.

Theorem 23A. If the two sides of one triangle are respectively equal to the two sides of another, and if the base of the first triangle is greater than the base of the second triangle, then the vertical angle of the first triangle shall be greater than the vertical angle of the second.

Theorem 24A. In any polygon the sum of the interior angles together with four right angles is equal to twice as many right angles as the figure has sides.

Theorem 25A. If the sides of a convex polygon be produced in order the sum of the exterior angles so formed is equal to four right angles.

PARALLELOGRAMS.

Theorem 26A. In any parallelogram, opposite sides are equal, and opposite angles are equal ; also each diagonal divides the parallelogram into two congruent triangles.

Theorem 27A. (i) If the opposite sides of a quadrilateral are equal, it is a parallelogram ; also (ii) if the opposite angles of a quadrilateral are equal, it is a parallelogram.

Theorem 28A. If in a quadrilateral two opposite sides are equal and parallel, then the quadrilateral is a parallelogram, and the other two sides are also equal and parallel.

Theorem 29A. The diagonals of a parallelogram bisect each other.

Theorem 30A. If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and have also the angles included between those sides equal, then the two parallelograms shall be congruent.

Theorem 31A. If there are three or more parallel straight lines and if the intercepts made by them on any one transversal are equal, then the intercepts made by them on any other transversal are also equal.

Theorem 32A. (i) The line which joins the middle points of the sides of a triangle is parallel to, and half the length of, the base ; and (ii) the line drawn parallel to the base, from the middle point of one side, bisects the other side and is half the length of the base.

PROBLEMS ON ANGLES, TRIANGLES, QUADRILATERALS, AND PARALLEL AND PERPENDICULAR LINES.

Problem 1A. To bisect a given angle, that is, to divide it into two equal parts.

Problem 2A. At a given point in a given straight line to construct an angle equal to a given angle.

Problem 3A. To construct a triangle, given its three sides.

Problem 4A. To construct a triangle, given two sides and the included angle.

Problem 5A. To construct a triangle, given two angles and the side between them.

Problem 6A. To construct a triangle, given two angles and the side opposite to one of them.

Problem 7A. To construct a triangle, given two sides and the angle opposite to one of them.

Problem 8A. To construct a quadrilateral, given three sides and the two included angles.

Problem 9A. To construct a quadrilateral, given the four sides and one diagonal.

Problem 10A. To construct a quadrilateral, given the four sides and one angle.

Problem 11A. To draw a line through a given point parallel to a given straight line.

Problem 12A. To draw a line perpendicular to a given straight line from a given point within it.

Problem 13A. To bisect a given straight line.

Problem 14A. To draw a line perpendicular to a given straight line from a given point without the line.

Problem 15A. To divide a given straight line into five equal parts.

Problem 16A. To construct a square, given one side.

LOCI.

Theorem 33A. The locus of a point which is at a given distance from a given straight line, on one side of it, is a straight line drawn parallel to the given line through any point which lies at the given distance from the given line.

Theorem 34A. The locus of a point which is equidistant from two given points is the straight line which bisects at right angles the line joining the two given points.

Theorem 35A. The locus of a point which is equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the given lines.

AREAS.

Theorem 36A. Parallelograms on the same base and between the same parallels are equal in area.

Theorem 37A. Parallelograms on equal bases and between the same parallels are equal in area.

Theorem 38A. Triangles on the same base and between the same parallels are equal in area.

Theorem 39A. Triangles on equal bases and between the same parallels are equal in area.

Theorem 40A. If a parallelogram and a triangle be on the same base and between the same parallels then the area of the parallelogram shall be double that of the triangle.

Theorem 41A. Equal triangles on equal bases in the same straight line, and on the same side of it, are between the same parallels.

THE RIGHT-ANGLED TRIANGLE.

Theorem 42A. In a right-angled triangle, the square described on the hypotenuse is equal in area to the sum of the squares described on the other two sides.

Theorem 43A. If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, the angle contained by these sides is a right angle.

PROBLEMS ON AREAS.

Problem 17A. To construct a triangle equal to a given quadrilateral.

Problem 18A. To construct a triangle equal to a given convex polygon.

Problem 19A. To construct a triangle equal to a given convex polygon, given one side and an adjacent angle of the required triangle.

Problem 20A. To construct a rectangle equal to a given convex polygon.

THE CIRCLE: CHORDS AND TANGENTS.

Theorem 1B. If a diameter of a circle bisects a chord which is not a diameter, it is perpendicular to that chord; and conversely, if the diameter is perpendicular to the chord, the diameter bisects the chord.

Theorem 2B. The line which bisects any chord of a circle at right angles passes through the centre.

Theorem 3B. It is possible to draw one and only one circle through three points which do not lie in a straight line.

Theorem 4B. In any circle equal chords are equidistant from the centre; and conversely, chords which are equidistant from the centre are equal.

Theorem 5B. In any circle, if two chords are equal they subtend equal angles at the centre; and conversely, chords which subtend equal angles at the centre are equal.

Theorem 6B. In any circle the diameter is the greatest chord. Also of any other chords a chord which is nearer to the centre is greater than one which is more remote; and conversely a chord which is greater than another is nearer to the centre.

Theorem 7B. The line drawn through any point on the circumference of a circle at right angles to the radius through that point is a tangent to the circle; and any other line drawn through that point is a secant.

Theorem 8B. If two tangents are drawn to a circle from the same point outside the circle, they are equal in length; also the line joining this point to the centre bisects the angle between the tangents, and the angle between the radii to the points of contact of the tangents, and it also bisects at right angles the line joining the two points of contact.

Theorem 9B. If two circles cut one another the line joining their centres bisects the common chord at right angles.

Theorem 10B. If two circles touch one another then the join of their centres passes through the point of contact.

THE CIRCLE: ANGULAR PROPERTIES.

Theorem 11B. The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining arc of the circumference.

Theorem 12B. Angles in the same segment of a circle are equal.

Theorem 13B. If the line joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

Theorem 14B. If a quadrilateral be inscribed in a circle the sum of each pair of opposite angles is equal to two right angles.

Theorem 15B. If two opposite angles of a quadrilateral are together equal to two right angles, then the vertices of the quadrilateral are concyclic.

Theorem 16B. (i) Any angle in a semi-circle is a right angle. (ii) Any angle in a segment greater than a semi-circle is less than a right angle. (iii) Any angle in a segment less than a semi-circle is greater than a right angle.

Theorem 17B. If a chord be drawn from the point of contact of a tangent to a circle, then each of the angles which the chord makes with the tangent is equal to the angle in the alternate segment of the circle.

Theorem 18B. If a line drawn through one extremity of a chord of a circle makes an angle on one side of the chord equal to the angle in the alternate segment, then this line is a tangent to the circle.

Theorem 19B. In equal circles, if two arcs subtend equal angles at the centres they are equal.

Theorem 20B. In equal circles, if two arcs are equal they subtend equal angles at the centres.

PROBLEMS ON CIRCLES.

Problem 1B.—To bisect a given arc.

Problem 2B. Given an arc of a circle, to find the centre of the circle.

Problem 3B. From a given point to draw a tangent to a given circle.

Problem 4B. To draw the direct common tangents to two given unequal circles.

Problem 5B. To draw the transverse common tangents to two given circles.

Problem 6B. To circumscribe a circle to a given triangle.

Problem 7B. To inscribe a circle in a given triangle.

Problem 8B. To draw a circle touching three given straight lines which are neither concurrent nor all parallel.

Problem 9B. From a given circle to cut off a segment to contain a given angle.

Problem 10B. On a given base to draw a segment of a circle to contain a given angle.

Problem 11B. In a given circle to inscribe a triangle equiangular to a given triangle.

Problem 12B. To circumscribe to a given circle a triangle equiangular to a given triangle.

CONCURRENT LINES IN CONNECTION WITH TRIANGLES.

Theorem 21B. The three lines which bisect the sides of a triangle at right angles are concurrent.

Theorem 22B. The three lines which bisect the angles of a triangle are concurrent.

Theorem 23B. In any triangle the three medians are concurrent.

Theorem 24B. The three lines drawn from the vertices of a triangle perpendicular to the opposite sides are concurrent.

THE COMPOSITION OF RECTANGLES.

Theorem 1C. If a rectangle be divided by lines parallel to its sides then any of the quadrilaterals so formed is a rectangle.

Theorem 2C. The rectangle contained by any given line and the sum of two other given lines is equal to the sum of the rectangles contained by the first line and each of the other two lines.

Theorem 3C. If a line be divided internally at any point, then the square on the line is equal to the sum of the squares on the segments plus twice the rectangle contained by the segments.

Theorem 4C. If a line be divided *externally* at any point, then the square on the line is equal to the sum of the squares on the segments *minus* twice the rectangle contained by the segments.

Theorem 5C. The difference between the squares on two given lines is equal to the rectangle contained by the sum and the difference of the two given lines.

RECTANGLE-THEOREMS ON TRIANGLES.

Theorem 6C. In an obtuse-angled triangle the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle plus twice the rectangle contained by one of these two sides and the projection of the other upon it.

Theorem 7C. In any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the two sides containing that angle minus twice the rectangle contained by one of these two sides and the projection of the other upon it.

RECTANGLE-THEOREMS ON CIRCLES.

Theorem 8C. If two chords of a circle intersect at a point within the circumference, then the rectangle contained by the segments of the one chord shall be equal to the rectangle contained by the segments of the other.

Theorem 9C. If two lines divide each other internally so that the rectangle contained by the segments of the one line is equal to that contained by the segments of the other, then the four extremities of the two lines are concyclic points.

Theorem 10C. If two chords of a circle intersect at a point without the circumference, then the rectangle contained by the segments of the one chord shall be equal to the rectangle contained by the segments of the other.

Theorem 11C. If a chord of a circle is divided externally by a tangent, the rectangle contained by the segments of the chord is equal to the square on the tangent.

Theorem 12C. If two lines divide each other externally so that the rectangle contained by the segments of the one line is equal to that contained by the segments of the other, the four extremities of the two lines are concyclic points.

Theorem 13C. If a chord of a circle is divided externally at a given point, and if a line is drawn from this point to the circumference such that the square on this line is equal to the rectangle contained by the segments of the chord, then this line touches the circle.

PROBLEMS ON RECTANGLES.

Problem 1C. To construct a square equal to a given rectangle.

Problem 2C. To divide a given line internally so that the rectangle contained by the segments is equal to the square on a given line.

Problem 3C. To divide a given line externally so that the rectangle contained by the segments is equal to the square on a given line.

Problem 4C. To divide a given straight line (internally) into two parts so that the rectangle contained by the whole and one part is equal to the square on the other part.

Problem 5C. To construct an isosceles triangle having each base angle double of the vertical angle, given one side of the triangle.

Problem 6C. To draw a circle to pass through two given points and to touch a given straight line.

Problem 7C. To draw a circle to pass through a given point and to touch two given straight lines.

REGULAR POLYGONS.

Theorem 14C. If the circumference of a circle be divided into any number of equal parts, and if the points of division be joined in order, then the joining lines form a regular polygon inscribed in the circle.

Theorem 15C. If the circumference of a circle be divided into any number of equal parts, and if tangents to the circle be drawn at the points of division, then these tangents form a regular polygon circumscribed about the circle.

Theorem 16C. The bisectors of the angles of a regular polygon all meet in a point: also with this point as centre circles can be circumscribed to, and inscribed in, the polygon.

RATIO AND PROPORTION.

Theorem 1D. A ratio of greater inequality is diminished and one of lesser inequality is increased by adding the same quantity to each term.

Theorem 2D. If the ratio $A : B$ is equal to the ratio $C : D$, then shall the ratio $B : A$ be equal to the ratio $D : C$.

Theorem 3D. If four quantities A, B, C, D are all of the same kind, and if the ratio $A : B$ is equal to the ratio $C : D$, then shall the ratio $A : C$ be equal to the ratio $B : D$.

Theorem 4D. If the ratio $A : B$ is equal to the ratio $C : D$, then shall the ratio $A + B : B$ be equal to the ratio $C + D : D$, and the ratio $A - B : B$ be equal to the ratio $C - D : D$.

Theorem 5D. If the quantities A, B, C, D, E, F are all of the same kind, and if $A : B = C : D = E : F$, then shall each of these ratios be equal to the ratio $A + C + E : B + D + F$.

Theorem 6D. If four numbers are proportionals, the product of the extremes is equal to the product of the means ; and conversely, if the product of one pair of numbers is equal to the product of another pair, the four numbers shall be proportionals if taken in such an order that one pair forms the means and the other pair the extremes.

Theorem 7D. If three numbers are proportionals the ratio of the first to the third shall be equal to the duplicate ratio of the first to the second.

Theorem 8D. If the ratio of two numbers is equal to the ratio of two other numbers then shall their duplicate, triplicate, . . . ratios be equal.

Theorem 9D. If the ratio $A : B$ is equal to the ratio $P : Q$, and the ratio $B : C$ is equal to the ratio $Q : R$, then shall the ratio $A : C$ be equal to the ratio $P : R$.

Theorem 10D. If the ratio $A : B$ is equal to the ratio $C : D$, and the ratio $E : B$ is equal to the ratio $F : D$, then shall the ratio $A + E : B$ be equal to the ratio $C + F : D$.

Theorem 11D. If a series of parallel lines divides each of two transversals into segments, then any two segments on the one line shall be proportional to the corresponding two segments on the other.

Theorem 12D. If a straight line is drawn parallel to one side of a triangle it shall divide the other sides (internally or externally) in the same ratio.

Theorem 13D. If two sides of a triangle are divided (internally or externally) in the same ratio, then the line joining the points of section shall be parallel to the third side.

Theorem 14D. The line which bisects the interior vertical angle of a triangle divides the base internally in the ratio of the sides ; and conversely.

Theorem 15D. The line which bisects the exterior vertical angle of a triangle divides the base externally in the ratio of the sides ; and conversely.

Theorem 16D. Given the base of a triangle and the ratio of the sides, the locus of the vertex is a circle ; or:—

The locus of a point which moves so that its distances from two fixed points are in a given ratio is a circle.

Theorem 17D. In equal circles the arcs are proportional to their angles.

SIMILAR FIGURES.

Theorem 18D. If two triangles are equiangular they are similar.

Theorem 19D. If in two triangles two sides of the one are proportional to two sides of the other, and if also the angles included between these proportional sides are equal, then the triangles shall be similar.

Theorem 20D. If the three sides of one triangle are respectively proportional to the three sides of another, the two triangles shall be similar.

Theorem 21D. If the hypotenuse and one side of one right-angled triangle are respectively proportional to the hypotenuse and one side of another right-angled triangle, then the two triangles shall be similar.

Theorem 22D. If two sides of one triangle are proportional to two sides of another, and if the angles opposite to one pair of corresponding sides are equal, then shall the angles opposite to the other pair of corresponding sides be either equal or supplementary.

Theorem 23D. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.

Theorem 24D. Two similar polygons can always be divided into similar triangles.

Theorem 25D. Two similar polygons can always be placed so as to have a centre of similitude.

Theorem 26D. If two parallel straight lines are cut by a series of transversals which all pass through one point, then the intercepts formed on the two parallel straight lines are proportional.

Theorem 27D. The locus of a point within a given angle whose distances from the arms of the angle are in a constant ratio is a straight line through the vertex of the angle.

PROPORTION THEOREMS ON AREAS.

Theorem 28D. Triangles of equal altitude are proportional to their bases; and triangles on equal bases are proportional to their altitudes.

Theorem 29D. If two triangles have one angle of the one equal to one angle of the other, their areas are in the ratio of the products of the sides containing the equal angles.

Theorem 30D. Similar triangles are proportional to the squares on corresponding sides.

Theorem 31D. Similar polygons are proportional to the squares on corresponding sides.

Theorem 32D. In a right-angled triangle, any rectilinear figure described on the hypotenuse is equal to the sum of the two similar rectilinear figures described on the sides containing the right angle.

Theorem 33D. If four lines are proportional the rectangle contained by the extremes is equal to the rectangle contained by the means; and conversely.

Theorem 34D. In equal circles the areas of sectors are proportional to their angles and also to their arcs.

LIST OF PROPOSITIONS.

RECTANGLE THEOREMS.

Theorem 35D. If a quadrilateral be inscribed in a circle, the sum of the rectangles contained by the two pairs of opposite sides is equal to the rectangle contained by its diagonals.

Theorem 36D. If from the vertical angle of a triangle a perpendicular is drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by this perpendicular and the diameter of the circle described about the triangle.

Theorem 37D. If the vertical angle of a triangle is bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base *plus* the square on the line which bisects the vertical angle.

Theorem 38D. If the vertical angle of a triangle is bisected externally by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base *minus* the square on the line which bisects the vertical angle externally.

PROBLEMS ON RATIO AND PROPORTION AND SIMILAR FIGURES.

Problem 1D. To divide a straight line proportionally to a given divided line.

Problem 2D. To find a fourth proportional to three given lines.

Problem 3D. To find the mean proportional to two given lines.

Problem 4D. On a given straight line as base to describe a polygon similar to a given polygon.

Problem 5D. To construct a polygon similar to one polygon and equal to another.

STRAIGHT LINES AND PLANES.

Theorem 1E. Only one plane can be drawn through three given points which are not collinear.

Theorem 2E. If two planes meet their intersection is a straight line.

Theorem 3E. If three planes intersect in pairs their three lines of intersection are coincident, concurrent, or parallel.

PARALLELS AND PARALLELEPIPEDS.

Theorem 4E. If two intersecting planes pass respectively through two parallel lines, their intersection is parallel to each of these lines.

Theorem 5E. If two straight lines are each parallel to a third they are parallel to each other.

Theorem 6E. If two intersecting lines in one plane are respectively parallel to two intersecting lines in another, then (i) the two planes are parallel, and (ii) the two pairs of lines contain equal angles.

Theorem 7E. If a plane intersects two parallel planes, the intersections are parallel lines.

Theorem 8E. In any parallelepiped (i) each face is a parallelogram, (ii) opposite faces are congruent, and (iii) any two diagonals bisect each other.

Theorem 9E. If two planes are parallel, any straight line in one plane is parallel to the other plane.

Theorem 10E. If two lines are parallel, then either of them is parallel to any plane through the other.

Theorem 11E. If a given straight line is parallel to a given plane, and if a plane through the given line intersects the given plane, then the line of intersection is parallel to the given line.

Theorem 12E. If a straight line is parallel to each of two intersecting planes, it is parallel to their line of intersection.

Theorem 13E. If two planes are each parallel to a third plane, they are parallel to one another.

Theorem 14E. If two straight lines are cut by three parallel planes, they are cut proportionally.

LINES PERPENDICULAR TO PLANES.

Theorem 15E. If a given straight line is perpendicular to two straight lines in a given plane which meet it, it shall be perpendicular to the given plane.

Theorem 16E. If a straight line is perpendicular to each of three concurrent straight lines at their point of intersection, these three straight lines shall lie in one plane.

Theorem 17E. Through a given point in a given plane it is always possible to draw one and only one straight line perpendicular to the given plane.

Theorem 18E. Planes to which the same straight line is perpendicular are parallel to one another.

Theorem 19E. If two planes are parallel any line which is perpendicular to one is also perpendicular to the other.

Theorem 20E. If two straight lines are parallel, and one is perpendicular to a plane, the other is perpendicular to the same plane.

Theorem 21E. If two lines are perpendicular to the same plane they are parallel to each other.

Theorem 22E. If a straight line is perpendicular to two intersecting lines which it does not meet, it is perpendicular to the plane containing them.

Problem 1E. To draw a straight line perpendicular to a given plane from a given point without it.

Problem 2E. Through two given straight lines which do not meet and are not parallel, to draw a straight line which is perpendicular to both.

DIHEDRAL, TRIHEDRAL, AND POLYHEDRAL ANGLES.

Theorem 23E. If a straight line is perpendicular to a given plane, then any plane passing through this straight line is also perpendicular to the given plane.

Theorem 24E. If two planes are perpendicular any line in one plane perpendicular to their intersection is perpendicular to the other plane.

Theorem 25E. If two intersecting planes are each perpendicular to a third, then the line of intersection of these two planes is perpendicular to the third.

Problem 3E. Through any given straight line to draw a plane perpendicular to a given plane.

Theorem 26E. If a straight line intersects a plane, then the acute angle which the straight line makes with its projection on the plane is less than the angle which it makes with any other straight line in the plane which meets it.

Theorem 27E. In any trihedral angle any two of the face angles are together greater than the third.

Theorem 28E. In any trihedral or convex polyhedral angle the sum of the face angles is less than four right angles.

POLYHEDRA.

Theorem 29E. There cannot be more than five different types of regular polyhedra.

Theorem 30E. If F , E , and C are the numbers of faces, edges, and corners respectively of any convex polyhedron, then $E + 2 = F + C$.

Theorem 31E. The two lines which join any two vertices of a tetrahedron to the centroids of the opposite faces divide one another in the ratio 3 : 1.

Theorem 32E. In a regular tetrahedron the perpendicular from any angular point to the opposite face meets that face at its centroid.

Theorem 33E. In any tetrahedron any two lines which join the middle points of opposite edges bisect one another.

Problem 4E. About any tetrahedron to circumscribe a parallelepiped such that opposite edges of the tetrahedron are diagonals of opposite faces of the parallelepiped.

Theorem 34E. If in a tetrahedron two pairs of opposite edges consist of perpendicular lines, then the third pair of opposite edges consists of perpendicular lines.

Theorem 35E. If each edge of a tetrahedron is perpendicular to the opposite edge then the tetrahedron is orthocentric.

Theorem 36E. Any plane section of a pyramid taken parallel to the base is similar to the base, and the area of such a section varies as the square of its distance from the vertex.

SOLIDS OF REVOLUTION : THE SPHERE, CYLINDER, AND CONE.

Theorem 37E. Any plane section of a sphere is a circle.

Theorem 38E. The plane drawn through a point on the surface of a sphere perpendicular to the radius at that point is a tangent plane.

Problem 5E. To find the locus of a point which is equidistant from three given points.

Problem 6E. To draw a sphere through four given points which do not lie in one plane.

Problem 7E. To find the locus of a point which is equidistant from two given intersecting planes.

Problem 8E. To inscribe a sphere in a given tetrahedron.

SURFACE AREAS AND VOLUMES OF SIMPLE SOLIDS.

Theorem 39E. If a section of a pyramid be made by a plane parallel to its base the volume of the pyramid thus cut off is to the volume of the whole pyramid as the ratio of the cubes of corresponding edges, and also as the ratio of the cubes of their altitudes.

HARMONIC SECTION.

Theorem 1F. If the points **P** and **Q** divide the line **AB** harmonically, then (1) the points **A** and **B** divide the line **PQ** harmonically, (2) the lengths **AQ**, **AB**, and **AP** are in harmonical progression, and the lengths **QB**, **QP**, and **QA** are in harmonical progression.

Theorem 2F. If a straight line **AB** is bisected at **O**, and if **P** and **Q** are harmonic conjugates with respect to **A** and **B**, then $OP \cdot OQ = OB^2$, and conversely if $OP \cdot OQ = OB^2$, then **P** and **Q** are harmonic conjugates with respect to **A** and **B**.

Theorem 3F. If $(O, APBQ)$ is a harmonic pencil, then the rays **OP**, **OQ** cut off equal segments on a line through **B** parallel to **OA**; and conversely.

Theorem 4F. Any transversal cuts a harmonic pencil in a harmonic range.

CENTRES OF SIMILITUDE.

Theorem 5F. If in two circles two parallel radii are drawn, then the line joining their outer extremities meets the line joining their centres in a centre of similitude.

Theorem 6F. If a variable line through a centre of similitude, S , of two circles meets the circles in P, Q, P', Q' , where P corresponds to P' and Q to Q' , then $SP \cdot SQ' = SP' \cdot SQ =$ a constant.

Theorem 7F. If a variable circle touches two fixed circles the line joining their points of contact passes through a centre of similitude.

Theorem 8F. If a line joining a fixed point S to a variable point P on a given circle is divided in a given ratio at P' , the locus of P' is a circle, and S is a centre of similitude of the two circles.

INVERSION.

Theorem 9F. The inverse of a circle which passes through the centre of inversion is a straight line perpendicular to the diameter through the centre of inversion, and conversely, the inverse of a straight line (not passing through the centre of inversion) is a circle through the centre of inversion such that its diameter through the centre of inversion is perpendicular to the given line.

Theorem 10F. The inverse of a circle which does not pass through the centre of inversion is another circle: also the centre of inversion is a centre of similitude.

Theorem 11F. If A and B are the inverse points of C and D with respect to O , then AB and CD are antiparallels with reference to the angle COD .

THE RADICAL AXIS.

Theorem 12F. The radical axis of two circles is a straight line perpendicular to the join of their centres.

Problem 1F. To construct the radical axis of two given circles.

Theorem 13F. The radical axes of three circles taken in pairs are concurrent.

Theorem 14F. The difference between the squares on the tangents from a given point to two given circles is double of the rectangle contained by the join of the centres and the distance of the given point from the radical axis.

THE THEOREMS OF CEVA AND MENELAUS.

Theorem 15F. (*Ceva's Theorem*). If three lines drawn from the vertices of a triangle are concurrent, then the product of the ratios of the segments into which these lines divide the opposite sides is equal to $+1$: and conversely.

Theorem 16F. (*Menelaus' Theorem*). If three points taken on the three sides of a triangle are collinear, then the product of the ratios of the segments into which they divide the sides is -1 : and conversely.

THE NINE POINTS CIRCLE.

Theorem 17F. In any triangle the following nine points are concyclic:—the middle points of the sides, the middle points of the lines joining the orthocentre to the vertices, and the feet of the perpendiculars from the orthocentre to the sides.

Theorem 18F. In any triangle the following four points are collinear:—the orthocentre, the centre of the nine-points circle, the intersection of medians, and the circumcentre.

Theorem 19F. In any triangle the radius of the nine-points circle is half the radius of the circum-circle; also the orthocentre and the intersection of medians are the centres of similitude of the circum-circle and the nine-points circle.

POLES AND POLARS.

Theorem 20F. If the polar of **P** passes through **Q**, then the polar of **Q** passes through **P**.

Theorem 21F. If a point lies outside a circle its polar is the chord of contact of tangents drawn from the given point to the circle.

Theorem 22F. If a variable chord passes through a fixed point, then the locus of the intersection of the tangents at the extremities of the chord is the polar of the point.

Theorem 23F. (*Salmon's Theorem*). The distances of any two points from the centre of a circle are in the same ratio as the two distances of each from the polar of the other.

Theorem 24F. Any chord of a circle drawn through a given point is cut harmonically by the point and its polar.

Theorem 25F. The mid points of the three diagonals of a complete quadrilateral are collinear.

THE TANGENCY PROBLEMS.

PPP. See Problem 6B.

LLL. See Problems 7B, 8B.

PPL. To draw a circle passing through two given points and touching a given straight line.

PLL. To draw a circle passing through a given point and touching two given straight lines.

PFC. To draw a circle passing through two given points and touching a given circle.

PCC. To draw a circle passing through a given point and touching two given circles.

PLC. To draw a circle passing through a given point and touching a given line and a given circle.

LLC. To draw a circle touching two given lines and a given circle.

CCC. To draw a circle touching three given circles.

LCC. To draw a circle touching a given line and two given circles.

ANSWERS.

Exercises I.

1. (i) $\frac{5}{8}$; (ii) $\frac{3}{4}$; (iii) $\frac{9}{10}$; (iv) This is not a ratio. 2. (i) 0.366; (ii) 0.692; (iii) This is not a ratio; (iv) 0.189. 3. (i) $\frac{4}{9}$; (ii) $\frac{2}{3}$; (iii) $\frac{27}{64}$; (iv) $\frac{1}{12}$. 4. (i) 1.25; (ii) 3.75; (iii) 0.231; (iv) 0.208. 5. (i) 8 in.; (ii) 4 sq. ft.; (iii) 2 sq. yds.; (iv) 40 sq. kilom. 6. 5:6; 2:3; 4:11.

Exercises V.

1. 1.17". 2. 14.93 cm. 3. 1.39". 4. 4.62 cm. 5. 2.65 cm. 6. (i) 25.9; (ii) 6.72; (iii) 2.65; (iv) .575. 7. 4.3 cm. 8. 1.86, 2.34, 10.47. 9. (i) 1.68; (ii) 5.14; (iii) 26.4. 10. 7.24 cm. 11. 9.98 cm. 12. 1.83". 13. 2.04".

Exercises VI.

1. The lines are concurrent. 2. (a) Six; (b) Nine. 3. Eight. 4. Four. 5. Six. 6. Seven. 7. Eight.

Exercises VII.

7. An infinite number; see Theorem 9E.

Exercises IX.

19. Eight.

Exercises X.

7. 1.63". 8. 54.7°, 70.5°. 9. 1.41". 10. 1.58", 4.74", 3.67"; 1.58", 4.74", 3.67".

11. (a) When the two opposite edges are perpendicular; (b) when in addition the distances of the plane of section from these two edges are proportional to the edges.

15. (i) Yes; (ii) No; (iii) No.

Exercises XI.

6. $1\cdot22''$, $\cdot41''$. 7. $2''$. 9. $2\cdot43''$.

Exercises XII.

1. (i) 600 sq. inches; (ii) 1116 sq. inches; (iii) $2[a^2 + b^2 + c^2 + 3(bc + ca + ab)]$; (iv) $4(bc + ca + ab) - 2(a^2 + b^2 + c^2)$.
 2. (i) 280 cubic inches; (ii) 1080 cubic inches; (iii) 60 cubic feet; (iv) 5400 cubic cm.; (v) $\frac{2}{3}$ cubic metre; (vi) $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - 2abc - a^3 - b^3 - c^3$.
 3. 600 sq. inches. 4. 100 sq. inches. 5. £3 6s.
 6. $6\sqrt{3}$ cubic feet. 7. $18\frac{3}{4}$ feet. 8. 433 cubic feet 22 cubic inches.
 9. $(128 + 16\sqrt{3})$ sq. inches. 10. 98 cubic inches.
 11. $35\sqrt{5}$ square inches. 12. $11\frac{5}{22}$ feet. 13. $310\frac{1}{2}$ tons nearly
 14. 1848000 cubic feet. 15. $150\frac{9}{7}$ square inches.
 16. $269\frac{1}{2}$ cubic feet. 17. $\frac{2}{3}\pi^2r^3$; $\pi r^2(\sqrt{4\pi^2 + 1} + 1)$.
 18. 88 cubic feet. 19. 201062000 sq. miles.
 20. 268083000000 cubic miles. 21. 50 : 9.
 22. $377\frac{1}{7}$ square inches.

Exercises XIII.

8. (a) One centre of similitude lies within both circles.
 (b) Both within the inner circle. (c) One is at the point of contact. (d) One is at infinity. (e) The two extremities of the diameter perpendicular to the straight line.
 24. The radical axis of the two given circles.

INDEX TO DEFINITIONS AND TERMS.

- ADDENDO, 12
 Alternando, 10
 Angular points of polyhedron, 98
 Antecedent, 2
 Antiparallels, 205
 Apollonius, Circle of, 25
 Axis of a cone, 168
 ,, cylinder, 167
 ,, prism, 145
 ,, section of a sphere, 166
 ,, radical, 209
- CENTRE of a sphere, 166
 ,, direct similitude, 200
 ,, similarity, 32
 ,, similitude, 32, 199
 ,, transverse similitude, 200
- Ceva's Theorem, 216
 Circle of Apollonius, 25
 Coaxial, 210
 Commensurable, 2, 3
 Complete plane, 87
 ,, quadrilateral, 220
 Componendo, 11
 Compounding of ratios, 4
 Cone, 168
 Conjugates, harmonic, 6, 195
 Consequent, 2
 Convex polyhedral angle, 132
 Co-planar lines, 88
 Corners of polyhedron, 98
 Corresponding terms, 5
 Cube, 99
 Cuboid, 99
 Cylinder, 167
- DIAGONAL of a parallelepiped, 99
 ,, polyhedron, 145
 Diameter of a sphere, 166
 Dihedral angle, 131, 132
 Dividendo, 11
 Dodecahedron, 98
 ,, regular, 147
 Duplicate ratio, 4
- EDGES of polyhedron, 98
 ,, solid angle, 132
 Equiangular, 31
 Euler's Theorem, 152
 Ex aequali, 15
 Extreme and mean ratio, 7
 Extremes, 5
- FACE angles of a solid angle, 132
 Faces of polyhedron, 98
 Fourth proportional, 5
 Frustum of a cone, 168
 ,, pyramid, 146
 ,, sphere, 167
- GAUCHE quadrilateral, 93
 Great circle, 166
- HARMONIC conjugates, 6
 ,, mean, 195
 ,, points, 228
 ,, range, 6
 Harmonical progression, 195
 Hexahedron, 98
 Homologous, 56
- ICOSAHDREDON, 98

Icosahedron, regular, 147
 Incommensurable, 2, 3
 Inverse, 204
 " points, 205
 " proportion, 5
 Inversion, 204
 " , centre of, 204
 " , constant of, 204
 " , origin of, 204
 " , radius of, 204
 Invertendo, 9

 MEANS, 5
 Medial section, 7
 Menelaus' Theorem, 216, 217

 NET of a regular polyhedron, 150
 Nine-points circle, 219
 Normal, 116

 OCTAHEDRON, 98
 " , regular, 147

 PARALLELEPIPED, 99
 Parts of a trihedral angle, 132
 Pencil, 195
 Pentahedron, 98
 Plane, 87
 " , complete, 87
 Platonic bodies, 99
 Polar, 223
 Pole, 223
 Poles, 166
 Polyhedral angle, 132
 Polyhedron, 98
 " , regular, 99
 Prism, 107, 145
 Prismoidal formula, 186
 Projection, 116
 " , conical, 199
 " , orthogonal, 199
 Proportion, 5
 " , continued, 6
 " , inverse, 5
 Proportional, 5
 " , fourth, 5
 " , mean, 6
 " , third, 6

Ptolemy's Theorem, 69
 Pyramid, 146

 RADICAL axis, 209
 " centre, 210
 Radian, 27
 Radius of a sphere, 166
 Range, 195
 " , harmonic, 6, 195
 Ray, 195
 Ratio, 1
 " , duplicate, 4
 " , extreme and mean, 7
 " of equality, 5
 " , greater inequality, 5
 " , less inequality, 5
 " , similitude, 31
 " , sub-duplicate, 4
 " , sub-triplicate, 4
 " , terms of, 2
 " , triplicate, 4
 Reciprocally proportional, 33
 Rectangular solid, 99
 Regular polyhedra, 99

 SALMON'S Theorem, 226
 Sector of a sphere, 167
 Segment of a sphere, 167
 Similar, 31
 " polyhedra, 145
 Similarity, centre of, 32
 Similarly situated, 33
 Similitude, centre of, 32, 199
 " , , , direct, 2 0
 " , , , transverse, 200
 " , , ratio of, 31
 Skew lines, 88, 90
 " quadrilateral, 93
 Small circle, 166
 Solid angle, 132
 Species, 32
 Sphere, 166
 Sub-duplicate ratio, 4
 Sub-triplicate " , 4

 TANGENCY problems, 230
 Tangent line of a sphere, 167
 " plane " , 167
 Terms of a ratio, 2

Tetrahedron, 98, 146
 ,, , orthocentric, 147
 ,, , regular, 146
 Third diagonal, 228
 Trihedral angle, 132
 Triplicate ratio, 4
 Truncated cone, 168
 ,, cylinder, 168
 ,, prism, 146
 ,, pyramid, 146

VERTEX of a cone, 168
 ,, , pencil, 195
 ,, , pyramid, 146
 ,, , solid angle, 132
 Vertices of complete quadri-
 lateral, 228
 ,, , polyhedron, 98
 ZONE, 167

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