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ELEMENTARY ALGEBRA FOR SCHOOLS

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ELEMENTARY ALGEBRA FOR SCHOOLS

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PART I

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PREFACE

MANY years spent in teaching Elementary Mathematics have convinced the authors of this book that the difficulties which some pupils find in acquiring a sound knowledge of Algebra are due, in the first place, to a failure to realise that, in the main, its operations are identical with those to which they are already accustomed in their Arithmetic ; and, secondly, to an impression that its treatment depends on arbitrary rules, concerned with signs and brackets, for which they see no adequate explanation

The early part of the book deals with the ordinary processes of Arithmetic applied to algebraical expressions consisting of a single term, and is restricted to those sections of Arithmetic which are likely to have come within the scope of the pupils' knowledge. It is hoped that these chapters will prove useful in establishing the principles that underlie their methods of calculation in both subjects With this object in view the formulation of "rules" has been avoided as far as possible, as it has been found that though these may provide an easy path over an immediate difficulty it is often at the expense of much mental confusion in subsequent work For the same reason the dangerous word "cancel" is never used.

Negative Numbers are postponed until Chapter VI, and up to this point the negative sign is used solely to denote subtraction.

Special stress has been laid upon those parts of the subject which usually cause errors—such as the manipulation of brackets, the treatment of fractions preceded by a negative sign, and the order of operations

Revision sets of examples on such matters are provided, to which pupils may be referred if it is found at a later stage that these points have not been properly grasped

Factors have been dealt with by the introduction of a skeleton framework which shows their relation to the formation of the product The method throws light on what is too often merely a mechanical process, and has proved especially illuminating in the solution of quadratic equations.

For convenience, the graphical work has been collected in a single chapter, but parts of it can be used at different stages if considered desirable The authors believe that when once the method of graphic representation has been properly grasped, practice in its applications is better obtained by occasional questions in weekly papers or in homework rather than by the working of numerous examples consecutively. Those teachers who agree with them may prefer to use the last sets of examples in Chapter VIII in this manner.

In the Miscellaneous Examples an attempt has been made to combine those of the usual straightforward types with others of a more interesting character involving reference to a figure or the construction of an algebraic formula The sets marked B are slightly more difficult than those marked A

Chapter XII provides material for the more able pupils and for those whose later reading is likely to be of a specialist nature.

A few historical notes have been introduced in places. For these the main authority is Cajori's *History of Elementary Mathematics*, but other sources have been consulted.

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R C F
H C B

CLIFTON, 1928

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"*Pupil* And I to your authoritie my witte doe subdue,
Whatsoever you say, I take it for true

Master Though I might of my scholler some credence require,
Yet though I shew reason, I do it not desire"

R RECORDE, *The Grounde of Artes*, A D 1540

Says Brahmagupta (*circa* A D 628) "These problems are proposed simply for pleasure : the wise man can invent a thousand others, or he can solve the problems of others from the rules given here As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more if he solves them"

(See CAJORI, pp 100, 184)

ELEMENTARY ALGEBRA

FOR SCHOOLS

PART I

CHAPTER I

Introduction

ALGEBRA deals with the use of symbols to represent numbers. It enables us to generalise and extend the knowledge we have gained from arithmetic, and it is found that its use saves time and trouble in the arrangement of our work, and helps us to avoid many unnecessary calculations.

Thus, in arithmetic, we learn that $2+3=5$ and also that $3+2=5$ Similarly whatever pair of numbers we use the total is the same, whatever the order in which we add them In algebra we use letters to stand for numbers and express this fact by saying $x+y=y+x$, which means that for any two numbers x and y, the first added to the second gives the same total as the second added to the first. We have here made a general statement which includes the result given above, viz $3+2=2+3$, as well as all others of the same kind, such as $5+6=6+5$, $3+4=4+3$, etc., and we shall see later (Chap VI) how this simple generalisation is used in extending our power of dealing with numbers.

As an instance of the use of algebra to save time, consider the following examples. If we are asked to add the first 12 numbers from 1 to 12 we can find by writing them all down that the total is 78. So also if we add the 18 numbers from 8 to 25 we find the total is 297. The work is easy, but if we had a number of such sums to work out the calculations would be tedious By the use of algebra we can discover what is known as a *formula* which enables us to deal with these two examples and all similar examples much more quickly.

It is $s = \dfrac{n(a+l)}{2}$, which means that if the first number is called a and the last l, then if a is added to l and the total multiplied by n (the number of the figures we are adding together), then this result divided by 2 gives us s, the sum required.

In the first case $a=1$, $l=12$, $n=12$, and the answer (s) is $\dfrac{(12+1)12}{2} = 78$; in the second case s is $\dfrac{(8+25)\times 18}{2} = 297$.

Example.—Show that the sum of the 20 numbers from 21 to 40 is 610.

Substitution: Evaluation of Formulæ

As in arithmetic, we use $+$ (plus) to show Addition, $-$ (minus) to show Subtraction, $\times$ for Multiplication, and $\div$ for Division, but in algebra there is one important difference. When we write 23 in arithmetic we mean *twenty added to* 3, but when we write two letters side by side, such as xy, we mean $x \times y$; and if we write $2ab$ we mean $2 \times a \times b$.

Sometimes a dot on the line is used to show multiplication; thus $3 \times x$, $3x$, $3 . x$ all mean that x and 3 are multiplied together. When a letter is multiplied by a figure the figure is written *before* the letter, thus: $3x$, *not* $x3$

In arithmetic we write 2^2 for 2×2, so we write x^2 (x squared), meaning $x \times x$, and x^3 (x cubed) for $x \times x \times x$. x^4 is read "x to the 4th." So also $\sqrt{4}$ means the square root of 4 and $\sqrt{a}$ means the square root of the number represented by a.

Illustrations

1. The sum of 2 and 3 is written $2+3$, similarly the sum of x and y is $x+y$.

2. The difference between 5 and 2 is written $5-2$, so also the difference between x and y is $x-y$, if x is greater than y.

3. The product of 3 and 5 is 3×5, and the product of a and b is $a \times b$ or ab.

4. The quotient when 6 is divided by 2 is $\frac{6}{2}$, and the result of dividing p by q is $\dfrac{p}{q}$.

5. When $a=2$, the value of $3a$ is $3 \times 2=6$.

6. When $a=3$, the value of a^3 is $3 \times 3 \times 3=27$, and the value of $2a^3$ is $2 \times 3 \times 3 \times 3=54$. *N B.*—$2a^3$ means a^3 multiplied by 2, it does not mean that $2a$ is to be cubed; this would be written $(2a)^3=2a \times 2a \times 2a=8a^3$.

7. When $x=2$ and $y=3$, x^2+3xy equals $2 \times 2+3 \times 2 \times 3 =4+18=22$.

8. Just as $2(3+5)=2 \times 3+2 \times 5=6+10$, so also $2(x+y)$ means $2x+2y$.

9. When $a=9$, $\sqrt{a}$ means the square root of 9, that is $\sqrt{9}=3$; and $2\sqrt{a}$ equals $2 \times 3=6$. Another value of $\sqrt{9}$ will be explained in Chapter XI.

Exercise I (a)

When $x=2$, $y=3$, write down the values of

1. $x+y$, $y-x$, xy, $2xy$.

2. $2x+y$, $2x+3y$, $3x+2y$, $3x-2y$.

3. $3(x+y)$, $2(y-x)$, $x(x+y)$, $5(x+2y)$.

4. $\dfrac{x}{y}$, $\dfrac{2x}{y}$, $\dfrac{x}{2y}$, $\dfrac{x+2y}{y}$.

5. x^2, $3x^2$, $2y^2$, x^2+y^2.

6. $2x^2y$, $3y+2x^2$, $x+2y^2$, xy^2.

7. x^3, $4x^3$, $(2x)^3$, x^3y, xy^3, $2x^2y^3$.

When $x=4$, $y=9$, evaluate

8. $\sqrt{x}$, $\sqrt{y}$, $\sqrt{x}+\sqrt{y}$, $2\sqrt{x}$.

9. $5\sqrt{x}+2\sqrt{y}$, $x\sqrt{y}$, $y\sqrt{x}$.

When $x=3$ and $y=2$, find the difference between

10. $3x^2$ and $(3x)^2$

11. $2y^2$ and $(2y)^2$.

12. $3x^2y$ and $3xy^2$.

13. $5x+y$ and $5xy$

14. $3(x+y)$ and $3x+y$.
15. $5(2x+3y)$ and $10x+3y$.
16. $(x+y)^2$ and x^2+y^2.
17. $(x-y)^2$ and x^2-y^2.
18. x^2+x and x^3.
19. $(3xy)^2$ and $3x^2y^2$.
20. $5x^2y^2$ and $(5xy)^2$.
21. $2(x+y)^2$ and $(2x+2y)^2$
22. $3(x-y)^2$ and $(3x-3y)^2$.
23. $3(x-y)^2$ and $3x^2-3y^2$
24. $5x^2+y$ and $5x^2y$.

When $a=4$, $b=9$, $c=3$, $d=1$, write down the values of

25. $2a+b+c$
26. $3a^2c+2b$
27. b^2+d
28. $c\sqrt{a}+\sqrt{b}$
29. $\sqrt{ab}$.
30. $3\sqrt{a}+2\sqrt{b}$.
31. $\frac{3a}{c}+b$.
32. $\frac{2b}{c}+c^2$.
33. $\frac{d}{\sqrt{b}}+\frac{1}{a}$.
34. $3(2a+b+c)$.
35. $6a+3b+3c$.
36. $4(b+c-d)$
37. $4b-4c-4d$.

When $p=2$, $q=1$, $r=0$, $s=5$, find the values of

38. $3p+2q+5r$.
39. $pq+qr+rs$.
40. $p^2+q^2+r^2$.
41. $p(q+2r+s)$
42. $p^2q+q^2r+r^2s$.
43. $(2p+q)(r+2s)$.
44. $(p+q)r+(q+r)s$.
45. $3p+2q(3r+2s)$.

Using the letters x, y, and z, make general statements to represent the following facts .

46. $3\times2=2\times3$; $4\times6=6\times4$; $5\times7=7\times5$.
47. $5(3+6)=15+30$; $4(2+5)=8+20$; $6(2+1)=12+6$
48. $2\times2=2^2$, $3\times3=3^2$, $4\times4=4^2$.
49. $2+2=2(2)$; $3+3=2(3)$; $4+4=2(4)$
50. $3^2-2^2=(3+2)(3-2)$, $4^2-3^2=(4+3)(4-3)$; $5^2-3^2=(5+3)(5-3)$.

Exercise I (b)

1. Use the formula $s=t\left(\frac{u+v}{2}\right)$ to find s when $u=10$, $v=20$, $t=4$.

2. Given $C=2\pi r$, find C when $\pi=\frac{22}{7}$, $r=14$.

3. If $s=\frac{n(n+1)(2n+1)}{6}$, find s when $n=12$.

4. From the formula $v^2=u^2+2fs$, find v when $u=6$, $f=2$, $s=7$.

5. Given the formula $V=\frac{4}{3}\pi r^3$, find V when $r=3$, $\pi=\frac{22}{7}$.

6. The simple interest on £P for n years at r per cent. is £$\frac{P.n.r}{100}$ Find the interest on £120 for 3 years at 4 per cent.

7. Use the formula $(a^2-b^2)=(a+b)(a-b)$ to find the difference between the square of 19 and the square of 17.

8. If two telegraph posts are l ft apart the wire halfway between them will dip d ft., given by the formula $d=\frac{l^2w}{8s}$ where the weight of a foot of wire is w lb. and the pull in the wire is s lb. Find the dip when $l=100$, $w=$ 07, and $s=250$.

9. An embankment h ft. high, l yd long, x ft wide at the top, and y ft wide at the bottom contains $\frac{(x+y)lh}{18}$ cubic yards of earth. Find the volume of earth contained in an embankment for which $h=15$, $x=12$, $y=34$, $l=30$.

10. The R A C rating formula for the horse-power of a motor engine is H.P $=\frac{D^2N}{2\ 5}$ where D is the diameter of the cylinders in inches and N is the number of cylinders What will be the horse-power of an engine which has 6 cylinders of diameter 3 2 in ?

CHAPTER II

Simple Rules

Terms.—Expressions which are to be added or subtracted are called **terms.**

In the expression $2x+3y$, $2x$ and $3y$ are called terms.

The number which multiplies a letter is called its **coefficient.** Thus 3 in the expression $3x$ is the coefficient of x The coefficient may be another letter; thus a is the coefficient of x in the expression ax If the coefficient is 1 it is usually omitted; thus x means $1x$.

Addition.—If we add 3 horses to 2 horses we have 5 horses, so also $3x+2x=5x$ and $5x-2x=3x$.

If we add 3 horses to 2 cows we get neither 5 horses nor 5 cows, but simply 3 horses + 2 cows.

So if we add $3x$ to $2y$ the result is $3x+2y$.

Like terms such as $3x$ and $2x$, $3a^2$ and $2a^2$ may be combined by addition or subtraction and the results written thus . $3x+2x=5x$, $3a^2+2a^2=5a^2$, $3x-2x=x$, $3a^2-2a^2=a^2$

Unlike terms such as $3x$ and $2y$ when added are written $3x+2y$; when subtracted, $3x-2y$ So also a^2 added to a is a^2+a.

Note.—a^2+a does *not* equal a^3; for a^3 means $a \times a \times a$, i.e. $(a \times a) \times a$ or $a^2 \times a$

If $a=3$, $a^2+a=9+3=12$, whereas $a^3=3\times 3\times 3=27$.

Exercise II (a)

Add

1. ___ and $2x$	**2.** $3y$ and $6y$.	**3.** $2a$ and $3a$
4. $2ab$ and $5ab$.	**5.** $2x$ and $3y$.	**6.** $3a$ and $4b$.
7. $2a^2$ and $3a^2$.	**8.** x^2 and $2x$.	**9.** $3a^3$ and a^2.
10. $6xy$ and $2xy$.	**11.** $2a$ and $3b$.	**12.** $2a^2$ and a.

Subtract

13. $2x$ from $5x$. **14.** x from $3x$. **15.** $2x^2$ from $3x^2$.

16. $2ab$ from $5ab$ **17.** x from y. **18.** $2x$ from $3y$

19. a^2 from a. **20.** a from a^2. **21.** x^2 from x^3

22. x^3 from $3x^3$. **23.** xy from $3xy$. **24.** $2a$ from $3b$.

Simplify

25. $5y+2y$ **26.** $3z+7z$. **27.** $x+5x$.

28. p^2+2p^2 **29.** $3q^2+2q^2$ **30.** $xy+3xy$

31. $x+y+2x$. **32.** $3x+2y+x-y$. **33.** $5p+6q+2p+4q$

34. x^2+x+3x^2. **35.** $5x^2-y^2+x^2-2y^2$. **36.** $3y^2+y-y$.

If $x=4$, $y=3$, find the difference between

37. x^2+x and x^3 **38.** $3x+y$ and $3xy$ **39.** $2y^2-y$ and y^2.

40. x^2-x and $2x-x$ **41.** x^3-x^2 and x **42.** $x+y^2$ and xy^2.

Simplify

43. $\frac{1}{4}+\frac{1}{3}$. **44.** $\dfrac{a}{4}+\dfrac{a}{3}$ **45.** $\frac{1}{2}-\frac{1}{3}$ **46.** $\dfrac{b}{2}-\dfrac{b}{3}$.

47. $\dfrac{a^2}{2}+\dfrac{a^2}{3}$. **48.** $\dfrac{ab}{4}-\dfrac{ab}{5}$. **49.** $\dfrac{a}{3}+\dfrac{a}{2}-\dfrac{a}{6}$.

50. $\dfrac{a^2}{4}+\dfrac{a^2}{3}-\dfrac{a^2}{12}$ **51.** $\dfrac{b^2}{3}+\dfrac{b^2}{4}-\dfrac{5b^2}{12}$. **52.** $1{\cdot}2x+2{\cdot}8x$.

53. $1{\cdot}6x+3{\cdot}2x-4x$. **54.** $2{\cdot}6x^2-3\ 8x^2+2{\cdot}2x^2$.

55. If the sum of two numbers is x and one of them is y, what is the other ?

56. The difference between two numbers is x; if the smaller is y, find the greater ; if the greater is y, find the smaller.

57. The sum of two expressions is x^2+5x. If one is $3x$, find the other.

58. Write down expressions containing x which represent three consecutive numbers. Add them and show that the total is always a multiple of 3.

59. The difference between two expressions is $2n$. If the smaller is $n+1$, find the larger. If the larger is $3n+2$, find the smaller.

60. The sum of three expressions is n^2+3n+5; if two of them are n^2+1 and $2n+3$, find the third

Multiplication.—If we simplify $3\times2\times4$ the answer is 24. This is the same as multiplying 2×4 by 3. It does *not* mean $3\times2\times3\times4$.

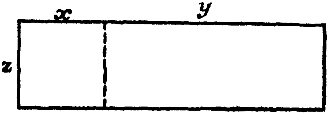
So also $3xy$ is the same as $3\times x\times y$, it does *not* mean $3x\times3y$.

If, however, there is a bracket containing two or more *terms*, each term is multiplied by any factor before the bracket. thus $2(3+5)$ means 2×8, which is the same as $2\times3+2\times5=6+10=16$.

So also $2(x+y)$ is the same as $2x+2y$.

N.B.—$2(xy)$ means $2xy$ as there is only *one* term, xy, in the bracket.

If we draw a rectangle with one side z, and the other side divided into any two parts x and y, the area of the whole rectangle is $z(x+y)$ But this area is the same as the sum of two areas, one being xz, and the other zy



$\therefore\ z(x+y)=zx+zy$

A term such as x^3 is said to be of the 3rd **degree**, x^5 is a term of the 5th degree x^2, x^3, x^4, etc, are **powers** of x The figure which indicates the power is called an **index.**

In arithmetic the numbers 2^2, 2^3, 2^4 . . . are called **powers** of 2. so in algebra x^3 is the third power of x, it is also said to be a term of the 3rd degree. x^5 is a term of the 5th degree.

A term such as $3x^2y^4$ is said to be of the 6th degree (2 degrees in x and 4 in y).

Note.—(1) $a^2\times a^3$ means $(a\times a)\times(a\times a\times a)=a^5$.

(2) $2x^2\times y$ means $2\times(x\times x)\times y=2x^2y$;

$2x\times xy$ means $(2\times x)\times(xy)=2\times x\times x\times y=2x^2y$;

$2\times x^2y$ means $2\times(x\times x\times y)=2x^2y$;

so that $2x^2\times y$, $2x\times xy$, $2\times x^2y$ all equal $2x^2y$.

(3) In arithmetic $3\times2\times5=5\times2\times3=2\times5\times3$; so also in algebra $3a^2b\times2b^2a^3=3\times2\times b\times b^2\times a^2\times a^3=6b^3a^5$ or $6a^5b^3$.

Exercise II (b)

Multiply

1. $3x$ by y
2. 3 by xy
3. a^2 by b
4. ab by a.
5. x by x^2y
6. x^2 by xy
7. $2x^2$ by $3y$.
8. $2x^2y$ by 3
9. a^2b by c.
10. ab by ac
11. abc by a
12. p^2 by q
13. ab by 3.
14. xy by xz.
15. $(a+b)$ by 3.
16. $(2a+b)$ by 5.
17. $2ab$ by 5
18. $(2a+b)$ by a.
19. $2ab$ by a.
20. $(x+2y)$ by xy
21. $(x-y)$ by x^2.
22. xy by x^2.

Write expressions without brackets equivalent to the following :

23. $a(xy)$.
24. $a(x+y)$
25. $3(x^2y)$
26. $3(x^2+y)$.
27. $3(x^2-y)$.
28. $a(bc)$.
29. $a(b+c)$.
30. $a(bc-b)$

Simplify

31. $x^3 \times x$
32. $y^2 \times y^3$
33. $y^8 \times y^4$
34. $2x^3 \times 3x^3$.
35. $4y^3 \times 3y^2$
36. $x^6 \times x^4$
37. $2x^3y \times x^2y^2$
38. $x^a \times x^b$
39. $x^a \times x^2$.

Write without brackets .

40. $a(a^2+a)$.
41. $x^2(x+y)$.
42. $x^2(x^3y)$.
43. $x^2(x^3+y)$
44. $x^2y^2(x^3y^3)$
45. $x^2y^2(x^3+y^3)$
46. $xy(x-y)$.
47. $2x(3x+3y)$
48. $a^2x^2(a^2-x^2)$.
49. $4xy(2x+3y)$.
50. $3x^2(2x^2y^2)$.

51. Multiply $2xy$ by $2xy$: write down a square root of $4x^2y^2$.

52. Multiply a^3b^2 by a^3b^2. What is a square root of a^6b^4 ?

53. Square $4x^2y^2$. What is a square root of $25x^6y^4$?

54. If $x=2$, $y=3$, what is the value of $5x^3y$ and of $25x^6y^2$? What is the square of $5x^3y$? State a square root of $25x^6y^2$.

55. Cube x^2y. What is the cube root of $8x^3y^6$?

56. Write without a root sign the values of $\sqrt{x^2y^4}$, $\sqrt[3]{x^3y^9}$, $\sqrt{x^6y^8}$, $\sqrt{9x^4y^2}$.

57. Square $\frac{a}{2}$. Draw a line and call its length a. On it draw a line of length $\frac{a}{2}$ and on it describe a square. Compare the sizes of the two squares.

58. How many squares of side $\frac{a}{3}$ will equal a square of side a?

59. Write (using a bracket) the squares of $(2a+b)$, $(a-b)$, $(a+b)$, $(3a^2-b)$.

60. Write (with a bracket) the square roots of $(a+b)^2$, $(2a-b)^2$, $(a^2-b^2)^4$, $(3a^2-b)^6$.

61. Write (with a bracket) the squares of $(2a+b)$, $(3a-b)$, $4(2a+b)$, and the square roots of $4(2a+b)^2$, $9(2x-3y)^2$, $16(x-2y^2)^4$.

62. Write down the cube roots of $27(x+y)^6$, $8(x-2y)^9$.

Factors.—To factorise a number in arithmetic we find those numbers which multiplied together will make the given number, thus the factors of 35 are 5 and 7

Similarly the factors of the expression a^2b are ab and a or a^2 and b.

Since $x^2+2x=x(x+2)$ the factors of x^2+2x are x and $(x+2)$.

Note.—The factors of $2a^2b+ab$ are written $ab(2a+1)$. Although it would be true to say they are also $b(2a^2+a)$, it is usual to put the factor of highest degree possible outside the bracket.

Exercise II (c)

Write down the factors of

1. $ab+b$.	**2.** $5x+10$.	**3.** $x+2xy$.
4. a^2+a.	**5.** $5a^2+5a$.	**6.** $6b^2+2b$.
7. $6x^2-3x$	**8.** $2y-y^2$.	**9.** $4xy-4x^2$.
10. $2x^2-3x$	**11.** $3x^2y+2xy^2$.	**12.** $6x^2y-12y$.
13. $3x^3+2x^2$.	**14.** $5a^2b+10b^2$.	**15.** $4a^2+2ab+6a$
16. $2x^2+xy+x$.	**17.** $3x^3+6x^2y+9x^2$.	**18.** $x^4+2x^3-x^2$.
19. x^2y-xy^2.	**20.** $2ax+4a^2x^2+2ax^3$.	

Highest Common Factor.—Any number which will divide exactly into 12 is a factor of 12; thus all the numbers 12, 6, 4, 3, 2, 1 are factors of 12.

So also the expression $3a^2b$ can be written $3\times(a^2b)$ or $3a\times(ab)$ or $3a^2\times b$ or $3ab\times a$; all these pairs of expressions are factors of $3a^2b$.

The *highest* common factor (H.C.F.) of $12a^2b$ and $9a^3b^2c$ can be seen by inspection to be $3a^2b$, for 3 is the H.C.F. of 12 and 9, and there is no power of a higher than a^2 or of b higher than b, which is a *common* factor of both expressions, and c is not a factor of both.

The given expressions can be written thus:—$(3a^2b)(4)$ and $(3a^2b)(3abc)$.

Lowest Common Multiple.—The lowest common multiple (L.C.M.) of $6x^2y$, $3xy^2$, and $2x^3y^3z$ is seen by inspection to be $6x^3y^3z$.

The least multiple of the coefficients 6, 3, and 2 is 6.

The lowest multiple of x^2, x, and x^3 is x^3, and of y, y^2, and y^3 is y^3.

The L.C.M. must contain a factor z in order that it may be a multiple of $2x^3y^3z$

$\therefore$ the L.C.M. is $6x^3y^3z$.

This L.C.M. contains $6x^2y$, xy^2z times.
" " $3xy^2$, $2x^2yz$ times.
" " $2x^3y^3z$, 3 times.

Exercise II (d)

Find the H.C.F. and L.C.M. of the following, and write each term as two factors, one of which is the H.C.F.

1. $6a$, $4a$, $8a$.
2. $12x^2$, $4x$.
3. $2a^2b$, ab^2.
4. $3ab$, $6a^2bc$.
5. ab, abc, bc.
6. pqr, qrt.
7. $3ab$, $2a^2b$, $4ab^2$.
8. $6a^2b$, $12ab^3c$, $4ab^2c^2$.
9. $4xy$, $6x^2y$, $3xy^2$.
10. $2a^3b$, $3a^2b^2$, $2ab^3$.
11. p^2q, $4pq$, $6pq^2$.
12. x^2yz, xy^2z, xyz^2.
13. $6a^2b^3$, $8b^4c$, $12b^2c^2$.
14. $6x^2y^2$, $16x^4$, $12x^3y$.
15. $3xy^2$, $4x^2y^2$, $2x^3y^3$.
16. $2x^3y^2$, $3x^2y^3$, $4x^2yz$.

Note.—The lowest common multiple in algebra means the expression of *lowest degree* in the letters involved which is exactly divisible by each of the given expressions. It may be noticed that if the values of the letters are known, the result is not necessarily the same as the least common multiple in arithmetic. Thus, algebraically the L.C.M. of a^2b, $4ab$, $6ab^2$ is $12a^2b^2$, but if $a=2$, $b=1$, these expressions become 4, 8, 12, whose arithmetical L.C.M. is 24, whereas $12a^2b^2=48$.

Simplification of Fractions. Division

In arithmetic a fraction such as $\frac{9}{12}$ can be simplified by dividing numerator and denominator by the common factor 3.

Thus
$$\frac{9}{12}=\frac{\cancel{3}\times 3}{\cancel{3}\times 4}=\frac{3}{4}.$$

So in algebra fractions may be simplified by dividing the numerator and denominator by their H C.F

Thus
$$\frac{a^2bc}{ab^2}=\frac{(\cancel{ab})(ac)}{(\cancel{ab})(b)}=\frac{ac}{b}$$

$$\frac{10a+10b}{5}=\frac{\overset{2}{\cancel{10}}(a+b)}{\cancel{5}}=2(a+b).$$

Division.—The method given above also enables us to perform simple division thus .

$$3x^2y-xy=\frac{3\times x\times x\times y}{x\times y}$$
$$=3x.$$
$$a^5\div a^2=\frac{a\times a\times a\times a\times a}{a\times a}$$
$$=a\times a\times a=a^3.$$

If there is more than one term in the numerator, remember that *all* the terms have to be divided by the denominator, thus :

$$\frac{6+4}{2}=3+2=5 \quad \text{and} \quad \frac{ax+bx^2}{x}=\frac{ax}{x}+\frac{bx^2}{x}=a+bx.$$

Exercise II (e)

Simplify

1. $\frac{2x^2y}{x}$.
2. $\frac{3x^2a}{ax}$.
3. $\frac{6ay^2x}{2ay}$
4. $\frac{9a^4}{3a^2}$.
5. $\frac{8x^5}{2x^3}$.
6. $\frac{4p^2q}{6pq^2}$.
7. $\frac{25x^{10}}{5x^5}$
8. $\frac{18x^9}{3x^3}$.
9. $\frac{6x}{12x^2}$.
10. $\frac{2y^2}{6y^6}$.
11. $\frac{a^2bc}{ab^2}$.
12. $\frac{3p^4}{9p^{12}q}$.
13. $\frac{12a+12b}{4}$
14. $\frac{6a^2+6a}{12a}$
15. $\frac{5x^2-5x}{5x}$.
16. $\frac{a^2-a}{a}$.
17. $\frac{2b^2-b^3}{b^2}$.
18. $\frac{3a^3+6a^2}{3a^2}$.
19. $\frac{x^3-x^2}{2x^2}$.
20. $\frac{p^2q-pq^2}{pq}$.
21. $\frac{a^2x+ax^2}{ax}$
22. $\frac{3b^2x-2bx^2}{bx}$.
23. $\frac{4x^2+2}{4}$.
24. $\frac{6x^2y^2-4xy^2}{2xy}$
25. $\frac{6xy^2-3y^3}{3y^2}$.

26. One factor of $2x^2+x$ is x, find the other One factor of $6x^2-12xy$ is $6x$, find the other One factor of $4x^2+6xy+2x^2y$ is $2x$, find the other

27. The product of two expressions is $4a^2-6ab$ If one of them is $2a$, find the other.

Divide

28. $3x$ by x.
29. $2x^2$ by x
30. $3x^2y$ by y
31. $3xy^2$ by y^2.
32. a^2b^3 by ab
33. $10a^2$ by $2a$
34. x^2yz^3 by xz^2.
35. $12xy^2$ by $3xy$.
36. $15ab^3$ by $3b^2$
37. $4x^5$ by x^3.
38. $25x^7$ by $5x^2$.
39. $9a^4$ by $3a$.
40. a^2b^2c by abc.
41. x^a by x.
42. x^n by x^2.
43. $2a^5+4a^4+6a^3$ by $2a$
44. $3a^3b^2+6a^2b^3$ by $3a^2b$.
45. $10x^5y^3+15x^2y^6$ by $5x^2y^3$
46. $12p^3q+15pq^3+9p^2qr$ by $3q$.
47. $8x^6+10x^4+4x^2$ by $2x^2$.

48. $\dfrac{24(x+3)}{2}$. **49.** $\dfrac{a(x+y)}{a}$ **50.** $\dfrac{bc(b+c)}{b}$

51. $\dfrac{b^2c+b^3c^2}{bc}$. **52.** $\dfrac{10x^2y+5x}{5x}$.

Fractions (Multiplication and Division)

Fractions are multiplied as in arithmetic thus ;

$$\frac{2}{3}\times\frac{5}{7}=\frac{2\times5}{3\times7}=\frac{10}{21}.$$

So also
$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}=\frac{ac}{bd}.$$

Frequently the result may be simplified by dividing numerator and denominator by the same factor thus

$$\frac{3}{6}\times\frac{4}{12}=\frac{\cancel{3}}{\cancel{3}\times2}\times\frac{\cancel{4}}{\cancel{4}\times3}=\frac{1}{6}$$

So also
$$\frac{a^2x}{by}\times\frac{b^2y^2}{ax^3}=\frac{\overset{a}{\cancel{a^2}}\times\cancel{x}\times\overset{b}{\cancel{b^2}}\times\overset{y}{\cancel{y^2}}}{\cancel{b}\times\cancel{y}\times\cancel{a}\times\underset{x^2}{\cancel{x^3}}}=\frac{aby}{x^2}.$$

Here we have divided numerator and denominator by the common factors a, b, y, and x.

It is often necessary to perform the reverse process and multiply numerator and denominator of a fraction by a common factor , this operation, of course, does not alter the value of the fraction.

Thus
$$\frac{x}{a}=\frac{x}{a}\times\frac{b}{b}=\frac{xb}{ab}.$$

When there are several fractions it is sometimes required to express them so that all the denominators are the same as the L.C.M. of the given denominators.

Thus the L.C.M. of the denominators of the fractions $\frac{x}{a}$, $\frac{y}{b}$, $\frac{xy^2}{a^2b}$ is a^2b.

Now
$$\frac{x}{a}=\frac{x}{a}\times\frac{ab}{ab}=\frac{xab}{a^2b},$$
$$\frac{y}{b}=\frac{y}{b}\times\frac{a^2}{a^2}=\frac{ya^2}{a^2b}.$$

$\therefore$ the given fractions are equivalent to
$$\frac{xab}{a^2b},\quad\frac{ya^2}{a^2b},\quad\frac{xy^2}{a^2b}.$$

Division by Fractions.—Division by a fraction is performed as in arithmetic, thus: $6\div\frac{1}{2}=6\times\frac{2}{1}=12$.

So also
$$a\div\frac{1}{b}=a\times\frac{b}{1}=ab,$$
$$\frac{a}{2}\div\frac{3}{b}=\frac{a}{2}\times\frac{b}{3}=\frac{ab}{6}.$$

Exercise II (f)

Simplify the following:

1. $\frac{p}{q}\times\frac{x}{y}$. 2. $a\times\frac{b}{c}$. 3. $a^2\times\frac{b}{a}$. 4. $\frac{a^2b}{c}\times\frac{c^2}{b}$.

5. $\frac{4xy}{x^2}\times\frac{x}{y^2}$. 6. $a-a$. 7. $a\times\frac{1}{a}$. 8. $\frac{1}{a}\div\frac{1}{x}$.

9. $\frac{1}{a^2x}\div\frac{1}{xa}$. 10. $px-\frac{x^2}{p^2}$. 11. $\frac{6a^2x}{y}-\frac{3a}{y^2}$. 12. $\frac{x^2y}{z}\times\frac{z^2}{x}$.

13. $\frac{3xy^3}{a^2}\times\frac{a^3b}{y^2}$. 14. $\frac{12x^2y}{3z}\div\frac{4xy}{z^2}$. 15. $\frac{x}{2}\times\frac{x}{4}$.

16. $\frac{x}{2}\div\frac{x}{4}$. 17. $\frac{2}{x}\times\frac{4}{x}$. 18. $\frac{2}{x}\div\frac{4}{x}$.

19. $\frac{x^2}{y^2}-\frac{x}{y}$. 20. $\frac{x^2}{y^2}\times\frac{x}{y}$. 21. $\frac{2x^2}{y}\div\frac{4x}{y^2}$.

22. $\frac{15x^2y}{4ab}\div\frac{3x}{2b^2}$. 23. $5(a+b)\times\frac{1}{10(a+b)^2}$.

24. $6(x+y)-2(x+y)$. 25. $4(x+y)^2\times\frac{1}{12(x+y)}$.

Write the following fractions with the expression in brackets as their denominator.

26. $\frac{1}{a}, \frac{1}{b}, \frac{1}{ab}$ (ab)

27. $\frac{1}{2a}, \frac{1}{3b}, \frac{1}{6ab}$ $(6ab)$

28. $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{x^2y^2}$ (x^2y^2)

29. $\frac{1}{2x}, \frac{1}{3y}, \frac{1}{4xy}$ $(12xy)$

30. $\frac{1}{x^2y}, \frac{1}{xy^2}, \frac{2}{x^2y^2}$ (x^2y^2).

31. $\frac{1}{x^3}, \frac{1}{y^3}, \frac{1}{x^3y^3}$ (x^3y^3)

In each of the following, find the L.C.M of the denominators and then express each fraction with this L C.M as the denominator.

32. $\frac{x}{2}, \frac{x}{3}$.

33. $\frac{x}{4}, \frac{x}{6}$

34. $\frac{x}{8}, \frac{x}{12}, \frac{x}{6}$

35. $\frac{1}{x}, \frac{1}{2x}$.

36. $\frac{3}{x}, \frac{5}{3x}$.

37. $\frac{1}{2x}, \frac{1}{6x}$.

38. $\frac{1}{2x^2}, \frac{1}{3x^2}$

39. $\frac{1}{x}, \frac{1}{y}$

40. $\frac{1}{xy}, \frac{1}{x^2y}$.

41. $\frac{1}{x^2}, \frac{1}{x^2y}$

42. $\frac{3}{a}, \frac{2}{b}$

43. $1, \frac{1}{x}$.

44. $1, \frac{1}{x^2}$

45. $a, \frac{1}{a}$.

46. $\frac{1}{x^3y^2}, \frac{3}{x^2y^3}$.

47. $\frac{2}{a^2b^4}, \frac{3}{ab^3}, \frac{1}{a^3b^3}$

48. $\frac{1}{a^2bc}, \frac{1}{ab^2c}, \frac{1}{abc^2}$.

49. $\frac{1}{x^2y}, \frac{1}{xy^2}, \frac{2}{x^3y}$.

50. $1, \frac{1}{ab}, \frac{1}{abc}$.

Fractions (Addition and Subtraction)

Fractions are added or subtracted as in arithmetic by expressing each fraction so that its denominator is the L.C.M. of the denominators of the given fractions.

Thus $$\tfrac{1}{2}+\tfrac{1}{3}=\tfrac{3}{6}+\tfrac{2}{6}=\tfrac{5}{6}.$$

So also $$\frac{1}{a}+\frac{1}{b}=\frac{b}{ab}+\frac{a}{ab}=\frac{b+a}{ab},$$

since $$\frac{1}{a}=\frac{1}{a}\times\frac{b}{b}=\frac{b}{ab}, \quad\text{and}\quad \frac{1}{b}=\frac{1}{b}\times\frac{a}{a}=\frac{a}{ab}.$$

Exercise II (g)

Simplify

1. $\frac{x}{2}+\frac{x}{3}$.
2. $\frac{x}{4}-\frac{x}{6}$.
3. $\frac{x^2}{2}+\frac{x^2}{5}$.
4. $\frac{ab}{2}-\frac{ab}{3}$.
5. $\frac{p^3}{5}-\frac{p^3}{8}$.
6. $\frac{x}{6}+\frac{x}{2}-\frac{x}{4}$
7. $\frac{a^2}{3}+\frac{a^2}{5}-\frac{2a^2}{15}$
8. $\frac{2x}{3}-\frac{3x}{5}$.
9. $x-\frac{x}{6}$
10. $x^2-\frac{x^2}{4}+\frac{x^2}{3}$
11. $\frac{1}{x}+\frac{1}{2x}$
12. $\frac{1}{2x}+\frac{1}{3x}$
13. $\frac{1}{3a}+\frac{1}{5a}$.
14. $\frac{1}{2x^2}+\frac{1}{x}$.
15. $\frac{1}{x}+\frac{1}{y}$.
16. $\frac{1}{3x}+\frac{1}{6y}$.
17. $\frac{2}{a}+\frac{3}{b}$.
18. $1+\frac{x}{y}$.
19. $\frac{1}{2a}-\frac{1}{3b}$.
20. $x-\frac{1}{y}$
21. $\frac{1}{2x^2}+\frac{1}{3x}$.
22. $\frac{1}{2xy}+\frac{1}{4x^2y}$.
23. $1+\frac{1}{6a^2}$.
24. $\frac{1}{x}+\frac{1}{3x^2}$.
25. $\frac{1}{a}-\frac{1}{b}$.
26. $\frac{a}{b}+\frac{b}{c}$.
27. $\frac{a}{x}-\frac{b}{y}$.
28. $\frac{x}{a^2}-\frac{y}{b^2}$.
29. $\frac{x}{y}-1$.
30. $1+\frac{a^2}{b^2}$.
31. $1-\frac{x^2}{a^2}$.

Write each of the following as two separate fractions :

32. $\frac{a+b}{x}$
33. $\frac{x-x^2}{x^3}$.
34. $\frac{a^2-ba}{a^2}$.
35. $\frac{3b+2a}{2ab}$
36. $\frac{x^2-xy}{y^2}$.
37. $\frac{x^2y-xy^2}{xy}$.
38. $\frac{a^2b+ab^2}{a^3b^3}$.
39. $\frac{2x-3y}{6xy}$
40. $\frac{3a-4b}{12a^2b^2}$.
41. $\frac{x+y}{x}$.
42. $\frac{a^2-x^2}{x^2}$

CHAPTER III

Order of Operations: Brackets. Unitary Method

Order of Operations.—The symbols $+$, $-$, $\times$, $\div$, $\surd$, etc , are called symbols of operation, as they indicate that an operation (addition, subtraction, etc.) is required to be performed.

An expression often contains more than one of these symbols The parts of an expression which are separated by the signs $+$ and $-$ are called *terms*, and it is always understood that unless there is a bracket the terms are to be simplified first before the additions or subtractions are performed.

For instance $a+bc$ contains two terms and means that the product bc has first to be evaluated and the result added to a

If terms are bracketed together they are to be considered as a single expression , thus $(a+b)c+d$ means that d is to be added to $(a+b)c$, whereas $a+b(c+d)$ means that $b(c+d)$ is to be added to a

It is very important to distinguish between such expressions as

(i) $a+b(c+d)$, (ii) $(a+b)c+d$; (iii) $(a+b)(c+d)$

Each is often loosely read as $a+b$ multiplied by $c+d$ but a different form of words should be found to suit each case

E.g. (i) $a+b(c+d)$ means that a is to be added to b times the sum of c and d.

(ii) $(a+b)c+d$ means that c times the sum of a and b is to be added to d.

(iii) $(a+b)(c+d)$ means that the sum of a and b is to be multiplied by the sum of c and d [1]

A fraction is sometimes written in the form 3/4 as in

[1] $(a+b)(c+d)$ should be read as $(a+b)$ **times** $(c+d)$, not as $(a+b)$ **into** $(c+d)$

arithmetic. In such cases only the term adjacent to the stroke forms the numerator of the fraction, thus:

$$a+b/c=a+\frac{b}{c},$$

whereas $$(a+b)/c=\frac{a+b}{c}.$$

Examples.

$$2+3\times 5=2+15=17.$$
$$(2+3)5=5\times 5=25.$$
$$6+4\div 2=6+\tfrac{4}{2}=8.$$
$$(6+4)\div 2=\tfrac{10}{2}=5.$$
$$6\div 2+4=\tfrac{6}{2}+4=7.$$
$$6\div(2+4)=\tfrac{6}{6}=1.$$

$6\div 2\times 4$ is ambiguous and should never be written; it is usually taken to mean $6\times\frac{1}{2}\times 4=12$, but it is preferable to insert a bracket and write it $(6\div 2)\times 4$.

Exercise III (a)

1. Evaluate

(i) $3+6\times 2$; (ii) $3+6-2$; (iii) $(3+6)2$; (iv) $3\times 6+2$, (v) $3\div(6+2)$; (vi) $3-6+2$; (vii) $27+14\times 3$; (viii) $(27+14)\times 3$.

2. Express in words the meaning of the following expressions, and evaluate them when $v=1$, $x=2$, $y=3$, $z=4$:

(i) $xy+z$; (ii) $x+yz$; (iii) $(x+y)z$; (iv) $x+y(v+z)$, (v) $(x+y)v+z$; (vi) $x+yv+z$; (vii) $(x+y)(v+z)$.

3. Evaluate when $a=4$, $b=3$, $c=2$:

(i) $a+b\div c$; (ii) $(a+b)\div c$; (iii) $a\div b+c$; (iv) $a\div(b+c)$; (v) $a+b\div b+c$; (vi) $(a+b)\div b+c$; (vii) $a+b\div(b+c)$; (viii) $(a+b)-(b+c)$.

4. Simplify (i) $\frac{x}{2}+\frac{x}{3}\times\frac{1}{2}$; (ii) $\left(\frac{x}{2}+\frac{x}{3}\right)\frac{1}{2}$,

(iii) $\dfrac{x}{2}+\dfrac{x}{3}\times\dfrac{1}{2}+\dfrac{x}{4}$, (iv) $\left(\dfrac{x}{2}+\dfrac{x}{3}\right)\dfrac{1}{2}+\dfrac{x}{4}$

5. Simplify (i) $\dfrac{x}{2}\times x-\dfrac{x}{3}\times\dfrac{x}{4}$; (ii) $x\left(\dfrac{x}{2}-\dfrac{x}{3}\right)\dfrac{x}{4}$.

6. Express in words the meaning of the following, and evaluate them when $a=4$, $b=3$, $c=2$, $d=1$:

(i) $4a-bc+d$, (ii) $(a-b)c+d$;
(iii) $3a-b(c+d)$, (iv) $(a-b)(c-d)$.

7. Simplify (i) $x-2+x\times\frac{1}{3}$; (ii) $x\times\frac{1}{2}-x\div 3$

8. Reduce to their simplest form :

(i) $\dfrac{a^2x}{b^2y}\div\dfrac{a^2}{b^2}+\dfrac{bx}{2ay}\times\dfrac{a}{b}$, (ii) $\dfrac{3ax}{2y}\times\dfrac{1}{a}-\dfrac{2by}{3y}\times\dfrac{1}{b}$

9. Simplify

(i) $x\div b+x\div a$; (ii) $(x+b)-x-b-x$; (iii) $(xa+xb)\div(a+b)$

10. Simplify (i) $x^2b-x+b\times x$;

(ii) $\dfrac{1}{x}-a+\dfrac{1}{a}\div x$, (iii) $\left(\dfrac{1}{x}+a\right)-\left(\dfrac{1}{a}+x\right)$

11. Simplify (i) $y/x+y/q$; (ii) $(p+q)/p-q/p$.

12. (i) $1+y/z-(z+y)/z$; (ii) $(1+x)/y+1-x/y\times(1+x)/x$

Brackets.—The insertion and removal of brackets play a much greater part in algebra than they do in arithmetic Some types need especial care

a−(b+c).—In arithmetic $5-(2+1)$ means that we are to subtract $2+1$, i e 3, from 5, obtaining the result 2.

This could be written $5-2-1$, since both the 2 and the 1 have to be subtracted from 5.

This can also be seen from a diagram

If a man walks 5 miles from O to A and then returns first of all 2 miles to B and then another 1 mile to C, his distance from O is then 3 miles less than it was when he was at A. This distance

may therefore be represented either by $5-2-1$ or by $5-3$, *i.e.* $5-(2+1)$. Thus $5-2-1$ and $5-(2+1)$ represent the same number. Similarly $a-(b+c)$ has the same meaning as $a-b-c$, since both the b and the c are to be subtracted from a.

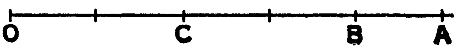
Note.—It will be seen that reversing the direction of motion corresponds to a change from + to −, for if he had continued to walk in the same direction his final distance from O would have been $5+2+1$.

a−(b−c).—The expression $5-(3-2)$ means "subtract $(3-2)$, *i.e.* 1, from 5" and the result is 4.

When we remove the bracket and write down $5-3$ we have subtracted 2 too much from 5, and the result is therefore 2 too small.

Hence $5-(3-2)$ is 2 *greater* than $5-3$, and therefore equals $5-3+2$.

If a man walks 5 miles from O to A, and then wishes to reduce his distance from O by $(3-2)$ miles, he will finally be $5-(3-2)$ miles from O, *i.e.* at B. Suppose that by mistake he returned 3 miles to C, he would then be $5-3$ miles from O and would have reduced his distance from O by 2 miles more than he intended. To arrive at B he must now return 2 miles and will then be $5-3+2$ miles from O, at B. Hence $5-(3-2)=5-3+2$. In a similar way $a-(b-c)$ means $a-b+c$. Here we wish to subtract $(b-c)$ from a. When we write $a-b$ we have subtracted b from a, *i.e.* c *more* than was required. Our result is therefore too *small* by c, and we must *add* c to $a-b$ to give the value of $a-(b-c)$. Hence $a-(b-c)=a-b+c$.



Note.—We have proved that $a-b-c$ is equivalent to subtracting $(b+c)$ from a, and that $a-b+c$ is equivalent to subtracting $(b-c)$ from a.

Example.—Write with brackets the result of subtracting the difference between 5 and 3 from the sum of 2 and 6.

Answer.—$(2+6)-(5-3)$. The result is $8-2=6$. which is the same as $2+6-5+3$.

Example 1.—Subtract the difference between a and b, $(a>b)$, from the sum of p and q.

N.B.—$a>b$ means "a is greater than b."
$a<b$ means "a is less than b."

Note that in both cases the sign widens towards the greater term.

Answer.—The result is $(p+q)-(a-b)$. When the brackets are removed this is equal to $p+q-a+b$.

Example 2.—Remove the brackets from the expression $2(x+y)-3(a-b)$. Just as $2(5+3)$ is the same as $2(5)+2(3)$ and $3(6-4)$ is the same as $3(6)-3(4)$, so $2(x+y)=2x+2y$ and $3(a-b)=3a-3b$.

Hence $$2(x+y)-3(a-b)=(2x+2y)-(3a-3b)$$
$$=2x+2y-3a+3b.$$

Note the difference between this and $2(x+y)-3(a+b)$. In this case we have to subtract 3 times $(a+b)$ from $2(x+y)$, *i.e.* we have to subtract both $3a$ and $3b$.

Hence $$2(x+y)-3(a+b)=(2x+2y)-(3a+3b).$$
$$=2x+2y-3a-3b.$$

Examples.—Insert brackets round the first pair of letters and also round the second pair in the following cases :

$$a+b-x-y\,; \qquad a+b-x+y.$$

In the expression $a+b-x-y$ we have to subtract both x and y from $a+b$. Hence $a+b-x-y=a+b-(x+y)$.

As a test we can put $a=5$, $b=2$, $x=3$, $y=1$, then
$$a+b-x-y=5+2-3-1=3$$
and $$(a+b)-(x+y)=7-4=3.$$

In the expression $a+b-x+y$ the result is to be y greater than if we subtract x only from $a+b$; thus we must altogether subtract y less than x, *i.e.* $(x-y)$ from $(a+b)$.

$$\therefore\ a+b-x+y=a+b-(x-y).$$

Testing: $$a+b-x+y=5+2-3+1=5.$$
$$(a+b)-(x-y)=(5+2)-(3-1)=7-2=5.$$

Exercise III (b)

In Questions 1–7 the answers should be written first with brackets and then with the brackets removed. The results should be tested by substituting the numbers given in both forms of the answer.

1. Find the remainder when the sum of a and b is subtracted from p. [Take $a=2$, $b=3$, $p=7$.]

2. Subtract the difference between a and b, ($a>b$), from c. [$a=3$, $b=2$, $c=5$.]

3. Subtract a from b and then subtract the result from $c+d$. [$c=3$, $d=2$, $a=1$, $b=4$]

4. What is the number of shillings left out of £x after paying two bills, one of y pounds and the other of x shillings? [$x=5$, $y=2$]

5 Subtract the sum of p and q from the difference between x and y ($x>y$). [$x=9$, $y=2$, $p=4$, $q=1$]

6. Subtract 3 times the sum of x and y from twice the difference between a and b ($a>b$). [$a=12$, $b=1$, $x=2$, $y=3$]

7. What must be added to the sum of $3x$ and $2y$ to make $4z$? [$z=5$, $x=2$, $y=1$.]

8. Write the following expressions with a bracket round the second and third terms. Evaluate them both before and after putting in the brackets

(i) $a+b-c$; (ii) $a-b-c$, (iii) $a-b+c$; (iv) $a+b+c$.

[$a=6$, $b=3$, $c=2$.]

9. Treat in a similar way

(i) $2a+3b-c$, (ii) $a-2b+c$, (iii) $2a-2b-3c$.

10. Rewrite the following with a bracket round the first and second terms, and a second bracket round the third and fourth terms.

(i) $a-b+c-d$; (ii) $a+b-c+d$; (iii) $a-b-c-d$;
(iv) $2a-b-3c+d$; (v) $a+2b-2c-3d$; (vi) $3a-2b+2c-d$;
(vii) $a-2b-2c+3d$.

[$a=10$, $b=3$, $c=2$, $d=1$.]

11. Remove the brackets from the following

(i) $(a-b)+(c+d)$; (ii) $a-b-(c+d)$, (iii) $(a-b)-(c-d)$, (iv) $(a-b)+(c-d)$.
[$a=6$, $b=1$, $c=3$, $d=2$.]

12. Remove the brackets from

(i) $3(a+b)+2(c+d)$, (ii) $3(a+b)+2(c-d)$, (iii) $3(a+b)-2(c+d)$, (iv) $3(a+b)-2(c-d)$; (v) $2(a-b)-(c+d)$, (vi) $2(a-2b)-3(c-2d)$.

13. Rewrite the following so that each contains a bracket with a numerical factor outside it:

(i) $3a+2b+2c$, (ii) $3a+2b-2c$, (iii) $3a-2b+2c$, (iv) $3a+2b+3c$, (v) $3a-2b+3c$; (vi) $3a-2b-3c$.

14. Treat in a similar manner

(i) $2p-5q+2r$, (ii) $3p+5q+6r$, (iii) $3p-5q-5r$, (iv) $6p-5q+4r$, (v) $5p-3q-3r$, (vi) $5p-2q-4r$.

* **Application of Brackets to Simplification of Fractions.**—When the numerators of fractions contain more than one term it is often necessary to insert brackets in the process of adding or subtracting.

Example.—Simplify $\dfrac{a+2b}{6}+\dfrac{3a-b}{4}$.

Expressing both fractions with 12 as the denominator (since 12 is the L C M of 6 and 4),

$$\begin{aligned}\frac{a+2b}{6}+\frac{3a-b}{4}&=\frac{2(a+2b)}{12}+\frac{3(3a-b)}{12}\\&=\frac{(2a+4b)+(9a-3b)}{12}\\&=\frac{2a+4b+9a-3b}{12}\\&=\frac{11a+b}{12}.\end{aligned}$$

* The rest of the chapter may be postponed, if desired, until after Chapter IV.

Example.—Simplify $\frac{x+y}{2}-\frac{x-y}{3}$.

Since 6 is the L C M of 2 and 3 we first rewrite both fractions with 6 as their denominators.

$$\frac{x+y}{2}-\frac{x-y}{3}=\frac{3(x+y)}{6}-\frac{2(x-y)}{6}$$
$$=\frac{3(x+y)-2(x-y)}{6}$$
$$=\frac{3x+3y-2x+2y}{6}$$
$$=\frac{x+5y}{6}.$$

The question tells us to subtract $(x-y)$ *thirds* from $(x+y)$ *halves*, *i e.* $2(x-y)$ *sixths* from $3(x+y)$ *sixths.*

The result is

$$\frac{3x+3y-(2x-2y)}{6}, \quad \text{NOT} \quad \frac{3x+3y-2x-2y}{6},$$

which would mean subtracting both $\frac{2x}{6}$ and $\frac{2y}{6}$, *i e.* $\frac{2x+2y}{6}$,

since $$\frac{3x+3y-2x-2y}{6}=\frac{(3x+3y)-(2x+2y)}{6}.$$

Example —Simplify $a-\frac{a-b}{2}$.

Here we are to subtract $(a-b)$ halves from $2a$ halves

$$a-\frac{a-b}{2}=\frac{2a}{2}-\frac{a-b}{2}$$
$$=\frac{2a-(a-b)}{2}$$
$$=\frac{2a-a+b}{2}$$
$$=\frac{a+b}{2}.$$

Note that the expression $\frac{2a-a-b}{2}$ would mean subtract both $\frac{a}{2}$ and $\frac{b}{2}$, *i.e.* $\frac{2a-(a+b)}{2}$.

Exercise III (c)

Simplify

1. $\dfrac{x-2}{3}+\dfrac{x-1}{4}$

2. $\dfrac{x-1}{4}-\dfrac{x-2}{3}.$

3. $\dfrac{a-b}{2}-\dfrac{a+b}{3}.$

4. $\dfrac{a-b}{3}-\dfrac{a+b}{4}.$

5. $x-\dfrac{x+y}{2}.$

6. $x+\dfrac{x-y}{3}$

7. $1-\dfrac{a-1}{2}.$

8. $1-\dfrac{x+2}{3}.$

9. $a-\dfrac{a+b}{4}.$

10. $\dfrac{p-q}{4}-\dfrac{p+q}{6}.$

11. $\dfrac{2p-q}{3}-\dfrac{p+2q}{4}.$

12. $1-\dfrac{a-b}{a}$

13. $a-\dfrac{a^2-x^2}{a}.$

14. $\dfrac{2a-b}{3a}-\dfrac{3a-b}{2a}.$

15. $1-\dfrac{a-2}{3}+\dfrac{a+3}{2}$

16. $2x-y-\dfrac{2x-3y}{3}.$

17. $5x+\dfrac{3x-y}{2}.$

18. $\dfrac{x+2y}{3a}-\dfrac{2x-y}{6a}$

19. $p-\dfrac{p-q}{2}+\dfrac{p+q}{3}.$

20. $\dfrac{x-a}{a}-\dfrac{x-b}{b}.$

In the following examples the expressions are to be simplified and the answers tested by substituting $x=4$, $y=2$, $z=3$:

21. $2x-\dfrac{3x-y}{3}.$

22. $3-\dfrac{x-y}{y}.$

23. $\dfrac{x+2y}{2x}-\dfrac{x+3y}{3x}$

24. $4x-3y-\dfrac{2x-y}{3}.$

25. $\dfrac{x}{2}-(y-1)$

26. $\dfrac{x+y}{z}-\dfrac{x-y}{2z}.$

27. $2-\dfrac{3-y}{2y}.$

28. $\dfrac{x}{2y}+\dfrac{2x-y}{3y}.$

29. $\dfrac{3y+2z}{4}-\dfrac{3y-x}{2}+\dfrac{x+y-z}{3}.$

30. $1-\dfrac{x-z}{y}.$

31. $\dfrac{2z-y}{u}-\dfrac{2z-x}{x}.$

32. $(x+y)/x-(y-x)/y.$

Revision Exercise III (d)

1. When we write down the simplified value of $6-3-1$, by how much is the result less than 6 ?

Have we subtracted $3+1$ or $3-1$ from 6 ?

Rewrite the expression with a bracket round the last two figures.

When we write the value of $6-3+1$, by how much is the result less than 6 ?

Have we subtracted $3+1$ or $3-1$ from 6 ?

Write down the expression with a bracket round the last two figures.

Does $a-b-c$ mean that $b-c$ or $b+c$ is subtracted from a ? Justify your answer

Does $a-b+c$ mean that $b-c$ or $b+c$ is subtracted from a ? Justify your answer

2. Explain why $8-2-3$ is the same as $8-(2+3)$.

Why is $a-b-c$ the same as $a-(b+c)$?

Explain why $8-3+2$ is the same as $8-(3-2)$ Show that $x-y+z$ is the same as $x-(y-z)$.

3. Write down expressions for the total wealth in shillings of a boy in the following cases, putting all terms except the first in a bracket. Evaluate each when $x=10$, $y=6$, $z=4$

(i) Having x shillings he is given 2 presents of y shillings each and 3 of z pence each.

(ii) Having £x he is given 2 presents of y shillings each and spends z shillings

(iii) Having £x he spends $3y$ shillings and loses $2z$ pence.

(iv) Having £x he loses y shillings and earns $6z$ pence.

4. Write down and simplify the expressions which show the following operations :

(i) Half the sum of a and b is to be subtracted from two-thirds of the difference between a and b. $(a>b)$

(ii) Half the difference between a and b is to be subtracted from 1.

(iii) One-third of the sum of $2a$ and b is to be subtracted from half the difference between $2a$ and b. $(2a>b.)$

(iv) One-third of the difference between a and b is to be subtracted from half the sum of a and b. $(a>b.)$

5. Simplify $x-\dfrac{y-z}{2}$ Test your answer by putting $x=5$, $y=3$, $z=1$.

6. When $p=5$, $q=6$, $r=4$, find the values of

(i) $p+2q-r$, (ii) $(2p+q)\div r$; (iii) $p\div(q+r)$;
(iv) $2p-q-r+2$; (v) $2p-q\div(r+2)$.

7. Express without brackets

(i) $3(a+b)-2(a-b)$, (ii) $5(a-c)-4(b-d)$; (iii) $a+b-2(c-d)$;
(iv) $a-(b-c)d$, (v) $a+b-\frac{1}{2}(b-c)$.

8. In the following, bracket together those terms which have a common numerical factor.

(i) $2x-y+2z$, (ii) $2x+3y-3z$, (iii) $3x-2y-2z$,
(iv) $3a-4b-4c+3d$, (v) $a+2b-c+2d$; (vi) $2a-3b+2c-3d$.

9. The sum of p and $2q$ is to be subtracted from the amount by which $5x$ exceeds $3y$ Express the result (i) in a form containing two brackets, (ii) without brackets.

10. Simplify

(i) $\dfrac{a-b}{3}-\dfrac{a+b}{5}$; (ii) $p-\dfrac{p-q}{3}+\dfrac{p+2q}{4}$;

(iii) $\dfrac{1}{2}\left(\dfrac{p}{2}+\dfrac{q}{3}\right)-\dfrac{1}{4}\left(p-\dfrac{q}{3}\right)$, (iv) $\left(\dfrac{a}{2}+\dfrac{b}{3}\right)\div\left(\dfrac{2}{a}+\dfrac{3}{b}\right)$.

Miscellaneous Examples I

1

1. What is the difference in meaning between $2x^2$ and $(2x)^2$? Evaluate each when $x=3$.

2. Simplify (i) $2x^2\times 3x^3$; (ii) $6x^3y^2\div 2xy$;

(iii) $\dfrac{6x^6}{2x^2}$; (iv) $\sqrt{16x^{16}}$.

3. Given $A=\pi r^2$, find A when $\pi=\dfrac{22}{7}$, $r=2$.

4. Write down the results of (i) adding $2x$ to $3y$, ; (ii) multiplying $2xy$ by $3x^2y^2$; (iii) subtracting $4xy$ from $6xy$.

5. Simplify (i) $\frac{x}{3}+\frac{x}{2}$; (ii) $\frac{x}{2}-\frac{x}{3}$; (iii) $\frac{x}{3}\times\frac{x}{2}$; (iv) $\frac{x}{3}\div\frac{x}{2}$.

6. By how much is $a-(b-c)$ greater than $a-(b+c)$ if $b=6$ and $c=2$?

7. Write the number 36 in terms of x and y if $x=3$ and $y=6$

2

1. What is the difference in meaning between $6x+y$ and $6xy$? Evaluate when $x=2$, $y=1$

2. Divide $(3xy)^2$ by $3x^2y^2$ and multiply $3x+2y$ by $2x$.

3. If $s=\frac{n}{2}(a+l)$, find s when $a=2$, $l=14$, $n=8$.

4. Find the L C M. and H C F. of $12a^2bc$, $8ab^2c$, $6abc^2$.

5. Simplify (i) $\frac{1}{a}+\frac{1}{b}$; (ii) $\frac{1}{a}\times\frac{1}{b}$; (iii) $\frac{1}{a}\div\frac{1}{b}$.

6. Write the following with a bracket round the last two terms · $a-2b-3c$, $a+2b-3c$, $a-2b+3c$

7. What is the distance in feet round a rectangle whose length is y yd and width x ft ?

3

1. Remove the brackets from

(i) $3x(2x+y)$, (ii) $3x-(2x+y)$; (iii) $3x(2xy)$.

2. Factorise (i) $6x^2-4xy$, (ii) a^2b-ab^2 ; (iii) a^3+2a^2-a.

3. Simplify (i) $\frac{4x-4y}{4}$; (ii) $\frac{3x^2-x}{x}$; (iii) $\frac{2a^2+4}{4}$.

4. Divide $10x^2y+5xy^2+5xy^3$ by $5xy$

5. Write the following so that each term has the same denominator : (i) $1-\frac{x}{y}$; (ii) $\frac{1}{x}-\frac{1}{y}$.

6. Simplify (i) $\frac{3x+2y}{4}+\frac{2x-y}{6}$; (ii) $\frac{3x+2y}{4}-\frac{2x-y}{6}$.

7. Write down an expression which shows that the difference between x and y is to be subtracted from the sum of a and b.

4

1. When $x=2$ and $y=1$, find the values of $(x-y)^2$ and x^2-y^2.

2. Simplify $p(p-2q)+q(2p-q)$.

3. Write down (i) with brackets, (ii) without brackets, an expression stating that $y+z$ is to be subtracted from x.

4. Given $A=2\pi r(r+h)$, evaluate A when $\pi=\frac{22}{7}$, $r=3$, $h=4$

5. Draw a square on a line of length a. Show from your diagram that this square is four times the square on $\frac{a}{2}$.

6. Simplify (i) $\frac{x^2}{y^2}+\frac{x}{y}$; (ii) $\frac{x^2}{y^2}\times\frac{x}{y}$; (iii) $\frac{x^2}{y^2}\div\frac{x}{y}$.

7. If $x=3$, $y=2$, $z=1$, write an expression in x, y, and z whose value is 321.

5

1. Given that $(a+b)^2=a^2+2ab+b^2$, find by using this formula the square of 23, *i.e* $(20+3)$. Use the same method to find the square of 107.

2. Multiply by $5a$ (i) $3a+b$; (ii) $3ab$.

3. What expression multiplied by $3ax$ will make $6a^2x^2+3ax^3$?

4. Simplify (i) $\frac{1}{y}+\frac{1}{x}$; (ii) $\frac{1}{y}\times\frac{1}{x}$; (iii) $\frac{1}{y}-\frac{1}{x}$.

5. Write each of the following as two separate fractions:

$$\text{(i) } \frac{a^2+ab}{b^2}; \qquad \text{(ii) } \frac{p^2-q^2}{pq}.$$

6. Write down (i) with brackets, (ii) without brackets, an expression which means that $p-q$ is subtracted from r.

7. The area of a circle is πr^2. Write down without simplification an expression for the area between two concentric circles of radii x and $x+d$.

6

1. What expression divided by $3xy$ will give $2x+3y$?

2. Simplify (i) $3x^2+2x-xy-x^2+x+xy$;
(ii) $3a^2-a+2-a^2-2a-1$.

3. Factorise (i) $2a^3-a^2-2a$; (ii) x^2+x; (iii) x^2y^2-xy.

4. Find the H.C.F. and L.C.M. of $4a^2b^2c$, $6a^3b^3c^2$, and $8abc^3$.

5. If $s=ut+\frac{1}{2}gt^2$, find s when $u=10$, $t=3$, $g=32$.

6. Simplify (i) $\frac{a+b}{3}-\frac{a-b}{4}$; (ii) $2x-\frac{x-y}{3}$.

7. Fill in the gaps in the following :

(i) $\frac{x}{a}=\frac{bx}{?}$; (ii) $5-3x+3y=5-3(\quad ? \quad)$;

(iii) $p-\frac{3q}{2}-\frac{3r}{2}=p-\frac{3}{2}(\quad ? \quad)$.

7

1. What expression subtracted from $2x-3y$ gives $x-2y$? Prove your answer.

2. Write down the square of $4x^2y^2$ and the square root of $6a^4b^4$.

3. Two of the angles in a triangle are x degrees and y degrees What is the third angle ?

4. Simplify (i) $\frac{9abc}{9abc}$; (ii) $\frac{1}{2ab}-\frac{1}{4a^2b^2}$; (iii) $9abc-9abc$

5. State which of the following are factors and which are multiples of $2a^2b$.

(i) $4a^2b^2$; (ii) $2ab$; (iii) $6ab^2$; (iv) a^2;
(v) $3a^2b$; (vi) $2a$; (vii) ab; (viii) $6a^3b^3$.

6. The relation between F, the reading on a Fahrenheit thermometer, and C, the same temperature on a Centigrade thermometer, is given by $F=\frac{9}{5}C+32$. Find F when (i) $C=100$; (ii) $C=0$.

7. Write in shillings the sum of £x and y shillings.

8

1. Write without brackets
(i) $3ab(2b+3a)$; (ii) $3ab(2ab)$; (iii) $\frac{1}{2}(4x+8)$.

2. Simplify (i) $a(a-b)-b(b-a)$; (ii) $\dfrac{2x-y}{2}-\dfrac{x-2y}{3}$.

3. Find the perimeter of a square whose area is $16x^2$.

4. A stone thrown upwards with a velocity of u ft. per sec will travel s ft in t secs. where $s=ut-16t^2$. Find s when $u=200$ and $t=2$.

5. Simplify (i) $\dfrac{1}{x}\div y+\dfrac{1}{y}\div x$; (ii) $\dfrac{1}{x}\times y+\dfrac{1}{y}\times x$

6. If p is greater than q, will $x-p$ be greater or less than $x-q$? By how much?

7. I rule $2n+1$ parallel lines on a page, n being any whole number. What is the number of the middle line?

9

1. Simplify (i) $4-\dfrac{6}{x^2y^2}$, (ii) $\dfrac{2x}{3y}-\dfrac{x}{6y}$

2. Find the L.C M. of the denominators of the following fractions, and write each fraction with this L.C M as the denominator:

$$\frac{ab}{c^2},\ \frac{ac}{b^2},\ \frac{bc}{a^2}.$$

3. A father whose age is x years is 31 years older than his son What is his son's age?
What will be the sum of their ages in 4 years' time?

4. Find a value of x which makes

(i) $x+10=27$; (ii) $3x=27$; (iii) $\dfrac{x}{3}=4$

5. If $\dfrac{1}{f}=\dfrac{1}{u}+\dfrac{1}{v}$, find f when $u=10$, $v=12$.

6. Simplify (i) $a-\dfrac{a}{2}$; (ii) $a\times\dfrac{a}{2}$, (iii) $a\div\dfrac{a}{2}$.

7. Given that $a^2-b^2=(a+b)(a-b)$, find the difference between the squares of 19 and 17 without squaring either of them.

10

1. By what expression must $3xy$ be (i) multiplied, (ii) divided, to give 1 as the answer ?

2. Write without brackets

(i) $2x(2x+x^2)$; (ii) $\frac{1}{4}(2x+4)$; (iii) $\frac{1}{4}(4xy)$.

3. Write down three consecutive numbers of which the middle one is $2n$ Which of the three will be odd and which even ?

4. Write down the square of $9x^2y^2$ and the square root of $\dfrac{16x^4y^2}{9z^2}$

5. Fill in the gaps in the following :

(i) $\dfrac{x}{y}=\dfrac{x^2}{?}$; (ii) $\dfrac{2}{x}=\dfrac{?}{x^2}$; (iii) $\dfrac{x+2}{a}=\dfrac{?}{3a}$.

6. How many minutes are there between x minutes past 10 and x minutes to 11 ?

7. The last term l of a series of n numbers is given by $l=a+(n-1)d$, where a is the first term and d is the difference between consecutive numbers of the series. Find the 20th term if $a=3$ and $d=2$.

11

1. How much is left over when $(x+2)y$ is subtracted from $x(y+2)$?

2. The perimeter of a quadrilateral is l in. One of the sides is a in., and two of the other sides are b in. each. What is the length of the fourth side ?

3. How many times is the sum of $\dfrac{a}{2}$ and $\dfrac{a}{3}$ contained in $10a$?

4. A photographer finds that he makes 3d. profit on each of the first twelve prints that he sells and 6d. profit on each additional copy over and above 12. What is his profit if he sells 20 copies? How much profit does he make if he sells x copies where $x > 12$?

5. Simplify $(a^2b^3c \times a^3b^2c^3) \div a^4b^4c^4$.

6. Show that $6(2a+b)+3(a-2b)=15a$ How much is left over if $3(a-2b)$ is subtracted from $15a$?

7. Write the number 553 in the form $100x+y$. A number of the form $100x+y$ is known to be divisible by 7 if $2x+y$ is divisible by 7. Test the number 553 in this way. Is the number 861 divisible by 7?

12

1. What is the difference between 4^x and x^4 (i) when $x=3$, and (ii) when $x=1$?

2. Simplify (i) $x + \dfrac{x}{3} - 2\left(x - \dfrac{x}{3}\right)$; (ii) $x + \dfrac{x^2}{3} \div 2\left(x - \dfrac{x}{3}\right)$.

3. The postage for a foreign letter is $2\frac{1}{2}$d for the first ounce and $1\frac{1}{2}$d for each additional ounce What will be the charge for a letter weighing x ounces? The answer may be given in halfpennies

4. Divide 15 shillings between A and B so that A gets 3 shillings more than B If x shillings is divided between A and B so that A gets 3 shillings more than B, how much does each get?

5. If a boy saves 1d the first week, 3d the second week, 5d. the third week, and so on, the total amount he will have saved in x weeks is x^2 pence. Show that this is true (i) when $x=4$, (ii) when $x=10$.

6. Find the result of adding the product of $3a$ and $(2a+b)$ to the product of $2b$ and $(a+3b)$.

7. Simplify as much as possible

$$\text{(i) } \frac{9}{21} + \frac{35}{49}; \qquad \text{(ii) } \frac{a^2}{ab} + \frac{bc}{b^2}.$$

CHAPTER IV

Symbolic Expressions

THE solution of problems in algebra requires considerable facility in the writing of statements which are similar to those required in arithmetic, but involve the use of letters instead of numbers.

The student will find little difficulty in the following exercises if he substitutes numbers for letters and then copies the operations which are required in the solution of the arithmetical example

Example.—The sum of two numbers is x. If one is y, find the other.

A similar example is : The sum of two numbers is 12. If one is 4, find the other.

The answer to the latter question is $12-4$, and to the former it is $x-y$

Oral Examples

1. The sum of two numbers is x, the smaller is 5. What is the greater ?

2. The difference of two numbers is y, the smaller is p. What is the greater ?

3. How many halfpence are there in x shillings ?

4. How many pence are required to pay for two articles, one of which costs £x and the other y shillings ?

5. A man who has a miles to go, travels for x hours at 3 miles per hour. How much of his journey remains ?

6. How many trees are there in an orchard if there are x rows with y trees in each and b rows with c trees in each ?

7. What length of paper is left if x pieces each y in long are cut from a strip z in. long ?

8. The product of two numbers is x; one is 4. What is the other?

9. How many jugs each containing x pints can be filled from a cask which contains y gallons? (Try a numerical example first.)

10. How much will be left in a cask which originally contained y gallons when z jugs each containing x pints have been filled from it?

11. If n is an even number, what are the next two even numbers?

12. If n is an odd number, what is the next even number? What is the preceding odd number?

Exercise IV (a)

1. How many shillings are left after deducting x florins from y pounds?

2. Find an expression for the total area A (sq ft.) of two rectangles, one x ft. by y ft. and the other p yd. by q ft.

3. What is the number of pence left after paying x shillings out of y pounds?

4. Find the number of books in a bookcase which has x shelves each containing y books and p shelves each containing q books.

5. Find the length of a rectangle whose area is 24 sq. yd. and width 4 yd.

What is the length of a rectangle whose area is A sq. yd. and width b yd.?

6. A boy is x years old. How old was he y years ago? If his brother is z years older, how old will this brother be in b years' time?

7. Write down an expression for the number of pence left out of x shillings after buying n buns at y pence each.

8. By what number must x be divided to obtain the quotients (i) 3; (ii) y?

9. The breadth of a rectangle is y ft. less than its length. What is its area if its length is x ft.?

10. If the sum of two numbers is 10 and one is x, what is the other ?

11. If the product of two numbers is 6 and one is x, find the other.

12. A boy is allowed pocket money at the rate of x pence for each weekday and y pence for each Sunday. How many shillings does he receive in a term of 10 weeks ?

13. If x is any whole number, write down (i) the next consecutive number ; (ii) the previous number.

14. Write down 3 consecutive whole numbers of which x is (i) the least ; (ii) the second ; (iii) the greatest.

15. What is an even number ?

Using the letter x to represent any whole number, write down an expression containing x which will always represent (i) an even number ; (ii) an odd number.

16. How many times is $\frac{1}{3}$ contained (i) in 5 ; (ii) in x ?

How many heaps each weighing $\frac{3}{8}$ lb. can be made out of x lb ?

17. A piece of paper is $\frac{1}{n}$ in. thick. How many such pieces would make a pile (i) 1 in high ; (ii) x in. high ?

18. How many articles worth $\frac{3}{4}$d. can be bought for x pence ?

19. If n books each y in. wide will just go on a shelf, how many books of width x in. will fill it ?

20. If the perimeter of a rectangle is 12 in. and one side is x in., what is the other ?

21. If the perimeter of a square is x ft , what is the area ?

22. The perimeter of a rectangle is 12 in and one side is x in. ; find the area.

23. If the area of a square is x^2 sq. in., what is the length of (i) each side , (ii) the perimeter ?

24. Write down an expression containing x which will (i) always, (ii) never, be a multiple of 3, whatever the value of x.

25. What is the total cost in pounds of x yd. at p shillings per yard and y ft. at q shillings per foot ?

26. A boy is y years old and his brother is x years older, what will be their total ages in 4 years' time ?

27. A sheet of paper has x lines, y in. apart, drawn across it parallel to the top ; the lines start 1 in. from the top and end 1 in. from the bottom. What is the length of the sheet ?

28. A chapter in a book starts at page x and ends on page y. On an average there are z letters per page. How many letters are there in the chapter ?

29. The carriage paid for excess luggage is x pence for every pound above y lb. How many shillings have to be paid for z cwt. of luggage ?

30. A rectangular lawn x ft by y ft. has a path z ft. wide all round it. What is the area of the path ?

Construction of Formulæ

If you walk at 3 miles an hour for 2 hours you travel 6 miles. If you cycle at 10 miles per hour for 4 hours you travel 40 miles The distance travelled in miles is always obtained by multiplying the number of miles per hour at which you travel by the number of hours you travel. This fact is expressed in letters by a formula, $d=st$; where d represents the distance travelled in miles, s the speed in miles per hour, and t the time taken in hours.

To construct a formula in algebra we consider first the corresponding arithmetical example, thus . Find a formula for the number of pence (p) in x shillings added to £y. The number of pence in (3 shillings + £2) is clearly $12\times 3+240\times 2$.
$\therefore$ for (x shillings + £y) the number of pence is

$$12\times x+240\times y=12x+240y.$$

$\therefore$ $p=12x+240y$ is the required formula.

Exercise IV (b)

Find formulæ for the following —

1. The number of books (b) in a bookcase which has s shelves with n books on each.

2. The number of books (b) in a bookcase having x shelves each holding y books, and p shelves each holding q books.

3. The number of miles (m) travelled in x hours at the rate of y miles per hour (*cf.* the number of miles travelled in 5 hours at 4 miles per hour).

4. The number of hours (h) it takes to travel m miles at y miles per hour (*cf.* the time taken to travel 20 miles at 4 miles per hour).

5. The number of pence (p) in x shillings.

6. The number of pence (p) in x shillings $+ y$ sixpences

7. The number of feet (f) in x yd.

8. The number of inches (z) in x yd. $+ y$ ft.

9. The number of shillings (s) in x pence.

10. The number of pounds (p) in $3x$ shillings $+ y$ pence.

11. The cost (c) in shillings of borrowing n books from a library at x pence each if y shillings is charged as an entrance fee.

12. The number of pence (p) you have left if originally you had x shillings and then buy 4 buns each costing y pence.

13. The number of square yards (A) in the area of a rectangle whose length is l yd and breadth b yd.

14. The length in feet (l) of a rectangle whose area is A sq. yd and breadth b ft

15. The time in minutes (t) required to walk x yd. at y ft per second

16. The profit £p made by buying n books for £x and selling them for y shillings each.

17. The area A sq ft. of the walls of a room whose length is l ft., breadth b ft , and height h ft.

18. The amount A paid in pounds by a customer who buys an article marked x shillings and is allowed y pence in the shilling as discount.

19. The speed v ft. per second of a train after t sec. if it starts with a speed of u ft. per second and increases its speed by a ft per second every second.

20. The amount of tax (T) in pounds paid on an income of £P if £x is untaxed and the rest is taxed at y shillings per pound.

Examples in Unitary Method.

A man walks x miles in p hours. How far will he walk in y hours at the same rate ?

A similar example in arithmetic would be : A man walks 10 miles in 3 hours How far will he walk in 7 hours ?

In three hours he walks 10 miles.

$\therefore$ in 1 hour ,, ,, $\frac{10}{3}$,,

$\therefore$,, 7 hours ,, ,, $\frac{10}{3}\times 7$ miles.

So in algebra

In p hours he walks x miles

$\therefore$ in 1 hour ,, ,, $\frac{x}{p}$,,

$\therefore$,, y hours ,, ,, $\frac{x}{p}\times y$ miles.

$=\frac{xy}{p}$ miles.

N.B.—Be careful to state your units. The letters represent numbers. Do *not* say a man walks a distance x.

If $x=3$, a distance x might mean 3 ft. or 3 yd. or 3 kilometres etc. When a number is used with a unit such as 3 miles the expression is called a "quantity": thus 3 is a number, 3 miles is a quantity.

Exercise IV (c)

1. If apples cost 2 shillings a dozen, find in pence the price of 7 apples. If apples cost x shillings a dozen, find in pence the price of y apples.

2. If a train travels at 30 miles per hour, find how long it will take to go 80 miles. If a train travels at x miles per hour, find how long it will take to go y miles.

3. If a yd. of cloth cost b shillings, how many pence should c yd. cost ?

4. If x men can do a job in y days, how many men would be required to do it in z days ?

5. The fare for a journey of x miles is y pence. What should be the fare for a journey of z miles at the same rate ?

6. If $\frac{5}{6}$ of a stick is x ft. long, what is the length of the complete stick ?

7. A bath holds x gallons. How long will it take to fill it at the rate of y gallons per minute ?

8. How long does it take to travel $(60-x)$ miles at y miles per hour ?

9. How far can you go in x hours at $(y-5)$ miles per hour ?

10. If p cu. ft. of water weigh q lb , what is the volume of 1 ton of water ?

11. A train travelling at x miles per hour travels for y hours. How far did it go ? How long would a train travelling 4 miles an hour faster take over the same journey ?

12. A journey of x miles takes y hours How long would a journey of $x+5$ miles take at the same rate ?

13. If m men have provisions for x days, how long should their provisions last if y additional men are added to the party ?

14. One clock gains x minutes a day and another loses y minutes a day. If they are both put to the right time, in how many days will they differ by z minutes ?

15. One boy runs at x yd. per minute and a slower one runs at y yd. per minute. If the slow one is given z yd. start, in how many minutes will he be caught ?

16. One rectangle's length is x ft. and breadth y ft. If another rectangle which is 2 ft. longer has the same area, what is its breadth ?

17. If a road rises x ft. in height for every y yd. of its length, how much will it rise in m miles ?

18. If x sheep can be put in a square pen whose side is b yd., how many can be put in a square pen whose side is c yd. ?

19. How many eggs at x pence per dozen can be bought for y shillings ?

20. A sheet of paper has x lines, y in. apart, drawn across it parallel to the top the lines start 1 in. from the top and end 1 in from the bottom. If the lines were z in. apart, how many would there be ?

CHAPTER V

Equations

In the following examples x represents a number; find its value by guessing:

1. $x+10=15$. 2. $x-5=6$. 3. $2x=12$.

4. $\frac{x}{4}=3$. 5. $2x+4=8$. 6. $3x+9=12$.

7. $2x-5=7$. 8. $3x-21=9$. 9. $\frac{x}{3}+\frac{x}{2}=\frac{5}{6}$

The exercises above are known as "equations."

An equation is a statement of an equality between two expressions which contain a letter representing an unknown number and is only true for a special value of that letter. The process of finding this value is called "solving the equation."

When a statement of equality is true for *all* values of the letters involved, it is called an **Identity**: *e g* $3(a+b)=3a+3b$.

The sign $\equiv$ is sometimes used between the two sides of an identity, thus, $2x+3(2y+z)\equiv 2(x+3y)+3z$

Many problems may be solved by writing the given information algebraically in the form of an equation in which the unknown number which is to be discovered is denoted by a letter, generally but not necessarily, x.

Example.—Find a number which when increased by 12 becomes 18. Let x be the required number, then the question tells us that $x+12=18$. Since $6+12=18$, it is clear that $x=6$.

When the question is more complicated, some simplifications may be necessary before the answer is obtained.

Example.—A certain number is doubled and the product increased by 6; if the result is 9 more than the original number, find that number. Let x be the required number; when doubled it is $2x$. Hence from the question we know that $2x+6=x+9$

The expressions on either side of the sign of equality are called the "*sides*" of the equation

Since the sides are equal to one another, the remainders will be equal if we subtract the same number from each side.

Subtract x from each side, then $x+6=9$

Subtract 6 from each side, then $x=3$, which is the required number.

Check —If $x=3$, $2x+6=6+6=12$,
and $x+9=3+9=12$.

If a letter is used instead of a number the method is unaltered; *e g.* solve $2x+a=x+b$.

Subtract x from each side. $\therefore x+a=b$.

Subtract a from each side $\therefore x=b-a$.

In simplifying an equation it is convenient to carry out operations which will eventually bring x alone on one side of the equation. The following axioms are constantly used for this purpose.

(i) *If equals are added to or subtracted from equals, the results are equal*

(ii) *If two equals are multiplied or divided by the same number, the results are equal.*

The example below is worked out in full to illustrate the process and to show a convenient method of arranging the work.

Example.—Solve $5(x-4)+11=3(x+1)+2$.

$\therefore 5x-20+11=3x+3+2$ [Clearing of brackets.]

$\therefore 5x-9=3x+5$ [Collecting like terms.]

$\therefore 5x=3x+14$ [Adding 9 to each side.]

$\therefore 2x=14$ [Subtracting $3x$ from each side.]

$\therefore x=7$ [Dividing each side by 2.]

Check.—When $x=7$,

L.H.S. (left-hand side) $=5(7-4)+11=5\times3+11=26$.

R.H.S. (right-hand side) $=3(7+1)+2=3\times8+2=26$.

Note.—(1) Do *not* check by substituting $x=7$ in the equations thus:

$$5(7-4)+11=3(7+1)+2.$$
$$\therefore\ 15+11=24+2.$$
$$\therefore\ 26=26.$$

Nobody asks us to prove $26=26$.

(2) Use the sign $\therefore$ (therefore) before each step of the solution.

Exercise V (a)

In the following equations the operations should in all cases be stated in words and the answers checked:

1. The sum of x and 7 is 10. Find x.

2. The sum of $4x$ and 3 is 15. Find x.

3. The excess of $3x$ over 7 is 5. Find x.

4. What number is next above x? If the sum of these two consecutive numbers is 29, find x

5. If the sum of two numbers is 18 and the smaller is x, what is the greater? If the greater is twice the smaller, find x.

6. If the difference between two numbers is 6 and the smaller is x, what is the greater? If the greater is three times the smaller, find x.

7. If the sum of three consecutive numbers is 24, find them.

8. Three times a number is 11 greater than 7 Find it.

9. If x is an odd number, what is the next greater odd number? If their sum is 28, find them

10. Twenty-seven is 3 less than 5 times x Find x.

11. If x is the middle of three consecutive numbers, what are the other two? If the sum of the three numbers is 27, what are they?

Solve the equations:

12. $3x=2x+5$. **13.** $3x+6=2x+8$.

14. $5x-10=2x+2$. **15.** $5x-7+3x=9$.

16. $4(5x-3)=68$. **17.** $3(2x-4)=8$.

18. $2(x+3)+5(2x-3)=39$.

19. $3(x-2)+4(x-1)=5(2x+7)-54$. **20.** $2x-3=5(2x-7)$.

21. $2+y=2(y+3)-3(y-2)$. **22.** $3p+24-5p=30-8p$.

23. $a-3=5-a.$ **24.** $3x+(x-7)=21.$

25. $3(x-8)=x.$ **26.** $3(x+2)=2(x-3)+17.$

27. $5x-4(x+13)=4(x-13)-7x.$

28. $20x-3(x+6)=3(2-x)+16.$

29. $3(t-2)-2(t+1)+5=0.$ **30.** $7-(t+2)=2(1-t)+3.$

31. $3-2(t-1)+3(t-4)+7t=0.$

32. $2(t-1)-8(3-t)=3-(2t-1).$

33. $4(1+y)-6(2-y)=2-5(1-y).$

34. $8m+2(3m-2)-5(2m+1)=0.$

35. $3(8x+3)-4x=68-3(9x+4)$

36. $3{\cdot}6x-9{\cdot}6=1{\cdot}2x-2{\cdot}4$ **37.** $2(3x-1{\cdot}8)+5(x-3{\cdot}8)=4{\cdot}9.$

38. $3(x+1{\cdot}25)=4{\cdot}25x-0{\cdot}125.$

39. To a certain number I add 3·4, and then multiply the sum by 2 If the result is 11, find the number.

40. If I subtract 2·8 from a certain number and then multiply the difference by 5, the result is the same as if I had subtracted 1·2 from the original number. Find the number.

Equations involving Fractions

Example.—Find a number such that, when one-half of it is added to one-third of it, the result is 25.

Let the number be x.

Then $$\frac{x}{2}+\frac{x}{3}=25.$$

* Multiply both sides *of the equation* by 6 (the L C M. of 2 and 3).

$$\therefore\ 3x+2x=150$$
$$\therefore\ 5x=150.$$
$$\therefore\ x=30.$$

Check.—L H.S. $=\frac{30}{2}+\frac{30}{3}=15+10$
$=25.$

* *Note.*—(1) This step gives a new equation equivalent to the original one and without fractions. The words in italics should always be included to prevent confusion with the process of addition, such as $\frac{x}{2}+\frac{x}{3}=\frac{3x}{6}+\frac{2x}{6}=\frac{5x}{6}$.

(2) Do not write $=3x+2x=150$; $3x+2x$ is *not* equal to $\frac{x}{2}+\frac{x}{3}$.

Example.—Solve $\dfrac{x+1}{3}=\dfrac{x+3}{5}$.

Multiply both sides *of the equation* by 15 (the L.C.M. of the denominators).

$$\therefore \frac{15(x+1)}{3}=\frac{15(x+3)}{5}.$$

$$\therefore 5(x+1)=3(x+3).$$
$$\therefore 5x+5=3x+9.$$
$$\therefore 2x=4.$$
$$\therefore x=2.$$

Check.—$\dfrac{x+1}{3}=\dfrac{2+1}{3}=1.$

$\dfrac{x+3}{5}=\dfrac{2+3}{5}=1.$

Example —Solve $\dfrac{3x-1}{4}-\dfrac{3x-2}{5}-\dfrac{9}{20}=0.$

Multiply both sides *of the equation* by 20.

$$\therefore 5(3x-1)-4(3x-2)-9=0.$$
$$\therefore 15x-5-12x+8-9=0.$$
$$\therefore 3x=6.$$
$$\therefore x=2.$$

Check.

$$\text{L.H.S.}=\frac{6-1}{4}-\frac{6-2}{5}-\frac{9}{20}$$
$$=\frac{25-16-9}{20}=0.$$

Exercise V (b)

Solve the equations 1–14.

1. $\frac{3}{5}x=\frac{2}{3}$. **2.** $\frac{4}{5}x=\frac{7}{15}$. **3.** $3\frac{1}{3}x=4$.

4. $3\frac{1}{3}x=\frac{5}{9}$. **5.** $16=\dfrac{4x}{5}$. **6.** $33=2\frac{1}{5}x$.

7. $\dfrac{4}{z}=3$. **8.** $\frac{3}{4}y+\frac{2}{3}=3\frac{1}{3}$. **9.** $\dfrac{x+2}{3}=1{\cdot}2$.

10. $\dfrac{x-2}{4}=1{\cdot}5$. **11.** $\frac{3}{4}t+\frac{1}{2}t=35$. **12.** $\dfrac{3t}{5}-\dfrac{t}{2}=1$.

13. $\dfrac{x+2}{3}=\dfrac{x+7}{5}$. **14.** $\dfrac{x}{3}+\dfrac{x-2}{4}=\dfrac{5}{4}$.

15. Simplify $\dfrac{m-1}{2}-\dfrac{m-2}{3}$. **16.** Solve $\dfrac{2(x-1)}{5}-\dfrac{x-3}{7}=1\frac{4}{7}$.

17. Simplify $1-\dfrac{4-x}{6}$. **18.** Solve $p-\dfrac{p-3}{2}=5\frac{1}{2}$.

19. One-third of a number added to one-quarter of it equals 14. Find the number

20. The difference between half a number and one-third of it equals 3. Find the number.

21. Half a number added to one-third of it is 28 less than twice the number. Find it.

22. One-fourth of a number subtracted from two-thirds of it is 6 more than one quarter of it. Find the number.

23. What value of x will make $\frac{3x+1}{2}-\frac{2x+7}{3}$ equal to $\frac{2x+1}{6}$?

24. What value of x will make $\frac{x+5}{6}-\frac{x+1}{9}+\frac{x-3}{4}$ equal to 0?

Solve

25. $\frac{x}{2}-\frac{2x}{3}+\frac{3x}{4}-\frac{7}{2}=0.$

26. $\frac{3y+2}{6}-\frac{1}{3}=\frac{y+1}{4}$)12

27. $\frac{2x-9}{4}-\frac{3}{5}(2x+\frac{1}{2})+\frac{1}{3}(4x+1)=0.$

28. $\frac{1}{4}(x+7)-\frac{1}{10}(x-3)=\frac{2(x-9)}{3}-\frac{(x-7)}{5}.$

29. $1-\frac{5-x}{4}=\frac{(x-1)}{6}-\frac{3x-8}{12}.$

30. $10-\frac{x+3}{4}=\frac{2(x-3)}{7}+\frac{3x-5}{3}.$

31. Simplify $\frac{x-2}{2}-\frac{x-3}{3}.$

32. Solve $\frac{x-2}{2}-\frac{x-3}{3}=2.$

33. Simplify $1-\frac{x-2}{3}+\frac{2x-7}{4}.$

34. What value of x will make $1-\frac{x-2}{3}+\frac{2x-7}{4}$ equal to $\frac{x+3}{12}$?

35. Simplify $\frac{7(x-3)}{4}-\frac{3(x-2)}{2}.$

36. Solve $\frac{7(x-3)}{4}-\frac{3(x-2)}{2}=8.$

Equations involving Quantity

Example.—I bought a number of books at 5 shillings each. If the price had been 4s. 6d. each I could have bought one more book by spending another 6d. How many books did I buy ?

Let x be the number of books bought, then the amount actually spent is $5x$ shillings.

N.B.—Do not say "Let x=the books." x stands for a number, not a quantity (see p. 40).

To buy $(x+1)$ books at 4s. 6d. each would cost $\frac{9}{2}(x+1)$ shillings The question tells us that this is 6d , or $\frac{1}{2}$ shilling more than $5x$ shillings.

$$\therefore \; \tfrac{9}{2}(x+1)=5x+\tfrac{1}{2}.$$

Multiply both sides *of the equation* by 2.

$$\therefore \; 9(x+1)=10x+1.$$
$$\therefore \; 9x+9=10x+1.$$
$$\therefore \; x=8.$$

The number of books bought was 8.

Test.—8 books at 5s. cost 40s.
9 books at 4s. 6d. cost 40s. 6d.

Example.—A man takes 7 hours over a journey of 60 miles. Part of the time he walks at 4 miles per hour and the rest he cycles at 12 miles per hour How far did he walk ?

It is often helpful to draw a diagram.

Let x=the number of miles he walked

Then $60-x$=number of miles he cycled.

	← 60 miles →	
Distance	x	$60-x$
Time	$\frac{x}{4}$ hours	$\frac{60-x}{12}$ hours
	← 7 hours →	

Time taken to walk x miles at 4 miles per hour $=\frac{x}{4}$ hours.

Time taken to cycle $(60-x)$ miles at 12 miles per hour

$$=\frac{60-x}{12}\text{ hours.}$$

But since the total time is 7 hours we have $\frac{x}{4}+\frac{60-x}{12}=7$.

Multiply both sides *of the equation* by 12.

$$\therefore\ 3x+(60-x)=84.$$
$$\therefore\ 2x=24$$
$$\therefore\ x=12.$$

Test.—12 miles at 4 miles per hour takes 3 hours.
48 miles at 12 miles per hour takes 4 hours. Total time, 7 hours.

Alternative Method—If we had taken x to be the number of hours of walking, then $7-x$=number of hours cycling.

Distance walked$=4x$ miles. Distance cycled$=(7-x)12$ miles.

$$\therefore\ 4x+12(7-x)=60.$$
$$\therefore\ -8x=-24.$$
$$\therefore\ 8x=24$$
$$\therefore\ x=3.$$

[Which of the two methods do you think is easier ?]

Exercise V (c)

1. I bought some cards at 3d. each and twice as many at 2d. each. If the total cost was 4s. 1d., how many of each did I buy ?

2. I sold one set of books at 2s. each, and another set which contained 2 volumes more than the first at 2s 6d. each. The total receipts were £2, 10s. How many books did I sell ?

3. A is four times as old as his son, and the difference between their ages is 36 years Find their ages

4. The length of a rectangular field is twice its width. If its perimeter is 360 yd , find its dimensions

5. A mixed school has 340 pupils. If there are 40 more boys than girls, find the number of each.

6. The angles of a triangle are in the ratio 1 : 2 : 3. Find them.

7. A school of 764 boys is divided into an upper and a lower school. The number of boys in the upper school is 34 less than 5 times the number in the lower school. How many boys are in each part ?

8. A shopkeeper with a stock of 120 bats sells some at 30s. each and the rest at 15s each. By disposing of his whole stock he obtains £150. How many bats did he sell at 15s each ?

9. A has twice as much money as B. B gives A £5 and then A has 5 times as much as B. What had they at first ?

10. To pay a bill of 8d. I give a tradesman £1. The change consists of an equal number of half-crowns, florins, and pence, together with half that number of sixpences How many coins did he give me ?

11. A straight line 25 in. long is divided into two parts, one of which is $\frac{2}{3}$ of the other Find the length of each part.

12. A cycles from X to Y at 10 miles per hour, but B goes by car at 25 miles per hour. If B takes 3 hours less than A on the journey, find the distance between X and Y

13. I travel to a neighbouring village by a bus whose speed is 15 miles per hour, and return by tram at 10 miles per hour The return journey takes me 12 minutes longer than the outward journey What is the distance to the village ?

14. The ages of two men differ by a fifth of that of the younger. In six years they will differ by a seventh of that of the elder. What are their present ages ?

15. A boy cycled from A to B at 10 miles per hour, remained there 2 hours and came back by train going 25 miles per hour by a route 15 miles shorter than the road If the total time taken was 7 hours, find the distance by road from A to B.

16. A man walked to a town at 4 miles per hour and returned by the same road in a car at 25 miles per hour. The double journey took 4 hours 21 minutes How far away was the town ?

17. A monarch who came to the throne at the age of 30 reigned for $\frac{5}{11}$ of his life. How long did he reign ?

18. A boy walked for $2\frac{1}{2}$ hours, starting at 3 miles per hour, but after some time he increased his speed to 4 miles per hour. If the times he walked at the two speeds had been interchanged the total distance gone would have been $\frac{1}{2}$ mile farther. How long did he walk at each speed ?

19. In a certain railway carriage the width of seat each person has when 4 sit on each side is $4\frac{1}{2}$ in. more than if 5 sit on each side. Find the width of the seat.

20. A motorist starts from a town X for another town Y, 135 miles distant from X, at the same time as another, travelling 6 miles per hour faster, leaves Y for X. If they meet after $2\frac{1}{2}$ hours, what is the speed of each car ?

21. A boy's cricket average after his first x completed innings is 17. In his next five innings he makes only 40 runs, and his average consequently drops to 12 What is the value of x ?

22. 20 c.c. of a liquid weighing 0 8 gram per c.c. are mixed with another liquid weighing 1·2 grams per c c. If the mixture weighs 1·1 gram per c c., how much of the second liquid was used ?

23. A is employed at an initial salary of £120 with annual increments of £20 B at the same time starts at £300 with a yearly rise of £5 After how many years will their annual salary be the same ?

24. A school of 750 boys had 360 boarders. The following term the number of day boys remained unaltered, but the boarders increased so that they formed half the school. Find the increase in boarders.

25. A man spends x shillings a day during July and 5 shillings more each day during August. How much does he spend in July and August together ? During September his daily expenses are 1 shilling a day less than in July, and his expenses for this month come to £14, 1s. less than the whole cost for July and August together. How much a day does he spend in July ?

Revision Exercise V (d)

1. If $3x=5$, does $x=5-3$? Give reasons for your answer
If $ax=b$, does $x=b-a$?
If $ax=b$, what is the correct value of x?

2. Does $\frac{5}{2}-\frac{3}{4}=10-3$?

Is $\frac{2x}{3}-\frac{3}{5}=10x-9$? Correct these two statements.

What multiple of $\frac{2x}{3}-\frac{3}{5}$ is equal to $10x-9$?

If $\frac{2x}{3}-\frac{3}{5}=4$, what is the value of x?

3. Is $2-\frac{4(x-1)}{5}=\frac{10}{5}-\frac{4x+4}{5}$?

How many fifths have been subtracted from 2 in this statement? What is the correct number to be subtracted?

4. Is $3-\frac{5-1}{4}$ equal to $12-(5-1)$? If $3-\frac{5-1}{4}$ is replaced by $12-(5-1)$, what operation has been performed?

If $a-\frac{b-c}{4}=x$, what does $4a-b+c$ equal?

5. Multiply ab by 3. If 2×5 is multiplied by 3, express the result in the form of *two* factors.

What is the result of multiplying $\frac{2}{3}(x-5)$ by 9?

6. Multiply $\frac{1}{2}(x-\frac{1}{3})$ by 6. By what must you multiply the given expression to obtain the result $3x(6x-2)$?

7. Multiply $\frac{1}{3}-\frac{3}{4}(2x-\frac{2}{3})$ by 12, without removing the bracket.

Solve the following equations:

8. $\frac{5}{4}(x-3)-\frac{2}{5}(x+\frac{1}{2})=x-5.$

9. $\frac{x-\frac{1}{2}}{3}-\frac{2x-5}{8}=5\frac{1}{2}-\frac{3x-1\frac{1}{2}}{4}$

10. $\frac{1}{4}(2x+5)-\frac{1}{5}(x+3\frac{1}{2})=2(2-x)$.

11. $x+1=\frac{4}{3}(x+3)-\frac{4}{5}(2x-1)$.

12. $3x-\dfrac{4x-9}{3}=5+\frac{7}{4}(x-2)$.

Historical Note

The name Algebra is derived from one of the processes adopted for the solution of equations. The Arabic mathematician Al Khwarizmi wrote a treatise about A.D. 830, which he called "Al-jebr w'al muquabala." By "Al-jebr," or "Restoration," he meant the transferring of negative quantities from one side of an equation to the other side; by "muquabala," or "Opposition," the removal of such like terms as occur on both sides of the equation. Thus the equation $5x-3=2x+12$ when treated by "Al-jebr" becomes $5x=2x+3+12$, and the further operation of "muquabala" reduces it to $3x=3+12$ (by the removal of $2x$ from each side). The verb from which Al-jebr was derived meant to "make whole" or "restore that which was broken." It was adopted into the Spanish language in its original sense, so that to-day in Spain "algebraista" means a bonesetter or a surgeon.

One of the earliest equations known in algebra is the Egyptian

which means $x(\frac{2}{3}+\frac{1}{2}+\frac{1}{7}+1)=33$.

In the Middle Ages, Z=zensus or census stood for x^2, and R=res stood for x; p was plus and m minus. Thus $1Zp5Rm4$ is x^2+5x-4. Later on this was written $1\overset{2}{\smile}p5\overset{1}{\smile}m4$, but there was no method universally adopted and progress was hindered by the absence of a suitable shorthand.

The signs + and − came into general use in the 16th century; the sign = was introduced by Robert Recorde in his algebra called "the Whetstone of Witte" (1557). The use of x and y for unknown quantities and a, b, . . . for known quantities is due to René Descartes (1637).

Miscellaneous Examples II

A 13

1. A has x pence, B has y pence more than A, and C has twice as much as A and B together. How much have they between them ?

2. Express t hours and m minutes in seconds.

3. Solve the equation $26-3x=38-7x$, testing your result.

4. What is the result of multiplying x^2+2x by $3x$? If this is 8 less than that of multiplying $2x+3$ by $2x^2$, what is the value of x ?

5. Simplify $3(a+2b)-5(b+2c)+2(a-3c)$ and check your answer by putting $a=5$, $b=4$, $c=1$.

6. What is the H.C.F. of 12, 20, 36 ? Find the H.C.F. of $2ab^2$, $4b^2c$, $6bc^2$

7. Divide 29 shillings between two boys so that the elder boy gets 7 shillings less than twice the younger boy's share.

A 14

1. Find the values of $(a+2b)c$, $(a+2)(b+c)$, and $a+2(b+c)$ when $a=4$, $b=3$, $c=2$

2. What is the L C M. of $3ab^2$ and $2bc^2$? How many times does each divide into the L.C.M. ?

3. Subtract $\frac{1}{3}x+\frac{1}{4}y$ from $\frac{1}{2}x+\frac{1}{3}y$.

4. If $4x+3=5x-a$ when $x=7$, what is the value of a ?

5. How much is left when $3x^2(2x^2y+5xy^2)$ is subtracted from $4x^3y(2x+7y)$?

6. Solve the equations

$$\text{(i)}\ \frac{x}{2}=5-\frac{x}{3}; \qquad \text{(ii)}\ 2(3x+5)+3(4x-2)=40.$$

7. A piece of cord 28 in. long is divided by two knots into three parts. The middle part is 2 in. longer than the first part and 3 in. shorter than the last part. What is the length of each part ?

A 15

1. Rewrite in its simplest form

$$2a^2+8ab+5a-4a^2+7a+3a^2-6ab.$$

2. Show that $x-3y \div y+z$ and $(x-3y)\div(y+z)$ have different values when $x=12$, $y=2$, $z=1$. Is there any value of x for which they are equal when $y=2$, $z=1$?

3. Solve the equations (i) $\frac{3}{x}=\frac{4}{5}$; (ii) $\frac{x-1}{6}+\frac{2x}{9}=\frac{x+1}{4}$.

4. If $x=3y+2$ and $z=4y+3$, show that $3z-4x=1$.

5. A party of x boys divide a number of cherries between them. Each boy gets y cherries and there are z left over. How many cherries were there in all? Test your answer by putting $x=5$, $y=13$, $z=3$.

6. If a tons of steel are required to make x girders, how much steel would be wanted to make y such girders?

7. A has four times as much money as B If he gives B £8 he will then have only twice as much as B How much did each have at first?

A 16

1. Subtract $3a-2b-c$ from $5a+3b+2c$, testing your answer by putting $a=5$, $b=3$, $c=1$

2. What is the total cost in shillings of buying a lb of salt at p shillings per lb. and y oz. of pepper at z oz. for a shilling?

3. Add together $\frac{2a+3b}{3}$, $\frac{3a-2b}{4}$, $\frac{5a+3b}{2}$.

4. Which of the numbers 1, 2, 3, . . 10, when substituted for x will make the expression $\frac{3x+5}{4}$ a whole number? Can you *guess* the next value of x which will also do this?

5. Solve

(i) $2(5z-3)-3(4z-5)=14-3(z+1)$; (ii) $\frac{x+a}{2}=b-x$

6. Remove the brackets from $2(x+3y)-3(2a-b)$. Bracket together the terms which are divisible by 3 and 5 in the expressions $5a-3b-10c+9d$ and $5a-3b+10c-9d$

7. An excursion party consists of 40 persons To meet the expenses, which came to £2, 1s., each adult contributes 4 shillings and each juvenile pays 6d How many adults were there?

A 17

1. Distinguish between the meanings which are given to x^5 and $5x$. What is the actual difference of their values when $x=2$?

2. Subtract the sum of $\frac{a}{3}+\frac{b}{2}$ and $\frac{a}{2}+\frac{b}{4}$ from $3a+2b$.

3. If $xy+yz=21$, what is the value of y when $x=2$ and $z=5$?

4. Solve the equations

   (i) $x-a=\frac{2}{3}(x+a)$; (ii) $y-3=\frac{2}{5}(y-1)+\frac{1}{3}(y+1)$.

5. A cart can hold c cwt. of sand. How much will it have carried after x journeys when three-quarters full and y journeys when only half full ?

6. A dog sleeps for x hours out of the 24 hours of the day. If his waking hours are 4 less than his sleeping hours, find the value of x.

7. A tennis club buys a new set of nets and raises the cost by a levy of 12 shillings a head on its members If the number of members had been 10 greater the cost per head would have been only 9 shillings. How many members were there ?

A 18

1. What is the difference in the values of $x^2 \times x^3$ and x^2+x^3 when $x=2$?

2. Find the factors of $2a^3+4a^2b$ and of $5ab+10b^2$. Have they any common factor ?

3. Simplify $2x^2y \div \frac{x}{y^2}$ and $6x\left(\frac{3}{2x}-\frac{1}{3}\right)$.

4. If $a-2b=1$, $3b-2c=1$, $4c-3d=1$, find the value of $5a-4b+3c-2d$ when $d=5$.

5. Solve

   (i) $3x-2(x-5)=57-4(x+3)$; (ii) $x+\frac{9}{2}=\frac{4}{3}(x+4)-2(1-x)$.

6. Insert a bracket round the last two terms of each of the expressions $2x+y-3z$, $2x-y-3z$, and $2x-y+3z$. Find the value of each when $x=5$, $y=3$, $z=1$ both before and after the insertion of the brackets.

7. In the year 1928 A was 55 years old and B was 25. In what year will A be twice as old as B ?

A 19

1. When $a=5$, $b=2$, find the values of

$$a^3-b^3, \qquad (a-b)^3, \qquad \frac{a^2}{2b}-\frac{2b^2}{a}.$$

2. Find the H C F. and L.C.M. of $4a^2b^3c$, $6a^3b^2d$, $8a^4bc^2$

3. The equation $\frac{x}{3}-\frac{x-2}{4}=1+\frac{1}{x+5}$ is satisfied by one of the three values 3, 5, 7. Which is it ?

4. Divide $8a^4b^2+6a^2b^4+12a^3b^3$ by $2a^2b$.

5. A tumbler h in high, x in. in diameter across the top, and y in. across the bottom holds $\frac{h}{133}(x^2+xy+y^2)$ pints What are the contents of such a tumbler for which $x=3$, $y=2$, $h=3\frac{1}{2}$?

6. What value of x will make $\frac{2}{5}(x+3\frac{1}{2})+\frac{1}{3}(x+4\frac{1}{2})$ equal to $x+2\frac{1}{2}$?

7. A house agent bought a house at an auction and resold it again for £1620 after having spent on redecoration, etc , a sum equal to a quarter of the amount he paid for the house. If he thereby gained one-tenth of the purchase price of the house, at what price did he buy it ?

A 20

1. Express in words the difference in meaning between $a+b(c+d)$ and $(a+b)c+d$. What is the difference in value when $a=3$, $b=1$, $c=2$, and $d=0$?

2. Simplify (i) $\frac{a^2b}{c^2}\times\frac{c^3}{ab^2}$; (ii) $\frac{2a^2}{b^2c}\div\frac{ac^2}{2b^3}$.

3. Solve

$$\text{(i) } 5\left(\frac{x}{2}-3\right)=3\left(\frac{x}{4}+2\right); \qquad \text{(ii) } 2x-\frac{x-5}{4}=20-\frac{x}{3}.$$

4. After riding along a straight road for x miles at y miles per hour a cyclist has a puncture and walks back home at z miles per hour. How many hours was he away from home ?

5. The whole area of a circular tin is $2\pi r^2+2\pi rh$ where r is the radius of the base, h is the height, and $\pi=3\frac{1}{7}$. Rewrite this expression in factors and find its value when $r=3$, $h=4$

6. Simplify (i) $\frac{2a+3}{8a}-\frac{3a+2}{12a}$; (ii) $2-\frac{a+3}{5}+\frac{3a+1}{4}$.

Test each answer by putting $a=1$.

7. A can walk at x miles per hour, B can walk $\frac{1}{2}$ mile per hour faster. If they start from two places $12\frac{1}{4}$ miles apart and meet in $1\frac{3}{4}$ hours, what is the value of x?

A 21

1. Divide $2x^3y^2+4x^2y^3$ by $2x^2y$ What are the factors of your answer?

2. If $\frac{4}{5}(2x-9)-\frac{1}{2}(3x-a)=2$ is satisfied by $x=7$, what must be the value of a?

3. The area of a rectangle is A sq. in and its longest side is l in. What is the length of the shorter side? If the perimeter is s in, write down a formula connecting s, l, A

4. Find the H.C F. and L.C M. of $6p^3q^2$, $18p^2q^3$, $9pq^4$.

5. What is the value of $\frac{6x^2-13x-5}{3x+1}$ when $x=2{\cdot}6$?

6. Solve the equations

$$\text{(i) } 7-\frac{x-4}{3}=5; \qquad \text{(ii) } \frac{3x+8}{2x}=2\tfrac{1}{2}.$$

7. The sum of n terms of the series x, $x+2$, $x+4$, $x+6$, . . . is $n(x+n-1)$. What is the value of x if the first 12 of these numbers add up to 240?

A 22

1. If $xy=120$, find the value of y when $x=10$. What are then the values of $\frac{x}{y}$ and x^2y?

2. Subtract the product of $5x^2+2x+10$ and $3x$ from that of $3x^2+4x+6$ and $5x$. For what value of x is the result equal to 1400?

3. One angle of a triangle is half the sum of the other two. If these two differ by 12°, what is the size of each angle ?

4. What value of x will make $\frac{9}{2}-\frac{3}{x}$ equal to $\frac{3}{x}-\frac{9}{2}$?

5. By choosing any values you like for p and q (p should be greater than q), verify that the result of adding the squares of p^2-q^2 and $2pq$ is the same as that of finding the square of p^2+q^2

6. Rewrite in a form free from brackets

$a^3b^3(a^2+b^2)$, $a^3b^3(a^2b^2)$, $a^3b^3-(a^2b^2)^2$, $(a^3b^3)^3$, $(a^3b^2+a^2b^3)-a^2b^2$.

7. A school is divided into 18 forms. Of these x have an average of 15 boys each, 3 others have an average of $(x+4)$ boys each, and the remainder have 10 boys each If the total number of boys in the school is 258, what is the value of x ?

A 23

1. If $a=2b$, express the value of $\frac{5a+8b}{3}-\frac{3a+2b}{4}$ in terms of b.

2. If the equation $3x^2-ax+12=0$ is satisfied when $x=3$, what must be the value of a ?

3. Simplify (i) $\frac{a}{x}-\frac{b}{x^2}\div\frac{c}{x}$; (ii) $\left(\frac{a}{x}-\frac{b}{x^2}\right)\div\frac{c}{x}$.

4. The area of any triangle can be found from the formula $\frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}$ where a, b, c are the lengths of the sides. Calculate the area of a triangle whose sides are 6 in., $1\frac{3}{4}$ in., $6\frac{1}{4}$ in.

5. Solve the equations

$$\text{(i) } 5-\frac{3}{x}=\frac{4}{x}-9\,; \qquad \text{(ii) } a-\frac{b}{x}=\frac{c}{x}-d.$$

6. The raw material for a certain machine costs £a and the expense of its manufacture is £b. What will be the total cost of making x such machines ? If the price of the raw material rises by £p and the cost of manufacture drops £q, how many machines can be made for the same total cost as before ?

7. A motorist allows himself 4 hours to complete a journey of 75 miles. After going $2\frac{1}{2}$ hours he finds himself behind time and increases his speed by 2 miles per hour. If he arrives punctually, at what speed did he start on his journey ?

A 24

1. A has £p, B has £q more than A, C has £r more than half B's amount. How much have A and B together more than C has ?

2. Rewrite the fractions $\frac{a}{3b}, \frac{3}{4a}, \frac{5}{6ab}$ in terms of a common denominator.

3. Show that $x=5$ and $x=3\frac{1}{2}$ both satisfy the equation $2x^2-17x+35=0$.

4. Bracket together the terms which have a common factor in $2ab-3cd+5ac+7bd$. In how many ways can this be done ?

5. Show that $\left\{\frac{n(n+1)}{2}\right\}^2-\left\{\frac{n(n-1)}{2}\right\}^2$ has the same value as n^3 when $n=2$. Is this also true when $n=3$?

6. Simplify

(i) $\frac{3a}{2b}-3\left(\frac{a}{6b}-\frac{a}{b}\right)$; (ii) $\frac{3a^2+2a-5}{4}-\frac{2a^2+3a-4}{6}$.

7. A tradesman spent £8 in buying one kind of cloth at a certain price per yard and £24 on another kind costing twice as much per yard. Altogether he bought 50 yd. What was the price per yard of the cheaper cloth ?

B 25.

1. Find the values of $a+2b$ and $(a+2)b$ when $a=3$, $b=4$.

2. A hawker buys one barrel containing $4x$ lb. of apples for 2d. per lb. and another barrel containing $2y$ lb. for 3d. per lb. If he mixes the two lots together and sells the whole at $3\frac{1}{2}$d. per lb., how much will he gain ?

3. If $x=y+3t$, $y=z+4t$, $z=t+2$, find x, y, z when $t=4$ and show that $2y-x-z=4$.

4. Simplify $2(3a+5b)-5(a+2b)+7(b+3c)$.

5. There are three pieces of machinery weighing $(2x^3+3x^2+4)$ tons, $(5x^2+2x)$ tons, and $(3x^3+5x+8)$ tons respectively. What is their joint weight? By how much does the weight of the last two together exceed that of the first?

6. Solve the equation

$$(x+1)+2(x+2)+3(x+3)=4(x+4).$$

7. Divide £53 between A, B, C so that B gets £3 more than A and C gets £2 more than B.

B 26

1. A, B, C divide $(n^2+5n+10)$ pounds between them. If A gets £$(n+2)$ and B gets twice as much as A, how much is left for C? Test your answer by putting $n=3$

2. Find the values of x^2-y^2, $(x-y)^2$, and $(x+y)(x-y)$ when $x=6$, $y=4$.

3. If the number of fine days in June is x, what is the expression for the number of wet days in that month? If these are two less than the number of fine days, what is the value of x?

4. Solve the equations

(i) $\frac{x}{3}+\frac{x}{4}=2+\frac{x}{2}$; (ii) $5(3x-7)=28-2(2x+3)$.

5. Express the difference between the square of a number and the square of its double in terms of the original number. [Take this to be n.] If this difference is equal to 192, what was the original number?

6. Simplify (i) $12\left(\frac{x}{3}+\frac{y}{2}\right)-8\left(\frac{x}{2}-\frac{y}{3}\right)$; (ii) $3x\left(\frac{2}{x}-\frac{1}{3}\right)$.

7. One girl buys 30 yd. of ribbon at x pence a yard. Her sister buys 10 yd. less but pays twice as much per yard. Between them they spend 8s 9d. How much did the ribbon cost per yard?

B 27

1. Multiply $2ab+3b^2$ by $3a^2$ and a^2+2ab by $2b^2$. What is the sum of the two answers?

2. Prove that a rectangular block whose edges are l in., b in., h in. has a total area of $2(lb+bh+hl)$ sq in What is the value of this when $l=8$, $b=5$, $h=7$?

3. Find the values of $3x^3+4x^2$, $3x^3\times 4x^2$, $3x^3\div 4x^2$, $(3x+4)^2$ when $x=2$.

4. How many pence are there in £a, bs. cd. ? What should I have left in pence out of this sum if I paid away a half-crowns and b florins and received c sixpences in change ?

5. Solve the equations

(i) $5(3x+5)-4(2x+8)=28$; (ii) $\frac{1}{4}(x+3)-\frac{1}{5}(x+1)=\frac{1}{7}(x-2.)$

6. If a car takes p hours over a journey when its speed is x miles per hour, what time will it take when it is driven at z miles per hour ?

7. A hammer is made up of a wooden handle and an iron head. The total weight of the two parts is $6\frac{1}{2}$ lb., and the handle alone weighs 1 lb. less than half the weight of the head. What is the weight of each part ?

B 28

1. One boy can count the number of cards in a pack at the rate of x cards per second. A slower boy can count only at the rate of y cards per second. By what time will the first boy beat the second in counting z cards ?

2. Find the values of $(3a-2b)^2$ and $9a^2-4b^2$ when $a=3$, $b=4$.

3. Evaluate $(p-q)r-t$ and $p-q(r-t)$, when $p=10$, $q=3$, $r=4$, $t=2$. Rewrite each expression without brackets.

4. Express 48 as a product of two factors in as many ways as you can. Can you discover values of x and y which make $2x+y=20$ and $xy=48$?

5. Simplify $x-\dfrac{x-3}{4}$. Check your answer by putting $x=7$.

6. Solve the equations

(i) $\dfrac{3}{2x}+\dfrac{5}{x}=6\frac{1}{4}-\dfrac{6}{x}$; (ii) $\frac{1}{2}\left(6-\dfrac{2x}{3}\right)=\dfrac{x}{2}\left(1-\dfrac{4}{x}\right)$.

7. A garden bed is 10 ft. long and x ft. broad If 2 ft. is cut off both the length and the breadth the area would be diminished by 32 sq. ft. What was the original breadth of the garden ?

B 29

1. A ship travels x miles on its first day out of port, and each following day exceeds the preceding day's run by y miles. How far will it have gone in 5 days ? If on the 5th day it goes a quarter of the whole distance then travelled, show that $y = \frac{x}{6}$.

2. Simplify $\frac{2a+5b}{4} - \frac{3a-b}{6} + \frac{a+2b}{8}$.

3. Rearrange $3a-4b-6c+8d$ so that the first and third terms may be bracketed together, and also the second and fourth terms Find the value of the expression both before and after the rearrangement, given $a=8$, $b=7$, $c=2$, $d=3$.

4. Solve the equation $\frac{1}{4}(2x+5)-\frac{1}{6}(x+4\frac{1}{2})=\frac{1}{5}(x+3\frac{1}{2})$.

5. Find three consecutive *odd* numbers whose sum is 291

6. Simplify $\frac{2a^3b^2c^5}{3a^2c} \times \frac{6a}{bc^3} \div \frac{8b^3c}{a^4}$.

7. What is the distance between two neighbouring towns if a traveller saves $10\frac{1}{2}$ minutes by taking the bus which runs between them at 18 miles per hour, instead of a tram whose average speed is 12 miles per hour ?

B 30

1. Subtract the sum of $x(x-2y)$ and $y(x+2y)$ from $3(x^2+y^2)$, and check your result by putting $x=5$, $y=2$.

2. If $a=2b$, $b=3c+1$, $c=d+2$, find the values of a, b, c when $d=3$, and calculate the value of $3a+4b-5c$.

3. Simplify $\frac{3}{2x} - \frac{2x+1}{5x}$.

4. Convert the two sums of money ps. qd. and qs. pd. both into pence, and show that the result of adding the sum and difference of the two amounts is twice the larger of the original sums [Take $p>q$]

5. Solve the equation $7x-\frac{13-2x}{5}=29-\frac{x+2}{5}$.

6. In t seconds a stone dropped from the top of a tower will fall through $16t^2$ ft How far does a stone fall in 7 seconds ? How far will it fall *during* the 7th second ?

7. A museum charges 3d for entrance on Tuesdays and Fridays, and 1d. on each other weekday In a certain week for which the admission fees amounted to £6, 3s 4d. it was noticed that the average number of visitors per day on the cheap days was 250 more than on the other two days. What was the average attendance on Tuesday and Friday ?

B 31

1. A motor car uses g gallons of spirit in running n miles What is the yearly cost in pounds of the spirit if its daily run averages x miles, and the price of the spirit is y shillings per gallon ?

2. Simplify $\frac{a-b}{b}-\frac{b-c}{c}$.

3. Express in their simplest forms

$$\sqrt{9x^2y^4}, \qquad \sqrt[3]{x^{12}}, \qquad \frac{a^2b}{3c^3}\times\frac{2b^2c}{ax^3}-\frac{ab^2x}{6c^2}$$

4. Solve the equation $\frac{1}{7}(x-1\frac{1}{2})+\frac{1}{5}(x-\frac{1}{3})=1\frac{13}{30}$

5. Rewrite $\frac{6a^2-8bc}{12ab}$ as the difference of two fractions in their lowest terms, and simplify $\left(\frac{2}{x}-\frac{4}{x^2}\right)\div\frac{3}{x}$.

6. A's marks for the term are a, and B's marks are b. In the examination A gains three times as many marks as B, and his total marks for term and examination together are thus twice B's. Find an expression in terms of a and b for the number of marks B gained in the examination. Show that your answer is correct by putting $a=80$, $b=48$.

7. A train has to travel D miles in h hours. The average speed for the first d miles is m miles per hour. For the rest of the journey it travels at x miles per hour and thereby makes up for lost time. Prove $x=m(\mathrm{D}-d)/(mh-d)$

B 32

1. A chain is made up of a section of $4x$ links each $\frac{1}{2}$ in. long, followed by a section of $12x$ links each $\frac{1}{4}$ in. long What is the whole length of the chain ? If it is hung over a rail so that $6x$ of the links are on one side and the remainder (all small links) are on the other, what is the difference in depths between the two ends of the chain ?

2. A railway engine burns p tons of coal for each mile that it runs. How many hundredweight does it consume per hour if it makes a journey of b miles in c hours ?

3. The middle angle of a triangle is 7 degrees larger than the smallest angle and 4 degrees less than the greatest Find the size of each angle in degrees.

4. Show that $\dfrac{ab+bc}{a+c}$ can be written in a simpler form. If you are given $ab=20$, $bc=30$, $a+c=40$, can you find the values of a, b, c ?

5. Find the factors common to $6p^3q+9p^2q^2$ and $6p^2q^2+12pq^3$. What is their H.C.F. ?

6. Bracket together in each of the following expressions those terms which have a common numerical factor .

$2a-4b+5c$, $\quad 3a-4b+6c$, $\quad 5a-3b-6c$, $\quad 4a-5b-6c$.

7. Two towns P and Q are 84 miles apart. A car starting from P at 9 a.m. meets at 11.45 p m. another car which started from Q at 9.30 a.m. and travelled 4 miles per hour faster. What were the speeds of the two cars ?

B 33

1. A boy spends s shillings in buying envelopes at pd. per dozen. How many does he buy? If the price of envelopes falls by 2d. per dozen, find an expression for the additional number he would be able to buy for the same expenditure.

2. Simplify

$$\frac{3a}{9b} \times 6ab, \qquad \frac{3a}{9b} \div 6ab, \qquad \frac{3a+9}{b} \times 6ab, \qquad \frac{3a+9}{b} \div 6ab.$$

3. Express $a + \frac{b}{c}$ as a single fraction and $\frac{pq^2+qr^2}{q^3}$ as the sum of two separate fractions

4. Show that the difference between any number and the square of that number is equal to the product of the original number and the result of diminishing it by unity. [Take the original number to be n.]

5. If $x=1-2s^2$ and $y=2cs$, find the value of x^2+y^2 when $s=0 6$ and $c=0{\cdot}8$

6. Solve the equation $\frac{3}{2}(x+\frac{1}{3})+\frac{2}{3}\left(\frac{x}{3}+1\right)=16\frac{2}{3}$.

7. During the War a village contained 120 men, 385 women, and a number of children. The number of girls was 73 more than the number of boys, and the total number of females was twice the number of males. How many boys were there?

B 34

1. Simplify $\left(\frac{a}{3}-\frac{a}{4}+\frac{a}{5}-\frac{a}{6}\right) \div \left(\frac{a}{3}+\frac{a}{4}\right)$.

2. Divide $12x^3y^2-15xy^4$ by $3y^2$, and $6x^4y-4x^2y^3$ by $2xy$. Subtract your second answer from the first.

3. Find the value of $\frac{6x^2-5x-21{\cdot}3}{2x-3}$ when $x=2{\cdot}4$.

4. The sum of the cubes of the first n numbers is given by the formula $S=\frac{n^2(n+1)^2}{4}$. Find from this formula the sum

of the cubes of the numbers from 1 to 6 and also that of the numbers from 15 to 20

5. A regiment loses one-fifth of the number of men engaged in each of two successive battles. What will be its final strength if it starts with a men and is reinforced by b men between the two battles ?

6. Write down expressions for the squares of $3(p-q)$, $x^2(x+2y)$, $2a^3b^2$, where necessary employing brackets. Also rewrite $(4p-8q)^2$ as a numerical coefficient multiplying the square of an expression in brackets.

7. A father is twice as old as his elder son, who is himself twice as old as his younger brother Three years hence their combined ages will be 100 How old is each now ?

B 35

1. If $\dfrac{3x-2a}{5}$ and $\dfrac{2x-4b}{3}$ have the same value when $x=8$, show that $b=\dfrac{3a+4}{10}$.

2. What is the total cost of p things at 5d. a score and q things at 8d. per dozen ? Express your answer in shillings. What is the average cost of each thing ?

3. Can the numerator or denominator of $\dfrac{4a^3+6a^2}{2a^2+3a}$ be simplified in any manner ? Can the whole fraction be simplified ?

4. Simplify (i) $p^2-q^2 \div \dfrac{p}{q}$, (ii) $\left(\dfrac{1}{p}+2q\right)\div\left(\dfrac{1}{2q}+p\right)$.

5. If $y^2=4ax$ and $x=at^2$, find an expression for y in terms of a and t.

6. Solve the equations

$$\text{(i) } 3(x-1{\cdot}2)-\tfrac{1}{5}(3x-4{\cdot}4)=4;$$
$$\text{(ii) } 3x-\tfrac{1}{3}(4x-9)=5+\tfrac{7}{4}(x-2).$$

7. A boy receives a fixed sum as pocket money at the beginning of every week. In each week he spends two-thirds of all he had at the beginning of the week. At the end of the 3rd week he had 1s. 1d. left. What is his weekly allowance ?

B 36

1. Solve the equations

(i) $\cdot 4x + \cdot 7x - 1\cdot 4 = \cdot 35x + \cdot 85$; (ii) $x - \dfrac{x+a}{2} = \dfrac{x+b}{3}$.

2. For a train rounding a curve at V miles per hour the outer rail should be $\dfrac{4d\mathrm{V}^2}{5\mathrm{R}}$ in. higher than the inner rail; R being the radius of the curve in feet and d the distance between the rail in feet. Find the difference in heights of the rails when $\mathrm{R}=600$, $d=5$, for a train travelling at 44 ft. per second.

3. It is known that $(a-b)^2=a^2-2ab+b^2$ By putting $a=40$ show that this relation can be used to find 39^2 In a similar manner find 199^2 and $(49\cdot 9)^2$.

4. Water is pouring from a tap into a tank at the rate of n cu. ft. per second. At a given moment the quantity of water in the tank was V cu ft. How much water was in the tank (i) t seconds later; (ii) x seconds earlier? How much water was added in the interval?

5. Simplify $\dfrac{(6p-4q)^2}{3p-2q}$, $\dfrac{(2a-4b)^2}{(a-2b)^2}$, $\dfrac{2ax-4bx}{3ay-6by}$.

6. One iron tool is one-sixth as heavy again as a second tool. If 4 lb. is taken from the weight of each the first will now be one-fifth as heavy again as the second. Find the weights of each.

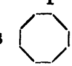
7. The value of a motor car diminishes by a quarter of its purchase price during the first year of use and by one-tenth of the purchase price during each subsequent year. After 6 years its value is estimated as £120. What was the price originally paid for it?

CHAPTER VI

Negative Numbers

The substance of this chapter should be taken orally by the teacher. The pupil is not expected to be able to reproduce the argument.

It has already been pointed out that algebra enables us to express in general terms results which have previously only been proved for special cases. It often happens, however, that a result or formula which is to be regarded as true " generally "—or for every possible case—cannot always be interpreted by the use of positive numbers only.

Thus $x-y$ represents the result of subtracting y from x ; but if y is greater than x the result cannot be given by a positive number. It has therefore been found necessary to invent another set of numbers, viz. -1, -2, -3, etc., called Negative Numbers. Such inventions are common in the history of mathematics, and many of them have proved of very great value in furthering the development of the subject. One such invention was that of the sign for zero. When the Arabic figures we now use were first introduced into the West (about the twelfth century) the sign for 0 had not been invented. The absence of a figure was indicated only by a dot, *e.g.* 7.1 (now 701). This was easily rubbed out by careless or unscrupulous merchants, so its place was filled by a figure made thus , which in time was modified to 0.

The idea of a " negative " number is probably not entirely new to you. For instance, the negative numbers on a Centigrade thermometer scale are probably familiar. The temperature of freezing water is marked 0° and that of boiling water 100°. For temperatures below 0° the graduations are marked -1, -2, etc. Here -1 does not mean that 1 is subtracted from any number, but that the temperature indicated is one degree less than that shown by 0°.

It will be clear that $-2°$ is a *higher* temperature than $-3°$, and that we are justified in saying that 0 is greater than -1, -1 greater than -2, etc. Such a fact is usually expressed by saying that although 3 is numerically greater than 2, yet -3 is algebraically less than -2.

If no sign is written before a letter or figure it is assumed to be positive. At first negative numbers will be enclosed in a bracket.

When negative numbers were adopted for use in mathematics it became necessary to find rules for the methods of operating with them. It is not, for instance, immediately obvious what is even meant by the operations of multiplication and division by a negative number such as $2\times(-3)$ and $6-(-?)$.

In searching for these rules it is clearly desirable that the introduction of new symbols should not alter the principles already established for ordinary numbers, though they may extend the meaning to be given to them. Thus, rules like $x+y=y+x$ or $x\times y=y\times x$, which have been proved true for positive numbers, will be taken to hold true for negative numbers as well.

Addition of Negative Numbers.—To add 3 to 2 on the scale shown we begin at 2 and travel 3 divisions upwards to 5, which represents $2+3$.

Similarly, to add 3 to (-2) we start at (-2) and travel 3 divisions upwards, arriving at 1. $\therefore\ (-2)+3=1$.

So far we have only added positive numbers; we have now to find a method of adding negative numbers.

For positive numbers it has been shown that $x+y=y+x$. Such a law should therefore hold good for negative numbers.

$$\therefore\ 3+(-2)=(-2)+3=1.$$

Hence to add (-2) to 3 we start at 3 on the scale and *descend* 2 divisions.

Similarly $4+(-2)=(-2)$.

Addition of positive numbers, thus, is performed by *ascending* the scale; for negative numbers we *descend* the scale.

We have shown that $3+(-2)=1$.

But we know $3-2=1$. $\therefore\ 3+(-2)$ is the same as $3-2$.

Again, $4+(-2)=2$ and $4-2=2$.

Hence our calculations may be performed by removing the bracket and taking the combination of the sign of addition with a negative sign as equivalent to a minus sign, *e g* $a+(-b)=a-b$ and $3p+(-2q)=3p-2q$.

Subtraction of Negative Numbers.—A question in subtraction can always be transformed into one which involves addition only: thus, Subtract 2 from 5 corresponds to the question "What number *added* to 2 makes 5?"

Example 1.—Subtract (-3) from 2. Otherwise, "What number added to (-3) makes 2?" From our scale we see that the answer is 5.

$$\therefore\ 2-(-3)=5.$$

A temperature of 2° is 5 degrees higher than a temperature of $-3°$, hence the difference between 2° and $-3°$ is 5°.

Example 2.—Subtract 2 from (-3). Otherwise what number added to 2 makes (-3)?

Starting at 2 on the scale we must *descend* 5 divisions to reach (-3). But we have seen that to *descend* 5 divisions is equivalent to adding (-5).

$$\therefore (-3)-2=(-5).$$

In Example 1 we have shown that $2-(-3)=5$: but we know that $2+3=5$, so that we can obtain the required answer by considering that the combination of the sign of subtraction with a negative sign is equivalent to $+$, *e g.*

$$a-(-b)=a+b, \qquad 2p-(-3q)=2p+3q.$$

These results may also be justified by the following illustration:

Suppose the money in a man's actual possession to be regarded as a positive quantity, while that which may be owed by him is taken to be a negative quantity.

Then a man who possesses £5 and owes £2 may be said to have $5+(-2)$ pounds. His total worth, however, is clearly $5-2$ or £3. Thus $5+(-2)=5-2=3$.

Exercise VI (a)

1. Add 3 to (-2) Use the identity $x+y=y+x$ to show the result of adding (-2) to 3. What is the value of $3-2$ and of $-2+3$?

2. What rise of temperature above $-5°$ will give a temperature of 2°? What is the difference between a temperature of 2° and $-5°$? Subtract (-5) from 2. Evaluate $2-(-5)$

3. What rise of temperature above $-4°$ will give a temperature of $-1°$? How much is -1 greater than -4? Subtract (-4) from (-1). Evaluate $(-1)-(-4)$. Evaluate $-1+4$.

4. The height of a place 200 ft. below sea-level being given as -200 ft., state the difference in level between A whose height is 300 ft. and B whose height is -100 ft. Evaluate $300-(-100)$ and $300+100$.

5. What is the height of a place which is 500 ft. below A, A's height above the sea-level being 200 ft.?

6. I ascend 200 ft. above a place whose height is -150 ft. What is then my height?

7. A boy's place in his form at the beginning of term was

8th. If this is taken as his standard position, what is meant by saying that his place at half term was (i) $+6$; (ii) -9?

8. In what other way could you date an event which is described as happening in the year (-57 A.D.)? What would be the meaning of (-1928 B.C.)?

9. A boy finds that his average mark on a weekly grammar paper is 62. What are his actual marks in two weeks when he gets a mark which he thinks of as (i) $+17$; (ii) -23?

10. Simplify $6-(-2)+3$, $(-2)+(-3)-(-2)$, $4-(-6)-5$.

11. Write without brackets the expressions

$a-(-b)+c$, $\quad a-(b-c)$, $\quad a-2(b+c)$, $\quad a-b-(-c)$

12. What number added to -3 makes 5? Subtract (-3) from 5.

13. What number added to -2 makes -4? Subtract (-2) from (-4). Evaluate $(-4)-(-2)$.

14. If $a=-2$, $b=1$, $c=-3$, find the values of $a-2b+c$, $-a+b-c$, $-a-b-c$, $a-(b-c)$, $a+(b-c)$.

15. The temperature falls from 12° to 6°. What is the change in temperature? What is the change if it falls from $a°$ to $b°$?

Evaluate this result if $a=-6$, $b=-12$. What is the value of $(-6)-(-12)$?

16. What is the rise in temperature if it changes from (i) 4° to 8°; (ii) $a°$ to $b°$?

Evaluate the latter if $a=-8$, $b=-4$. What is the value of $(-4)-(-8)$?

17. How many feet is A higher than B (i) if A is 500 ft. and B 200 ft. above sea-level; (ii) if A is x ft. and B y ft. above sea-level?

Evaluate the latter expression when $x=-200$, $y=-300$. What is the value of $-200-(-300)$?

18. How far is A to the right of B on the given line (i) if A is at 3 and B is at 1, (ii) if A is at x and B at y?

Evaluate the latter when $x=-3$, $y=-1$. What is the value of $-3-(-1)$? Explain how this agrees with the diagram.

Evaluate the expression in (ii) if $x=-3$, $y=+2$. What is the value of $(-3)-(+2)$?

19. A walks to a place 5 miles to the east of X. B walks 3 miles to the east of X and then returns 1 mile (i) How far is B finally to the east of X ? (ii) How far is A to the east of B ?

Evaluate $5-(3-1)$ and explain how the result is illustrated by the question just answered

Multiplication by a Negative Number.—Multiplication is repeated addition: thus 3 multiplied by 4 means $3+3+3+3=12$.

Similarly $(-3)\times 4$ means $(-3)+(-3)+(-3)+(-3)=(-12)$.

For positive numbers it has been shown that $x\times y=y\times x$.

This rule is also to hold good when negative numbers are involved. $\therefore 4\times(-3)=(-3)\times 4$.

But we have seen that $(-3)\times 4=(-12)$. $\therefore 4\times(-3)=(-12)$.

We have now to find the result of multiplying a negative number by another negative number, e g $(-4)\times(-3)$

We know that $4\times 3=12$, and that not only does $4\times(-3)=-12$, but also $(-4)\times 3=(-12)$.

It thus appears that the effect of multiplying a positive number by a negative number is to change the sign of the result obtained by multiplying it by the same positive number; and a similar change of sign occurs if we multiply a negative number by a positive number. It would therefore appear likely that if we multiply a negative number (-4) by another negative number (-3), the result will have a changed sign from that obtained from $(-4)\times 3$: in other words

$$(-4)\times(-3)=+12$$

This conclusion can be confirmed in another way.

For all positive numbers $x(y+z)=xy+xz$, and a similar result should be true if any of the numbers are negative, *i.e.*

$$(-3)[7+(-2)]=(-3)(7)+(-3)(-2).$$

Consider the left hand side:

$$7+(-2)=5 \quad\text{and}\quad (-3)(5)=(-15).$$
$$\therefore (-3)[7+(-2)]=(-15).$$

Now consider the right hand side which must also equal (-15). We know that the first term $(-3)(7)=(-21)$, so that the second term must equal $+6$, *i.e.* $(-3)(-2)=+6$.

Division by a Negative Number.—To divide 6 by 2 is equivalent to asking the question: "What number multiplied by 2 makes 6?"

To divide 6 by (-2) we ask the question: "What number multiplied by (-2) makes 6?" and we have just seen that the answer is (-3). $\therefore \frac{6}{(-2)}=(-3)$.

Similarly, to divide (-6) by (-2) we ask the question: "What number multiplied by (-2) makes (-6)?"

Now $\qquad 3\times(-2)=(-6)$. $\quad \therefore \frac{(-6)}{(-2)}=3$.

The results we have illustrated may be classified as follows:

$$\begin{aligned} a+(-b)&=a-b,\\ a-(-b)&=a+b,\\ a\times(-b)&=-ab,\\ (-a)\times(-b)&=ab,\end{aligned}$$

$$\frac{a}{-b}=-\frac{a}{b}, \qquad \frac{-a}{-b}=\frac{a}{b}.$$

Exercise VI (b)

1. Show how to obtain a meaning for (-4) multiplied by 3 Using the identity $xy=yx$, prove that 3 multiplied by (-4) $=-12$.

2. A temperature drops 2° per minute for 3 minutes. If it was originally 0°, what is it finally? If it was originally 2°, what is it finally?

Evaluate $0+3(-2)$, $2+3(-2)$.

3. Find the square of (-2); also the cube of (-2).

4. Show that the square root of 4 can have two values. What are the square roots of (i) 9; (ii) a^2, (iii) b^4?

5. State the cube root of (i) 8; (ii) (-8); (iii) a^3, (iv) $-a^3$.

6. A has £x, B has £x but owes £a. How much better off is A than B? Use this to illustrate the identity $x-(x-a)=a$.

7. Evaluate $6(3-2)$, $6\times3+6(-2)$, $6\times3-6\times2$.

8. Write a general formula which is illustrated by the following results.

$(-3)(7)=-(3)(7)$, $\qquad (-4)(3)=-(4)(3)$, $\qquad (-5)(6)=-(5)(6)$.

9. Simplify (i) $(-1)(-1)(-1)$; (ii) $(-1)(-2)(-3)$.

10. A body's velocity is 20 ft per second, and it changes at the rate of (-5) ft. per second What will be its velocity in 6 seconds? How do you explain the result?

11. The velocity v of a body is given by $v=20+at$, where a is the increase in velocity per second. If $a=5$, what is v when $t=0$, $t=4$? If $a=-5$, what is v when $t=6$? Can you explain the result given when $a=-5$, $t=-4$?

12. The velocity v of a body thrown upwards with velocity 128 ft per second is given after t seconds by $v=128-32t$. What is v when $t=4$; $t=5$? After what time did the body reach its highest point? How do you explain the sign of the result when $t=5$?

13. Evaluate $(-1)^2$, $(-1)^3$, $(-1)^4$. What is the value of $(-1)^n$ (i) if n is odd; (ii) if n is even?

14. By what must (-2) be multiplied to give 6? By what must $-a$ be multiplied to give b?

15. One factor of b is $(-a)$ What is the other?

16. What number divided by a gives $-b$ as a quotient?

17. What is the meaning of x^2 when $x=-2$? When $x=-2$, find the value of $5x^2$ and also the value of $(5x)^2$.

18. When $x=-3$, evaluate

$$2x^2,\quad (2x)^2,\quad x^3,\quad 5x^3,\quad (5x)^3,\quad 4x-x^2.$$

When $a=3$, $b=-2$, $c=1$, $x=-4$, evaluate the following:

19. a^2-b^2, $(a-b)^2$, a^2-2b^2, $a^2-(2b)^2$, $(a-2b)^2$.

20. $3ab-2bc+ca$, $3a^2-2b^2+c^2$, $a^3+b^3+c^3$.

21. a^2x+bx^2, $cx^3-bc^3+ab^3$, $(cx-b)^2$, $c^2x^2-b^2$.

22. If $x=-2$, find the values of

$3x^2-5x+2$, $(3x-5)(x+2)$, $3x^2-5(x+2)$, $(3x-5)x+2$.

23. Evaluate $(a-2b)^2-(b-3c)^2$ when $a=3$, $b=2$, $c=-1$.

Notes.—(1) In Chapter III we have seen that if the letters employed represent positive numbers, then $a-(b+c)=a-b-c$ and $a-(b-c)=a-b+c$, and an example will show that these results hold good when the letters represent negative numbers.

Let $a=4$, $b=-2$, $c=-1$. Now $a-(b+c)$ means that the sum of b and c is to be subtracted from a. The sum of b and $c=(-2)+(-1)=(-3)$ and $a-(b+c)=4-(-3)=7$.

Also, $a-b-c=4-(-2)-(-1)=4+2+1=7$.

Whatever the values of the letters, it appears that a negative sign before a bracket changes the signs inside the bracket, while a positive sign in front leaves the signs in the bracket unchanged.

(2) A negative number need not be written inside a bracket unless another sign comes before it; thus, $-6+8$ equals 2, whether it stands for $(-6)+(+8)$ or for $8-6$.

So also, instead of writing $\dfrac{4}{(-2)} = (-2)$, we may write $\dfrac{4}{-2} = -2$,

i.e. $\dfrac{4}{-2} = -\dfrac{4}{2}$.

Similarly $\dfrac{a}{-b} = -\dfrac{a}{b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$.

(3) $-(a-b)$. If we change the sign of the expression $(a-b)$ it becomes $-(a-b)$, *i.e.* $-a+b$ or $b-a$. Consider, for instance, the expression $6-2$, which equals 4. If we change its sign it becomes $-(6-2)$, *i.e.* $-6+2=-4$.

It is a common error to say that when the sign of $(a-b)$ is changed it becomes $a+b$, but it is clear that here we have only changed the sign of b and have left the sign of a unaltered.

(4) If one factor either in the numerator or in the denominator of a fraction has its sign changed, the sign of the whole fraction is altered.

Thus
$$\frac{(3)(2)}{(7)(5)} = \frac{6}{35},$$

but
$$\frac{(3)(-2)}{(7)(5)} = -\frac{6}{35} \quad\text{and}\quad \frac{(3)(2)}{(-7)(5)} = -\frac{6}{35}.$$

$$\therefore \frac{(3)(2)}{(7)(5)} = -\frac{(3)(-2)}{(7)(5)} = -\frac{(3)(2)}{(-7)(5)}.$$

If, however, two factors have their signs changed, the sign of the fraction is unaltered; thus

$$\frac{(-3)(-2)}{(7)(5)} = \frac{6}{35} \quad\text{and}\quad \frac{(-3)(2)}{(-7)(5)} = \frac{-6}{-35} = \frac{6}{35}.$$

$$\therefore \frac{(3)(2)}{(7)(5)} = \frac{(-3)(-2)}{(7)(5)} = \frac{(-3)(2)}{(-7)(5)}.$$

Similarly
$$\frac{a}{a-b} = -\frac{a}{(b-a)},$$

but
$$x-y = \frac{y-x}{b-a}.$$

(5) When desirable, both sides of an equation may be multiplied or divided by -1, thus:

If $$-x=a,$$
$$\therefore\ x=-a;$$
or if $$-x+b=-c+d,$$
$$\therefore\ x-b=c-d,$$
$$\therefore\ x=b+c-d.$$

When possible, it is usual to write an answer with a positive term first, thus:

If $x=-a+b$, write $x=b-a$,

or if $x=\dfrac{-a+b}{c-d}$, write $x=\dfrac{b-a}{c-d}$ or $\dfrac{a-b}{d-c}$.

Multiplication by 0.—3 times $0=0$, *i.e.* $0\times 3=0$; but since $a\times x$ is always to equal $x\times a$,

$$\therefore\ 3\times 0=0\times 3. \qquad \therefore\ 0 \text{ times } 3=0.$$

Similarly $a\times 0=0$.

Division by 0.—To find the result of dividing 6 by 2 we ask the question: "What number multiplied by 2 equals 6?" If, when asked to divide 6 by 0 we change the question to "What number multiplied by 0 makes 6?" we see that no answer can be given, for any number we can suggest when multiplied by 0 makes 0 and cannot produce 6.

We can, however, look upon the question in another way. If we evaluate the fractions $\dfrac{6}{1}, \dfrac{6}{\cdot 1}, \dfrac{6}{\cdot 01}, \dfrac{6}{\cdot 001}, \dfrac{6}{\cdot 00001}$ the answers are 6, 60, 600, 6000, 600,000. The smaller the denominator the larger the quotient.

If we make the denominator approach nearer to 0 the greater will the quotient become and we say that it eventually becomes infinitely great. The symbol for infinity is ∞, so that the only answer we can give to $\frac{6}{0}$ is to say that if the denominator can be regarded as steadily approaching 0, the quotient approaches ∞.

We can write the result $\underset{x\to 0}{\dfrac{a}{x}\to\infty}$.

Such expressions will not occur in the earlier parts of the book.

Exercise VI (c)

1. When $a=-3$, $b=-2$, $c=-1$, evaluate

(i) $2a-3b+c$; (ii) $2(a-3b)+(a-b)$,
(iii) $2(a-3b)-2(a-b)$; (iv) $2(b-c)+3(a-3c)$;

(v) $a^2-b^2+c^2$; (vi) $(a-b)^2+(c-a)^2$;
(vii) $2(a-b)^2+3(c+a)^2$, (viii) $2(a+b)^2-(c+a)^2$,
(ix) $3b^2-c^3-a$, (x) $2b^2+3c^2-a^3$.

2. When $a=-3$, $b=2$, $c=-1$, $d=0$, evaluate

(i) $\frac{ab}{c}$; (ii) $\frac{a}{c}$; (iii) $\frac{-a}{c}$; (iv) $bc-ad$

(v) $\frac{-c}{b}+\frac{d}{a}$; (vi) $ac-\frac{b}{c}$; (vii) $ad-bc$, (viii) $\frac{ab}{bc}$,

(ix) $\frac{a+c}{b+d}$; (x) $\frac{-a^2c}{ab}$.

3. When $a=3$, $b=-2$, $c=2$, $d=-1$, evaluate

(i) $\frac{a+b}{b-c}$; (ii) $\frac{ab}{cd}$; (iii) $\frac{a-b}{c+d}$; (iv) $\frac{(-a)(c)}{(b)(-d)}$;

(v) $\frac{a-b}{b-c}$; (vi) $\frac{b-a}{c-b}$; (vii) $\frac{a+b}{c-d}$; (viii) $-\frac{a+b}{d-c}$;

(ix) $\frac{bd-ac}{bc}$; (x) $\frac{c(b-d)-ab}{cb-(bd-ac)}$.

4. Fill in the gaps in the following :

(i) $\frac{a}{b}=\frac{(-a)}{(\quad)}$; (ii) $\frac{a}{b}=-\frac{(a)}{(\ )}$;

(iii) $\frac{a-b}{b}=\frac{b-a}{(\quad)}$; (iv) $\frac{x-y}{x}=-\frac{(x-y)}{(\quad)}$;

(v) $\frac{x-y}{a-b}=\frac{(\quad)}{(b-a)}$; (vi) $\frac{x-y}{a-b}=-\frac{(y-x)}{(\quad)}$;

(vii) $\frac{x-y}{a-b}=\frac{-(x-y)}{(\quad)}$; (viii) $\frac{ab}{cd}=\frac{a(-b)}{(\ )(d)}$;

(ix) $\frac{ab}{cd}=-\frac{(a)(b)}{(c)(\ )}$; (x) $\frac{ab}{cd}=\frac{ab}{(-c)(\ )}$;

(xi) $\frac{x-y}{a-b}=\frac{-(x-y)}{-(\quad)}$; (xii) $\frac{x-y}{a-b}=\frac{-(y-x)}{-(\quad)}$;

(xiii) $\frac{x-y}{a-b}=\frac{-(y-x)}{(\quad)}$; (xiv) $\frac{x-y}{a-b}=\frac{(x-y)}{-(\quad)}$.

5. If $x=3$, $y=-2$, $z=-1$, find the values of

$$x+\frac{y}{z}, \quad y-\frac{z}{x}, \quad z+\frac{x}{y}, \quad \frac{x}{y}-\frac{z}{x}, \quad 2+\frac{y}{x}-\frac{x}{z}.$$

6. Multiply

$$(x-y), \quad (x-y)(a+b), \quad (x-y)(b-a), \quad (a-b)(c-d)(x-y)$$

each by -1, giving as many forms for each answer as you can

7. Divide each of the expressions in Question 7 by -1, again giving as many forms as you can for each answer.

8. If $\dfrac{x-y}{a-b}=\dfrac{3}{2}$, write down the values of

(i) $\dfrac{x-y}{b-a}$, (ii) $\dfrac{y-x}{b-a}$, (iii) $\dfrac{x-y}{-(a-b)}$; (iv) $\dfrac{-(y-x)}{a-b}$.

9. If $x=4$, $b=-2$, $y=3$, $c=-1$, evaluate

(i) $\dfrac{x-b}{y-c}$; (ii) $\dfrac{b-x}{c-y}$, (iii) $-\dfrac{x-b}{c-y}$; (iv) $-\dfrac{x+b}{y+c}$.

10. If $a=3$, $b=-2$, $c=1$, $d=-1$, evaluate

(i) $\dfrac{ab}{cd}$, (ii) $\dfrac{a(-b)}{cd}$; (iii) $-\dfrac{(-a)b}{(-c)d}$; (iv) $\dfrac{(-a)(-b)}{(-c)d}$.

11. Simplify $\dfrac{(x+1)^3}{(1+x)^2}$ and $\dfrac{(x-1)^3}{(1-x)^2}$.

12. Evaluate $x^2-(-x)^2$, $x^3-(-x)^3$.

13. Discuss whether or no $(a-b)^2$ is the same as $(b-a)^2$; and whether or no $(a-b)^3$ is the same as $(b-a)^3$ Find the values of each when $a=3$, $b=2$.

14. Is either of the following statements correct:

(i) $(a-b)^2-(b-a)^2=2(a-b)^2$; (ii) $(a-b)^3-(b-a)^3=0$?

Justify your answer.

15. Write in its simplest form $\dfrac{(a-b)^3}{(b-a)^2}+\dfrac{(a+b)^3}{(b+a)^2}$.

CHAPTER VII

Simultaneous Equations : Literal Equations

A PROBLEM is sometimes solved more easily by taking two unknowns, x and y, than by taking x only.

Example.—5 tons of coal and 3 tons of coke cost £17, 5s.

2 tons of coal and 4 tons of coke cost £12, 10s.

Find the cost of coal and of coke per ton.

Let one ton of coal cost £x and one ton of coke cost £y.

Then $$5x+3y=17\tfrac{1}{4} \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$2x+4y=12\tfrac{1}{2} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Multiply both sides of equation (i) by 2, and both sides of equation (ii) by 5, then :

$$10x+6y=34\tfrac{1}{2}$$
$$10x+20y=62\tfrac{1}{2}.$$

If we subtract $10x+6y$ from $10x+20y$ the result will equal the difference between $62\frac{1}{2}$ and $34\frac{1}{2}$.

$$\therefore\ 14y=28.$$
$$\therefore\ \underline{y=2.}$$

The operation above has been performed in order to "*eliminate*" (*i.e.* get rid of) x and so obtain an equation involving only one unknown, y

We can now obtain x by substituting 2 for y in either equation (i) or (ii).

From equation (i) we have

$$5x+6=17\tfrac{1}{4}.$$
$$\therefore\ 5x=11\tfrac{1}{4}.$$
$$\therefore\ \underline{x=2\tfrac{1}{4}.}$$

Testing our result by substituting in (ii) as we have already used (i), we have $2(\frac{9}{4})+4(2)=4\frac{1}{2}+8=12\frac{1}{2}$.

Hence coal cost £2, 5s. per ton and coke £2 per ton.

We could also obtain x by eliminating y from (i) and (ii) by multiplying (i) by 4 and (ii) by 3, and then subtracting.

Indeterminate Equations.—When we have only one equation involving two unknowns, we can obtain any number of values of x and y which satisfy it. If, for instance, $3x+2y=5$, then when $x=0$, $2y=5$ or $y=\frac{5}{2}$ When $x=1$, then $3+2y=5$ or $y=1$, and so on For any value of x we can obtain a corresponding value for y.

If, however, we are given $3x+2y=5$ and also $5x-3y=2$, there will be only one value of x and one value of y for which both equations are true. The equations are called **simultaneous equations,** because the same values of x and y must simultaneously satisfy both.

Example.—Given

$$3x+2y=5 \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$5x-3y=2 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

to find x and y.

Multiply both sides of equation (i) by 5 and both sides of equation (ii) by 3, then

$$15x+10y=25$$
$$15x-9y=6$$

Since the x and y in the first equation are to have the same values as in the second equation, we may subtract and so obtain

$$19y=19.$$
$$\therefore \underline{y=1}.$$

Substituting $y=1$ in equation (i), we have

$$3x+2=5.$$
$$\therefore \underline{x=1}$$

It will be found that $x=1$ and $y=1$ also satisfy equation (ii), for $5x-3y=5(1)-3(1)=2$.

$$\therefore \left.\begin{array}{l} x=1 \\ y=1 \end{array}\right\}.$$

Exercise VII (a)

N.B—The graphical method of solution will be found in Chapter VIII.

Solve the simultaneous equations :

1. $3x+5y=8.$ $4x+3y=7.$
2. $x+y=11.$ $x-y=3.$
3. $3x+4y=14.$ $4x-3y=2.$
4. $6p-7q=1.$ $11p-13q=1.$
5. $8x-6y=72.$ $3y=2x.$
6. $2x-3y=30.$ $5x+4y=6.$
7. $2x+y=9.$ $x-3y=1.$
8. $x+3y=-2.$ $4x+5y=13.$
9. $3x+8y=17.$ $2x-y=5.$
10. $9y-5z=17.$ $13z-2y=20.$
11. $12x+3y=6.$ $2x=14-7y.$
12. $12x=8y+4.$ $10y=3-x.$
13. $5u-7v=20.$ $9u-11v=44.$
14. $7x+4y=8.$ $9x-6y=1.$
15. $8p=6-9q.$ $3p=23+7q.$
16. $3x=7y+8{\cdot}5.$ $0{\cdot}5x+0{\cdot}2y+2=0.$
17. $3x+4y-6=0.$ $12x+5y+9=0.$

In the following examples, before attempting to solve the equations, both sides should first be multiplied by suitable numbers so as to obtain whole numbers for coefficients.

18. $\dfrac{x}{2}+\dfrac{2y}{5}=4.$ $\tfrac{2}{3}x+\dfrac{y}{2}=5\tfrac{1}{6}.$
19. $\dfrac{3x}{2}-\dfrac{y}{3}=4.$ $\dfrac{x}{3}+\dfrac{y}{4}=2\tfrac{5}{6}.$
20. $x=\tfrac{1}{2}(y-3).$ $y=\tfrac{1}{4}(x+4).$
21. $\tfrac{1}{3}(x-1)=\tfrac{1}{2}(y-1).$ $x-y=1.$
22. $\dfrac{x+2}{3}+8y=31.$ $\dfrac{y+5}{4}+2x=40.$
23. $\dfrac{x}{3}+3y=7.$ $\dfrac{4x-2}{5}=3y-4.$
24. $\tfrac{2}{7}(2x-1)=\tfrac{1}{3}(y+1).$ $\dfrac{5}{x+1}=\dfrac{4}{y-1}.$
25. $\dfrac{3x}{2}+\dfrac{3y}{2}=4x-y.$ $3x-2y=1.$

26. $24-3x=y=10+\frac{x}{2}$.

27. $\frac{x+2}{3}=\frac{y+2}{4}$.

$\frac{y-2x}{2y-3x}=4$.

28. $x-2y=1=3x+5y$.

29. $2x-3y=0=x+2y-4$.

30. $\frac{x-y}{2}=3=\frac{2x}{3}+\frac{y}{2}-1$.

Example 1.—The relation between the load (W) raised by an effort (P) applied to a set of pulleys is given by $P=aW+b$ where a and b are constants When $W=120$, $P=64$, and when $W=200$, $P=80$; determine a and b, and find what effort will raise 500 lb.

From the data we have

$$64=a(120)+b \quad . \quad . \quad . \quad \text{(i)}$$

and

$$80=a(200)+b \quad . \quad . \quad . \quad \text{(ii)}$$

These are two simultaneous equations in which the unknowns are a and b.

By subtraction $16=80a$. $\therefore\ a=\frac{1}{5}=\cdot 2$.

Substituting this value in (i), $64=24+b$. $\therefore\ b=40$.

Hence the relation is $P=\cdot 2W+40$.

When $W=500$ lb., $P=\cdot 2(500)+40=140$ lb.

Example 2.—Find a number the sum of whose digits is 11 such that the digits are interchanged when 27 is added to it.

Let x be the digit in the tens place and y the digit in the units place.

The number will be $10x+y$.

When the digits are reversed, the number will be $10y+x$

The sum of the digits is $x+y$.

Now $x+y=11$ and $10x+y+27=10y+x$.

$$\left.\begin{aligned}\therefore\ x+y&=11 \text{ (i)}.\\ 9x-9y&=-27 \text{ (ii)}.\end{aligned}\right\} \qquad \left.\begin{aligned}\therefore\ 9x+9y&=99.\\ 9x-9y&=-27.\end{aligned}\right\}$$

$$\therefore\ 18y=126.$$

$$\underline{y=7} \text{ and } \underline{x=4}$$

The number is therefore 47

Exercise VII (b)

1. An article in a magazine occupied $7\frac{1}{2}$ pages and contained 11,700 words. In large type a page contained 1200 words and in small type 1800 words. Of how many pages of each type did the article consist ?

2. A man, who went to business by tram 4 miles and by rail 10 miles, paid 1s. 7d. in fares. By another route, going 3 miles by tram and 12 miles by rail, the fare was 1s. 9d. What were the train and tram fares per mile ?

3. A submarine travelled at 20 knots (*i.e.* sea miles per hour) on the surface and at 14 knots when submerged. A journey of 65 sea miles was made in 4 hours. How much of the distance travelled was under water ?

4. A boy gives his age in years and months. Twice the number of months was 1 more than the number of years, and five times the number of years was 1 more than 9 times the number of months. Find his age.

5. The road from my house to the golf links is 5 miles in length and is all either uphill or downhill. My speed on a cycle is 8 miles per hour when going uphill and 12 miles per hour when going downhill. It takes me 30 minutes to ride out to the links. How long will it take me to return ?

6. In a school of 600 children, boys, girls, and infants, there are 18 more boys than there are girls, and the number of boys and girls together is double the number of infants. How many boys are there ?

7. For a certain machine the effort P required to lift a load W is given by $P=aW+b$, where a and b are constants. When $W=24$, $P=18$, and when $W=60$, $P=27$; find a and b. What is P when $W=100$?

8. The difference of the two digits of a number is 4. If the number itself is added to the number formed by reversing the digits the result is 110. Find the number.

9. Tea worth 2s. a lb. is mixed with another brand worth 2s. 8d. a lb. to make 56 lb. worth 2s. 6d. a lb. How much of each kind is taken ?

10. If $ax+by$ equals -12 when $x=3$, $y=-2$ and equals -4 when $x=-2$, $y=1$, find a and b and the value of $ax+by$ when $x=1$, $y=-1$.

11. A servant who is sent with £1 to buy 6 oz. of wool and 4 oz. of silk buys by mistake 4 oz. of wool and 6 oz. of silk. The change she brings home in consequence is 1s. instead of 5s. 3d. Find the cost of wool and of silk per oz.

12. A book of 2 volumes contains 1189 chapters. The number of chapters in the first volume is 111 less than 4 times the number in the second volume. Find the number of chapters in each volume.

13. A firm buys 200 tons of coal for £369. If some was bought at 36s. per ton and the remainder at 40s. per ton, how much was bought at each price?

14. A cattle show lasted for 6 days During the first two days the charge for admittance was 2s. 6d., and on subsequent days 1s. Altogether 56,000 people visited the show, and the total takings at the gates were £3160. How many visitors were there during the first two days?

15. A man invested £1200, partly at 4 per cent. simple interest and the rest at 5 per cent. If his income from the investments was £52, 16s., how much was invested at each rate?

16. A boy's bowling average was 21 If he had taken 7 more wickets and had 53 less runs scored off him it would have been 17. How many wickets did he actually take?

17. Bronze is an alloy of copper and tin. A cubic centimetre of copper weighs 8·8 grams, and a cubic centimetre of tin weighs 7·4 grams. If 100 c.c. of bronze weighs 857·6 grams, how much of each metal does it contain?

18. In a football match the winning side scored as many goals and tries respectively as the losing side scored tries and goals and won by 2 points. If one side had converted all the tries into goals and the other none, the difference would have been 6 points. Find the scores. (A goal $=5$ points, a try $=3$ points.)

19. A carrier charged 4d. each for all parcels not exceeding a certain weight. For heavier parcels an additional charge is made for each pound above that weight. A parcel weighing 10 lb. costs 7d. and the charge for 15 lb. was twice as much as for 9 lb. What was the scale of charges ?

20. The sum of the two digits of a number is 12. When 18 is added to the number the digits are reversed. Find the number.

21. A government office which employed 5 clerks of grade A and 6 of grade B paid £3, 10s more per week in wages in consequence of an increase in the salary scales Another office, employing 4 clerks of grade A and 8 of grade B, paid an extra £4 per week What was the weekly increase of salary for each grade of clerk ?

22. A man who smoked 2 oz. of tobacco and 50 cigarettes each week found that he could save 2d. a week by changing the brands bought. If his consumption had been 1 oz. of tobacco and 100 cigarettes his saving would have been 10d. What was the difference in the two prices paid for 1 oz. of tobacco and the two prices for 50 cigarettes ?

[Take x pence to be the saving on 1 oz. of tobacco, and y pence the saving on 50 cigarettes. One of your answers will be negative. What does this imply ?]

23. A traveller to Dublin had the choice of two routes—one 222 miles by rail and 56 miles by sea, the other 180 miles by rail and 64 miles by sea. A change in the rates charged made an increase of 6s. 11d. in the fare for the first journey and of 4s. 10d. for the second. What were the alterations in the fares per mile by rail and by sea ?

24. A man bought 20 shares in a rubber company and 50 oil shares. At the same time a friend bought 50 of the rubber shares and 10 oil shares. They sold out on the same date, the first man making £105 profit and the second losing £25. What change had taken place in the price of each kind of share ?

25. A motorist said that with one of his cars he averaged 40 miles per hour in the country and 12 miles per hour through towns, and that a journey to London took 4 hours. In another car his averages were 30 miles per hour and 10 miles per hour respectively, and the journey took 5 hours 12 minutes. How far was the journey ?

26. Tin appears to lose one-seventh of its weight when weighed in water. Lead appears to lose one-twelfth of its weight under the same conditions. An alloy of tin and lead which weighs 180 lb. in air appears to weigh only 160 lb when weighed in water. How much of each metal does it contain?

27. At an election the successful candidate had a majority of 88, but if 1 out of every 7 of his supporters had voted for his opponent he would have been in a minority of 18. How many votes were recorded for each candidate ?

28. A boy proposed to cycle a certain distance in 10 hours, but when he had gone half-way he increased his speed by 2 miles per hour and consequently arrived 50 minutes before he was expected. Find the distance and the rate at which he started.

29. The ratio of a number of two digits to that formed by reversing the digits is 4 7 If the units digits had been 3 larger, the ratio obtained would have been 3 . 8. Find the number.

30. The value of gold mined in the British Empire to the value of silver mined was in the ratio 7 . 1. In the rest of the world the ratio was 4 : 7, and for the whole world the ratio was 5 : 4. Find the ratio of the value of the gold mined in the British Empire to that mined in the rest of the world

Literal Equations. Change of Subject.—Sometimes letters instead of numbers are used in an equation as coefficients of x and also for the terms which do not contain x , the method of solution is in no way altered.

Example.—Solve $ax+b=cx+d$.

It is necessary to get all the terms which contain x on one side of the equation and the rest of the terms on the other side. If the equation were $5x+2=3x+4$ the procedure would be :

Subtract $3x$ and 2 from both sides of the equation, then

$$5x-3x=4-2.$$

So in the given equation we subtract cx and b from both sides.

Then $$ax-cx=d-b.$$

c x's are to be subtracted from a x's. $\therefore$ L.H S. $=(a-c)x$.

$$\therefore\ (a-c)x=d-b.$$

Divide both sides of the equation by $(a-c)$, then $x=\dfrac{d-b}{a-c}$.

Change of Subject.—The formula $l=a+(n-1)d$ gives the last term of a series of n numbers, each of which differs by d from the preceding one, a being the first term of the series. For instance 3, 5, 7, 9, 11, 13, 15, 17 is such a series in which $a=3$, $l=17$, $d=2$, $n=8$. From the formula we could find the 8th term if we knew only that $a=3$ and $d=2$, for

$$l=3+(8-1)2=17.$$

If it is required to find a formula for d, knowing a, n, and l, we change the subject of the formula from l to d by the same methods as those used in solving equations.

Since $$l=a+(n-1)d,$$
subtract a from both sides of the equation, then

$$l-a=(n-1)d.$$

Dividing both sides by $(n-1)$ we have

$$\frac{l-a}{n-1}=d.$$

If then $l=17$, $a=3$, $n=8$, $d=\dfrac{17-3}{8-1}=2$.

Example.—Change the subject of the formula $A=\pi r^2$ from A to r.

Since $$A=\pi r^2,$$

$$\therefore\ \frac{A}{\pi}=r^2.$$

Take the square root of both sides.

$\therefore\ \sqrt{\dfrac{A}{\pi}}=r$, which is a formula giving the radius of a circle in terms of the area.

Exercise VII (c)

Solve

1. $\dfrac{x}{3}=a.$ **2.** $\dfrac{x}{a}=b.$ **3.** $ax=b.$

4. $\dfrac{a}{x}=b.$ **5.** $2x-a=x.$ **6.** $3x+a=2x+b.$

7. $ax+b=c.$ **8.** $ax-b=c$ **9.** $\dfrac{x}{a}+\dfrac{x}{b}=1.$

10. $px-r=qx-s.$ **11.** $\dfrac{a}{x}=\dfrac{b}{c}.$ **12.** $a+\dfrac{x}{b}=c.$

13. $\dfrac{a}{x}+1=b.$ **14.** $\dfrac{x+a}{p}+x=b.$ **15.** $b-\dfrac{x}{a}=c+\dfrac{x}{d}.$

16. $a(x-b)=c.$ **17.** $(a+b)x=b-cx.$ **18.** $\dfrac{x}{a}-b=c.$

19. $p(a-x)=q(a-x).$ **20.** $\dfrac{a}{x}+\dfrac{b}{x}=c.$

In Examples 21–25, express y in terms of x given :—

21. $3x+5y=12$ **22.** $xy-6=0.$ **23.** $ax+by=c$

24. $x^2+y^2=4.$ **25.** $xy+x=a.$

26. Given $P=aW+b$, find W. **27.** If $u^2+2fs=v^2$, find s.

28. If $A=2\pi r^2+2\pi rh$, find h. **29.** If $W=\dfrac{2bt^2}{l}$, find t.

30. If $\dfrac{1}{v}+\dfrac{1}{u}=\dfrac{1}{f}$, make f the subject.

31. Express x in terms of y, regarding the other letters as constants.

(i) $xy=a$; (ii) $x+y=b$; (iii) $xy=x+y$;

(iv) $3ax+2by=c$; (v) $\dfrac{a-x}{a}=\dfrac{y-b}{b}$; (vi) $x/y=x+y$;

(vii) $\dfrac{1}{x}+\dfrac{1}{y}=2$; (viii) $x^2+y=a.$

32. Given $\pi x(x+y)=\text{A}$, find a formula for y.

33. Make t the subject of the formula $x=\dfrac{m(1-t)}{0{\cdot}4}$.

34. If $v_0(1+at)=v$, find a formula for a.

35. Make r the subject of the formula $\text{A}-\dfrac{\text{B}}{r^2}=e$.

36. Given $\text{A}=\pi r^2$ and $\text{C}=2\pi r$ Find r (i) in terms of A; (ii) in terms of C. Hence find (i) A in terms of C; (ii) C in terms of A.

37. The velocity v ft./sec. of a body thrown upwards with a velocity u ft./sec. is given after t seconds by the formula $v=u-gt$; write this with t as the subject.

Given $u=192$, $g=32$, find t when (i) $v=0$; (ii) $v=100$.

38. The Fahrenheit (F°) and Centigrade (C°) temperatures of a body are connected by the formula $5\text{F}-160=9\text{C}$ Express this with F as the subject For what temperature are the Fahrenheit and Centigrade readings the same?

39. Simple interest is given by the formula $\text{I}=\dfrac{\text{P}rt}{100}$. Write this with t as the subject.

How many years (t) will it take for the interest on a principal (P) of £450 to amount to £48 at a rate (r) of 4 per cent. per annum?

40. If $d=\sqrt{\dfrac{x-a}{x-c}}$, find x

41. Make P the subject of the formula $\text{W}=\sqrt{\dfrac{\text{P}}{(\text{P}-d)a}}$.

42. Find x if $\dfrac{2}{x}=\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}$.

Simultaneous equations with literal coefficients are solved by the same methods as those having numerical coefficients. It is often more convenient to find both unknowns independently than to use the process of substitution.

Example i.—Solve
$$px+qy=r \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$ax-by=c \quad . \quad . \quad . \quad . \quad \text{(ii)}$$
Multiplying both sides of equation (i) by b and both sides of equation (ii) by q, we get
$$bpx+bqy=br,$$
$$aqx-bqy=cq.$$
By addition $(bp+aq)x=br+cq$, whence $x=\dfrac{br+cq}{bp+aq}$.

Again multiplying both sides of (i) by a and both sides of (ii) by p we get
$$apx+aqy=ar,$$
$$apx-bpy=cp.$$
By subtraction $(aq+bp)y=ar-cp$, whence $y=\dfrac{ar-cp}{aq+bp}$.

Thus $x=\dfrac{br+cq}{bp+aq}$, $y=\dfrac{ar-cp}{bp+aq}$ is the required solution.

When the simultaneous equations contain reciprocals of the unknowns they may be solved as follows:

Example ii.—Solve
$$\frac{1}{x}+\frac{1}{y}=7 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$\frac{3}{x}+\frac{4}{y}=25 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$
Consider the unknowns to be $\frac{1}{x}$ and $\frac{1}{y}$ (write p for $\frac{1}{x}$ and q for $\frac{1}{y}$ if you like).

Multiply both sides of (i) by 3 and subtract (ii):
$$\frac{3}{x}+\frac{3}{y}=21.$$
$$\frac{3}{x}+\frac{4}{y}=25.$$
$$\therefore \quad \frac{1}{y}=4. \quad \therefore \quad y=\tfrac{1}{4}.$$
By substitution $x=\frac{1}{3}$.

The following example is added to show how the use of a little ingenuity will at times reduce the amount of working involved.

Example iii.—Solve $13x+19y=2$ . . . (i)

$19x+13y=62$ . . . (ii)

By addition $32x+32y=64$.

$\therefore\ x+y=2$ . . . . (iii)

Subtracting (i) from (ii), $6x-6y=60$.

$\therefore\ x-y=10$ . . . . (iv)

Working with equations (iii), (iv) we get by addition $2x=12$ and by subtraction $2y=-8$.

$$\therefore\ x=6,\quad y=-4.$$

Notes.—(1) A convenient form of giving the answer to a simultaneous equation is

x	6
y	-4

(2) The continental method of solution is to find one of the unknowns in terms of the other unknown.

Thus Example iii above would be solved by rewriting the equations (i), (ii) in the forms $y=\dfrac{2-13x}{19}$ and $y=\dfrac{62-19x}{13}$ and then solving the equation

$$\frac{2-13x}{19}=\frac{62-19x}{13}.$$

(3) In Literal Equations and Change of Subject it will be found useful at first always to give the argument for each step of the work, *e.g.* in solving $px+q=a-bx$.

$\therefore\ px+bx=a-q$ [Adding $bx-q$ to each side.]

$\therefore\ (p+b)x=a-q$ [Collecting like terms.]

$\therefore\ x=\dfrac{a-q}{p+b}$ [Dividing both sides by $(p+b)$.]

Avoid the use of "getting bx to the other side."

Exercise VII (d)

Solve the equations

1. $\frac{3}{x} + \frac{4}{y} = 3.$
$\frac{6}{x} - \frac{2}{y} = 1.$

2. $\frac{1}{x} + \frac{3}{y} = \frac{4}{3x} + \frac{2}{y} = 1.$

3. $x - ay = b.$
$x + by = a.$

4. $ax + by = a + b.$
$x + cy = 1 + c.$

5. $6x + \frac{12}{y} = 5x + \frac{40}{y} = 45.$

6. $ax = by.$
$x + y = 1/a + 1/b.$

7. $px + qy = pq.$
$qx - py = q^2.$

8. $ax + by + c = px + qy + r = 0.$

9. $\frac{3}{x+y} - 3 = \frac{1}{x-y}.$
$\frac{5}{x+y} - 1 = \frac{3}{x-y}.$

10. $\frac{8}{x} + \frac{5}{y} = 31.$
$\frac{5}{x} - \frac{3}{y} = 1.$

11. $(a+b)x - (a-b)y = a^2 + 2ab - b^2.$
$bx + ay = 0.$

12. $\frac{x}{a} + \frac{y}{b} = a - b.$
$ax - by = a^3 + b^3.$

13. $a(x+y) = b(x-y) = 2ab.$

14. $qx + py = 2p.$
$pq(x-y) = p^2 - pq.$

15. Rewrite the equation $y = \frac{x-a}{x-b}$ so that x is the subject.

16. In $\frac{1}{R} = \frac{a}{b} - 1$ make R the subject

17. If $\sqrt{x^2 + a^2} = b$, find x.

18. If $y = \sqrt{\frac{x-a}{bx}}$, find x.

19. Given $ar = \sqrt{\frac{y-b}{y+c}}$, make y the subject.

20. If $\frac{1}{\ } = \frac{1}{\ } - \frac{1}{\ }$, make v the subject.

CHAPTER VIII

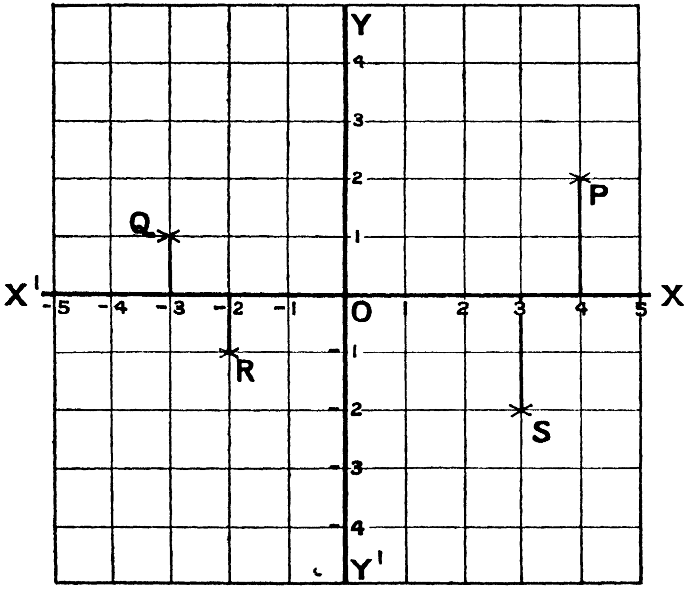
Co-ordinates: Graphs

THE position of any point on a plane is determined if we know its distances from two fixed lines on the plane (generally taken at right angles to one another)

These lines are called axes, OX being the X-axis and OY the Y-axis.

Their point of intersection, O, is called the *origin*.

Equal distances are marked off along these axes and numbered positively when they are to the right along OX or upwards along OY, and negatively when they are to the left along OX′ or downwards along OY′.



P is 4 units from OY and 2 units from OX : these distances are called the *co-ordinates* of P.

It will be seen that we can arrive at P from O by moving along OX to 4 and then upwards through 2 units parallel to OY.

A point which has p and q for its co-ordinates is called the point (p, q). Thus P is the point (4, 2). Note that the x co-ordinate (*i.e.* the one parallel to OX) is always written first.

Q is the point (−3, 1), R is (−2, −1), S is (3, −2).

The resemblance of this method to that employed in the index of a geographical atlas should be noticed.

Exercise VIII (a)

(Graphical)

1. A is the point (10, 6). Join A to the origin O and bisect OA at right angles. Read off the co-ordinates of the points where this bisector cuts the axes.

2. Draw a circle with centre at the origin and radius 10 units (take 2 small divisions to represent the unit). Two lines are drawn parallel to OY and 6 units from it, one on each side of O. What are the co-ordinates of the points where they cut the circle ?

3. The plan of a field, ABCDE, is drawn by joining the points A (12, 11), B (17, −1), C (−6, −19), D (−11, −7), E (−5, 10). A tree is known to stand where the line BD cuts the line AC. What are the co-ordinates of the position of the tree ?

4. The 3 points A (7, 7), B (21, 7), C (2, −2) are 3 corners of a parallelogram ABCD. Find the co-ordinates of D if AD is parallel to BC What is the area of the parallelogram ?

5. A quadrilateral, ABCD, has vertices at A (6, 10), B (13, −3), C (−2, −5), and D (−5, 3). Through A and C lines are drawn parallel to OX and through B and D lines parallel to OY. What is the area of each of the four triangles which now appear on the figure ? What is the area of the quadrilateral ABCD ?

6. The vertices of a triangle are at A (2, 3), B (4, 7), C (8, 5).

Through the vertices draw AL, BM, CN perpendicular to OX. Find the areas of the trapeziums ALMB, BMNC, ALNC, and deduce the area of the triangle ABC.

Graphs of Statistics

The information provided by a table of statistical results can often be shown more clearly by "plotting" the data on squared paper The diagram so obtained is known as a graph.

Example.—The amount of coal exported from this country during the period 1913–1925 is given in the table below The figures are given to the nearest million tons.

Year . . .	1913	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923	1924	1925
Exports in millions of tons . .	94	79	57	51	46	41	47	39	36	81	97	80	67

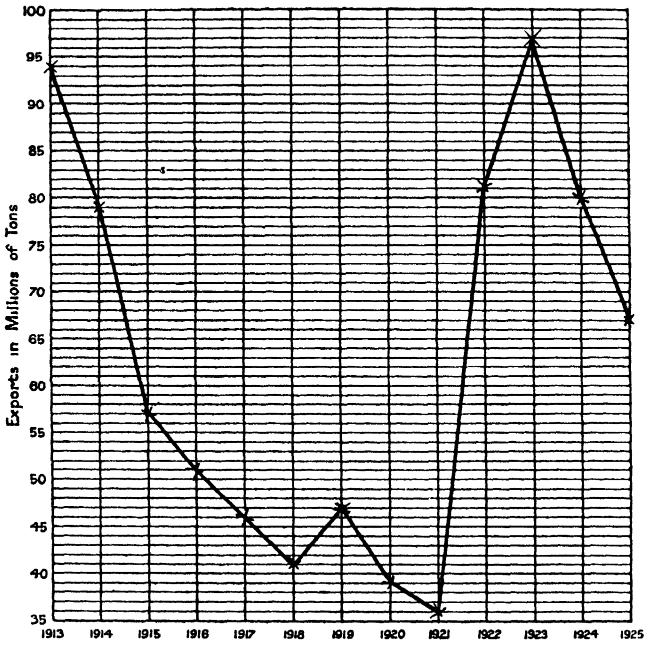
The amounts exported range between 36 millions and 97 millions, *i.e.* a difference of 61 millions. For the sake of preserving round numbers it is convenient to arrange to represent from 30 millions to 100 millions If the size of our paper permits, we take 10 divisions to represent 10 millions , if not, we can take 5 divisions to represent 10 millions.

The dates are represented along the "horizontal" axis, the amounts exported along the "vertical" axis.

For any year (say 1915) we place a small cross on the vertical line denoting that year at a point level with the number of millions, viz. 57, exported in that year.

The graph shows at once a rapid fall in exports during the War years, 1914–1918, and the failure to return to the normal during the first few years of peace , a sudden rise during the years 1921–1924 when there were stoppages in the German coalfields, followed by another drop.

The amount exported in different years depends on many circumstances and there is no means of predicting the increase



Graph showing the exports of coal from 1913–1925.

or decrease from year to year. In such cases it is usual to join the points by *straight* lines, but these are to be interpreted only as helps for making the changes more conspicuous to the eye.

Note.—It is most important to select scales so that the values can be read at a glance. The sides of the large squares on the paper are usually divided into 5 or 10 subdivisions. Each such subdivision should represent 1 unit or some simple multiple of the unit. It would be unnecessarily inconvenient, for instance, to take 10 divisions to represent 12 millions instead of 10 millions. Graphs should always be drawn as large as the paper will allow consistently with the condition that the scales chosen are reasonable. At first they should be made to fill most of the sheet.

Always state what the numbers on the axes represent.

Exercise VIII (b)

Represent the following statistics graphically. The points shown may be joined by straight lines.

1. The value of the coal exported from Great Britain was

Year . .	1916	1917	1918	1919	1920	1921	1922	1923
Millions of £'s	51	52	52	92	100	43	73	100

Plot a graph, taking 1 in. along OY for every 10 million pounds from 50 to 100 Is it possible to estimate from the graph the value of the coal exports in 1924 ?

2. The mean temperature in degrees Fahrenheit for the different months of the year are as shown.

Jan.	Feb.	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec.
39	39	42	47	53	59	63	62	57	50	44	40

At what times of the year does the mean temperature (i) increase, (ii) decrease, most rapidly ? Plot the months along OX.

3. The "index prices" of meat and sugar for the period 1919–1925 are given below (The price in 1913 is taken as 100 in each case)

Draw the two graphs on the same axes. At what periods was the percentage rise in each the same ?

Year .	1919	1920	1921	1922	1923	1924	1925
Meat .	214	264	220	185	161	159	163
Sugar .	300	432	162	152	203	197	157

4. The world's gold production in millions of pounds at different dates was as given below.

Year . . .	1912	1914	1916	1918	1920	1922	1924
Value of gold produced	95 9	90·4	93 5	79 0	69 3	65 5	80 0

In what period was the decrease in the amount produced most rapid ?

5. An explorer crossing an unknown river takes soundings of its depth at intervals of 10 yd From the results given below draw a plan of the section of the river at the place of crossing

Distance from bank in yards . .	10	20	30	40	50	60	70	80	90	100
Depth in feet .	4	7	9	13	25	21	17	11	5	3

6. Compare graphically the Birth Rates and Death Rates in England and Wales for the dates given below.

Year . .	1912	1914	1916	1918	1920	1922	1924
Birth rate .	23 9	23 8	20 9	17 7	25 5	20 4	18 8
Death rate .	13 3	14 0	14 4	17 6	12 4	12 8	12 2

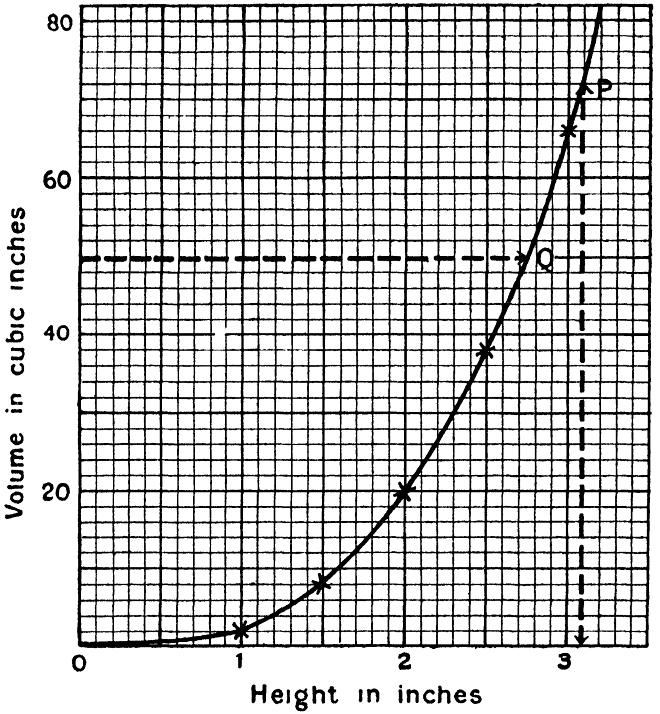
7. Represent the following running records graphically :

Distance in yards .	100	120	150	200	440	600	880	1760
Time in seconds .	$9\frac{3}{5}$	$11\frac{1}{4}$	$14\frac{1}{5}$	19	47	$70\frac{4}{5}$	$112\frac{1}{5}$	$250\frac{2}{5}$

Show that the lines joining each point to the origin represent the *average* speeds.

Example 1.—A set of flower-pots of the same shape but of different sizes hold V cu in. of earth when the height is h in given by the following table .

Height (h) .	1	1 5	2	2 5	3
Volume (V) .	2 5	8 4	20	39	67



Volume and height of flower-pots

From the nature of the question we see that when $h=0$ then $V=0$.

In this example the height and the volume of the pots are two quantities which vary, hence they are called *variables.* The volume depends upon the height selected, and is called the *dependent* variable.

The dependent variable should always be plotted along OY.

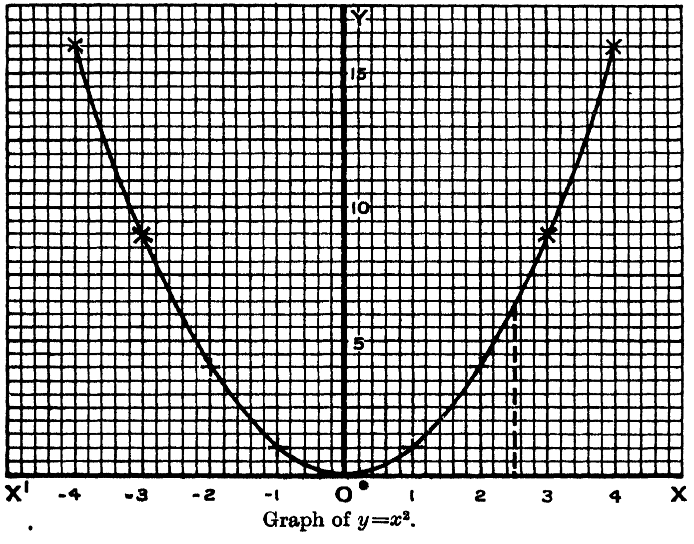
Here the two variables are connected by a law, so that it would be possible to find the volume corresponding to any given height of pot. In such cases the points plotted should be joined by a smooth curve.

From the graph find (i) the cubic contents of a pot 3 1 in. deep, and (ii) the depth of a pot holding 50 cu in.

From the point P on the graph we see that when $h=3\ 1$, $V=72$, and from the point Q we have $h=2{\cdot}75$ when $V=50$

Example 2.—Draw a graph to show the squares of all numbers from -4 to $+4$

Numbers	-4	-3	-2	-1	0	1	2	3	4
Squares	16	9	4	1	0	1	4	9	16



Graph of $y=x^2$.

Since the square of a number is never negative, the scale for the numbers, viz. -4 to $+4$, may be taken at the bottom of the paper.

If y represents the square of x, then $y=x^2$; this relation is called the equation of the curve given by the graph. In this case there is a definite law connecting the two variables x and y, and consequently the points of the graph should be connected by a smooth curve. Any square not actually plotted, e g. $(2{\cdot}5)^2$, can be read off from the graph by considering the point on the graph whose x is $2{\cdot}5$. The y of this point is $6{\cdot}25$. Thus $(2{\cdot}5)^2=6{\cdot}25$.

Points to be remembered when drawing graphs ·

1. Write your name on the paper.
2. Select scales that are easily read. They need not extend much beyond the range of the variables given in the question.
3. The independent variable (*i.e.* nearly always the one whose values in the table increase by equal amounts) should be represented on the horizontal axis.
4. The graph should cover most of the paper used.
5. Mark the units on the axes and state what they represent.
6. Mark the given points with a small ×.
7. Draw the curve neatly in pencil.
8. If the shape of the curve between two points is at all doubtful, calculate the position of an intermediate point, when this is possible.
9. Print a title to the graph.

Exercise VIII (c)

1. An electric lamp at a height H cm. above the centre of a desk provides at the blotting-pad an amount of light L given by

H . .	20	30	40	50	60	75	90	100
L .	57	79	96	107	113	115	112	106

Plot these, taking 1 in. for 10 units on both axes, and from the graph determine the best height for the lamp

2. A rectangular flower-bed of area A sq. ft. is to be laid out against the wall of a garden If the width parallel to the wall is x ft., what length of wire-netting will be required to protect the three exposed sides ? Taking A to be 200, draw a graph to show the value of your expression for values of x ranging from 5 to 50, and discover what width will enable you to purchase the netting at least cost.

3. When £100 is invested at 5 per cent. compound interest, the amounts to which it accumulates at different periods is

Years . .	0	5	10	15	20	25	30	35
Amount in £'s .	100	128	163	208	265	339	432	552

Plot these values, taking 5 years to the inch along OX and £100 to the inch along OY. Read off the amount at the end of 18 years and the number of years required for £100 to exceed £300.

4. The distances in feet travelled by a train in t seconds after the application of the brakes are

"t" in seconds	0	10	20	30	40	50	60	70	80
Distance in ft	0	350	680	990	1280	1550	1800	2030	2240

How long will the train take to run a quarter of a mile ?

5. The following table shows the expectation of life for males :

Age in years . .	20	30	40	50	60	70	80
Expectation in years .	39	32	25	19	13	8	5

From a graph discover how long a man aged 62 years may expect to live, and at what age the expectation of life is a further 20 years.

6. The number of cubic feet of water left in a tank t seconds after the opening of an escape tap is given by the formula $\left(8-\dfrac{t}{10}\right)^2$. Write down the values of this expression when $t=10$, 20, 30, etc. Draw a graph to show the volume of water left in the tank for intervals of 10 seconds after the opening of the escape tap. What is the original amount of water in the tank, and how long is required to empty it? At what moment was it half full?

7. A small water-trough for a bird-cage is to be made out of a rectangular sheet of lead 20 cm long and 10 cm broad. Four equal squares are cut off from the corners, and the side pieces left are then bent up so as to form the vertical sides of the trough. Calculate the volume of the trough when each side of the squares is 1 cm, and obtain the corresponding values when the sides of the squares are 1·5 cm., 2 0 cm, 2·5 cm., etc.

Draw a graph to illustrate your results What is the largest amount of water that such a trough can contain, and what is the length cut off at each corner in this case?

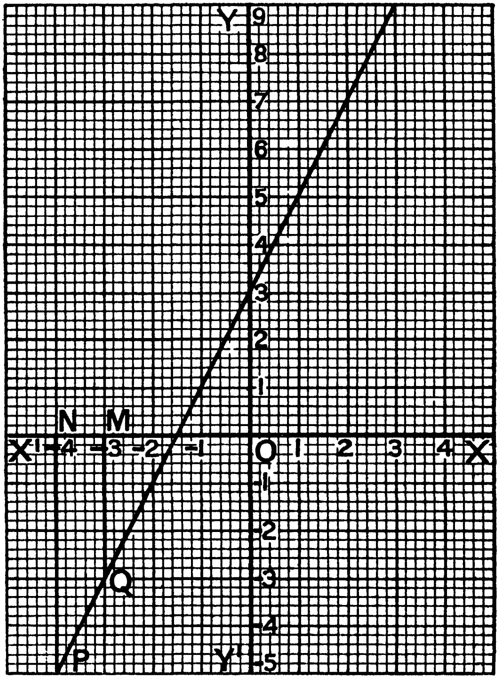
Functions.—An expression whose value depends on the value of x is called a *function* of x. Thus $3x+5$, $2x^2-7x$, $\dfrac{4x-5}{2x+3}$ are all functions of x. The value of the expression itself is often denoted by y When the law of a function is known, we can plot its graph by giving x a series of numerical values and obtaining the corresponding values of y. The number of values to be given to x will depend on the nature of the curve thus obtained. Care should be taken so that the points through which the graph has to be drawn are not too far apart.

Two cases of especial importance are the functions whose laws are of the form $ax+b$ and ax^2+bx+c The former is called the *Linear Function* of x, and is shown in works on Algebraic Geometry to be always represented graphically by a straight line The latter, known as the *Quadratic Function*, gives a curve which is called a Parabola (see, for instance, the graph of $y=x^2$ on p 101) One example of a parabola is the path of a stone thrown up into the air.

Graph of 2x+3.—Make a table of corresponding values for x and $2x+3$ thus :

x	-4	-3	-2	-1	0	1	2	3
$2x+3$	-5	-3	-1	1	3	5	7	9

Here x ranges from -4 to 3 and $2x+3$ ranges from -5 to 9. The axes should be drawn in such a position on the paper that all these values can be shown. (In most cases of graphs you will *not* put your origin in the middle of the paper.)



NP(-5) represents the value of $2x+3$ when $x=-4$.
MQ represents the value of $2x+3$ when $x=-3$.

It will be found that all the points so obtained, such as P, Q, etc., lie on a straight line. For every point on it the y co-ordinate equals twice the x co-ordinate $+3$

If $2x+3$ is called y, then $y=2x+3$ is the *equation* of the straight line shown in the graph.

Exercise VIII (d)

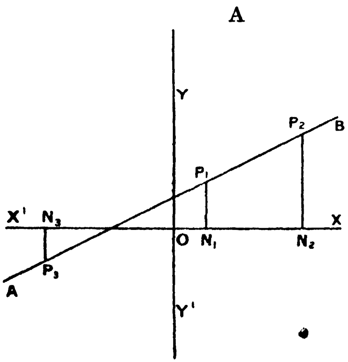
1. Draw on the same axes the graphs of $3x$ and of $3x+4$, from $x=-5$ to $x=+5$. What resemblance is there between them? What does your graph give for the value of $3x+4$ when $x=2{\cdot}3$?

2. Draw a graph of $y=\frac{x}{2}-3$ from $x=-4$ to $x=+4$ How could you now draw also the graph of $y=\frac{x}{2}$ without calculating any values of x?

3. Draw a graph of $2x-5$ from $x=-4$ to $x=+4$ What is the increase in $2x-5$ between $x=-1$ and $x=3$?

4. If 1 in. is taken to be 2 5 cm., how many centimetres are there in x in? If this number is called y, write down the relation between y and x. Draw a graph for converting inches into centimetres.

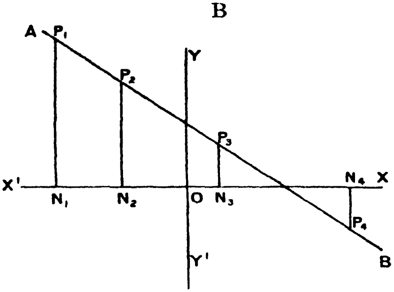
5. If F° Fahrenheit is the same temperature as C° Centigrade it is known that $F=\frac{9}{5}C+32$. Draw a graph for converting from one scale to the other. From it read off 10 4 C° in F°. What Centigrade temperature $=0°$ F.?



6. The line AB is the graph of $\frac{x}{2}+1$.

If $ON_1=1$, what is N_1P_1?
If $ON_2=4$, what is N_2P_2?
If $ON_3=-4$, what is N_3P_3?

7. The line AB is the graph of $-\frac{2x}{3}+2$.



If $ON_1=-4$, find N_1P_1.
If $ON_2=-2$, find N_2P_2.
If $ON_3=1$, find N_3P_3.
If $ON_4=6$, find N_4P_4.

8. Given 1 kg. $=2\cdot2$ lb., draw a graph showing the relation between pounds and kilograms from 1 kg. to 5 kg.

If there are y lb. in x kg., what is the relation between y and x? What is the equation of your graph? From it find (i) number of kilograms in 8 lb.; (ii) number of pounds in 3·4 kg.

9. Show that 60 miles per hour $=88$ feet per second. If x is a speed in miles per hour and y the same speed in feet per second, find the relation between x and y. Draw a graph to show this relation for speeds from 10 miles per hour to 100 miles per hour.

What is the equation of the graph? From it convert 35 miles per hour into feet per second and 60 feet per second into miles per hour.

10. Draw a graph which will reduce marks running from 123 to 317 to a range of 0 to 100. You may assume it will be a straight line. What ranged mark will correspond to 200?

11. Draw a graph that will give the results of dividing any number from 1 to 100 by 5·9. (Note $\frac{1}{5\cdot9}=0\cdot17$ approx.) From it find $\frac{74}{5\cdot9}$.

12. A function is equal to $2x$ for values of x between 0 and 5, to $10-x$ for values between 5 and 10, and to $2x-10$ for values between 10 and 15. Draw its graph, taking one small division of your paper as the unit.

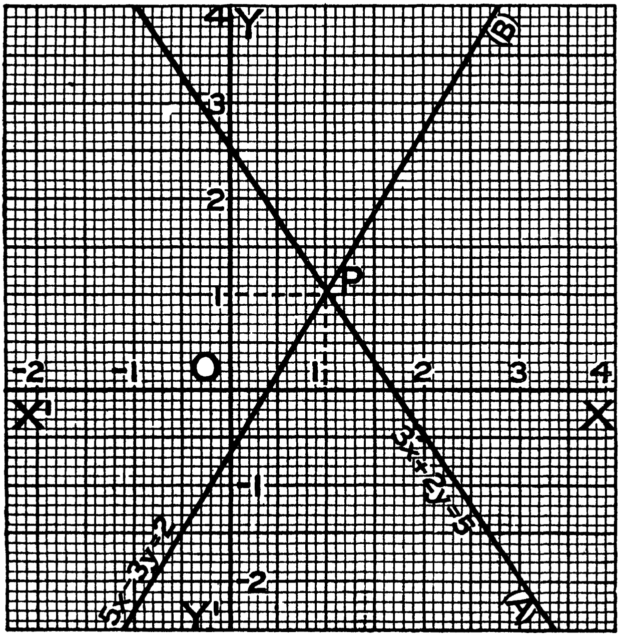
[Such curious functions as this do actually occur in practical problems.]

Graphical Solution of Simultaneous Equations.

Example.—Solve $3x+2y=5$, $5x-3y=2$ graphically.

We have seen that the single equation $3x+2y=5$ is satisfied by any number of values of x and y. If some of these values are obtained by taking $x=-2$, -1, etc , and finding the corresponding values of y, the results can be tabulated thus :

x	-1	0	1	2	3
y	4	$2{\cdot}5$	1	$-0\ 5$	-2



Similarly for $5x-3y=2$ we get

x	-2	-1	0	1	2
y	-4	$-2\frac{1}{3}$	$-\frac{2}{3}$	$+1$	$+2\frac{2}{3}$

Now plot both equations on the same axes.

The co-ordinates of all points on line A satisfy $3x+2y=5$, nd of all points on line B satisfy $5x-3y=2$.

$\therefore$ the co-ordinates of the point P where the lines meet nust satisfy both equations

Thus the required solution is $x=1$, $y=1$.

Note.—To tabulate the values quickly it is sometimes a con-enience to rewrite the equations in the forms $y=\dfrac{5-3x}{2}$ and $=\dfrac{5x-2}{3}$ before substituting for x.

Exercise VIII (e)

Note.—Plot the following from $x=-2$ to $x=3$.

1. Draw the graphs of $y=2x-4$ and $5x+2y=13$ on the ame axes, and find where they intersect. [Take one large livision of your paper to represent the unit along each axis.]

2. The graphs of $5x-6y=5{\cdot}4$ and $4x+5y=12$ are drawn o a scale of 1 in to the unit on each axis. What are the o-ordinates of the point at which they cut ?

Solve graphically the following simultaneous equations :

3. $3x-5y=3{\cdot}5$.
$2x+3y=7{\cdot}4$.

4. $\dfrac{x}{2}+\dfrac{y}{3}=1$.
$x=2y-3$.

5. $2x+y=2$.
$4x-3y=10$.

6. $3x+5y=10$.
$x-\frac{3}{4}y=1$.

7. $3x+2y=10$.
$y=\frac{7}{5}(x-1)$.

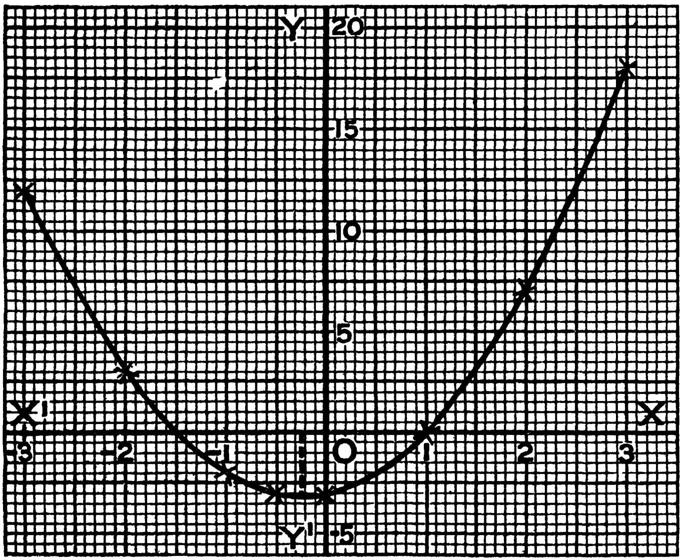
8. $6x=4y+5$.
$x+y+2=0$.

For further examples, *see* Exercise VII (a).

Graph of $2x^2+x-3$.—From $x=-3$ to $x=+3$.

x x^2	-3 9	-2 4	-1 1	0 0	1 1	2 4	3 9
$2x^2$ x -3	18 -3 -3	8 -2 -3	2 -1 -3	0 0 -3	2 1 -3	8 2 -3	18 3 -3
$2x^2+x-3$ or y	12	3	-2	-3	0	7	18

Here we have to represent values of $2x^2+x-3$ from 18 to -3, so that it is convenient to take a smaller scale for $2x^2+x-3$ than for x.



Graph of $2x^2+x-3$.

Let y represent the values of $2x^2+x-3$. From the symmetry of the graph we see that the lowest value of y occurs when x is *less* than 0

To get a more accurate drawing we also calculate y when $x=-\frac{1}{4}$ and $x=-\frac{1}{2}$.

We find that when $x=-\frac{1}{4}$, $y=-3\frac{1}{8}$; and when $x=-\frac{1}{2}$, $y=-3$: so that $-3\frac{1}{8}$ is the *least* value of y.

As x increases from large negative values to $x=-\frac{1}{4}$, y diminishes. As x increases from $-\frac{1}{4}$, y increases.

N.B.—There are two values of x for which $2x^2+x-3=0$, viz. $x=1$ and $-1{\cdot}5$. These are called the *roots* or *solutions* of the equation $2x^2+x-3=0$. There are also *two* values of x which make $2x^2+x-3$ have a value such as 10. These values are called the *roots* of the equation $2x^2+x-3=10$.

N.B.—A useful check for the numerical values of an expression in which the values of x increase by regular steps is to subtract each value of the expression from the preceding one.

Thus, the values found for $2x^2+x-3$ are

$$\begin{array}{ccccccccccccccccc}
25, & & 12, & & 3, & & -2, & & -3, & & 0, & & 7, & & 18, & & 33 \\
& 13 & & 9 & & 5 & & 1 & & -3 & & -7 & & -11 & & -15 & \\
& & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & &
\end{array}$$

If the calculated values are correct, the numbers in the third line should all be the same when the function is a quadratic one. For a linear function the numbers would be found to be all equal in the second line.

Exercise VIII (f)

1. Make up a table of the values of $3x^2+4x-2$ from $x=-3$ to $x=+3$.

2. Make up a table of the values of $4x^2-6x+7$ from $x=-4$ to $x=+4$.

3. What is the value of x^2-4x+3 when $x=0$ and when $x=2$? Show from your results that the equation $x^2-4x+3=0$ has a root between 0 and 2.

4. Draw a graph of the values of $3-4x+2x^2$ from $x=-3$ to $x=+3$. What is the least value of the expression? For what values of x will the expression equal 5?

5. Draw a graph of x^2-3x+1 from $x=-3$ to $x=5$. For what values of x will the expression equal zero?

6. Find from a graph the values of x for which $6x-x^2$ is negative.

Find for what values of x, $6x-x^2=2$ and $6x-x^2-4=0$.

7. With the same axes draw the graphs of $2x+1$ and x^2. From your graph state the values of x which make $x^2=2x+1$.

8. A cricket-ball thrown in the air describes a curve given by $y=x-\dfrac{x^2}{120}$. Calculate y when $x=0$, 20, 40, 60, etc., and draw a graph. What is the greatest height the ball reaches? Where does it hit the ground?

9. Draw the graph of $y=x^2-3x-2$ for values of x between -1 and $+4$, taking 1 in to represent unity on each axis. From your graph read off (i) the greatest negative value of y; (ii) the values of x which make $x^2-3x=3$.

10. If $(x+3)^2=5(y+2)$, express y in terms of x, and draw its graph for values of x between -8 and $+2$.

11. The resistance (R) to a train's motion in pounds per ton is given by $R=6+0{\cdot}01V^2$, where V is the speed in miles per hour. Draw a graph from $V=5$ to $V=40$, and from it read off the value of V when $R=20$.

Revision Exercise VIII (g)

1. Plot on the same axes the graphs of $5y=2x+7$ and $y=1{\cdot}5x-2$ for values of x from -1 to $+3$ Show how your graphs can be used to solve the simultaneous equations $5y-2x=7$, $1{\cdot}5x-y=2$

2. When a weight W lb. is hung on the end of an elastic string 3 ft long, the string stretches to a length l ft. given by $W=\frac{2}{3}(l-3)$. Draw a graph showing the relation between W and l for weights not greater than 6 lb. What is the stretched length when the weight is $1\frac{3}{4}$ lb., and what weight would stretch it an additional foot?

3. Plot the graphs of $\dfrac{2x-5}{3}$ and $\dfrac{7-3x}{4}$ on the same axes. From your graph read off the values of x for which $\dfrac{2x-5}{3}-\dfrac{7-3x}{4}$

is equal to (i) 0; (ii) $+2$; (iii) -2 Check your results by calculation.

4. An expression is equal to $2x$ when x lies between 0 and 2 (inclusive), but to $x^2-7x+14$ when x lies between 2 and 5 (inclusive), and to $14-2x$ when x lies between 5 and 7. Draw its graph between $x=0$ and $x=7$, and find the values of x for which it is equal to 3.

5. Draw the graph of $2-\left(4x-\dfrac{3x^2}{5}\right)$ between $x=-1$ and $x=7$. Hence solve the equation $2-4x+\dfrac{3x^2}{5}=0$ For what values of x does $4x-\dfrac{3x^2}{5}$ equal 4 ?

6. Show the graphs of $3x^2-7x$ and $x+3$ on the same axes. From your figure discover the values of x for which

(i) $(3x^2-7x)-(x+3)=0$; (ii) $(3x^2-7x)-(x+3)=2$.

7. If $x=3p-2$ and $y=7-2p$, find the values of x and y as p changes from 1 to 5, and show graphically that the points represented by the connected pairs of values of x and y all lie on a straight line. What is its equation ?

8. Draw the graphs of x^2 and $\dfrac{48}{3x+2}$ for values of x between 0 and 6. For what value of x in this interval are the expressions equal in value ?

9. On the same axes draw the graphs of x^2+3x and $1+x+2x^2$. Show that the two curves *touch* one another, and find the values of x for which the difference of their ordinates is $\frac{1}{2}$.

10. Calculate the values of $\dfrac{1}{x}$ when $x=\cdot001$, $\cdot01$, $\cdot1$, 1, 10, 100, 10,000. Plot the values of $\dfrac{1}{x}$ from $x=1$ to $x=5$. What do you consider will be the values of $\dfrac{1}{x}$ (i) when $x=0$; (ii) when $x=\infty$?

11. Without drawing a graph show that $2x^2-\cdot84x-\cdot693=0$ has a root between 0·84 and 0·83.

12. Find the values of $x^2-2\cdot82x+1\cdot98$ when $x=1\cdot3$ and when $x=1\cdot4$. Plot these values on a large-scale graph showing only these two points. Assuming that for this magnification the curve between these points is practically a straight line, find the approximate value of the root of $x^2-2\cdot82x+1\cdot98=0$ which lies between $1\cdot3$ and $1\cdot4$.

13. Draw the graph of $\dfrac{x^3-13x}{4}$ between the values -4 and $+4$ of x, and use it to solve the equation $x^3-13x=8$.

14. Draw a graph to show the values of $\dfrac{(x+2)(x-1)(3-x)}{4}$ between $x=-4$ and $x=+4$. What values of x make the expression have the same value as (i) x; (ii) $6-2x$?

Miscellaneous Examples III

A 37

1. Divide $8a^2b-10ab^2$ by $-2a$, and $6a^3b+9ab^3$ by $-3ab$. Subtract your second result from the first.

2. Find the values of $3p^3+5p^2$, $3p^3\times5p^2$, $3p^3\div5p^2$, $(3p+5)^2$ when $p=-4$

3. Solve the equations $\left.\begin{aligned}3x+7y&=4.\\7x+3y&=16.\end{aligned}\right\}$

4. Simplify $\dfrac{x-3a}{a}-\left(\dfrac{3x-a}{2a}-\dfrac{x+a}{3a}\right)$. Is there any value of x for which the result would be equal to $\frac{2}{3}$?

5. Rewrite $p+q-r+t$ with (i) brackets round the last three terms, and (ii) brackets round the last two terms. Test each result by putting $p=5$, $q=6$, $r=3$, $t=1$.

6. It costs 16s. 8d. to go a journey of 120 miles partly by car at 2d. a mile and partly by train at $1\frac{1}{2}$d a mile. How much of the journey is made by car and how much by train ?

A 38

1. Find the value of $(a-2b)(c-3d)$ when $a=5$, $b=-3$, $c=2$, $d=-4$

2. Find the H.C.F. and L.C.M. of $12a^3b^2c^4$ and $20a^4b^2c^3$, and show that the product of the H.C.F. and L.C.M. is equal to the product of the two numbers themselves.

3. Simplify (i) $1-\dfrac{x+y}{2x}-\dfrac{x+2y}{3x}$; (ii) $\dfrac{p-q}{2r}-\dfrac{p-2q}{3r}$.

4. The expression ax^2-bx is equal to 40 when $x=4$, and to 16 when $x=-2$, find the values of a and b.

5. Find the values of the letters shown in the diagram if each line and each diagonal add up to the same number.

1	21	20	a
16	b	x	13
c	14	15	7
19	3	2	d

6. A manufacturer stocks a number of grinding wheels all of the same thickness. Their diameters in inches and weights in pounds are given in the table below.

Diameter .	3	4	5	6	8	10	12	14
Weight .	1 72	3·00	4 80	6 70	12 6	19·5	28·2	37·4

Draw a graph to show the relation between the weight and the diameter, and read off from it the weights of wheels whose diameters are 7 in. and 11 in.

A 39

1. Evaluate $\dfrac{a^2}{b}+\dfrac{c^2}{d}$, $\dfrac{(a+c)^2}{b+d}$, $\dfrac{2a^3}{b^2}-\dfrac{c^3}{2d^2}$, $\dfrac{(a-c)^3}{b^2-d^2}$ when $a=3$, $b=-2$, $c=-4$, $d=1$.

2. A substance which lo... x oz. in every pound of its weight on being dried finally weighs y lb. What was its weight before being dried ?

3. A, B, C, D are four points in order on a straight line such that AB$=1$, BC$=2$, CD$=3$. The direction from A to D being regarded as positive, how would you describe the distance (i) BD; (ii) CA ? Prove that AB$+$BC$+$CA$=0$. Can you invent any illustration to explain this result ?

4. Express $2ax-3by+9ay-4bx$ in terms of two expressions in brackets (i) with coefficients 2 and 3; (ii) with coefficients a and b.

5. A poulterer sells 6 ducks and 15 chickens for £7, the price of each duck being 1s 2d. more than that of a chicken. How much does he charge for each ?

6. A, B, C are at (2, 3), (5, 7), (9, 1) respectively. Through A a line DAP is drawn parallel to OY, and through B a line PBQ parallel to OX; through C, two lines CD, CQ are drawn, CD being parallel to OX and CQ parallel to OY. A rectangle PQCD is thus formed What is its area ? What are the areas of (i) the triangles PAB, QCB, ADC, and (ii) the original triangle ABC ?

A 40

1. Divide $12a^2b^5-8a^5b^2$ by $-4a^2b$, and add your result to the product of $2a^2-3ab$ and $-3a^2$.

2. Show that the equation $6x^2-13x-5=0$ is satisfied both by $x=2\frac{1}{2}$ and by $x=-\frac{1}{3}$.

3. A boy obtains 5 marks for every sum which he gets right at first trial, 2 marks for every sum which he gets wrong at first but afterwards corrects, and loses 3 marks for every sum left uncorrected. If he tries x sums and gets y of them correct at first and afterwards corrects half of the others, how many marks will he get ?

4. Find the values of x and y from the two equations

$$5-\frac{2x}{7}=\frac{x-y}{3}, \quad \frac{2x+3y}{8}=6-\frac{3x-2y}{5}.$$

5. Which of the fractions $\frac{ab}{bc}$, $\frac{a+b}{b+c}$, $\frac{ab+bc}{bc}$, $\frac{a+b}{bc}$ can be expressed in simpler form? Find the value of each when $a=3$, $b=2$, $c=5$.

6. A class of boys cut a series of regular hexagons out of cardboard and find their weights The results are

Length of side in cm.	4	4 5	5	6	6 5	7	8
Weight in grams	26	32 9	40 6	58 5	68 7	79 6	104

Draw a graph to show the relation between these results, and read off (i) the probable weight of a hexagon whose side is 5·5 cm., and (ii) the side of a hexagon whose weight is 90 grams.

A 41

1. Find the values of a^2-b^2, $(a-b)^2$, $(a+b)^3$ when $a=-2$, $b=-5$.

2. A train goes x miles in y hours; how far would it go in 2 hours less at the same speed?

3. What value of x will make $\frac{1}{3}(x-2)-\frac{1}{5}(x-6)$ three times the value of $\left\{\frac{x-4}{4}-\frac{x-8}{6}\right\}$?

4. Simplify $a \div \frac{b}{c} + c \div \frac{b}{a}$ and $\left(a+\frac{b}{c}\right)\div\left(c+\frac{b}{a}\right)$.

5. The straight line joining the points (5, 2) and (x, -2) cuts the axis OX in the point (2, 0) By drawing a figure find the value of x and prove your answer correct by the help of congruent triangles.

6. One of two persons is a years old and the other is b years younger. How many years must elapse before the elder is twice as old as the younger? What does your answer become (i) when $a=28$, $b=16$; and (ii) when $a=37$, $b=14$? Explain your result in the latter case.

A 42

1. When $a=x$, $b=-x$, $c=x^2$, express the values of $(a\div b)\times c$, $a\div(bc)$, $a-b\div c$, $(a-b)\div c$, $a-b^3\div c$ in their simplest forms.

2. Simplify $\dfrac{b-a}{ab}-\dfrac{b-c}{bc}+\dfrac{a+c}{ac}$.

3. Bracket together the terms divisible by 2 and the terms divisible by 3 in the expression $3x-2y+4z-9t$. Check your answer by putting $x=5$, $y=-3$, $z=-1$, $t=2$.

4. Solve the equations $\begin{cases} \dfrac{x+1}{2}-\dfrac{y+3}{5}=3. \\ 3x+y=1. \end{cases}$

5. The equation $\dfrac{\mathrm{A}}{x-5}-\dfrac{\mathrm{B}}{x+2}=7$ is satisfied both when $x=8$ and when $x=-3$. Find A and B.

6. If $x=3a^2p^3$, $y=2ap^2$, show that the value of $\dfrac{ay^3}{x^2}$ does not depend on the values of a and p.

A 43

1. If $xy+2yz-3zx=x+2y-3z$, find (i) x when $y=3$, $z=-2$, and (ii) y when $x=-1$, $z=-3$.

2. Solve the equations

(i) $3x-10-\frac{1}{4}(x-6)=\frac{1}{3}(5x+4)-\frac{1}{12}$;

(ii) $2x-3y=3x-2y=1$.

3. After paying income tax on his salary at 4 shillings in the pound a man's net income is £A. What was his salary?

4. Find the sum of $\dfrac{1}{2pq}$ and $\dfrac{1}{3p^2}$ as a single fraction, and rewrite $\dfrac{2p^2-3q^2}{6pq}$ as the difference of two separate fractions.

5. The area of an equilateral triangle whose side is l can be proved to be $\dfrac{l^2\sqrt{3}}{4}$. A regular hexagon, each of whose sides is a, has its three alternate sides produced so as to form an equilateral triangle. What are the areas of this triangle and of the three smaller triangles so formed ? Deduce the area of the original hexagon.

6. A firm sends out 2000 circulars, paying $\frac{1}{2}$d. postage each for some of them and $1\frac{1}{2}$d. each for the others. If their stamp bill came to £10, how many of their circulars were posted with $\frac{1}{2}$d. stamps ?

A 44

1. Write down two common multiples of $2a^3b$ and $3a^2b^3$, and two common factors of $10x^2y^3$ and $15y^2z^3$. What is the L C M of these four expressions ?

2. Find the value of $a-b+2\left\{\dfrac{a^2-(b+c)^2}{a+b+c}\right\}$ when $a=2$, $b=-1$, $c=-2$.

3. Solve the equations

$$\text{(i) } \tfrac{1}{6}(2x-3)-\tfrac{1}{3}(4x-19)=5x-\tfrac{9}{10}(3x+8);$$

$$\text{(ii) } 7x-10y=5, \quad \frac{3x+1}{4}-\frac{4x-2y}{7}=x-y.$$

4. If $v^2=u^2+2fs$, express f in terms of the other letters and find its value if $v=12$, $u=20$, and $s=8$.

5. A cross Channel passage consists of two stages which are supposed to be made at 15 knots and 12 knots respectively, in which case the time taken over the whole distance would be 7 hours. Owing to bad weather the speeds over the two stages are reduced to 12 knots and 8 knots and the steamer is consequently $2\frac{1}{4}$ hours late in arriving at its destination. Find the distances of the two sections of the journey. [1 knot=1 sea mile per hour.]

6. A is the point $(-6, 3)$ and B is $(10, -9)$. What are the co-ordinates of the mid-point of AB ? If with this point as centre a circle is drawn of radius 10, at what points will it cut (i) the line $x=8$, (ii) the line $y=5$?

A 45

1. What is the value of $\frac{p}{q}+\frac{q}{r}-\frac{r}{p}$ when $p=-2$, $q=3$, $r=-1$?

2. Simplify

$x+(-x)$, $x+(-x)^2$, $x\times(-x)$, $x\div(-x)^3$, $(-x)^2-(-x)^3$.

3. A rectangular field is a yd. long and b yd. broad. A path everywhere x ft wide is constructed round the *outside* of the field. Draw a rough diagram and calculate the area of the path in square feet.

4. Three places, A, B, C, lie on a straight road, A being 15 miles east of B and C 20 miles west of B. A man starting from A at 9 a.m. cycles towards C at 10 miles per hour. How far (i) east of B; (ii) west of B; (iii) west of C, will he be at 11.30 a.m.?

5. The plan of an orchard can be drawn by joining in order the points (10, 3), (5, 8), (−2, 3), (−5, −7), and (10, −2). Draw it on squared paper, taking each small division of your paper as the unit. What is the area of the estate if each small division represents 10 yd? See p. 95, Ex. 5.

6. After a meeting a silver collection was made to defray the expenses. Each person present gave either half a crown or a shilling. There were 53 persons present and the collection amounted to £4. How many gave half-crowns?

A 46

1. Find the value of $\frac{6-7x-20x^2}{4x+3}$ when $x=-1{\cdot}1$.

2. The distance between the two points whose co-ordinates are (x_1, y_1) and (x_2, y_2) can be proved to be $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. What is the distance between (5, −2) and (−4, 10)?

3. Solve the equation $\frac{ax}{b}=c-\frac{(c-a)x}{b}$.

4. Find the value of p if the three equations $0{\cdot}2x-0{\cdot}3y=-2$,

$\frac{x}{4}+\frac{y}{5}=6\cdot7$, and $\frac{y+x}{y-x}=p$ are all satisfied by the same pair of values for x and y.

5. If $x-y=3$ and $x-z=1$, what are the values of $(x-y)(x-z)$; $y-x$, $(z-x)^2$, $(y-x)(z-x)$, and $y-z$?

6. A car bought originally for £450 loses £120 of its sale value during the first year of use. For the next 10 years it deteriorates by £20 a year and subsequently by £10 a year. Draw a graph to show its value during 15 years of use When could it first be bought for (i) less than £150, (ii) less than £100 ?

A 47

1. Find the difference between $a-2b\div b+3c$ and $(a-2b)\div(b+3c)$ when $a=5$, $b=-2$, $c=1$.

2. Solve the equations

(i) $2y-\frac{3y+5}{4}=\frac{7y-4}{5}$; (ii) $2x-\frac{y}{3}=12$, $3x=\frac{2y}{3}+11$.

3. Express $20\left\{1-\frac{x-2}{5}\right\}-12\{2-\frac{3}{4}(x+1)\}$ in its simplest form, and test your answer by putting x equal to 7.

4. n lines parallel to one edge of a sheet of paper are ruled y in apart, beginning and ending 1 in. from the edge Find the length of the sheet.

5. A housekeeper buys 17 lb. of cooking apples and 9 lb. of eating apples for 11s. 3d. Had she bought 13 lb. of the former and 7 lb. of the latter the cost would have been 8s. 8d. What was the price of each kind per lb.?

6. A battleship B steaming b miles per hour pursues an enemy vessel C which is d miles ahead and is steaming at c miles per hour. If B's guns can hit at a distance of h miles, find in what time C will be within range of B.

A 48

1. Remove the brackets from $p(5a-2b)-q(5x-2y)$, and express your answer in the form 2(   )−5(   ).

2. The diameter AB of a circle is produced to O. If $AO=x$ $BO=y$, find the distance from O of the centre of the circle

3. If $(x+y)(x-y)=7$ and $(x-y)=3$, find the values of x and y.

4. A party of boy scouts on arriving at camp find that the number of tents is too few by a to allow 3 boys to a tent while if 4 boys are put in each tent there will be b tents left empty. How many tents were there and what was the number of scouts ?

5. Find a value of x such that $2x-1$, $5x-7$, $21-2x$ may be in ascending order of magnitude. Is there more than one such value ?

6. The horse-power obtainable for 3d. an hour with single and triple cylinder engines is given below .

Price of coal per ton .	20	30	40	50 shillings.
H.P. (single cylinder) .	11·2	7·5	5·6	4·4
H.P. (triple cylinder) .	18 6	12 4	9 3	7·5

Draw a graph showing the relation between horse-power and the cost of coal for the two cases At what price per ton does the triple cylinder give 4 H.P. more than the single cylinder ?

B 49

1. A boy buys p articles at q pence each. How many of the articles could he have bought for the same money if each had cost r pence more ?

2. Find the value of $\dfrac{5x-4}{1+3x}-\dfrac{4x-3}{3x-1}$ when $x=-\frac{2}{3}$.

3. Simplify $3x^5\times 5x^3$, $4x^8\div 2x^4$, $x^a\times x^b$, $x^{n^2}\div x^n$, $\sqrt[2]{(x^{4n})^3}$.

4. If $A=2h(l+b)$, express l in terms of A, h, b Verify your answer by putting $A=600$, $h=10$, $b=12$, both in the original formula and in your answer.

5. A non-stop train runs from B to L (120 miles), passing through R which is 80 miles from B. At t minutes after noon it will be $(\frac{5}{8}t-30)$ miles *beyond* R. How far is it from R at 1 p.m.? At what times did it (i) pass through R ; (ii) leave B ; (iii) arrive at L ?

6. Tabulate the values of $2x+3$ and $3x-2$ for values of x between -2 and $+3$. Deduce the values of $(2x+3)(3x-2)$ and $\frac{2x+3}{3x-2}$ between the same limits.

B 50

1. The sum of the first n odd numbers is equal to n^2 Show that this is true in the case (i) when $n=3$, (ii) when $n=7$.

2. Find the values of $(x+3)^{x+2}-(x+2)^{x+3}$ when x has the values -1, 0, $+1$.

3. At the present time B is twice as old as A. Prove that when A is as old as B is now, B's age will be only half as much again as A's.

4. If $y=\frac{a+3}{2}$, express the value of $\frac{y-2a}{a-2y}$ in its simplest form.

5. If $x>1$ prove that $x^2>x$. Is there any value of x for which x^2 is less than x? Find a value of x for which $(x-2)^2<(x-2)$.

6. A seaside golf course offers daily tickets and monthly tickets to holiday visitors. A visitor who played 3 days a week for 4 weeks would save 6 shillings by taking a monthly ticket instead of daily tickets. If he played only twice a week over the same period he would waste 4 shillings by taking the monthly ticket. What is the price of each kind of ticket ?

B 51

1. If $a=3$, $b=-2$, and $a+b+c=0$, show that

(i) $a^2-b^2+c^2=abc$ and (ii) $a^3+b^3+c^3=3abc$.

2. A rectangular field is p yd. long and q yd. broad. A path everywhere x ft. wide is constructed round the inside of the field. Draw a rough diagram and calculate the area of the path in square feet.

3. By how much are the first and last of the three expressions $3a-2b$, $4b-a$, $b-4a$ greater than the average of the three? Verify by putting $a=6$, $b=5$, and explain your result.

4. Which of the fractions $\frac{pq}{qr}$, $\frac{p-q}{q-r}$, $\frac{pq-qr}{pq}$, $\frac{p-q}{rp}$ can be expressed in simpler form? Find the value of each when $p=3$, $q=-2$, $r=-1$.

5. A and B, touring by car between two towns, choose different routes. A travelling by the eastern route at 20 miles per hour takes half an hour longer over the journey than B, who goes by the western route at 24 miles per hour. If the routes chosen had been interchanged B would have beaten A by 1 hour 36 minutes. What is the length of each route?

6. The relation between the reading (F°) of a Fahrenheit thermometer and that of a Centigrade thermometer is $C=\frac{5}{9}(F-32)$. Draw a graph for converting C into F and read off

(i) the Fahrenheit temperature corresponding to 64° C.;
(ii) ,, Centigrade ,, ,, 80° F.

B 52

1. Simplify a^2+2x^2, $(a+2x)^2$, a^2-2x^2, a^3-x^3, $(a-x)^3$ when $x=-a/2$.

2. If $p-q=3$ and $y-z=2$, find the values of $(p-q)(z-y)$, $(q-p)^2$, $(z-y)^3$, and $\frac{2q-2p}{(3z-3y)^2}$.

3. The expression ax^3+bx^2 is equal to -40 when $x=-2$, and to 45 when $x=3$. What is its value when $x=1$?

4. Clear the equation $\frac{1}{u}+\frac{1}{v}=\frac{1}{3}$ of fractions, and hence express v in terms of u. Find the value of v when $u=12$, both from the given relation and from your first answer.

5. One year the expenses of a factory for raw material and labour came to £50,000. In the following year the amount paid for raw material diminished by 10 per cent., while the cost of labour increased by 20 per cent. The saving on the year in consequence was £2000. How much was paid for raw material and for labour in each year ?

6. If $\frac{x-a}{x-b}=\frac{1}{2}$ and $\frac{x-b}{x-c}=\frac{2}{3}$, prove that $a+c=2b$.

B 53

1. If $z=(x-3)(y+2-)(x+2)(y-3)$, find the value of z when $x=-4$ and $y=-3$. What value of y will give z this same value when $x=-2$?

2. At the end of a year a boy is given x pence for every shilling he has saved during the year. After this he possesses £y. What was the original amount of his savings in pounds ?

3. If a point is taken on each side of a figure of n sides, and these points joined in every possible manner by straight lines, the number of parts into which the figure is divided is given by the expression $1+\frac{n(n-1)}{24}\{n^2-5n+18\}$. What is the number of parts when the original figure is (i) a triangle ; (ii) a quadrilateral ? Verify by drawing a figure for each case.

4. Rewrite the expression $\frac{3(5-x)}{(2+x)}+\frac{2(1-x)}{(2-x)}$, so that in each bracket the x term comes before the numerical term.

5. A map is drawn to show the ups and downs of a switch-back road between two places A and F each 100 ft above sea-level B (1, 300) ; C ($3\frac{1}{2}$, -250) ; D (4, -150) ; E (5, -100) represent the tops and bottoms of the intermediate hills, referred to AF as horizontal axis and A as origin. The first co-ordinate is measured in miles and the second in feet. Draw a sketch on squared paper, taking 1 in. to the mile horizontally and $\frac{1}{2}$ in. to each 100 ft. vertically. Which is the steepest section of the road ?

6. A sum of money is expressed in shillings and pence Show that if the difference between it and the new amount obtained by interchanging the numbers which express the shillings and the pence is given in pence, the number so obtained is always divisible by 11.

B 54

1. Find the value of $\dfrac{3x^2-7xy-6y^2}{2x^2-7xy+3y^2}$ when $x=-1{\cdot}5$, $y=-2$.

2. Simplify $\dfrac{a^3b^2}{4}\left(\dfrac{3x}{a^2b}-\dfrac{5y}{ab^2}\right)-\dfrac{ax^2}{4}\left(\dfrac{3b}{x}-\dfrac{ay}{x^2}\right)$.

3. If £a will buy p articles, how many shillings does each article cost? Write down an expression for the number of such articles which can be bought for £a when the price of each has gone up b shillings. Verify your answer by putting $a=12$, $p=16$, $b=5$.

4. Bracket together the odd terms and the even terms in the expression $\dfrac{2x}{3}-\dfrac{3y}{4}-\dfrac{5z}{6}+\dfrac{7w}{8}$, giving your answer in the form $\frac{1}{6}(\quad)-\frac{1}{8}(\quad)$.

5. A train which departs at $2x$ minutes past x o'clock and arrives at x minutes past $2x$ o'clock is 4 hours 55 minutes on the road. What is the value of x?

6. What are the values of a and m if the equation $y=mx+\dfrac{a}{m}$ is satisfied both when $x=2$, $y=3$, and when $x=3$, $y=4$?

B 55

1. It is known that an equation whose graph is a straight line can always be written in the form $y=ax+b$. Find in this form the equation of the line which joins the two points $(3, -2)$ and $(-4, 7)$.

2. Simplify

$$5y^3\times 3y^2,\quad py^q\times qy^p,\quad (-2x^3)^3,\quad (-x^4)^{2n},\quad \sqrt[3]{64x^{27}}.$$

3. Solve the equations $\dfrac{x}{2y+3}=5$, $\dfrac{y}{2x+3}=\dfrac{1}{4}$.

4. If $T=R\left(\frac{l-a}{a}\right)$, find an expression for a in terms of R, l, and T. What is its value when $l=4$, $R=\frac{1}{3}$, $T=2$?

5. An aeroplane on its first trial over a measured course completed the distance in 2 hours, which implied a speed of 5 miles per hour less than that expected by the makers. On a second trial, after slight alterations, it travelled the course in $1\frac{1}{2}$ hours, and the calculated speed in consequence proved to be 25 miles per hour better than before What were the length of the course and the speed originally expected ?

6. A series of square table-covers is made from material costing 3 shillings per *square* yard. Each is provided with a fringe costing 1 shilling per linear yard. What is the total cost of such a cover whose side is x yards in length ? Draw a graph for values of x increasing by $\frac{1}{4}$ at a time from $x=1$ to $x=3$, and read off from your figure the costs of a cover (i) $1\frac{1}{2}$ yards square ; (ii) $2\frac{1}{2}$ yards square.

B 56

1. Subtract $\frac{7}{5}(a-2b)-\frac{2}{3}(a+2b)$ from $\frac{2}{5}(a-2b)+\frac{1}{3}(a+2b)$.

2. If the expression $x^3-(x-1)^3$ is denoted by the symbol $f(x)$, find the value of $f(5)-f(1)$. Find also the value of $f(2)-f(-2)$.

3. Solve the equations

$$2x-3y=(a+b)c, \qquad \frac{x}{a}+\frac{y}{b}=\frac{c}{6}.$$

4. Given $x-3y=1$, express y in terms of x and deduce the value of $\frac{x-2y}{x+2y}$ in terms of x, giving your answer in its simplest form.

5. A body moving in a straight line has travelled a distance $(15t+3t^2)$ ft at the end of the first t seconds of its motion. How far does it travel (i) in 2 seconds , (ii) in 3 seconds , (iii) *during* the third second ? Show that during the 6th second it travels twice as far as during the 2nd second.

6. ABCD is a parallelogram, and P is the meet of the diagonals. If B is at $(-2, 5)$, C at $(6, -1)$, and P at $(2, -2)$, what will be the position of A? [Remember that the diagonals of a parallelogram bisect one another.] What is the position of D?

B 57

1. Simplify $\left(\frac{1}{2a^2} - \frac{9}{ab}\right) \div \left(\frac{-3b}{a}\right)$.

2. If $x-y=a$, $x+y=b$, and $y=px$, express p in terms of a and b.

3. Three armies A, B, C are making arrangements for a combined offensive, and agree to commence operations at different times, based on the hour at which B will be ready to start, which is known as the zero hour. On this scheme A is timed to begin at -15 hours 20 minutes and C at $+9$ hours 30 minutes. If B decides that the zero hour shall be 5.30 a.m. on 1st August, at what times do A and C begin their moves?

4. A machine bought for £A deteriorates in value by £30 per year during the first 10 years after its purchase and by £15 during each year afterwards. What is the value of the machine (i) 7 years; (ii) 13 years after its manufacture?

Find a formula for its value after n years (i) when n is less than 10; (ii) when n is greater than 10.

5. Rewrite the two equations $v=10(2-f)$ and $v^2=20f$, so that f may be the subject of the sentence in each of the two new equations. If the two equations are true simultaneously, give in a form cleared of fractions an equation from which v could be found. [You are not asked to solve it.]

6. The height of a cricket-ball t seconds after it leaves the thrower's hand is given by the formula $a.t+b.t^2$. Find the values of a and b if the ball reaches a height of 64 ft. after 1 second and 96 ft. after 3 seconds. What will be its height after $2\frac{1}{2}$ seconds, and when will it reach the ground?

B 58

1. The equation of the line which joins the two points $(x_1 y_1)$ and $(x_2 y_2)$ is known to be $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$. What does this become when $x_1=3$, $y_1=-5$, and $x_2=-1$, $y_2=-7$? Express your result in as simple a form as possible.

2. The area of a trapezium is 12 sq. in. and one of the parallel sides is $3\frac{1}{2}$ in. If the other parallel side were increased by 1 in the area would be increased to $13\frac{1}{2}$ sq in. Find the height of the trapezium and the length of the longer parallel side. [Area of trapezium $=\frac{1}{2}h(a+b)$.]

3. If $x-y=2(y-z)$ and $y+z=3(z-x)$, find y and z in terms of x and show that $4z=5y$.

4. If $\dfrac{2ax+b}{ax^2+bx+a}$ is equal to 2 when $x=\frac{1}{2}$, show that it will be equal to $\frac{1}{2}$ when $x=2$ and to $-\frac{1}{3}$ when $x=-3$.

5. The circumference of a circle is given by the formula $2\pi r$ and its area by the formula πr^2. What are the radius and area of a circle whose circumference is a in.?

If a piece of paper a in. long by b in broad is rolled up so as to form a cylinder, what will be the volume of this cylinder? Show that there are two possible answers, and find the difference between them when $a=5$, $b=4$. [Take $\pi=\frac{22}{7}$.]

6. The amount of useful work done by a water-wheel is measured by $240n-\dfrac{5n^3}{2}$, where n is the number of revolutions made by the wheel in 1 minute. Draw a graph of the expression for values of n between 0 and 10, and find the number of revolutions per minute when the wheel is doing most work.

B 59

1. Simplify $\frac{1}{2}(2nD-x)-\frac{1}{3}\{nD+(-1)^nx\}$, where $D=180$ and $x=60$ in the two cases (i) when $n=1$; (ii) when $n=2$.

2. A traveller is paid a fixed salary of £400, and also receives 15 per cent. on the amount by which his total sales exceed £500. If he sells goods worth £x (where $x>500$) in any one year and receives £y pay, find the relation between x and y.

3. In the series 135, 132, 129, . . . each number is 3 less than the preceding number. Find an expression for the nth number, and verify your answer by putting $n=5$. What is the 50th number ?

4. Solve the equations

$$\text{(i) } \frac{x}{2}-3=\frac{5}{2}-\frac{3x}{5}; \qquad \text{(ii) } \frac{x}{a}-b=\frac{c}{a}-\frac{bx}{c}.$$

5. If the average value of x shillings and y pennies is 10d., find x in terms of y, and deduce that the average value of y shillings and x pence would be 3d.

6. A and B are x miles apart and are riding towards one another A is riding a miles per hour and they meet in $\dfrac{x}{a+b}$ hrs How long would A have to ride before he caught B if they had both been riding in the same direction ? $(a>b)$

B 60

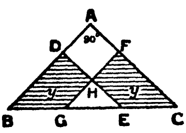
1. By putting $\dfrac{1}{7x+y}=a$, and $\dfrac{1}{3x-y}=b$, solve the equations

$$\frac{5}{7x+y}-\frac{3}{3x-y}=1, \qquad \frac{2}{7x+y}+\frac{7}{3x-y}=25,$$

finding the values of a and b and deducing the corresponding values of x and y.

2. Find by trial the integral values of x less than 30 for which $7x+1$ is a perfect square. Is there any such value for which $5x+1$ is also a perfect square ? [An integer is a whole number.]

3. The nth term (T_n) of a series of numbers is given by the formula $T_n \equiv n^2-1$. Write down the first five numbers of the series. Is the $(2n)$th number double the nth number ?



4. In the diagram AB=AC=12, GF is parallel to AB and DE parallel to AC at equal distances x from AB and AC respectively. Express BD, GF, GH in terms of x, and then obtain expressions in terms of x only for each of the four areas shown. Draw a graph showing y in terms of x, and find for what value of x, y is a maximum.

5. O, A, B are 3 points on a straight line such that $OA=2$, $AB=3$. Find the positions of two points P, Q on the line which are such that $\frac{AP}{BQ}=2$ and $PQ=5$.

6. A man has £p deposited in one banking account and £q in another If he wishes to withdraw a total sum of £r, how much should he withdraw from each account so that (i) the amounts left in the two accounts may be equal, (ii) the amount left in the first may be half that left in the second ?

CHAPTER IX

Binomial Expressions : Factors : Remainder Theorem

Column Addition and Subtraction.—In the preceding chapters easy cases of multiplication and division have been of frequent occurrence and have been treated by first principles and in connection with the interpretation of brackets. In dealing with more complicated examples it is convenient to adopt a formal arrangement of the work similar to that employed for long multiplication and division in arithmetic.

Addition by Columns.—When several expressions, each containing a number of terms, are to be added, they may be written in lines under one another as in arithmetic, the like terms being placed in the same columns.

Thus : add $2x^2+x+3y-2$, $-3x^2+2y+1$, $4x+y+y^2$.

$$\begin{array}{rrrrr} 2x^2 & +\ x & +3y & -2 & \\ -3x^2 & & +2y & +1 & \\ & 4x & +y & & +y^2 \\ \hline -x^2 & +5x & +6y & -1 & +y^2 \\ \hline \end{array}$$

The answer could be rewritten so that the terms of higher degree come first, viz. $-x^2+y^2+5x+6y-1$

Subtraction by Columns.—As we have seen in Chapter III, the result of subtracting $2a-b$ from $3a+2b$ may be written $(3a+2b)-(2a-b)$, and this is equal to $3a+2b-2a+b$ or $a+3b$.

The negative sign in front of the second bracket causes a change of sign in each of its terms.

If we arrange the work in columns as in arithmetic, we must therefore change mentally the signs of each term of the lower line and then proceed as in addition.

$$\begin{array}{ll} \text{Thus} & 3a+2b \\ & 2a-\ b \\ \hline \text{Difference} & a+3b \end{array}$$

In order to subtract the lower line from the upper we change the $2a$ mentally into $-2a$ and add $3a$ to $-2a$. So also $-b$ is changed mentally into $+b$ and $2b$ added to $+b$.

Exercise IX (a)

Add together, employing the "Column Method":

1. $a-b,\ \ a+b,\ \ 2a+b,\ \ a-2b.$
2. $3x-2y,\ \ x+y,\ \ 2x-y,\ \ x-3y.$
3. $2a-b,\ \ -a+2b,\ \ 3b-2a,\ \ b+a.$
4. $2x+y-4,\ \ x-3y+2,\ \ 3x-y+1.$
5. $x^2+x+y+2,\ \ 2x^2-2y+1,\ \ x-3y+2.$
6. $3x^3-2x^2+4,\ \ x^3+x-5,\ \ 2x^2-2x.$

Find the sum of the following expressions:

7. $-(a-b)+3,\ \ -a+2(b-2)$
8. $a-2(b-c),\ \ 3(a-2c)+b,\ \ -(a-2b)+c$
9. $x^2-x(y+z),\ \ xy-x(z-x).$
10. $rp+q(r-s),\ \ -r(p-q)+qs,\ \ rq-p(s-r)$

Subtract both by the "Column Method" and the "Bracket Method":

11. $x-y$ from $x+y$. **12.** $2x-y$ from $x+2y$.
13. $a-b+c$ from $a+b-c$. **14.** $2x-3$ from $-x+4$.
15. $1-x$ from $x-2$. **16.** $2x^2-x+1$ from x^2+2x-2.
17. $2a-3b+c$ from $a+2b-3c$.

Subtract, employing the "Column Method" only:

18. $a-b$ from $b-c$. **19.** $p-q+r$ from $2p+q$.
20. x^2-2x+1 from $2x^2$ **21.** $x-(y+z)$ from $(2x-y)-z$.
22. $-(a-b)+3$ from $-a+2(b-1)$.
23. $x^2-x(y+z)$ from $xy-x(z-x)$.

In the following examples find the remainder when the second expression is subtracted from the first :

24. $2m(n-t)+nt$ and $mn-2t(m+n)$.

25. $\frac{2}{3}a-\frac{1}{2}(b-c)$ and $\dfrac{c}{3}-\frac{1}{2}(a-b)$

26. $-\frac{1}{2}(a-b)+\dfrac{c}{3}$ and $\dfrac{a}{3}-\frac{1}{2}(b-c)$

27. $\frac{3}{4}x-\frac{1}{2}(y+z)$ and $\dfrac{y}{3}-\frac{1}{2}(z-x)$

Multiplication by a Single Term.

Multiply $2x^2+4x-4$ by $-3x^2$

$$\begin{array}{r} 2x^2+4x-4 \\ -3x^2 \qquad\quad \\ \hline -6x^4-12x^3+12x^2 \\ \hline \end{array}$$

This is the same as $-3x^2(2x^2+4x-4)$.

Division by a Single Term.

Divide $4x^3-6x^2y-12x$ by $-2x$.

$$\begin{array}{r|l} -2x) & 4x^3-6x^2y-12x \\ \hline & -2x^2+3xy+6 \\ \hline \end{array}$$

Since $-2x(-2x^2+3xy+6)=4x^3-6x^2y-12x$

Exercise IX (b)

Multiply

1. $1-x+x^2$ by x^2.

2. $a+b-c$ by abc

3. $a^2+b^2-c^2$ by ab.

4. x^2+2x+1 by $-3x$.

5. $-x^2+x-1$ by -1.

6. x^n by x, x^{n+1} by x, x^{n-1} by x.

7. x^p+x^q by x, $x^{p-1}+x^{q-1}$ by x^2, x^m+x^n by x^r.

Divide

8. x^3-x^2+x-1 by -1.

9. a^2b-ab^2+ab by ab.

10. $4x^2y-6xy^2+8xy$ by $-2xy$.

11. $12a^2b-6ab^2+9b^3$ by $3b$.

12. x^n by x, x^{n-1} by x, x^{n-1} by x^2.

13. x^p by x^2, x^p by x^q. **14.** x^p+x^q by x, x^m-x^n by x^2.
15. x^m+x^n by x^r, x^a by $-x$.
16. x^a-x^b by $-x^2$. **17.** $x^{n+2}+x^{n+1}$ by x^n.

Multiplication by a Binomial Expression

A Binomial expression is one which contains two terms, *e.g* $2x+3y$. An expression consisting of one term only, *e.g.* $3xy$, is called a Monomial.

In the previous set of examples on multiplication our multipliers have always been monomials. The method used could be extended to binomial multipliers thus, since by definition $(a+b)\times(x+y)$ means the sum of $(a+b)$ multiplied by x and $(a+b)$ multiplied by y.

$$\begin{aligned}\therefore\ (a+b)(x+y) &= (a+b)x+(a+b)y\\ &= ax+bx+ay+by.\end{aligned}$$

In many cases, however, it is convenient to adopt the following arrangement of the work :—

Multiply $2x+3$ *by* $3x+2$.

$$\begin{array}{l} 2x+3 \\ 3x+2 \\ \hline 6x^2+9x \\ \qquad\ \ 4x\ +6 \\ \hline 6x^2+13x+6 \\ \hline \end{array}$$

This corresponds exactly to the process used in arithmetic, as is clear from the following example :—

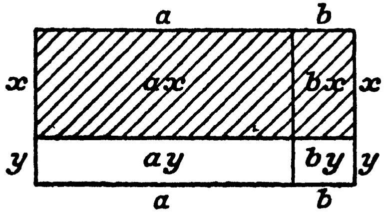
Multiply 23 *by* 32.

$$23=20+3 \qquad \begin{array}{l} 20+3 \\ 30+2 \\ \hline 600+90 \\ \qquad\ \ 40\ +6 \\ \hline 600+130+6=736 \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} 23 \\ 32 \\ \hline 690 \\ 46 \\ \hline 736 \\ \hline \end{array}$$
$$32=30+2$$

Geometrical Illustration.—The area of a rectangle whose sides are $a+b$ and $x+y$ is the product $(a+b)(x+y)$.

The area of the shaded part is $(a+b)x$ and that of the rest is $(a+b)y$. The total area is $ax+bx+ay+by$.

Compare with the two lines of the working in



$$\begin{array}{l}
a+b \\
x+y \\
\hline
ax+bx \\
\qquad\quad +ay+by \\
\hline
ax+bx+ay+by \\
\hline
\end{array}$$

Note.—When the method has been mastered, the working of longer examples may be shortened by writing the coefficients only: thus instead of

$$\begin{array}{rrr}
3x^2 & -x & +2 \\
 & x & +2 \\
\hline
3x^3- & x^2+2x & \\
 & 6x^2-2x & +4 \\
\hline
3x^3+5x^2 & & +4 \\
\hline
\end{array}
\qquad \text{we may write} \qquad
\begin{array}{llll}
3 & -1 & +2 & \\
1 & +2 & & \\
\hline
3 & -1 & +2 & \\
 & +6 & -2 & +4 \\
\hline
3x^3 & +5x^2 & & +4 \\
\hline
\end{array}$$

When a term is missing a zero must be supplied in its place, thus $(x^3+x-2)\times(x^2+3)$ is written

$$\begin{array}{llllll}
1 & 0 & +1 & -2 & & \\
1 & 0 & +3 & & & \\
\hline
1 & 0 & +1 & -2 & & \\
 & & +3 & 0 & +3 & -6 \\
\hline
x^5 & & +4x^3 & -2x^2 & +3x & -6 \\
\hline
\end{array}$$

This is known as the method of "*Detached Coefficients.*"

Exercise IX (c)

Multiply

1. $a-b$ by $x-y$.
2. $a+2b$ by $2x+y$.
3. $p+q$ by $p-q$.
4. $x-2$ by $x+3$.
5. $a-4$ by $a-5$.
6. $2a-b$ by $x+2y$.
7. $a+b$ by $a+b$.
8. $a-b$ by $a-b$.
9. x^2+1 by x^2-1.

10. $3x-2y$ by $a+b$.
11. $3x+2y$ by $2x+3y$.
12. $a+2b$ by $a-3b$.
13. a^2+b^2 by $a+b$.
14. a^2-ab+b^2 by $a+b$.
15. a^2+ab+b^2 by $a-b$.
16. $x^2+2xy+y^2$ by $x+y$.
17. a^2+ab+b^2 by a^2-ab+b^2.
18. $3x^2+x-7$ by $x-3$.
19. x^3+3x^2+3x+1 by x^2-x+1
20. $(2x+3)(3x+2)(2x-7)$.
21. $4x^2-3x+5$ by $2x^2+3x+2$
22. $3x^2-4x+7$ by $3x^2-2$.
23. x^3+2x^2+x-3 by $2x^2-5x+1$.
24. $2x^3-x+6$ by x^2+2x-3.

Find *by inspection* the coefficient of x in the following products:

25. $(x+2)(x+3)$. **26.** $(x-5)(x+2)$. **27.** $(x+a)(x+b)$.
28. $(x+a)(x-b)$. **29.** $(3x-1)(x+2)$. **30.** $(2x-3)(5x+4)$.
31. $(2x+a)(3x-b)$. **32.** $(x-2a)(2x-b)$.

Find *by inspection* the coefficient of x^2 in the following products:

33. $(2x^2+x-3)(x-2)$.
34. $(x^2-2x+3)(x-3)$.
35. $(x^2+ax+b)(x+a)$.
36. $(x^2-x+1)(x+1)$.
37. $(ax^2+bx+c)(ax+b)$.
38. $(x+1)(x+2)(x+3)$.
39. $(x-1)(x+2)(x-3)$.
40. $(3x^2+2x+1)(x^2+3)$.
41. $(2x^3+x^2-3)(x^2-2x+1)$.

Factors

Monomial Factors.—If all the terms of an expression contain a common factor, the expression may be factorised by inspection.

Factorise x^3+2x^2-x.

Here x is a factor of each term.

$$\therefore\ x^3+2x^2-x=x(x^2+2x-1).$$

N.B.—Always remove a numerical factor first, thus:

$$3x^2+6x+12=3(x^2+2x+4).$$

Exercise IX (d)

Factorise

1. $ax-bx$. **2.** x^2+xy **3.** m^2-2mn.
4. x^3-x^2-2x. **5.** $3y^2-6y+12$. **6.** a^3-2a^2+ab.
7. $xa-xb+xc$ **8.** $6a^4-3a^3+9ab$ **9.** $abd-bcd+acd$.

10. $acb-2a^2c+ac^2d$. **11.** $x^2yz-xyz^2+xy^3$.
12. $7ab-7bc-14bx$. **13.** $14x^3-7x^2y+21xy^2$.
14. $2x^3-4x^2+6x$. **15.** $3x^2-6xy+9xy^2$.

By reversing the operation shown on p 135 we can find the factors of any expression which has been obtained by multiplying two binomial expressions together.

Factorise $ax-bx-ay+by$.—Make up in stages the multiplication which has resulted in this product.

To get the term ax the two factors should begin with a and x respectively.

Stage (i)
$$\begin{array}{ll} a & ? \\ x & ? \\ \hline ax & \\ \hline \end{array}$$

The term $-bx$ would come by multiplying $(-b)$ by x, so we get

Stage (ii)
$$\begin{array}{l} a-b \\ x\ \ ? \\ \hline \end{array}$$

The term $-ay$ is the result of multiplying a by $-y$, thus we have

Stage (iii)
$$\begin{array}{ll} a-b & \\ x-y & \\ \hline ax-bx & \\ & -ay+by \\ \hline ax-bx-ay+by & \\ \hline \end{array}$$

The work is checked by noticing that $(-b)\times(-y)$ gives $(+by)$, the term whose factors have not been guessed. Thus the expression $=(a-b)(x-y)$

The result may also be obtained by taking together those terms which have a common factor, thus :

$$ax-bx-ay+by=x(a-b)-y(a-b).$$

This means that $(a-b)$ is to be multiplied by x and also by $-y$, *i e* altogether by $x-y$. $\therefore$ the expression $=(a-b)(x-y)$.

Exercise IX (e)

Factorise

1. $ax+ay+bx+by$.

2. $xy+2x+3y+6$.

3. $xa-xb-ya+yb$.

4. $ab-3b+4a-12$.

5. $3pq+2q-12p-8$.

6. x^3+x^2+x.

7. $3x^2-3x$.

8. $a^2+ab+2a+2b$.

9. $p(x+y)-q(x+y)$.

10. $x^2+xy+x+y$.

11. $3cd+2c-18d-12$

12. $x^2-ax+x-a$

13. $xy+7y+2x+14$.

14. $x(a-b)-y(a-b)$.

15. $8xy-20y+6x-15$

16. $2ab^2+2b^2+4a+4$.

17. $4x^2-4xy-4x+4y$.

18. $ab-2a+3b-6$

19. $3x^3+3x^2+3x+3$

20. $x^2y^2+x^2+1+y^2$.

21. $a^2-ab-ac+bc$.

22. x^3-3x^2+2x-6.

23. $a^2+b(c-a)-ac$.

24. $x^2-x(2-y)-2y$.

25. $x^2-x(y-z)-yz$.

26. $x^3-x(2x+3)+6$.

27. $3ax+3bx-2ay-2by$.

28. $bc-ab+ac-a^2$.

29. $x^2y^2+3y^2-5x^2y-15y$.

30. $a(b-c)+d(c-b)$

31. $p(q-r)-s(r-q)$.

Factors of x^2+ax+b.

Example —Factorise x^2-5x+6

Each factor must begin with the term x.

The other terms marked ? are such that their product is 6, hence they are either 3 and 2, or 6 and 1.

$$\begin{array}{ll} x & ? \\ x & ? \\ \hline x^2 & \\ & \\ \hline x^2-5x+6 & \end{array}$$

Since their product is +, they are both + or both −.

Since the middle term in the product is $-5x$ and is obtained by adding the terms above it in the partial products, these terms are clearly $-3x$ and $-2x$

The missing signs can now be supplied and the factors are $(x-3)(x-2)$.

$$\begin{array}{ll} x & 3 \\ x & 2 \\ \hline x^2 & 3x \\ & 2x+6 \\ \hline x^2-5x+6 & \end{array}$$

Factors of ax^2+bx+c.

Example.—Factorise $6x^2-5x-6$.

The required factors begin either with $6x$ and x or else with $3x$ and $2x$, and end either with 3 and 2 or else with 6 and 1. Also the last pair must have different signs as their product, viz. -6, is negative. Thus we have the four possibilities called A, B, C, D below:

A	$6x$	3 or 2	B	$6x$	6 or 1
	x	2 or 3		x	1 or 6
C	$2x$	3 or 2	D	$2x$	2 or 3
	$3x$	2 or 3		$3x$	3 or 2

If one of the cases in A is chosen for trial, an inspection of the multiplication shows that we have made a wrong selection as we ought to obtain by the cross multiplication a pair of terms which, taken together with opposite signs, make $-5x$.

$$\begin{array}{lrr} 6x & & 3 \\ x & & 2 \\ \hline 6x^2 & 3x & \\ & 12x & -6 \\ \hline 6x^2 & & -6 \end{array}$$

A little practice enables us to rule out the unfavourable combinations, and we note that one of the possibilities will give $9x$ and $4x$

Of these two the greater must be negative, hence we must have $-9x$ and $4x$.

The missing signs can now be supplied and the factors are $2x-3$ and $3x+2$.

$$\begin{array}{lrr} 2x & -3 & \\ 3x & +2 & \\ \hline 6x^2 & -9x & \\ & +4x & -6 \\ \hline 6x^2 & -5x & -6 \end{array}$$

Exercise IX (f)

Factorise

1. x^2-2x+1.	**2.** x^2-3x+2.	**3.** x^2+5x+4.
4. x^2-6x+8.	**5.** x^2-x-12.	**6.** $y^2-xy-6x^2$.
7. $a^2+6ab-7b^2$.	**8.** x^2+x-20.	**9.** x^2-x-2.
10. $x^2-10x+25$.	**11.** x^2+x-42.	**12.** $3x^2-3x-6$.
13. x^2+x-6.	**14.** $x^2-x-110$.	**15.** x^2-x-20.

16. $2x^2-6x-20$. **17.** $1-3x+2x^2$. **18.** $1-7x+12x^2$.
19. $12-7x+x^2$. **20.** x^2-8x+7. **21.** $a^2-2ab-8b^2$.
22. $x^2+xy-6y^2$. **23.** $p^2-pq-2q^2$. **24.** $72+x-x^2$.
25. $5x^2-12x+4$. **26.** $4x^2-10x+4$. **27.** $6x^2-11x+3$
28. $6x^2-x-2$. **29.** $3x^2-7x+2$. **30.** $2x^2+5x-3$.
31. $4x^2-8x+3$. **32.** $3x^2-7x-6$. **33.** $3x^2+11x+6$.
34. $15x^2-29x+12$. **35.** $15x^2+11x-12$. **36.** $8x^2-14x+3$
37. $3-11x+6x^2$. **38.** $6-19x+10x^2$. **39.** $4x^2-16x+15$.
40. $12x^2-7x-12$.

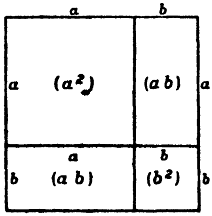
Squares.—$(a+b)^2$, $(a-b)^2$.

By multiplication

$$\begin{array}{lll} a+b & & \\ a+b & & \\ \hline a^2+ab & & \\ \quad ab & +b^2 & \\ \hline a^2+2ab+b^2 & & \end{array} \qquad \begin{array}{lll} a-b & & \\ a-b & & \\ \hline a^2-ab & & \\ \quad -ab & +b^2 & \\ \hline a^2-2ab+b^2 & & \end{array}$$

$$\therefore\ \mathbf{(a+b)^2=a^2+2ab+b^2}$$

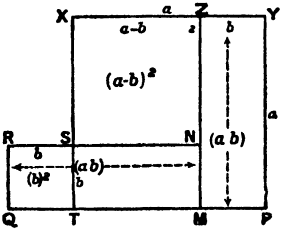
and $$\mathbf{(a-b)^2=a^2-2ab+b^2}.$$



Geometrical Illustration—(i) The square on the line whose length is $(a+b)$ is made up of two squares, a^2 and b^2, together with two rectangles each of area ab.

$$\therefore\ (a+b)^2=a^2+b^2+2ab.$$

(ii) XY is of length a and ZY of length b. $\therefore$ XZ$=a-b$. Add a square of side b, viz. RSTQ, to the figure XP, which is the square on a.



The area of XYPQR is a^2+b^2. If from this we subtract the two rectangles RM and ZP, each of area ab, we obtain XN, *i.e.*

$$a^2+b^2-2ab=(a-b)^2.$$

The squares of all binomial expressions may be written down from the results obtained by squaring $(a+b)$ and $(a-b)$ as above.

Note that there are two terms, a *and* b, *in the expression* a+b. *The square of* (a+b) *contains the first term squared+twice the product of the two terms+the square of the second term. The square of* (a−b) *contains* (a²)−(*twice the product of* a *and* b)+(b²).

Thus

$$(2p+3q)^2=(2p)^2+2(2p)(3q)+(3q)^2=4p^2+12pq+9q^2.$$

Here the two terms are $2p$ and $3q$. Similarly

$$\begin{aligned}(ax-by)^2&=(ax)^2-2(ax)(by)+(by)^2\\&=a^2x^2-2abxy+b^2y^2.\end{aligned}$$

$$\begin{aligned}a+b+c+d)^2&=\{(a+b)+(c+d)\}^2\\&=(a+b)^2+2(a+b)(c+d)+(c+d)^2\\&=a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2\\&=a^2+b^2+c^2+d^2+2ab+2bc+2ca+2ad+2bd+2cd.\end{aligned}$$

Exercise IX (g)

Write down the squares of

1. $a-2b$. **2.** $2x-y$. **3.** $2x-3y$. **4.** $px+qy$.
5. $ax-by$, **6.** $2ax+3b$. **7.** $3p+2q$ **8.** x^2-y^2
9. $3p-4q$. **10.** a^2+b^2. **11.** a^2-2b^2. **12.** $3a-x^2$.
13. $xy-pq$. **14.** $2x^2-3y^2$. **15.** $3ab-2c$. **16.** a^3+b^3.

Add a term to each of the following to make the expression a perfect square, and state the square root :

17. x^2+2x. **18.** a^2-2a. **19.** x^2-2xy. **20.** x^2+6x.
21. x^2-8x **22.** x^2-4xy. **23.** $4x^2+4x$. **24.** $9x^2-12xy$
25. $4x^2-16x$ **26.** x^2+x. **27.** x^2-3x. **28.** x^2+5x.
29. y^2-y. **30.** a^6+3a^3.

Square

31. $x+a+c$. **32.** $x-a+c$. **33.** $x+y-z$.
34. $x+a+y+b$. **35.** $x-a-y+b$. **36.** $2x-3y+a+b$.

State the square roots of:

37. $4x^2-4xy+y^2$. **38.** $9x^2+6x+1$. **39.** $a^2-4ab+4b^2$.

40. $a^4-a^2b^2+\frac{b^4}{4}$. **41.** $4a^2-4a+1$. **42.** $a^2-3ab+\frac{9b^2}{4}$.

43. $x^2+xy+\frac{y^2}{4}$. **44.** $p^2-5p+\frac{25}{4}$. **45.** $x^4+3x^2y^2+\frac{9y^4}{4}$.

The difference of two squares: $\mathbf{a^2-b^2}$.

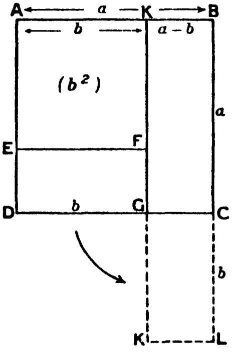
By multiplication

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\; -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

$\therefore$ **$\mathbf{a^2-b^2=(a+b)(a-b)}$.**

Geometrical Illustration.—The area of ADCB is a^2 and AEFK is b^2.

$\therefore a^2-b^2=$rect. EG+rect. KC.



Place EG in the position GL.

$\therefore a^2-b^2=$rect. KC+rect GL
$=$rect. KL
$=(a-b)(a+b)$.

Note the difference between

$(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2$
and $a^2-b^2=(a+b)(a-b)$.

$(a-b)^2$ is given in factors, and is required to be expanded.

a^2-b^2 is required to be written in factors.

If $a=6$, $b=2$

$$(a-b)^2=(6-2)^2=4^2=16.$$
$$a^2-b^2=6^2-2^2=36-4=32$$

Or better $=(6+2)(6-2)=(8)(4)=32$

Example 1.—Factorise $16x^2-9y^2$.

Since $16x^2=(4x)^2$ and $9y^2=(3y)^2$, the expression $16x^2-9y^2$ is the difference of two squares.

Comparing with a^2-b^2, we see that $4x$ stands for a and $3y$ stands for b.

Hence $4x+3y$ represents $a+b$ and $4x-3y$ represents $a-b$.

$$\therefore\ 16x^2-9y^2=(4x)^2-(3y)^2=(4x+3y)(4x-3y).$$

At first it is useful to write the two terms which are squared at the side of the page.

The two factors are obtained, one by adding these two terms, the other by subtracting them, thus:

$$\begin{aligned}16x^2-9y^2&=(4x)^2-(3y)^2\\&=(4x+3y)(4x-3y).\end{aligned}\qquad\begin{bmatrix}\mathbf{4x}\\\mathbf{3y}\end{bmatrix}$$

In more complicated examples it is necessary to rearrange the terms in order to show that the expression is equivalent to the difference of two squares.

Example 2.—Factorise $1-x^2+2xy-y^2$.

Here the term $2xy$ suggests that it is to be taken with the x^2 and y^2 terms, being part of either $x^2+2xy+y^2=(x+y)^2$, or of $(x^2-2xy+y^2)=(x-y)^2$.

Bracketing the last three terms together, we have

$$\begin{aligned}1-x^2+2xy-y^2&=1-(x^2-2xy+y^2)\\&=(1)^2-(x-y)^2.\end{aligned}$$

Here a is represented by 1 and b by $(x-y)$:

$$\begin{aligned}&=[\underbrace{(1)+(x-y)}_{a+b}][\underbrace{(1)-(x-y)}_{a-b}]\\&=[1+x-y][1-x+y].\end{aligned}\qquad\begin{bmatrix}\mathbf{1}\\\mathbf{x-y}\end{bmatrix}$$

Exercise IX (h)

Factorise

1. x^2-4y^2.	**2.** $25x^2-9y^2$.	**3.** $16x^2-y^2$.
4. $6x^2-6y^2$.	**5.** $1-x^2$.	**6.** $25a^2-1$.
7. $2x^2-8y^2$.	**8.** $\frac{x^2}{9}-1$.	**9.** $49a^2b^2-b^4$.
10. $x^2-\frac{y^2}{4}$.	**11.** $\pi a^2-\pi b^2$.	**12.** x^4-y^4.
13. a^4-4b^4.	**14.** $5x^2-20y^2$.	**15.** $1-16x^4$.

Evaluate

16. 15^2-12^2. **17.** 45^2-44^2. **18.** 113^2-111^2.
19. $\pi(16{\cdot}4)^2-\pi(13{\cdot}6)^2$. **20.** 99^2-1. **21.** 216^2-16.

Factorise

22. $(x+y)^2-a^2$. **23.** $4(x-y)^2-a^2$. **24.** $(2x+y)^2-9z^2$.
25. $x^2-(y+z)^2$. **26.** $x^2-(y-z)^2$. **27.** $1-9(a+b)^2$.
28. $4a^2-(b-c)^2$. **29.** $1-(x-y)^2$. **30.** $(a-b)^2-(x+y)^2$.

Factorise

31. $a^2+2ab+b^2-x^2$. **32.** $x^2-a^2-2ab-b^2$.
33. $x^2-a^2+2ab-b^2$. **34.** $4-x^2-2xy-y^2$.
35. $1-x^2+2xy-y^2$ **36.** $a^2-b^2+2bc-c^2$.
37. $1-4a^2-8ab-4b^2$. **38.** $25x^2-4y^2+8yz-4z^2$.

Factorise and simplify

39. $(a+b)^2-4a^2$. **40.** $(a+b)^2-(a-b)^2$.
41. $x^2-(x-y)^2$. **42.** $1-(1-x)^2$.
43. $(a+b-c)^2-(a-b+c)^2$. **44.** $(a+b+c)^2-(a-b-c)^2$.
45. $16x^2-(4x-3y)^2$. **46.** $x^2-(x-y+z)^2$.
47. $(1-x)^2-9(1-y)^2$. **48.** $(2x+3)^2-(3x+2)^2$.
49. $100-(x-10)^2$. **50.** $9(x+y)^2-4(x-y)^2$.
51. $16(2x+y)^2-9(2x-y)^2$. **52.** $5x^3y-5xy^3$.
53. $a^2-b^2-y^2+2ab$ **54.** $1-4(x-1)^2$.
55. $9-(2x-3)^2$.

Division by a Binomial Expression.—The method is exactly similar to that used in arithmetic. Detached co-efficients may conveniently be used.

Example.—$2x-y)2x^3+x^2y+3xy^2-2y^3(x^2+xy+2y^2$

$$\begin{array}{l}
\underline{2x^3-x^2y} \\
\qquad 2x^2y+3xy^2 \\
\qquad \underline{2x^2y-xy^2} \\
\qquad\qquad 4xy^2-2y^3 \\
\qquad\qquad \underline{4xy^2-2y^3}
\end{array}$$

or

$$\begin{array}{l}
2-1)2+1+3-2(1+1+2 \\
\quad\;\; \underline{2-1} \qquad\quad x^2+xy+2y^2 \\
\qquad\quad 2+3 \\
\qquad\quad \underline{2-1} \\
\qquad\qquad 4-2 \\
\qquad\qquad \underline{4-2}
\end{array}$$

N.B.—It is necessary to arrange the terms of the dividend in descending powers of x if the first term of the division contains x.

Remainder Theorem.—If a number N is divided by another number D and the quotient is Q, then, when there is no remainder,

$$N=D.Q, \text{ e.g. } \tfrac{12}{3}=4. \qquad \therefore\ 12=(3)(4).$$

If there is a remainder R, then

$$N=DQ+R, \text{ e.g. } \tfrac{11}{3}=3+\text{rem } 2. \quad \therefore\ 11=(3)(3)+2.$$

If x^3+x^2-2x+1 is divided by $x-3$ until the remainder does not contain x, let the quotient be called Q and the remainder R; then

$$x^3+x^2-2x+1=(x-3)Q+R.$$

This is true whatever be the value of x Let $x=3$, then

$$3^3+3^2-2(3)+1=0+R \qquad \text{(R remains unaltered since it does not contain } x\text{)}$$

$$\therefore\ \underline{31=R}. \qquad \text{(Check this by division)}$$

If the divisor be $x+a$, we have

$$x^3+x^2-2x+1=(x+a)Q+R.$$

Putting $x=-a$, $\quad -a^3+a^2+2a+1=R.$

Application to Factors.—If $R=0$, it follows that the divisor is a factor of the given expression.

Example 1. $\quad x^3-3x^2+x+1=(x-1)Q+R.$

Put $x=1$, $\quad 1-3+1+1=R.$

$$\underline{0=R.}$$

$$\therefore\ x-1 \text{ is a factor.}$$

Note.—The result of putting $x=1$ in any expression is to reduce the value of each term to that of its coefficient. If, then, the coefficients (with any constant term) add up to zero, $x-1$ is a factor of the expression.

This method is often useful in obtaining factors.

Example 2.—Factorise x^3+x^2-4x-4.

Trying $x=1$ we have 1^3+1^2-4-4, which $\neq 0$.

" $x=-1$ " $-1+1+4-4$, *i.e.* 0

$\therefore\ \underline{x+1}$ is a factor.

Trying $x=2$ we have $2^3+2^2-4(2)-4$, *i.e* 0.

$\therefore\ \underline{x-2}$ is a factor.

When one factor has been found, the others can be found by division A convenient arrangement of the work is as follows, *e g.* to find the factors of x^3-7x-6.

Having by trial discovered that $x+1$ is a factor, we write

$$(x+1),\quad (x+1),\quad (x+1).$$

To make x^3 we multiply the first factor by x^2 .

$$x^2(x+1),\quad (x+1),\quad (x+1)$$

This gives also x^2, and as we require no x^2 in the result, we multiply the 2nd bracket by $-x$

$$x^2(x+1)-x(x+1),\quad (x+1).$$

This gives also $-x$, and as we require $-7x$, we multiply the last bracket by -6

$$\begin{aligned} x^2(x+1)-x(x+1)-6(x+1) \\ =(x+1)(x^2-x-6) \\ =(x+1)(x-3)(x+2) \end{aligned}$$

Exercise IX (i)

Divide

1. x^3-3x^2+3x-2 by $x-2$.

2. $2x^3+5x^2-x+6$ by $x+3$

3. x^3-a^3 by $x-a$.

4. x^3+a^3 by $x+a$

5. $a^4+a^2b^2+b^4$ by a^2-ab+b^2.

6. $x^3-2ax^2+a^3$ by $x-a$

7. $4a^3+4a^2b-5ab^2-3b^3$ by $2a+b$.

8. $x^2+2xz-y^2+z^2$ by $x-y+z$.

9. $x^2+(a+b)x+ab$ by $x+a$.

10. $x^2-xy-xz+yz$ by $x-y$.

11. a^3-b^3 by a^2+ab+b^2.

12. a^6-b^6 by a^2+ab+b^2.

13. $a^3+b^3+c^3-3abc$ by $a+b+c$.

Find the remainder in the following cases:

14. $\dfrac{x^3-2x^2-x+3}{x-1}$

15. $\dfrac{2x^3-x^2+2x+1}{x+2}$.

16. $\dfrac{3x^3-x^2+x-2}{x+3}$.

17. $\dfrac{x^2-ax+a^2}{x-b}$.

Factorise

18. $x^3-6x^2+11x-6$.

19. x^3+2x^2-x-2.

20. x^3-3x-2.

21. x^3+x^2-4x-4.

22. x^3+x^2-9x-9.

23. x^3-7x-6

24. $2x^3+7x^2+7x+2$.

25. $2x^3+x^2-13x+6$.

26. $x^3+3ax^2-10a^2x-24a^3$.

27. $x^3-2x^2y-xy^2+2y^3$.

28. $x^3+x^2y+2xy^2-4y^3$.

29. $2x^3+3x^2-11x-6$.

30. For what value of a is $3x^4+ax^2-5x+3$ exactly divisible by $x-1$?

31. For what value of a is $2x^3+3x^2-ax+4$ exactly divisible by $x+1$?

32. For what value of p is x^3-2x^2+x-p exactly divisible by $x+2$?

33. For what value of a is $x^3+2x^2y-3axy^2+y^3$ exactly divisible by $x-y$?

34. For what values of a and b is $x^4-5x^3+ax^2+5x+b$ divisible by $x-1$ and by $x-2$?

What are then the remaining factors?

Revision Exercise IX (j)

Factorise

1. $a^3+2a^2b+ab^2$.

2. $10pq-6q+35p-21$.

3. $a^2-bc-b+a^2c$.

4. $2a(3b-2)-5(3b-2)$.

5. $x(2y-3)+3-2y$.

6. $x^2-\frac{1}{9}$

7. $a^2+p(p+1)a+p^3$.

8. $\dfrac{a^2}{b^2}+\dfrac{2a}{b}+1$.

9. $xy+y-ax-a$.

10. a^4-1.

11. $x^5+x(x^3+1)+1$.

12. $81x^{18}-16y^{16}$.

13. $4(a-3b)^2-9(2b-c)^2$.

14. $ab(x^2-y^2)+xy(a^2-b^2)$.

15. $a^4-4a^3-5a^2$.

16. $x^2y+3xy-5x-15$.

17. $3x^4-12(x-4)^2$.

18. $p^2-q^2-4qr-4r^2$.

19. $p^3-p(3p+4)+12$.

20. $(x+2)^2+2(x+2)(y-1)+(y-1)^2$.

21. $x^3-13x+12$.

22. $2(3x-y)^2+(9x-3y)^2$.

23. $(a-b)^2+2(b-c)(b-a)$.

24. $p^4-4p^2+4q^2-p^2q^2$.

25. $p^4-4p^2+4-p^2q^2$.

26. $a(x-b)^2-b(x^2-b^2)$.

27. $x(2x+5)-y(2y-5)$.

28. $a^2-b^2+x^2-y^2+2(ax-by)$.

29. y^8-1.

30. $4(a^2-c^2)-3b(4a-3b)$.

31. $a(4x^2-9y^2)+xy(4a^2-9)$.

32. $2x^3-5x^2+x+2$.

33. $(a+2)(a-1)(a+3)^2-(a-1)^2(a+4)(a+3)$

34. $x(y^2-1)-y(x^2-1)$.

35. $x^2-2xy+y^2-x+y$.

36. $4\pi(2a)^2-4\pi\left(\frac{b}{3}\right)^2$.

37. $(a-2)^2+(a-2)(b+3)-6(b+3)^2$.

38. $1-(p^2+q^2)+2pq$.

39. $2x^3-3x^2+2x-1$.

40. $9x^4-32x^2y^2-16y^4$.

41. $3x^3-8x^2-x+6$.

42. x^3-y^3.

43. x^3+y^3.

44. $20(x^2-y^2)-9xy$.

45. $(a-b)^3-c^2a+c^2b$

46. $a^2-a^3bc-(ab^3c-b^2)$

47. $3x^3+6x^2-189x$

48. $(a-b)(a+b)^2(a+x)-(a-b)^2(a+b)(a-x)$.

49. $2(x-a)^2+(x-a)(x-b)-3(x-b)^2$.

50. $x^3-\frac{4}{x}$.

CHAPTER X

H.C.F., L.C.M., and Fractions

H.C.F. and L.C.M. of Binomials.—The method of obtaining the H C.F and L C.M of algebraic expressions has been given in Chapter II for cases where the factors of the expressions are obvious In more difficult cases the expressions must first be factorised by the processes of the last chapter.

Example.—Find the H.C.F. and L.C.M. of $2a^2+2ab$, $6(a^2-b^2)$.

We have

$$2a^2+2ab=2a(a+b).$$
$$6(a^2-b^2)=6(a+b)(a-b).$$

The H C F of these is clearly $2(a+b)$

The multiple of lowest degree which will contain both is $6a(a+b)(a-b)$ and this is the required L C.M

Check.—$2a(a+b)$ is contained by the L C.M $3(a-b)$ times,

since $$\frac{6a(a+b)(a-b)}{2a(a+b)}=3(a-b).$$

Similarly, $6(a+b)(a-b)$ is contained a times.

Exercise X (a)

Find the H C.F. and L.C.M. of

1. a^2-b^2, $ab+b^2$.
2. $2(x^2-a^2)$, $4(x-a)$.
3. $4x+4y$, $6x^2(x^2-y^2)$.
4. a^2b-ab^2, a^3-ab^2.
5. x^2-2x-3, $x^2-7x+12$, x^2-x-6
6. a^2-4x^2, a^2+2ax
7. x^2+x, $(x+1)^2$.
8. x^2+x, x^2-1.
9. $2p^2-p-1$, $3p^2-p-2$.
10. $x^2-ax-bx+ab$, x^2-a^2.
11. x^2+x-2, x^2+4x+4.
12. $(x^2+x)^2$, x^3-x^2.
13. $x(x-1)-2$, $x(x-6)+8$.

14. $a^2-b^2,\ b-a.$ **15.** $[2(a-b)]^2,\ b^2-a^2.$
16. $4a^2-b^2,\ (2ab+b^2)^2.$ **17.** $1-x,\ x^3-x,\ (x-1)^2.$
18. $2b-2a,\ 2a+2b,\ 4(a^2-b^2).$ **19.** $x+2,\ 3x+6,\ 4-x^2.$
20. $4a^2(a^2-b^2)^2,\ 6(a-b)^3.$

Reduction of Fractions to Lowest Terms.—Factorise numerator and denominator and divide both by any common factors, thus:

$$\frac{a^2-b^2}{(a-b)^2}=\frac{\cancel{(a-b)}(a+b)}{\cancel{(a-b)}(a-b)}=\frac{a+b}{a-b}.$$

Exercise X (b)

Reduce to lowest terms:

1. $\dfrac{a^2-ab}{ba-b^2}.$ 2. $\dfrac{(a+b)^2}{a^2-b^2}.$ 3. $\dfrac{2ax+3x^2}{3ax+2x^2}.$

4. $\dfrac{x^2-2x-3}{x^2-5x+6}.$ 5. $\dfrac{x^2-y^2}{(x+y)^2}.$ 6. $\dfrac{4a^2+4ab}{6ab+6b^2}$

7. $\dfrac{x^2y^2-4z^2}{(xy+2z)^2}.$ 8. $\dfrac{9x^4-4}{3x^2+2}.$ 9. $\dfrac{4a^2-b^2}{2ax-bx}.$

10. $\dfrac{a^3+a^2b}{a^2+2ab+b^2}.$ 11. $\dfrac{a^2-9}{12-a-a^2}.$ 12. $\dfrac{6x^2-5x-4}{6x^2+x-12}.$

13. $\dfrac{a-b}{b-a}.$ 14. $\dfrac{a^2-b^2}{b-a}.$ 15. $\dfrac{6x^2-6y^2}{3y-3x}$

16. $\dfrac{5a^2-5ax}{x^2-a^2}.$ 17. $\dfrac{1-3a+2a^2}{2-a-a^2}.$ 18. $\dfrac{x^2-7x+10}{x^2-x-2}.$

Multiplication and Division of Fractions.—The process is similar to that given in Chapter II.

Example.—Simplify $\dfrac{x^2-x-6}{x^2+3x-4}\div\dfrac{x^2+x-12}{x^2+x-2}.$

$$\frac{x^2-x-6}{x^2+3x-4}\div\frac{x^2+x-12}{x^2+x-2}=\frac{\cancel{(x-3)}(x+2)}{(x+4)\cancel{(x-1)}}\times\frac{(x+2)\cancel{(x-1)}}{(x+4)\cancel{(x-3)}}$$
$$=\frac{(x+2)^2}{(x+4)^2}.$$

Exercise X (c)

Simplify

1. $\dfrac{a^2+2ab}{a^2+4b^2} \times \dfrac{ab-2b^2}{a^2-4b^2}$.

2. $\dfrac{a^2+b^2}{a^2-b^2} \times \dfrac{a-b}{a+b}$.

3. $\dfrac{x^2-4a^2}{ax+2a^2} \times \dfrac{2a}{x-2a}$.

4. $\dfrac{a^2-81}{a^2-4} \div \dfrac{a-9}{a-2}$.

5. $\dfrac{x^2+5x+6}{x^2-1} \times \dfrac{x^2-2x-3}{x^2-9}$.

6. $\dfrac{25a^2-b^2}{9a^2x^2-4x^2} \div \dfrac{5a-b}{x(3a-2)}$.

7. $\dfrac{x^2y^2+2xy}{4x^2-1} \div \dfrac{xy+2}{2x+1}$.

8. $\dfrac{a^2-b^2}{a^2+ab} \div \dfrac{a-b}{a}$.

9. $\dfrac{ax-a^2}{[x(x+a)]^2} \times \dfrac{x^2+2ax+a^2}{ax^2-a^3} \div \dfrac{1}{x^3+x^2a}$.

10. $\dfrac{6x^2-7x-20}{2x^2+x-15} \times \dfrac{x^2+2x-3}{x^2+6x-7} \div \dfrac{3x^2-2x-8}{x^2+5x-14}$.

Addition and Subtraction

Example.—Simplify $\dfrac{x^2}{x^2-y^2} + \dfrac{x^2}{x^2-xy} - \dfrac{y^2}{xy+y^2}$.

The denominators should first be factorised whenever possible.

$$\frac{x^2}{(x-y)(x+y)} + \frac{x^2}{x(x-y)} - \frac{y^2}{y(x+y)}$$

$$= \frac{x^2}{(x-y)(x+y)} + \frac{x}{x-y} - \frac{y}{x+y} \quad \text{(simplifying the fractions).}$$

The L.C.M. of the denominators is $(x-y)(x+y)$.

Now $\dfrac{x}{x-y} = \dfrac{x(x+y)}{(x-y)(x+y)}$, $\quad \dfrac{y}{x+y} = \dfrac{y(x-y)}{(x+y)(x-y)}$,

$$\therefore \frac{x^2}{(x-y)(x+y)} + \frac{x(x+y)}{(x-y)(x+y)} - \frac{y(x-y)}{(x+y)(x-y)}$$

$$= \frac{x^2+x(x+y)-y(x-y)}{(x-y)(x+y)}$$

$$= \frac{x^2+x^2+xy-xy+y^2}{(x-y)(x+y)}$$

$$= \frac{2x^2+y^2}{(x-y)(x+y)}.$$

Exercise X (d)

Express in their simplest form

1. $\frac{1}{a-b}+\frac{1}{a+b}$.
2. $\frac{1}{a-b}-\frac{1}{a+b}$.
3. $1+\frac{x+y}{x}$.
4. $1-\frac{x-y}{x}$.
5. $a-\frac{a-b}{2}$.
6. $\frac{1}{2a-2b}-\frac{1}{3a-3b}$.
7. $\frac{1}{x^2}-\frac{1}{x^2+x}$.
8. $a-\frac{ab}{a+b}$.
9. $\frac{1}{x}+\frac{1}{x+y}$.
10. $\frac{y}{x-y}+\frac{x}{x+y}$.
11. $\frac{x}{x-y}-1$.
12. $\frac{a^2}{a-b}-a$.
13. $1-\frac{a^2+ba}{a^2-b^2}$.

Simplify as much as possible

14. $\frac{a}{a-b}-\frac{b}{a+b}$.
15. $\frac{a-b}{a+b}+\frac{ab}{a^2-b^2}$.
16. $\frac{x^2-xy}{x^2-y^2}-1$.
17. $\frac{1+\frac{1}{p}}{q}-\frac{1+\frac{1}{q}}{p}$.
18. $\frac{1}{x^2+2x-8}-\frac{1}{x^2+7x+12}$.
19. $\frac{x^2+8x+15}{x^2+7x+10}-\frac{x-1}{x+2}$.
20. $\frac{p}{pq-q^2}-\frac{q}{p^2-pq}$.
21. $\frac{a^2}{a-2b}+\frac{2a^2b}{(a-2b)^2}$.
22. $\frac{3}{2x-4y}-\frac{x-y}{(x-2y)^2}$
23. $\frac{x+7}{x+2}+\frac{25}{x^2-x-6}$.
24. $\frac{5}{3x+12}+\frac{x+\frac{2}{3}}{2x+8}$.
25. $\left(\frac{1}{p}+\frac{1}{q}\right)\left(a+b\right)-\left(\frac{a+b}{p}-\frac{a-b}{q}\right)$.

Find the value of

26. $\dfrac{1}{2x^2-3x+1}+\dfrac{1}{2x^2-5x+3}-\dfrac{4}{4x^2-8x+3}.$

27. $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{y}{x-y}+\dfrac{x}{x+y}.$

28. $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}+\dfrac{a^2+x^2}{a^2-x^2}.$

29. $\dfrac{x-a}{(x+a)^2}+\dfrac{x+a}{(x-a)^2}-\dfrac{2x}{(x^2-a^2)}.$

30. $\dfrac{1}{6(1-a)}+\dfrac{1}{6(1+a)}+\dfrac{1}{4(1+a^2)}.$

31. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^2y-x^3}{y(x^2-y^2)}.$

Prove the truth of the following identities :

32. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}\equiv\dfrac{2}{x+3}.$

33. $\dfrac{a+b}{2(a-b)}-\dfrac{a-b}{2(a+b)}+\dfrac{2b^2}{a^2-b^2}\equiv\dfrac{2b}{a-b}.$

In some cases it is desirable to rewrite the expressions contained in some of the fractions so that the order of the letter involved may be the same throughout the question.

Example 1.—Simplify $\dfrac{1}{8-2x}\div\dfrac{1}{3x-12}.$

Here $\qquad 8-2x=2(4-x)=-2(x-4)$

and $\qquad 3x-12\qquad =3(x-4).$

$$\therefore\ \text{expression}=\frac{1}{-2(x-4)}\times 3(x-4)=-\frac{3}{2}.$$

Example 2.—Simplify $\dfrac{b}{a-2b}-\dfrac{2b}{2b+a}-\dfrac{ab}{4b^2-a^2}.$

Here $2b+a=+(a+2b).\qquad \therefore\ -\dfrac{2b}{2b+a}=-\dfrac{2b}{a+2b}.$

$4b^2-a^2=-(a^2-4b^2).\qquad \therefore\ -\dfrac{ab}{4b^2-a^2}=+\dfrac{ab}{a^2-4b^2}.$

$\therefore$ expression

$$= \frac{b}{(a-2b)} - \frac{2b}{(a+2b)} + \frac{ab}{(a-2b)(a+2b)}$$

$$= \frac{b(a+2b)}{(a-2b)(a+2b)} - \frac{2b(a-2b)}{(a-2b)(a+2b)} + \frac{ab}{(a-2b)(a+2b)}$$

$$= \frac{(ab+2b^2)-(2ab-4b^2)+(ab)}{(a-2b)(a+2b)}.$$

$$= \frac{ab+2b^2-2ab+4b^2+ab}{(a-2b)(a+2b)}.$$

$$= \frac{6b^2}{a^2-4b^2}.$$

Exercise X (e)

Simplify

1. $\dfrac{a}{6a-9b} + \dfrac{b}{6b-4a}$.

2. $\dfrac{1}{ax-a^2} + \dfrac{1}{ax-x^2}$.

3. $\dfrac{7p}{7p-3q} + \dfrac{3q}{3q-7p}$.

4. $\dfrac{1}{(x-1)^2} - \dfrac{1}{1-x^2}$.

5. $\dfrac{a}{2-a} - \dfrac{4-2a}{(a-2)^2}$.

6. $(a+x) \div \left(1 - \dfrac{a^2}{x^2}\right)$.

Express in their simplest form

7. $\dfrac{1}{6+x-x^2} \div \dfrac{1}{x^2+x-12}$.

8. $\dfrac{3x-4}{x-2} + \dfrac{12-5x}{2-x}$.

9. $\left(\dfrac{p}{q} - \dfrac{q}{p}\right) - \left(\dfrac{1}{p} - \dfrac{1}{q}\right)$.

10. $\dfrac{2}{x-y} + \dfrac{x+3y}{y^2-x^2}$.

11. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}} \times (a-b)$.

12. $\dfrac{(2x-y)^2}{x^2-y^2} + \dfrac{y-x}{y+x}$.

13. $\dfrac{b}{b-3x} - \dfrac{b^2}{(3x-b)^2}$.

Find the values of

14. $\dfrac{1}{x-a}-\dfrac{1}{x-b}$ when $x=\dfrac{a+b}{2}$.

15. $\dfrac{1-\dfrac{1}{x}}{\dfrac{1}{x}-\dfrac{1}{2}}$ when $x=\dfrac{2a-3b}{a-b}$.

16. $\left\{\dfrac{1}{y+a+b}-\dfrac{1}{a-b-y}\right\}\div\dfrac{1}{a+b-y}$ when $y=\dfrac{3a+b}{3}$.

Simplify as much as possible

17. $\left(\dfrac{2}{x}-\dfrac{x}{2}\right)\div\left(\dfrac{x}{2}-4+\dfrac{6}{x}\right)$.

18. $\dfrac{1}{x-3}-\dfrac{2}{3+x}-\dfrac{1}{3-x}$.

19. $\dfrac{3}{x^2+3x+2}-\dfrac{6}{x^2+x-2}-\dfrac{4}{1-x^2}$.

20. $\dfrac{2x}{4-x^2}+\dfrac{1}{x-2}+\dfrac{1}{x+2}$.

21. $\dfrac{1}{a^2-5ab+6b^2}-\dfrac{5}{a^2-ab-6b^2}-\dfrac{6}{4b^2-a^2}$.

22. $\left(x-\dfrac{4}{4-x}\right)\times\left\{2x-5+\dfrac{6(3-x)}{x-2}\right\}$.

23. $\dfrac{4a^2+a-3}{a^2-1}-\dfrac{2-5a+3a^2}{(1-a)^2}$.

24. $\left(\dfrac{1}{x^2}-x^2\right)\div\left(1-\dfrac{1}{x^2}\right)^2$.

25. $\dfrac{b}{ab-a^2}+\dfrac{b-c}{(a-b)(a-c)}-\dfrac{c}{ac-a^2}$.

26. $\dfrac{(b-c)^2-a^2}{b^2-(c-a)^2}-\dfrac{(c+a)^2-b^2}{(a-b)^2-c^2}$.

27. $(a+b/x)\times 1/(ax+b)$.

28. $1-a/x\times x/(a+b)$. **29.** $(1-a/x)\times x/(a+b)$.

30. If $a=\dfrac{p+q}{1+pq}$, prove that $\dfrac{1-a}{1+a}=\dfrac{(1-p)(1-q)}{(1+p)(1+q)}$.

31. Find the value of $\dfrac{ay}{a+y}$ when $\dfrac{1}{y}=\dfrac{1}{x}-\dfrac{1}{a}$.

Solution of Fractional Equations

Example 1.—Solve $\dfrac{x}{x+3}-\dfrac{2}{3x+9}=\dfrac{4}{15}$.

First rewrite the equation with the denominators in factors.

$$\therefore \frac{x}{x+3}-\frac{2}{3(x+3)}=\frac{4}{15}$$

Multiply both sides *of the equation* by $15(x+3)$, *i.e.* the L.C.M. of the denominators.

$$\therefore 15x-10=4(x+3).$$
$$\therefore 11x=22.$$
$$\therefore x=2.$$

Example 2.—Solve

$$\frac{2x-2}{x-3}-\frac{4x-3}{2x-4}=\frac{4-x}{x^2-5x+6}.$$

$$\therefore \frac{2x-2}{x-3}-\frac{4x-3}{2(x-2)}=\frac{4-x}{(x-2)(x-3)}.$$

Multiply both sides *of the equation* by $2(x-2)(x-3)$

$$2(2x-2)(x-2)-(4x-3)(x-3)=2(4-x).$$
$$\therefore 4x^2-12x+8-(4x^2-15x+9)=8-2x.$$
$$\therefore 3x-1=8-2x.$$
$$\therefore 5x=9.$$
$$\therefore x=1\tfrac{4}{5}.$$

It may be noted that what is apparently an equation may have no solution or may have an infinite number of solutions.

Thus the equation $\frac{4}{x-1}-\frac{3}{x+1}=\frac{x+6}{x^2-1}$ on clearing of fractions gives us $4(x+1)-3(x-1)=x+6$,

$$\text{i.e. } 4x+4-3x+3=x+6,$$
$$\text{i.e } x+7=x+6,$$

which leads to $1=0$, and the equation has no solution.

On the other hand, $\frac{3}{x-2}-\frac{2}{x+2}=\frac{x+10}{x^2-4}$ would give

$$3(x+2)-2(x-2)=x+10,$$
$$\text{i.e. } 3x+6-2x+4=x+10,$$
$$\text{i.e } x+10=x+10,$$

which is clearly satisfied by all values of x

The relation $\frac{3}{x-2}-\frac{2}{x+2}=\frac{x+10}{x^2-4}$ is actually an identity, the right-hand side being merely the result of adding the fractions on the left side, and is true whatever value is given to x.

Exercise X (f)

Solve the following equations :

1. $\frac{3x+8}{x+2}-\frac{2x+7}{x+4}=1.$

2. $\frac{3}{x^2+x-2}-\frac{2}{x^2+2x-3}=\frac{5}{x^2+5x+6}.$

3. $\frac{x-4}{x+1}+\frac{x+2}{x-1}=2.$

4. $\frac{4}{x+3}-\frac{5}{x-1}=\frac{9x-19}{x^2+2x-3}.$

5. $\frac{x+a}{x+b}=\frac{a-2}{b-2}.$

6. $\frac{4}{2x-1}+\frac{3}{6x+8}=\frac{5}{2x}.$

7. $\frac{ax+c}{d-ax}=\frac{c-bx}{bx+d}.$

8. $\frac{x+5}{x+4}=1+\frac{4}{3x^2+10x-8}.$

9. $\frac{7}{x+1}-\frac{6}{x+3}=\frac{1}{x-4}.$

10. $\frac{10x-40}{x^2-4x+3}=\frac{2x-8}{2x^2+x-21}+\frac{9}{x-3}.$

11. $\frac{2}{5x^2+4x-12}+\frac{1}{x^2-4}=\frac{3}{5x^2-16x+12}.$

12. $\frac{5}{x+4}-\frac{8}{2x-6}=\frac{6}{3x+12}-\frac{10}{4x-12}.$

13. $\frac{3x}{3x-6}-\frac{2x+4}{2x-2}+\frac{1}{x+2}=0.$

14. $\frac{3x+6}{x^2+x-2}-\frac{2x+2}{x^2-x-2}=\frac{x-4}{x^2-3x+2}$

15. $\frac{1}{x-1}-\frac{1}{x+2}=\frac{1}{x+3}-\frac{1}{x+6}.$

CHAPTER XI

Quadratic Equations

An equation which contains the square of the unknown quantity is called a *Quadratic Equation.* A simple example is the equation $x^2=9$.

Since $3^2=9$, it is clear that $x=3$ is a solution of the equation, but since $(-3)^2=9$, it follows that $x=-3$ must also be a solution. Thus the equation has two solutions which can be written together in the form $x=\pm 3$ (read + or − 3). By taking the square root of both sides of the equation $x^2=9$, we should get either $+x=+3$ or $+x=-3$, or $-x=+3$ or $-x=-3$, but if $-x=-3$, it follows that $+x=+3$, and if $-x=+3$, it follows that $+x=-3$: hence the statement $x=\pm 3$ includes all the possible solutions.

Similarly if $(x+2)^2=16$.

Taking the square root of both sides, we have

$$x+2=\pm 4, \quad i.e. \quad x+2=+4 \quad \text{or} \quad x+2=-4.$$

Hence $x=2$ or -6.

All quadratic equations have *two* possible solutions, and these are often called the *roots* of the equation. A root which is an integer (whole number) is called an integral solution of the equation.

Exercise XI (a)

Solve the quadratics :

1. $x^2=16$. **2.** $x^2=1$. **3.** $(x-1)^2=4$.
4. $(x-2)^2=9$. **5.** $(x-a)^2=b^2$. **6.** $x^2=a^2$.
7. $(x-3)^2=(2x-1)^2$. **8.** $(x-1)^2=1$. **9.** $5p^2=5$.
10. $2x^2-50=0$.

A useful method of solving easy quadratics can be derived from the following considerations. When the product of two unknown numbers is given, we can as a rule find an infinite number of pairs of numbers which have the given product; *e.g.* if $pq=12$, then $p=2$, $q=6$; or $p=3$, $q=4$; or $p=7$, $q=1\frac{5}{7}$, etc., all satisfy the condition $pq=12$.

If, however, the product is 0, so that $pq=0$, then either p or q must be 0.

Example.—Solve $3x^2-7x+2=0$

The factors of the left-hand side are $(3x-1)(x-2)$

Hence $(3x-1)(x-2)=0$ either $3x-1=0$ or $x-2=0$, *i e.* either $x=\frac{1}{3}$ or $x=2$ Both these values of x satisfy the equation

If $x=2$, $3x^2-7x+2=12-14+2=0$.
If $x=\frac{1}{3}$, $3x^2-7x+2=\frac{1}{3}-\frac{7}{3}+2=0$.

This method of solving by factors requires that the right-hand side of the equation must always be zero. To solve the equation $5x^2=13-8x$, it must first be written in the form

$$5x^2+8x-13=0,$$

then

$$(5x+13)(x-1)=0.$$

$$\text{either } 5x+13=0 \quad \text{or} \quad x-1=0,$$

$$\text{i e. } x=-2\tfrac{3}{5} \quad \text{or} \quad 1.$$

N.B.—It should be noted that when x is a factor of each term of the equation the value $x=0$ is a solution; *e g.*

$$x^2-5x=0. \quad \therefore\ x(x-5)=0. \quad \therefore\ x=0 \text{ or } 5.$$

The reverse operation enables us to write down the quadratic whose roots are given To form the quadratic whose roots are $-2\frac{3}{5}$ or 1, we note that these solutions come from the statement that $x=-2\frac{3}{5}$ or $x=1$.

$$\therefore\ x+\tfrac{13}{5}=0 \quad \text{or} \quad x-1=0.$$

$$\therefore\ (5x+13)(x-1)=0.$$

$$\therefore\ 5x^2+8x-13=0.$$

Find the quadratic whose roots are a and b.

Since $x=a$ or $x=b$,

$$\therefore\ x-a=0 \text{ or } x-b=0.$$

$$\therefore\ (x-a)(x-b)=0.$$

$$\therefore\ x^2-x(a+b)+ab=0.$$

Note that *when the coefficient of* x^2 *is* 1, the coefficient of x is $-(a+b)$, *i.e.* the *sum of the roots with their sign changed,* and the constant term is ab, *i.e.* the *product of the roots.*

This is a convenient test for the correctness of a solution.

In the previous example

$$5x^2+8x-13=0,$$

$$i.e.\ x^2+\tfrac{8}{5}x-\tfrac{13}{5}=0.$$

The roots are $-\frac{13}{5}$ and 1.

Their sum $=-\frac{8}{5}$, which is the coefficient of x with the sign changed.

Their product $=-\frac{13}{5}$, which is the constant term.

An equation of the third degree in x is called a *cubic equation.* If the left-hand side can be factorised when the right-hand side is zero, the roots can be seen by inspection.

Example.—Solve $(x-1)(x+2)(x+3)=0$.

Evidently the left-hand side equals zero if either

$$x-1=0 \text{ or } x+2=0 \text{ or } x+3=0,$$

$$i.e.\ x=1 \text{ or } -2 \text{ or } -3.$$

Exercise XI (b)

Solve the following quadratics:

1. $(x-a)(x-b)=0$.

2. $x^2-3x+2=0$

3. $x^2+2x-3=0$.

4. $x^2-1=0$.

5. $x^2-4x+3=0$.

6. $x^2+3x=0$.

7. $3x^2+x-2=0$.

8. $2x^2=3x+20$.

9. $x^2-2ax=0$.

10. $2(x+2)^2=46-7x$.

11. $1=\frac{12}{x}-20x.$ **12.** $(3x-1)(2x+3)=30$

13. $\frac{3x+20}{9}=\frac{7}{x}.$ **14.** $(x-2)(x-3)=(2x+1)(5x+6)$

15. $3x^2-2bx=0.$ **16.** $2x^2=5ax.$

Find the roots of the following equations :

17. $3x^2+5x=12.$ **18.** $x+10=\frac{24}{x}.$

19. $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{x+4}.$ **20.** $\frac{1}{3x}-\frac{2}{5}=\frac{1}{2x+1}.$

21. $x^2-px-qx+pq=0.$ **22.** $a^2x^2+x(ab-ac)=bc.$

Form the equations whose roots are

23. 3 and 2 **24.** 5 and -1. **25.** $\frac{2}{3}$ and 7.

26. $-\frac{3}{4}$ and $-\frac{1}{5}$ **27.** 0 and $3\frac{1}{2}$. **28.** $2a$ and $2b$.

29. 1, 2, and 3. **30.** 0, -1, and 2.

31. Show that $x=2$ is one root of $99x^2-97x-202=0$. What is the other ?

32. Show that $x=\frac{1}{2}$ is one root of $114x^2-119x+31=0$. Find the other root.

33. If $x=1$ is a solution of $ax^2+bx-c=0$, show that $a+b=c$.

34. If there is an integral solution of $x^3+x^2-4x=4$ between 0 and 3, find it, and find also the other two solutions.

35. What values of x will make the expression $4x^2-(2x-3)$ have the same value as $x^2+2x+10$?

36. Find two values of x which will make $5x^2-3x+2$ and $2x^2+7x+10$ have equal values.

37. If $x=-2$ is a solution of $3x^2+ax-5=0$, show that $a=3\frac{1}{2}$

38. If $x=-2$ and $x=\frac{1}{2}$ are solutions of $2x^2+bx+c=0$, prove that $b=3$ and $c=-2$.

39. Find two values of x for which $2x^2+x+1$ will be twice the value of $2x+3$.

Solution by Completing the Square.—If the left-hand side of the equation cannot easily be factorised when the right-hand side is zero, we proceed as follows:

Solve $x^2+x-4=0$.

Bring the constant term (*i e.* the term which does not contain x) to the right-hand side, then $x^2+x=4$.

By trial multiplication find a term which, added to x^2+x, makes a perfect square (see p. 142). The places marked ? must clearly be filled by $+\frac{x}{2}$, since the terms are equal, and when added are to make x.

$$\begin{array}{ll} x & \\ x & \\ \hline x^2 & ? \\ & ? \\ \hline x^2+x & \end{array}$$

$\therefore$ the term required to make x^2+x a perfect square is $\frac{1}{4}$.

Adding $\frac{1}{4}$ to both sides of the equation, we have

$$x^2+x+\tfrac{1}{4}=4\tfrac{1}{4}=\tfrac{17}{4}.$$

$$\begin{array}{lll} x & +\frac{1}{2} & \\ x & +\frac{1}{2} & \\ \hline x^2 & +\frac{1}{2}x & \\ & \frac{1}{2}x & +\frac{1}{4} \\ \hline x^2 & +\ x & +\frac{1}{4} \end{array}$$

Taking the square root of both sides, we have

$$x+\frac{1}{2}=\pm\frac{\sqrt{17}}{2}=\pm\frac{4\cdot123}{2}.$$

$$\therefore\ x=+2\cdot06-\cdot5=1\cdot56,$$

$$\text{or}\quad x=-2\cdot06-\cdot5=\underline{-2\cdot56}.$$

Test.

Sum of roots $=-1$. Product $=-3\cdot99$. *Cf* $x^2+1.x-4=0$

If the coefficient of x^2 is not 1, a preliminary step is necessary.

Example.—Solve $3x^2-x=2$

Here we cannot conveniently complete the square as the equation stands, since our first term would be $\sqrt{3}x$.

$$\begin{array}{ll} \sqrt{3}x- & ? \\ \sqrt{3}x- & ? \\ \hline 3x^2 & \end{array}$$

To avoid this complication we first divide both sides of the equation by 3.

Then $$x^2 - \frac{x}{3} = \frac{2}{3}$$

To make the L H S. a perfect square, we add $\frac{1}{36}$ to both sides.

Then $$x^2 - \frac{x}{3} + \frac{1}{36} = \frac{2}{3} + \frac{1}{36} = \frac{25}{36}.$$

Taking the square root of both sides,

$$x - \tfrac{1}{6} = \pm\tfrac{5}{6} \quad \therefore\ x = -\tfrac{2}{3} \quad \text{or} \quad +1$$

$$\begin{array}{l} x \ - \ , \\ x \ - \\ \hline x^2 - \frac{x}{6} \\ \quad - \frac{x}{6} + \\ \hline x^2 - \frac{x}{3} \\ \hline \end{array}$$

Test.

Sum of roots $=\frac{1}{3}$. Product $=-\frac{2}{3}$ *Cf* $x^2-\frac{1}{3}x-\frac{2}{3}=0$

Note that the solution of the equation $3x^2-x-2=0$ enables us to factorise the expression $3x^2-x-2$ Since the roots are $-\frac{2}{3}$ and 1, the equation may be written :

$$(x+\tfrac{2}{3})(x-1)=0,$$

i e. $$(3x+2)(x-1)=0$$

Hence $$3x^2-x-2=(3x+2)(x-1).$$

Example —Solve $x^2-2x+4=0$.

$$\therefore\ x^2-2x=-4$$

Completing the square,

$$\therefore\ x^2-2x+1=-4+1=-3.$$

$$\therefore\ (x-1)^2=-3.$$

No square can be negative, so it is impossible to find the square root of -3. In such cases the solution of the equation is said to be *unreal*.

Exercise XI (c)

Add a number to the following expressions to make them perfect squares, and then give their square roots :

1. x^2+6x. **2.** x^2+3x **3.** x^2-5x. **4.** $x^2+\frac{2}{3}x$

5. $x^2-\frac{1}{2}x$. **6.** $x^2+\frac{4}{5}x$. **7.** $4x^2+24x$ **8.** $9x^2-6x$.

9. $4x^2-4x$. **10.** x^2+ax.

Solve the following equations by completing the square. Approximate answers should be given to 2 decimal places. Take $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$, $\sqrt{5}=2\ 236$, $\sqrt{7}=2\ 646$, $\sqrt{11}=3{\cdot}317$.

11. $x^2+6x=7$. **12.** $x^2-5x=24$. **13.** $3x^2+2x=8$.
14. $2x^2+6x+1=0$. **15.** $9x^2-24x+11=0$. **16.** $2x^2-x=5\frac{7}{8}$.
17. $3x^2+2x=3\frac{1}{3}$. **18.** $x-1=\dfrac{3}{2x}$ **19.** $3x^2-x+1=0$

In the following the necessary square roots may be found from tables :

20. $4x^2-7x+2=0$. **21.** $3x^2+4x-5=0$
22. $x(2x+3)=\frac{1}{2}$. **23.** $10x^2+x-2=0$
24. $3x^2-12x+2=0$ **25.** $(3x-1)(x-1)=3$
26. $2(x+2)=x(x+6)$ **27.** $\left(\dfrac{1}{x}-3\right)\left(\dfrac{1}{3}-x\right)=1$.
28. $\cdot 9x^2+2{\cdot}7x=52\ 8$ **29.** $\cdot 5x^2+2x=4\ 5$.
30. $\cdot 2x^2-\cdot 78x+\cdot 7=0$ **31.** $(x-1)(x+3)=140$

Graphical Solution of Quadratics

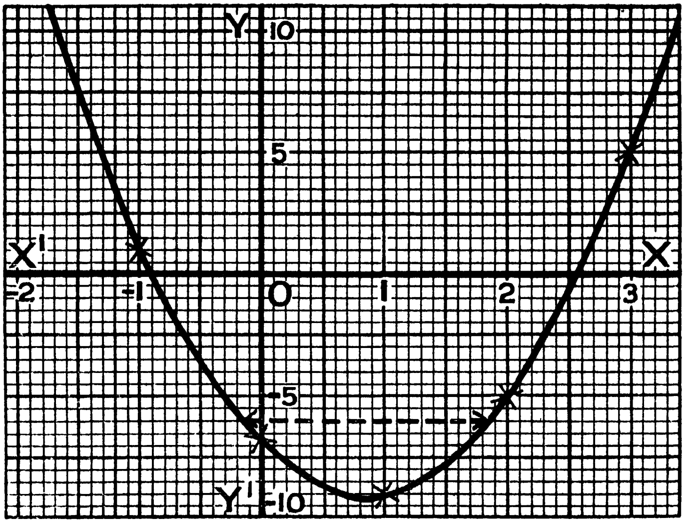
Solve graphically $3x^2-5x-7=0$
Draw a graph of the expression $3x^2-5x-7$ (see p 110).

x x^2	-2 4	-1 1	0 0	1 1	2 4	3 9
$3x^2$ $-5x$ -7	12 10 -7	3 5 -7	0 0 -7	3 -5 -7	12 -10 -7	27 -15 -7
$3x^2-5x-7$ or y	1·5	1	-7	-9	-5	5

On tabulating the values we see the expression is positive when $x=-1$ and negative when $x=0$

Again the expression is negative when $x=2$, positive when $x=3$.

$\therefore$ the expression is zero for a value of x between -1 and 0 and between 2 and 3.



Graph of $3x^2-5x-7$

From the graph $3x^2-5x-7=0$ when $x=-\;9$ and $2{\cdot}57$.

Example —From this graph find the solution of

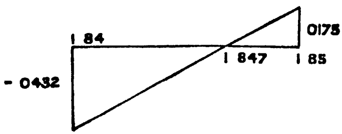
$$3x^2-5x-1=0.$$

If $3x^2-5x-1=0$, then $3x^2-5x=1$ and $3x^2-5x-7=-6$. From the graph $3x^2-5x-7=-6$ when $x=-{\cdot}18$ and $1{\cdot}85$.

Magnification.—Greater accuracy may be obtained by magnifying the portion of the graph in the neighbourhood of the root.

When $x=1{\cdot}85$, $3x^2-5x-1=10{\cdot}2675-9{\cdot}25-1= \cdot0175$.

When $x=1{\cdot}84$, $3x^2-5x-1=10{\cdot}1568-9{\cdot}20-1=-\cdot0432$.



Between these values the magnified graph will be approximately a straight line and the expression is zero when $x=1{\cdot}847$.

Exercise XI (d)

1. Plot the graph of $(x+2)(x-3)$ from $x=-6$ to $x=+6$ Read off from your figure the values of x which make the expression equal to 10 Of what equation have you thus obtained the solution ?

2. Draw the graph of $8-5x-4x^2$ between $x=-3$ and $x=+2$, taking 1 in. to the unit along OX and 0·1 in. to the unit along OY. For what values of x does the expression equal (i) $+5$; (ii) -10 ? From the graph solve the equation $4x^2+5x-8=0$ If $4x^2+5x-10=0$, what is the value of $8-5x-4x^2$? Hence solve the equation $4x^2+5x-10=0$

3. Draw the graph of $3x-2x^2$ between -2 and $+4$ Hence solve the equations $2x^2-3x-7=0$ and $4x^2-6x=15$

4. Employ the graph of x^2-3x to solve the equations

$$x^2-3x+1=0, \qquad x^2-3x-5=0, \qquad x^2=3x+9.$$

Solve graphically the following equations :

5. $3x^2-4x-2=0$.

6. $(x-2)(2x+7)=5$

7. $2x^2-10x+7=0$.

8. $2x^2+\dfrac{3x}{2}=2{\cdot}5$.

9. Show by means of a graph that one root of the equation $3x^2-x-5=0$ lies between 1 and 2. By "magnifying" the part of the graph in this neighbourhood, find this root to two decimal places.

10. If $3x^2-8x+4=0$, show that $8-3x=\dfrac{4}{x}$, and hence solve the original equation by means of drawing two graphs

Employ the method of Ex. 10 to solve the equations

11. $5x^2-8x+2=0$ **12.** $2x^2+9x+3=0.$

13. By drawing the graph of x^3-12x, show that one root of $x^3-12x=9$ lies between 3 and 4, and obtain its value to one decimal place.

14. Between what values of x will the expression x^2-x-2 be negative ?

15. What is the condition that $2x^2-x$ should be greater than 6 ?

Example.—Solve the equation $\dfrac{3}{x-2}-\dfrac{5}{x-1}=\dfrac{8}{7}.$

Multiply both sides *of the equation* by $7(x-1)(x-2)$.

$$\therefore\ 21(x-1)-35(x-2)=8(x-1)(x-2)$$
$$\therefore\ 21x-21-35x+70=8x^2-24x+16$$
$$\therefore\ 8x^2-10x-33=0.$$
$$\therefore\ (2x+3)(4x-11)=0.$$
$$\therefore\ x=-1\tfrac{1}{2} \text{ or } 2\tfrac{3}{4}.$$

Example—A train which is timed to run a distance of 126 miles in 2 hours 55 minutes finds when it has gone half-way that it is behind time By increasing its speed by 18 miles per hour it just arrives at the time specified. At what speeds did it run the two portions of the journey ?

Let the original speed be x miles per hour the time taken over the first half of the journey is $\dfrac{63}{x}$ hours

The final speed is $x+18$ miles per hour. .. the time over this section $=\dfrac{63}{x+18}$ hours.

$$\therefore\ \frac{63}{x}+\frac{63}{x+18}=2\tfrac{11}{12},$$

$$i\,e.\ \frac{63}{x}+\frac{63}{x+18}=\frac{35}{12},$$

$$\text{or}\quad \frac{9}{x}+\frac{9}{x+18}=\frac{5}{12} \text{ (dividing both sides by 7).}$$

Multiply both sides *of the equation* by $12x(x+18)$.

$$\therefore\ 108(x+18)+108x=5x(x+18).$$
$$\therefore\ 5x^2-126x-1944=0.$$
$$\therefore\ (x-36)(5x+54)=0$$
$$\therefore\ x=36 \text{ (or } -10\tfrac{4}{5}\text{, which is inadmissible).}$$

$\therefore$ the original speed is 36 miles per hour, and the later speed $36+18$, or 54, miles per hour.

Examples Leading to Quadratics

Exercise XI (e)

Solve the equations

1. $\dfrac{2}{1-x}=\dfrac{2x}{3}-\dfrac{3(1+x)}{4}.$

2. $\dfrac{x}{x+1}-\dfrac{1}{4x}=\dfrac{2}{3}.$

3. $\dfrac{14}{x+1}-\dfrac{15}{x+2}=\dfrac{2}{3}.$

4. $\dfrac{1}{x}-\dfrac{3}{8}=\dfrac{2}{2x+3}.$

5. $\dfrac{2}{x-4}=\dfrac{3}{2x-9}-\dfrac{2}{x-3}.$

6. $\dfrac{x-1}{3x+2}=2-\dfrac{3x+2}{x-1}.$

7. Two numbers differ by 15 If their product is 406, find the numbers.

8. The perimeter of a triangle is 21 cm. and the longest side is 9 cm. If the sum of the squares on the three sides is 155 sq. cm., what are the lengths of the two smaller sides ?

9. A piece of wire 26 cm. long is bent up so as to form a rectangle whose area is 36 sq. cm. Find the sides of the rectangle.

10. Find two numbers differing by unity whose product is $15\frac{3}{4}$.

11. The equation $y=mx+\dfrac{18}{m}$ is satisfied when $x=-1$ and $y=3$ What are the possible values of m ?

12. The cost of levelling a square plot at 1 shilling per square yard and then enclosing it with palings at 2s 6d. per linear yard is £60. Find the length of each side.

13. A circular summer-house is surrounded by a verandah 6 ft. wide. The floor-space of the verandah is $5\frac{1}{4}$ times that of the summer-house. What is the diameter of the summer-house?

14. A rectangular yard is surrounded on three sides by a wall of total length 36 yd. If the area of the yard is 160 sq. yd., what is the length of each wall?

15. The length of a rectangular field is 22 yd. more than its breadth. If the area is 2 acres, find the breadth.

16. A ship sails $(x+4)$ miles due north, then $(x+1)$ miles west, and finally $(x-3)$ miles south If it is now 25 miles from its starting-point, what is the value of x? Can you interpret both of your answers?

17. A train is due in at the station at x minutes past x^2 o'clock By mistake a man arrives to meet it at x^2 minutes past x o'clock. He consequently has to wait x minutes less than x hours. When was the train due?

18. When two chords AB, CD of a circle cut at O, it can be proved by geometry that the area of a rectangle whose sides are OA and OB is equal to that of another rectangle whose sides are OC and OD A chord 10 cm long passes through the midpoint of another chord 7 cm. long. What are the lengths of the segments of the first chord?

19. Divide a line 10 in long into two parts such that the square on the longer part may be twice the square on the smaller.

20. A line AB of length 8 cm. is bisected at D and divided unequally at C. If $\frac{AD}{DC} \div \frac{AC}{CB} = 6$, find the lengths of AC and CB.

21. Given a quadrant of a circle of radius 5 cm, find the radius of the circle which can be drawn within the quadrant so as to touch each of the two radii and also the circumference.

22. Two numbers differ by 4. When each is divided into 80, the difference of the quotients is $4\frac{1}{2}$. Find the number.

23. A rectangular field is 100 yd long by 40 yd broad. A gate in one of the longer sides is twice as far from one of the opposite corners as it is from the other. How far is it from the nearer corners ?

24. A tourist could save $\frac{1}{4}$ hour in getting to a place 36 miles away by travelling 2 miles per hour faster At what speed does he actually travel ?

25. A novel contains 360 pages Reading the second half of the book at 20 pages per hour more than the first half, I complete the whole book in $7\frac{1}{2}$ hours. How many pages do I read in an hour ?

26. By purchasing an Empire tobacco which costs him 5d. per ounce less than his usual American mixture, a man can buy $\frac{1}{2}$ lb more than before for each £1 that he spends Find the price per ounce of each of the two tobaccos

27. A country farmer having spent £1 in the market in buying a number of baskets finds that he has not obtained enough, and the next day has to spend another £1 in purchasing an extra supply from a travelling van For these he has to pay 2d apiece more than the market price Altogether the number bought was 54 At what prices did he buy ?

28. B takes 8 minutes more than A to do a piece of work. If they worked together the whole work could be done in $7\frac{1}{2}$ minutes How long would each take separately ?

29. A cathedral enclosure in the form of a rectangle 100 yd. by 60 yd. has a covered cloister walk constructed all round the inside The remainder is covered with grass and used as a graveyard. What is the width of the cloister if the area of the grass is exactly 1 acre ?

30. A boy who is plotting the graph of $\dfrac{x^2}{2x+1}$ finds that there are two values of x differing by $\frac{3}{4}$ for which the expression has the same value What are these values of x ?

31. Two billiard balls get jammed against each other in the corner of the table, each touching a different side. The distances of the corner from the two points where the balls touch the sides are 2·2 in. and 2·6 in. respectively. Find the radius of each ball.

32. The speed of a river being $1\frac{1}{2}$ miles per hour, a man finds that it takes him 48 minutes to row to a place $2\frac{1}{4}$ miles down stream and then to return against the current. At what speed would he propel the boat in still water ?

Example.—Solve $\sqrt{x-3}=2x-7$.

Squaring both sides of the equation we obtain

$$x-3=4x^2-28x+49 \quad . \quad . \quad . \quad \text{(i)}$$
$$\therefore\ 4x^2-29x+52=0.$$
$$\therefore\ (x-4)(4x-13)=0.$$
$$\therefore\ x=4 \text{ or } 3\tfrac{1}{4} \quad . \quad . \quad . \quad \text{(ii)}$$

If we test these solutions we find that when $x=4$ both sides of the equation are equal to 1, and therefore $x=4$ satisfies the original equation, but that when $x=3\frac{1}{4}$, the L H S is equal to $+\frac{1}{2}$ and the R H.S to $-\frac{1}{2}$, and that therefore $3\frac{1}{4}$ does not satisfy the original equation Thus, there is really only one solution, viz. $x=4$. It is easy to see that the equation (i) would come from squaring $-\sqrt{x-3}=2x-7$ as well as from the original equation (i), and that therefore the two values given in (ii) are the solutions of the *two* equations $\sqrt{x-3}=2x-7$ and $-\sqrt{x-3}=2x-7$ The test shows that the first value 4 belongs to the given equation, and the extraneous value $3\frac{1}{4}$ to the second equation.

The only solution to the given equation is $x=4$.

Note.—You should never square both sides of an equation containing a square root until it has been rearranged so that the expression containing the square root stands on one side of the equation by itself; *e.g.* to solve

$$2x-\sqrt{x-3}=7,$$

first rearrange, $\therefore\ 2x-7=\sqrt{x-3}$,

and now proceed as above.

Simultaneous Quadratics

Simultaneous equations, of which one is quadratic and the other linear, are conveniently solved by eliminating one of the unknowns.

Example 1.—Solve

$$\begin{cases} x^2+8y=5xy+36 \\ 3x+y=8. \end{cases}$$

From the simple equation $y=8-3x$.

Substituting in the quadratic we have

$$x^2+8(8-3x)=5x(8-3x)+36.$$
$$\therefore\ x^2+64-24x=40x-15x^2+36$$
$$\therefore\ 16x^2-64x+28=0.$$
$$\therefore\ 4(2x-7)(2x-1)=0.$$
$$\therefore\ \text{either } x=\tfrac{7}{2} \quad \text{or} \quad x=\tfrac{1}{2}.$$

From the simple equation

$$y=8-\tfrac{21}{2}=-\tfrac{5}{2} \quad \text{or} \quad 8-\tfrac{3}{2}=\tfrac{13}{2}.$$
$$\therefore\ x=\tfrac{7}{2},\ y=-\tfrac{5}{2}, \quad \text{or} \quad x=\tfrac{1}{2},\ y=\tfrac{13}{2}.$$

A similar method may be used in the following example:

Example 2.—Solve $\begin{cases} xy=28 & \text{(i)} \\ x^2+y^2=65 & \text{(ii)} \end{cases}$

From (i) $$y=\frac{28}{x}.$$

Substituting in (ii) we have $x^2+\dfrac{784}{x^2}=65.$

$$\therefore\ x^4-65x^2+784=0.$$
$$\therefore\ (x^2-49)(x^2-16)=0.$$
$$\begin{aligned} \therefore\ x&=\pm7 \\ \text{and } \therefore\ y&=\pm4 \end{aligned}\Big\} \quad \begin{aligned} &\text{or} \quad \pm4 \\ &\text{or} \quad \pm7 \end{aligned}\Big\} \text{ from 1.}$$

Another method is illustrated by the following:

Example 3.—Solve $\begin{cases} x+y=3 & \text{(i)} \\ xy=2 & \text{(ii)} \end{cases}$

Square (i) and subtract 4 times (ii), then

$$x^2+y^2-2xy=1.$$
$$\therefore\ x-y=\pm1.$$

Also $$x+y=3.$$

$$\therefore\ x=2,\ y=1; \quad \text{or} \quad x=1,\ y=2.$$

Exercise XI (f)

Solve the simultaneous equations

1. $x^2-2xy=51$, $2x+y=1$.
2. $x+y=3$, $xy=-10$.
3. $3x^2-2y=19$, $x+3y=9$.
4. $5x^2+3xy-2y^2=20$, $5x-2y=10$.
5. $\dfrac{x^2}{9}+\dfrac{y^2}{4}=25$, $2x-3y=6$.
6. $5x-3y=-1$, $xy=2$.
7. $x^2-4y^2=42$, $x-2y=7$.
8. $x^3+y^3=19$, $x+y=1$.
9. $3x^2-5xy-2y^2=24$, $4x+3y=-1$
10. $2x^2+3y^2=56$, $3y-2x=8$.

Solve the following equations, in each case testing the values you obtain and rejecting those which do not satisfy the actual equation given :

11. $\sqrt{2x+6}=x-1$.
12. $\sqrt{18+3x}=2x+9$.
13. $4x-\sqrt{2x+11}=6$.
14. $3+2\sqrt{x}=x$.
15. $\sqrt{5x-19}=2\sqrt{x-3}$.
16. $5x=8+2\sqrt{5x+7}$.
17. $\sqrt{29+19x-2x^2}-2x=3$
18. $\sqrt{3x+4}-3x-4=0$
19. $3\sqrt{4x+9}-2x=7$.

Solution by Formula

Since any quadratic can be expressed in the form $ax^2+bx+c=0$, we could write down the solution of any numerical example by remembering the result obtained for this equation.

We have $ax^2+bx+c=0$.

$$\therefore\ x^2+\frac{b}{a}x=-\frac{c}{a}\ \text{[dividing both sides by } a\text{]}.$$

$$\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2\ \text{[completing the square on the L.H.S.]}.$$

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}.$$

$$\therefore\ x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Thus $x=-\dfrac{b}{2a}+\dfrac{\sqrt{b^2-4ac}}{2a}$ or $-\dfrac{b}{2a}-\dfrac{\sqrt{b^2-4ac}}{2a}$,

which are generally combined in the single formula

$$\mathbf{x}=\frac{-\mathbf{b}\pm\sqrt{\mathbf{b}^2-\mathbf{4ac}}}{\mathbf{2a}}.^*$$

Example.—To solve $3x^2-2x=5$.

N.B.—Rewrite in the form $3x^2-2x-5=0$ to correspond with $ax^2+bx+c=0$.

Here $a=3$, $b=-2$, $c=-5$. $\therefore\ b^2-4ac=4+60=64$.

$$x=\frac{+2\pm\sqrt{64}}{6}$$
$$=\frac{2+8}{6}\ \text{or}\ \frac{2-8}{6}$$
$$=\tfrac{5}{3}\ \text{or}\ -1.$$

Exercise XI (g)

Without extracting the square roots, write down the roots of the following equations

1. $5x^2-7x-3=0$
2. $2x^2-8x+5=0$
3. $7x^2+4x-1=0$
4. $3x^2+2x=9$
5. $4x^2+11x+3=0$
6. $3-5x-4x^2=0$.
7. $2x^2-5x=1$.
8. $6x^2+7x=6$.

* An eighteenth-century mnemonic for this runs :

"From square of b
Take $4ac$,
Square Root extract
And b subtract,
Divide by $2a$
You'll get x alway."

By means of the formula, solve the equations

9. $ax^2-(3a-2b)x+2(a-b)=0.$

10. $pqx^2-2(p^2-q^2)x-4pq=0.$

11. $ax^2+(a^2-b)x-ab=0.$

Find approximate numerical values for a and b when

12. $4x^2+11x+3\equiv 4(x-a)(x-b).$

13. $2x^2-5x-1=2(x-a)(x-b).$

HISTORICAL NOTE

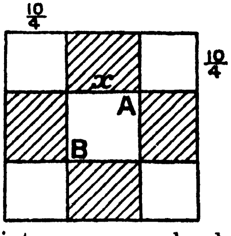
The solution of quadratic equations presented itself to the Greeks as a geometrical problem, and was effected by means of a set of constructions based on the theorem corresponding to $(a+b)^2=a^2+2ab+b^2$

Diophantus (*circa* A.D. 250) was the first to arrange the work in an algebraic form, but found it necessary to limit the problem to the three cases corresponding to $ax^2+bx=c$, $ax^2=bx+c$, $ax^2+c=bx$, where a, b, c are all positive numbers. As he always ignores any negative root which may occur, it would seem as if he regarded $ax^2+bx+c=0$ (both of whose roots are negative) as insoluble.

The influence of geometry is still traceable in the time of Al Kharismi (*circa* A.D. 825), whose solution of the equation $x^2+10x=39$, if modernised, would be much as follows:

Represent the square of the unknown (*i.e* x^2) by the figure AB, and add a border of width $\frac{10}{4}$ all round the square.

Then the figure AB + the 4 shaded rectangles $=x^2+10x$
$=39.$



Adding to x^2+10x the four corner squares whose area is $4(\frac{10}{4})^2$ or 25, then the area of the completed square is $39+25=64$.
$\therefore$ the side of this enlarged square is 8.
$\therefore$ $x=8-2(\frac{10}{4})=8-5=3.$

It will be seen that this method only gives the positive root, although the equation also possesses another solution, viz. $x=-13$.

The Hindus were apparently the first to unify the three cases given by Diophantus into one general rule for $ax^2+bx+c=0$, where a, b, c may be either positive or negative; but though they were able to discover negative roots they were unable to interpret their meaning. Thus Bhaskara (A.D. 1150), after correctly finding the roots of $x^2-45x=250$ to be $+50$ and -5, adds, "but the second value in this is not to be taken, for it is inadequate; people do not approve of negative roots."

It was not until much work had been done on equations of the 3rd and higher degrees in the sixteenth century (especially by the Italians), and until after the work of Descartes in the first quarter of the seventeenth century, that the understanding of negative roots became common.

Miscellaneous Examples IV

A 61

1. From $5x-3y$ subtract the sum of $2x-3y+5z$, $4x-2y-3z$, and $y-3x+2z$.

2. Find all the possible values of $\left\{\frac{2\sqrt{x}}{y}-\frac{x}{\sqrt{y}}\right\}^3$ when $x=9$, $y=4$.

3. Express in as simple a form as possible

$$\text{(i)}\quad [(x-a)(y+b)]-[(x+a)(y-b)],$$
$$\text{(ii)}\quad [(x-a)(y+b)]\times[(x+a)(y-b)].$$

4. Find the factors of

$$10x^2+29x+21,\quad 10x^2+29x-21,\quad 10x^2-90.$$

5. Solve the equations

$$\text{(i)}\quad (4x+7)(3x-5)=24-6(2x-1)(3-x);$$
$$\text{(ii)}\quad (4x+7)(3x-5)=24.$$

6. Prove that the product of the first and last of three consecutive numbers is always one less than the square of the middle number.

7. Simplify (i) $\dfrac{1+\frac{a}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$; (ii) $\dfrac{3}{2x-2}-\dfrac{x+1}{x^2-x}$.

A 62

1. Simplify $\frac{3}{4}(a-b+2c)-\frac{2}{3}(b-c-2a)+\frac{1}{2}(c-a-2b)$.

2. Find the values of $(2a^4b^3)^2$, $\sqrt{16x^{14}}$, $(-3x^2)^3$

3. Find the H.C F. and L.C M. of $6(x-2)^2$, $4(x-2)(x+3)$, and $(2x^2-8)$.

4. Pick out the coefficients of yz in $(3x+2y-5z)(2x-4y+3z)$, and of x^2 in $(2x^2-3x+1)(4x^2+5x-3)$.

5. A, B, C, D are 4 points in order on a straight line. If $AB=a$, $BC=b$, $CD=c$, show that $AB.CD+BC.AD=AC.BD$.

6. Construct an equation of the form $ax^2+bx+c=0$ whose roots will be -3 and $+1\frac{1}{2}$.

7. Solve the equations

$$\text{(i) } 2x^2=5x\,; \qquad \text{(ii) } 2x^2=5x+8.$$

A 63

1. Express in words the meanings of $(a-b)c+d$ and $a-b(c+d)$; and find the value of each when $a=5$, $b=-2$, $c=1$, $d=-3$.

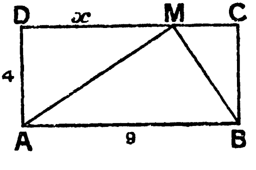
2. Arrange in descending powers of x the result of subtracting $3(2x^2+5x-3)(x^2-2x)$ from $2(3x^2-4)(2x^2-3x+2)$

3. Simplify $x-\dfrac{x-2y}{3}$, $\left(x-\dfrac{x}{3}\right)\div 2y$, $\dfrac{3}{x-2y}-\dfrac{1}{x}$.

4. Find the factors of

$$2x^2-9x-35, \quad x^4-x^2, \quad 6x^2+xyz-y^2z^2.$$

5. Prove that the product of the first and last of four consecutive integers is always 2 less than the product of the two middle numbers.



6. ABCD is a rectangle · M a point on DC such that $DM=x$. Prove that $AM^2+MB^2=2x^2-18x+113$. For what value of x is AMB a right angle ?

A 64

1. Prove that $(ac-bd)^2+(ab+cd)^2=(a^2+d^2)(b^2+c^2)$, and obtain an arithmetical relation by putting $a=4$, $b=3$, $c=2$, $d=1$.

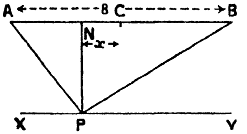
2. Divide $(2x+3)^2-3(x+6)$ by $4x-3$. Test your answer by putting $x=-2$.

3. One solution of the equation $3x^2-ax+12=0$ is $x=3$. What is the value of a and what is the other solution?

4. Simplify (i) $\dfrac{5}{x-\dfrac{y}{2}}-\dfrac{3}{x+\dfrac{y}{2}}$; (ii) $\dfrac{6x^2+7x-5}{9x^2-25}$.

5. Express as the product of two factors:

(i) $x^2+2xy-3xz-6yz$, (ii) $x^3+2x^2+5x+10$.



6. AB=8 cm. and is 2 cm. from XY, which is parallel to AB. From P any point in XY, PN is drawn perpendicular to AB, so that NC$=x$, where C is the midpoint of AB. Find an equation for x when $PB^2=3PA^2$, and solve it.

A 65

1. Multiply $3x^2y-5y^3$ by $2x-3y$, and divide $6x^3-25x^2+20x-4$ by $3x-2$.

2. If $a=\frac{3}{5}$, $b=-\frac{4}{5}$, $c=\frac{2}{5}$, find the values of

$$\frac{a}{b},\quad \frac{ab}{c^2},\quad \frac{2a+b}{c},\quad \frac{a^2b}{c^3}.$$

3. Simplify $\dfrac{4x+2}{6x+3}-\dfrac{3x+6}{2x+4}$.

4. A room is a ft long and b ft. broad When it is carpeted a border c ft. wide is left all the way round. What are the areas of the carpet and of the border? Find the cost of staining the border at x shillings per square yard.

5. What must be added to the expressions x^2+8x, x^2-7x, $4x^2+5x$ in order that the results may be perfect squares? In each case state the expression whose square has been obtained.

6. The father of a family buying tickets for a railway journey notices that he has bought x tickets at x shillings each If the amount he paid for the tickets was £3, 12s. 3d., what is the value of x?

A 66

1. If $\dfrac{2x^2-5x+6}{x-4}$ is written as a mixed number in the form $ax+b+\dfrac{c}{x-4}$, where a, b, c are numerical, what are their values ?

2. If $x=\dfrac{a+b}{2}$ and $y=\dfrac{a-b}{3}$, find the value of

$$4x^2-12xy+9y^2$$

3. The diagonal of a rectangle is 3 in and its perimeter is 8·4 in. Find the lengths of its sides to the nearest tenth of an inch.

4. Express in their simplest form

$$\sqrt{36x^{36}}, \quad \sqrt[3]{8x^6}, \quad (-2x^3y^2)^5, \quad \sqrt[3]{\frac{x^6}{8y^9}} - \sqrt[2]{\frac{9x^4}{4y^6}}$$

5. Show that values of a and b can be found such that the equations

$$\frac{x-a}{3}=\frac{b-y}{4}=1 \quad \text{and} \quad \frac{x+2a}{9}=\frac{b+y}{2}=1$$

have a common solution

6. Draw two graphs to illustrate the growths of two investments of £100, one increasing regularly by £12 each year, and the other increasing each year by one-tenth of its value at the beginning of that year. When will the second sum overtake the first ?

A 67

1. Find the values of $a^{2n+1}-b^{2n}$ when

(i) $a=3$, $b=2$, $n=\frac{1}{2}$, (ii) $a=-3$, $b=4$, $n=1$

2. Simplify

(i) $3-6x\div6x-3$; (ii) $(3-6x)-(6x-3)$; (iii) $3-6x-(6x-3)$

3. Solve the equations

(i) $2x^2+9x-35=0$; (ii) $x^3-x^2-2x=0$.

4. By what expression must the product of $2x^2+x-6$ and

$6x^2-11x+3$ be multiplied so that the result may be a perfect square ?

5. Two numbers differ by two. If 24 is divided by each, the sum of the two quotients is 7. Find the numbers.

6. Prove that the area of the ring between two concentric circles of radii r and $r+x$ is $\pi x(2r+x)$.

A 68

1. Expand $\left(\frac{a^2}{4b}+\frac{2b}{a^2}\right)^2$, $(2a-b-3c)^2$, $(2x+y)^2-(2x-y)^2$.

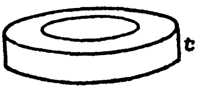
2. Solve the equation $a(a-x)=2b(x+2b)$.

3. Solve $\frac{8x+7}{2x+1}-\frac{3x+3}{x+2}=1$, first expressing each fraction in the form of a mixed number.

4. Find the factors of

(i) $2p^2-4pq+2q^2$; (ii) $2p^2-4pq+3pr-6qr$;
(iii) $(2p-q)^2-(q-2r)^2$.

5. A piece of wire 36 cm. long is bent up to form a rectangle of area 72 sq. cm. What are the lengths of the sides ?



6. The internal diameter of a circular washer is d and the width of the ring is n. If the thickness is t, prove that the volume is $\pi nt(n+d)$.

A 69

1. Simplify

$(-2a^3b^2)^3$, $\sqrt{4x^{16}}$, $x^{p+q}\times x^{2p-q}$, $(17{\cdot}3)^2-(2{\cdot}\dot{7})^2$.

2. Find the value of

$$2(x-y)-3\left\{\frac{x^2-(y+z)^2}{x+y+z}\right\} \text{ when } x=-2,\ y=3,\ z=-5.$$

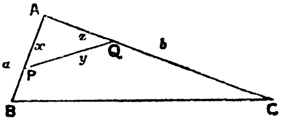
3. In how many different ways can you write the result of multiplying $(2x-y)(4x+y)$ by 5 ? Write down in factor form the result of dividing $(6x-8y)(6x-9y)$ by -6.

4. Solve the equations $\frac{15}{x}+\frac{4}{y}=11$, $\frac{3}{y}-\frac{1}{x}=4\frac{1}{6}$.

5. Express as the product of two factors

(i) $(x-a)^2-(a+y)^2$; (ii) $a^2-4ab+4b^2-c^2$.

6. In the figure $BAC=90°$ and PQ divides the triangle into two equal areas. Express y in terms of x and z. Find a relation between x, z, a, b, and show that $y^2=x^2+\dfrac{a^2b^2}{4x^2}$.



A 70

1. If $a+2b=0$, express $4(a-b)^2$, $4a^2-4b^2$, $(4-a)^2-4b^2$ in terms of a only.

2. Employ the formula $(a-b)^3=a^3-b^3-3ab(a-b)$ to calculate the value of $(9{\cdot}7)^3$.

3. Simplify

$$\frac{1}{2x-4}-\frac{1}{4+2x},\qquad \frac{1}{2x-4}-\frac{1}{4-2x},\qquad \frac{1}{2x-4}-\frac{1}{4-x^2},$$

and
$$\frac{1}{2x-4}-\frac{1}{(2-x)^2}.$$

4. At what rate per cent. of simple interest will £x amount to £y after n years? Check your answer by taking $x=100$, $y=150$, $n=5$.

5. If $x=\dfrac{y+a}{1+ay}$, find y in terms of x. What is its value when $x=-a$?

6. A series of points A, B, C . . . have co-ordinates (3, 2), (6, 6), (9, 10), (12, 14), and so on. What will be the co-ordinates of the next point? Show from a graph that all these points lie on a straight line. Can you *prove* this geometrically? [Draw lines parallel to the axes so as to form triangles ABS, BCK.]

A 71

1. Write down the square of $2a-7b$, the square root of $9x^2-30x+25$, and the quotient when $9a^2-30ac+25c^2-49b^2$ is divided by $3a+7b-5c$.

2. Rewrite $3ap-10bq+15bp-2aq$ (i) in the form ( )$-$( ) and (ii) in the form ( )$\times$( ).

3. Simplify

$$\text{(i)}\ \frac{2}{a^2-2ab}-\frac{1}{2b^2-ab}; \qquad \text{(ii)}\ \frac{2}{a^2-2ab}\div\frac{1}{2b^2-ab}.$$

4. Show that $x=2$ satisfies the equation $x^3-6x^2+7x+2=0$ What are the other two solutions ?

5. The product of two consecutive *odd* numbers is 1763 What are they ?

6. A man who has to travel by car to a railway statior 18 miles away from his home is delayed 9 minutes on starting and consequently has to drive his car at 4 miles per hour more than his usual speed in order to arrive at the station at the time he had originally intended. At what speed did he actually drive ?

A 72

1. Show that the product of $2x^2-5x-3$, $3x^2-10x+3$, and $6x^2+x-1$ is a perfect square, and write down its square roo in factor form.

2. By how much does $(99{\cdot}7)^2$ exceed $(99{\cdot}3)^2$? Write dowı an algebraic formula by means of which the calculation can b quickly effected.

3. Solve the equations

$$\text{(i)}\ \frac{5}{x+2}-\frac{6}{x+1}=\frac{6}{x}; \qquad \text{(ii)}\ \frac{5}{x+2}\div\frac{6}{x+1}=\frac{6}{x}.$$

4. Simplify where possible

$$\frac{a^2-b^2}{(a-b)^2},\quad \frac{a^2-ab}{bc-ac},\quad \frac{(a-b)^3}{(b-a)^2},\quad \frac{(a-b)(c-d)}{(b+a)(d-c)}.$$

5. The perimeter of an isosceles triangle is 20 in. and it altitude is 2 in. ; find the lengths of the sides. [Take th length of the base to be $2x$.]

6. Draw the graphs of $\frac{1}{3}(x-2)^2$ and $\frac{3}{4}(x-2)$. For wha values of x is the first expression greater than the second ?

A 73

1. If $2s=a+b+c$, prove that $s(s-a)+(s-b)(s-c)=bc$.

2. Evaluate $\dfrac{6x^2+11xy-10y^2}{2x^2-3xy-20y^2}$ when $x=0{\cdot}6$ and $y=-0{\cdot}1$.

3. If $F=\dfrac{P}{c}(\mu+d)$, find μ in terms of the other letters.

4. Solve the equation $\dfrac{3x-1}{x-1}-\dfrac{2x+1}{x+1}=x$.

5. If a landowner makes a profit of x per cent. by selling an estate for £a, what would be his profit per cent. if he sold it for £b?

6. The speed of a falling body increases from u ft. per second to v ft. per second while falling through a vertical height h ft. It is known that the distance fallen in the first half of the time is $\dfrac{(3u+v)h}{4(u+v)}$. Find the distance fallen in the second half. If this result is called b, find $\dfrac{v}{u}$ in terms of b and h.

A 74

1. Divide $(p+q)^2-(q-r)^2$ by $p+r$, and multiply the quotient by $p-2q+r$.

2. Find the factors of

(i) x^2-y^2-x+y, (ii) $x^2-4y^2-12y-9$.

3. Simplify the expression $\dfrac{4}{5x+5}-\dfrac{3}{4x-4}-\dfrac{1}{1-x^2}$.

4. Find the roots of the equation $2x^2-x-15=0$, and construct a new equation whose roots will be double those of this equation.

5. Tarpaulin sheets are made of material costing 5d. per square foot and the cost of binding their edges is 8d. per yard. Each sheet is twice as long as it is broad. If there are two such sheets whose lengths differ by 2 ft. and whose total costs differ by 8s. 10d., what are the dimensions of each ?

6. An article is bought for £a and sold for £b. Find the gain per cent. If buying price and selling price are both increased by £x, find the new gain per cent. By how much does it differ from the previous result ? Give your reasons for stating whether it is greater or less than before.

A 75

1. Multiply $3(2b-a)$ by -2, and $(x-2y)(p-q)$ by -3. In each case give three possible forms of the answer

2. Divide a^3-8b^3 by $a-2b$ and show that

$$\frac{a^3-8b^3}{a-2b}-(a-2b)^2 \equiv 6ab.$$

3. If $x-3$ divides $2x^3-x^2+Rx+3$ without a remainder, what must be the value of R ? When R has this value, what are the values of x which make $2x^3=x^2-Rx-3$?

4. Find two values of x which make $\left(\frac{1}{x}-\frac{x+1}{x^2}\right)\left(x-\frac{x^2}{x+1}\right)$ equal to $-\frac{1}{2}$.

5. A man who is out walking at 12 o'clock when it comes on to rain calculates that if he increases his speed by $\frac{1}{2}$ mile per hour he would get home by 3.30 p.m., or that if he left his speed unaltered and cut out one of the villages he had intended to visit (thereby reducing his journey by $3\frac{1}{2}$ miles) he would return by 3 p.m. Finally he decides both to shorten the journey and to quicken his pace. At what time does he arrive home ?

6. Given that $x=\frac{3-3t^2}{1+t^2}$, $y=\frac{6t}{1+t^2}$, find the values of x and y when t has the 7 values 0, ±1, ±2, ±3 Plot each corresponding pair of values of x and y on squared paper What does the position of your points suggest ? Calculate the value of x^2+y^2 from the expressions given for x and y.

B 76

1. Simplify $\left(x+3-\frac{6}{x-2}\right)\left(x+5+\frac{7}{x-3}\right)$.

2. Solve the equations

(i) $\frac{9}{x+3}=2+\frac{1}{x+1}$; (ii) $\frac{9}{x+3}=2-\frac{2}{x+1}$.

3. Express $\sqrt{4a^2+4a+1}-\sqrt{4a^2-a+\frac{1}{16}}$ in its simplest orm.

4. If $\frac{p}{qx+p}=\frac{q}{px-q}$, express x in terms of p and q.

5. A square metal plate of side a in. is lightened by the unching of 7 circular holes each of radius r in If the reluced weight is $\frac{7}{18}$ths of the original weight of the plate, find in terms of a. (Take $\pi=\frac{22}{7}$.)

6. Show that the triangle whose sides are x^2+4x+3, $2x+4$, nd x^2+4x+5 contains a right angle. If the length of the ypotenuse is 17, what are the lengths of the other two sides ?

B 77

1. Simplify $(3x+7y)^2-4(x+y)(3x+7y)-21(x+y)^2$.

2. What must be added to $9x^2+12x$ to make the result a erfect square ? Hence solve the equation $9x^2+12x=5$.

3. Simplify $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}\times(p-q)^2$.

4. If $P\equiv 7+3a$, $Q\equiv 5-2a$, what values of a will make

(i) $\frac{P+Q}{P-Q}=\frac{1}{4}$, (ii) $\frac{P}{Q}=\frac{Q}{P}$?

5. Write down an expression which will always denote an odd number. If any three consecutive odd numbers are chosen, prove that the sum of their squares increased by unity is always divisible by 12.

6. Last year one department (A) of a general stores made a profit of £3000, and it is calculated that its annual profits are increasing at the rate of £200 per year. Of another department (B) which last year made a profit of £5000, it is estimated that the profits are diminishing by £400 each year. Draw two graphs to show the probable profits of the two departments during the next 10 years When will the annual profits derived from A first exceed those from B, and in what year will it exceed them by £2000 ? When are A's profits double those of B's ?

B 78

1. What is the coefficient of x^5 in $\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right)^2$?

2. Simplify

(i) $\frac{3}{x^2-3xy} + \frac{1}{3y^2-xy}$, (ii) $\frac{3}{x^2-3xy} - \frac{1}{3y^2-xy}$.

3. Solve the equations $x+y=a+b$, $bx-ay=2(a+b)$.

4. Find numerical values of A and B such that $\frac{x^2}{(x-3)(x+4)}$ may be equivalent to $1 + \frac{A}{x-3} + \frac{B}{x+4}$.

5. What multiples of $\frac{(2x-1)(2-x)}{3}$ are

(i) $\frac{(4x-2)(4-2x)}{3}$; (ii) $3(2-4x)(4-2x)$,

and (iii) $\frac{2(2-4x)(3x-6)}{9}$?

6. A man who has bought a number of articles for £450 sells them all so that his profit on each article is 5 shillings If his total profit was the amount he received for 3 articles how many did he buy ?

B 79

1. Reduce $(3x+2)(2x-3)-(3x-1)(2x-1)$ to its simplest form. Is there any value of x which would leave the value of the given expression unaltered if the sign between the brackets were read as — instead of "minus"?

2. Find the equations whose roots are (i) $2\frac{1}{3}$ and $-\frac{1}{2}$; (ii) 3, 2, and -5.

3. The speed of a train t seconds after the application of the brakes is given by the formula $(44-\frac{4}{3}t)$ ft per second, and the distance it has run at that time is $(44t-\frac{2}{3}t^2)$ ft. How far does the train go in 30 seconds after the brakes have been applied, and at what speed will it then be running? How much longer and how much farther will it go before coming to a halt?

4. Solve the equations $\dfrac{1-x}{y}=a, \qquad \dfrac{1+x}{y}=\dfrac{1}{b}.$

5. If the price of sugar rose $1\frac{1}{2}$d per pound one could buy 6 lb. less than before for 9 shillings. What is the present price per pound?

6. Draw the graph of $\frac{1}{4}(2x-1)(2x+3)$ for values of x between -3 and $+2$ inclusive. For what values of x is the expression (i) negative, (ii) equal to $+2$?

B 80

1. If $z=(x-3)(y+4)-(x+3)(y-4)$, find an expression for the value of y in terms of x and z. If the ratio of x to z is equal to $3 : 4$, find the ratio $y : z$.

2. Solve the equation $\dfrac{x}{x-\dfrac{2}{x-1}}=1\frac{1}{2}.$

3. Simplify $2-x-(3-x)-(2-x)\div(3-x)$.

4. If $t^2=x(x+2r)$, find an expression for r in terms of t and x, and calculate its value when $t=3{\cdot}2$, $x=0{\cdot}8$.

5. Express $(a-b)(a^2+3ab+2b^2)-(a+b)(a^2-3ab+2b^2)$ as the product of three factors.

6. From a point h ft. above the sea-level the distance of a ship which is just visible on the horizon is approximately d miles where $d^2=\frac{3}{2}h$. How far away is a ship which I can just see from the upper gallery of a lighthouse situated 200 ft. above sea-level? If the ship is approaching the lighthouse directly, how fast is it travelling if I can just see it on the horizon 15 minutes later when I have descended 80 ft. to a lower gallery?

B 81

1. Factorise $(2x^2-6x+1)^2-(x-4)^2$, and hence find the values of x which satisfy the equation $(2x^2-6x+1)^2=(x-4)^2$.

2. Without solving, prove that the equation $3x^2-9x=7$ has a root between 3 and 4. Then solve it and give the roots correct to 2 decimal places.

3. Given that $\dfrac{2}{x}+\dfrac{1}{y}=\dfrac{2+m}{x+y}$, find m in terms of x and y.

4. If $x^2=x+1$, prove that $x^3=2x+1$ and $x^4=3x+2$, and obtain a similar expression for x^7.

5. Construct a table of values for $4-5x-2x^2$ as x changes from -3.5 to $+2.5$ by steps of 0.5 at a time. Hence, without drawing a graph, find the solutions of $4-5x-2x^2=0$.

6. The distance between two towns is 120 miles. Owing to a snowstorm a train running between the two towns has to reduce its usual speed by 10 miles per hour, and is consequently 36 minutes late on arrival. What is its usual speed?

B 82

1. Reduce to as simple a form as possible

$[(p-q)(q+r)]-[(p+q)(q-r)]$; $[(p-q)(q+r)]\div[(q-p)(r-q)]$;
$[(p-q)(q+r)]\times[(p+q)(q-r)]$; $[(p+q)(q-r)]+[(q+p)(r-p)]$.

2. What values of x and y will satisfy all *three* of the equations $x^2=4$, $y^2=9$, and $3y-2x=13$?

3. A series of terms is written down thus: $1.2+2.3+3.4+4.5+\ldots$ What is the nth of these terms? *Prove* that the average of *any* two consecutive terms of the series is always a perfect square Two consecutive terms add up to 1682. Which are they ?

4. Express $ab(c^2+d^2)+cd(a^2+b^2)$ as the product of two factors.

5. Divide 1 by $1-2x$ by the long-division process, stopping at the term of the quotient which involves x^2 and giving the remainder in the form of a fraction. What are the values of the quotient and remainder (i) when $x=0{\cdot}1$, and (ii) when $x=1$?

6. When the contents of a cylinder are poured into another cylinder whose radius is 2 in greater, the height reached in the second cylinder is half that reached in the first. Find the radius of each cylinder.

B 83

1. What are the H C F. and L.C.M. of $12a^3b^2$, $15a^2b^3c$, and $18a^3c^2$?

2. What number reduced by x per cent becomes y ? What does y become when it is increased by x per cent. ?

3. Simplify (i) $\dfrac{a^2+3a-4}{2a^2-7a+6}\times\dfrac{6a^2-5a-6}{3a^2-a-2}$,

(ii) $\dfrac{1}{(x-2)^2}+\dfrac{1}{4-x^2}$.

4. Given $(a+b)^3\equiv a^3+3a^2b+3ab^2+b^3$, write down the cubes of (i) $3x-2y$, and (ii) 102.

5. The diameter AB of a circle of radius 2 in. is produced to C so that BC$=x$ in. If CT is the tangent from C to the circle, find x when CT$=\sqrt{12}$ in.

6. In June and July a boy has 25 completed innings at cricket. At the end of June his average was 3 less than the number of innings he had played, but for the month of July his average is 22 more than the number of his innings during that month. If his final average is 25, how many innings did he have in June ?

B 84

1. Solve each of the equations

$$\text{(i)}\ \frac{9x-5}{3x-2} = \frac{6x-6}{3x+1} + 1;$$

$$\text{(ii)}\ \frac{9x-5}{3x-2} = \frac{6x+4}{3x+1} + 1;$$

$$\text{(iii)}\ \frac{9x-5}{3x-2} = \frac{6x+3}{3x+1} + 1;$$

accounting for any peculiarity that may occur.

2. Find the L.C.M. of $(2a^2-8b^2)^2$, $(2ab+4b^2)^2$, $6a^4-12a^3b$

3. Simplify $\dfrac{p-q}{1+pq}$ when $q = \dfrac{3p-2}{2p+3}$.

4. A shopkeeper buys a certain number of articles for £13, 10s, keeps 3 for his own use and sells the remainder at 3 shillings each more than he paid for them. If he thus receives back the full amount of his purchase-money, how many articles did he buy?

5. If $x^2+h^2=a^2$, $y^2+h^2=b^2$, and $x+y=c$, prove that $x-y = \dfrac{a^2-b^2}{c}$.

The base BC of a triangle ABC is 5 cm; AB=6 cm., and AC=4 cm. If the perpendicular from A on to BC is of length h and divides the base into two segments of lengths x and y find the values of x and h and deduce the area of the triangle.

6. Draw the graph of $\sqrt{25-x^2}$, taking the values ± 5, ± 4, ± 3, ± 2, ± 1, 0 for x. What is the shape of the graph? Can you account for the result? Draw the graph of $\dfrac{3x-1}{2}$ on the same axes, and hence solve the equation $\dfrac{3x-1}{2} = \sqrt{25-x^2}$.

B 85

1. Write down the square of x^2+3x-2 by putting $x^2 \equiv a$ and $3x-2 \equiv b$, and simplify the result

2. Find the factors of $(9x-6y)(a-2b)+(4b-2a)(3x-2y)$ and $(a^2+2bc)-(b^2+2ac)$.

3. Solve the equations $\frac{x}{a}+\frac{y}{c-a}=1$, $\frac{x}{b}+\frac{y}{c-b}=1$

4. Solve the equation $x=1+\frac{6}{x}$ If the roots of $x=a+\frac{b}{x}$ are double those of $x=1+\frac{6}{x}$, find the values of a and b.

5. A semicircle is described on the hypotenuse of a right-angled triangle so as to pass through the right angle. If the other two sides are a and b respectively, show that the area of the portions of the semicircle which lie outside the triangle is $\frac{11a^2-7ab+11b^2}{14}$. (Take $\pi=\frac{22}{7}$.)

6. A railway bus plies between two stations $5\frac{1}{2}$ miles apart. If the average speed of the journey is increased by $2\frac{3}{4}$ miles per hour, the time taken between the stations would be diminished by 4 minutes. What is the scheduled time ?

B 86

1. What are the factors of $a^2-2ab+b^2$? Use the result to simplify $(x^2-4x+3)^2+(x^2-3x+2)^2-2(x^2-4x+3)(x^2-3x+2)$.

2. Simplify $\frac{a}{b(a-b)}-\frac{b}{a(a-b)}-\frac{1}{a}$.

3. If $x=5$ and $x=-2$ both satisfy the equation $x^3+ax+b=0$, find the values of a and b. There is a third value of x which satisfies this equation. What is it ?

4. The rate of discharge of steam from a boiler is given by the formula $Q=3{\cdot}6\sqrt{\frac{p}{v}}$. From this, find a formula expressing p in terms of Q and v, and find its value when $Q=216$, $v=4$.

5. The price of wine having been raised 6 shillings a dozen, 6 bottles less than before can now be bought for 8 guineas. What is the present price per dozen ?

6. Draw the graphs of $\frac{7-4x}{6}$ and $\frac{2x^2}{3}$ on the same axes for values of x from -3 to $+3$. Read off from your graph the values of x (i) for which $\frac{7-4x}{6}=\frac{2x^2}{3}$, and (ii) for which $\frac{7}{6}-\frac{2x}{3}-\frac{2x^2}{3}$ is greatest.

B 87

1. Simplify $(x^2)^3(xy^2)^5\div(y^3)\left(\frac{x}{y^2}\right)^4$.

2. If $\frac{m-m'}{1+mm'}=\frac{3}{4}$, express m' in terms of m.

3. Show that there are two values of a for which x^2+x-2 and x^3+ax^2-5x+3 have a common factor.

4. Divide a line a in. long into two parts, such that the first is x per cent. longer than the second.

5. If $y=\frac{2x-1}{3}$, find the value of $3x^2-5xy+4y^2$ in terms of x.

Hence solve the equations

$$\begin{cases} 3y=2x-1, \\ 3x^2-5xy+4y^2=6. \end{cases}$$

6. ABCD is a rectangle having AB=5 in. and AD=3 in. Two *equal* circles are drawn to touch one another, and so that one of them touches AB and AD while the other touches BC and CD. Find their radii.

B 88

1. Two fishermen between them caught $(3x-1)$ fish of average weight $(x-3\frac{1}{2})$ lb. Of these, the first caught $(x+2)$ fish, and the average weight of his catch was $(x-3)$ lb. The average weight of the second man's catch was 1 lb. less. What is the value of x ?

2. Find the product of $(1+x)^2$ and $1-2x$, and divide it by $1+3x$ as far as the term involving x^2. If the remainder is given in the form of a fraction, what is its value (i) when $x=10$; (ii) when $x=\cdot01$?

3. Find the factors of $x^2-(4p-3q)x-12pq$ and of x^3-4p^2x.

4. A car will run 20 miles for each gallon of petrol when going uphill and 30 miles for each gallon when on the level. Running downhill the consumption may be neglected Going out to a distant place I use 4 gallons, and on my return by the same road 3 gallons. If half of each journey was on the level, how many miles of the journey out was uphill ?

5. If $y^2=x^3$, find the value of x when $y=8t^6$, and express the value of $\sqrt[4]{xy^2}$ in terms of t.

6. A wireless aerial is stretched from a point on a wall 40 ft. above the ground to the top of a pole erected at a point 60 ft. from the wall. The length of the wire if stretched tight would be three times the height of the pole. Find this height

B 89

1. Use the Remainder Theorem to show that $2x+3$ is a factor of $4x^3-5x+6$, and find the values of a and b if $x-1$ and $3x-1$ are both factors of $ax^4+bx^2+20x-4$.

2. Solve the equations
$$\frac{3}{x+1}+\frac{5}{y+2}=4$$
$$\frac{7}{y+2}-\frac{5}{x+1}=24$$

3. What are the roots of $6x^2-7x+2=0$? Find a new equation whose roots will be equal to their sum and difference.

4. Show that $\left\{\frac{n(n+1)}{2}\right\}^2-\left\{\frac{n(n-1)}{2}\right\}^2\equiv n^3$. Deduce the value of $\left\{\frac{(n+2)(n+3)}{2}\right\}^2-\left\{\frac{(n+2)(n+1)}{2}\right\}^2$.

5. If $A\equiv\frac{5x+1}{2x+3}$ and $B\equiv\frac{5x-2}{3x+2}$, solve the equation $A+B=AB$.

6. AOB is a diameter of a circle whose centre is O and radius r. A line NPQ parallel to AB meets the tangent at B in N and cuts the circle again at P and Q If $BN=a$, $NP=b$, prove that $r=\frac{a^2+b^2}{2b}$, and find an expression for NQ in terms of a and b Check your results by putting $a=4$, $b=2$.

B 90

1. If $a=p(1+q^2)$, $b=q(1+p^2)$, $c=(p+q)(1-pq)$, express $a+b+c$, $a+b-c$, $a-b+c$, $b+c-a$ in their simplest forms Hence find the area of a triangle whose sides a, b, c have these values by means of the formula

$$\tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}.$$

2. Find the factors of

(i) $(2x-y)^2+4(2x-y)-5$; (ii) $x^2+2xy+y^2-3x-3y-4$.

3. The perimeter of a rectangle is p in. What will be the increase in (i) its perimeter, (ii) its area, if the length and breadth are each increased by a in.? Verify your answer by means of a numerical example.

4. If $\frac{1}{r^2}=\frac{1}{a^2}+\frac{1}{b^2}$, find an expression for b in terms of a and r. From the given relation find r when $a=\frac{1}{3}$, $b=\frac{1}{4}$, and check your answer to the first part of the question by means of the result.

5. Eliminate y between the equations $x^2+y^2=25$ and $2x+y=5$, and hence find values of x and y which satisfy both equations.

6. A rides from P to Q at 10 miles per hour. B sets out from Q 10 minutes after A to ride to P at 8 miles per hour. If they meet 5 miles from half-way, what is the distance between P and Q?

B 91

1. Find the factors of

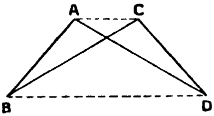
(i) $3(a+b)^2-5(a+b)-2$, (ii) $(x+2y-3z)^2-(x-2y+3z)^2$.

2. A and B have a common factor x. Show that the remainder, after subtracting any multiple of B from any multiple of A, also has x as a factor

3. If $S = ar^2$, $V = br^3$, find a relation between S and V which does not contain r.

4. A train takes s seconds to pass a man walking at a miles per hour, and t seconds to pass a man cycling at b miles per hour, both men travelling in the same direction as the train Find the length of the train in feet and its speed in miles per hour.

5. A cylindrical tin of radius a cm contains water to a depth of x cm If a solid cylinder of the same height as the tin but of radius b cm. is placed inside, prove that the water rises $\frac{b^2x}{a^2-b^2}$ cm



6. The figure represents 4 rods hinged at A, B, C, D $AB = CD = a$, $CB = AD = b$ If AM is drawn perpendicular to BD, prove $AD^2 - AB^2 = DM^2 - BM^2$ Hence show that if $AC = x$, $BD = y$, then for all positions of the framework xy is constant and equals $b^2 - a^2$

CHAPTER XII

Equations in Three Unknowns, Harder Factors and Graphs, Cyclic Order

Equations in Three Unknowns.—Equations involving three unknown variables are solved by eliminating one of the variables, and thereby reducing the question to that of solving *two* equations between *two* unknowns.

Example.—Solve

(i) $x+2y-3z=0$; (ii) $3x+3y-z=5$; (iii) $x-2y+2z=1$

First eliminate z from (i) and (ii).

We have $x+2y-3z=0$ from (i).

$9x+9y-3z=15$ „ (ii)$\times 3$.

$\therefore$ by subtraction $-8x-7y=-15$ . . . (iv)

Now eliminating z from (ii) and (iii),

$6x+6y-2z=10$ from (ii)$\times 2$.

$x-2y+2z=1$ „ (iii).

$\therefore$ by addition $7x+4y=11$ . . . (v)

The question is thus reduced to solving the two simultaneous equations (iv), (v) for x and y.

We have $56x+49y=105$ from (iv)$\times -7$.

$56x+32y=88$ „ (v)$\times 8$.

$\therefore\ 17y=17.$

$\therefore\ y=1.$

Hence from (v) $7x+4=11.$ $\therefore\ x=1.$

And from (i) $1+2-3z=0.$ $\therefore\ z=1.$

$$\therefore\ x=1, \quad y=1, \quad z=1.$$

Exercise XII (a)

Solve the equations

1. $3x-2y+z=13.$
$2x+5y+4z=-7.$
$x-3y-2z=9.$

2. $5x-4y+2z=-1.$
$3x+y-4z=27.$
$2x+3y+3z=0.$

3. $2x+3y+z=-9.$
$x+2y+2z=3.$
$2x+6y+z=0.$

4. $4x-3y+2z=18.$
$5x+2y+3z=21.$
$7y-4z=12.$

5. $2x+7y=3y+4z=5z-3x=27.$

6. $\dfrac{x}{2}=\dfrac{y-z}{3},\quad 3x-z=4y,\quad x-1=3(y+z).$

7. The expression $A+Bx+Cx^2$ is equal to 13 when $x=1$, to 10 when $x=2$, and to 3 when $x=3$. Find the values of A, B, C.

Squares and Square Roots of Trinomial Expressions

Example 1.

$$(a+b+c)^2=\{(a+b)+c\}^2=(a+b)^2+2(a+b)c+c^2$$
$$=a^2+2ab+b^2+2ac+2bc+c^2.$$
$$\therefore\ \mathbf{(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.}$$

Example 2.

$$(x-2y+3z)^2=\{(x-2y)+3z\}^2=x^2+4y^2+9z^2-12yz+6zx-4xy.$$

A study of the form of the result in Example 1 enables us to reverse the process, *e.g.* to find the square root of

$$x^4-2x^3y+11x^2y^2-10xy^3+25y^4.$$

There are three terms which are perfect squares, viz. x^4, x^2y^2, $25y^4$, which suggests that the square root contains the terms x^2, $\pm xy$, $\pm 5y^2$. A little trial enables us to write the expression in the form

$$(x^2)^2+(-xy)^2+(5y^2)^2+2(x^2)(-xy)+2(-xy)(5y^2)+2(x^2)(5y^2),$$

which from the result of Example 1 is seen to be $(x^2-xy+5y^2)^2$. Hence the required square root is $x^2-xy+5y^2$.

Example 3.—Find the square root of

$$4p^4-12p^3q+9p^2q^2+6pqr^2-4p^2r^2+r^4.$$

The expression can be written

$$(2p^2)^2+(-3pq)^2+(-r^2)^2+2(2p^2)(-3pq)$$
$$+2(-3pq)(-r^2)+2(2p^2)(-r^2),$$

which is of the form $(a+b+c)^2$, where $a=2p^2$, $b=-3pq$, $c=-r^2$
Hence the required square root is $2p^2-3pq-r^2$.

Exercise XII (b)

Write down the squares of

1. $3a+2b+c$. **2.** $2a-b-3c$. **3.** $2x^2-3x+1$
4. $p^2-q^2+r^2$. **5.** $x^2-2xy+3y^2$.

Find the square root of

6. $a^2+b^2+c^2-2bc-2ca+2ab$.
7. $9x^2+y^2+4z^2+4yz+12zx+6xy$
8. $x^4-6x^3+17x^2-24x+16$.
9. $x^4+6x^3y+5x^2y^2-12xy^3+4y^4$.
10. $4a^2-12ab+9b^2+12bc-8ac+4c^2$.

Cubes of (a+b) and (a−b)

We have

$$\mathbf{(a+b)^3=a^3+3a^2b+3ab^2+b^3.}$$
$$\mathbf{(a-b)^3=a^3-3a^2b+3ab^2-b^3.}$$

These results can be found by direct multiplication, or thus :

$$(a+b)^3=(a+b)(a^2+2ab+b^2)=a^3+3a^2b+3ab^2+b^3.$$
$$(a-b)^3=(a-b)(a^2-2ab+b^2)=a^3-3a^2b+3ab^2-b^3.$$

Example. $(2x-3y)^3=(2x)^3-3(2x)^2(3y)+3(2x)(3y)^2-(3y)^3$
$$=8x^3-36x^2y+54xy^2-27y^3.$$

Factors of a^3-b^3, a^3+b^3, and $a^4+a^2b^2+b^4$

By the Remainder Theorem, since $a^3-b^3=0$ when $a=b$, the expression $a-b$ is one factor of a^3-b^3. The other factor is found by actual division to be a^2+ab+b^2, or the result can be obtained by reconstructing a^3-b^3 thus .

$$a^3-b^3=a^2(a-b), \qquad (a-b), \qquad (a-b).$$

The first bracket provides a^3, but also $-a^2b$, therefore the second bracket must be multiplied by ab in order to provide a term $+a^2b$ to destroy the $-a^2b$. This, however, introduces a term $-ab^2$, therefore the third bracket must be multiplied by $+b^2$.

Hence $a^3-b^3=a^2(a-b)+ab(a-b)+b^2(a-b)$

$$\therefore\ \mathbf{a^3-b^3=(a-b)(a^2+ab+b^2).}$$

To find the factors of a^3+b^3, we see that $a+b$ is one factor, since $a^3+b^3=0$ when $a=-b$. Proceeding as above, we obtain

$$a^3+b^3=a^2(a+b)-ab(a+b)+b^2(a+b)$$

$$\therefore\ \mathbf{a^3+b^3=(a+b)(a^2-ab+b^2).}$$

Example.—Factorise $8a^3-27b^3$

$$\begin{aligned} 8a^3-27b^3&=(2a)^3-(3b)^3\\ &=(2a-3b)[(2a)^2+(2a)(3b)+(3b)^2]\\ &=(2a-3b)(4a^2+6ab+9b^2) \end{aligned}$$

A combination of the results for a^3+b^3 and a^3-b^3 gives us

$$\begin{aligned} a^6-b^6&=(a^3+b^3)(a^3-b^3)\\ &=(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2) \qquad . \quad \text{(i)} \end{aligned}$$

Factorising a^6-b^6 as the difference of two cubes, we have, however,

$$\begin{aligned} a^6-b^6&=(a^2)^3-(b^2)^3\\ &=(a^2-b^2)(a^4+a^2b^2+b^4)\\ &=(a+b)(a-b)(a^4+a^2b^2+b^4) \qquad \text{(ii)} \end{aligned}$$

A comparison of (i) and (ii) shows that

$$\mathbf{a^4+a^2b^2+b^4=(a^2-ab+b^2)(a^2+ab+b^2).}$$

This can also be obtained thus :

$$\begin{aligned} a^4+a^2b^2+b^4&=a^4+2a^2b^2+b^4-a^2b^2\\ &=(a^2+b^2)^2-(ab)^2\\ &=(a^2-ab+b^2)(a^2+ab+b^2). \end{aligned}$$

The method last employed can be used whenever the first and last terms of a trinomial are both perfect squares.

Example.—Find the factors of $4x^4+19x^2+25$.

We have

$$\begin{aligned}4x^4+19x^2+25 &= 4x^4+20x^2-x^2+25 \quad (N.B.—20=2\times 2\times 5.)\\ &=(2x^2+5)^2-(x)^2\\ &=(2x^2+x+5)(2x^2-x+5).\end{aligned}$$

Exercise XII (c)

Write down the expanded forms of

1. $(2x-y)^3$.
2. $(3a+2b)^3$.
3. $(2x+1)^3$.
4. $(5x+2y)^3$.
5. $(3x^2-x)^3$.
6. $\left(x^2+\dfrac{2}{x}\right)^3$.

Find the factors of

7. $125x^3+27y^3$.
8. $8x^3-1$.
9. $a^3-1000b^3$.
10. $3x^4-24xy^3$.
11. $x^{12}+y^{12}$.
12. $27a^3b^3+8c^3$.

Express as the product of two quadratic factors:

13. x^4+x^2+1.
14. x^4+2x^2+9.
15. $9x^4-7x^2+1$.
16. $a^4-7a^2b^2+b^4$.
17. $a^4+3a^2b^2+4b^4$.
18. x^4+4.

Factors of $ax^2+2hxy+by^2+2gx+2fy+c$

Under certain conditions expressions of this form can be written as the product of two trinomial factors. The method used will be clear from an example.

Find the factors of $10x^2-13xy-3y^2+16x+10y-8$.

Since $10x^2-13xy-3y^2=(2x-3y)(5x+y)$, suppose the expression to be equal to $(2x-3y+p)(5x+y+q)$. On multiplying this out we find it is necessary to make $5p+2q=16$, $p-3q=10$, $pq=-8$. These are *all* satisfied by $p=4$, $q=-2$.

Therefore the required factors are $(2x-3y+4)(5x+y-2)$.

Factors of $a^3+b^3+c^3-3abc$

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

This should be verified by multiplication.

Notice that the given expression is symmetrical; any two of the letters a, b, c can be interchanged without altering the expression. Consequently the factors must also be symmetrical.

E.g. if we put b for a and a for b, we get

$$b^3+a^3+c^3-3bac=(b+a+c)(b^2+a^2+c^2-ba-ac-cb),$$

which is the same result as before.

Note.—If $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

Hence $\quad (x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$.

Exercise XII (d)

Write down the factors of

1. $x^3-y^3+z^3+3xyz$. **2.** $8a^3+b^3+27c^3-18abc$.

3. $x^3+8y^3-1000z^3-60xyz$.

4. What do the factors of $a^3+b^3+c^3-3abc$ become (i) when $c=0$; (ii) when $c=0$, $b=-d$?

5. Prove that

$$(a+2b)^3+(b-3a)^3+(2a-3b)^3=3(a+2b)(b-3a)(2a-3b)$$

6. Show that

$$(x-2y)^3+(2y-z)^3-(x-z)^3+3(x-2y)(2y-z)(x-z)=0.$$

7. Show that $47^3+53^3-100^3=-300\times 2491$.

Factorise

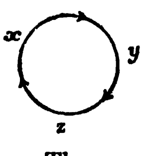
8. $2x^2+5xy-3y^2-3x+5y-2$. **9.** $3x^2-xy-2y^2-4x-y+1$.

10. $4x^2+2xy-12y^2-5x+10y$ [illegible].

Cyclic Order.—*Example.*—Simplify

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-z)(y-x)}+\frac{1}{(z-x)(z-y)}$$

Since $y-x=-(x-y)$, it will be seen that of the six factors in the denominators three can be expressed in terms of the remaining three. In deciding which forms to alter and which to retain, it is convenient to adopt what is known as "cyclic order," the differences being taken in the order in which the letters occur on a circular diagram.



Thus we keep $x-y$, $y-z$, $z-x$, and change $(x-z)$, $(y-x)$, $(z-y)$ into $-(z-x)$, $-(x-y)$, $-(y-z)$.

The expression may now be written

$$-\frac{1}{(x-y)(z-x)}-\frac{1}{(y-z)(x-y)}-\frac{1}{(z-x)(y-z)},$$

which is equal to $\dfrac{-(y-z)-(z-x)-(x-y)}{(x-y)(y-z)(z-x)}=0.$

Exercise XII (e)

Arrange in "cyclic order":

1. $\dfrac{a-b}{a-c}+\dfrac{b-c}{b-a}+\dfrac{c-a}{c-b}.$

2. $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-c)(b-a)}+\dfrac{a+b}{(c-a)(c-b)}$

3. $\dfrac{x-y}{(y-z)(x-z)}-\dfrac{y+z}{(x-y)(y-z)}+\dfrac{x+z}{(y-x)(z-x)}$

Find the value of

4. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}.$

5. $\dfrac{a+b-c}{(b-c)(c-a)}+\dfrac{b+c-a}{(c-a)(a-b)}-\dfrac{b-a-c}{(a-b)(b-c)}$

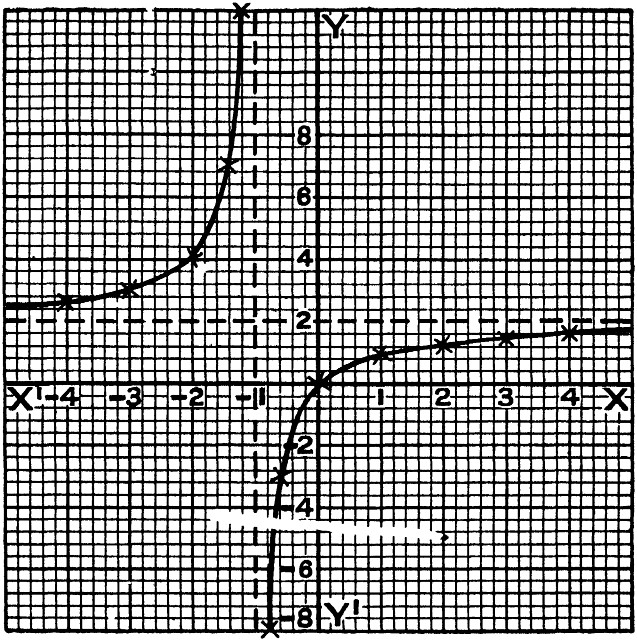
6. $\dfrac{xy}{(y-z)(z-x)}-\dfrac{yz}{(z-x)(y-x)}+\dfrac{zx}{(x-y)(y-z)}.$

7. $\dfrac{p-q}{(r+p)(r+q)}+\dfrac{q-r}{(p+q)(p+r)}-\dfrac{r-p}{(q+p)(q+r)}.$

Discontinuous Graphs.—*Example.*—Draw the graph of $\dfrac{2x}{x+1}$ between $x=-4$ and $x=+3$. Constructing the table of values we have

x	-4	-3	-3	-1	0	1	2	3
$2x$	-8	-6	-4	-2	0	2	4	6
$x+1$	-3	-2	-1	0	1	2	3	4
$\dfrac{2x}{x+1}$ or y	$2\frac{2}{3}$	3	4	$-\dfrac{2}{0}$	0	1	$1\frac{1}{3}$	$1\frac{1}{2}$

The meaning of $\frac{1}{0}$ is explained on p. 77; its value is generally denoted as "infinity." In drawing a graph in which infinite values occur, it is always necessary to discover the nature of the curve just before and just after the infinite value.



Graph of $\dfrac{2x}{x+1}$.

Just before $x = -1$ we get

x	$-1{\cdot}4$	$-1{\cdot}2$	$-1\ 1$
y	$+7$	$+12$	$+22$

and just after $x=-1$ we have

x	$-0{\cdot}9$	$-0{\cdot}8$	$-0{\cdot}6$
y	-18	-8	-3

Thus as x approaches -1 from the left, y steadily increases and tends to *plus* infinity. When x is greater than -1, we see that for values of x close to -1, y is steadily increasing, but negative; thus this part of the curve approaches *minus* infinity.

The graph is therefore as drawn.

It will be noticed that there are two distinct branches. Such branches are liable to occur whenever the denominator of the function is capable of assuming the value zero In our example the difficulty arose when $x=-1$, *i.e.* where $x+1=0$.

The dotted line whose equation is $x+1=0$ is called an *asymptote*. The curve approaches nearer and nearer to it without actually meeting it. If we rewrote the equation $y=\dfrac{2x}{x+1}$ in the equivalent form, $x=\dfrac{y}{2-y}$, we should find that $y-2=0$ is also an asymptote.

Exercise XII (f)

Draw the graphs of the following functions

1. $\dfrac{20}{x}$ between $x=-5$ and $x=+5$.

2. $\dfrac{6}{x}-\dfrac{3x}{2}$ between $x=-4$ and $x=+4$.

3. $\dfrac{2+x}{1-x}$ between $x=-2$ and $x=+7$.

4. Given $y^2=5-\dfrac{8}{x-2}$, plot a graph to show the changes in y for values of x between 0 and $+6$.

5. Draw the graph of the curve whose equation is $y=\dfrac{(1+x)(2+x)}{3+x}$ for values of x between -6 and $+2$.

6. Taking a scale of 1·0 in. to the unit along OX and 0·5 in. to the unit along OY, draw the graph of $\dfrac{x^3-1}{x}$ between $x=-2$ and $x=+2$.

Revision Exercise XII (g)

Simplify

1. $\left(\dfrac{x}{y}-1+\dfrac{y}{x}\right)\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$ **2.** $\left(x+2+\dfrac{4}{x}\right)\left(1-\dfrac{2}{x}\right)$.

3. $\dfrac{a-b}{a^2-ab+b^2}-\dfrac{a+b}{a^2+ab+b^2}$. **4.** $\left(\dfrac{x^2}{y}-\dfrac{y^2}{x}\right)-\left(\dfrac{x}{y}+\dfrac{y}{x+y}\right)$.

5. $\dfrac{a^3+b^3}{a^2-ab+b^2}-\dfrac{b^3-c^3}{b^2+bc+c^2}-\dfrac{c^3+a^3}{c^2-ac+a^2}$

6. $\dfrac{3}{x^3+1}-\dfrac{1}{(x+1)^3}$.

Find the square root of

7. $a^2-4ab+4b^2+4bc-2ca+c^2$. **8.** $x^4-x^3+\dfrac{5x^2}{4}-\dfrac{x}{2}+\dfrac{1}{4}$.

Factorise

9. $9x^4-16x^2y^2+4y^4$. **10.** $4x^4+11x^2y^2+9y^4$.

11. $x^2-xy-6y^2-xz+13yz-6z^2$.

12. $3x^2+4xy-4y^2-8x+8y-3$.

13. $(a^3-1)(b+1)-(a-1)(b^3+1)$.

14. $64p^3-27q^3-r^3-36pqr$.

15. $8x^3-\dfrac{y^3}{z^3}$. **16.** a^6+b^6.

17. If $x+\dfrac{1}{x}=z$, express in terms of z the values of $x^2+\dfrac{1}{x^2}$, $x^3+\dfrac{1}{x^3}$, $x^4+\dfrac{1}{x^4}$.

Miscellaneous Examples V

N.B.—These do not require a knowledge of Chapter XII.

B 92

1. Simplify (i) $\dfrac{1-x^2}{(1+ax)^2-(a+x)^2}$; (ii) $\dfrac{1}{(x-2)^2}+\dfrac{1}{4-x^2}$.

2. Solve by the method of factors:

(i) $2x^3+x^2-2x=1$; (ii) $(3x+5)(3x+2)=2+\dfrac{x}{2}$.

3. A and B start at the same time from two places m miles apart and walk towards one another, A at a miles per hour and B at b miles per hour. After how many hours will they meet? If B had walked 1 mile per hour faster and consequently reduced the time by c hours, prove $m=c(a+b)(a+b+1)$.

4. The difference between two whole numbers is 8 and their product is 713. Find the numbers.

5. Find to 2 places of decimals the roots of

$$\frac{1}{x}-\frac{1}{x+1}=\frac{1}{2x+1}.$$

6. A billiard-cue which tapers uniformly has a radius a at one end and b at the other. To find the area of the cross-section midway between the ends it is proposed to take the mean of the areas of the two ends. By how much is the result in error?

B 93

1. Solve the equation $6x^2-13x-5=0$. What is the sum of the roots and what is their product?

2. Express $\dfrac{4x+5}{x+1}$ in a form similar to that of a mixed number in arithmetic, and hence simplify $\dfrac{4x+5}{x+1}-\dfrac{4x+7}{x+2}$.

3. Show that $(y-z)^3+(z-x)^3+(x-y)^3$ equals zero when $y=z$. Hence find its factors.

4. Solve the equation $x^2-ax = \dfrac{a^2}{4b^2}(4-b^2)$.

5. Evaluate $\dfrac{a+x}{b+x} - \dfrac{a}{b}$.

What is the condition that $\dfrac{a+x}{b+x}$ should be greater than $\dfrac{a}{b}$ if all the letters are positive numbers? Show that if the same number is added to both numerator and denominator of a fraction $\dfrac{a}{b}$, then the fraction is increased if $a<b$ and decreased if $a>b$.

6. A train whose normal speed is v miles per hour is M minutes late If its speed is increased to V miles per hour, prove that D, the distance in which lost time is made up, is given by $\mathrm{D} = \dfrac{\mathrm{M}v\mathrm{V}}{60(\mathrm{V}-v)}$.

Rewrite this result to give V in terms of M, D, v

B 94

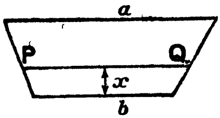
1. Factorise $3x^2-27x^4$, $6x^2-15xy-2xz+5yz$

2. Solve the equation $\dfrac{5x-1}{3x+1} - \dfrac{4x-3}{3x-1} = \dfrac{1}{3}$.

3. Use the fact that a squared expression is always positive to prove (i) $a^2+b^2>2ab$; (ii) $a+b>2\sqrt{ab}$.

4. The base of a triangle is 10 cm and its altitude is 5 cm. If one of the remaining sides is double the other, find their lengths to the nearest millimetre. [Take x as one of the segments of the base and use Pythagoras.]

5. The internal and external diameters of a tube are d and D. Show that if the thickness of the metal is n times the internal diameter, then $\mathrm{D}=(2n+1)d$. Substitute this in $\dfrac{\mathrm{D}^2-d^2}{\mathrm{D}^2+d^2}$ and simplify Evaluate when $n=0\cdot1$.



6. The two parallel sides of a trapezium are a and b, and its height is h. Find the length y of a line PQ parallel to the base b and at a distance x from it, which cuts the non-parallel sides at P and Q.

B 95

1. The expressions $2x^2+x-6$ and $4x^3-4x^2-7x+6$ have a common factor. What is it ?

2. Simplify (i) $(p-2q)^2-(2q-p)^2$; (ii) $(p-2q)^2\div(2q-p)^2$; (iii) $(p-2q)^2\times(2q-p)^3$.

3. A gardener increases the width of a bed 10 yd. by 25 yd. by digging a border x yd. wide round two adjacent edges. What value of x doubles the area of the bed ?

4. Find m if both the following equations are satisfied for the same value of x :

(i) $8-(x-2)/3=7$; (ii) $(m-1)x^2-(3m-4)x-(8m+1)=0$.

Show that for this value of m there is another value of x which will satisfy (ii) but not (i). Find it.

5. ABC is a triangle. AB=13, AC=15, BC=14. AH is drawn perpendicular to BC. If BH $=x$, write down two expressions for AH, and hence find BH and HC.

6. The cross-section of a trough PQBA is a trapezium in which PQ is parallel to the base AB. PQ=16 in., AB=12 in., AP=BQ, the depth is 10 in. and the length of trough 25 in. Find the volume of water contained when the depth is x in., and calculate the depth when the trough contains 1400 cu. in.

B 96

1. Simplify $\dfrac{a-c+ab^2-b^2c}{1+ac+b^2+ab^2c}$.

2. Factorise (i) $4a^2-7ab-4c^2+7bc$; (ii) $2ab+1-a^2-b^2$

3. Find the value of x for which $\dfrac{p-q}{x}=\dfrac{p}{q}-\dfrac{q}{p}$.

4. Solve the equations

(i) $\frac{x-3}{4x-3}+\frac{4x-3}{x-3}=2$; (ii) $\frac{x-3}{4x-3}+\frac{4x-3}{x-3}=-2$.

5. A basket containing x strawberries is divided between 5 boys as follows: The first takes 1 strawberry and one-sixth of the remainder. The second takes 2 berries and one-sixth of the remainder. The third takes 3 and one-sixth of the remainder, and so on. Express in terms of x the number of strawberries taken by the first and second boys. Show that for a certain value of x *all* the boys get the same number, and find it.

6. The volume of a tumbler of circular cross-section is given by $V=0{\cdot}262h(D^2+Dd+d^2)$, where h is the height, D and d the top and bottom diameters Find V when $h=10$, $D=8$, $d=6$. If such a vessel had to be made with the same height, 10 cm., but with a volume of 500 c.c., it could be done by increasing the above diameters 8 and 6 by x cm., where x is given by $x^2+14x=14{\cdot}3$. Prove this and find x.

B 97

1. If $F(x)\equiv x^3-(x-1)^3$, find the value of $F(x)-F(x-1)$. If this is denoted by $f(x)$, show that $f(x)-f(x-1)=6$.

2. If a man gains a per cent by selling for £x, what must he sell for to gain b per cent.?

3. Two tangents PQ, PQ′ are drawn from P to a circle of radius a. If P is distant x from the centre, prove that

$$QQ' = \frac{2a\sqrt{x^2-a^2}}{x}.$$

Show algebraically that as x increases this expression approaches the value $2a$.

4. A spherical ball of radius r is dropped into a cone, the diameter of whose base is x and height h. The centre of the ball rests a distance d above the vertex of the cone. Prove

$$h^2 = \frac{x^2}{4}\frac{(d-r)(d+r)}{r^2}.$$

5. When three numbers are added together two at a time, the results are a, b, and c respectively. Find the numbers.

6. A point P is situated in the plane of a triangle ABC at a distance p from A, the line AP being parallel to BC and in the same sense. A line is drawn through P cutting AC in Q, and CB produced in R, so as to make the triangle QRC equal in area to ABC. If the perpendiculars from A and Q to BC are p_1 and p_2 respectively, prove $\frac{p_1}{p_2}=\frac{x+p}{x}$ where $CR=x$, and if $CB=a$, prove $x^2-ax-ap=0$.

B 98

1. The following formulæ occur in works on hydraulics —

(i) $z=a\left(1+\frac{b}{v}\right)$; (ii) $p=\sqrt{al+b}$, (iii) $\frac{1}{c}=\frac{1}{a}+\frac{b}{a\sqrt{m}}$.

Rewrite all three with a as the subject of the formula in each case.

2. Show that if x is greater than 2 and less than 10, the value of $(x-2)(x-10)$ is negative. For what range of values of x is $2x^2+5x-3$ negative ?

3. A square and an equilateral triangle have equal perimeters. Show that their areas are approximately in the ratio 100 : 77.

4. If a, b are two different numbers such that $(a-2)(a-3)=(b-2)(b-3)$, show that their sum is equal to 5

5. If $\frac{1}{a}=\frac{1}{x}+\frac{1}{y}$ and $\frac{1}{b}=\frac{1}{x}-\frac{1}{y}$, find the value of $\frac{a+b}{a-b}$.

6. The area of an estate is found to be A sq. in. on a map drawn to a scale of x miles to the inch What is its actual area ? What would be the area on a map drawn to a scale of y miles to the inch ?

B 99

1. Show that a root of $x^2-3{\cdot}9x+3{\cdot}4=0$ lies between 1·3 and 1·4. Find its value correct to 2 decimal places.

2. Simplify (i) $\dfrac{p^2x^{m+2}-q^2x^m}{q-px}$, (ii) $\dfrac{(p-q)^5}{(q-p)^3}$.

3. A cylindrical tin, radius a, contains water to a depth x A solid cylinder of the same length and of radius b is placed upright inside it. If the water is then up to the top, prove that the length of the cylinders is $\dfrac{a^2x}{a^2-b^2}$.

4. On the same side of a line ABC, where AB=BC, 3 semicircles are described with diameters AB, BC, AC. A circle of radius r is drawn to touch all three If AB=2R, prove $(R+r)^2-(2R-r)^2=R^2$, and hence find r in terms of R.

5. ABC are 3 consecutive railway stations, B being a miles from A and C being b miles farther along the line A non-stop express travelling at uniform speed passes through A at x a.m and through C at y p m At what time did it pass through B? If this time is 12 midday, prove $bx+ay=12b$.

6. When spheres are piled up to make a pyramid with 1 in the top layer, 3 in the second, 6 in the third, and so on, the total number of spheres at any time is given by the expression an^3+bn^2+cn, where n is the number in the side of the triangular base. Find a, b, and c.

B 100

1. Given $V=\pi r^2h$ and $S=\pi r^2+2\pi rh$, find S in terms of V and h, and also S in terms of V and r.

2. A sphere of radius R is placed so that it rests on the top of a vertical hollow cylinder whose radius is r and height h Show that if a is the height of the centre of the sphere above the bottom of the cylinder, then $r^2=(R+a-h)(R-a+h)$

3. The expression $\dfrac{ax^2+b}{x}$ equals $\frac{1}{2}$ when $x=1$, and equals 7 when $x=2$. Find a and b. Find also the value of the expression when $x=-1$.

4. The speed of an aeroplane in still air is u miles per hour If the speed of the wind is v miles per hour and the average speed for a journey of d miles with the wind and a return of d miles against the wind is a miles per hour, prove $\frac{1}{a}=\frac{1}{2}\left(\frac{1}{u+v}+\frac{1}{u-v}\right)$.

5. If e is a small positive number and $A=l+e$, $B=m-e$, show that either of the fractions $\frac{A}{m}$, $\frac{l}{B}$ is nearer to $\frac{l}{m}$ than $\frac{A}{B}$ is, where A, B, l, m are all positive.

6. A man cycles from A to B and back at the same rate, but the clock at B is 15 minutes slower than that at A, so he appears to go out at 9 miles per hour and to return at 6 miles per hour. What was his actual rate and the distance ?

B 101

1. What is the remainder when b hours a minutes is subtracted from a hours b minutes ($a>b$) ? If you now add to your answer the result of interchanging in it the number of hours and the number of minutes, show that the sum is independent of a and b.

2. If $a=\frac{x-y}{x+y}$, $b=\frac{y-z}{y+z}$, $c=\frac{z-x}{z+x}$, prove that $\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\cdot\frac{1+c}{1-c}=1$

3. Find the values of $(x+2)(y-1)$ and x from the equations

$$3(x+2)(y-1)-7x=26$$
$$2(x+2)(y-1)+3x=48,$$

and hence solve for x and y

4. A point P is taken on a line AB, produced if necessary, such that $AP^2-PB^2=nAB^2$. Show that $\frac{AP}{BP}=\frac{n+1}{n-1}$. Taking $AB=4$ in., find the position of P corresponding to (i) $n=2$; (ii) $n=-\frac{1}{2}$.

5. A crown of weight W is made partly of gold and partly of silver, and is found to displace a volume V of a liquid in which it is immersed. A weight W of gold would displace a volume V_1 of this liquid, and a weight W of silver would displace V_2. If W_1 and W_2 are the weights of the gold and silver in the crown, prove that $W_1/W_2=(V_2-V)/(V-V_1)$.

6. Draw the graph of $xy+2x-y-5=0$. What is the value of x when $y=-2$?

B 102

1. How many times is (*a*) contained in (*b*) in each of the following cases ?

(i) (*a*) y shillings, (*b*) £x;

(ii) (*a*) $\frac{1}{y}$ shillings, (*b*) £$\frac{1}{x}$,

(iii) (*a*) z pence, (*b*) £$x+y$ shillings.

2. If $\dfrac{(a+b)^2+7(a+b)+12}{(a+b)^2+4(a+b)+3}=\dfrac{1}{2}$ and $a-b=5$, find the values of a and b.

3. If a, b, c are three unequal quantities such that

$$a-\frac{bc}{a}=b-\frac{ac}{b},$$

find c in terms of a and b, and prove that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=0$.

4. By writing $x^2-5x=y$, find the four solutions of the equation $x^2-5x=\dfrac{36}{x^2-5x}$

5. What digit must d represent if the result of multiplying the number $d5$ by the number $3d$ is $2dd5$?

6. Draw the graph of $2x^3-x^2-2x+1$ from $x=-2$ to $x=+2$. What are the solutions of the equation

$$2x^3-x^2=2x-1 \text{ ?}$$

B 103

1. If $\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{2}{r}$ and $u-p=r=v-q$, prove $\dfrac{1}{p}+\dfrac{1}{q}=-\dfrac{2}{r}$.

2. A block of ice, originally a cube of 40 cm. edge, melts so that after t hours (for all values of t) it is still a cube whose volume is $10t^2+at+b$ cu. cm. Find b, and if the block melts completely in 80 hours, find a. Show that after 40 hours the edge of the cube is still more than 25 cm

3. If $s = \dfrac{a+b+c}{2}$, prove that $a+b-c=2(s-c)$.

Given $a^2=b^2+c^2-2bcp$, find and factorise the values of $1+p$ and $1-p$. Show that $1-p^2 = \dfrac{4s(s-a)(s-b)(s-c)}{b^2c^2}$.

4. If $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{p}$ and $x+y=q$, find $x-y$

5. A cubical block of edge a in. is cut into n equal cubical blocks. Find the length of edge of each block, and find for what value of n the total surface of the n blocks is twice the area of the surface of the original block

6. A bucket has a circular mouth of diameter 14 in. and a circular bottom of diameter 10 in. The vertical depth is 16 in., and when water is poured in to a depth of x in , the area of the water surface is A sq. in. Find the formula connecting A and x.

TABLE OF SQUARE ROOTS

											Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1·0	1·000	1·005	1·010	1·015	1·020	1·025	1·030	1·034	1·039	1·044	0	1	1	2	2	3	3	4	4
1·1	1·049	1·054	1·058	1·063	1·068	1·072	1·077	1·082	1·086	1·091	0	1	1	2	2	3	3	4	4
1·2	1·095	1·100	1·105	1·109	1·114	1·118	1·122	1·127	1·131	1·136	0	1	1	2	2	3	3	4	4
1·3	1·140	1·145	1·149	1·153	1·158	1·162	1·166	1·170	1·175	1·179	0	1	1	2	2	3	3	3	4
1·4	1·183	1·187	1·192	1·196	1·200	1·204	1·208	1·212	1·217	1·2[illegible]1	0	1	1	2	2	2	3	3	4
1·5	1·225	1·229	1·233	1·237	1·241	1·245	1·249	1·253	1·257	1·261	0	1	1	2	2	2	3	3	4
1·6	1·265	1·269	1·273	1·277	1·281	1·285	1·288	1·292	1·296	1·300	0	1	1	2	2	2	3	3	3
1·7	1·304	1·308	1·311	1·315	1·319	1·323	1·327	1·330	1·334	1·338	0	1	1	2	2	2	3	3	3
1·8	1·342	1·345	1·349	1·353	1·356	1·360	1·364	1·367	1·371	1·375	0	1	1	1	2	2	3	3	3
1·9	1·378	1·382	1·386	1·389	1·393	1·396	1·400	1·404	1·407	1·411	0	1	1	1	2	2	3	3	3
2·0	1·414	1·418	1·421	1·425	1·428	1·432	1·435	1·439	1·442	1·446	0	1	1	1	2	2	2	3	3
2·1	1·449	1·453	1·456	1·459	1·463	1·466	1·470	1·473	1·476	1·480	0	1	1	1	2	2	2	3	3
2·2	1·483	1·487	1·490	1·493	1·497	1·500	1·503	1·507	1·510	1·513	0	1	1	1	2	2	2	3	3
2·3	1·517	1·520	1·523	1·526	1·530	1·533	1·536	1·539	1·543	1·546	0	1	1	1	2	2	2	3	3
2·4	1·549	1·552	1·556	1·559	1·562	1·565	1·568	1·572	1·575	1·578	0	1	1	1	2	2	2	3	3
2·5	1·581	1·584	1·587	1·591	1·594	1·597	1·600	1·603	1·606	1·609	0	1	1	1	2	2	2	3	3
2·6	1·612	1·616	1·619	1·622	1·625	1·628	1·631	1·634	1·637	1·640	0	1	1	1	2	2	2	2	3
2·7	1·643	1·646	1·649	1·652	1·655	1·658	1·661	1·664	1·667	1·670	0	1	1	1	2	2	2	2	3
2·8	1·673	1·676	1·679	1·682	1·685	1·688	1·691	1·694	1·697	1·700	0	1	1	1	1	2	2	2	3
2·9	1·703	1·706	1·709	1·712	1·715	1·718	1·720	1·723	1·726	1·729	0	1	1	1	1	2	2	2	3
3·0	1·732	1·735	1·738	1·741	1·744	1·746	1·749	1·752	1·755	1·758	0	1	1	1	1	2	2	2	3
3·1	1·761	1·764	1·766	1·769	1·772	1·775	1·778	1·780	1·783	1·786	0	1	1	1	1	2	2	2	3
3·2	1·789	1·792	1·794	1·797	1·800	1·803	1·806	1·808	1·811	1·814	0	1	1	1	1	2	2	2	2
3·3	1·817	1·819	1·822	1·825	1·828	1·830	1·833	1·836	1·838	1·841	0	1	1	1	1	2	2	2	2
3·4	1·844	1·847	1·849	1·852	1·855	1·857	1·860	1·863	1·865	1·868	0	1	1	1	1	2	2	2	2
3·5	1·871	1·873	1·876	1·879	1·881	1·884	1·887	1·889	1·892	1·895	0	1	1	1	1	2	2	2	2
3·6	1·897	1·900	1·903	1·905	1·908	1·910	1·913	1·916	1·918	1·921	0	1	1	1	1	2	2	2	2
3·7	1·924	1·926	1·929	1·931	1·934	1·936	1·939	1·942	1·944	1·947	0	1	1	1	1	2	2	2	2
3·8	1·949	1·952	1·954	1·957	1·960	1·962	1·965	1·967	1·970	1·972	0	1	1	1	1	2	2	2	2
3·9	1·975	1·977	1·980	1·[illegible]	1·985	1·987	1·990	1·992	1·995	1·997	0	1	1	1	1	2	2	2	2
4·0	2·000	2·002	2·005	2·007	2·010	2·012	2·015	2·017	2·020	2·022	0	0	1	1	1	1	2	2	2
4·1	2·025	2·027	2·030	2·032	2·035	2·037	2·040	2·042	2·045	2·047	0	0	1	1	1	1	2	2	2
4·2	2·049	2·052	2·054	2·057	2·059	2·062	2·064	2·066	2·069	2·071	0	0	1	1	1	1	2	2	2
4·3	2·074	2·076	2·078	2·081	2·083	2·086	2·088	2·090	2·093	2·095	0	0	1	1	1	1	2	2	2
4·4	2·098	2·100	2·102	2·105	2·107	2·110	2·112	2·114	2·117	2·119	0	0	1	1	1	1	2	2	2
4·5	2·121	2·124	2·126	2·128	2·131	2·133	2·135	2·138	2·140	2·142	0	0	1	1	1	1	2	2	2
4·6	2·145	2·147	2·149	2·152	2·154	2·156	2·159	2·161	2·163	2·166	0	0	1	1	1	1	2	2	2
4·7	2·168	2·170	2·173	2·175	2·177	2·179	2·182	2·184	2·186	2·189	0	0	1	1	1	1	2	2	2
4·8	2·191	2·193	2·195	2·198	2·200	2·202	2·205	2·207	2·209	2·211	0	0	1	1	1	1	2	2	2
4·9	2·214	2·216	2·218	2·220	2·223	2·225	2·227	2·229	2·232	2·234	0	0	1	1	1	1	2	2	2
5·0	2·236	2·238	2·241	2·243	2·245	2·247	2·249	2·252	2·254	2·256	0	0	1	1	1	1	2	2	2
5·1	2·258	2·261	2·263	2·265	2·267	2·269	2·272	2·274	2·276	2·278	0	0	1	1	1	1	2	2	2
5·2	2·280	2·283	2·285	2·287	2·289	2·291	2·293	2·296	2·298	2·300	0	0	1	1	1	1	2	2	2
5·3	2·302	2·304	2·307	2·309	2·311	2·313	2·315	2·317	2·319	2·322	0	0	1	1	1	1	2	2	2
5·4	2·324	2·326	2·328	2·330	2·332	2·335	2·337	2·339	2·341	2·343	0	0	1	1	1	1	1	2	2

	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
10	3·162	3·178	3·194	3·209	3·225	3·240	3·256	3·271	3·286	3·302	2	3	5	6	8	9	11	12	14
11	3·317	3·332	3·347	3·362	3·376	3·391	3·406	3·421	3·435	3·450	1	3	4	6	7	9	10	12	13
12	3·464	3·479	3·493	3·507	3·521	3·536	3·550	3·564	3·578	3·592	1	3	4	6	7	8	10	11	13
13	3·606	3·619	3·633	3·647	3·661	3·674	3·688	3·701	3·715	3·728	1	3	4	5	7	8	10	11	12
14	3·742	3·755	3·768	3·782	3·795	3·808	3·821	3·834	3·847	3·860	1	3	4	5	7	8	9	11	12
15	3·873	3·886	3·899	3·912	3·924	3·937	3·950	3·962	3·975	3·987	1	3	4	5	6	8	9	10	11
16	4·000	4·012	4·025	4·037	4·050	4·062	4·074	4·087	4·099	4·111	1	2	4	5	6	7	9	10	11
17	4·123	4·135	4·147	4·-59	4·171	4·183	4·195	4·207	4·219	4·231	1	2	4	5	6	7	8	10	11
18	4·243	4·254	4·266	4·278	4·290	4·301	4·313	4·324	4·336	4·347	1	2	3	5	6	7	8	9	10
19	4·359	4·370	4·382	4·393	4·405	4·416	4·427	4·438	4·450	4·461	1	2	3	5	6	7	8	9	10
20	4·472	4·483	4·494	4·506	4·517	4·528	4·539	4·550	4·561	4·572	1	2	3	4	6	7	8	9	10
21	4·583	4·593	4·604	4·615	4·626	4·637	4·648	4·658	4·669	4·680	1	2	3	4	5	6	8	9	10
22	4·690	4·701	4·712	4·722	4·733	4·743	4·754	4·764	4·775	4·785	1	2	3	4	5	6	7	8	9
23	4·796	4·806	4·817	4·827	4·837	4·848	4·858	4·868	4·879	4·889	1	2	3	4	5	6	7	8	9
24	4·899	4·909	4·919	4·930	4·940	4·950	4·960	4·970	4·980	4·990	1	2	3	4	5	6	7	8	9
25	5·000	5·010	5·020	5·030	5·040	5·050	5·060	5·070	5·079	5·089	1	2	3	4	5	6	7	8	9
26	5·099	5·109	5·119	5·128	5·138	5·148	5·158	5·167	5·177	5·187	1	2	3	4	5	6	7	8	9
27	5·196	5·206	5·215	5·225	5·235	5·244	5·254	5·263	5·273	5·282	1	2	3	4	5	6	7	8	9
28	5·292	5·301	5·310	5·320	5·329	5·339	5·348	5·357	5·367	5·376	1	2	3	4	5	6	7	7	8
29	5·385	5·394	5·404	5·413	5·422	5·431	5·441	5·450	5·459	5·468	1	2	3	4	5	5	6	7	8
30	5·477	5·486	5·495	5·505	5·514	5·523	5·532	5·541	5·550	5·559	1	2	3	4	4	5	6	7	8
31	5·568	5·577	5·586	5·595	5·604	5·612	5·621	5·630	5·639	5·648	1	2	3	3	4	5	6	7	8
32	5·657	5·666	5·675	5·683	5·692	5·701	5·710	5·718	5·727	5·736	1	2	3	3	4	5	6	7	8
33	5·745	5·753	5·762	5·771	5·779	5·788	5·797	5·805	5·814	5·822	1	2	3	3	4	5	6	7	8
34	5·831	5·840	5·848	5·857	5·865	5·874	5·882	5·891	5·899	5·908	1	2	3	3	4	5	6	7	8
35	5·916	5·025	5·933	5·941	5·950	5·958	5·967	5·975	5·983	5·992	1	2	2	3	4	5	6	7	8
36	6·000	6·008	6·017	6·025	6·033	6·042	6·050	6·058	6·066	6·075	1	2	2	3	4	5	6	7	7
37	6·083	6·091	6·099	6·107	6·116	6·124	6·132	6·140	6·148	6·156	1	2	2	3	4	5	6	7	7
38	6·164	6·173	6·181	6·189	6·197	6·205	6·213	6·221	6·229	6·237	1	2	2	3	4	5	6	6	7
39	6·245	6·253	6·261	6·269	6·277	6·285	6·293	6·301	6·309	·317	1	2	2	3	4	5	6	6	7
40	6·325	6·332	6·340	6·348	6·356	6·364	6·372	6·380	6·387	6·395	1	2	2	3	4	5	6	6	7
41	6·403	6·411	6·419	6·427	6·434	6·442	6·450	6·458	6·465	6·473	1	2	2	3	4	5	5	6	7
42	6·481	6·488	6·496	6·504	6·512	6·519	6·527	6·535	6·542	6·550	1	2	2	3	4	5	5	6	7
43	6·557	6·565	6·573	6·580	6·588	6·595	6·603	6·611	6·618	6·626	1	2	2	3	4	5	5	6	7
44	6·633	6·641	6·648	6·656	6·663	6·671	6·678	6·686	6·693	6·701	1	2	2	3	4	5	5	6	7
45	6·708	6·716	6·723	6·731	6·738	6·745	6·753	6·760	6·768	6·775	1	1	2	3	4	4	5	6	7
46	6·782	6·790	6·797	6·804	6·812	6·819	6·826	6·834	6·841	6·848	1	1	2	3	4	4	5	6	7
47	6·856	6·863	6·870	6·877	6·885	6·892	6·899	6·907	6·914	6·921	1	1	2	3	4	4	5	6	7
48	6·928	6·935	6·943	6·950	6·957	6·964	6·971	6·979	6·986	6·993	1	1	2	3	4	4	5	6	6
49	7·000	7·007	7·014	7·021	7·029	7·036	7·043	7·050	7·057	7·064	1	1	2	3	4	4	5	6	6
50	7·071	7·078	7·085	7·092	7·099	7·106	7·113	7·120	7·127	7·134	1	1	2	3	4	4	5	6	6
51	7·141	7·148	7·155	7·162	7·169	7·176	7·183	7·190	7·197	7·204	1	1	2	3	4	4	5	6	6
52	7·211	7·218	7·225	7·232	7·239	7·246	7·253	7·259	7·266	7·273	1	1	2	3	3	4	5	6	6
53	7·280	7·287	7·294	7·301	7·308	7·314	7·321	7·328	7·335	7·342	1	1	2	3	3	4	5	5	6
54	7·348	7·355	7·362	7·369	7·376	7·382	7·389	7·396	7·403	7·409	1	1	2	3	3	4	5	5	6

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
5·5	2·345	2·347	2·349	2·352	2·354	2·356	2·358	2·360	2·362	2·364	0	0	1	1	1	1	1	2	2
5·6	2·366	2·369	2·371	2·373	2·375	2·377	2·379	2·381	2·383	2·385	0	0	1	1	1	1	1	2	2
5·7	2·387	2·390	2·392	2·394	2·396	2·398	2·400	2·402	2·404	2·406	0	0	1	1	1	1	1	2	2
5·8	2·408	2·410	2·412	2·415	2·417	2·419	2·421	2·423	2·425	2·427	0	0	1	1	1	1	1	2	2
5·9	2·429	2·431	2·433	2·435	2·437	2·439	2·441	2·443	2·445	2·447	0	0	1	1	1	1	1	2	2
6·0	2·449	2·452	2·454	2·456	2·458	2·460	2·462	2·464	2·466	2·468	0	0	1	1	1	1	1	2	2
6·1	2·470	2·472	2·474	2·476	2·478	2·480	2·482	2·484	2·486	2·488	0	0	1	1	1	1	1	2	2
6·2	2·4900	2·492	2·494	2·496	2·498	2·500	2·502	2·504	2·506	2·508	0	0	1	1	1	1	1	2	2
6·3	2·510	2·512	2·514	2·516	2·518	2·520	2·522	2·524	2·526	2·528	0	0	1	1	1	1	1	2	2
6·4	2·530	2·532	2·534	2·536	2·538	2·540	2·542	2·544	2·546	2·548	0	0	1	1	1	1	1	2	2
6·5	2·550	2·551	2·553	2·555	2·557	2·559	2·561	2·563	2·565	2·567	0	0	1	1	1	1	1	2	2
6·6	2·569	2·571	2·573	2·575	2·577	2·579	2·581	2·583	2·585	2·587	0	0	1	1	1	1	1	2	2
6·7	2·588	2·590	2·592	2·594	2·596	2·598	2·600	2·602	2·604	2·606	0	0	1	1	1	1	1	2	2
6·8	2·608	2·610	2·612	2·613	2·615	2·617	2·619	2·621	2·623	2·625	0	0	1	1	1	1	1	2	2
6·9	2·627	2·629	2·631	2·632	2·634	2·636	2·638	2·640	2·642	2·644	0	0	1	1	1	1	1	2	2
7·0	2·646	2·648	2·650	2·651	2·653	2·655	2·657	2·659	2·661	2·663	0	0	1	1	1	1	1	2	2
7·1	2·665	2·666	2·668	2·670	2·672	2·674	2·676	2·678	2·680	2·681	0	0	1	1	1	1	1	1	2
7·2	2·683	2·685	2·687	2·689	2·691	2·693	2·694	2·696	2·698	2·700	0	0	1	1	1	1	1	1	2
7·3	2·702	2·704	2·706	2·707	2·709	2·711	2·713	2·715	2·717	2·718	0	0	1	1	1	1	1	1	2
7·4	2·720	2·722	2·724	2·726	2·728	2·729	2·731	2·733	2·735	2·737	0	0	1	1	1	1	1	1	2
7·5	2·739	2·740	2·742	2·744	2·746	2·748	2·750	2·751	2·753	2·755	0	0	1	1	1	1	1	1	2
7·6	2·757	2·759	2·760	2·762	2·764	2·766	2·768	2·769	2·771	2·773	0	0	1	1	1	1	1	1	2
7·7	2·775	2·777	2·778	2·780	2·782	2·784	2·786	2·787	2·789	2·791	0	0	1	1	1	1	1	1	2
7·8	2·793	2·795	2·796	2·798	2·800	2·802	2·804	2·805	2·807	2·809	0	0	1	1	1	1	1	1	2
7·9	2·811	2·812	2·814	2·816	2·818	2·820	2·821	2·823	2·825	2·827	0	0	1	1	1	1	1	1	2
8·0	2·828	2·830	2·832	2·834	2·835	2·837	2·839	2·841	2·843	2·844	0	0	1	1	1	1	1	1	2
8·1	2·846	2·848	2·850	2·851	2·853	2·855	2·857	2·858	2·860	2·862	0	0	1	1	1	1	1	1	2
8·2	2·864	2·865	2·867	2·869	2·871	2·872	2·874	2·876	2·877	2·879	0	0	1	1	1	1	1	1	2
8·3	2·881	2·883	2·884	2·886	2·888	2·890	2·891	2·893	2·895	2·897	0	0	1	1	1	1	1	1	2
8·4	2·898	2·900	2·902	2·903	2·905	2·907	2·909	2·910	2·912	2·914	0	0	1	1	1	1	1	1	2
8·5	2·915	2·917	2·919	2·921	2·922	2·924	2·926	2·927	2·929	2·931	0	0	1	1	1	1	1	1	2
8·6	2·933	2·934	2·936	2·938	2·939	2·941	2·943	2·944	2·946	2·948	0	0	1	1	1	1	1	1	2
8·7	2·950	2·951	2·953	2·955	2·956	2·958	2·960	2·961	2·963	2·965	0	0	1	1	1	1	1	1	2
8·8	2·966	2·968	2·970	2·972	2·973	2·975	2·977	2·978	2·980	2·982	0	0	1	1	1	1	1	1	2
8·9	2·983	2·985	2·987	2·988	2·990	2·992	2·993	2·995	2·997	2·998	0	0	1	1	1	1	1	1	2
9·0	3·000	3·002	3·003	3·005	3·007	3·008	3·010	3·012	3·013	3·015	0	0	0	1	1	1	1	1	1
9·1	3·017	3·018	3·020	3·022	3·023	3·025	3·027	3·028	3·030	3·032	0	0	0	1	1	1	1	1	1
9·2	3·033	3·035	3·036	3·038	3·040	3·041	3·043	3·045	3·046	3·048	0	0	0	1	1	1	1	1	1
9·3	3·050	3·051	3·053	3·055	3·056	3·058	3·059	3·061	3·063	3·064	0	0	0	1	1	1	1	1	1
9·4	3·066	3·068	3·069	3·071	3·072	3·074	3·076	3·077	3·079	3·081	0	0	0	1	1	1	1	1	1
9·5	3·082	3·084	3·085	3·087	3·089	3·090	3·092	3·094	3·095	3·097	0	0	0	1	1	1	1	1	1
9·6	3·098	3·100	3·102	3·103	3·105	3·106	3·108	3·110	3·111	3·113	0	0	0	1	1	1	1	1	1
9·7	3·114	3·116	3·118	3·119	3·121	3·122	3·124	3·126	3·127	3·129	0	0	0	1	1	1	1	1	1
9·8	3·130	3·132	3·134	3·135	3·137	3·138	3·140	3·142	3·143	3·145	0	0	0	1	1	1	1	1	1
9·9	3·146	3·148	3·150	3·151	3·153	3·154	3·156	3·158	3·159	3·161	0	0	0	1	1	1	1	1	1

	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
55	7·416	7·423	7·430	7·436	7·443	7·450	7·457	7·463	7·470	7·477	1	1	2	3	3	4	5	5	6
56	7·483	7·490	7·497	7·503	7·510	7·517	7·523	7·530	7·537	7·543	1	1	2	3	3	4	5	5	6
57	7·550	7·556	7·563	7·570	7·576	7·583	7·589	7·596	7·603	7·609	1	1	2	3	3	4	5	5	6
58	7·616	7·622	7·629	7·635	7·642	7·649	7·655	7·662	7·668	7·675	1	1	2	3	3	4	5	5	6
59	7·681	7·688	7·694	7·701	7·707	7·714	7·720	7·727	7·733	7·740	1	1	2	3	3	4	4	5	6
60	7·746	7·752	7·759	7·765	7·772	7·778	7·785	7·791	7·797	7·804	1	1	2	3	3	4	4	5	6
61	7·810	7·817	7·823	7·829	7·836	7·842	7·849	7·855	7·861	7·868	1	1	2	3	3	4	4	5	6
62	7·874	7·880	7·887	7·893	7·899	7·906	7·912	7·918	7·925	7·931	1	1	2	3	3	4	4	5	6
63	7·937	7·944	7·950	7·956	7·962	7·969	7·975	7·981	7·987	7·994	1	1	2	3	3	4	4	5	6
64	8·000	8·006	8·012	8·019	8·025	8·031	8·037	8·044	8·050	8·056	1	1	2	2	3	4	4	5	6
65	8·062	8·068	8·075	8·081	8·087	8·093	8·099	8·106	8·112	8·118	1	1	2	2	3	4	4	5	6
66	8·124	8·130	8·136	8·142	8·149	8·155	8·161	8·167	8·173	8·179	1	1	2	2	3	4	4	5	5
67	8·185	8·191	8·198	8·204	8·210	8·216	8·222	8·228	8·234	8·240	1	1	2	2	3	4	4	5	5
68	8·246	8·252	8·258	8·264	8·270	8·276	8·283	8·289	8·295	8·301	1	1	2	2	3	4	4	5	5
69	8·307	8·313	8·319	8·325	8·331	8·337	8·343	8·349	8·355	8·361	1	1	2	2	3	4	4	5	5
70	8·367	8·373	8·379	8·385	8·390	8·396	8·402	8·408	8·414	8·420	1	1	2	2	3	4	4	5	5
71	8·426	8·432	8·438	8·444	8·450	8·456	8·462	8·468	8·473	8·479	1	1	2	2	3	4	4	5	5
72	8·485	8·491	8·497	8·503	8·509	8·515	8·521	8·526	8·532	8·538	1	1	2	2	3	3	4	5	5
73	8·544	8·550	8·556	8·562	8·567	8·573	8·579	8·585	8·591	8·597	1	1	2	2	3	3	4	5	5
74	8·602	8·608	8·614	8·620	8·626	8·631	8·637	8·643	8·649	8·654	1	1	2	2	3	3	4	5	5
75	8·660	8·666	8·672	8·678	8·683	8·689	8·695	8·701	8·706	8·712	1	1	2	2	3	3	4	5	5
76	8·718	8·724	8·729	8·735	8·741	8·746	8·752	8·758	8·764	8·769	1	1	2	2	3	3	4	5	5
77	8·775	8·781	8·786	8·792	8·798	8·803	8·809	8·815	8·820	8·826	1	1	2	2	3	3	4	4	5
78	8·832	8·837	8·843	8·849	8·854	8·860	8·866	8·871	8·877	8·883	1	1	2	2	3	3	4	4	5
79	8·888	8·894	8·899	8·905	8·911	8·916	8·922	8·927	8·933	8·939	1	1	2	2	3	3	4	4	5
80	8·944	8·950	8·955	8·961	8·967	8·972	8·978	8·983	8·989	8·994	1	1	2	2	3	3	4	4	5
81	9·000	9·006	9·011	9·017	9·022	9·028	9·033	9·039	9·044	9·050	1	1	2	2	3	3	4	4	5
82	9·055	9·061	9·066	9·072	9·077	9·083	9·088	9·094	9·099	9·105	1	1	2	2	3	3	4	4	5
83	9·110	9·116	9·121	9·127	9·1[illegible]2	9·138	9·143	9·149	9·154	9·160	1	1	2	2	3	3	4	4	5
84	9·165	9·171	9·176	9·182	9·187	9·192	9·198	9·203	9·209	9·214	1	1	2	2	3	3	4	4	5
85	9·220	9·225	9·230	9·236	9·241	9·247	9·252	9·257	9·263	9·268	1	1	2	2	3	3	4	4	5
86	9·274	9·279	9·284	9·290	9·295	9·301	9·306	9·311	9·317	9·322	1	1	2	2	3	3	4	4	5
87	9·327	9·333	9·338	9·343	9·349	9·354	9·359	9·365	9·370	9·375	1	1	2	2	3	3	4	4	5
88	9·381	9·386	9·391	9·397	9·402	9·407	9·413	9·418	9·423	9·429	1	1	2	2	3	3	4	4	5
89	9·434	9·439	9·445	9·450	9·455	9·460	9·466	9·471	9·476	9·482	1	1	2	2	3	3	4	4	5
90	9·487	9·492	9·497	9·503	9·508	9·513	9·518	9·524	9·529	9·534	1	1	2	2	3	3	4	4	5
91	9·539	9·545	9·550	9·555	9·560	9·566	9·571	9·576	9·581	9·586	1	1	2	2	3	3	4	4	5
92	9·592	9·597	9·602	9·607	9·612	9·618	9·623	9·628	9·633	9·638	1	1	2	2	3	3	4	4	5
93	9·644	9·649	9·654	9·659	9·664	9·670	9·675	9·680	9·685	9·690	1	1	2	2	3	3	4	4	5
94	9·695	9·701	9·706	9·711	9·716	9·721	9·726	9·731	9·737	9·742	1	1	2	2	3	3	4	4	5
95	9·747	9·752	9·757	9·762	9·767	9·772	9·778	9·783	9·788	9·793	1	1	2	2	3	3	4	4	5
96	9·798	9·803	9·808	9·813	9·818	9·823	9·829	9·834	9·839	9·844	1	1	2	2	3	3	4	4	5
97	9·849	9·854	9·859	9·864	9·869	9·874	9·879	9·884	9·889	9·894	1	1	1	2	3	3	4	4	5
98	9·899	9·905	9·910	9·915	9·920	9·925	9·930	9·935	9·940	9·945	0	1	1	2	2	3	3	4	4
99	9·950	9·955	9·960	9·965	9·970	9·975	9·980	9·985	9·990	9·995	0	1	1	2	2	3	3	4	4

ANSWERS

ELEMENTARY ALGEBRA
FOR SCHOOLS

PART I

ANSWERS

Exercise I (a) (p. 3)

1. 5, 1, 6, 12. **2.** 7, 13, 12, 0. **3.** 15, 2, 10, 40.
4. $\frac{2}{3}$, $\frac{4}{5}$, $\frac{1}{3}$, $2\frac{2}{3}$ **5.** 4, 12, 18, 13. **6.** 24, 17, 20, 18.
7. 8, 32, 64, 24, 54, 216 **8.** 2, 3, 5, 4 **9.** 16, 12, 18
10. 54. **11.** 8. **12.** 18 **13.** 13 **14.** 4. **15.** 24.
16. 12. **17.** 4 **18.** 15. **19.** 216. **20.** 720. **21.** 50.
22. 6 **23.** 12. **24.** 43 **25.** 20. **26.** 162. **27.** 82.
28. 9 **29.** 6 **30.** 12. **31.** 13 **32.** 15 **33.** $\frac{7}{12}$.
34. 60. **35.** 60 **36.** 44. **37.** 20. **38.** 8. **39.** 2.
40. 5. **41.** 12. **42.** 4. **43.** 50. **44.** 5. **45.** 26.
46. $xy=yx$. **47.** $x(y+z)=xy+xz$. **48.** $x\times x=x^2$.
49. $x+x=2x$ **50.** $x^2-y^2=(x+y)(x-y)$.

Exercise I (b) (p. 4)

1. 60. **2.** 88. **3.** 650. **4.** 8. **5.** $113\frac{1}{7}$.
6. £14, 8s. **7.** 72. **8.** $\frac{7}{20}$ ft. **9.** 1150 cu. yd. **10.** 24 6 H.P

Exercise II (a) (p. 6)

1. $6x$. **2.** $9y$. **3.** $5a$. **4.** $7ab$. **5.** $2x+3y$.
6. $3a+4b$. **7.** $5a^2$. **8.** x^2+2x. **9.** $3a^3+a^2$. **10.** $8xy$.
11. $2a+3b$. **12.** $2a^2+a$. **13.** $3x$. **14.** $2x$. **15.** x^2.
16. $3ab$. **17.** $y-x$. **18.** $3y-2x$. **19.** $a-a^2$. **20.** a^2-a.
21. x^3-x^2. **22.** $2x^3$. **23.** $2xy$. **24.** $3b-2a$. **25.** $7y$.
26. $10z$. **27.** $6x$. **28.** $3p^2$. **29.** $5q^2$. **30.** $4xy$.
31. $3x+y$. **32.** $4x+y$. **33.** $7p+10q$. **34.** $4x^2+x$. **35.** $6x^2-3y^2$.

36. $3y^2$. 37. 44. 38. 21. 39. 6. 40. 8
41. 44. 42. 23. 43. $\frac{7}{12}$. 44. $\frac{7a}{12}$. 45. $\frac{1}{6}$.
46. $\frac{b}{6}$. 47. $\frac{5a^2}{6}$. 48. $\frac{ab}{20}$. 49. $\frac{2a}{3}$. 50. $\frac{a^2}{2}$.
51. $\frac{b^2}{6}$. 52. $4x$. 53. 0 $8x$. 54. x^2. 55. $x-y$.
56. $x+y$; $y-x$. 57. x^2+2x. 58. x, $x+1$, $x+2$: sum is $3(x+1)$.
59. $3n+1$; $n+2$. 60. $n+1$.

Exercise II (b) (p. 9)

1. $3xy$. 2. $3xy$. 3. a^2b. 4. a^2b. 5. x^3y.
6. x^3y. 7. $6x^2y$. 8. $6x^2y$. 9. a^2bc 10. a^2bc.
11. a^2bc. 12. p^2q 13. $3ab$. 14. x^2yz. 15. $3a+3b$.
16. $10a+5b$. 17. $10ab$. 18. $2a^2+ab$. 19. $2a^2b$.
20. x^2y+2xy^2. 21. x^3-x^2y. 22. x^3y. 23. axy.
24. $ax+ay$. 25. $3x^2y$. 26. $3x^2+3y$. 27. $3x^2-3y$.
28. abc. 29. $ab+ac$. 30. $abc-ab$. 31. x^4.
32. y^5. 33. y^{12}. 34. $6x^6$. 35. $12y^5$.
36. x^{10}. 37. $2x^5y^3$. 38. x^{a+b}. 39. x^{a+2}.
40. a^3+a^2. 41. x^3+x^2y. 42. x^5y. 43. x^5+x^2y.
44. x^5y^5. 45. $x^5y^2+x^2y^5$. 46. x^2y-xy^2. 47. $6x^2+6xy$.
48. $a^4x^2-a^2x^4$. 49. $8x^2y+12xy^2$. 50. $6x^4y^2$. 51. $4x^2y^2$, $2xy$.
52. a^6b^4, a^3b^2. 53. $16x^4y^4$, $5x^3y^2$. 54. 120, 14400, $25x^6y^2$, $5x^3y$
55. x^6y^3, $2xy^2$. 56. xy^2, xy^3, x^3y^4, $3x^2y$. 57. $\frac{a^2}{4}$; 4 : 1.
58. 9. 59. $(2a+b)^2$, $(a-b)^2$, $(a+b)^2$, $(3a^2-b)^2$.
60. $a+b$, $2a-b$, $(a^2-b^2)^2$, $(3a^2-b)^3$.
61. $(2a+b)^2$, $(3a-b)^2$, $16(2a+b)^2$; $2(2a+b)$, $3(2x-3y)$, $4(x-2y^2)^2$.
62. $3(x+y)^2$, $2(x-2y)^3$.

Exercise II (c) (p. 10)

1. $b(a+1)$. 2. $5(x+2)$. 3. $x(1+2y)$. 4. $a(a+1)$.
5. $5a(a+1)$. 6. $2b(3b+1)$. 7. $3x(2x-1)$. 8. $y(2-y)$.
9. $4x(y-x)$. 10. $x(2x-3)$. 11. $xy(3x+2y)$. 12. $6y(x^2-2)$.
13. $x^2(3x+2)$. 14. $5b(a^2+2b)$. 15. $2a(2a+b+3)$.
16. $x(2x+y+1)$. 17. $3x^2(x+2y+3)$. 18. $x^2(x^2+2x-1)$.
19. $xy(x-y)$. 20. $2ax(1+2ax+x^2)$.

Exercise II (d) (p. 11)

1. $2a$, $24a$. 2. $4x$, $12x^2$. 3. ab, $2a^2b^2$. 4. $3ab$, $6a^2bc$.
5. b, abc. 6. qr, $pqrt$. 7. ab, $12a^2b^2$. 8. $2ab$, $12a^2b^3c^2$.
9. xy, $12x^2y^2$. 10. ab, $6a^3b^3$. 11. pq, $12p^2q^2$. 12. xyz, $x^2y^2z^2$
13. $2b^2$, $24a^2b^4c^2$. 14. $2x^2$, $48x^4y^2$. 15. xy^2, $12x^3y^3$. 16. x^2y, $12x^3y^3z$.

Exercise II (e) (p 13)

1. $2xy$. 2. $3x$. 3. $3xy$. 4. $3a^2$. 5. $4x^2$.
6. $\frac{2p}{3q}$. 7. $5x^5$. 8. $6x^6$. 9. $\frac{1}{2x}$. 10. $\frac{1}{3y^4}$.
11. $\frac{ac}{b}$. 12. $\frac{1}{3p^8q}$. 13. $3a+3b$. 14. $\frac{a+1}{2}$. 15. $x-1$.
16. $a-1$ 17. $2-b$ 18. $a+2$ 19. $\frac{x-1}{2}$. 20. $p-q$.
21. $a+x$. 22. $3b-2x$. 23. $\frac{2x^2+1}{2}$ 24. $3xy-2y$. 25. $2x-y$.
26. $2x+1$, $x-2y$, $2x+3y+xy$ 27. $2a-3b$.
28. 3. 29. $2x$. 30. $3x^2$ 31. $3x$. 32. ab^2.
33. $5a$. 34. xyz. 35. $4y$. 36. $5ab$. 37. $4x^2$.
38. $5x^5$. 39. $3a^3$ 40. ab 41. x^{a-1} 42. x^{n-2}.
43. $a^4+2a^3+3a^2$. 44. $ab+2b^2$. 45. $2x^3+3y^3$ 46. $4p^3+5pq^2+3p^2r$.
47. $4x^4+5x^2+2$. 48. $12(x+3)$. 49. $x+y$.
50 $c(b+c)$. 51. $b+b^2c$. 52. $2xy+1$.

Exercise II (f) (p 15)

1. $\frac{px}{qy}$. 2. $\frac{ab}{c}$. 3. ab. 4. a^2c. 5. $\frac{4}{y}$. 6. 0.
7. 1. 8. $\frac{x}{a}$. 9. $\frac{1}{a}$. 10. $\frac{p^3}{x}$. 11. $2axy$. 12. xyz.
13. $3abxy$. 14. xz. 15. $\frac{x^2}{8}$. 16. 2. 17. $\frac{8}{x^2}$. 18. $\frac{1}{2}$.
19. $\frac{x}{y}$. 20. $\frac{x^3}{y^3}$. 21. $\frac{xy}{2}$. 22. $\frac{5bxy}{2a}$. 23. $\frac{1}{2(a+b)}$.
24. 3. 25. $\frac{x+y}{3}$. 26. $\frac{b}{ab}$, $\frac{a}{ab}$, $\frac{1}{ab}$. 27. $\frac{3b}{6ab}$, $\frac{2a}{6ab}$, $\frac{1}{6ab}$.
28. $\frac{y^2}{x^2y^2}$, $\frac{x^2}{x^2y^2}$, $\frac{1}{x^2y^2}$ 29. $\frac{6y}{12xy}$, $\frac{4x}{12xy}$, $\frac{3}{12xy}$. 30. $\frac{y}{x^2y^2}$, $\frac{x}{x^2y^2}$, $\frac{2}{x^2y^2}$.

31. $\frac{y^3}{x^3y^3}, \frac{x^3}{x^3y^3}, \frac{1}{x^3y^3}$. **32.** $\frac{3x}{6}, \frac{2x}{6}$ **33.** $\frac{3x}{12}, \frac{2x}{12}$. **34.** $\frac{3x}{24}, \frac{2x}{24}, \frac{4x}{24}$.

35. $\frac{2}{2x}, \frac{1}{2x}$. **36.** $\frac{9}{3x}, \frac{5}{3x}$. **37.** $\frac{3}{6x}, \frac{1}{6x}$. **38.** $\frac{3}{6x^2}, \frac{2}{6x^2}$.

39. $\frac{y}{xy}, \frac{x}{xy}$. **40.** $\frac{x}{x^2y}, \frac{1}{x^2y}$. **41.** $\frac{y}{x^2y}, \frac{1}{x^2y}$. **42.** $\frac{3b}{ab}, \frac{2a}{ab}$.

43. $\frac{x}{x}, \frac{1}{x}$. **44.** $\frac{x^2}{x^2}, \frac{1}{x^2}$ **45.** $\frac{a^2}{a}, \frac{1}{a}$.

46. $\frac{y}{x^3y^3}, \frac{3x}{x^3y^3}$ **47.** $\frac{2a}{a^3b^4}, \frac{3a^2b}{a^3b^4}, \frac{b}{a^3b^4}$ **48.** $\frac{bc}{a^2b^2c^2}, \frac{ac}{a^2b^2c^2}, \frac{ab}{a^2b^2c^2}$.

49. $\frac{xy}{x^3y^2}, \frac{x^2}{x^3y^2}, \frac{2y}{x^3y^2}$ **50.** $\frac{abc}{abc}, \frac{c}{abc}, \frac{1}{abc}$.

Exercise II (g) (p. 17)

1. $\frac{5x}{6}$. **2.** $\frac{x}{12}$. **3.** $\frac{7x^2}{10}$. **4.** $\frac{ab}{6}$. **5.** $\frac{3p^3}{40}$.

6. $\frac{5x}{12}$. **7.** $\frac{2a^2}{5}$. **8.** $\frac{x}{15}$. **9.** $\frac{5x}{6}$. **10.** $\frac{13x^2}{12}$.

11. $\frac{3}{2x}$. **12.** $\frac{5}{6x}$. **13.** $\frac{8}{15a}$. **14.** $\frac{2x+1}{2x^2}$. **15.** $\frac{x+y}{xy}$

16. $\frac{x+2y}{6xy}$. **17.** $\frac{3a+2b}{ab}$ **18.** $\frac{x+y}{y}$ **19.** $\frac{3b-2a}{6ab}$. **20.** $\frac{xy-1}{y}$.

21. $\frac{2x+3}{6x^2}$. **22.** $\frac{2x+1}{4x^2y}$. **23.** $\frac{6a^2+1}{6a^2}$. **24.** $\frac{3x+1}{3x^2}$. **25.** $\frac{b-a}{ab}$.

26. $\frac{ac+b^2}{bc}$. **27.** $\frac{ay-bx}{xy}$ **28.** $\frac{b^2x-a^2y}{a^2b^2}$ **29.** $\frac{x-y}{y}$ **30.** $\frac{a^2+b^2}{b^2}$.

31. $\frac{a^2-x^2}{a^2}$. **32.** $\frac{a}{x}+\frac{b}{x}$. **33.** $\frac{1}{x^2}-\frac{1}{x}$. **34.** $1-\frac{b}{a}$. **35.** $\frac{3}{2a}+\frac{1}{b}$

36. $\frac{x^2}{y^2}-\frac{x}{y}$. **37.** $x-y$. **38.** $\frac{1}{ab^2}+\frac{1}{a^2b}$. **39.** $\frac{1}{3y}-\frac{1}{2x}$.

40. $\frac{1}{4ab^2}-\frac{1}{3a^2b}$. **41.** $1+\frac{y}{x}$. **42.** $\frac{a^2}{x^2}-1$.

Exercise III (a) (p 19)

1. (i) 15 ; (ii) 6 ; (iii) 18 ; (iv) 20 ; (v) $\frac{3}{8}$; (vi) $2\frac{1}{2}$; (vii) 69 ; (viii) 123.

2. (i) 10 ; (ii) 14 ; (iii) 20 ; (iv) 17 ; (v) 9 ; (vi) 9 , (vii) 25.

3. (i) $5\frac{1}{2}$; (ii) $3\frac{1}{2}$; (iii) $3\frac{1}{3}$, (iv) $\frac{4}{5}$; (v) 7 ; (vi) $4\frac{1}{3}$, (vii) $4\frac{2}{5}$, (viii) $1\frac{3}{8}$.

4. (i) $\frac{2x}{3}$; (ii) $\frac{5x}{12}$; (iii) $\frac{11x}{12}$; (iv) $\frac{2x}{3}$. 5. (i) $\frac{5x^2}{12}$, (ii) $\frac{x^3}{24}$.

6. (i) 11; (ii) 3; (iii) 3; (iv) 1. 7. (i) $\frac{5x}{6}$; (ii) $\frac{x}{6}$.

8. (i) $\frac{3x}{2y}$; (ii) $\frac{9x-4y}{6y}$. 9. (i) $\frac{(a+b)x}{ab}$, (ii) 1, (iii) x.

10. (i) $2bx$, (ii) $\frac{2}{ax}$; (iii) $\frac{a}{x}$. 11. (i) $\frac{(q+x)y}{qx}$, (ii) 1.

12. (i) 0, (ii) 1.

Exercise III (b) (p. 23)

1. $p-a-b$ (2) 2. $c-a+b$ (4) 3. $c+d-b+a$ (2)
4. $19x-20y$ (55 sh). 5. $x-y-p-q$ (2). 6. $2a-2b-3x-3y$ (7).
7. $4z-3x-2y$ (12) 8. (i) $a+(b-c)$ (7), (ii) $a-(b+c)$ (1); (iii) $a-(b-c)$ (5), (iv) $a+(b+c)$ (11).
9. (i) $2a+(3b-c)$ (19), (ii) $a-(2b-c)$ (2), (iii) $2a-(2b+3c)$ (0).
10. (i) $(a-b)+(c-d)$ (8); (ii) $(a+b)-(c-d)$ (12), (iii) $(a-b)-(c+d)$ (4), (iv) $(2a-b)-(3c-d)$ (12); (v) $(a+2b)-(2c+3d)$ (9); (vi) $(3a-2b)+(2c-d)$ (27); (vii) $(a-2b)-(2c-3d)$ (3).
11. (i) $a-b+c+d$ (10); (ii) $a-b-c-d$ (0); (iii) $a-b-c+d$ (4); (iv) $a-b+c-d$ (6).
12. (i) $3a+3b+2c+2d$; (ii) $3a+3b+2c-2d$, (iii) $3a+3b-2c-2d$; (iv) $3a+3b-2c+2d$; (v) $2a-2b-c-d$; (vi) $2a-4b-3c+6d$.
13. (i) $3a+2(b+c)$, (ii) $3a+2(b-c)$; (iii) $3a-2(b-c)$; (iv) $3(a+c)+2b$, (v) $3(a+c)-2b$, (vi) $3(a-c)-2b$.
14. (i) $2(p+r)-5q$, (ii) $3(p+2r)+5q$, (iii) $3p-5(q+r)$; (iv) $2(3p+2r)-5q$; (v) $5p-3(q+r)$; (vi) $5p-2(q+2r)$

Exercise III (c) (p. 26)

1. $\frac{7x-11}{12}$. 2. $\frac{5-x}{12}$. 3. $\frac{a-5b}{6}$. 4. $\frac{a-7b}{12}$. 5. $\frac{x-y}{2}$.
6. $\frac{4x-y}{3}$. 7. $\frac{3-a}{2}$. 8. $\frac{1-x}{3}$. 9. $\frac{3a-b}{4}$. 10. $\frac{p-5q}{12}$.
11. $\frac{5p-10q}{12}$. 12. $\frac{b}{a}$ 13. $\frac{x^2}{a}$ 14. $\frac{b-5a}{6a}$. 15. $\frac{a+19}{6}$.
16. $\frac{4x}{3}$. 17. $\frac{13x-y}{2}$. 18. $\frac{5y}{6a}$. 19. $\frac{5p+5q}{6}$. 20. $\frac{bx-ax}{ab}$.
21. $\frac{3x+y}{3}$. 22. $\frac{4y-x}{y}$ 23. $\frac{1}{6}$. 24. $\frac{10x-8y}{3}$.

25. $\frac{x-2y+2}{2}$. **26.** $\frac{x+3y}{2z}$. **27.** $\frac{5y-3}{2y}$. **28.** $\frac{7x-2y}{6y}$.
29. $\frac{10x-5y+2z}{12}$. **30.** $\frac{y+z-x}{y}$. **31.** $\frac{2xz-2yz}{xy}$. **32.** $\frac{x^2+y^2}{xy}$.

Exercise III (d) (p. 27)

3. (i) $x+\left(2y+\frac{z}{4}\right)$, (23 sh.); (ii) $20x+(2y-z)$, (208 sh.);
(iii) $20x-\left(3y+\frac{z}{6}\right)$, ($181\frac{1}{3}$ sh.); (iv) $20x-\left(y-\frac{z}{2}\right)$, (196 sh).
4. (i) $\frac{a-7b}{6}$; (ii) $\frac{2-a+b}{2}$; (iii) $\frac{2a-5b}{6}$; (iv) $\frac{a+5b}{6}$.
5. $\frac{2x-y+z}{2}$, 4. **6.** (i) 8; (ii) 4, (iii) $\frac{1}{2}$, (iv) $10\frac{1}{2}$, (v) 9.
7. (i) $a+5b$, (ii) $5a-4b-5c+4d$, (iii) $a+b-2c+2d$;
(iv) $a-bd+cd$; (v) $a+\frac{b}{2}+\frac{c}{2}$.
8. (i) $2(x+z)-y$; (ii) $2x+3(y-z)$, (iii) $3x-2(y+z)$;
(iv) $3(a+d)-4(b+c)$; (v) $a-c+2(b+d)$, (vi) $2(a+c)-3(b+d)$.
9. (i) $(5x-3y)-(p+2q)$; (ii) $5x-3y-p-2q$.
10. (i) $\frac{2a-8b}{15}$; (ii) $\frac{11p+10q}{12}$; (iii) $\frac{q}{4}$; (iv) $\frac{ab}{6}$.

Miscellaneous Examples I (p. 28)

1

1. 18, 36. **2.** (i) $6x^5$; (ii) $3x^2y$, (iii) $3x^4$; (iv) $4x^8$.
3. $12\frac{4}{7}$. **4.** (i) $2x+3y$; (ii) $6x^3y^3$; (iii) $2xy$.
5. (i) $\frac{5x}{6}$; (ii) $\frac{x}{6}$; (iii) $\frac{x^2}{6}$; (iv) $\frac{5}{6}$. **6.** 4. **7.** $10x+y$.

2

1. 13, 12. **2.** 3, $6x^2+4xy$. **3.** 64.
4. $24a^2b^2c^2$, $2abc$. **5.** (i) $\frac{a+b}{ab}$; (ii) $\frac{1}{ab}$; (iii) $\frac{b}{a}$.
6. (i) $a-(2b+3c)$; (ii) $a+(2b-3c)$; (iii) $a-(2b-3c)$. **7.** $(2x+6y)$ ft.

3

1. (i) $6x^2+3xy$; (ii) $x-y$; (iii) $6x^2y$.
2. (i) $2x(3x-2y)$; (ii) $ab(a-b)$; (iii) $a(a^2+2a-1)$.

3. (i) $x-y$; (ii) $3x-1$; (iii) $\frac{a^2+2}{2}$. **4.** $2x+y+y^2$.

5. (i) $\frac{y}{y}-\frac{x}{y}$, (ii) $\frac{y}{xy}-\frac{x}{xy}$. **6.** (i) $\frac{13x+4y}{12}$; (ii) $\frac{5x+8y}{12}$.

7. $a+b-x+y$.

4

1. 1, 3. **2.** p^2-q^2. **3.** (i) $x-(y+z)$; (ii) $x-y-z$.

4. 132. **6.** (i) $\frac{x^2+xy}{y^2}$; (ii) $\frac{x^3}{y^3}$; (iii) $\frac{x}{y}$. **7.** $100x+10y+z$.

5

1. 529, 11449. **2.** (i) $15a^2+5ab$; (ii) $15a^2b$. **3.** $2ax+x^2$.

4. (i) $\frac{x+y}{xy}$; (ii) $\frac{1}{xy}$; (iii) $\frac{x}{y}$. **5.** (i) $\frac{a^2}{b^2}+\frac{a}{b}$; (ii) $\frac{p}{q}-\frac{q}{p}$.

6. (i) $r-(p-q)$; (ii) $r-p+q$. **7.** $\pi(x+d)^2-\pi x^2$.

6

1. $6x^2y+9xy^2$. **2.** (i) $2x^2+3x$; (ii) $2a^2-3a+1$.

3. (i) $a(2a^2-a-2)$; (ii) $x(x+1)$; (iii) $xy(xy-1)$.

4. $2abc$, $24a^3b^3c^3$. **5.** 174.

6. (i) $\frac{a+7b}{12}$; (ii) $\frac{5x+y}{3}$. **7.** (i) ab ; (ii) $(x-y)$; (iii) $(q+r)$.

7

1. $x-y$. **2.** $16x^4y^4$, $4a^2b^2$.

3. $180-x-y$. **4.** (i) 1 ; (ii) $\frac{2ab-1}{4a^2b^2}$; (iii) 0.

5. Factors (ii), (iv), (vi), (vii) ; multiples (i), (viii).

6. (i) 212 ; (ii) 32. **7.** $20x+y$.

8

1. (i) $6ab^2+9a^2b$; (ii) $6a^2b^2$; (iii) $2x+4$.

2. (i) a^2-b^2 ; (ii) $\frac{4x+y}{6}$. **3.** $16x$. **4.** 336.

5. (i) $\frac{2}{xy}$; (ii) $\frac{x^2+y^2}{xy}$. **6.** Less, $p-q$. **7.** $n+1$.

9

1. (i) $\frac{2x^2y^2}{3}$; (ii) $\frac{x}{2y}$. **2.** $\frac{a^3b^3}{a^2b^2c^2}$, $\frac{a^3c^3}{a^2b^2c^2}$, $\frac{b^3c^3}{a^2b^2c^2}$.

3. $x-31$, $2x-23$. **4.** (i) 17 ; (ii) 9 ; (iii) 12.

5. $5\frac{5}{11}$. **6.** (i) $\frac{a}{2}$; (ii) $\frac{a^2}{2}$; (iii) 2. **7.** 72.

10

1. (i) $\frac{1}{3xy}$; (ii) $3xy$. **2.** (i) $4x^2+2x^3$, (ii) $\frac{x}{2}+1$, (iii) xy.
3. $2n-1$, $2n$, $2n+1$. Second even.
4. $81x^4y^4$, $\frac{4x^2y}{3z}$. **5.** (i) xy; (ii) $2x$; (iii) $3x+6$
6. $60-2x$. **7.** 41.

11

1. $2x-2y$. **2.** $l-a-2b$. **3.** 12. **4.** 7 sh, $\left(\frac{x}{2}-3\right)$ sh.
5. ab. **6.** $6(2a+b)$. **7.** $x=5$, $y=53$. Yes.

12

1. (i) 17; (ii) 3. **2.** (i) 0; (ii) $\frac{5x}{4}$. **3.** $(3x+2)$ halfpence.
4. $\frac{x+3}{2}$, $\frac{x-3}{2}$. **6.** $6a^2+5ab+6b^2$. **7.** (i) $\frac{8}{7}$, (ii) $\frac{a+c}{b}$.

Exercise IV (a) (p. 36)

1. $20y-2x$. **2.** $A=xy+3pq$. **3.** $240y-12x$ **4.** $xy+pq$.
5. $\frac{A}{b}$ yd. **6.** $x-y$, $x+z+b$. **7.** $12x-ny$. **8.** (i) $\frac{x}{3}$, (ii) $\frac{x}{y}$.
9. (x^2-xy) sq. ft. **10.** $10-x$. **11.** $\frac{6}{x}$. **12.** $\frac{30x+5y}{6}$.
13. (i) $x+1$; (ii) $x-1$.
14. (i) x, $x+1$, $x+2$, (ii) $x-1$, x, $x+1$; (iii) $x-2$, $x-1$, x
15. (i) 2x; (ii) $2x+1$ or $2x-1$. **16.** (i) $3x$; (ii) $\frac{5x}{3}$.
17. (i) n; (ii) nx. **18.** $\frac{4x}{3}$. **19.** $\frac{ny}{x}$. **20.** $6-x$.
21. $\frac{x^2}{16}$. **22.** $x(6-x)$. **23.** (i) x; (ii) $4x$.
24. (i) $3x$; (ii) $3x\pm$any non-multiple of 3 **25.** $\frac{px+qy}{20}$.
26. $x+2y+8$ **27.** $(x-1)y+2$. **28.** $(y-x+1)z$.
29. $\frac{(112z-y)x}{12}$ sh. **30.** $(x+2z)(y+2z)-xy$.

Exercise IV (b) (p. 38)

1. $b=ns$.	**2.** $b=xy+pq$.	**3.** $m=xy$.	**4.** $n = \frac{m}{y}$.
5. $p=12x$	**6.** $p=12x+6y$.	**7.** $f=3x$.	**8.** $z=36x+12y$.
9. $s=\frac{x}{12}$.	**10.** $p=\frac{3x}{20}+\frac{y}{240}$.	**11.** $c=y+\frac{nx}{12}$.	**12.** $p=12x-4y$
13. $A=lb$.	**14.** $l=9\frac{A}{b}$.	**15.** $\frac{x}{20y}$.	**16.** $\frac{ny}{20}-x$.
17. $A=2h(l+b)$	**18.** $\frac{x}{20}-\frac{xy}{240}$.	**19.** $v=u+at$	**20.** $T=\frac{(P-x)y}{20}$.

Exercise IV (c) (p 40)

1. xy pence	**2.** $\frac{y}{x}$ hrs.	**3.** $\frac{12bc}{x}$.	**4.** $\frac{xy}{z}$.
5. $\frac{yz}{x}$ pence	**6.** $\frac{6x}{5}$ ft	**7.** $\frac{x}{y}$ min.	**8.** $\frac{60-x}{y}$ hrs
9. $x(y-5)$ miles	**10.** $\frac{2240p}{q}$ cu ft	**11.** xy miles ; $\frac{xy}{x+4}$ hrs	
12. $\frac{(x+5)y}{x}$ hrs	**13.** $\frac{mx}{m+y}$ days	**14.** $\frac{z}{x+y}$.	**15.** $\frac{z}{x-y}$ min
16. $\frac{xy}{x+2}$ ft	**17.** $\frac{1760mx}{y}$ ft.	**18.** $\frac{c^2x}{b^2}$.	**19.** $\frac{144y}{x}$.
20. $\frac{(x-1)y}{z}+1$.			

Exercise V (a) (p. 44)

1. 3	**2.** 3	**3.** 4.	**4.** 14	**5.** 6	**6.** 3.
7. 7, 8, 9	**8.** 6	**9.** 13, 15	**10.** 6	**11.** 8, 9, 10	**12.** 5
13. 2.	**14.** 4.	**15.** 2	**16.** 4.	**17.** $3\frac{1}{3}$	**18.** 4
19. 3	**20.** 4.	**21.** 5.	**22.** 1.	**23.** 4	**24.** $2\frac{1}{4}$.
25. 12.	**26.** 5.	**27.** 0.	**28.** 2	**29.** 3	**30.** 0.
31. $\frac{7}{8}$.	**32.** $2\frac{1}{2}$	**33.** 1	**34.** $2\frac{1}{4}$	**35.** 1	**36.** 3
37. $2\frac{1}{2}$.	**38.** 3 1.	**39.** 2·1.	**40.** 3 2.		

Exercise V (b) (p. 46)

1. $1\frac{1}{9}$.	**2.** $\frac{7}{12}$	**3.** 1 2.	**4.** $\frac{1}{6}$.	**5.** 20.	**6.** 15.
7. $1\frac{1}{3}$.	**8.** $3\frac{5}{9}$.	**9.** 1 6	**10.** 8.	**11.** 28.	**12.** 10.
13. $5\frac{1}{2}$.	**14.** 3.	**15.** $\frac{m+1}{6}$.	**16.** 6	**17.** $\frac{x+2}{6}$.	**18.** 8.

19. 24. **20.** 18. **21.** 24. **22.** 36. **23.** 4. **24.** $\frac{1}{11}$.
25. 6. **26.** 1. **27.** $3\frac{1}{2}$. **28.** 21. **29.** $2\frac{1}{4}$. **30.** $7\frac{3}{5}$.
31. $\frac{x}{6}$. **32.** 12. **33.** $\frac{2x-1}{12}$. **34.** 4. **35.** $\frac{x-9}{4}$. **36.** 41.

Exercise V (c) (p. 49)

1. 7, 14. **2.** 22. **3.** 48, 12.
4. 120 yd., 60 yd. **5.** 190 boys, 150 girls. **6.** 30°, 60°, 90°.
7. 631 upper, 133 lower. **8.** 40. **9.** £20, £10.
10. 14. **11.** 10 in., 15 in. **12.** 50 miles.
13. 6 miles. **14.** 36 years, 30 years **15.** 40 miles.
16. 15 miles. **17.** 25 years.
18. $1\frac{1}{2}$ hr. at 3 m.p.h., 1 hr. at 4 m.p h. **19.** $7\frac{1}{2}$ ft.
20. 24 m.p.h., 30 m.p.h. **21.** 4. **22.** 60 c.c.
23. 12 years. **24.** 30 **25.** 3 sh.

Exercise V (d) (p. 52)

7. $4-3(6x-2)$. **8.** 7. **9.** $6\frac{1}{2}$. **10.** $1\frac{1}{2}$. **11.** 3. **12.** 18.

Miscellaneous Examples II (p. 54)

A 13

1. $6x+3y$. **2.** $3600t+60m$. **3.** 3. **4.** $3x^3+6x^2$; 2.
5. $5a+b-16c$; 13. **6.** 4; $2b$. **7.** 17 sh., 12 sh.

A 14

1. 20, 30, 14. **2.** $6ab^2c^2$; $2c^2$, $3ab$. **3.** $\frac{1}{6}x+\frac{1}{12}y$.
4. 4. **5.** $13x^3y^2+2x^4y$. **6.** (i) 6; (ii) 2.
7. 7, 9, 12.

A 15

1. $a^2+12a+2ab$. **2.** 10, 2: when $x=0$. **3.** (i) $3\frac{3}{4}$; (ii) 3.
5. $xy+z$; 68. **6.** $\frac{ay}{x}$. **7.** £48, £12.

A 16

1. $2a+5b+3c$; 28. **2.** $\left(ap+\frac{y}{z}\right)$ sh. **3.** $\frac{47a+24b}{12}$.
4. 1, 5, 9; 13. **5.** (i) 2; (ii) $\frac{2b-a}{3}$.
6. $5(a-2c)-3(b-3d)$; $5(a+2c)-3(b+3d)$. **7.** 6.

A 17

1. 32, 10. **2.** $\frac{13a}{6} + \frac{5b}{4}$. **3.** 3. **4.** (i) $5a$; (ii) 11.
5. $\frac{(3x+2y)c}{4}$. **6.** 14. **7.** 30.

A 18

1. 32, 12. **2.** $2a^2(a+2b)$, $5b(a+2b)$, $a+2b$ **3.** $2xy^3$; $9-2x$.
4. 25. **5.** (i) 7; (ii) $\frac{1}{2}$. **6.** 10, 4, 10. **7.** 1933.

A 19

1. 117, 27, $4\frac{13}{20}$. **2.** $2a^2b$; $24a^4b^3c^2d$ **3.** 7
4. $4a^2b+3b^3+6ab^2$. **5.** $\frac{1}{2}$ pint. **6.** $1\frac{1}{2}$ **7.** £1200.

A 20

1. 5 and 8 **2.** (i) $\frac{ac}{b}$; (ii) $\frac{4ab}{c^3}$. **3.** (i) 12, (ii) 9. **4.** $\left(\frac{x}{y}+\frac{x}{z}\right)$ hrs
5. $2\pi r(r+h)$, 132. **6.** (i) $\frac{5}{24a}$; (ii) $\frac{11a+33}{20}$. **7.** $3\frac{1}{4}$.

A 21

1. $xy+2y^2$, $y(x+2y)$. **2.** 17. **3.** $s=2l+\frac{2A}{l}$. **4.** $3pq^2$; $18p^3q^4$.
5. 0 2. **6.** (i) 10, (ii) 4. **7.** 9.

A 22

1. 12; $\frac{5}{8}$; 1200. **2.** $14x^2$; 10. **3.** 66°, 60°, 54°.
4. $\frac{2}{3}$. **6.** $a^5b^3+a^3b^5$; a^5b^5; $\frac{1}{ab}$; a^9b^9; $a+b$. **7.** 12.

A 23

1. $4b$. **2.** 13. **3.** (i) $\frac{ac-b}{cx}$; (ii) $\frac{ax-b}{cx}$. **4.** $5\frac{1}{4}$ sq. in.
5. (i) $\frac{1}{2}$; (ii) $\frac{b+c}{a+d}$. **6.** $(a+b)x$, $\frac{(a+b)x}{a+b+p-q}$. **7.** 18 m.p.h.

A 24

1. $\frac{p+q-2r}{2}$. **2.** $\frac{4a^2,\ 9b,\ 10}{12ab}$.
4. $a(2b+5c)-d(3c-7b)$; $a(2b+5c)+d(7b-3c)$; $b(2a+7d)-c(3d-5a)$; $b(2a+7d)+c(5a-3d)$.
5. Always. **6.** (i) $\frac{4a}{b}$; (ii) $\frac{5a^2-7}{12}$. **7.** 8 sh.

B 25

1. 11, 20. **2.** $(6x+y)$ pence **3.** 34, 22, 6 **4.** $a+7b+21c$.
5. $5x^3+8x^2+7x+12$; x^3+2x^2+7x+4. **6.** 1 **7.** 15, 18, 20.

B 26

1. n^2+2n+4. **2.** 20, 4, 20 **3.** 16 **4** (i) 24, (ii) 3.
5. $3n^2$, 8. **6.** (i) $\frac{26y}{3}$; (ii) $6-x$. **7.** $1\frac{1}{2}$d. per yd.

B 27

1. $6a^3b+11a^2b^2+4ab^3$ **2.** 262. **3.** 40, 384, $1\frac{1}{2}$, 100
4. $240a+12b+c$, $210a-12b+7c$. **5.** (i) 5, (ii) 9
6. px/z. **7.** $1\frac{1}{2}$ lb., 5 lb.

B 28

1. $\frac{(x-y)z}{xy}$. **2.** 1, 17. **3.** 26; 4.
4. 5 ways. $x=6$, $y=8$ or $x=4$, $y=12$. **5.** $(3x+3)/4$.
6. (i) 2; (ii) 6 **7.** 8.

B 29

2. $(30a+40b)/24$. **3.** $3(a-2c)-4(b-2d)$; 8.
4. $1\frac{1}{2}$. **5.** 95, 97, 99. **6.** $\frac{a^6}{2b^2}$. **7.** $6\frac{3}{10}$ miles.

B 30

1. $2x^2+xy+y^2$; 64. **2.** 32, 16, 5; 135. **3.** $\frac{13-4x}{10x}$.
5. $4\frac{2}{19}$. **6.** 784 ft., 208 ft. **7.** 48.

B 31

1. £$\frac{73gxy}{4n}$. **2.** $\frac{ac-b^2}{bc}$. **3.** $3xy^2$, x^4, $\frac{4b}{x^4}$.
4. 5. **5.** $\frac{a}{2b}-\frac{2c}{3a}$; $\frac{2x-4}{3x}$ **6.** $2b-a$.

B 32

1. Zero. **2.** $\frac{20bp}{c}$. **3.** 65°, 61°, 54°.
4. b; $b=1\frac{1}{4}$, $c=24$, $a=16$. **5.** 3 pq
6. $2(a-2b)+5c$; $3(a+2c)-4b$; $5a-3(b+2c)$; $2(2a-3c)-5b$.
7. 15 m.p.h.; 19 m p h.

B 33

1. $\dfrac{144s}{p}$, $\dfrac{144s}{p-2} - \dfrac{144s}{p}$. 2. $2a^2$; $\dfrac{1}{18b^2}$; $18a(a+3)$; $\dfrac{a+3}{2ab^2}$.

3. $\dfrac{ac+b}{c}$; $\dfrac{p}{q} + \dfrac{r^2}{q^2}$. 5. 1. 6. 9. 7. 218.

B 34

1. $\frac{1}{8}$. 2. $4x^3-5xy^2$; $3x^3-2xy^2$; x^3-3xy^2.

3. 0 7. 4. 441, 33075 5. $\dfrac{16a+20b}{25}$.

6. $9(p-q)^2$; $x^4(x+2y)^2$; $4a^6b^4$; $16(p-2q)^2$ 7. 52, 26, 13.

B 35

2. $\dfrac{3p+8q}{144}$ sh, $\dfrac{3p+8q}{12(p+q)}$ pence. 3. $2a$. 4. (i) $\dfrac{p^3-q^3}{p}$, (ii) $\dfrac{2q}{p}$.

5. $2at$ 6. (i) 2 8, (ii) 18 7. 2s 3d

B 36

1. (i) 3; (ii) $3a+2b$ 2. 6 in 3. 1521, 39,601, 2490 01

4. $V+nt$; $V-nx$; $n(t+x)$ 5. $4(3p-2q)$; 4, $\dfrac{2x}{3y}$.

6. 24 lb., 28 lb 7. £480

Exercise VI (a) (p. 71)

2. 7°. 3. 3°. 4. 400 ft 5. (−300) ft.
6. 50 ft. 7. (i) 2nd, (ii) 17th. 8. 57 B.C, 1928 A.D
9. (i) 79, (ii) 39. 10. 11, (−3), 5
11. $a+b+c$, $a-b+c$, $a-2b-2c$, $a-b+c$. 12. 8.
13. (−2). 14. −7; 6, 4, −6, 2
15. **drop** 6°; **drop** $(a-b)°$; +6°. 16. (i) 4°, (ii) $(b-a)°$; 4°.
17. (i) +300 ft., (ii) $+(x-y)$ ft.; +100.
18. (i) +2, (ii) $x-y$; −2; A is 2 spaces to the **left** of B; −5.
19. (i) 2; (ii) 3.

Exercise VI (b) (p. 74)

2. −6°, −4. 3. +4, −8 4. ±2; (i) ±3; (ii) $\pm a$; (iii) $\pm b^2$
5. (i) 2; (ii) (−2), (iii) a, (iv) $(-a)$. 6. £a. 7. 6, 6, 6
8. $(-x)$ $y=-xy$ 9. (i) (−1); (ii) (−6).
10. −10 ft. per sec. **Descending.**
11. 20, 40, −10, 4 seconds previously it was 40 ft. per sec.
12. 0, −32, 4 sec.; passed its highest point and is **dropping**.

13. $+1$, -1, $+1$; (i) -1; (ii) $+1$.
14. -3, $-\frac{b}{a}$.
15. $-\frac{b}{a}$.
16. $-ab$.
17. $+4$; 20, 100.
18. 18, 36, -27, -135, -3375, -21.
19. 5, 25, 1, -7, 49.
20. -11; 20, 20.
21. -68, -86, 4, 12
22. 24, 0, 12, 24.
23. -24.

Exercise VI (c) (p. 77)

1. (i) -1; (ii) 5; (iii) 8; (iv) -2; (v) 6; (vi) 5; (vii) 50; (viii) 34, (ix) 16; (x) 38.
2. (i) 6; (ii) 3; (iii) -3; (iv) -2; (v) $\frac{1}{2}$, (vi) 5; (vii) 2; (viii) 3; (ix) -2; (x) $-1\frac{1}{2}$.
3. (i) $-\frac{1}{4}$; (ii) 3; (iii) 5; (iv) 3; (v) $-\frac{5}{4}$; (vi) $-\frac{5}{4}$; (vii) $\frac{1}{3}$; (viii) $\frac{1}{3}$; (ix) 1; (x) ∞.
4. (i) $(-b)$; (ii) $(-b)$; (iii) $(-b)$; (iv) $(-x)$; (v) $(y-x)$; (vi) $(a-b)$; (vii) $(b-a)$; (viii) $(-c)$; (ix) $(-d)$; (x) $(-d)$; (xi) $(a-b)$; (xii) $(b-a)$; (xiii) $(a-b)$; (xiv) $(b-a)$.
5. 5, $-1\frac{2}{3}$, $-2\frac{1}{2}$, $-1\frac{1}{6}$, $4\frac{1}{3}$.
8. (i) $-\frac{3}{2}$; (ii) $+\frac{3}{2}$; (iii) $-\frac{3}{2}$; (iv) $\frac{3}{2}$.
9. (i) $1\frac{1}{2}$; (ii) $1\frac{1}{2}$; (iii) $1\frac{1}{2}$; (iv) -1.
10. (i) 6; (ii) -6; (iii) -6; (iv) -6.
11. $x+1$, $x-1$.
12. 0, $2x^3$.
13. Yes. 1 and 1; No. 1 and -1.
14. (i) No; (ii) No.
15. $2a$.

Exercise VII (a) (p. 82)

1. (1, 1).
2. (7, 4).
3. (2, 2).
4. (6, 5).
5. (18, 12).
6. (6, -6).
7. (4, 1).
8. (7, -3)
9. (3, 1).
10. (3, 2).
11. (0, 2).
12. ($\frac{1}{2}$, $\frac{1}{4}$).
13. (11, 5).
14. ($\frac{2}{3}$, $\frac{5}{6}$).
15. (3, -2).
16. (-3, -2 5).
17. (-2, 3).
18. (4. 5).
19. (4, 6).
20. ($-\frac{8}{7}$, $\frac{5}{7}$).
21. (4, 3).
22. (19, 3).
23. (3, 2).
24. (4, 5).
25. (1, 1).
26. (4, 12).
27. (7, 10).
28. ($\frac{7}{11}$, $-\frac{2}{11}$)
29. ($\frac{12}{7}$, $\frac{8}{7}$).
30. (6, 0).

Exercise VII (b) (p. 84)

1. 3 large, $4\frac{1}{2}$ small.
2. Tram 1d., rail $1\frac{1}{2}$d.
3. 35 miles.
4. 11 years 6 months.
5. $32\frac{1}{2}$ min.
6. 209 boys.
7. ($\frac{1}{4}$, 12); 37.
8. 73.
9. 14 at 2s., 28 at 2s. 8d.
10. -16.
11. $7\frac{1}{2}$d., 2s. 9d.

12. 929, 260. **13.** 155, 45. **14.** 4800.
15. £480 at 5 per cent., £720 at 4 per cent. **16.** 43.
17. Copper 84 c.c., tin 16 c.c. **18.** 2 goals 1 try, and 1 goal 2 tries.
19. 1d. for each lb. over 7 lb. **20.** 57. **21.** A 5s., B 7s. 6d.
22. Price of cigarettes falls 6d. for 50; price of tobacco rises 2d. per oz.
23. Rail fares up ½d. per mile; sea fare down ½d. per mile.
24. Rubber £1 down; oil £2, 10s. up. **25.** 132 miles.
26. Tin 84 lb.; lead 96 lb. **27.** 371, 283.
28. 100 miles, 10 miles per hour. **29.** 24. **30.** 133 : 92.

Exercise VII (c) (p. 89)

1. $3a$. **2.** ab. **3.** $\frac{b}{a}$. **4.** $\frac{a}{b}$. **5.** a.

6. $b-a$. **7.** $\frac{c-b}{a}$. **8.** $\frac{b+c}{a}$. **9.** $\frac{ab}{a+b}$. **10.** $\frac{r-s}{p-q}$.

11. $\frac{ac}{b}$. **12.** $b(c-a)$. **13.** $\frac{a}{b-1}$ **14.** $\frac{pb-a}{p+1}$. **15.** $\frac{ad(b-c)}{a+d}$.

16. $b+\frac{c}{a}$. **17.** $\frac{b}{a+b+c}$. **18.** $a(b+c)$. **19.** a. **20.** $\frac{a+b}{c}$.

21. $\frac{12-3x}{5}$. **22.** $\frac{6}{x}$. **23.** $\frac{c-ax}{b}$. **24.** $\pm\sqrt{4-x^2}$. **25.** $\frac{a-x}{x}$.

26. $\frac{P-b}{a}$. **27.** $\frac{v^2-u^2}{2f}$. **28.** $\frac{A}{2\pi r}-r$. **29.** $\sqrt{\frac{Wl}{2b}}$. **30.** $f=\frac{uv}{u+v}$.

31. (i) $\frac{a}{y}$; (ii) $b-y$; (iii) $\frac{y}{y-1}$; (iv) $\frac{c-2by}{3a}$; (v) $2a-\frac{ay}{b}$;
(vi) $\frac{y^2}{1-y}$; (vii) $\frac{y}{2y-1}$; (viii) $\pm\sqrt{a-y}$.

32. $y=\frac{A}{\pi x}-x$. **33.** $t=1-\frac{2x}{5m}$. **34.** $a=(v-v_0)/v_0t$.

35. $r=\pm\sqrt{\frac{B}{A-e}}$. **36.** $\sqrt{\frac{A}{\pi}}$, $\frac{C}{2\pi}$, $\frac{C^2}{4\pi}$, $\sqrt{4\pi A}$. **37.** $\frac{u-v}{g}$; 6; $2\frac{7}{8}$.

38. $F=32+1{\cdot}8\,C.$; -40. **39.** $t=\frac{100I}{P\,.\,r}$; $2\frac{2}{3}$ years.

40. $\frac{d^2c-a}{d^2-1}$. **41.** $P=\frac{adW^2}{aW^2-1}$. **42.** $\frac{2abc}{bc+ca-ab}$.

Exercise VII (d) (p. 93)

1. (3, 2). **2.** (2, 6). **3.** $\left(\frac{a^2+b^2}{a+b}, \frac{a-b}{a+b}\right)$.

4. (1, 1). **5.** (7, 4). **6.** $\left(\frac{1}{a}, \frac{1}{b}\right)$.

7. $(q,\ 0)$. **8.** $\dfrac{br-cq}{aq-bp}$; $\dfrac{cp-ar}{aq-bp}$. **9.** $(\frac{5}{12},\ \frac{1}{12})$.
10. $(\frac{1}{2},\ \frac{1}{3})$. **11.** $(a,\ -b)$ **12.** $(a^2,\ -b^2)$.
13. $(a+b,\ b-a)$ **14.** $\left(\dfrac{p}{q},\ 1\right)$ **15.** $x = \dfrac{by-a}{y-1}$.
16. $R = \dfrac{b}{a-b}$. **17.** $\pm\sqrt{b^2-a^2}$. **18.** $\dfrac{a}{1-by^2}$.
19. $y = \dfrac{b+a^2cr^2}{1-a^2r^2}$. **20.** $v = \dfrac{uf}{u+f}$.

Exercise VIII (a) (p. 95)

1. (6 8, 0), (0, 11 3). **2.** $(\pm 6,\ \pm 8)$.
3. (3, −4). **4.** (−12, −2), 126 square units.
5. $38\frac{1}{2}$, $45\frac{1}{2}$, 15, 12; 159. **6.** 10, 24, 24; 10.

Exercise VIII (b) (p. 98)

2. April to June, Sept to Oct.
3. 1920–1921, 1922–1923, 1924–1925 **4.** 1916–1918.

Exercise VIII (c) (p. 102)

1. About 70 cm **2.** $x + \dfrac{2A}{x}$; 20 ft.
3. £241, 23 years. **4.** 41 4 sec.
5. 12 years, æt 48 **6.** 64 cu. ft.; 80 sec.; after 23 4 sec.
7. $4x(5-x)(10-x)$ cu. cm.; 193 cu. cm.; 2·11 cm.

Exercise VIII (d) (p. 106)

1. 10·9. **3.** 8.
5. 50 7° F.; −17 8° C. **6.** 1 5; 3; −1.
7. +4 7; +3 3; +1 3; −2. **8.** $y=2{\cdot}2x$; 3 6 kg.; 7 5 lb.
9. $15y=22x$; 51·3 feet per sec.; 41 miles per hour.
10. 40. **11.** 12 5.

Exercise VIII (e) (p. 109)

1. (2·3, 0 7). **2.** (2 02, 0·78). **3.** (2·5, 0·8). **4.** (0 75, 1·87).
5. (1 6, −1 2). **6.** (1·72, 0 96). **7.** (2·2, 1 7). **8.** (−0 3, −1·7).

Exercise VIII (f) (p. 111)

1.

x	-3	-2	-1	0	1	2	3
y	13	2	-3	-2	5	18	37

2.

x	-4	-3	-2	-1	0	1	2	3	4
y	95	61	35	17	7	5	11	25	47

4. 1 ; -0 4 and $+2$ 4 **5.** $+0$ 38 and 2·62.
6. $x>6$ and $x<0$; 5 65 and 0 35 ; 5 24 and 0 76
7. 2 41 and -0 41. **8.** 30 ft ; 120 ft
9. $-4\frac{1}{4}$, $+3$ 79 and -0 79. **10.** $\dfrac{(x+3)^2-10}{5}$. **11.** 37 4.

Exercise VIII (g) (p 112)

1. (3 09, 2 63) **2.** 5·6 ft , 2 4 lb
3. 2 41, 3·82, 1 **4.** 1 5, 2 38, 4 62, 5 5
5. (6·12, 0 54) ; (1 22, 5 44). **6.** 3 and $-\frac{1}{3}$; 3 19 and -0 52.
7. $2x+3y=17$. **8.** 2 32 **9.** 1 71 and 0 29. **10.** ∞, 0.
11. $f(8\ 3)=-\cdot0124$.
$f(8\cdot4)=+\ 0126$
12. 1 33
13. -0 63, -3 25, $+3$ 88 **14.** -1 46 ; $+3$, $+2$ 7, -3 7

Miscellaneous Examples III (p 114)

A 37

1. $2a^2-4ab+8b^2$. **2.** (i) -112 , (ii) -15360 ; (iii) -2 4 , (iv) 49.
3. $(2\frac{1}{2},\ -\frac{1}{2})$. **4.** $-\dfrac{x+13a}{6a}$, $-17a$.
5. $p+(q-r+t)$; $p+q-(r-t)$. **6.** 40 miles, 80 miles.

A 38

1. 154. **2.** $4a^3b^2c^3$, $60a^4b^2c^4$ **3.** (i) $\dfrac{x-7y}{6x}$; (ii) $\dfrac{p+q}{6r}$.
4. 3, 2. **5.** $a=4$, $b=8$, $c=10$, $d=22$, $x=9$ **6.** 9 0, 23·4.

A 39

1. $11\frac{1}{3}$, -1, $45\frac{1}{2}$, $114\frac{1}{3}$. **2.** $\frac{16y}{16-x}$ lb. **3.** $(+5, \; -3)$

4. (i) $2x(a-2b)-3y(b-3a)$, (ii) $a(2x+9y)-b(4x+3y)$

5. Chicken 6s. 4d., duck 7s. 6d **6.** 42; (i) 6, 12, 7; (ii) 17.

A 40

1. $-6a^4+11a^3b-3b^4$. **3.** $\frac{11y-x}{2}$. **4.** $(7, \; -2)$.

5. (i) $\frac{a}{c}$, No, $\frac{a+c}{c}$, No; (ii) $\frac{3}{5}$, $\frac{5}{7}$, $1\frac{3}{5}$, $\frac{1}{2}$. **6.** 49·1 gm.; 7·4 cm.

A 41

1. -21, 9, -343. **2.** $\frac{x(y-z)}{y}$. **3.** -4. **4.** $\frac{2ac}{b}$, $\frac{a}{c}$.

5. -1 **6.** (i) $2b-a$, (ii) 4 years, -9 years ago.

A 42

1. $-x^2$, $-\frac{1}{x^2}$, $\frac{x^2+1}{x}$, $\frac{2}{x}$, $2x$. **2.** $\frac{2}{a}$. **3.** $3(x-3t)-2(y-2z)$.

4. 3, -8. **5.** 24, 10. **6.** $\frac{8}{9}$.

A 43

1. (i) 3, (ii) $-1\frac{8}{9}$. **2.** (i) 9, (ii) $+\frac{1}{5}$, $-\frac{1}{5}$. **3.** £$\frac{5A}{4}$.

4. $\frac{3p+2q}{6p^2q}$; $\frac{p}{3q} - \frac{q}{2p}$ **5.** $\frac{9a^2\sqrt{3}}{4}$, $\frac{a^2\sqrt{3}}{4}$, $\frac{3a^2\sqrt{3}}{2}$ **6.** 600.

A 44

1. $30a^3b^3x^2y^3z^3$ **2.** 13. **3.** (i) 4; (ii) 5, 3 **4.** -16.

5. 75 miles, 24 miles.

6. $(2, -3)$; (i) $(8, 5)$ or $(8, -11)$, (ii) $(8, 5)$ or $(-4, 5)$.

A 45

1. $-4\frac{1}{6}$. **2.** 0, $x+x^2$, $-x^2$, $-\frac{1}{x^2}$, x^2+x^3. **3.** $6ax+6bx+4x^2$.

4. -10, $+10$, -10. **5.** 2·63 acres (12,750 sq. yd.). **6.** 18.

A 46

1. 7·5. **2.** 15. **3.** b. **4.** 15.

5. $+3$, -3, 1, 3, -2. **6.** After (i) 10 years, (ii) 14 years.

A 47

1. 3. 2. (i) -3; (ii) 13, 42. 3. $5x+13$.

4. $\{(n-1)y+2\}$ in. 5. $4\frac{1}{2}$d., $6\frac{1}{2}$d. 6. $\dfrac{d-h}{b-c}$ hours.

A 48

1 $2(qy-bp)-5(ap+qx)$ 2. $(x+y)/2$. 3. $2\frac{2}{3}$, $-\frac{1}{3}$.

4. $3a+4b$, $12a+12b$ 5. Any value between 2 and 4. 6. 38 sh.

B 49

1. $\dfrac{pq}{q+r}$. 2. $5\frac{4}{5}$. 3. $15x^8$, $2x^4$, x^{a+b}, x^{n^2-n}, x^{6n}.

4. $l=\dfrac{A}{2h}-b$.

5. 20 miles, (i) 12 36 p m; (ii) 11 a m.; (iii) 1 24 p m.

B 50

2. 1, $+1$, -17. 4. $\dfrac{a-1}{2}$. 5. Between 0 and 1; between 2 and 3.

6. 2s. 6d., 24 sh

B 51

2. $6(p+q)x-4x^2$. 3. $\dfrac{11a-9b}{3}$; $-\dfrac{10a}{3}$.

4. 1st and 3rd, -3, -5, $1\frac{1}{3}$, $-1\frac{2}{3}$.

5. East 120 miles; West 132 miles. 6. (i) 147 2° F.; (ii) 26·7° C.

B 52

1. $\dfrac{3a^2}{2}$, 0, 2, $\dfrac{9a^3}{8}$, $\dfrac{27a^3}{8}$ 2. -6, 9, -8, $-\frac{1}{8}$.

3. $a=3$, $b=-4$; -1. 4. $\dfrac{3u}{u-3}$; 4.

5. (*a*) £40,000, £10,000; (*b*) £36,000, £12,000.

B 53

1. -5, -1. 2. £$12y/(12+x)$. 3. (i) 4; (ii) 8.

4. $\dfrac{2(x-1)}{x-2}-\dfrac{3(x-5)}{x+2}$. 5. D to E.

B 54

1. 8 5. 2. $-\dfrac{a^2y}{4}$. 3. $\dfrac{20a}{p}$ sh.; $\dfrac{20ap}{20a+bp}$.

4. $\frac{1}{6}(4x-5z)-\frac{1}{8}(6y-7w)$. 5. 5. 6. 1, 1.

B 55

1. $y = -\frac{9}{7}x + \frac{13}{7}$. **2.** $15y^5$, pqy^{p+q}, $-8x^9$, $+x^{8n}$, $4x^9$.

3. $-5\frac{5}{8}$, $-2\frac{1}{16}$. **4.** $\frac{\mathrm{R}\ l}{\mathrm{T}+\mathrm{R}}$; $\frac{4}{7}$.

5. 150 miles ; 80 miles per hour. **6.** (i) 12s. 9d. ; (ii) 28s. 9d.

B 56

1. $4b$. **2.** 60, -12. **3.** $\frac{ac}{2}$, $-\frac{bc}{3}$. **4.** $\frac{x-1}{3}$, $\frac{x+2}{5x-2}$.

5. (i) 42 ft ; (ii) 72 ft. ; (iii) 30 ft. **6.** A(-2, -3) ; D(6, -9).

B 57

1. $\frac{3}{b^2} - \frac{1}{6ab}$. **2.** $\frac{b-a}{b+a}$.

3. A, 2.10 p m. 31st July ; C, 3 p.m. 1st August.

4. (i) A-210, (ii) A-345 ; (i) A$-30n$, (ii) A$-15n-150$.

5. $v^2+2v=40$. **6.** 80, -16 ; 100 ft , 5 sec

B 58

1. $x-2y=13$. **2.** Height 3 in., side $4\frac{1}{2}$ in. **3.** $2x$, $\frac{5x}{2}$.

4. a is equal to 0. **5.** $\frac{a}{2\pi}$, $\frac{a^2}{4\pi}$; $\frac{a^2b}{4\pi}$ and $\frac{ab^2}{4\pi}$; $1\frac{13}{22}$.

6. 5 7 revolutions.

B 59

1. (i) 35 ; (ii) 25. **2.** $y=325+\frac{3x}{20}$. **3.** $138-3n$; -12.

4. (i) 5 ; (ii) c. **5.** $x=\frac{3}{2}y$. **6.** $\frac{a}{a-b}$ hours.

B 60

1. $x=+\frac{1}{12}$, $y=-\frac{1}{12}$. **2.** 5, 9, 24 ; $x=24$.

3. 0, 3, 8, 15, 24. No.

4. x^2, $\frac{1}{2}(6-x)^2$, $\frac{3x}{2}(8-x)$; y is maximum when $x=4$.

5. P is 2 units to the left of O.

6. (i) $\frac{p-q+r}{2}$, $\frac{r-p+q}{2}$; (ii) $\frac{2p-q+r}{3}$, $\frac{2r-2p+q}{3}$.

Exercise IX (a) (p. 133)

1. $5a-b$. **2.** $7x-5y$. **3.** $5b$.
4. $6x-3y-1$. **5.** $3x^2+2x-4y+5$. **6.** $4x^3-x-1$.
7. $-2a+3b-1$. **8.** $3a+3b-3c$. **9.** $2x^2-2xz$.
10. $pr+3qr-ps$. **11.** $2y$. **12.** $-x+3y$.
13. $2b-2c$. **14.** $-3x+7$. **15.** $2x-3$. **16.** $-x^2+3x-3$.
17. $-a+5b-4c$. **18.** $-a+2b-c$. **19.** $p+2q-r$. **20.** x^2+2x-1.
21. x. **22.** $b-5$. **23.** $2xy$. **24.** $mn+3nt$
25. $\frac{7a}{6}-b+\frac{c}{6}$. **26.** $-\frac{5a}{6}+b-\frac{c}{6}$. **27.** $\frac{x}{4}-\frac{5y}{6}$.

Exercise IX (b) (p. 134)

1. $x^2-x^3+x^4$. **2.** $a^2bc+ab^2c-abc^2$ **3.** $a^3b+ab^3-abc^2$
4. $-3x^3-6x^2-3x$. **5.** x^2-x+1. **6.** x^{n+1}, x^{n+2}, x^n.
7. $x^{p+1}+x^{q+1}$, $x^{p+1}+x^{q+1}$, $x^{m+r}+x^{n+r}$. **8.** $-x^3+x^2-x+1$.
9. $a-b+1$ **10.** $-2x+3y-4$ **11.** $4a^2-2ab+3b^2$.
12. x^{n-1}, x^{n-2}, x^{n-3}. **13.** x^{p-2}, x^{p-q}. **14.** $x^{p-1}+x^{q-1}$, $x^{m-2}-x^{n-2}$.
15. $x^{m-r}+x^{n-r}$, $-x^{a-1}$. **16.** $-x^{a-2}+x^{b-2}$. **17.** x^2+x.

Exercise IX (c) (p. 136)

1. $ax-bx-ay+by$. **2.** $2ax+4bx+ay+2by$.
3. p^2-q^2. **4.** x^2+x-6. **5.** $a^2-9a+20$.
6. $2ax-bx+4ay-2by$ **7.** $a^2+2ab+b^2$. **8.** $a^2-2ab+b^2$.
9. x^4-1. **10.** $3ax-2ay+3bx-2by$.
11. $6x^2+13xy+6y^2$. **12.** $a^2-ab-6b^2$.
13. $a^3+ab^2+a^2b+b^3$. **14.** a^3+b^3. **15.** a^3-b^3.
16. $x^3+3x^2y+3xy^2+y^3$. **17.** $a^4+a^2b^2+b^4$.
18. $3x^3-8x^2-10x+21$. **19.** $x^5+2x^4+x^3+x^2+2x+1$.
20. $12x^3-16x^2-79x-42$ **21.** $8x^4+6x^3+9x^2+9x+10$.
22. $9x^4-12x^3+15x^2+8x-14$. **23.** $2x^5-x^4-7x^3-9x^2+16x-3$.
24. $2x^5+4x^4-7x^3+4x^2+15x-18$. **25.** 5. **26.** -3.
27. $(a+b)$ **28.** $(a-b)$. **29.** 5. **30.** -7.
31. $(3a-2b)$. **32.** $-(4a+b)$. **33.** -3. **34.** -5.
35. $2a$. **36.** 0. **37.** $2ab$. **38.** 6.
39. -2. **40.** 10. **41.** -2.

Exercise IX (d) (p. 137)

1. $x(a-b)$. **2.** $x(x+y)$. **3.** $m(m-2n)$.
4. $x(x^2-x-2)$. **5.** $3(y^2-2y+4)$. **6.** $a(a^2-2a+b)$.

7. $x(a-b+c)$. 8. $3a(2a^3-a^2+3b)$ 9. $d(ab-bc+ac)$.
10. $ac(b-2a+cd)$. 11. $xy(xz-z^2+y^2)$. 12. $7b(a-c-2x)$.
13. $7x(2x^2-xy+3y^2)$ 14. $2x(x^2-2x+3)$. 15 $3x(x-2y+3y^2)$.

Exercise IX (e) (p. 139)

1. $(a+b)(x+y)$. 2. $(x+3)(y+2)$. 3. $(x-y)(a-b)$
4. $(a-3)(b+4)$. 5. $(3p+2)(q-4)$. 6. $x(x^2+x+1)$.
7. $3x(x-1)$ 8. $(a+2)(a+b)$. 9 $(p-q)(x+y)$
10. $(x+1)(x+y)$ 11. $(c-6)(3d+2)$. 12. $(x-a)(x+1)$.
13. $(x+7)(y+2)$. 14. $(x-y)(a-b)$ 15. $(2x-5)(4y+3)$
16. $2(a+1)(b^2+2)$ 17. $4(x-y)(x-1)$. 18. $(a+3)(b-2)$
19. $3(x^2+1)(x+1$, 20. $(x^2+1)(y^2+1)$ 21. $(a-b)(a-c)$
22. $(x-3)(x^2+2)$ 23. $(a-c)(a-b)$. 24. $(x-2)(x+y)$.
25. $(x-y)(x+z)$ 26. $(x-2)(x^2-3)$. 27. $(a+b)(3x-2y)$
28. $(b+a)(c-a)$. 29. $y(x^2+3)(y-5)$. 30. $(a-d)(b-c)$.
31. $(p+s)(q-r)$

Exercise IX (f) (p. 140)

1. $(x-1)^2$. 2. $(x-1)(x-2)$. 3. $(x+1)(x+4)$.
4. $(x-4)(x-2)$. 5. $(x-4)(x+3)$. 6. $(y-3x)(y+2x)$
7. $(a+7b)(a-b)$. 8. $(x+5)(x-4)$ 9. $(x-2)(x+1)$.
10. $(x-5)^2$. 11. $(x+7)(x-6)$. 12. $3(x-2)(x+1)$.
13. $(x+3)(x-2)$ 14. $(x-11)(x+10)$ 15. $(x-5)(x+4)$.
16. $2(x-5)(x+2)$ 17. $(1-2x)(1-x)$ 18. $(1-4x)(1-3x)$.
19. $(4-x)(3-x)$ 20. $(x-7)(x-1)$. 21. $(a-4b)(a+2b)$
22. $(x+3y)(x-2y)$ 23. $(p-2q)(p+q)$ 24. $(9-x)(8+x)$.
25. $(5x-2)(x-2)$. 26. $2(2x-1)(x-2)$. 27. $(3x-1)(2x-3)$.
28. $(3x-2)(2x+1)$. 29. $(3x-1)(x-2)$ 30. $(x+3)(2x-1)$
31. $(2x-3)(2x-1)$. 32. $(3x+2)(x-3)$. 33. $(3x+2)(x+3)$.
34. $(3x-4)(5x-3)$ 35. $(3x+4)(5x-3)$. 36. $(2x-3)(4x-1)$.
37. $(3-2x)(1-3x)$. 38. $(2-5x)(3-2x)$. 39. $(2x-5)(2x-3)$.
40. $(4x+3)(3x-4)$.

Exercise IX (g) (p. 142)

1. $a^2-4ab+4b^2$. 2. $4x^2-4xy+y^2$. 3. $4x^2-12xy+9y^2$
4. $p^2x^2+2pqxy+q^2y^2$ 5. $a^2x^2-2abxy+b^2y^2$. 6. $4a^2x^2+12abx+9b^2$.
7. $9p^2+12pq+4q^2$ 8. $x^4-2x^2y^2+y^4$. 9. $9p^2-24pq+16q^2$.
10. $a^4+2a^2b^2+b^4$. 11. $a^4-4a^2b^2+4b^4$. 12. $9a^2-6ax^2+x^4$.
13. $x^2y^2-2pqxy+p^2q^2$. 14. $4x^4-12x^2y^2+9y^4$. 15. $9a^2b^2-12abc+4c^2$.

16. $a^6+2a^3b^3+b^6$. **17.** $+1,\ x+1$. **18.** $+1,\ a-1$.

19. $+y^2,\ x-y$. **20.** $+9,\ x+3$ **21.** $+16,\ x-4$

22. $+4y^2,\ x-2y$. **23.** $+1,\ 2x+1$ **24.** $+4y^2,\ 3x-2y$

25. $+16,\ 2x-4$. **26.** $+\frac{1}{4},\ x+\frac{1}{2}$. **27.** $+2\frac{1}{4},\ x-\frac{3}{2}$.

28. $+6\frac{1}{4},\ x+\frac{5}{2}$. **29.** $+\frac{1}{4},\ y-\frac{1}{2}$. **30.** $+\frac{9}{4},\ a^3+\frac{3}{2}$.

31. $x^2+2ax+2cx+a^2+2ac+c^2$ **32.** $x^2-2ax+2cx+a^2-2ac+c^2$.

33. $x^2+y^2+z^2+2xy-2xz-2yz$.

34. $x^2+y^2+2xy+2ax+2bx+2ay+2by+a^2+2ab+b^2$.

35. $x^2+y^2-2xy-2ax+2bx+2ay-2by+a^2-2ab+b^2$

36. $4x^2-12xy+9y^2+4ax+4bx-6ay-6by+a^2+2ab+b^2$.

37. $2x-y$ **38.** $3x+1$. **39.** $a-2b$.

40. $a^2-\dfrac{b^2}{2}$. **41.** $2a-1$. **42.** $a-\dfrac{3b}{2}$.

43. $x+\dfrac{y}{2}$. **44.** $p-\frac{5}{2}$. **45.** $x^2+\frac{3}{2}y^2$.

Exercise IX (h) (p 144)

1. $(x-2y)(x+2y)$ **2.** $(5x-3y)(5x+3y)$ **3.** $(4x-y)(4x+y)$

4. $6(x-y)(x+y)$. **5.** $(1-x)(1+x)$. **6.** $(5a-1)(5a+1)$

7. $2(x-2y)(x+2y)$ **8.** $\left(\dfrac{x}{3}-1\right)\left(\dfrac{x}{3}+1\right)$. **9.** $b^2(7a-b)(7a+b)$

10. $\left(x-\dfrac{y}{2}\right)\left(x+\dfrac{y}{2}\right)$. **11.** $\pi(a-b)(a+b)$.

12. $(x-y)(x+y)(x^2+y^2)$ **13.** $(a^2-2b^2)(a^2+2b^2)$

14. $5(x-2y)(x+2y)$. **15.** $(1-2x)(1+2x)(1+4x^2)$

16. 81. **17.** 89. **18.** 448. **19.** 264. **20.** 9800 **21.** 46640

22. $(x+y+a)(x+y-a)$ **23.** $(2x-2y+a)(2x-2y-a)$.

24. $(2x+y+3z)(2x+y-3z)$ **25.** $(x+y+z)(x-y-z)$.

26. $(x+y-z)(x-y+z)$. **27.** $(1+3a+3b)(1-3a-3b)$.

28. $(2a+b-c)(2a-b+c)$ **29.** $(1-x+y)(1+x-y)$.

30. $(a-b+x+y)(a-b-x-y)$ **31.** $(a+b+x)(a+b-x)$

32. $(x+a+b)(x-a-b)$ **33.** $(x+a-b)(x-a+b)$

34. $(2+x+y)(2-x-y)$ **35.** $(1+x-y)(1-x+y)$.

36. $(a+b-c)(a-b+c)$ **37.** $(1+2a+2b)(1-2a-2b)$

38. $(5x+2y-2z)(5x-2y+2z)$. **39.** $(3a+b)(b-a)$.

40. $4ab$ **41.** $y(2x-y)$. **42.** $x(2-x)$.

43. $4a(b-c)$. **44.** $4a(b+c)$. **45.** $3y(8x-3y)$

46. $(y-z)(2x-y+z)$ **47.** $(4-x-3y)(3y-x-2)$.

48. $5(1+x)(1-x)$ **49.** $x(20-x)$. **50.** $(5x+y)(x+5y)$

51. $(14x+y)(2x+7y)$. **52.** $5xy(x+y)(x-y)$ **53.** $(a-b+y)(a-b-y)$

54. $(2x-1)(3-2x)$. **55.** $4x(3-x)$.

Exercise IX (i) (p. 147)

1. x^2-x+1. 2. $2x^2-x+2$. 3. x^2+ax+a^2.
4. x^2-ax+a^2. 5. a^2+ab+b^2. 6. x^2-ax-a^2.
7. $2a^2+ab-3b^2$. 8. $x+y+z$. 9. $x+b$.
10. $x-z$. 11. $a-b$. 12. $a^4-a^3b+ab^3-b^4$.
13. $a^2+b^2+c^2-ab-bc-ca$. 14. $+\dfrac{1}{x-1}$.
15. $-\dfrac{23}{x+2}$. 16. $-\dfrac{95}{x+3}$. 17. $\dfrac{a^2-ab+b^2}{x-b}$.
18. $(x-1)(x-2)(x-3)$. 19. $(x+2)(x-1)(x+1)$
20. $(x+1)^2(x-2)$. 21. $(x+1)(x-2)(x+2)$.
22. $(x+1)(x-3)(x+3)$. 23. $(x+1)(x+2)(x-3)$.
24. $(x+1)(x+2)(2x+1)$ 25. $(x-2)(x+3)(2x-1)$.
26. $(x+2a)(x+4a)(x-3a)$ 27. $(x-y)(x+y)(x-2y)$.
28. $(x-y)(x^2+2xy+4y^2)$. 29. $(x-2)(x+3)(2x+1)$.
30. -1. 31. -5. 32. -18.
33. $1\frac{1}{3}$. 34. $a=5,\ b=-6,\ (x-3)(x+1)$.

Exercise IX (j) (p. 148)

1. $a(a+b)^2$. 2. $(5p-3)(2q+7)$. 3. $(a^2-b)(c+1)$.
4. $(2a-5)(3b-2)$. 5. $(x-1)(2y-3)$. 6. $(x-\frac{1}{3})(x+\frac{1}{3})$
7. $(a+p)(a+p^2)$. 8. $\left(\dfrac{a}{b}+1\right)^2$. 9. $(x+1)(y-a)$.
10. $(a-1)(a+1)(a^2+1)$ 11. $(x+1)(x^4+1)$. 12. $(9x^9-4y^8)(9x^9+4y^8)$.
13. $(2a-3c)(2a-12b+3c)$. 14. $(ax-by)(bx+ay)$.
15. $a^2(a-5)(a+1)$ 16. $(xy-5)(x+3)$.
17. $3(x^2-2x+8)(x+4)(x-2)$. 18. $(p+q+2r)(p-q-2r)$.
19. $(p-2)(p+2)(p-3)$. 20. $(x+y+1)^2$.
21. $(x-1)(x-3)(x+4)$. 22. $11(3x-y)^2$.
23. $(a-b)(a-3b+2c)$. 24. $(p-q)(p+q)(p-2)(p+2)$.
25. $(p^2+pq-2)(p^2-pq-2)$. 26. $(x-b)(ax-bx-ab-b^2)$.
27. $(x+y)(2x-2y+5)$. 28. $(a+b+x+y)(a-b+x-y)$.
29. $(y-1)(y+1)(y^2+1)(y^4+1)$. 30. $(2a-3b+2c)(2a-3b-2c)$
31. $(4ax-9y)(x+ay)$. 32. $(x-1)(x-2)(2x+1)$.
33. $2(a-1)(a+3)(a+5)$ 34. $(y-x)(xy+1)$.
35. $(x-y)(x-y-1)$. 36. $4\pi\left(2a+\dfrac{b}{3}\right)\left(2a-\dfrac{b}{3}\right)$.
37. $(a+3b+7)(a-2b-8)$. 38. $(1+p-q)(1-p+q)$.
39. $(x-1)(2x+1)(x-1)$. 40. $(x-2y)(x+2y)(9x^2+4y^2)$.
41. $(x-1)(3x^2-5x-6)$. 42. $(x-y)(x^2+xy+y^2)$.

43. $(x+y)(x^2-xy+y^2)$.
44. $(4x-5y)(5x+4y)$.
45. $(a-b)(a-b+c)(a-b-c)$.
46. $(a^2+b^2)(1-abc)$.
47. $3x(x-7)(x+9)$.
48. $2a(a-b)(a+b)(b+x)$.
49. $(b-a)(5x-2a-3b)$.
50. $\frac{1}{x}(x^2-2)(x^2+2)$.

Exercise X (a) (p 150)

1. $a+b$; $b(a^2-b^2)$.
2. $2(x-a)$; $4(x^2-a^2)$.
3. $2(x+y)$; $12x^2(x^2-y^2)$.
4. a^2-ab; $ab(a^2-b^2)$
5. $(x-3)$, $(x-4)(x-3)(x+1)(x+2)$.
6. $a+2x$; $a(a^2-4x^2)$.
7. $x+1$; $x(x+1)^2$.
8. $x+1$; $x(x^2-1)$.
9. $(p-1)$; $(p-1)(2p+1)(3p+2)$.
10. $(x-a)$; $(x-a)(x+a)(x-b)$.
11. $x+2$, $(x+2)^2(x-1)$.
12. x^2, $x^2(x-1)(x+1)^2$.
13. $(x-2)$, $(x+1)(x-2)(x-4)$.
14. $(a-b)$; (a^2-b^2).
15. $a-b$, $4(a-b)^2(a+b)$.
16. $2a+b$; $b^2(2a-b)(2a+b)^2$.
17. $x-1$; $x(x-1)^2(x+1)$.
18. 2; $4(a^2-b^2)$.
19. $x+2$; $3(4-x^2)$.
20. $2(a-b)^2$; $12a^2(a-b)^3(a+b)^2$.

Exercise X (b) (p 151)

1. $\frac{a}{b}$
2. $\frac{a+b}{a-b}$.
3. $\frac{2a+3x}{3a+2x}$.
4. $\frac{x+1}{x-2}$.
5. $\frac{x-y}{x+y}$.
6. $\frac{2a}{3b}$.
7. $\frac{xy-2z}{xy+2z}$.
8. $3x^2-2$.
9. $\frac{2a+b}{x}$.
10. $\frac{a^2}{a+b}$.
11. $-\frac{a+3}{a+4}$.
12. $\frac{2x+1}{2x+3}$.
13. -1.
14. $-(a+b)$.
15. $-2(x+y)$.
16. $-\frac{5a}{x+a}$.
17. $\frac{-2a}{2+a}$.
18. $\frac{x-5}{x+1}$.

Exercise X (c) (p. 152)

1. $\frac{ab}{a^2+4b^2}$.
2. $\frac{a^2+b^2}{(a+b)^2}$.
3. 2.
4. $\frac{a+9}{a+2}$.
5. $\frac{x+2}{x-1}$.
6. $\frac{5a+b}{x(3a+2)}$.
7. $\frac{xy}{2x-1}$.
8. 1.
9. 1.
10. 1.

Exercise X (d) (p. 153)

1. $\frac{2a}{a^2-b^2}$.
2. $\frac{2b}{a^2-b^2}$.
3. $\frac{2x+y}{x}$.
4. $\frac{y}{x}$.
5. $\frac{a+b}{2}$.
6. $\frac{1}{6(a-b)}$.
7. $\frac{1}{x^3+x^2}$.
8. $\frac{a^2}{a+b}$.

9. $\dfrac{2x+y}{a(x+y)}$. 10. $\dfrac{x^2+y^2}{x^2-y^2}$. 11. $\dfrac{y}{x-y}$. 12. $\dfrac{ab}{a-b}$.
13. $-\dfrac{b}{b-a}$. 14. $\dfrac{a^2+b^2}{a^2-b^2}$ 15. $\dfrac{a^2-ab+b^2}{a^2-b^2}$. 16. $-\dfrac{y}{x+y}$.
17. $\dfrac{p-q}{pq}$. 18. $\dfrac{5}{(x-2)(x+3)(x+4)}$. 19. $\dfrac{4}{x+2}$.
20. $\dfrac{p+q}{pq}$. 21. $\dfrac{a^3}{(a-2b)^2}$. 22. $\dfrac{x-4y}{2(x-2y)^2}$. 23. $\dfrac{x+2}{x-3}$
24. $\frac{1}{2}$. 25. $\dfrac{2a}{q}$. 26. 0. 27. $\dfrac{2x}{(x+y)}$.
28. $\dfrac{3(a^2+x^2)}{a^2-x^2}$. 29. $\dfrac{8a^2x}{(x^2-a^2)^2}$. 30. $\dfrac{7+a^2}{12(1-a^4)}$ 31. $\dfrac{y}{x+y}$.

Exercise X (e) (p. 155)

1. $\frac{1}{6}$. 2. $\dfrac{1}{ax}$. 3. 1. 4. $\dfrac{2x}{(x+1)(x-1)^2}$. 5. -1
6. $\dfrac{x^2}{x-a}$. 7. $-\dfrac{x+4}{x+2}$. 8. 8 9. $-(p+q)$. 10. $\dfrac{1}{x+y}$.
11. $-(a+b)$. 12. $\dfrac{3x^2-2xy}{x^2-y^2}$ 13. $\dfrac{-3bx}{(3x-b)^2}$. 14. $\dfrac{4}{b-a}$.
15. $\dfrac{2a-4b}{b}$. 16. $\dfrac{3a+4b}{6a+4b}$. 17. $\dfrac{x+2}{6-x}$. 18. $\dfrac{12}{x^2-9}$.
19. $\dfrac{1}{(x+1)(x+2)}$. 20. 0 21. $\dfrac{2}{a^2-4b^2}$. 22. $2x^2-11x+14$
23. 1. 24. $\dfrac{x^2(1+x^2)}{1-x^2}$. 25. 0. 26. $\dfrac{2b}{b+c-a}$
27. $\dfrac{1}{x}$. 28. $\dfrac{b}{a+b}$ 29. $\dfrac{x-a}{a+b}$. 31. x.

Exercise X (f) (p. 158)

1. $-3\frac{1}{3}$. 2. $2\frac{1}{2}$. 3. 4. 4. 0 5. -2.
6. -5. 7. 0. 8. 2. 9. 9. 10. -5.
11. 4. 12. 10. 13. 10. 14. Identity. 15. $-2\frac{1}{2}$.

Exercise XI (a) (p. 160)

1. ± 4. 2. ± 1. 3. 3 or -1. 4. -1 or 5. 5. $a-b$ or $a+b$.
6. $\pm a$. 7. $1\frac{1}{3}$ or -2. 8. 2 or 0. 9. ± 1. 10. ± 5.

Exercise XI (b) (p. 162)

1. $a, \; b$ 2. $1, \; 2$. 3. $1, \; -3$ 4. ± 1.
5. $1, \; 3$ 6. $0, \; -3$ 7. $-1, \; +\frac{2}{3}$. 8. 4 or $-2\frac{1}{2}$.
9. $0, \; 2a$ 10. 2 or $-9\frac{1}{2}$ 11. $\frac{3}{4}$ or $-\frac{4}{5}$. 12. $1\frac{5}{6}$ or -3.
13. $2\frac{1}{3}$ or -9. 14. 0 or $-2\frac{4}{9}$. 15. 0 or $\frac{2b}{3}$. 16. 0 or $\frac{5a}{2}$.
17. $1\frac{1}{3}$ or -3. 18. 2 or -12. 19. ± 2. 20. $\frac{1}{3}$ or $-1\frac{1}{4}$
21. p o. q. 22. $-\frac{b}{a}$ or $+\frac{c}{a}$ 23. $x^2-5x+6=0$ 24. $x^2-4x-5=0$.
25. $3x^2-23x+14=0$ 26. $20x^2+19x+3=0$ 27. $2x^2-7x=0$.
28. $x^2-2(a+b)x+4ab=0$. 29. $x^3-6x^2+11x-6=0$ 30. $x^3-x^2-2x=0$.
31. $-\frac{101}{90}$. 32. $\frac{31}{57}$. 34. $2, \; -1, \; -2$.
35. $-1, \; +2\frac{1}{3}$. 36. $4, \; -\frac{2}{3}$. 39. $-1, \; 2\frac{1}{2}$.

Exercise XI (c) (p. 165)

1. 9 ; $x+3$. 2. $2\frac{1}{4}$; $x+\frac{3}{2}$. 3. $6\frac{1}{4}$; $x-2\frac{1}{2}$ 4. $\frac{1}{9}$, $x+\frac{1}{3}$.
5. $+\frac{1}{16}$; $x-\frac{1}{4}$ 6. $\frac{4}{25}$, $x+\frac{2}{5}$. 7. 36 , $2x+6$ 8. 1 , $3x-1$.
9. 1 ; $2x-1$. 10. $\frac{a^2}{4}$, $x+\frac{a}{2}$ 11. 1 or -7. 12. 8 or -3
13. -2 or $\frac{4}{3}$. 14. $-2·82$ or $-0·18$ 15. $2·08$ or $0·59$
16. $1·98$ or $-1·48$ 17. $0·77$ or $-1·44$ 18. $1·82$ or $-0·82$
19. Roots unreal 20. $1·39$ or $0·36$ 21. $-2·12$ or $0·79$
22. $+0·15$ or $-1·65$ 23. $0·4$ or $-0·5$ 24. $3·83$ or $0·17$
25. $1·72$ or $-0·39$ 26. $0·83$ or $-4·83$ 27. $0·87$ or $0·13$
28. $6·30$ or $-9·30$. 29. $1·61$ or $-5·61$. 30. $1·4$ or $2·5$
31. 11 or -13.

Exercise XI (d) (p 168)

1. $-3·53, \; +4·53$; $x^2-x-16=0$
2. (i) $-1·69$ or $0·44$, (ii) $-2·84, \; +1·59$; (iii) $-2·17, \; +0·92$, (iv) -2 ; (v) $-2·33, \; +1·08$.
3. (i) $-1·26, \; +2·76$, (ii) $-1·33, \; +2·83$.
4. (i) $2·62, \; +0·38$, (ii) $-1·19, \; +4·19$; (iii) $4·85$ or $-1·85$.
5. $-0·39, \; +1·72$. 6. $-3·92, \; +2·42$ 7. $0·84, \; 4·16$
8. $0·80, \; -1·55$. 9. $1·47$. 10. 2 or $0·67$.
11. $0·31, \; 1·29$. 12. $-0·36, \; -4·14$. 13. $3·79$.
14. Between -1 and $+2$. 15. $x>2$ or $x<-1·5$.

Exercise XI (e) (p. 170)

1. 3 or -11. 2. 3 or $-\frac{1}{4}$. 3. $2\frac{1}{3}$ or -7
4. $1·39$ or $-2·89$. 5. 5 or $3·6$. 6. $-1·5$ (twice).

7. 14, 29. **8.** 5 cm., 7 cm. **9.** 4 cm., 9 cm.
10. $3\frac{1}{2}$, $4\frac{1}{2}$. **11.** 3 or -6. **12.** 30 yd.
13. 8 ft. **14.** 16 yd, 10 yd., 10 yd. **15.** ?8 yd.
16. 3 or -5 **17.** 4.2 p.m. **18.** 8 57 cm., 1 43 cm.
19. 5 86 in, 4 14 in. **20.** 4 51 cm., 3 49 cm. or $-1{\cdot}18$ cm, 9 18 cm.
21. 2·07 cm. **22.** $6\frac{2}{3}$, $10\frac{2}{3}$. **23.** 20 yd., 80 yd
24. 16 miles per hour. **25.** 40. 60
26. 10 pence, 15 pence. **27.** 8d, 10d
28. 12 min, 20 min. **29.** 1 90 yd
30. $-\frac{1}{4}$ and $\frac{1}{2}$ or $-\frac{3}{2}$ and $-\frac{3}{4}$ **31.** 1' in
32. 6 miles per hour.

Exercise XI (f) (p 175)

1. $(-3,\ 7)$ or $(3\ 4,\ -5{\cdot}8)$. **2.** $(5,\ -2)$ or $(-2,\ 5)$.
3. $(-3,\ 4)$ or $\left(\frac{25}{9}, \frac{56}{27}\right)$. **4.** $(2,\ 0)$.
5. $(12,\ 6)$ or $(-9,\ -8)$. **6.** $(-1\frac{1}{5},\ -1\frac{2}{5})$ or $(1,\ 2)$.
7. $(6\frac{1}{2},\ -\frac{1}{4})$ **8.** $(3,\ -2)$ or $(-2,\ 3)$.
9. $(2,\ -3)$ or $(-1\ 98,\ +2\ 31)$. **10.** $(2,\ 4)$ or $(-5\ 2,\ -0{\cdot}8)$.
11. 5. **12.** -3. **13.** $2\frac{1}{2}$ **14.** 9. **15.** 7.
16. 3 6. **17.** $-1\frac{1}{3}$ or $+2\frac{1}{2}$ **18.** $-\frac{1}{3}$ or -1. **19.** -2 or 4

Exercise XI (g) (p. 176)

1. $\frac{7\pm\sqrt{109}}{10}$. **2.** $\frac{4\pm\sqrt{6}}{2}$. **3.** $\frac{-2\pm\sqrt{11}}{7}$. **4.** $\frac{-1\pm\sqrt{28}}{3}$.
5. $\frac{-11\pm\sqrt{73}}{8}$. **6.** $\frac{-5\pm\sqrt{73}}{8}$. **7.** $\frac{5\pm\sqrt{33}}{4}$. **8.** $\frac{-7\pm\sqrt{193}}{12}$.
9. 1 and $\frac{2(a-b)}{a}$. **10.** $\frac{2p}{q}$ and $\frac{-2q}{p}$ **11.** $-a$ and $\frac{b}{a}$.
12. $4(x-0{\cdot}31)(x+2\ 44)$. **13.** $2(x-2{\cdot}69)(x+0{\cdot}19)$

Miscellaneous Exercises IV (p. 178)

A 61

1. $2x+y-4z$. **2.** ±27, ±216. **3.** (i) $2(bx-ay)$; (ii) $(x^2-a^2)(y^2-b^2)$.
4. $(5x+7)(2x+3)$, $(5x-3)(2x+7)$, $10(x-3)(x+3)$.
5. (i) $1\frac{4}{45}$; (ii) 2·17 or $-2{\cdot}26$. **7.** (i) $\frac{a^2b}{b-a}$; (ii) $\frac{x-2}{2x(x-1)}$.

A 62

1. $\frac{1}{12}\{19a-29b+32c\}$. **2.** $4a^8b^6$, $\pm 4x^8$, $-27x^6$.
3. H.C F $2(x-2)$, L.C.M. $12(x-2)^2(x+2)(x+3)$. **4.** 26, -17.
6. $2x^2+3x-9=0$. **7.** (i) 0, $2\frac{1}{2}$; (ii) 3·61 or -1 11.

A 63

1. 4, 1. **2.** $6x^4-21x^3+35x^2+6x-16$ **3.** $\dfrac{2(x+y)}{3}$, $\dfrac{x}{3y}$, $\dfrac{2(x+y)}{x(x-2y)}$.
4. $(x-7)(2x+5)$, $x^2(x-1)(x+1)$, $(2x+yz)(3x-yz)$. **6.** 6 56, 2 44.

A 64

1. $5^2+14^2=13\times 17$ **2.** $x+3$. **3.** $a=13$, $1\frac{1}{3}$.
4. (i) $\dfrac{8(x+2y)}{4x^2-y^2}$, (ii) $\dfrac{2x-1}{3x-5}$ **5.** (i) $(x+2y)(x-3z)$, (ii) $(x+2)(x^2+5)$.
6. 1 37 or 14 63.

A 65

1. (i) $6x^3y-9x^2y^2-10xy^3+15y^4$, (ii) $2x^2-7x+2$.
2. $-\frac{3}{4}$, -3, 1, $-4\frac{1}{2}$ **3.** $-\frac{5}{6}$.
4. $(a-2c)(b-2c)$ sq ft, $2c(a+b-2c)$ sq ft, $\dfrac{2cx(a+b-2c)}{9}$ sh.
5. 16, $12\frac{1}{4}$, $1\frac{9}{10}$; $x+4$, $x-\frac{7}{2}$, $2x+\frac{5}{4}$. **6.** $8\frac{1}{2}$.

A 66

1. $a=2$, $b=3$, $c=18$. **2.** $4b^2$. **3.** 1 8 in and 2 4 in.
4. $\pm 6x^{18}$, $2x^2$, $-32x^{15}y^{10}$, $-\dfrac{x^2}{y^3}$ **5.** $a=2$, $b=3$, $x=5$, $y=-1$.
6. In the 5th year.

A 67

1. (i) 7, (ii) -43 **2.** (i) -1, (ii) -1; (iii) $\dfrac{4x-3}{2x-1}$.
3. (i) $2\frac{1}{2}$ or -7; (ii) 0, -1, $+2$.
4. $3x^2+5x-2$. **5.** (6, 8) or ($\frac{6}{7}$, $-1\frac{1}{7}$).

A 68

1. $\dfrac{a^4}{16b^2}+1+\dfrac{4b^2}{a^4}$, $4a^2+b^2+9c^2-4ab+6bc-12ac$, $8xy$. **2.** $a-2b$.
3. -1. **4.** (i) $2(p-q)^2$; (ii) $(2p+3r)(p-2q)$; (iii) $4(p-r)(p-q+r)$.
5. 6 cm., 12 cm.

A 69

1. $-8a^9b^6$, $\pm 2x^8$, x^3p, 292 **2.** -10 **3.** $(3x-4y)(3y-2x)$

4. 3, $\frac{2}{3}$ **5.** (i) $(x+y)(x-y-2a)$, (ii) $(a-2b-c)(a-2b+c)$

6. $2xz=ab$.

A 70

1. $9a^2$, $3a^2$, $16-8a$ **2.** 912 673

3. $\dfrac{2}{x^2-4}$, $\dfrac{1}{x-2}$, $\dfrac{x+4}{2(x^2-4)}$, $\dfrac{x-4}{2(x-2)^2}$ **4.** $\dfrac{100(y-x)}{xn}$

5. $\dfrac{x-a}{1-ax}$, $\dfrac{-2a}{1+a^2}$ **6.** (15, 18)

A 71

1. $4a^2-28ab+49b^2$, $3x-5$, $3a-7b-5c$

2. $3p(a+5b)-2q(a+5b)$, $(3p-2q)(a+5b)$

3. (i) $\dfrac{a+2b}{ab(a-2b)}$, (ii) $-\dfrac{2b}{a}$ **4.** 4 24 and -0 24 **5.** 41, 43.

6. 24 miles per hour.

A 72

1. $(x-3)(2x+1)(3x-1)$ **2.** 79 6 **3.** (i) -3 or $-\frac{4}{7}$, (ii) 8 or $-1\frac{4}{5}$

4. $\dfrac{a+b}{a-b}$, $-\dfrac{a}{c}$, $a-b$, $\dfrac{b-a}{b+a}$. **5.** 9 6 in, 5 2 in, 5 2 in.

6. When not between 2 and $4\frac{1}{4}$.

A 73

2. 2 **3.** $\dfrac{cF}{P}-d$ **4.** 0, -1 56, 2.56

5. $\dfrac{100(b-a)+bx}{a}$. **6.** $\dfrac{(u+3v)h}{4(u+v)}$, $\dfrac{4b-h}{3h-4b}$.

A 74

1. $p+2q-r$, $p^2-4q^2+4qr-r^2$

2. (i) $(x-y)(x+y-1)$; (ii) $(x+2y+3)(x-2y-3)$. **3.** $\dfrac{x-11}{20(x^2-1)}$.

4. 3, $-2\frac{1}{2}$; $x^2-x=30$. **5.** 10 ft. by 5 ft.; 8 ft. by 4 ft.

6. $\dfrac{100(b-a)}{a}$ per cent.; $\dfrac{100(b-a)x}{a(a+x)}$ per cent. less.

A 75

2. $a^2+2ab+4b^2$. **3.** -16; 3, 0 19, -2 69 **4.** 1 and -2.
5. 2.37½ p m. **6.** 9.

B 76

1. $(x+4)^2$. **2.** (i) 0, (ii) 3 or $-1\frac{1}{2}$ **3.** $1\frac{1}{4}$.
4. $(2pq)/(p^2-q^2)$. **5.** $r=a/6$. **6.** 8, 15.

B 77

1. $-8x(3x+5y)$ **2.** 4, $\frac{1}{3}$ or $-\frac{5}{3}$ **3.** q^2-p^2
4. (i) 46, (ii) -12 or $-\frac{2}{3}$ **6.** 4th year, 7th year, 7th year

B 78

1. $-\frac{5}{8}$. **2.** (i) $-\dfrac{1}{xy}$, (ii) $-\dfrac{3y}{x}$ **3.** $a+2$, $b-2$.
4. $A=1\frac{2}{7}$, $B=-2\frac{2}{7}$. **5.** (i) 4, (ii) -36, (iii) $+4$.
6. 75 things at £6 each

B 79

1. $+7$, 0 81 and 0 026 **2.** (i) $6x^2-11x-7$, (ii) $x^3-19x+30=0$
3. 20 ft, 4 ft. per sec 3 sec., 6 ft.
4. $x=\dfrac{1-ab}{1+ab}$, $y=\dfrac{2b}{1+ab}$ **5.** 4½d. per lb.
6. (i) between $-1\frac{1}{2}$ and $+\frac{1}{2}$; (ii) 1 23 and -2 23.

B 80

1. $y=\dfrac{8x-z}{6}$, 5·6. **2.** 3 or -2 **3.** $\dfrac{4-2x}{3-x}$. **4.** 6
5. $4b(a-b)(a+b)$ **6.** 17 3 miles, 15 6 miles per hour.

B 81

1. $(2x+1)(2x-5)(x-1)(x-3)$, $-\frac{1}{2}$, 1, $2\frac{1}{2}$, 3 **2.** -0 64, $+3$ 64.
3. $\dfrac{x}{y}+1+\dfrac{2y}{x}$. **4.** $13x+8$. **5.** 0 64, -3 14.
6. 50 miles per hour.

B 82

1. $2(pr-q^2)$; $\dfrac{q+r}{q-r}$; $(p^2-q^2)(q^2-r^2)$; q^2-p^2.
2. $x=-2$, $y=+3$ **3.** $n(n+1)$; 28th and 29th **4.** $(ac+bd)(ad+bc)$.
5. $\dfrac{8x^3}{1-2x}$, (i) 1 24 and 01; (ii) 7 and -8 **6.** 4 83 in. and 6 83 in.

B 83

1. H.C.F. $3a^2$: L.C.M. $180\ a^3b^3c^2$.

2. $\dfrac{100y}{100-x}$; $\dfrac{(100+x)y}{100}$. 3. (i) $\dfrac{a+4}{a-2}$; (ii) $\dfrac{4}{(x-2)^2(x+2)}$.

4. (i) $27x^3-54x^2y+36xy^2-8y^3$; (ii) 1061208. 5. 2. 6. 10.

B 84

1. (i) $\frac{5}{9}$; (ii) $\frac{5}{3}$; (iii) no solution (other than $x \to \infty$).

2. $12a^3b^2(a^2-4b^2)^2$. 3. $\frac{4}{3}$. 4. 18.

5. $x=4\ 5$, $h=3\ 968$, area $=9\ 92$ sq cm. 6. A semicircle : 3.

B 85

1. $x^4+6x^3+5x^2-12x+4$. 2. $(3x-2y)(a-2b)$; $(a-b)(a+b-2c)$.

3. $\dfrac{ab}{c}$, $\dfrac{(c-a)(c-b)}{c}$. 4. 3, -2; $+2$, $+24$. 6. 24 min.

B 86

1. $(1-x)^2$. 2. $\dfrac{1}{b}$. 3. $a=-19$, $b=-30$; -3.

4. $\dfrac{Q^2v}{12\ 96}$, 14,400. 5. 48 sh.

6. (i) 0 91 or $-1\ 91$; (ii) when $x=-0\ 5$.

B 87

1. x^7y^{15}. 2. $\dfrac{4m-3}{3m+4}$. 3. 1 or $-1\frac{1}{4}$.

4. $\dfrac{100a}{200+x}$, $\dfrac{(100+x)a}{200+x}$. 5. $\dfrac{13x^2-x+4}{9}$,

x	2	-1 92
[illegible]	1	$-1{\cdot}61$

6. 1·26 in.

B 88

1. 5. 2. (i) $1-3x+6x^2-\dfrac{20x^3}{1+3x}$; (ii) $-645\ 2$, $-\ 00002$.

3. $(x+3q)(x-4p)$; $x(x+2p)(x-2p)$

4. Outwards 40 miles ; return 20 miles. 5. $4t^4$, $\pm 4t^4$.

6. Nearly 21 ft.

B 89

1. $a=9$, $b=-25$. 2. $-\frac{3}{2}$, $-\frac{3}{2}$. 3. $\frac{2}{3}$ and $\frac{1}{2}$; $36x^2-48x+7=0$.

4. $(n+2)^3$. 5. $\frac{2}{29}$. 6. a^2/b.

B 90

1. $pq(p+q)(1-pq)$.
2. (i) $(2x-y+5)(2x-y-1)$; (ii) $(x+y-4)(x+y+1)$.
3. (i) $4a$; (ii) $a^2+\frac{ap}{2}$.
4. $\pm\frac{ar}{\sqrt{a^2-r^2}}$, $r=\pm\frac{1}{2}$.
5.

x	4	0
y	-3	5

6. $76\frac{2}{3}$ miles.

B 91

1. (i) $(3a+3b+1)(a+b-2)$; (ii) $4x(2y-3z)$.
3. $b^2S^3=a^3V^2$
4. $\frac{22(a-b)st}{15(s-t)}$ ft; $\frac{as-bt}{s-t}$ miles per hour.

Exercise XII (a) (p. 199)

1. $x=2,\ y=-3,\ z=1$.
2. $x=3,\ y=+2,\ z=-4$
3. $x=-11,\ y=3,\ z=4$.
4. $x=6,\ y=0,\ z=-3$
5. $x=-4,\ y=5,\ z=3$.
6. $x=10,\ y=9,\ z=-6$.
7. $A=12,\ B=3,\ C=-2$.

Exercise XII (b) (p. 200)

1. $9a^2+4b^2+c^2+12ab+4bc+6ca$.
2. $4a^2+b^2+9c^2-4ab+6bc-12ca$.
3. $4x^4-12x^3+13x^2-6x+1$.
4. $p^4+q^4+r^4-2p^2q^2-2q^2r^2+2r^2p^2$
5. $x^4-4x^3y+10x^2y^2-12xy^3+9y^4$.
6. $a+b-c$.
7. $3x+y+2z$.
8. x^2-3x+4.
9. $x^2+3xy-2y^2$.
10. $2a-3b-2c$.

Exercise XII (c) (p. 202)

1. $8x^3-12x^2y+6xy^2-y^3$.
2. $27a^3+54a^2b+36ab^2+8b^3$
3. $8x^3+12x^2+6x+1$.
4. $125x^3+150x^2y+60xy^2+8y^3$.
5. $27x^6-27x^5+9x^4-x^3$.
6. $x^6+6x^3+12+\frac{8}{x^3}$.
7. $(5x+3y)(25x^2-15xy+9y^2)$.
8. $(2x-1)(4x^2+2x+1)$.
9. $(a-10b)(a^2+10ab+100b^2)$.
10. $3x(x-2y)(x^2+2xy+4y^2)$.
11. $(x^4+y^4)(x^8-x^4y^4+y^8)$.
12. $(3ab+2c)(9a^2b^2-6abc+4c^2)$
13. $(x^2+x+1)(x^2-x+1)$.
14. $(x^2-2x+3)(x^2+2x+3)$.
15. $(3x^2-x-1)(3x^2+x-1)$.
16. $(a^2-3ab+b^2)(a^2+3ab+b^2)$.
17. $(a^2+ab+2b^2)(a^2-ab+2b^2)$
18. $(x^2+2x+2)(x^2-2x+2)$.

Exercise XII (d) (p. 203)

1. $(x-y+z)(x^2+y^2+z^2+xy+yz-zx)$.
2. $(2a+b+3c)(4a^2+b^2+9c^2-2ab-3bc-6ca)$.
3. $(x+2y-10z)(x^2+4y^2+100z^2-2xy+20yz+10zx$
4. (i) $(a^3+b^3)=(a+b)(a^2-ab+b^2)$; (ii) $(a^3-d^3)=(a-d)(a^2+ad+d^2)$.
8. $(x+3y-2)(2x-y+1)$. **9.** $(3x+2y-1)(x-y-1)$.
10. $(4x-6y+3)(x+2y-2)$.

Exercise XII (e) (p. 204)

4. 0. **5.** 0. **6.** -1. **7.** $\dfrac{2(p-r)}{(p+q)(q+r)}$.

Exercise XII (g) (p. 207)

1. $\dfrac{x^3+y^3}{x^2y^2}$. **2.** $\dfrac{x^3-8}{x^2}$. **3.** $\dfrac{-2b^3}{a^4+a^2b^2+b^4}$. **4.** $\dfrac{x^2-y^2}{x}$.
5. 0. **6.** $\dfrac{2x^2+7x+2}{(1+x^3)(1+x)^2}$ **7.** $a-2b-c$. **8.** $x^2-\dfrac{x}{2}+\dfrac{1}{2}$.
9. $(3x^2-2xy-2y^2)(3x^2+2xy-2y^2)$ **10.** $(2x^2+xy+3y^2)(2x^2-xy+3y^2)$.
11. $(x-3y+2z)(x+2y-3z)$ **12.** $(3x-2y+1)(x+2y-3)$
13. $(a-1)(b+1)(a+b)(a-b+1)$.
14. $(4p-3q-r)(16p^2+9q^2+r^2+12pq-3qr+4rp)$.
15. $\left(2x-\dfrac{y}{z}\right)\left(4x^2+\dfrac{2xy}{z}+\dfrac{y^2}{z^2}\right)$ **16.** $(a^2+b^2)(a^4-a^2b^2+b^4)$
17. $z^2-2,\ z^3-3z,\ z^4-4z^2+2$.

Miscellaneous Examples V (p. 208)

B 92

1. (i) $\dfrac{1}{1-a^2}$; (ii) $\dfrac{4}{(x-2)^2(x+2)}$. **2.** (i) $-1,\ -\frac{1}{2},\ +1$; (ii) $-\frac{1}{2},\ -1\frac{7}{9}$.
4. 23, 31. **5.** 1 62 or -0 62. **6.** $\pi\left(\dfrac{a-b}{2}\right)^2$.

B 93

1. $-\frac{1}{3},\ 2\frac{1}{2}$; $2\frac{1}{6},\ -\frac{5}{6}$. **2.** $\dfrac{2x+3}{(x+1)(x+2)}$. **3.** $3(x-y)(y-z)(z-x)$.
4. $\dfrac{a}{2}\pm\dfrac{a}{b}$. **5.** $b>a$ **6.** $\dfrac{60Dv}{60D-Mv}$.

B 94

1. (i) $3x^2(1-3x)(1+3x)$; (ii) $(3x-z)(2x-5y)$. **2.** $\frac{11}{18}$.
4. 5·1 cm. and 10·2 cm. ; or 9 2 cm. and 18 4 cm.
5. $\dfrac{2n(n+1)}{2n^2+2n+1}$, 0·181. **6.** $\dfrac{hb+ax-bx}{h}$.

B 95

1. $2x-3$. **2.** (i) 0, (ii) 1, (iii) $(2q-p)^5$. **3.** 6·08
4. (i) $m=3$; (ii) $x=-2\frac{1}{3}$. **5.** 5, 9. **6.** $5x^2+300x$; 4 35 in.

B 96

1. $\dfrac{a-c}{1+ac}$. **2.** (i) $(a-c)(4a-7b+4c)$; (ii) $(1+a-b)(1-a+b)$.
3. $\dfrac{pq}{p+q}$ **4.** (i) 0 ; (ii) $1\frac{1}{3}$. **5.** $\dfrac{x+5}{6}$, $\dfrac{5x+55}{36}$, 25.
6. 387 8, 0 96 cm.

B 97

1. $6(x-1)$ **2.** $\left(\dfrac{100+b}{100+a}\right)x$. **5.** $\dfrac{a+b-c}{2}$, $\dfrac{b+c-a}{2}$, $\dfrac{c+a-b}{2}$.

B 98

1. (i) $a=\dfrac{vz}{v+b}$; (ii) $a=\dfrac{p^2-b}{l}$; (iii) $a=c+\dfrac{bc}{\sqrt{m}}$.
4. Between +0·5 and −3. **5.** $-\dfrac{y}{x}$. **6.** Ax^2 sq. miles ; $\dfrac{Ax^2}{y^2}$ sq. in.

B 99

1. 1 32. **2.** (i) $-x^m(px+q)$; (ii) $-(q-p)^2$. **4.** $\frac{2}{3}$R.
5. $\dfrac{bx+ay+12a}{a+b}$ a.m. **6.** $a=\frac{1}{6}$, $b=\frac{1}{2}$, $c=\frac{1}{3}$.

B 100

1. (i) $S=\dfrac{V}{h}+2\sqrt{\pi hV}$; (ii) $S=\pi r^2+\dfrac{2V}{r}$. **3.** $4\frac{1}{2}$, -4 ; $-\frac{1}{2}$.
6. 9 miles ; $7\frac{1}{2}$ miles per hour.

B 101

1. 59 hours 59 minutes. **3.** $x=4$, $y=4$.
4. (i) P is 2 in. beyond B ; (ii) P is 1 in. from A and between A and B.
6. $x=-2-\varepsilon$, $y\to-\infty$; $x=-2+\varepsilon$, $y\to+\infty$.

B 102

1. (i) $\frac{20x}{y}$, (ii) $\frac{20y}{x}$, (iii) $\frac{240x+12y}{z}$.

2. -1, -6

3. $\frac{-ab}{a+b}$.

4. 6, 3, 2, -1.

5. 7

6. -1, $+\frac{1}{2}$, $+$

B 103

2. $b=64{,}000$, $a=-1600$.

3. $\frac{2s(s-a)}{bc}$, $\frac{2(s-b)(s-c)}{bc}$.

4. $\pm\sqrt{q^2-4pq}$

5. 8

6. $A=\pi\left(5+\frac{x}{8}\right)$

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